Reeb flows transverse to foliations

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Let $\mathcal{F}$ be a cooriented C^2 foliation on a closed, oriented 3–manifold. We show that $T\mathcal{F}$ can be perturbed to a contact structure with Reeb flow transverse to $\mathcal{F}$ if and only if $\mathcal{F}$ does not support an invariant transverse measure. The resulting Reeb flow has no contractible orbits. This answers a question of Colin and Honda. The main technical tool in our proof is leafwise Brownian motion which we use to construct good transverse measures for $\mathcal{F}$; this gives a new perspective on the Eliashberg–Thurston theorem.

1 Introduction

In their seminal work, Eliashberg and Thurston [1998] proved that a C^2 foliation of a closed, oriented 3–manifold not homeomorphic to $S^1 \times S^2$ may be C^0 approximated by positive or negative contact structures. When the foliation is taut, the approximating contact structures are universally tight and weakly symplectically fillable. This theorem, along with its subsequent generalizations [Kazez and Roberts 2017; Bowden 2016a], serves as a bridge between contact topology and the theory of foliations. In one direction, one can export genus detection results from Gabai's theory [1983] of sutured manifolds to the world of Floer homology [Ozsváth and Szabó 2004]. In the other direction, a uniqueness result of Vogel [2016] for the approximating contact structure implies that invariants of the approximating contact structure become invariants of the deformation class of the foliation [Bowden 2016b].

Colin and Honda asked when the approximating contact structure can be chosen so that its Reeb flow is transverse to the foliation. When this is the case, the foliation can be used to control the Reeb dynamics. In particular, a transverse Reeb flow can have no contractible Reeb orbits. A contact form with this property is called *hypertight*. The hypertight condition is useful for defining and computing pseudoholomorphic curve invariants. A motivating example for us is cylindrical contact homology, an invariant of contact structures that is well-defined when the contact structure supports at least one hypertight contact form [Bao and Honda 2018; Hutchings and Nelson 2016; 2022].

Colin and Honda [2005] constructed such transverse Reeb flows for sutured hierarchies. They recently extended their methods to finite-depth foliations on closed 3–manifolds [Colin and Honda 2022]. In this setting, although the Reeb flow cannot be made transverse to the closed leaf, it is transverse to a related essential lamination and hence has no contractible orbits. The goal of the present paper is to give a complete answer to the existence question for transverse Reeb flows for all C^2 foliations.

A closed leaf is an obstruction to a transverse Reeb flow. Suppose that Σ is a closed, oriented surface and α is the 1–form representing a contact structure ξ. By Stokes' theorem, $\int_\Sigma d\alpha = 0$, and it follows that $d\alpha = 0$ at some point on Σ. The Reeb flow must be tangent to Σ at this point. In particular, this implies that foliations with transverse Reeb flows have no compact leaves, and hence are taut.

More generally, an invariant transverse measure for $\mathcal{F}$ is an obstruction to a transverse Reeb flow. We prove that, for C^2 foliations, this is the only obstruction.

Theorem 1 *Let $\mathcal{F}$ be a coorientable C^2 foliation on a closed, oriented 3–manifold M. Then $T\mathcal{F}$ can be perturbed to a contact structure with Reeb flow transverse to $\mathcal{F}$ if and only if $\mathcal{F}$ does not support an invariant transverse measure.*

When $\mathcal{F}$ is taut, an invariant transverse measure gives rise to a nontrivial class in $H^1(M, \mathbb{R})$. Therefore, we have the following corollary (cf Conjecture 1.5 from [Colin and Honda 2005]):

Corollary 2 *If M is a closed, oriented 3–manifold with a cooriented C^2 taut foliation and $M \not\cong S^1 \times S^2$, then M supports a hypertight contact structure.*

Proof If $H^1(M, \mathbb{R}) = 0$, then the foliation has no invariant transverse measure and Theorem 1 applies. Otherwise, M supports a finite-depth taut foliation. In this case, Theorem 3.14 of [Colin and Honda 2022] gives the desired hypertight contact structure.   □

One should think of invariant transverse measures as exceptional. In fact, under some conditions, it follows from a result of Bonatti–Firmo that nonexistence of invariant transverse measures is generic for C^∞ foliations.

