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We give a complete classification of all κ–noncollapsed, complete ancient solutions to the Kähler Ricci flow with nonnegative bisectional curvature.

53E20, 53E30

1 Introduction

In this paper, we study the κ–solutions to the Ricci flow. Recall that a κ–solution $(M^n, g(t))$ is an ancient solution to the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t))$$

defined for $t \in (-\infty, 0]$ such that for each t, $(M^n, g(t))$ is a κ–noncollapsed, complete Riemannian manifold with weakly PIC_2. In addition, the curvature operator Rm is uniformly bounded on the spacetime $M \times (-\infty, 0]$. For a precise definition of κ–solutions, see Definition 2.1.

The concept of κ–solutions plays a vital role in the study of the singularity in the Ricci flow. In Perelman's celebrated work [42; 44; 43] on the Poincaré conjecture and the geometrization conjecture, it is proved that the high-curvature regions of the Ricci flow on closed 3–manifolds are modeled on κ–solutions. By investigating the geometric and topological properties of the 3–dimensional κ–solutions, Perelman proved a canonical neighborhood theorem for the high curvature part and was able to continue the Ricci flow by performing the surgery.

In general, any ancient solution from a finite-time singularity of a compact Ricci flow is κ–noncollapsed by the monotonicity formula of Perelman's entropy [42]. Therefore, to investigate the ancient solution, it is natural to add the κ–noncollapsing condition, which excludes the complicated local collapsing phenomenon at spatial or time infinity.

In dimension 3, it follows from the famous Hamilton–Ivey pinching that any ancient solution must have nonnegative sectional curvature. For the higher-dimensional cases, Chen [22] proved that the scalar curvature of any ancient solution is nonnegative. However, nonnegative scalar curvature alone is usually too weak to do any further analysis. Therefore, in many articles, various positivity conditions for the curvature are imposed.

The curvature assumption we consider here is the weakly PIC_2 condition, which was introduced by Brendle and Schoen in [13]. In the 3–dimensional case, the concept of weakly PIC_2 is equivalent to the nonnegativity of the curvature operator. However, the former is weaker than the latter in the higher-dimensional case and hence the κ–solution defined here has a weaker assumption than that defined by Perelman [42]. Many important properties of κ–solutions still hold under the weakly PIC_2 assumption; see for instance Brendle [5; 7] and Cabezas-Rivas and Wilking [15].

In dimension 2, all κ–solutions are $\mathbb{R}^2$, S^2 and $\mathbb{RP}^2$, proved by Perelman [42]. In recent breakthrough works, Brendle [9] in the noncompact case and Brendle, Daskalopoulos and Sesum [10] in the compact case gave the complete classification of all 3–dimensional κ–solutions. More precisely, they show that shrinking cylinders, the Bryant soliton, shrinking spheres, Perelman's solution, and their quotients are all possibilities. The classification of higher-dimensional κ–solutions remains open. For recent developments in higher dimensions, see for example Brendle, Huisken and Sinestrari [11], Li and Zhang [34], Brendle and Naff [12] and Cho and Li [24].

The analogous question is to classify all κ–solutions to the Kähler Ricci flow, which by definition are κ–noncollapsed, complete ancient solutions to the Kähler Ricci flow with bounded and nonnegative bisectional curvature (Definition 3.1). Notice that the weakly PIC_2 condition is replaced by nonnegative bisectional curvature, which is weaker and more natural in the Kähler setting. In the compact case, all κ–solutions to the Kähler Ricci flow are classified by Deng and Zhu in [27].

The main result of the paper is the following classification theorem for the general case.

Theorem 1.1 Let $(M^m, g(t))_{t \in (-\infty, 0]}$ be a κ–noncollapsed, complete ancient solution to the Kähler Ricci flow with nonnegative bisectional curvature. Then it is isometrically biholomorphic to a finite quotient of $N^k \times \mathbb{C}^{m-k}$, where N is a simply connected, compact, Hermitian symmetric space.

Theorem 1.1 extends our earlier result (Cho and Li [24, Theorem 1.3]) regarding the classification of κ–solutions on Kähler surfaces. In [24], a key observation is that any complete, complex 2–dimensional ancient solution to the Kähler Ricci flow with nonnegative bisectional curvature automatically has nonnegative curvature operator [24, Lemma 4.6]. Here, we obtain a similar curvature improvement (see Theorem 3.3), which states that any complete ancient solution to the Kähler Ricci flow with nonnegative bisectional curvature has weakly PIC_2. Therefore, any κ–solution to the Kähler Ricci flow is also a κ–solution to the (real) Ricci flow.

Another important observation is that any κ–solution to the Kähler Ricci flow must be of Type I (see Lemma 3.2), which was initially proved by Deng and Zhu [27, Lemma 2.5] in the compact case. Essentially, this follows from the classical point-picking argument and the nonexistence of a κ–noncollapsed, nonflat, steady Kähler Ricci soliton with nonnegative bisectional curvature; see Deng and Zhu [28]. Here, the Kähler condition is crucial, and one cannot expect the same conclusion to hold for the real case (eg the Bryant soliton).

Therefore, to prove Theorem 1.1, it is essential to understand all Type I κ–solutions to the Ricci flow. One motivation is the similar classification result of Ricci shrinkers; see Munteanu and Wang [38] and Li and Ni [33]. More precisely, it was proved in [33, Theorem 3.1(ii)] that any Ricci shrinker with weakly PIC_2 is locally symmetric. On the one hand, Li and Wang [35] proved that any Ricci shrinker is κ–noncollapsed, a result previously established by Naber [39] for Ricci shrinkers with bounded curvature. On the other hand, it is well-known that any Ricci shrinker with bounded curvature can be regarded as a self-similar, Type I, ancient solution to the Ricci flow. Therefore, it is natural to expect that any Type I κ–solution is also locally symmetric. We confirm this by the following theorem; see also Zhang [48] and Hallgren [31] for the 3–dimensional case.

Theorem 1.2 *Let* $(M^n, g(t))_{t \in (-\infty, 0]}$ *be a* κ–*noncollapsed, Type I, complete ancient solution to the Ricci flow with weakly* PIC_2. *Then it is isometric to a finite quotient of* $N^k \times \mathbb{R}^{n-k}$, *where* N *is a simply connected, compact symmetric space.*

Theorem 1.1 follows from Theorem 1.2 immediately, provided that the curvature is uniformly bounded (Theorem 3.4). To complete the proof of Theorem 1.1, we next show that any solution in Theorem 1.1 must have bounded curvature. Here, we follow the strategy in [24] to prove first a canonical neighborhood theorem, which states that any high curvature region in the flow is modeled on a κ–solution to the Kähler Ricci flow (Theorem 3.18). The key point is to show that any Kähler manifold, whose high curvature regions are modeled on κ–solutions, must have uniformly bounded curvature; see Proposition 3.15. Notice that in the 3–dimensional case, a similar conclusion is proved by the fact that any complete manifold with nonnegative sectional curvature cannot have smaller and smaller ϵ–necks; see Chen and Zhu [23, Proposition 2.2]. In our case, we show that the high curvature part locally has a singular fibration structure on an orbifold, where each fiber is a simply connected, compact, Hermitian symmetric space N or its quotient. By analyzing the Kähler form, we show that the volumes of any two regular fibers are uniformly comparable and hence a global bound for the curvature can be obtained; see Proposition 3.15. With the canonical neighborhood theorem, the rest of the proof follows from similar arguments in [24] by showing the curvature is bounded first for each time slice and then for any compact time interval. Therefore, the global curvature bound follows from the Harnack inequality.

Now, we briefly discuss the proof of Theorem 1.2. Notice that the conclusion on compact manifolds has been proved by Ni [41], under the strictly PIC_2 condition. By using similar arguments, one can easily prove Theorem 1.2 on compact manifolds; see Theorem 2.10. For the noncompact case, we analyze the asymptotic Ricci shrinker, which is helpful to understand the geometric behavior as $t \to -\infty$. For this purpose, we consider the moduli space $\mathcal{M}$ (see Definition 2.5) such that any asymptotic Ricci shrinker of a Type I κ–solution belongs to $\mathcal{M}$.

To prove Theorem 1.2, we may assume the κ–solution $(M^n, g(t))$ is simply connected and irreducible, by Lemma 2.11 and an inductive argument. If the solution is nonflat, then we can derive a contradiction.

Indeed, we fix a point p and a time sequence $t_i = -4^i$ such that the geometry on $B_i = B_{g(t_i)}(p, \epsilon^{-1}\sqrt{|t_i|})$ is modeled on an element in $\mathcal{M}$, if i is sufficiently large. By a careful analysis of $\mathcal{M}$ and the structure of $(M^n, g(t))$, we prove that all B_i are close to a fixed $N^k \times \mathbb{R}^{n-k}$. The difficulty here is to prove the uniqueness of the model space and exclude any possible quotient. In particular, there exists a local fibration structure on B_i. It is clear from the distance comparison (see Lemma 2.9(i)) that $\{B_i\}$ forms an exhaustion of M. Therefore, a standard argument yields a global fibration on M with fiber N. On the other hand, M is known to be diffeomorphic to $\mathbb{R}^n$; see [15, Theorem 1.2]. Therefore, we obtain a contradiction, since $\mathbb{R}^n$ does not admit a fibration with nontrivial compact fibers; see Borel and Serre [3].

From Theorem 1.2, we know that any Type I κ–solution is essentially a Ricci shrinker. Notice that for κ–solutions, one can rescale each factor in the decomposition of N, but for Ricci shrinkers, the scaling is fixed so that each factor has the same Einstein constant. For Type II κ–solutions, the classification is much more difficult. It is still unclear how a Type II κ–solution is related to a steady soliton.

This paper is organized as follows. In Section 2, we review some basic properties of Type I κ–solutions and Ricci shrinkers. Then we discuss the structure of the singular fibration and prove Theorem 1.2. In Section 3, we obtain the curvature improvement for the Kähler Ricci flow and prove Theorem 1.1. In Section 4, we propose a conjecture for general Type I ancient solutions.

2 Classification of Type I κ–solutions

Definition 2.1 Let $(M^n, g(t))_{t \in (-\infty, 0]}$ be a complete ancient solution to the Ricci flow.

(i) $(M^n, g(t))_{t \in (-\infty, 0]}$ is of Type I if there exists a constant $C_0 > 0$ such that

$$(2\text{-}1) \qquad |\mathrm{Rm}|(x, t) \leq \frac{C_0}{1 + |t|} \quad \text{for any } (x, t) \in M \times (-\infty, 0].$$

(ii) Given $\kappa > 0$, we say that $(M^n, g(t))_{t \in (-\infty, 0]}$ is κ–noncollapsed if for any ball $B_{g(t)}(x, r)$ satisfying $R(y, t) \leq r^{-2}$ for all $y \in B_{g(t)}(x, r)$, we have

$$|B_{g(t)}(x, r)|_{g(t)} \geq \kappa r^n.$$

(iii) $(M^n, g(t))_{t \in (-\infty, 0]}$ has weakly PIC_2 if for any $(x, t) \in M \times (-\infty, 0]$ and any $\zeta, \eta \in T_x M \otimes \mathbb{C}$, we have at (x, t),

$$\mathrm{Rm}(\zeta, \eta, \bar{\zeta}, \bar{\eta}) \geq 0.$$

For other equivalent definitions of weakly PIC_2, see [7, Chapter 7].

(iv) (κ–solution to the Ricci flow) $(M^n, g(t))_{t \in (-\infty, 0]}$ is called a κ–solution if it has weakly PIC_2, uniformly bounded curvature and is κ–noncollapsed.

Next, we recall the following definition of Ricci shrinkers.

