Geometry & Topology

Volume 28 (2024)

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Geometry & Topology **28**:7 (2024) 3001–3022
DOI: 10.2140/gt.2024.28.3001
Published: 25 November 2024

Holomorphic 1–forms on the moduli space of curves

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Since the 1960s it has been well known that there are no nontrivial closed holomorphic 1–forms on the moduli space $\mathcal{M}_g$ of smooth projective curves of genus $g > 2$. We strengthen this result, proving that for $g \geq 5$ there are no nontrivial holomorphic 1–forms. With this aim, we prove an extension result for sections of locally free sheaves $\mathcal{F}$ on a projective variety X. More precisely, we give a characterization for the surjectivity of the restriction map $\rho_D\colon H^0(\mathcal{F}) \to H^0(\mathcal{F}|_D)$ for divisors D in the linear system of a sufficiently large multiple of a big and semiample line bundle $\mathcal{L}$. Then we apply this to the line bundle $\mathcal{L}$ given by the Hodge class on the Deligne–Mumford compactification of $\mathcal{M}_g$.

14H15; 32L10

Introduction

Let X be a n–dimensional smooth irreducible projective variety defined over an algebraically closed field $\Bbbk$. We will say that a vector bundle $\mathcal{F}$ over X is *liftable with respect to a line bundle* $\mathcal{L}$ (or $\mathcal{L}$*–liftable* in short) if there exists a positive integer m_0 such that the restriction map

$$\rho_D\colon H^0(\mathcal{F}) \to H^0(\mathcal{F}|_D)$$

is an isomorphism for any divisor $D \in |\mathcal{L}^m|$ and for $m \geq m_0$ (see Definition 1.2). Surjectivity for m large enough is not guaranteed in general: further positivity assumptions on $\mathcal{L}$ are needed. For instance, $\mathcal{F}$ is $\mathcal{L}$–liftable as soon as $\mathcal{L}$ is ample, by Serre's criteria of vanishing and duality. One can furthermore relax this up to $(n-2)$–ampleness (see [Sommese 1978b, Definition 1.3]). The first intent of this paper is to characterize $\mathcal{L}$–liftability for big and semiample line bundles. We recall that $\mathcal{L}$ is semiample if, for some suitable $d > 0$, $\varphi_{|\mathcal{L}^d|}\colon X \to \mathbb{P}H^0(\mathcal{L}^d)^* = \mathbb{P}^N$ is a morphism. Furthermore, $\mathcal{L}$ semiample is $(n-2)$–ample if it has no divisors contracted to points by $\varphi_{|\mathcal{L}^d|}$. We will show that, in the general case, the divisors contracted to points play a crucial role in Theorem 1.3, which can be stated as:

Theorem A *Let $\mathcal{L}$ be a big and semiample line bundle on X and let E be the divisor of X contracted to points by $\varphi_{|\mathcal{L}^d|}$ for d large enough. Then a locally free sheaf $\mathcal{F}$ on X is $\mathcal{L}$–liftable if and only if, for all $m > 0$, the maps $H^0(\mathcal{F}) \to H^0(\mathcal{F}(mE))$ are surjective.*

The proof uses the theorem of formal functions [Hartshorne 1977, III.11]. This theorem holds for an algebraically closed field, while the other result that we will present from Section 2 onward will essentially be over the complex numbers. It is not surprising then that in the classical case, that is, when $\Bbbk$ is the field of complex numbers, the above statement translates into a sort of concavity result. We borrow the terminology from complex analysis and geometry (see for example [Andreotti 1963; Andreotti and Grauert 1962; Sommese 1978a]) and say that $\mathscr{F}$ is $\mathscr{L}$–concave if, for any divisor $D \in |\mathscr{L}^a|$ with $a \geq 1$ and any open connected neighborhood U of D, the restriction map

$$\rho_U : H^0(\mathscr{F}) \to H^0(\mathscr{F}|_U)$$

is surjective and therefore an isomorphism (see Definition 1.6). The open subset U, indeed, behaves in a similar way to that of a concave set in an analytic space [Andreotti 1963].

We have the following (see Theorem 1.7):

Theorem B *Let X be a smooth complex projective variety and let $\mathscr{L}$ be a big and semiample line bundle on X. Then a vector bundle $\mathscr{F}$ on X is $\mathscr{L}$–liftable if and only if it is $\mathscr{L}$–concave.*

In Section 2 we give examples of surfaces to add value to the above results. We investigate more precisely the cotangent bundle Ω^1_S of a smooth projective surface S and show that such a sheaf can be either $\mathscr{L}$–concave or not with respect to a big and semiample line bundle $\mathscr{L}$. We furthermore raise some questions about surfaces in the Noether–Lefschetz locus of $\mathbb{P}^3$ (see Question 2.5). The importance of the cotangent bundle in this paper is much deeper and will appear evident in a moment.

We are aware, also in view of the results of [Totaro 2013; Ottem 2012], that it could be really interesting to drop the assumption of semiampleness. This seems to us technically difficult at the moment and not necessary to tackle the problem that motivated all these studies.

Let us introduce our motivating problem. Let $\pi : \mathscr{C} \to B$ be a smooth holomorphic family of compact Riemann surfaces of genus g. During the preparation of [Biswas et al. 2021], Indranil Biswas explained to the second author of this article that $\mathscr{C}^\infty$–families of projective structures on $C_t = \pi^{-1}(t)$, with $t \in B$, are in one-to-one correspondence with $\bar{\partial}$–closed $\mathscr{C}^\infty$ $(1,1)$–forms on B with fixed cohomology class, modulo *holomorphic* $(1,0)$*–forms* of B. For details, see [Biswas et al. 2021, Section 3]. He then raised the problem of the existence of global holomorphic forms on $\mathscr{M}_g$, the moduli space of compact Riemann surfaces, that is, of smooth complex projective curves, of genus g.

It is well known, at least since [Mumford 1967], that there are no closed holomorphic 1–forms on $\mathscr{M}_g$, and a proof of this will be outlined in Section 3. We could not find any result in the literature concerning nonclosed holomorphic forms. Our result, which can be seen as a concavity result, is the following (see Theorem 3.1):

Theorem C *Let $\mathscr{M}_g^{\mathrm{o}} \subset \mathscr{M}_g$ be the smooth locus. Then, for $g \geq 5$, $\mathscr{M}_g^{\mathrm{o}}$ has no holomorphic* 1*–forms; that is,* $H^0(\Omega^1_{\mathscr{M}_g^{\mathrm{o}}}) = 0$.

The proof of this uses the Deligne–Mumford compactification $\overline{\mathcal{M}}_g^{\mathrm{DM}}$ and the Satake map $\overline{\tau}\colon \overline{\mathcal{M}}_g^{\mathrm{DM}} \to \overline{\mathcal{A}}_g^{\mathrm{Sat}}$. Since $\overline{\mathcal{M}}_g^{\mathrm{Sat}} := \overline{\tau}(\overline{\mathcal{M}}_g^{\mathrm{DM}})$ is a projective variety, we intersect $\overline{\mathcal{M}}_g^{\mathrm{Sat}}$ with $3g-5$ and $3g-4$ suitably general hyperplanes, respectively. By taking the inverse image on $\overline{\mathcal{M}}_g^{\mathrm{DM}}$, we reduce our problem to curves and surfaces in $\overline{\mathcal{M}}_g^{\mathrm{DM}}$. We call these, respectively, H–surfaces and H–curves. It is easy to show that, for $g > 3$, the general H–curve is contained in $\mathcal{M}_g^{\mathrm{o}}$ and a general H–surface intersects the boundary of $\overline{\mathcal{M}}_g^{\mathrm{DM}}$ only on Δ_1, the locus of stable curves with an elliptic tail. We apply Theorem 1.3 to a general H–surface S and $\mathcal{L} = \mathcal{O}_S(C)$, with C a general H–curve contained in S. Using the fact that the contracted divisor is exactly $E = \Delta_1 \cap S$, we show that Ω^1_S is $\mathcal{L}$–liftable and $H^0(\Omega^1_S) = 0$.

We strengthen the above result by proving the following theorem, which has the flavor of a concavity result (see Theorem 3.10):

Theorem D *Let C be a H–curve and let $U \subset \mathcal{M}_g^{\mathrm{o}}$ be a connected open neighborhood of C for the classical topology. Then, for $g \geq 5$, $H^0(\Omega^1_U) = 0$.*

Our last result, contained in Section 3.4, is an extension of Theorem 3.1 to the case of moduli of marked curves. More precisely, if $\mathcal{M}^o_{g,n}$ is the smooth locus of $\mathcal{M}_{g,n}$, we have the following (see Theorem 3.11):

Theorem E *Let $g \geq 5$. Then $H^0(\Omega^1_{\mathcal{M}^o_{g,n}}) = 0$, for all $n \geq 0$.*

The proof of Theorem E is straightforward, but uses the extra ingredient of the infinitesimal variation of Hodge structures. We would also like to mention that the existence of possibly nonclosed holomorphic forms in a neighborhood of a compact curve plays a subtle role in the infinitesimal variation of its periods [Pirola and Torelli 2020; González-Alonso et al. 2019; González-Alonso and Torelli 2021; Favale and Torelli 2017]. Similar results should hold at least for many families of curves; for instance, the case of smooth plane curves is treated implicitly in [Favale et al. 2018; Pirola and Torelli 2020].

As a remarkable consequence, Theorems C and E solve the corresponding problems at the level of moduli stack of curves (possibly with marked points), by interpreting $\mathcal{M}^o_{g,n}$ as an open subset of the stack.

Corollary *For $g \geq 5$, the moduli stack of curves with $n \geq 0$ marked points has no holomorphic 1–forms.*

To conclude, the most interesting problem arising from our result would probably be to consider holomorphic p–forms on the moduli space of curves. The methods used for 1–forms seem to us insufficient at the moment to deal with these more general cases.

Acknowledgements

We would like to thank Indranil Biswas for introducing this interesting problem to us, Francesco Bonsante and Riccardo Salvati Manni for helpful remarks and discussions on the Teichmüller space and on the compactification of the moduli space of curves, and Alessandro Ghigi for enlightening explanations on the relations between concavity and extensions of holomorphic functions. We also want to express their gratitude to the referee for the helpful remarks and suggestions.

All authors are partially supported by INdAM-GNSAGA. Favale was partially supported by "2017–ATE-0253" (Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca) and is now partially supported by the INdAM-GNSAGA project *Classification problems in algebraic geometry*: *Lefschetz properties and moduli spaces* (CUP_E55F22000270001). Pirola is partially supported by PRIN 2017 *Moduli spaces and Lie theory* and by (MIUR): Dipartimenti di Eccellenza program (2018–2022) – Dept. of Math., Univ. of Pavia. Torelli is supported by the Alexander von Humboldt Foundation.

