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We determine the local geometric structure of two-dimensional metric spaces with curvature bounded above as the union of finitely many properly embedded/branched immersed Lipschitz disks. As a result, we obtain a graph structure of the topological singular point set of such a singular surface.

53C20, 53C23

1 Introduction

Let X be a locally compact, geodesically complete Alexandrov space with curvature bounded above. In this paper, we are concerned with the local structure of X. In general X may have very complicated local geometry. For instance, X may have no polyhedral structure even in local. There is such a two-dimensional space constructed by Kleiner; see also Nagano [21]. In the present paper, we completely describe the local geometry of such spaces in dimension two.

The study of metric spaces with curvature bounded above began with the work of Alexandrov [2]. For the dimensions of such spaces X, Kleiner [15] proved that the topological dimension coincides with the maximal dimension of topological manifolds embedded in X. For *geodesically complete* metric spaces X with curvature bounded above, Otsu and Tanoue [25] implicitly showed that the topological dimension coincides with the Hausdorff dimension. This has been verified via a different method by recent work due to Lytchak and Nagano [17], which has also clarified that the local geometric properties of geodesically complete metric spaces X with curvature bounded above have a lot of analogues to those of Alexandrov spaces with curvature bounded below; see also Remarks 1.5 and 1.7 below. Lytchak and Stadler [19] have recently proved that for every convex open ball in a CAT(κ)–space there exists a complete CAT(-1)–metric on the ball that is locally bi-Lipschitz to the original CAT(κ)–metric; in particular, in local considerations on topological properties of CAT(κ)–spaces, we may assume κ to be -1.

For basic textbooks in this subject, there are several general references, and we refer to Ballmann [8], Bridson and Haefliger [10], Burago, Burago and Ivanov [11] and Alexander, Kapovitch and Petrunin [5].

Now let us consider our main concern, the two-dimensional such spaces. The study in this particular dimension began with a classical deep work due to Alexandrov and Zalgaller [3] on two-dimensional topological manifolds with more general curvature bound, called the *bounded curvature*. They constructed the curvature measure on such surfaces and established the Gauss–Bonnet theorem. See also Reshetnyak [28] for the work from an analytic point of view. Generalizing [3] and following the works of Ballmann and Buyalo [9] and Arsinova and Buyalo [7], Burago and Buyalo [12] established the theory of two-dimensional polyhedra with curvature bounded above.

Here it should be emphasized that there were no general results determining local structure even in dimension two. The purpose of this paper is to determine the general local geometric structure of two-dimensional geodesically complete metric spaces with curvature bounded above.

Let X be a two-dimensional locally compact, geodesically complete metric space with curvature $\le \kappa$ for a constant κ. For every $p \in X$, the space of directions $\Sigma_p = \Sigma_p(X)$ is the disjoint union of finitely many points and connected finite graphs. Since we are interested in the local structure, we assume the most essential case when Σ_p is a connected graph, called a CAT(1)–graph; see Section 2. We shall determine the geometry of the closed r–ball $B(p,r)$ around p for small enough $r > 0$ as follows.

Let $\mathcal{S}(X)$ denote the set of all topological singular points in X. For $\ell \ge 2\pi$ and $r > 0$, we denote by $D^2(\ell; r)$ the closed disk of radius r around the vertex O in the Euclidean cone over the circle of length ℓ. A map $f : D^2(\ell; r) \to B(p,r)$ is called *proper* if $f^{-1}(\partial B(p,r)) = \partial D^2(\ell; r)$. Let $\tau_p(r)$ denote a function depending on p and r satisfying $\lim_{r\to 0} \tau_p(r) = 0$. Let $S(p,r)$ denote the metric sphere $\partial B(p,r)$.

The main result in this paper is stated as follows.

Theorem 1.1 *For every $p \in X$ such that Σ_p is a connected graph, there exists a positive number r_0 such that for every $0 < r \le r_0$, $B(p,r)$ is a union of images* Im f_i *of finitely many proper Lipschitz immersions $f_i : D^2(\ell_i; r) \to B(p,r)$ for some $\ell_i \ge 2\pi$, possibly with branch point $f_i^{-1}(p) = \{O\}$ satisfying the following*:

(1) *With respect to the length metric induced from X,* Im f_i *are* CAT(κ)*–spaces.*

(2) *Either f_i is an embedding, or else $f_i(\partial D^2(\ell_i; r))$ is the union of two circles of length $\ge 2\pi r$ connected by an arc, which could be a point. In the latter case, $\ell_i \ge 4\pi$.*

(3) *The bi-Lipschitz constant of f_i is less than $1+\tau_p(r)$ when f_i is an embedding. If f_i is a branched immersion, the local bi-Lipschitz constant of f_i except at $\{O\}$ is less than $1+\tau_p(r)$.*

Moreover, $\mathcal{S}(X)\cap B(p,r)$ consists of finitely many simple Lipschitz arcs starting from p and reaching $S(p,r)$.

Remark 1.2 One might ask if it is possible to fill the ball $B(p,r)$ with those $\operatorname{Im} f_i$ that are convex in X or properly embedded disks. However, both are impossible in general. For example, take the Euclidean cone X over the union of two circles of length 2π joined by an arc. No metric ball around the vertex of X can be written as a union of properly embedded disks as described in Theorem 1.1. For an example showing the impossibility of filling the ball via convex properly embedded $\mathrm{CAT}(\kappa)$–disks, see Example 4.5.

From the proof of Theorem 1.1, we actually have the following.

Corollary 1.3 *Let $r=r_p$ be sufficiently small as in Theorem 1.1. Then for any locally injective continuous map $\zeta\colon [a,b]\to\Sigma_p(X)$, there is a closed subset E of X containing p such that*

(1) *E is a $\mathrm{CAT}(\kappa)$–space with respect to the length metric,*

(2) $\Sigma_p(E)=\operatorname{Im}(\zeta)$,

(3) *$\partial E\subset S(p,r)$, except possibly the segments from p directing to the endpoints of ζ. Here ∂E denotes the set of points of E where local geodesically completeness of E fails.*

The set E is the image of a locally almost isometric, branched immersion, except at the branch locus $\{p\}$, from the closed disk of radius r around the vertex in the Euclidean cone over the interval of length $L(\zeta)$. When ζ is surjective in addition, this provides another description of $B(p,r)$.

Using Theorem 1.1, we can define a metric graph structure on $\mathcal{S}(X)$ in a generalized sense (Definition 6.7), and we have:

Corollary 1.4 *Suppose that Σ_p is a connected graph for every $p\in X$. Then with respect to the induced length structure, $\mathcal{S}(X)$ is isometric to a metric graph having (possibly locally uncountably many vertices, but) the vertices of locally finite order.*

Remark 1.5 In the general dimension, Lytchak and Nagano [17] characterized the singular set in the k–dimensional part as a countably $(k-1)$–rectifiable set. Corollary 1.4 gives a refinement of this result in dimension two.

Recall that a compact metric graph Σ is a $\mathrm{CAT}(\mu)$–graph ($\mu>0$) if every noncontractible loop in Σ has length $\geq 2\pi/\sqrt{\mu}$.

Corollary 1.6 *For a given $p \in X$ such that Σ_p is a connected graph, there exists a positive number r_p such that for every $0 < r \le r_p$, $S(p,r)$ with the interior metric is a* CAT$(\mu_\kappa(r))$*–graph having the same homotopy type as Σ_p, where $\mu_\kappa(r)$ is given by the sharp constant*

$$\mu_\kappa(r) = \begin{cases} \left(\dfrac{\sin\sqrt{\kappa}r}{\sqrt{\kappa}}\right)^{-2} & \text{if } \kappa > 0,\\ r^{-2} & \text{if } \kappa = 0,\\ \left(\dfrac{\sinh\sqrt{-\kappa}r}{\sqrt{-\kappa}}\right)^{-2} & \text{if } \kappa < 0. \end{cases}$$

Remark 1.7 A result in [17] shows that for every small r, $S(p,r)$ has the same homotopy type as Σ_p in the general dimension. Corollary 1.6 gives a refinement of this result in dimension two.

Remark 1.8 All the results in this paper are local. Therefore they are also valid under the assumption of local geodesic completeness of X.

The idea of the proof of the main result is as follows. We know the structure of the space Σ_p of directions at p, which is completely characterized as a CAT(1)–graph without endpoints. If we rescale the metric of X by the factor $1/r$, then $((1/r)X, p)$ converges to the tangent cone (K_p, o_p) at p as $r \to 0$ with respect to the pointed Gromov–Hausdorff topology. Let Σ_p^{sing} be a small neighborhood of the vertices of the graph Σ_p, and Σ_p^{reg} the complement of Σ_p^{sing}. Now the convergence theorem (see Nagano [23]) applied to the unit cone $K_1(\Sigma_p^{\mathrm{reg}})$ over Σ_p^{reg} yields the existence of a Lipschitz domain $B^{\mathrm{reg}}(p,r)$ of $B(p,r)$ consisting of finitely many sectors corresponding to sectors of $K_1(\Sigma_p^{\mathrm{reg}})$. One can consider $B^{\mathrm{reg}}(p,r)$ as a regular part of $B(p,r)$. The main problem is to determine the structure of the singular part $B^{\mathrm{sing}}(p,r)$, the complement of $B^{\mathrm{reg}}(p,r)$ in $B(p,r)$. To carry out this, we consider finitely many thin ruled surfaces, say S_{ij} here, and fill $B^{\mathrm{sing}}(p,r)$ using them. A key is to show that those ruled surfaces are CAT(κ)–spaces with respect to the *interior metrics* and are homeomorphic to a disk. According to Alexandrov's result in [1], every ruled surface in a CAT(κ)–space is also a CAT(κ)–space with respect to the *pullback metric*. Obviously, the interior metric and the pullback metric are completely different from each other in general. Therefore we have to show that in our thin ruled surfaces, pullback metrics coincide with the interior metrics. After achieving this, it turns out that the topological singular point set $\mathcal{S}(X)$ locally arises from the intersections of those thin ruled surfaces S_{ij}. We investigate how those ruled surfaces meet each other to get the structure of $\mathcal{S}(X) \cap B(p,r)$ as the union of finitely many Lipschitz curves. Combining the structures of both $B^{\mathrm{reg}}(p,r)$ and $B^{\mathrm{sing}}(p,r)$ and considering the graph structure of Σ_p, we define the embeddings or the branched immersions $f_i : D^2(\ell_i; r) \to B(p,r)$ as described in Theorem 1.1.

As related studies on ruled surfaces, Petrunin-Stadler [26] have proved that for metric minimizing disks in CAT(0)–spaces, the pullback metrics on the disks are CAT(0), which is a generalization of Alexandrov's result [1] on ruled surfaces in the CAT(0)–setting. According to Stadler [29, Theorem 2], for any Jordan triangle in a CAT(0)–space, every minimal disk filling of the triangle is an embedded disk that is CAT(0) with respect to the interior metric.

The organization of the paper is as follows.

In Section 2, we recall and verify basic results for locally compact, geodesically complete Alexandrov spaces with curvature bounded above.

In Section 3, we give basic properties of a ruled surface S in a CAT(κ)–space. We discuss the pullback metric, the induced metric, the interior metric of S and their relations. In the original argument in Alexandrov [1], there are several unclear points for the authors — for instance, there is no description of quasicontinuous monotone representations. We make clear all these points.

In Section 4, which is a key section, we investigate a thin ruled surface S in a two-dimensional space, and prove that S actually admits the induced metric and therefore becomes a CAT(κ)–space with respect to the interior metric. Then we obtain the crucial property that S is homeomorphic to a disk.

In Section 5, we fill $B(p,r)$ via those embedded/branched immersed disks using thin ruled surfaces essentially. We prove Theorem 1.1(1)–(3), with the exception of (1) for branched immersed disks.

In Section 6, we describe $\mathcal{S}(X) \cap B(p,r)$ as a union of finitely many Lipschitz curves starting from p and reaching points of $S(p,r)$. The structure of generalized metric graph of $\mathcal{S}(X)$ is also discussed there.

In Section 7, we provide the proof of Theorem 1.1(1) for branched immersed disks as well as Corollary 1.3.

In the appendix, following the basic idea of [1], we give the proof of Alexandrov's result on ruled surfaces in CAT(κ)–spaces based on the results proved in Section 3.

Burago and Buyalo [12] gave a complete characterization of two-dimensional polyhedra of curvature bounded above. In the second part [24] of our work, we show the following:

(a) We provide sufficient conditions for two-dimensional metric spaces to have curvature bounded above, which shows that the results in this paper completely characterize the local structure of two-dimensional metric spaces with curvature bounded above.

(b) Every pointed two-dimensional geodesically complete locally CAT(κ)–space (X, p) can be approximated by a sequence of two-dimensional pointed geodesically complete, polyhedral locally CAT(κ)–spaces (X_n, p_n) having *the same homotopy type as* X with respect to the pointed Gromov–Hausdorff topology. This solves a problem raised by Burago and Buyalo [12].

(c) We establish a Gauss–Bonnet type theorem for two-dimensional geodesically complete locally CAT(κ)–spaces.

Most results in the present paper were announced in [31].

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2 Basic properties of CAT(κ)–spaces

For some basic results in this section, we refer to Bridson and Haefliger [10] and Burago, Burago and Ivanov [11].

The distance between two points x and y in a metric space X is denoted by $|x,y|$ or $|x,y|_X$, and sometimes $d(x,y)$ or $d_X(x,y)$. The metric r–ball around p is denoted by $B(p,r)$. We sometimes use $B^X(p,r)$ to emphasize the metric ball in X. Let X be a locally compact, complete geodesic space with curvature $\le \kappa$. By definition, for each point $p \in X$, there exists a positive number $r > 0$ with $r \le \pi/2\sqrt{\kappa}$ when $\kappa > 0$ such that the ball $B(p,r)$ is convex and has the following properties: Let M_κ^2 be the simply connected complete surface of constant curvature κ, called the *κ–plane* for short. For any geodesic triangle $\triangle xyz$ in $B(p,r)$ with vertices x, y and z, we denote by $\tilde{\triangle} xyz$ a *comparison triangle* in M_κ^2 having the same side lengths as $\triangle xyz$. Then the natural mapping $\tilde{\triangle} xyz \to \triangle xyz$ is nonexpanding. A convex domain with this property is called a CAT(κ)*–domain*. Such a space X with curvature $\le \kappa$ is called a locally CAT(κ)–space, and X is called a CAT(κ)–space if X itself is a CAT(κ)–domain. Although all geodesics have constant speed by definition, most geodesics are assumed to have unit speed unless otherwise stated. For arbitrary x and y in $B(p,r)$, let $\gamma_{x,y}\colon [0,|x,y|] \to X$ denote a unique minimal geodesic joining x to y. We say that a curve is *shortest* if its length is minimal among all curves joining the endpoints. The angle between the geodesics $\gamma_{y,x}$ and $\gamma_{y,z}$ is denoted by $\angle xyz$, and the corresponding angle of $\tilde{\triangle} xyz$ by $\tilde{\angle} xyz$. The space of directions and the tangent cone of X at p are denoted by $\Sigma_p = \Sigma_p(X)$ and $K_p = K_p(X)$, respectively. We shall occasionally use the identification $\Sigma_p = \Sigma_p \times \{1\} \subset K_p$. We denote by $\dot\gamma_{x,y}(0)$ or $\uparrow_x^y$ the direction at x defined by $\gamma_{x,y}$. For every $\xi \in \Sigma_p(X)$, γ_ξ denotes a geodesic with $\dot\gamma_\xi(0) = \xi$. For a path-connected subset $S \subset X$ and $x, y \in S$, we denote by $\gamma_{x,y}^S$ a shortest curve in S joining x to y if it exists. Occasionally, we identify a geodesic with its image, and write $x \in \gamma$ for instance. The length metric of S induced from X is denoted by d_S or $|\ ,\ |_S$.

For a closed subset A of X and for an accumulation point p of A, the set of all directions $\xi \in \Sigma_p(X)$ such that there is a sequence a_i in $A \setminus \{p\}$ satisfying $a_i \to p$ and $\dot\gamma_{p,a_i}(0) \to \xi$ is denoted by $\Sigma_p(A)$ and called the space of directions of A at p.

The upper semicontinuity of angle is fundamental in the geometry of spaces with curvature bounded above.

Lemma 2.1 *Suppose that sequences p_i, q_i and r_i converge to p, q and r, respectively, in a* CAT(κ)*–domain. Then we have* $\limsup_{i\to\infty} \angle p_i q_i r_i \le \angle pqr$.

Next we briefly discuss the connectivity of a small neighborhood of a given point in X. For each point $p \in X$, the set of components of Σ_p are in one-to-one correspondence with the set of components of $B(p,r) \setminus \{p\}$ if $B(p,r)$ is a CAT(κ)–domain; see [16]. We call the number of components of $\Sigma_p(X)$ *the order* of p.

Now we state the gluing theorem proved by [27], which is convenient to construct spaces with curvature bounded above. The proof is also found in [10, page 347].

Theorem 2.2 *Let D_i for $i = 1, 2$ be a closed convex subset in an Alexandrov space X_i with curvature $\le \kappa$. If there is an isometry $f : D_1 \to D_2$, then the identification space $X_1 \cup_f X_2$ is an Alexandrov space with curvature $\le \kappa$ with respect to the natural length metric.*

From now, we assume X to be *geodesically complete*. That is, every geodesic segment in X can be extended to a geodesic defined on $\mathbb{R}$.

The following lemma follows from [17, Corollary 13.3].

Lemma 2.3 *For a point $p \in X$, suppose that $\Sigma_p(X)$ has no isolated points. Then there exists a positive number r such that every point x in $B(p,r) \setminus \{p\}$ has order one.*

Let d_p denote the distance function from p. For every $x \ne p$, let us denote by $(\nabla d_p)(x)$ the set of all directions $\xi \in \Sigma_x(X)$ such that $\angle(\xi, \uparrow_x^p) = \pi$. For simplicity, we set $-(\nabla d_p)(x) = \uparrow_x^p$. The following lemma, which describes local geometry around a given point, is basic in our study of local structure of surfaces with curvature bounded above.

Let $\tau_p(\epsilon_1, \dots, \epsilon_k)$ be a function depending on p and $\epsilon_1, \dots, \epsilon_k$ such that $\lim_{\epsilon_1,\dots,\epsilon_k \to 0} \tau_p(\epsilon_1, \dots, \epsilon_k) = 0$.

Lemma 2.4 *For every $p \in X$, there exists a positive number r_0 such that for every r with $0 < r \le r_0$, the ball $B(p,r)$ satisfies the following:*

(1) $\operatorname{diam}((\nabla d_p)(x)) < \tau_p(r)$ *for every $x \in B(p,r) \setminus \{p\}$.*

(2) *For any two geodesics γ_1 and γ_2 starting at p with angle θ, and for every $s \in [0,r]$, the geodesic $\sigma_s(t)$ joining $\gamma_1(s)$ to $\gamma_2(s)$ satisfies*

$$\left|\angle\big(-(\nabla d_p)(\sigma_s(t)), \dot\sigma_s(t)\big) - \tfrac{1}{2}\pi\right| < \tau_p(\theta, s).$$

Proof (1) is due to [25]; see also [17, Proposition 7.3]. (2) easily follows from (1), and hence the proof is omitted. □

The following lemma is fundamental, and plays an important role as in the case of Alexandrov space with curvature bounded below [13]. For the proof, see [23, Lemma 3.6].

Lemma 2.5 (Jack lemma) *For every $p \in X$, there is a positive number r_0 such that if $x \neq y \in B(p, r_0)$ and q satisfy that $\tilde{\angle} pxq > \pi - \epsilon$ and $|x, y| < \epsilon \min\{|p, x|, |q, x|\}$, then we have*

$$|\angle pxy - \tilde{\angle} pxy| < \tau_p(|p, x|, \epsilon).$$

In the study of spaces of curvature bounded below, the theory of the Gromov–Hausdorff convergence has been useful. We apply it in our case of curvature bounded above.

We denote by $\mathcal{H}^n$ the n–dimensional Hausdorff measure, and set $\omega_n := \mathcal{H}^n(S^n(1))$, where $S^n(1)$ is the unit n–sphere.

Theorem 2.6 ([23], compare [30]) *For each positive integer n, there is a positive number ϵ_n satisfying the following: Let X_i for $i = 1, 2, \dots$ and X be n–dimensional locally compact, geodesically complete, pointed Alexandrov spaces with curvature $\le \kappa$, and suppose that a compact $\mathrm{CAT}(\kappa)$–domain U_i of X_i converges to a compact $\mathrm{CAT}(\kappa)$–domain U of X with respect to the Gromov–Hausdorff distance. Then for every compact domain V in $\operatorname{int} U$ satisfying $\mathcal{H}^{n-1}(\Sigma_x(X)) < \omega_{n-1} + \epsilon$ with $\epsilon \le \epsilon_n$ for all $x \in V$, there are a compact domain V_i in $\operatorname{int} U_i$ and a $\tau(\epsilon, 1/i)$–almost isometry $\varphi_i : V_i \to V$ in the sense that*

$$\left| \frac{|\varphi_i(x), \varphi_i(y)|}{|x, y|} - 1 \right| < \tau\Big(\epsilon, \frac{1}{i}\Big) \quad \text{for all } x, y \in V_i.$$

Lemma 2.7 *For every $p \in X$ and arbitrary $x, y \in B(p, r)$ we have*

$$\tilde{\angle} xpy - \angle xpy < \tau_p(r), \quad \tilde{\angle} pxy - \angle pxy < \tau_p(r), \quad \tilde{\angle} pyx - \angle pyx < \tau_p(r). \tag{2-1}$$

Proof For the proof, it suffices to show that for every $\epsilon > 0$ there is an r such that for arbitrary $x, y \in B(p, r)$ we have (2-1) for ϵ in place of $\tau_p(r)$. Fix a constant $C \ge 1$.

Case 1 $C^{-1} \le |p, x|/|p, y| \le C$.

We show (2-1) for $\epsilon = \tau_{p,C}(r)$, where $\tau_{p,C}(\cdot)$ is a function depending on p, C with $\lim_{r \to 0} \tau_{p,C}(r) = 0$. Suppose $|x, y| < \zeta |p, x|$ for $\zeta > 0$. Then Lemma 2.5 implies

$$\tilde{\angle} pxy - \angle pxy < \tau_p(r, \zeta) \quad \text{and} \quad \tilde{\angle} pyx - \angle pyx < \tau_p(r, \zeta).$$

Since $\tilde{\angle} xpy < \tau(\zeta)$, (2-1) holds when $r \le r_0$ and $\zeta \le \zeta_0$ for some r_0 and ζ_0. Next, suppose $|x, y| \ge \zeta_0 |p, x|$. We proceed by contradiction. Suppose the lemma does not hold in this case. Then there are $x_n, y_n \to p$

with $|x_n, y_n| \ge \zeta_0 |p, x_n|$. Choose z_n such that $x_n \in \gamma_{p,z_n}$ and $|z_n, x_n| = |p, x_n|$. Consider the rescaling limit

$$\left(\frac{1}{|p, x_n|} X, p\right) \to (K_p, o_p).$$

Since $\lim_{n\to\infty}(\angle p x_n y_n + \angle z_n x_n y_n) = \pi$, we have

$$\lim_{n\to\infty} \angle p x_n y_n = \angle o_p x_\infty y_\infty = \lim_{n\to\infty} \tilde{\angle} p x_n y_n,$$

where x_∞ and y_∞ are the limits of x_n and y_n. Similarly, we have $\lim_{n\to\infty} \angle p y_n x_n = \lim_{n\to\infty} \tilde{\angle} p y_n x_n$. Since obviously we have

$$\lim_{n\to\infty} \angle x_n p y_n = \angle x_\infty o_p y_\infty = \lim_{n\to\infty} \tilde{\angle} x_n p y_n,$$

we derive a contradiction.

Case 2 $|p, x| < C^{-1}|p, y|$.

We show (2-1) for $\epsilon = \tau_p(r) + \tau(C^{-1})$. First note that

$$\tilde{\angle} p y x < \tau(C^{-1}). \tag{2-2}$$

Thus considering large C, we only have to consider the angles at p and x. Take z with $x \in \gamma_{p,z}$ and $|z, p| = |y, p|$. Then we have

$$\tilde{\angle} y p z \ge \tilde{\angle} y p x \ge \angle y p x. \tag{2-3}$$

From Case 1, we have

$$\tilde{\angle} y p z - \angle y p x < \tau_{p,1}(r). \tag{2-4}$$

Combining (2-3) and (2-4), we certainly have

$$\tilde{\angle} y p x - \angle y p x < \tau_{p,1}(r). \tag{2-5}$$

From $|p, x| < C^{-1}|p, y|$, we have

$$|\tilde{\angle} x y z - \tilde{\angle} p y z| < \tau(C^{-1}) \quad \text{and} \quad |\tilde{\angle} x z y - \tilde{\angle} p z y| < \tau(C^{-1}). \tag{2-6}$$

From (2-5) and (2-4), we have

$$|\tilde{\angle} x p y - \tilde{\angle} z p y| < \tau_{p,1}(r).$$

From (2-2) and the first inequality in (2-6), we have

$$|\tilde{\angle} p y x + \tilde{\angle} x y z - \tilde{\angle} p y z| < \tau(C^{-1}).$$

Now consider the quadrangle $\tilde{p}\tilde{x}\tilde{z}\tilde{y}$ on M^2_κ which is the union of the triangles $\tilde{\triangle} p x y$ and $\tilde{\triangle} x y z$ glued along the edge $\tilde{x}\tilde{y}$ corresponding to xy. We estimate the deviation of the angle of the quadrangle $\tilde{p}\tilde{x}\tilde{z}\tilde{y}$ at $\tilde{x}$ from π. Combining the last two inequalities and the second inequality in (2-6), we have

$$|\tilde{\angle} p x y + \tilde{\angle} y x z - \pi| < \tau_{p,1}(r) + \tau(C^{-1}).$$

Since

$$|\angle p x y + \angle y x z - \pi| < \tau_p(r),$$

the last two inequalities yield $\tilde{\angle} p x y - \angle p x y < \tau_p(r) + \tau(C^{-1})$, as required. □

A point p in X is called a *topological singular point* of X if any neighborhood of p is not homeomorphic to a disk, and the set of all topological singular point of X is denoted by $\mathcal{S}(X)$. It is proved in [17] that if $\dim X = n$, then $\dim_H \mathcal{S}(X) \le n-1$. In particular $X \setminus \mathcal{S}(X)$ has full measure with respect to $\mathcal{H}^n$ [25].

Two-dimensional case By a result of Otsu-Tanoue [25], the Hausdorff dimension of every relatively compact open domain of X is an integer. See [17] for a different proof. It is also known that $\Sigma_p(X)$ is a compact geodesically complete CAT(1)–space for every $p \in X$.

Obviously, if X is 1–dimensional, then it is a locally finite graph without endpoints. Now we assume X has dimension 2. Then any component Σ of Σ_p has dimension ≤ 1. If $\dim \Sigma = 1$, then Σ has the structure of a finite graph without endpoints. Furthermore Σ is a so called CAT(1)–*graph* without endpoints in the sense that each simple closed curve in Σ has length at least 2π. If $\dim \Sigma = 0$, then Σ is a point and the component of $B(p,r) \setminus \{p\}$ corresponding to Σ is an arc for any small enough r. Thus, a small neighborhood of any point $p \in X$ is the gluing at p of several purely 2–dimensional spaces with all links connected graphs and a ball around the vertex in the cone over finitely many points. Therefore the study of local structure around p reduces to the case when Σ_p is a connected CAT(1)–graph without endpoints.

Lemma 2.8 *A neighborhood of $p \in X$ is homeomorphic to a two-dimensional disk if and only if $\Sigma_p(X)$ is a circle.*

Proof This follows from [22, Proposition 3.1, Remark 3.4]. □

Lemma 2.9 *Let $p \in \mathcal{S}(X)$. Then $\Sigma_p(\mathcal{S}(X))$ coincides with the set $V(\Sigma_p(X))$ of all vertices of the graph $\Sigma_p(X)$.*

Proof For every $v \in \Sigma_p(\mathcal{S}(X))$, take a sequence x_i in $\mathcal{S}(X)$ converging to p which is such that $\lim_{i\to\infty} \angle(\dot\gamma_{p,x_i}(0), v) = 0$. If v is not a vertex of $\Sigma_p(X)$, choose $\epsilon > 0$ such that the ϵ–neighborhood of v contains no vertices of $\Sigma_p(X)$. Let $\delta_i := |x_i, p|$. Theorem 2.6 applied to the convergence

$$\left(\frac{1}{\delta_i} X, x_i\right) \to (K_p(X), v)$$

yields that a small neighborhood of x_i is almost isometric to a neighborhood in $\mathbb{R}^2$. This is a contradiction.

Conversely, suppose there is $v \in V(\Sigma_p(X))$ that is not contained in $\Sigma_p(\mathcal{S}(X))$. Choose $\epsilon_0 > 0$ and $\delta_0 > 0$ such that the cone neighborhood

$$C(v, \delta_0, \epsilon_0) := \{x \mid \angle(\uparrow_p^x, v) \le \delta_0, |p, x| \le \epsilon_0\} \tag{2-7}$$

is included in $\mathcal{R}(X)$. Take three distinct directions $\xi_1, \xi_2, \xi_3 \in \Sigma_p(X)$ having angle $\delta_0/2$ with v, and set $x_i(\epsilon) := \gamma_{\xi_i}(\epsilon)$, where $\epsilon \le \epsilon_0$, $1 \le i \le 3$. Note that the geodesic $[x_1(\epsilon), x_2(\epsilon)]$ converges to the geodesic $[\xi_1, \xi_2]$ in $K_p(X)$ under the convergence

$$\left(\frac{1}{\epsilon} X, p\right) \to (K_p(X), o_p) \quad \text{as } \epsilon \to 0.$$

Let $y(\epsilon)$ be a nearest point of $[x_1(\epsilon), x_2(\epsilon)]$ from $x_3(\epsilon)$. Since $y(\epsilon) \in \mathcal{R}(X)$, $\Sigma_{y(\epsilon)}(X)$ must be a circle of length almost equal to 2π with $\angle(\nabla d_p(y(\epsilon)), -\nabla d_p(y(\epsilon))) = \pi$. However, Lemma 2.4 shows that the angle $\angle(\nabla d_p(y(\epsilon)), \eta_i(\epsilon))$ is almost $\pi/2$, where $\eta_i := \uparrow_{y(\epsilon)}^{x_i(\epsilon)}$ for $1 \le i \le 3$, which implies $\angle(\eta_1(\epsilon), \eta_2(\epsilon))$ is almost equal to 0. This is a contradiction since $\angle(\eta_1(\epsilon), \eta_2(\epsilon)) = \pi$. □

Remark 2.10 In place of the above geometric argument of the second half of the proof of Lemma 2.9, we can also use more general topological result in [14, Theorem 2.1].

Lemma 2.11 *Let $p \in \mathcal{S}(X)$. For any $x \in \mathcal{S}(X) \cap (B(p,r) \setminus \{p\})$, $V(\Sigma_x(X))$ is contained in the $\tau_p(r)$–neighborhood of $\{(-\nabla d_p)(x), (\nabla d_p)(x)\}$.*

Therefore there is a positive integer $m \ge 3$ such that the Gromov–Hausdorff distance between $\Sigma_x(X)$ and the spherical suspension over m points is less than $\tau_p(r)$.

Proof This follows from [17, Proposition 6.6, Corollary 13.3]. □

As an immediate consequence of Lemmas 2.9 and 2.11, we have

Corollary 2.12 *Let $p \in \mathcal{S}(X)$. For every $x \in \mathcal{S}(X) \cap (B(p,r) \setminus \{p\})$, $\Sigma_x(\mathcal{S}(X))$ is contained in a $\tau_p(r)$–neighborhood of $\{(-\nabla d_p)(x), (\nabla d_p)(x)\}$.*

Finally in this subsection, we shortly discuss the cardinality of singular points in a two-dimensional manifold X with curvature $\le \kappa$. Let $\epsilon > 0$. We say that $x \in X$ is an *ϵ–singular* point if $L(\Sigma_x(X)) \ge 2\pi + \epsilon$. We also say that x is a *singular* point if it is ϵ–singular for some $\epsilon > 0$.

Lemma 2.13 (see [3] and [12, Proposition 4.5]) *For a domain D of a two-dimensional manifold X with curvature $\le \kappa$, the set of all singular points contained in D is at most countable.*

Proof By Lemma 2.4(1), the set of all ϵ–singular points contained in a bounded set is finite for every $\epsilon > 0$, which immediately yields the conclusion of the lemma. □

3 Basic properties of ruled surfaces

We recall the notion of ruled surfaces in metric spaces introduced by Alexandrov [1]. The metric on a ruled surface discussed in [1] is the pullback metric defined below, although an explicit definition was not given in [1]. See also Remark 3.4. In this section, we provide some fundamental properties of the pullback metric, most of which are not contained in [1]. These are used in the proof of Alexandrov's result (Theorem 3.17), which is presented in the appendix. There are related results in [26, Section 2].

