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Weak del Pezzo surfaces with global vector fields

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We classify smooth weak del Pezzo surfaces with global vector fields over an arbitrary algebraically closed field k of arbitrary characteristic $p \geq 0$. We give a complete description of the configuration of (-1)– and (-2)–curves on these surfaces and calculate the identity component of their automorphism schemes. It turns out that there are 53 distinct families of such surfaces if $p \neq 2, 3$, while there are 61 such families if $p = 3$ and 75 such families if $p = 2$. Each of these families has at most one moduli. As a byproduct of our classification, it follows that weak del Pezzo surfaces with nonreduced automorphism schemes exist over k if and only if $p \in \{2, 3\}$.

14E07, 14J26, 14J50, 14L15

1 Introduction

Recall that a weak del Pezzo surface over an algebraically closed field k is a smooth projective surface X with anticanonical divisor class $-K_X$ big and nef, or, equivalently, X is $\mathbb{P}^1 \times \mathbb{P}^1$, the second Hirzebruch surface $\mathbb{F}_2$, or the blowup of at most eight points in $\mathbb{P}^2$ in almost general position. More classically, weak del Pezzo surfaces appear as the minimal resolution of surfaces of degree d in $\mathbb{P}^d$ which are neither cones nor projections of surfaces of minimal degree d in $\mathbb{P}^{d+1}$; see Dolgachev [4, Definition 8.1.5].

By a result of Matsumura and Oort [9], the automorphism functor Aut_X of a proper variety X over k is representable by a group scheme locally of finite type over k. Since Aut_X is well known for surfaces of minimal degree (that is, for quadric surfaces, the Veronese surface and rational normal scrolls [4, Corollary 8.1.2]), weak del Pezzo surfaces form the first class of smooth projective surfaces for which the study of Aut_X is interesting. We are concerned here with the identity component Aut_X^0 of Aut_X, which can be nonreduced in positive characteristic.

While this nonreducedness phenomenon does not occur for smooth projective curves, we will see that it appears for one of the first nontrivial classes of smooth projective surfaces, namely for weak del Pezzo surfaces (see also Neuman [10]), at least in characteristic 2 and 3. This means that for a weak del Pezzo surface X in characteristic 2 and 3 we may have $h^0(X, T_X) > \dim \mathrm{Aut}_X^0$; that is, X may have more global vector fields than expected.

More classically, automorphisms of (weak) del Pezzo surfaces are being studied in the context of the plane Cremona group, ie the group of birational automorphisms of $\mathbb{P}^2$. The main reason for this is that automorphisms of (weak) del Pezzo surfaces yield birational automorphisms of $\mathbb{P}^2$ that do not necessarily extend to biregular automorphisms. For the action of Aut_X^0 on a weak del Pezzo surface X, the situation is very different, since this action always descends to an action on the whole minimal model of X by Blanchard's lemma (Lemma 2.10).

This special feature of the connected automorphism scheme Aut_X^0 will enable us to calculate it explicitly for all weak del Pezzo surfaces that are blowups of $\mathbb{P}^2$ in terms of stabilizers as a subgroup scheme of PGL_3. Using this, we will classify all weak del Pezzo surfaces X with nontrivial Aut_X^0 and determine their configurations of (-2)– and (-1)–curves, as well as their number of moduli:

Main Theorem *Let X be a weak del Pezzo surface over an algebraically closed field. If $h^0(X, T_X) \neq 0$, then X is one of the surfaces in Tables 1, 2, 3, 4, 5 or 6. All cases exist and have an irreducible moduli space of the stated dimension.*

In Tables 1, 3, 4, 5 and 6, the figure describing the configuration of (-2)– and (-1)–curves (lines) on these surfaces is given in column 2. In these figures, a thick curve denotes a (-2)–curve, while a thin curve denotes a (-1)–curve. The intersection multiplicity of two such curves is no more than 3 at every point; intersection multiplicities 1 and 2 will be clear from the picture, whereas we write a small 3 next to the point of intersection if the intersection multiplicity is 3. Recall that the dual graph of all (-2)–curves on a weak del Pezzo surface is a union of Dynkin diagrams of types A_n, D_n and E_n. This graph can be read off from the corresponding figure, but for ease of reference we give its Dynkin type in column 3. For the same reason, in column 4 we list the number of (-1)–curves on these surfaces. In column 5 we describe a general S–valued point of Aut_X^0, where S is a k–scheme. In particular, the dimension of $H^0(X, T_X) = \mathrm{Aut}_X^0(k[\epsilon]/(\epsilon^2))$ can be read off from this description and is listed in column 6 for the convenience of the reader. Comparing this with the dimension of Aut_X^0, it can be checked whether Aut_X^0 is smooth or not. This is done in column 7. If there is more than one weak del Pezzo surface with the configuration of curves and with the automorphism scheme as in the previous columns, we give the dimension of a modular family of such surfaces in column 8. If, instead, there is a unique surface of this type, we write "{pt}" in column 8 in order to emphasize that the surface is unique. Finally, in column 9, we give the characteristic(s) in which the respective surface(s) exist(s).

In particular, our classification also gives a complete list of weak del Pezzo surfaces with nonreduced automorphism schemes. In the following corollary, we list the characteristics p and degrees d for which every weak del Pezzo surface of degree d in characteristic p has reduced automorphism scheme.

Corollary 1.1 *Let k be an algebraically closed field of characteristic $p \geq 0$. Then every weak del Pezzo surface X of degree d over k has reduced automorphism scheme if and only if one of the following three conditions holds*:

(1) $p \neq 2, 3$,

(2) $p = 3$ *and* $d \geq 4$,

(3) $p = 2$ *and* $d \geq 5$.

Moreover, if Aut_X *is nonreduced, then the number of* (-2)*–curves on* X *is at least* $7 - d$.

In particular, the above corollary recovers the result that the automorphism scheme of every del Pezzo surface (where $-K_X$ is ample) is smooth, which is in fact easier to prove and has already been observed by Dolgachev and Duncan (see [5, Theorem 2.4.]).

case	figure	(-2)–curves	#{lines}	$\mathrm{Aut}_X^0 \subseteq \mathrm{PGL}_3$	$h^0(X, T_X)$	Aut_X^0 smooth?	moduli	char(k)
				degree 9				
9A		$\varnothing$	0	PGL_3	8	✓	{pt}	any
				degree 8				
8A	2	$\varnothing$	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & h & i \end{pmatrix}$	6	✓	{pt}	any
				degree 7				
7A	1	$\varnothing$	3	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix}$	4	✓	{pt}	any
7B	9	A_1	2	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix}$	5	✓	{pt}	any
				degree 6				
6A	1	$\varnothing$	6	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	{pt}	any
6B	8	A_1	4	$\begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix}$	3	✓	{pt}	any
6C	1	A_1	3	$\begin{pmatrix} 1 & & c \\ & 1 & f \\ & & i \end{pmatrix}$	3	✓	{pt}	any
6D	8	$2A_1$	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix}$	4	✓	{pt}	any
6E	21	A_2	2	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^2 \end{pmatrix}$	4	✓	{pt}	any
6F	21	$A_2 + A_1$	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix}$	5	✓	{pt}	any
				degree 5				
5A	1	A_1	7	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
5B	8	$2A_1$	5	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	{pt}	any
5C	6	A_2	4	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix}$	2	✓	{pt}	any
5D	8	$A_2 + A_1$	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix}$	3	✓	{pt}	any
5E	21	A_3	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix}$	3	✓	{pt}	any
5F	26	A_4	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix}$	4	✓	{pt}	any

Table 1: Weak del Pezzo surfaces of degree ≥ 5 that are blowups of $\mathbb{P}^2$.

case	(−2)–curves	#{lines}	Aut_X^0	$h^0(X,T_X)$	Aut_X^0 smooth?	moduli	char(k)
$\mathbb{P}^1\times\mathbb{P}^1$	$\varnothing$	0	$\mathrm{PGL}_2\times\mathrm{PGL}_2$	6	✓	{pt}	any
$\mathbb{F}_2$	A_1	0	$(\mathrm{Aut}_{\mathbb{P}(1,1,2)})_{\mathrm{red}}=(\mathbb{G}_a^3\rtimes\mathrm{GL}_2)/\mu_2$	7	✓	{pt}	any

Table 2: Weak del Pezzo surfaces of degree 8 that are not blowups of $\mathbb{P}^2$.

Remark 1.2 Since every Jacobian rational (quasi)elliptic surface X' is the blowup of a weak del Pezzo surface X of degree 1 in the unique basepoint of its anticanonical linear system, Lemma 2.11 yields an isomorphism $\mathrm{Aut}_{X'}^0\cong\mathrm{Aut}_X^0$. In particular, our Main Theorem gives a complete classification of Jacobian rational (quasi)elliptic surfaces with global vector fields. The non-Jacobian case is more involved and will be treated by the second-named author in an upcoming article.

case	figure	(−2)–curves	#{lines}	$\mathrm{Aut}_X^0\subseteq\mathrm{PGL}_3$	$h^0(X,T_X)$	Aut_X^0 smooth?	moduli	char(k)
4A	4	$2A_1$	8	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	1 dim	any
4B	5	$3A_1$	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
4C	5	A_2+A_1	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
4D	6	A_3	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
4E	17	A_3	4	$\begin{pmatrix}1&&c\\&1&\\&&1\end{pmatrix}$	1	✓	{pt}	$\neq 2$
4F	7	$4A_1$	4	$\begin{pmatrix}1&&\\&e&\\&&i\end{pmatrix}$	2	✓	{pt}	any
4G	7	A_2+2A_1	4	$\begin{pmatrix}1&&\\&e&\\&&i\end{pmatrix}$	2	✓	{pt}	any
4H	17	A_3+A_1	3	$\begin{pmatrix}1&&c\\&1&\\&&i\end{pmatrix}$	2	✓	{pt}	any
4I	20	A_4	3	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}$	2	✓	{pt}	any
4J	25	D_4	2	$\begin{pmatrix}1&&c\\&e&\\&&e^2\end{pmatrix}$	2	✓	{pt}	$\neq 2$
4K	20	A_3+2A_1	2	$\begin{pmatrix}1&&\\&e&f\\&&i\end{pmatrix}$	3	✓	{pt}	any
4L	28	D_5	1	$\begin{pmatrix}1&&c\\&e&f\\&&e^3\end{pmatrix}$	3	✓	{pt}	$\neq 2$
4M	17	A_3	4	$\begin{pmatrix}1&&c\\&1&\\&&i\end{pmatrix}, i^2=1$	2	×	{pt}	$=2$
4N	25	D_4	2	$\begin{pmatrix}1&&c\\&1&f\\&&1\end{pmatrix}$	2	✓	{pt}	$=2$
4O	25	D_4	2	$\begin{pmatrix}1&&c\\&e&f\\&&e^2\end{pmatrix}$	3	✓	{pt}	$=2$
4P	28	D_5	1	$\begin{pmatrix}1&b&c\\&1&f\\&&1\end{pmatrix}$	3	✓	{pt}	$=2$
4Q	28	D_5	1	$\begin{pmatrix}1&b&c\\&e&f\\&&e^3\end{pmatrix}$	4	✓	{pt}	$=2$

Table 3: Weak del Pezzo surfaces of degree 4.

case	figure	(-2)–curves	#{lines}	$\mathrm{Aut}^0_X \subseteq \mathrm{PGL}_3$	$h^0(X, T_X)$	Aut^0_X smooth?	moduli	char(k)
3A	4	$2A_2$	7	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	1 dim	any
3B	6	D_4	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
3C	4	$2A_2+A_1$	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
3D	4	A_3+2A_1	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
3E	16	A_4+A_1	4	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
3F	19	A_5	3	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	{pt}	$\neq 3$
3G	25	D_5	3	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}$	1	✓	{pt}	$\neq 2$
3H	7	$3A_2$	3	$\begin{pmatrix}1&&\\&e&\\&&i\end{pmatrix}$	2	✓	{pt}	any
3I	19	A_5+A_1	2	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}$	2	✓	{pt}	any
3J	29	E_6	1	$\begin{pmatrix}1&&c\\&e&\\&&e^3\end{pmatrix}$	2	✓	{pt}	$\neq 2,3$
3K	19	A_5	3	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}, e^3=1$	2	×	{pt}	$=3$
3L	29	E_6	1	$\begin{pmatrix}1&&c\\&1&f\\&&1\end{pmatrix}$	2	✓	{pt}	$=3$
3M	29	E_6	1	$\begin{pmatrix}1&&c\\&e&f\\&&e^3\end{pmatrix}$	3	✓	{pt}	$=3$
3N	13	A_4	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	{pt}	$=2$
3O	25	D_5	3	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	{pt}	$=2$
3P	25	D_5	3	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}$	2	✓	{pt}	$=2$
3Q	29	E_6	1	$\begin{pmatrix}1&b&c\\&1&b^2+b\\&&1\end{pmatrix}$	2	✓	{pt}	$=2$
3R	29	E_6	1	$\begin{pmatrix}1&b&c\\&e&b^2e\\&&e^3\end{pmatrix}$	3	✓	{pt}	$=2$

Table 4: Weak del Pezzo surfaces of degree 3.

Remark 1.3 Independently, shortly after the upload of this article to arXiv and using a completely different approach, Cheltsov and Prokhorov [2] classified RDP del Pezzo surfaces Y over an algebraically closed field k of characteristic 0 such that $\mathrm{Aut}_Y(k)$ is infinite. Now, $\mathrm{Aut}_Y(k)$ is infinite if and only if $\mathrm{Aut}^0_Y(k)$ is infinite, which holds if and only if $\mathrm{Aut}^0_X(k)$ is infinite, where X is the weak del Pezzo surface that is the minimal resolution of Y. Since Aut^0_X is always smooth in characteristic 0 by Cartier's theorem (see eg Perrin [11, Corollaire 4.2.8]), $\mathrm{Aut}^0_X(k)$ is infinite if and only if X admits global vector fields. So, the classification in [2] is equivalent to the characteristic-0 part of our Main Theorem.

case	figure	(-2)–curves	#{lines}	$\mathrm{Aut}_X^0 \subseteq \mathrm{PGL}_3$	$h^0(X, T_X)$	Aut_X^0 smooth?	moduli	$\mathrm{char}(k)$
2A	3	$2A_3$	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	1 dim	any
2B	15	D_5+A_1	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
2C	27	E_6	4	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}$	1	✓	{pt}	$\neq 2$
2D	3	$2A_3+A_1$	4	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
2E	3	D_4+3A_1	4	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
2F	16	A_5+A_2	3	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
2G	24	D_6+A_1	2	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}$	1	✓	{pt}	$\neq 2$
2H	24	A_7	2	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	{pt}	$\neq 2$
2I	30	E_7	1	$\begin{pmatrix}1&&\\&e&\\&&e^3\end{pmatrix}$	1	✓	{pt}	$\neq 2, 3$
2J	18	A_6	4	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}, e^3=1$	1	×	{pt}	$=3$
2K	23	D_6	3	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}, e^3=1$	1	×	{pt}	$=3$
2L	30	E_7	1	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	{pt}	$=3$
2M	30	E_7	1	$\begin{pmatrix}1&&\\&e&f\\&&e^3\end{pmatrix}$	2	✓	{pt}	$=3$
2N	11	A_5	7	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	1 dim	$=2$
2O	15	D_5	8	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	{pt}	$=2$
2P	12	A_5+A_1	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	{pt}	$=2$
2Q	11	A_5+A_1	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	{pt}	$=2$
2R	23	D_6	3	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	1 dim	$=2$
2S	27	E_6	4	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}, f^2=0$	2	×	{pt}	$=2$
2T	24	D_6+A_1	2	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	{pt}	$=2$
2U	24	D_6+A_1	2	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}$	2	✓	{pt}	$=2$
2V	24	A_7	2	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}, e^4=1$	2	×	{pt}	$=2$
2W	30	E_7	1	$\begin{pmatrix}1&&c\\&1&\\&&1\end{pmatrix}$	1	✓	{pt}	$=2$
2X	30	E_7	1	$\begin{pmatrix}1&b&c\\&1&b^2\\&&1\end{pmatrix}$	2	✓	{pt}	$=2$
2Y	30	E_7	1	$\begin{pmatrix}1&b&c\\&e&b^2e\\&&e^3\end{pmatrix}$	3	✓	{pt}	$=2$

Table 5: Weak del Pezzo surfaces of degree 2.

case	figure	(-2)–curves	#{lines}	$\mathrm{Aut}_X^0 \subseteq \mathrm{PGL}_3$	$h^0(X, T_X)$	Aut_X^0 smooth?	moduli	char(k)
1A	2	$2D_4$	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	1 dim	any
1B	15	E_6+A_2	4	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	{pt}	any
1C	27	E_7+A_1	3	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}$	1	✓	{pt}	$\neq 2$
1D	31	E_8	1	$\begin{pmatrix}1&&\\&e&\\&&e^3\end{pmatrix}$	1	✓	{pt}	$\neq 2, 3$
1E	22	D_7	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^3=1$	1	×	{pt}	$=3$
1F	26	E_7	5	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}, e^3=1$	1	×	{pt}	$=3$
1G	17	A_8	3	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}, e^3=1$	1	×	{pt}	$=3$
1H	31	E_8	1	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	{pt}	$=3$
1I	31	E_8	1	$\begin{pmatrix}1&&\\&e&f\\&&e^3\end{pmatrix}$	2	✓	{pt}	$=3$
1J	13	E_6	13	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	1 dim	$=2$
1K	13	E_6+A_1	8	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	{pt}	$=2$
1L	10	A_7	8	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	1 dim	$=2$
1M	26	E_7	5	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}, f^2=0$	1	×	{pt}	$=2$
1N	10	D_6+2A_1	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	{pt}	$=2$
1O	10	A_7+A_1	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2=1$	1	×	{pt}	$=2$
1P	27	E_7+A_1	3	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}, f^2=0$	2	×	{pt}	$=2$
1Q	24	D_8	2	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	1 dim	$=2$
1R	24	D_8	2	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}, e^4=1$	2	×	{pt}	$=2$
1S	31	E_8	1	$\begin{pmatrix}1&&c\\&1&\\&&1\end{pmatrix}$	1	✓	{pt}	$=2$
1T	31	E_8	1	$\begin{pmatrix}1&b&c\\&e&b^2e\\&&e^3\end{pmatrix}, b^4=0$	3	×	{pt}	$=2$

Table 6: Weak del Pezzo surfaces of degree 1.

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2 Generalities

This section provides the necessary background on our two main topics: weak del Pezzo surfaces and automorphism schemes. Throughout, we will be working over an algebraically closed field k.

2.1 Geometry of weak del Pezzo surfaces and their "height"

We recall that every weak del Pezzo surface X (except $X = \mathbb{P}^1 \times \mathbb{P}^1$ and the second Hirzebruch surface $X = \mathbb{F}_2$) is a successive blowup of $\mathbb{P}^2$ satisfying certain properties (see Lemmas 2.5 and 2.7), and we define the notion of "height", which is a measure for the complexity of X. We describe the set of all (-2)– and (-1)–curves on X in terms of a realization of X as a blowup of $\mathbb{P}^2$.

Definition 2.1 A *weak del Pezzo surface* is a smooth projective surface X with nef and big anticanonical class $-K_X$. The number $\deg(X) = K_X^2$ is called the *degree of* X.

