Geometry & Topology

Volume 28 (2024)

The string coproduct "knows" Reidemeister/Whitehead torsion

FLORIAN NAEF

msp

msp

Geometry & Topology **28**:8 (2024) 3643–3659
DOI: 10.2140/gt.2024.28.3643
Published: 20 December 2024

The string coproduct "knows" Reidemeister/Whitehead torsion

FLORIAN NAEF

We show that the string coproduct is not homotopy invariant. More precisely, we show that the (reduced) coproducts are different on $L(1,7)$ and $L(2,7)$. Moreover, the coproduct on $L(k,7)$ can be expressed in terms of the Reidemeister torsion and hence transforms with respect to the Whitehead torsion of a homotopy equivalence. The string coproduct can thereby be used to compute the image of the Whitehead torsion under the Dennis trace map.

55P50

1 Introduction

Given a compact oriented manifold M of dimension n, Chas and Sullivan [1999; 2004] defined a number of operations on the homology of the free loop space $LM = \mathrm{Map}(S^1, M)$. The most prominent ones are the string product, which is an operation of the type

$$\star\colon H_\bullet(LM \times LM) \to H_{\bullet-n}(LM),$$

the string coproduct

$$\Delta\colon H_\bullet(LM, M) \to H_{\bullet-n+1}(LM \times LM, M \times LM \cup LM \times M),$$

and the circle action

$$B\colon H_\bullet(LM) \to H_{\bullet+1}(LM).$$

The string product and coproduct are defined in terms of intersections of chains satisfying a certain transversality condition. In particular, it is not a priori clear whether or not they depend on the manifold structure beyond its homotopy type. Or, said differently, one can ask whether a homotopy equivalence $f\colon M_1 \to M_2$ that preserves the orientation classes induces a map $f\colon H_\bullet(LM_1) \to H_\bullet(LM_2)$ that intertwines all the above operations, ie $(\star, \Delta, B)$. The operator B is clearly homotopy-invariant. For the string product $\star$, it is shown in [Cohen et al. 2008; Crabb 2008; Gruher and Salvatore 2008] (or could be deduced from [Cohen and Jones 2002]) that it is homotopy-invariant. We show in this short note that this is *not* true for the string coproduct. To that extent, we compute enough string coproducts on lens spaces to show that it is sensitive to Reidemeister torsion and transforms with respect to Whitehead torsion. In particular, string topology can tell $L(1,7)$ and $L(2,7)$ apart. Moreover, we verify (in a certain range) the transformation formula

$$\Delta f(x) = f(\Delta(x)) + f(x \star d\log \tau(f)), \tag{1}$$

where Δ is the string coproduct, $\star$ is the string product, and $\tau(f)$ is the Whitehead torsion under the Dennis trace map, which we denote by $d\log$. Naturally, one is led to conjecture that this formula is true in full generality, ie for all closed manifolds M and all $f \in \pi_*(\mathrm{aut}(M))$. Such a transformation formula is not entirely unexpected considering the following. Naef and Willwacher [2019] showed that the natural comparison map (over the reals) between loop space cohomology and Hochschild homology of the cochain algebra $C^*(M)$ can be made to intertwine coproducts. The description of the coproduct on the algebraic side, however, depends on the 1–loop contributions of the partition function of a Chern–Simons type field theory. It is moreover easy to see that not every Com_∞–automorphism of $C^*(M)$ (the algebraic analogue of a homotopy equivalence) preserves the coproduct since it might change the 1–loop part. In particular, the algebraic analogue of the above transformation formula is true, where the Whitehead torsion term is defined as the action on the 1–loop part. Stretching the analogy a bit, we would like to think that this 1–loop part merely computes (a certain expansion of) the Reidemeister torsion as in the cellular model in [Cattaneo et al. 2020].

The structure of the paper is as follows. First we compute the integral homology of the free loop space of a lens space $M = L(k, 7)$ and give generators. We proceed to compute all the string coproducts of generators in $H_3(LM)$ in terms of these generators. We then show that, after quotienting out certain "inconvenient" classes, we can write particularly succinct formulas for the previous calculation and that, even after "forgetting" these classes, we can still detect Reidemeister torsion and get the correction terms as in (1). Finally, we show for one particularly striking example that the transformation formula (1) is also true with the "inconvenient" classes intact.

Acknowledgements

I am grateful to Pavel Mnev and Konstantin Wernli for the discussions that inspired the note, and to Pavel Safronov for helpful comments. I moreover thank the referees for a careful reading and useful suggestions. I also would like to thank Nathalie Wahl for discussions, suggestions and support.

2 Lens spaces

In the following, M will be a lens space of the form $L(k, 7)$. That is, let $e^t = \mathrm{e}^{2\pi\sqrt{-1}t}$ and consider the $\mathbb{Z}_7$–action on $S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ generated by

$$(z_1, z_2) \mapsto (e^{1/7} z_1, e^{k/7} z_2).$$

There is a residual action of the two-torus $S^1 \times S^1 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\left(\frac{1}{7}\mathbb{Z}\right)$ given by

$$(z_1, z_2) \mapsto (e^t z_1, e^{kt+s} z_2)$$

for $(s, t) \in [0, 1] \times \left[0, \frac{1}{7}\right]$. Note that this action is free away from the two circles

$$K_1 = \{z_1 = 0\} \quad \text{and} \quad K_2 = \{z_2 = 0\}.$$

Let r denote the inverse of $k \in \mathbb{Z}_7^\times$.

2.1 Homology

Components of the free loop space LM are in one-to-one correspondence with conjugacy classes of $\pi_1(M) = \mathbb{Z}_7$. Let $L_l M$ denote the component corresponding to $l \in \mathbb{Z}_7$. There is a fibration

$$\Omega_l M \to L_l M \to M, \tag{2}$$

where $\Omega_l M$ is the component of the based loop space corresponding to l and $L_l M \to M$ is the evaluation map. Since ΩM is group-like, the component $\Omega_l M$ is homotopy-equivalent to $\Omega_0 M$, the component of contractible loops. Comparing homotopy groups, we see that the map

$$\Omega S^3 \to \Omega_0 M$$

induced by the covering map $S^3 \to M$ is a homotopy equivalence. Hence, the (integral) homology of $\Omega_l M$ is given by

$$H_*(\Omega_l M) = \begin{cases} \mathbb{Z} & \text{for } * = 0, 2, 4, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

The differential on the E_2–page of the Serre spectral sequence associated to the fibration (2) is zero for degree reasons. The E_3–page is

$$\begin{array}{cccc} \vdots & & & \\ \mathbb{Z} & \mathbb{Z}_7 & 0 & \mathbb{Z} \\ 0 & 0 & 0 & 0 \\ \mathbb{Z} & \mathbb{Z}_7 & 0 & \mathbb{Z} \\ 0 & 0 & 0 & 0 \\ \mathbb{Z} & \mathbb{Z}_7 & 0 & \mathbb{Z} \end{array}$$

where the only possibly nonzero differentials are indicated. From the residual torus action, one can see that the map $H_*(L_l M) \to H_*(M)$ is onto (for the component of the contractible loop it is onto for any space). We will see this in more detail below by exhibiting sections $\rho_{l,m}$ of $L_l M \to M$. Hence, the differential $H_3(M) \to H_0(M, H_2(\Omega_l M))$ on the E_3–page vanishes and we obtain for the homology

$$H_0(L_l M) = \mathbb{Z}, \quad H_1(L_l M) = \mathbb{Z}_7, \quad H_2(L_l M) = \mathbb{Z}, \quad H_3(L_l M) = \mathbb{Z} \oplus \mathbb{Z}_7, \quad H_4(L_l M) = \mathbb{Z}, \quad \dots.$$