For our purpose, it is sufficient to consider ruled surfaces in spaces with curvature bounded above. Throughout this section, let X be a locally compact, complete geodesic space with curvature $\le \kappa$ with metric d_X, where we do not need the dimension restriction, nor geodesic completeness.

We fix a rectangle $R := [0, \ell] \times [0, 1]$ in this section.

Ruled surfaces

Definition 3.1 A continuous map $\sigma\colon R \to X$ is called a *ruled surface* in X if

(1) for every $s \in [0, \ell]$ the t–curve $\lambda_s\colon [0, 1] \to X$ of σ defined as $\lambda_s(t) := \sigma(s, t)$ is a minimal geodesic in X from $\sigma(s, 0)$ to $\sigma(s, 1)$, and

(2) for some continuous function $\xi\colon [0, \ell] \to [0, 1]$, the curve $\Sigma(s) = \sigma(s, \xi(s))$, where $0 \le s \le \ell$, is rectifiable with respect to d_X.

As usual, the subset S of X defined as $S := \sigma(R)$ is also called a *ruled surface in* X. For each $s \in [0, \ell]$, the minimal geodesic $\lambda_s\colon [0, 1] \to X$ is called a *generator* of σ, or a *ruling geodesic* of σ.

For each $t \in [0, 1]$, the curve $\sigma_t\colon [0, \ell] \to X$ is called a *directrix of* σ *at* t.

Pullback metrics and induced metrics on ruled surfaces

Let $\sigma\colon R \to X$ be a ruled surface in X defined as above. We denote by $\mathrm{Sing}(\sigma)$ (resp. by $\mathrm{Reg}(\sigma)$) the set of all $s \in [0, \ell]$ such that λ_s are constant (resp. nonconstant). For $s \in [0, \ell]$, we set

$$I_s := \{s\} \times [0, 1] \subset R.$$

Definition 3.2 We say that a (not necessarily continuous) map $c\colon [a, b] \to R$ is *monotone*

- if $p_1 \circ c$ is monotone nondecreasing or monotone nonincreasing, where $p_1\colon R \to [0, \ell]$ is the projection to the first factor, and
- if $p_1 \circ c(t) = p_1 \circ c(t') = s$ with $t < t'$, then $p_2 \circ c|_{[t,t']}$ is monotone, where $p_2\colon R \to [0, 1]$ is the projection to the second factor.

Similarly, c is said to be *strictly monotone* if $p_1 \circ c$ is strictly monotone.

We say that a monotone map $c\colon [a, b] \to R$ is a *quasicontinuous curve* if the following hold:

(1) $p_1 \circ c([a, b])$ is a closed interval, and

(2) c is continuous on the set of all t with $p_1 \circ c(t) \in \mathrm{Reg}(\sigma) \cup \mathrm{int}\,\mathrm{Sing}(\sigma)$.

We define the *pullback metric* e_σ on R induced from σ as

$$e_\sigma(u, u') := \inf_c L(\sigma \circ c), \tag{3-1}$$

where c runs over all quasicontinuous curves in R from u to u', and L denotes the length of curves with respect to d_X. Note that the metric e_σ is certainly finite since our ruled surface σ has the rectifiable curve Σ.

We denote by R_* the quotient metric space

$$(R_*, e_\sigma) := (R, e_\sigma)/\{e_\sigma = 0\}.$$

Let $\pi\colon R \to R_*$ be the projection.

Example 3.3 Let $\sigma_k : [0, 1/\pi] \to \mathbb{R}^2$ for $k = 0, 1$ be the curve defined by

$$\sigma_0(s) = \left(s, -\left|s \cos \frac{1}{s}\right|\right) \quad \text{and} \quad \sigma_1(s) = \left(s, \left|s \sin \frac{1}{s}\right|\right).$$

For $R := [0, 1/\pi] \times [0, 1]$, define the ruled surface $\sigma : R \to \mathbb{R}^2$ as in the above definition, where we have $\mathrm{Sing}(\sigma) = \{0\}$. For $u = (0, 0)$, $u' = (1/\pi, 0)$, consider the map $c : [0, 1/\pi] \to R$ defined by

$$c(s) = \begin{cases} \sigma^{-1}(s, 0) & \text{if } 0 < s \le 1/\pi, \\ (0, 0) & \text{if } s = 0. \end{cases}$$

Since c oscillates infinitely many times near $\{0\} \times [0, 1]$, it is quasicontinuous, but realizes the distance $e_\sigma(u, u')$.

We remark that there is no continuous curve realizing $e_\sigma(u, u')$ in Example 3.3. Note also that $\pi \circ c$ is always continuous for every quasicontinuous curve c. These are the reasons why we employ the notion of quasicontinuous curves in the definition (3-1) of the pullback metric e_σ.

Obviously, $e_\sigma(u, u') = 0$ implies $\sigma(u) = \sigma(u')$. Therefore, we can define a map $\sigma_* : R_* \to X$ such that $\sigma = \sigma_* \circ \pi$. Note that $\sigma_* : R_* \to X$ is continuous. The properties of the projection $\pi : R \to R_*$ depend on those of the end s–curves σ_0 and σ_1. If σ_0 and σ_1 are Lipschitz continuous, then so is σ, and hence $\pi : R \to R_*$ is continuous. However, in the general case, $\pi : R \to R_*$ is not necessarily continuous; see Example 3.5 below.

Remark 3.4 Comparing the conditions of ruled surfaces given in [1] and ours:

(1) Some ruling geodesics λ_s of σ may be constant geodesics for all s in an interval of $[0, \ell]$. This case was excluded in [1] as the conditions of ruled surfaces defined there.

(2) The existence of continuous arc in the preimage of any point of R_* by π, which is a more restrictive property than the existence of quasicontinuous curve given in Corollary 3.8, was assumed in [1] as one of the conditions of the metric on the ruled surface under consideration.

Example 3.5 Let $\sigma_1(s)$, where $0 \le s \le \ell$, be a continuous parametrization of a Koch curve on the unit sphere $\mathbb{S}^2(1) \subset \mathbb{R}^3$. Letting $\sigma_0(s) = O$, we define the ruled surface $\sigma : [0, \ell] \times [0, 1] \to \mathbb{R}^3$ by $\sigma(s, t) = t\sigma_1(s)$. Note that

$$e_\sigma((s, t), (s', t')) = \begin{cases} |t - t'| & \text{if } s = s', \\ t + t' & \text{if } s \ne s'. \end{cases}$$

Note also that $\pi : R \to R_*$ is continuous only at $\{t = 0\}$.

For $s \in [0, \ell]$, we set

$$I_s^* := \pi(I_s).$$

For a continuous curve $c_* : [a_0, b_0] \to R_*$ (resp. $\gamma : [a_0, b_0] \to S$), we simply say that a quasicontinuous curve $c : [a, b] \to R$ is a *lift* of c_* (resp. of γ) if $c_* = \pi \circ c$ (resp. $\gamma = \sigma \circ c$) up to monotone parametrization.

From now, we fix arbitrary $u, u' \in R$ with $u = (s_0, t_0), u' = (s_0', t_0')$ and $s_0 < s_0'$. Take a sequence of quasicontinuous curves $c_n : [0,1] \to R$ from u to u' such that $e_\sigma(u,u') = \lim_{n\to\infty} L(\sigma \circ c_n)$, where we may assume that c_n is monotone. By the Arzelà–Ascoli theorem, passing to a subsequence we may assume that a Lipschitz parametrization γ_n of $\sigma \circ c_n$ converges to a Lipschitz curve γ in S from $\sigma(u)$ to $\sigma(u')$. Note that

$$L(\gamma) \le e_\sigma(u,u'). \tag{3-2}$$

We set

$$J := [s_0, s_0'], \quad J_{\mathrm{reg}} := J \cap \mathrm{Reg}(\sigma), \quad J_{\mathrm{sing}} := J \cap \mathrm{Sing}(\sigma).$$

In the following proposition, we show the equality in (3-2).

Proposition 3.6 *Under the above situation, there is a lift c of γ in R from u to u'.*

In particular, $\pi \circ c$ provides a (continuous) shortest curve c_* in R_* from $\pi(u)$ to $\pi(u')$, and we have

$$L(c_*) = L(\gamma) = e_\sigma(\pi(u), \pi(u')).$$

Example 3.7 Let $\gamma : [0,1] \to X$ be a minimal geodesic between distinct two points in a CAT(κ)–space. Consider the ruled surface $\sigma : [0,1] \times [0,1] \to X$ defined as $\sigma(s,t) = \gamma(t)$. Then $e_\sigma((0,0),(1,1)) = L(\gamma)$. Note that any curve $c(t) = (x(t), y(t))$ such that $x(t)$ and $y(t)$ are monotone from 0 to 1 is a lift of γ from $(0,0)$ to $(1,1)$.

The above simple example suggests that in Proposition 3.6, one cannot construct a lift of the limit γ only from γ, and one needs to take a subsequence of c_n properly to obtain a limit, which is expected as a lift of γ. In the proof of Proposition 3.6 below, we proceed in this way.

Proof of Proposition 3.6 We show that the monotone curve c_n converges to a monotone quasicontinuous curve c, up to monotone parametrization, except on $\mathrm{Sing}(\sigma) \times [0,1]$. By the Arzelà–Ascoli theorem, this is obvious if the length of $c_n|_{J_{\mathrm{reg}} \times [0,1]}$ is uniformly bounded. However, when one of the end curves $\sigma_0(s)$ and $\sigma_1(s)$ is not rectifiable, one cannot expect that the length of $c_n|_{J_{\mathrm{reg}} \times [0,1]}$ is even finite.

In the argument below, we use the idea of the proof of the Arzelà–Ascoli theorem taking the monotonicity of c_n into account. Since each c_n is continuous, for any $s \in J$ there is $t_n(s) \in [0,1]$ satisfying $c_n(t_n(s)) \in I_s$. Let J_0 be a countable dense subset of J. Take a subsequence $\{m\} \subset \{n\}$ such that $c_m(t_m(s))$ converges to a point $x(s) \in I_s$ for every $s \in J_0$.

Roughly speaking, the limit curve c is defined via the limit set of the sequence $\{\mathrm{Im}(c_m)\}_m$. For every $s \in J_{\mathrm{reg}}$, let us consider the subset $E_s \subset I_s$ defined as the set of all points $x \in I_s$ with $\lim_{i\to\infty} c_{m_i}(t_i) = x$ for a subsequence $\{m_i\} \subset \{m\}$ and $t_i \in [0,1]$. We set

$$J_{\mathrm{reg},1} := \{s \in J_{\mathrm{reg}} \mid E_s \text{ is a single point}\}, \quad J_{\mathrm{reg},2} := J_{\mathrm{reg}} \setminus J_{\mathrm{reg},1},$$
$$J^0_{\mathrm{reg},1} := J_{\mathrm{reg},1} \cap J_0 \qquad J^0_{\mathrm{reg},2} := J_{\mathrm{reg},2} \cap J_0.$$

For $s \in J_{\mathrm{reg},1}$, we define $x(s)$ by

$$E_s = \{x(s)\}.$$

Note also that $J^0_{\mathrm{reg},1}$ or $J^0_{\mathrm{reg},2}$ may be empty.

(1) We first show that $x(s)$ is continuous on $J_{\mathrm{reg},1}$.

This is obvious since if $s_i \in J_{\mathrm{reg},1}$ converges to $s \in J_{\mathrm{reg},1}$, then any limit of $\{x(s_i)\}$ must belong to $E_s = \{x(s)\}$.

(2) Next we show that E_s is an interval for every $s \in J_{\mathrm{reg},2}$.

For arbitrary $y, y' \in E_s$, choose subsequences $\{m_i\}$ and $\{m_{i'}\}$ of $\{m\}$ such that $c_{m_i}(t_i) \to y$ and $c_{m_{i'}}(t'_i) \to y'$ as $i, i' \to \infty$ for some $t_i, t'_i \in [0,1]$. Take $s_j \in J^0_{\mathrm{reg}}$ with $s < s_j$ converging to s. Note that $x(s_j) = \lim_{i\to\infty} c_{m_i}(t_{m_i}(s_j)) = \lim_{i'\to\infty} c_{m_{i'}}(t_{m_{i'}}(s_j))$. Passing to a subsequence, we may assume that $x(s_j)$ converges to a point $z \in E_s$ as $j \to \infty$. As $i, i' \to \infty$ and then $j \to \infty$, the arcs $c_{m_i}([t_i, t_{m_i}(s_j)])$ and $c_{m_{i'}}([t_{i'}, t_{m_{i'}}(s_j)])$ converge to $[y, z]$ and $[y', z]$, respectively. Since $[y,z] \cup [y',z] \subset E_s$, we obtain $[y, y'] \subset E_s$.

(3) For $s_i < s$ (resp. $s_i > s$) with $s_i \in J_{\mathrm{reg}}$, $s \in J_{\mathrm{reg},2}$, let s_i converge to s. In what follows, we show that $x(s_i)$ converges to an endpoint of E_s (resp. the other endpoint of E_s).

Let $\{y, y'\} = \partial E_s$.

(a) We assume $s_i < s$. The other case is similar. Suppose that $x(s_{i_k})$ converges to an interior point v of E_s as $k \to \infty$, for a subsequence $\{i_k\}$ of $\{i\}$. We also have subsequences $\{m_\ell\}$ and $\{m_{\ell'}\}$ of $\{m\}$ such that $c_{m_\ell}(t_\ell) \to y$ and $c_{m_{\ell'}}(t_{\ell'}) \to y'$ for some $t_\ell, t_{\ell'} \in [0,1]$. As $\ell, \ell' \to \infty$ and then $k \to \infty$, the arcs $c_{m_\ell}([t_{m_\ell}(s_{i_k}), t_\ell])$ and $c_{m_{\ell'}}([t_{m_{\ell'}}(s_{i_k}), t_{\ell'}])$ converge to the subarcs $[v, y]$ and $[v, y']$, respectively. Now take a sequence $s_\alpha \in J^0_{\mathrm{reg}}$ with $s_\alpha > s$ such that $x(s_\alpha)$ converges to a point $w \in I_s$ as $\alpha \to \infty$. Note that

$$x(s_\alpha) = \lim_{\ell\to\infty} c_{m_\ell}(t_{m_\ell}(s_\alpha)) = \lim_{\ell'\to\infty} c_{m_{\ell'}}(t_{m_{\ell'}}(s_\alpha)).$$

We see that as $\ell, \ell' \to \infty$ and then $k \to \infty$, the arcs $c_{m_\ell}([t_{m_\ell}(s_{i_k}), t_{m_\ell}(s_\alpha)])$ and $c_{m_{\ell'}}([t_{m_{\ell'}}(s_{i_k}), t_{m_{\ell'}}(s_\alpha)])$ converge to the unions $[v, y] \cup [y, w]$ and $[v, y'] \cup [y', w]$, respectively. However, considering $\sigma \circ c_{m_\ell}$ or $\sigma \circ c_{m_{\ell'}}$, we have a contradiction since $\sigma \circ c_m$ is a sequence minimizing $e_\sigma(u, u')$.

(b) We show that as $s_\alpha < s$ converges to s, then $x(s_\alpha)$ converges to a unique endpoint of E_s. Suppose that for subsequences $s_i \to s$ and $s_{i'} \to s$ with $s_i, s_{i'} < s$, $x(s_i)$ (resp. $x(s_{i'})$) converges to y (resp. to y'). Choose large i and $i' = i'(i)$ with $i' \gg i$. Then as $m \to \infty$, the arc $c_m([t_m(s_i), t_m(s_{i'})])$ oscillates many times near E_s, which implies $\lim_{m\to\infty} L(\sigma \circ c_m) = \infty$. This is a contradiction.

(c) We show that as $s_i \to s$ and $s_{i'} \to s$ with $s_i < s < s_{i'}$, if $x(s_i)$ converges to y, then $x(s_{i'})$ converges to y'. Otherwise, as $m \to \infty$ and $i, i' \to \infty$, the arc $c_m([t_m(s_i), t_m(s_{i'})])$ converges to the union $[y, y'] \cup [y', y]$, which is a contradiction to the hypothesis that $\sigma \circ c_m$ is a minimizing sequence.

(4) We show that $J_{\mathrm{reg},2}$ is at most countable, and

$$\sum_{s \in J_{\mathrm{reg},2}} L(\sigma(E_s)) \le L(\gamma).$$

For an arbitrary finite set $s_1 < s_2 < \cdots < s_k$ of $J_{\mathrm{reg},2}$, the argument in (3c) shows that some subarcs of c_m are so close to the union $E_{s_1} \cup \cdots \cup E_{s_k}$ for any large m. Thus, $\sigma(E_{s_1}) \cup \cdots \cup \sigma(E_{s_k})$ is the union of finite subarcs of γ. Therefore we have

$$\sum_{i=1}^{k} L(\sigma(E_s)) < L(\gamma).$$

The conclusion follows immediately.

(5) For $s \in J_{\mathrm{sing}}$, let $x(s) := (s, a) \in I_s$ for any fixed constant $a \in [0,1]$, for instance. Let L_0 be the total sum of $L(\sigma(E_s))$ for all $s \in J_{\mathrm{reg},2}$. Now we consider the collection $\mathcal{C}$ consisting of points $\{x(s) \mid s \in J_{\mathrm{sing}} \cup J_{\mathrm{reg},1}\}$ and the intervals E_s for all $s \in J_{\mathrm{reg},2}$. In view of (1)–(4), it is possible to parametrize $\mathcal{C}$ as a quasicontinuous curve $c\colon [s_0, s_0' + L_0] \to R$ from u to u'. For details, see (2) in the proof of Proposition 3.14.

From construction, we see that c is a lift of γ.

The second half of the assertion of the proposition is immediate, completing the proof of Proposition 3.6. □

As an immediate consequence of Proposition 3.6, we have:

Corollary 3.8 *If $e_\sigma(u, u') = 0$, then there is a strictly monotone quasicontinuous curve $c\colon [0,1] \to R$ joining u to u' such that $\pi(c) = \pi(u) = \pi(u')$.*

In particular, if $\mathrm{Sing}(\sigma)$ is empty, $\pi^{-1}(\pi(u))$ is a strictly monotone (continuous) curve.

The following example shows that it is impossible to take a monotone (continuous) curve c in Corollary 3.8 as well as in Proposition 3.6.

Example 3.9 Let X be the one-point union of two copies, say $\mathbb{R}^2_0$ and $\mathbb{R}^2_1$, of $\mathbb{R}^2$ at the origin O. Let $\sigma_k(u)$ be straight lines on $\mathbb{R}^2_k$ with $\sigma_k(0) = O$ for $k = 0, 1$. Consider strictly monotone continuous parametrizations $\sigma_k(\varphi_k(s))$ of $\sigma_k(t)$ with $\varphi_k(0) = 0$. Joining $\sigma_0(\varphi_0(s))$ and $\sigma_1(\varphi_1(s))$ by the minimal geodesics, we define a ruled surface $\sigma\colon \mathbb{R} \times [0,1] \to X$. Note that $\mathrm{Sing}(\sigma) = \{0\}$. For each $s \in \mathbb{R} \setminus \{0\}$, let $t(s) \in (0,1)$ be such that $\lambda_s(t(s)) = O$. Thus we have

$$\sigma^{-1}(O) = \{(s, t(s)) \mid s \in \mathbb{R} \setminus \{0\}\} \cup I_0.$$

Now choosing the two parameters $\varphi_0(s)$ and $\varphi_1(s)$ properly, we can let the function $t(s)$ oscillate as $s \to 0$. In that case, for arbitrary $u, u' \in \mathbb{R} \times [0,1]$ with $\sigma(u) = \sigma(u') = O$ and $p_1(u) < 0 < p_1(u')$, there is no continuous curve in $\sigma^{-1}(O)$ joining u and u' but quasicontinuous one.

Next, using the procedure in the proof of Proposition 3.6, we provide a condition for a curve c_* in R_* to have a lift c in R.

Definition 3.10 For $x \in R_*$, we set

$$s(x) := \{s \in [0,\ell] \mid x \in I_s^*\} = p_1(\pi^{-1}(x)), \quad s_{\min}(x) := \min s(x), \quad s_{\max}(x) := \max s(x).$$

Write $s(x) < s(y)$ when $s_{\max}(x) < s_{\min}(y)$, and write $s(A) := \{s \in [0,\ell] \mid A \cap I_s^* \neq \varnothing\}) = p_1(\pi^{-1}(A))$ for a subset A of R_*.

Lemma 3.11 *For any continuous curve $c_*\colon [a,b] \to R_*$, $s(c_*([a,b]))$ is connected.*

Proof Choose $u \in \pi^{-1}(c_*(a))$ and $u' \in \pi^{-1}(c_*(b))$. We may assume $p_1(u) < p_1(u')$. Let $\gamma := \sigma_* \circ c_*$. Since $\pi^{-1}(c_*([a,b])) = \sigma^{-1}(\gamma([a,b]))$, $p_1(\pi^{-1}(c_*([a,b])))$ is closed. Suppose that $p_1(\pi^{-1}(c_*([a,b])))$ is not connected. Then there are some $s_- < s_+$ in $[p_1(u), p_1(u')]$ satisfying

$$\pi^{-1}(c_*([a,b])) \subset [0,s_-] \times [0,1] \cup [s_+,\ell] \times [0,1].$$

Set $R_- := [0,s_-] \times [0,1]$ and $R_+ := [s_+,\ell] \times [0,1]$. In view of Corollary 3.8, we may assume that $\pi^{-1}(c_*(a)) \subset R_-$ and $\pi^{-1}(c_*(b)) \subset R_+$. Let us consider

$$t_- := \sup\{t \mid \pi^{-1}(c_*([a,t])) \subset R_-\}.$$

Note that $\pi^{-1}(c_*(t_-)) \subset R_-$. Take $t_n > t_-$ with $t_n \to t_-$ such that $\pi^{-1}(c_*(t_n)) \subset R_+$. Choose a point $x_n \in \pi^{-1}(c_*(t_n))$. Passing to a subsequence, we may assume that x_n converges to a point $x_\infty \in R_+$. This is a contradiction since $x_\infty \in \pi^{-1}(c_*(t_-))$. □

Definition 3.12 For a continuous curve c_* in R_*, we say that a subset $A_* \subset I_s^*$ is *c_*–convex* if whenever $c_*(t), c_*(t') \in A_*$ with $t \le t'$, then $c_*([t,t']) \subset A_*$. For a continuous curve γ in S, the notion of *γ–convexity* of a subset $\Lambda \subset \lambda_s$ is similarly defined.

Let $c_*\colon [a,b] \to (R_*, e_\sigma)$ be a continuous curve from $\pi(u)$ to $\pi(u')$, and put $s_0 = p_1(u)$, $s_0' = p_1(u')$. For any $s \in [s_0, s_0']$, we consider

$$E_s^* := I_s^* \cap c_*([a,b]),$$

which is nonempty by Lemma 3.11.

Lemma 3.13 *For a continuous curve $c_*\colon [a,b] \to (R_*, e_\sigma)$ with $s_{\min}(c_*(a)) \le s_{\max}(c_*(b))$, suppose that E_s^* is c_*–convex for every $s \in p_1(c_*([a,b]))$. Then we have the monotonicity for all $t < t'$ in $[a,b]$,*

$$s_{\min}(c_*(t)) \le s_{\max}(c_*(t')).$$

Proof Suppose that there are $t_1 < t_2$ such that $s_{\min}(c_*(t_1)) > s_{\max}(c_*(t_2))$. If $s_{\max}(c_*(b)) \ge s_{\min}(c_*(t_1))$, then Lemma 3.11 shows the existence of $t_3 \in [t_2, b]$ such that $s(c_*(t_3))$ meets $s(c_*(t_1))$. This contradicts the c_*–convexity of I_s^* for $s \in s(c_*(t_1)) \cap s(c_*(t_3))$, since $c_*(t_1), c_*(t_3) \in I_s^*$ and $c_*(t_2) \notin I_s^*$. If $s_{\max}(c_*(b)) < s_{\min}(c_*(t_1))$, then we have $s(c_*(b)) < s(c_*(t_1))$ with $s_{\min}(c_*(a)) \le s_{\max}(c_*(b))$. Therefore similarly, we have a contradiction. □

Proposition 3.14 *Let $c_*\colon [a,b] \to (R_*, e_\sigma)$ be a continuous curve with $L(c_*) < \infty$ from $\pi(u)$ to $\pi(u')$. We assume that for each $s \in s(c_*([a,b]))$,*

(1) *E_s^* is c_*–convex, and*

(2) *the restriction $c_*|_{E_s^*}$ is monotone for every E_s^* that is an interval.*

Then there is a lift of c_* in R from u to u'.

Proof (1) Since we only need to construct a lift c on $\mathrm{Reg}(\sigma) \times [0,1]$, we may assume $\mathrm{Sing}(\sigma)$ is empty. If $s(c_*(a))$ meets $s(c_*(b))$, then c_* is a geodesic subarc of I_s^* for $s \in s(c_*(a)) \cap s(c_*(b))$, and hence certainly has a lift in R by Corollary 3.8.

Thus we may assume $s(c_*(a)) < s(c_*(b))$. We denote by $\mathcal{E}_+^*$ (resp. $\mathcal{E}_0^*$) the collection of all E_s^* having positive length (resp. zero length, that is, points). Since $L(c_*) < \infty$, the set $\mathcal{E}_+^*$ is at most countable, and

$$L_0 := \sum_{E_s^* \in \mathcal{E}_+^*} L(E_s^*) \le L(c_*).$$

For each $E_s^* \in \mathcal{E}_0^*$, by Corollary 3.8, $\pi^{-1}(E_s^*)$ is a continuous strictly monotone arc, denoted by $c_{E_s^*}$.

For $E_s^* \in \mathcal{E}_+^*$ with endpoints $c_*(t)$ and $c_*(t')$ with $t < t'$, from the convexity condition together with Lemma 3.13, we have

$$s_{\min}(c_*(t)) \le s_{\max}(c_*(t')). \tag{3-3}$$

Let $a(t)$ and $b(t')$ be the endpoints of $E_s := \pi^{-1}(E_s^*) \cap I_s$ corresponding to $c_*(t)$ and $c_*(t')$, respectively. Let $a_{\min}(t) \in \pi^{-1}(c_*(t))$ and $b_{\max}(t') \in \pi^{-1}(c_*(t'))$ be such that $p_1(a_{\min}(t)) = s_{\min}(c_*(t))$ and $p_1(b_{\max}(t')) = s_{\max}(c_*(t'))$. Then let us denote by $c_{E_s^*}$ the union of the subarc of $\pi^{-1}(c_*(t))$ from $a_{\min}(t)$ to $a(t)$, E_s^* and the subarc of $\pi^{-1}(c_*(t'))$ from $b(t')$ to $b_{\max}(t')$.

Let $\mathcal{E}^*$ be the union of the collections $\mathcal{E}_0^*$ and $\mathcal{E}_+^*$. Note that from construction, the family of p_1–images $\{p_1(c_{E_s^*}) \mid E_s \in \mathcal{E}^*\}$ is pairwise disjoint, and all the union coincides with $[s_0, s_0']$. In particular, we can define the natural order on the set $\mathcal{E}^*$.

(2) We are now ready to parametrize the union of all those arcs $c_{E_s^*}$ for $E_s^* \in \mathcal{E}^*$ to construct a lift $c\colon [s_0, s_0' + L_0] \to R$ of c_*. For each $E_s^* \in \mathcal{E}^*$, let $\mathcal{E}_+^*(s)$ denote the set of all $E_{s'}^* \in \mathcal{E}_+^*$ with $E_{s'}^* < E_s^*$. We set

$$\ell(E_s^*) := \sum_{E_{s'}^* \in \mathcal{E}_+^*(s)} L(E_{s'}^*).$$

For $E_s^* \in \mathcal{E}_0^*$, let a, b be the endpoints of the arc $c_{E_s^*}$ with $p_1(a) \le p_1(b)$. We parametrize $c_{E_s^*}$ on $[\ell(E_s^*) + p_1(a), \ell(E_s^*) + p_1(b)]$ by the condition

$$p_1(c_{E_s^*}(\ell(E_s^*) + t)) = t \quad \text{for } t \in [p_1(a), p_1(b)].$$

For $E_s^* \in \mathcal{E}_+^*$ with endpoints $c_*(t), c_*(t')$ with $t < t'$, define $a(t), b(t') \in \partial E_s^*$ and $a_{\min}(t), b_{\max}(t')$ as in the previous paragraph. Then we parametrize $c_{E_s^*}$ on $[\ell(E_s^*) + p_1(a_{\min}(t)), \ell(E_s^*) + L(E_s^*) + p_1(b_{\max}(t))]$ by the conditions that

$$p_1(c_{E_s^*}(\ell(E_s^*) + t)) = t$$

for $t \in [p_1(a_{\min}(t)), p_1(a(t))] \cup [L(E_s^*) + p_1(b(t')), L(E_s^*) + p_1(b_{\max}(t'))]$, and

$$c_{E_s^*}(\ell(E_s^*) + p_1(a(t)) + t) = E_s^*(t)$$

for $t \in [0, L(E_s^*)]$, where $E_s^*(t)$ is the arclength parameter from $a(t)$ to $b(t')$.

Finally we observe the continuity of the family $\{c_{E_s^*} \mid E_s^* \in \mathcal{E}^*\}$ in the following sense: Let $\{E_{s_i}^*\} \in \mathcal{E}_0^*$ be a Cauchy sequence in R_* satisfying $E_{s_i}^* < E_s^*$ (resp. $E_{s_i}^* > E_s^*$) such that its limit meets E_s^*. Let a and b be the initial and terminal points of $c_{E_s^*}$, respectively. Then $c_{E_{s_i}^*}$ converges to a (resp. to b). This follows from the conditions (1), (2) and (3-3), and the details are omitted here.

Thus we can define the curve $c\colon [s_0, s_0 + L_0] \to R$ as the union of all $c_{E_s^*}$ with $E_s^* \in \mathcal{E}^*$. It is easy to see that c is a continuous and monotone lift of c_*. This completes the proof. □

Remark 3.15 To consider the problem of lifting a curve γ in S, we need an extra condition on σ or γ, which will be discussed later in Proposition 3.24.

By Proposition 3.14, we immediately have the following.

Proposition 3.16 *Let $c_*\colon [a,b] \to (R_*, e_\sigma)$ be a shortest curve from $\pi(u)$ to $\pi(u')$. Then there is a lift c of c_* from u to u'.*

Alexandrov proved the following result, which plays a crucial role in the present paper.

Theorem 3.17 [1, Theorem 2] *Let S be a ruled surface in a* CAT(κ)*–space X with parametrization $\sigma\colon R \to X$. Then (R_*, e_σ) is a* CAT(κ)*–space.*

The proof of Theorem 3.17 is deferred to the appendix.

One might expect to define the induced "metric" d_σ on S along σ as

$$d_\sigma(x,y) := \inf\{e_\sigma(u,v) \mid \sigma(u) = x \text{ and } \sigma(v) = y\}.$$

However, d_σ does not necessarily satisfy the triangle inequality. See Remark 4.3. Even if (S, d_σ) becomes a metric space in certain cases, it could be far from the notion of "induced metric", as described in the following example.

Example 3.18 Let us consider the curve $\alpha\colon [0,5\pi]\to\mathbb{C}$ on $\mathbb{C}=\mathbb{R}^2$ defined as

$$\alpha(s)=\begin{cases} e^{\sqrt{-1}s} & \text{if } 0\le s\le \pi/2,\\ (0,2s/\pi) & \text{if } \pi/2\le s\le \pi,\\ (0,4-2s/\pi) & \text{if } \pi\le s\le 3\pi/2,\\ e^{\sqrt{-1}(s-\pi)} & \text{if } 3\pi/2\le s\le 5\pi.\end{cases}$$

We define the ruled surface $\sigma\colon [0,5\pi]\times[0,1]\to\mathbb{R}^3$ by $\sigma(s,t)=(\alpha(s),t)$. In this case, d_σ is a distance on the image S of σ. Actually d_σ coincides with the interior metric of S defined in Definition 3.23.

On the other hand, if we consider the restriction σ' of σ to $[0,3\pi]\times[0,1]$, then $d_{\sigma'}$ is not the distance on the ruled surface S' defined by σ'.

Lemma 3.19 *Suppose that we have for all $u,v\in R$,*

$$\sigma(u)=\sigma(v)\iff e_\sigma(u,v)=0. \tag{3-4}$$

Then (S,d_σ) is a metric space, and $\sigma_*\colon (R_*,e_\sigma)\to(S,d_\sigma)$ is an isometry.