Recall that every birational morphism $\pi\colon X' \to X$ of smooth projective surfaces can be factored as

$$\pi\colon X' \xrightarrow{\varphi} X'^{(n)} \xrightarrow{\pi^{(n-1)}} X'^{(n-1)} \xrightarrow{\pi^{(n-2)}} \cdots \xrightarrow{\pi^{(1)}} X'^{(1)} \xrightarrow{\pi^{(0)}} X'^{(0)} = X,$$

where φ is an isomorphism and each $\pi^{(i)}\colon X'^{(i+1)} \to X'^{(i)}$ is the blowup of a number of distinct closed points on $X'^{(i)}$. The isomorphism φ can be neglected by identifying X' with $X'^{(n)}$ via φ. Then the above factorization becomes unique (up to unique isomorphism for every $n \geq i \geq 1$) if in each step we blow up the maximal number of distinct closed points of $X'^{(i)}$. In this case, we call the above factorization of π *minimal*.

Definition 2.2 Let X and X' be two smooth projective surfaces.

- For every birational morphism $\pi\colon X' \to X$, let $\pi = \pi^{(0)} \circ \cdots \circ \cdots \pi^{(n-1)}$ be its minimal factorization. The *height of* π is defined as
$$\mathrm{ht}(\pi) := n.$$
- If X' admits some birational morphism to X, we define the *height of* X' *over* X as
$$\mathrm{ht}(X'/X) := \min_{\pi\colon X' \to X} \{\mathrm{ht}(\pi)\},$$
where the minimum is taken over all birational morphisms $\pi\colon X' \to X$.
- If X is a weak del Pezzo surface which is a successive blowup of $\mathbb{P}^2$, then we define
$$\mathrm{ht}(X) := \mathrm{ht}(X/\mathbb{P}^2),$$
and if X is not a blowup of $\mathbb{P}^2$, we set $\mathrm{ht}(X) = 0$.

Remark 2.3 The reader should compare our notion of height with the height function on the bubble space of X considered in [4, Section 7.3.2].

Notation 2.4 Let $\pi\colon X \to \mathbb{P}^2$ be a birational morphism of height n, and let $\pi = \pi^{(0)} \circ \cdots \circ \pi^{(n-1)}$ be its minimal factorization. Then we fix the following notation:

- For each $0 \leq i < n$, we let $p_{1,i}, \dots, p_{n_i,i} \in X^{(i)}$ be the points blown up under $\pi^{(i)}$.
- The exceptional divisor $(\pi^{(i)})^{-1}(p_{j,i}) \subseteq X^{(i+1)}$ over a closed point $p_{j,i} \in X^{(i)}$ will be denoted by $E_{j,i}$ for $j = 1, \dots, n_i$.
- For every $0 \leq i \leq k \leq n$, the strict transform of a curve $C \subseteq X^{(i)}$ along $\pi^{(i)} \circ \cdots \circ \pi^{(k-1)}$ is denoted by $C^{(k)}$.

Using this notation, we can now state a necessary and sufficient criterion for a successive blowup of $\mathbb{P}^2$ to be a weak del Pezzo surface.

Lemma 2.5 [3; 4, Section 8.1.3] *With Notation 2.4, let $\pi\colon X \to \mathbb{P}^2$ be a birational morphism of height n. Then X is a weak del Pezzo surface if and only if the following three conditions hold*:

- *On each $E_{j,i}$ there is at most one $p_{k,i+1}$.*
- *For every line $\ell \subseteq \mathbb{P}^2$ there are at most three $p_{j,i}$ with $p_{j,i} \in \ell^{(i)}$, where i ranges over $0, \dots, n-1$.*
- *For every irreducible conic $Q \subseteq \mathbb{P}^2$ there are at most six $p_{j,i}$ with $p_{j,i} \in Q^{(i)}$, where i ranges over $0, \dots, n-1$.*

Notation 2.6 By Lemma 2.5, there is at most one $p_{k,i+1}$ on each $E_{j,i}$. Therefore, it makes sense to rename the $p_{k,i+1}$ so that $p_{k,i+1}$ lies on $E_{k,i}$. We will adopt this convention from now on.

If the above three conditions of Lemma 2.5 are satisfied, we say that the points $p_{j,i}$ are in *almost general position*. Using this terminology, there is the following well-known characterization of weak del Pezzo surfaces:

Lemma 2.7 [4, Section 8.1.3] *If X is a weak del Pezzo surface*, *then*

(i) $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, *or*

(ii) $X \cong \mathbb{F}_2$, *the* ***second Hirzebruch surface***, *or*

(iii) *X is the successive blowup of $\mathbb{P}^2$ in $n \leq 8$ points in* ***almost general position***.

In particular, $1 \leq \deg(X) \leq 9$, *and* $\mathrm{ht}(X) = 0$ *if and only if* $X \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2\}$.

All classes of (-2)– and (-1)–curves in the odd unimodular lattice $\mathrm{Pic}(X) = \mathrm{I}_{1,9-\deg(X)}$ of signature $(1, 9-\deg(X))$ are well known and described in [4, Proposition 8.2.7; 6, Definition 23.7, Proposition 26.1]. This lattice-theoretic description can be translated into geometry (see [6, Theorem 26.2(ii)] for the case of del Pezzo surfaces). A straightforward adaption of Manin's approach to our situation of *weak* del Pezzo surfaces yields the following description of (-2)– and (-1)–curves on X:

Lemma 2.8 *Let X be a weak del Pezzo surface and let $\pi : X = X^{(n)} \to \mathbb{P}^2$ be a birational morphism of height n.*

(i) *A curve on X is a (-2)–curve if and only if it is of one of the following four types*:
- *the strict transform $E_{j,i}^{(n)}$ of an exceptional curve such that there is exactly one $p_{j,i+1}$ on $E_{j,i}$,*
- *the strict transform $\ell^{(n)}$ of a line $\ell \subseteq \mathbb{P}^2$ such that there are exactly three $p_{j,i}$ with $p_{j,i} \in \ell^{(i)}$,*
- *the strict transform $C^{(n)}$ of an irreducible conic $C \subseteq \mathbb{P}^2$ such that there are exactly six $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, or*
- *the strict transform $C^{(n)}$ of an irreducible singular cubic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that one of the $p_{j,0}$ is the singular point of C.*

(ii) *A curve on X is a (-1)–curve if and only if it is of one of the following seven types*:
- *the strict transform $E_{j,i}^{(n)}$ of an exceptional curve such that there is no $p_{k,i+1}$ on $E_{j,i}$,*
- *the strict transform $\ell^{(n)}$ of a line $\ell \subseteq \mathbb{P}^2$ such that there are exactly two $p_{j,i}$ with $p_{j,i} \in \ell^{(i)}$,*
- *the strict transform $C^{(n)}$ of an irreducible conic $C \subseteq \mathbb{P}^2$ such that there are exactly five $p_{j,i}$ with $p_{j,i} \in C^{(i)}$,*
- *the strict transform $C^{(n)}$ of an irreducible singular cubic $C \subseteq \mathbb{P}^2$ such that there are exactly seven $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that one of the $p_{j,0}$ is the singular point of C,*
- *the strict transform $C^{(n)}$ of an irreducible singular quartic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that exactly three of the $p_{j,i}$ are double points of $C^{(i)}$,*
- *the strict transform $C^{(n)}$ of an irreducible singular quintic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that exactly six of the $p_{j,i}$ are double points of $C^{(i)}$, or*
- *the strict transform $C^{(n)}$ of an irreducible singular sextic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that exactly seven of the $p_{j,i}$ are double points of $C^{(i)}$ and exactly one of the $p_{j,0}$ is a triple point of C.*

Remark 2.9 The criterion given in Lemma 2.5 simply tells us that a successive blowup of $\mathbb{P}^2$ in at most eight points is a weak del Pezzo surface if and only if we have never blown up a point on a (-2)–curve.

2.2 Automorphism schemes of blowups of smooth surfaces

By a result of Matsumura and Oort [9], the automorphism functor Aut_X^0 of a proper variety over k is representable, and it is well known that the tangent space of Aut_X^0 can be identified naturally with $H^0(X, T_X)$. The main tool in our study of automorphism schemes of weak del Pezzo surfaces is the following lemma of Blanchard (see [1, Theorem 7.2.1]):

Lemma 2.10 (Blanchard's lemma) *Let $f : Y \to X$ be a morphism of proper schemes over k with $f_*\mathcal{O}_Y = \mathcal{O}_X$. Then f induces a homomorphism of group schemes $f_* : \mathrm{Aut}_Y^0 \to \mathrm{Aut}_X^0$. If f is birational, then f_* is a closed immersion.*

Thus, if f is birational, we can and will identify Aut_Y^0 with its image under f_* in the following. If f is the blowup of a smooth surface X in a closed point p, it is possible to describe the image of f_*; see [7, Proposition 2.7; 10, Lemma 1.1].

Lemma 2.11 *Let $f\colon Y \to X$ be the blowup of a smooth projective surface X in n distinct points $p_1,\dots,p_n \in X$. Then $\mathrm{Aut}_Y^0 = \big(\bigcap_{i=1}^n \mathrm{Stab}_{p_i}^0\big)^0$.*

Proof We prove the claim by induction on n with the case $n=0$ being trivial. For the inductive step, let Y' be the blowup of X in $p_1,\dots,p_{n-1}$. Then $f'\colon Y \to Y'$ is the blowup in p_n and we have $\mathrm{Aut}_{Y'}^0 = \big(\bigcap_{i=1}^{n-1} \mathrm{Stab}_{p_i}^0\big)^0$ by the induction hypothesis. Note that the identity component of the stabilizer of $p_n \in Y'$, with respect to the action of $\mathrm{Aut}_{Y'}^0$, is precisely $\big(\bigcap_{i=1}^n \mathrm{Stab}_{p_i}^0\big)^0$. By [7, Remark 2.8], the Aut_Y^0–action on Y preserves the exceptional divisor of f', hence Aut_Y^0, being connected, is contained in $\big(\bigcap_{i=1}^n \mathrm{Stab}_{p_i}^0\big)^0$. Conversely, by [7, Proposition 2.7], the $\big(\bigcap_{i=1}^n \mathrm{Stab}_{p_i}^0\big)^0$–action on Y' lifts to Y, and since $\big(\bigcap_{i=1}^n \mathrm{Stab}_{p_i}^0\big)^0$ is connected, it actually lifts to a subgroup scheme of Aut_Y^0. □

Let $\pi\colon X'^{(n)} \to X$ be a birational morphism of smooth projective surfaces X and $X'^{(n)}$. Let $E \subseteq X'^{(n)}$ be a π–exceptional irreducible curve. Recall that the left-action of Aut_X^0 on Hilb_X is given on S–valued points by

$$\mathrm{Aut}_X^0(S) \times \mathrm{Hilb}_X(S) \xrightarrow{\rho(S)} \mathrm{Hilb}_X(S), \quad (g\colon X_S \to X_S, \iota\colon Z \hookrightarrow X_S) \mapsto (Z \times_{\iota, X_S, g^{-1}} X_S \hookrightarrow X_S),$$

where $X_S := X \times S$, and this induces a natural action ρ of $\mathrm{Aut}_{X'^{(n)}}^0 \subseteq \mathrm{Aut}_X^0$ on Hilb_X. For a *pencil* (that is, a 1–dimensional linear system) $f\colon \mathcal{C} \to \mathbb{P}^1 \subseteq \mathrm{Hilb}_X$ of curves on X, we will identify a point $p \in \mathbb{P}^1(S)$ with its fiber $\mathcal{C}_p$ under f. Let $V \subseteq \mathbb{P}^1$ be an open subset such that any two fibers $\mathcal{C}_p$ and $\mathcal{C}_q$ with $p,q \in V$ (as well as their strict transforms in all the $X'^{(i)}$) have the same multiplicity at the $p_{j,i}$. Then the rational map

$$\mathbb{P}^1 \supseteq V \to \mathrm{Hilb}_E, \quad p \mapsto \mathcal{C}_p^{(n)} \cap E, \tag{2-1}$$

can be extended to a morphism φ from $\mathbb{P}^1$, since every irreducible component of Hilb_E is proper.

Definition 2.12 Let $\pi\colon X'^{(n)} \to X$ be a birational morphism of smooth projective surfaces X and $X'^{(n)}$. Let $E \subseteq X'^{(n)}$ be a π–exceptional irreducible curve. A pencil of curves $f\colon \mathcal{C} \to \mathbb{P}^1$ is called *adapted* to E and π (or *E–adapted*), if the morphism φ of (2-1) factors through an isomorphism $\mathbb{P}^1 \xrightarrow{\cong} E \subseteq \mathrm{Hilb}_E$.

For an adapted pencil $\mathcal{C} \to \mathbb{P}^1$, we can transfer the $\mathrm{Aut}_{X'^{(n)}}^0$–action on E via φ to an action on the pencil. Over V, we can describe this action explicitly on S–valued points as follows. For $\mathcal{C}_p \in V(S) \subseteq \mathbb{P}^1(S)$ with embedding $\iota\colon \mathcal{C}_p \to X_S$, an element $g \in \mathrm{Aut}_{X'^{(n)}}(S)$ sends $\mathcal{C}_p$ to the unique curve $\mathcal{C}_{g(p)} \in \mathbb{P}^1(S)$ such that $(\mathcal{C}_p \times_{\iota, X_S, g^{-1}} X_S)^{(n)} \cap E_S = \varphi(\mathcal{C}_{g(p)})$. The action of $\mathrm{Aut}_{X'^{(n)}}^0$ transferred from E to the pencil is the unique extension of the above action from V to $\mathbb{P}^1$. In particular, orbits and stabilizers of the $\mathrm{Aut}_{X'^{(n)}}^0$–action on E can be calculated on $\mathbb{P}^1$, which we exploit throughout.

Remark 2.13 In most of the cases occurring in our classification we can choose the adapted pencil $\mathcal{C} \to \mathbb{P}^1$ to be stable under the natural action of $\mathrm{Aut}_{X'^{(n)}}^0$ on Hilb_X. In this case, $\mathcal{C}_{g(p)} = \mathcal{C}_p \times_{\iota, X_S, g^{-1}} X_S$.

Example 2.14 $\mathrm{Aut}^0_{X'^{(n)}}$–stable adapted pencils do not always exist, even for blowups of $\mathbb{P}^2$:

Consider the morphism $\pi\colon X'^{(2)} \to \mathbb{P}^2$ of height 2 given by blowing up the points $p_{1,0} = [1:0:0]$, $p_{2,0} = [0:1:0]$, $p_{3,0} = [1:1:0]$ and $p_{1,1} := \ell_y^{(1)} \cap E_{1,0}$, where $\ell_y = V(y)$. Then $X'^{(2)}$ is surface 5C in Table 1. In the classification in Section 4 (see Case 5C), we use an $E_{1,1}$–adapted pencil which is not $\mathrm{Aut}^0_{X'^{(2)}}$–stable to show that

$$\mathrm{Aut}^0_{X'^{(2)}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$$

acts on $E_{1,1}$ as $[\lambda:\mu] \mapsto [\lambda : i^2\mu]$. For this morphism π, there is no $E_{1,1}$–adapted pencil which is also $\mathrm{Aut}^0_{X'^{(2)}}$–stable.

Indeed, seeking a contradiction, assume that there exists such a pencil whose fiber over $[\lambda:\mu] \in \mathbb{P}^1$ is $C_{\lambda,\mu} = V(\lambda f_1 + \mu f_2)$ with f_1 and f_2 homogeneous of the same degree. By the previous paragraph, the subgroup scheme $\mathbb{G}_a \subseteq \mathrm{Aut}^0_{X'^{(2)}}$ of automorphisms with $i = 1$ acts trivially on $E_{1,1}$. By Remark 2.13, this implies that every $C_{\lambda,\mu}$ is stable under this $\mathbb{G}_a$–action. In particular, every $C_{\lambda,\mu}$ is a union of orbits of the $\mathbb{G}_a$–action on $\mathbb{P}^2$. The closures of the $\mathbb{G}_a$–orbits are the lines through $[1:0:0]$ except $V(z)$, and every point on $V(z)$. Therefore, each $C_{\lambda,\mu}$ is a union of lines through $[1:0:0]$, hence $\varphi(C_{\lambda,\mu}) = n(\ell_y^{(2)} \cap E_{1,1})$ for some $n \geq 0$, and thus the pencil is not $E_{1,1}$–adapted, contradicting our assumption.

Remark/Notation 2.15 If $X = \mathbb{P}^2$, and f_1 and f_2 are homogeneous equations of the same degree, we say that $\lambda f_1 + \mu f_2$ is *adapted* (to π and E) if the pencil spanned by $C_1 = \mathcal{V}(f_1)$ and $C_2 = \mathcal{V}(f_2)$ is adapted to π and E and if, in addition, we identified C_1 and C_2 with $[1:0]$ and $[0:1]$ in $\mathbb{P}^1$, respectively. We will use this choice of coordinates to determine the orbits and stabilizers of the $\mathrm{Aut}^0_{X'^{(n)}}$–action on E explicitly by reducing it to a calculation on the pencil $[\lambda:\mu]$.

3 Strategy of proof

For the proof of our Main Theorem we argue inductively by going through all possible weak del Pezzo surfaces with nontrivial connected automorphism scheme in the order given by their height. We start with del Pezzo surfaces of height 0, which are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_2$. Then, by Lemma 2.7, to study del Pezzo surfaces of height 1 we have to study blowups of $\mathbb{P}^2$ in a number of distinct "honest" points. After that, for height 2, we have to consider del Pezzo surfaces that arise as blowups of points on exceptional divisors of blowups of points in $\mathbb{P}^2$ (sometimes we will also refer to such points as *infinitely near* points of the first order, as was introduced in [4, Section 7.3.2, page 307]). Continuing this pattern, increasing the height by 1 means that we have to study those surfaces that arise as blowups of points on the "latest exceptional divisor".

In this subsection, we further specify our strategy of proof and explain why the classification of weak del Pezzo surfaces with nontrivial vector fields obtained via our inductive procedure is indeed complete.

3.1 Inductive strategy

Assume we have a complete set $\mathcal{L}_i = \{X_k\}_{k \in K_i}$, for some index set K_i, of representatives of weak del Pezzo surfaces of height i that are blowups of $\mathbb{P}^2$ with $H^0(X_k, T_{X_k}) \neq 0$, where for every X_k we have fixed a birational morphism $\psi_k\colon X_k \to \mathbb{P}^2$ of height i. Further assume that we have calculated $(\psi_k)_*(\mathrm{Aut}^0_{X_k}) \subseteq \mathrm{PGL}_3$ (see Lemma 2.11) for every k. If $i = 0$, such a list is given by $\mathcal{L}_0 = \{\mathbb{P}^2\}$ with $\mathrm{Aut}^0_{\mathbb{P}^2} = \mathrm{PGL}_3$. Using the list $\mathcal{L}_i$, we produce a list $\mathcal{L}_{i+1}$ as follows:

Procedure 3.1 **Step 1** Choose $X \in \mathcal{L}_i$ with $\psi\colon X \to \mathbb{P}^2$ and let

$$\psi\colon X \xrightarrow{\psi^{(i-1)}} X^{(i-1)} \xrightarrow{\psi^{(i-2)}} \cdots \xrightarrow{\psi^{(0)}} X^{(0)} = \mathbb{P}^2$$

be the minimal factorization of ψ.

Step 2 If $i = 0$, let $E := X = \mathbb{P}^2$. Otherwise, let

$$E := \left(\mathrm{Exc}(\psi^{(i-1)}) - \bigcup_{j=0}^{i-2} \mathrm{Exc}(\psi^{(j)})\right) - D,$$

where D is the union of all (-2)–curves on X. Note that, if $i > 0$, then E is the set of points on the "latest" exceptional divisors that do not lie on (-2)–curves. Using the description of Aut^0_X as a subgroup scheme of PGL_3, we calculate the orbits and stabilizers of the action of Aut^0_X on E using $E_{j,i-1}$–adapted pencils.