Let us be more precise about these identifications and give more explicit descriptions for H_0, H_1 and H_3. We identify $H_0(LM)$ with $\mathbb{Z}[\mathbb{Z}_7]$. Moreover, let $\Omega^1 = \Omega^1(\mathbb{Z}[\mathbb{Z}_7]) = \mathbb{F}_7[t]/(t^7 - 1)\, dt/t$ denote the vector space of formal de Rham 1–forms and identify

$$\mathbb{F}_7[t]/(t^7 - 1)\, \frac{dt}{t} \to H_1(LM) = \bigoplus_{l \in \mathbb{Z}_7} H_1(L_l M) \cong \bigoplus_{l \in \mathbb{Z}_7} H_1(M),$$

$$(c_0 + c_1 t + \dots c_6 t^6)\, \frac{dt}{t} \mapsto (c_0, \dots, c_6).$$

Remark 2.1 The reason we identify H_1 with Ω^1 and not with $\mathbb{F}_7[\mathbb{Z}_7]$ is first and foremost to make the formulas later more appealing. One can justify this identification at this point by appealing to the fact that $H_1(LM) = \mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7]) = \Omega^1$, where HH_1 is Hochschild homology. Thus, the identification is in particular natural with respect to automorphisms of $\pi_1 = \mathbb{Z}_7$. Furthermore, the circle action $H_0(LM) \to H_1(LM)$ can now be written as the "de Rham differential"

$$\mathbb{Z}[\mathbb{Z}_7] \to \Omega^1(\mathbb{Z}[\mathbb{Z}_7]), \quad c_0 + c_1 t + \cdots + c_6 t^6 \mapsto (c_1 t + 2c_2 t^2 + \cdots + 6c_6 t^6)\,\frac{dt}{t}.$$

This is a repackaging of the calculation that the composition $H_0(L_l M) \to H_1(L_l M) \to H_1(M) = \pi_1(M)^{\mathrm{ab}}$ sends 1 to l.

Let us also define $\overline{\Omega}^1 = \Omega^1/\mathbb{F}_7\, dt/t$, so that we can identify

$$\overline{\Omega}^1 \cong H_1(LM, M).$$

Similarly, $\overline{\mathbb{Z}}[\mathbb{Z}_7] = \mathbb{Z}[\mathbb{Z}_7]/\mathbb{Z}1$, so

$$\overline{\mathbb{Z}}[\mathbb{Z}_7] \cong H_0(LM, M).$$

For $H_3(L_l M)$, the spectral sequence gives us a short exact sequence

$$0 \to H_1(M, H_2(\Omega_l M)) \to H_3(L_l M) \to H_3(M) \to 0.$$

From the residual two-torus action, we can construct a number of classes in $H_3(L_l M)$ that map to the fundamental class in $H_3(M)$. Consider the following S^1–actions. For given integers (l, m) we define

$$(t, z_1, z_2) \mapsto \rho_{l,m}(t, z_1, z_2) := (e^{lt} z_1, e^{(kl+7m)t} z_2) \quad \text{for } t \in \left[0, \tfrac{1}{7}\right].$$

We view $\rho_{l,m}$ as a map $M \to LM$ and denote the image of the fundamental class by $[\rho_{l,m}] \in H_3(LM)$. The class of $\rho_{l,m}$ lies in the component corresponding to l,

$$[\rho_{l,m}] \in H_3(L_l M),$$

for all m, as can be seen for instance by setting $z_2 = 0$. We will argue below that these classes span all of $H_3(LM)$.

3 String coproduct

3.1 Definition of the string coproduct

The string coproduct is informally defined as an operation

$$H_\bullet(LM, M) \to H_{\bullet-n+1}((LM, M) \times (LM, M)) := H_{\bullet-n+1}(LM \times LM, LM \times M \cup M \times LM)$$

given by "cutting transverse loops at self-intersections". We refer to [Hingston and Wahl 2023] for a more formal definition and to [loc. cit., Proposition 3.7] for the statement that the two definitions coincide under suitable assumptions (a proof of the special case that is used below is given in Appendix B as Proposition B.2).

We now recall how the string coproduct is computed using Proposition B.2. Suppose that the homology class $[\alpha] \in H_p(LM)$ is represented by a map $N \to LM$, where N is an oriented closed p–dimensional manifold. That is, we are given a map

$$\alpha\colon S^1 \times N \to M, \quad (t,n) \mapsto \alpha(t,n).$$

The self-intersection locus is defined by

$$V = \{(t,n) : \alpha(t,n) = \alpha(0,n),\, t \neq 0\} \subset S^1 \times N.$$

We are assuming that V is compact and the intersection is transverse (in the sense of Lemma B.1), so that V is an oriented submanifold of $S^1 \times N$ of dimension $p+1-\dim(M)$. Splitting the loops at intersection points, we obtain

$$\Delta(\alpha)\colon V \to LM \times LM, \quad (t,n) \mapsto \big(s \mapsto \alpha(st,n),\, s \mapsto \alpha(t+(1-t)s,n)\big).$$

This gives a class $[\Delta(\alpha)] \in H_{p-n+1}(LM \times M)$, which we project onto $H_{p-n+1}((LM,M)\times(LM,M)) := H_{p-n+1}(LM \times LM, LM \times M \cup M \times LM)$. By Proposition B.2, we have

$$\Delta([\alpha]) = [\Delta(\alpha)].$$

3.2 String coproduct on $L(k,7)$

For $M = L(k,7)$ we will describe the string coproduct map $H_*(LM) \to H_{*+1-3}(LM/M \times LM/M)$ (ie the composition with the forgetting map $H_*(LM) \to H_*(LM,M)$). Moreover, we consider only the component

$$\Delta\colon H_3(LM) \to H_{3+1-3}(LM/M \times LM/M) \to H_1(LM,M) \otimes H_0(LM,M).$$

Under the identifications from the previous section, the string coproduct gives a map

$$H_3(LM) \to \overline{\Omega}^1 \otimes \overline{\mathbb{Z}}[\mathbb{Z}_7] = \mathbb{F}_7[t,t_2]\,\frac{dt}{t} \Big/ \Big((t^7-1, t_2^7-1) \oplus \Omega^1 \cdot 1 \oplus \frac{dt}{t}\,\mathbb{Z}[t_2]\Big).$$

More concretely, we identify a monomial $t^p t_2^q\, dt/t$ with the class in $H_1(L_pM) \otimes H_0(L_qM)$ whose image under $H_1(L_pM) \otimes H_0(L_qM) \to H_1(M) \otimes H_0(L_qM) \cong \mathbb{Z}_7$ is the canonical generator.