1 Surjectivity of restriction maps

Throughout this section, X will be a smooth projective variety of dimension $n \geq 2$ over an algebraically closed field $\Bbbk$. For any big and semiample line bundle $\mathscr{L}$ on X, let $d_0 \in \mathbb{N}$ be a positive integer such that, for any $d \geq d_0$, the morphism

$$\psi_{|\mathscr{L}^d|}\colon X \to \mathbb{P}H^0(\mathscr{L}^d)^* = \mathbb{P}^{N_d}$$

is birational onto its image and the map $\psi_{|\mathscr{L}^d|}\colon X \to \psi_{|\mathscr{L}^d|}(X)$ does not depend on d. We set $Y = \psi_{|\mathscr{L}^d|}(X)$, $\psi\colon X \to Y$ the induced morphism, $E \subset X$ the divisor contracted to points and E_i the connected divisor contracted to the point p_i. Notice that, in particular, the divisor E does not depend on d.

Remark 1.1 Notice that $\mathscr{L} = \psi^*(\mathscr{L}')$, where $\mathscr{L}'$ is an ample line bundle on Y. Indeed, by assumption, the map induced by $\mathscr{L}^d$ and $\mathscr{L}^{d+1}$ are the same and so $\mathscr{L}^d = \psi^*\mathscr{O}_{\mathbb{P}^{N_d}}(1)|_Y$ for any $d \geq d_0$. Therefore, $\mathscr{L} = \mathscr{L}^{d+1} \otimes \mathscr{L}^{-d} = \psi^*(\mathscr{O}_{\mathbb{P}^{N_{d+1}}}(1)|_Y \otimes \mathscr{O}_{\mathbb{P}^{N_d}}(1)^{-1}|_Y)$, as claimed.

Let $\mathscr{F}$ be a locally free sheaf on X. For a large enough, consider $D_a \in |\mathscr{L}^a|$ and take the short exact sequence induced by $\mathscr{O}_X(-D_a) \subset \mathscr{O}_X$ twisted with $\mathscr{F}$,

$$0 \to \mathscr{F}(-D_a) \to \mathscr{F} \to \mathscr{F}|_{D_a} \to 0. \tag{1}$$

Definition 1.2 We say that $\mathscr{F}$ is $\mathscr{L}$–liftable if the map

$$\rho_a\colon H^0(\mathscr{F}) \to H^0(\mathscr{F}|_{D_a}) \tag{2}$$

induced by (1) is an isomorphism for all a large enough and any $D_a \in |\mathscr{L}^a|$.

Consider also the short exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}(mE) \to \mathscr{F}(mE)|_{mE} \to 0. \tag{3}$$

This section is dedicated to proving the following theorem:

Theorem 1.3 *$\mathscr{F}$ is $\mathscr{L}$–liftable if and only if, for $m \geq 0$, the map $\tau_m\colon H^0(\mathscr{F}) \to H^0(\mathscr{F}(mE))$ induced by (3) is surjective.*

Observe that injectivity holds as soon as a is large enough since $H^0(\mathcal{F}(-D_a)) = 0$. Hence, $\mathcal{L}$–liftability is a property concerning the surjectivity of that map. We thus have to study the injectivity of the map $H^1(\mathcal{F}(-D_a)) \to H^1(\mathcal{F})$, which is equivalent, by Serre duality, to the surjectivity of $H^{n-1}(\mathcal{E}) \to H^{n-1}(\mathcal{E} \otimes \mathcal{L}^a)$, where we write $\mathcal{E} = \mathcal{F}^* \otimes \omega_X$. We first compute $H^{n-1}(\mathcal{E} \otimes \mathcal{L}^a)$.

Lemma 1.4 *For a large enough, $H^{n-1}(\mathcal{E} \otimes \mathcal{L}^a) \simeq H^0(R^{n-1}\psi_*\mathcal{E})$. Moreover, $\mathcal{G} = R^{n-1}\psi_*\mathcal{E}$ is a sum of skyscraper sheaves $\mathcal{G}_i$ supported on the images p_i of the divisors E_i contracted to points by ψ.*

Proof Recall that $\mathcal{L} = \psi^*(\mathcal{L}')$ with $\mathcal{L}'$ ample, as observed in Remark 1.1. Applying the projection formula, we have

$$R^p\psi_*(\mathcal{E} \otimes \mathcal{L}^a) = R^p\psi_*(\mathcal{E} \otimes \psi^*(\mathcal{L}')^a) = R^p\psi_*(\mathcal{E}) \otimes (\mathcal{L}')^a.$$

As a is large enough, we can apply Serre's criterion to obtain

$$H^i(R^p\psi_*(\mathcal{E} \otimes \mathcal{L}^a)) = H^i(R^p\psi_*(\mathcal{E}) \otimes (\mathcal{L}')^a) = 0$$

for $i > 0$ and any p. Then all terms in the Leray spectral sequence are zero except for $H^0(R^{n-1}\psi_*(\mathcal{E}\otimes\mathcal{L}^a))$, which is mapped to zero by the differential. So we get the isomorphism

$$H^{n-1}(\mathcal{E} \otimes \mathcal{L}^a) \simeq H^0(R^{n-1}\psi_*(\mathcal{E} \otimes \mathcal{L}^a)).$$

Consider the sheaf $\mathcal{G} = R^{n-1}\psi_*(\mathcal{E} \otimes \mathcal{L}^a)$. If $y \in Y$ is different from p_i for all i, then the fiber of ψ over y is a subvariety of X of codimension at least 2. Hence, the stalk of $\mathcal{G}$ in y is 0. This proves also that $\mathcal{G}$ is a sum of skyscraper sheaves $\mathcal{G}_i$ with support on the points p_i. But now $\mathcal{L}|_{E_i} = (\psi^*\mathcal{L}')|_{E_i} = \mathcal{O}_{E_i}$, so $\mathcal{G} = R^{n-1}\psi_*(\mathcal{E} \otimes \mathcal{L}^a) = R^{n-1}\psi_*\mathcal{E}$. □

Observe now that, for $a \ge d_0$, for all $D_a \in |\mathcal{L}^a|$ we have $\mathcal{F}(mE)|_{D_a} \simeq \mathcal{F}|_{D_a}$. Indeed, every D_a is the inverse image of a hyperplane and so the general D_a is disjoint from E. Therefore, $\mathcal{O}_{D_a}(mE) \simeq \mathcal{O}_{D_a}$. Notice that we can conclude the same for all D_a by the seesaw theorem. Thus, using the map $\mathcal{O}_X(-mE) \hookrightarrow \mathcal{O}_X$, the short exact sequence (1) and its twist by $\mathcal{O}_X(mE)$ for $m \ge 0$ fit into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} \otimes \mathcal{L}^{-a} & \longrightarrow & \mathcal{F} & \xrightarrow{\rho_a} & \mathcal{F}|_{D_a} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\simeq} & & \\ 0 & \longrightarrow & \mathcal{F}(mE) \otimes \mathcal{L}^{-a} & \longrightarrow & \mathcal{F}(mE) & \longrightarrow & \mathcal{F}|_{D_a} & \longrightarrow & 0 \end{array} \tag{4}$$

which induces in cohomology

$$\begin{array}{ccccccccc} H^0(\mathcal{F} \otimes \mathcal{L}^{-a}) & \hookrightarrow & H^0(\mathcal{F}) & \xrightarrow{\rho_a} & H^0(\mathcal{F}|_{D_a}) & \longrightarrow & H^1(\mathcal{F} \otimes \mathcal{L}^{-a}) & \xrightarrow{\eta_a} & H^1(\mathcal{F}) \\ \big\downarrow & & {\scriptstyle\tau_m}\big\downarrow & & \big\downarrow{\scriptstyle\simeq} & & & & \\ H^0(\mathcal{F}(mE) \otimes \mathcal{L}^{-a}) & \hookrightarrow & H^0(\mathcal{F}(mE)) & \xrightarrow{\rho'_a} & H^0(\mathcal{F}|_{D_a}) & & & & \end{array} \tag{5}$$

Let τ_m^* and η_a^* be the maps obtained from τ_m and η_a by Serre duality, respectively. Then τ_m^* fits into the following exact sequence induced by $\mathcal{O}_X(-mE) \hookrightarrow \mathcal{O}_X$ twisted by $\mathcal{E}$:

$$\cdots \to H^{n-1}(\mathcal{E}) \xrightarrow{\delta_m} H^{n-1}(\mathcal{E}|_{mE}) \to H^n(\mathcal{E}(-mE)) \xrightarrow{\tau_m^*} H^n(\mathcal{E}) \to \cdots . \tag{6}$$

We summarize two results which stem from the above discussion in the following lemma:

Lemma 1.5 *The following equivalent conditions hold*:

(1) *The surjectivity of ρ_a is equivalent to the surjectivity of $\eta_a^*\colon H^{n-1}(\mathcal{E}) \to H^{n-1}(\mathcal{E} \otimes \mathcal{L}^a)$.*

(2) *The surjectivity of τ_m is equivalent to the surjectivity of $\delta_m\colon H^{n-1}(\mathcal{E}) \to H^{n-1}(\mathcal{E}|_{mE})$.*

Proof of Theorem 1.3 Assume first that ρ_a is surjective for $a \gg 0$. We prove that τ_m is surjective for any $m > 0$. By diagram (5),

$$\begin{array}{ccccc} & & H^0(\mathcal{F}) & \xrightarrow{\rho_a} & H^0(\mathcal{F}|_{D_a}) \\ & & \downarrow{\scriptstyle \tau_m} & & \downarrow{\scriptstyle \simeq} \\ H^0(\mathcal{F}(mE) \otimes \mathcal{L}^{-a}) & \hookrightarrow & H^0(\mathcal{F}(mE)) & \xrightarrow{\rho_a'} & H^0(\mathcal{F}|_{D_a}) \end{array}$$

As ρ_a is surjective, also $\rho_a' \circ \tau_m$ is surjective. But now the kernel of ρ_a' is $H^0(\mathcal{F}(mE) \otimes \mathcal{L}^{-a})$ and it is trivial for a large enough. Hence τ_m is surjective.

Let us now prove the other implication. Assume that τ_m is surjective for any $m > 0$. We prove that ρ_a is surjective for $a \gg 0$. Equivalently, by Lemma 1.5, it is enough to prove that $\eta_a^*\colon H^{n-1}(\mathcal{E}) \to H^{n-1}(\mathcal{E} \otimes \mathcal{L}^a)$ is surjective. By Lemma 1.4, we can write $H^{n-1}(\mathcal{E} \otimes \mathcal{L}^a) = H^0(R^{n-1}\psi_*\mathcal{E}) = \bigoplus_i H^0((R^{n-1}\psi_*\mathcal{E})_{p_i})$ because $(R^{n-1}\psi_*\mathcal{E})_{p_i}$ is a skyscraper supported on the points p_i that are images of contracted divisors and so the map η_a^* from Lemma 1.5, up to isomorphism, is

$$\eta_a^*\colon H^{n-1}(\mathcal{E}) \to \bigoplus_i H^0((R^{n-1}\psi_*\mathcal{E})_{p_i}). \tag{7}$$

To prove the surjectivity of (7), we use the machinery of inverse limits and the theorem of formal functions (see [Grothendieck 1961, (4.1.5), (4.2.1); Hartshorne 1977, III, Theorem 11.1]). This theorem gives the isomorphisms

$$(R^{n-1}\psi_*\mathcal{E})^{\wedge}_{p_i} \simeq \varprojlim H^{n-1}(\mathcal{E}|_{kE_i})$$

and defines then a map

$$\hat{\eta}_a^*\colon H^{n-1}(\mathcal{E}) \to \bigoplus_i \varprojlim H^{n-1}(\mathcal{E}|_{kE_i}). \tag{8}$$

We will prove that $\hat{\eta}_a^*$ is surjective and that this implies that η_a^* is surjective.