Step 3 Choose a set of points $\{p_{1,i}, \dots, p_{n_i,i}\} \subseteq E$ such that $\left(\bigcap_{j=1}^{n_i} \mathrm{Stab}^0_{p_{j,i}}\right)^0$ is nontrivial and such that the blowup $\psi'\colon X' \to X$ in these points is still a weak del Pezzo surface (see the criterion given in Lemma 2.8). In particular, since there is at most one of the $p_{j,i}$ on every exceptional curve, we may assume that $p_{j,i} \in E_{j,i-1}$. Note that we obtain isomorphic surfaces if we replace a point $p_{j,i}$ by a point in the same orbit under the action of $\bigcap_{k \neq j} \mathrm{Stab}_{p_{k,i}} \subseteq \mathrm{Aut}_X$.

Step 4 If X' is isomorphic to a surface already contained in $\mathcal{L}_j$ for some $j \leq i+1$, discard this case. Otherwise, add X' to $\mathcal{L}_{i+1}$, choose the blowup realization $\psi \circ \psi'\colon X' \to \mathbb{P}^2$, and calculate

$$(\psi \circ \psi')_*(\mathrm{Aut}^0_{X'}) = (\psi_*)\left(\bigcap_{j=1}^{n_i} \mathrm{Stab}^0_{p_{j,i}}\right)^0 \subseteq \mathrm{PGL}_3 .$$

We do this by describing the group $\mathrm{Aut}^0_{X'}(R)$ for an arbitrary local k–algebra R (see Section 3.2).

Step 5 Repeat Steps 3 and 4 for all possible point combinations $\{p_{1,i}, \dots, p_{n_i,i}\}$.

Step 6 Repeat Steps 1–5 for all $X \in \mathcal{L}_i$.

Lemma 3.2 *For every i, Procedure 3.1 yields a complete set $\mathcal{L}_{i+1} = \{X_k\}_{k \in K_{i+1}}$ of representatives of isomorphism classes of weak del Pezzo surfaces of height $i+1$ with nontrivial global vector fields that are blowups of $\mathbb{P}^2$.*

Proof We prove the claim by induction on the height i. The case $i = 0$ with $\mathcal{L}_0 = \{\mathbb{P}^2\}$ is clear by Lemma 2.11. Therefore, assume that the claim holds for $i - 1 \geq 0$ and that we have a list $\mathcal{L}_i$.

Let X' be a weak del Pezzo surface of height $i+1$ with $h^0(X', T_{X'}) \neq 0$. Choose a birational morphism $\pi: X' \to \mathbb{P}^2$ with minimal factorization

$$\pi: X' = X'^{(i+1)} \xrightarrow{\pi^{(i)}} X'^{(i)} \xrightarrow{\pi^{(i-1)}} \cdots \xrightarrow{\pi^{(0)}} X'^{(0)} = \mathbb{P}^2$$

such that, for every birational morphism $\pi': X' \to \mathbb{P}^2$, the number of exceptional curves for $\pi'^{(i)}$ is at least as great as the number of exceptional curves for $\pi^{(i)}$, ie such that the number of points blown up by the last step $\pi^{(i)}$ is minimal. By Lemma 2.10, there is an inclusion

$$(\pi^{(i)})_*(\mathrm{Aut}^0_{X'}) \subseteq \mathrm{Aut}^0_{X'^{(i)}}.$$

In particular, $h^0(X'^{(i)}, T_{X'^{(i)}}) \neq 0$, since $\mathrm{Aut}^0_{X'} \neq \{\mathrm{id}\}$ and $(\pi^{(i)})_*$ is a closed immersion. Hence, by the induction hypothesis, there is $X \in \mathcal{L}_i$ such that there exists an isomorphism $\phi: X'^{(i)} \to X$ and X comes with a birational morphism $\psi: X \to \mathbb{P}^2$.

To prove the claim, it suffices to show that $\phi \circ \pi^{(i)}$ is the blowup of X in a set of points $p_{1,i}, \dots, p_{n_i,i}$ on E, defined as in Procedure 3.1. Indeed, once we prove this, it will follow from Lemma 2.11 and the assumption $h^0(X', T_{X'}) \neq 0$ that $\mathrm{Aut}^0_{X'} = \left(\bigcap_{j=1}^{n_i} \mathrm{Stab}^0_{p_{j,i}}\right)^0$ is nontrivial.

Now, note that the condition that the $p_{j,i}$ lie on E is equivalent to $\phi \circ \pi^{(i)}$ being the first step in the minimal factorization of

$$\psi' := \psi \circ \phi \circ \pi^{(i)}: X' \to X'^{(i)} \to X \to \mathbb{P}^2.$$

Thus, we take the minimal factorization of ψ' and let $\psi'^{(i)}: X' \to X''$ be the first morphism in the minimal factorization of ψ'. Since X has height i, the morphism $\phi \circ \pi^{(i)}: X' \to X$ factors through $\psi'^{(i)}$, which means there is a morphism $f: X'' \to X$ such that $f \circ \psi'^{(i)} = \phi \circ \pi^{(i)}$. In particular, the number of points blown up under $\psi'^{(i)}$ is at most the number of points blown up under $\pi^{(i)}$. As we chose the number of points blown up under $\pi^{(i)}$ to be minimal, this shows that f is an isomorphism. In fact, since f is an isomorphism over $\mathbb{P}^2$, this isomorphism is unique, and we can identify X'' with X. □

One technical question that arises in Procedure 3.1 is how one checks, in Step 4, whether X' is isomorphic to a surface in one of our lists $\mathcal{L}_j$ with $j \leq i+1$. Clearly a necessary condition for this is that X' has the same configuration of negative curves as one of the surfaces $X_k \in \mathcal{L}_j$ for some $j \leq i+1$. By Lemma 3.2, we have the following converse:

Corollary 3.3 *Let X' be a weak del Pezzo surface with nontrivial global vector fields that arises in Step 3 of Procedure 3.1. Assume that X' has the same configuration of negative curves as a surface in $\mathcal{L}_j$ for some $j < i+1$. Then X' is isomorphic to a surface already contained in $\mathcal{L}_j$.*

Proof If X' has the same configuration as a surface in $\mathcal{L}_j$, then there is a sequence of contractions of (-1)–curves on X' that realizes X' as a weak del Pezzo surface of height $j < i+1$, and then Lemma 3.2 shows that X' is isomorphic to a surface in $\mathcal{L}_j$. □

Remark 3.4 If, instead, X' has the same configuration of negative curves as a surface in $\mathcal{L}_{i+1}$, then we cannot immediately use Lemma 3.2, since the list $\mathcal{L}_{i+1}$ is not yet complete at that point. Whenever this happens in Section 4, we will describe an explicit way of blowing down X' to a surface with the same configuration as (hence, by Lemma 3.2, isomorphic to) some $X_k \in \mathcal{L}_i$ in such a way that the image of the exceptional locus lies in the set $E \subseteq X_k$. If Steps 1–5 of Procedure 3.1 have already been carried out for $X_k \in \mathcal{L}_i$, this implies that X' is isomorphic to a surface already contained in $\mathcal{L}_{i+1}$.

Since we distinguish the families of weak del Pezzo surfaces with global vector fields according to their configuration of negative curves and automorphism schemes, once we know that X' is isomorphic to a surface in $\mathcal{L}_j$, we can determine the family to which it belongs by describing its configuration of negative curves and by computing its automorphism scheme.

3.2 On the calculation of stabilizers

Before starting our classification, let us explain how to calculate the scheme-theoretic stabilizers of the points $p_{j,i} \in E_{j,i-1}$ occurring in Step 4 of Procedure 3.1. First, recall the definition of the scheme-theoretic stabilizer:

Definition 3.5 Let $\rho\colon G \times X \to X$ be an action of a group scheme G on a scheme X over k. Let $p\colon \operatorname{Spec} k \to X$ be a k–valued point. The stabilizer $\operatorname{Stab}_p \subseteq G$ of p with respect to ρ is defined as

$$\operatorname{Stab}_p\colon (\mathrm{Sch}/k) \to (\mathrm{Sets}), \quad S \mapsto \{g \in G(S) \mid g(p_S) = p_S\},$$

where $p_S\colon S \to \operatorname{Spec} k \to X$.

The stabilizer $\operatorname{Stab}_p \subseteq G$ is a closed subgroup scheme of G. As mentioned in Step 4 of Procedure 3.1, we will describe only the R–valued points of the stabilizers occurring in our classification, where R is a local k–algebra. This is sufficient, since in each case — all the conditions on the matrices in $\mathrm{PGL}_3(R)$ of Tables 1 and 3–6 being given by polynomial equations which respect the group structure on PGL_3 — there will be an obvious closed subgroup scheme G of PGL_3 that admits the same R–valued points as the given stabilizer. The group scheme G will then be equal to the stabilizer because of the following well-known lemma:

Lemma 3.6 *Let $Z_1, Z_2 \subseteq X$ be two closed subschemes of a scheme X over a field k. If $Z_1(R) = Z_2(R) \subseteq X(R)$ for all local k–algebras R, then $Z_1 = Z_2$ as closed subschemes of X.*

The advantage of only considering R–valued points of PGL_n lies in the fact that R–valued points $\mathbb{P}^n$ are simply given by $(n+1)$–tuples of elements in R, up to units in R, such that at least one of the elements in the $(n+1)$–tuple is a unit. This allows us to describe the action of $\mathrm{Aut}^0_X(R)$ on $E_{j,i-1}(R) \cong \mathbb{P}^1(R)$ explicitly using adapted pencils, so that the calculation of the scheme-theoretic stabilizer of a k–valued point $p_{j,i} \in E_{j,i-1}$ becomes straightforward (by Lemma 3.6). Thus, R will denote a local k–algebra from now on.

4 Proof of Main Theorem: classification

In this section, we will carry out Procedure 3.1 to obtain the classification of weak del Pezzo surfaces with global regular vector fields and prove our Main Theorem.

Firstly, note that there are two weak del Pezzo surfaces which do not fit into the framework of Procedure 3.1, namely those which are not blowups of $\mathbb{P}^2$. By Lemma 2.7, these are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_2$. As is well known, $\mathrm{Aut}_{\mathbb{P}^1\times\mathbb{P}^1} = \mathrm{PGL}_2 \times \mathrm{PGL}_2$. As for $\mathrm{Aut}_{\mathbb{F}_2}$, we make use of the fact that this group scheme is smooth and connected by [8, Theorem 1 and Lemma 10]. An explicit description of this group scheme is given in [8]. Alternatively, one can blow down the unique (-2)–curve on $\mathbb{F}_2$ to obtain the weighted projective plane $\mathbb{P}(1,1,2)$ and use the fact that $(\mathrm{Aut}_{\mathbb{P}(1,1,2)})_{\mathrm{red}}$ fixes the unique singular point on $\mathbb{P}(1,1,2)$. Hence, this action lifts to $\mathbb{F}_2$ and we get $\mathrm{Aut}_{\mathbb{F}_2} = (\mathrm{Aut}_{\mathbb{P}(1,1,2)})_{\mathrm{red}}$. These results are summarized in Table 2.

For the remaining cases we can apply Procedure 3.1, and we will subdivide the proof into subsections according to the height of our weak del Pezzo surfaces. Throughout, we write $\ell_f := \mathcal{V}(f)$ for the line given by $f = 0$ in $\mathbb{P}^2$. Recall that in the following figures a thick curve denotes a (-2)–curve, while a thin curve denotes a (-1)–curve. The intersection multiplicity of two such curves is at most 3 at every point; intersection multiplicities 1 and 2 will be clear from the picture, whereas we write a small 3 next to the point of intersection if the intersection multiplicity is 3.

4.1 Height 0

We have $\mathcal{L}_0 = \{X_{9A}\}$, where $X_{9A} := \mathbb{P}^2$ with $\mathrm{Aut}_{\mathbb{P}^2} = \mathrm{PGL}_3$.

4.2 Height 1

Case 9A In this case, $X = \mathbb{P}^2$ and $\psi = \mathrm{id}$. We have $E = \mathbb{P}^2$, and the action of $\mathrm{Aut}_X^0 = \mathrm{PGL}_3$ on E is transitive. Now, note that if $p_{1,0}, \dots, p_{n_0,0} \in \mathbb{P}^2$ are points such that at least four of them are in general position, then

$$\mathrm{Aut}^0_{X'} = \left(\bigcap_{j=1}^{n_0} \mathrm{Stab}^0_{p_{j,0}}\right) = \{*\}.$$

On the other hand, according to Lemma 2.5, to guarantee that X' is a weak del Pezzo surface, no more than three of the $p_{j,0}$ may be on a line. Up to isomorphism, this leaves five possibilities for $p_{1,0}, \dots, p_{n_0,0}$:

(1) $n = 4$, and $p_{1,0}$, $p_{2,0}$ and $p_{4,0}$ are on a line ℓ with $p_{3,0} \notin \ell$. Using the action of PGL_3, we may assume that $p_{1,0} = [1:0:0]$, $p_{2,0} = [0:1:0]$, $p_{3,0} = [0:0:1]$, $p_{4,0} = [1:1:0]$ and $\ell = \ell_z$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$

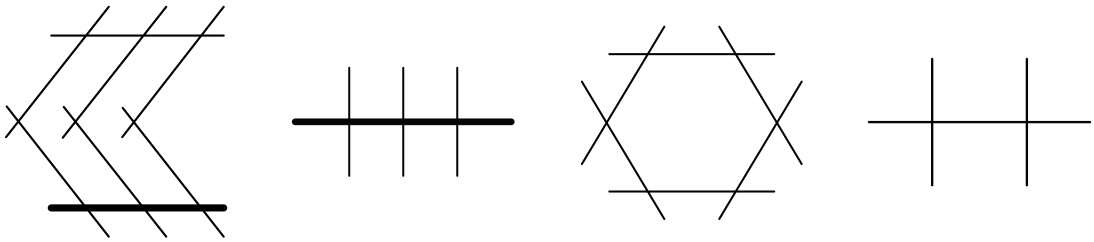


Figure 1

- We have a (-2)–curve $\ell_z^{(1)}$ and (-1)–curves $E_{1,0}$, $E_{2,0}$, $E_{3,0}$, $E_{4,0}$, $\ell_x^{(1)}$, $\ell_y^{(1)}$ and $\ell_{x-y}^{(1)}$, with configuration as in 5A of Figure 1.

This is case 5A.

(2) $n = 3$ and all points are on a line ℓ. We may assume that $p_{1,0} = [1:0:0]$, $p_{2,0} = [0:1:0]$, $p_{3,0} = [1:1:0]$ and $\ell = \ell_z$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have a (-2)–curve $\ell_z^{(1)}$ and (-1)–curves $E_{1,0}$, $E_{2,0}$ and $E_{3,0}$, with configuration as in 6C of Figure 1.

This is case 6C.

(3) $n = 3$ and not all points are on a line. We may assume that $p_{1,0} = [1:0:0]$, $p_{2,0} = [0:1:0]$ and $p_{3,0} = [0:0:1]$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have no (-2)–curves and (-1)–curves $E_{1,0}$, $E_{2,0}$, $E_{3,0}$, $\ell_x^{(1)}$, $\ell_y^{(1)}$ and $\ell_z^{(1)}$, with configuration as in 6A of Figure 1.

This is case 6A.

(4) $n = 2$. We may assume that $p_{1,0} = [1:0:0]$ and $p_{2,0} = [0:1:0]$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have no (-2)–curves and (-1)–curves $E_{1,0}$, $E_{2,0}$ and $\ell_z^{(1)}$, with configuration as in 7A of Figure 1.

This is case 7A.

Figure 2

(5) $n = 1$. We may assume that $p_{1,0} = [1:0:0]$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & h & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have no (-2)–curves and (-1)–curve $E_{1,0}$, with configuration as in 8A of Figure 2.

This is case 8A.

Summarizing, we obtain $\mathcal{L}_1 = \{X_{5A}, X_{6C}, X_{6A}, X_{7A}, X_{8A}\}$.

4.3 Height 2

Case 5A We have $E = \left(\bigcup_{j=1}^4 E_{j,0}\right) - \ell_z^{(1)}$. Recall that the R–valued points of Aut^0_X are given by

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

We calculate the action of Aut^0_X on the $E_{j,0}$ using adapted pencils:

- $\lambda y + \mu z$ is $E_{1,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu] \mapsto [\lambda : i\mu]$.
- $\lambda x + \mu z$ is $E_{2,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu] \mapsto [\lambda : i\mu]$.
- $\lambda x + \mu y$ is $E_{3,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu] \mapsto [\lambda : \mu]$.
- $\lambda(x-y) + \mu z$ is $E_{4,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu] \mapsto [\lambda : i\mu]$.

In particular, there is one unique point with nontrivial stabilizer on each of $E \cap E_{1,0}$, $E \cap E_{2,0}$ and $E \cap E_{4,0}$. Since $p_{1,0}$, $p_{2,0}$ and $p_{4,0}$ can be interchanged by automorphisms of $\mathbb{P}^2$ preserving $p_{3,0}$, we have ten possibilities for $p_{1,1}, \dots, p_{n,1}$:

(1) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$, $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$, $p_{3,1} = E_{3,0} \cap \ell_{x+\alpha y}^{(1)}$ with $\alpha \notin \{0,-1\}$ and $p_{4,1} = E_{4,0} \cap \ell_{x-y}^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$, $E_{2,0}^{(2)}$, $E_{3,0}^{(2)}$, $E_{4,0}^{(2)}$, $\ell_x^{(2)}$, $\ell_y^{(2)}$, $\ell_z^{(2)}$ and $\ell_{x-y}^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,1}$, $E_{4,1}$ and $\ell_{x+\alpha y}^{(2)}$, with configuration as in 1A of Figure 2.

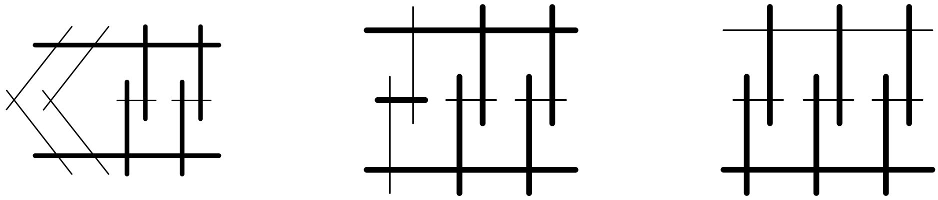


Figure 3

This is case 1A and we see that we get a 1–dimensional family of such surfaces $X_{1A,\alpha}$ depending on the parameter α.

(2) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$, $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$ and $p_{3,1} = E_{3,0} \cap \ell_{x+\alpha y}^{(1)}$ with $\alpha \notin \{0,-1\}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,1}$, $E^{(2)}_{4,0}$, $\ell^{(2)}_{x-y}$ and $\ell^{(2)}_{x+\alpha y}$, with configuration as in 2A of Figure 3.

This is case 2A and we see that we get a 1–dimensional family of such surfaces $X_{2A,\alpha}$ depending on the parameter α.

(3) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$, $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$ and $p_{3,1} = E_{3,0} \cap \ell_{x-y}^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$, $\ell^{(2)}_z$ and $\ell^{(2)}_{x-y}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,1}$ and $E^{(2)}_{4,0}$, with configuration as in 2D of Figure 3.

This is case 2D.

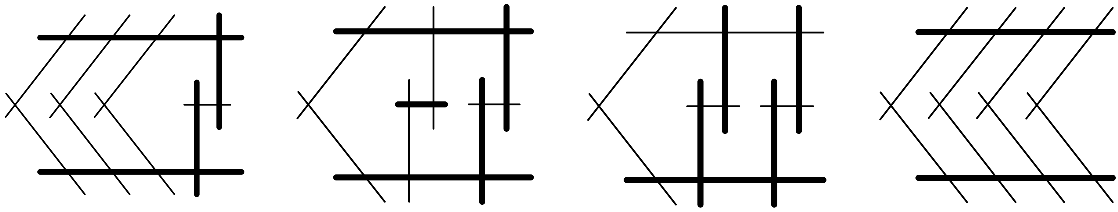
(4) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$, $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$ and $p_{4,1} = E_{4,0} \cap \ell_{x-y}^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{4,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$, $\ell^{(2)}_z$ and $\ell^{(2)}_{x-y}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{4,1}$ and $E^{(2)}_{3,0}$, with configuration as in 2E of Figure 3.