Proposition 3.1 *Under the above identifications, the string coproduct of the classes $[\rho_{l,m}]$ for l and m positive coprime integers is given by the formula*

$$\Delta([\rho_{l,m}]) = (t t_2^{l-1} + t^2 t_2^{l-2} + \cdots + t^{l-1} t_2)\,\frac{dt}{t} + r(t^r t_2^{(kl+7m-1)r} + t^{2r} t_2^{(kl+7m-2)r} + \cdots + t^{(kl+7m-1)r} t_2^r)\,\frac{dt}{t}, \tag{3}$$

where r is the multiplicative inverse of k mod 7.

Remark 3.2 With some more care, the condition that l and m be coprime can be dropped (see [Naef et al. 2023]).

Proof We will compute the coproduct of the class $[\rho_{l,m}]$ for (l,m) positive coprime integers using Proposition B.2, using the above description.

That is, we first find

$$V = \{(t, z_1, z_2) \in (0,1) \times M : \rho_{l,m}(t, z_1, z_2) = (z_1, z_2)\},$$

then argue that it is compact and transversely cut out to be able to apply Proposition B.2. Away from the circles $K_1 = \{z_1 = 0\}$ and $K_2 = \{z_2 = 0\}$, the action is free, since in that case the above equation reads as

$$\begin{pmatrix} l \\ kl + 7m \end{pmatrix} t \in \mathbb{Z}^2 + \frac{1}{7}\begin{pmatrix} 1 \\ k \end{pmatrix}\mathbb{Z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{Z} \oplus \frac{1}{7}\begin{pmatrix} 1 \\ k \end{pmatrix}\mathbb{Z},$$

whose solutions are $t \in \frac{1}{7}\mathbb{Z}$ since l and m are coprime. Hence, we only need to consider the self-intersection loci on the circles $K_1 = \{z_1 = 0\}$ and $K_2 = \{z_2 = 0\}$. Let us write

$$V = V_1 \sqcup V_2,$$

where

$$V_i = \{(t, z_1, z_2) \in (0,1) \times M : (t, z_1, z_2) \in V \text{ and } (z_1, z_2) \in K_i\}.$$

For V_2, we choose the (orientation-preserving) coordinates (α, z) around K_2 via the assignment

$$(\alpha, z) \mapsto (e^{\alpha}\sqrt{1 - |z|^2}, z),$$

which defines an orientation-preserving diffeomorphism

$$(S^1 \times \{z \in \mathbb{C} : |z| < 1\})/\mathbb{Z}_7 \to M \setminus K_1,$$

where the $\mathbb{Z}_7$–action on the left is given by $(\alpha, z) \mapsto \left(\alpha + \frac{1}{7}, e^{k/7} z\right)$.

In these coordinates, the action $\rho_{l,m}$ reads as

$$\rho^c_{l,m}(t, \alpha, z) = (\alpha + lt, e^{(kl+7m)t} z).$$

We obtain that the self-intersection locus V_2 is

$$V_2 = \{(t, \alpha, z) : \rho^c_{l,m}(t, \alpha, z) = \rho^c_{l,m}(0, \alpha, z)\} = \left\{(t, \alpha, 0) : t = \frac{1}{7l}, \frac{2}{7l}, \dots, \frac{l-1}{7l}\right\},$$

which is a disjoint union of circles. To verify that the intersection is indeed transverse, we compute the derivative of $f(t, \alpha, z) := \tilde{\rho}^c_{l,m}(t, \alpha, z) - \tilde{\rho}^c_{l,m}(0, \alpha, z)$ at $(n/7l, \alpha, 0)$, where $\tilde{\rho}^c$ is the induced map on the universal covers $\mathbb{R} \times \{|z| < 1\}$. We obtain

$$df\left(\frac{n}{7l}, \alpha, 0\right) = \begin{pmatrix} l & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{(kl+7m)n/7l} - e^{kn/7} \end{pmatrix}.$$

From this we see that the intersection V_2 is indeed transversely cut out in the sense of Lemma B.1 and moreover oriented such that $s \mapsto (n/7l, e^s, 0)$ is an orientation-preserving map. Each component of V_2 gives a term in $H_1(LM) \otimes H_0(LM)$. To identify these terms we only need to know which connected component it belongs to and what the image under $H_1(LM) \to H_1(M)$ is. The term belonging to

$(n/7l, \alpha, 0)$ lies in the connected component associated to $t^n t_2^{l-n}$ and, as we saw, the coefficient is given by the element in $H_1(M)$ that corresponds to $s \mapsto (e^s, 0)$, which is the generator. Thus, the contribution from V_2 is

$$(t t_2^{l-1} + t^2 t_2^{l-2} + \cdots + t^{l-1} t_2)\, \frac{dt}{t}.$$

Similarly, we obtain the contribution coming from V_1; here the equation to solve is

$$(0, e^{(kl+7m)t} z) = (0, z),$$

which gives

$$V_1 = \left\{(t, 0, z_2) : t = \frac{1}{7(kl+7m)}, \frac{2}{7(kl+7m)}, \ldots, \frac{kl+7m-1}{7(kl+7m)}\right\},$$

and the same argument as above shows that the intersection is transverse and oriented such that $\alpha \to (n/7(kl+7m), 0, e^\alpha)$ is orientation-preserving. The contributions can thus again be expressed in terms of the class $\alpha \to (0, e^\alpha)$, which in $H_1(M)$ corresponds to r, where r is the multiplicative inverse of $k \bmod 7$ and the contribution is

$$r(t^r t_2^{(kl+7m-1)r} t^{2r} t_2^{(kl+7m-2)r} + \cdots + t^{(kl+7m-1)r} t_2^r)\, \frac{dt}{t}. \qquad \square$$

We can now finally conclude that the classes $[\rho_{l,m}]$ indeed span all of $H_3(L_l M)$. Note that the classes $[\rho_{l,m}]$ are lifts of the fundamental class along the map $L_l M \to M$, which we found was a semidirect product of $\mathbb{Z}$ by $\mathbb{Z}_7$. Thus, any two such lifts span $H_3(L_l M)$ as long as they are not equal. From the above calculation, we obtain

$$\Delta([\rho_{l,m+n}] - [\rho_{l,m}]) = rn t^{kl+7m}(t t_2^6 + \cdots + t^6 t_2 + t^7)\, \frac{dt}{t}.$$

Thus, we see that any two classes $[\rho_{l,m_1}]$ and $[\rho_{l,m_2}]$ for $m_1 \neq m_2 \bmod 7$ with both coprime to l are nonzero and not equal and hence span $H_3(L_l M)$. Alternatively, we actually see that the classes $[\rho_{l,1+nl}]$ for $n = 1, \ldots, 7$ are all the lifts of the fundamental class along $H_3(L_l M) \to H_3(M)$ if $l \neq 0$. For $l = 0$, take the classes $[\rho_{7,n}]$ for $n = 1, \ldots, 6$ and $[\rho_{0,0}]$.