To be more explicit, consider the following two inverse systems of $\Bbbk$–vector spaces $\mathfrak{B}$ and $\mathfrak{C}$. The first one, $\mathfrak{B} = (B_k)_{k\in\mathbb{N}}$, is the constant inverse system with $B_k = H^{n-1}(\mathcal{E})$ and maps $b_k = \mathrm{id}\colon B_k \to B_{k-1}$

for all k. In order to define the second one, $\mathfrak{C} = (C_k)_{k\in\mathbb{N}}$, we set $C_k = H^{n-1}(\mathscr{E}|_{kE})$ and we construct the maps $c_k\colon C_k \to C_{k-1}$ as follows. Consider for any $k \geq 1$ the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E}(-kE) & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{E}|_{kE} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow \scriptstyle{\mathrm{id}} & & \big\downarrow \scriptstyle{c'_k} & & \\ 0 & \longrightarrow & \mathscr{E}(-(k-1)E) & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{E}|_{(k-1)E} & \longrightarrow & 0 \\ & & \big\downarrow & & & & & & \\ & & \mathscr{E}(-(k-1)E)|_E & & & & & & \end{array} \tag{9}$$

where the two rows are defined by $\mathcal{O}_X(-lE) \subset \mathcal{O}_X$ twisted with $\mathscr{E}$ for $l = k$ and $l = k-1$, respectively, the first column is given by $\mathcal{O}_X(-E) \subset \mathcal{O}_X$, the second column by the identity and the map c'_k in the third column is that one that makes the diagram commutative. As the vertical arrow in the first column is an injection and that in the second column is the identity, c'_k is surjective and its kernel is isomorphic to $\mathscr{E}(-(k-1)E)|_E$. Hence, the last column induces the exact sequence

$$\cdots \to H^{n-1}\big(\mathscr{E}(-(k-1)E)|_E\big) \to H^{n-1}(\mathscr{E}|_{kE}) \xrightarrow{c_k} H^{n-1}(\mathscr{E}|_{(k-1)E}) \to 0, \tag{10}$$

where the last 0 follows as $\mathscr{E}(-(k-1)E)|_E$ is supported on a divisor. So the maps c_k are epimorphisms.

Now we want to prove that there is a surjection between the limits of the inverse systems $\mathfrak{B}$ and $\mathfrak{C}$ induced by the morphism $\delta = (\delta_m)_{m\in\mathbb{N}}\colon \mathfrak{B} \to \mathfrak{C}$, where

$$\delta_m\colon H^{n-1}(\mathscr{E}) \to H^{n-1}(\mathscr{E}|_{mE}) \tag{11}$$

is the morphism in Lemma 1.5. These are surjective for $m \geq 0$ by Lemma 1.5, since, by assumption, $\tau_m\colon H^0(\mathscr{F}) \to H^0(\mathscr{F}(mE))$ is surjective for $m \geq 0$. In particular, we have a surjective map between the limits

$$H^{n-1}(\mathscr{E}) = \varprojlim \mathfrak{B} \twoheadrightarrow \varprojlim \mathfrak{C} = \varprojlim H^{n-1}(\mathscr{E}|_{kE}). \tag{12}$$

Observe that $\varprojlim H^{n-1}(\mathscr{E}|_{kE}) = \bigoplus_i \varprojlim H^{n-1}(\mathscr{E}|_{kE_i})$. By [Hartshorne 1977, III, Proposition 8.5 and Theorem 11.1], the vector space $H^{n-1}(\mathscr{E}|_{kE_i})$ has a natural structure of an $\mathcal{O}_{Y,p_i}$–module, which is compatible with the structure of a $\Bbbk$–vector space. Thus we can apply the theorem of formal functions (see [Grothendieck 1961, (4.1.5),(4.2.1); Hartshorne 1977, III, Theorem 11.1]) to $R^{n-1}\psi_*\mathscr{E}$ to conclude that

$$(R^{n-1}\psi_*\mathscr{E})\widehat{_{p_i}} = \varprojlim H^{n-1}(\mathscr{E}|_{kE_i}) = \varprojlim \mathfrak{C}. \tag{13}$$

Using (12) and (13), one can define the morphism

$$\hat{\eta}_a^*\colon H^{n-1}(\mathscr{E}) \to \bigoplus_i (R^{n-1}\psi_*\mathscr{E})\widehat{_{p_i}}$$

in (8), which is surjective by construction.

In order to conclude that

$$\eta_a^*\colon H^{n-1}(\mathscr{E}) \to \bigoplus_i H^0((R^{n-1}\psi_*\mathscr{E})_{p_i})$$

is surjective, notice that $(R^{n-1}\psi_*\mathscr{E})\widehat{_{p_i}}$ is naturally isomorphic to $H^0((R^{n-1}\psi_*\mathscr{E})_{p_i})$. Indeed, one has

$$(R^{n-1}\psi_*\mathscr{E})\widehat{_{p_i}} = (R^{n-1}\psi_*\mathscr{E})_{p_i} \otimes \widehat{\mathscr{O}}_{p_i} = (R^{n-1}\psi_*\mathscr{E})_{p_i} \otimes \mathscr{O}_{p_i} = (R^{n-1}\psi_*\mathscr{E})_{p_i} = H^0((R^{n-1}\psi_*\mathscr{E})_{p_i}),$$

where the second equality follows from the isomorphism $\widehat{\mathscr{O}}_{p_i} \simeq \mathscr{O}_{p_i}$ (which holds since $\mathscr{O}_{p_i}$ is Artinian) and the last equality follows since the sheaf is supported on the point. □

1.1 A concavity result

From now on we assume $\Bbbk = \mathbb{C}$. We will prove a result, which is of an analytic kind, by applying the algebraic results stated in Theorem 1.3. Let $\mathscr{L}$ be a line bundle on a smooth projective variety X of dimension n. Motivated by the classical notion of concavity (see [Andreotti 1963; Andreotti and Grauert 1962; Sommese 1978a]), we give the following definition:

Definition 1.6 We say that a locally free sheaf $\mathscr{F}$ on X is *$\mathscr{L}$–concave* if, for any $D \in |\mathscr{L}^a|$ with $a \geq 1$ and any open connected neighborhood U of D with respect to the analytical topology, the restriction map gives the equality $H^0(\mathscr{F}) = H^0(\mathscr{F}|_U)$.

Theorem 1.7 *Let $\mathscr{L}$ be a big and semiample line bundle. Then a locally free sheaf $\mathscr{F}$ is $\mathscr{L}$–liftable if and only if $\mathscr{F}$ is $\mathscr{L}$–concave.*

Proof Assume first that $\mathscr{F}$ is $\mathscr{L}$–liftable. Take $D \in |\mathscr{L}^a|$ and let U be an open connected neighborhood of D with respect to the analytical topology. Since $\mathscr{F}$ is $\mathscr{L}$–liftable, there is $m_0 \in \mathbb{N}$ such that the restriction map induces the isomorphism $H^0(\mathscr{F}) \simeq H^0(\mathscr{F}|_{mD})$ for all $m \geq m_0$. In particular, the restriction map induces isomorphisms $H^0(\mathscr{F}|_{m_0 D}) \simeq H^0(\mathscr{F}|_{(m_0+k)D})$ for $k \geq 0$. We have to show that the restriction $H^0(\mathscr{F}) \to H^0(\mathscr{F}|_U)$ is surjective. Fix a section $\omega \in H^0(\mathscr{F}|_U)$ and let $\omega_m \in H^0(\mathscr{F}|_{mD})$ be its restriction. If $\alpha \in H^0(\mathscr{F})$ is the lift of ω_{m_0}, then, for all $m \geq m_0$, $\alpha|_U - \omega \in H^0(\mathscr{F}|_U)$ restricts to zero in $H^0(\mathscr{F}|_{mD})$. Thus, the series expansion of $\alpha|_U - \omega$ in a local coordinate neighborhood of the generic point of U vanishes. Therefore, since U is connected, $\alpha|_U - \omega = 0$.

Now assume that $\mathscr{F}$ is not $\mathscr{L}$–liftable. Since $\mathscr{L}$ is big and semiample, by Theorem 1.3 there is a nonzero effective divisor E contracted by $\varphi_{|\mathscr{L}^d|}$ for $d \gg 0$, and a section $\alpha \in H^0(\mathscr{F}(mE)) \setminus H^0(\mathscr{F})$ for some integer $m > 0$. Set $U = X \setminus E$ and $D \in |\mathscr{L}^d|$ such that $D \cap E = \varnothing$. Then U is an open connected neighborhood of D. By construction, the restriction $\alpha|_U$ defines a section of $H^0(\mathscr{F}|_U)$ that cannot be extended to $H^0(\mathscr{F})$. □

2 Examples in dimension two

In this section we analyze Theorem 1.3 when the variety is a surface S over an algebraically closed field of characteristic 0 and $\mathscr{F} = \Omega^1_S$ is its cotangent bundle. Theorem 1.3, in this framework, can be restated here as follows:

> *Let S be as surface and let $\mathscr{L}$, and E be as in Theorem 1.3. Then Ω^1_S is not $\mathscr{L}$–liftable if and only if there exists $m > 0$ such that $h^0(\Omega^1_S(mE)) > h^0(\Omega^1_S)$.*

We will give examples of surfaces S for which Ω^1_S is $\mathscr{L}$–liftable and cases for which it is not.

2.1 The nonliftable case: projective bundles over curves and coverings

We will use well-known results about projective bundles. The reader can refer to [Hartshorne 1977, Section V.2] for details. Let B be a smooth projective curve of genus $g \geq 2$. We fix a globally generated line bundle M of degree $d > 0$ with $h^1(M) > 0$. Then one has $0 < d \leq 2g-2$ and $d = 2g-2$ if and only if $M = \omega_B$. We can consider the vector bundle $\mathscr{V} = \mathbb{O}_B \oplus M^{-1}$ on B. It is the only decomposable normalized[1] vector bundle of rank 2 on B such that $c_1(\mathscr{V}) = M^{-1}$. Let $S = \mathbb{P}(\mathscr{V}) \xrightarrow{f} B$ and consider the section $\sigma\colon B \to S$ induced by $\mathscr{V} \twoheadrightarrow M^{-1}$. Its image is an effective curve E which is isomorphic to B via σ. Moreover, by construction, if $\mathscr{N}$ is a line bundles on B, then $f^*(\mathscr{N})|_E$ corresponds to $\mathscr{N}$ via σ. In particular,

$$\mathbb{O}_E(E) = f^*(M^{-1})|_E$$

and so $E^2 = -d$. Set $\mathscr{L} = \mathbb{O}_S(E) \otimes f^*M$ and take $H \in |\mathscr{L}|$. Notice that, by construction, $\mathbb{O}_E(H) = \mathbb{O}_E$, so $H \cdot E = 0$.