Proof First note that $e_\sigma(u,u')=0$ implies $\sigma(u)=\sigma(v)$. Suppose (3-4) holds for all $u,v\in R$. Then we have $d_\sigma(x,y)=e_\sigma(u,v)$ for all $x,y\in S$ and $u\in\sigma^{-1}(x)$, $v\in\sigma^{-1}(y)$. This implies that d_σ is a metric on S. It is also obvious that $\sigma_*\colon (R_*,e_\sigma)\to(S,d_\sigma)$ is an isometry. □

Definition 3.20 We say that S has the *induced metric* from σ if $\sigma_*\colon R_*\to S$ is injective. This is the case when (3-4) holds for all $u,v\in R$, and therefore $\sigma_*\colon (R_*,e_\sigma)\to(S,d_\sigma)$ is an isometry by Lemma 3.19. In this case, d_σ is called the induced metric from σ.

Corollary 3.21 *Let S be a ruled surface in a* CAT(κ)*–space X with parametrization $\sigma\colon R\to X$. If S has the induced metric from σ, then (S,d_σ) is a* CAT(κ)*–space.*

From now, in the rest of this section, we consider curves γ in S with respect to the topology of S induced from X.

Lemma 3.22 *If S has the induced metric from σ, then $s(\gamma([a,b]))$ is an interval for any continuous curve $\gamma\colon [a,b]\to S$.*

Proof Let $J:=[a,b]$. If the conclusion does not hold, we have $s_-<s_+$ in $s(\gamma(J))$ such that (s_-,s_+) does not meet $s(\gamma(J))$. Set $R_-:=[0,s_-]\times[0,1]$ and $R_+:=[s_+,\ell]\times[0,1]$. Let J_+ and J_- be the set of all $t\in J$ such that the arc $\sigma^{-1}(\gamma(t))$ is contained in R_- and R_+, respectively. Since S has the induced metric from σ, Corollary 3.8 implies that $J=J_+\cup J_-$. We show that J_- and J_+ are open, yielding a contradiction. Suppose J_- is not open, for instance, and choose $t\in J_-\setminus \operatorname{int} J_-$ and a sequence t_n in J_+ converging to t. Choose any $x_n\in\sigma^{-1}(\gamma(t_n))$ converging to a point $x_\infty\in R_+$. Since $\sigma(x_n)=\gamma(t_n)\to\sigma(x_\infty)$ as $n\to\infty$, we have $\gamma(t)=\sigma(x_\infty)$. It turns out that $\sigma^{-1}(\gamma(t))\in R_+$. This is a contradiction to $t\in R_-$. □

Interior metrics on ruled surfaces

Let S be a ruled surface in X with parametrization $\sigma\colon R \to X$.

Definition 3.23 We denote by d_S the *interior metric* on S associated with d_X defined as

$$d_S(x_0, x_1) := \inf\{L(\gamma) \mid \gamma \text{ is a curve in } S \text{ from } x_0 \text{ to } x_1\}.$$

Due to the Arzelà–Ascoli theorem, (S, d_S) is a geodesic space.

We discuss the problem of lifting curves in S. For a subset $A \subset S$, we set

$$s(A) := \{s \mid \lambda_s \cap A \neq \varnothing\}.$$

Note that $s(A) = p_1(\sigma^{-1}(A))$. In particular, for every $x \in S$, we define $s(x)$, $s_{\max}(x)$, $s_{\min}(x)$ in this way as in Definition 3.10.

Proposition 3.24 *Let $\gamma\colon [a,b] \to S$ be a continuous curve of finite length from $\sigma(u)$ to $\sigma(u')$ with $p_1(u) < p_1(u')$. Set $J := [p_1(u), p_1(u')]$, and*

$$\Lambda_s^* := \sigma_*^{-1}(\gamma([a,b])) \cap I_s^* \quad \text{for } s \in J.$$

We assume the following:

1. *For arbitrary $t < t'$ in $[a,b]$, $s(\gamma([t,t']))$ is connected.*
2. *$\sigma_*(\Lambda_s^*)$ is γ–convex for each $s \in J$.*
3. *γ is monotone on $\sigma_*(\Lambda_s^*)$ that is an interval.*

Then there is a lift of γ in R from u to u'.

Lemma 3.25 *For a continuous curve $\gamma\colon [a,b] \to S$ with $s_{\min}(\gamma(a)) \leq s_{\max}(\gamma(b))$, suppose assumptions (1) and (2) of Proposition 3.24 hold for γ. Then we have monotonicity for all $t < t'$ in $[a,b]$,*

$$s_{\min}(\gamma(t)) \leq s_{\max}(\gamma(t')).$$

Proof Using condition (1) of Proposition 3.24 in place of Lemma 3.11, we can proceed in the same manner as the proof of Lemma 3.13 in our setting, to get the conclusion. □

Proof of Proposition 3.24 In view of conditions (2) and (3) and Lemma 3.25, using Λ_s^* in place of E_s^*, we construct the family of continuous arcs $c_{\Lambda_s^*}$ in the same manner as in Proposition 3.14. Then parametrize them and take the union of those arcs to obtain a lift of γ in R. Since the procedure is the same, we omit the details. □

Theorem 3.26 *Let S be a ruled surface in a* CAT(κ)*–space X with parametrization $\sigma\colon R \to X$. If S has the induced metric from σ, then we have $d_S = d_\sigma$, and (S, d_S) is a* CAT(κ)*–space.*

Proof Since $d_S \le d_\sigma$, to see $d_S = d_\sigma$ it suffices to show $d_S(x, x') \ge d_\sigma(x, x')$ for arbitrary $x, x' \in S$. Take a d_S–shortest curve $\gamma\colon [a, b] \to S$ from x to x'. Since γ is d_S–shortest, the conditions (2) and (3) in Proposition 3.24 certainly hold for γ. By Lemma 3.22, $s(\gamma([t_1, t_2]))$ is an interval, and condition (1) in Proposition 3.24 holds too. Therefore by Proposition 3.24, we have a lift of γ in R. Thus we have $d_S(x, x') = L(\gamma) \ge d_\sigma(x, x')$. Finally Corollary 3.21 implies that (S, d_S) is a CAT(κ)–space. □

4 Thin ruled surfaces

Let X be a locally compact, geodesically complete two-dimensional space with curvature $\le \kappa$, and fix $p \in X$. It is known that $\Sigma_p(X)$ is a finite metric graph without endpoints. For a vertex v of $\Sigma_p(X)$, take $\nu_1, \nu_2 \in \Sigma_p(X)$ with equal distance to v such that $\angle(\nu_1, v) + \angle(v, \nu_2) = \angle(\nu_1, \nu_2)$ and v is the unique vertex contained in the shortest geodesic joining ν_1 and ν_2 in $\Sigma_p(X)$. We set

$$\delta := \angle(\nu_1, v) = \angle(\nu_2, v), \tag{4-1}$$

where δ is assumed to be small enough and will be determined later on in Section 4. Let $\alpha_i\colon [0, \ell] \to X$ be geodesics in the directions ν_i for $i = 1, 2$. Joining $\alpha_1(s)$ to $\alpha_2(s)$ by the minimal geodesic $\lambda_s\colon [0, 1] \to X$, we have a ruled surface S in X. Let $B(p, r)$ be a small ball, and we assume $\ell = 2r$. Set $R = [0, \ell] \times [0, 1]$. Let $\sigma\colon R \to S$ be the map that defines S:

$$\sigma(s, t) = \lambda_s(t).$$

See Figure 1.

We define the *boundary* and the *interior* of S as

$$\partial S := \alpha_1 \cup \alpha_2 \cup \lambda_\ell \quad \text{and} \quad \operatorname{int} S := S \setminus \partial S. \tag{4-2}$$

The purpose of this section is to prove the following:

Theorem 4.1 *There exists an $r_p > 0$ such that for every $r \in (0, r_p]$, S with length metric is a* CAT(κ)*–space homeomorphic to a two-disk.*

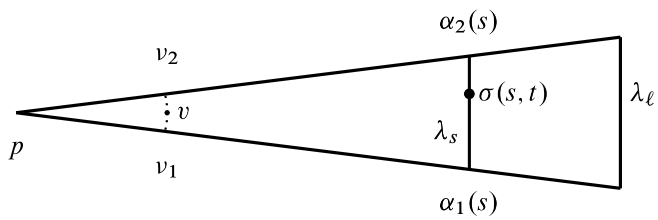

Figure 1

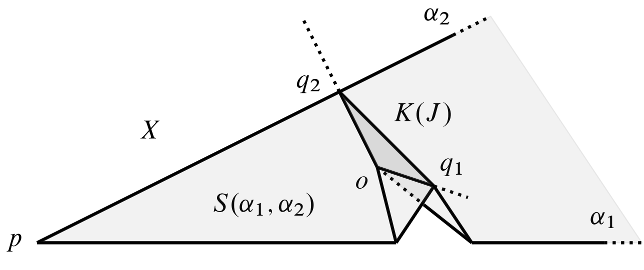

Figure 2

The proof of Theorem 4.1 is completed in Section 4.4. As shown in the following example, Theorem 4.1 does not hold for a general ruled surface even in a two-dimensional ambient space.

Example 4.2 For any $0 < a < \pi/2$, let X_0 be the complement of the domain $\{(x, y) \mid |y| < (\tan a)x\}$ on the xy–plane. For b with

$$a < b < a + \tfrac{1}{4}\pi, \tag{4-3}$$

consider the Euclidean cone $K(I)$ over a closed interval I of length $2b$. Let X_1 be the gluing of X_0 and $K(I)$ along their boundaries, where the origin o of X_0 is identified with the vertex of $K(I)$. Let ξ be the midpoint of I and let γ_ξ denote the geodesic ray of X_1 from o in the direction ξ. Next consider the Euclidean cone $K(J)$ over an interval J of length θ with $\pi - (b - a) \le \theta < \pi$. Let X be the gluing of X_1 and $K(J)$ in such a way that $\partial K(J)$ is identified with γ_ξ and $L := \{(x, 0) \mid x \le 0\} \subset X_0$ in the obvious way. It is easy to see that X is a locally compact, geodesically complete, two-dimensional CAT(0)–space. Let $p = (0, -10) \in X_0 \subset X$, and let σ_+ (resp. σ_-) be the geodesic ray starting from o defined by the ray $y = (\tan a)x$ (resp. by the ray $y = -(\tan a)x$). Note that the geodesic in X joining p and $\sigma_+(1)$ intersects $\sigma_- \setminus \{o\}$ because of (4-3). Let $\ell := 2d(p, \sigma_+(1))$, and let $\alpha_1 : [0, \ell] \to X$ be the geodesic starting from p through $\sigma_+(1)$. Let q_1 be the intersection point of α_1 with γ_ξ. Let q_2 be the point of L such that $d(p, q_1) = d(p, q_2)$. Letting $\alpha_2 : [0, \ell] \to X$ be the geodesic starting from p through q_2, consider the ruled surface $S = S(\alpha_1, \alpha_2)$ in X. Let Δ_1 (resp. Δ_2) be the geodesic triangle region in X_1 (resp. in $K(J)$) with vertices p, $\alpha_1(\ell)$ and $\alpha_2(\ell)$ (resp. o, q_1 and q_2). Obviously, S is the gluing of Δ_1 and Δ_2 along the geodesic segments oq_1 and oq_2. In particular S is not homeomorphic to a disk. See Figure 2.

Remark 4.3 (1) In Example 4.2 we have $\operatorname{diam}((\nabla d_p)(o)) = \pi$, which never happens in a small neighborhood of p by Lemma 2.4. This suggests the validity of Theorem 4.1, which is verified in the argument below.

(2) In Example 4.2, take two points x and y from the distinct components of $S \setminus \Delta_2$, respectively. Then if x and y are sufficiently close to the point o, we have $d_\sigma(x, y) > d_\sigma(x, o) + d_\sigma(o, y)$. Thus d_σ is not a distance for Example 4.2.

4.1 Behavior of ruling geodesics

In this subsection, we start the study of the behavior of ruling geodesics of S. We begin with two examples, which help us to understand the argument in the rest of the paper.

Example 4.4 (Kleiner; cf Nagano [23]) First consider a smooth nonnegative function $f\colon \mathbb{R} \to \mathbb{R}_+$ such that $\{f = 0\} = \{1/n \mid n = 1, 2, \dots\} \cup [1, \infty) \cup (-\infty, 0]$. Let $\Omega := \{(x, y) \mid |y| \le f(x), x \in \mathbb{R}\}$, equipped with the natural length metric induced from that of $\mathbb{R}^2$. We set

$$I_n^+ := \left\{(x, +f(x)) \;\middle|\; \frac{1}{n+1} \le x \le \frac{1}{n}\right\}, \quad I_n^- := \left\{(x, -f(x)) \;\middle|\; \frac{1}{n+1} \le x \le \frac{1}{n}\right\},$$
$$L_+ := \{(x, 0) \mid x \ge 1\}, \qquad L_- := \{(x, 0) \mid x \le 0\}.$$

Let ℓ_n denote the length of $I_n^\pm$, and let κ_n be the maximum of absolute geodesic curvature of $I_n^\pm$. We choose f satisfying

$$\sum \ell_n < \infty \quad \text{and} \quad \sum \kappa_n \ell_n < 2\pi. \tag{4-4}$$

By these conditions, one can take a closed domain H in $\mathbb{R}^2$ such that:

(1) ∂H is smooth, connected and concave in the sense that the geodesic curvature is nonpositive everywhere.

(2) There are consecutive points $p_1, p_2, \dots$ on ∂H such that the subarc K_n between p_n and p_{n+1} of ∂H has length equal to ℓ_n.

(3) If we denote by p_∞ the limit of p_n, the closure of the complement of the arc between p_1 and p_∞ in ∂H consists of two geodesic rays, say R_+ and R_-, in $\mathbb{R}^2$ with $p_1 \in R_+$ and $p_\infty \in R_-$.

(4) The absolute geodesic curvature of K_n is greater than or equal to κ_n everywhere.

Take four copies $H_1, \dots, H_4$ of H, and for $1 \le \alpha \le 4$ let $K_n^{(\alpha)}$ and $R_\pm^{(\alpha)}$ denote the respective copies of $K_n, R_\pm \subset \partial H_\alpha$. We put

$$\partial_+\Omega := \left(\bigcup_{n=1}^\infty I_n^+\right) \cup L_+ \cup L_- \quad \text{and} \quad \partial_-\Omega := \left(\bigcup_{n=1}^\infty I_n^-\right) \cup L_+ \cup L_-.$$

Now glue H_1, H_2 and Ω along their boundaries ∂H_1, ∂H_2 and $\partial_+\Omega$ in such a way that I_n, L_+ and L_- are respectively glued with $K_n^{(\alpha)}$, $R_+^{(\alpha)}$ and $R_-^{(\alpha)}$ for $\alpha = 1, 2$ in an obvious way. Similarly glue H_3, H_4 and Ω along their boundaries ∂H_3, ∂H_4 and $\partial_-\Omega$. See Figure 3.

Let X be the result of these gluings equipped with natural length metric, which is a two-dimensional locally compact, geodesically complete space. Let $\iota\colon \Omega \to X$ be the natural inclusion, and let $O = (0, 0) \in \Omega$. Note that no neighborhood of $p := \iota(O)$ in X has a triangulation. Approximating f by functions f_k for $k = 1, 2, \dots$ that are 0 near 0, we have polyhedral spaces X_k in a similar way which approximate X in the sense of Gromov–Hausdorff distance. Applying a result in [12], we see that X_k are CAT(0)–spaces. Thus the limit space X is also a CAT(0)–space. Note that $\mathcal{S}(X)$ consists of the two curves $\iota(\partial_+\Omega)$ and $\iota(\partial_-\Omega)$.

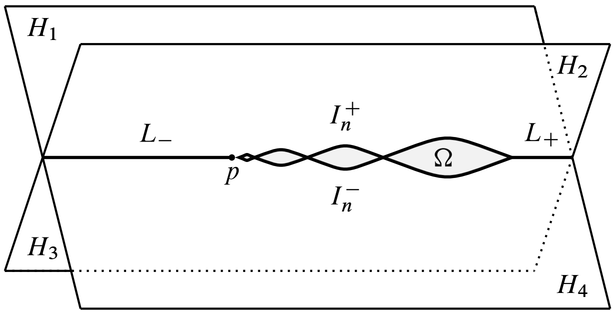

Figure 3

Example 4.5 This example is based on Example 4.4. The point is we perform different gluings. This time we glue H_1, H_2, H_3, H_4 and Ω along their boundaries as follows:

(1) $K_n^{(1)}$ is glued with I_n^+ for all n.

(2) $K_n^{(2)}$ is glued with I_n^+ if $n \not\equiv 2 \pmod 4$ and with I_n^- if $n \equiv 2 \pmod 4$.

(3) $K_n^{(3)}$ is glued with I_n^+ if $n \equiv 0, 1 \pmod 4$ and with I_n^- if $n \equiv 2, 3 \pmod 4$.

(4) $K_n^{(4)}$ is glued with I_n^+ if $n \equiv 3 \pmod 4$ and with I_n^- if $n \not\equiv 3 \pmod 4$.

Here, R_+^α and R_-^α for $1 \le \alpha \le 4$ are glued with L_+ and L_-, respectively, in those gluings. The result Y of these gluings equipped with the natural length metric is a two-dimensional locally compact, geodesically complete CAT(0)–space. Let $\iota\colon \big(\bigsqcup_{i=1}^4 H_i\big) \sqcup \Omega \to Y$ be the identification map. Note that

$$\text{for all } 1 \le \alpha \ne \beta \le 4, \quad \iota(K_n^\alpha) = \iota(K_n^\beta) \quad \text{for some } n. \tag{4-5}$$

Let $p := \iota(O)$, where O is the origin of Ω, and let v denote the direction at p defined by the union of all $I_n^\pm$ for $n = 1, 2, \ldots$. For small $\epsilon > 0$, take sufficiently small $r > 0$ and choose $a_i \in S(p, r) \cap \iota(H_i)$, for $1 \le i \le 4$, such that $\angle(\dot\gamma_{p,a_i}(0), v) = \epsilon$. Let $S(a_i, a_j)$ be the ruled surface defined by the geodesic segments γ_{p,a_i} and γ_{p,a_j}. Then it follows from (4-5) that $S(a_i, a_j)$ are not convex in Y for all $i \ne j$.

Remark 4.6 (1) In Example 4.4, if we take a_i in a way similar to Example 4.5, then $S(a_i, a_j)$ for $i = 1, 2$ and $j = 3, 4$ are convex in X, while $S(a_1, a_2)$ and $S(a_3, a_4)$ are not convex. Considering the other vertex of $\Sigma_p(X)$, it is possible to fill a neighborhood of the singular set $B(p, r) \cap \mathcal{S}$ via those convex ruled surfaces. This is not the case of Example 4.5.

(2) In Example 4.5, it is impossible to fill the ball $B(p, r)$ for any $r > 0$ via properly embedded convex disks. More strongly, there is no such convex disk properly embedded in $B(p, r)$. If there were such a convex disk D, from the convexity of D we could take some $a_i \ne a_j$ in ∂D. The convexity of D would also imply that $S(a_i, a_j) \subset D$, and hence $S(a_i, a_j)$ must be convex. However this is impossible, as indicated in Example 4.5.

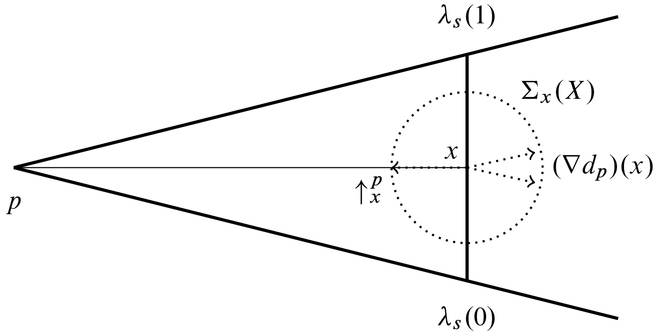

Figure 4

For $x \in S$ with $x = \lambda_s(t)$, from Lemma 2.4, we have

$$|\angle(\pm\dot{\lambda}_s(t), (\nabla d_p)(x)) - \pi/2| < \tau_p(|p,x|, \delta). \tag{4-6}$$

For $x \in S$, let $\Sigma_x(S)$ denote the set of all directions $\xi \in \Sigma_x(X)$ such that $\xi = \lim_{i\to\infty} \uparrow_x^{x_i}$ for some sequence $x_i \in S$ with $|x, x_i|_X \to 0$, as in Section 2. We call $\Sigma_x(S)$ the *extrinsic space of directions of S at x*. See Figure 4.

In this paper, we use the following terminology. We call a direction $\xi \in \Sigma_x(X)$

- **horizontal** if $\angle(\xi, \pm(\nabla d_p)(x)) \le 3\pi/10$,
- **vertical** if $\angle(\xi, \pm(\nabla d_p)(x)) \ge \pi/5$,
- **medial** if it is horizontal and vertical.

We also call a direction $\xi \in \Sigma_x(X)$

- **negative** if $\angle(\xi, -(\nabla d_p)(x)) < \pi/2$,
- **positive** if $\angle(\xi, (\nabla d_p)(x)) < \pi/2$.

We say that an open subset $\Omega \subset \Sigma_x(X)$ is in the *positive side* (resp. *negative side*) of $\Sigma_x(X)$ if every element of Ω is positive (resp. negative).

Assume that a Lipschitz curve $c\colon [a,b] \to B(p,r) \setminus \{p\}$ has the right and left directions $\dot{c}_+(t)$ and $\dot{c}_-(t)$ respectively at every $t \in [a,b]$. We say that such a curve c is *vertical* (resp. *horizontal* or *medial*) if both $\dot{c}_+(t)$ and $\dot{c}_-(t)$ are vertical (resp. horizontal or medial) for every $t \in [a,b]$.

Recall that for every $x \in S$,

$$s(x) = \{s \in [0,\ell] \mid x \in \lambda_s\}, \quad s_{\max}(x) = \max s(x), \quad s_{\min}(x) = \min s(x).$$

For every $x \in \operatorname{int} S$, we set

$$+\dot{\boldsymbol{\lambda}}(x) := \{\uparrow_x^{\lambda_s(1)} \mid s \in s(x)\} \quad \text{and} \quad -\dot{\boldsymbol{\lambda}}(x) := \{\uparrow_x^{\lambda_s(0)} \mid s \in s(x)\}.$$

We show that $s(x)$ is a closed interval later in Lemma 4.28.

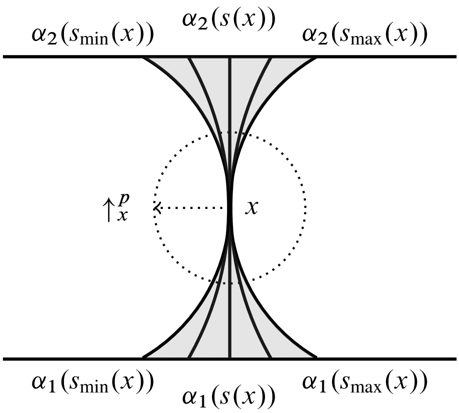

Figure 5

Lemma 4.7 *For every $x \in \operatorname{int} S$, we have*

(1) $\operatorname{diam}(s(x))/|p,x| < \tau_p(|p,x|)$,

(2) $\operatorname{diam}(\pm\dot{\boldsymbol{\lambda}}(x)) < \tau_p(|p,x|))$.

Proof Suppose that the conclusion does not hold. Then we have a sequence $x_i \in \operatorname{int} S$ and a positive constant c such that one of the following holds:

(i) $\operatorname{diam} s(x_i)/|p,x_i| \geq c$, or

(ii) $\operatorname{diam}(\pm\dot{\lambda}(x_i)) \geq c$.

Let $x_i = \lambda_{s_i}(t_i)$. Note that $0 < t_i < \ell$.

We may assume $t_i = \min\{t_i, 1-t_i\}$, since the other case is similar. For any other $s_i' \in s(x_i)$, from $|x_i, p| \to 0$, we have

$$\lim_{i\to\infty} \angle x_i \lambda_{s_i}(0) p = \tfrac{1}{2}\pi - \delta \quad \text{and} \quad \lim_{i\to\infty} \angle x_i \lambda_{s_i'}(0) p = \tfrac{1}{2}\pi - \delta.$$

We may assume that $s_i' < s_i$ without loss of generality. Note also that

$$\lim_{i\to\infty} \angle x_i \lambda_{s_i'}(0)\lambda_{s_i}(0) = \tfrac{1}{2}\pi + \delta,$$

$$\lim_{i\to\infty} \big(\tilde{\angle}\lambda_{s_i}(0)x_i\lambda_{s_i'}(0) + \tilde{\angle}x_i\lambda_{s_i}(0)\lambda_{s_i'}(0) + \tilde{\angle}\lambda_{s_i}(0)\lambda_{s_i'}(0)x_i\big) = \pi.$$

It follows that

$$\lim_{i\to\infty} \angle\lambda_{s_i}(0)x_i\lambda_{s_i'}(0) \leq \lim_{i\to\infty} \tilde{\angle}\lambda_{s_i}(0)x_i\lambda_{s_i'}(0) = \lim_{i\to\infty} (\pi - \tilde{\angle}x_i\lambda_{s_i}(0)\lambda_{s_i'}(0) - \tilde{\angle}\lambda_{s_i}(0)\lambda_{s_i'}(0)x_i) = 0.$$

Thus we conclude that $\operatorname{diam}(-\dot{\lambda}(x_i)) \to 0$. Therefore the assumption (ii) does not hold. Note that

$$|s_i - s_i'| \leq |p, x_i| t_i \tilde{\angle}\lambda_{s_i}(0)x_i\lambda_{s_i'}(0),$$

Therefore, from $\lim_{i\to\infty} \tilde{\angle}\lambda_{s_i}(0)x_i\lambda_{s_i'}(0) = 0$, we see that the assumption (i) does not hold either. □

Now, for any $x \in S$ and $s \in (0, \ell)$ with $x \notin \lambda_s$, let $y \in \lambda_s$ be such that $|x, y| = |x, \lambda_s|$. By Lemma 2.4, we have either

$$\angle(\uparrow_y^x, -\nabla d_p) < \tau_p(\delta, r) \quad \text{or} \quad \angle(\uparrow_y^x, \nabla d_p) < \tau_p(\delta, r).$$

Remark 4.8 From the proof of Lemma 4.7 by contradiction, one might think the function $\tau_p(\,\cdot\,)$ depends also on S. However, the number of possible S at p is finite. Therefore it is possible to find such a function depending only on p. From now on, we use this convention.

Lemma 4.9 *Under the above situation, if $\angle(\uparrow_y^x, -\nabla d_p) < \tau_p(\delta, r)$ (resp. $\angle(\uparrow_y^x, \nabla d_p) < \tau_p(\delta, r)$), then we have*

$$\angle(\uparrow_x^y, \nabla d_p) < \tau_p(\delta, r) \quad (\text{resp. } \angle(\uparrow_x^y, -\nabla d_p) < \tau_p(\delta, r)).$$

Proof We assume $\angle(\uparrow_y^x, -\nabla d_p) < \tau_p(\delta, r)$. The proof of the other case is similar. By Lemma 2.4, we have $\angle pyx < \tau_p(\delta, r)$. Since $\angle xpy < 2\delta$, it follows from Lemma 2.7 that $\angle pxy > \pi - \tau_p(\delta, r)$, which implies $\angle(\uparrow_x^y, \nabla d_p) < \tau_p(\delta, r)$. □

Lemma 4.10 *For $x \in \operatorname{int} S$, fix s_0 and t_0 with $x = \lambda_{s_0}(t_0)$. Then for every $u \in \Sigma_x(S)$ with*

$$\angle(u, \pm\dot{\lambda}_{s_0}(t_0)) \ge \tfrac{1}{3}\pi, \tag{4-7}$$

there exists a shortest path $\xi_\infty\colon [-1, 1] \to \Sigma_x(X)$ satisfying

$$\begin{cases} u \in \xi_\infty([-1, 1]), \\ \angle_X(\xi_\infty(\pm 1), \pm\dot{\lambda}_{s_0}(t_0)) < \tau_p(r), \\ \xi_\infty([-1, 1]) \subset \Sigma_x(S). \end{cases} \tag{4-8}$$

See Figure 6 below. We need a sublemma.

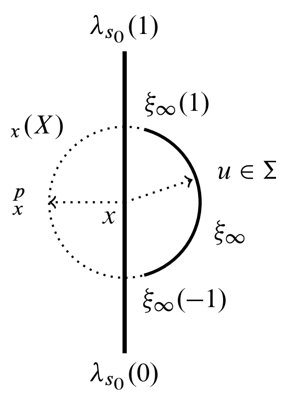

Figure 6

Sublemma 4.11 *There is a uniform positive constant c satisfying the following: For any $x \in S$ and any horizontal direction $\xi \in \Sigma_x(S)$, let $y_i \in S$ be a sequence such that $y_i \to x$ and $\uparrow_x^{y_i} \to \xi$. There is an i_0 such that if λ_{s_i} meets y_i for some $i \geq i_0$, then we have*

$$|x, \lambda_{s_i}| \geq c|x, y_i|.$$

Proof Note that λ_{s_i} does not pass through x for any large enough i since otherwise ξ would be a direction tangent to λ_{s_i} at x, which is a contradiction. Take $z_i \in \lambda_{s_i}$ such that $|x, z_i| = |x, \lambda_{s_i}|$. By Lemmas 2.4 and 2.11, we have

$$|\angle x z_i y_i - \tfrac{1}{2}\pi| < \tau_p(\delta, r).$$

It follows that $\angle(\uparrow_{z_i}^x, (-\nabla d_p)) < \tau_p(\delta, r)$ or $\angle(\uparrow_{z_i}^x, \nabla d_p) < \tau_p(\delta, r)$. We assume the former since the latter case is similar. It follows from Lemma 4.9 that $\angle(\uparrow_x^{z_i}, \nabla d_p) < \tau_p(\delta, r)$. Lemma 2.5 implies that

$$|\tilde{\angle} p z_i y_i - \tfrac{1}{2}\pi| < \tau_p(\delta, r). \tag{4-9}$$

By Lemma 2.7, we have

$$\tilde{\angle} p z_i x < \tau_p(\delta, r). \tag{4-10}$$

Let x_i be the point on the geodesic $z_i p$ satisfying $|z_i, x_i| = |z_i, x|$. By (4-9), we have $|\tilde{\angle} x_i z_i y_i - \pi/2| < \tau_p(\delta, r)$. It follows from (4-10) that

$$|\tilde{\angle} x z_i y_i - \tfrac{1}{2}\pi| < \tau_p(\delta, r). \tag{4-11}$$

Now let us consider the convergence

$$\left(\frac{1}{|x, z_i|} X, x\right) \to (K_x(X), o_x)$$

as $i \to \infty$. Let $z_\infty \in \Sigma_x \subset K_x(X)$ be the limit of z_i under this convergence. Since $\angle(z_\infty, \nabla d_p) < \tau_p(\delta, r)$ and since $\uparrow_x^{y_i} \to v$, the limit y_∞ of y_i under the above convergence certainly exists, and we have

$$\angle y_\infty o_x z_\infty < \tfrac{1}{2}\pi - \tfrac{1}{3}\pi + \tau_p(\delta, r) < \tfrac{1}{5}\pi. \tag{4-12}$$

By (4-11), we also have

$$|\angle o_x z_\infty y_\infty - \tfrac{1}{2}\pi| < \tau_p(\delta, r), \tag{4-13}$$

and (4-12) and (4-13) imply that $|o_x, z_\infty| \geq c|o_x, y_\infty|$ for some uniform constant $c > 0$. This yields the conclusion of the lemma via contradiction. □

Proof of Lemma 4.10 Let $y_i \in S$ be such that $|x, y_i|_X \to 0$ and $\uparrow_x^{y_i}$ converges to u. Take $s_i \in (0, \ell)$, $t_i \in (0, 1)$ such that $y_i = \lambda_{s_i}(t_i)$. Let $\epsilon_i := |x, y_i|_X$, and consider the convergence

$$\left(\frac{1}{\epsilon_i} X, x\right) \to (K_x(X), o_x) \quad \text{as } i \to \infty.$$

Note that the minimal geodesic $\hat{\lambda}_{s_i}(t) := \lambda_{s_i}(t_i + \epsilon_i t)$, where $-t_i/\epsilon_i < t < (1-t_i)/\epsilon_i$, has a uniformly bounded speed for $(1/\epsilon_i)X$ independent of i. Therefore, passing to a subsequence, we may assume that $\hat{\lambda}_{s_i}(t)$ converges to a minimal geodesic $\hat{\lambda}_\infty(t)$ in $K_x(X)$ defined on $(-\infty,\infty)$, where this convergence is uniform on every bounded interval. Note that $\hat{\lambda}_\infty(0) = u$. From Sublemma 4.11 and (4-7), the geodesic $\hat{\lambda}_\infty$ does not pass through o_x. Consider the curve

(4-14) $$\hat{\xi}_\infty(t) := \frac{\hat{\lambda}_\infty(t)}{|\hat{\lambda}_\infty(t)|}.$$

Obviously, $\hat{\xi}_\infty$ is a shortest path in $\Sigma_x(X)$, and $\hat{\xi}((-\infty,\infty)) \subset \Sigma_x(S)$. Let $\xi_\infty\colon [-1,1] \to \Sigma_x(S)$ be a reparametrization of the extension $\check{\xi}_\infty\colon [-\infty,\infty] \to \Sigma_x(S)$ of $\hat{\xi}_\infty$.