4 Relation to Reidemeister and Whitehead torsion

We wish to write the above formulas in a more convenient way. As we have seen, $H_3(L_i M)$ is an extension of $H_3(M) = \mathbb{Z}$ by $\mathbb{Z}_7$. As we have seen in the calculation above, there is not much variation in the coproduct of the $\mathbb{Z}_7$ summand, so we are modding it out to simplify notation. To that effect, let us denote the kernel of the map

$$H_3(LM) \to \bigoplus_{i \in \mathbb{Z}_7} H_3(M)$$

by K. Thus, we can identify $H_3(LM)/K$ with $\mathbb{Z}[\mathbb{Z}_7]$ and our the formulas define a map

$$H_3(LM)/K \to H_1(LM, M) \otimes H_0(LM, M)/\Delta(K),$$

which we identify with

$$\mathbb{Z}[\mathbb{Z}_7] \to \overline{\Omega}^1 \otimes \overline{\mathbb{Z}}[\mathbb{Z}_7]/\Delta(K).$$

Our formulas actually lift to a map

$$\mathbb{Z}[\mathbb{Z}_7] \to \Omega^1 \otimes \mathbb{Z}[\mathbb{Z}_7]/\Delta(K),$$

which we will describe and, at the very end, project to $\overline{\Omega}^1 \otimes \overline{\mathbb{Z}}[\mathbb{Z}_7]/\Delta(K)$. The target can be identified with the quotient of $\mathbb{F}_7[t,t_2]/(t^7-1,t_2^7-1)\,dt/t$ by the subvector space spanned by

$$\left\{t^l\,\frac{dt}{t}, t_2^l\,\frac{dt}{t}, t^l(tt_2^6+\cdots+t^6t_2+t^7)\,\frac{dt}{t}\right\}_{l\in\mathbb{Z}}.$$

Call that quotient $Q_{\mathbb{F}_7}$.

4.1 String coproduct in terms of Reidemeister torsion

To rewrite the formulas in a more convenient way, let us introduce a rational version of the above target space. Let $\mathbb{Q}[t,t_2]$ be the polynomial algebra in two variables. We consider the ideal $I = (t^7-1, t_2^7-1, tt_2^6+\cdots+t^7)$ and define $A = \mathbb{Q}[t,t_2]/I$. This has the convenient effect that now $t^l - t_2^l$ are units in A, since $(t-t_2)(t^6+2t^5t_2+\cdots+7t_2^6) = -7+(tt_2^6+\cdots+t^6t_2+t^7)$. Let then $Q_{\mathbb{Q}}$ denote the vector space obtained by taking the quotient of A by the subvector space spanned by the elements t^l and t_2^l and formally adjoin a symbol dt/t. Similarly, there is an integral version of said space, $Q_{\mathbb{Z}} \subset Q_{\mathbb{Q}}$. The above formulas define a map

$$\Delta\colon \mathbb{Z}[\mathbb{Z}_7] \to Q_{\mathbb{Z}} \subset Q_{\mathbb{Q}}.$$

After reduction mod 7, this the component of the string coproduct (the rationalization is merely to write down the formulas in a more convenient way). The map $\Delta\colon \mathbb{Z}[\mathbb{Z}_7] \to Q_{\mathbb{Z}} \subset Q_{\mathbb{Q}}$ is defined by formulas

$$\begin{aligned}
t^l &\mapsto (tt_2^{l-1}+t^2t_2^{l-2}+\cdots+t^{l-1}t_2)\,\frac{dt}{t} + r(t^rt_2^{(kl+7m-1)r}+t^{2r}t_2^{(kl+7m-2)r}+\cdots+t^{(kl+7m-1)r}t_2^r)\,\frac{dt}{t}\\
&= (tt_2^{l-1}+t^2t_2^{l-2}+\cdots+t^{l-1}t_2+t^l)\,\frac{dt}{t}\\
&\qquad + r(t^rt_2^{(kl+7m-1)r}+t^{2r}t_2^{(kl+7m-2)r}+\cdots+t^{(kl+7m-1)r}t_2^r+t^{(kl+7m)r})\,\frac{dt}{t}\\
&= (t_2^{l-1}+tt_2^{l-2}+\cdots+t^{l-2}t_2+t^{l-1})\,dt\\
&\qquad + (t_2^{(kl+7m-1)r}+t^rt_2^{(kl+7m-2)r}+\cdots+t^{(kl+7m-2)r}t_2^r+t^{(kl+7m-1)r})\,dt^r\\
&= \frac{t^l-t_2^l}{t-t_2}\,dt + \frac{t^{r(kl+7m)}-t_2^{r(kl+7m)}}{t^r-t_2^r}\,dt^r\\
&= \frac{t^l-t_2^l}{t-t_2}\,dt + \frac{t^l-t_2^l}{t^r-t_2^r}\,dt^r\\
&= (t^l-t_2^l)\,d\log((t^r-t_2^r)(t-t_2))\\
&= (t^l-t_2^l)\,d\log(R),
\end{aligned}$$

where $R \in Q_{\mathbb{Z}}$ is the homogenized Reidemeister torsion

$$R = (t^r - t_2^r)(t - t_2).$$

We refer to the lecture notes [Mnev 2014, equation (58)] or [Milnor 1966] for the fact that this is indeed the Reidemeister torsion (our convention differs slightly). We summarize our findings in the following:

Proposition 4.1 *The string coproduct descends to*

$$\begin{array}{ccc}
K & \xrightarrow{\quad\Delta\quad} & \Delta(K) \\
\downarrow & & \downarrow \\
H_3(LM) & \xrightarrow{\quad\Delta\quad} & H_1(LM,M)\otimes H_0(LM,M) \\
\downarrow & & \downarrow \\
\bigoplus_{i\in\mathbb{Z}_7} H_3(M) & \xrightarrow{\quad\mathcal{R}\quad} & \bigoplus_{i\neq 0\in\mathbb{Z}_7} H_1(M)\otimes H_0(LM,M)/\Delta(K)
\end{array}$$

where $\mathcal{R}$ is the map $t^l \mapsto (t^l - t_2^l)\, d\log(R)$, where R is the homogenized Reidemeister torsion and the term $(t^l - t_2^l)\, d\log(R)$ is evaluated as explained above.

Remark 4.2 For us, Reidemeister torsion is merely an expression of the form $(t^p - t_2^p)(t^q - t_2^q)$. We do not fully explain here what the exact space of these expressions is. We will only need that the Whitehead group acts on these expressions faithfully.

Example 4.3 Let us give the calculation of $\mathcal{R}$ for $L(1,7)$ and $L(2,7)$:

L(1,7), k = 1, r = 1	L(2,7), k = 2, r = 4
$t^0 \mapsto 0,$	$t^0 \mapsto 0,$
$t^1 \mapsto 0,$	$t^1 \mapsto 4t^4t_2^4\,\frac{dt}{t},$
$t^2 \mapsto 2t^1t_2^1\,\frac{dt}{t},$	$t^2 \mapsto 5t^1t_2^1 + 4t^4t_2^5 + 4t^5t_2^4\,\frac{dt}{t},$
$t^3 \mapsto 2t^1t_2^2 + 2t^2t_2^1\,\frac{dt}{t},$	$t^3 \mapsto 5t^1t_2^2 + 5t^2t_2^1 + 4t^4t_2^6 + 4t^5t_2^5 + 4t^6t_2^4\,\frac{dt}{t},$
$t^4 \mapsto 2t^1t_2^3 + 2t^2t_2^2 + 2t^3t_2^1\,\frac{dt}{t},$	$t^4 \mapsto 5t^1t_2^3 + 5t^2t_2^2 + 5t^3t_2^1 + 4t^5t_2^6 + 4t^6t_2^5\,\frac{dt}{t},$
$t^5 \mapsto 2t^1t_2^4 + 2t^2t_2^3 + 2t^3t_2^2 + 2t^4t_2^1\,\frac{dt}{t},$	$t^5 \mapsto 2t^1t_2^4 + 5t^2t_2^3 + 5t^3t_2^2 + 2t^4t_2^1 + 4t^6t_2^6\,\frac{dt}{t},$
$t^6 \mapsto 2t^1t_2^5 + 2t^2t_2^4 + 2t^3t_2^3 + 2t^4t_2^2 + 2t^5t_2^1\,\frac{dt}{t},$	$t^6 \mapsto 2t^1t_2^5 + 2t^2t_2^4 + 5t^3t_2^3 + 2t^4t_2^2 + 2t^5t_2^1\,\frac{dt}{t}.$