Proposition 2.1 *The line bundle $\mathscr{L}$ is big and $|\mathscr{L}|$ is basepoint-free. Moreover, E is contracted by $\varphi_{|\mathscr{L}|}$ and Ω^1_S is not $\mathscr{L}$–liftable.*

Proof Let $p \in S$. Since $|M|$ is basepoint-free by assumption and $E + |f^*M|$ is a subsystem of $|\mathscr{L}|$, if p is a basepoint of $|\mathscr{L}|$, then necessarily $p \in E$. On the other hand, by the projection formula,

$$f_*(\mathbb{O}_S(E)) = \mathscr{V}, \quad f_*(\mathbb{O}_S(H-E)) = M, \quad f_*(\mathbb{O}_S(H)) = \mathscr{V} \otimes M,$$

so

$$H^0(\mathbb{O}_S(H-E)) = H^0(f^*M) \simeq H^0(M), \quad H^0(\mathbb{O}_S(H)) \simeq H^0(\mathscr{V} \otimes M) \simeq H^0(\mathbb{O}_B) \oplus H^0(M).$$

Hence, from the exact sequence

$$(14) \qquad 0 \to H^0(\mathbb{O}_S(H-E)) \to H^0(\mathbb{O}_S(H)) \xrightarrow{\alpha} H^0(\mathbb{O}_E(H)) \to H^1(\mathbb{O}_S(H-E)) \to \cdots,$$

[1] Recall that a vector bundle $\mathscr{E}$ on a curve B is normalized if $h^0(\mathscr{E}) \neq 0$ and, for all line bundle M of negative degree, $h^0(\mathscr{E} \otimes M) = 0$.

one has that α is surjective since we have shown that $H^0(\mathcal{O}_S(H-E))$ has codimension 1 in $H^0(\mathcal{O}_S(H))$ and $H^0(\mathcal{O}_E(H)) = H^0(\mathcal{O}_E)$ is 1–dimensional. This also shows that there exists a section s of $\mathcal{L} = \mathcal{O}_S(H)$ that is not zero at any point of E. Hence $|\mathcal{L}|$ is basepoint-free, as claimed. This also shows that H is nef and, since $H^2 = d > 0$, we have that H is big. As observed before, $H \cdot E = 0$, so the morphism $\varphi_{|\mathcal{L}|}$ contracts E.

We have $R^1 f_* \mathcal{O}_S = 0$ since the fibers of f are projective lines. Since S is ruled, $h^0(\Omega^1_S) = g$. Hence, in order to show that Ω^1_S is not $\mathcal{L}$–liftable, it is enough to show that $h^0(\Omega^1_S(E)) > g$. Consider the relative cotangent sheaf $\Omega^1_{S/B}$. It is a line bundle on S such that

$$\Omega^1_{S/B} = \omega_S \otimes f^*\omega_B^{-1} = \mathcal{O}_S(-2E) \otimes f^*M^{-1} = \mathcal{O}_S(-H-E).$$

If we twist the cotangent sequence by E, we obtain

$$0 \to H^0(\mathcal{O}_S(E) \otimes f^*\omega_B) \to H^0(\Omega^1_S(E)) \to H^0(\Omega^1_{S/B}(E)) = H^0(\mathcal{O}_S(-H)) = 0$$

and so $H^0(\Omega^1_S(E)) \simeq H^0(\mathcal{O}_S(E) \otimes f^*\omega_B)$. Then, as $f_*\mathcal{O}_S(E) = \mathcal{V}$, we can write

$$h^0(\Omega^1_S(E)) = h^0(\omega_B) + h^0(\omega_B \otimes M^{-1}) = g + h^1(M).$$

As $h^1(M) > 0$ by assumption, we have proved the claim. □

Now we want to produce other examples for which the liftability property of the cotangent sheaf does not hold. Let S be as before and consider a generically finite projective morphism $\pi : \widehat{S} \to S$ such that the branch divisor D is smooth and different from E. Set $\widehat{E} = \pi^*E$, $\widehat{H} = \pi^*H$ and $\widehat{\mathcal{L}} = \pi^*\mathcal{L}$.

Proposition 2.2 *The line bundle $\widehat{\mathcal{L}}$ is big, $|\widehat{\mathcal{L}}|$ is basepoint-free and $\varphi_{|\widehat{\mathcal{L}}|}$ contracts $\widehat{E}$. Moreover, $\Omega^1_{\widehat{S}}$ is not $\widehat{\mathcal{L}}$–liftable.*

Proof Since π is surjective and generically finite, $\widehat{\mathcal{L}}$ is big and the linear system $|\widehat{\mathcal{L}}|$ is basepoint-free. Since π^* commutes with the intersection product, $0 = \pi^*(H \cdot E) = \pi^*(H) \cdot \pi^*(E) = \widehat{H} \cdot \widehat{E}$, so $\widehat{E}$ is contracted by $\varphi_{|\widehat{\mathcal{L}}|}$. As the branch divisor of π is different from E, if $\eta \in H^0(\Omega^1_S(E)) \setminus H^0(\Omega^1_S)$, then $\pi^*\eta \in H^0(\Omega^1_{\widehat{S}}(E)) \setminus H^0(\Omega^1_{\widehat{S}})$. This proves that $\Omega^1_{\widehat{S}}$ is not $\widehat{\mathcal{L}}$–liftable. □

We conclude this subsection by constructing examples of cyclic covering of any S as before. These give elliptic fibrations and surfaces of general type. We will use some results about cyclic coverings which can be found in [Barth et al. 1984, Section I.17].

Recall that $\mathcal{L}$ is basepoint-free. Then, for each $n \geq 1$, there exists a smooth irreducible curve $C_n \in |\mathcal{L}^n|$ by Bertini's theorem (see [Lazarsfeld 2004, Theorem 3.3.1]) and using that $\varphi_{|\mathcal{L}|}(S)$ has dimension 2. Hence, for all $n \geq 1$, we can construct a cyclic covering $\pi : \widehat{S} \to S$ of degree n with branch C_n. Since C_n is smooth and it does not intersect E for all n, we can apply Proposition 2.2 in order to prove that $\Omega^1_{\widehat{S}}$ is not $\widehat{\mathcal{L}}$–liftable.

Proposition 2.3 *Consider the cyclic coverings above with branch C_n. Then*:

(a) *For all $n \geq 1$, $\hat{f} = f \circ \pi$ is a fibration.*

(b) *The general fiber of $\hat{f}$ is smooth of genus $\frac{1}{2}(n-1)(n-2)$.*

(c) *$\widehat{S}$ is an elliptic fibration for $n = 3$ and is canonically polarized for $n \geq 4$.*

Proof (a) The line bundle $\mathscr{L}$ restricted to the fibers of f has positive degree (more precisely, $H \cdot F = 1$) and

$$\hat{f}_* \mathbb{O}_{\widehat{S}} = f_*(\pi_* \mathbb{O}_{\widehat{S}}) = f_*\left(\bigoplus_{k=0}^{n-1} \mathscr{L}^{-k}\right) = \bigoplus_{k=0}^{n-1} f_*(\mathscr{L}^{-k}) = f_* \mathbb{O}_S = \mathbb{O}_B.$$

Hence $\hat{f}$ is proper and surjective and has connected fibers as well, ie it is a fibration.

(b) The general fiber F of S intersects C_n transversally in $nH \cdot F = nE \cdot F = n$ points, so $\widehat{F} = \pi^{-1}(F)$ is a covering of F totally ramified on n points and unramified outside these n points. From Riemann–Hurwitz, we obtain that the genus of the general fiber $\widehat{F}$ of $\hat{f}$ is

$$g(\widehat{F}) = \tfrac{1}{2}(n-1)(n-2).$$

Notice, in particular, that $g(\widehat{F}) = 1$ if $n = 3$.

(c) By (b), $\widehat{S}$ is an elliptic fibration for $n = 3$. Assume now $n \geq 4$. We want to show that $\omega_{\widehat{S}}$ is ample. As $\widehat{S}$ is a cyclic covering of S of order n, we have $\omega_{\widehat{S}} = \pi^*(\omega_S \otimes \mathscr{L}^{n-1})$. Since

$$\omega_S \otimes \mathscr{L}^{n-1} = \mathbb{O}_S((n-3)E) \otimes f^*(\omega_B \otimes M^{n-2}),$$

by [Hartshorne 1977, Section V.2, Proposition 2.20] we have that $\omega_S \otimes \mathscr{L}^{n-1}$ is ample as soon as $n \geq 4$. Then, since π is finite and surjective, $\omega_{\widehat{S}}$ is ample and $\widehat{S}$ is canonically polarized. □

2.2 The liftable case: surfaces in the Noether–Lefschetz locus

For this example we restrict to the case $k = \mathbb{C}$. We analyze some surfaces S in $\mathbb{P}^3$ with Ω^1_S that is $\mathscr{L}$–liftable for a suitable big and semiample line bundle $\mathscr{L}$. In order to find interesting examples one needs to consider surfaces in the Noether–Lefschetz locus as, otherwise, all big line bundles on S would be multiples of the hyperplane class (see [Voisin 2003, Chapter I, Section 3.3; Lopez 1991] for details).

More precisely, S will be a very general surface in the Noether–Lefschetz locus of sextic surfaces which contains a general quartic plane curve E. These surfaces have Picard rank 2 and $\mathrm{NS}(S) \simeq \mathrm{Pic}(S)$ is spanned by the hyperplane class H and E (see [Lopez 1991]). An extremal ray for the cone of effective curves on S is E itself as $E^2 = -4$ whereas the other one is the residual intersection R in S of the hyperplane containing E. By construction, R is a conic with self-intersection $R^2 = -6$. It is easy to see that $\mathbb{O}_S(H+E)$ and $\mathbb{O}_S(4H-E)$ (contracting E and R, respectively) are, up to multiples, the only line bundles which are big and semiample (actually, globally generated) but not ample. Moreover, the classes of $H+E$ and $4H-E$ span the nef cone of S.

Proposition 2.4 *The cotangent bundle Ω^1_S is $\mathscr{L}$–liftable for $\mathscr{L}$ equal to $\mathbb{O}_S(H+E)$ or $\mathbb{O}_S(4H-E)$.*

Proof We will prove $\mathscr{L}$–liftability of Ω^1_S for $\mathscr{L} = \mathbb{O}_S(H+E)$. With the same techniques, one can prove the result for $\mathbb{O}_S(4H-E)$.