Definition 2.2 (Ricci shrinker) A Ricci shrinker (M^n, g, f) is a complete Riemannian manifold equipped with a smooth potential function f satisfying

$$\text{Ric} + \text{Hess}\, f = \tfrac{1}{2}g,$$

where f is normalized so that

(2-2) $$R + |\nabla f|^2 = f.$$

Any Ricci shrinker can be regarded as a self-similar ancient solution to the Ricci flow; see [26, Chapter 4]. It is proved in [35] that any Ricci shrinker is κ–noncollapsed for some $\kappa > 0$. Moreover, we have the following classification of Ricci shrinkers with weakly PIC_2 from [33]; see also [38, Corollary 4].

Lemma 2.3 [33, Theorem 3.1(ii)] *Let* (M^n, g, f) *be a Ricci shrinker with weakly* PIC_2. *Then* (M^n, g) *is isometric to a finite quotient of* $N^k \times \mathbb{R}^{n-k}$, *where* N *is a simply connected, compact symmetric space.*

For Ricci shrinker $(N^k \times \mathbb{R}^{n-k}, \bar{g})$, the metric $\bar{g} = g_N \times g_E$, where g_N is an Einstein metric with Einstein constant $\tfrac{1}{2}$ and g_E is the Euclidean metric. If $n - k \geq 1$, then the potential function f of the Ricci shrinker is $f = \tfrac{1}{4}|z|^2 + \tfrac{1}{2}k$, where $z \in \mathbb{R}^{n-k}$. If $n = k$, then the Ricci shrinker is compact and $f = \tfrac{1}{2}n$. In the following, we will use $\bar{g}$ to denote both the metric of the Ricci shrinker $N^k \times \mathbb{R}^{n-k}$ and the induced metric on the quotient $(N^k \times \mathbb{R}^{n-k})/\Gamma$. Also, for any Ricci shrinker $(N^k \times \mathbb{R}^{n-k})/\Gamma$, the associated Ricci flow $((N^k \times \mathbb{R}^{n-k})/\Gamma, \bar{g}(t))_{t \in (-\infty, 0)}$, where $\bar{g}(-1) = \bar{g}$, is implicitly understood.

Next, we recall that the fundamental group of any Ricci shrinker is always finite; see [36; 49; 29; 47]. In fact, the order can be controlled by the κ–noncollapsing condition; see [25, Proposition 3]. More precisely:

Lemma 2.4 *For any* κ–*noncollapsed Ricci shrinker* (M^n, g, f), *there exists a positive constant* $C = C(n, \kappa) < \infty$ *such that the order of the fundamental group satisfies*

$$|\pi_1(M)| \leq C.$$

Now, we define the following moduli space.

Definition 2.5 The moduli space $\mathcal{M}(n, \kappa)$ consists of all κ–noncollapsed, nonflat, Ricci shrinkers in the form of $(N^k \times \mathbb{R}^{n-k})/\Gamma$, where N is a simply connected, compact symmetric space and $\Gamma \subset \text{Iso}(N^k \times \mathbb{R}^{n-k})$ is a finite subgroup which acts freely on $N^k \times \mathbb{R}^{n-k}$.

It follows from Lemma 2.4 that for any $(N^k \times \mathbb{R}^{n-k})/\Gamma \in \mathcal{M}(n, \kappa)$,

(2-3) $$|\Gamma| \leq C(n, \kappa).$$

Lemma 2.6 *There are finitely many isometry classes in the moduli space $\mathcal{M}(n, \kappa)$.*

Proof For any simply connected, compact symmetric space N, we have by de Rham's decomposition theorem [32, Chapter IV, Theorem 6.2]

$$N = N_1 \times N_2 \times \cdots \times N_l,$$

where each N_i is a simply connected, irreducible, compact symmetric space. Moreover, each factor N_i has the same Einstein constant $\frac{1}{2}$. From the classification, see [2, Chapter 7], of all simply connected, irreducible, compact symmetric spaces, there are finitely many possible N with dimension not greater than n and Einstein constant $\frac{1}{2}$.

We fix a space $N^k \times \mathbb{R}^{n-k} \in \mathcal{M}(n, \kappa)$ and consider all its possible quotient $(N^k \times \mathbb{R}^{n-k})/\Gamma$. For any $\sigma \in \Gamma$, we can write $\sigma = (\sigma_1, \sigma_2) \in \mathrm{Iso}(N^k) \times \mathrm{Iso}(\mathbb{R}^{n-k})$ from the uniqueness of de Rham's decomposition theorem. If we denote the projections of Γ to $\mathrm{Iso}(N^k)$ and $\mathrm{Iso}(\mathbb{R}^{n-k})$ by Γ_0 and Γ_1, respectively, then both Γ_0 and Γ_1 are finite groups. From the center of mass construction, there exists a point $x \in \mathbb{R}^{n-k}$ fixed by Γ_1. Therefore, by a translation, we can regard

$$\Gamma \subset \mathrm{Iso}(N^k) \times O(n-k).$$

Since $\mathrm{Iso}(N^k) \times O(n-k)$ is a compact group, there are only finitely many conjugacy classes of any finite subgroup with fixed order; see, for example, [14, Corollary 8.1.7]. From (2-3), it is easy to see that there are finitely many quotients $(N^k \times \mathbb{R}^{n-k})/\Gamma$. $\qquad\square$

For any $(N^k \times \mathbb{R}^{n-k})/\Gamma \in \mathcal{M}(n, \kappa)$, we assume $n - k \geq 1$ and analyze its fibration structure. As above, we denote the projection of Γ to $\mathrm{Iso}(\mathbb{R}^{n-k})$ by Γ_1. Therefore, one can regard $(N^k \times \mathbb{R}^{n-k})/\Gamma$ as a singular fibration over the orbifold $\mathbb{R}^{n-k}/\Gamma_1$ such that each fiber is a quotient of N. More precisely, the fibration is induced by the natural projection map

$$(N^k \times \mathbb{R}^{n-k})/\Gamma \xrightarrow{\ \pi\ } \mathbb{R}^{n-k}/\Gamma_1.$$

For any connected open set $U \subset \mathbb{R}^{n-k}/\Gamma_1$, there exist a connected open set $\widetilde{U} \subset \mathbb{R}^{n-k}$, which projects onto U, and a subgroup $G \subset \Gamma_1$ such that $\widetilde{U}$ is invariant under the action of G. Moreover, U is isometric to $\widetilde{U}/G$. It is clear that the following diagram is commutative:

$$
\begin{array}{ccc}
N \times \widetilde{U} & \longrightarrow & \pi^{-1}(U) \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi} \\
\widetilde{U} & \xrightarrow{\ \mathrm{pr}\ } & U
\end{array}
$$

We define Λ to be the preimage of G for the projection $\Gamma \to \Gamma_1$. Then the projection map p_2 in the above diagram is (Λ, G)–equivariant. Moreover, it is clear that $(N \times \widetilde{U})/\Lambda$ is isometric to $\pi^{-1}(U)$. Notice that for any $x \in (N^k \times \mathbb{R}^{n-k})/\Gamma$, there is a natural decomposition of the tangent space at x into horizontal and vertical directions from the fibration structure.

For any $p \in \mathbb{R}^{n-k}/\Gamma_1$, the fiber $\pi^{-1}(p)$ is a nontrivial quotient of N if p is a singular point. Moreover, there exists a small constant $r > 0$ such that we can choose $U = B_d(p, r)$ and $\tilde{U} = B_{g_E}(\tilde{p}, r)$, where $\tilde{p}$ is a lift of p and d is the induced distance in $\mathbb{R}^{n-k}/\Gamma_1$. In this case, the group $G = G_{\tilde{p}}$ is the isotropy group of $\tilde{p}$ in Γ_1. We call the open set $B_{g_E}(\tilde{p}, r)/G_{\tilde{p}}$ an orbifold chart, and the open set $(N \times B_{g_E}(\tilde{p}, r))/\Lambda$ a fundamental chart.

For later applications, we construct a family of fundamental charts for any space in $\mathcal{M}(n, \kappa)$.

Lemma 2.7 *For any n and $\kappa > 0$, there exists a constant $\delta_0 = \delta_0(n, \kappa) \in (0, 1)$ satisfying the following property.*

For any $\bar{r} > 0$ and $(N^k \times \mathbb{R}^{n-k})/\Gamma \in \mathcal{M}(n, \kappa)$ with $n - k \geq 1$, there are countably many fundamental charts $(N \times B_{g_E}(\tilde{p}_i, 100r_i))/\Lambda_i$ such that $r_i \in [\delta_0 \bar{r}, \bar{r}]$ and

$$(N^k \times \mathbb{R}^{n-k})/\Gamma = \bigcup_{i \geq 1}(N \times B_{g_E}(\tilde{p}_i, r_i))/\Lambda_i.$$

Proof From Lemma 2.6, we only need to prove the lemma for a fixed space $(N^k \times \mathbb{R}^{n-k})/\Gamma \in \mathcal{M}(n, \kappa)$. As before, we set Γ_1 to be the projection of Γ to $\mathrm{Iso}(\mathbb{R}^L)$, where $L := n - k$. Since Γ_1 is finite, we may assume $\Gamma_1 \subset O(L)$ by a translation. Therefore, the orbifold $\mathbb{R}^L/\Gamma_1$ can be regarded as a cone over the compact orbifold S^{L-1}/Γ_1 and hence we may assume $\bar{r} = 1$ by rescaling.

Claim *If δ is sufficiently small, any point $x \in \mathbb{R}^L$ is contained in a ball $B_{g_E}(\tilde{p}, r)$ with $r \in [\delta, 1]$ such that $B_{g_E}(\tilde{p}, 100r)/G_{\tilde{p}}$ is an orbifold chart.*

Suppose the claim does not hold. Then there exists a sequence $x_i \in \mathbb{R}^L$ such that x_i is not contained in any open set in the form of $B_{g_E}(\tilde{p}, r)$ with $r \in [\delta_i, 1]$, where $B_{g_E}(\tilde{p}, 100r)/G_{\tilde{p}}$ is an orbifold chart and $\delta_i \to 0$.

Notice that $x_i \to \infty$. Otherwise, we have $\lim_{i \to \infty} x_i = x$ by taking a subsequence, and $B_{g_E}(x, r_0)/G_x$ is an orbifold chart for a small constant $r_0 > 0$, which contradicts our assumption. Moreover, there exists $\gamma_i \in \Gamma_1$ such that

$$(2\text{-}4) \qquad\qquad \gamma_i(x_i) \neq x_i \quad \text{and} \quad \lim_{i \to \infty} |\gamma_i(x_i) - x_i| = 0,$$

since otherwise, by taking a subsequence, $B_{g_E}(x_i, r_1)/G_{x_i}$ is an orbifold chart for a small constant $r_1 > 0$. Since Γ_1 is finite, we assume all γ_i are the same and denote it by γ. By taking a subsequence if necessary, we assume

$$\lim_{i \to \infty} \frac{x_i}{|x_i|} = y \in S^{L-1},$$

and hence by (2-4) and the fact that $x_i \to \infty$,

$$\gamma(y) = y.$$

There exists a number $s_0 > 0$ such that $B_{g_E}(y, s_0)/G_y$ is an orbifold chart. If we set $y_i = |x_i|y$, then it is clear that y_i is also fixed by γ and $G_{y_i} = G_y$. Therefore, $B_{g_E}(y_i, ts_0)/G_{y_i}$ is an orbifold chart for any $1 \le t \le |x_i|$. By our assumption, we conclude that

$$\lim_{i \to \infty} |x_i - y_i| = \infty \quad \text{and} \quad \lim_{i \to \infty} \frac{|x_i - y_i|}{|x_i|} = 0.$$

Next, by taking a subsequence, we assume

$$\lim_{i \to \infty} \frac{x_i - y_i}{|x_i - y_i|} = z \in S^{L-1}.$$

Then it is clear that z is perpendicular to y, and $\gamma(z) = z$. If we set $z_i = y_i + |x_i - y_i|z$, then we can repeat the above argument, with 0, y_i and Γ_1 replaced with y_i, z_i and G_y, respectively. Notice that even though the y_i are different based points, $G_{y_i} = G_y$ are the same. By iterating at most L times, one can conclude that γ is the identity map on $\mathbb{R}^L$, which is a contradiction.