This is case 2E.

(5) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$ and $p_{3,1} = E_{3,0} \cap \ell_{x+\alpha y}^{(1)}$ with $\alpha \notin \{0,-1\}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.



3A 3C 3D 4A

Figure 4

- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_y$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{3,1}$, $E^{(2)}_{2,0}$, $E^{(2)}_{4,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_{x-y}$ and $\ell^{(2)}_{x+\alpha y}$, with configuration as in 3A of Figure 4.

This is case 3A and we see that we get a 1–dimensional family of such surfaces $X_{3A,\alpha}$ depending on the parameter α.

(6) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_y$ and $p_{3,1} = E_{3,0} \cap \ell^{(1)}_x$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{3,1}$, $E^{(2)}_{2,0}$, $E^{(2)}_{4,0}$ and $\ell^{(2)}_{x-y}$, with configuration as in 3C of Figure 4.

This is case 3C.

(7) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_y$ and $p_{2,1} = E_{2,0} \cap \ell^{(1)}_x$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E^{(2)}_{3,0}$, $E^{(2)}_{4,0}$ and $\ell^{(2)}_{x-y}$, with configuration as in 3D of Figure 4.

This is case 3D.

(8) $p_{3,1} = E_{3,0} \cap \ell^{(1)}_{x+\alpha y}$ with $\alpha \notin \{0, -1\}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{3,0}$ and $\ell^{(2)}_z$ and (-1)–curves $E_{3,1}$, $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{4,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$, $\ell^{(2)}_{x-y}$ and $\ell^{(2)}_{x+\alpha y}$, with configuration as in 4A of Figure 4.

This is case 4A and we see that we get a 1–dimensional family of such surfaces $X_{4A,\alpha}$ depending on the parameter α.

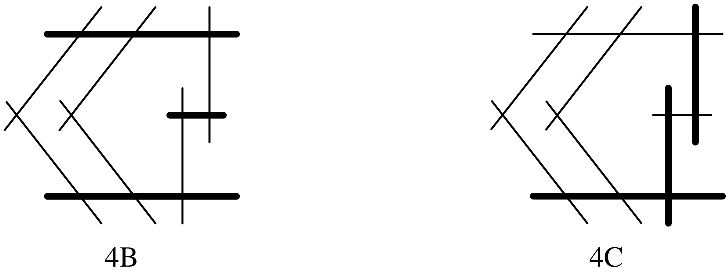


Figure 5

(9) $p_{3,1} = E_{3,0} \cap \ell_y^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{3,0}^{(2)}$, $\ell_y^{(2)}$ and $\ell_z^{(2)}$ and (-1)–curves $E_{3,1}$, $E_{1,0}^{(2)}$, $E_{2,0}^{(2)}$, $E_{4,0}^{(2)}$, $\ell_x^{(2)}$ and $\ell_{x-y}^{(2)}$, with configuration as in 4B of Figure 5.

This is case 4B.

(10) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$, $\ell_y^{(2)}$ and $\ell_z^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,0}^{(2)}$, $E_{3,0}^{(2)}$, $E_{4,0}^{(2)}$, $\ell_x^{(2)}$ and $\ell_{x-y}^{(2)}$, with configuration as in 4C of Figure 5.

This is case 4C.

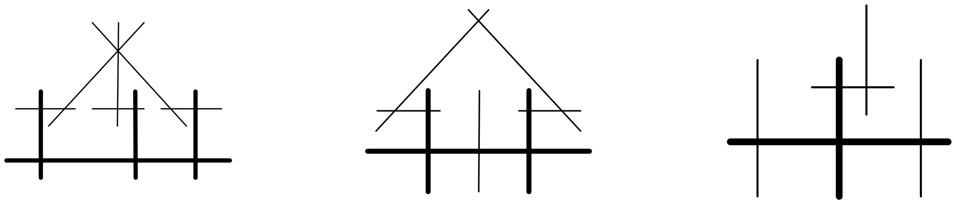
Case 6C We have $E = \left(\bigcup_{j=1}^3 E_{j,0}\right) - \ell_z^{(1)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda y + \mu z$ is $E_{1,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + f\lambda]$.
- $\lambda x + \mu z$ is $E_{2,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + c\lambda]$.
- $\lambda(x - y) + \mu z$ is $E_{3,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + (c - f)\lambda]$.

Since $p_{1,0}$, $p_{2,0}$ and $p_{3,0}$ can be interchanged by automorphisms of $\mathbb{P}^2$ and the action of Aut^0_X is transitive on every $E \cap E_{i,0}$, we have three possibilities for $p_{1,1}, \dots, p_{n,1}$:

(1) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$, $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$ and $p_{3,1} = E_{3,0} \cap \ell_{x-y}^{(1)}$.



3B 4D 5C

Figure 6

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{3,0}$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,1}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$ and $\ell^{(2)}_{x-y}$, with configuration as in 3B of Figure 6.

This is case 3B.

(2) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_y$ and $p_{2,1} = E_{2,0} \cap \ell^{(1)}_x$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_x$ and $\ell^{(2)}_y$, with configuration as in 4D of Figure 6.

This is case 4D.

(3) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_y$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(2)}_{1,0}$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E^{(2)}_{2,0}$, $E^{(2)}_{3,0}$ and $\ell^{(2)}_y$, with configuration as in 5C of Figure 6.

This is case 5C.

Case 6A We have $E = \bigcup_{j=0}^3 E_{j,0}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda y + \mu z$ is $E_{1,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i\mu]$.
- $\lambda x + \mu z$ is $E_{2,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.
- $\lambda x + \mu y$ is $E_{3,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e\mu]$.

Since $p_{1,0}$, $p_{2,0}$ and $p_{3,0}$ can be permuted arbitrarily by automorphisms of $\mathbb{P}^2$, we have nine possibilities for $p_{1,1},\dots,p_{n,1}$:

(1) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{y-z}$, $p_{2,1} = E_{2,0} \cap \ell^{(1)}_{z}$ and $p_{3,1} = E_{3,0} \cap \ell^{(1)}_{x}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_x$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,1}$, $\ell^{(2)}_y$ and $\ell^{(2)}_{y-z}$, with configuration as in Figure 4, case 3C.

Blowing down the two right-most (-1)–curves in Figure 4 (3C), we see that X' arises as a blowup of X_{5A} in two points on E and $X' \cong X_{3C}$ by Remark 3.4.

(2) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{y-z}$, $p_{2,1} = E_{2,0} \cap \ell^{(1)}_{z}$ and $p_{3,1} = E_{3,0} \cap \ell^{(1)}_{y}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_y$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,1}$, $\ell^{(2)}_x$ and $\ell^{(1)}_{y-z}$, with configuration as in Figure 4, case 3D.

Blowing down the two (-1)–curves in Figure 4 (3D) that are not adjacent to any other (-1)–curve, we see that X' arises as a blowup of X_{5A} in two points on E and $X' \cong X_{3D}$ by Remark 3.4.

(3) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{z}$, $p_{2,1} = E_{2,0} \cap \ell^{(1)}_{x}$ and $p_{3,1} = E_{3,0} \cap \ell^{(1)}_{y}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$ and $E_{3,1}$, with configuration as in 3H of Figure 7.

This is case 3H.

(4) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{y-z}$ and $p_{2,1} = E_{2,0} \cap \ell^{(1)}_{z}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E^{(2)}_{1,0}$, $E^{(2)}_{2,0}$ and $\ell^{(2)}_z$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E^{(2)}_{3,0}$, $\ell^{(2)}_x$, $\ell^{(2)}_y$ and $\ell^{(2)}_{y-z}$, with configuration as in Figure 5, case 4C.

Blowing down the (-1)–curve in Figure 5 (4C) that is not adjacent to any other (-1)–curve, we see that X' arises as a blowup of X_{5A} in one point on E and hence $X' \cong X_{4C}$ by Remark 3.4.

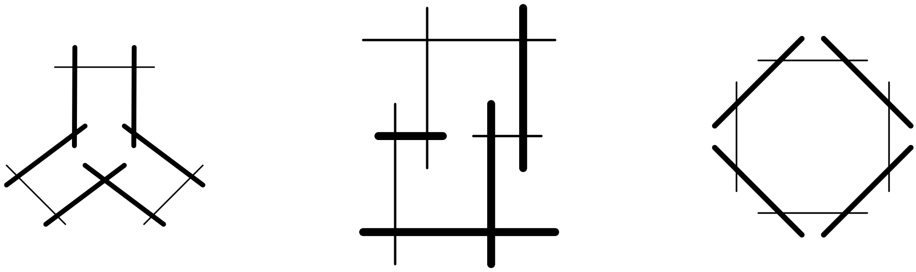


Figure 7

(5) $p_{1,1} = E_{1,0} \cap \ell_{y-z}^{(1)}$ and $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$, $E_{2,0}^{(2)}$ and $\ell_x^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,0}^{(2)}$, $\ell_y^{(2)}$, $\ell_z^{(2)}$ and $\ell_{y-z}^{(2)}$, with configuration as in Figure 5, case 4B.

Blowing down one of the (-1)–curves in Figure 5 (4B) that is not adjacent to any other (-1)–curve, we see that X' arises as a blowup of X_{5A} in one point on E and $X' \cong X_{4B}$ by Remark 3.4.

(6) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$ and $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$, $E_{2,0}^{(2)}$, $\ell_x^{(2)}$ and $\ell_z^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,0}^{(2)}$ and $\ell_y^{(2)}$, with configuration as in 4G of Figure 7.

This is case 4G.

(7) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$ and $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$, $E_{2,0}^{(2)}$, $\ell_x^{(2)}$ and $\ell_y^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $E_{3,0}^{(2)}$ and $\ell_z^{(2)}$, with configuration as in 4F of Figure 7.

This is case 4F.

(8) $p_{1,1} = E_{1,0} \cap \ell_{y-z}^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- We have a (-2)–curve $E_{1,0}^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,0}^{(2)}$, $E_{3,0}^{(2)}$, $\ell_x^{(2)}$, $\ell_y^{(2)}$, $\ell_z^{(2)}$ and $\ell_{y-z}^{(2)}$, with configuration as in Figure 1, case 5A.

By Corollary 3.3, we have $X' \cong X_{5A}$.

(9) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$ and $\ell_z^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,0}^{(2)}$, $E_{3,0}^{(2)}$, $\ell_x^{(2)}$ and $\ell_y^{(2)}$, with configuration as in 5B of Figure 8.

This is case 5B.

Case 7A We have $E = E_{1,0} \cup E_{2,0}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda y + \mu z$ is $E_{1,0}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i\mu + f\lambda]$.
- $\lambda x + \mu z$ is $E_{2,0}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + c\lambda]$.

Since $p_{1,0}$ and $p_{2,0}$ can be interchanged by an automorphism of $\mathbb{P}^2$, we have four possibilities for $p_{1,1}, \dots, p_{n,1}$:

(1) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$ and $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$ and $E_{2,0}^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$, $\ell_x^{(2)}$, $\ell_y^{(2)}$ and $\ell_z^{(2)}$, with configuration as in Figure 8, case 5B.

Blowing down the (-1)–curve in Figure 8 (5B) that is not adjacent to any other (-1)–curve, we see that X' arises as a blowup of X_{6A} in one point on E and $X' \cong X_{5B}$ by Remark 3.4.

(2) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$ and $p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$, $E_{2,0}^{(2)}$ and $\ell_z^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,1}$ and $\ell_x^{(2)}$, with configuration as in 5D of Figure 8.

This is case 5D.

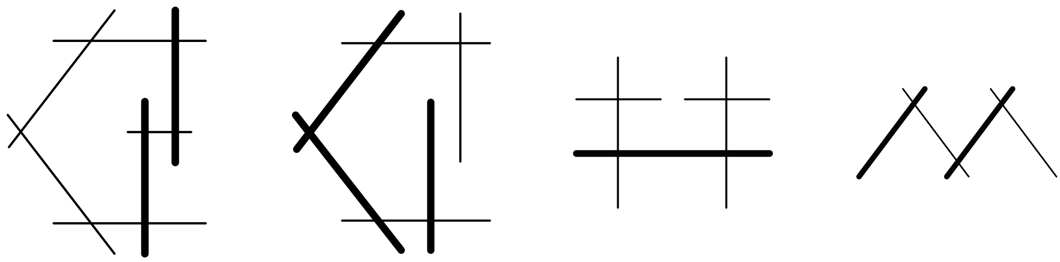


Figure 8

(3) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have a (-2)–curve $E_{1,0}^{(2)}$ and (-1)–curves $E_{1,1}$, $E_{2,0}^{(2)}$, $\ell_y^{(2)}$ and $\ell_z^{(2)}$, with configuration as in 6B of Figure 8.

This is case 6B.

(4) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(2)}$ and $\ell_z^{(2)}$ and (-1)–curves $E_{1,1}$ and $E_{2,0}^{(2)}$, with configuration as in 6D of Figure 8.

This is case 6D.

Case 8A We have $E = E_{1,0}$ and

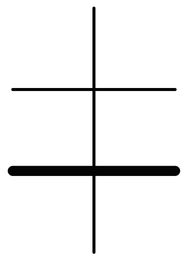
$$\mathrm{Aut}^0_{X}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & h & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda y + \mu z$ is $E_{1,0}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda + h\mu : i\mu + f\lambda]$.

Therefore, there is a unique possibility for $p_{1,1}, \dots, p_{n,1}$ up to isomorphism:

(1) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.



7B

Figure 9

- We have a (-2)–curve $E_{1,0}^{(2)}$ and (-1)–curves $E_{1,1}$ and $\ell_z^{(2)}$, with configuration as in 7B of Figure 9.

This is case 7B.

Summarizing, we obtain

$$\mathcal{L}_2 = \{X_{1A,\alpha}, X_{2A,\alpha}, X_{2D}, X_{2E}, X_{3A,\alpha}, X_{3C}, X_{3D}, X_{4A,\alpha}, X_{4B}, X_{4C}, X_{3B}, X_{4D}, X_{5C}, X_{3H}, X_{4G}, X_{4F}, X_{5B}, X_{5D}, X_{6B}, X_{6D}, X_{7B}\}.$$

4.4 Height 3

Case 2A We have $E = \bigcup_{j=1}^3 E_{j,1} - \big(\bigcup_{j=1}^3 E_{j,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)}\big)$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $E_{2,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.
- $\lambda y^2 + \mu(x + \alpha y)z$ is $E_{3,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that X has degree 2. Therefore we are only allowed to blow up one more point, $p_{j,2}$. Moreover, the involution $x \leftrightarrow \alpha y$ of $\mathbb{P}^2$ lifts to an involution of X interchanging $E_{1,1}$ and $E_{2,1}$, thus we may assume without loss of generality that $j = 1$ or $j = 3$. Finally, if $j = 3$, then the stabilizer of $p_{3,2} \in E \cap E_{3,1}$ is trivial unless $p_{3,2}$ lies on the strict transform of $\ell_{x+\alpha y}$. Moreover, Aut_X^0 acts transitively on $E \cap E_{1,1}$. Hence, we have two possibilities:

(1) $p_{3,2} = E_{3,1} \cap \ell_{x+\alpha y}^{(2)}$ with $\alpha \notin \{0, -1\}$.

- $$\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{3,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $\ell_z^{(3)}$ and $\ell_{x+\alpha y}^{(3)}$ and (-1)–curves $E_{3,2}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $E_{4,0}^{(3)}$ and $\ell_{x-y}^{(3)}$, with configuration as in Figure 2, case 1A.

By Corollary 3.3, we have $X' \cong X_{1A,\alpha'}$ for some α'.

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{3,1}^{(3)}$, $E_{4,0}^{(3)}$, $\ell_{x-y}^{(3)}$, $\ell_{x+\alpha y}^{(3)}$, $C_1^{(3)}$ and $C_2^{(3)}$ with $\alpha \notin \{0, -1\}$ and $C_2 = \mathcal{V}(x^3y + xy^3 + x^2z^2 + \alpha^2 y^2 z^2)$, with configuration as in 1L of Figure 10.

This is case 1L and we see that we get a 1–dimensional family of such surfaces $X_{1L,\alpha}$ depending on the parameter α.

Case 2D We have $E = \bigcup_{j=1}^3 E_{j,1} - \left(\bigcup_{j=1}^3 E_{j,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)} \cup \ell_{x-y}^{(2)}\right)$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

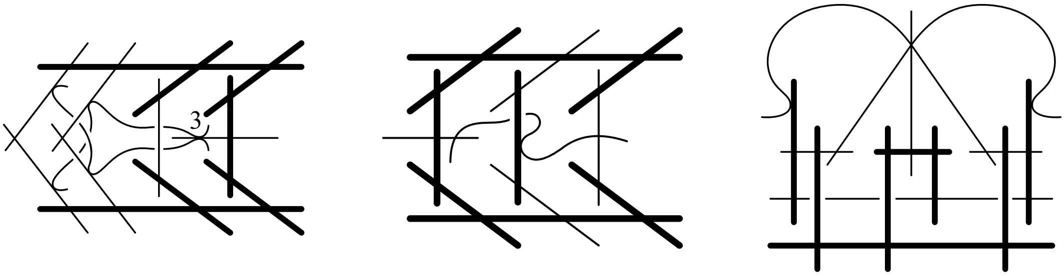
- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $E_{2,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.
- $\lambda y^2 + \mu(x-y)z$ is $E_{3,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that X has degree 2, thus we are only allowed to blow up one more point, $p_{j,2}$. Next, note that the stabilizer of every point on $E \cap E_{3,1}$ is trivial, and hence we may assume $j = 1$ or $j = 2$. Similar to Case $2A$, the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an involution of X interchanging $E_{1,1}$ and $E_{2,1}$, thus we may assume without loss of generality that $j = 1$. Hence, there is a unique choice for $p_{j,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $\ell_z^{(3)}$ and $\ell_{x-y}^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{3,1}^{(3)}$, $E_{4,0}^{(3)}$ and $C^{(3)}$, with configuration as in 1O of Figure 10.

This is case 1O.



1L 1O 1N

Figure 10

Case 2E We have $E = (E_{1,1} \cup E_{2,1} \cup E_{4,1}) - (E_{1,0}^{(2)} \cup E_{2,0}^{(2)} \cup E_{4,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)} \cup \ell_{x-y}^{(2)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $E_{2,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.
- $\lambda(x-y)x + \mu z^2$ is $E_{4,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.

Note that X has degree 2, thus we are only allowed to blow up one more point, $p_{j,2}$. Next, the automorphisms of $\mathbb{P}^2$ interchanging $p_{1,0}$, $p_{2,0}$ and $p_{4,0}$ and preserving $p_{3,0}$ lift to X and interchange $E_{1,1}$, $E_{2,1}$ and $E_{4,1}$, thus we may assume $j = 1$. Finally, Aut_X^0 acts transitively on $E \cap E_{1,1}$, and hence we have a unique choice for $p_{j,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{4,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $\ell_z^{(3)}$ and $\ell_{x-y}^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{4,1}^{(3)}$, $E_{3,0}^{(3)}$, $C_1^{(3)}$ and $C_2^{(3)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$, with configuration as in 1N of Figure 10.

This is case 1N.