Corollary 3 *Suppose that M is a closed, oriented, atoroidal 3–manifold. Then any C^∞ taut foliation on M is C^0 close to a hypertight contact structure.*

Corollary 4 *Cylindrical contact homology is an invariant of the taut deformation class of C^∞ taut foliations on closed, oriented, atoroidal 3–manifolds.*

We defer explanations of the last two corollaries to Section 3.5.

Our method contrasts with prior constructions of contact approximations. The strategy of Eliashberg and Thurston begins with identifying some closed curves in leaves of $\mathcal{F}$ with attracting holonomy. They produce a contact perturbation in the neighborhood of these curves, and then "flow" the contactness to the rest of the 3–manifold. It is not clear how to control the Reeb vector field during this flow operation. In the case of sutured hierarchies, Colin and Honda give an explicit inductive construction of the contact perturbation. At each step of the sutured hierarchy, they can ensure that the Reeb flow has a good standard form near the boundary compatible with the sutures.

Our construction begins with an arbitrary transverse measure for $\mathcal{F}$ and smooths it out by *logarithmic diffusion*. This process is similar to the leafwise heat flow studied by Garnett [1983]. The advantage of our diffusion process is that we can show that in finite time, the transverse measure becomes *log superharmonic*. Roughly speaking, this means that the transverse distance between two nearby leaves is the exponential of a superharmonic function. This gives a global picture of the holonomy of $\mathcal{F}$. Finally, we show that there is a canonical way to deform a log superharmonic transverse measure into a contact structure. Since this perturbation is done in one shot, we have full control over the Reeb flow.

Acknowledgements

2 Examples

In this section, we give some concrete examples of foliations and how one may deform them to contact structures. The reader should keep these examples in mind while reading the rest of the paper. We expect that this section can be read with minimal background, but the reader may wish to review Sections 3.1 and 3.3 for definitions of foliations and contact structures, and Sections 3.4 and 3.6 for a discussion of holonomy.

Example 5 (creating contact regions using holonomy) Consider $\mathbb{R}^3$ foliated by planes $z = \mathrm{const}$ and endowed with the Riemannian metric $g = dx^2 + dy^2 + 2^{-2x}\,dz^2$. This is a local model for a foliation with holonomy. One may produce a contact structure by rotating the tangent planes to the foliation by a Riemannian angle of ε around an axis parallel to the x–axis. Any line parallel to the x–axis is a Legendrian. Traveling along such a Legendrian in the positive x–direction, the Riemannian angles are constant, so the Euclidean angles are increasing. This twisting of contact planes along a Legendrian is the hallmark of a contact structure.

In equations, the contact form is $\alpha := 2^{-x}\,dz + \varepsilon\,dy$ and

$$d\alpha = -2^{-x}\,dx \wedge dz.$$

In this case, the Reeb flow is not transverse to the foliation but points in the y–direction. This construction may equally well be done in the quotient of $\mathbb{R}^3$ by the isometry $(x, y, z) \mapsto (x + 1, y, 2z)$. Here we have seen the general rule that "Legendrians (in the characteristic foliation on a leaf of the foliation) flow in the direction of contracting holonomy".

This kind of construction is visualized in Figure 1. Three boxes are shown, each equipped with both a product foliation transverse to the z–direction and an approximating contact structure. The top and bottom
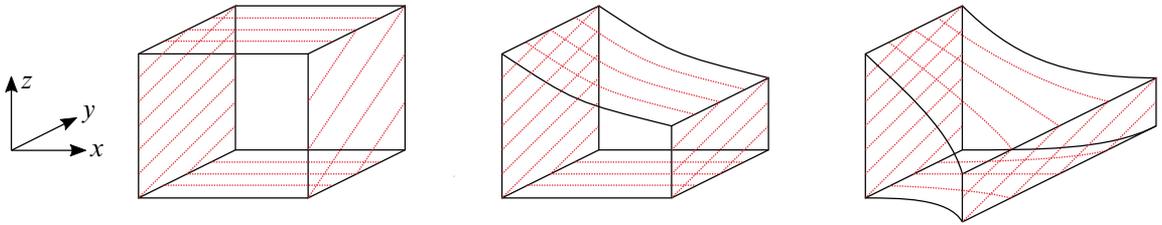
Figure 1

faces of each box are leaves. The red dotted lines show the characteristic foliation of the contact structure on some of the boxes' faces. Observe that in the first box, the slope of the characteristic foliation on the left face is greater than the slope on the right face. These ever-increasing slopes are unsustainable in a closed manifold. The second and third boxes (corresponding with Examples 5 and 6) are diffeomorphic to the first, but we have introduced some contraction of the leaves of the foliation. Now the slopes on the left and right faces can be made identical, facilitating gluing up to a closed manifold.