Now the conclusion follows immediately from the claim and a covering argument. $\square$

Remark 2.8 For any $(N^k \times \mathbb{R}^{n-k})/\Gamma \in \mathcal{M}(n, \kappa)$, it follows from the Bonnet–Myers theorem that the diameter of the fiber is at most $\sqrt{2(n-1)}\pi$. Therefore, in practice, we will always assume that $\delta_0 \bar{r} \ge n^{10}$ so that the base of a fundamental chart has a much larger size than the fiber.

Now, we recall the distance comparison and existence of the asymptotic Ricci shrinker for a Type I κ–solution.

Lemma 2.9 *Let $(M^n, g(t))_{t \in (-\infty, 0]}$ be a nonflat, Type I, κ–solution satisfying (2-1). Then we have the following.*

(i) *For any $t_2 < t_1 < 0$ and $p, q \in M$, we have*

$$d_{g(t_2)}(p, q) - 8(n-1)C_0(\sqrt{|t_2|} - \sqrt{|t_1|}) \le d_{g(t_1)}(p, q) \le d_{g(t_2)}(p, q).$$

(ii) *For any $p \in M$ and $\tau_i \to \infty$, if we set $g_i(t) = \tau_i^{-1} g(\tau_i t)$, then the sequence of Ricci flows $(M, g_i(t), p)_{t \in (-\infty, 0)}$ subconverges smoothly to a Ricci flow associated with a Ricci shrinker in $\mathcal{M}(n, \kappa)$.*

Proof Assertion (i) follows directly from the Type I assumption and [42, Lemma 8.3]. For assertion (ii), the existence of the asymptotic Ricci shrinker follows from [39, Theorem 3.1]. Notice that the limit must be nonflat by [19, Theorem 4.1] and hence it belongs to $\mathcal{M}(n, \kappa)$ by Lemma 2.3. $\square$

Now we classify all compact, Type I, κ–solutions.

Theorem 2.10 *Let $(M^n, g(t))_{t \in (-\infty, 0]}$ be a compact, Type I, κ–solution. Then $(M^n, g(t))$ is isometric to a finite quotient of a compact symmetric space N^n.*

Proof   We first prove that the universal cover of $(M, g(t))$, denoted by $(\widetilde{M}, \widetilde{g}(t))$, is also compact. Indeed, for a fixed point p and any sequence $\tau_i \to \infty$, it follows from Lemma 2.9(ii) that $(M, \tau_i^{-1} g(-\tau_i), p)$ subconverges smoothly to a Ricci shrinker $(M_\infty, g_\infty, p_\infty) \in \mathcal{M}(n, \kappa)$. Moreover, it follows from the distance estimate in Lemma 2.9(i) that M_∞ is compact and hence is diffeomorphic to M. Therefore, the universal cover of M_∞ is compact since the fundamental group of M_∞ is finite by (2-3).

We may assume the compact ancient solution $(\widetilde{M}, \widetilde{g}(t))$ is irreducible since otherwise, we can prove the conclusion for each factor. Therefore, we have the following possible cases by Berger's holonomy classification.

  (a) $(\widetilde{M}, \widetilde{g}(t))$ is a compact symmetric space. In this case, there is nothing to prove.

  (b) $\mathrm{Hol}(\widetilde{M}, \widetilde{g}(t)) = \mathrm{SO}(n)$. In this case, one can prove that $(\widetilde{M}, \widetilde{g}(t))$ has strictly PIC$_2$ by using the strong maximum principle developed in [13]. For details, refer to [8, Proposition 6.6, Case 1]. Therefore, $(\widetilde{M}, \widetilde{g}(t))$ is isometric to the standard sphere S^n; see [41, Corollary 0.4].

  (c) $\mathrm{Hol}(\widetilde{M}, \widetilde{g}(t)) = U(n/2)$, where n is even. In this case, $(\widetilde{M}, \widetilde{g}(t))$ is a κ–noncollapsed ancient solution to the Kähler Ricci flow. From the weakly PIC$_2$ condition, we derive, in particular, that $(\widetilde{M}, \widetilde{g}(t))$ has nonnegative bisectional curvature. Therefore, $(\widetilde{M}, \widetilde{g}(t))$ is isometric to a compact Hermitian symmetric space from [27, Theorem 1.1].

  (d) $\mathrm{Hol}(\widetilde{M}, \widetilde{g}(t)) \neq \mathrm{SO}(n)$ or $U(n/2)$. In this case, $(\widetilde{M}, \widetilde{g}(t))$ is an Einstein manifold [2, Chapter 10]. Therefore, $(\widetilde{M}, \widetilde{g}(t))$ is a symmetric space by [6, Theorem 1].

In sum, $(\widetilde{M}, \widetilde{g}(t))$ is a compact symmetric space, and the proof is complete. $\square$

Now, we turn our attention to noncompact κ–solutions. First, we prove the following result, which indicates that we only need to consider the simply connected case.

Lemma 2.11   *Let $(M^n, g(t))$ be a κ–solution whose universal cover is $N^k \times \mathbb{R}^{n-k}$, where N is a simply connected, compact symmetric space. Then the fundamental group of M is finite and*

$$|\pi_1(M)| \leq C(n, \kappa),$$

where $C(n, \kappa)$ is the same constant as in Lemma 2.4.

Proof   We assume the fundamental group of M is infinite and derive a contradiction.

From our assumption, $(M^n, g(t))$ is isometric to $(N^k \times \mathbb{R}^{n-k})/\Gamma$, where $\Gamma \subset \mathrm{Iso}(N^k \times \mathbb{R}^{n-k})$ is an infinite discrete subgroup which acts freely on $N^k \times \mathbb{R}^{n-k}$. Moreover, we may assume $L := n - k \geq 1$, since otherwise, the universal cover is compact.

If we denote the projections of Γ to $\mathrm{Iso}(N)$ and $\mathrm{Iso}(\mathbb{R}^L)$ by Γ_0 and Γ_1, respectively, then it is clear that Γ_1 is infinite, since otherwise $(N^k \times \mathbb{R}^{n-k})/\Gamma$ has smaller Hausdorff dimension than n. As before, we can regard $(N^k \times \mathbb{R}^{n-k})/\Gamma$ as a singular fibration over the orbifold $\mathbb{R}^L/\Gamma_1$ such that each fiber is a quotient of N.

Since Γ_1 is discrete, after a change of origin in $\mathbb{R}^L$, there exists a normal subgroup Γ_1' consists of translating elements, such that the index is finite, see [46, Theorem 3.2.8]. If we set V to be the subspace generated by elements in Γ_1', then $l := \dim V \geq 1$ since Γ_1 is infinite.

For Ricci flow $((N^k \times \mathbb{R}^L)/\Gamma, \overline{g}(t))_{t \in (-\infty, 0]}$, it is easy to see that as $t \to -\infty$, the metric on the $\mathbb{R}^L$ direction remains unchanged, but the metric on the fiber direction is rescaled by a factor $|t|$. In addition, the scalar curvature of $(((N^k \times \mathbb{R}^L)/\Gamma, \overline{g}(t))$ is identically $C|t|^{-1}$. On the other hand, it is clear from our analysis above that for any $x \in (N^k \times \mathbb{R}^L)/\Gamma$,

$$\frac{|B_{g(-r^2)}(x, r)|_{g(-r^2)}}{r^{n-l}} \leq C \quad \text{for any } r \geq 1.$$

However, it contradicts our κ–noncollapsing condition if $r \to \infty$.

To summarize, we have shown that Γ is finite. Now, it is easy to see that any asymptotic Ricci shrinker of $(M^n, g(t))$ is $(N^k \times \mathbb{R}^{n-k})/\Gamma \in \mathcal{M}(n, \kappa)$. The control of the order of Γ thus follows from Lemma 2.4. $\square$

Definition 2.12 Let (M_1, g_1, x_1) and (M_2, g_2, x_2) be two pointed Riemannian manifolds. We say (M_1, g_1, x_1) is ϵ–close to (M_2, g_2, x_2) if $B_{g_1}(x_1, \epsilon^{-1})$ and $B_{g_2}(x_2, \epsilon^{-1})$ are ϵ–close in $C^{[\epsilon^{-1}]}$–topology. More precisely, there exist open neighborhoods U_i such that $B_{g_i}(x_i, \epsilon^{-1} - \epsilon) \subset U_i \subset \overline{U}_i \subset B_{g_i}(x_i, \epsilon^{-1})$ for $i = 1, 2$. Moreover, there exists a diffeomorphism $\varphi: U_1 \to U_2$ such that $\varphi(x_1) = x_2$ and

$$\sup_{U_1} \left(|\varphi^* g_2 - g_1|^2 + \sum_{i=1}^{[\epsilon^{-1}]} |\nabla_{g_1}^i (\varphi^* g_2)|^2 \right) \leq \epsilon^2.$$

Lemma 2.13 *For any n and $\kappa > 0$, there exists a small constant $\epsilon_0 = \epsilon_0(n, \kappa) > 0$ satisfying the following property.*

Suppose $(N_i^{k_i} \times \mathbb{R}^{n-k_i})/\Gamma_i \in \mathcal{M}(n, \kappa)$ for $i = 1, 2$ such that $((N_1^{k_1} \times \mathbb{R}^{n-k_1})/\Gamma_1, x_1)$ is ϵ–close to $((N_2^{k_2} \times \mathbb{R}^{n-k_2})/\Gamma_2, x_2)$ with $\epsilon \leq \epsilon_0$. Then N_1 is isometric to N_2.

Proof From Lemma 2.6, we only need to prove the lemma for two fixed spaces $(N_i^{k_i} \times \mathbb{R}^{n-k_i})/\Gamma_i \in \mathcal{M}(n, \kappa)$ for $i = 1, 2$.

From Lemma 2.7, for $i \in \{1, 2\}$ there is a fundamental chart $(N_i \times B_{g_E}(0, 100r_i))/\Lambda_i$ in $(N_i^{k_i} \times \mathbb{R}^{n-k_i})/\Gamma_i$ such that $x_i \in (N_i \times B_{g_E}(0, r_i))/\Lambda_i$. As discussed in Remark 2.8, we assume $r_i \in [n^{10}, \delta_0^{-1} n^{10}]$ and, without loss of generality, $r_1 \leq r_2$. Moreover, we denote the projection map from $N_i \times B_{g_E}(0, 100r_i)$ to $(N_i \times B_{g_E}(0, 100r_i))/\Lambda_i$ by π_i.

By our assumption, there exists a map φ from a neighborhood of x_1 to that of x_2, which is almost isometric. We denote a lift of x_i to $N_i \times B_{g_E}(0, 100r_i)$ by $\tilde{x}_i$. Since $N_1 \times B_{g_E}(0, r_1)$ is simply connected, we can lift φ to a smooth map

$$\tilde{\varphi}: N_1 \times B_{g_E}(0, r_1) \to N_2 \times B_{g_E}(0, 100r_2)$$

such that $\tilde{\varphi}(\tilde{x}_1) = \tilde{x}_2$. Now we claim that $\tilde{\varphi}$ is a smooth embedding. It is clear that $\tilde{\varphi}$ is a local diffeomorphism since φ is a diffeomorphism. Therefore, we only need to prove that $\tilde{\varphi}$ is injective.

