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When does the zero fiber of the moment map have rational singularities?

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Let G be a complex reductive group and V a G -module. There is a natural moment mapping $\mu : V \oplus V^* \rightarrow \mathfrak{g}^*$ and we denote $\mu^{-1}(0)$ (the shell) by N_V . We use invariant theory and results of Mustață (Invent. Math. 145 (2001) 397–424) to find criteria for N_V to have rational singularities and for the categorical quotient $N_V // G$ to have symplectic singularities, the latter results improving upon our earlier work (Herbig et. al., Compos. Math. 156 (2020) 613–646). It turns out that for “most” G -modules V , the shell N_V has rational singularities. For the case of direct sums of classical representations of the classical groups, N_V has rational singularities and $N_V // G$ has symplectic singularities if N_V is a reduced and irreducible complete intersection. Another important special case is $V = p\mathfrak{g}$ (the direct sum of p copies of the Lie algebra of G) where $p \geq 2$. We show that N_V has rational singularities and that $N_V // G$ has symplectic singularities, improving upon previous results of Aizenbud–Avni, Budur, Glazer–Hendel, and Kapon. Let $\pi = \pi_1(\Sigma)$, where Σ is a closed Riemann surface of genus $p \geq 2$. Let G be semisimple and let $\text{Hom}(\pi, G)$ and $\mathcal{X}(\pi, G)$ be the corresponding representation variety and character variety. We show that $\text{Hom}(\pi, G)$ is a complete intersection with rational singularities and that $\mathcal{X}(\pi, G)$ has symplectic singularities. If $p > 2$ or G contains no simple factor of rank 1, then the singularities of $\text{Hom}(\pi, G)$ and $\mathcal{X}(\pi, G)$ are in codimension at least four and $\text{Hom}(\pi, G)$ is locally factorial. If, in addition, G is simply connected, then $\mathcal{X}(\pi, G)$ is locally factorial.

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1 Introduction

Let G be a reductive complex group and V a G -module. Then there is a natural G -invariant symplectic form ω on $V \oplus V^*$ such that V and V^* are isotropic subspaces. Let \mathfrak{g} denote the Lie algebra of G . Then there is a canonical moment mapping $\mu: V \oplus V^* \simeq T^*V \rightarrow \mathfrak{g}^*$. If $(v, v^*) \in V \oplus V^*$ and $A \in \mathfrak{g}$, then $\mu(v, v^*)(A) = v^*(A(v))$. Let N_V denote $\mu^{-1}(0)$, which we call *the shell*. In [Herbig et al. 2020] we found criteria for $N_V // G$ to have symplectic singularities. In some examples, eg, if G is a torus, proving that $N_V // G$ had symplectic singularities involved first proving that N_V had rational singularities. This was also established in [Cape et al. 2016] for $G = \mathrm{SO}_n(\mathbb{C})$ and $V = k\mathbb{C}^n$, $k \geq n$. The main theorem of [Budur 2021] proves that N_V has rational singularities when $V = p\mathfrak{g}$, $p \geq 2$, and $G = \mathrm{GL}_n(\mathbb{C})$. Aizenbud and Avni [2016] prove the same for G semisimple, but require larger p . The bounds were improved in [Kapon 2019] and to $p \geq 4$ in [Glazer and Hendel 2024]. This all sparked our interest in determining when a general N_V has rational singularities.

Budur uses work of Mustařă [2001] which gives a criterion for a local complete intersection variety to have rational singularities in terms of its jet schemes. We use this criterion throughout our paper. A first consequence is:

Theorem A (Corollary 3.7, Remark 3.8) *Let G be semisimple and consider G -modules V such that $V^G = 0$ and each irreducible factor of V is an almost faithful G -module. Then there are only finitely many isomorphism classes of such G -modules such that N_V does not have rational singularities.*

The theorem above, however, is not very useful when one is presented with a specific G and V .

We say that N_V has *FPIG* (finite principal isotropy groups) if it has an open dense subset of closed orbits with finite isotropy group. When N_V is a complete intersection with FPIG and rational singularities, we say that N_V is *CIFR*. We use [Mustařă 2001] and the symplectic slice theorem of [Herbig et al. 2020] to find criteria for N_V to be CIFR in terms of the symplectic slice representations of N_V . For more about the following see the discussion preceding Theorem 2.2. Let $x \in N_V$ such that Gx is closed. Then the isotropy group $H = G_x$ is reductive and we have a symplectic slice representation $H \rightarrow \mathrm{GL}(S)$, where $S \subset V \oplus V^*$ is an H -submodule and the restriction ω_S of ω to S is nondegenerate (and H -invariant). We can decompose S as $S^H \oplus S_0$ where ω_S is nondegenerate on S_0 and admits Lagrangian H -submodules, ie, isotropic submodules of dimension $\frac{1}{2} \dim S_0$. Let W_0 be such a submodule and let $N_0 := N_{W_0}$ denote the shell of $W_0 \oplus W_0^*$. The shell N_0 only depends upon the restriction of ω_S to S_0 and not the choice of W_0 (Lemma 2.1). Let $\mathcal{N}(S_0)$ denote the null cone of S_0 , the union of the H -orbits with 0 in their closure, and let $\mathcal{N}(N_0) = N_0 \cap \mathcal{N}(S_0)$. Let $(N_0)_m$ denote the m^{th} jet scheme of N_0 and let $\rho_m: (N_0)_m \rightarrow N_0$ denote the natural projection; see Section 2.4. Let $(N_0)'_m$ denote the closure of $\rho_m^{-1}(N_0 \setminus (\mathcal{N}(N_0) \cap (N_0)_{\mathrm{sing}}))$, where $(N_0)_{\mathrm{sing}}$ denotes the singular points of N_0 . We say that $\mathcal{N}(N_0)$ is *irrelevant* if $(N_0)'_m = (N_0)_m$ for all $m \geq 1$.

We then have the following criterion for the shell to have rational singularities.

Theorem B (Theorem 2.27) *Let V be a G -module and $N = N_V$. Assume that N is a normal complete intersection with FPIG. Then N has rational singularities, hence is CIFR, if and only if $\mathcal{N}(N_0)$ is irrelevant for every symplectic slice representation (S, H) of N .*

For many cases of G and V , the dimensions of the $\rho_m^{-1}(\mathcal{N}(N_0))$ are already small enough to establish the irrelevance of the $\mathcal{N}(N_0)$, and we frequently use this approach. The main novel technique of this paper is to use bounds on the dimension of linear subspaces of the null cone $\mathcal{N}(W_0)$ to show that $\rho_m^{-1}(\mathcal{N}(N_0))$ is nowhere dense in the jet scheme $(N_0)_m$; this implies the irrelevance of $\mathcal{N}(N_0)$ and hence that N has rational singularities by Theorem B. More specifically, let $m_0(W_0)$ denote the maximal dimension of a linear subspace of $\mathcal{N}(W_0)$. One of our main results is:

Theorem C (see Theorems 2.11 and 3.14) *Let V be a G -module and suppose that for all symplectic slice representations $(S = S^H \oplus S_0, H)$ of N_V where $\dim H > 0$ there is a choice of Lagrangian H -submodule W_0 of S_0 such that either*

- (1) *the group H^0 is a torus and W_0 has FPIG, or*
- (2) *$m_0(W_0) < \dim W_0 - \dim H$.*

Then each $\mathcal{N}(N_0)$ is irrelevant and N_V is CIFR.

In general, the calculation of $m_0(W_0)$ is difficult. However, if W_0 is an orthogonal H -module, we may replace (2) by

$$(2') \quad \dim H < \frac{1}{2}(\dim W_0 - \dim W_0^T), \text{ where } T \text{ is a maximal torus of } H.$$

This follows from the following general fact.

Proposition D (Proposition 3.15) *Let V be an orthogonal G -module. Then*

$$m_0(V) = \frac{1}{2}(\dim V - \dim V^T),$$

where T is a maximal torus of G .

If $V = \mathfrak{g}$ where G is simple, then the proposition says that $m_0(\mathfrak{g})$ is the dimension of a maximal nilpotent subalgebra of \mathfrak{g} . This result is due to Gerstenhaber [1958] for \mathfrak{sl}_n and Meshulam and Radwan [1998] in general. See also the work of Draisma, Kraft and Kuttler [Draisma et al. 2006]. This paper provided a key idea in our proof of Proposition D. If V is an orthogonal G -module, then in Theorem C we may always choose W_0 to be an orthogonal H -module (Lemma 3.16 and Proposition 3.17).

Condition (2) of Theorem C is stronger than the equivalence given in Theorem B; for instance, the SL_2 -module $V = 3C^2$ does not satisfy (2) yet N_V is CIFR. However, Theorem C along with condition (2') in the orthogonal case (and some ad hoc arguments) is sufficient to establish that the classical representations of the classical groups, as well as several other important cases of G and V , have shells that are CIFR whenever the shells are complete intersection varieties, ie, whenever Mustața's criterion for rational singularities can even be applied; see Section 6. In addition, it allows us to handle the case of two or

more copies of the adjoint representation, yielding the following theorem. We say that a complex affine variety is *factorial* if $\mathbb{C}[X]$ is a UFD.

Theorem E (Theorem 4.7) *Assume that G is semisimple and let $V = \mathfrak{p}\mathfrak{g}$ where $p \geq 2$. Then:*

- (1) N_V is CIFR.
- (2) If $p > 2$ or G contains no simple factor of rank 1, then N_V and $N_V // G$ are factorial.

Theorem F (see Theorem 5.2) *Let V be as in Theorem C or E. Then $N_V // G$ is graded Gorenstein with symplectic singularities.*

Representations of quivers have associated moment mappings as well, and our criteria can be applied to show that the corresponding zero fibers have rational singularities in many cases.

Let $\pi = \pi_1(\Sigma)$, where Σ is a compact Riemann surface of genus $p > 1$. Let G be semisimple, let $\text{Hom}(\pi, G)$ be the corresponding representation variety, and let $\mathcal{X}(\pi, G)$ be the corresponding character variety. Let $\rho_0 \in \text{Hom}(\pi, G)$ denote the trivial homomorphism. We use Theorem E to show that the tangent cone to $\text{Hom}(\pi, G)$ at a closed G -orbit has rational singularities and we establish:

Theorem G (Theorem 7.1) *Let G and π be as above.*

- (1) $\text{Hom}(\pi, G)$ is CIFR and each irreducible component has dimension $(2p - 1) \dim G$.
- (2) $\mathcal{X}(\pi, G)$ has symplectic singularities and each irreducible component has dimension $(2p - 2) \dim G$.

Now suppose that $p > 2$ or that every simple component of G has rank at least 2.

- (3) *The singularities of $\text{Hom}(\pi, G)$ are in codimension at least four and $\text{Hom}(\pi, G)$ is locally factorial.*
- (4) *The singularities of $\mathcal{X}(\pi, G)$ are in codimension at least four and the irreducible component containing $G\rho_0$ is locally factorial. In particular, if G is simply connected, then $\mathcal{X}(\pi, G)$ is locally factorial.*

As in [Budur 2021, Theorem 1.10], using results of Simpson [1994, page 69], Theorem G shows that the Betti, de Rham and Dolbeault representation spaces of principal G -bundles on our Riemann surface Σ have rational singularities and that the corresponding moduli spaces have symplectic singularities. Here G is semisimple. We also prove a version of Theorem G that handles the case of G reductive (Theorem 7.4). Compare [Lawton and Manon 2016, Theorem 1.1] for similar results in the case when π is a free group as well as the related work [Kapon 2019] on graph varieties and [Glazer and Hendel 2024] on word maps.

For a topological group Γ , let $r_n(\Gamma)$ denote the number of n -dimensional irreducible continuous complex representations of Γ (the *growth sequence of Γ*) and let $R_n(\Gamma) = \sum_{i=1}^n r_i(\Gamma)$. The *representation zeta function* $\zeta_\Gamma(s)$ of Γ is given by

$$\zeta_\Gamma(s) = \sum_{i=1}^{\infty} r_n(\Gamma) n^{-s}.$$

The abscissa of convergence of $\zeta_\Gamma(s)$ is

$$\alpha(\Gamma) := \limsup_{n \rightarrow \infty} \frac{\log R_n(\Gamma)}{\log n}.$$

Then $\zeta_\Gamma(s)$ converges absolutely for $\{\operatorname{Re}(s) > \alpha(\Gamma)\}$.

We now apply some remarkable results of Aizenbud and Avni relating growth sequences of groups and rational singularities of varieties $\operatorname{Hom}(\pi, G)$. From Theorem G(1) and [Aizenbud and Avni 2016, Theorem IV] we have:

Theorem H *Let \mathbb{k} be a finitely generated field of characteristic zero. Let G be a semisimple group defined over \mathbb{k} and \mathbb{F} a local field containing \mathbb{k} . Let Γ be a compact open subgroup of $G(\mathbb{F})$. Then $\alpha(\Gamma) < 2$.*

The above result is in [Aizenbud and Avni 2016; 2018] with a larger bound on $\alpha(\Gamma)$ that was improved in [Glazer and Hendel 2024, Corollary 1.9]. For $G = \operatorname{SL}_n$, the estimate above was established by Budur [2021, Theorem 1.7]. Similarly, Aizenbud and Avni give bounds for abscissae related to arithmetic subgroups of high rank semisimple groups. Theorem G allows one to improve these bounds. For example, one has the following version of [Aizenbud and Avni 2018, Theorem B] (established for SL_n in [Budur 2021, Theorem 1.4]).

Theorem I *Let G be an affine group scheme over \mathbb{Z} whose generic fiber $G_{\mathbb{Q}}$ is (almost) \mathbb{Q} -simple, connected, simply connected and of \mathbb{Q} -rank at least 2. Then $\alpha(G(\mathbb{Z})) \leq 2$.*

Remark 1.1 Suppose that Γ is one of the groups considered above and $\alpha(\Gamma) \leq 2$. Then using a generalization of Faulhaber's formula [McGown and Parks 2007] one gets estimates on the growth of $r_n(\Gamma)$; namely, $r_n(\Gamma) = O(n^{1+\epsilon})$ for every $\epsilon > 0$. If $\alpha(\Gamma) < 2$, one gets that $r_n(\Gamma) = O(n^{1-\epsilon})$ for some $\epsilon > 0$.