In particular, we see that they cannot possibly be isomorphic. In $L(1,7)$, there are two i's such that $H_3(L_iM) \to H_1(LM/M)\otimes H_0(LM/M)$ has rank one (or rank zero after quotienting out $\Delta(K)$). In $L(2,7)$, there is only one such i. Since all the $H_3(L_iM)$ have images in different components, we see that the ranks of the maps $H_3(L_iM) \to H_1(LM,M)\otimes H_0(LM,M)$ differ.

We summarize the result of the above example in:

Proposition 4.4 *The string coproduct coalgebras on $L(1,7)$ and $L(2,7)$ are nonisomorphic. More precisely, they are told apart by the dimension of the kernel of $\Delta\colon H_3(LM,M)\to H_1(LM/M\times LM/M)$. For $M=L(2,7)$, the coproduct is injective on $H_3(LM,M)$, while for $M=L(1,7)$, the kernel is spanned by the class $[\rho_{1,0}]$.*

4.2 Transformation formula in terms of Whitehead torsion

Let $f\colon L(1,7)\to L(2,7)$ be a homotopy equivalence. Let $\tau(f)\in \mathrm{Wh}(\mathbb{Z}_7)=(\mathbb{Z}[\mathbb{Z}_7])^\times/\mathbb{Z}_7$ be its Whitehead torsion. We denote by the same symbol its image under the map

$$\mathrm{Wh}(\mathbb{Z}_7)\to \mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7])/\mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7],\mathbb{Z})=H_1(LM/M)\to H_1(LM/M\times LM/M)\xrightarrow{1\times\sigma} H_1(LM/M\times LM/M),$$

where $\sigma\colon LM\to LM$ is given by precomposing with the orientation-reversing diffeomorphism of S^1. Let us recall the definition of the Dennis trace map in our case. The Hochschild homology computes as $\mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7])=\Omega^1$ and $\mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7],\mathbb{Z})=\mathbb{Z}_7$. Under these identifications, the Dennis trace map is then given by

$$\mathrm{Wh}(\mathbb{Z}_7)\to \mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7])/\mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7],\mathbb{Z}),\quad \alpha\mapsto \alpha^{-1}\,d\alpha = d\log\alpha.$$

The calculation in the previous section partially verifies the formula

$$\Delta f(x)=f(\Delta(x))+f(x\star d\log\tau(f)),$$

where $\star$ is the string product (applied to both factors as a derivation). Namely, recall that $R_{2,7}=f(R_{1,7}\tau(f))$ and that, moreover, $\tau(f)\in\mathbb{Z}[\mathbb{Z}_7]^\times$. We then have

$$\begin{aligned}\Delta f(t^l)&=(t^{f(l)}-t_2^{f(l)})\,d\log R_{2,7}\\&=(t^{f(l)}-t_2^{f(l)})\big(f(d\log R_{1,7})+d\log f(\tau(f))\big)\\&=f(\Delta(t^l))+(t^{f(l)}-t_2^{f(l)})\,d\log f(\tau(f))\\&=f(\Delta(t^l))+f((t^l-t_2^l)\,d\log\tau(f)),\end{aligned}$$

where we used the calculation of the string product in Proposition A.1.

Remark 4.5 We used the following in the previous calculation. Let $I\to\mathbb{F}_7[\mathbb{Z}_7]\to\mathbb{F}_7$ be the augmentation ideal. Then $(t-1)\in I$ is not a zero-divisor in the algebra I, hence neither is $R_1=R_{1,7}=(t-1)^2$ nor $R_2=f(R_{2,7})=(t^2-1)(t^4-1)$. Let $u\in\mathbb{F}_7[\mathbb{Z}_7]^\times$ be such that $R_1=R_2u$. Then, to show the identity

$$(t^l-1)\,d\log(R_1)-(t^l-1)\,d\log(R_2)=(t^1-1)\,du\,u^{-1}\mod\Sigma,$$

it is clearly enough to show that it is true after multiplying with $R_1R_2=R_2R_2u$. Doing this, we obtain

$$(t^l-1)(dR_1\,R_2-dR_2\,R_2u)=(t^l-1)(R_2R_2\,du)=R_2R_2(t^l-1)\,du\,u^{-1}.$$

Specializing to $l = 1$, we obtain

$$\Delta f(t) = f((t - t_2)\, d\log \tau(f)).$$

Example 4.6 It is known that there exists a homotopy equivalence $f\colon L(1,7) \to L(2,7)$ that sends the preferred generator t to t^2. Its Whitehead torsion (in our convention) is thus

$$\begin{aligned}\tau(f) &= \frac{(t^4-1)(t^2-1)}{(t-1)^2} = \ (t^3+t^2+t+1)(t+1) - \Sigma \ = t + t^2 + t^3 - t^5 - t^6,\\ \tau(f)^{-1} &= \frac{(t^8-1)(t^8-1)}{(t^4-1)(t^2-1)} = (1+t^4)(1+t^2+t^4+t^6) - \Sigma = t^4 - t^5 + t^6,\end{aligned}$$

where $\Sigma = 1 + t + t^2 + \cdots + t^6$. Its image under the Dennis trace is

$$d\log(\tau(f)) = (1 + 2t + 3t^2 - 5t^4 - 6t^5)(t^4 - t^5 + t^6)\, dt = (6 + 5t + 6t^2 + t^3 + 2t^4 + t^5 + 2t^6)\, dt,$$

and hence (after homogenizing again to match notation from above)

$$(t - t_2)\, d\log \tau(f) = (4 + 2tt_2^6 + 6t^3t_2^4 + 2t^5t_2^2)\, dt - (1 + tt_2^6 + t^2t_2^5 + t^3t_2^4 + t^4t_2^3 + t^5t_2^2 + t^6t_2)\, dt,$$

and finally

$$\begin{aligned}f((t - t_2)\, d\log \tau(f)) &= (4 + 2t^2t_2^5 + 6t^6t_2 + 2t^3t_2^4)d(t^2)\\ &= (4 + 2t^2t_2^5 + 6t^6t_2 + 2t^3t_2^4)2t\, dt\\ &= (t^2 + 4t^4t_2^5 + 5tt_2 + 4t^5t_2^4)\, \frac{dt}{t},\end{aligned}$$

where we dropped multiples of Σ.