Example 6 (Anosov flows) Example 5 can be modified so that the Reeb flow is transverse to the leaves. The Legendrian flow will be Anosov, ie spreading out in the y–direction as well as contracting in the z–direction. The relevant metric is $g = dx^2 + 2^{2x} \, dy^2 + 2^{-2x} \, dz^2$ with the flow parallel to the x–axis.

It is instructive to write this example in different coordinates. Let

$$\mathbb{H} = \{(x, y) \mid y > 0\}$$

be the upper half-plane. Consider the manifold $\mathbb{H} \times \mathbb{R}$, foliated by planes of the form $\mathbb{H} \times \{pt\}$. Endow the manifold $\mathbb{H} \times \mathbb{R}$ with coordinates x, y, t, where x and y correspond with the coordinates on $\mathbb{H}$ and t parametrizes the $\mathbb{R}$–factor. Let $f(x, y, t) = y$. The 1–form $f \, dt$ defines the horizontal foliation and is an example of a harmonic transverse measure. The flow parallel to the y–axis oriented in the negative y–direction is an Anosov flow. The 1–form $\alpha = f \, dt - \varepsilon \star d \log f$ is a contact perturbation of our foliation. Indeed,

$$\alpha = y \, dt + \varepsilon \frac{1}{y} \, dx,$$

$$\alpha \wedge d\alpha = \varepsilon \left(2 \frac{1}{y} \, dx \wedge dy \wedge dt \right) + O(\varepsilon^2).$$

This time, the Reeb flow is transverse to the foliation since $d\alpha$ evaluates positively on $\mathcal{F}$. The moral here is that "spreading of the Legendrians" in the characteristic foliation can contribute to contactness, and also help to make the Reeb flow transverse to the foliation.

More generally, the stable/unstable foliations of Anosov flows have no invariant transverse measures, and so are candidates for the application of Theorem 1. However, these stable/unstable foliations are in general only C^1 so our theorem does not directly apply.
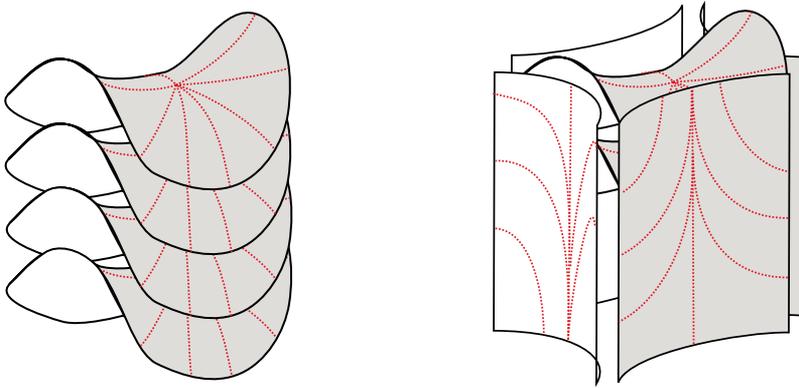
Figure 2: This is a foliation by saddles of the solid torus with four longitudinal sutures. The S^1–direction is vertical. The limiting annular boundary leaves are added on the right. The red dotted lines denote the characteristic foliation of an approximating contact structure.

Example 7 (a coarsely harmonic transverse measure)   Here is an example of a foliation with holonomy that one can keep in mind while reading the rest of the paper. Let P be a pair of pants with boundary components a, b, and c. Let φ be a diffeomorphism of P exchanging b and c. Let M° be the mapping torus of φ. The manifold M° has a foliation by parallel copies of P. It has two toroidal boundary components, each with a horizontal foliation and one "twice as long" as the other. Glue these boundary components together by a map $\theta \mapsto 2\theta$, where θ parametrizes the direction transverse to P. Call the resulting foliated manifold M.