Take an arbitrary $w_+ \in (\nabla d_p)(x)$ and set $w_- = -(\nabla d_p)(x)$. Consider the sets $\{w_+, \dot{\lambda}_{s_0}(t_0), w_-, -\dot{\lambda}_{s_0}(t_0)\}$ and $\{w_+, \xi_\infty(1), w_-, \xi_\infty(-1)\}$. They are on a circle C in $\Sigma_x(X)$, in these orders, where

(4-15) $$|L(C) - 2\pi| < \tau_p(r).$$

Since $|\angle(\pm w, \xi_\infty(\pm 1)) - \pi/2| < \tau_p(\delta, r)$ and $|\angle(\pm w, \pm\dot{\lambda}_{s_0}(t_0)) - \pi/2| < \tau_p(\delta, r)$, we have the conclusion (4-8). □

A direction $\xi \in \Sigma_x(X)$ is called *regular* if $\xi \notin \mathcal{S}(\Sigma_x(X))$.

Lemma 4.12 *For $x \in \operatorname{int} S$, let $\xi_1, \xi_2 \in \Sigma_x(S)$ be positively horizontal (resp. negatively horizontal). Assume that ξ_1 is regular. Take an X–geodesic γ_1 such that $\dot{\gamma}_1(0) = \xi_1$, and a sequence $x_i \in S$ such that $|x, x_i|_X \to 0$ and $\uparrow_x^{x_i} \to \xi_2$. Then for s_0 with $x \in \lambda_{s_0}$, there exists an $\epsilon > 0$ such that if some ruling geodesic λ_s with $|s - s_0| < \epsilon$ passes through x_i for a sufficiently large i, then it passes through γ_1, too.*

See Figure 7.

Proof Suppose that the conclusion does not hold. Then there exists a sequence $s_i \to s_0$ such that λ_{s_i} meets x_i while λ_{s_i} does not pass through γ_1. Applying Lemma 4.10 to $x = \lambda_{s_0}(t_0)$ and ξ_2, we have a shortest arc $\xi_\infty\colon [-1,1] \to \Sigma_x(X)$ joining two points close to $\pm\dot{\lambda}_{s_0}(t_0)$ such that $\xi_2 \in \xi_\infty([-1,1])$. Similarly, applying Lemma 4.10 to ξ_1, we have a shortest arc $\bar{\xi}_\infty\colon ([-1,1]) \to \Sigma_x(X)$ joining two points close to $\pm\dot{\lambda}_{s_0}(t_0)$ such that $\xi_1 \in \bar{\xi}_\infty([-1,1])$.

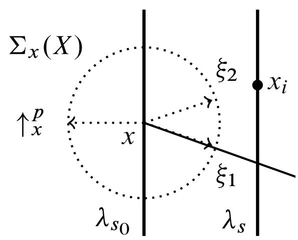

Figure 7

Note that both ξ_∞ and $\bar\xi_\infty$ pass through the positive side of $\Sigma_x(X)$ and connect points close to $\pm\dot\lambda_{s_0}(t_0)$. From construction, ξ_∞ and $\bar\xi_\infty$ pass through horizontal directions ξ_2 and ξ_1 respectively. Furthermore, both intersections $\xi_\infty([-1,-1+\epsilon_1])\cap\bar\xi_\infty([-1,-1+\bar\epsilon_1])$ and $\xi_\infty([1-\epsilon_2,1])\cap\bar\xi_\infty([1-\bar\epsilon_2,1])$ are not empty for some small $\epsilon_i,\bar\epsilon_i>0$ for $i=1,2$, since they are in the regular parts of $\Sigma_x(X)$ by Corollary 2.12. Therefore by the uniqueness of geodesics in the CAT(1)–space $\Sigma_x(X)$, we conclude that $\xi_\infty([-1+\epsilon_3,1-\epsilon_3])=\bar\xi_\infty([-1+\bar\epsilon_3,1-\bar\epsilon_3])$ for some small $\epsilon_3,\bar\epsilon_3>0$, and in particular ξ_∞ and $\bar\xi_\infty$ pass through both ξ_1 and ξ_2.

Take $\xi_3,\xi_4\in\Sigma_x(X)$ close to ξ_1 such that every element of the arc $[\xi_3,\xi_4]$ in $\Sigma_x(X)$ is regular and ξ_1 is the midpoint of $[\xi_3,\xi_4]$. Let $\gamma_i\colon[0,\epsilon]\to X$ be X–geodesics with $\dot\gamma(0)=\xi_i$ for $i=3,4$, and γ_5 the X–minimal geodesic joining $\gamma_3(\epsilon)$ to $\gamma_4(\epsilon)$. If $\epsilon>0$ is small enough, then the triangle $\triangle(\gamma_3,\gamma_4,\gamma_5)$ bounds a domain in X homeomorphic to a two-disk D. Let $\operatorname{int} D$ denote the interior of the disk D. Note that $\operatorname{int} D$ is open in X and that $\gamma_1([0,\epsilon_1])\subset D$ for a small $\epsilon_1<\epsilon$. Since $\operatorname{int} D$ is open in X and since ξ_∞ is constructed by (4-14), λ_{s_i} really passes through γ_1 for large i, which is a contradiction. This completes the proof. □

4.2 Canonical balls

In this subsection, we introduce the notion of canonical balls, which turns out to be useful to have better understanding of the behavior of ruling geodesics of S.

We denote by $\mathcal{R}(X)$ the set of topological regular points, $\mathcal{R}(X)=X\setminus\mathcal{S}(X)$.

Definition 4.13 For $x\in B(p,r)$, a ball $B(x,\epsilon)$ is called *canonical* if for every $y\in B(x,\epsilon)\setminus\{x\}$ with vertical $\uparrow_x^y$, we have $y\in\mathcal{R}(X)$.

Lemma 4.14 *There exists an $r=r_p>0$ such that there is a canonical ball around every point in $B(p,r)\setminus\{p\}$.*

Lemma 4.14 is a direct consequence of the following Lemma 4.15, which is immediate from Corollary 2.12.

Lemma 4.15 *For every $x\in\mathcal{S}(X)\cap B(p,r)\setminus\{p\}$, we have*

$$\sup\{\angle(\xi,\nabla d_p)\mid \xi\in\Sigma_x(\mathcal{S}(X))\text{ is positive}\}<\tau_p(|p,x|),$$
$$\sup\{\angle(\xi,-\nabla d_p)\mid \xi\in\Sigma_x(\mathcal{S}(X))\text{ is negative}\}<\tau_p(|p,x|).$$

Definition 4.16 For $x\in\operatorname{int} S$, let $B(x,\epsilon)$ be a canonical ball. We set

$$U_+(x,\epsilon):=\{y\in B(x,\epsilon)\mid \angle(\uparrow_x^y,\dot{\boldsymbol\lambda}(x))<\tfrac14\pi\},$$
$$U_-(x,\epsilon):=\{y\in B(x,\epsilon)\mid \angle(\uparrow_x^y,-\dot{\boldsymbol\lambda}(x))<\tfrac14\pi\}.$$

Note that both $U_+(x,\epsilon)$ and $U_-(x,\epsilon)$ are convex in X for small $\epsilon>0$.

In Lemma 4.21, we show that $U_\pm(x,\epsilon)\subset S$ for a small $\epsilon>0$.

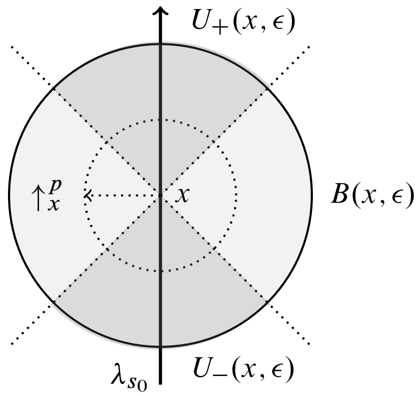

Figure 8: A canonical ball around x.

We denote by $|A|$ the cardinality of a set A.

Lemma 4.17 *Let $\gamma\colon [0,1]\to X$ be a vertical X–geodesic in $B(p,r)$. Then we have $|\gamma\cap\mathcal{S}(X)|<\infty$*

Proof Suppose that the lemma does not hold. Then we would have an accumulation point $x=\gamma(t_0)$ of $\gamma\cap\mathcal{S}(X)$. It turns out that either $\dot\gamma(t_0)$ or $-\dot\gamma(t_0)$ is in $\Sigma_x(\mathcal{S}(X))$, which is a contradiction to the existence of a canonical ball around x. □

Remark 4.18 At this stage, we do not know yet a uniform bound on $|\gamma\cap\mathcal{S}(X)|$ for all the vertical geodesics γ. In Section 6, we give a uniform bound (see Sublemma 6.8).

The following is a key lemma.

Lemma 4.19 (no-return lemma) *For every s_0, there exists an $\epsilon>0$ such that for any $s_1\in(s_0-\epsilon,s_0)$ (resp. any $s_1\in(s_0,s_0+\epsilon)$), there are no $t_0,t_1\in[0,1]$ satisfying that $\uparrow_{\lambda_{s_0}(t_0)}^{\lambda_{s_1}(t_1)}$ is positively horizontal (resp. negatively horizontal) of $\Sigma_{\lambda_{s_0}(t_0)}(X)$.*

Proof Suppose the conclusion does not hold. Then we have some sequence $s_i<s_0$ with $\lim_{i\to\infty}s_i=s_0$ such that

$$\uparrow_{\lambda_{s_0}(t_i)}^{\lambda_{s_i}(u_i)} \text{ is positively horizontal for some } t_i,u_i\in(0,1). \tag{4-16}$$

See Figure 9. We show that both $\lambda_{s_i}((0,u_i))$ and $\lambda_{s_i}((u_i,1))$ meet λ_{s_0}, which yields a contradiction to the minimality of λ_{s_0}.

From Lemmas 4.14 and 4.17, it is possible to cover λ_{s_0} by finitely many canonical balls $B(x_\alpha,\epsilon_\alpha)$, with $1\le\alpha\le N$, where $x_\alpha=\lambda_{s_0}(t_\alpha)$ and $t_\alpha<t_{\alpha+1}$. Taking smaller ϵ_α if necessary, we may further assume that for any large i,

(1) $\lambda_{s_i}\subset\bigcup_{\alpha=1}^N B(x_\alpha,\epsilon_\alpha)$,

(2) $B(x_\alpha,\epsilon_\alpha)\cap B(x_{\alpha+1},\epsilon_{\alpha+1})\cap\lambda_{s_i}\subset U_+(x_\alpha,\epsilon_\alpha)\cap U_-(x_{\alpha+1},\epsilon_{\alpha+1})$ for each α.

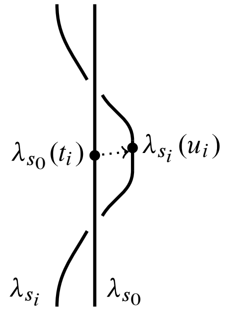

Figure 9: $\uparrow_{\lambda_{s_0}(t_i)}^{\lambda_{s_i}(u_i)}$ is positively horizontal.

Note that $\lambda_{s_0} \cap \mathcal{S}_X \subset \{x_\alpha\}_{\alpha=1}^N$, and that $U_+(x_\alpha, \epsilon_\alpha) \cap U_-(x_{\alpha+1}, \epsilon_{\alpha+1})$ is convex in X and homeomorphic to a disk. Suppose that $\lambda_{s_i}((0, u_i))$ does not meet λ_{s_0}. Take a maximal interval I_α^i in $[0, 1]$ such that $\lambda_{s_0}(I_\alpha^i) \subset B(x_\alpha, \epsilon_\alpha)$, and set

$$\xi_\alpha^i(t) := \uparrow_{x_\alpha}^{\lambda_{s_i}(t)} \quad \text{for } t \in I_\alpha^i.$$

From the assumption, $\xi_\alpha^i(I_\alpha^i)$ is in either the negative side or the positive side of $\Sigma_{x_\alpha}(X)$. Note that $\xi_1^i(I_1^i)$ is in the negative side of $\Sigma_{x_1}(X)$. Let $\alpha_0 = \alpha_0(i)$ be such that $\lambda_{s_0}(t_i) \in B(x_{\alpha_0}, \epsilon_{\alpha_0})$. From (4-16), $\xi_{\alpha_0}^i(I_{\alpha_0}^i)$ is in the positive side of $\Sigma_{x_{\alpha_0}}(X)$. Therefore for some $\alpha \le \alpha_0$, $\xi_{\alpha-1}^i(I_{\alpha-1}^i)$ is in the negative side of $\Sigma_{x_{\alpha-1}}(X)$ and $\xi_\alpha^i(I_\alpha^i)$ is in the positive side of $\Sigma_{x_\alpha}(X)$. Now λ_{s_0} divides the disk domain $U_+(x_{\alpha-1}, \epsilon_{\alpha-1}) \cap U_-(x_\alpha, \epsilon_\alpha)$ into two disk domains D_- and D_+, where we may assume that $\lambda_{s_i}(t_-) \in D_-$ and $\lambda_{s_i}(t_+) \in D_+$ for some $t_- \in I_{\alpha-1}$ and $t_+ \in I_\alpha$. Thus $\lambda_{s_i}([t_-, t_+])$ must meet λ_{s_0}.

Similarly, we would have another intersection point of $\lambda_{s_i}((u_i, 1))$ and λ_{s_0}. This completes the proof. $\square$

The following lemma is a global version of Lemma 4.19.

Lemma 4.20 *For arbitrary $s_1 < s_2$, there are no $t_1, t_2 \in [0, 1]$ such that $\uparrow_{\lambda_{s_1}(t_1)}^{\lambda_{s_2}(t_2)}$ (resp. $\uparrow_{\lambda_{s_2}(t_2)}^{\lambda_{s_1}(t_1)}$) is negatively horizontal in $\Sigma_{\lambda_{s_1}(t_1)}(X)$ (resp. positively horizontal in $\Sigma_{\lambda_{s_2}(t_2)}(X)$).*

Proof Let $I(s_1)$ be the set of all $s \in (s_1, s_2]$ such that there are no $t_1, t \in [0, 1]$ such that $\uparrow_{\lambda_{s_1}(t_1)}^{\lambda_s(t)}$ is negatively horizontal in $\Sigma_{\lambda_{s_1}(t_1)}(X)$. By Lemma 4.19, $(s_1, s_0) \subset I(s_1)$ for some $s_0 \in (s_1, s_2)$. Let u be the supremum of those s_0. From the continuity of the map $\sigma\colon R \to S$, $(s_1, s_2] \setminus I(s_1)$ is open in $(s_1, s_2]$. It follows that $u \in I(s_1)$. Suppose that $u < s_2$. Then we have a sequence of positive numbers ϵ_i with $\epsilon_i \to 0$ such that $u_i := u + \epsilon_i \notin I(s_1)$. Namely we have sequences t_i and t_i' satisfying that $\uparrow_{\lambda_{s_1}(t_i)}^{\lambda_{u_i}(t_i')}$ is negatively horizontal in $\Sigma_{\lambda_{s_1}(t_i)}(X)$. Set $x_i := \lambda_{u_i}(t_i')$, and let $y_i := \lambda_{s_1(t_i)}$. Take $z_i \in \lambda_{u_i}$ and $w_i \in \lambda_u$ such that

$$|y_i, z_i| = |y_i, \lambda_{u_i}| \quad \text{and} \quad |z_i, w_i| = |z_i, \lambda_u|.$$

Since $\uparrow_{y_i}^{x_i}$ is horizontal, we have $y_i \ne z_i$. By (4-6), we obtain

$$\angle(\uparrow_{z_i}^{y_i}, \nabla d_p) < \tau_p(\delta, r) \quad \text{or} \quad \angle(\uparrow_{z_i}^{y_i}, -\nabla d_p) < \tau_p(\delta, r).$$

We show that

$$\angle(\uparrow_{z_i}^{y_i}, \nabla d_p) < \tau_p(\delta, r). \tag{4-17}$$

Otherwise, we have $\angle(\uparrow_{z_i}^{y_i}, -\nabla d_p) < \tau_p(\delta, r)$. In view of Lemma 4.9, it turns out that

$$\angle x_i y_i z_i > \tfrac{2}{3}\pi - \tau_p(\delta, r),$$

and hence

$$\angle x_i z_i y_i < \pi - \angle x_i y_i z_i - \angle y_i x_i z_i + \tau_p(r) < \tfrac{1}{3}\pi + \tau_p(\delta, r).$$

This is a contradiction to the choice of z_i.

Next note that $w_i \neq z_i$. Because if $w_i = z_i$, then $\uparrow_{y_i}^{w_i}$ must be negatively horizontal by (4-17), which contradicts $u \in I(s_1)$. Now by Lemma 4.19, $\uparrow_{w_i}^{z_i}$ is positively horizontal. In view of Lemma 4.9, we have

$$\angle(\uparrow_{z_i}^{w_i}, -\nabla d_p) < \tau_p(\delta, r). \tag{4-18}$$

It follows from (4-17) and (4-18) that $\angle y_i z_i w_i > \pi - \tau_p(\delta, r)$, which implies $\angle(\uparrow_{y_i}^{w_i}, -\nabla d_p) < \tau_p(\delta, r)$. In particular $\uparrow_{y_i}^{w_i}$ is negatively horizontal. This contradicts $u \in I(s_1)$. Thus we conclude $u = s_2$.

Similarly we see that there are no t_1, t_2 satisfying that $\uparrow_{\lambda_{s_2}(t_2)}^{\lambda_{s_1}(t_1)}$ is positively horizontal. This completes the proof. □

Lemma 4.21 *For every $x \in \operatorname{int} S$, there exists an $\epsilon > 0$ such that*

$$U_+(x, \epsilon) \subset S \quad \text{and} \quad U_-(x, \epsilon) \subset S.$$

Proof Let $B(x, \epsilon_0)$ be a canonical ball. Take the positively horizontal $v_+ \in \Sigma_x(X)$ (resp. negatively horizontal $v_- \in \Sigma_x(X)$) such that $\angle(v_\pm, \dot\lambda(x)) = \pi/4$. Let $\gamma_\pm$ be X–geodesics starting from x with $\dot\gamma_\pm(0) = v_\pm$.

Sublemma 4.22 *For any $0 < \epsilon < \epsilon_0$, there are $s_- \in (0, s_{\min}(x))$ and $s_+ \in (s_{\max}(x), r)$ such that $\lambda_{s_\pm}$ pass through $\gamma_\pm((0, \epsilon])$, respectively.*

See Figure 10.

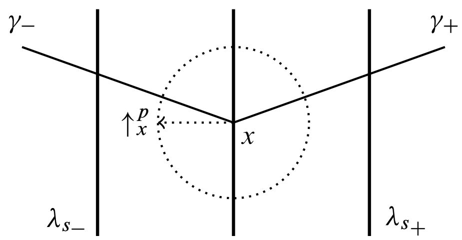

Figure 10

Proof Suppose that there is no such $s_- < s_{\min}(x)$. Then we have a sequence $s_i < s_{\min}(x)$ converging to $s_{\min}(x)$ such that λ_{s_i} does not pass through $\gamma_-((0,\epsilon])$ for some $\epsilon > 0$ and all i. As in the proof of Lemma 4.10 together with Lemma 4.19, the curves $\xi_i(t) = \uparrow_x^{\lambda_{s_i}(t)}$, where $t \in [0,1]$, in $\Sigma_x(X)$ pass through $\dot\gamma_-(0)$ for all i. This shows in particular that $\dot\gamma_-(0) \in \Sigma_x(S)$. Since $\dot\gamma_-(0)$ is regular, it follows from Lemma 4.12 that λ_{s_i} meets $\gamma_-((0,\epsilon])$ for every large enough i. This is a contradiction.

Similarly, we see that λ_s meets $\gamma_+((0,\epsilon])$ for any $s > s_+$ sufficiently close to s_+. □

Take a sufficiently small $0 < \epsilon_1 < \epsilon_0$ such that

(4-19) the triangle $\Delta\gamma_+(\epsilon_1)x\gamma_-(\epsilon_1)$ spans a disk domain in X.

Let $s_\pm$ be as in Sublemma 4.22 for ϵ_1, and set $I := [s_-, s_+]$ and $\gamma_{\epsilon_1} = \gamma_+([0,\epsilon_1]) \cup \gamma_-([0,\epsilon_1])$. It follows from the continuity of σ, Lemma 4.19 and (4-19) that λ_s meets γ_{ϵ_1} for all $s \in I$. Now define $\varphi\colon I \to \gamma_{\epsilon_1}$ by $\varphi(s) = \lambda_s \cap \gamma_{\epsilon_1}$. Let $\gamma_\pm(\epsilon_\pm) := \varphi(s_\pm)$. Since φ is continuous, the intermediate-value theorem implies that

$$\gamma_{\epsilon_2} \subset \operatorname{Im}\varphi \subset S,$$

where $\epsilon_2 := \min\{\epsilon_+, \epsilon_-\}$.

For any $0 < \epsilon \le \epsilon_2$, let μ_ϵ be the X–geodesic joining $\gamma_-(\epsilon)$ to $\gamma_+(\epsilon)$. Put

$$\hat\gamma_{\epsilon,\epsilon_2} := \gamma_-([\epsilon,\epsilon_2]) \cup \mu_\epsilon \cup \gamma_+([\epsilon,\epsilon_2]).$$

Similarly, we can define the map $\psi\colon I \to \hat\gamma_{\epsilon,\epsilon_2}$ by $\psi(s) = \lambda_s \cap \hat\gamma_{\epsilon,\epsilon_2}$. Again, since ψ is continuous, the intermediate-value theorem implies that ψ is surjective, and hence $\mu_\epsilon \subset S$. Now we can take $\epsilon_3^+ > 0$ such that

$$U_+(x,\epsilon_3^+) \subset \bigcup_{0 \le \epsilon \le \epsilon_2} \mu_\epsilon \subset S.$$

Similarly we have $U_-(x,\epsilon_3^-) \subset S$ for some $\epsilon_3^- > 0$. This completes the proof of Lemma 4.21. □

4.3 Spaces of directions

In this subsection, we determine the structure of the space of directions of S at each point of S.

Lemma 4.23 *For every $x \in S$, let $\xi \in \Sigma_x(S)$ be regular in $\Sigma_x(X)$, and let γ be an X–geodesic with $\dot\gamma(0) = \xi$. Then $\gamma([0,\epsilon]) \subset S$ for a small $\epsilon > 0$. Furthermore, ϵ can be taken locally uniformly for ξ.*

Proof First assume $x \in \operatorname{int} S$, and let $x = \lambda_s(t)$. From Lemma 4.21, we may assume $\angle(\xi, \pm\dot\lambda_s(t)) \ge \frac{1}{3}\pi$. If ξ is positive, Lemmas 4.12 and 4.19 imply that for some $s_1 > s_{\max}(x)$, λ_s meets γ at, say, $\gamma(t(s))$ for every $s \in [s_{\max}(x), s_1]$. Since ξ is horizontal, $t(s)$ is unique and continuous in s, and $t(s) > 0$ if $s > s_{\max}(x)$. Therefore $\gamma([0, t(s_1)]) \subset S$. From this argument, the local uniformity of $t(s_1)$ for ξ is clear. The case when ξ is negative is similar. If $x \in \partial S \setminus \{p\}$, the proof is similar.

Finally we consider the case $x = p$. For small enough $\epsilon > 0$, let $\xi_\pm \in \Sigma_p(X)$ be such that

$$\angle(\xi_+, \xi_-) = \angle(\xi_+, \xi) + \angle(\xi, \xi_-) = 2\angle(\xi_+, \xi) = 2\epsilon,$$

and let $\gamma_\pm$ be X–geodesics with $\dot\gamma_\pm(0) = \xi_\pm$. For a small $\eta > 0$, let $U(\eta)$ denote the domain bounded by $\gamma_\pm$ and $S(p, \eta)$. If η is small enough, then $U(\eta)$ is homeomorphic to a disk. Take $x_i \in S$ with $x_i \to p$ such that $\uparrow_p^{x_i} \to \xi$. Then for a large N, we have $x_i \in U(\eta)$ for all $i \geq N$. If $x_i = \lambda_{s_i}(t_i)$, λ_{s_i} must meet $\gamma_\pm$. The intermediate-value theorem then yields that the subdomain of $U(\eta)$ bounded by $\gamma_\pm$ and λ_{s_N} is contained in S. In particular, $\gamma([0, \epsilon_1]) \subset S$ for small $\epsilon_1 > 0$. □

Remark 4.24 If $\xi \in \Sigma_x(S)$ is singular in $\Sigma_x(X)$, Lemma 4.23 does not hold in general. See Examples 4.4 and 4.5.

Lemma 4.25 *Let $x \in S$.*

(1) *If $x \in \operatorname{int} S$, then $\Sigma_x(S)$ is a circle of length $< 2\pi + \tau_p(r)$.*

(2) *If $x \in \partial S$, then $\Sigma_x(S)$ is an arc.*

Proof (1) First we show that $\Sigma_x(S)$ contains a circle C. Take an $s_0 \in s(x)$ and t_0 with $x = \lambda_{s_0}(t_0)$. Obviously

$$C_0 := \{\xi \in \Sigma_x(X) \mid \angle(\pm\dot\lambda_{s_0}(t_0), \xi) \leq \tfrac{1}{3}\pi\}$$

consists of two arcs in the regular part of $\Sigma_x(X)$. It follows from Lemma 4.21 that C_0 is contained in $\Sigma_x(S)$. For a positively horizontal direction $v_+ \in \Sigma_x(S)$, we apply Lemma 4.10 to obtain a minimal arc C_+ in $\Sigma_x(S)$ joining two points close to $\pm\dot\lambda_{s_0}(t_0)$ and containing v_+. Similarly, for a negatively horizontal direction $v_- \in \Sigma_x(S)$, we apply Lemma 4.10 to obtain a minimal arc C_- in $\Sigma_x(S)$ joining two points close to $\pm\dot\lambda_{s_0}(t_0)$ and containing v_-. Obviously the union of C_0, C_+ and C_- forms a circle C in $\Sigma_x(S)$. It follows from Lemma 4.7 and (4-8) that

$$|L(C_\pm) - \pi| < \tau_p(r) \quad \text{and} \quad L(C \setminus (C_+ \cup C_-)) < \tau_p(r),$$

which implies $|L(C) - 2\pi| < \tau_p(r)$.

Suppose next that $\Sigma_x(S) \setminus C$ is not empty, and take a w in $\Sigma_x(S) \setminus C$. Since $\angle(w, \pm\dot\lambda_{s_0}(t_0)) \geq \frac{1}{3}\pi$, we can apply Lemma 4.10 to obtain a minimal arc C_1 in $\Sigma_x(S)$ joining two points close to $\pm\dot\lambda_{s_0}(t_0)$ and containing w. Note that the complement C_1' of a small neighborhood of $\pm\dot\lambda_{s_0}(t_0)$ in C_1 is contained in C, and w must be contained in C_1', which is a contradiction.

(2) If $x = \lambda_s(0)$ with $0 < s < \ell$ (resp. $s = \ell$), then $\Sigma_x(S)$ is an arc with endpoints $\pm\dot\alpha_1(s)$ (resp. $-\dot\alpha_1(\ell)$ and $\dot\lambda_\ell(0)$) through $\dot\lambda_s(0)$ (recall (4-2)). The case $x = \lambda_s(1)$ with $0 < s \leq \ell$ is similar. Next consider the case $x = p$. Let v and ν_1, ν_2 be as in (4-1). We show that $\Sigma_p(S)$ coincides with the arc $[\nu_1, \nu_2]$ in $\Sigma_p(X)$. Let η_i be any interior point of $[\nu_i, v]$, and let σ_i be X–geodesics with $\dot\sigma_i(0) = \eta_i$. If $s > 0$ is small enough, then λ_s meets both σ_1 and σ_2. This implies that $[\nu_1, \eta_1] \cup [\eta_2, \nu_2]$ is contained in $\Sigma_p(S)$.

Letting $\eta_1, \eta_2 \to v$, we obtain that $[\nu_1, \nu_2] \subset \Sigma_p(S)$. Conversely, for any $\xi \in \Sigma_p(S)$, take $x_i \in S$ with $|p, x_i| \to 0$ and $\uparrow_p^{x_i} \to \xi$. Since x_i can be written as $x_i = \lambda_{s_i}(t_i)$ with $s_i \to 0$, it is obvious that $\xi \in [\nu_1, \nu_2]$. Thus we have $\Sigma_p(S) = [\nu_1, \nu_2]$. □

Definition 4.26 For $x \in S$, let $\Sigma_x(S^{\text{int}})$ denote the *intrinsic space of directions* of S at x, which is defined as the completion of the set of all equivalence classes of S–geodesics starting from x equipped with the upper angle $\angle^S$ for the induced interior metric of S.

Lemma 4.27 *$\Sigma_x(S)$ is isometric to $\Sigma_x(S^{\text{int}})$.*

Proof First assume $x \in \operatorname{int} S$. Let

$$\Omega := \Sigma_x(S) \cap \mathcal{S}(\Sigma_x(X)).$$

Note that $|\Omega| < \infty$. We first show that each component Σ of $\Sigma_x(S) \setminus \Omega$ is isometrically embedded in $\Sigma_x(S^{\text{int}})$. Take ξ_1 and ξ_2 from Σ with $|\xi_1, \xi_2| < \pi$. Let $\mu_i : [0, \epsilon] \to X$ be an X–geodesic with $\dot\mu_i(0) = \xi_i$. Then for small ϵ, we have from Lemma 4.23 that

(1) $\mu_i \subset S$, and

(2) every X–geodesic joining $\mu_1(t)$ and $\mu_2(t)$ is contained in S for every $t \in [0, \epsilon]$.

Thus we conclude that $\angle^X(\xi_1, \xi_2) = \angle^S(\xi_1, \xi_2)$.

Next, for any $v \in \Omega$, take $\xi_3, \xi_4 \in \Sigma_x(S) \setminus \Omega$ close to v such that ξ_3, v and ξ_4 are in this order on the circle $\Sigma_x(S)$. Take X–geodesics γ_i, for $i = 3, 4$, in the direction ξ_i. By Lemma 4.12, we can find a sequence s_i such that $s_i \to s_0 \in s(x)$ and λ_{s_i} meets both γ_3 and γ_4. This implies that $\angle^X(\xi_3, \xi_4) = \angle^S(\xi_3, \xi_4)$. This completes the proof for this case.

The case $x \in \partial S$ is similar, and hence we omit the proof. □

4.4 Proof of Theorem 4.1

In this subsection, we first prove Theorem 4.1. Then we control the difference between the geometries of S and X.

Lemma 4.28 *For every $x \in S$, we have that*

(1) *$s(x)$ is either a point or a closed interval, and*

(2) *$\sigma^{-1}(x)$ is a strictly monotone arc in R.*

Proof Suppose that the conclusion (1) does not hold. Then we would have $s_- < s_+$ such that $s_\pm \in s(x)$ and (s_-, s_+) does not meet $s(x)$. By Lemma 2.4, we may assume $x \neq p$. Choose an S–geodesic $\gamma : [0, a) \to S$ starting from x such that $\dot\gamma(0)$ is a positive, horizontal and regular direction. Let us write

$$I = \{s \in (s_-, s_+) \mid \lambda_s \text{ passes through } \gamma \setminus \{x\}\}.$$

From Lemmas 4.19 and 4.12, there is $\epsilon_0 > 0$ such that $(s_-, s_- + \epsilon) \subset I$ for every $0 < \epsilon \le \epsilon_0$. Since $x \in \lambda_{s_+}$, this is a contradiction to Lemma 4.20.

Conclusion (2) follows immediately from (1) and the injectivity of $\sigma|_{I_s}$ for each $s \in (0, \ell]$. □

Proof of Theorem 4.1 By Lemma 4.28, we have (3-4) for all $u, v \in R$. Thus S has the induced metric from σ. Theorem 3.26 then implies that (S, d_S) is a CAT(κ)–space.

We set $S^{\mathrm{int}} := (S, d_S)$.