The outline of this paper is as follows. In Section 2 we provide background material as well as criteria for N_V to be a normal complete intersection with FPIG. We introduce Mustařa's criterion for rational singularities and establish Theorem B. In Section 3 we study the jet schemes of N_V and establish Theorems A and C. In Section 4 we apply our results to the case of copies of the adjoint representation and establish Theorem E. In Section 5 we establish Theorem F. In Section 6 we apply Theorem C to classical representations of the classical groups. In Section 7 we consider applications to representation varieties and character varieties and establish Theorem G. The appendix is devoted to showing that certain maps arising in Section 7 are moment mappings.

We use some standard and not so standard abbreviations in this paper. Particularly important are three already mentioned in the introduction, which we summarize here. The notion of a G -variety having FPIG (finite principal isotropy groups) is defined in Section 2.1, the notion of a G -module having some property UTCLS (up to choosing a Lagrangian submodule) is given in Definition 2.7, and a G -variety is CIFR if it is a complete intersection with FPIG and rational singularities.

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2 Background

2.1 G -modules and the shell

Let G be a reductive complex group and X an affine G -variety. (We assume that varieties are reduced but not necessarily irreducible except in Section 2.4.) The categorical quotient $Z = X//G$ is the affine variety with coordinate ring $\mathbb{C}[X]^G$ and $\pi : X \rightarrow Z$, the quotient mapping, is dual to the inclusion $\mathbb{C}[X]^G \subset \mathbb{C}[X]$. The categorical quotient parametrizes the closed orbits in X . A subset of X is said to be G -saturated if it is a union of fibers of π . If $x \in X$ and Gx is closed, then the isotropy group G_x is reductive. For a reductive subgroup H of G , we let $Z_{(H)}$ denote the set of closed orbits in X whose isotropy groups are in the conjugacy class (H) of H . We write $(H) < (H')$ if H is conjugate to a proper subgroup of H' . The $Z_{(H)}$ are called *isotropy strata* of Z . They are locally closed in the Zariski topology. Assume that Z is irreducible. Then there is an open dense stratum Z_{pr} , the *principal stratum*. Corresponding closed orbits are called *principal orbits* and any corresponding reductive subgroup H of G is called a *principal isotropy group*. We let X_{pr} denote $\pi^{-1}(Z_{\text{pr}})$. If $Z_{(H')}$ is not empty, then $(H) \leq (H')$, ie, a principal isotropy group H is conjugate to a subgroup of H' . The G -action on X is *stable* if $\pi^{-1}(Z_{\text{pr}})$ consists of closed orbits. We say that X has FPIG if the principal isotropy groups are finite.

Let V be a G -module and $x \in V$ such that Gx is closed. Let $H = G_x$. Then, as an H -module, $V = T_x(Gx) \oplus W$ where W is an H -module, which is called the *slice representation of H* [Luna 1973]. Since $T_x(Gx) \simeq \mathfrak{g}/\mathfrak{h}$ where \mathfrak{h} denotes the Lie algebra of H , W is completely determined by H .

Let V be a G -module and set $U = V \oplus V^*$. Then the *standard symplectic form ω on U* and *standard moment mapping $\mu : U \rightarrow \mathfrak{g}^*$* are given by

$$\begin{aligned} \omega((v, v^*), (w, w^*)) &= w^*(v) - v^*(w), & (v, v^*), (w, w^*) &\in V \oplus V^*, \\ \mu(v, v^*)(A) &= v^*(A(v)), & (v, v^*) &\in V \oplus V^*, A \in \mathfrak{g}. \end{aligned}$$

Of course, moment mappings are only unique up to a constant in $\text{Ann}[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}^*$, so our μ is standard in the sense that $\mu(0) = 0$.

Lemma 2.1 *Let (U, ω) be a symplectic G -module with a moment mapping vanishing at $0 \in U$. If U admits a Lagrangian G -submodule V , then there is an isomorphism $U \simeq V \oplus V^*$ under which the symplectic form and moment mapping become standard.*

Proof Since V is Lagrangian, ω induces a nondegenerate pairing $V \times (U/V) \rightarrow \mathbb{C}$ so that $U/V \simeq V^*$. Since G is reductive, $U \simeq V \oplus V^*$ where V^* is Lagrangian and the usual pairing of V and V^* is induced by ω . With U viewed as $V \oplus V^*$, ω and μ are standard. \square

The shell N_V of V , which we denote simply N if V is clear from the context, is the fiber $\mu^{-1}(0) \subset U$. The shell may be neither reduced nor irreducible. In [Herbig et al. 2020] we called N_V the *complex shell* since we were also dealing with real moment mappings. In this paper we only deal with the complex case. A big part of our task is to deduce properties of N_V from those of V .

We now recall (a weak form of) the symplectic slice theorem of [Herbig et al. 2020, Section 3.4]. Let $x \in N_V$ such that Gx is closed. The isotropy group $H = G_x$ is reductive and we let E denote $T_x(Gx)$. Then E is isotropic and ω is nondegenerate on E^\perp/E , where \perp denotes perpendicular with respect to ω . Since H is reductive, there is an H -module S such that $E^\perp = S \oplus E$ and $U \simeq E \oplus E^* \oplus S$ where ω induces a symplectic form ω_S on S .

Note that H determines S since $E \simeq \mathfrak{g}/\mathfrak{h} \simeq E^*$. By [Herbig et al. 2020, Lemma 3.10], S admits a Lagrangian H -submodule W so that ω_S is standard on $S \simeq W \oplus W^*$. Let N_W denote the corresponding shell in S .

Recall that if $\phi: X \rightarrow Y$ is a G -equivariant mapping of G -varieties, then for any $x \in X$, $G_x \subset G_{\phi(x)}$. The mapping ϕ is *isovariant* if it is equivariant and $G_x = G_{\phi(x)}$ for all $x \in X$.

Theorem 2.2 (symplectic slice theorem) *There is an H -saturated affine neighborhood Q of $0 \in E^* \oplus S$ and an étale G -isovariant mapping $\phi: G \times^H Q \rightarrow V \oplus V^*$ where $[e, 0]$ is sent to x . The image of ϕ is G -saturated and open. Pulling back the standard symplectic form and moment mapping on $V \oplus V^*$ by ρ , the shell of $G \times^H Q$ is $G \times^H (Q \cap (\{0\} \times N_W))$. The induced mapping*

$$\rho^{-1}(N_V) \simeq G \times^H (Q \cap (\{0\} \times N_W)) \rightarrow N_V$$

is an étale mapping of affine schemes and is the restriction of a G -isovariant map.

Remark 2.3 The theorem in [Herbig et al. 2020] is stated in the case that N_V is an irreducible variety but the proof does not need this and implies the version above.

Corollary 2.4 *Let $Gx \subset N_V$ be a closed orbit with symplectic slice representation (S, H) . Then*

$$\dim_x N_V = \dim G - \dim H + \dim_0 N_W.$$

Corollary 2.5 *Let (P) be one of the following conditions: reduced, smooth, normal or rational singularities. Then N_V satisfies (P) at x if and only if N_W satisfies (P) at 0.*

Remark 2.6 Let (P) be one of conditions normal or rational singularities. Suppose that an affine G -variety X satisfies (P) along every closed orbit. Let $\Omega = \{x \in X \mid (P) \text{ holds at } x\}$. Then Ω is open and G -stable. If $\Omega \neq X$, then Ω^c is G -stable and closed, hence contains a closed G -orbit, which is a contradiction. Hence X satisfies (P).

2.2 Conditions on affine G -varieties

Let X be an affine G -variety whose quotient Z is irreducible. For $k \geq 0$, we say that X is k -principal if $\text{codim } X \setminus X_{\text{pr}} \geq k$. Let $X_{(r)}$ denote the locally closed set of points with isotropy group of dimension r . We say that X is k -modular if $\text{codim}_X X_{(r)} \geq r + k$ for $1 \leq r \leq \dim G$. Note, in particular, if X is k -modular, then $X_{(0)} \neq \emptyset$. We say that X is k -large if it is k -principal, k -modular, and has FPIG. See [Schwarz 1995] for more background on these concepts, and note that references sometimes differ about whether FPIG is required as part of the definition of k -modular or k -principal. We say that a G -module V is *orthogonal* if V admits a nondegenerate symmetric G -invariant bilinear form. Note that if V is orthogonal, then it is stable [Luna 1973].

Definition 2.7 Let V be a G -module and $U = V \oplus V^*$ with the standard symplectic form. We say that V has property (P) *UTCLS (up to choosing a Lagrangian submodule)* if there is a Lagrangian G -submodule V' of U with property (P).

By Lemma 2.1, $U \simeq V' \oplus (V')^*$ where the shell remains the same.

We recall the following; see [Schwarz 1995, Proposition 9.4; Herbig et al. 2020, Proposition 3.2], and for (6), [Panyushev 1994, Theorem 2.4; Avramov 1981, Proposition 6].

Proposition 2.8 *Let $U = V \oplus V^*$ with the standard symplectic form and shell $N = N_V$. Let*

$$R = \{x \in U \mid G_x \text{ is finite}\}.$$

- (1) *The shell N is a complete intersection (hence Cohen–Macaulay) if and only if V is 0-modular if and only if $\dim N = 2 \dim V - \dim G$.*

Now assume that V is 0-modular.

- (2) *The set R equals $\{x \in U \mid d\mu \text{ has maximal rank at } x\}$.*
- (3) *The set of smooth points N_{sm} of N is $N \cap R$.*
- (4) *The shell is normal if and only if $N \setminus R$ has codimension at least two in N .*
- (5) *The shell is reduced and irreducible if and only if V is 1-modular.*
- (6) *The shell is factorial if and only if V is 2-modular.*

Remark 2.9 Let V' be a Lagrangian G -submodule of U . Then the proposition and Lemma 2.1 show that V is k -modular if and only if V' is k -modular for $k \leq 2$. This holds for any $k \geq 0$ by [Panyushev 1994, Theorem 2.4] which shows that k -modularity of V is equivalent to a homological condition on $\mathbb{C}[N_V]$. However, this is not the case for the properties of stable, k -principal, k -large, or orthogonal as illustrated by the following example. See also the examples in Section 6.

Example 2.10 Let V be the \mathbb{C}^\times -module with weight vector $(1, 1)$. Then as $V_{(1)} = \{0\}$, V is 1-modular. It is clear that V is not orthogonal, and as the only closed orbit in V is the origin, V is not stable and does not have FPIG, so is not 1-large. However, $V \oplus V^*$ has weight vector $(1, 1, -1, -1)$ and hence is isomorphic to $V' \oplus (V')^*$, where V' is a Lagrangian submodule with weight vector $(1, -1)$. As V' is stable, orthogonal, and 1-large, V is stable, orthogonal and 1-large UTCLS.

In the real case (see [Herbig et al. 2020, Section 2]), changing the Lagrangian submodule can drastically change the (real) shell.

Torus actions often have shells which are CIFR [Herbig et al. 2020, Proposition 5.4]. Note that N_V depends only on G^0 and if V has FPIG, then it is stable.

Theorem 2.11 *Let V be a G -module with FPIG where G^0 is a torus. Then the shell $N_V \subset V \oplus V^*$ is CIFR.*

Remark 2.12 If G^0 is a torus and V is a 1-modular G^0 -module with finite kernel, then it can be shown that, UTCLS, V has FPIG. Hence, the hypotheses of Theorem 2.11 can be relaxed to assume that G^0 is a torus and V is a 1-modular G -module with finite kernel. This is elaborated in the forthcoming paper [Herbig et al. 2023].

2.3 Normality and FPIG

Let V be a G -module and $U = V \oplus V^*$ with the canonical symplectic structure and moment mapping. The null cones $\mathcal{N}(V)$ and $\mathcal{N}(U)$ are the unions of the G -orbits with closure containing the origin. Note that $N = N_V$, $\mathcal{N}(V)$ and $\mathcal{N}(U)$ only depend upon V as a G^0 -module. We will use the following construction and notation throughout the paper. Let (S, H) be a symplectic slice representation of N with the induced symplectic form ω_S . Write $S = S^H \oplus S_0$. Then ω_S is nondegenerate on S^H and S_0 . There is a Lagrangian H -submodule $W = W^H \oplus W_0$ of S where $W^H \subset S^H$ and $W_0 \subset S_0$ are Lagrangian. Let $N_0 = N_{W_0}$ (resp. N_W) denote the shell in S_0 (resp. S). Then $N_W = S^H \times N_0$. Note that any irreducible component of N_0 has dimension at least $\dim S_0 - \dim H$. Let $\mathcal{N}(N_0) = \mathcal{N}(S_0) \cap N_0$ and let $\mathcal{N}(N_0)_{\text{sg}}$ denote $\mathcal{N}(N_0) \cap (N_0)_{\text{sing}}$, the points in $\mathcal{N}(N_0)$ which are singular points of N_0 .

Definition 2.13 (property (F)) We say that a symplectic slice representations (S, H) of N has property (F) if H is finite or $\dim \mathcal{N}(N_0) < \dim S_0 - H$.

Proposition 2.14 *The following are equivalent:*

- (1) *Every symplectic slice representation of N has property (F).*
- (2) *The shell N is a reduced complete intersection and each irreducible component has FPIG.*

Proof Suppose that (1) holds. Let (S, H) be a symplectic slice representation of N where $\dim H > 0$. Let N_{red} denote N with its reduced structure and let $\pi: N_{\text{red}} \rightarrow Z$ denote the quotient by G . By the symplectic slice theorem, $Z_{(H)}$ and S^H have the same dimension and the fibers of π over $Z_{(H)}$ are isomorphic to $G \times^H \mathcal{N}(N_0)$. It follows that

$$\begin{aligned} \dim \pi^{-1}(Z_{(H)}) &= \dim S^H + \dim G - \dim H + \dim \mathcal{N}(N_0) \\ &< \dim S + \dim G - 2 \dim H \\ &= 2 \dim V - 2 \dim G/H + \dim G - 2 \dim H \\ &= 2 \dim V - \dim G. \end{aligned}$$

Let N' be an irreducible component N . Then $\dim N' \geq 2 \dim V - \dim G > \dim \pi^{-1}(Z_{(H)})$. This inequality holds for all strata $Z_{(H)}$ where $\dim H > 0$. Thus N' contains a closed orbit Gx with G_x finite. It follows that $\dim_x N' = 2 \dim V - \dim G$. Then by Proposition 2.8, x is a smooth point of N' , N is a reduced complete intersection and (2) holds.