Case 3A We have $E = (E_{1,1} \cup E_{3,1}) - (E_{1,0}^{(2)} \cup E_{3,0}^{(2)} \cup \ell_y^{(2)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.
- $\lambda y^2 + \mu(x + \alpha y)z$ is $E_{3,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is one unique point with nontrivial stabilizer on $E \cap E_{3,1}$, while Aut^0_X acts transitively on $E \cap E_{1,1}$. Hence, we have three choices up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{3,2} = E_{3,1} \cap \ell^{(2)}_{x+\alpha y}$ with $C_1 = \mathcal{V}(xy + z^2)$ and $\alpha \notin \{0, -1\}$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{3,0}$, $E^{(3)}_{1,1}$, $E^{(3)}_{3,1}$, $\ell^{(3)}_y$, $\ell^{(3)}_z$ and $\ell^{(3)}_{x+\alpha y}$ and (-1)–curves $E_{1,2}$, $E_{3,2}$, $E^{(3)}_{2,0}$, $E^{(3)}_{4,0}$, $\ell^{(3)}_x$, $\ell^{(3)}_{x-y}$, $C^{(3)}_2$ and $C^{(3)}_3$ with
  $$C_2 = \mathcal{V}(x^2y + xz^2 + \alpha yz^2) \quad \text{and} \quad C_3 = \mathcal{V}(x^2y + xz^2 + \alpha yz^2 + y^3),$$
  with configuration as in Figure 10, case 1L.

Blowing down the right-most (-1)–curve in Figure 10 (1L), we see that X' is the blowup of some $X_{2A,\alpha}$ in one point on E and $X' \cong X_{1L,\alpha'}$ for some α' by Remark 3.4.

(2) $p_{3,2} = E_{3,1} \cap \ell^{(2)}_{x+\alpha y}$ with $\alpha \notin \{0, -1\}$.

- $$\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{3,0}$, $E^{(3)}_{3,1}$, $\ell^{(3)}_y$, $\ell^{(3)}_z$ and $\ell^{(3)}_{x+\alpha y}$ and (-1)–curves $E_{3,2}$, $E^{(3)}_{1,1}$, $E^{(3)}_{2,0}$, $E^{(3)}_{4,0}$, $\ell^{(3)}_x$ and $\ell^{(3)}_{x-y}$, with configuration as in Figure 3, case 2A.

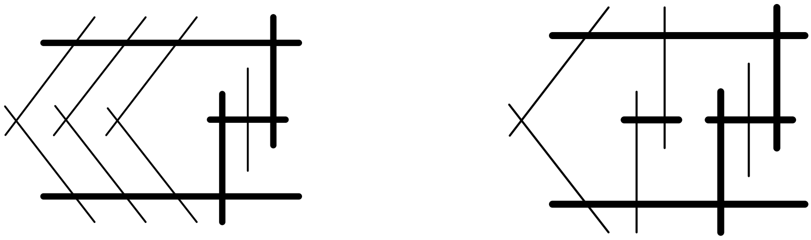
By Corollary 3.3, we have $X' \cong X_{2A,\alpha'}$ for some α'.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{3,0}$, $E^{(3)}_{1,1}$, $\ell^{(3)}_y$ and $\ell^{(3)}_z$ and (-1)–curves $E_{1,2}$, $E^{(3)}_{3,1}$, $E^{(3)}_{2,0}$, $E^{(3)}_{4,0}$, $\ell^{(3)}_x$, $\ell^{(3)}_{x-y}$ and $\ell^{(3)}_{x+\alpha y}$ with $\alpha \notin \{0, -1\}$, with configuration as in 2N of Figure 11.

This is case 2N and we see that we get a 1–dimensional family of such surfaces $X_{2N,\alpha}$ depending on the parameter α.



2N 2Q

Figure 11

Case 3C We have $E = (E_{1,1} \cup E_{3,1}) - (E_{1,0}^{(2)} \cup E_{3,0}^{(2)} \cup \ell_y^{(2)} \cup \ell_x^{(2)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.
- $\lambda xz + \mu y^2$ is $E_{3,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : \mu]$.

Note that the stabilizer of every point in $E \cap E_{3,1}$ is trivial while Aut_X^0 acts transitively on $E \cap E_{1,1}$. Hence, we have a unique choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{3,1}^{(3)}$, $E_{2,0}^{(3)}$, $E_{4,0}^{(3)}$ and $\ell_{x-y}^{(3)}$, with configuration as in 2Q of Figure 11.

This is case 2Q.

Case 3D We have $E = (E_{1,1} \cup E_{2,1}) - (E_{1,0}^{(2)} \cup E_{2,0}^{(2)} \cup \ell_y^{(2)} \cup \ell_x^{(2)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $E_{2,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.

Note that the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an involution of X interchanging $E_{1,1}$ and $E_{2,1}$. Moreover, Aut_X^0 acts transitively and with finite stabilizers on both $E \cap E_{1,1}$ and $E \cap E_{2,1}$. Hence, we have three possibilities for $p_{1,2}, \dots, p_{n,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ and $p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $\ell_z^{(3)}$ and $C^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,0}^{(3)}$, $E_{4,0}^{(3)}$ and $\ell_{x-y}^{(3)}$, with configuration as in Figure 10, case 1O.

Blowing down the left-most (-1)–curve in Figure 10 (1O), we see that X' is a blowup of X_{2D} in one point on E and $X' \cong X_{1O}$ by Remark 3.4.

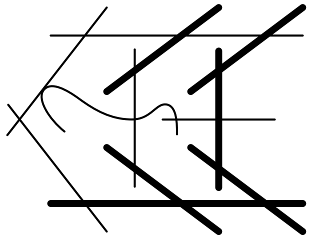
(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$, $C_2 = \mathcal{V}(xy + \alpha z^2)$ and $\alpha \notin \{0, 1\}$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,0}^{(3)}$, $E_{4,0}^{(3)}$, $\ell_{x-y}^{(3)}$, $C_1^{(3)}$, $C_2^{(3)}$ and $C_3^{(3)}$ with $C_3 = \mathcal{V}(x^3y^2 + x^2y^3 + xz^4 + \alpha^2 yz^4)$, with configuration as in Figure 10, case 1L.

Blowing down the right-most (-1)–curve in Figure 10 (1L), we see that X' is the blowup of some $X_{2A,\alpha}$ in one point on E and $X' \cong X_{1L,\alpha'}$ for some α' by Remark 3.4.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.



2P

Figure 12

- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{3,0}^{(3)}$, $E_{4,0}^{(3)}$, $\ell_{x-y}^{(3)}$ and $C^{(3)}$, with configuration as in 2P of Figure 12.

This is case 2P.

Case 4A We have $E = E_{3,1} - E_{3,0}^{(2)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda y^2 + \mu(x+\alpha y)z$ is $E_{3,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda : i\mu]$.

Note that there is one unique point on $E \cap E_{3,1}$ with nontrivial stabilizer, leading to a unique choice for $p_{3,2}$:

(1) $p_{3,2} = E_{3,1} \cap \ell_{x+\alpha y}^{(2)}$ with $\alpha \notin \{0,-1\}$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E_{3,0}^{(3)}$, $E_{3,1}^{(3)}$, $\ell_z^{(3)}$ and $\ell_{x+\alpha y}^{(3)}$ and (-1)–curves $E_{3,2}$, $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{4,0}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_{x-y}^{(3)}$, with configuration as in Figure 4, case 3A.

By Corollary 3.3, we have $X' \cong X_{3A,\alpha'}$ for some α'.

Case 4B We have $E = E_{3,1} - (E_{3,0}^{(2)} \cup \ell_y^{(2)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2 + \mu yz$ is $E_{3,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda : i\mu]$.

There is no point on $E \cap E_{3,1}$ with nontrivial stabilizer, so we get no new cases by further blowing up X.

Case 4C We have $E = E_{1,1} - (E_{1,0}^{(2)} \cup \ell_y^{(2)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.

In particular, Aut_X^0 acts transitively on $E \cap E_{1,1}$. We get a unique choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{4,0}^{(3)}$, $\ell_x^{(3)}$ and $\ell_{x-y}^{(3)}$, with configuration as in 3N of Figure 13.

This is case 3N.

Case 3B We have $E = \bigcup_{j=1}^3 E_{j,1} - \bigcup_{j=1}^3 E_{j,0}^{(2)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $E_{2,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.
- $\lambda(x-y)x + \mu z^2$ is $E_{3,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.

Note that automorphisms of $\mathbb{P}^2$ fixing $[0:0:1]$ and interchanging the $p_{j,0}$ lift to automorphisms of X interchanging the $E_{j,1}$. Moreover, since X has degree 3, we are only allowed to blow up two more points. Finally, on every $E \cap E_{j,1}$, the action of Aut_X^0 has two orbits and one of them is a fixed point. Hence, we get six possibilities for $p_{1,2}, \dots, p_{3,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap C_1^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.

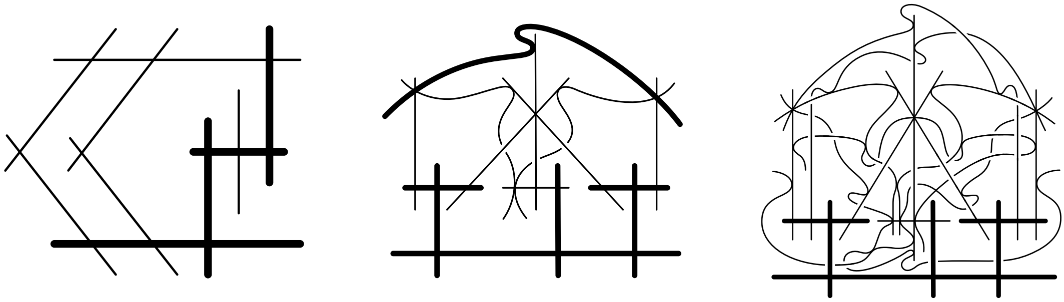


Figure 13

- We have (−2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_z^{(3)}$ and $C_1^{(3)}$ and (−1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $\ell_{x-y}^{(3)}$, $C_2^{(3)}$ and $C_3^{(3)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$ and $C_3 = \mathcal{V}(xy + x^2 + z^2)$, with configuration as in 1K of Figure 13.

This is case 1K.

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$, $C_2 = \mathcal{V}(xy + \alpha z^2)$ and $\alpha \notin \{0, 1\}$

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.

- We have (−2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$ and $\ell_z^{(3)}$ and (−1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $\ell_{x-y}^{(3)}$, $C_1^{(3)}$, $C_2^{(3)}$, $C_3^{(3)}$, $C_4^{(3)}$, $C_5^{(3)}$, $C_6^{(3)}$ and $C_7^{(3)}$ with $C_3 = \mathcal{V}(xy + y^2 + z^2)$, $C_4 = \mathcal{V}(xy + x^2 + \alpha z^2)$, $C_5 = \mathcal{V}(x^2y^2 + xy^3 + \alpha y^2 z^2 + z^4)$, $C_6 = \mathcal{V}(x^2y^2 + x^3y + x^2z^2 + \alpha^2 z^4)$ and $C_7 = \mathcal{V}(x^3y^2 + x^2y^3 + xz^4 + \alpha^2 yz^4)$, with configuration as in 1J of Figure 13.

This is case 1J and we see that we get a 1–dimensional family of such surfaces $X_{1J,\alpha}$ depending on the parameter α.

Remark 4.1 Figure 13 (1J) is by far the most complicated configuration that occurs in our classification. To make Figure 13 (1J) easier to digest for the reader, we will now break our habit of describing the curve configuration only via an intuitive picture, and also describe the *dual graph* of the configuration. Each white vertex in the dual graph corresponds to a (−1)–curve and each black vertex corresponds to a (−2)–curve. The number of edges between two vertices corresponding to curves C_1 and C_2 is equal to the intersection number of C_1 and C_2. With these conventions, the dual graph of Figure 13 (1J) is given in Figure 14.

In general, a dual graph carries less information than the nondual picture. In our case, we see from Figure 13 (1J) that every simply laced triangle of vertices in Figure 14 corresponds to three curves

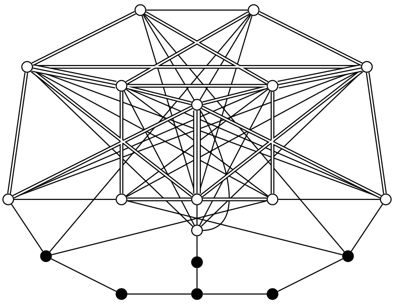


Figure 14: Dual graph of Figure 13 (1J).

meeting in a single point, every double edge corresponds to two curves meeting in a single point with multiplicity 2, and every triple edge corresponds to two curves meeting in two distinct points with multiplicities 2 and 1. While the symmetry group of Figure 14 is the dihedral group D_{12} of order 12, the interested reader can use the additional information from Figure 13 (1J) to check that the only involution in D_{12} that can actually come from an automorphism of X is the unique central involution. And indeed, the pencil of cubic curves through the eight points $p_{i,j}$ contains the curve $\mathcal{V}(z^3)$ and the smooth curve $\mathcal{V}(z^3+z^2x+\alpha z^2y+x^2y+xy^2)$, and hence it is an elliptic pencil and the inverse in the group structure on the generic fiber of the associated rational elliptic surface (classically called the "Bertini involution" associated to the points $p_{i,j}$) induces the central $\mathbb{Z}/2\mathbb{Z}$–symmetry of the graph in Figure 14.

(3) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$ with $C_1 = \mathcal{V}(xy+z^2)$.

- $$\operatorname{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p=2$. Therefore, we assume $p=2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,1}^{(3)}$, $\ell_y^{(3)}$, $\ell_{x-y}^{(3)}$, $C_1^{(3)}$, $C_2^{(3)}$ and $C_3^{(3)}$ with $C_2 = \mathcal{V}(xy+y^2+z^2)$ and $C_3^{(3)} = \mathcal{V}(x^2y^2+xy^3+z^4)$, with configuration as in Figure 13, case 1K.

Blowing down the left-most and the right-most (-1)–curve in Figure 13 (1K), we see that X' is a blowup of X_{3B} in two points on E which do not lie on the intersection of E with the other (-1)–curves on X_{3B} and $X' \cong X_{1K}$ by Remark 3.4.

(4) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$ and $p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$.

- $$\operatorname{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

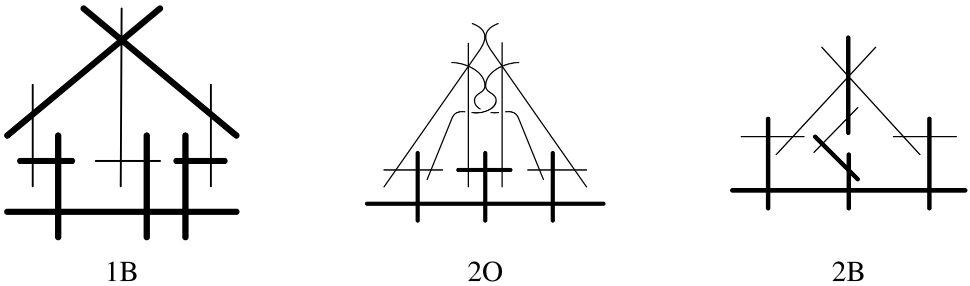


Figure 15

- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,1}^{(3)}$ and $\ell_{x-y}^{(3)}$, with configuration as in 1B of Figure 15.

This is case 1B.

(5) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $$\operatorname{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.

- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{3,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $\ell_{x-y}^{(3)}$, $C_1^{(3)}$ and $C_2^{(3)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$, with configuration as in 2O of Figure 15.

This is case 2O.

(6) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$.

- $$\operatorname{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{3,1}^{(3)}$, $\ell_x^{(3)}$ and $\ell_{x-y}^{(3)}$, with configuration as in 2B of Figure 15.

This is case 2B.

Case 4D We have $E = \bigcup_{j=1}^{2} E_{j,1} - \bigcup_{j=1}^{2} E_{j,0}^{(2)}$ and

$$\operatorname{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $E_{2,1}$–adapted and $\operatorname{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.

Note that automorphisms of $\mathbb{P}^2$ fixing $[0:0:1]$ and interchanging $p_{1,0}$ and $p_{2,0}$ lift to automorphisms of X interchanging $E_{1,1}$ and $E_{2,1}$. Moreover, Aut_X^0 has two orbits on each $E \cap E_{j,1}$, one of which is a fixed point. Hence, we get six possibilities for $p_{1,2}, p_{2,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ and $p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_z^{(3)}$ and $C^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,0}^{(3)}$, $\ell_x^{(3)}$ and $\ell_y^{(3)}$, with configuration as in Figure 11, case 2Q.

Blowing down the right-most (-1)–curve in Figure 11 (2Q), we see that X' is a blowup of X_{3C} in one point on E and $X' \cong X_{2Q}$ by Remark 3.4.

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$, $C_2 = \mathcal{V}(xy + \alpha z^2)$ and $\alpha \notin \{0, 1\}$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,0}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$, $C_1^{(3)}$ and $C_2^{(3)}$, with configuration as in Figure 11, case 2N.

Blowing down the right-most (-1)–curve in Figure 11 (2N), we see that X' is a blowup of some $X_{3A,\alpha}$ in one point on E and $X' \cong X_{2N,\alpha'}$ for some α' by Remark 3.4.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ and $p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,0}^{(3)}$, $\ell_y^{(3)}$ and $C^{(3)}$, with configuration as in Figure 11, case 2Q.

2F 3E

Figure 16

Blowing down the right-most (-1)–curve in Figure 11 (2Q), we see that X' is a blowup of X_{3C} in one point on E and $X' \cong X_{2Q}$ by Remark 3.4.

(4) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$ and $p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$ and $E_{3,0}^{(3)}$, with configuration as in 2F of Figure 16.

This is case 2F.

(5) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$$
  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{3,0}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $C^{(3)}$, with configuration as in Figure 13, case 3N.

Blowing down the right-most (-1)–curve in Figure 13 (3N), we see that X' is a blowup of X_{4C} in one point on E and $X' \cong X_{3N}$ by Remark 3.4.

(6) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$, $E_{3,0}^{(3)}$ and $\ell_x^{(3)}$, with configuration as in 3E of Figure 16.

This is case 3E.

Case 5C We have $E = E_{1,1} - E_{1,0}^{(2)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$.

Note that this is the first case in which there exists no Aut_X^0–stable $E_{1,1}$–adapted pencil (see Example 2.14). We remind the reader that we explained how to calculate the Aut_X^0–action on exceptional curves using not necessarily Aut_X^0–stable adapted pencils after Definition 2.12. From now on, we will no longer explicitly point out when a non-Aut_X^0–stable adapted pencil is used and assume that the reader is familiar with the techniques explained in Section 2.2.

Since Aut_X^0 has two orbits on $E \cap E_{1,1}$, we get two possibilities for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  We describe the configurations of negative curves on X' for $p \neq 2$ and $p = 2$ simultaneously:

- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{1,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$ and $\ell_y^{(3)}$, with configuration as in 4E and 4M of Figure 17.

This is case 4E if $p \neq 2$, and case 4M if $p = 2$.

(2) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$.

- $$\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,0}^{(3)}$ and $E_{3,0}^{(3)}$, with configuration as in 4H of Figure 17.

This is case 4H.

Case 3H We have $E = \bigcup_{j=1}^3 E_{j,1} - \left(\bigcup_{j=1}^3 E_{j,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)} \cup \ell_z^{(2)}\right)$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

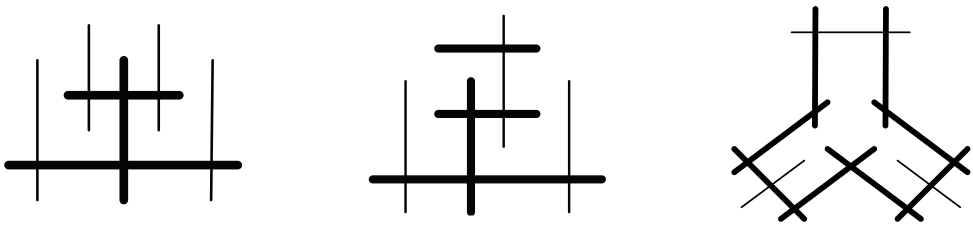


Figure 17

- $\lambda xz + \mu y^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$.
- $\lambda xy + \mu z^2$ is $E_{2,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$.
- $\lambda yz + \mu x^2$ is $E_{3,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [ei\lambda : \mu]$.