Lemma 4.29 *The set* $\operatorname{int} S^{\mathrm{int}}$ *is locally geodesically complete.*

Proof This is immediate from Lemma 4.25 in a straightforward way. See [10, Proposition II.5.12] and [18, Theorem 1.5] together with [16, Theorem A] for general considerations. □

We prove that S^{int} is a topological two-manifold with boundary. In view of Lemmas 2.8, 4.25, 4.27 and 4.29, it suffices to show that a small S–ball around any point $x \in \partial S$ is homeomorphic to a half-disk. Suppose $x = p$. The other cases are similar. The argument is standard. Logically, we proceed as follows. For a positive integer m with $m \ge [\pi/2\delta] + 1$, gluing m copies of S in order around p, we have a sector T with sector angle $\ge \pi$ at p, which is a CAT(κ)–space by Theorem 2.2. Glue two copies of T along their edges to obtain a CAT(κ)–space W for which $L(\Sigma_p(W)) \ge 2\pi$. Then Lemma 2.8 shows that p has an open disk neighborhood, which implies that p has a half-disk neighborhood in S.

Finally, from the CAT(κ)–property of S, the contractibility of S is immediate since we may assume that the diameter of S for the metric d_S is less than $\pi/\sqrt{\kappa}$ when $\kappa > 0$. This completes the proof of Theorem 4.1. □

In the rest of this section, we present a few results that control the difference between the geometries of X and S. These will be needed in Sections 5 and 7.

Lemma 4.30 *For arbitrary distinct* $x, y \in S$*, let* $(-\nabla^S d_x)(y)$ *denote* $\dot\gamma^S_{y,x}(0)$*, where* $\gamma^S_{y,x}$ *is the* S*–geodesic from* y *to* x*. Then we have*

(1) $\angle(\dot\gamma^S_{x,y}(0), \dot\gamma^X_{x,y}(0)) < \tau_x(|x, y|_X)$, *and*
(2) $\angle((-\nabla^S d_x)(y), (-\nabla d_x)(y)) < \tau_x(|x, y|_X)$.

For the proof, we need a sublemma.

Sublemma 4.31 *For every* $x \in S$*, we have*

$$\sup_{y \in B^S(x,s) \setminus \{x\}} \frac{|x, y|_S}{|x, y|_X} < 1 + \tau_x(s).$$

When $x \in S \setminus \mathcal{S}(X)$, Sublemma 4.31 and Lemma 4.30 are clear.

Proof of Sublemma 4.31 If the sublemma does not hold, there would exist a sequence x_n in S converging to x such that

$$\frac{|x, x_n|_S}{|x, x_n|_X} > 1 + c > 1 \tag{4-20}$$

for some constant $c > 0$ independent of n. Passing to a subsequence, we may assume that $\uparrow_x^{x_n}$ converges to a direction $v \in \Sigma_x(X)$. It is easily seen from (4-20) that v is a vertex of $\Sigma_x(X)$. Take a small enough $\epsilon > 0$ compared with c and an $s_n \in s(x_n)$. Let y_n be an element of λ_{s_n} with $|x_n, y_n| = \epsilon |x, x_n|_X$. From Lemma 4.25, the X–geodesic joining x and y_n is contained in S for any large n. It follows from the triangle inequality that

$$\frac{|x, x_n|_S}{|x, x_n|_X} \le \frac{|x, y_n|_S + |y_n, x_n|_S}{|x, x_n|_X} \le \frac{|x, x_n|_X + 2|y_n, x_n|_X}{|x, x_n|_X} = 1 + 2\epsilon < 1 + c,$$

which is a contradiction. □

Proof of Lemma 4.30 If the lemma did not hold, there would be a sequence x_i of S converging to x such that

$$\angle(\dot\gamma^S_{x,x_i}(0), \dot\gamma^X_{x,x_i}(0)) > c > 0 \tag{4-21}$$

or

$$\angle((-\nabla^S d_x)(x_i), (-\nabla d_x)(x_i)) > c > 0, \tag{4-22}$$

where c is a uniform positive constant. From now, we assume $x \in \operatorname{int} S$. The other case is similar. We may assume that $\xi_i^X := \dot\gamma^X_{x,x_i}(0)$ and $\xi_i^S := \dot\gamma^S_{x,x_i}(0)$ converge to $\xi^X \in \Sigma_x(X)$ and $\xi^S \in \Sigma_x(S) \subset \Sigma_x(X)$ respectively. Note that $\xi^X \in \Sigma_x(S)$.

(1) First we assume (4-21). Then we have $\angle(\xi^X, \xi^S) \ge c$. We show $\xi^X \in \Sigma_x(S) \cap V(\Sigma_x(X))$. Actually, by Lemma 4.23, if $\xi^X \in \Sigma_x(S) \setminus V(\Sigma_x(X))$, we have $\epsilon > 0$ such that $\gamma^X_{\xi_i^X}([0, \epsilon]) \subset S$ for any large i. It turns out that $\gamma^X_{x,x_i} \subset S$, which is a contradiction to (4-21). Similarly, we have $\xi^S \in \Sigma_x(S) \cap V(\Sigma_x(X))$. Since $\Sigma_x(S) \cap V(\Sigma_x(X))$ is a point, it follows that $\xi^X = \xi^S$. This is a contradiction.

(2) Next assume (4-22). We set $\xi := \xi^X = \xi^S$, and $\Sigma := \Sigma_x(S)$. From the above argument of (1) and (4-22), we have $\xi \in \Sigma \cap V(\Sigma_x(X))$. Letting $t_i^X := |x_i, x|_X$, consider the convergence

$$\left(\frac{1}{t_i^X} X, x_i\right) \to (K_x(X), \xi^X).$$

Similarly, letting $t_i^S := |x_i, x|_S$, from Lemma 4.27, we have the convergence

$$\left(\frac{1}{t_i^S} S, x_i\right) \to (K(\Sigma), \xi^S).$$

Let μ_1, μ_2 be elements of $\Sigma \setminus V(\Sigma_x(X))$ near ξ such that ξ is in the interior of the shortest arc between μ_1 and μ_2. Take any $s_i \in s(x_i)$, and let y_i and z_i be the intersections of λ_{s_i} with γ_{μ_1} and γ_{μ_2}, respectively.

Let $y_\infty \in K(\Sigma)$ and $z_\infty \in K(\Sigma)$ be the respective limits of y_i and z_i under the above rescaling limit. By Lemma 2.1, we have

$$\limsup_{i\to\infty} \angle^X y_i x_i x \le \angle y_\infty \xi o_x, \quad \limsup_{i\to\infty} \angle^X z_i x_i x \le \angle z_\infty \xi o_x,$$
$$\limsup_{i\to\infty} \angle^S y_i x_i x \le \angle y_\infty \xi o_x, \quad \limsup_{i\to\infty} \angle^S z_i x_i x \le \angle z_\infty \xi o_x.$$

It follows from

$$\angle^X y_i x_i x + \angle^X z_i x_i x \ge \pi, \quad \angle^S y_i x_i x + \angle^S z_i x_i x \ge \pi \quad \text{and} \quad \angle y_\infty \xi o_x + \angle z_\infty \xi o_x = \pi$$

that

$$\begin{aligned} \lim_{i\to\infty} \angle^X y_i x_i x &= \angle y_\infty \xi o_x = \lim_{i\to\infty} \angle^S y_i x_i x, \\ \lim_{i\to\infty} \angle^X z_i x_i x &= \angle z_\infty \xi o_x = \lim_{i\to\infty} \angle^S z_i x_i x. \end{aligned} \tag{4-23}$$

Now let $w_i \in \Sigma_{x_i}(S)$ be the nearest point of $\Sigma_{x_i}(S)$ from $\dot\gamma^X_{x_i,x}(0)$. If $\dot\gamma^X_{x_i,x}(0) \in \Sigma_{x_i}(S)$, then (4-23) implies that $\angle(\dot\gamma^X_{x_i,x}(0), \dot\gamma^S_{x_i,x}(0)) \to 0$ as $i \to \infty$. This contradicts (4-22). Suppose $\dot\gamma^X_{x_i,x}(0) \notin \Sigma_{x_i}(S)$. Then $w_i \in V(\Sigma_{x_i}(X))$. It follows from Lemma 2.11 that $\angle(\dot\gamma^X_{x_i,x}(0), w_i) < \tau_x(|x, x_i|_X)$. In what follows, we may assume that $\angle(\dot\gamma_{x_i,y_i}(0), w_i) \ge \angle(\dot\gamma_{x_i,y_i}(0), \dot\gamma^S_{x_i,x}(0))$ without loss of generality by replacing y_i by z_i if necessary. Then using Lemma 4.25 and (4-23), we obtain

$$\begin{aligned} \angle(\dot\gamma^X_{x_i,x}(0), \dot\gamma^S_{x_i,x}(0)) &= \angle(\dot\gamma^X_{x_i,x}(0), w_i) + \angle(w_i, \dot\gamma^S_{x_i,x}(0)) \\ &= \angle(\dot\gamma^X_{x_i,x}(0), w_i) + \angle(w_i, \dot\gamma_{x_i,y_i}(0)) - \angle(\dot\gamma^S_{x_i,x}(0), \dot\gamma_{x_i,y_i}(0)) \\ &\le 2\angle(\dot\gamma^X_{x_i,x}(0), w_i) + \angle(\dot\gamma^X_{x_i,x}(0), \dot\gamma_{x_i,y_i}(0)) - \angle(\dot\gamma^S_{x_i,x}(0), \dot\gamma_{x_i,y_i}(0)) \\ &< \tau_x(|x, x_i|_X) + o_i, \end{aligned}$$

where $\lim_{i\to\infty} o_i = 0$. This is a contradiction to (4-22). □

Lemma 4.32 *For $x, y \in S$, suppose that the S–geodesic $\gamma^S_{x,y}\colon [0,1] \to S$ from x to y is vertical. Then $\gamma^S_{x,y}$ is an X–geodesic.*

Proof For any $t \in [0,1]$, let $\epsilon > 0$ be chosen as in Lemma 4.21 for $z := \gamma^S_{x,y}(t)$. Choose $t_n \to t$, and set $z_n := \gamma^S_{x,y}(t_n)$. Let γ^X_n be the X–geodesic from z to z_n. In view of Lemmas 4.25 and 4.21, we have

$$\gamma^X_n \subset U_\pm(z, \epsilon) \subset S.$$

Thus γ^X_n must be a subarc of $\gamma^S_{x,y}$, and hence $\gamma^S_{x,y}$ is an X–geodesic. □

Lemma 4.33 *For $x, y \in S$ with $x \in \alpha_1$ and $y \in \alpha_2$ satisfying*

$$\big||p, x|_X - |p, y|_X\big| < \tfrac{1}{100}|x, y|_X,$$

the S–geodesic joining x and y is an X–geodesic.

Proof It follows from the assumption that

$$\big||p,x|_S - |p,y|_S\big| < \tfrac{1}{100}|x,y|_X \le \tfrac{1}{100}|x,y|_S.$$

Using Lemma 2.5 in S, we have

$$|\angle p\gamma^S_{x,y}(t)x - \pi/2| < \tfrac{1}{3}\pi \quad \text{and} \quad |\angle p\gamma^S_{x,y}(t)y - \tfrac{1}{2}\pi| < \tfrac{1}{3}\pi$$

for all $t \in (0,1)$, where $\gamma^S_{x,y} \colon [0,1] \to S$ is the S–geodesic joining x to y. This implies that $\gamma^S_{x,y}$ is vertical. The lemma then follows from Lemma 4.32. □

In a way similar to Lemma 4.33, we have the following.

Lemma 4.34 *For arbitrary $x, y \in S$ such that*

$$\big||p,x|_S - |p,y|_S\big| < \tfrac{1}{100}|x,y|_S,$$

the S–geodesic joining x and y is an X–geodesic.

5 Filling via CAT(κ)–disks

Let v be a vertex of Σ_p of order N, and let $v_1,\dots,v_N$ be the set of all points of Σ_p with $d(v_i,v)=\delta$ for a sufficiently small positive number δ. Take a small enough $r>0$ and points $a_1,\dots,a_N$ of $S(p,2r)$ with $\dot\gamma_{p,a_i}(0)=v_i$ and $r\le r_p$. For simplicity, we denote by $S(a_i,a_j)$ the ruled surface $S(\gamma_{p,a_i},\gamma_{p,a_j})$ spanned by γ_{p,a_i} and γ_{p,a_j}. Let $V(\Sigma_p)$ be the set of all vertices of the graph Σ_p. Since $V(\Sigma_p)$ is finite, we have a positive number r_p such that for any $0<r\le r_p$, all the $S(a_i,a_j)$, when v runs over $V(\Sigma_p)$, satisfy the conclusion of Theorem 4.1. Then obviously $\mathcal{S}(X)\cap B(p,r)$ is contained in the union of all $S(a_i,a_j)$ when v runs over $V(\Sigma_p)$.

Sector correspondence We fix $S := S(a_i,a_j)$ for a moment, and set

$$\Omega(S,r)^X := B^X(p,r)\cap S \quad \text{and} \quad C^X := S^X(p,r)\cap S.$$

From here on, we use the symbols $B^X(p,r)$ and $S^X(p,r)$ to emphasize the metric ball and the metric circle *in* X. Note that $\Omega(S,r)^X$ is bounded by the two geodesics γ_{p,a_i}, γ_{p,a_j} and C^X.

To show Theorem 1.1(3), we need the following lemma.

Lemma 5.1 *For any small enough $r\le r_p$, the sector $\Omega(S,r)^X$ is $\tau_p(r)$–almost isometric to a Euclidean sector.*

Proof By Lemma 4.30, $\angle(\dot\gamma^X_{x,p}(0), \dot\gamma^S_{x,p}(0)) < \tau_p(r)$ for every $x \in C^X$. Since $\angle(\dot\gamma^X_{x,p}(0), \dot C^X) = \frac{1}{2}\pi$, it follows that

$$\left|\angle(\dot\gamma^S_{x,p}(0), \dot C^X) - \tfrac{1}{2}\pi\right| < \tau_p(r). \tag{5-1}$$

Consider the rescaling limit of the CAT(κ)–space: $((1/r)S, p) \to (K_p(S), o_p)$ as $r \to 0$. By Theorem 2.6, we have a $\tau_p(r)$–almost isometry $\varphi\colon \Omega(S,r)^X \to \mathrm{image}(\varphi) \subset \mathbb{R}^2$. It suffices to show that image(φ) is $\tau_p(r)$–almost isometric to a Euclidean sector. Although the argument below is elementary and standard, we present a proof for completeness since we do not find a reference.

We may assume $\varphi(p) = (0,0) = O$. For $k = 1, 2$, let L_k be the line segment from O to $\varphi(\gamma_k(r))$. We express L_k in the polar coordinates as

$$L_k(x) = (x, \theta_k) \quad \text{for } 0 \le x \le x_k(r),$$

where θ_k is a constant and $x_k(r) := |\varphi(\gamma_k(r)), O|$. Let θ_0 be the direction representing the midpoint of $\dot L_1(0)$ and $\dot L_2(0)$. We may assume that $\theta_1 < \theta_0 = 0 < \theta_2$, and let L_0 be the line segment from O in the direction θ_0: $L_0(x) = (x, 0)$. Let $\varphi(C^X)$ intersect L_0 with $(r_0, 0)$. Set $q_k := L_k(x_k(r))$.

Let U_k (resp. D_k) be the domain bounded by L_0, $\varphi(\gamma_k)$ and $\varphi(C^X)$ (resp. by L_0, L_k and $\varphi(C^X)$). Let $\Omega(L_1, L_2; r)$ denote the Euclidean sector bounded by the rays in the directions L_1, L_2 and the circle of radius r. In the first step, we deform image$(\varphi) = U_1 \cup U_2$ to $D_1 \cup D_2$ via a $\tau_p(r)$–almost isometry. In the second step, we deform $D_1 \cup D_2$ to $\Omega(L_1, L_2; r)$ via a $\tau_p(r)$–almost isometry.

Step 1 Choose a point $q_0 \in L_0$ such that $\angle\frac{1}{4}\pi \le Oq_kq_0 \le \frac{1}{3}\pi$ for $k = 1, 2$. Note that $[q_k, q_0] \subset U_k$. Let J_k denote the union $[O, q_0] \cup [q_0, q_k]$. Let $\widehat U_k$ (resp. $\widehat D_k$) be the domain bounded by L_0, $\varphi(\gamma_k)$ and $[q_k, q_0]$ (resp. by L_0, L_k and $[q_k, q_0]$). We first show that $\widehat U_k$ is $\tau_p(r)$–almost isometric to $\widehat D_k$.

Let $J_k(x)$ for $0 \le x \le L(J_k)$ be the arclength parameter of J_k with $J_k(0) = O$. For every $x \in [0, L(J_k)]$, let $\zeta_k(x, s)$ for $0 \le s \le 2$ be the segment such that

- $\zeta_k(x, 0) = J_k(x)$ and $\zeta_k(x, 1) \in L_k$,
- $|O, \zeta_k(x, s)| = |O, \zeta_k(x, 0)|$ for all $s \in [0, 2]$,
- $s \mapsto \zeta_k(x, s)$ is proportional to arclength.

Then $\zeta_k(x, s)$ with $0 \le x \le L(J_k)$ and $0 \le s \le 1$ defines a parametrization of $\widehat D_k$, and it is differentiable except at $x = x_0$, where $J_k(x_0) = q_0$. Take a unique $t_k(x) \in (0, 2)$ such that

$$\zeta_k(x, t_k(x)) \in \mathrm{Im}(\varphi_k \circ \gamma_k).$$

Now, we define $\psi_k\colon \widehat U_k \to \widehat D_k$ for $k = 1, 2$ by

$$\psi_k(\zeta_k(x, s)) := \zeta_k\left(x, \frac{s}{t_k(x)}\right).$$

Obviously, $t_k(x)$ is locally Lipschitz, and hence differentiable on a set $\Omega \subset [0, L(J_k)]$ with full measure since $\zeta_k(x, s)$ defines a locally bi-Lipschitz embedding. See Figure 11.

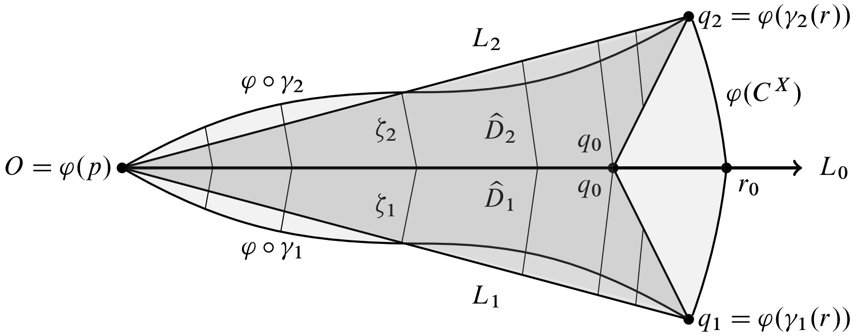

Figure 11

Sublemma 5.2 *Each $\psi_k : \widehat{U}_k \to \widehat{D}_k$ is a $\tau_p(r)$–almost isometry.*

Proof In the expression $\psi_k(x,s) := \psi_k \circ \zeta_k(x,s) = \zeta_k(x, s/t_k(x))$, we have on $\Omega \times [0,1]$

$$\frac{\partial \psi_k}{\partial s} = \frac{1}{t_k(x)} \frac{\partial \zeta_k}{\partial s} \quad \text{and} \quad \frac{\partial \psi_k}{\partial x} = \frac{\partial \zeta_k}{\partial x} + \left(\frac{-s t_k'(x)}{t_k(x)^2} \right) \frac{\partial \zeta_k}{\partial s}. \tag{5-2}$$

It is easily checked that for some uniform positive constants $c_1, \dots, c_4$,

$$\begin{cases} 0 < c_1 < \left| \dfrac{\partial \zeta_k}{\partial x} \right| < c_2, \\ 0 < c_3 < \angle\left(\dfrac{\partial \zeta_k}{\partial s}, \dfrac{\partial \zeta_k}{\partial x} \right) < \pi - c_4. \end{cases} \tag{5-3}$$

Note also that

$$|t_k(x) - 1| < \tau_p(r). \tag{5-4}$$

By the property of φ, we see that any tangent vector to $\varphi \circ \gamma_k$ is $\tau_p(r)$–almost parallel to the radial direction. Now consider the curve $\eta_k(x) = \zeta_k(x, t_k(x))$ parametrizing $\varphi \circ \gamma_k$. It follows from the expression

$$\frac{d\eta_k}{dx}(x) = \frac{\partial \zeta_k}{\partial x}(x, t_k(x)) + \frac{\partial \zeta_k}{\partial s}(x, t_k(x)) t_k'(x)$$

that

$$\left| \frac{\partial \zeta_k}{\partial s}(x,s) t_k'(x) \right| < \tau_p(r). \tag{5-5}$$

Let

$$v := \frac{\partial \zeta_k}{\partial x}, \quad V := d\psi_k(v) \quad \text{and} \quad w := \left| \frac{\partial \zeta_k}{\partial s} \right|^{-1} \frac{\partial \zeta_k}{\partial s}, \quad W := d\psi_k(w).$$

Combining (5-2)–(5-5), we have

$$\big| |V| - |v| \big| < \tau_p(r), \quad \big| |W| - |w| \big| < \tau_p(r), \quad |\langle V, W \rangle - \langle v, w \rangle| < \tau_p(r).$$

Together with (5-3), this implies $\big| |d\psi_k(u)| - 1 \big| < \tau_p(r)$ for each unit tangent vector u on $\Omega \times [0,1]$. This completes the proof of Sublemma 5.2. □

Obviously, the $\tau_p(r)$–almost isometry $\psi_k\colon \hat{U}_k \to \hat{D}_k$ extends to a $\tau_p(r)$–almost isometry $\psi_k\colon U_k \to D_k$. Combining ψ_1 and ψ_2, we obtain a $\tau_p(r)$–almost isometry ψ between the image of φ and $D_1 \cup D_2$:

$$\psi\colon \mathrm{Im}(\varphi) \to D_1 \cup D_2 \subset \mathbb{R}^2.$$

Step 2 Finally we deform $D_1 \cup D_2$ to the Euclidean sector $\Omega(L_1, L_2; r)$. Let $\varphi(C^X)$ be parametrized as $\varphi(C^X) = (r(t), \theta(t))$ for $0 \le t \le 1$. For every $0 \le r' \le r(t)$, let us define

$$\phi(r', \theta(t)) = \left(\frac{r}{r(t)} r', \theta(t)\right),$$

which defines a $\tau_p(r)$–almost isometry

$$\phi\colon D_1 \cup D_2 \to \Omega(L_1, L_2; r).$$

Thus the composition $\phi \circ \psi \circ \varphi\colon \Omega(S, r)^X \to \Omega(L_1, L_2; r)$ is a $\tau_p(r)$–almost isometry. This completes the proof of Lemma 5.1. □

Lemma 5.3 *$S \cap B(p, r)$ is a* CAT(κ)*–space with respect to the interior metric.*

Proof It suffices to show that every point $q \in S \cap S(p, r)$ has a neighborhood U in $S \cap B(p, r)$ such that any S–geodesic triangle region whose vertices are in U is contained in $S \cap B(p, r)$. To achieve this, we only have to show that $S \cap B(p, r)$ is boundary convex, in the sense that for arbitrary $x, y \in S \cap S(p, r)$, any S–minimal geodesic $\gamma^S_{x,y}$ joining them is contained in $S \cap B(p, r)$. We may assume that $\gamma^S_{x,y}$ is vertical, and therefore it is an X–geodesic (see also Lemma 6.3). Hence the conclusion follows from the X–convexity of $B(p, r)$. □

Filling ball

Now we fill the ball $B(p, r)$ via properly embedded/branched immersed CAT(κ)–disks. For a vertex v of Σ_p of order N, let $\nu_1, \dots, \nu_N$ and $a_1, \dots, a_N$ be as in the beginning of Section 5. For every pair (i, j) with $1 \le i < j \le N$, we want to take a simple loop in $\Sigma_p(X)$ passing through ν_i, v and ν_j. Since this is not possible in general, we consider the two cases.

Case I There is a simple loop ζ in $\Sigma_p(X)$ through ν_i, v and ν_j.

Consider the ruled surface $S(a_i, a_j)$ as well as the other ruled surfaces defined around other points of ζ which are vertices of $\Sigma_p(X)$ (if they exist). By Lemma 5.1, considering the regular part of ζ as well, we can define a proper Lipschitz embedding $f^v_{ij}\colon D^2(\ell; r) \to B(p, r)$ with $f^v_{ij}(O) = p$ satisfying $\Sigma_p(\mathrm{Im}(f^v_{ij})) = \zeta$, where ℓ is the length of ζ.

Proof of Theorem 1.1(1) for embedded disks Lemma 5.3 together with the gluing procedure as discussed after Lemma 4.29 implies that $\mathrm{Im}(f^v_{ij})$ is a CAT(κ)–space. Note that f^v_{ij} has bi-Lipschitz constant $< 1 + \tau_p(r)$. □

Case II There are no simple loops in $\Sigma_p(X)$ containing ν_i, v and ν_j.

Claim 5.4 *There is an immersion $g\colon S^1 \to \Sigma_p(X)$ such that*:

(1) *If W is the set of multiple points of g, then $g^{-1}(W)$ consists of two arcs W_1 and W_2 (they may be points), and each restriction $g|_{W_a}\colon W_a \to W$ for $a = 1, 2$ is injective.*

(2) *There is an arc I of S^1 such that $g(I)$ coincides with the arc between ν_i and ν_j containing v.*

Proof In view of the present case, there are noncontractible loops C_i and C_j at v, freely homotopic to a circle, such that $\nu_i \in C_i$, $\nu_j \in C_j$, $\nu_j \notin C_i$ and $\nu_i \notin C_j$. If both C_i and C_j are simple, we can define a desired immersion $g\colon S^1 \to \Sigma_p(X)$ with $W = \{v\}$. Suppose C_i is not simple. Then C_i contains a simple loop $\widetilde{C}_i$ at a point u_i such that C_i is the union of $\widetilde{C}_i$ and the arc $[v, u_i]$. If C_j is also not simple, then we consider the union of simple loops $\widetilde{C}_i$, $\widetilde{C}_j$ and the arc $[u_i, u_j]$. If only C_i is not simple, then we consider the union of simple loops $\widetilde{C}_i$, C_j and the arc $[u_i, v]$. This observation provides a desired immersion $g\colon S^1 \to \Sigma_p(X)$ with $W = [u_i, u_j]$ or $W = [u_i, v]$. □

First suppose $W = \{v\}$ and find $\nu_k \in C_i$ and $\nu_\ell \in C_j$ with $1 \le k, \ell \le N$ and $k, \ell \ne i, j$. Chasing on $g(I)$ in the order

$$\nu_i \to v \to \nu_j \to \nu_\ell \to v \to \nu_k \to \nu_i,$$

we consider the ruled surfaces $S(a_i, a_j)$ and $S(a_k, a_\ell)$ as well as the other ruled surfaces defined around other points of $g(I)$ which are vertices of $\Sigma_p(X)$ (if they exist). By Lemma 5.1, considering the regular part of $g(I)$ as well, we can define a proper Lipschitz immersion $f_{ij}^v\colon D^2(\ell; r) \to B(p, r)$ with branched point $(f_{ij}^v)^{-1}(p) = \{O\}$ satisfying $\Sigma_p(\mathrm{Im}(f_{ij}^v)) = g(S^1)$, in a way similar to Case I. Note that any multiple point $q \in \mathrm{Im}\, f_{ij}^v$ lies in a direction close to v.

Next suppose $W = [u_i, v]$ and find $\nu_\ell \in C_j$ with $1 \le \ell \le N$ and $\ell \ne i, j$. Chasing on $g(I)$ in the order

$$\nu_i \to v \to \nu_j \to \nu_\ell \to v \to \nu_i \to u_i \to \widetilde{C}_i \to \nu_i,$$

we similarly consider the ruled surfaces $S(a_i, a_j)$, $S(a_j, a_\ell)$ as well as the other ruled surfaces defined around other points of $g(I)$ which are vertices of $\Sigma_p(X)$ (if they exist). In a way similar to the previous case, we can define a desired proper Lipschitz immersion $f_{ij}^v\colon D^2(\ell; r) \to B(p, r)$ branched at the point $(f_{ij}^v)^{-1}(p) = \{O\}$ satisfying $\Sigma_p(\mathrm{Im}(f_{ij}^v)) = g(S^1)$.

The other case is similar, and hence omitted.

Note that f_{ij}^v has bi-Lipschitz constant (resp. local bi-Lipschitz constant except the origin) less than $1 + \tau_p(r)$ in Case I (resp. in Case II).

Lemma 5.5
$$B(p, r) = \bigcup_{v \in V(\Sigma_p(X))} \biggl(\bigcup_{1 \le i < j \le N} \mathrm{Im}\, f_{ij}^v \biggr).$$

Proof First note that from construction, $\Sigma_p(X)$ coincides with all the union of $\Sigma_p(\mathrm{Im}\, f_{ij}^v)$. Suppose there is a point $x \in B(p, r)$ which is not contained in any image $\mathrm{Im}\, f_{ij}^v$. Let $\xi := \uparrow_p^x$. Take some $\mathrm{Im}\, f_{ij}^v$

such that $\xi \in \Sigma_p(\mathrm{Im}\, f_{ij}^v)$. We may assume that ξ is close to the vertex v, since if ξ is far from any vertex of $\Sigma_p(X)$, then $x = \gamma_\xi(|p,x|_X)$ is certainly contained in the union of all the images $\mathrm{Im}\, f_{ij}^v$, which is a contradiction.

Let γ be a geodesic in $\mathrm{Im}\, f_{ij}^v$ starting from p in the direction ξ. Note that γ reaches the metric sphere $S(p,r)$; see also Sublemma 4.31. Let x' be the point of γ_ξ such that $|p,x'|_X = |p,x|_X$. Consider the geodesic $\gamma^X_{x,x'}$. If we extend $\gamma^X_{x,x'}$ through x', it meets γ_{p,a_k} for some k. Similarly, if we extend $\gamma^X_{x,x'}$ through x, it meets γ_{p,a_ℓ} for some ℓ. Lemma 4.33 yields that $x \in S(a_k,a_j)$. This is a contradiction. □

Combining Lemma 5.5 and the above discussion, we complete the proof of Theorem 1.1(1)–(3) except for (1) for the branched immersed disks that occur from the above Case II.

The proof of Theorem 1.1(1) for the branched immersed disks is deferred to Section 7.

6 Graph structure of singular set

Our next step is to characterize $\mathcal{S}(X) \cap B(p,r)$ as a union of finitely many Lipschitz curves.

For a subset A of X, we denote by ∂A the complement in $\bar{A}$ of the set of all points a of A such that there is a neighborhood of a homeomorphic to an open disk and contained in A.

For distinct $1 \le i,j,k \le N$, we set

$$C_{ij;k} := (\partial(S(a_i,a_j) - S(a_j,a_k)) - \partial S(a_i,a_j)) \cap B(p,r).$$

Lemma 6.1 *$C_{ij;k}$ is a simple Lipschitz arc in $\mathcal{S}(X)$ such that*

(1) *it starts from p and reaches a point of $\partial B(p,r)$,*

(2) *its length is less than $(1+\tau_p(r))r$, and*

(3) *each point of $\Sigma_x(C_{ij;k})$ is a vertex of $\Sigma_x(X)$ for every $x \in C_{ij;k}$. In particular, $C_{ij;k}$ has definite directions everywhere, and*

$$\frac{|d_p(x) - d_p(y)|}{|x,y|_X} \ge 1 - \tau_p(r) \quad \text{for all } x,y \in C_{ij;k}.$$

Proof For each $s \in [0,2r]$, consider the ruling geodesic $\lambda_s(t)$ with $0 \le t \le 1$ of $S(a_k,a_j)$ joining $\gamma_{p,a_k}(s)$ to $\gamma_{p,a_j}(s)$ in X. Let $t_0 \in (0,1)$ be the first parameter at which λ_s meets $S(a_i,a_j)$.

We claim that

$$\lambda_s([t_0,1]) \subset S(a_i,a_j). \tag{6-1}$$

Since $z_s := \lambda_s(t_0)$ is a topological singular point of X, by Lemma 4.25 we can take a direction ξ_0 in $\Sigma_{z_s}(S(a_i,a_j))$ with $\angle(\xi_0,\lambda_s'(t_0)) = \pi$. A geodesic γ_{ξ_0} in $S(a_i,a_j)$ with direction ξ_0 reaches γ_{p,a_i}. Take $\xi_1 \in \Sigma_{z_s}(S(a_i,a_j))$ with $\angle(\xi_0,\xi_1) = \pi$. Similarly, a geodesic γ_{ξ_1} in $S(a_i,a_j)$ with direction ξ_1

reaches γ_{p,a_j}. It follows from Lemma 4.34 that γ_{ξ_0} and γ_{ξ_1} form a geodesic in X. In particular, γ_{ξ_0} is a geodesic in X, and therefore γ_{ξ_0} and $\lambda_s([t_0,1])$ form a geodesic, say γ, in X, Lemma 4.33 implies that γ is contained in $S(a_i,a_j)$, and so is $\lambda_s([t_0,1])$.