Summarizing our findings, we conclude with:

Proposition 4.7 *The string coproduct on the lens spaces $L(k,7)$ detects Whitehead torsion. More precisely, the restriction of the string coproduct to $H_3(LM)$, after taking the quotient described in Proposition 4.1, transforms according to formula (1) and two elements in $\mathrm{Wh}(\mathbb{Z}_7)$ give the same correction term if and only if they are equal under the Dennis trace map.*

5 More details on an example

Let us make the previous example more concrete and show that the formula in the introduction is still true even without modding out K (ie dropping the multiples of Σ). To that extent, recall that the homotopy equivalence f is constructed as

$$L(1,7) \to S^3 \vee L(1,7) \xrightarrow{\mathrm{id} \vee (z_1^2, z_2^4)} S^3 \vee L(2,7) \xrightarrow{\Phi} L(2,7),$$

where $\Phi\colon S^3 \to L(2,7)$ is any map of degree -7 (see [Mnev 2014, Section 6.4]).

Lemma 5.1 $$f([\rho_{1,0}]) = [\rho_{2,3}].$$

Proof Let us first try to compare the maps

$$f \circ \rho_{1,0} \colon S^1 \times L(1,7) \to L(2,7) \quad \text{and} \quad \rho_{2,3} \circ (\mathrm{id} \times f) \colon S^1 \times L(1,7) \to L(2,7)$$

using obstruction theory. Away from a neighborhood of a point in $L(1,7)$, the two maps are given by

$$f \circ \rho_{1,0} \colon (t, z_1, z_2) \mapsto (e^{2t} z_1^2, e^{4t} z_2^4), \quad \rho_{2,3} \circ (\mathrm{id} \times f) \colon (t, z_1, z_2) \mapsto (e^{2t} z_1^2, e^{(4+7\cdot 3)t} z_2^4).$$

Recalling the standard cell decomposition of $L(1,7)$ as

$$\begin{aligned} e_0 &= \{(1,0)\}, & e_1 &= \{(e^s, 0) : s \in (0, \tfrac{1}{7})\}, \\ e_2 &= \{(z_1, r) : r \in (0,1)\}, & e_3 &= \{(z_1, z_2) : z_2 = e^s r \text{ for } s \in (0, \tfrac{1}{7})\}, \end{aligned}$$

we see that the two maps already coincide on $A := \{0\} \times L(1,7) \cup S^1 \times e_0$. The first obstruction for these two maps being homotopic relative to A lies in

$$H^3\big(M/A; \pi_3(L(2,7))\big) = \mathbb{Z}_7.$$

The obstruction is computed by comparing the two maps on the 3–cell $I \times e_2$. Since the maps coincide on the boundary of that cell, they fit together to a map $S^3 \to L(2,7)$, ie an element in $\pi_3(L(2,7)) = \mathbb{Z}$, where the identification is by computing the degree and dividing by 7. Thus, it is enough to show that the degrees of the two maps restricted to $I \times e_2$ are equal mod 49. For the map $f \circ \rho_{1,0}$, we note that $\rho_{1,0}$ maps the cell $I \times e_2$ homeomorphically onto e_3. Since f has degree 1, we get a contribution of 1. For the map $\rho_{2,3} \circ (\mathrm{id} \times f)$, we note that f is given by (z_1^2, z_2^4) on the cell $I \times e_2$ and hence we are computing the degree of the map

$$(0, \tfrac{1}{7}) \times e_2 \to L(2,7), \quad (t, z_1, r) \mapsto (e^{2t} z_1^2, e^{(4+7\cdot 3)t} r^4).$$

This map has degree $2 \cdot 25 = 50$ (it has the same degree as its 7–fold cover $S^1 \times e_2 \to S^3$ given by the same formula but now $t \in [0,1]$). We see that the obstruction vanishes since $1 \cong 50 \bmod 49$. We conclude that the two maps in question are homotopic at least up to the 3–skeleton of $S^1 \times L(1,7)$. They could still potentially differ on their 4–cell $I \times e_3$ by an element in

$$H^4\big(M/A; \pi_4(L(2,7))\big) = \mathbb{Z}_2.$$

We can view $\rho_{1,0}$ as an element in $\pi_1\big(\mathrm{aut}_1(L(1,7))\big)$, where $\mathrm{aut}_1(L(1,7))$ is the monoid of self-equivalences homotopic to the identity. Under this identification, the action of $H^4\big(M/A; \pi_4(L(2,7))\big) = \mathbb{Z}_2$ corresponds to multiplication by the element

$$\begin{aligned} S^1 \times L(1,7) \to (S^1 \times L(1,7)) \vee S^4 \to (S^1 \times L(1,7)) \vee S^3 \\ \to L(1,7) \vee L(1,7) \to L(1,7) \in \pi_1\big(\mathrm{aut}_1(L(1,7))\big), \end{aligned}$$

which is an element of order 2. However, it follows directly from the definition of the string product that

$$\pi_1\big(\mathrm{aut}_1(L(1,7))\big) \to \big(H_3(LL(1,7)), \star\big)$$

is a morphism of monoids. Moreover, the image is contained in $\bigcup_{i\in\mathbb{Z}_7} H_3(LM)$ and maps to the fundamental class $[M]$ under $H_3(LM)\to H_3(M)$. We also saw that all these classes are of the form $\rho_{l,m}$. Thus, we conclude that the image of $\pi_1\big(\mathrm{aut}_1(L(1,7))\big)$ is a group of order 49 and hence any element of order 2 gets sent to zero. This shows that, indeed,

$$f([\rho_{1,0}]) = [\rho_{2,3}] \in H_3(LL(2,7)),$$

where $M = L(2,7)$. □

We are now ready to evaluate (1) for $x = [\rho_{1,0}]$. The left-hand side is given by

$$\begin{aligned}\Delta f([\rho_{1,0}]) &= \Delta[\rho_{2,3}]\\ &= tt_2\,\frac{dt}{t} + 4\cdot 3(t + t^2t_2^6 + t^3t_2^5 + t^4t_2^4 + t^5t_2^3 + t^6t_2^2 + t_2)\,\frac{dt}{t} + 4(t^4t_2^5 + tt_2 + t^5t_2^4)\,\frac{dt}{t}\\ &= 12(t + t^2t_2^6 + t^3t_2^5 + t^4t_2^4 + t^5t_2^3 + t^6t_2^2 + t_2)\,\frac{dt}{t} + (4t^4t_2^5 + 5tt_2 + 4t^5t_2^4)\,\frac{dt}{t},\end{aligned}$$

reading off formula (3). For the right-hand side we obtain

$$f(\Delta[\rho_{0,1}]) + f([\rho_{1,0}]\star d\log\tau(f)) + f([\rho_{1,0}]\star d\log\tau(f)) = f((t-t_2)\,d\log\tau(f)),$$

using that $[\rho_{0,1}]$ has no self-intersections and the calculation of the string product in Appendix A and introducing a Koszul sign. We already calculated the image of the Whitehead torsion under the Dennis trace map in the example above, that is,

$$d\log\tau(f) = (6 + 5t + 6t^2 + t^3 + 2t^4 + t^5 + 2t^6)\,dt \in \mathrm{HH}_1(\mathbb{Z}[\mathbb{Z}_7]) = H_1(LM),$$

the map $H_1(LM)\to H_1(LM\times LM)\to H_1(LM)\otimes H_0(LM)$ given by the diagonal and reversing the circle on the second factor sends a monomial $t^l\,dt/t$ to $t^lt_2^{-l}\,dt/t$ and hence is homogenization. We can thus compute, as in the above example (this time without dropping Σ terms), that

$$\begin{aligned}f((t-t_2)\,d\log\tau(f)) &= (4+2t^2t_2^5+6t^6t_2+2t^3t_2^4)d(t^2)+6(1+tt_2^6+t^2t_2^5+t^3t_2^4+t^4t_2^3+t^5t_2^2+t^6t_2)d(t^2)\\ &= (4+2t^2t_2^5+6t^6t_2+2t^3t_2^4)2t\,dt+12(1+tt_2^6+t^2t_2^5+t^3t_2^4+t^4t_2^3+t^5t_2^2+t^6t_2)\,dt\\ &= (t^2+4t^4t_2^5+5tt_2+4t^5t_2^4)\,\frac{dt}{t}+12(t+t^2t_2^6+t^3t_2^5+t^4t_2^4+t^5t_2^3+t^6t_2^2+t_2)\,\frac{dt}{t}.\end{aligned}$$