Note that all automorphisms of $\mathbb{P}^2$ inducing cyclic permutations of $p_{1,0}$, $p_{2,0}$, and $p_{3,0}$ lift to automorphisms of X, and since X has degree 3, we can only blow up two additional points. Moreover, Aut_X^0 acts transitively on every $E \cap E_{j,1}$. Hence, we get two possibilities for $p_{1,2}, \dots, p_{3,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xz + y^2)$ and $C_2 = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

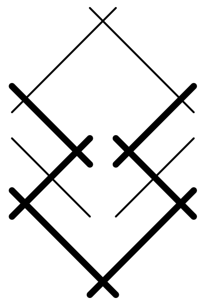
  Hence, X' has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$ and $E_{3,1}^{(3)}$, with configuration as in 1G of Figure 17.

This is case 1G.

(2) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{3,0}^{(3)}$, $E_{1,1}^{(3)}$, $\ell_x^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$ and $E_{3,1}^{(3)}$, with configuration as in Figure 16, case 2F.

Blowing down the left-most and the right-most (-1)–curve in Figure 16 (2F), we see that X' is a blowup of X_{4D} in two points on E and $X' \cong X_{2F}$ by Remark 3.4.



2J

Figure 18

Case 4G We have $E = \bigcup_{j=1}^{2} E_{j,1} - \big(\bigcup_{j=1}^{2} E_{j,0}^{(2)} \cup \ell_z^{(2)} \cup \ell_x^{(2)}\big)$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xz + \mu y^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$.
- $\lambda xy + \mu z^2$ is $E_{2,1}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$.

Since Aut_X^0 acts transitively on every $E \cap E_{j,1}$, we get the following three possibilities for $p_{1,2}, p_{2,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xz + y^2)$ and $C_2 = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

  Hence, X' has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E_{3,0}^{(3)}$ and $\ell_y^{(3)}$, with configuration as in 2J of Figure 18.

This is case 2J.

(2) $p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{2,2}$, $E_{1,1}^{(3)}$, $E_{3,0}^{(3)}$ and $\ell_y^{(3)}$, with configuration as in Figure 16, case 3E.

Blowing down the left-most (-1)–curve in Figure 16 (3E), we see that X' is a blowup of X_{4D} in one point on E and $X' \cong X_{3E}$ by Remark 3.4.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{2,0}$, $E^{(3)}_{1,1}$, $\ell^{(3)}_x$ and $\ell^{(3)}_z$ and (-1)–curves $E_{1,2}$, $E^{(3)}_{2,1}$, $E^{(3)}_{3,0}$ and $\ell^{(3)}_y$, with configuration as in Figure 16, case 3E.

Blowing down the left-most (-1)–curve in Figure 16 (3E), we see that X' is a blowup of X_{4D} in one point on E and $X' \cong X_{3E}$ by Remark 3.4.

Case 4F We have $E = (E_{1,1} \cup E_{2,1}) - (E^{(2)}_{1,0} \cup E^{(2)}_{2,0} \cup \ell^{(2)}_x \cup \ell^{(2)}_y)$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $E_{2,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$.

Note that the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an involution of X interchanging $E_{1,1}$ and $E_{2,1}$. Moreover, Aut^0_X acts transitively on both $E \cap E_{1,1}$ and $E \cap E_{2,1}$, but the stabilizer of every point on $E \cap E_{1,1}$ acts trivially on $E \cap E_{2,1}$. Hence, we have three possibilities up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ and $p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{2,0}$, $E^{(3)}_{1,1}$, $E^{(3)}_{2,1}$, $\ell^{(3)}_x$, $\ell^{(3)}_y$ and $C^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E^{(3)}_{3,0}$ and $\ell^{(3)}_z$, with configuration as in Figure 3, case 2D.

By Corollary 3.3, we have $X' \cong X_{2D}$.

(2) $p_{1,2} = E_{1,1} \cap C^{(2)}_1$ and $p_{2,2} = E_{2,1} \cap C^{(2)}_2$ with $C_1 = \mathcal{V}(xy + z^2)$, $C_2 = \mathcal{V}(xy + \alpha z^2)$ and $\alpha \notin \{0, 1\}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{2,0}$, $E^{(3)}_{1,1}$, $E^{(3)}_{2,1}$, $\ell^{(3)}_x$ and $\ell^{(3)}_y$ and (-1)–curves $E_{1,2}$, $E_{2,2}$, $E^{(3)}_{3,0}$, $\ell^{(3)}_z$, $C^{(3)}_1$ and $C^{(3)}_2$, with configuration as in Figure 3, case 2A.

By Corollary 3.3, we have $X' \cong X_{2A,\alpha'}$ for some α'.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{2,0}$, $E^{(3)}_{1,1}$, $\ell^{(3)}_x$ and $\ell^{(3)}_y$ and (-1)–curves $E_{1,2}$, $E^{(3)}_{2,1}$, $E^{(3)}_{3,0}$, $\ell^{(3)}_z$ and $C^{(3)}$, with configuration as in Figure 4, case 3D.

By Corollary 3.3, we have $X' \cong X_{3D}$.

Case 5B We have $E = E_{1,1} - (E^{(2)}_{1,0} \cup \ell^{(2)}_z)$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xz + \mu y^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$.

Since Aut^0_X acts transitively on $E \cap E_{1,1}$, we have a unique choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{1,1}$ and $\ell^{(3)}_z$ and (-1)–curves $E_{1,2}$, $E^{(3)}_{2,0}$, $E^{(3)}_{3,0}$, $\ell^{(3)}_x$ and $\ell^{(3)}_y$, with configuration as in Figure 6, case 4D.

By Corollary 3.3, we have $X' \cong X_{4D}$.

Case 5D We have $E = \bigcup_{j=1}^2 E_{j,1} - \left(\bigcup_{j=1}^2 E^{(2)}_{j,0} \cup \ell^{(2)}_z\right)$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xz + \mu y^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$.
- $\lambda xy + \mu z^2$ is $E_{2,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$.

Note that Aut^0_X acts transitively on $E \cap E_{1,1}$, and with two orbits, one of which is a fixed point, on $E \cap E_{2,1}$. Hence, we have five choices for $p_{1,2}$ and $p_{2,2}$ up to isomorphism:

Figure 19

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ and $p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xz + y^2)$ and $C_2 = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 3, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

  We describe the configurations of negative curves on X' for $p \neq 3$ and $p = 3$ simultaneously:
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,2}$ and $\ell_x^{(3)}$, with configuration as in 3F and 3K of Figure 19.

This is case 3F if $p \neq 3$, and case 3K if $p = 3$.

(2) $p_{1,2} = E_{1,1} \cap C^{(2)}$ and $p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$ with $C = \mathcal{V}(xz + y^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$ and $E_{2,2}$, with configuration as in 3I of Figure 19.

This is case 3I.

(3) $p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{2,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{2,2}$, $E_{1,1}^{(3)}$ and $\ell_x^{(3)}$, with configuration as in Figure 17, case 4H.

Blowing down the (-1)–curve in the middle of Figure 17 (4H), we see that X' is a blowup of X_{5C} in one point on E and $X' \cong X_{4H}$ by Remark 3.4.

Figure 20

(4) $p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{2,1}^{(3)}$, $\ell_x^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{2,2}$ and $E_{1,1}^{(3)}$, with configuration as in 4K of Figure 20.

This is case 4K.

(5) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{2,0}^{(3)}$, $E_{1,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,1}^{(3)}$ and $\ell_x^{(3)}$, with configuration as in 4I of Figure 20.

This is case 4I.

Case 6B We have $E = E_{1,1} - E_{1,0}^{(2)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy + \mu z^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$.

Since Aut^0_X has two orbits on $E \cap E_{1,1}$, we have two choices for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- We have (-2)–curves $E_{1,0}^{(3)}$ and $E_{1,1}^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,0}^{(3)}$, $\ell_y^{(3)}$ and $\ell_z^{(3)}$, with configuration as in Figure 6, case 5C.

By Corollary 3.3, we have $X' \cong X_{5C}$.

(2) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{1,1}^{(3)}$ and $\ell_y^{(3)}$ and (-1)–curves $E_{1,2}$, $E_{2,0}^{(3)}$ and $\ell_z^{(3)}$, with configuration as in Figure 8, case 5D.

By Corollary 3.3, we have $X' \cong X_{5D}$.

Case 6D We have $E = E_{1,1} - (E_{1,0}^{(2)} \cup \ell_z^{(2)})$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xz + \mu y^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$.

Since Aut^0_X acts transitively on $E \cap E_{1,1}$, there is only one choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(3)}$, $E_{1,1}^{(3)}$ and $\ell_z^{(3)}$ and (-1)–curves $E_{1,2}$ and $E_{2,0}^{(3)}$, with configuration as in 5E of Figure 21.

This is case 5E.

Case 7B We have $E = E_{1,1} - E_{1,0}^{(2)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xz + \mu y^2$ is $E_{1,1}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$.

Since Aut^0_X has two orbits on $E \cap E_{1,1}$, there are two choices for $p_{1,2}$ up to isomorphism:

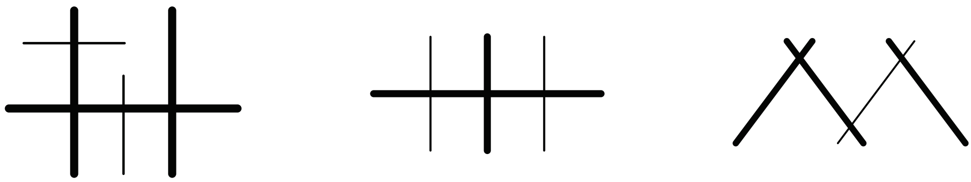


Figure 21

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(3)}_{1,0}$ and $E^{(3)}_{1,1}$ and (-1)–curves $E_{1,2}$ and $\ell^{(3)}_z$, with configuration as in 6E of Figure 21.

This is case 6E.

(2) $p_{1,2} = E_{1,1} \cap \ell^{(2)}_z$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(3)}_{1,0}$, $E^{(3)}_{1,1}$ and $\ell^{(3)}_z$ and (-1)–curve $E_{1,2}$, with configuration as in 6F of Figure 21.

This is case 6F.

Summarizing, we obtain

$$\mathcal{L}_3 = \{X_{1L,\alpha}, X_{1O}, X_{1N}, X_{2N,\alpha}, X_{2Q}, X_{2P}, X_{3N}, X_{1K}, X_{1J,\alpha}, X_{1B}, X_{2O}, X_{2B}, X_{2F}, X_{3E}, X_{4E}, X_{4M}, X_{4H}, X_{1G}, X_{2J}, X_{3F}, X_{3K}, X_{3I}, X_{4K}, X_{4I}, X_{5E}, X_{6E}, X_{6F}\}.$$

4.5 Height 4

Case 2N This case exists only if $p = 2$. We have $E = E_{1,2} - E^{(3)}_{1,1}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\}.$$

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with nontrivial stabilizer, hence we have a unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{3,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_y^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{3,1}^{(4)}$, $E_{2,0}^{(4)}$, $E_{4,0}^{(4)}$, $\ell_x^{(4)}$, $\ell_{x-y}^{(4)}$, $\ell_{x+\alpha y}^{(4)}$, $C_1^{(4)}$, $C_2^{(4)}$, $C_3^{(4)}$, $C_4^{(4)}$, $C_5^{(4)}$ and $C_6^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$, $C_3 = \mathcal{V}(x^2y + xz^2 + \alpha yz^2)$, $C_4 = \mathcal{V}(x^2y + xz^2 + y^3 + \alpha yz^2)$, $C_5 = \mathcal{V}(x^2y^2 + x^2z^2 + x^3y + \alpha^2y^2z^2)$, $C_6 = \mathcal{V}(xy^3 + x^2z^2 + x^3y + \alpha^2y^2z^2)$ and $\alpha \notin \{0, -1\}$, with configuration as in Figure 13, case 1J.

By Corollary 3.3, we have $X' \cong X_{1J,\alpha'}$ for some α'.

Case 2Q This case exists only if $p = 2$. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\}.$$

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with nontrivial stabilizer, hence we have a unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{3,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_x^{(4)}$, $\ell_y^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{3,1}^{(4)}$, $E_{2,0}^{(4)}$, $E_{4,0}^{(4)}$, $\ell_{x-y}^{(4)}$, $C_1^{(4)}$, $C_2^{(4)}$ and $C_3^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$ and $C_3 = \mathcal{V}(xz^2 + x^2y + y^3)$, with configuration as in Figure 13, case 1K.

By Corollary 3.3, we have $X' \cong X_{1K}$.

Case 2P This case exists only if $p = 2$. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\}.$$

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : \mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with nontrivial stabilizer, hence we have the following unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_x^{(4)}$, $\ell_y^{(4)}$, $\ell_z^{(4)}$ and $C_1^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,1}^{(4)}$, $E_{3,0}^{(4)}$, $E_{4,0}^{(4)}$, $\ell_{x-y}^{(4)}$ and $C_2^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$, with configuration as in Figure 10, case 1N.

By Corollary 3.3, we have $X' \cong X_{1N}$.

Case 3N This case exists only if $p = 2$. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\}.$$

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with nontrivial stabilizer, hence we have the following unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_y^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,0}^{(4)}$, $E_{3,0}^{(4)}$, $E_{4,0}^{(4)}$, $\ell_x^{(4)}$, $\ell_{x-y}^{(4)}$, $C_1^{(4)}$ and $C_2^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$, with configuration as in Figure 15, case 2O.

By Corollary 3.3, we have $X' \cong X_{2O}$.

Case 2O This case exists only if $p = 2$. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

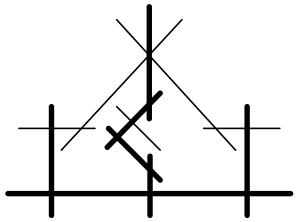
$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\}.$$

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with nontrivial stabilizer, hence we have a unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^2 = 1 \right\}$.



1E

Figure 22

- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{3,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_z^{(4)}$, $C_1^{(4)}$ and $C_2^{(4)}$ with $C_2 = \mathcal{V}(xy+y^2+z^2)$ and (-1)–curves $E_{1,3}$, $E_{2,1}^{(4)}$, $E_{3,1}^{(4)}$, $\ell_x^{(4)}$, $\ell_y^{(4)}$ and $\ell_{x-y}^{(4)}$, with configuration as in Figure 10, case 1N.

By Corollary 3.3, we have $X' \cong X_{1N}$.

Case 2B We have $E = E_{1,2} - (E_{1,1}^{(3)} \cup \ell_y^{(3)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2 y + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3 \mu]$.

Hence, we have the following unique choice for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2 y + z^3)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

  Hence, X' has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{3,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_y^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,1}^{(4)}$, $E_{3,1}^{(4)}$, $\ell_x^{(4)}$ and $\ell_{x-y}^{(4)}$, with configuration as in 1E of Figure 22.

This is case 1E.

Case 2F We have $E = (E_{1,2} \cup E_{2,2}) - (E_{1,1}^{(3)} \cup E_{2,1}^{(3)} \cup \ell_x^{(3)} \cup \ell_y^{(3)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2 y + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3 \mu]$.
- $\lambda x y^2 + \mu z^3$ is $E_{2,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3 \mu]$.

Note that the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an automorphism of X interchanging $E_{1,2}$ and $E_{2,2}$. Moreover, since X has degree 2, we are only allowed to blow up one more point. Hence, we have a unique choice for $p_{1,3}$ and $p_{2,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2y + z^3)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

  Hence, X' has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- We have (-2)–curves $E^{(4)}_{1,0}$, $E^{(4)}_{2,0}$, $E^{(4)}_{1,1}$, $E^{(4)}_{2,1}$, $E^{(4)}_{1,2}$, $\ell^{(4)}_x$, $\ell^{(4)}_y$ and $\ell^{(4)}_z$ and (-1)–curves $E_{1,3}$, $E^{(4)}_{2,2}$ and $E^{(4)}_{3,0}$, with configuration as in Figure 17, case 1G.

By Corollary 3.3, we have $X' \cong X_{1G}$.

Case 3E We have $E = E_{1,2} - (E^{(3)}_{1,1} \cup \ell^{(3)}_y)$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2 y + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3 \mu]$.

Hence, we have a unique choice for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2y + z^3)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; i^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

  Hence, X' has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- We have (-2)–curves $E^{(4)}_{1,0}$, $E^{(4)}_{2,0}$, $E^{(4)}_{1,1}$, $E^{(4)}_{1,2}$, $\ell^{(4)}_y$ and $\ell^{(4)}_z$ and (-1)–curves $E_{1,3}$, $E^{(4)}_{2,1}$, $E^{(4)}_{3,0}$ and $\ell^{(4)}_x$, with configuration as in Figure 18, case 2J.

By Corollary 3.3, we have $X' \cong X_{2J}$.

Case 4E This case exists only if $p \neq 2$. We have $E = E_{1,2} - E^{(3)}_{1,1}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu - c\lambda]$.

In particular, the stabilizer of every point on $E \cap E_{1,2}$ is trivial, and hence this case does not lead to additional weak del Pezzo surfaces with global vector fields.

Case 4M This case exists only if $p = 2$. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\}.$$

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + c\lambda]$.

In particular, Aut_X^0 acts transitively on $E \cap E_{1,2}$, so there is a unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,0}^{(4)}$, $E_{3,0}^{(4)}$, $\ell_y^{(4)}$, $C_1^{(4)}$ and $C_2^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$, with configuration as in Figure 13, case 3N.

By Corollary 3.3, we have $X' \cong X_{3N}$.

Case 4H We have $E = E_{1,2} - (E_{1,1}^{(3)} \cup \ell_y^{(3)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2y + \mu z^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3\mu]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,2}$, there is a unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2y + z^3)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 3, \\ \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

  We describe the configurations of negative curves on X' for $p \neq 3$ and $p = 3$ simultaneously:
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_y^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,0}^{(4)}$ and $E_{3,0}^{(4)}$, with configuration as in Figure 19, case 3F or 3K.

By Corollary 3.3, we have $X' \cong X_{3F}$ if $p \neq 3$, and $X' \cong X_{3K}$ if $p = 3$.

Case 2J This case exists only if $p=3$. We have $E=(E_{1,2}\cup E_{2,2})-(E_{1,1}^{(3)}\cup E_{2,1}^{(3)})$ and

$$\mathrm{Aut}^0_X(R)=\left\{\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}\in \mathrm{PGL}_3(R)\;\middle|\; e^3=1\right\}.$$

- $\lambda(x^2z+xy^2)+\mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu]\mapsto[e^2\lambda:\mu]$.
- $\lambda(xy^2+yz^2)+\mu z^3$ is $E_{2,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu]\mapsto[e^2\lambda:\mu]$.

Note that X has degree 2, and hence we are only allowed to blow up one more point. Moreover, there is a unique point on $E\cap E_{1,2}$ and on $E\cap E_{2,2}$ with nontrivial stabilizer. Therefore, we have two possibilities for $p_{1,3}$ and $p_{2,3}$:

(1) $p_{2,3}=E_{2,2}\cap C^{(3)}$ with $C=\mathcal{V}(xy+z^2)$.

- $\mathrm{Aut}^0_{X'}(R)=\left\{\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}\in \mathrm{PGL}_3(R)\;\middle|\; e^3=1\right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{2,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{2,3}$, $E_{1,2}^{(4)}$, $E_{3,0}^{(4)}$, $\ell_y^{(4)}$ and $C^{(4)}$, with configuration as in Figure 22, case 1E.