Conversely, suppose that (2) holds while (1) fails. Then there is a stratum $Z_{(H)}$ where $\dim H > 0$ and $\dim \pi^{-1}(Z_{(H)}) = 2 \dim V - \dim G = \dim N$. Thus $\pi^{-1}(Z_{(H)})$ contains an irreducible component of N which does not have FPIG, a contradiction. \square

Corollary 2.15 *If N is a complete intersection, then every N_W is a complete intersection. Equivalently, every Lagrangian H -submodule of S_0 is 0-modular.*

Proof By Corollary 2.4,

$$\dim N_W = \dim N - \dim G + \dim H = 2 \dim V - 2 \dim G + \dim H = \dim S - \dim H.$$

Thus $N_W = S^H \times N_0$ is a complete intersection. \square

Definition 2.16 (property (N)) We say that a slice representations (S, H) has property (N) if either H is finite or

$$\dim \mathcal{N}(N_0)_{\text{sg}} \leq \dim S_0 - \dim H - 2.$$

Proposition 2.17 *The following are equivalent:*

- (1) *The shell N is a normal complete intersection with FPIG.*
- (2) *Every symplectic slice representation of N has properties (N) and (F).*

Proof Let (S, H) be a symplectic slice representation of N . If N has the properties in (1), then by Corollaries 2.5 and 2.15, so does N_0 . Hence (S, H) has property (F) by Proposition 2.14, and property (N) follows from normality. Conversely, if (2) holds, then by Proposition 2.14, N is a reduced complete intersection each of whose irreducible components has FPIG. By induction we may assume that $N_0 \setminus \mathcal{N}(N_0)$ is normal since every closed orbit there has a symplectic slice representation (S', H') where $(H') < (H)$. Then $N_0 \setminus \mathcal{N}(N_0)$ has singularities in codimension two and by (N), so does N_0 . Hence N_0 is normal and by induction, N is normal. Thus (1) holds. \square

Remarks 2.18 We don't know of any examples where N is normal and does not have FPIG. If (S, H) is a symplectic slice representation such that, UTCLS, $S_0 \simeq W_0 \oplus W_0^*$ where W_0 is 1-large, then W_0 has FPIG, hence so do N_W and N . If H is semisimple and W_0 is 1-modular, then W_0 is stable [Popov 1970; Luna and Vust 1973], hence 1-large.

Now we introduce an important definition.

Definition 2.19 Let V be a G -module. Define

$$m_0(V) = \max \{ \dim L \mid L \subset \mathcal{N}(V), L \text{ is linear} \}.$$

Remark 2.20 From the proof of [Herbig et al. 2020, Lemma 3.4] we see that $m_0(V) = m_0(V^*)$ for any G -module V .

Lemma 2.21 Let $(S \simeq W \oplus W^*, H)$ be a symplectic slice representation of N where $\dim H > 0$. Let U be a maximal unipotent subgroup of H^0 and let

$$\delta = \dim W_0 - \dim H - m_0(W_0).$$

Then

- (1) $\dim \mathcal{N}(W_0) \leq \dim W_0 - \delta - \dim H/U$.
- (2) $\dim \mathcal{N}(S_0) \leq \dim S_0 - 2\delta - \dim H - \dim H/U$.

Proof Let $\lambda: \mathbb{C}^\times \rightarrow H$ be a 1-parameter subgroup. Let $Z_\lambda(W_0)$ denote the sum of the positive weight spaces of λ on W_0 . Then the dimension of $\mathcal{N}(W_0)$ is at most the maximum over λ of $\dim Z_\lambda(W_0) + \dim U$. Thus as each $\dim Z_\lambda(W_0) \leq m_0(W_0)$,

$$\dim \mathcal{N}(W_0) \leq \dim W_0 - \dim H - \delta + \dim U = \dim W_0 - \delta - \dim H/U$$

which is (1). Since $\dim Z_\lambda(S_0) \leq 2m_0(W_0)$, we similarly get that

$$\dim \mathcal{N}(S_0) \leq \dim S_0 - 2 \dim H - 2\delta + \dim U,$$

giving (2). \square

Using Lemma 2.21, Theorem 2.11 and Propositions 2.8 and 2.17, we obtain the following.

Proposition 2.22 *Assume that for every symplectic slice representation (S, H) of N with $\dim H > 0$, one of the following holds (possibly UTCLS).*

- (1) N_0 is a normal complete intersection with FPIG.
- (2) H^0 is a torus and W_0 has FPIG.
- (3) Properties (F) and (N).
- (4) $m_0(W_0) < \dim W_0 - \dim H$.

Then each N_0 is a normal complete intersection with FPIG and each Lagrangian H -submodule of S_0 is 1-modular. In particular, N is a normal complete intersection with FPIG and V is 1-modular.

2.4 Mustață's criterion for rational singularities

In this section, varieties will be assumed to be reduced and irreducible. Recall that a complex variety X has rational singularities if X is normal and for every resolution of singularities $f: Y \rightarrow X$, $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$; see [Kovács 2000, page 188]. This condition is local (and open), so one can talk about X having a rational singularity at $x \in X$. We let X_{sm} (resp. X_{sing}) denote the smooth (resp. singular) points of X .

Let $f \in \mathbb{C}[x_1, \dots, x_n]$. For any $m \geq 1$, define new variables x_i^j , $j = 1, \dots, m$, $i = 1, \dots, n$. For a variable t , let $x_i(t) = x_i + tx_i^1 + \dots + t^m x_i^m$ and let $f = 0$, $f_{(1)} = 0$, \dots , $f_{(m)} = 0$ be the coefficients of t^i , $i = 0, \dots, m$, in the equation $f(x_1(t), \dots, x_n(t)) = 0 \pmod{t^{m+1}}$. If $X \subset \mathbb{C}^n$ is an affine variety defined by an ideal (f^1, \dots, f^k) , then the m^{th} jet scheme X_m of X is defined by the ideal of the polynomials f^1, \dots, f^k , $f_{(1)}^1, \dots, f_{(1)}^k$, \dots , $f_{(m)}^1, \dots, f_{(m)}^k$. There is a natural projection $\rho_m: X_m \rightarrow X$. This construction is local, so one can define X_m and $\rho: X_m \rightarrow X$ if X is a variety that is not affine.

The main criteria for rational singularities we use is the following result of Mustață.

Theorem 2.23 ([Mustață 2001, Theorem 0.1 and Proposition 1.4]) *Let X be a local complete intersection variety over an algebraically closed field of characteristic 0 and let $m \geq 1$.*

- (1) *The closure of $\rho_m^{-1}(X_{\text{sm}})$ is an irreducible component of X_m of dimension $(m+1) \dim X$.*

The following are equivalent:

- (2) X_m is irreducible.
- (3) $\dim \rho_m^{-1}(X_{\text{sing}}) < (m+1) \dim X$.

The equivalent conditions (2) and (3) imply that $\dim X_m = (m+1) \dim X$. Moreover, X has rational singularities if and only if (2) and (3) hold for every $m \geq 1$.

Let V be a G -module. Then by Proposition 2.8(5), N is a reduced and irreducible complete intersection if and only if V is 1-modular, which we now assume. Hence we can apply Theorem 2.23, which we restate in this context as:

Proposition 2.24 *Let G be a complex reductive group, V a 1-modular G -module and $N = N_V \subset V \oplus V^*$ the shell. The following are equivalent for $m \geq 1$:*

- (1) N_m is irreducible.
- (2) $\dim \rho_m^{-1}(N_{\text{sing}}) < (m+1) \dim N$.

The equivalent conditions (1) and (2) imply that $\dim N_m = (m+1) \dim N$. Moreover, N has rational singularities if and only if (1) and (2) hold for all $m \geq 1$.

Definition 2.25 Let V be a 1-modular G -module and $N = N_V$. We say that $\mathcal{N}(N)$ is irrelevant if N_m is the closure of $\rho_m^{-1}(N \setminus \mathcal{N}(N)_{\text{sg}})$ for any $m \geq 1$.

Proposition 2.26 *Let V be a 1-modular G -module and $N = N_V$. If $N \setminus \mathcal{N}(N)$ has rational singularities and $\mathcal{N}(N)$ is irrelevant, then N has rational singularities.*

Proof Let $m \geq 1$ and let $N' = N \setminus \mathcal{N}(N)_{\text{sg}}$. By hypothesis, N' has rational singularities and N_m is the closure of $\rho_m^{-1}(N')$ where the latter is irreducible of dimension $(m+1) \dim N$. Hence N has rational singularities. □

Recall that the shell N is CIFR if it is a complete intersection with FPIG and rational singularities.

Theorem 2.27 *Let V be a G -module and $N = N_V$. We assume that N is a normal complete intersection with FPIG. Then N is CIFR if and only if $\mathcal{N}(N_0)$ is irrelevant for every symplectic slice representation $(S = S^H \oplus S_0, H)$ of N .*

Proof Let $(S = S^H \oplus S_0, H)$ be a symplectic slice representation of N and let $\rho_m: (N_0)_m \rightarrow N_0$ denote the projection. By Proposition 2.17, N_0 is a normal complete intersection with FPIG. If N has rational singularities, then so does N_0 by Corollary 2.5 and we may apply Proposition 2.24 which gives that

$$\dim \rho_m^{-1}((N_0)_{\text{sing}}) < (m+1) \dim N_0.$$

Since any irreducible component of $(N_0)_m$ has dimension at least $(m+1) \dim N_0$, the above shows that $\mathcal{N}(N_0)$ is irrelevant.

Conversely, suppose that each $\mathcal{N}(N_0)$ is irrelevant. Given a particular (S, H) , we may assume by induction that for any symplectic slice representation (S', H') with $(H') < (H)$, the shell N'_0 has rational singularities. (The induction starts with the case that H is a principal isotropy group, in which case N_0 is smooth.) By Corollary 2.5 and Remark 2.6, $N_0 \setminus \mathcal{N}(N_0)$ has rational singularities. By Proposition 2.26, N_0 has rational singularities. It follows that N_0 is CIFR, hence N is CIFR. □

3 The jet schemes of the shell

Let G be a reductive complex group and V a 1–modular G –module. Let $N = N_V$ be the shell and let $m \geq 1$. Then N_m is a subscheme of $V^{m+1} \oplus (V^*)^{m+1}$. As explained in Section 2.4, N_m is the subscheme defined by the following system of equations where $(x_0, \dots, x_m) \in V^{m+1}$, $(\xi_0, \dots, \xi_m) \in (V^*)^{m+1}$, and A runs through a basis of \mathfrak{g} :

$$\begin{aligned}
 (3-0) \quad & \xi_0(A(x_0)) = 0, \\
 (3-1) \quad & \xi_0(A(x_1)) + \xi_1(A(x_0)) = 0, \\
 (3-2) \quad & \xi_0(A(x_2)) + \xi_1(A(x_1)) + \xi_2(A(x_0)) = 0, \\
 (3-3) \quad & \xi_0(A(x_3)) + \xi_1(A(x_2)) + \xi_2(A(x_1)) + \xi_3(A(x_0)) = 0, \\
 & \vdots \\
 (3-m) \quad & \xi_0(A(x_m)) + \xi_1(A(x_{m-1})) + \dots + \xi_{m-1}(A(x_1)) + \xi_m(A(x_0)) = 0.
 \end{aligned}$$

Remark 3.1 If N has rational singularities, then each N_m has dimension $(m+1) \dim N$ and hence by counting equations is a complete intersection. Moreover, each N_m is a variety [Mustařa 2001, Proposition 1.5].

We observe the following consequences of the description of N_m above.

Lemma 3.2 *The equivalent conditions (1) and (2) of Proposition 2.24 hold with $m = 1$ if and only if N is 1–modular as a G –variety.*

Proof By Proposition 2.8(3), $N_{\text{sing}} = \bigcup_{r=1}^{\dim G} N_{(r)}$. If $(x_0, \xi_0) \in N_{(r)}$ with $r \geq 1$, then $\rho_1^{-1}(x_0, \xi_0)$ is defined by the linear system (3-1) in (x_1, ξ_1) , which has rank $\dim G - r$. Therefore, $\rho_1^{-1}(N_{(r)})$ is a vector bundle over $N_{(r)}$ of rank $2 \dim V - \dim G + r = \dim N + r$, and

$$\dim \rho_1^{-1}(N_{(r)}) = \dim N_{(r)} + \dim N + r.$$

Hence $\dim \rho_1^{-1}(N_{(r)}) < 2 \dim N$ if and only if $\dim N - \dim N_{(r)} \geq r + 1$, which holds for each $r \geq 1$ if and only if N is 1–modular. □

3.1 The shell has rational singularities for generic V

Let $\pi : N \rightarrow N//G$ denote the categorical quotient. The next several results follow from the techniques in [Budur 2021].

Lemma 3.3 *Let V be a 1–modular G –module and $N = N_V$. Then for any $m \geq 1$ and $(x_0, \xi_0) \in N$, $\dim \rho_m^{-1}(x_0, \xi_0) \leq \dim \rho_m^{-1}(0, 0)$. If $m = 1$, then $\rho_1^{-1}(0, 0) \simeq V \oplus V^*$ and if $m \geq 2$, then*

$$\rho_m^{-1}(0, 0) \simeq N_{m-2} \times V \times V^*.$$

Proof Since N_m and N are cones, the fiber $\rho_m^{-1}(x_0, \xi_0)$ has dimension at most that of $\rho_m^{-1}(0, 0)$. The latter is given by equations

$$\sum_{i+j=k} \xi_i A(x_j) = 0, \quad A \in \mathfrak{g}, \quad i, j \geq 1, \quad k = 2, \dots, m-1,$$

where $x_1, \dots, x_m \in V$ and $\xi_1, \dots, \xi_m \in V^*$. Note that there are no conditions on x_m and ξ_m and that the equations on the remaining variables define a copy of N_{m-2} . \square

Theorem 3.4 *Let V be 1-modular and $N = N_V$. Let $N' \subset N$ be a closed subvariety such that $N \setminus N'$ has rational singularities. If $\text{codim}_N(N' \cap N_{\text{sing}}) > \dim G$, then each N_m is irreducible, hence N has rational singularities.*

Proof We may replace N' by $N' \cap N_{\text{sing}}$. We leave it to the reader to show that N_1 is irreducible, so assume that $m \geq 2$. By induction we may assume that N_{m-2} is irreducible of dimension $(m-1) \dim N$. Then

$$\dim \rho_m^{-1}(N_{\text{sing}} \setminus N') < (m+1) \dim N,$$

and

$$\begin{aligned} \dim \rho_m^{-1}(N') &\leq \dim N' + \dim N_{m-2} + 2 \dim V \\ &< \dim N - \dim G + (m-1) \dim N + 2 \dim V = (m+1) \dim N. \end{aligned}$$

It follows that N_m is irreducible. Then N has rational singularities by Proposition 2.24. \square

Corollary 3.5 *Let V be 1-modular and $N = N_V$. If $\text{codim}_N N_{\text{sing}} > \dim G$, then N has rational singularities.*

Corollary 3.6 *Let V be 1-modular and $N = N_V$. Suppose that $N \setminus \mathcal{N}(N)$ has rational singularities and $\text{codim}_N \mathcal{N}(N)_{\text{sg}} > \dim G$. Then $\mathcal{N}(N)$ is irrelevant and N has rational singularities.*

Corollary 3.7 *If V is $(\dim G)$ -modular, then $N = N_V$ has rational singularities.*

Proof Note that N_{sing} is the union of the $N_{(r)}$ for $r \geq 1$. Now

$$N_{(r)} \subset \bigcup_{s,t \geq r} V_{(s)} \times V_{(t)}^*$$

and it follows that

$$\dim N_{(r)} \leq 2 \dim V - 2r - 2 \dim G = \dim N - 2r - \dim G$$

and Corollary 3.5 applies. \square

Remark 3.8 By [Herbig et al. 2020, Theorem 3.6], if G is semisimple, then among G -modules V such that $V^G = \{0\}$ and such that each irreducible component of V is a faithful \mathfrak{g} -module, all but finitely many are k -modular, up to isomorphism, for any k . Hence, Corollary 3.7 demonstrates that among such G -modules, the shell has rational singularities in all but finitely many cases.