Since E is a general plane curve and S is assumed to be very general, the morphism $\varphi_{|\mathscr{L}|}$ cannot contract other curves besides E (since the effective cone is spanned by E and R). As $h^0(\Omega^1_S) = 0$, by Theorem 1.3, we have to show that $h^0(\Omega^1_S(mE)) = 0$ for all $m \geq 0$. Notice that the sequence $\{h^0(\Omega^1_S(mE))\}_{m\geq 0}$ is nondecreasing. We claim that it is stationary from $m = 1$ onwards. To see this we consider the cotangent bundle sequence of E in S and twist it with mE, ie the sequence $0 \to \mathbb{O}_E((m-1)E) \to \Omega^1_S(mE)|_E \to \omega_E(mE) \to 0$. This yields

$$0 \to H^0\big(\mathbb{O}_E((m-1)E)\big) \to H^0(\Omega^1_S(mE)|_E) \to H^0(\omega_E(mE)) \to \cdots .$$

For $m \geq 2$, both $\deg_E((m-1)E)$ and $\deg_E(\omega_E(mE))$ are negative, so $h^0(\Omega^1_S(mE)|_E) = 0$. Hence, from the exact sequence

$$0 \to \mathbb{O}_S((m-1)E) \to \mathbb{O}_S(mE) \to \mathbb{O}_E(mE) \to 0$$

twisted by Ω^1_S, we get $h^0(\Omega^1_S(E)) = h^0(\Omega^1_S(mE))$ for all $m \geq 1$. Hence, it is enough to show that $h^0(\Omega^1_S(E)) = 0$.

The restriction of the Euler sequence on $\mathbb{P}^3$ to S twisted by E yields an exact sequence

$$0 \to H^0(\Omega^1_{\mathbb{P}^3}(E)|_S) \to H^0(\mathbb{O}_S(-H+E))^{\oplus 4} \to H^0(\mathbb{O}_S(E)) \to \cdots ,$$

which gives us $h^0(\Omega^1_{\mathbb{P}^3}(E)|_S) = 0$ as $-H+E = -R$ is not effective. If we denote by $N^*_{S/\mathbb{P}^3}$ the conormal bundle of S in $\mathbb{P}^3$, we can write the cotangent sequence of S in $\mathbb{P}^3$ twisted by $\mathbb{O}_S(E)$ as

$$0 \to N^*_{S/\mathbb{P}^3}(E) \to \Omega^1_{\mathbb{P}^3}(E)|_S \to \Omega^1_S(E) \to 0.$$

As $h^0(\Omega^1_{\mathbb{P}^3}(E)|_S) = 0$, this gives an injection $H^0(\Omega^1_S(E)) \hookrightarrow H^1(N^*_{S/\mathbb{P}^3}(E))$. Hence, we are done if we prove that $H^1(N^*_{S/\mathbb{P}^3(E)}) = 0$. Notice that

$$H^1(N^*_{S/\mathbb{P}^3}(E)) = H^1(\mathbb{O}_S(-6H+E)).$$

The divisor $6H-E$ is ample since it is in the interior of the nef cone, which is spanned, as observed before, by $H+E$ and $4H-E$. Hence, by Kodaira vanishing, $H^1(N^*_{S/\mathbb{P}^3}(E)) = H^1(\mathbb{O}_S(-6H+E)) = 0$, as desired. □

Besides the sextic surfaces containing a plane quartic, we have studied other components of the Noether–Lefschetz locus. We could not find any pair $(S, \mathscr{L})$ for which Ω^1_S is not $\mathscr{L}$–liftable. Motivated by this, we pose the following question:

Question 2.5 Is there any surface S in $\mathbb{P}^3$ with a big and semiample line bundle $\mathscr{L}$ for which Ω^1_S is not $\mathscr{L}$–liftable?

3 Holomorphic one forms on $\mathcal{M}_g^o$

Let $\mathcal{M}_g$ be the coarse moduli space of smooth complex projective curves of genus $g \geq 2$. We will use some classical results about $\mathcal{M}_g$ and its compactifications. The reader can refer to [Arbarello et al. 2011; Harris and Morrison 1998]. The variety $\mathcal{M}_g$ is quasiprojective and it is singular on points parametrizing curves with nontrivial automorphism group for $g \geq 4$. We denote by $\mathcal{M}_g^o \subset \mathcal{M}_g$ the locus of points parametrizing curves with trivial automorphism group, which coincides with the smooth locus for $g \geq 4$. Furthermore, the singular locus $\mathcal{M}_g^{sing} = \mathcal{M}_g \setminus \mathcal{M}_g^o$ has codimension $g-2$, since its largest subvariety is the hyperelliptic locus.

We are interested in studying the cotangent bundle $\Omega^1_{\mathcal{M}_g^o}$. Our result is the following:

Theorem 3.1 *For $g \geq 5$, $\mathcal{M}_g^o$ has no holomorphic forms; that is, $H^0(\Omega^1_{\mathcal{M}_g^o}) = 0$.*

It is well known (see [Mumford 1967]) that $\mathcal{M}_g^o$ has no closed holomorphic 1–forms with respect to the de Rham differential. For completeness, we will briefly recall this in Theorem 3.4. Nevertheless, $H^0(\Omega^1_{\mathcal{M}_g^o})$ could still be nonzero since $\mathcal{M}_g^o$ is not compact and so holomorphic forms are not automatically closed.

To prove the result, we need to use two classical compactifications of $\mathcal{M}_g$ that we recall now. The first one is the Deligne–Mumford compactification $\overline{\mathcal{M}}_g^{\mathrm{DM}}$ (see [Harris and Morrison 1998; Arbarello et al. 2011]), which is defined as the coarse moduli space of stable curves of genus g. It is a projective variety and the boundary $\partial\,\overline{\mathcal{M}}_g^{\mathrm{DM}} = \overline{\mathcal{M}}_g^{\mathrm{DM}} \setminus \mathcal{M}_g$ is

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_{[g/2]}.$$

It is built up as union of divisors Δ_i for $i = 0, 1, \ldots, \left[\frac{1}{2}g\right]$, characterized as follows. The generic point of Δ_0 represents the class of an irreducible nodal curve with a single node and arithmetic genus g. The generic point of Δ_i with $i \geq 1$ represents the class of a nodal curve with two smooth components of genus i and $g-i$, respectively, meeting at a single node. Let λ be the Hodge class and recall that Δ_1 is divisible by 2 in the Picard group of $\overline{\mathcal{M}}_g^{\mathrm{DM}}$.

The canonical divisor of $\overline{\mathcal{M}}_g^{\mathrm{DM}}$ can be written (see [Harris and Morrison 1998, page 160, equation (3.113)]) as

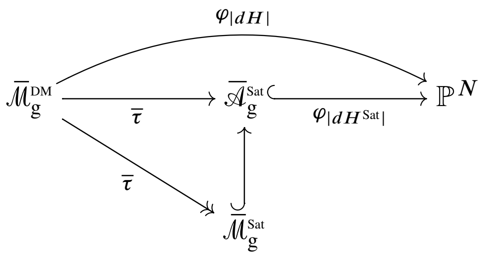
$$K_{\overline{\mathcal{M}}_g^{\mathrm{DM}}} = 13\lambda - \tfrac{3}{2}\Delta_1 - 2\Delta_0 - 2\sum_{i=2}^{[g/2]} \Delta_i = 13\lambda - 2\Delta + \tfrac{1}{2}\Delta_1.$$

Remark 3.2 Recall that the canonical divisor $K_{\overline{\mathcal{M}}_g^{\mathrm{DM}}}$ of the coarse moduli space is related to the canonical divisor K of the moduli stack by the formula $\pi^* K_{\overline{\mathcal{M}}_g^{\mathrm{DM}}} = K + \delta_1$, where π denotes the projection from the moduli stack to the coarse moduli space and δ_1 is the class of the divisor Δ_1 in the moduli stack. For a general point $[C] \in \Delta_1$, we have in fact that the versal deformation space of C is a two-sheeted cover onto its image in $\overline{\mathcal{M}}_g^{\mathrm{DM}}$, ramified over Δ_1. So in the language of stacks the morphism π is ramified over Δ_1 and therefore Δ_1 is 2–divisible. For a complete explanation, see [Harris and Morrison 1998, page 160].

The second compactification we are interested in is the Satake compactification $\overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{Sat}}$. It is constructed by considering the Satake compactification $\overline{\mathcal{A}}_{\mathrm{g}}^{\mathrm{Sat}}$ of $\mathcal{A}_g$ (see [Satake 1956; Baily and Borel 1966; Igusa 1967]) and the morphism $\overline{\tau}\colon \overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{DM}} \to \overline{\mathcal{A}}_{\mathrm{g}}^{\mathrm{Sat}}$, defined set-theoretically by sending a stable curve to the Jacobian of its normalization. Then $\overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{Sat}}$ is defined as the image of $\overline{\tau}$ and it is projective since $\overline{\mathcal{A}}_{\mathrm{g}}^{\mathrm{Sat}}$ is projective (see [Baily and Borel 1966; Igusa 1967]). Let H^{Sat} be the ample class on $\overline{\mathcal{A}}_{\mathrm{g}}^{\mathrm{Sat}}$ giving the Satake embedding and consider its pullback H on $\overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{DM}}$, which is therefore big and semiample since the Torelli morphism is given by a multiple of λ.

Furthermore, $\overline{\tau}$ is a birational morphism that is injective on $\mathcal{M}_g$ by the Torelli theorem and contracts the divisor Δ_1 to a subvariety of $\overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{Sat}}$ of codimension 2, whereas, if $i \neq 1$, Δ_i is contracted to a subvariety having codimension 3.

In other words, the morphism induced by a suitable multiple of H factors as in the diagram



We give the following definition:

Definition 3.3 Let $\varphi_{|dH|}\colon \overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{DM}} \to \mathbb{P}^N$ be the map introduced above and let $L \subset \mathbb{P}^N$ be a linear subspace of codimension c. We set $X_L = \varphi_{|dH|}^{-1}(L)$ and say that X_L is an H–variety if $\dim X_L = 3g-3-c$. In particular, for $c = 3g-5$, X_L is an H–surface, and for $c = 3g-4$ it is an H–curve.

3.1 Closed holomorphic forms on $\mathcal{M}_g^o$

We denote by $\Omega^i_{\mathcal{M}_{\mathrm{g}}^{\mathrm{o}},c}$ the kernel of $d\colon \Omega^i_{\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}} \to \Omega^{i+1}_{\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}}$, where d is the holomorphic de Rham differential. Hence $\Omega^i_{\mathcal{M}_{\mathrm{g}}^{\mathrm{o}},c}$ is just the sheaf of d–closed holomorphic 1–forms on $\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}$.

Theorem 3.4 *If $g \geq 4$, then $H^0(\mathcal{O}_{\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}}) = \mathbb{C}$, $H^1(\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}, \mathbb{C}) = 0$ and $H^0(\Omega^1_{M_g^o,c}) = 0$.*

Proof We first prove $H^0(\mathcal{O}_{\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}}) = \mathbb{C}$. Consider a point $p \in \mathcal{M}_{\mathrm{g}}^{\mathrm{o}}$; then we can cut out a smooth projective curve C_q in $\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}$ passing through it and a general point $q \in \mathcal{M}_g^o$ by using hyperplanes of $\mathcal{M}_{\mathrm{g}}^{\mathrm{o}} \subset \overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{DM}}$. This can be done since we are assuming $g \geq 4$ and so the complement of $\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}$ inside $\overline{\mathcal{M}}_{\mathrm{g}}^{\mathrm{DM}}$ has codimension at least 2. Then, for any $f \in H^0(\mathcal{O}_{\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}})$, since C_q is projective, $f|_{C_q}$ is constant. Then $f(p) = f(q)$, so f is constant on $\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}$.