Thus, we see that the two sides of (1) coincide up to the term $t^2\,dt/t$, which corresponds to an element in $H_1(LM)\otimes H_0(M)$ and is hence zero in $H_1(LM,M)\otimes H_0(LM,M)$.

Appendix A String product

Let

$$\star\colon H_3(LM)\otimes H_1(LM)\to H_1(LM)$$

denote the string product for $M = L(k,7)$.

Proposition A.1 *For any $\omega \in \Omega^1 \cong H_1(LM)$, we have*

$$[\rho_{l,m}] \star \omega = t^l \omega.$$

Proof We check the formula for the generators $t^n\, dt/t$. We only need the following two formal properties of the string product, which one can readily see from the definition given in [Hingston and Wahl 2023], for instance. The first property is that it is "additive" on path components. That is, the product of two classes in $H_\bullet(L_\alpha M)$ and $H_\bullet(L_\beta M)$ lies in $\bigoplus_{\gamma\in\alpha\cdot\beta} H_\bullet(L_\gamma M)$, where α and β are conjugacy classes in π_1 and $\alpha\cdot\beta$ is the set of conjugacy classes obtained by taking products of elements in α and β, respectively. From this we see that

$$[\rho_{l,m}] \star t^n \frac{dt}{t} \in H_1(L_{l+n}M),$$

which is hence a multiple of $t^{l+n}\, dt/t$. To determine the coefficient, we use that the projection map $H_\bullet(LM) \to H_\bullet(M)$ intertwines the string product and the homological intersection product, which gives

$$[\rho_{l,m}] \star t^n \frac{dt}{t} = t^{l+n} \frac{dt}{t}. \qquad \square$$

Appendix B Transverse string topology

We show that the transverse calculation of the string coproduct is indeed the invariantly defined string coproduct by comparing it with the definition in [Naef and Willwacher 2019] (see also [Naef et al. 2023, Proposition 4.4] for the equivalence with the definition of [Hingston and Wahl 2023]). Recall that in [Naef and Willwacher 2019] the string coproduct was defined by the zigzag of spaces

$$\begin{array}{ccccccc} \dfrac{LM}{M} & \overset{\text{suspend}}{\dashrightarrow} & \dfrac{I\times LM}{\partial I\times LM\cup I\times M} & \xrightarrow{s} & \dfrac{\mathrm{Map}(\bigcirc_2)}{F} & \longrightarrow & \dfrac{\mathrm{Map}(\bigcirc_2)/\mathrm{Map}'(\bigcirc_2)}{F/F|_{UTM}} \\ & & & & & \nearrow_{\simeq} & \\ \dfrac{\mathrm{Map}(8)/\mathrm{Map}'(8)}{F/F|_{UTM}} & & \overset{\text{Th}}{\dashrightarrow} & & \dfrac{\mathrm{Map}(8)}{F} & \longrightarrow & \dfrac{LM\times LM}{LM\times M\cup M\times LM}, \end{array} \tag{4}$$

where dashed arrows are only defined on homology and we have used the following notation:

- (Iterated) quotients denote (iterated) cofibers, that is, cone constructions.
- UTM is the unit tangent bundle.
- $FM_2(M)$ is the compactified configuration space of two points, namely it is obtained from $M\times M$ by a real oriented blowup along the diagonal. It is a manifold with boundary UTM and homotopy-equivalent to $M\times M\setminus M$ and fits into the commuting diagram

$$\begin{array}{ccc} UTM & \longrightarrow & FM_2(M) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M\times M \end{array}$$

  where $M \to M\times M$ is the diagonal map.

- $\mathrm{Map}(\bigcirc_2)$ is simply LM thought of as a fibration over $M \times M$ given by evaluating the loop at times 0 and $\frac{1}{2}$.
- Taking the pullback of $\mathrm{Map}(\bigcirc_2)$ along the above square, we obtain

$$\begin{array}{ccc} \mathrm{Map}'(8) & \longrightarrow & \mathrm{Map}'(\bigcirc_2) \\ \downarrow & & \downarrow \\ \mathrm{Map}(8) & \longrightarrow & \mathrm{Map}(\bigcirc_2) \end{array}$$

- F is $LM \cup LM$ thought of as the subspace of $\mathrm{Map}(8)$ where at least one of the ears is mapped to M constantly.
- The map s reparametrizes a loop. It takes a parameter $t \in I$ and a loop γ and reparametrizes it in such a way that the path $\gamma_{[0,t]}$ is run through on the interval $\left[0, \frac{1}{2}\right]$ and the path $\gamma_{[t,1]}$ is run through on the interval $\left[\frac{1}{2}, 1\right]$.
- The map Th is capping with the Thom class in $H^n(M, UTM)$.

Let us first formulate the following:

Lemma B.1 *Let $\alpha\colon N \to LM$ be transverse in the sense that*:

(i) *$\partial\alpha(u,t)/\partial t$ is nonzero at $t = 0$.*

(ii) *The map $\overline{\alpha}\colon N \times (0,1) \to M \times M$ given by $(n,t) \mapsto (\alpha(n,0), \alpha(n,t))$ intersects the diagonal transversely in a compact submanifold $V \subset N \times (0,1)$,*

Then there is a unique map $\widehat{\alpha}\colon \widehat{N \times I}^V \to FM_2(M)$ from the real oriented blowup of $N \times I$ at V, denoted by $\widehat{N \times I}^V$, to the compactified configuration space of two points such that

$$\begin{array}{ccc} \widehat{N \times I}^V & \longrightarrow & FM_2(M) \\ \downarrow & & \downarrow \\ N \times I & \longrightarrow & M \times M \end{array}$$

commutes. Moreover, $\widehat{\alpha}$ identifies the unit normal bundle of V in $N \times (0,1)$ with $\overline{\alpha}|_V^* UTM$.