Since the curve $c(s) := z_s$ is continuous, its image coincides with $C_{ij;k}$. By Corollary 2.12, we have

$$\angle((\nabla d_p)(c(s)), \Sigma_{c(s),+}(C_{ij;k})) < \tau_p(r) \quad \text{and} \quad \angle((-\nabla d_p)(c(s)), \Sigma_{c(s),-}(C_{ij;k})) < \tau_p(r),$$

where $\Sigma_{c(s),+}(C_{ij;k})$ (resp. $\Sigma_{c(s),-}(C_{ij;k})$) denotes the space of directions of $C_{ij;k}$ at $c(s)$ in the positive direction (resp. negative direction).

Now we take another parametrization $\varphi(s)$ of $C_{ij;k}$ defined as $\varphi(s) = C_{ij;k} \cap S(p,s)$, where $S(p,s)$ denotes the metric circle of radius s with respect to d_X. If s' is close enough to s, then we have $\angle(\uparrow_{\varphi(s)}^{\varphi(s')}, \nabla d_p(\varphi(s))) < \tau_p(r)$, which implies that

$$\lim_{s' \to s} \frac{|\varphi(s),\varphi(s')|_X}{|s-s'|} \le 1 + \tau_p(r). \tag{6-2}$$

Thus φ is Lipschitz with Lipschitz constant $\le 1+\tau_p(r)$, therefore of length $L(\varphi) = L(C_{ij;k}) \le (1+\tau_p(r))r$. Equation (6-2) also implies the inequality in (3). □

Lemma 6.1 claims that the closure of $S(a_j,a_k) - S(a_i,a_j)$ "transversally" intersects $S(a_i,a_j)$ with the Lipschitz curve $C_{ij;k}$. In particular, we have:

Lemma 6.2
$$C_{ij;k} = C_{ji;k} = C_{jk;i}.$$

In view of Lemma 6.2, we use the notation

$$C_{ijk} := C_{ij;k}.$$

Using the discussion in the proof of Lemma 6.1, we show the following refined version of Lemma 4.33, which is not a direct consequence of Lemma 4.34.

Lemma 6.3 *For arbitrary $x, y \in S = S(a_i,a_j)$ such that*

$$\big||p,x|_X - |p,y|_X\big| < \tfrac{1}{1000}|x,y|_X,$$

the X–geodesic joining x and y is an S–geodesic.

Proof Consider the geodesic $\gamma_{x,y}^X$ and extend it in both directions until it reaches γ_{p,a_k} and γ_{p,a_ℓ} for some k,ℓ at $w_k \in \gamma_{p,a_k}$ and $w_\ell \in \gamma_{p,a_\ell}$ respectively. That is,

$$[w_k,w_\ell]_X = [w_k,x]_X \cup [x,y]_X \cup [y,w_\ell]_X.$$

Let z (resp. u) be the first point at which $[w_k,x]$ (resp. $[w_\ell,y]$) meets $R(a_i,a_j)$. As in the proof of Lemma 6.1, we have points $w_i \in \gamma_{p,a_i}$ and $w_j \in \gamma_{p,a_j}$ such that

$$[w_i,w_j]_X = [w_i,z]_X \cup [z,x]_X \cup [x,y]_X \cup [y,w_j]_X.$$

From the hypothesis, we have $\big||p,w_i|-|p,w_j|\big| < \frac{1}{100}|w_i,w_j|_X$. Lemma 4.33 then implies that $[w_i,w_j]_X$ is an S–geodesic. Thus we conclude that $[x,y]_X$ is an S–geodesic, as required. □

For a vertex v of $\Sigma_p(X)$, suppose that $a_1,\dots,a_N \in S(p,2r)$ are as in Section 5, where $N = N_v$. Let $S(a_1,\dots,a_N;r)$ be the closed domain of $B(p,r)$ bounded by γ_{p,a_i} for $1 \le i \le N$, and $S(p,r)$. Note that $S(a_1,\dots,a_N;r)$ is the union of all the ruled surfaces $S(a_i,a_j)$ and $B(p,r)$.

Corollary 6.4 *For a vertex v of $\Sigma_p(X)$, the union of all C_{ijk} coincides with $\mathcal{S}(X) \cap S(a_1,\dots,a_N;r)$.*

Proof Since every element of $\mathcal{S}(X) \cap S(a_1,\dots,a_N;r)$ comes from the intersection of some $S(a_i,a_j)$ and $S(a_k,a_\ell)$, it suffices to show that $\partial(S(a_i,a_j) \cap S(a_k,a_\ell)) \setminus S(p,r)$ is contained in $C_{ijk} \cup C_{ij\ell}$. For every $x \in \partial(S(a_i,a_j) \cap S(a_k,a_\ell))$, take an s such that the ruling geodesic λ_s joining $\gamma_{p,a_i}(s)$ to $\gamma_{p,a_j}(s)$ goes through x, say at $\lambda_s(t_0) = x$. Since $x \in \mathcal{S}(X)$, Lemma 2.4(2), Theorem 4.1 and Corollary 2.12 imply the existence of a direction $\xi \in \Sigma_x(S(a_k,a_\ell))$ such that $\angle(\xi,\lambda_s'(t_0)) = \pi$. Then a geodesic γ_ξ in $S(a_k,a_\ell)$ with direction ξ must reach γ_{p,a_k} or γ_{p,a_ℓ}. Suppose it reaches γ_{p,a_k} for instance: $\gamma_\xi(t_1) \in \gamma_{p,a_k}$ for some $t_1 > 0$. An argument similar to that in the proof of Lemma 6.1 then implies that $\gamma_\xi([0,t_0])$ does not meet $S(a_i,a_j)$ except for x, and that the union $\gamma_\xi([0,t_0]) \cup \lambda_s([t_0,1])$ forms a geodesic in $S(a_k,a_j)$. This shows $x \in C_{ijk}$. □

Proof of the second half of Theorem 1.1 It is now an immediate consequence of Lemma 6.1 and Corollary 6.4. □

We call a curve C in $\mathcal{S}(X)$ *a singular curve*.

Remark 6.5 Each singular curve C contained in $S(a_1,\dots,a_N;r)$ has the direction v at p. From now on, we always consider the case when d_p is strictly increasing along C. In that case, for each interior point q of C, C has definite directions $\Sigma_q(C)$ consisting of two vertices of $\Sigma_q(X)$.

Structure of metric circles

Next we discuss the structure of $S(p,r)$.

Let $b_i := \gamma_{p,a_i}(r)$. For $0 < t \le r$, set

$$S(v;t) := \Big(\bigcup_{1\le i<j\le N} S(a_i,a_j)\Big) \cap S^X(p,t).$$

Lemma 6.6 *For each $0 < t \le r$, $S(v;t)$ is a tree with endpoints $\gamma_{p,a_i}(t)$ for $1 \le i \le N$.*

Proof For $3 \le k \le N$, put

$$S_k(v;t) := \Big(\bigcup_{1\le i<j\le k} S(a_i,a_j)\Big) \cap S^X(p,t).$$

Inductively we show that $S_k(v;t)$ is a tree for every $3 \le k \le N$. This is certainly true for $k = 3$. Assume that $S_{k-1}(v;t)$ is a tree. Set

$$S(a_i, a_k)(t) := S(a_i, a_k) \cap S^X(p,t).$$

Let $p_k(t) := \gamma_{p,a_k}(t)$. Let $q(t)$ be the point of $S_{k-1}(v,t)$ where the arc starting from $p_k(t)$ in $S_k(v,t)$ first meets $S_{k-1}(v,t)$. For every $1 \le i \ne j \le k-1$ with $q(t) \in S(a_i, a_j)(t)$, equation (6-1) implies that

$$S(a_i, a_k)(t) \setminus [p_k(t), q(t)] \subset S(a_i, a_j), \tag{6-3}$$

where $[p_k(t), q(t)]$ denotes the arc between $p_k(t)$ and $q(t)$ in $S_k(v,t)$. Equation (6-3) implies that $S_k(v,t) = S_{k-1}(v,t) \cup [p_k(t), q(t)]$. Thus $S_k(v,t)$ is a tree. □

Proof of Corollary 1.6 Let $r_p \ge r > 0$ be as in Theorem 1.1. By Lemma 6.6, for every vertex v of $\Sigma_p(X)$, $S(v;r)$ is a tree with endpoints b_i for $1 \le i \le N$. Therefore $S(p,r)$ has the same homotopy type as $\Sigma_p(X)$.

Let α be any noncontractible simple closed loop in $S(p,r)$. From the discussion in Case I and the proof of Theorem 1.1(1) in Section 5, there is a noncontractible simple closed loop ζ in $\Sigma_p(X)$ of length, say, $\ell \ge 2\pi$, and a properly embedded CAT(κ)–disk $f : D^2(\ell;r) \to B(p,r)$ associated with ζ such that $\Sigma_p(\mathrm{Im}(f)) = \zeta$ and $f(\partial D^2(\ell;r)) = \alpha$. For $v \in \zeta \cap V(\Sigma_p(X))$, let $\xi_i, \xi_j \in \zeta$ be points near v such that v is the midpoint of the arc $[\xi_i, \xi_j]$. Let S_{ij} be the ruled surface defined by ξ_i and ξ_j. Note that $B^{S_{ij}}(p,r) \subset S_{ij} \cap B^X(p,r)$. Since we may assume $r < \pi/2\sqrt{\kappa}$, the nearest-point map $S_{ij} \cap S^X(p,r) \to S^{S_{ij}}(p,r)$ is distance nonincreasing. Let $\widetilde{S}_{ij}$ be a sector in the model M_κ^2 with vertex $\tilde{p}$ bounded by two geodesics of length r and $S(\tilde{p}, r)$ such that the sector angle at $\tilde{p}$ is equal to $\angle(\xi_1, \xi_2)$. From the curvature condition, we have

$$L(S^{S_{ij}}(p,r)) \ge L(\widetilde{S}_{ij} \cap S(\tilde{p}, r)) = \angle(\xi_1, \xi_2)/\sqrt{\mu(\kappa, r)},$$

yielding $L(S_{ij} \cap S^X(p,r)) \ge \angle(\xi_1, \xi_2)/\sqrt{\mu(\kappa, r)}$. Applying a similar argument to the other parts of α and ζ, we conclude that

$$L(\alpha) \ge L(\zeta)/\sqrt{\mu(\kappa, r)} \ge 2\pi/\sqrt{\mu(\kappa, r)}.$$

This completes the proof. □

Now we define a metric graph structure of $\mathcal{S}(X)$ in a generalized sense as follows.

Definition 6.7 We consider the relative topology of $\mathcal{S}(X)$ with length metric. Let I be an open set of $\mathcal{S}(X)$. We call I an *open arc* in $\mathcal{S}(X)$ if it is open in $\mathcal{S}(X)$ and is isometric to an open interval. A maximal open arc I with respect to the inclusion is called an *open edge* of $\mathcal{S}(X)$. We denote by $E(\mathcal{S}(X))$ (resp. $|E(\mathcal{S}(X))|$) the set (resp. the union) of all open edges in $\mathcal{S}(X)$. We call each element of $\mathcal{S}(X) \setminus |E(\mathcal{S}(X))|$ a *vertex* of $\mathcal{S}(X)$. We denote by $V(\mathcal{S}(X))$ the set of all vertices of $\mathcal{S}(X)$. Let us denote by $V_*(\mathcal{S}(X)) \subset V(\mathcal{S}(X))$ the set of all accumulation points of $V(\mathcal{S}(X))$. The case $V_*(\mathcal{S}(X)) = V(\mathcal{S}(X))$

or $\mathcal{H}^1(V_*(\mathcal{S}(X))) > 0$ may happen; see Example 6.9. As usual, two vertices v_1 and v_2 of $\mathcal{S}(X)$ are *adjacent* if there is at least one open edge joining them. The *order* of a vertex v is defined as the limit of the number of components of $B^{\mathcal{S}(X)}(v,\epsilon) \setminus \{v\}$ as $\epsilon \to 0$.

Proof of Corollary 1.4 First note that by Theorem 1.1, $\mathcal{S}(X)$ is locally path-connected. For a given point $p \in \mathcal{S}(X)$ and $v \in V(\Sigma_p(X))$, let $N = N_v$ be the branching number of v in $\Sigma_p(X)$, and take $r = r_p$ as in Theorem 1.1. For $\delta > 0$ with $\delta \ll \min\{\angle(v,v') \mid v \neq v' \in V(\Sigma_p(X))\}$, let $\gamma_1,\dots,\gamma_N$ be geodesics from p with $\angle(\dot\gamma_i(0),v) = \delta$ and $\angle(\dot\gamma_i(0),\dot\gamma_j(0)) = 2\delta$ for $1 \le i \neq j \le N_v$. By Corollary 6.4, we have

$$\mathcal{S}(X) \cap U(v) = \bigcup_{1\le i<j<k\le N} C_{ijk},$$

where $U(v) := C(v,\delta,r)$ is the cone neighborhood around v; see (2-7). By Lemma 6.1(3), the distance function d_p is strictly monotone on each C_{ijk}. It follows from Corollary 6.4 that $V(\mathcal{S}(X))$ has locally finite order.

In what follows, we give an explicit sharp bound on the orders at the vertices in $\mathcal{S}(X) \cap B(p,r)$.

Sublemma 6.8 *$\mathcal{S}(X) \cap U(v)$ can be written as the union of at most $N_v - 2$ singular curves C starting from p in the direction v, and reaching $S(p,r)$ such that d_p is strictly increasing along C.*

Proof By Corollary 6.4, $\mathcal{S}(X) \cap U(v)$ coincides with the set of all topological singular points resulting from the intersections of distinct ruled surfaces S_{ij} and $S_{i'j'}$ for all $1 \le i < j \le N$ and $1 \le i' < j' \le N$ with $(i,j) \neq (i',j')$. For $2 \le k \le N$, let E_k be the union of all S_{ij} with $1 \le i < j \le k$. We inductively define singular curves C_j for $2 \le j \le N-1$ as the set of all points of E_j where geodesics almost perpendicularly starting from points of γ_{j+1} intersect E_j for the first time. Then it is obvious to see that $\mathcal{S}(X) \cap U(v) = C_2 \cup \dots \cup C_{N-1}$. From Lemma 6.1, d_p is strictly increasing along C. $\square$

Let $\Gamma := \mathcal{S}(X) \cap B(p,r)$. It follows from Sublemma 6.8 that

- the order at the vertex p of the graph $\Gamma \cap U(v)$ is at most $N_v - 2$,
- the order at any vertex y in $\Gamma \cap U(v) \setminus \{p\}$ is at most $2(N_v - 2)$.

Therefore the maximum of orders of vertices contained in Γ is at most

$$\max\Bigl\{\sum_{v\in V(\Sigma_p(X))}(N_v-2),\ \max_{v\in V(\Sigma_p(X))} 2(N_v-2)\Bigr\}.$$

This completes the proof of Corollary 1.4. $\square$

We exhibit the following example, which is another version of Example 4.4. Here we use the notion of ϵ–Cantor set (see [6]) to produce a two-dimensional CAT(0)–space X such that $V_*(\mathcal{S}(X))$ is one-dimensional. A similar construction for a boundary singular set of a limit space of manifolds with boundary was made in [32].

Example 6.9 For any $0 < \epsilon < 1$, set $\delta := 1 - \epsilon$. We define the so-called ϵ–Cantor set of $[0, 1]$ inductively as follows: We start with $I_0 := [0, 1]$, and remove from I_0 the open interval of length $\delta/2$ around the center of I_0. We denote by I_1 the result of this removing. Note that I_1 consists of 2^1 disjoint closed intervals $I_{1,j}$ for $j = 1, 2$ having the same length and that $L(I_1) = 1 - \delta/2$. Suppose that we have constructed I_k consisting of 2^k disjoint closed intervals $I_{k,j}$ with $1 \le j \le 2^k$ of the same length such that $L(I_k) = 1 - \delta/2 - \cdots - \delta/2^k$. Remove from each $I_{k,j}$ the open interval of length $\delta/2^{2k+1}$ around the center of $I_{k,j}$. We denote by I_{k+1} the result of this removing. Thus, inductively we have constructed I_n for every n. Finally we set

$$I_\infty := \bigcap_{n=0}^{\infty} I_n, \quad J_n := [0, 1] \setminus I_n, \quad J_\infty := \bigcup_{n=0}^{\infty} J_n = [0, 1] \setminus I_\infty.$$

Note that $\mathcal{H}^1(I_\infty) = \lim_{n\to\infty} L(I_n) = 1 - \delta = \epsilon$. The set I_∞ is called an ϵ–Cantor set.

Next, inductively we define smooth functions $f_n : \mathbb{R} \to [0, 1]$ for $n \in \mathbb{N}$ such that

- $\mathrm{supp}(f_n) = J_n$,
- $f_n = f_{n-1}$ on J_{n-1},
- if we set $\hat{J}_n^{\pm} := \{(x, \pm f_n(x)) \mid x \in J_n\}$, then the length ℓ_n and the maximum κ_n of absolute geodesic curvature of $\hat{J}_n^{\pm}$ satisfy (4-4).

Now we define the limit $f := \lim_{n\to\infty} f_n : \mathbb{R} \to [0, 1]$, which satisfies $\mathrm{supp}(f) = J_\infty$. Using f, we define the closed subset Ω of $\mathbb{R}^2$ by

$$\Omega := \{(x, y) \mid |y| \le f(x), x \in \mathbb{R}\},$$

equipped with the length metric. Set

$$\partial_\pm \Omega := \{(x, y) \mid y = \pm f(x), x \in \mathbb{R}\}.$$

Take closed concave domains $H_\pm$ in $\mathbb{R}^2$ homeomorphic to the half plane such that for certain isometries $g_\pm : \partial_\pm \Omega \to \partial H_\pm$ the absolute geodesic curvature of $g_\pm(J_n^\pm)$ is greater than κ_n. Take two copies $H_\pm^1, H_\pm^2$ of $H_\pm$, and make a gluing of $H_+^1, H_+^2, H_-^1, H_-^2$ and Ω along their boundaries via $g_\pm$ as in Example 4.4 to get a two-dimensional locally compact, geodesically complete CAT(0)–space X. Note that $V_*(\mathcal{S}(X)) = V(\mathcal{S}(X)) = I_\infty$ and therefore $\mathcal{H}^1(V_*(\mathcal{S}(X))) = \epsilon > 0$.

7 Approximations by polyhedral spaces

In this section, we give the proof of Theorem 1.1(1) for branched immersed disks. We need to recall the notion of turn, which was first defined in the context of surfaces with bounded curvature in [3]; see also [28].

Definition 7.1 For a moment, let X be a surface with bounded curvature. In X, we have the notion of angles between geodesics starting from a point, and use the same notation for spaces of directions, etc; see [3, Theorem II.10].

Let F be a domain in X with boundary C. For an open arc e of C, we assume that e has definite directions at the endpoints a, b and the spaces of directions of F at a and b have positive lengths. Then the *turn* (rotation) $\tau_F(e)$ of e (see [3, Chapter VI]) from the side of F is defined as follows: Let γ_n be a broken geodesic in $F \setminus e$ except for the endpoints, joining a and b and converging to e as $n \to \infty$. Let Γ_n be the domain bounded by e and γ_n. We denote by α_n and β_n the sector angle of Γ_n at a and b, respectively. Let θ_{ni} for $1 \le i \le N_n$ denote the sector angle at the break points of γ_n, viewed from $F \setminus \Gamma_n$. Let

$$\tilde{\tau}_F(\gamma_n) := \sum_{i=1}^{N_n} (\pi - \theta_{ni}) + \alpha_n + \beta_n.$$

Then the turn $\tau_F(e)$ is defined as

$$\tau_F(e) := \lim_{n\to\infty} \tilde{\tau}_F(\gamma_n),$$

where the existence of the above limit is shown in [3, Theorem VI.2].

For an interior point c of e having definite two directions $\Sigma_c(e)$, the turn of e at c from the side F is defined as

$$\tau_F(c) := \pi - L(\Sigma_c(F)).$$

We now assume the following additional conditions for all $c \in e$:

(1) $L(\Sigma_c(F)) > 0$,

(2) e has definite two directions $\Sigma_c(e)$.

Consider the constant $\mu_F(e) \in [0, \infty]$ defined by

$$\mu_F(e) := \sup_{\{a_i\}} \sum_{i=1}^{n-1} |\tau_F((a_i, a_{i+1}))| + \sum_{i=2}^{n-1} |\tau_F(a_i)|, \tag{7-1}$$

where $\{a_i\} = \{a_i\}_{i=1,\dots,n}$ runs over all the consecutive points on e. The constant $\mu_F(e)$ is called the *turn variation* of e from the side F, and e has *finite turn variation* when $\mu_F(e) < \infty$. For general treatments of curves with finite turn variation in CAT(κ)–spaces, see for instance [4].

Let e be a simple arc on X. One can define the notion of sides F_+ and F_- of e. Under the corresponding assumptions, we define the turns $\tau_{F_+}(e)$ and $\tau_{F_-}(e)$ of e from F_+ and F_- respectively, as above. Similarly, we define the turn variations $\mu_{F_+}(e)$ and $\mu_{F_-}(e)$ from F_+ and F_-. We say e has finite turn variation if $\mu_{F_+}(e) < \infty$ and $\mu_{F_-}(e) < \infty$ (actually, both are finite if one is [3, Lemma IX.1]). When e has finite turn variation, τ_{F_+} and τ_{F_-} provide signed Borel measures on e; see [3, Theorem IX.1].

The structure of the union of ruled surfaces

Let $p \in \mathcal{S}(X)$, and $r = r_p > 0$ be as in Theorem 1.1. From now, we work on $B(p,r)$. Fix any $v \in V(\Sigma_p(X))$, and let $N = N_v$ be the branching number of $\Sigma_p(X)$ at v. For small enough $\delta_p > 0$, let $\gamma_1, \dots, \gamma_N$ be the geodesics from p with $\angle(\dot\gamma_i(0), v) = \delta_p$ and $\angle(\dot\gamma_i(0), \dot\gamma_j(0)) = 2\delta_p$ for $1 \le i \ne j \le N$. For $2 \le k \le N$, we define E_k as the union of ruled surfaces S_{ij} determined by γ_i and γ_j for all $1 \le i \ne j \le k$.

Let C denote the union of all singular curves $C_{ij\ell}$ for $1 \le i < j < \ell \le k$. By Corollary 6.4, C coincides with the set of all topological singular points resulting from the intersections of distinct ruled surfaces S_{ij} and $S_{i'j'}$ for all $1 \le i < j \le k$, $1 \le i' < j' \le k$ with $(i,j) \ne (i',j')$.

Note that a singular curve in the direction v not included in C might meet E_k. We consider the graph structure of C inherited from that of $\mathcal{S}(X)$, which is not the one of C itself introduced as in Definition 6.7. Thus we set

$$E(C) := C \cap E(\mathcal{S}(X)), \quad V(C) := C \cap V(\mathcal{S}(X)), \quad V_*(C) := C \cap V_*(\mathcal{S}(X)),$$

and call $E(C)$ and $V(C)$ the set of edges and the set of vertices of C respectively. Remember that all edges are assumed to be open (Definition 6.7).

Definition 7.2 We say that a vertex point $x \in V(C)$ is *singular* if either $x \in V_*(C)$ or there are two singular curves C_1, C_2 in C starting from x such that

(1) $\angle_x(C_1, C_2) = 0$,

(2) C_1 and C_2 have no intersections near x other than x.

The direction $v \in \Sigma_x(C)$ determined by the above C_1 and C_2 as well as $v = \lim_{i\to\infty} \uparrow_x^{x_i}$ with $V(C) \ni x_i \to x$ is also called *singular*. The set of singular vertices of C is denoted by $V_{\mathrm{sing}}(C)$. The set of singular directions at $x \in V_{\mathrm{sing}}(C)$ is denoted by $\Sigma_x^{\mathrm{sing}}(C)$.

We set $r(x) := d_p(x)$ for simplicity.

For $x \in C \setminus \{p\}$, let $\Sigma_{x,+}(C) := \Sigma_x(C \setminus \operatorname{int} B(p, r(x)))$ and $\Sigma_{x,-}(C) := \Sigma_x(C \cap B(p, r(x)))$. By Lemma 2.11, we may assume that for all $x \in C \setminus \{p\}$,

$$\begin{cases} \angle(\nabla d_p(x), \Sigma_{x,+}(C)) < 10^{-10}, & \operatorname{diam}(\Sigma_{x,+}(C)) < 10^{-10}, \\ \angle(-\nabla d_p(x), \Sigma_{x,-}(C)) < 10^{-10}, & \operatorname{diam}(\Sigma_{x,-}(C)) < 10^{-10}. \end{cases} \tag{7-2}$$

The following lemma is clear.

Lemma 7.3 *For every $x \in C \setminus \{p\}$, $\Sigma_x(E_k)(\subset \Sigma_x(X))$ coincides with the union of all circles $\Sigma_x(S_{ij})$ such that $x \in S_{ij} \subset E_k$, where the circles $\Sigma_x(S_{ij})$ are attached at the points of $\Sigma_{x,\pm}(C)$.*

The following is the main result of this section.

Theorem 7.4 *E_k is a* CAT(κ)*–space.*

Remark 7.5 Recently, we learned that Theorem 7.4 is a direct consequence of the main result of Lytchak and Stadler [20]. However, in what follows, we present our original proof, which provides deep insights on the local geometry of X, and will also be used in [24] as one of key methods.

The basic strategy of the proof of Theorem 7.4 is to use the results [12, Theorems 0.5 and 0.6] on the characterizations for polyhedral spaces to be CAT(κ)–spaces.

Let $\mathcal{F}_\kappa$ be the family of two-dimensional polyhedral locally CAT(κ)–spaces F possibly with boundary ∂F such that any edge of ∂F has finite turn variation. For a collection $\{F_i\}$ of $\mathcal{F}_\kappa$, let X be the polyhedron resulting from certain gluing of $\{F_i\}$ along their edges. We always consider the intrinsic metric of X induced from those of F_i. We consider the following two conditions:

(A) For any Borel subset B of an arbitrary edge e of X, and arbitrary faces F_i and F_j adjacent to e, we have
$$\tau_{F_i}(B) + \tau_{F_j}(B) \le 0.$$

(B) For any vertex x of X, $\Sigma_x(X)$ is CAT(1).

Theorem 7.6 [12, Theorem 0.5] *A polyhedron X resulting from a certain gluing of $\{F_i\} \subset \mathcal{F}_\kappa$ along their edges belongs to $\mathcal{F}_\kappa$ if and only if the conditions* (A) *and* (B) *are satisfied.*

Theorem 7.7 [12, Theorem 0.6] *Each polyhedron X in $\mathcal{F}_\kappa$ can be glued from the faces $\{F_i\}$ contained in $\mathcal{F}_\kappa$ along their edges in such a way that the conditions* (A) *and* (B) *are satisfied.*

In particular, each edge of $\mathcal{S}(X)$ has finite turn variation.

Note that E_k is not a polyhedral space in general. Even in that case, we have some difficulty mentioned below. From these reasons, we shall do surgeries to get a polyhedral space $\widetilde{E}_k$ which approximates E_k in the Gromov–Hausdorff sense. The point is, we can apply Theorems 7.6 and 7.7 to $\widetilde{E}_k$ to conclude that it is CAT(κ). Finally taking the limit, we will obtain the conclusion.

From now on, we set $E := E_k$ for simplicity. We need some preliminary argument on the local geometry of E.

Lemma 7.8 *For every $x \in E$, $\Sigma_x(E)$ is isometric to the intrinsic space of directions $\Sigma_x(E^{\mathrm{int}})$ in the sense of Definition 4.26.*

Proof The basic idea of the proof is the same as that of Lemma 4.27. Obviously, we may assume $x \in C$. We only consider the case $x \neq p$. We first show that each component Σ of $\Sigma_x(E) \setminus \Sigma_x(C)$ is isometrically embedded in $\Sigma_x(E^{\mathrm{int}})$. For $\xi_1, \xi_2 \in \Sigma$ with $|\xi_1, \xi_2| < \pi$, let μ_n be an X–geodesic with $\dot\mu_n(0) = \xi_n$ for $n = 1, 2$. Then for small ϵ, we have $\mu_1([0,\epsilon]) \subset S_{ij}$ and $\mu_2([0,\epsilon]) \subset S_{k\ell}$ for some $S_{ij}, S_{k\ell}$ in E. Note that the X–geodesic $\gamma^X_{\mu_1(t),\mu_2(t)}$ joining $\mu_1(t)$ and $\mu_2(t)$ does not meet C, and hence $\gamma^X_{\mu_1(t),\mu_2(t)}$ is contained in the same ruled surface $S_{ij} = S_{k\ell} \subset E$. This implies that $\angle^X(\xi_1, \xi_2) = \angle^{S_{ij}}(\xi_1, \xi_2)$. From $\angle^X \le \angle^E \le \angle^{S_{ij}}$, we conclude that $\angle^X(\xi_1, \xi_2) = \angle^E(\xi_1, \xi_2)$ and the existence of an isometric embedding $\iota\colon \Sigma \to \Sigma_x(E^{\mathrm{int}})$.

Next, for any $v \in \Sigma_x(C)$, take $\xi_3, \xi_4 \in \Sigma_x(E) \setminus \Sigma_x(C)$ close to v such that the segment $[\xi_3, \xi_4]$ in $\Sigma_x(E)$ meets $\Sigma_x(C)$ only at v. Take X–geodesics α_3, α_4 in the directions ξ_3, ξ_4, and choose $S_{ij}, S_{i'j'}$ in E such that $\alpha_3(t) \subset S_{ij}$ and $\alpha_4(t) \subset S_{i'j'}$. Let $\alpha\colon [0,1] \to X$ be the X–geodesic from $\alpha_3(t)$ to $\alpha_4(t)$. By (7-2), α is vertical. We extend α until it reaches ∂E. We can choose such an extension that $\alpha(t_-) \in \gamma_\ell$ and $\alpha(t_+) \in \gamma'_\ell$ with $\ell \in \{i, j\}$ and $\ell' \in \{i', j'\}$ for some $t_- < 0 < 1 < t_+$. Thus, we have $\alpha([t_-, t_+]) \subset S_{\ell\ell'}$.

By Lemma 4.12, we can find a sequence s_n such that the ruling geodesics λ_{s_n} of $S_{\ell\ell'}$ meets both α_3 and α_4, and λ_{s_n} converges to a ruling geodesic through x as $n \to \infty$. This implies that $\angle^X(\xi_3, \xi_4) = \angle^{S_{ij}}(\xi_3, \xi_4)$, and hence $\angle^X(\xi_3, \xi_4) = \angle^E(\xi_3, \xi_4)$. This completes the proof. □

Lemma 7.9 *For arbitrary $x, y \in E$, let $\gamma := \gamma^E_{x,y}\colon [0, |x,y|_E] \to E$ be an E–shortest curve between x and y. Suppose that the set of accumulation points of $\gamma \cap \mathcal{S}(X)$ is finite. Then γ is an X–geodesic.*

Proof Set $\Gamma := \gamma \cap \mathcal{S}(X)$. We only have to consider the case when Γ has a unique accumulation point $\gamma(u)$ with $\Gamma = \{\gamma(t_i), \gamma(s_j), \gamma(u) \mid i, j = 1, 2, \dots\}$ with $0 \le t_1 < t_2 < \dots < t_i < \dots < u < \dots < s_j < \dots < s_2 < s_1 \le |x,y|_E$ and $\lim_{i\to\infty} t_i = \lim_{j\to\infty} s_j = u$.

For each i and any small enough $\epsilon > 0$, $\gamma([t_{i-1}+\epsilon, t_i-\epsilon])$ is contained in the surface $X \setminus \mathcal{S}(X)$. Therefore $\gamma|_{[t_{i-1}+\epsilon, t_i-\epsilon]}$ is locally X–minimizing, and hence X–minimizing. Thus, $\gamma|_{[t_{i-1},t_i]}$ is X–minimizing.

By Lemma 7.8, we have

$$\angle^X(\dot\gamma^E_{\gamma(t_i),\gamma(t_{i-1})}, \dot\gamma^E_{\gamma(t_i),\gamma(t_{i+1})}) = \angle^E(\dot\gamma^E_{\gamma(t_i),\gamma(t_{i-1})}, \dot\gamma^E_{\gamma(t_i),\gamma(t_{i+1})}) = \pi.$$

Hence $\gamma|_{[t_{i-1},t_{i+1}]}$ is an X–geodesic, which implies that $\gamma|_{[0,u]}$ is an X–geodesic. Similarly, $\gamma|_{[u,|x,y|_E]}$ is an X–geodesic. In a way similar to the above, we have $\angle^X(\dot\gamma^E_{\gamma(u),\gamma(0)}(0), \dot\gamma^E_{\gamma(u),\gamma(|x,y|_E)}(0)) = \pi$. It follows that γ is an X–geodesic. □

Remark 7.10 Lemma 7.9 does not hold in case a subarc of γ is contained in $\mathcal{S}(X)$; see Example 4.4.