Blowing down the (-1)–curve in Figure 22 (1E) that is not adjacent to any other (-1)–curve, we see that X' is a blowup of X_{2B} in one point on E and $X'\cong X_{1E}$ by Remark 3.4.

(2) $p_{1,3}=E_{1,2}\cap C^{(3)}$ with $C=\mathcal{V}(xz+y^2)$.

- $\mathrm{Aut}^0_{X'}(R)=\left\{\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}\in \mathrm{PGL}_3(R)\;\middle|\; e^3=1\right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,2}^{(4)}$, $E_{3,0}^{(4)}$, $\ell_y^{(4)}$ and $C^{(4)}$, with configuration as in Figure 22, case 1E.

Blowing down the (-1)–curve in Figure 22 (1E) that is not adjacent to any other (-1)–curve, we see that X' is a blowup of X_{2B} in one point on E and $X'\cong X_{1E}$ by Remark 3.4.

Case 3F This case exists only if $p\neq 3$. We have $E=(E_{1,2}\cup E_{2,2})-(E_{1,1}^{(3)}\cup E_{2,1}^{(3)})$ and

$$\mathrm{Aut}^0_X(R)=\left\{\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}\in \mathrm{PGL}_3(R)\right\}.$$

- $\lambda(x^2z+xy^2)+\mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu]\mapsto[\lambda:\mu-2f\lambda]$.
- $\lambda(xy^2+yz^2)+\mu z^3$ is $E_{2,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu]\mapsto[\lambda:\mu-f\lambda]$.

If $p\neq 2$, then Aut^0_X acts simply transitively on both $E\cap E_{1,2}$ and $E\cap E_{2,2}$, and hence we cannot blow up X any further and still obtain a weak del Pezzo surface with global vector fields. If $p=2$, then Aut^0_X

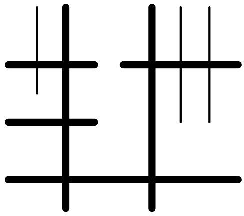


Figure 23

still acts transitively on $E \cap E_{2,2}$, but now it acts trivially on $E \cap E_{1,2}$. This leads to the following possibilities for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + \alpha y^3)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, 3, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- We have (-2)–curves $E^{(4)}_{1,0}$, $E^{(4)}_{2,0}$, $E^{(4)}_{1,1}$, $E^{(4)}_{2,1}$, $E^{(4)}_{1,2}$ and $\ell^{(4)}_z$ and (-1)–curves $E_{1,3}$, $E^{(4)}_{2,2}$ and $\ell^{(4)}_x$, with configuration as in 2R of Figure 23.

This is case 2R and we see that we get a 1–dimensional family of such surfaces $X_{2R,\alpha}$ depending on the parameter α.

Case 3K This case exists only if $p = 3$. We have $E = (E_{1,2} \cup E_{2,2}) - (E^{(3)}_{1,1} \cup E^{(3)}_{2,1})$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, e^3 = 1 \right\}.$$

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : \mu - 2ef\lambda]$.
- $\lambda(xy^2 + yz^2) + \mu z^3$ is $E_{2,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : \mu - ef\lambda]$.

Note that Aut^0_X acts transitively on both $E \cap E_{1,2}$ and $E \cap E_{2,2}$. The stabilizer of every point on $E \cap E_{1,2}$ is isomorphic to μ_3, and this μ_3 has a unique fixed point on $E \cap E_{2,2}$. This leads to three possibilities for $p_{1,3}$ and $p_{2,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}_1$ and $p_{2,3} = E_{2,2} \cap C^{(2)}_2$ with $C_1 = \mathcal{V}(xz + y^2)$ and $C_2 = \mathcal{V}(xy + z^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, e^3 = 1 \right\}.$$

- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{1,2}^{(4)}$, $E_{2,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,3}$, $\ell_x^{(4)}$, $C_2^{(4)}$ and $C_3^{(4)}$ with $C_3 = \mathcal{V}(x^2y^2 + x^3z + z^4)$, with configuration as in Figure 22, case 1E.

Blowing down the (-1)–curve in Figure 22 (1E) that is not adjacent to any other (-1)–curve, we see that X' is a blowup of X_{2B} in one point on E and $X' \cong X_{1E}$ by Remark 3.4.

(2) $p_{2,3} = E_{2,2} \cap C^{(3)}$ with $C = \mathcal{V}(xy + z^2)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\}.$
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{2,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{2,3}$, $E_{1,2}^{(4)}$, $\ell_x^{(4)}$ and $C^{(4)}$, with configuration as in Figure 18, case 2J.

By Corollary 3.3, we have $X' \cong X_{2J}$.

(3) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\}.$
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{1,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,2}^{(4)}$ and $\ell_x^{(4)}$, with configuration as in 2K of Figure 23.

This is case 2K.

Case 3I We have $E = (E_{1,2} \cup E_{2,2}) - (E_{1,1}^{(3)} \cup E_{2,1}^{(3)} \cup \ell_x^{(3)})$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [e^2\lambda : e^3\mu - 2ef\lambda]$.
- $\lambda xy^2 + \mu z^3$ is $E_{2,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [e^2\lambda : e^6\mu]$.

Note that Aut_X^0 acts transitively on $E \cap E_{2,2}$. If $p \neq 2$ (resp. $p = 2$), then Aut_X^0 acts transitively (resp. with two orbits) on $E \cap E_{1,2}$. We have five possibilities for $p_{1,3}$, $p_{2,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ and $p_{2,3} = E_{2,2} \cap C_2^{(3)}$ with $C_1 = \mathcal{V}(x^2z + xy^2 + y^3)$, $C_2 = \mathcal{V}(xy^2 + \alpha z^3)$ and $\alpha \neq 0$.

- $\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2. \end{cases}$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.

- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{1,2}^{(4)}$, $E_{2,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$ and $E_{2,3}$, with configuration as in 1Q of Figure 24.

This is case 1Q and we see that we get a 1–dimensional family of such surfaces $X_{1Q,\alpha}$ depending on the parameter α.

(2) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ and $p_{2,3} = E_{2,2} \cap C_2^{(3)}$ with $C_1 = \mathcal{V}(xz+y^2)$ and $C_2 = \mathcal{V}(xy^2+z^3)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^4 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  Hence, X' has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.

- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{1,2}^{(4)}$, $E_{2,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$ and $E_{2,3}$, with configuration as in 1R of Figure 24.

This is case 1R.

(3) $p_{2,3} = E_{2,2} \cap C^{(3)}$ with $C = \mathcal{V}(xy^2+z^3)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^4 = 1 \right\} & \text{if } p = 2. \end{cases}$$

  We describe the configurations of negative curves on X' for $p \neq 2$ and $p = 2$ simultaneously:

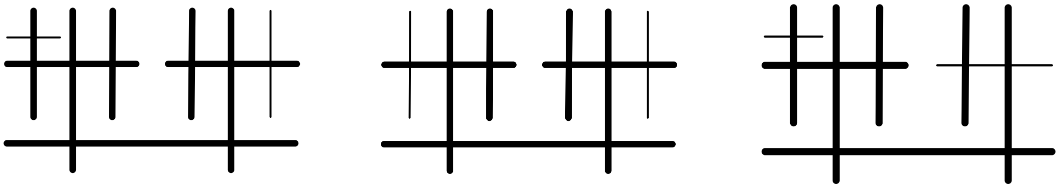
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{2,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{2,3}$ and $E_{1,2}^{(4)}$, with configuration as in 2H and 2V of Figure 24.

This is case 2H if $p \neq 2$, and case 2V if $p = 2$.

(4) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz+y^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2. \end{cases}$$

  We describe the configurations of negative curves on X' for $p \neq 2$ and $p = 2$ simultaneously:



1Q and 1R 2H and 2V 2G, 2T and 2U

Figure 24

- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$ and $E_{2,2}^{(4)}$, with configuration as in 2G and 2U of Figure 24.

This is case 2G if $p \neq 2$, and case 2U if $p = 2$.

(5) $p = 2$ and $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{2,1}^{(4)}$, $E_{1,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$ and $E_{2,2}^{(4)}$, with configuration as in 2T of Figure 24.

This is case 2T.

Case 4K We have $E = E_{2,2} - (E_{2,1}^{(3)} \cup \ell_x^{(3)})$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda xy^2 + \mu z^3$ is $E_{2,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : i^3\mu]$.

Since Aut^0_X acts transitively on $E \cap E_{2,2}$, there is a unique possibility for $p_{2,3}$ up to isomorphism:

(1) $p_{2,3} = E_{2,2} \cap C^{(3)}$ with $C = \mathcal{V}(xy^2 + z^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^2 = i^3 \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{2,1}^{(4)}$, $E_{2,2}^{(4)}$, $\ell_x^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{2,3}$ and $E_{1,1}^{(4)}$, with configuration as in Figure 19, case 3I.

By Corollary 3.3, we have $X' \cong X_{3I}$.

Case 4I We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^3\mu - 2ef\lambda]$.

If $p \neq 2$, then Aut_X^0 acts transitively on $E \cap E_{1,2}$, while if $p = 2$, then Aut_X^0 has two orbits on $E \cap E_{1,2}$. Hence, if $p = 2$, there is only one possibility for $p_{1,3}$ and if $p = 2$, there are two possibilities up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2. \end{cases}$$

  We describe the configurations of negative curves on X' for $p \neq 2$ and $p = 2$ simultaneously:

- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,1}^{(4)}$ and $\ell_x^{(4)}$, with configuration as in 3G and 3P of Figure 25.

This is case 3G if $p \neq 2$, and case 3P if $p = 2$.

(2) $p = 2$ and $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + y^3)$.

- $$\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{2,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$, $E_{2,1}^{(4)}$ and $\ell_x^{(4)}$, with configuration as in 3O of Figure 25.

This is case 3O.

Case 5E We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^3\mu - 2ef\lambda]$.

As in the previous case, if $p \neq 2$, there is only one possibility for $p_{1,3}$ up to isomorphism, and if $p = 2$, there are two possibilities up to isomorphism:

3G, 3O and 3P 4J, 4N and 4O

Figure 25

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2, \\ \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2. \end{cases}$$

  We describe the configurations of negative curves on X' for $p \neq 2$ and $p = 2$ simultaneously:
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$ and $E_{2,0}^{(4)}$, with configuration as in 4J and 4O of Figure 25.

This is case 4J if $p \neq 2$, and case 4O if $p = 2$.

(2) $p = 2$ and $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(4)}$, $E_{1,1}^{(4)}$, $E_{1,2}^{(4)}$ and $\ell_z^{(4)}$ and (-1)–curves $E_{1,3}$ and $E_{2,0}^{(4)}$, with configuration as in 4N of Figure 25.

This is case 4N.

Case 6E We have $E = E_{1,2} - E_{1,1}^{(3)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^3\mu - be^2\lambda - 2ef\lambda]$.

Since Aut^0_X acts transitively on $E \cap E_{1,2}$, there is a unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & -2fe^{-1} & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(4)}_{1,0}$, $E^{(4)}_{1,1}$ and $E^{(4)}_{1,2}$ and (-1)–curves $E_{1,3}$ and $\ell^{(4)}_z$, with configuration as in Figure 21, case 5E.

By Corollary 3.3, we have $X' \cong X_{5E}$.

Case 6F We have $E = E_{1,2} - (E^{(3)}_{1,1} \cup \ell^{(3)}_z)$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2 z + \mu y^3$ is $E_{1,2}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^3\mu]$.

Since Aut^0_X acts transitively on $E \cap E_{1,2}$, there is a unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2 z + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(4)}_{1,0}$, $E^{(4)}_{1,1}$, $E^{(4)}_{1,2}$ and $\ell^{(4)}_z$ and (-1)–curve $E_{1,3}$, with configuration as in 5F of Figure 26.

This is case 5F.

Summarizing, we obtain

$$\mathcal{L}_4 = \{X_{1E}, X_{2R,\alpha}, X_{2K}, X_{1Q,\alpha}, X_{1R}, X_{2H}, X_{2V}, X_{2G}, X_{2U}, X_{2T}, X_{3G}, X_{3P}, X_{3O}, X_{4J}, X_{4O}, X_{4N}, X_{5F}\}.$$

4.6 Height 5

Case 2R This case exists only if $p = 2$. We have $E = E_{1,3} - E^{(4)}_{1,2}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x + \alpha y)^2(xz + y^2 + \alpha yz) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as
$$[\lambda : \mu] \mapsto [\lambda : \mu + (\alpha f + f^2)\lambda].$$

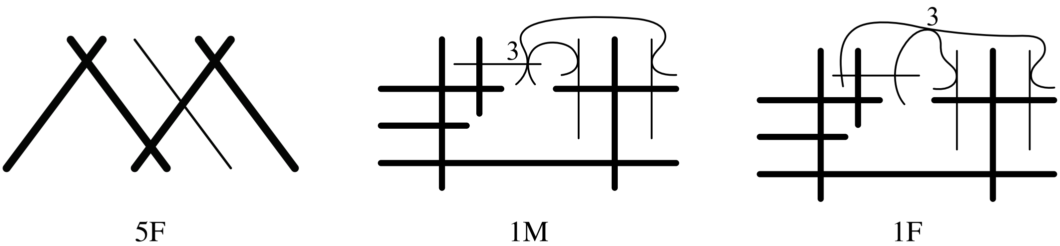


Figure 26

Therefore, if $\alpha \neq 0$, then the identity component of the stabilizer of every point on $E \cap E_{1,3}$ is trivial, hence there is no way of further blowing up X and still obtaining a weak del Pezzo surface with global vector fields. If $\alpha = 0$, then there is the following unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C_1^{(4)}$ with $C_1 = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; f^2 = 0 \right\}$.
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{2,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{2,1}^{(5)}$, $E_{1,2}^{(5)}$, $E_{1,3}^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{1,4}$, $E_{2,2}^{(5)}$, $\ell_x^{(5)}$, $C_1^{(5)}$ and $C_2^{(5)}$ with $C_2 = \mathcal{V}(x^2y^2 + x^3z + z^4)$, with configuration as in 1M of Figure 26.

This is case 1M.

Case 2K This case exists only if $p = 3$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\}.$$

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e\mu]$.

Note that there is a unique point on $E \cap E_{1,3}$ with nontrivial stabilizer. This leads to a unique possibility for $p_{1,4}$:

(1) $p_{1,4} = E_{1,3} \cap C_1^{(4)}$ with $C_1 = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\}$.
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{2,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{2,1}^{(5)}$, $E_{1,2}^{(5)}$, $E_{1,3}^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{1,4}$, $E_{2,2}^{(5)}$, $\ell_x^{(5)}$ $C_1^{(5)}$ and $C_2^{(5)}$ with $C_2 = \mathcal{V}(x^2y^2 + x^3z + z^4 + 2xyz^2)$, with configuration as in 1F of Figure 26.

This is case 1F.

Case 2H This case exists only if $p \neq 2$. We have $E = E_{2,3} - E_{2,2}^{(4)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda y(xy^2 + z^3) + \mu z^4$ is $E_{2,3}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu - 2f\lambda]$.

In particular, since $p \neq 2$, the stabilizer of every point on $E \cap E_{2,3}$ is trivial, and hence there is no way of further blowing up X and obtaining a weak del Pezzo surface with global vector fields.

Case 2V This case exists only if $p = 2$. We have $E = E_{2,3} - E_{2,2}^{(4)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^4 = 1 \right\}.$$

- $\lambda y(xy^2 + z^3) + \mu z^4$ is $E_{2,3}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [e^3\lambda : \mu]$.

This leads to two possibilities for $p_{1,4}$:

(1) $p_{2,4} = E_{2,3} \cap C^{(4)}$ with $C = \mathcal{V}(xy^3 + yz^3 + \alpha z^4)$ and $\alpha \neq 0$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{2,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{2,1}^{(5)}$, $E_{2,2}^{(5)}$, $E_{2,3}^{(5)}$, $\ell_x^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{2,4}$ and $E_{1,2}^{(5)}$, with configuration as in Figure 24, case 1Q.

By Corollary 3.3, we have $X' \cong X_{1Q,\alpha'}$ for some α'.

(2) $p_{2,4} = E_{2,3} \cap C^{(4)}$ with $C = \mathcal{V}(xy^2 + z^3)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^4 = 1 \right\}$.
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{2,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{2,1}^{(5)}$, $E_{2,2}^{(5)}$, $E_{2,3}^{(5)}$, $\ell_x^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{2,4}$ and $E_{1,2}^{(5)}$, with configuration as in Figure 24, case 1R.

By Corollary 3.3, we have $X' \cong X_{1R}$.

Case 2G This case exists only if $p \neq 2$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^2\mu]$.

Since $p \neq 2$, there is a unique point on $E \cap E_{1,3}$ such that the identity component of its stabilizer is nontrivial. This leads to a unique possibility for $p_{1,4}$:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(5)}_{1,0}$, $E^{(5)}_{2,0}$, $E^{(5)}_{1,1}$, $E^{(5)}_{2,1}$, $E^{(5)}_{1,2}$, $E^{(5)}_{1,3}$, $\ell^{(5)}_x$ and $\ell^{(5)}_z$ and (-1)–curves $E_{1,4}$, $E^{(5)}_{2,2}$ and $C^{(5)}$, with configuration as in 1C of Figure 27.

This is case 1C.

Case 2U This case exists only if $p = 2$. We have $E = E_{1,3} - E^{(4)}_{1,2}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^4\mu + f^2\lambda]$.

Since Aut^0_X acts transitively on $E \cap E_{1,3}$, there is a unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; f^2 = 0 \right\}$.
- We have (-2)–curves $E^{(5)}_{1,0}$, $E^{(5)}_{2,0}$, $E^{(5)}_{1,1}$, $E^{(5)}_{2,1}$, $E^{(5)}_{1,2}$, $E^{(5)}_{1,3}$, $\ell^{(5)}_x$ and $\ell^{(5)}_z$ and (-1)–curves $E_{1,4}$, $E^{(5)}_{2,2}$ and $C^{(5)}$, with configuration as in 1P of Figure 27.

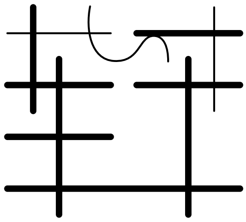
This is case 1P.

Case 2T This case exists only if $p = 2$. We have $E = E_{1,3} - E^{(4)}_{1,2}$ and

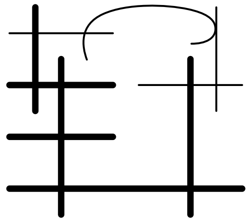
$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x + y)(x^2z + xy^2 + y^3 + y^2z) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (f + f^2)\lambda]$.

Note that the identity component of the stabilizer of every point on $E \cap E_{1,3}$ is trivial, hence we cannot blow up further and still obtain a weak del Pezzo surface with global vector fields.



1C and 1P



2C and 2S

Figure 27

Case 3G This case exists only if $p \neq 2$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^2\mu]$.

Since $p \neq 2$, there is a unique point on $E \cap E_{1,3}$ for which the identity component of the stabilizer is nontrivial. This leads to a unique possibility for $p_{1,4}$:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{2,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{1,2}^{(5)}$, $E_{1,3}^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{1,4}$, $E_{2,1}^{(5)}$, $\ell_x^{(5)}$ and $C^{(5)}$, with configuration as in 2C of Figure 27.

This is case 2C.

Case 3P This case exists only if $p = 2$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^4\mu + f^2\lambda]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,3}$, there is a unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; f^2 = 0 \right\}.$

- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{2,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{1,2}^{(5)}$, $E_{1,3}^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{1,4}$, $E_{2,1}^{(5)}$, $\ell_x^{(5)}$ and $C^{(5)}$, with configuration as in 2S of Figure 27.

This is case 2S.