Suppose there are $a, b \in N_1 \times B_{g_E}(0, r_1)$ with $a \neq b$ such that $\widetilde{\varphi}(a) = \widetilde{\varphi}(b) = c$. From our construction, we set $z := \pi_1(a) = \pi_1(b)$ and $w := \varphi(z) = \pi_2(c)$. We connect a with b by a geodesic segment $\widetilde{\gamma}_1$ contained in $N_1 \times B_{g_E}(0, r_1)$ so that the loop $\gamma_1 := \pi_1 \circ \widetilde{\gamma}_1$ represents a nontrivial element in Λ_1. In addition, if we set $\widetilde{\gamma}_2 := \widetilde{\varphi} \circ \widetilde{\gamma}_1$ and $\gamma_2 := \pi_2 \circ \widetilde{\gamma}_2$, then γ_2 is a trivial element in Λ_2. Therefore, there exists a homotopy F of γ_2 to w such that the image of F is contained in $B(x_2, 10r_1)$. Since φ is a diffeomorphism that is almost isometric, we can pull back the homotopy F, and it implies that γ_1 is homotopic to z in $(N_1 \times B_{g_E}(0, 100r_1))/\Lambda_1$, which is a contradiction.

To summarize, $\widetilde{\varphi}$ is an embedding that is almost isometric. In particular, $\widetilde{\varphi}$ almost preserves the parallel directions $\mathbb{R}^{n-k_1}$. If ϵ is sufficiently small, it is easy to see that $k_1 = k_2$, and $\widetilde{\varphi}$ almost preserves the vertical direction as well. Therefore, if we denote the copies of N_1 and N_2 through $\widetilde{x}_1$ and $\widetilde{x}_2$ by N_1' and N_2', respectively, then the normal projection of $\widetilde{\varphi}(N_1')$ to N_2' is almost an isometry, see for instance [20, Appendix 2]. Therefore, N_1 is isometric to N_2 if ϵ is sufficiently small. $\qquad\square$

Remark 2.14 The proof of Lemma 2.13 indicates that if ϵ is sufficiently small, then φ injects Λ_1' into Λ_2, where Λ_1' is the group of isometries obtained by restriction of Λ_1 on $N_1 \times B_{g_E}(0, r_1)$.

Now we prove the main result of this section.

Theorem 2.15 *Let $(M^n, g(t))_{t \in (-\infty, 0]}$ be a simply connected, noncompact, Type I, κ–solution. Then $(M^n, g(t))$ is isometric to $N^k \times \mathbb{R}^{n-k}$, where N is a simply connected, compact symmetric space.*

Proof We prove the theorem by induction. If $n = 2$, the conclusion obviously holds, since all 2–dimensional κ–solutions are S^2, $\mathbb{RP}^2$ and $\mathbb{R}^2$, proved by Perelman [42]. Now we assume the conclusion holds for any dimension smaller than n.

Since $(M^n, g(t))$ has weakly PIC_2, it follows from [15, Theorem 1.2] that $(M^n, g(t))$ is isometric to $\Sigma \times F$, where Σ is the k–dimensional soul and F is diffeomorphic to $\mathbb{R}^{n-k}$. If $k \geq 1$, it is clear that $(M^n, g(t))$ splits as Type I κ–solutions on Σ and F. Therefore, the conclusion follows from our inductive assumption.

We assume $k = 0$ and hence M is diffeomorphic to $\mathbb{R}^n$. If $(M, g(t))$ is flat, then the proof is complete. For the rest of the proof, we assume that $(M, g(t))$ is nonflat and derive a contradiction.

Let p be a fixed point in M and $\delta = \delta(\epsilon)$ a positive function such that $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$. In the following proof, δ may differ line by line.

Claim 1 *For any $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that for any $\tau \geq T$, $(M, \tau^{-1}g(\tau t), p)$ is ϵ–close to a space $(M_\infty, g_\infty(t), p_\infty)$ for $t \in \left[-10, -\frac{1}{10}\right]$, where $(M_\infty, g_\infty(-1)) \in \mathcal{M}(n, \kappa)$.*

Indeed, suppose the claim does not hold for some ϵ and a sequence $\tau_i \to \infty$, then by Lemma 2.9, $(M, \tau_i^{-1}g(\tau_i t), p)$ subconverges smoothly to a Ricci flow associated with a Ricci shrinker in $\mathcal{M}(n, \kappa)$, which is a contradiction.

Next we define a sequence $\tau_i := 4^i$ and set $g_i(t) = \tau_i^{-1} g(\tau_i t)$ and $g_i = g_i(-1)$. Then it is clear that

$$(2\text{-}5) \qquad\qquad g_{i-1} = \tau_{i-1}^{-1} g(-\tau_{i-1}) = 4\tau_i^{-1} g\left(-\tfrac{1}{4}\tau_i\right) = 4g_i\left(-\tfrac{1}{4}\right).$$

We set I to be the smallest integer such that $\tau_{I-1} \geq T(\epsilon)$. For any $i \geq I$, it follows from Claim 1 that (M, g_i, p) is ϵ–close to a Ricci shrinker $(N_i^{k_i} \times \mathbb{R}^{n-k_i})/\Gamma_i, x_i) \in \mathcal{M}(n, \kappa)$. By Definition 2.12, there exists a diffeomorphism φ_i from an open neighborhood containing $B_{g_i}(p, 4\epsilon^{-1}/5)$ onto an open neighborhood containing $B(x_i, 4\epsilon^{-1}/5) \subset (N_i^{k_i} \times \mathbb{R}^{n-k_i})/\Gamma_i$ such that $\varphi_i(p) = x_i$. As discussed earlier, we regard $(N_i^{k_i} \times \mathbb{R}^{n-k_i})/\Gamma_i$ as a singular fibration $(N_i^{k_i} \times \mathbb{R}^{n-k_i})/\Gamma_i \xrightarrow{\pi_i} \mathbb{R}^{n-k_i}/\Gamma_{1,i}$. If we denote the induced distance function on $\mathbb{R}^{n-k_i}/\Gamma_{1,i}$ by d_i, then we define

$$W_i := B_{d_i}\left(\pi_i(x_i), \tfrac{1}{4}\epsilon^{-1}\right), \quad W_i' := B_{d_i}\left(\pi_i(x_i), \tfrac{1}{2}\epsilon^{-1}\right),$$

$$V_i := \varphi_i^{-1}(\pi_i^{-1}(W_i)), \qquad U_i := \varphi_i^{-1}(\pi_i^{-1}(W_i')).$$

In particular, we obtain a singular fibration structure on U_i.

Claim 2   *N_i is isometric to N_{i-1} for all $i \geq I$, if ϵ is sufficiently small.*

From the fibration structure, there exist an open set $\widetilde{W}_i \subset \mathbb{R}^{n-k_i}$, two subgroups $G_i \subset \Gamma_{1,i}$ and $\Lambda_i \subset \Gamma_i$, and a map ψ_i such that $\widetilde{W}_i/G_i$ is an orbifold chart and $\psi_i \colon (N_i \times \widetilde{W}_i)/\Lambda_i \to \pi_i^{-1}(W_i)$ is an isometry. Since G_i is finite, there exists a point $y_i \in \widetilde{W}_i$ fixed by G_i. For simplicity, we assume $y_i = 0$ and define

$$\widetilde{W}_i' = \{2x \mid x \in \widetilde{W}_i\}.$$

Therefore, we have the diffeomorphism

$$(N_i \times \widetilde{W}_i')/\Lambda_i \xrightarrow{\iota_i} (N_i \times \widetilde{W}_i)/\Lambda_i$$

such that ι_i multiplies the horizontal coordinates by $\tfrac{1}{2}$. From Claim 1 and (2-5), we immediately conclude that (V_i, g_{i-1}, p) is δ–close to $((N_i \times \widetilde{W}_i')/\Lambda_i, z_i)$, where $z_i := (\psi_i \circ \iota_i)^{-1}(x_i)$. Here, we have used the fact that the vertical direction in $4g_i\left(-\tfrac{1}{4}\right)$ is almost unchanged.

Since (M, g_{i-1}, p) is ϵ–close to $((N_{i-1} \times \mathbb{R}^{n-k_{i-1}})/\Lambda_{i-1}, x_{i-1})$, it is clear from Lemma 2.13 that N_i is isometric to N_{i-1} for any $i \geq I$, if ϵ is sufficiently small. From now on, we denote all N_i by a common $N = N^k$.

Claim 3   *V_i is simply connected for any $i \geq I$, if ϵ is sufficiently small.*

To prove Claim 3, we only need to show that each fiber in V_i is simply connected. Indeed, if all fibers are simply connected, V_i is a regular fibration over $B_{g_E}\left(\pi_i(x_i), \tfrac{1}{4}\epsilon^{-1}\right)$ with fiber N. Since $B_{g_E}\left(\pi_i(x_i), \tfrac{1}{4}\epsilon^{-1}\right)$ is contractible, we conclude that V_i is diffeomorphic to $N \times B_{g_E}(0, 1)$.

Suppose the conclusion does not hold for some $j - 1 \geq I$, then there exists a point $x \in V_{j-1}$ such that the fiber F_{j-1} containing x_{j-1} is not simply connected. Therefore, there exists a loop γ_{j-1} based at x_{j-1}, which is contained in F_{j-1} and represents a nontrivial element in $\pi_1(F_{j-1})$. In particular, it follows from the structure of the fibration that γ_{j-1} also represents a nontrivial element in $\pi_1(U_{j-1})$.

On the one hand, since M is simply connected, there exists a homotopy $\Psi(u,v)\colon [0,1]\times[0,1]\to M$ such that $\Psi(0,v)=\gamma_{j-1}(v)$ and $\Psi(1,v)=\Psi(u,0)=\Psi(u,1)=x_{j-1}$ for any $(u,v)\in[0,1]\times[0,1]$. Since the image of Ψ is compact, we assume that

$$\text{Image}(\Psi)\subset B_{g_{j-1}}(p,K)\quad\text{for some large constant }K>0.$$

From the above arguments, we know that (M,g_{j-1},p) is δ–close to $((N\times\widetilde{W}'_j)/\Lambda_j, z_j)$. As in the proof of Lemma 2.13 (see Remark 2.14), we conclude that there exists a loop γ_j based at a point $x_j\in V_j$ such that γ_j is contained in the fiber F_j through x_j and γ_j represents a nontrivial element in $\pi_1(F_j)$. Moreover, γ_{j-1} is homotopic to $\eta_j*\gamma_j$, where η_j is a geodesic segment with respect to g_{j-1} connecting x_{j-1} with x_j and $*$ is the joining of two curves. Besides, the image of the homotopy Ψ_j is contained in $B_{g_{j-1}}(x_{j-1},L)$ for some constant L depending only on n and κ. By iteration, for any $i\geq j$, we can successively define a loop γ_i which is based at x_i and contained in the fiber F_i through x_i and a homotopy Ψ_i between γ_{i-1} and $\eta_i*\gamma_i$, where η_i is a geodesic segment with respect to g_{i-1} connecting x_{i-1} with x_i. Moreover, the image of Ψ_i is contained in $B_{g_{i-1}}(x_{i-1},L)$ and γ_i represents a nontrivial element in $\pi_1(F_i)$. Notice that γ_i also represents a nontrivial element in $\pi_1(U_i)$.