Lemma 7.11 *For a fixed $x \in E$, we have for every $y \in E$ with $y \ne x$,*

$$\frac{|x,y|_E}{|x,y|_X} < 1 + \tau_x(|x,y|_X).$$

Proof We may assume $x \in E \cap \mathcal{S}(X)$. Suppose the conclusion does not hold. Then we have a sequence $y_n \in E$ converging to x such that $|x,y_n|_E / |x,y_n|_X > 1 + c$ for some positive constant c. Passing to a subsequence, we may assume that all $y_n \in S_{ij}$ for some S_{ij}. This is a contradiction to Sublemma 4.31 since $|x,y_n|_E \le |x,y_n|_{S_{ij}}$. □

Proof of Theorem 7.4 For each edge $e \in E(C)$, let D_i for $1 \le i \le m(e)$ be open half-disks in X with $\partial D_i = e$ such that

(7-3) $$\bigcup_{i=1}^{m(e)} D_i \text{ is an open neighborhood of } e \text{ in } X.$$

Let τ_{D_i} be the turn of e from the side D_i. We want to apply Theorem 7.7 to the completion of the components of $E \setminus C$. Let A be such a completion containing some D_i. However here are some difficulties: The domain A might be too thin to define $\tau_{D_i}(e)$ because of the presence of singular vertices. In particular, we do not know if e has finite turn variation in A. We also have to care about $V_*(C)$. To overcome these difficulties, we do surgeries around points of $V_{\text{sing}}(C)$. At this moment, we can apply Theorem 7.7 to e locally. Each point of e has a convex neighborhood P in $\bigcup_{i=1}^{m(e)} D_i$ such that ∂P consists of broken geodesics joining the endpoints of $e \cap P$. It follows from Theorem 7.7 that we have

(7-4) $$\tau_{D_i}(e \cap P) + \tau_{D_j}(e \cap P) \le 0 \quad \text{for all } 1 \le i \ne j \le m(e),$$

and e has locally finite turn variation in D_i.

Let ϵ_0 be any positive number. For $x \in V_{\text{sing}}(C)$, we assume that the singularity of x occurs from the positive direction. Namely, there is $v \in \Sigma_{x,+}^{\text{sing}}(C)$. The other case $v \in \Sigma_{x,-}^{\text{sing}}(C)$ is similarly discussed. Let $C(v)$ denote the union of singular curves in C starting at x in the direction v.

Choose $\delta = \delta_x > 0$ and $\epsilon = \epsilon_x > 0$ with $\delta, \epsilon \le \epsilon_0$ and $\epsilon \ll \delta$ such that

(7-5) $$\text{the } \{B^{\Sigma_x(X)}(v, 2\delta)\} \text{ for } v \in \Sigma_x^{\text{sing}}(C) \text{ are mutually disjoint,}$$

(7-6) $$C(v, \delta, 2\epsilon) \text{ (see (2-7)) covers } C(v) \cap B_+(x, 2\epsilon),$$

(7-7) $$E \cap S(p, r(x) + \epsilon) \text{ does not meet } V(C),$$

where $B_+(x,\epsilon) := B(x,\epsilon) \setminus \operatorname{int} B(p, r(x))$. By Lemma 6.6, $C(v,\delta,\epsilon) \cap E \cap S(p, \delta(x)+\epsilon)$ is a tree, say $\hat{T}(x,v)$. Replacing each edge of $\hat{T}(x,v)$ by the X–geodesic between the endpoints, we obtain a geodesic tree $T(x,v)$. By Lemma 4.32, we have $T(x,v) \subset E$. Let $K(x,v)$ be a closed domain of E bounded by $T(x,v)$ and the X–geodesic segments between x and the endpoints of the tree $T(x,v)$. Note that such X–geodesics between x and the endpoints of $T(x,v)$ are contained in E. Taking smaller δ and ϵ if necessary, we may assume

(7-8) $$\tilde{\angle}^X yxy' < \epsilon_0 \quad \text{for all } y, y' \in T(x,v).$$

For each vertex $y \in V(T(x,v))$, take the X–geodesic $\gamma_{x,y}^X$ between x and y. For each edge $e \in E(T(x,v))$ with endpoints y and y', let $\triangle_e^X$ denote the X–geodesic triangle consisting of $\gamma_{x,y}^X \cup \gamma_{x,y'}^X \cup \gamma_{y,y'}^X$.

Let $\tilde{\blacktriangle}_e^X$ be the triangular region bounded by $\tilde{\triangle}_e^X$. Gluing $\{\tilde{\blacktriangle}_e^X \mid e \in E(T(x,v))\}$ properly, we obtain a polyhedral space $\tilde{K}(x,v)$ corresponding to $K(x,v)$. See Figure 12.

We provide a relation between $K(x,v)$ and $\tilde{K}(x,v)$.

Lemma 7.12 (1) *Let y and y' be arbitrary endpoints of $T(x,v)$. For arbitrary $z \in \gamma_{x,y}^X = \gamma_{x,y}^E$ and $z' \in \gamma_{x,y'}^X = \gamma_{x,y'}^E$, assuming $|x,z|_X \le |x,z'|_X$, we have*

$$\big||z,z'|_E - |\tilde{z},\tilde{z}'|\big| < \tau(\epsilon_0)|x,z|_X,$$

where $\tilde{z}, \tilde{z}' \in \partial\tilde{K}(x,v)$ are the points corresponding to z and z', respectively.

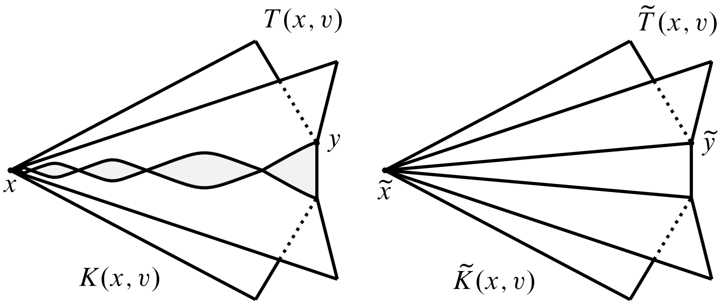

Figure 12

(2) *For arbitrary $y \in T(x,v)$ and $z \in \partial K(x,v)$, we have*

$$\big||y,z|_E - |\tilde y,\tilde z|\big| < \tau(\epsilon_0)|x,y|_X,$$

where $\tilde y$ and $\tilde z$ are the points of $\partial \tilde K(x,v)$ corresponding to y and z.

Proof (1) Let $\hat z \in \gamma^X_{x,z'}$ be the point such that $|x,\hat z|_X = |x,z|_X$. Note that $\gamma^X_{z,\hat z}$ is vertical, and therefore contained in E. By triangle inequality, we have $\big||z,z'|_E - |z',\hat z|_E\big| \le |\hat z, z|_E = |\hat z, z|_X$. Equation (7-8) implies $\tilde\angle^X zx\hat z < \epsilon_0$, and hence $|z,\hat z|_X < \tau(\epsilon_0)|x,z|_X$. In view of $\gamma^X_{x,y} = \gamma^E_{x,y}$ and $\gamma^X_{x,y'} = \gamma^E_{x,y'}$, we have

$$\big||z,z'|_E - (|x,z'|_X - |x,z|_X)\big| \le \tau(\epsilon_0)|x,z|_X.$$

Since $\angle \tilde z\tilde x\tilde z' < \tau(\epsilon_0)$, similarly we have $\big||\tilde z,\tilde z'| - (|x,z'|_X - |x,z|_X)\big| \le \tau(\epsilon_0)|x,z|_X$. Combining the last two inequalities, we obtain the required inequality.

(2) Choose $w \in \partial T(x,v)$ such that $z \in \gamma^X_{x,w}$. From (7-8), we have $|y,w|_E = |y,w|_X < \tau(\epsilon_0)|x,y|_X$. Therefore by triangle inequality, we obtain $\big||y,z|_E - |z,w|_E\big| \le |y,w|_E < \tau(\epsilon_0)|x,y|_X$. Since $\angle \tilde y\tilde x\tilde z < \epsilon_0$, by a similar consideration on the triangle $\triangle \tilde y\tilde x\tilde z$ in M^2_κ, we have $\big||\tilde y,\tilde z| - |\tilde z,\tilde w|\big| < \tau(\epsilon_0)|x,y|_X$, where $\tilde w$ is the point of $\partial \tilde T(x,v)$ corresponding to w. Since $|z,w|_E = |z,w|_X = |\tilde z,\tilde w|$, combining the last two inequalities, we obtain the required inequality. □

In E, we do surgeries by removing $K(x,v)$ from E, and gluing $E \setminus K(x,v)$ and $\tilde K(x,v)$ along their isometric boundaries to get a new space, say $\tilde E_{x,v}$.

Proposition 7.13 *For each vertex $\tilde y$ of $\tilde K(x,v)$, $\Sigma_{\tilde y}(\tilde E_{x,v})$ is* CAT(1).

We begin with:

Lemma 7.14 *For each vertex $y \in V(T(x,v))$, $\Sigma_{\tilde y}(\tilde E_{x,v})$ is* CAT(1), *where $\tilde y \in V(\tilde T(x,v))$ is the vertex corresponding to y.*

Proof The lemma is clear when y is an endpoint of $T(x,v)$. From now, we assume that y is an interior vertex of $T(x,v)$. Let us consider

$$\Sigma^-_y := \Sigma_y(K(x,v)) \subset \Sigma_y(X), \quad \Sigma^+_y := \Sigma_y(E \setminus \operatorname{int} K(x,v)) \subset \Sigma_y(X), \quad \tilde\Sigma^-_y := \Sigma_{\tilde y}(\tilde K(x,v)).$$

Note that $\Sigma_y(E) = \Sigma_y^- \cup \Sigma_y^+$ is a subgraph of $\Sigma_y(X)$ without endpoints, and hence it is $\mathrm{CAT}(1)$. Since $\Sigma_{\tilde y}(\tilde E_{x,v}) = \tilde\Sigma_y^- \cup \Sigma_y^+$, it suffices to show:

Claim 7.15 *There is an expanding map $\Sigma_y^- \to \tilde\Sigma_y^-$.*

Proof Let $\gamma := \gamma_{y,x}^X : [0, |y,x|_X] \to X$, and set $u := \dot\gamma(0)$. Choose any $\xi \in \Sigma_y(T(x,v)) \subset \Sigma_y^-$, and let $w \in \Sigma_y^-$ be the direction of C. Let $\tilde\xi$ and $\tilde u$ be the directions in $\tilde\Sigma_y^-$ corresponding to ξ and u, respectively.

Case (i) $u \in \Sigma_y^-$.

Since X is locally $\mathrm{CAT}(\kappa)$, we have $\angle^X(\xi,u) \le \angle(\tilde\xi,\tilde u)$. Therefore the correspondence $\xi \to \tilde\xi$, $u \to \tilde u$ gives rise to the desired expanding map $\Sigma_y^- \to \tilde\Sigma_{\tilde y}^-$.

Case (ii) $u \notin \Sigma_y^-$.

This is the case when γ leaves E after $y = \gamma(0)$ at least for a short time. From (7-7), y is contained in an open edge in $E(C)$. Therefore, for small enough $t > 0$, the X–geodesic starting from $\gamma(t)$ to $\gamma_\xi(t)$ must meet C. This implies

$$\angle^X(\xi,w) \le \angle^X(\xi,u) \le \angle(\tilde\xi,\tilde u). \tag{7-9}$$

Therefore the correspondence $\xi \to \tilde\xi$, $w \to \tilde u$ gives rise to the desired expanding map $\Sigma_y^- \to \tilde\Sigma_{\tilde y}^-$. Note that by (7-3), $\Sigma_y(X)$ is homeomorphic to the suspension with vertices $\Sigma_x(C)$, from which (7-9) also follows. □

This completes the proof of Lemma 7.14. □

For the proof of Proposition 7.13, it suffices to show the following.

Lemma 7.16 $\Sigma_{\tilde x}(\tilde E_{x,v})$ *is* $\mathrm{CAT}(1)$.

Proof Let σ_i for $1 \le i \le m$ be the X–geodesics joining x to the points of $\partial T(x,v)$, and set $\nu_i := \dot\sigma_i(0)$ for $1 \le i \le m$. Remember that $\Sigma_x(K(x,v))$ consists of m segments from the vertex v to ν_i of length δ. Since $\Sigma_x(X)$ is $\mathrm{CAT}(1)$, it suffices to show

$$\angle(\tilde\nu_i,\tilde\nu_j) \ge 2\delta \quad \text{for all } 1 \le i \ne j \le m, \tag{7-10}$$

where $\tilde\nu_i$ denotes the direction at $\tilde x$ corresponding to ν_i.

For arbitrary $y, y' \in V_{\mathrm{int}}(T(x,v))$ adjacent to $\partial T(x,v)$, assume $z_1,\dots,z_\ell \in \partial T(x,v)$ (resp. $z_{1'},\dots,z_{n'} \in \partial T(x,v)$ with $1' < \dots < n'$) are the set of $\partial T(x,v)$ adjacent to y (resp. to y') with $z_i \in \sigma_i$ (resp. $z_{i'} \in \sigma_{i'}$). Set $v_y := \dot\gamma_{x,y}^X(0)$ and $\tilde v_y := \dot\gamma_{\tilde x,\tilde y}(0) \in V(\Sigma_{\tilde x}(\tilde K(x,v)))$. Using the angle comparison for $\triangle_e^X$, we have for any $1 \le i \ne j \le \ell$,

$$\angle(\tilde\nu_i,\tilde\nu_j) = \angle(\tilde\nu_i,\tilde v_y) + \angle(\tilde v_y,\tilde\nu_j) \ge \angle^X(\nu_i,v_y) + \angle^X(v_y,\nu_j) = 2\delta.$$

Let $\bigcup_{\alpha=1}^k [y_{\alpha-1}, y_\alpha]$ be the shortest path from y to y' in $T(x,v)$ with $y = y_0$, $y' = y_k$, $y_\alpha \in V_{\mathrm{int}}(T(x,v))$ and $[y_{\alpha-1}, y_\alpha] \in E(T(x,v))$. Then for arbitrary $1 \le i \le \ell$ and $1' \le j' \le n'$, we have

$$\angle(\tilde{\nu}_i, \tilde{\nu}_{j'}) = \angle(\tilde{\nu}_i, \tilde{v}_y) + \sum_{\alpha=1}^k \angle(\tilde{v}_{y_{\alpha-1}}, \tilde{v}_{y_\alpha}) + \angle(\tilde{v}_{y'}, \tilde{\nu}_{j'})$$
$$\ge \angle^X(\nu_i, v_y) + \sum_{\alpha=1}^k \angle^X(v_{y_{\alpha-1}}, v_{y_\alpha}) + \angle^X(v_{y'}, \nu_{j'}) \ge 2\delta.$$

This completes the proof of Lemma 7.16. □

Note that in $\tilde{E}_{x,v}$, the subarc $[x,y]$ of C is replaced by the geodesic $[\tilde{x}, \tilde{y}] := \gamma_{\tilde{x},\tilde{y}}$. On the singular locus $\tilde{C}(x,v)$ of $\tilde{E}_{x,v}$, we consider the graph structure inherited from C (and hence from $\mathcal{S}(X)$), except that $\tilde{x}, \tilde{y} \in V(\tilde{C}(x,v))$ and $(\tilde{x}, \tilde{y}) \in E(\tilde{C}(x,v))$.

After all the surgeries at x possibly in the both positive and negative singular directions, we obtain a new space, denoted by $\tilde{E}_x$. Note that the point $\tilde{x} \in \tilde{E}_x$ replacing x is no longer singular in the graph structure of the new singular locus $\tilde{C}(x) \subset \tilde{E}_x$.

In what follows, we shall perform such surgeries finitely many times consistently in the directions of $\Sigma_x^{\mathrm{sing}}(X)$ at points $x \in V_{\mathrm{sing}}(C)$ so that the surgery parts cover $V_{\mathrm{sing}}(C)$.

First take $\epsilon = \epsilon_p > 0$ satisfying (7-5)–(7-8) for $x = p$, and set $\delta_0 = \epsilon_p$. Remember that $S(p, \delta_0)$ does not meet $V(C)$. We enlarge the radius of the ball $B(p, \delta_0)$, and choose $r_1 > \delta_0$ such that during the enlarging, $S(p, r_1)$ first meets $V_{\mathrm{sing}}(C)$, say at x, after $S(p, \delta_0)$. We call r_1 a *critical radius* in the surgeries. Now we do the above surgery at x, either in the negative direction $-\nabla d_p(x)$, where the surgeries should be carried out inside the annulus $A(p, \delta_0, r_1) = B(p, r_1) \setminus \operatorname{int} B(p, \delta_0)$, or in the positive direction $\nabla d_p(x)$ to resolve the singularity at x.

We again perform such surgeries at all points $x \in S(p, r_1) \cap V_{\mathrm{sing}}(C)$. Here, taking the smallest constant $\epsilon = \epsilon_x$ among all x and all singular directions there, we may assume that those surgeries are carried out based on a common metric sphere around p. More precisely, for some $0 < \delta_1 < r_1 - \delta_0$, we have $V(T(x,v)) \subset S(p, r_1 + \delta_1)$ (resp. $V(T(x,v)) \subset S(p, r_1 - \delta_1)$) for all $x \in S(p, r_1) \cap V_{\mathrm{sing}}(C)$ and $v \in \Sigma_{x,+}^{\mathrm{sing}}(C)$ (resp. $v \in \Sigma_{x,-}^{\mathrm{sing}}(C)$). We call δ_1 (resp. δ_0) the *surgery radius* at $S(p, r_1)$ (resp. at p).

Then we again enlarge the radius of $B(p, r_1 + \delta_1)$ until the next critical radius r_2. Repeating this procedure, we have a possibly infinite sequence of critical radii r_i,

$$0 < r_1 < r_2 < \cdots < r_i < \cdots,$$

and surgery radii δ_i at $S(p, r_i)$ with

$$r_i + \delta_i < r_{i+1} - \delta_{i+1}$$

such that the X–annulus $A^X(p, r_i + \delta_i, r_{i+1} - \delta_{i+1})$ does not meet $V_{\mathrm{sing}}(C)$. Note also that the number of surgeries at points of $S(p, \delta_i)$ is bounded by the uniform constant $N_v - 2$.

We show that one can cover $V_{\mathrm{sing}}(C)$ after performing surgeries as above finitely many times. Suppose $r_* = \lim_{i\to\infty} r_i < r$. From construction, $S(p, r_*)$ meets $V_{\mathrm{sing}}(C)$. We again do surgeries at points of $S(p, r_*) \cap V_{\mathrm{sing}}(C)$. For the surgeries in the negative direction at those points, we can make them consistent with the previous surgeries since our procedure is done based on metric spheres around p. This shows that after finitely many such surgeries, we can resolve all singular vertices in $V(C)$. Let $0 < r_1 < r_2 < \cdots < r_J < r$ be critical radii, and δ_i for $0 \le i \le J$ surgery radii, where we may assume that $A(p, r_J + \delta_J, r)$ does not meet $V_{\mathrm{sing}}(C)$ by taking slightly larger r if necessary.

Let $\mathcal{K} := \{K_{n,i} := K(x_n, v_{n,i}) \mid 1 \le n \le M, 1 \le i \le L_n\}$ be the set of all conelike domains in E constructed as above for $x_n \in V_{\mathrm{sing}}(C)$ and $v_{n,i} \in \Sigma_x^{\mathrm{sing}}(C)$ which arise in the course of the surgeries. Set $I_0 := [0, \delta_0]$, $I_j := [r_j - \delta_j, r_j + \delta_j]$ and $A_j := d_p^{-1}(I_j)$ for $1 \le j \le J$. From construction, we have the following for every $K_n \in \mathcal{K}$:

- $K_n \in \mathcal{K}$ is convex in E.
- K_n is contained in some A_j.
- K_n and $K_{n'}$ do not have intersection in their interiors for all $n \ne n'$.
- The number of K_n contained in A_j is at most $N_v - 2$ for each $1 \le j \le J$.

Let $\widetilde{E}$ be the result of those surgeries, and let $\widetilde{C}$ be the singular locus of $\widetilde{E}$, with graph structure $V(\widetilde{C})$, $E(\widetilde{C})$ defined as above. Note that $V(\widetilde{C})$ is finite and $V_{\mathrm{sing}}(\widetilde{C})$ is empty.

Lemma 7.17 *$\widetilde{E}$ is a* CAT(κ)*–space.*

Proof From the construction and Proposition 7.13, we have

- for every edge e of $E(\widetilde{C})$, the condition (A) holds and e has finite turn variation, and
- $\Sigma_{\tilde{y}}(\widetilde{E})$ is CAT(1) for every $\tilde{y} \in V(\widetilde{C})$.

Consider any triangulation of $\widetilde{E}$ extending $V(\widetilde{C})$ and $E(\widetilde{C})$ by adding geodesic edges if necessary. Now, we are ready to apply Theorem 7.6 to this triangulation to conclude that $\widetilde{E}$ is CAT(κ). □

Now we are going to show the Gromov–Hausdorff convergence $\widetilde{E} \to E$ as $\epsilon_0 \to 0$.

For each $K(x_n, v_{n,i}) \in \mathcal{K}$, we fix any element $y_{n,i} \in V(T(x_n, v_{n,i}))$, and let $\gamma_{n,i} : [0, |x_n, y_{n,i}|_E] \to E$ be an E–geodesic from x_n to $y_{n,i}$.

Define $\varphi \colon \widetilde{E} \to E$ as follows. Let φ be identical outside the surgery part. For every $\tilde{z} \in \operatorname{int} \widetilde{K}(x_n, v_{n,i})$, we let

$$\varphi(\tilde{z}) := \gamma_{n,i}(|\tilde{x}_n, \tilde{z}|).$$

Since $\operatorname{diam}(K(x_n, v_{n,i})) < \tau(\epsilon_0)$, the image of φ is $\tau(\epsilon_0)$–dense in E.

For arbitrary $\tilde{z}, \tilde{z}' \in \tilde{E}$, set $z = \varphi(\tilde{z})$, $z' = \varphi(\tilde{z}')$, and choose an E–shortest curve $\gamma\colon [0, |z,z'|_E] \to E$ between z and z'. Suppose first that $d_p(\gamma(t))$ takes a local minimum or local maximum. Then we see that γ is vertical, and hence an X–geodesic. Moreover, γ intersects C almost perpendicularly with at most $N_v - 2$ points (Sublemma 6.8). This implies that γ meets at most $N_v - 2$ elements of $\mathcal{K}$. Therefore from Lemma 7.12, we have

$$\big||z,z'|_E - |\tilde{z},\tilde{z}'|\big| < \tau(\epsilon_0)(r + N_v - 2).$$

Now we assume that $d_p(\gamma(t))$ is strictly monotone. Let $\mathcal{K}_\gamma$ be set of all $K(x_n, v_{n,i}) \in \mathcal{K}$ meeting γ. For simplicity, we renumber elements of $\mathcal{K}_\gamma$ as $\mathcal{K}_\gamma = \{K_i \mid 1 \le i \le I\}$. Let $\mathcal{K}_j$ be the set of all $K_n \in \mathcal{K}_\gamma$ contained in A_j. If γ meets $K_n \in \mathcal{K}_j$ with $\{z_n, z_n'\} = \gamma \cap \partial K_n$, then from Lemma 7.12 we have

$$|\varphi^{-1}(z_n), \varphi^{-1}(z_n')|_{\tilde{E}} - |z_n, z_n'|_E| < 2\tau(\epsilon_0)\delta_j.$$

It follows that

$$\big||z,z'|_E - |\tilde{z},\tilde{z}'|\big| < 2r(N_v - 1)\tau(\epsilon_0).$$

In this way, we conclude that $\tilde{E}$ converges to E as $\epsilon_0 \to 0$ with respect to the Gromov–Hausdorff distance, which yields that E is a CAT(κ)–space. This completes the proof of Theorem 7.4. □

Proof of Theorem 1.1(1) in Case II We consider Case II in the subsection of filling ball of Section 5. We only have to apply Theorem 7.4 for $k = 4$ to Case II. The rest of the argument is similar to that in Case I given in Section 5, and hence omitted. □

The proof of Corollary 1.3 is similar to that of Theorem 1.1(1) in Case II, and hence omitted.

Using Theorem 7.4, we also have the following.

Theorem 7.18 *In Theorem 1.1, every union* $\operatorname{Im} f_{i_1} \cup \cdots \cup \operatorname{Im} f_{i_k}$ *is a* CAT(κ)*–space.*

Proof The basic idea of the proof of Theorem 7.18 is the same as that of Theorem 1.1(1) for branched immersed disks. Set $\Sigma_{p,i_j} := \Sigma_p(\operatorname{Im}(f_{i_j}))$ for $1 \le j \le k$, and consider $\Sigma := \Sigma_{p,i_1} \cup \cdots \cup \Sigma_{p,i_k}$. For each $v \in V(\Sigma_p(X))$ contained in Σ, we construct a ruled surface S for which we may assume CAT(κ) by taking smaller r. Let $S(v)$ denote the union of all such ruled surfaces S. By Theorem 7.4, $S(v)$ is CAT(κ). The rest of the argument is the same as before, and hence omitted. □

Appendix Alexandrov's result on ruled surfaces

Following the ideas of Alexandrov in [1], we prove Theorem 3.17. As mentioned in Section 1, it also follows from [26] in the CAT(0)–setting.

We denote by D_κ the diameter of M_κ^2. Recall that a CAT(κ)–space is defined as a D_κ–geodesic space in which every triangle with perimeter $< 2D_\kappa$ is not thicker than its comparison triangle in M_κ^2 with the same side lengths, where a D_κ–geodesic space means a metric space in which any two points with distance $< D_\kappa$ can be joined by a minimal geodesic. Throughout this appendix, let X be a CAT(κ)–space.

A.1 Finite sequences of ruling geodesics

Let S be a ruled surface in X with parametrization $\sigma\colon R \to X$, where $R = [0,\ell]\times[0,1]$. Let $\pi\colon R \to R_*$ and $p_1\colon R \to [0,\ell]$ be as in Section 3.

We give an explicit formulation of the pullback metric e_σ. For $u = (s_0, t_0)$ and $u' = (s_0', t_0')$ with $s_0 < s_0'$ in R, let $\Delta\colon s_0 \le s_1 \le \cdots \le s_n = s_0'$ be a decomposition of $[s_0, s_0']$, and set $|\Delta| = \max\{|s_i - s_{i-1}| \mid 1 \le i \le n\}$. We consider

$$e_\sigma^\Delta(\pi(u), \pi(u')) := \inf\left\{ \sum_{i=1}^n |x_{i-1}, x_i| \;\middle|\; x_0 = \sigma(u), x_n = \sigma(u'), x_i \in \lambda_{s_i} \right\}.$$

Choose a sequence $\{x_i\}_{i=0,1,\dots,n}$ in X such that $x_0 = \sigma(u)$, $x_n = \sigma(u')$, $x_i \in \lambda_{s_i}$ for all $i \in \{1,\dots,n-1\}$, and

$$e_\sigma^\Delta(\pi(u), \pi(u')) = \sum_{i=1}^n |x_{i-1}, x_i|.$$

We call such a sequence $\{x_i\}_{i=0,1,\dots,n}$ a *Δ–minimizing chain along S from $\sigma(u)$ to $\sigma(u')$*. Notice that possibly we have $x_{i-1} = x_i$ for some $i \in \{1,\dots,n\}$. We set $\gamma^\Delta := \bigcup x_{i-1}x_i$, and call it a *$\Delta$–minimizing broken geodesic* in X from $\sigma(u)$ to $\sigma(u')$, which realizes $L(\gamma^\Delta) = e_\sigma^\Delta(\pi(u), \pi(u'))$.

Lemma A.1 *Under the above situation, the following hold:*

(1) *We have*

$$e_\sigma(\pi(u), \pi(u')) = \sup_\Delta e_\sigma^\Delta(\pi(u), \pi(u')),$$

where Δ runs over all decompositions of $[s_0, s_0']$.

(2) *For any sequence Δ_n of decompositions of $[s_0, s_0']$ satisfying $\lim_{n\to\infty} |\Delta_n| = 0$, we have*

$$e_\sigma(\pi(u), \pi(u')) = \lim_{n\to\infty} e_\sigma^{\Delta_n}(\pi(u), \pi(u')).$$

Proof By Proposition 3.6, there is a shortest curve $c_{0*}\colon [0,1] \to (R_*, e_\sigma)$ from $\pi(u)$ to $\pi(u')$ together with its lift c_0. Set $\gamma_0(t) := \sigma_* \circ c_{0*}(t)$. For any decomposition $\Delta = \{s_i\}_{i=1}^N$ of $[s_0, s_0']$, take $t_i \in [0,1]$ such that $\gamma_0(t_i) \in \lambda_{s_i}$. Then in view of Proposition 3.16, we have

$$e_\sigma^\Delta(\pi(u), \pi(u')) \le \sum_{i=1}^N |\gamma_0(t_{i-1}), \gamma_0(t_i)| \le L(\gamma_0) = L(c_{0*}) = e_\sigma(\pi(u), \pi(u')).$$

Thus we have $\sup_\Delta e_\sigma^\Delta(\pi(u), \pi(u')) \le e_\sigma(\pi(u), \pi(u'))$.

Let $\{\Delta_n\}$ be a sequence of decompositions of $[s_0, s_0']$ with $\lim_{n\to\infty} |\Delta_n| = 0$ such that

$$\lim_{n\to\infty} e_\sigma^{\Delta_n}(\pi(u), \pi(u')) = \liminf_{|\Delta|\to 0} e_\sigma^\Delta(\pi(u), \pi(u')).$$

Let $\gamma_n\colon [0,1] \to X$ be a Δ_n–minimizing broken geodesic in X. Passing to a subsequence, we may assume that γ_n converges to a curve $\gamma\colon [0,1] \to X$. From $|\Delta_n| \to 0$, it follows that $\gamma([0,1]) \subset S$.

Sublemma A.2 *The limit curve γ has a lift in R from u to u'.*

Proof We may assume $\mathrm{Sing}(\sigma)$ is empty. Let

$$\Delta_n : s_0 = s_{n,0} \le s_{n,1} \le \cdots \le s_{n,k_n} = s_0',$$

and $\gamma_n = \gamma^{\Delta_n} := \bigcup_{i=1}^{k_n} x_{n,i-1}x_{n,i}$ with $x_{n,i} \in \lambda_{s_{n,i}}$. Choose $a_{n,i} \in I_{s_{n,i}}$ with $\sigma(a_{n,i}) = x_{n,i}$ for $1 \le i \le k_n - 1$, and consider the Euclidean broken geodesic $c_n := \bigcup_{i=1}^{k_n} a_{n,i-1}a_{n,i}$. Note that c_n is monotone, and $\sigma \circ c_n$ also converges to γ as $n \to \infty$. We show that a subsequence of c_n converges to a curve c, which is a lift of γ. We do not know if $L(\sigma \circ c_n)$ is uniformly bounded or even if it is finite, which is the only difference from Proposition 3.6.

Since the basic strategy is the same as the proof of Proposition 3.6, we present only an outline. Let J_0 be a countable dense subset of $J = [s_0, s_0']$. For each $s \in J$, choose a point $c_n(t_n(s))$ of c_n with $c_n(t_n(s)) \in I_s$. Now we have a subsequence $\{m\}$ of $\{n\}$ such that $c_m(t_m(s))$ converges to a point $x(s) \in I_s$ for every $s \in J_0$. We consider the limit set, say $\mathrm{LS}(\{c_m\})$, of the sequence $\{\mathrm{Im}(c_m)\}_m$, and set

$$E_s := \mathrm{LS}(\{c_m\}) \cap I_s,$$

as in the proof of Proposition 3.6. Then we have the decomposition $J = J_1 \cup J_2$, where

$$J_1 = \{s \in J \mid E_s \text{ is a single point}\} \quad \text{and} \quad J_2 = J \setminus J_1.$$

In the same way, we have the conclusions (1)–(4) in the proof of Proposition 3.6. Here it should be remarked that the following holds as well:

$$\sum_{s \in J_2} \sigma(E_s) \le L(\gamma).$$

Thus as before, we obtain a monotone continuous parametrization on the union of points and segments $\{E_s \mid s \in J\}$, which provides a lift of γ from u to u'. □

By Sublemma A.2, we conclude that

$$e_\sigma(\pi(u), \pi(u')) \le L(\gamma) \le \lim_{n\to\infty} L(\gamma_n) = \liminf_{n\to\infty} e_\sigma^{\Delta_n}(\pi(u), \pi(u')).$$

This completes the proof of Lemma A.1. □

From the choice of a Δ–minimizing chain along S, we derive the following:

Lemma A.3 *In the setting discussed above, let $\{x_i\}_{i=0,1,\ldots,n}$ be a Δ–minimizing chain along S from $\sigma(u)$ to $\sigma(u')$. Then for each $i \in \{1, \ldots, n-1\}$ and for each $t \in \{0, 1\}$, we have*

$$\angle x_{i-1}x_i\lambda_{s_i}(t) + \angle \lambda_{s_i}(t)x_ix_{i+1} \ge \pi$$

whenever $|x_{i-1}, x_i|, |x_i, x_{i+1}| < D_\kappa$, and the angles $\angle x_{i-1}x_i\lambda_{s_i}(t)$ and $\angle\lambda_{s_i}(t)x_ix_{i+1}$ can be defined.