3.2 Linear subspaces of the null cone and rational singularities

Let V be a G -module and $N = N_V$. We assume that $V^G = 0$. Some of our criteria above for N to be CIFR depended upon estimating the dimensions of $\mathcal{N}(N_0)_{\text{sg}}$ for symplectic slice representations $(S = S^H \oplus S_0, H)$ of N . We develop another approach which relies upon estimates of $m_0(W_0^*)$ for W_0 a Lagrangian H -submodule of S_0 .

Let $n = \dim V$. Fix $\vec{x} = (x_0, \dots, x_m) \in V^{m+1}$ and let

$$(\diamond) \quad Y_{\vec{x}} = \{\vec{\xi} \in (V^*)^{m+1} \mid (\vec{x}, \vec{\xi}) \in N_m\}.$$

Note that $Y_{\vec{x}}$ is defined by linear equations.

Let E_0 denote $\mathfrak{g}(x_0)$ and let \mathfrak{g}_0 denote $\text{Lie}(G_{x_0})$. For $i = 0, \dots, m - 1$, inductively define the subspace $E_{i+1} = \mathfrak{g}_i(x_{i+1}) + E_i$ and the Lie subalgebra \mathfrak{g}_{i+1} as the set of $A \in \mathfrak{g}_i$ such that $A(x_{i+1}) \in E_i$, ie, the kernel of the map $\mathfrak{g}_i \rightarrow V/E_i$ sending A to $A(x_{i+1}) + E_i$. Note that each E_i is \mathfrak{g}_i -stable. Choose linear subspaces $\mathfrak{p}_0, \dots, \mathfrak{p}_m$ of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g}_i = \mathfrak{g}_{i+1} \oplus \mathfrak{p}_{i+1}$ for $0 \leq i < m$. Set $E'_i = \mathfrak{p}_i(x_i)$, $0 < i \leq m$. Then $E_i = E_{i-1} \oplus E'_i$, $0 < i \leq m$. Let $r_i = \dim E_i$, $i = 0, \dots, m$.

Lemma 3.9 *Let $\vec{\xi} \in (V^*)^{m+1}$. Then $\vec{\xi} \in Y_{\vec{x}}$ if and only if:*

- (1) $\xi_0 \in \text{Ann } E_m$.
- (2) For $0 < i \leq m$, ξ_i restricted to E_{m-i} satisfies linear equations of the form $\xi_i(A(x_{m-i})) = C_{m,i}$ where the right hand sides $C_{m,i}$ are determined by ξ_0, \dots, ξ_{i-1} .

Proof The proof proceeds by induction on m . For $m = 0$, equation (3-0) gives (1) and part (2) is vacuous. Assume the lemma holds for $m - 1$. Now $\mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{p}_m \oplus \mathfrak{p}_{m-1} \oplus \dots \oplus \mathfrak{p}_0$ where for $A \in \mathfrak{g}_m$, equation (3- m) vanishes. For $A \in \mathfrak{p}_m$, equation (3- m) shows that ξ_0 vanishes on E'_m . Since we already know that ξ_0 vanishes on E_{m-1} , it follows that $\xi_0 \in \text{Ann } E_m$ and we have (1). For $0 < i \leq m$, equation (3- m) with $A \in \mathfrak{p}_{m-i}$ becomes

$$\xi_i(A(x_{m-i})) = -\xi_{i-1}(A(x_{m-i+1})) - \dots - \xi_0(A(x_m)).$$

Thus ξ_i restricted to E'_{m-i} is uniquely determined by ξ_0, \dots, ξ_{i-1} . Since the same is true for ξ_i restricted to E_{m-i-1} we have (2). □

Let $\tau: Y_{\vec{x}} \rightarrow V^*$ be the projection sending $\vec{\xi}$ to ξ_0 .

Corollary 3.10 *With $Y_{\vec{x}}$ defined as in (\diamond) , we have:*

- (1) *The dimension of $Y_{\vec{x}}$ is $(m + 1)n - \sum_{i=0}^m r_i$.*
- (2) *The projection τ has image $\text{Ann } E_m$. Its fibers are affine subspaces of dimension $mn - \sum_{i=0}^{m-1} r_i$.*

Let $R_m = \{\vec{r} = (r_0, \dots, r_m) \in \mathbb{Z}^{m+1} \mid 0 \leq r_0 \leq r_1 \leq \dots \leq r_m \leq \dim G\}$. For each $\vec{r} \in R_m$, let $X_{\vec{r}}$ be the set of $\vec{x} \in V^{m+1}$ such that $\dim E_i = r_i$, $i = 0, \dots, m$. Let X be an irreducible component of $X_{\vec{r}}$ and let Y denote the solutions to (3-0)–(3- m) with $\vec{x} \in X$. By Corollary 3.10, Y is irreducible of dimension $\dim X + (m + 1)n - \sum_{i=0}^m r_i$. If $r_0 = \dim G$, then for any $(\vec{x}, \vec{\xi}) \in Y$, the isotropy group of (x_0, ξ_0) is finite, so by Proposition 2.8(3), $(x_0, \xi_0) \in N_{\text{sm}}$.

Corollary 3.11 *Let V be a 1–modular G –module and $N = N_V$. Suppose that, UTCLS, for each $\vec{r} \in R_m$ and each irreducible component X of $X_{\vec{r}}$ where $r_0 < \dim G$, the codimension of X in V^{m+1} is greater than $(m+1) \dim G - \sum_{i=0}^m r_i$. Then N has rational singularities.*

Proof By Corollary 3.10(1) and our hypothesis, $\dim Y < (m+1) \dim N$. It follows that

$$\dim \rho_m^{-1}(N_{\text{sing}}) < (m+1) \dim N,$$

hence N has rational singularities by Proposition 2.24. □

Recall the definition of $m_0(V)$ (Definition 2.19).

Corollary 3.12 *Let V be a 1–modular G –module with $V^G = 0$. Suppose that, UTCLS,*

$$m_0(V) < \dim V - \dim G.$$

Then $\mathcal{N}(N)$ is irrelevant.

Proof Let N'_m be the closure of $\rho_m^{-1}(N \setminus \mathcal{N}(N)_{\text{sg}})$. Let $\vec{x} \in V^{m+1}$ and let $Y_{\vec{x}}$ be as in (\diamond). Since $\dim E_m \leq \dim G$,

$$\dim \text{Ann } E_m \geq \dim V - \dim G > m_0(V^*)$$

and $Y_{\vec{x}}$ contains a dense open set of points $\vec{\xi}$ where $\xi_0 \notin \mathcal{N}(V^*)$, hence $(x_0, \xi_0) \notin \mathcal{N}(N)$. Thus $(\{\vec{x}\} \times Y_{\vec{x}}) \cap N'_m$ is dense in $\{\vec{x}\} \times Y_{\vec{x}}$, hence $\{\vec{x}\} \times Y_{\vec{x}} \subset N'_m$. Thus $N_m = N'_m$ and $\mathcal{N}(N)$ is irrelevant. □

Remark 3.13 The argument in the proof of Corollary 3.12 shows that $\mathcal{N}(N)$ is irrelevant if we can find any reason that no $\text{Ann } E_m$ is contained in $\mathcal{N}(V^*)$.

Theorem 3.14 *Let V be a G –module and $N = N_V$. Suppose that for any symplectic slice representation $(S = W \oplus W^*, H)$ of N , where $\dim H > 0$, one of the following holds, UTCLS:*

- (1) N_0 is CIFR.
- (2) H^0 is a torus and W_0 has FPIG.
- (3) $\dim(\mathcal{N}(N_0)_{\text{sg}}) < \dim S_0 - 2 \dim H$ and $\dim(\mathcal{N}(N_0)) < \dim S_0 - H$.
- (4) $m_0(W_0) < \dim W_0 - \dim H$.

Then N is CIFR.

Proof By Proposition 2.22, N is a normal complete intersection with FPIG. By Theorem 2.27 we need only show that each $\mathcal{N}(N_0)$ is irrelevant. This is clear in (1), (2) and (4), and in (3) we may assume by induction that $N_0 \setminus \mathcal{N}(N_0)$ has rational singularities and apply Corollary 3.6. \square

We can say more when V is an orthogonal G -module, ie, if V admits a nondegenerate symmetric G -invariant bilinear form. Let K be a maximal compact subgroup of G so that $G = K_{\mathbb{C}}$ is the complexification of K . Then $V = W \otimes_{\mathbb{R}} \mathbb{C}$ where W is a real K -module [Schwarz 1980, Proposition 5.7]. Let T be the complexification of a maximal torus T_0 of K . Let Λ be the nonzero weights of V relative to T . Then $\Lambda = -\Lambda$. If $\lambda \in \Lambda$, let V_{λ} denote the corresponding weight space of V . Since V is orthogonal, $\dim V_{\lambda} = \dim V_{-\lambda}$.

Proposition 3.15 *If V is orthogonal, then $m_0(V) = \frac{1}{2}(\dim V - \dim V^T)$.*

Proof Let $\mu_0(V) = \frac{1}{2}(\dim V - \dim V^T)$ and let $L \subset \mathcal{N}(V)$ be linear of dimension m . First note that if $\mathcal{N}(V)$ contains a linear subspace of dimension m , then $\mathcal{N}(V)$ contains a linear T -stable subspace of dimension m . This is shown using an argument from the proof of [Draisma et al. 2006, Lemma 1], which we now recall. Let $\text{Gr}_m(V)$ denote the Grassmann variety of m -dimensional subspaces of V . Let Z_m denote the elements of $\text{Gr}_m(V)$ which lie in $\mathcal{N}(V)$. Then Z_m is a nonempty closed G -stable subvariety of $\text{Gr}_m(V)$. By Borel's fixed point theorem [1991, III.10.4], Z_m contains a T -fixed point. Thus we may assume that L is T -stable so that L is the direct sum of weight spaces L_{λ} , $\lambda \in \Lambda$, although some L_{λ} may be zero.

Now each $V_{\lambda} \oplus V_{-\lambda}$ is $W'_{\lambda} \otimes_{\mathbb{R}} \mathbb{C}$ where W'_{λ} is a T_0 -stable subspace of W . Suppose that $m > \mu_0(V)$. Then for some $\lambda \in \Lambda$, $\dim L_{\lambda} + \dim L_{-\lambda} > \dim V_{\lambda}$ and

$$\text{codim}_{(V_{\lambda} + V_{-\lambda})}(L_{\lambda} + L_{-\lambda}) < \dim V_{\lambda}.$$

The real dimension of W'_{λ} is $2 \dim V_{\lambda}$. Thus W'_{λ} and L have a positive-dimensional intersection. But all points of W'_{λ} lie on closed K -orbits, hence on closed G -orbits [Birkes 1971]. Thus L is not contained in $\mathcal{N}(V)$, a contradiction. Hence $m_0(V) \leq \mu_0(V)$.

Let $\rho: \mathbb{C}^{\times} \rightarrow T$ be a 1-parameter subgroup which acts nontrivially on every V_{λ} , $\lambda \neq 0$. Let

$$L = \text{Span}\{V_{\lambda} \mid \rho \text{ has strictly positive weight on } V_{\lambda}\}.$$

Then $L \subset \mathcal{N}(V)$ and $\dim L = \frac{1}{2}(\dim V - \dim V^T) = \mu_0(V)$. Thus $m_0(V) = \mu_0(V)$. \square

As far as we know, the question of the dimension of linear subspaces of $\mathcal{N}(V)$ has only been investigated when G is reductive and $V = \mathfrak{g}$. Proposition 3.15 was then established by Gerstenhaber [1958] for $G = \text{SL}_n(\mathbb{C})$ and for general semisimple G by Meshulam and Radwan [1998]. See also [Draisma et al. 2006]. These works also consider more general fields than \mathbb{C} and the question of conjugacy of maximal-dimensional linear subspaces of $\mathcal{N}(\mathfrak{g})$.