Now let f be a function on P which takes the value 1 on b and c and the value 2 on a. We can further arrange that f is invariant under φ and has only a single critical point. Then f pulls back to M°. After some smoothing near the cuffs, the 1–form $f\,d\theta$ is a smooth transverse measure on M. With respect to this transverse measure, the manifold has contracting holonomy along paths from a to b or a to c.

We will return to the contact perturbation in this case in Example 35 in Section 5.

Example 8 (sutured manifold)   The foliation in Example 7 has every leaf dense, but we will also need to consider foliations whose leaves accumulate on sublaminations. The example to keep in mind is a taut sutured manifold [Gabai 1983]. One of the simplest sutured manifolds is the solid torus T_n with $2n$ longitudinal sutures, $n \geq 2$. It is foliated by a stack of monkey saddles, each of which accumulates on the boundary leaves. For a natural choice of Riemannian metric, the foliation has contracting holonomy along every path to infinity in a leaf. This foliation may be perturbed into a contact structure whose characteristic foliation on each leaf consists of radial lines emanating from a single elliptic singularity to infinity. See Figure 2. Notice that the curves in the characteristic foliation travel in the direction of contracting holonomy. The Reeb flow has a single periodic orbit along the core of the solid torus; every other orbit enters along a positively oriented boundary leaf and leaves along a negatively oriented one.

3 Preliminaries

3.1 Foliations and their regularity

A *foliation* on a 3–manifold M is a decomposition of M into surfaces, locally modeled on $\mathbb{R}^2 \times \mathbb{R}$. A foliation is specified by a covering of M by charts such that their transition maps preserve the decomposition of $\mathbb{R}^3$ into horizontal planes. The surfaces in the decomposition are called leaves. A foliation is oriented if the transition maps preserve the orientations of the leaves and cooriented if the transition maps preserve the orientation of the transverse $\mathbb{R}$–factor.

For $k \geq l$, we say that a foliation is $C^{k,l}$ if the mixed partial derivatives of the transition maps up to order l in the transverse direction and up to order k in total exist and are continuous. Calegari [2001] showed that a $C^{1,0}$ foliation can be improved to a $C^{\infty,0}$ foliation by a topological isotopy. His method extends to show that a $C^{l,l}$ foliation can be improved to a $C^{\infty,l}$ foliation by a C^l isotopy. (His Lemma 3.2 holds for $C^{l,l}$ foliations, so one need only check that Lemma 3.3 in [Calegari 2001] works using C^l isotopies.) However, it might not be possible to improve the transverse regularity class of a foliation; the holonomy maps of a $C^{k,0}$ foliation are C^0 functions which might not be conjugate to smooth functions. A foliation is called C^l when it is $C^{l,l}$. We will also use the notation $C^{k,l}$ to denote the regularity class of functions on M.

In this paper, all manifolds are closed and oriented. All foliations are oriented and cooriented. In light of Calegari's result, we will assume from now on that our foliations are $C^{\infty,2}$ unless stated otherwise.

3.2 Taut foliations

An oriented $C^{\infty,2}$ foliation $\mathcal{F}$ on an oriented 3–manifold M is *taut* if any of the following equivalent conditions hold:

(1) For each point $x \in M$, there is a closed curve transverse to $\mathcal{F}$ passing through x.

(2) There is a volume-preserving flow transverse to $\mathcal{F}$.

(3) There is a closed 2–form ω evaluating positively on $T\mathcal{F}$.

Taut foliations enjoy a number of good properties:

(1) Loops transverse to $\mathcal{F}$ are not contractible.

(2) Leaves of $\mathcal{F}$ are π_1–injective.

If we exclude the exceptional case $M \cong S^1 \times S^2$, we can say more:

(1) M is irreducible.

(2) The universal cover $\widetilde{M}$ is $\mathbb{R}^3$.

(3) Leaves of $\widetilde{F}$ are properly embedded planes in $\widetilde{M}$.

If a foliation has no compact leaves, then it is taut.