Proof First we show the conclusion in the case $t = 0$. Set

$$\theta_i^- := \angle x_{i-1} x_i \lambda_{s_i}(0) \quad \text{and} \quad \theta_i^+ := \angle \lambda_{s_i}(0) x_i x_{i+1}.$$

Take $t_i \in (0,1]$ with $x_i = \lambda_{s_i}(t_i)$, where we may assume $t_i \neq 0$. If we put

$$h(\epsilon) := |\lambda_{s_i}(t_i - \epsilon), x_{i-1}| + |\lambda_{s_i}(t_i - \epsilon), x_{i+1}|$$

for small $\epsilon > 0$, then by the first variation formula (see eg [10, Corollary II.3.6]) together with the Δ–minimizing property of $\{x_i\}_{i=0,1,\dots,n}$, we have

$$0 \le \lim_{\epsilon \to 0+} \frac{h(\epsilon) - h(0)}{\epsilon} = -(\cos\theta_i^- + \cos\theta_i^+).$$

This implies $\theta_i^- + \theta_i^+ \ge \pi$. Similarly, we see the inequality for $t = 1$. □

Let $u_* := \pi(u), v_* := \pi(v), w_* := \pi(w)$ be distinct points in R_*. Assume for a while that

$$p_1(u) \le p_1(v) \le p_1(w),$$

and choose a decomposition $\Delta = \{s_i\}_{i=0,1,\dots,n}$ of $[p_1(u), p_1(w)]$ such that for some $m \in \{1, \dots, n-1\}$ we have $p_1(v) = s_m$. Let $\Delta' := \{s_i\}_{i=0,1,\dots,m}$ be the decomposition of $[p_1(u), p_1(v)]$, and $\Delta'' := \{s_{m+i}\}_{i=0,1,\dots,n-m}$ the decomposition of $[p_1(v), p_1(w)]$. Take a Δ'–minimizing chain $\{y_i\}_{i=0,1,\dots,m}$ along S from $\sigma(u)$ to $\sigma(v)$, a Δ''–minimizing chain $\{y_{m+i}\}_{i=0,1,\dots,n-m}$ along S from $\sigma(v)$ to $\sigma(w)$, and a Δ–minimizing chain $\{z_i\}_{i=0,1,\dots,n}$ along S from $\sigma(u)$ to $\sigma(w)$. Assume in addition that we have

$$e_\sigma^{\Delta'}(u_*, v_*) + e_\sigma^{\Delta''}(v_*, w_*) + e_\sigma^{\Delta}(w_*, u_*) < 2D_\kappa.$$

Set $x := \sigma(u)$, $y := \sigma(v)$ and $z := \sigma(w)$. Let $B^\Delta(xy)$ be the broken geodesic $\bigcup_{i=1}^m y_{i-1} y_i$ in X joining x and y, $B^\Delta(yz)$ the broken geodesic $\bigcup_{i=1}^{n-m} y_{m+i-1} y_{m+i}$ in X joining y and z, and $B^\Delta(zx)$ the broken geodesic $\bigcup_{i=1}^n z_{i-1} z_i$ in X joining z and x. We denote by $P^\Delta(xyz)$ the polygon in X defined by

$$P^\Delta(xyz) := B^\Delta(xy) \cup B^\Delta(yz) \cup B^\Delta(zx),$$

and we call $P^\Delta(xyz)$ the *Δ–minimizing chain triple along S*. We denote by $\theta_x^\Delta(y,z)$ the angle at x in X between $B^\Delta(xy)$ and $B^\Delta(zx)$, by $\theta_y^\Delta(z,x)$ the angle at y in X between $B^\Delta(yz)$ and $B^\Delta(xy)$, and by $\theta_z^\Delta(x,y)$ the angle at z in X between $B^\Delta(zx)$ and $B^\Delta(yz)$. See Figure 13.

In the model surface M_κ^2, we define a comparison polygon $\widetilde{P}^\Delta(xyz)$ for $P^\Delta(xyz)$ as follows: Let $\triangle \tilde{x}\tilde{y}_1\tilde{z}_1$ and $\triangle \tilde{y}_{n-1}\tilde{z}_{n-1}\tilde{z}$ be comparison triangles in M_κ^2 for $\triangle x y_1 z_1$ and for $\triangle y_{n-1} z_{n-1} z$, respectively. For each $i \in \{1, \dots, n-1\}$, take comparison triangles $\triangle \tilde{y}_i \tilde{y}_{i+1} \tilde{z}_i$ and $\triangle \tilde{y}_{i+1} \tilde{z}_i \tilde{z}_{i+1}$ in M_κ^2 for $\triangle y_i y_{i+1} z_i$ and for $\triangle y_{i+1} z_i z_{i+1}$, respectively, and then glue all the comparison triangles in M_κ^2 along $\tilde{y}_i\tilde{z}_i$, and along $\tilde{y}_{i+1}\tilde{z}_i$, for all $i \in \{1, \dots, n-1\}$. Let $\widetilde{B}^\Delta(xy)$ be the broken geodesic $\bigcup_{i=1}^m \tilde{y}_{i-1}\tilde{y}_i$ in M_κ^2 joining $\tilde{x}$ and $\tilde{y}$, $\widetilde{B}^\Delta(yz)$ the broken geodesic $\bigcup_{i=1}^{n-m} \tilde{y}_{m+i-1}\tilde{y}_{m+i}$ in M_κ^2 joining $\tilde{y}$ and $\tilde{z}$, $\widetilde{B}^\Delta(zx)$ the broken geodesic $\bigcup_{i=1}^n \tilde{z}_{i-1}\tilde{z}_i$ in M_κ^2 joining $\tilde{z}$ and $\tilde{x}$. Then we put

$$\widetilde{P}^\Delta(xyz) := \widetilde{B}^\Delta(xy) \cup \widetilde{B}^\Delta(yz) \cup \widetilde{B}^\Delta(zx),$$

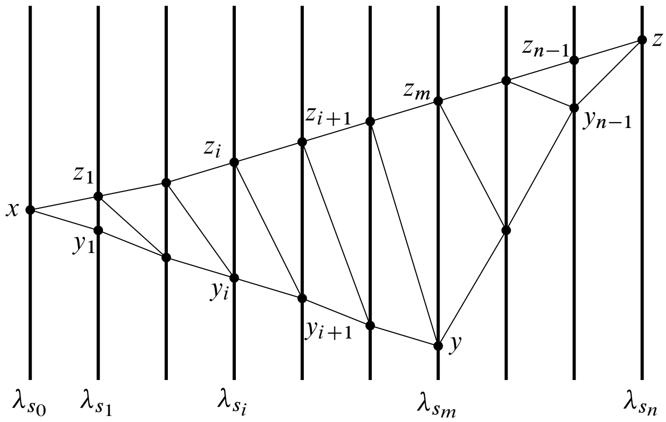

Figure 13

and we call $\widetilde{P}^\Delta(xyz)$ a *comparison Δ–minimizing chain triple in M^2_κ for $P^\Delta(xyz)$*. We denote by $\widetilde{\theta}^\Delta_x(y,z)$ the angle at $\tilde{x}$ in M^2_κ between $\widetilde{B}^\Delta(xy)$ and $\widetilde{B}^\Delta(zx)$, by $\widetilde{\theta}^\Delta_y(z,x)$ the angle at $\tilde{y}$ in M^2_κ between $\widetilde{B}^\Delta(yz)$ and $\widetilde{B}^\Delta(xy)$, and by $\widetilde{\theta}^\Delta_z(x,y)$ the angle at $\tilde{z}$ in M^2_κ between $\widetilde{B}^\Delta(zx)$ and $\widetilde{B}^\Delta(yz)$. Note that

$$\theta^\Delta_x(y,z) \le \widetilde{\theta}^\Delta_x(y,z), \quad \theta^\Delta_y(z,x) \le \widetilde{\theta}^\Delta_y(z,x), \quad \theta^\Delta_z(x,y) \le \widetilde{\theta}^\Delta_z(x,y).$$

From Lemma A.3 we derive the following concavity of $\widetilde{P}^\Delta(xyz)$ except at the vertices $\tilde{x},\tilde{y},\tilde{z}$: namely, for each $i \in \{1,\dots,n-1\}$ with $i \neq m$ the inner angle at $\tilde{y}_i$ in $\widetilde{P}^\Delta(xyz)$ is at least π; moreover, for each $i \in \{1,\dots,n-1\}$, the inner angle at $\tilde{z}_i$ in $\widetilde{P}^\Delta(xyz)$ is at least π.

By stretching the comparison Δ–minimizing chain triple $\widetilde{P}^\Delta(xyz)$ at the concave vertices, we obtain a triangle $\triangle\bar{x}\bar{y}\bar{z}$ in M^2_κ, whose side-lengths satisfy

$$|\bar{x},\bar{y}| = e^{\Delta'}_\sigma(u_*,v_*), \quad |\bar{y},\bar{z}| = e^{\Delta''}_\sigma(v_*,w_*), \quad |\bar{z},\bar{x}| = e^{\Delta}_\sigma(w_*,u_*).$$

We call $\triangle\bar{x}\bar{y}\bar{z}$ a *comparison Δ–minimizing stretched triangle in M^2_κ for $P^\Delta(xyz)$*, and we denote it by $\bar{P}^\Delta(xyz)$. We denote by $\bar{\theta}^\Delta_x(y,z)$ the angle $\angle\bar{y}\bar{x}\bar{z}$ at $\bar{x}$ in M^2_κ between $\overline{\bar{x}\bar{y}}$ and $\overline{\bar{z}\bar{x}}$, by $\bar{\theta}^\Delta_y(z,x)$ the angle $\angle\bar{z}\bar{y}\bar{x}$ at $\bar{y}$ in M^2_κ between $\overline{\bar{y}\bar{z}}$ and $\overline{\bar{x}\bar{y}}$, and by $\bar{\theta}^\Delta_z(x,y)$ the angle $\angle\bar{x}\bar{z}\bar{y}$ at $\bar{z}$ in M^2_κ between $\overline{\bar{z}\bar{x}}$ and $\overline{\bar{y}\bar{z}}$. Let $\bar{y}_i \in \overline{\bar{x}\bar{y}}$ and $\bar{z}_i \in \overline{\bar{x}\bar{z}}$ for $i \in \{1,\dots,n-1\}$ be the points corresponding to $\tilde{y}_i$ and to $\tilde{z}_i$, respectively. Since $\widetilde{P}^\Delta(xyz)$ is concave except the vertices, the Alexandrov stretching lemma (see eg [10, Lemma I.2.16]) leads to the following:

Lemma A.4 *Under the setting discussed above, we have*

$$\widetilde{\theta}^\Delta_x(y,z) \le \bar{\theta}^\Delta_x(y,z), \quad \widetilde{\theta}^\Delta_y(z,x) \le \bar{\theta}^\Delta_y(z,x), \quad \widetilde{\theta}^\Delta_z(x,y) \le \bar{\theta}^\Delta_z(x,y).$$

Moreover, for all $i \in \{1,\dots,n-1\}$, we have $|y_i,z_i| \le |\bar{y}_i,\bar{z}_i|$.

Let $y_j \in B^\Delta(xy) \setminus \{x,y\}$ be a broken point for $j \in \{1,\dots,m-1\}$, $y_k \in B^\Delta(yz) \setminus \{y,z\}$ a broken point for $k \in \{m+1,\dots,n-1\}$, and $z_l \in B^\Delta(zx) \setminus \{z,x\}$ a broken point for $l \in \{1,\dots,n-1\}$.

Assume that the broken points y_j, y_k, and z_l are distinct to each other. Choose four Δ–minimizing chain triples $P^\Delta(xy_jz_l)$, $P^\Delta(y_jyy_k)$, $P^\Delta(z_ly_kz)$, and $P^\Delta(y_jy_kz_l)$ along S, and take comparison Δ–minimizing stretched triangles $\bar{P}^\Delta(xy_jz_l)$, $\bar{P}^\Delta(y_jyy_k)$, $\bar{P}^\Delta(z_ly_kz)$, and $\bar{P}^\Delta(y_jy_kz_l)$ in M_κ^2 for $P^\Delta(xy_jz_l)$, $P^\Delta(y_jyy_k)$, $P^\Delta(z_ly_kz)$, and $P^\Delta(y_jy_kz_l)$, respectively.

From Lemma A.4 we derive the following monotonicity:

Lemma A.5 *Under the setting discussed above, we have*

$$\bar{\theta}_x^\Delta(y_j,z_l)\le\bar{\theta}_x^\Delta(y,z),\quad \bar{\theta}_y^\Delta(y_k,y_j)\le\bar{\theta}_y^\Delta(z,x),\quad \bar{\theta}_z^\Delta(z_l,y_k)\le\bar{\theta}_z^\Delta(x,y).$$

Proof Gluing the triangles $\bar{P}^\Delta(xy_jz_l)=\triangle\bar{x}\bar{y}_j\bar{z}_l$, $\bar{P}^\Delta(y_jyy_k)=\triangle\bar{y}_j\bar{y}\bar{y}_k$, $\bar{P}^\Delta(z_ly_kz)=\triangle\bar{z}_l\bar{y}_k\bar{z}$, and $\bar{P}^\Delta(y_jy_kz_l)=\triangle\bar{y}_j\bar{y}_k\bar{z}_l$ in M_κ^2 along the edges $\bar{y}_j\bar{y}_k$, $\bar{y}_k\bar{z}_l$, and $\bar{z}_l\bar{y}_j$, we obtain a hexagon $\bar{x}\bar{y}_j\bar{y}\bar{y}_k\bar{z}\bar{z}_l$ in M_κ^2 whose side-lengths satisfy $|\bar{x},\bar{y}_j|+|\bar{y}_j,\bar{y}|=e_\sigma^{\Delta'}(u_*,v_*)$, $|\bar{y},\bar{y}_k|+|\bar{y}_k,\bar{z}|=e_\sigma^{\Delta''}(v_*,w_*)$, and $|\bar{z},\bar{z}_l|+|\bar{z}_l,\bar{x}|=e_\sigma^{\Delta}(w_*,u_*)$. By Lemmas A.3 and A.4, we have

$$\pi\le\theta_{y_j}^\Delta(x,z_l)+\theta_{y_j}^\Delta(z_l,y_k)+\theta_{y_j}^\Delta(y_k,y)\le\bar{\theta}_{y_j}^\Delta(x,z_l)+\bar{\theta}_{y_j}^\Delta(z_l,y_k)+\bar{\theta}_{y_j}^\Delta(y_k,y).$$

Similarly, we have

$$\pi\le\bar{\theta}_{y_k}^\Delta(y,y_j)+\bar{\theta}_{y_k}^\Delta(y_j,z_l)+\bar{\theta}_{y_k}^\Delta(z_l,z)\quad\text{and}\quad\pi\le\bar{\theta}_{z_l}^\Delta(z,y_k)+\bar{\theta}_{z_l}^\Delta(y_k,y_j)+\bar{\theta}_{z_l}^\Delta(y_j,x).$$

By stretching the hexagon $\bar{x}\bar{y}_j\bar{y}\bar{y}_k\bar{z}\bar{z}_l$ at the concave vertices $\bar{y}_j$, $\bar{y}_k$ and $\bar{z}_l$, we obtain a comparison Δ–minimizing stretched triangle $\bar{P}^\Delta(xyz)$ in M_κ^2 for $P^\Delta(xyz)$. The Alexandrov stretching lemma (see eg [10, Lemma I.2.16]) leads to the desired inequalities. □

From Lemma A.4 we also derive the following:

Lemma A.6 *Let $u_*,u'_*\in R_*$ be distinct points. Assuming $p_1(u)\le p_1(u')$, we choose a decomposition $\Delta=\{s_i\}_{i=0,1,\dots,n}$ of $[p_1(u),p_1(u')]$. If $e_\sigma^\Delta(u_*,u'_*)<D_\kappa$, then a Δ–minimizing chain $\{x_i\}_{i=0,1,\dots,n}$ along S from $\sigma(u)$ to $\sigma(u')$ is uniquely determined.*

Proof Let $x:=\sigma(u)$ and $x':=\sigma(u')$, and suppose that two distinct Δ–minimizing chains $\{x_i\}_{i=0,1,\dots,n}$ and $\{y_i\}_{i=0,1,\dots,n}$ along S from x to x' satisfy $x_m\ne y_m$ for $m\in\{1,\dots,n-1\}$. Then for the Δ–minimizing chain triple $P^\Delta(xy_mx')$ along S we see that a comparison Δ–minimizing stretched triangle $\bar{P}^\Delta(xy_mx')$ degenerates in M_κ^2. Hence we have $\bar{x}_m=\bar{y}_m$. On the other hand, Lemma A.4 implies $|x_m,y_m|\le|\bar{x}_m,\bar{y}_m|$. This is a contradiction. □

A.2 Curvature bounds on ruled surfaces

Let $\hat{\triangle}u_*v_*w_*$ be a geodesic triangle in (R_*,e_σ) with distinct vertices and with perimeter $<2D_\kappa$ determined by $\hat{\triangle}u_*v_*w_*=\widehat{u_*v_*}\cup\widehat{v_*w_*}\cup\widehat{w_*u_*}$, where $\widehat{u_*v_*}$, $\widehat{v_*w_*}$, and $\widehat{w_*u_*}$ are the edges of $\hat{\triangle}u_*v_*w_*$.

We denote by $\triangle\tilde{u}_*\tilde{v}_*\tilde{w}_*$ a comparison triangle in M_κ^2 for $\hat{\triangle}u_*v_*w_*$ with the same side-lengths, and by $\tilde{\theta}_{u_*}(v_*,w_*)$ the angle $\angle\tilde{v}_*\tilde{u}_*\tilde{w}_*$ at $\tilde{u}_*$ between $\tilde{u}_*\tilde{v}_*$ and $\tilde{u}_*\tilde{w}_*$.

To complete the proof of Theorem 3.17, it suffices to show the following; see eg [10, Proposition II.1.7].

Lemma A.7 *Every geodesic triangle $\widehat{\triangle} u_* v_* w_*$ in (R_*, e_σ) as above satisfies the convexity of angle κ–comparison: namely, for all $w'_* \in \widehat{u_* v_*} \setminus \{u_*, v_*\}$, $u'_* \in \widehat{v_* w_*} \setminus \{v_*, w_*\}$ and $v'_* \in \widehat{w_* u_*} \setminus \{w_*, u_*\}$, we have the following monotonicity:*

$$\widetilde{\theta}_{u_*}(v'_*, w'_*) \le \widetilde{\theta}_{u_*}(v_*, w_*), \quad \widetilde{\theta}_{v_*}(w'_*, u'_*) \le \widetilde{\theta}_{v_*}(w_*, u_*), \quad \widetilde{\theta}_{w_*}(u'_*, v'_*) \le \widetilde{\theta}_{w_*}(u_*, v_*).$$

Before proving Lemma A.7, we show the following sublemma. By Proposition 3.16, for every minimal geodesic c_* in (R_*, e_σ) there exists a monotone curve c in R with $\pi \circ c = c_*$ up to monotone parametrization.

Sublemma A.8 *In the same setting as in Lemma A.7, let u_* and u'_* be distinct points in (R_*, e_σ) with $e_\sigma(u_*, u'_*) < D_\kappa$, and let c_* be a minimal geodesic in (R_*, e_σ) from u_* to u'_*. Assume $p_1(u) \le p_1(u')$, and choose a sequence $\{\Delta_n\}_{n \in \mathbb{N}}$ of decompositions $\Delta_n = \{s_i\}_{i=0,1,\dots,n}$ of $[p_1(u), p_1(u')]$ satisfying $\lim_{n\to\infty} |\Delta_n| = 0$. For $n \in \mathbb{N}$, let $\{x_i\}_{i=0,1,\dots,n}$ be the Δ_n–minimizing chain along S from $x := \sigma(u)$ to $x' := \sigma(u')$, and take a sequence $\{y_i\}_{i=0,1,\dots,n}$ in the image of $\gamma := \sigma_* \circ c_*$ in such a way that $y_0 = x$, $y_n = x'$, and $y_i \in \lambda_{s_i}$ for all $i \in \{1, \dots, n-1\}$. Then the following hold:*

(1) *We have*

$$e_\sigma(u_*, u'_*) = \lim_{n\to\infty} \sum_{i=1}^{n} |y_{i-1}, y_i|.$$

(2) *For every $s \in [p_1(u), p_1(u')]$, and for every sequence $\{s_{i_n}\}_{n\in\mathbb{N}}$ converging to s with $s_{i_n} \in \Delta_n$, we have*

$$\lim_{n\to\infty} |x_{i_n}, y_{i_n}| = 0.$$

Proof (1) From Lemma 3.19, we derive that $e_\sigma(u_*, u'_*) = \lim_{n\to\infty} \sum_{i=1}^{n} |x_{i-1}, x_i|$; moreover, we get $e_\sigma(u_*, u'_*) = \lim_{n\to\infty} \sum_{i=1}^{n} |y_{i-1}, y_i|$. Indeed, we have

$$\begin{aligned} e_\sigma(u_*, u'_*) = \lim_{n\to\infty} \sum_{i=1}^{n} |x_{i-1}, x_i| &\le \liminf_{n\to\infty} \sum_{i=1}^{n} |y_{i-1}, y_i| \\ &\le \limsup_{n\to\infty} \sum_{i=1}^{n} |y_{i-1}, y_i| \le \limsup_{n\to\infty} \sum_{i=1}^{n} e_\sigma(y_{i-1}, y_i) = e_\sigma(u_*, u'_*). \end{aligned}$$

(2) For $n \in \mathbb{N}$, let $P_n = \left(\bigcup_{i=1}^{n} x_{i-1}x_i\right) \cup \left(\bigcup_{i=1}^{n} y_{i-1}y_i\right)$ be the polygon in X. In the model surface M_κ^2, we construct a comparison $(n+1)$–gon $\bar{P}_n$ for P_n as follows: Let $\triangle \tilde{x}\tilde{x}_1\tilde{y}_1$ and $\triangle \tilde{x}_{n-1}\tilde{y}_{n-1}\tilde{x}'$ be comparison triangles in M_κ^2 for $\triangle x x_1 y_1$ and $\triangle x_{n-1} y_{n-1} x'$, respectively. For each $i \in \{1, \dots, n-1\}$, take comparison triangles $\triangle \tilde{x}_i \tilde{x}_{i+1} \tilde{y}_i$ and $\triangle \tilde{x}_{i+1} \tilde{y}_i \tilde{y}_{i+1}$ in M_κ^2 for $\triangle x_i x_{i+1} y_i$ and $\triangle x_{i+1} y_i y_{i+1}$, respectively, and then glue all the comparison triangles in M_κ^2 along $\tilde{x}_i \tilde{y}_i$, and along $\tilde{x}_{i+1} \tilde{y}_i$, for all $i \in \{1, \dots, n-1\}$. Then we put $\widetilde{P}_n := \left(\bigcup_{i=1}^{n} \tilde{x}_{i-1}\tilde{x}_i\right) \cup \left(\bigcup_{i=1}^{n} \tilde{y}_{i-1}\tilde{y}_i\right)$. From Lemma A.3 it follows that

for each $i \in \{1, \dots, n-1\}$, the inner angle at $\tilde{x}_i$ in $\tilde{P}_n$ is at least π. By stretching the polygon $\tilde{P}_n$ at the concave vertices, we obtain an $(n+1)$–gon $\bar{P}_n = \bar{x}\bar{x}' \cup \bigl(\bigcup_{i=1}^n \bar{y}_{i-1}\bar{y}_i\bigr)$ in M_κ^2 whose side-lengths satisfy $|\bar{x}, \bar{x}'| = e_\sigma^{\Delta_n}(u_*, u'_*)$ and $|\bar{y}_{i-1}, \bar{y}_i| = |y_{i-1}, y_i|$ for all $i \in \{1, \dots, n\}$. Let $\bar{x}_i \in \bar{x}\bar{x}'$ for $i \in \{1, \dots, n-1\}$ be the points corresponding to $\tilde{x}_i$. The Alexandrov stretching lemma (see eg [10, Lemma I.2.16]) leads to $|x_i, y_i| \le |\bar{x}_i, \bar{y}_i|$ for all $i \in \{1, \dots, n-1\}$.

Suppose that the second half of the sublemma is false. Then we find $s \in (p_1(u), p_1(u'))$, and a sequence $\{s_{i_n}\}_{n\in\mathbb{N}}$ converging to s such that for all $n \in \mathbb{N}$ we have $s_{i_n} \in \Delta_n$, and we have $|x_{i_n}, y_{i_n}| \ge C$ for some $C > 0$. Then for the points $\bar{x}_{i_n}, \bar{y}_{i_n}$ on the comparison $(n+1)$–gon $\bar{P}_n$ for P_n, we have

$$C \le \liminf_{n\to\infty} |x_{i_n}, y_{i_n}| \le \liminf_{n\to\infty} |\bar{x}_{i_n}, \bar{y}_{i_n}|.$$

On the other hand, since

$$e_\sigma(u_*, u'_*) = \lim_{n\to\infty} \sum_{i=1}^n |x_{i-1}, x_i| \quad \text{and} \quad e_\sigma(u_*, u'_*) = \lim_{n\to\infty} \sum_{i=1}^n |y_{i-1}, y_i|,$$

the comparison $(n+1)$–gon $\bar{P}_n$ degenerates in M_κ^2 as $n \to \infty$. This yields a contradiction. □

Proof of Lemma A.7 Without loss of generality, we may assume that

$$p_1(u) \le p_1(v) \le p_1(w).$$

For each $n \in \mathbb{N}$, choose a decomposition $\Delta_n = \{s_i\}_{i=0,1,\dots,n}$ of $[p_1(u), p_1(w)]$ with $\lim_{n\to\infty} |\Delta_n| = 0$ such that $p_1(v) = s_m$ for some $m \in \{1, \dots, n-1\}$. Let $\Delta'_n := \{s_i\}_{i=0,1,\dots,m}$ be the decomposition of $[p_1(u), p_1(v)]$, and let $\Delta''_n := \{s_{m+i}\}_{i=0,1,\dots,n-m}$ be the decomposition of $[p_1(v), p_1(w)]$. Set $x := \sigma(u)$, $y := \sigma(v)$, $z := \sigma(w)$ and take the (unique) Δ'_n–minimizing chain $\{y_i\}_{i=0,1,\dots,m}$ along S from x to y, and the Δ''_n–minimizing chain $\{y_{m+i}\}_{i=0,1,\dots,n-m}$ along S from y to z, and the Δ_n–minimizing chain $\{z_i\}_{i=0,1,\dots,n}$ along S from x to z.

Let $P^{\Delta_n}(xyz)$ be the Δ_n–minimizing chain triple along S defined by

$$P^{\Delta_n}(xyz) := B^{\Delta_n}(xy) \cup B^{\Delta_n}(yz) \cup B^{\Delta_n}(zx),$$

where $B^{\Delta_n}(xy)$ is the broken geodesic $\bigcup_{i=1}^m y_{i-1}y_i$ in X joining x and y, $B^{\Delta_n}(yz)$ is the broken geodesic $\bigcup_{i=1}^{n-m} y_{m+i-1}y_{m+i}$ in X joining y and z, and $B^{\Delta_n}(zx)$ is the broken geodesic $\bigcup_{i=1}^n z_{i-1}z_i$ in X joining z and x. Set $x' := \sigma(u')$, $y' := \sigma(v')$ and $z' := \sigma(w')$. By Sublemma A.8, we can take sequences $\{y_{j_n}\}_{n\in\mathbb{N}}$, $\{y_{k_n}\}_{n\in\mathbb{N}}$, $\{z_{l_n}\}_{n\in\mathbb{N}}$ of broken points on $P^{\Delta_n}(xyz) \setminus \{x, y, z\}$ satisfying

$$\lim_{n\to\infty} |y_{j_n}, z'| = 0, \quad \lim_{n\to\infty} |y_{k_n}, x'| = 0, \quad \lim_{n\to\infty} |z_{l_n}, y'| = 0,$$

where $j_n \in \{1, \dots, m-1\}$, $k_n \in \{m+1, \dots, n-1\}$, $l_n \in \{1, \dots, n-1\}$.

Let $\bar{P}^{\Delta_n}(xyz) = \triangle\bar{x}\bar{y}\bar{z}$ be a comparison Δ_n–minimizing stretched triangle in M_κ^2 for $P^{\Delta_n}(xyz)$ whose side-lengths satisfy

$$|\bar{x}, \bar{y}| = e_\sigma^{\Delta'_n}(u_*, v_*), \quad |\bar{y}, \bar{z}| = e_\sigma^{\Delta''_n}(v_*, w_*), \quad |\bar{z}, \bar{x}| = e_\sigma^{\Delta_n}(w_*, u_*).$$

Set

$$\overline{\theta}_x^{\Delta_n}(y,z) := \angle \overline{y}\overline{x}\overline{z}, \quad \overline{\theta}_y^{\Delta_n}(z,x) := \angle \overline{z}\overline{y}\overline{x}, \quad \overline{\theta}_z^{\Delta_n}(x,y) := \angle \overline{x}\overline{z}\overline{y}.$$

Choose the three Δ_n–minimizing chain triples $P^{\Delta_n}(xy_jz_l)$, $P^{\Delta_n}(y_jyy_k)$, and $P^{\Delta_n}(z_ly_kz)$ along S, and take comparison Δ_n–minimizing chain triangles $\overline{P}^{\Delta_n}(xy_jz_l)$, $\overline{P}^{\Delta_n}(y_jyy_k)$, and $\overline{P}^{\Delta_n}(z_ly_kz)$ in M_κ^2 for $P^{\Delta_n}(xy_jz_l)$, $P^{\Delta_n}(y_jyy_k)$, and $P^{\Delta_n}(z_ly_kz)$, respectively. As shown in Lemma A.5, we have the monotonicity

$$\overline{\theta}_x^{\Delta_n}(y_j,z_l) \le \overline{\theta}_x^{\Delta_n}(y,z), \quad \overline{\theta}_y^{\Delta_n}(y_k,y_j) \le \overline{\theta}_y^{\Delta_n}(z,x), \quad \overline{\theta}_z^{\Delta_n}(z_l,y_k) \le \overline{\theta}_z^{\Delta_n}(x,y).$$

From the choices of the sequences $\{y_{j_n}\}_{n\in\mathbb{N}}$, $\{y_{k_n}\}_{n\in\mathbb{N}}$ and $\{z_{l_n}\}_{n\in\mathbb{N}}$, it follows that $\overline{P}^{\Delta_n}(xy_{j_n}z_{l_n})$, $\overline{P}^{\Delta_n}(y_{j_n}yy_{k_n})$ and $\overline{P}^{\Delta_n}(z_{l_n}y_{k_n}z)$ converge to comparison triangles in M_κ^2 for triangles $\widehat{\triangle}u_*w'_*v'_*$, $\widehat{\triangle}w'_*v_*u'_*$, and $\widehat{\triangle}v'_*u'_*w_*$ in (R_*, e_σ), respectively. Notice that $\overline{P}^{\Delta_n}(xyz)$ converges to a comparison triangle in M_κ^2 for the triangle $\widehat{\triangle}u_*v_*w_*$. Therefore we obtain

$$\widetilde{\theta}_{u_*}(w'_*, v'_*) = \lim_{n\to\infty} \overline{\theta}_x^{\Delta_n}(y_{j_n}, z_{l_n}) \le \lim_{n\to\infty} \overline{\theta}_x^{\Delta_n}(y,z) = \widetilde{\theta}_{u_*}(v_*, w_*).$$

Similarly, we see $\widetilde{\theta}_{v_*}(u'_*, w'_*) \le \widetilde{\theta}_{v_*}(w_*, u_*)$ and $\widetilde{\theta}_{w_*}(v'_*, u'_*) \le \widetilde{\theta}_{w_*}(u_*, v_*)$. Thus $\widehat{\triangle}u_*v_*w_*$ satisfies the convexity of angle κ–comparison. □

From Lemma A.7 we conclude that (R_*, e_σ) is a CAT(κ)–space, proving Theorem 3.17. □

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