Lemma 3.16 *Let V be an orthogonal G -module and let $v \in V$ lie on a closed orbit with isotropy group H .*

- (1) *The slice representation (W, H) is orthogonal.*
- (2) *The symplectic slice representation at $(v, 0) \in N \subset V \oplus V^*$ is $(W \oplus W^*, H)$.*

Proof Let K be a maximal compact subgroup of G . There is a real K -module U such that $V \simeq U \otimes_{\mathbb{R}} \mathbb{C}$. By [Schwarz 1980, Proposition 5.8], the isotropy groups of closed G -orbits in V are conjugate to the complexifications of those occurring in the K -module U . Hence W is orthogonal. As an H -module, $V \simeq W \oplus (\mathfrak{g}/\mathfrak{h})$ and

$$V \oplus V^* \simeq W \oplus W^* \oplus \mathfrak{g}/\mathfrak{h} \oplus (\mathfrak{g}/\mathfrak{h})^*.$$

Thus the symplectic slice representation at $(v, 0)$ is $(W \oplus W^*, H)$. □

Proposition 3.17 *Assume that V is orthogonal and let $N \subset V \oplus V^*$ be the shell. Then the symplectic slice representations are precisely those of the form $(W \oplus W^*, H)$, where (W, H) is a slice representation of V .*

Proof Let K and U be as in the proof of Lemma 3.16. Then $V \oplus V^* \simeq V \oplus V$ is the complexification of $U \oplus U$. The isotropy groups of K which occur in $U \oplus U$ are all in slice representations of the isotropy groups which occur in a single copy of U . Thus any symplectic slice representation of $V \oplus V^*$ is in turn a symplectic slice representation of one at a point $(v, 0)$ or $(0, v^*)$. By Lemma 3.16 the latter symplectic slice representations are of the form $(W \oplus W^*, H)$ where (W, H) is a slice representation of V . By induction, the proposition holds for (W, H) . Since any slice representation of (W, H) is also a slice representation of (V, G) , the proposition follows. □

Theorem 3.18 *Let V be a G -module and $N = N_V$. Suppose that for any symplectic slice representation $(S = W \oplus W^*, H)$ of N , where $\dim H > 0$ and T is a maximal torus of H , one of the following holds, UTCLS:*

- (1) N_0 is CIFR.
- (2) H^0 is a torus and W_0 has FPIG.
- (3) $\dim(\mathcal{N}(N_0)_{\text{sg}}) < \dim S_0 - 2 \dim H$ and $\dim \mathcal{N}(N_0) < \dim S_0 - H$.
- (4) S_0 has an orthogonal Lagrangian H -submodule W_0 and $\dim H < \frac{1}{2}(\dim W_0 + \dim W_0^T)$.

Then N is CIFR.

Proof Clearly we only have to consider (4). We may assume that there is a W_0 as described. By Proposition 3.15, $m_0(W_0) = \frac{1}{2}(\dim W_0 - \dim W_0^T)$. Then

$$\dim W_0 - \dim H > \dim W_0 - \frac{1}{2}(\dim W_0 + \dim W_0^T) = \frac{1}{2}(\dim W_0 - \dim W_0^T) = m_0(W_0),$$

which is (4) of Theorem 3.14. Hence N is CIFR. □

4 Copies of the adjoint representation

In this section, we consider the case that $V = p\mathfrak{g}$ with $p > 1$, ie, V is given by copies of the adjoint representation. We first need the following preliminaries. Recall that a group acts *almost faithfully* if the kernel of the action is finite.

Lemma 4.1 *Let V be an orthogonal G -module.*

- (1) V is 2-principal if and only if $V//G$ has no codimension one strata.
- (2) A slice representation (W, H) corresponds to a codimension one stratum if and only if $\dim W_0//H = 1$, where W_0 is an H -module and $W^H \oplus W_0 = W$.

Now suppose that $G = T$ is a torus and T acts almost faithfully on V .

- (3) The module V is k -principal if and only if it is k -large, $k \geq 1$.
- (4) If (W, H) is the slice representation of a codimension one stratum, then $\dim H = 1$ and $\dim W_0 = 2$.

Proof By [Schwarz 1980, Corollary 7.4], any stratum of $V//G$ of codimension at least 2 has inverse image in V of codimension at least 2 and (1) follows. Part (2) is obvious and (3) is [Schwarz 1995, Proposition 10.1]. Let (W_0, H) be as in (4). Then $\dim W_0//H = 1$ and W_0 is just the nontrivial part of V as an H -module. By Lemma 3.16(1), (W_0, H) is orthogonal. Since the weights of V occur in pairs $\pm\nu$, if H is finite, then $\dim W_0 \geq 2$ and $\dim W_0/H \geq 2$, a contradiction. Since (W_0, H) is almost faithful, $\dim W_0 \geq 2 \dim H$ so that $\dim W_0//H \geq \dim H$. Hence $\dim H = 1$ and $\dim W_0 = 2$. \square

Proposition 4.2 *Let G be simple of rank at least 2. Let T be a maximal torus of G and Φ the corresponding set of roots of \mathfrak{g} . Let T act on the span V of the roots spaces \mathfrak{g}_α , $\alpha \in \Phi$. Then V is a 2-large T -module.*

Proof If V is not 2-principal, then there is a subgroup H of T of dimension 1 with slice representation W such that $\dim W_0 = 2$. Let $\alpha_1, \dots, \alpha_\ell$ be the simple roots of \mathfrak{g} . Since $\dim W_0 = 2$, at most one α_i does not vanish on \mathfrak{h} , say α_1 . Since $\text{rank } G > 1$, there is a positive root $\alpha = \sum_i n_i \alpha_i$, where $n_1 > 0$ and $n_j > 0$ for some $j > 1$. Then $\alpha(\mathfrak{h}) \neq 0$ which implies that $\dim W_0 \geq 4$, a contradiction. \square

Corollary 4.3 *Let (V, T) be as in Proposition 4.2. Let $N \subset V \oplus V^*$ be the shell. Then N is CIFR. Moreover, for any slice representation (W, H) of V where $\dim H > 0$, set $S = W \oplus W^*$ and we have*

$$\text{codim}_{N_0}(\mathcal{N}(N_0)_{\text{sg}}) \geq 4.$$

Proof We already know from Theorem 2.11 that every N_0 is CIFR. Let $\mathcal{N}_0 = \mathcal{N}(S_0)$. Then \mathcal{N}_0 is a (finite) union of linear subspaces $Z_\lambda(S_0)$ where $\lambda: \mathbb{C}^\times \rightarrow H$ is a 1-parameter subgroup. Since H is abelian, each $Z_\lambda(W_0)$ is an H -module. A given Z_λ is maximal (for set inclusion) if and only if $W_0^\lambda = 0$,

so we only need to consider such “generic” λ . Then $W_0 \simeq Z_\lambda(W_0) \oplus Z_\lambda(W_0)^*$. Since W_0 is 2–modular, as in Example 2.10, Lemma 2.1 tells us that $Z_\lambda(W_0) \oplus Z_\lambda(W_0)^*$, which is an irreducible component of \mathcal{N}_0 , is 2–modular. Thus $(\mathcal{N}_0)_{(r)}$ has codimension at least 3 in \mathcal{N}_0 for $r > 0$. Hence

$$\begin{aligned} \text{codim}_{\mathcal{N}_0} \mathcal{N}(N_0)_{\text{sg}} &\geq \dim N_0 - \dim \mathcal{N}_0 + 3 \\ &= 2 \dim W_0 - \dim H - \dim W_0 + 3 \\ &= \dim W_0 - \dim H + 3 \geq 4. \end{aligned} \quad \square$$

Lemma 4.4 *Let G be simple and $V = \mathfrak{p}\mathfrak{g}$ for $p > 1$. Let N denote the shell of $V \oplus V^*$ and let (S, H) be a symplectic slice representation of N with $\dim H > 0$ and orthogonal Lagrangian submodule $W = W^H \oplus W_0$ of S . Consider the conditions*

- (1) $m_0(W_0) < \dim W_0 - \dim H$,
- (2) H^0 is a torus and W_0 has FPIG (hence N_0 is CIFR),
- (3) $\text{codim}_{\mathcal{N}_0} \mathcal{N}(N_0)_{\text{sg}} \geq 4$,
- (4) $\text{codim}_{W_0} \mathcal{N}(W_0) \geq 2$.

Then (1) or (2) always hold. Parts (3) and (4) fail if and only if $p = 2$ and $\text{rank } G = 1$.

Proof It is easy to see that V has FPIG, so all (W_0, H) have FPIG. By Proposition 3.17, $(W_0 \oplus W^H, H)$ is a slice representation of V . As an H –module, $V = \mathfrak{p}\mathfrak{h} \oplus \mathfrak{p}(\mathfrak{g}/\mathfrak{h})$, hence

$$W \simeq \mathfrak{p}\mathfrak{h} \oplus (p-1)(\mathfrak{g}/\mathfrak{h}).$$

Write $H^0 = ZH_s$, where H_s is semisimple and Z is a central torus. Then Z acts trivially on $\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{z}$ and almost faithfully and orthogonally on $\mathfrak{g}/\mathfrak{h}$. Let \mathfrak{m} denote the H –complement to $(\mathfrak{g}/\mathfrak{h})^H$ in $(\mathfrak{g}/\mathfrak{h})$. Then $W_0 = \mathfrak{p}\mathfrak{h}_s \oplus (p-1)\mathfrak{m}$, where $\dim \mathfrak{m} \geq 2 \dim Z$. Let $B = TU$ be a Borel subgroup of H_s and let \mathfrak{t} and \mathfrak{u} be the corresponding subalgebras of \mathfrak{h}_s where $\dim \mathfrak{t} = \ell$. Since W_0 is orthogonal,

$$m_0(W_0) \leq \frac{1}{2}p(2 \dim \mathfrak{u}) + \frac{1}{2}(p-1) \dim \mathfrak{m},$$

and

$$\begin{aligned} \dim W_0 - \dim H - m_0(W_0) &\geq p(2 \dim \mathfrak{u} + \ell) + (p-1) \dim \mathfrak{m} - (2 \dim \mathfrak{u} + \ell) - \dim Z - m_0(W_0) \\ &= \delta := (p-2) \dim \mathfrak{u} + (p-1)\ell + \left(\frac{1}{2}(p-1) \dim \mathfrak{m} - \dim Z\right). \end{aligned}$$

The first two terms of δ are nonnegative, as is the third since $\dim \mathfrak{m} \geq 2 \dim Z$. If $p > 2$, one easily sees that $\delta \geq 2$ so that (1) holds, and (3) and (4) hold by Lemma 2.21. Now assume that $p = 2$ so that $\delta = \ell + \frac{1}{2} \dim \mathfrak{m} - \dim Z$. If H_s is not trivial, then (1), (3) and (4) hold since $\delta \geq 1$ and $\dim H/U \geq 2$. So we are left with the case that $H^0 = Z$ is a torus. Since W_0 is an almost faithful Z –module and (W_0, Z) is orthogonal, hence stable, W_0 has FPIG. Theorem 2.11 then shows that N_0 is CIFR and (2) holds. We may assume that $Z \subset T$ where now T is the maximal torus of G . Then (W_0, Z) is a slice representation of (\mathfrak{g}, T) and if $\text{rank } G > 1$, then (3) and (4) hold by Proposition 4.2 and Corollary 4.3.

If $\mathfrak{g} = \mathfrak{sl}_2$ and $p = 2$, then $V = 2\mathfrak{g}$ has a slice representation (W_0, \mathbb{C}^\times) where the weight vector is $(2, -2)$. Then $\text{codim}_{W_0} \mathcal{N}(W_0) = 1$, and N_0 has dimension three and has a singular point at the origin. Thus (3) and (4) fail. \square

Corollary 4.5 *Suppose that $p > 2$ or that $\text{rank } G > 1$. Let (S, H) be a symplectic slice representation of N .*

- (1) N_0 is factorial. In particular, N_V is factorial.
- (2) $\mathbb{C}[N_0]^* = \mathbb{C}^\times$.
- (3) If the character group $\chi(H)$ is trivial, then $N_0//H$ is factorial. In particular, $N_V//G$ is factorial.

Proof It follows from Lemma 4.4(3) and induction over symplectic slice representations that $(N_0)_{\text{sing}}$ has codimension at least 4 in N_0 . Since N_0 is a complete intersection, N_0 is locally factorial [SGA 2 2005, exposé XI, corollaire 3.14]. Since N_0 is a cone, it follows that $\mathbb{C}[N_0]$ is a UFD [Hartshorne 1977, Chapter 2, Example 6.3(d)], and we have (1). Let $f \in \mathbb{C}[N_0]^*$. Then \mathbb{C}^\times acts by the scalar action on N_0 , and by the argument in [Knop et al. 1989, Proposition 1.3], $f \circ t = \chi(t)f$ for a character χ of \mathbb{C}^\times . Hence f is homogeneous. This forces f to be a constant and we have (2). Now let $f \in \mathbb{C}[N_0]^H$. Then f factors uniquely in $\mathbb{C}[N_0]$ as a product $f_0 \cdot f_1 \cdots f_m$, where f_0 is a unit (hence a constant), and f_1, \dots, f_m are irreducible and transform by elements of $\chi(H)$. If $\chi(H)$ is trivial, the f_i are H -invariant and we have (3). \square

Corollary 4.6 *If $p > 2$ or $\text{rank } G > 1$, then $V//G$ has no codimension one strata.*

Proof Let (W, H) be the slice representation of a codimension one stratum where $\dim H > 0$. Then $\dim W_0//H = 1$ so that $\text{codim}_{W_0} \mathcal{N}(W_0) = 1$ which contradicts Lemma 4.4(4). Hence H is finite cyclic and $\dim W_0 = 1$. But H is a subgroup of a maximal torus T of G , in which case $\dim W_0$ is even, a contradiction. \square

In the following two results we consider the adjoint representation of a semisimple group G . Since the action of G on \mathfrak{g} factors through the adjoint group, there is no harm in assuming that G is a product of simple groups.

Theorem 4.7 *Let G be semisimple and $V = p\mathfrak{g}$ for $p \geq 2$. Let $N = N_V \subset V \oplus V^*$ be the shell.*

- (1) N is CIFR and V is 1-large.
- (2) If $p > 2$ or G contains no simple factor of rank 1, then N and $N//G$ are factorial and V is 2-large.

Proof We may assume that $G = G_1 \times \cdots \times G_m$ is a decomposition of G into simple factors. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ and correspondingly $N = N_1 \times \cdots \times N_m$. By Lemma 4.4 and Theorem 3.14, each N_i is CIFR, hence so is N . Since V is orthogonal and 1-modular, it is 1-large and we have (1).

Now assume the hypotheses of (2). By Corollary 4.5, each N_i is factorial, hence so are N and $N//G$. Moreover, by Proposition 2.8(6) and Corollary 4.6, each $p\mathfrak{g}_i$ is 2-modular and 2-principal. Hence V is 2-large. \square

We also have the following, which we will need in Section 7.1.