Let $\mathcal{T}_g$ be the Teichmüller space of Riemann surfaces of genus g and consider the mapping class group Γ_g. The proof that $H^1(\mathcal{M}_{\mathrm{g}}^{\mathrm{o}}, \mathbb{C}) = 0$ relies on the result about the abelianization of the mapping class group Γ_g; it is trivial as soon as $g \geq 3$ (see [Mumford 1967; Harer 1983]). We recall that $\mathcal{T}_g$ is contractible and that Γ_g acts properly discontinuously on $\mathcal{T}_g$ with quotient $\mathcal{M}_g$. Let $\pi\colon \mathcal{T}_g \to \mathcal{T}_g/\Gamma_g = \mathcal{M}_g$ be the quotient map. Set $\mathcal{T}_g^o = \pi^{-1}(\mathcal{M}_g^o)$, and let $\pi^o \colon \mathcal{T}_g^o \to \mathcal{M}_g^o$ be the restriction of π. The action of Γ_g on $\mathcal{M}_g^o$ is free, and, for $g \geq 4$, $\mathcal{T}_g \setminus \mathcal{T}_g^o$ has codimension in $\mathcal{T}_g$ equal to $g-2 \geq 2$. Therefore the fundamental groups of $\mathcal{T}_g^o$ and of $\mathcal{T}_g$ are isomorphic. It follows that π^o is the universal covering of $\mathcal{M}_g^o$. Then $\Pi_1(\mathcal{M}_g^o, x_0) \simeq \Gamma_g$ and $H_1(\mathcal{M}_g^o, \mathbb{C}) \simeq H^1(\mathcal{M}_g^o, \mathbb{C}) = 0$.

Let η be a closed holomorphic form on $\mathcal{M}_g^o$. Then, by the above result, η is also exact; that is, there exists f such that $\eta = df$. Since η is holomorphic, f is a holomorphic function and thus it is constant. Consequently, $\eta = 0$. □

Corollary 3.5 *Let $\nu \colon M' \to \overline{\mathcal{M}}_g^{DM}$ be a resolution of singularities such that $\nu^{-1}(\mathcal{M}_g^o) \simeq \mathcal{M}_g^o$. Then $H^0(\Omega^1_{M'}) = 0$, $H^1(\mathcal{O}_{M'}) = 0$ and $H^1(M', \mathbb{C}) = 0$.*

Proof Let $\eta \in H^0(\Omega^1_{M'})$. Since M' is projective, then η is d–closed and so its restriction $\eta|_{\nu^{-1}(\mathcal{M}_g^o)} = \eta|_{\mathcal{M}_g^o}$ to $\mathcal{M}_g^o$ is d–closed; that is, $\eta|_{\mathcal{M}_g^o} \in H^0(\Omega^1_{M_g^o,c})$. By Theorem 3.4, $\eta|_{\mathcal{M}_g^o} = 0$ and then $\eta = 0$. Since M' is smooth, we have also $H^1(\mathcal{O}_{M'}) = H^0(\Omega^1_{M'}) = 0$. □

3.2 H–surfaces

From now on S will be a general H–surface in $\overline{\mathcal{M}}_g^{DM}$, ie it is a general complete intersection of $3g-5$ hypersurfaces whose classes are suitable multiples of H. This surface S will play a central role in the proof of Theorem 3.1. We now give a list of properties of S:

Proposition 3.6 *For $g \geq 5$, a general H–surface S satisfies the following properties*:

(a) *S is smooth and contained in the open set of smooth points of $\overline{\mathcal{M}}_g^{DM}$. Moreover, $\mathcal{O}_S(n\lambda) \simeq \mathcal{O}_S(H)$ for suitable $n > 0$, where λ denotes the restriction of the Hodge class to S.*

(b) *We have $S \cap \Delta_i = \varnothing$ for $i \neq 1$ and $E = S \cap \Delta_1$ is an effective divisor which is a disjoint union of smooth curves of genus $g-1$.*

(c) *The canonical sheaf of S is $\omega_S = \mathcal{O}_S\big(k\lambda + \frac{3}{2}E\big)$ for some suitable $k > 0$.*

(d) *We have $H^0(\Omega^1_S) = 0$.*

(e) *The morphism $\overline{\tau}|_S$ is birational, contracts E to a finite number of points, is an isomorphism outside E, and $H \cdot E = 0$.*

(f) *Fix a general point $p \in \mathcal{M}_g^o$ and a general vector $v \in T_p\,\mathcal{M}_g^o$. Then there exists an H–surface S such that $v \in T_p S$.*

In particular, S is a smooth regular surface in $(\mathcal{M}_g^o \cup \Delta_1) \setminus \mathrm{Sing}(\overline{\mathcal{M}}_g^{DM})$.

Proof We recall that $\dim \overline{\tau}(\Delta_i) = 3g-6$ for $i \neq 1$ and $\dim \overline{\tau}(\Delta_1) = 3g-5$. The general point of Δ_1 is a curve B with one node and two smooth components given by an elliptic curve D and a smooth curve C of genus $g-1 \geq 4$ that we can take without nontrivial automorphisms. Then $\overline{\tau}(B) = D \times JC$ and the fiber of $\overline{\tau}$ over this point, by Torelli's theorem, is described as the curves obtained by gluing D and C at a point. Because of the translations on D, the fiber over $D \times JC$ has dimension 1 and can be identified with C.

Consider the locus $\Delta_{1,s}$ of singular points of $\overline{\mathcal{M}}_g^{\mathrm{DM}}$ lying in Δ_1, ie $\Delta_{1,s} = \mathrm{Sing}(\overline{\mathcal{M}}_g^{\mathrm{DM}}) \cap \Delta_1$, and set

$$Y = \overline{\tau}(\Delta_0 \cup \Delta_{1,s} \cup \Delta_2 \cup \cdots \cup \Delta_{[g/2]} \cup \mathrm{Sing}(\mathcal{M}_g)).$$

We claim now that, under the assumption $g \geq 5$, Y has codimension 3 in $\overline{\mathcal{M}}_g^{\mathrm{Sat}}$. It is enough to show that the image of $\widetilde{Y} = \left(\Delta_{1,s} \setminus \bigcup_{i \neq 1} \Delta_i\right)$ has codimension 3. Moreover, we can restrict to the subspace of $\widetilde{Y}$ represented by curves with at most two nodes (since curves with more than two nodes define loci of dimension $3g-6$). If B is one such curve, we have three possible cases:

(1) B is a curve which is the union of a smooth curve C of genus $g-1$ and an elliptic curve meeting at a point P. We distinguish two subcases:

- (1.a) $\mathrm{Stab}_{\mathrm{Aut}(C)}(P) \neq \{\mathrm{id}\}$.
- (1.b) $\mathrm{Stab}_{\mathrm{Aut}(D)}(P) \neq \{\pm \mathrm{id}\}$.

(2) B is a curve which is the union of a smooth curve C of genus $g-2$ and two disjoint elliptic curves E_1 and E_2 such that $E_i \cap C = Q_i$ is a point.

We denote by $\widetilde{Y}_{(1.\mathrm{a})}$, $\widetilde{Y}_{(1.\mathrm{b})}$ and $\widetilde{Y}_{(2)}$ the corresponding loci in $\overline{\mathcal{M}}_g^{\mathrm{DM}}$. First of all, notice that $\overline{\tau}(\widetilde{Y}_{(2)})$ has dimension $3g-7$. The locus $\widetilde{Y}_{(1.\mathrm{a})}$ has dimension at most $2g-2$ (which is the dimension of the locus of hyperelliptic curves of genus $g-1$ plus the dimension of the moduli of elliptic curves) and so $\overline{\tau}(\widetilde{Y}_{(1.\mathrm{a})})$ has codimension more than 3. Finally, notice that $\widetilde{Y}_{(1.\mathrm{b})}$ has dimension $3g-5$ but it is also true that all the fibers of $\overline{\tau}$ of points of $\widetilde{Y}_{(1.\mathrm{b})}$ are contained in $\widetilde{Y}_{(1.\mathrm{b})}$, so $\overline{\tau}(\widetilde{Y}_{(1.\mathrm{b})})$ has dimension $3g-6$.

(a) To prove that a general S is smooth, we can assume first $\overline{\tau}(S) \cap Y = \varnothing$. It follows in particular that S is disjoint from $\mathrm{Sing}(\overline{\mathcal{M}}_g^{\mathrm{DM}})$. Moreover, since H is semiample on $\overline{\mathcal{M}}_g^{\mathrm{DM}}$, the general S does not have singularities by Bertini's theorem. In order to see that $\mathcal{O}_S(H)$ is a positive multiple of $\mathcal{O}_S(\lambda)$, it is enough to observe that S is disjoint from Δ_0. Indeed, this implies that the closure of $\overline{\tau}(S \setminus \Delta_1)$ in $\overline{\mathcal{A}}_g^{\mathrm{Sat}}$ is contained in $\mathcal{A}_g$. Then the claim follows since the Picard group of $\mathcal{A}_g$ is spanned by the Hodge class $\lambda_{\mathcal{A}_g}$ (whose pullback is λ, by definition).

(b) The argument above shows that for S general, $S \cap \Delta_i = \varnothing$ for $i \neq 1$. As H^{Sat} is ample, the image $\overline{\tau}(S)$ needs to intersect $\overline{\tau}(\Delta_1)$ in a finite number of points $q_1, \ldots, q_r$. Hence, S intersects Δ_1 in a divisor, which we will denote by E, whose components are a finite number of disjoint smooth curves of genus $g-1$ (again one uses Bertini's theorem for the restriction $\overline{\tau}|_{\Delta_1}$). Then, by construction, the divisor E is 2–divisible, since Δ_1 is 2–divisible, as observed in Remark 3.2.

(c) As S is a complete intersection in the nonsingular locus of $\overline{\mathcal{M}}_g^{\mathrm{DM}}$ and since $\mathcal{O}_S(H) = \mathcal{O}_S(n\lambda)$ with suitable $n > 0$ (as proved in (a)), by adjunction we have

$$\omega_S = \mathcal{O}_S\left(13\lambda - \tfrac{3}{2}\Delta_1 - 2\sum_{i \neq 1}^{[g/2]} \Delta_i + mH\right) = \mathcal{O}_S\left(k\lambda - \tfrac{3}{2}E\right),$$

where $m > 0$ and $k > 0$.

(d) Let $\nu\colon M' \to \overline{\mathcal{M}}_g^{\mathrm{DM}}$ be a desingularization which induces an isomorphism $\nu^{-1}(\mathcal{M}_g^{\mathrm{o}}) \simeq \mathcal{M}_g^{\mathrm{o}}$. Then, by Corollary 3.5, $H^1(\mathcal{O}_{M'}) = 0$. Let H' be the pullback of H with respect to f. As S is a complete intersection in $(\mathcal{M}_g^{\mathrm{o}} \cup \Delta_1) \setminus \mathrm{Sing}(\overline{\mathcal{M}}_g^{\mathrm{DM}})$, we can realize S as a complete intersection in M' by using multiples of the big line bundle H'. The statement follows by this application of the Kawamata–Viehweg vanishing theorem [Kawamata 1982]:

> *Let Z be a smooth variety of dimension $\dim(Z) \geq 3$ with $h^1(\mathcal{O}_Z) = 0$ and let H be a big and nef divisor on Z. Then the general element $Y \in |H|$ is smooth and is such that $h^1(\mathcal{O}_Y) = 0$.*

Indeed, starting from M' and cutting with multiples of H' to obtain S,

$$h^0(\Omega_S^1) = h^1(\mathcal{O}_S) = \cdots = h^1(\mathcal{O}_{M' \cap m_1 H'}) = 0.$$

(e) The last statement follows immediately by the construction.