Now we define a sequence $\{a_i\}_{i\geq j-1}$ such that $a_{j-1}=0$ and $a_{i+1}=4(n-1)C_0+\frac{a_i+L}{2}$. Then it follows immediately from Lemma 2.9 (i) that for any $i\geq j$,

$$\text{Image}(\Psi_j*\cdots*\Psi_i)\subset B_{g_i}(x_{j-1},a_i),$$

where we use $*$ to denote the joining of homotopies. Clearly, the sequence $a_i\leq L+8(n-1)C_0$ and hence γ_{j-1} is homotopic to γ_i in U_i for any $i\geq j$, if ϵ is sufficiently small. In particular, γ_{j-1} represents a nontrivial element in $\pi_1(U_i)$.

On the other hand, it follows from Lemma 2.9(i) that

$$B_{g_{j-1}}(p,K)\subset B_{g_i}\left(p,\tfrac{1}{8}\epsilon^{-1}\right)$$

if i is sufficiently large. Therefore, the image of the homotopy Ψ is contained in U_i and hence the γ_{j-1} is trivial in $\pi_1(U_i)$. From this, we derive a contradiction.

In sum, we have shown that the fibration structure V_i is trivial. In particular, for any $i\geq I$, there exists a trivial fibration

$$N\hookrightarrow V_i\xrightarrow{\ f_i\ } B_{g_E}\left(0,\tfrac{1}{4}\epsilon^{-1}\right).$$

Moreover, (V_i,g_i,p) is δ–close to $N^k\times\mathbb{R}^{n-k}$. From Lemma 2.9(i), it is clear that $V_I\subset V_{I+1}\subset\cdots$ is an exhaustion of M. Therefore, it follows from a standard argument — see for example [21, Lemma 1.4] — that we can modify and glue all those fibrations so that we obtain a global fibration

$$N\hookrightarrow M\xrightarrow{\ f\ } Y,$$

where Y is an open manifold.

Since M is diffeomorphic to $\mathbb{R}^n$, we obtain a contradiction from [3], which states that for any fibration on $\mathbb{R}^n$ with compact fibers, the fiber must be a single point. $\qquad\square$

Proof of Theorem 1.2 It follows immediately from Theorem 2.10, Lemma 2.11 and Theorem 2.15. □

From [1, Lemma 4.2] — see also [33, Proposition 6.2] — we know that any complete ancient solution to the Ricci flow with weakly PIC_1 automatically has weakly PIC_2. Therefore, the following corollary is immediate from Theorem 1.2.

Corollary 2.16 *Let* $(M^n, g(t))_{t \in (-\infty, 0]}$ *be a* κ*–noncollapsed, Type I, complete ancient solution to the Ricci flow with weakly* PIC_1*. Then it is isometric to a finite quotient of* $N^k \times \mathbb{R}^{n-k}$*, where* N *is a simply connected, compact symmetric space.*

3 Proof of the main theorem

We recall the following definitions in the Kähler Ricci flow. In the following, we always denote the complex dimension by m and the real dimension by $n = 2m$.

Definition 3.1 Suppose that $(M^m, g(t))_{t \in (-\infty, 0]}$ is a complete ancient solution to the Kähler Ricci flow $\partial_t g = -2 \operatorname{Ric}$ on a complex manifold M.

(i) $(M, g(t))_{t \in (-\infty, 0]}$ has nonnegative bisectional curvature ($\mathrm{BK} \geq 0$ *for short*) if for any $(x, t) \in M \times (-\infty, 0]$ and any $Z, W \in T_x^{1,0} M$, we have at (x, t),

$$\operatorname{Rm}(Z, \overline{W}, \overline{Z}, W) \geq 0.$$

(ii) (κ–solution to the Kähler Ricci flow) $(M, g(t))_{t \in (-\infty, 0]}$ is called a κ–solution if it has $\mathrm{BK} \geq 0$, uniformly bounded curvature and is κ–noncollapsed.

Notice that the Kähler Ricci flow defined here agrees with the real Ricci flow, but differs by a constant 2 from the convention $\partial_t g_{i\bar{j}} = -R_{i\bar{j}}$. Moreover, the curvature operator here is different by a sign from that in most articles in Kähler geometry. The concepts of scalar curvature, Laplacian, etc, always refer to the concepts in real Riemannian geometry.

3.1 BK ≥ 0 implies weakly PIC_2

First, we prove that any κ–solution to the Kähler Ricci flow is of Type I. The proof is similar to [24, Lemma 6.2].

Lemma 3.2 *Let* $(M^m, g(t))_{t \in (-\infty, 0]}$ *be a* κ*–solution to the Kähler Ricci flow. Then there exists a constant* $C_0 > 0$ *such that*

$$|\operatorname{Rm}|(x, t) \leq \frac{C_0}{1 + |t|}.$$

for any $(x, t) \in M \times (-\infty, 0]$.

Proof Suppose the $(M^m, g(t))_{t\in(-\infty,0]}$ is of Type II. We take any $T_i \to -\infty$ and $\delta_i \to 0^+$ and choose $(x_i, t_i) \in M \times [T_i, 0]$ so that

$$|t_i|(t_i - T_i)R(x_i, t_i) \geq (1 - \delta_i) \sup_{M \times [T_i, 0]} |t|(t - T_i)R(x, t).$$

Notice that the choice is possible since the curvature is assumed to be uniformly bounded on $M \times (-\infty, 0]$. For the rescaled metric $g_i(t) := Q_i g(t_i + Q_i^{-1} t)$, where $Q_i := R(x_i, t_i)$, it follows by direct computation — see [26, Proposition 8.20] — that $(M, g_i(t), x_i)$ converges smoothly to $(M_\infty, g_\infty(t), x_\infty)_{t\in(-\infty,\infty)}$, which is an eternal κ–solution to the Kähler Ricci flow. Moreover, $R_{g_\infty}(x, t) \leq 1$ on $M_\infty \times (-\infty, \infty)$ and $R_{g_\infty}(x_\infty, 0) = 1$ by the construction. After taking the universal cover of M_∞ and using Cao's dimension-reduction argument [18, Theorem 2.1], we may assume that M_∞ is simply connected and has positive Ricci curvature. Therefore, it follows from [17, Theorem 1.3] that (M_∞, g_∞) is a nonflat, κ–noncollapsed Kähler Ricci steady soliton with nonnegative bisectional curvature. However, such a Kähler steady soliton is excluded by [28, Theorem 1.2]. $\square$

Next, we obtain the following curvature improvement for κ–solutions. Notice that a stronger curvature improvement was proved for Kähler surfaces in [24, Lemma 4.6].

Theorem 3.3 Let $(M^m, g(t))_{t\in(-\infty,0]}$ *be a complete ancient solution to the Kähler Ricci flow with* BK ≥ 0. *Then it has weakly* PIC$_2$.

Proof By taking the universal cover, we may assume M is simply connected. We prove the conclusion by induction for the dimension m. If $m = 1$, the conclusion trivially holds. We assume the conclusion is true for any dimension smaller than m.

If the Ricci curvature vanishes at some spacetime point, then it follows from [18, Theorem 2.1] that $(M^m, g(t))$ is isometrically biholomorphic to $\Sigma \times \mathbb{C}^k$ for some $k \geq 1$ such that Σ has positive Ricci curvature. Notice that the boundedness of the curvature is not necessary for the proof of [18, Theorem 2.1]. Therefore, $(M^m, g(t))$ splits as two ancient solutions to the Kähler Ricci flow with BK ≥ 0. Then the conclusion follows from the inductive assumption.

For this reason, we may assume the Ricci curvature is positive on the spacetime. From the strong maximum principle (see [37, Proposition 1.1]) we conclude that $(M^m, g(t))$ has positive holomorphic sectional curvature. If $(M^m, g(t))$ is symmetric, then the conclusion obviously holds. Otherwise, the holonomy group of M is $U(m)$ and the proof of [30, Theorem 1.2], which is based on the strong maximum principle in [13], shows that $(M^m, g(t))$ has BK > 0.

If we regard the curvature operator Rm as a symmetric operator on 2–forms $\Lambda^2 = \Lambda^2 M$, then BK > 0 is equivalent to

$$(3\text{-}1) \qquad \qquad \text{Rm}(v, \overline{v}) > 0$$

for any $v = Z \wedge \overline{W} \in \Lambda^{1,1}$ with $v \neq 0$, where $Z, W \in \Lambda^{1,0}$.

Next, define a function λ on $M \times (-\infty, 0]$ as

$$\lambda(x, t) = \inf_{v \in S(x,t)} \mathrm{Rm}(x, t)(v, \overline{v}),$$

where

(3-2) $$S(x, t) = \{v \mid v = Z \wedge \overline{W}, \ Z, W \in \Lambda_x^1 \text{ with } |P(Z)|_t = |P(W)|_t = 1\}.$$

Here $P \colon \Lambda^1 \to \Lambda^{0,1}$ is the projection and the norm $|\cdot|_t$ comes from the Hermitian inner product on Λ^1 induced by $g(t)$.

At a spacetime point $(x_0, t_0) \in M \times (-\infty, 0]$ such that $\lambda(x_0, t_0) \leq 0$, we claim that a minimizer v for λ exists. To prove this, we set $v_i = Z_i \wedge \overline{W}_i$ to be a minimizing sequence such that

$$\lim_{i \to \infty} \mathrm{Rm}(x_0, t_0)(v_i, \overline{v}_i) = \lambda(x_0, t_0).$$

Now we set $Z_i = a_i + \overline{b}_i$ and $W_i = c_i + \overline{d}_i$, where $a_i, b_i, c_i, d_i \in \Lambda_{x_0}^{1;0}$. From our definition, we have $|b_i| = |d_i| = 1$. By taking a subsequence, we may assume

$$\lim_{i \to \infty} b_i = b \quad \text{and} \quad \lim_{i \to \infty} d_i = d$$

where $b, d \in \Lambda_{x_0}^{1;0}$ and $|b| = |d| = 1$. From a direct calculation,

(3-3) $$\mathrm{Rm}(Z_i, \overline{W}_i, \overline{Z}_i, W_i) = \mathrm{Rm}(a_i, \overline{c}_i, \overline{a}_i, c_i) + 2\, \mathfrak{Re}(\mathrm{Rm}(a_i, \overline{c}_i, b_i, \overline{d}_i)) + \mathrm{Rm}(b_i, \overline{d}_i, \overline{b}_i, d_i).$$

By taking a subsequence if necessary, there are three possible cases.

Case 1 $(\lim_{i \to \infty} |a_i| |c_i| = 0)$ In this case, it is easy to see from (3-1) and (3-3) that

$$\lambda = \lim_{i \to \infty} \mathrm{Rm}(Z_i, \overline{W}_i, \overline{Z}_i, W_i) = \mathrm{Rm}(b, \overline{d}, \overline{b}, d) > 0,$$

which contradicts our assumption that $\lambda(x_0, t_0) \leq 0$.

Case 2 $(\lim_{i \to \infty} |a_i| |c_i| = +\infty)$ In this case, we rewrite the right-hand side of (3-3) as

$$(|a_i| |c_i|)^2 \mathrm{Rm}(a_i', \overline{c}_i', \overline{a}_i', c_i') + 2|a_i| |c_i| \, \mathfrak{Re}(\mathrm{Rm}(a_i', \overline{c}_i', b_i, \overline{d}_i)) + \mathrm{Rm}(b_i, \overline{d}_i, \overline{b}_i, d_i),$$

where $a_i' = a_i / |a_i|$ and $c_i' = c_i / |c_i|$. Therefore, it follows immediately from (3-1) that

$$\lambda = \lim_{i \to \infty} \mathrm{Rm}(Z_i, \overline{W}_i, \overline{Z}_i, W_i) = +\infty,$$

which is impossible.