Corollary 4.8 *Let $G = G_1 \times \cdots \times G_m$ be as above and $p \geq 2$. Let $H = H_1 \times \cdots \times H_m$ be a reductive subgroup of G where $H_i \subset G_i, i = 1, \dots, m$. Let $W = p\mathfrak{h} \oplus (p-1)(\mathfrak{g}/\mathfrak{h})$ and define W_0 and N_0 as usual. Then every symplectic slice representation of N_0 satisfies (1) or (2) of Lemma 4.4, hence N_0 is CIFR.*

Proof It is enough to consider the case that G is simple so that we are in the situation of Lemma 4.4. The proof of the lemma shows that (1) or (2) holds even if $(W \oplus W^*, H)$ is not a symplectic slice representation of N . The same is true for any symplectic slice representation of N_0 . By Theorem 3.14, N_0 is CIFR. \square

5 $N_V//G$ has symplectic singularities

Let $N = N_V$. We present conditions that are sufficient for $N//G$ to have symplectic singularities, different than our criteria in [Herbig et al. 2020] where we required V to be 3-large or 2-large with $(N//G)_{\text{pr}} = (N//G)_{\text{sm}}$. Henceforth, $(*)$ will denote the following condition:

- $(*)$ Let (S, H) be a symplectic slice representation of N . Then UTCLS we have either
 - (1) $m_0(W_0) < \dim W_0 - \dim H$, or
 - (2) H^0 is a torus and W_0 has FPIG.

Remark 5.1 By Proposition 2.22, $(*)$ implies that V is 1-modular and N is a normal complete intersection with FPIG.

By Theorems 2.11 and 3.14, $(*)$ implies that N is CIFR, hence $\dim N//G = 2 \dim V - 2 \dim G$ and $N//G$ has rational singularities [Boutot 1987].

Let X denote $N//G$. The algebra $\mathbb{C}[X]$ is graded, and it is normal and Cohen–Macaulay since X has rational singularities. Recall that X is *graded Gorenstein* if the canonical module ω_X (which has a grading) is a free $\mathbb{C}[X]$ -module with generator of degree $\dim X$.

Theorem 5.2 *If $(*)$ holds then X is graded Gorenstein and has symplectic singularities.*

We give a proof of the theorem after some preliminary results.

Proposition 5.3 *Assume $(*)$. Then X is graded Gorenstein.*

Proof Let $U = X_{\text{pr}} \subset X_{\text{sm}}$. Then $\text{codim}_X(X \setminus U) \geq 2$ (since all strata of X are even-dimensional). Now $L = \bigwedge^{\dim X} (T^*U)$ is a line bundle, and in [Herbig et al. 2020, proof of Theorem 4.6] we construct a nowhere-vanishing section σ of L . It follows that $\Gamma(U, L) \simeq \mathbb{C}[U] = \mathbb{C}[X]$, so that X is Gorenstein. By construction, the degree of σ equals the dimension of X , so that X is graded Gorenstein. \square

Assume that $N = N_V$ has principal isotropy groups, eg, V is 1–modular. Let $(S = W \oplus W^*, H)$ be a symplectic slice representation of N . We say that (S, H) is *proper* if $H \neq G$ and we say that (S, H) and the stratum $N_{(H)}$ of N are *subprincipal* if every proper symplectic slice representation of (S, H) is principal.

Lemma 5.4 *Let (S, H) be as in (*). If the corresponding stratum is subprincipal, then (W, H) is 1–large.*

Proof Let $W = W^H \oplus W_0$ be a Lagrangian H –submodule of S . We know that W_0 is 1–modular and N has FPIG. Since (S, H) is subprincipal, the complement of the principal orbits in W_0 is $\mathcal{N}(W_0)$. By (*), $\mathcal{N}(W_0) \neq W_0$. It follows that W_0 has FPIG and is 1–principal, hence 1–large. \square

Corollary 5.5 *If (*) holds, then $X_{\text{sm}} = X_{\text{pr}}$ and X_{sm} carries a symplectic form.*

Proof Let $(S, H) = (W \oplus W^*, H)$ be a subprincipal symplectic slice representation of N . Then W is 1–large and by [Herbig et al. 2015, Lemma 2.3], $N_W // H$ is singular along $(N_W // H)_{(H)}$. Thus X is singular along the union of the subprincipal strata, hence along their closure, which is the complement of X_{pr} . Thus $X_{\text{sm}} = X_{\text{pr}}$. Finally, [Herbig et al. 2020, Corollary 3.18] shows that X_{pr} carries a (holomorphic) symplectic form. \square

Proof of Theorem 5.2 We know that X is (graded) Gorenstein with rational singularities and has a symplectic form on its smooth locus. By [Namikawa 2001, Theorem 6], X has symplectic singularities. \square

6 Classical representations of the classical groups

We consider 1–modular classical representations (V, G) of the classical groups (and some not so classical) as in [Herbig and Schwarz 2013, Theorem 3.5]. Let $N = N_V \subset V \oplus V^*$. We show that for any symplectic slice representation $(S = W \oplus W^*, H)$ of N the shell N_W is CIFR. We leave it as an exercise for the reader to show that each $N // G$ has symplectic singularities. These results improve upon those of [Herbig et al. 2020], where V is required to be at least 2–large for $N // G$ to have symplectic singularities.

The following improves upon Theorem 5.1 of [Cape et al. 2016].

Theorem 6.1 *Let $(V, G) = (k\mathbb{C}^n, \text{SO}_n(\mathbb{C}))$, $n \geq 2$. Then V is 1–modular if and only if $k \geq n - 1$, in which case the shell $N \subset V \oplus V^*$ is CIFR.*

Proof For the statement about 1–modularity see [Schwarz 1995, Theorem 11.18]. So assume that $k \geq n - 1$. Since V is stable, it is also 1–large, so that we already know that N is a complete intersection

with FPIG. By Proposition 3.17 every nontrivial symplectic slice representation of N is, up to trivial factors, of the same form as $(V \oplus V^*, G)$ (with a smaller $n' < n$). Hence we may assume by induction on n that $N \setminus \mathcal{N}(N)$ has rational singularities. If n is odd, then Theorem 3.18(4) applies. Hence it is enough to consider the case $(V, G) = ((2n - 1)\mathbb{C}^{2n}, \text{SO}_{2n})$ when $\text{Ann } E_m \subset \mathcal{N}(V^*)$, with E_m as defined in Section 3.2, which implies that $\dim E_m = \dim G = \frac{1}{2} \dim V$. Then $\dim \text{Ann } E_m = \frac{1}{2} \dim V$ so that $\text{Ann } E_m$ is a maximal isotropic subspace of V^* . Thus E_m is maximally isotropic, hence $E_m \subset \mathcal{N}(V)$. Note that the projection of E_m to any component copy of \mathbb{C}^{2n} has dimension n .

Let $x_0 = (y_1, \dots, y_{2n-1})$, where $y_i \in \mathbb{C}^{2n}$ for all i . Let us assume that some y_i is not zero, say y_1 . Let W denote the first copy of \mathbb{C}^{2n} . Then the projection of Gx_0 to W has dimension $2n - 1 > n$. This is bigger than the projection of E_m . Thus we must have $x_0 = 0$. Similarly, $x_1 = \dots = x_m = 0$. Hence $\text{Ann } E_m \subset \mathcal{N}(V^*)$ is not possible. By Remark 3.13 $\mathcal{N}(N)$ is irrelevant, hence N has rational singularities and is CIFR by Proposition 2.26. \square

Now we consider copies of the $(G = \text{Sp}_{2n})$ -module \mathbb{C}^{2n} . Let K be a maximal compact subgroup of G . Let T be a maximal torus of G which is the complexification of a maximal torus T_0 of K . Then the weights of V are $\pm \epsilon_i$, $i = 1, \dots, n$, where the ϵ_i are the usual weights of $T \simeq (\mathbb{C}^\times)^n$ acting on \mathbb{C}^n .

Lemma 6.2 *Let $(V, G) = (k\mathbb{C}^{2n}, \text{Sp}_{2n})$. If $k \geq 2$, then $m_0(V) \leq kn$.*

Proof Let L be a linear subspace of $\mathcal{N}(V)$. If k is even, then V is orthogonal and this follows from Proposition 3.15. So assume that k is odd and $\dim L > kn$. As in the proof of Proposition 3.15, we may assume that L is T -stable so that

$$L = \bigoplus_{i=1}^n (L_{\epsilon_i} + L_{-\epsilon_i}).$$

Then for some $\epsilon = \epsilon_i$, $\dim(L_\epsilon + L_{-\epsilon}) > k$. Let V' denote the sum of two copies of \mathbb{C}^{2n} in V and let $\pi: V \rightarrow V'$ be the G -equivariant projection. Let $L' = \pi(L)$. Then for some choice of V' , $\dim(L'_\epsilon + L'_{-\epsilon}) > 2$. Note that L' lies in $\mathcal{N}(V')$ and that V' is orthogonal. Now $V' \simeq W \otimes_{\mathbb{R}} \mathbb{C}$, where W is a real K -module and $V'_\epsilon + V'_{-\epsilon}$ is the complexification of a real two-dimensional T_0 -stable subspace W_0 of W . Since $\dim(L'_\epsilon + L'_{-\epsilon}) > 2$, L' intersects W_0 nontrivially. But any nonzero point of W does not lie in the nullcone [Birkes 1971]. This is a contradiction. Hence $\dim L \leq kn$. \square

Theorem 6.3 *Let $(V, G) = (k\mathbb{C}^{2n}, \text{Sp}_{2n})$. Then V is 1-modular if and only if $k \geq 2n + 1$, in which case the shell N is CIFR.*

Proof By [Schwarz 1995, Theorem 11.20], V has FPIG only if $k \geq 2n$ and is 1-modular if and only if $k \geq 2n + 1$, which we now assume. The symplectic slice representations of N are all, up to trivial factors, of the same form as $V \oplus V^*$, so we may again assume that $N \setminus \mathcal{N}(N)$ has rational singularities. If k is even, then V is orthogonal, $k \geq 2n + 2$ and Theorem 3.18(4) applies. So assume that k is odd. By Lemma 6.2, $m_0(V) \leq kn$. If $k > 2n + 1$, then Theorem 3.14(4) applies so we only need consider

the case $k = 2n + 1$. Then $\text{Ann } E_m \subset \mathcal{N}(V^*)$ implies that $\dim E_m = \dim \text{Ann } E_m = \frac{1}{2} \dim V = \dim G$. Hence $\text{Ann } E_m$ is maximal isotropic in V^* so that E_m is maximal isotropic in V , and $E_m \subset \mathcal{N}(V)$. Thus we may assume that $E_0 = k\mathbb{C}^n \subset V$ is the subspace corresponding to the weights $\epsilon_1, \dots, \epsilon_n$. As in the proof of Theorem 6.1, this implies that we cannot have $\text{Ann } E_m \subset \mathcal{N}(V^*)$ so that N is CIFR. \square

Note that special cases of the two results above are $(2\mathbb{C}^3, \text{SO}_3(\mathbb{C})) = (2\mathfrak{sl}_2, \text{SL}_2(\mathbb{C}))$ and $(3\mathbb{C}^2, \text{SL}_2(\mathbb{C}))$.

Theorem 6.4 *Let $(V, G) = (k\mathbb{C}^7, G_2)$. Then V is 1–modular if and only if $k \geq 4$, in which case N is CIFR.*

Proof From [Schwarz 1995, Theorem 11.21] we see that V is not 1–modular if $k \leq 3$. We assume that $k \geq 4$, in which case V is 2–large. The nontrivial parts of the proper nontrivial slice representations of V are

$$((k-1)(\mathbb{C}^3 \oplus (\mathbb{C}^3)^*), \text{SL}_3(\mathbb{C})) \quad \text{and} \quad (2(k-2)\mathbb{C}^2, \text{SL}_2(\mathbb{C})),$$

and Theorem 3.18(4) applies for all the slice representations. \square

Theorem 6.5 *Let $(V, G) = (k\mathbb{C}^8, \text{Spin}_7(\mathbb{C}))$. Then V is 1–modular if and only if $k \geq 5$, in which case N is CIFR.*

Proof From [Schwarz 1995, Theorem 11.21] we see that V is not 1–modular if $k \leq 4$, so we assume that $k \geq 5$, in which case V is 2–large. The proper slice representations of V are, up to trivial factors, the slice representations of $((k-1)\mathbb{C}^7, G_2)$, so we may assume that $N \setminus \mathcal{N}(N)$ has rational singularities.

If $k \geq 6$, then Theorem 3.18(4) applies, so the only interesting case is $k = 5$. Everything is OK if $\dim E_m \leq 20$ for then $\dim \text{Ann } E_m \geq 20 = \frac{1}{2} \dim V$ and we can argue as above (a minimal nonzero orbit in \mathbb{C}^8 has dimension 7). That leaves the case $\dim E_m = 21 = \dim G$, in which case $\dim \text{Ann } E_m = 19 < \frac{1}{2} \dim V$. If $\text{Ann } E_m \subset \mathcal{N}(V^*)$, then the projection of $\text{Ann } E_m$ to each copy of $(\mathbb{C}^8)^*$ has dimension 3 or 4 so that the projection of E_m to each copy of \mathbb{C}^8 has dimension 4 or 5. As before, this leads to a contradiction. \square

We give a new proof of the following which is also established in [Herbig et al. 2020, Section 6.3].

Theorem 6.6 *Let $(V, G) = (R_1 + R_2, \text{SL}_2(\mathbb{C}))$. Then N is CIFR.*

Proof That V is 1–large follows from [Herbig and Schwarz 2013, Theorem 3.4]. Let $T = \mathbb{C}^\times \subset G$ be the diagonal torus. The only nontrivial nonprincipal symplectic slice representation of N is a T –module with nonzero weights $(1, 1, -1, -1)$, hence $N \setminus \mathcal{N}(N)$ has rational singularities. The dimension of $\mathcal{N}(V^* \simeq V)$ is 3, so that if $\dim E_m < 3$, then $\dim \text{Ann } E_m \geq 3$ and since $\mathcal{N}(V^*)$ is not a vector space, $\text{Ann } E_m$ contains points outside of $\mathcal{N}(V^*)$. Hence we only need to worry about the case $\dim E_m = 3$ and $L = \text{Ann } E_m \subset \mathcal{N}(V^*)$.