(f) Since p and v are generic (in $\mathcal{M}_g^{\mathrm{o}}$ and $T_p\mathcal{M}_g^{\mathrm{o}}$, respectively), the general H–surface has the desired property. □

3.3 Concluding the proof of Theorem 3.1

The following result is the main technical tool:

Proposition 3.7 *The sheaf Ω_S^1 is $\mathcal{O}_S(H)$–liftable.*

The proof of this proposition follows directly by applying Theorem 1.3, Proposition 3.6(e) and the following lemma:

Lemma 3.8 *With the above notation, $H^0(\Omega_S^1) \simeq H^0(\Omega_S^1(mE)) = 0$ for all $m \geq 0$.*

Proof By Proposition 3.6(d), $h^0(\Omega_S^1) = 0$. So we only have to show that $H^0(\Omega_S^1) \simeq H^0(\Omega_S^1(mE))$ for all m. We perform this in two steps: we prove first that $H^0(\Omega_S^1) \simeq H^0(\Omega_S^1(E))$ and then that $H^0(\Omega_S^1(mE)) \simeq H^0\big(\Omega_S^1((m+1)E)\big)$ for $m \geq 1$.

We start with the first claim. Consider the exact sequences

$$0 \to \Omega_S^1 \to \Omega_S^1(E) \to \Omega_S^1(E)|_E \to 0, \tag{I}$$

$$0 \to \mathcal{O}_E \to \Omega_S^1(E)|_E \to \omega_E(E) \to 0. \tag{II}$$

By Proposition 3.6(d), $E = \sum_{i=1}^k E_i$, where E_i are smooth disjoint curves of genus $g-1$. Since $\overline{\tau}|_S$ contracts E_i, we have $H|_S \cdot E_i = 0$ for all i and so $H|_S \cdot E = 0$. Then, by Proposition 3.6(c) and by adjunction,

$$\omega_E = \omega_S \otimes \mathcal{O}_E(E) = \mathcal{O}_E\big(k\lambda - \tfrac{3}{2}E + E\big) = \mathcal{O}_E\big(-\tfrac{1}{2}E\big).$$

Since E_i is effective and $H|_S \cdot E_i = \lambda|_S \cdot E_i = 0$, by the Hodge index theorem we have $E_i^2 < 0$ and so $H^0\big(\mathbb{O}_{E_i}\big(\frac{1}{2}E_i\big)\big) = 0$. Since E_i and E_j are disjoint, $E_i \cdot E_j = 0$ and $\mathbb{O}_E = \bigoplus_i \mathbb{O}_{E_i}$. In particular, $\omega_E(E) = \bigoplus_i \omega_{E_i}(E_i) = \bigoplus_i \mathbb{O}_{E_i}\big(\frac{1}{2}E_i\big)$, so $H^0(\omega_E(E)) = 0$.

Using this and the exact sequence (II),

$$H^0(\mathbb{O}_E) \xrightarrow{\alpha} H^0(\Omega^1_S(E)|_E)$$

is an isomorphism. Consider the sequence

$$\text{(III)} \qquad 0 \to \Omega^1_S \to \Omega^1_S(\log(E)) \xrightarrow{\text{res}} \mathbb{O}_E \to 0,$$

where $\Omega^1_S(\log(E))$ is the bundle of logarithmic differentials with poles along E. We have the following commutative diagram given by (I), (II) and (III):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\Omega^1_S) & \overset{\iota}{\hookrightarrow} & H^0(\Omega^1_S(\log E)) & \longrightarrow & H^0(\mathbb{O}_E) & \xrightarrow{\partial} & H^1(\Omega^1_S) \\
 & & \| & & \downarrow & & \alpha \downarrow \simeq & & \| \\
0 & \longrightarrow & H^0(\Omega^1_S) & \hookrightarrow & H^0(\Omega^1_S(E)) & \longrightarrow & H^0(\Omega^1_S(E)|_E) & \xrightarrow{\partial'} & H^1(\Omega^1_S)
\end{array}$$

Then we obtain $\partial' = \partial \circ \alpha$. One has to show that ∂' is injective or, equivalently, that ∂ is injective. As observed above, $H^0(\mathbb{O}_E) = \bigoplus_i^k H^0(\mathbb{O}_{E_i})$ and we can write

$$\partial\colon \bigoplus_i^k H^0(\mathbb{O}_{E_i}) \to H^1(\Omega^1_S).$$

By [Griffiths and Harris 1978, pages 458–459], this is just obtained by the Atiyah–Chern class [Atiyah 1957] via the residues computation

$$(a_1, \dots, a_k) \mapsto \sum a_i c_1(E_i).$$

The E_i are effective and disjoint divisors with negative self-intersection, as previously observed. It follows that their first Chern classes are independent and hence that ∂ is injective. In conclusion, $\iota\colon H^0(\Omega^1_S) \hookrightarrow H^0(\Omega^1_S(E))$ given by (I) is an isomorphism and $0 = h^0(\Omega^1_S) = h^0(\Omega^1_S(E))$.

We now prove the second part: for $m \geq 1$, $H^0(\Omega^1_S(mE)) \simeq H^0\big(\Omega^1_S((m+1)E)\big)$. Consider the exact sequences

$$\text{(I}'\text{)} \qquad 0 \to \Omega^1_S(mE) \to \Omega^1_S((m+1)E) \to \Omega^1_S((m+1)E)|_E \to 0,$$

$$\text{(II}'\text{)} \qquad 0 \to \mathbb{O}_E(mE) \to \Omega^1_S((m+1)E)|_E \to \omega_E((m+1)E) \to 0$$

obtained from (I) and (II), respectively. We have seen that $\omega_E = \mathbb{O}_E\big(-\frac{1}{2}E\big)$, so both $\omega_E((m+1)E) = \mathbb{O}_E\big(\big(m+\frac{1}{2}\big)E\big)$ and $\mathbb{O}_E(mE)$ have negative degree for $m \geq 1$. Then $H^0\big(\Omega^1_S((m+1)E)|_E\big) = 0$. This yields the desired result from the exact sequence (I′) and induction. □

We can now conclude the proof of Theorem 3.1.

Proof of Theorem 3.1 Let $\eta \in H^0(\Omega^1_{\mathcal{M}_g^o})$. If $\eta \not\equiv 0$, then, for a general point p in $\mathcal{M}_g^o$, $\eta_p \in \Omega^1_{\mathcal{M}_g^o,p}$ is not identically zero. Let v be a general element in $T_p\mathcal{M}_g^o$. Hence we can assume $\eta_p(v) \neq 0$. By Proposition 3.6(f), we can find a general H–surface S which passes through p and is such that $v \in T_pS$. Consider the open subset $U = S \setminus E$. Recall that $\mathcal{O}_S(H)$ is big and semiample and $E \cdot H|_S = 0$, so U is an open neighborhood of a general curve in $|dH|_S$. By construction, the restriction η_U of η to U defines a nontrivial element of $H^0(\Omega^1_S|_U)$. But now, by Proposition 3.7, Ω^1_S is $\mathcal{O}_S(H)$–liftable. Then, by Theorem 1.7, we can conclude that Ω^1_S is $\mathcal{O}_S(H)$–concave, so $H^0(\Omega^1_S|_U) \simeq H^0(\Omega^1_S) \neq 0$. On the other hand, this yields a contradiction since, by Proposition 3.6(d), $H^0(\Omega^1_S|_U) = 0$. □

Remark 3.9 The assumption $g \geq 5$ is necessary in order to have S smooth. Indeed, if $g = 4$, the general S meets the hyperelliptic locus (which has codimension 2 in the moduli space) in a finite number of points, so the general S has a finite number of nodes as singularities. Nevertheless, the theorem should follow just by blowing up the points. When $g = 2$, $\mathcal{M}_g^o = \varnothing$, and, when $g = 3$, one has to remove the hyperelliptic divisor that is ample. The vanishing of all holomorphic forms on the open set of the smooth locus could still hold. Some analysis of the singularities and of the fixed points for the action of the mapping class group is however necessary.

We now give a version of Theorem 3.4 for certain open analytic subsets containing an H–curve, which are not necessarily open for the Zariski topology. In this sense this provides a strengthening of the theorem.

Theorem 3.10 *Let $g \geq 5$ and let C be an H–curve in $\mathcal{M}_g^o$. Let $U \subseteq \mathcal{M}_g^o$ be a connected open neighborhood of C for the classical topology. Then $H^0(\Omega^1_U) = 0$.*

Proof Assume by contradiction that $H^0(\Omega^1_U) \neq 0$ and fix $\eta \in H^0(\Omega^1_U)$ with $\eta \neq 0$. Recall that C is a complete intersection of $3g-5$ general hypersurfaces in $|aH|$. More precisely, we can find $L \in \mathbb{G}(|aH|, 3g-5)$ general such that $\varphi^{-1}_{|aH|}(L) = C$. Since being contained in U gives an open condition in $\mathbb{G}(|aH|, 3g-5)$ (in the analytical topology), by moving L we can find U' with $C \subset U' \subseteq U$ such that U' is covered by H–curves and the tangent vectors of those curves span the tangent space of U' at the general point $p \in U'$. We can then find a general H–curve $C' \subset U'$ corresponding to $L' \in \mathbb{G}(|aH|, 3g-5)$ such that $\eta_{C'} \neq 0$, where $\eta_{C'} \in H^0(\Omega^1_{C'})$ is the restriction of η. Then the general element of $\mathbb{G}(|aH|, 3g-4)$ that contains L' yields a smooth H–surface S that contains C'. Let U_S be the connected component of $U \cap S$ that contains C'. The restriction $\eta_{U_S} \in H^0(\Omega^1_{U_S})$ of η is a fortiori nonzero. But Ω^1_S is H–liftable and, by Theorem 1.7, H–concave. Since $C' \in |aH|$, by Proposition 3.6 we get $0 = H^0(\Omega^1_S) \simeq H^0(\Omega^1_{U_S})$. This implies $\eta_{U_S} = 0$, which gives a contradiction. □

3.4 Holomorphic one-forms on moduli spaces of marked curves

In this subsection we extend the result of Theorem 3.1 to the moduli space of marked curves. We denote by $\mathcal{M}_{g,n}$ the coarse moduli space of n–marked smooth projective curves of genus g. Denote by $\mathcal{M}^o_{g,n} \subset \mathcal{M}_{g,n}$ the smooth locus of $\mathcal{M}_{g,n}$. We prove the following:

Theorem 3.11 *Let $g \geq 5$. Then $\mathcal{M}^o_{g,n}$ has no holomorphic 1–forms for any $n \geq 0$; that is, $H^0(\Omega^1_{\mathcal{M}^o_{g,n}}) = 0$ for any $n \geq 0$.*

Proof Consider the morphism $f^n \colon \mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ that forgets the last marked point, ie

$$f^n \colon [C, p_1, \dots, p_{n-1}, p_n] \mapsto [C, p_1, \dots, p_{n-1}].$$

Set $\mathcal{U}_{g,0} = \mathcal{M}^o_g$ and $\mathcal{U}_{g,n} = (f^n)^{-1}(\mathcal{U}_{g,n-1})$ for any $n \geq 1$. Note that $\mathcal{U}_{g,n} \subset \mathcal{M}^o_{g,n}$ is a Zariski open set parametrizing marked curves, whose underlying curve has trivial automorphism group. We now prove that $H^0(\Omega^1_{\mathcal{U}_{g,n}}) = 0$ for any $n \geq 0$. From this we conclude that $H^0(\Omega^1_{\mathcal{M}^o_{g,n}}) = 0$. In fact, $\mathcal{U}_{g,n} \subset \mathcal{M}^o_{g,n}$ is an open subset, so any nonvanishing form on $\mathcal{M}^o_{g,n}$ would restrict to the zero form on an open set $\mathcal{U}_{g,n}$, which is not possible.