Case 3 $(\lim_{i \to \infty} |a_i| |c_i| = \tau \in (0, \infty))$ In this case, we assume

$$\lim_{i \to \infty} \frac{a_i}{|a_i|} = a \quad \text{and} \quad \lim_{i \to \infty} c_i |a_i| = c$$

where $a, c \in \Lambda_{x_0}^{1;0}$ with $|a| = 1$ and $|c| = \tau$. Then it is clear from (3-3) that

$$\lambda = \lim_{i \to \infty} \mathrm{Rm}(Z_i, \overline{W}_i, \overline{Z}_i, W_i) = \mathrm{Rm}(a, \overline{c}, \overline{a}, c) + 2\, \mathfrak{Re}(\mathrm{Rm}(a, \overline{c}, b, \overline{d})) + \mathrm{Rm}(b, \overline{d}, \overline{b}, d)$$
$$= \mathrm{Rm}(Z, \overline{W}, \overline{Z}, W),$$

where $Z := a + \overline{b}$ and $W = c + \overline{d}$. In other words, $v := Z \wedge \overline{W} \in S(x_0, t_0)$ is a minimizer of λ. In particular, $\lambda > -\infty$.

Now we define $u = Z \wedge d$, $w = Z \wedge \overline{c}$ and $v(s) = u + sw$ for any $s \in \mathbb{R}$. It is clear that $v(s) \in S$ and $v = u + w = v(1)$. Since $\mathrm{Rm}(v(s), \overline{v}(s))$ attains the minimum at $s = 1$, by taking the derivative, we have

$$\mathfrak{Re}(\mathrm{Rm}(u, \overline{w})) + \mathrm{Rm}(w, \overline{w}) = 0$$

and hence

$$(3\text{-}4) \qquad \lambda = \mathrm{Rm}(u, \overline{u}) + 2 \mathfrak{Re}(\mathrm{Rm}(u, \overline{w})) + \mathrm{Rm}(w, \overline{w}) = \mathfrak{Re}(\mathrm{Rm}(v, \overline{u})).$$

From (3-4), we obtain

$$(3\text{-}5) \qquad |\lambda| \leq |\mathrm{Rm}(v, \overline{u})| = |\mathrm{Rm}(Z, \overline{W}, \overline{Z}, \overline{d})| = |\mathrm{Rm}(Z, \overline{W}, b, \overline{d})| \leq |\mathrm{Rm}(v)|,$$

where we have used the fact that $|b| = |d| = 1$.

Next, by applying Uhlenbeck's trick, the Riemannian curvature under the Ricci flow is deformed by

$$(3\text{-}6) \qquad (\partial_t - \Delta) \, \mathrm{Rm} = \mathrm{Rm}^2 + \mathrm{Rm}^{\#},$$

see [26, Lemma 2.58]. For each point (x_0, t_0) such that $\lambda(x_0, t_0) \leq 0$, we choose a minimizer $v(x_0, t_0) \in S(x_0, t_0)$ in the definition of λ. After one applies Uhlenbeck's trick by using a 1-parameter family of bundle isomorphism, the pullback of $g(t)$ is independent of t. Therefore, $v(x_0, t_0) \in S(x_0, t)$ from the definition (3-2). Extending v by parallel transport, one obtains a 2–form v locally defined on a spacetime neighborhood of (x_0, t_0) such that $v(x, t) \in S(x, t)$.

From the evolution equation of Rm (3-6) and (3-5), we conclude that in the barrier sense,

$$(3\text{-}7) \qquad (\partial_t - \Delta) \lambda \geq |\mathrm{Rm}(v)|^2 + \mathrm{Rm}^{\#}(v, \overline{v}) \geq \lambda^2 + \mathrm{Rm}^{\#}(v, \overline{v})$$

at (x_0, t_0). Now it follows from the same proof of [45, Theorem 1] that $\mathrm{Rm}^{\#}(v, \overline{v}) \geq 0$. Indeed, if we consider the Lie algebra $\mathfrak{g} = \mathfrak{u}(m) \otimes_{\mathbb{R}} \mathbb{C}$, then the set $S(x_0, t_0)$ in (3-2) is preserved by the adjoint representation of the Lie group with Lie algebra $\mathfrak{g}$. Therefore, the same proof of [45, Theorem 1] yields the claim.

Therefore, (3-7) implies that

$$(\partial_t - \Delta) \lambda^- \geq (\lambda^-)^2$$

in the barrier sense on $M \times (-\infty, 0]$, where $\lambda^- := \min\{\lambda, 0\}$. Now, it follows from [24, Corollary 2.4] that $\lambda \geq 0$ on $M \times (-\infty, 0]$.

From the definition of λ, we conclude that $(M^m, g(t))$ has weakly PIC_2. Indeed, from Definition 2.1(iii) we only need to prove

$$(3\text{-}8) \qquad \mathrm{Rm}(Z, \overline{W}, \overline{Z}, W) \geq 0$$

for any $Z, W \in \Lambda^1$. We set $Z = a + \overline{b}$ and $W = c + \overline{d}$ as before, where $a, b, c, d \in \Lambda^{1,0}$. If $|b||d| \neq 0$, then

$$\mathrm{Rm}(Z, \overline{W}, \overline{Z}, W) = (|b||d|)^2 \, \mathrm{Rm}(Z', \overline{W}', \overline{Z}', W') \geq 0,$$

where $Z' = Z/|b|$ and $W' = W/|d|$. If $b = 0$, then

$$\mathrm{Rm}(Z, \overline{W}, \overline{Z}, W) = \mathrm{Rm}(a, \overline{c}, \overline{a}, c) \geq 0$$

since $\mathrm{BK} \geq 0$. Similarly, (3-8) also holds for the case $d = 0$.

In sum, the proof is complete. $\qquad\square$

With the help of Theorem 1.2, Lemma 3.2 and Theorem 3.3, the classification of all κ–solutions is immediate.

Theorem 3.4 *Let $(M^m, g(t))_{t \in (-\infty, 0]}$ be a κ–solution to the Kähler Ricci flow. Then $(M^m, g(t))$ is isometrically biholomorphic to a finite quotient of $N^k \times \mathbb{C}^{m-k}$, where N is a simply connected, compact, Hermitian symmetric space.*

3.2 BK ≥ 0 implies bounded curvature

In this subsection, we aim to prove that any complete, κ–noncollapsed, ancient solution to the Kähler Ricci flow with BK ≥ 0 must have uniformly bounded curvature and hence is a κ–solution.

Definition 3.5 The moduli space $\mathcal{N} = \mathcal{N}(m, \kappa)$ consists of all noncompact, nonflat, κ–noncollapsed, Kähler manifolds in the form of $(N^k \times \mathbb{C}^{m-k}) / \Gamma$, where N is a simply connected, compact, Hermitian symmetric space and Γ is a finite subgroup in the holomorphic isometry group of $N^k \times \mathbb{C}^{m-k}$, which acts freely on $N^k \times \mathbb{C}^{m-k}$. Moreover, all Kähler manifolds in $\mathcal{N}$ are normalized so that the scalar curvature is identically 1.

It follows from Lemma 2.11 that for any $(N^k \times \mathbb{C}^{m-k}) / \Gamma \in \mathcal{N}(m, \kappa)$,

$$(3\text{-}9) \qquad\qquad |\Gamma| \leq C(m, \kappa) < \infty.$$

Moreover, from de Rham's decomposition [32, Chapter XI, Theorem 8.1], we have

$$(3\text{-}10) \qquad\qquad N = N_0 \times N_1 \times \cdots \times N_s,$$

where each N_i is an irreducible, simply connected, compact, Hermitian symmetric space. From Lemma 2.6, there are finitely many holomorphic isometry classes in $\mathcal{N}(m, \kappa)$, up to scaling on each factor N_i.

As discussed before, we have the singular fibration

$$(N^k \times \mathbb{C}^{m-k}) / \Gamma \xrightarrow{\pi} \mathbb{C}^{m-k} / \Gamma_1,$$

where Γ_1 is the projection of Γ on the holomorphic isometry group of $\mathbb{C}^{m-k}$. Since Γ_1 is finite, we may assume $\Gamma_1 \subset U(m-k)$. For orbifold $\mathbb{C}^{m-k} / \Gamma_1$, we define $\mathrm{pr} \colon \mathbb{C}^{m-k} \to \mathbb{C}^{m-k} / \Gamma_1$ to be the natural projection and denote the regular part and singular part of the orbifold by $\mathscr{R}$ and $\mathscr{S}$, respectively. Moreover, we set $\mathscr{S}_\epsilon$ to be the ϵ–neighborhood of $\mathscr{S}$ and $\mathscr{R}_\epsilon := (\mathbb{C}^{m-k} / \Gamma_1) \backslash \mathscr{S}_\epsilon$.

The following lemma is standard from the theory of Riemannian orbifolds; see eg [4, page 22, Theorem 3].

Lemma 3.6 *The regular part $\mathscr{R}$ of $\mathbb{C}^{m-k} / \Gamma_1$ is geodesically convex.*

In addition, we have:

Lemma 3.7 *There exists a number $\delta_1 = \delta_1(m, \kappa) > 0$ such that for any $\delta \leq \delta_1$ and any $z \in \mathbb{C}^{m-k} / \Gamma_1$,*

$$|B_d(z, 1) \cap \mathscr{S}_\delta| \leq \tfrac{1}{100} |B_d(z, 1)|.$$

Proof From (3-9), there are only finitely many possible orbifolds $\mathbb{C}^{m-k}/\Gamma_1$ and hence we only need to prove the conclusion for a fixed $\mathbb{C}^{m-k}/\Gamma_1$. Since $\Gamma_1 \subset U(m-k)$, the fixed set of any element in Γ_1 is a complex subspace with dimension smaller than $m-k$. Therefore, the set $\mathrm{pr}^{-1}(\mathscr{S})$ is the union of finitely many complex subspaces which has the Hausdorff dimension at most $2(m-k-1)$. Now, the existence of δ_1 follows immediately. $\qquad\square$

We set $r_0 = \delta_0\delta_1$, where δ_0 and δ_1 are the constants in Lemma 2.7 and Lemma 3.7, respectively. Notice that r_0 depends only on m and κ.

Lemma 3.8 *For any* $z \in \mathrm{pr}^{-1}(\mathscr{R}_{\delta_1})$, $B_{g_E}(z, r_0)$ *is disjoint from its translates by any nontrivial elements in* Γ_1.

Proof From Lemma 2.7, there exist $y \in \mathbb{C}^{m-k}$ and $r \in [\delta_0\delta_1, \delta_1]$ such that $z \in B_{g_E}(y, r)$ and such that $B_{g_E}(y, 100r)/G_y$ is an orbifold chart. Since $d_{g_E}(z, y) < r \leq \delta_1$, $\mathrm{pr}(y) \notin \mathscr{S}$ and hence $G_y = \mathrm{id}$. Therefore, the conclusion follows immediately. $\qquad\square$

Lemma 3.8 implies that for any $x \in \mathscr{R}_{\delta_1}$, $B_d(x, r_0)$ is isometric to $B_{g_E}(0, r_0)$ and hence

$$\pi^{-1}(B_d(x, r_0)) = N \times B_{g_E}(0, r_0).$$

Moreover, it follows from Lemma 3.7 that most points of $(N^k \times \mathbb{C}^{m-k})/\Gamma$ are contained in $\pi^{-1}(\mathscr{R}_{\delta_1})$.