Let (x, y) be a basis of R_1 with weights $(1, -1)$ relative to T . Let (e, h, f) be the basis of R_2 with weights $(2, 0, -2)$. Let L_0 denote $\mathbb{C} \cdot x \oplus \mathbb{C} \cdot e$. Then $\mathcal{N}(V^*)$ is $G \cdot L_0$. Let L_2 be the projection of L

to R_2 . If $\dim L_2 = 0$, then $L = R_1$. Otherwise, $\dim L_2 = 1$ since $\dim \mathcal{N}(R_2) = 2$ and $\mathcal{N}(R_2)$ is not linear. It follows that for some $g \in G$, $L = g \cdot L_0$. If $L = \text{Ann } E_m = R_1$, then E_m is R_2 and every x_i is in R_2 , in which case the corresponding points in N_m have dimension less than $(m+1) \dim N$. In the remaining case, we may assume that

$$E_m = \text{Ann } L_0 = \mathbb{C} \cdot y \oplus \mathbb{C} \cdot h \oplus \mathbb{C} \cdot f.$$

If $x_0 \neq 0$, then $E_0 \subset E_m$ forces x_0 to be a multiple of f and then $E_0 = \mathbb{C} \cdot h \oplus \mathbb{C} \cdot f$ and $\mathfrak{g}_0 = \mathbb{C} \cdot f$. Now \mathfrak{g}_0 maps $R_1 + R_2$ into E_0 . So $E_1 = E_2 = \dots = E_m = E_0$ and we never get $\dim E_m = 3$. Thus $x_0 = 0$, but then the same scenario repeats with x_1 , etc. Hence N is CIFR. \square

Let $(V, G) = (p\mathbb{C}^n \oplus q(\mathbb{C}^n)^*, \text{GL}_n(\mathbb{C}))$ where we may assume by choosing a Lagrangian submodule of $V \oplus V^*$ that $p \geq q$. Then V has nonfinite principal isotropy groups if $q < n$ and is 1-large if $q \geq n$. Using Lemma 2.1 and Remark 2.9 we see that V is 1-modular if and only if $p + q \geq 2n$.

Lemma 6.7 *Let V be as above where $p \geq q \geq n$. Let L be a linear subspace of $\mathcal{N}(V)$. Then $\dim L \leq np$. If $p > q$ and $\dim L = pn$, then $L = p\mathbb{C}^n$.*

Proof Since $W = q(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ is orthogonal, the projection of L to W has dimension at most qn by Proposition 3.15. Let $W' \simeq (p - q)\mathbb{C}^n$ be the complement of W . The projection of L to W' has dimension at most $(p - q)n$, so that $\dim L \leq pn$. If there is equality and $p > q$, then the projection of L to W' being surjective implies that the projection of L to $q(\mathbb{C}^n)^*$ is 0 so that $L = p\mathbb{C}^n$. \square

Theorem 6.8 *Let $(V, G) = (p\mathbb{C}^n \oplus q(\mathbb{C}^n)^*, \text{GL}_n(\mathbb{C}))$. Then V is 1-modular if and only if $p + q \geq 2n$, in which case N is CIFR.*

Proof By choosing a Lagrangian submodule we may assume that $p \geq q \geq n$. Any symplectic slice representation of N is either trivial or of the same form as $V \oplus V^*$, up to trivial factors and having a smaller n . Thus we may assume that $N \setminus \mathcal{N}(N)$ has rational singularities. Let $L = \text{Ann } E_m$ be a linear subspace of $\mathcal{N}(V^*)$. By Lemma 6.7,

$$\dim V^* - \dim G = n(p + q) - n^2 = n(p + q - n) \geq np \geq \dim L.$$

Thus $\dim L \leq \dim V - \dim G$ with equality only if $q = n$ and $\dim L = np$.

Suppose that $\text{Ann } E_m$ has dimension np and $q = n$. If $p > q$, then $\text{Ann } E_m = p(\mathbb{C}^n)^* \subset V^*$ so that $E_m = q(\mathbb{C}^n)^* \subset V$ and $\vec{x} \in V^{m+1}$ lies in $(q(\mathbb{C}^n)^*)^{m+1}$ which has codimension greater than $(m+1) \dim G$ in V^{m+1} . By Corollary 3.11, N has rational singularities. So we are left with the case $p = q = n$ and $\dim \text{Ann } E_m = n^2$. Then $\text{Ann } E_m$ is a maximal linear subspace on which the invariants of $V \simeq V^*$ vanish, hence so is E_m . The case that $E_m = n\mathbb{C}^n$ or $E_m = n(\mathbb{C}^n)^*$ is easy since then the codimension of $X_{\vec{r}}$ is greater than $(m+1) \dim G - \sum_{i=0}^m r_i$ unless $r_i = 0$ for all i , in which case $E_m = 0$, a contradiction. Thus E_m projects to a proper nonzero subspace of each copy of \mathbb{C}^n and $(\mathbb{C}^n)^*$. But any nonzero orbit of G on \mathbb{C}^n or its dual is the complement of the origin. This contradicts the form of E_m . Hence N is CIFR. \square

Finally, we consider the case $(V, G) = (p\mathbb{C}^n \oplus q(\mathbb{C}^n)^*, \mathrm{SL}_n(\mathbb{C}))$. Then V is 1-modular if and only if $p + q \geq 2n - 1$ [Schwarz 1995, Proposition 11.14], which we now assume.

Lemma 6.9 *Let (V, G) be as above where we also assume that $p \geq q$ and $p - q \leq 1$. Let L be a maximal linear subspace of $\mathcal{N}(V)$. Then $L = pW \oplus q \mathrm{Ann} W$, where W is a linear subspace of \mathbb{C}^n with $0 < \dim W < n$.*

Proof Let L_0 be the projection of L to $p\mathbb{C}^n$ and L_1 its projection to $q(\mathbb{C}^n)^*$. Since the determinant function vanishes on L_0 , $L_0 \subset pW$, where W is the span of a basis of L_0 and has dimension at most $n - 1$. It follows that a basis of L_1 spans a subspace of $\mathrm{Ann} W$. If $W \neq 0$, then L equals $pW \oplus q \mathrm{Ann} W$. If $W = 0$, then L is not maximal: as before, $L_1 \subset q \mathrm{Ann} W'$ where $\dim W' > 0$ and then $L \subset pW' \oplus q \mathrm{Ann} W'$. Thus the maximal L are as claimed. \square

Corollary 6.10 *Suppose that E is a linear subspace of V and $\mathrm{Ann} E \subset \mathcal{N}(V^*)$. Then*

$$\dim E \geq qn + (p - q).$$

Theorem 6.11 *Let (V, G) be as above. Then V is 1-modular if and only if $p + q \geq 2n - 1$, in which case N is CIFR.*

Proof It is easy to see that all proper nontrivial symplectic slice representations (S, H) of N are, up to trivial factors, of the same form with $H = \mathrm{SL}_{n'}$ for $n' < n$. By induction we may assume that $N \setminus \mathcal{N}(N)$ has rational singularities. By choosing a Lagrangian submodule, we may assume that $p \geq q$ and $p - q \leq 1$. We suppose that $L = \mathrm{Ann} E_m \subset \mathcal{N}(V^*)$ and by Corollary 6.10, $\dim E_m \geq qn + p - q$. If $q \geq n$ we get that $\dim E_m \geq n^2 > \dim G$, which is impossible. So the only interesting case to consider is $p = n$, $q = n - 1$. Then a maximal $L \subset \mathcal{N}(V^*)$ containing $\mathrm{Ann} E_m$ is of the form $n \mathrm{Ann} W \oplus (n - 1)W$, where $n - k = \dim W$ and $0 < k < n$, so that $E_m \supset nW \oplus (n - 1) \mathrm{Ann} W$ has dimension at least $n^2 - k$. Now suppose that x_0 projects nontrivially to $n\mathbb{C}^n$. Then E_0 projects surjectively onto some copy of \mathbb{C}^n . Since E_m already contains a copy of $n\mathbb{C}^{n-k}$, E_m has dimension at least $n^2 - k + k = n^2 > \dim G$. Thus $x_0 \in V_0 = (n - 1)(\mathbb{C}^n)^*$. Hence the nonrational locus of N lies in $(V_0 \times V^*) \cap N$. Reversing the role of V and V^* , we see that the nonrational locus of N lies in N' where N' is the null cone of the shell of V_0 . Since $\mathrm{codim}_N N' > \dim G$, Theorem 3.4, shows that N has rational singularities; hence N is CIFR. \square

7 Applications to representation and character varieties

Let Σ be a Riemann surface of genus $p \geq 2$ and let G be a reductive complex algebraic group. Let π denote $\pi_1(\Sigma)$. Then π is the quotient of the free group on generators $a_1, b_1, a_2, \dots, a_p, b_p$ by the normal subgroup generated by

$$[a_1, b_1][a_2, b_2] \cdots [a_p, b_p],$$

where $[a_i, b_i]$ is the commutator $a_i b_i a_i^{-1} b_i^{-1}$. Let $\text{Hom}(\pi, G)$ denote the set of homomorphisms from π to G . This has a scheme structure as $\Phi^{-1}(e)$, where

$$\Phi: G^{2p} \rightarrow G, \quad (g_1, h_1, g_2, \dots, g_p, h_p) \mapsto [g_1, h_1] \cdots [g_p, h_p].$$

Now Φ is G -equivariant where $g \in G$ acts on G by conjugation and on G^{2p} by conjugation on each component. Hence G acts on $\text{Hom}(\pi, G)$ and we denote the quotient by $\mathcal{X}(\pi, G)$. The quotient $\mathcal{X}(\pi, G)$ is called a *character variety* and $\text{Hom}(\pi, G)$ is called a *representation variety*, although they may not be varieties (but their irreducible components are). Let $\rho_0 \in \text{Hom}(\pi, G)$ denote the trivial homomorphism.

We will show the following.

Theorem 7.1 *Let π be as above and assume that G is semisimple.*

- (1) $\text{Hom}(\pi, G)$ is CIFR and each irreducible component has dimension $(2p - 1) \dim G$.
- (2) $\mathcal{X}(\pi, G)$ has symplectic singularities and has dimension $2(p - 1) \dim G$.

Now suppose that $p > 2$ or that every simple component of G has rank at least 2.

- (3) *The singularities of $\text{Hom}(\pi, G)$ are in codimension at least four and $\text{Hom}(\pi, G)$ is locally factorial.*
- (4) *The singularities of $\mathcal{X}(\pi, G)$ are in codimension at least four and the irreducible component containing $G\rho_0$ is locally factorial. In particular, if G is simply connected, then $\mathcal{X}(\pi, G)$ is locally factorial.*

Remark 7.2 By [Li 1993, Theorem 0.1], the number of irreducible components of $\text{Hom}(\pi, G)$ (and $\mathcal{X}(\pi, G)$) is the cardinality of the fundamental group $\pi_1(G)$ when G is connected and semisimple. More generally, when G is connected and reductive, the number of irreducible components of $\text{Hom}(\pi, G)$ is the cardinality of the fundamental group of a maximal connected semisimple subgroup of G by [Lawton and Ramras 2015, Proposition A.1].

From [Fu 2003, Corollary 1.3] we obtain the following. Note that locally factorial implies locally \mathbb{Q} -factorial.

Corollary 7.3 *Let C be an irreducible component of $\mathcal{X}(\pi, G)$ which is locally factorial with singularities in codimension at least 4. Let U be an open subset of C containing a singular point. Then U does not have a symplectic resolution. In particular, C does not have a symplectic resolution.*

Theorem 7.1(2) was established for reductive groups of type A in [Bellamy and Schedler 2023] while Theorem 7.1(1) was established for GL_n and SL_n in [Budur 2021]. The results for $\text{SL}_n(\mathbb{C})$ (and also $\text{GL}_n(\mathbb{C})$) rely on the theory of quivers. In fact, one proves the results for $\text{GL}_n(\mathbb{C})$ and then deduces they hold for $\text{SL}_n(\mathbb{C})$. For arbitrary reductive G we go in the opposite direction and prove our results first in the semisimple case and then deduce our results in the connected reductive case. We do not use the theory of quivers. As remarked before, Theorem 7.1(1), which is essential in our proof of Theorem 7.1(2), follows from [Aizenbud and Avni 2016], but requires values of p greater than 2.

We now return to the case of an arbitrary semisimple G . By Remark 2.6, to show that $Y = \text{Hom}(\pi, G)$ has rational singularities it is enough to show that Y has rational singularities at any $y \in Y$ such that Gy is closed. We show that the tangent cone at such y is the product of a vector space and the zero set of a moment mapping of the kind considered in Section 4. In particular, the tangent cone has rational singularities. It follows that $\text{Hom}(\pi, G)$ has rational singularities at y , hence has rational singularities. It is not hard to show (using the Campbell–Hausdorff formula) that the tangent cone at $\rho_0 \in \text{Hom}(\pi, G)$ is the zero set of the moment mapping corresponding to the G -module $p\mathfrak{g}$. We know that this has rational singularities. By [Aizenbud and Avni 2016, Theorem IV], this is enough to show that $\text{Hom}(\pi, G)$ has rational singularities. We give here a different and short proof using tangent cones as above. To show that $\mathcal{X}(\pi, G)$ has symplectic singularities we modify the arguments of Section 5 to apply to the case of character varieties.

7.1 The tangent cone

We establish Theorem 7.1(1). Let Y denote the affine scheme $\text{Hom}(\pi, G)$. Let $y \in Y$ where Gy is closed. We determine the tangent cone $\text{TC}_y(Y)$ of Y at y . We follow the approach of Goldman [1984; 1985]. Let A denote the Zariski closure of the subgroup of G generated by the components of y (as a point of G^{2p}). By [Richardson 1988, Theorem 3.6], Gy is closed if and only if A is reductive. Let H denote the stabilizer of y . Then $H = Z_G(A)$. The Zariski tangent space T_y to Y at y is $\text{Ker } d\Phi_y$. By the proof of [Goldman 1984, Proposition 3.7] or [Aizenbud and Avni 2016, Lemma 4.8], the image of $d\Phi_y$ in \mathfrak{g} is \mathfrak{h}^\perp , where the perpendicular is relative to the Killing form \mathcal{B} of \mathfrak{g} . Thus T_y is isomorphic to $(2p-1)\mathfrak{g} \oplus \mathfrak{h}$ as an H -module. Since Gy is a closed orbit in Y , there is a Luna slice S at y whose Zariski tangent space S_y at y is an H -stable complement to $B_y := T_y(Gy) \simeq \mathfrak{g}/\mathfrak{h}$ in T_y . Thus $S_y \simeq 2p\mathfrak{h} \oplus (2p-2)\mathfrak{g}/\mathfrak{h}$ as an H -module.