The proof is by induction on n. The case $n = 0$ is the content of Theorem 3.1. We now assume that $H^0(\Omega^1_{\mathcal{U}_{g,n-1}}) = 0$ and we prove that $H^0(\Omega^1_{\mathcal{U}_{g,n}}) = 0$.

For simplicity we define f to be the restriction of f^n to $\mathcal{U}_{g,n}$. Then $f \colon \mathcal{U}_{g,n} \to \mathcal{U}_{g,n-1}$ is a family of smooth projective curves of genus g over a smooth variety of dimension $3g - 3 + (n-1)$. Indeed, the fiber $f^{-1}([C, p_1, \dots, p_{n-1}])$ is naturally isomorphic to the curve C because C has trivial automorphism group. Consider the short exact sequence of relative differentials

$$0 \to f^*\Omega^1_{\mathcal{U}_{g,n-1}} \xrightarrow{df} \Omega^1_{\mathcal{U}_{g,n}} \to \Omega^1_{\mathcal{U}_{g,n}/\mathcal{U}_{g,n-1}} \to 0.$$

Since f is a fibration, the pushforward and the projection formula give the exact sequence

$$0 \to \Omega^1_{\mathcal{U}_{g,n-1}} \xrightarrow{df} f_*\Omega^1_{\mathcal{U}_{g,n}} \to f_*\Omega^1_{\mathcal{U}_{g,n}/\mathcal{U}_{g,n-1}} \xrightarrow{\partial} R^1 f_*\mathcal{O}_{\mathcal{U}_{g,n}} \otimes \Omega^1_{\mathcal{U}_{g,n-1}}.$$

We now show that $\ker(\partial) = 0$. Then, by induction, we get $0 = H^0(\Omega^1_{\mathcal{U}_{g,n-1}}) \simeq H^0(f_*\Omega^1_{\mathcal{U}_{g,n}}) \simeq H^0(\Omega^1_{\mathcal{U}_{g,n}})$ for all $n \geq 1$, which ends the proof.

Fix $x = [C, p_1, \dots, p_{n-1}, p_n] \in \mathcal{U}_{g,n}$. We can describe ∂_x as the homomorphism

$$\partial_x \colon H^0(\omega_C) \to H^1(\mathcal{O}_C) \otimes (\Omega^1_{\mathcal{U}_{g,n-1}})_x = \mathrm{Hom}(H^0(\omega_C), (\Omega^1_{\mathcal{U}_{g,n-1}})_x)$$

since $f_*\Omega^1_{\mathcal{U}_{g,n}/\mathcal{U}_{g,n-1}}$ is the Hodge bundle and $R^1 f_*\mathcal{O}_{\mathcal{U}_{g,n}}$ is its dual. The forgetful map $F \colon \mathcal{U}_{g,n} \to \mathcal{M}^o_g$ induces the sequence

$$0 \to F^*\Omega^1_{\mathcal{M}^o_g} \xrightarrow{dF} \Omega^1_{\mathcal{U}_{g,n}} \to \Omega^1_{\mathcal{U}_{g,n}/\mathcal{M}^o_g} \to 0.$$

In x, we have $(\Omega^1_{\mathcal{M}^o_g})_x = H^0(\omega_C^2)$ (see [Arbarello et al. 2011, Chapter XI]). Then we have an inclusion $\iota \colon H^0(\omega_C^2) \to (\Omega^1_{\mathcal{U}_{g,n-1}})_x$ given by $dF_x \circ (F^*)_x$. Note that, by construction, points in the fiber of f have the same image via F, so the Hodge bundle is constant along these fibers. Consider the multiplication map $\mu \colon H^0(\omega_C)^{\otimes 2} \to H^0(\omega_C^2)$ and define $\psi \colon H^0(\omega_C) \to \mathrm{Hom}(H^0(\omega_C), H^0(\omega_C^2))$ as $\psi(\alpha) = \mu(\alpha \otimes -)$.

Then, by construction, $\partial_x = \iota \circ \psi$, so we have a commutative diagram

$$\begin{array}{ccccc} H^0(\omega_C) & \xrightarrow{\quad\psi\quad} & H^1(\mathcal{O}_C) \otimes H^0(\omega_C^2) & = & \mathrm{Hom}(H^0(\omega_C), H^0(\omega_C^2)) \\ \| & & \downarrow{\scriptstyle \mathrm{id} \otimes \iota} & & \downarrow{\scriptstyle \iota \circ -} \\ H^0(\omega_C) & \xrightarrow{\quad\partial_x\quad} & H^1(\mathcal{O}_C) \otimes (\Omega^1_{\mathcal{U}_{g,n-1}})_x & = & \mathrm{Hom}(H^0(\omega_C), (\Omega^1_{\mathcal{U}_{g,n-1}})_x) \end{array}$$

In particular, since ψ and ι are injective by construction, ∂_x is injective. Then $\ker(\partial) = 0$, as claimed. □

References

[Andreotti 1963] **A Andreotti**, *Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves*, Bull. Soc. Math. France 91 (1963) 1–38 MR Zbl

[Andreotti and Grauert 1962] **A Andreotti**, **H Grauert**, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962) 193–259 MR Zbl

[Arbarello et al. 2011] **E Arbarello**, **M Cornalba**, **P A Griffiths**, *Geometry of algebraic curves, II*, Grundl. Math. Wissen. 268, Springer (2011) MR Zbl

[Atiyah 1957] **M F Atiyah**, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. 85 (1957) 181–207 MR Zbl

[Baily and Borel 1966] **W L Baily, Jr**, **A Borel**, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. 84 (1966) 442–528 MR Zbl

[Barth et al. 1984] **W Barth**, **C Peters**, **A Van de Ven**, *Compact complex surfaces*, Ergebnisse der Math. 4, Springer (1984) MR Zbl

[Biswas et al. 2021] **I Biswas**, **E Colombo**, **P Frediani**, **G P Pirola**, *A Hodge theoretic projective structure on compact Riemann surfaces*, J. Math. Pures Appl. 149 (2021) 1–27 MR Zbl

[Favale and Torelli 2017] **F F Favale**, **S Torelli**, *Covering of elliptic curves and the kernel of the Prym map*, Matematiche (Catania) 72 (2017) 155–182 MR Zbl

[Favale et al. 2018] **F F Favale**, **J C Naranjo**, **G P Pirola**, *On the Xiao conjecture for plane curves*, Geom. Dedicata 195 (2018) 193–201 MR Zbl

[González-Alonso and Torelli 2021] **V González-Alonso**, **S Torelli**, *Families of curves with Higgs field of arbitrarily large kernel*, Bull. Lond. Math. Soc. 53 (2021) 493–506 MR Zbl

[González-Alonso et al. 2019] **V González-Alonso**, **L Stoppino**, **S Torelli**, *On the rank of the flat unitary summand of the Hodge bundle*, Trans. Amer. Math. Soc. 372 (2019) 8663–8677 MR Zbl

[Griffiths and Harris 1978] **P Griffiths**, **J Harris**, *Principles of algebraic geometry*, Wiley, New York (1978) MR Zbl

[Grothendieck 1961] **A Grothendieck**, *Éléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, I*, Inst. Hautes Études Sci. Publ. Math. 11 (1961) 5–167 MR Zbl

[Harer 1983] **J Harer**, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. 72 (1983) 221–239 MR Zbl

[Harris and Morrison 1998] **J Harris**, **I Morrison**, *Moduli of curves*, Graduate Texts in Math. 187, Springer (1998) MR Zbl

[Hartshorne 1977] **R Hartshorne**, *Algebraic geometry*, Graduate Texts in Math. 52, Springer (1977) MR Zbl

[Igusa 1967] **J-i Igusa**, *A desingularization problem in the theory of Siegel modular functions*, Math. Ann. 168 (1967) 228–260 MR Zbl

[Kawamata 1982] **Y Kawamata**, *A generalization of Kodaira–Ramanujam's vanishing theorem*, Math. Ann. 261 (1982) 43–46 MR Zbl

[Lazarsfeld 2004] **R Lazarsfeld**, *Positivity in algebraic geometry, I: Classical setting: line bundles and linear series*, Ergebnisse der Math. 48, Springer (2004) MR Zbl

[Lopez 1991] **A F Lopez**, *Noether–Lefschetz theory and the Picard group of projective surfaces*, Mem. Amer. Math. Soc. 438, Amer. Math. Soc., Providence, RI (1991) MR Zbl

[Mumford 1967] **D Mumford**, *Abelian quotients of the Teichmüller modular group*, J. Anal. Math. 18 (1967) 227–244 MR Zbl

[Ottem 2012] **J C Ottem**, *Ample subvarieties and q–ample divisors*, Adv. Math. 229 (2012) 2868–2887 MR Zbl

[Pirola and Torelli 2020] **G P Pirola**, **S Torelli**, *Massey products and Fujita decompositions on fibrations of curves*, Collect. Math. 71 (2020) 39–61 MR Zbl

[Satake 1956] **I Satake**, *On the compactification of the Siegel space*, J. Indian Math. Soc. 20 (1956) 259–281 MR Zbl

[Sommese 1978a] **A J Sommese**, *Concavity theorems*, Math. Ann. 235 (1978) 37–53 MR Zbl

[Sommese 1978b] **A J Sommese**, *Submanifolds of abelian varieties*, Math. Ann. 233 (1978) 229–256 MR Zbl

[Totaro 2013] **B Totaro**, *Line bundles with partially vanishing cohomology*, J. Eur. Math. Soc. 15 (2013) 731–754 MR Zbl

[Voisin 2003] **C Voisin**, *Hodge theory and complex algebraic geometry, II*, Cambridge Stud. Adv. Math. 77, Cambridge Univ. Press (2003) MR Zbl

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Proposed: Mark Gross Received: 2 November 2020
Seconded: Richard P Thomas, Jim Bryan Revised: 10 March 2023

Geometry & Topology

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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GEOMETRY & TOPOLOGY

Volume 28 Issue 7 (pages 3001–3510) 2024