Now we fix $\overline{q} \in (N^k \times \mathbb{C}^{m-k})/\Gamma$ and set $\overline{x} = \pi(\overline{q})$. Moreover, we denote the metric on $(N^k \times \mathbb{C}^{m-k})/\Gamma$ by $\overline{g}$ and define the open sets

$$
\begin{aligned}
U' &:= B_{\overline{g}}(\overline{q}, \epsilon^{-1}) \cap \pi^{-1}\big(B_d\big(\overline{x}, \tfrac{1}{2}\epsilon^{-1}\big)\big), \\
V' &:= B_{\overline{g}}\big(\overline{q}, \tfrac{1}{10}\epsilon^{-1}\big) \cap \pi^{-1}\big(B_d\big(\overline{x}, \tfrac{1}{20}\epsilon^{-1}\big)\big), \\
\widehat{V}' &:= B_{\overline{g}}\big(\overline{q}, \tfrac{1}{1}\epsilon^{-1}0\big) \cap \pi^{-1}\big(B_d\big(\overline{x}, \tfrac{1}{20}\epsilon^{-1}\big) \cap \mathscr{R}\big), \\
\widehat{V}'_{\delta_1} &:= B_{\overline{g}}\big(\overline{q}, \tfrac{1}{10}\epsilon^{-1}\big) \cap \pi^{-1}\big(B_d\big(\overline{x}, \tfrac{1}{20}\epsilon^{-1}\big) \cap \mathscr{R}_{\delta_1}\big), \\
W' &:= B_{\overline{g}}\big(\overline{q}, \tfrac{1}{100}\epsilon^{-1}\big) \cap \pi^{-1}\big(B_d\big(\overline{x}, \tfrac{1}{200}\epsilon^{-1}\big)\big).
\end{aligned}
$$
(3-11)

From the decomposition (3-10), we assume N_0 has the maximal scalar curvature. In particular,

$$1 \geq R_{N_0} \geq \frac{1}{m},$$
(3-12)

and hence the diameter of N_0 is bounded above by $m\pi$. Notice that for any $q \in \widehat{V}'$, there exists a copy of N_0, denoted by $N_{0,q}$, which contains q.

For any $q_1, q_2 \in \widehat{V}'$, we define $N_{0,q_1} \overset{\circ}{=} N_{0,q_2}$ if either $N_{0,q_1} = N_{0,q_2}$ or there exists a smooth manifold T embedded in U' such that T is diffeomorphic to $N_0 \times [0, 1]$ and $\partial T = N_{0,q_1} \cup N_{0,q_2}$. Moreover, we say $N_{0,q_1} \equiv N_{0,q_2}$ if there exist $\{x_1, x_2, \cdots, x_s\} \subset \widehat{V}'$ such that $q_1 = x_1$, $q_2 = x_s$ and $N_{0,x_i} \overset{\circ}{=} N_{0,x_{i+1}}$ for any $1 \leq i \leq s-1$. It is clear from the definition that $\equiv$ is an equivalence relation.

Lemma 3.9 *With the above assumptions, $N_{0,q_1} \equiv N_{0,q_2}$ for any $q_1, q_2 \in \widehat{V}'$.*

Proof For any $q_1, q_2 \in \widehat{V}'$, we first assume that q_1 and q_2 are in the same fiber, that is, $\pi(q_1) = \pi(q_2) = x$ for some $x \in B_d(\overline{x}, \epsilon^{-1}/20) \cap \mathcal{R}$. From the definition of $\mathcal{R}$, there exist local coordinates such that $q_i = (a_i, b_i, x) \in N_0 \times N_0^c \times B_{g_E}(x, r)$ for $i = 1, 2$, where $N_0^c = N_1 \times \cdots \times N_s$ and r is a small number. If we consider a geodesic segment γ contained in N_0^c connecting b_1 with b_2, then it is easy to see $N_{0,q_1} \overset{\circ}{=} N_{0,q_2}$.

Next, we assume $\pi(q_i) = x_i \in B_d(\overline{x}, \epsilon^{-1}/20) \cap \mathcal{R}$ and $x_1 \neq x_2$. From Lemma 3.6, there exists a geodesic segment $\gamma_1(t)$ in $\mathcal{R}$ such that $\gamma_1(0) = x_1$ and $\gamma_1(1) = x_2$. It is clear that γ_1 is contained in $B_d(\overline{x}, \epsilon^{-1}/10)$. Hence, we can lift γ_1 to a curve $\widetilde{\gamma}_1$ in U' such that $\widetilde{\gamma}_1(0) = q_1$. If we set $\widetilde{\gamma}_1(1) = q_0$, then one easily derives that $N_{0,q_1} \overset{\circ}{=} N_{0,q_0}$ and hence $N_{0,q_1} \equiv N_{0,q_2}$. $\square$

Notice that the same conclusion also applies to any other factor N_i in the decomposition (3-10), provided that the size of N_i is much smaller than ϵ^{-1}.

Definition 3.10 Let (M_i, g_i, J_i, x_i) for $i = 1, 2$ be two pointed Kähler manifolds. (M_1, g_1, J_1, x_1) is said to be ϵ–close to (M_2, g_2, J_2, x_2) if there exist open neighborhoods U_i such that $B_{g_i}(x_i, \epsilon^{-1} - \epsilon) \subset U_i \subset \overline{U}_i \subset B_{g_i}(x_i, \epsilon^{-1})$ for $i = 1, 2$. Moreover, there exists a diffeomorphism $\varphi : U_1 \to U_2$ such that $\varphi(x_1) = x_2$ and

$$\sup_{U_1} \left(|\varphi^* J_2 - J_1|^2 + |\varphi^* g_2 - g_1|^2 + \sum_{i=1}^{[\epsilon^{-1}]} |\nabla_{g_1}^i(\varphi^* g_2)|^2 \right) \leq \epsilon^2.$$

Suppose a Kähler manifold (M, g, J, p) is ϵ–close to $((N^k \times \mathbb{C}^{m-k})/\Gamma, \overline{g}, \overline{J}, \overline{q}) \in \mathcal{N}(m, \kappa)$. Moreover, we denote their Kähler forms by ω and $\overline{\omega}$, respectively. From the diffeomorphism φ in Definition 3.10, we define from (3-11),

(3-13) $U = \varphi^{-1}(U'), \quad V = \varphi^{-1}(V'), \quad \widehat{V} = \varphi^{-1}(\widehat{V}'), \quad \widehat{V}_{\delta_1} = \varphi^{-1}(\widehat{V}'_{\delta_1}), \quad W = \varphi^{-1}(W').$

For any $p \in \widehat{V}$, there exists a submanifold Σ_p defined as $\Sigma_p = \varphi^{-1}(N_{0,\varphi(p)})$. Now, we prove:

Lemma 3.11 *There exists a function $\delta_2 = \delta_2(\epsilon, \kappa, m) > 0$, with $\lim_{\epsilon \to 0} \delta_2(\epsilon, \kappa, m) = 0$, such that for any $p \in \widehat{V}$,*

$$\left| \int_{\Sigma_p} \frac{\omega^L}{L!} - |N_0|_{\overline{g}} \right| \leq \delta_2.$$

Here, L is the dimension of N_0.

Proof It follows from a standard fact in Kähler geometry that for any $q \in \widehat{V}'$,

$$\int_{N_{0,q}} \frac{\overline{\omega}^L}{L!} = |N_0|_{\overline{g}},$$

since $N_{0,q}$ is a complex submanifold. Now the conclusion follows immediately from Definition 3.10 and the fact that $\omega(\cdot, \cdot) = g(J \cdot, \cdot)$. $\square$

It is clear from Lemma 3.9 that $\Sigma_{p_1} \equiv \Sigma_{p_2}$ for any $p_1, p_2 \in \widehat{V}$. Moreover, we have:

Lemma 3.12 *With the above assumptions, for any $p_1, p_2 \in \widehat{V}$,*

$$\int_{\Sigma_{p_1}} \frac{\omega^L}{L!} = \int_{\Sigma_{p_2}} \frac{\omega^L}{L!}.$$

Proof We only need to prove the case when $\Sigma_{p_1} \stackrel{\circ}{=} \Sigma_{p_2}$. From our definition, there exists an embedded submanifold $T \subset U$ such that $\partial T = \Sigma_{p_1} \cup \Sigma_{p_2}$ and T is diffeomorphic to $\Sigma_{p_1} \times [0, 1]$. From Stokes' theorem, and since ω is closed, we conclude

$$\int_{\Sigma_{p_2}} \omega^L - \int_{\Sigma_{p_1}} \omega^L = \int_T d\omega^L = 0. \qquad \square$$

Next, we have the following lemma similar to Lemma 2.13. In the proof we will add the subscript i to U, V, W, etc, to denote the corresponding sets in different charts.

Lemma 3.13 *For any m and $\kappa > 0$, there exists a number $\epsilon_1 = \epsilon_1(m, \kappa) > 0$ satisfying the following property.*

For a Kähler manifold (M, g, J), suppose (M, g, J, p_i) is ϵ–close to $((N_i^{k_i} \times \mathbb{C}^{m-k_i})/\Gamma_i, \overline{g}_i, \overline{J}_i, \overline{q}_i) \in \mathcal{N}(m, \kappa)$ for $i = 1, 2$, such that $W_1 \cap W_2 \neq \varnothing$ and $\epsilon \leq \epsilon_1$. Assume the decomposition of N_i is given by

$$N_i = N_{i,0} \times N_{i,1} \times \cdots \times N_{i,s_i}.$$

Then there exists $0 \leq j \leq s_2$ such that $N_{2,j}$, after a rescaling by a constant close to 1, is isometric to $N_{1,0}$.

Proof From the assumption, there exists a point $q \in \widehat{V}_{1,\delta_1} \cap \widehat{V}_{2,\delta_1}$. If we set $\psi = \varphi_2 \circ \varphi_1^{-1}$, then by using the local coordinates around q, we have

$$\psi : N_{1,0} \times X \times B_{g_E}(0, r_0) \longrightarrow (N_2 \times \mathbb{C}^{m-k_2})/\Gamma_2,$$

where X is a simply connected domain in $N_{1,0}^c$ such that X contains a geodesic ball of the radius $\epsilon^{-1}/100$. Now we lift ψ to $\widetilde{\psi}$ so that

$$\widetilde{\psi} : N_{1,0} \times X \times B_{g_E}(0, r_0) \longrightarrow N_2 \times \mathbb{C}^{m-k_2}.$$

By the same argument as in the proof of Lemma 2.13, if ϵ is sufficiently small, the map $\widetilde{\psi}$ is a smooth embedding, which is almost isometric. Therefore, there exists a factor $N_{2,j}$ in the decomposition of N_2 which is almost isometric to $N_{1,0}$. $\qquad \square$

By reordering the decomposition, we assume $N_{2,j} = N_{2,0}$. As discussed above, for any $p \in \widehat{V}_i$, there exists a submanifold $\Sigma_{i,p}$ containing p, which is a copy of $N_{i,0}$ for $i = 1, 2$. From Lemma 3.13, we know that $\Sigma_{1,q}$ is a small perturbation of $\Sigma_{2,q}$. Since $q \in \widehat{V}_{1,\delta_1} \cap \widehat{V}_{2,\delta_1}$, one can find a point q_1 close to q such that $\Sigma_{1,q_1} \stackrel{\circ}{=} \Sigma_{2,q}$, in which the connecting manifold T is easily constructed by the normal coordinates of $\Sigma_{2,q}$. Therefore, by the same proof of Lemma 3.12, we have the following result.

Lemma 3.14 *With the above assumptions, for any $p_i \in \hat{V}_i$ for $i = 1, 2$, we have*

$$\int_{\Sigma_{1,p_1}} \frac{\omega^L}{L!} = \int_{\Sigma_{2,p_2}} \frac{\omega^L}{L!}.$$

Now we prove the following proposition.