Let $\rho: \pi \rightarrow G$ be the homomorphism corresponding to y . Then π acts on \mathfrak{g} via ρ and the adjoint action of G . We denote the corresponding π -module by \mathfrak{g}_ρ . In terms of group cohomology [Goldman 1984, 1.3], $T_y = Z^1(\pi, \mathfrak{g}_\rho)$ and $B_y = B^1(\pi, \mathfrak{g}_\rho)$ so that $H^1(\pi, \mathfrak{g}_\rho) \simeq S_y$ as an H -module. Given $u, v \in H^1(\pi, \mathfrak{g}_\rho)$, their cup product is in $H^2(\pi, \mathfrak{g}_\rho \otimes \mathfrak{g}_\rho)$ which maps via \mathcal{B} to $H^2(\pi, \mathbb{C}) \simeq \mathbb{C}$. Goldman shows that the resulting alternating form ω is nondegenerate, i.e. it is a symplectic form on $H^1(\pi, \mathfrak{g}_\rho)$. We can also map $\mathfrak{g}_\rho \otimes \mathfrak{g}_\rho$ to \mathfrak{g}_ρ via Lie bracket. We get a symmetric bilinear form

$$u, v \in H^1(\pi, \mathfrak{g}_\rho) \mapsto [u, v] \in H^2(\pi, \mathfrak{g}_\rho) \simeq H^0(\pi, \mathfrak{g}_\rho^*)^* \simeq \mathfrak{h}.$$

The corresponding quadratic form is

$$(\#) \quad u \in H^1(\pi, \mathfrak{g}_\rho) \mapsto \frac{1}{2}[u, u] \in \mathfrak{h} \simeq \mathfrak{h}^*.$$

A necessary condition for u to be in $\text{TC}(Y)_y$ is that $[u, u]$ vanishes [Goldman 1985, Section 4]. Now S_y has an H -invariant orthogonal structure since \mathfrak{g} is an orthogonal H -module. By [Herbig et al. 2020, Lemma 3.10], there is a Lagrangian H -submodule W of S_y . Then $W \simeq p\mathfrak{h} \oplus (p-1)\mathfrak{g}/\mathfrak{h}$. We have the usual moment mapping $\mu: W \oplus W^* \rightarrow \mathfrak{h}^*$, and by Corollary 4.8, the shell $\mu^{-1}(0)$ is CIFR and has

dimension $(2p - 1) \dim \mathfrak{g} - \dim \mathfrak{g}/\mathfrak{h}$. In the appendix we show that (#) is a moment mapping, hence agrees with μ . It follows that $\mathrm{TC}_y(Y)$ is a subscheme of $N_y := \mu^{-1}(0) \times \mathfrak{g}/\mathfrak{h}$, where the latter has dimension $(2p - 1) \dim G$. Since N_y is reduced and irreducible and $\dim N_y \leq \dim_y Y$, it follows that $\mathrm{TC}(Y)_y = N_y$. Since N_y has rational singularities, $\mathrm{TC}(Y)_y$ does also and Y has rational singularities at y . By Remark 2.6, Y has rational singularities. Moreover, $\dim Y = (2p - 1) \dim G$ so that Y is a complete intersection. To establish Theorem 7.1(1) we only need to show that Y has FPIG.

Let $Y' = \{y \in Y \mid \text{the closed orbit in } \overline{Gy} \text{ has positive-dimensional isotropy}\}$. Then Y' is G -stable and closed. Let $y \in Y'$ such that Gy is closed. Then $\mathrm{TC}_y(Y') \subset \mathrm{TC}(Y)_y = N_y$ and, since $\mathrm{TC}_y(Y')$ is the limit of tangents of curves in Y' starting at y , $\mathrm{TC}_y(Y') \subset (N_y)'$, where $(N_y)'$ is defined relative to the action of $H = G_y$ on N_y . Since N_y has FPIG, $(N_y)'$ has positive codimension in N_y , hence Y' has positive codimension in Y and Y has FPIG. Now the principal isotropy group of each N_y is the center of G , which is therefore the principal isotropy group of each irreducible component of Y and Y has FPIG.

7.2 The character variety

We establish Theorem 7.1(2). We may assume that G is simple. Let $Y = \mathrm{Hom}(\pi, G)$ and $Z = Y//G = \mathcal{X}(\pi, G)$. By Theorem 7.1(1) and Boutot's theorem [1987], Z has rational singularities. Since Y has FPIG, each irreducible component of Z has dimension $2(p - 1) \dim G$. It remains to show that Z has symplectic singularities.

We argue along the lines of Section 5. Our first step is to show that $Z_{\mathrm{sm}} = Z_{\mathrm{pr}}$. The quotient of Y_{pr} is $Z_{\mathrm{pr}} \subset Z_{\mathrm{sm}}$. Let Gy be a closed orbit in Y such that the isotropy group G_y is not principal, ie, not the center of G . We need to show that the corresponding point $z \in Z$ is not smooth. Let \mathcal{O}_Y denote the structure sheaf of Y . Then $\mathrm{TC}_y(Y)$ is the variety of the associated graded ring of the local ring $\mathcal{O}_{Y,y}$. Let μ and the Luna slice S at y be as in Section 7.1. Then $\mathrm{TC}_y(S) = \mu^{-1}(0)$. By Luna's slice theorem, the mapping $S//G_y \rightarrow Z$ is étale at y . Now taking G_y -invariants commutes with taking the associated graded ring, hence the induced mapping $\mathrm{TC}_y(S)//G_y \rightarrow \mathrm{TC}_z(Z)$ is also étale at y . By Corollary 4.8 and Corollary 5.5, $\mathrm{TC}_y(S)//G_y$ is not smooth at y so that $\mathrm{TC}_z(Z)$ is not smooth at z , hence neither is Z . Thus $Z_{\mathrm{sm}} = Z_{\mathrm{pr}}$.

By [Goldman 1984, 1.7–1.9], there is a holomorphic symplectic form on Z_{pr} which agrees with ω on $H^1(\pi, \mathfrak{g}_\rho)$ where ρ is a principal point. Thus there is a nowhere-vanishing holomorphic volume form on Z_{sm} , which implies as in the proof of Proposition 5.3 that Z is Gorenstein. Then by [Namikawa 2001, Theorem 6], Z has symplectic singularities and we have established Theorem 7.1(2).

7.3 Codimension of the singular stratum

Suppose that $p > 2$ or that G contains no simple factor of rank 1. We prove parts (3) and (4) of Theorem 7.1. As in Theorem 4.7, the singularities of every tangent cone $\mathrm{TC}_y(Y)$ at a closed orbit in $Y = \mathrm{Hom}(\pi, G)$ are in codimension at least 4. Hence $\mathrm{TC}_y(Y)$ is factorial by [SGA 2 2005, exposé XI, corollaire 3.14] and Y

is factorial at y . Since the set of factorial points is open [Boissière et al. 2011] and the set of closed orbits is dense, Y is locally factorial and we have (3). It follows from (3) that the singularities of $Z = \mathcal{X}(\pi, G)$ are in codimension at least four. Let Y_0 denote the irreducible component of Y containing ρ_0 . Let L be a G -line bundle on Y_0 . Since G is semisimple, G acts trivially on the fiber L_{ρ_0} , hence G acts trivially on L in a neighborhood of ρ_0 , hence on all of Y_0 . Thus $\text{Pic}_G(Y_0) = \text{Pic}(Y_0)$, and it follows as in [Drézet 2004, Theorem 8.3] that $Y_0//G$ is locally factorial, giving (4).

7.4 The reductive case

Suppose that G is connected and reductive. Then $G = G_s C$, where G_s is semisimple and C is the connected center of G . Let $F = G_s \cap C$. The action of the subgroup F^{2p} on G_s^{2p} induces an action on $\text{Hom}(\pi, G_s)$. We have an isomorphism

$$(\dagger) \quad \text{Hom}(\pi, G) = \text{Hom}(\pi, (G_s \times C)/F) \simeq (\text{Hom}(\pi, G_s) \times C^{2p})/F^{2p},$$

where F^{2p} acts on $\text{Hom}(\pi, G_s)$ as above and on C^{2p} as a subgroup.

Theorem 7.4 *Let G be connected and reductive with $G = G_s C$ and $F = G_s \cap C$ as above. Then the conclusions of Theorem 7.1 hold with the following changes:*

$$\dim \text{Hom}(\pi, G) = (2p - 1) \dim G_s + 2p \dim C = (2p - 1) \dim G + \dim C,$$

$$\dim \mathcal{X}(\pi, G) = (2p - 2) \dim G_s + 2p \dim C = (2p - 2) \dim G + 2 \dim C,$$

and, in (4), $\mathcal{X}(\pi, G)$ is locally factorial if G_s is simply connected.

Proof In the following, (1), (2), etc refer to the parts of Theorem 7.1. Now C^{2p} is a symplectic variety via a symplectic form which is F^{2p} -invariant. The symplectic form on the smooth locus of $\mathcal{X}(\pi, G)$ is also F^{2p} -invariant since for $y, y' \in \text{Hom}(\pi, G_s)$ on the same F^{2p} -orbit, the corresponding π -modules \mathfrak{g}_ρ and $\mathfrak{g}_{\rho'}$ are the same. By (\dagger) and [Beauville 2000, Proposition 2.4], $\mathcal{X}(\pi, G)$ has symplectic singularities. With the changes in dimension, (1) and (2) follow. Part (3) is clear since $\text{Hom}(\pi, G)$ is a complete intersection with singularities in codimension at least 4, and then (4) follows since we are dividing by G_s which has trivial character group. \square

It was pointed out to us by Sean Lawton that an immediate consequence of Theorem 7.4 is the generalization of [Lawton and Sikora 2017, Corollary 8(4)] to the case of G connected and reductive. The argument is identical to that in [Lawton and Sikora 2017] using the fact that $\mathcal{X}(\pi, G)$ is normal. This in particular implies that the regular functions on the character variety are rational functions in characters.

Appendix

We summarize some results in [Goldman 1984, 1.7–1.8; Atiyah and Bott 1983] which we use to show that the mapping $(\#)$ of Section 7 is a moment mapping.

Let Σ be our compact Riemann surface of genus $p \geq 2$ with fundamental group π and universal cover $\tilde{\Sigma}$. Let G be semisimple, and let $x \in \text{Hom}(\pi, G)$ where Gx is closed. Let $\rho: \pi \rightarrow G$ be the corresponding homomorphism and let \mathfrak{g}_ρ denote \mathfrak{g} with the action $\text{Ad} \circ \rho$. Since $\tilde{\Sigma} \rightarrow \Sigma$ is a principal π -bundle we have various associated bundles. Let V_ρ denote the flat vector bundle $\tilde{\Sigma} \times^\pi \mathfrak{g}_\rho$ with flat connection ∇_ρ . We have a flat principal G -bundle $P = \tilde{\Sigma} \times^\pi G$ with flat connection ∇_P , where the action of $w \in \pi$ on G is via left-multiplication by $\rho(w)$. We have a flat bundle of groups $\tilde{G} = \tilde{\Sigma} \times^{\text{conj}_\pi} G$, where $w \in \pi$ acts on G via conjugation with $\rho(w)$. The global sections \mathcal{G} of \tilde{G} , the gauge group of P , act on the left of P as principal bundle automorphisms.

Let $A^*(\Sigma, V_\rho) = \mathcal{E}^*(\Sigma) \otimes \Gamma(V_\rho)$, where $\mathcal{E}^*(\Sigma)$ is the algebra of differential forms on Σ and $\Gamma(V_\rho)$ is the space of smooth sections of V_ρ . Then from ∇_ρ and the usual exterior derivative we get a differential $d_\nabla: A^i(\Sigma, V_\rho) \rightarrow A^{i+1}(\Sigma, V_\rho)$. Let $Z^i(\Sigma, V_\rho)$ and $B^i(\Sigma, V_\rho)$ denote the kernel and image, respectively, of d_∇ in $A^i(\Sigma, V_\rho)$, and let $H^i(\Sigma, V_\rho)$ denote the quotient. Then $H^1(\Sigma, V_\rho) \simeq H^1(\pi, \mathfrak{g}_\rho)$ and similarly for H^2 .

Let $\sigma, \tau \in A^1(\Sigma, V_\rho)$. Then $\sigma \wedge \tau \in A^2(\Sigma, V_\rho \otimes V_\rho)$, and using the killing form \mathcal{B} , we obtain a two form $\mathcal{B}_*(\sigma \wedge \tau) \in \mathcal{E}^2(\Sigma)$. Then $\int_\Sigma \mathcal{B}_*(\sigma \wedge \tau)$ gives a symplectic form $\omega^{(\mathcal{B})}$ on $A^1(\Sigma, V_\rho)$. Restricted to $Z^1(\Sigma, V_\rho)$, the form $\omega^{(\mathcal{B})}$ vanishes when either argument is in $B^1(\Sigma, V_\rho)$, and thus we obtain a 2-form (also denoted $\omega^{(\mathcal{B})}$) on $H^1(\Sigma, V_\rho) \simeq H^1(\pi, \mathfrak{g}_\rho)$ which agrees with the symplectic form ω constructed in Section 7.1.

Note that the Lie algebra of \mathcal{G} is $\tilde{\mathfrak{g}} = A^0(\Sigma, V_\rho)$. Let \mathfrak{A} denote the space of smooth connections on P and let $F(A) \in A^2(\Sigma, V_\rho) \simeq A^0(\Sigma, V_\rho)^* = \tilde{\mathfrak{g}}^*$ denote the curvature of $A \in \mathfrak{A}$. The kernel of F is the space of flat connections $\mathfrak{F} \subset \mathfrak{A}$. Now \mathfrak{A} is an affine space (infinite-dimensional) since $\mathfrak{A} = A + A^1(\Sigma, V_\rho)$ for any $A \in \mathfrak{A}$. Moreover, $\omega^{(\mathcal{B})}$ is invariant under translation by $A^1(\Sigma, V_\rho)$. Thus $\omega^{(\mathcal{B})}$ gives a closed nondegenerate 2-form on $T(\mathfrak{A})$. Now \mathcal{G} acts on \mathfrak{A} and by [Atiyah and Bott 1983, page 587], $F: \mathfrak{A} \rightarrow \tilde{\mathfrak{g}}^*$ is a moment mapping relative to $\omega^{(\mathcal{B})}$.

Now we consider what happens near ∇_P . Since $H = G_x$ and $\rho(G)$ commute, H is a subgroup of \mathcal{G} . The tangent space to the orbit $\mathcal{G} \cdot \nabla_P$ is $B^1(\Sigma, V_\rho)$. Since ∇_P is flat, F vanishes on $\nabla_P + B^1(\Sigma, V_\rho)$. Then for $u \in Z^1(\Sigma, V_\rho)$, we have by [Atiyah and Bott 1983, Lemma 4.5] that $F(\nabla_P + u) = \frac{1}{2}[u, u]$, where the bracket only depends upon the image of u in $H^1(\Sigma, V_\rho) \simeq H^1(\pi, \mathfrak{g}_\rho)$. This bracket is the same as taking the bracket we defined in (#). Thus (#) is a moment mapping.

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