Case 3O This case exists only if $p=2$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\operatorname{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x+y)(x^2z+xy^2+y^3+y^2z)+\mu y^4$ is $E_{1,3}$–adapted and $\operatorname{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda:\mu+(f+f^2)\lambda]$.

In particular, the identity component of the stabilizer of every point on $E \cap E_{1,3}$ is trivial, hence we cannot blow up further.

Case 4J This case exists only if $p \neq 2$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\operatorname{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(xz+y^2)+\mu y^4$ is $E_{1,3}$–adapted and $\operatorname{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda:e^2\mu+c\lambda]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,3}$, we have a unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz+y^2)$.

- $\operatorname{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{1,2}^{(5)}$, $E_{1,3}^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{1,4}$, $E_{2,0}^{(5)}$ and $C^{(5)}$, with configuration as in Figure 25, case 3G.

By Corollary 3.3, we have $X' \cong X_{3G}$.

Case 4O This case exists only if $p=2$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\operatorname{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(xz+y^2)+\mu y^4$ is $E_{1,3}$–adapted and $\operatorname{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [e^2\lambda:e^4\mu+(ce^2+f^2)\lambda]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,3}$, we have a unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & f^2e^{-2} \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{1,2}^{(5)}$, $E_{1,3}^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{1,4}$, $E_{2,0}^{(5)}$ and $C^{(5)}$, with configuration as in Figure 25, case 3P.

By Corollary 3.3, we have $X' \cong X_{3P}$.

Case 4N This case exists only if $p = 2$. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda(x + y)(x^2z + xy^2 + y^3 + y^2z) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (c + f + f^2)\lambda]$.

Since Aut^0_X acts transitively on $E \cap E_{1,3}$, we have a unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C_1^{(4)}$ with $C_1 = \mathcal{V}(x^2z + xy^2 + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & f + f^2 \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(5)}$, $E_{1,1}^{(5)}$, $E_{1,2}^{(5)}$, $E_{1,3}^{(5)}$ and $\ell_z^{(5)}$ and (-1)–curves $E_{1,4}$, $E_{2,0}^{(5)}$ and $C_2^{(5)}$ with $C_2 = \mathcal{V}(xz + yz + y^2)$, with configuration as in Figure 25, case 3O.

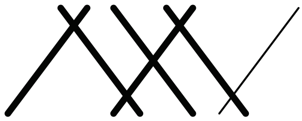
By Corollary 3.3, we have $X' \cong X_{3O}$.

Case 5F We have $E = E_{1,3} - E_{1,2}^{(4)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x(x^2z + y^3) + \mu y^4$ is $E_{1,3}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e\mu - 2b\lambda]$.

Therefore, if $p \neq 2$, we have one unique possibility for $p_{1,4} \in E \cap E_{1,3}$, while if $p = 2$, there are two possibilities:



4L, 4P and 4Q

Figure 28

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(x^2z + y^3)$.

- $$\mathrm{Aut}^0_{X'}(R) = \begin{cases} \left\{\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R)\right\} & \text{if } p \neq 2, \\ \left\{\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R)\right\} & \text{if } p = 2. \end{cases}$$

We describe the configurations of negative curves on X' for $p \neq 2$ and $p = 2$ simultaneously:

- We have (-2)–curves $E^{(5)}_{1,0}$, $E^{(5)}_{1,1}$, $E^{(5)}_{1,2}$, $E^{(5)}_{1,3}$ and $\ell^{(5)}_z$ and (-1)–curve $E_{1,4}$, with configuration as in 4L and 4Q of Figure 28.

This is case 4L if $p \neq 2$, and case 4Q if $p = 2$.

(2) $p = 2$ and $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(x^3z + xy^3 + y^4)$.

- $$\mathrm{Aut}^0_{X'}(R) = \left\{\begin{pmatrix} 1 & b & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R)\right\}.$$
- We have (-2)–curves $E^{(5)}_{1,0}$, $E^{(5)}_{1,1}$, $E^{(5)}_{1,2}$, $E^{(5)}_{1,3}$ and $\ell^{(5)}_z$ and (-1)–curve $E_{1,4}$, with configuration as in 4P of Figure 28.

This is case 4P.

Summarizing, we obtain

$$\mathcal{L}_5 = \{X_{1M}, X_{1F}, X_{1C}, X_{1P}, X_{2C}, X_{2S}, X_{4L}, X_{4Q}, X_{4P}\}.$$

4.7 Height 6

Case 2C This case exists only if $p \neq 2$. We have $E = E_{1,4} - E^{(5)}_{1,3}$ and

$$\mathrm{Aut}^0_X(R) = \left\{\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R)\right\}.$$

- $\lambda x^3(xz + y^2) + \mu y^5$ is $E_{1,4}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu]$.

Note that if $p \neq 3$, then there is a unique point on $E \cap E_{1,4}$ such that the identity component of its stabilizer is nontrivial. If $p = 3$, this identity component is nontrivial for every point. In all characteristics, the action of Aut_X^0 on $E \cap E_{1,4}$ has two orbits. Hence, we have two possibilities for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C_1^{(5)}$ with $C_1 = \mathcal{V}(x^4z + x^3y^2 + y^5)$.

- $$\mathrm{Aut}_{X'}^0(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2,3, \\ \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; e^3 = 1 \right\} & \text{if } p = 3. \end{cases}$$

  Hence, X' has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{2,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$ and $\ell_z^{(6)}$ and (-1)–curves $E_{1,5}$, $E_{2,1}^{(6)}$, $\ell_x^{(6)}$, $C_2^{(6)}$ and $C_3^{(6)}$ with $C_2 = \mathcal{V}(xz + y^2)$ and $C_3 = \mathcal{V}(xy^4 - xyz^3 - x^2y^2z + x^3z^2 - y^3z^2 - z^5)$, with configuration as in Figure 26, case 1F.

By Corollary 3.3, we have $X' \cong X_{1F}$.

(2) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(xz + y^2)$.

- $$\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$
- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{2,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$, $\ell_z^{(6)}$ and $C^{(6)}$ and (-1)–curves $E_{1,5}$, $E_{2,1}^{(6)}$ and $\ell_x^{(6)}$, with configuration as in Figure 27, case 1C.

By Corollary 3.3, we have $X' \cong X_{1C}$.

Case 2S This case exists only if $p = 2$. We have $E = E_{1,4} - E_{1,3}^{(5)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; f^2 = 0 \right\}.$$

- $\lambda x^3(xz + y^2) + \mu y^5$ is $E_{1,4}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu]$.

Since Aut_X^0 acts on $E \cap E_{1,4}$ with two orbits, we have two possibilities for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C_1^{(5)}$ with $C_1 = \mathcal{V}(x^4z + x^3y^2 + y^5)$.

- $$\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; f^2 = 0 \right\}.$$

- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{2,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$ and $\ell_z^{(6)}$ and (-1)–curves $E_{1,5}$, $E_{2,1}^{(6)}$, $\ell_x^{(6)}$, $C_2^{(6)}$ and $C_3^{(6)}$ with $C_2 = \mathcal{V}(xz+y^2)$ and $C_3 = \mathcal{V}(xy^4+x^3z^2+z^5)$, with configuration as in Figure 26, case 1M.

By Corollary 3.3, we have $X' \cong X_{1M}$.

(2) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(xz+y^2)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; f^2 = 0 \right\}$.
- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{2,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$, $\ell_z^{(6)}$ and $C^{(6)}$ and (-1)–curves $E_{1,5}$, $E_{2,1}^{(6)}$ and $\ell_x^{(6)}$, with configuration as in Figure 27, case 1P.

By Corollary 3.3, we have $X' \cong X_{1P}$.

Case 4L This case exists only if $p \neq 2$. We have $E = E_{1,4} - E_{1,3}^{(5)}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(x^2z+y^3)+\mu y^5$ is $E_{1,4}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda:\mu] \mapsto [e\lambda : e^3\mu - 3f\lambda]$.

In particular, if $p \neq 3$, then Aut^0_X acts transitively on $E \cap E_{1,4}$ and we have only one choice for $p_{1,5}$ up to isomorphism, and if $p = 3$, then Aut^0_X acts with two orbits on $E \cap E_{1,4}$. Hence we have two choices up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^2z+y^3)$.

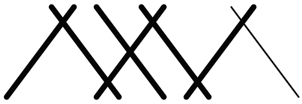
- $\mathrm{Aut}^0_{X'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2,3, \\ \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 3. \end{cases}$

  We describe the configurations of negative curves on X' for $p \neq 2,3$ and $p = 3$ simultaneously:
- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$ and $\ell_z^{(6)}$ and (-1)–curve $E_{1,5}$, with configuration as in 3J and 3M of Figure 29.

This is case 3J if $p \neq 2,3$, and case 3M if $p = 3$.

(2) Let $p = 3$ and $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^4z+x^2y^3+y^5)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.



3J, 3L, 3M, 3Q and 3R

Figure 29

- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$ and $\ell_z^{(6)}$ and (-1)–curve $E_{1,5}$, with configuration as in 3L of Figure 29.

This is case 3L.

Case 4Q This case exists only if $p=2$. We have $E = E_{1,4} - E_{1,3}^{(5)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(x^2z+y^3)+\mu y^5$ is $E_{1,4}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [e\lambda : e^3\mu + (b^2e+f)\lambda]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,4}$, there is a unique choice for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^2z+y^3)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$ and $\ell_z^{(6)}$ and (-1)–curve $E_{1,5}$, with configuration as in 3R of Figure 29.

This is case 3R.

Case 4P This case exists only if $p=2$. We have $E = E_{1,4} - E_{1,3}^{(5)}$ and

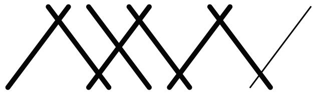
$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x(x^3z+xy^3+y^4)+\mu y^5$ is $E_{1,4}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda : \mu + (b+b^2+f)\lambda]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,4}$, we have a unique choice for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^3z+xy^3+y^4)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2+b \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.



2I, 2L, 2M, 2W, 2X and 2Y

Figure 30

- We have (-2)–curves $E_{1,0}^{(6)}$, $E_{1,1}^{(6)}$, $E_{1,2}^{(6)}$, $E_{1,3}^{(6)}$, $E_{1,4}^{(6)}$ and $\ell_z^{(6)}$ and (-1)–curve $E_{1,5}$, with configuration as in 3Q of Figure 29.

This is case 3Q.

Summarizing, we obtain

$$\mathcal{L}_6 = \{X_{3J}, X_{3M}, X_{3L}, X_{3R}, X_{3Q}\}.$$

4.8 Height 7

Case 3J This case exists only if $p \neq 2, 3$. We have $E = E_{1,5} - E_{1,4}^{(6)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^3(x^2z + y^3) + \mu y^6$ is $E_{1,5}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu + 2c\lambda]$.

Since $p \neq 2$, Aut_X^0 acts transitively on $E \cap E_{1,5}$, so there is a unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^2z + y^3)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$
- We have (-2)–curves $E_{1,0}^{(7)}$, $E_{1,1}^{(7)}$, $E_{1,2}^{(7)}$, $E_{1,3}^{(7)}$, $E_{1,4}^{(7)}$, $E_{1,5}^{(7)}$ and $\ell_z^{(7)}$ and (-1)–curve $E_{1,6}$, with configuration as in 2I of Figure 30.

This is case 2I.

Case 3M This case exists only if $p = 3$. We have $E = E_{1,5} - E_{1,4}^{(6)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^3(x^2z + y^3) + \mu y^6$ is $E_{1,5}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu + 2c\lambda]$.

As in the previous case, there is a unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^2z + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(7)}_{1,0}$, $E^{(7)}_{1,1}$, $E^{(7)}_{1,2}$, $E^{(7)}_{1,3}$, $E^{(7)}_{1,4}$, $E^{(7)}_{1,5}$ and $\ell^{(7)}_z$ and (-1)–curve $E_{1,6}$, with configuration as in 2M of Figure 30.

This is case 2M.

Case 3L This case exists only if $p = 3$. We have $E = E_{1,5} - E^{(6)}_{1,4}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x(x^4z + x^2y^3 + y^5) + \mu y^6$ is $E_{1,5}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + 2c\lambda]$.

As in the previous case, there is a unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^4z + x^2y^3 + y^5)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(7)}_{1,0}$, $E^{(7)}_{1,1}$, $E^{(7)}_{1,2}$, $E^{(7)}_{1,3}$, $E^{(7)}_{1,4}$, $E^{(7)}_{1,5}$ and $\ell^{(7)}_z$ and (-1)–curve $E_{1,6}$, with configuration as in 2L of Figure 30.

This is case 2L.

Case 3R This case exists only if $p = 2$. We have $E = E_{1,5} - E^{(6)}_{1,4}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^3(x^2z + y^3) + \mu y^6$ is $E_{1,5}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu]$.

Since Aut^0_X has two orbits on $E \cap E_{1,5}$, we have two choices for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^5z + x^3y^3 + y^6)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2 \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(7)}_{1,0}$, $E^{(7)}_{1,1}$, $E^{(7)}_{1,2}$, $E^{(7)}_{1,3}$, $E^{(7)}_{1,4}$, $E^{(7)}_{1,5}$ and $\ell^{(7)}_z$ and (-1)–curve $E_{1,6}$, with configuration as in 2X of Figure 30.

This is case 2X.

(2) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^2z + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(7)}_{1,0}$, $E^{(7)}_{1,1}$, $E^{(7)}_{1,2}$, $E^{(7)}_{1,3}$, $E^{(7)}_{1,4}$, $E^{(7)}_{1,5}$ and $\ell^{(7)}_z$ and (-1)–curve $E_{1,6}$, with configuration as in 2Y of Figure 30.

This is case 2Y.

Case 3Q This case exists only if $p = 2$. We have $E = E_{1,5} - E^{(6)}_{1,4}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2 + b \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(x^3z + xy^3 + y^4) + \mu y^6$ is $E_{1,5}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (b^2 + b)\lambda]$.

Since Aut^0_X acts transitively on $E \cap E_{1,5}$, we have a unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^3z + xy^3 + y^4)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(7)}_{1,0}$, $E^{(7)}_{1,1}$, $E^{(7)}_{1,2}$, $E^{(7)}_{1,3}$, $E^{(7)}_{1,4}$, $E^{(7)}_{1,5}$ and $\ell^{(7)}_z$ and (-1)–curve $E_{1,6}$, with configuration as in 2W of Figure 30.

This is case 2W.

Summarizing, we obtain

$$\mathcal{L}_7 = \{X_{2I}, X_{2M}, X_{2L}, X_{2X}, X_{2Y}, X_{2W}\}.$$

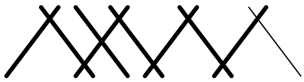
4.9 Height 8

Case 2I This case exists only if $p \neq 2, 3$. We have $E = E_{1,6} - E^{(7)}_{1,5}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^4(x^2z + y^3) + \mu y^7$ is $E_{1,6}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^4\mu]$.

Since $p \neq 2$, there is a unique point on $E \cap E_{1,6}$ whose stabilizer has nontrivial identity component. This leads to a unique choice for $p_{1,7}$ up to isomorphism:



1D, 1H, 1I, 1S and 1T

Figure 31

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^2z + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(8)}_{1,0}$, $E^{(8)}_{1,1}$, $E^{(8)}_{1,2}$, $E^{(8)}_{1,3}$, $E^{(8)}_{1,4}$, $E^{(8)}_{1,5}$, $E^{(8)}_{1,6}$ and $\ell^{(8)}_z$ and (-1)–curve $E_{1,7}$, with configuration as in 1D of Figure 31.

This is case 1D.

Case 2M This case exists only if $p = 3$. We have $E = E_{1,6} - E^{(7)}_{1,5}$ and

$$\mathrm{Aut}^0_X(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^4(x^2z + y^3) + \mu y^7$ is $E_{1,6}$–adapted and $\mathrm{Aut}^0_X(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^4\mu]$.

Since Aut^0_X acts with two orbits on $E \cap E_{1,6}$, we have two choices for $p_{1,7}$ up to isomorphism:

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^6z + x^4y^3 + y^7)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(8)}_{1,0}$, $E^{(8)}_{1,1}$, $E^{(8)}_{1,2}$, $E^{(8)}_{1,3}$, $E^{(8)}_{1,4}$, $E^{(8)}_{1,5}$, $E^{(8)}_{1,6}$ and $\ell^{(8)}_z$ and (-1)–curve $E_{1,7}$, with configuration as in 1H of Figure 31.

This is case 1H.

(2) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^2z + y^3)$.

- $\mathrm{Aut}^0_{X'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E^{(8)}_{1,0}$, $E^{(8)}_{1,1}$, $E^{(8)}_{1,2}$, $E^{(8)}_{1,3}$, $E^{(8)}_{1,4}$, $E^{(8)}_{1,5}$, $E^{(8)}_{1,6}$ and $\ell^{(8)}_z$ and (-1)–curve $E_{1,7}$, with configuration as in 1I of Figure 31.

This is case 1I.

Case 2L This case exists only if $p = 3$. We have $E = E_{1,6} - E_{1,5}^{(7)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^2(x^4z + x^2y^3 + y^5) + \mu y^7$ is $E_{1,6}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda:\mu + f\lambda]$.

Hence, the stabilizer of every point on $E \cap E_{1,6}$ is trivial, therefore we cannot blow up X further and still obtain a weak del Pezzo surface with global vector fields.

Case 2X This case exists only if $p = 2$. We have $E = E_{1,6} - E_{1,5}^{(7)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2 \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x(x^5z + x^3y^3 + y^6) + \mu y^7$ is $E_{1,6}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda:\mu + (b + b^4)\lambda]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,6}$, there is a unique choice for $p_{1,7}$ up to isomorphism:

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^5z + x^3y^3 + y^6)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.
- We have (-2)–curves $E_{1,0}^{(8)}$, $E_{1,1}^{(8)}$, $E_{1,2}^{(8)}$, $E_{1,3}^{(8)}$, $E_{1,4}^{(8)}$, $E_{1,5}^{(8)}$, $E_{1,6}^{(8)}$ and $\ell_z^{(8)}$ and (-1)–curve $E_{1,7}$, with configuration as in 1S of Figure 31.

This is case 1S.

Case 2Y This case exists only if $p = 2$. We have $E = E_{1,6} - E_{1,5}^{(7)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^4(x^2z + y^3) + \mu y^7$ is $E_{1,6}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda:\mu] \mapsto [\lambda: e^4\mu + b^4\lambda]$.

Since Aut_X^0 acts transitively on $E \cap E_{1,6}$, there is a unique choice for $p_{1,7}$ up to isomorphism:

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^2z + y^3)$.

- $\mathrm{Aut}_{X'}^0(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \;\middle|\; b^4 = 0 \right\}$.
- We have (-2)–curves $E_{1,0}^{(8)}$, $E_{1,1}^{(8)}$, $E_{1,2}^{(8)}$, $E_{1,3}^{(8)}$, $E_{1,4}^{(8)}$, $E_{1,5}^{(8)}$, $E_{1,6}^{(8)}$ and $\ell_z^{(8)}$ and (-1)–curve $E_{1,7}$, with configuration as in 1T of Figure 31.

This is case 1T.

Case 2W This case exists only if $p=2$. We have $E = E_{1,6} - E_{1,5}^{(7)}$ and

$$\mathrm{Aut}_X^0(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}.$$

- $\lambda x^3(x^3z + xy^3 + y^4) + \mu y^7$ is $E_{1,6}$–adapted and $\mathrm{Aut}_X^0(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + c\lambda]$.

In particular, the identity component of the stabilizer of every point on $E \cap E_{1,6}$ is trivial, hence we cannot blow up further and still obtain a weak del Pezzo surface with global vector fields.

Summarizing, we obtain

$$\mathcal{L}_8 = \{X_{1D}, X_{1H}, X_{1I}, X_{1S}, X_{1T}\}. \qquad \square$$

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