3.3 Contact structures

A *contact structure* on an oriented 3–manifold M is a plane field ξ such that there exists a 1–form α with $\xi = \ker(\alpha)$ and $\alpha \wedge d\alpha > 0$. The *Reeb flow* of α is a vector field R uniquely defined by the following properties:

$$d\alpha(R, -) = 0,$$
$$\alpha(R) = 1.$$

The Reeb flow is always transverse to ξ, and moreover is transverse to any tangent plane on which $d\alpha > 0$. The Reeb flow preserves α. In particular, it preserves the volume form $\alpha \wedge d\alpha$. Compare this with the second definition of taut foliation above; it follows that a foliation with a transverse Reeb flow is taut. If $\xi = \ker(\alpha)$ for some α whose corresponding Reeb flow has no contractible closed orbits, we say that ξ is *hypertight*.

Given a surface $\Sigma \subset M$, the *characteristic foliation* of ξ on Σ is the singular codimension–1 foliation on Σ defined by the intersection of ξ with Σ.

3.4 Transverse measures

A *transversal* to a foliation $\mathcal{F}$ is a smooth closed arc positively transverse to $\mathcal{F}$. When M is endowed with a Riemannian metric, we define an *orthogonal transversal* to be a transversal which is orthogonal to $\mathcal{F}$.

A *transverse measure* τ is an assignment of a nonnegative real number to each transversal to $\mathcal{F}$. We ask that the assignment be countably additive under concatenation of arcs.

Suppose that I_1 and I_2 are two transversals. We say that I_1 and I_2 are homotopic if there is a homotopy $H \colon [0,1] \times [0,1] \to M$ between them such that $H|_{[0,1] \times \{0\}} = I_1$, $H|_{[0,1] \times \{1\}} = I_2$, $H|_{\{0\} \times [0,1]}$ is contained in a leaf of $\mathcal{F}$, and $H|_{\{1\} \times [0,1]}$ is contained in a leaf of $\mathcal{F}$.

We say τ is an *invariant transverse measure* if $\tau(I_1) = \tau(I_2)$ whenever I_1 and I_2 are homotopic and τ is not the zero transverse measure.

A compact leaf λ gives rise to an invariant transverse measure which assigns to each transverse arc the number of intersections with λ. We will generally be concerned with $C^{k,l}$ *transverse measures of full support*. By this we mean a transverse measure which can be encoded as a nowhere-vanishing $C^{k,l}$ 1–form τ with $\ker(\tau) = T\mathcal{F}$. Note that a $C^{\infty,k}$ foliation always admits a $C^{\infty,k-1}$ transverse measure of full support. In a foliation chart with coordinates x, y, z, one may take the transverse measure to be $g \, dz$, where g is any positive $C^{\infty,k-1}$ function.

3.5 Perturbing away transverse measures

The following lemma was observed by Bowden and documented in [Zhang 2022, Lemma 7.2]:

Lemma 9 *Let M be an atoroidal 3–manifold and $\mathcal{F}$ a C^∞ taut foliation on M. Then $\mathcal{F}$ can be C^0 approximated by a C^∞ taut foliation $\mathcal{F}'$ such that either $\mathcal{F}'$ has no transverse invariant measure or the pair $(M, \mathcal{F}')$ is homeomorphic to a surface bundle over S^1 foliated by the fibers.*

Proof sketch   On C^2 foliations, invariant transverse measures are either supported on compact leaves or have full support. This is a consequence of Sacksteder's theorem [1965]; see also [Candel and Conlon 2000, Theorem 8.2.1]. In the former case, if $\mathcal{F}$ is C^∞, a result of Bonatti and Firmo [1994] allows us to perturb away compact leaves of genus ≥ 2. In the latter case, Tischler [1970] showed that $(Y, \mathcal{F})$ is a deformation of a fibration. See [Zhang 2022, Lemma 7.2] for more details.   □

Proof of Corollary 3   Given a C^∞ taut foliation $\mathcal{F}$ on an atoroidal manifold M, Lemma 9 yields a new foliation $\mathcal{F}'$ which is C^0 close to $\mathcal{F}$ that either has no invariant transverse measure or is a surface bundle over S^1. If $\mathcal{F}'$ has no invariant transverse measure, then Theorem 1 applies. Suppose instead that $\mathcal{F}'$ is a surface bundle over S^1. Since M is atoroidal, the fiber genus is at least 2 and the monodromy is pseudo-Anosov. By [Colin and Honda 2022, Theorem 3.17], the (unique) contact perturbation of $\mathcal{F}'$ is hypertight.   □