Proposition 3.15 *For any m and $\kappa > 0$, there exists a number $\epsilon_2 = \epsilon_2(m, \kappa) > 0$ satisfying the following property.*

Suppose (M^m, g, J, p_0) is a complete Kähler manifold such that $R(p_0) = 1$ and for any $p \in M$ with $R(p) \geq 10$, (M^m, g_p, J, p) is ϵ–close to an element in $\mathcal{N}(m, \kappa)$ and $\epsilon \leq \epsilon_2$, where $g_p = R(p)g$. Then for all $z \in M$,

$$R(z) \leq 20.$$

Proof In the proof, $\delta = \delta(\epsilon)$ represents a positive function such that $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$ and δ may be different line by line.

We prove the conclusion by contradiction. Suppose there exists $q_0 \in M$ with $R(q_0) > 20$. We consider a geodesic segment $\gamma(t)$ with $\gamma(0) = q_0$ and $\gamma(1) = p_0$. From the continuity, there exists an interval $[a, b] \subset [0, 1]$ such that $R(\gamma(a)) = 20$, $R(\gamma(b)) = 10$ and $R(\gamma(t)) \in [10, 20]$ for any $t \in [a, b]$.

By our assumption, for any $t \in [a, b]$, $(M^m, g_{\gamma(t)}, J, \gamma(t))$ is ϵ–close to an element in $\mathcal{N}(m, \kappa)$ and hence the corresponding sets $U, V, \hat{V}$ and W are defined. From the compactness, there exists a sequence $\{t_i\}_{i=1}^K \subset [a, b]$ with $p_i = \gamma(t_i)$ satisfying the following statements.

  (i) $p_1 = \gamma(a)$ and $p_K = \gamma(b)$.
 (ii) (M, g_{p_i}, J, p_i) is ϵ–close to $((N_i^{k_i} \times \mathbb{C}^{m-k_i})/\Gamma_i, \bar{g}_i, \bar{J}_i, \bar{q}_i) \in \mathcal{N}(m, \kappa)$ for all $1 \leq i \leq K$.
(iii) $W_i \cap W_{i+1} \neq \varnothing$ for any $1 \leq i \leq K - 1$.

From (ii) and (iii), we conclude that

$$\left| \frac{R(x)}{R(y)} - 1 \right| \leq \delta(\epsilon) \quad \text{for any } x \in U_i \text{ and } y \in U_{i+1}.$$

Hence g_{p_i} is δ–close to $g_{p_{i+1}}$ in $C^{[\delta^{-1}]}$–topology, on $U_i \cap U_{i+1}$. As discussed above, there exists an irreducible, simply connected, compact, Hermitian symmetric space $N_0 = N_0^L$ satisfying

(3-14) $$c_1 \leq |N_0|_{\bar{g}} \leq c_2$$

for two positive constants c_1 and c_2 depending only on m and κ; see (3-12). In addition, for any point $p \in \hat{V}_1$, there exists a submanifold $\Sigma_{1,p}$ containing p, which is a copy of N_0. Moreover, it follows from Lemma 3.12 that

$$\left| R^L(p_1) \int_{\Sigma_{1,p}} \frac{\omega^L}{L!} - |N_0|_{\bar{g}} \right| \leq \delta.$$

From Lemma 3.13, for any $p \in \widehat{V}_2$, we obtain a submanifold $\Sigma_{2,p}$ containing p, which is also a copy of N_0. By Lemma 3.14, we have

$$\int_{\Sigma_{1,p}} \frac{\omega^L}{L!} = \int_{\Sigma_{2,q}} \frac{\omega^L}{L!} \quad \text{for any } p \in \widehat{V}_1 \text{ and } q \in \widehat{V}_2.$$

By iteration, we can construct $\Sigma_{i,p}$ for any $p \in \widehat{V}_i$ and $1 \leq i \leq K$. Moreover, we have

(3-15)
$$\left| R^L(p_i) \int_{\Sigma_{i,p}} \frac{\omega^L}{L!} - |N_0|_{\overline{g}} \right| \leq \delta \quad \text{for any } p \in \widehat{V}_i \text{ and } 1 \leq i \leq K.$$

In addition,

(3-16)
$$\int_{\Sigma_{i,p}} \frac{\omega^L}{L!} = \int_{\Sigma_{i+1,q}} \frac{\omega^L}{L!} \quad \text{for any } p \in \widehat{V}_i, \ q \in \widehat{V}_{i+1} \text{ and } 1 \leq i \leq K-1.$$

Now we fix $x_1 \in \widehat{V}_1$ and $x_K \in \widehat{V}_K$. Combining (3-14), (3-15) and (3-16), we have

$$(20^L - 10^L)A \leq 2\delta \quad \text{and} \quad 10^L A \geq c_1 - \delta,$$

where

$$A = \int_{\Sigma_{1,x_1}} \frac{\omega^L}{L!} = \int_{\Sigma_{K,x_K}} \frac{\omega^L}{L!}.$$

Therefore, we obtain a contradiction if ϵ is sufficiently small. $\square$

The same proof of Proposition 3.15 yields the following more general result.

Theorem 3.16 *For any m, $\kappa > 0$ and $\delta > 0$, there exists a number $\overline{\epsilon} = \overline{\epsilon}(m, \kappa, \delta) > 0$ satisfying the following property.*

Let (M^m, g, J) be a complete Kähler manifold such that for any $p \in M$ with $R(p) \geq 1$, (M^m, g_p, J, p) is ϵ–close to an element in $\mathcal{N}(m, \kappa)$ and $\epsilon \leq \overline{\epsilon}$, where $g_p = R(p)g$. If there exists $p_0 \in M$ with $R(p_0) \geq 2$, then for all $z \in M$,

$$\left| \frac{R(z)}{R(p_0)} - 1 \right| \leq \delta.$$

Proof The proof follows verbatim from Proposition 3.15 by replacing 20 with $\max\{R(z), R(p_0)\}$ and 10 with $(1 - \delta)\max\{R(z), R(p_0)\}$. $\square$

Now, we can prove Theorem 1.1 following the same argument in [24, Section 5]. We first recall the following result from [40, Corollary 2.1(b)], which originates from [42, Corollary 11.6].

Lemma 3.17 *For every $w > 0$, there exist constants $C = C(w) < \infty$ and $\tau = \tau(w) > 0$ with the following properties. Let $(M^m, g(t))_{t \in [-T,0]}$ be a (possibly incomplete) Kähler Ricci flow solution with $BK \geq 0$. Suppose $B_{g(0)}(x_0, r_0)$ is compactly contained in M such that $|B_{g(0)}(x_0, r_0)| \geq wr_0^{2m}$ and $T \geq 2\tau r_0^2$. Then*

$$R(x, t) \leq Cr_0^{-2} \quad \text{for } (x, t) \in B_{g(0)}\left(x_0, \tfrac{1}{4}r_0\right) \times [-\tau r_0^2, 0].$$

Next, we have the following canonical neighborhood theorem.

Theorem 3.18 Let $(M^m, g(t))_{t \in [0,2]}$ *be a complete noncompact* κ*–noncollapsed Kähler Ricci flow solution with* $\mathrm{BK} \geq 0$. *For any* $\epsilon > 0$, *there exists a small number* $\bar{r} > 0$ *satisfying the following property.*

Suppose $(\bar{x}, \bar{t}) \in M \times [1, 2]$ *and* $R(\bar{x}, \bar{t}) = r^{-2} \geq \bar{r}^{-2}$, *then after rescaling the metric by the factor* r^{-2}, *the parabolic neighborhood* $B_{g(\bar{t})}(\bar{x}, \epsilon^{-1} r) \times [\bar{t} - \epsilon^{-1} r^2, \bar{t}]$ *is* ϵ*–close in* $C^{[\epsilon^{-1}]}$*–topology to a* κ*–solution.*

Proof The proof is similar to the proof of [24, Theorem 5.4], and we sketch it for readers' convenience.

Assume that there exists an $\bar{\epsilon} > 0$ such that the conclusion does not hold for a sequence $(x_i, t_i) \in M \times [1, 2]$ with $Q_k = R(x_k, t_k) \to \infty$. By a point-picking argument, we can assume that for any $A > 0$ and any $(y, t) \in B_{g(t_k)}(x_k, A Q_k^{-1/2}) \times [t_k - A Q_k^{-1}, t_k]$ with $R(y, t) \geq 2 Q_k$, the conclusion of the theorem holds.

Next, we consider the spacetime limit of $(M, g_k(t), x_k)$ for $t \leq 0$, where $g_k(t) = Q_k g(Q_k^{-1} t + t_k)$. From Lemma 3.17, one can show that the limit $(M_\infty, g_\infty, x_\infty)$ of $(M, g_k(0), x_k)$ is a complete Kähler manifold with $\mathrm{BK} \geq 0$ and is κ–noncollapsed; see [24, Theorem 5.4, Step 2] for details.

By our assumption, $(M_\infty, g_\infty, x_\infty)$ satisfies the assumption of Proposition 3.15, if $\bar{\epsilon}$ is sufficiently small. Therefore, we conclude that he curvature of the limit $(M_\infty, g_\infty, x_\infty)$ is uniformly bounded. Now, $(M_\infty, g_\infty(t))$ can be extended backward to an ancient solution with uniformly bounded curvature; see [24, Theorem 5.4, Step 4]. In other words, the limit $(M_\infty, g_\infty(t))_{t \in (-\infty, 0]}$ is a κ–solution to the Kähler Ricci flow.

In sum, we obtain a contradiction, and the proof is complete.   □

Now we can prove the boundedness of the curvature.

Proposition 3.19 Let $(M^m, g(t))_{t \in (-\infty, 0]}$ *be a* κ*–noncollapsed, noncompact, complete ancient solution to the Kähler Ricci flow with* $\mathrm{BK} \geq 0$. *Then the curvature of* $(M, g(t))_{t \in (-\infty, 0]}$ *is uniformly bounded.*

Proof The proof is similar to [24, Proposition 5.6], and we sketch it for readers' convenience.

From Theorem 3.3, we know that $(M, g(t))$ has weakly PIC_2. We first prove that for any $t_0 \leq 0$, the curvature at the time t_0 is uniformly bounded. Otherwise, there exists a sequence p_i such that $Q_i = R(p_i, t_0) \to \infty$. By applying Theorem 3.18 on $M \times [t_0 - 2, t_0]$, we conclude that $(M, Q_i g(t_0), p_i)$ converges smoothly to a κ–solution. However, this contradicts Proposition 3.15.

Since $(M, g(t))$ has bounded curvature on each time slice, it has bounded curvature on each compact time interval by [24, Lemma 5.5]. Therefore the trace Harnack inequality holds (see [16]), and hence the curvature is uniformly bounded since R is nondecreasing along t.   □

Proof of Theorem 1.1 It follows immediately from Theorem 3.4 and Proposition 3.19.   □

From [33, Theorem 1.3(i)], we know that any complete ancient solution to the Kähler Ricci flow with nonnegative orthogonal bisectional curvature automatically has BK ≥ 0. Moreover, a Kähler manifold has nonnegative orthogonal bisectional curvature if it has weakly PIC; see [7, Proposition 9.18]. Therefore, the following corollary is immediate from Theorem 1.1.

Corollary 3.20 *Let* $(M^m, g(t))_{t \in (-\infty, 0]}$ *be a* κ*–noncollapsed, complete ancient solution to the Kähler Ricci flow with nonnegative orthogonal bisectional curvature or weakly* PIC. *Then it is isometrically biholomorphic to a finite quotient of* $N^k \times \mathbb{C}^{m-k}$, *where* N *is a simply connected, compact, Hermitian symmetric space.*

4 Further discussion

In this section, we propose the following conjecture, of which Theorem 1.2 is a special case.

Conjecture 4.1 *Let* $(M^n, g(t))_{t \in (-\infty, 0]}$ *be a* κ*–noncollapsed, Type I, irreducible, complete ancient solution to the Ricci flow. Then it is isometric to a Ricci shrinker, up to scaling.*

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