Proof In local coordinates the map $M \times M \setminus M \to FM_2(M)$ looks like

$$\mathbb{R}^n \times \mathbb{R}^n \setminus \mathbb{R}^n \to \mathbb{R}^n \times S^{n-1} \times [0,\infty), \quad (x,y) \mapsto \left(x - y, \frac{x-y}{|x-y|}, |x-y|\right).$$

Composing with $\overline{\alpha}$, we readily see that condition (i) is sufficient (and necessary) to lift the map $N \times I \to M \times M$ to $FM_2(M)$ in a neighborhood of $N \times \partial I$. To obtain the statement away from the boundary, we observe that the function (in coordinates)

$$(n,t) \to \frac{\overline{\alpha}(n,t) - \overline{\alpha}(n,0)}{|\overline{\alpha}(n,t) - \overline{\alpha}(n,0)|}$$

smoothly extends from $N \times I \setminus V$ to $\widehat{N \times I}^V$. □

Such a transverse map $\alpha\colon N \to LM$ naturally defines a map

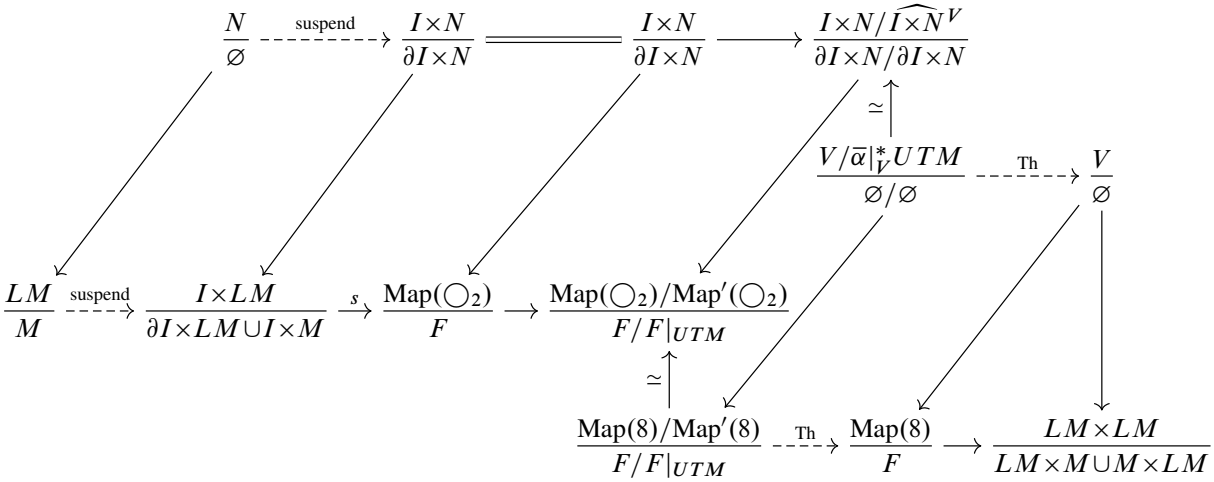
$$\Delta(\alpha)\colon V \to LM \times LM, \quad (n,t) \mapsto \big(s \mapsto \alpha(n,st),\, s \mapsto \alpha(n,(1-t)s+t)\big).$$

The following is a special case of [Hingston and Wahl 2023, Proposition 3.7] adapted to our notation:

Proposition B.2 *Let $\alpha\colon N \to LM$ be transverse in the sense of the previous lemma. Then*

$$\Delta(\alpha_*([N])) = (\Delta\alpha)_*([V]) \in H_*(LM \times LM,\, M \times LM \cup LM \times M).$$

Proof One checks that the following diagram commutes, where all the maps are the "obvious" ones:



The only thing left to show is that, after taking homology, the upper zigzag sends the fundamental class of N to the fundamental class of V. Namely, we have to show that, under

$$\begin{array}{ccccc} H_d(N) & \longrightarrow & H_{d+1}(I\times N, \partial I\times N) & \longrightarrow & H_{d+1}(I\times N, \widehat{I\times N}^V) \\ & & & & \simeq\uparrow \\ & & & & H_{d+1}(V, \bar\alpha|_V^* UTM) \xrightarrow{\text{Th}} H_{d+1-n}(V) \end{array}$$

where $d = \dim(N)$, the class $[N]$ gets sent to $[V]$. First note that the Thom isomorphism here is given by capping with a Thom class that is the pullback of the Thom class on $H_n(M, UTM)$ along $\bar\alpha|_V$. This Thom class is also the natural Thom class by considering $\bar\alpha|_V^* UTM$ as the oriented normal bundle of V in $I \times N$. Hence, apart from our insistence on avoiding tubular neighborhoods, we obtain the standard description of the intersection pairing, from which it follows that $[N]$ is sent to $[V]$. To see this more concretely, we note that it is enough to show that composing with $H_{d+1-n}(V) \to H_{d+1-n}(V, V\setminus\{x\})$ sends $[N]$ to the generator in $H_{d+1-n}(V, V\setminus\{x\})$. Thus, the situation is local and we can assume that $N = \mathbb{R}^d$ and $\bar\alpha\colon \mathbb{R}^{d+1} \to \mathbb{R}^n$ is a linear projection. In this case the statement follows directly from the definitions. □

References

[Cattaneo et al. 2020] **A S Cattaneo**, **P Mnev**, **N Reshetikhin**, *A cellular topological field theory*, Comm. Math. Phys. 374 (2020) 1229–1320 MR Zbl

[Chas and Sullivan 1999] **M Chas**, **D Sullivan**, *String topology*, preprint (1999) arXiv math/9911159

[Chas and Sullivan 2004] **M Chas**, **D Sullivan**, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, from "The legacy of Niels Henrik Abel" (O A Laudal, R Piene, editors), Springer (2004) 771–784 MR Zbl

[Cohen and Jones 2002] **R L Cohen**, **J D S Jones**, *A homotopy theoretic realization of string topology*, Math. Ann. 324 (2002) 773–798 MR Zbl

[Cohen et al. 2008] **R L Cohen**, **J R Klein**, **D Sullivan**, *The homotopy invariance of the string topology loop product and string bracket*, J. Topol. 1 (2008) 391–408 MR Zbl

[Crabb 2008] **M C Crabb**, *Loop homology as fibrewise homology*, Proc. Edinb. Math. Soc. 51 (2008) 27–44 MR Zbl

[Gruher and Salvatore 2008] **K Gruher**, **P Salvatore**, *Generalized string topology operations*, Proc. Lond. Math. Soc. 96 (2008) 78–106 MR Zbl

[Hingston and Wahl 2023] **N Hingston**, **N Wahl**, *Product and coproduct in string topology*, Ann. Sci. Éc. Norm. Supér. (4) 56 (2023) 1381–1447 MR Zbl

[Milnor 1966] **J Milnor**, *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1966) 358–426 MR Zbl

[Mnev 2014] **P Mnev**, *Lecture notes on torsions*, lecture notes, University of Zurich (2014) arXiv 1406.3705

[Naef and Willwacher 2019] **F Naef**, **T Willwacher**, *String topology and configuration spaces of two points*, preprint (2019) arXiv 1911.06202

[Naef et al. 2023] **F Naef**, **M Rivera**, **N Wahl**, *String topology in three flavors*, EMS Surv. Math. Sci. 10 (2023) 243–305 MR Zbl

Department of Mathematical Sciences, University of Copenhagen
Copenhagen, Denmark

Current address: *School of Mathematics, Trinity College Dublin*
Dublin, Ireland

naeffl@gmail.com

Proposed: Jesper Grodal
Seconded: Ciprian Manolescu, Stefan Schwede

Received: 4 July 2021
Revised: 14 November 2022

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2024 is US $805/year for the electronic version, and $1135/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 28 Issue 8 (pages 3511–3972) 2024