Proof of Corollary 4   Vogel proved that if $\mathcal{F}$ is a C^2 taut foliation on an atoroidal 3–manifold, all contact structures in a C^0 neighborhood of $T\mathcal{F}$ are isotopic. Moreover, pairs of taut foliations which are homotopic through taut foliations have isotopic contact approximations [Vogel 2016, Theorem 9.3]. By Corollary 3, this contact approximation is hypertight and hence has well-defined cylindrical contact homology.   □

3.6   Holonomy

Let M be an oriented 3–manifold with a taut, cooriented C^2 foliation, $\mathcal{F}$.

Let $\gamma \colon [0, T] \to M$ be a path which is contained in a leaf of $\mathcal{F}$ and supported in a single foliation chart U. Choose I_0, I_T, two transversals to $\mathcal{F}$ contained in U and passing through $\gamma(0), \gamma(T)$ respectively. We define the holonomy map $h_\gamma \colon I_0 \to I_T$ to be the map which preserves the transverse coordinate of the foliation chart; ie for all $x \in I_0$, x and $h_\gamma(x)$ lie on the same leaf in U. If γ' is another path in the same leaf of U and is homotopic rel endpoints to γ, then $h_\gamma = h_{\gamma'}$. The holonomy map can be defined for arbitrary paths γ tangent to $\mathcal{F}$ by composing holonomy maps for short subpaths. As γ grows in length, the domain of h_γ tends to shrink. From our perspective, we are usually just concerned with the germ at 0 of h_γ, so this shrinking of the domain does not matter.

Since $\mathcal{F}$ is a C^2 foliation, h_γ is C^2. Thus, we may take its derivative at $\gamma(0)$, which we call

$$h'_\gamma \colon T_{\gamma(0)}(I_0) \to T_{\gamma(T)}(I_T).$$

The embedding of I_0 in M gives an identification of $T_{\gamma(0)}(I_0)$ with $T^\perp_{\gamma(0)}\mathcal{F}$, and likewise an identification of $T_{\gamma(T)}I_T$ with $T^\perp_{\gamma(T)}\mathcal{F}$. Via this identification, we may regard h'_γ as a map from $T^\perp_{\gamma(0)}\mathcal{F}$ to $T^\perp_{\gamma(T)}\mathcal{F}$. This map depends only on γ and not on the choice of I_0 and I_T. Moreover, if γ and γ' are two paths in the same leaf of $\mathcal{F}$ and are homotopic rel endpoints, then

(1) $$h'_\gamma = h'_{\gamma'}.$$

Suppose now that $\mathcal{F}$ is equipped with a $C^{\infty,1}$ transverse measure τ, encoded as a 1–form with $\ker(\tau) = T\mathcal{F}$ and which evaluates positively on transversals to $\mathcal{F}$. Then we define

$$|h'_\gamma|_\tau = \frac{\tau(h'_\gamma(v))}{\tau(v)},$$

where v is any nonzero vector in $T^\perp_{\gamma(0)}\mathcal{F}$. In other words, $|h'_\gamma|_\tau$ is the factor by which holonomy along γ stretches τ–lengths. Note that this norm does not depend heavily on the choice of τ. Indeed, given any two such differentiable transverse measures τ, μ, we have $\tau = g\mu$ for some strictly positive function g on M. Then for any path γ in a leaf of $\mathcal{F}$, one can verify that

$$\inf_M g^2 \leq \frac{|h'_\gamma|_\mu}{|h'_\gamma|_\tau} \leq \sup_M g^2.$$

Given a leaf λ, let $\tilde{\lambda}$ be the universal cover of λ. At this point, we are thinking of $\tilde{\lambda}$, not as embedded in $\tilde{M}$. Choose a basepoint b in $\tilde{\lambda}$. We define the Radon–Nikodym derivative of τ to be the function $f_{\tau,\lambda,b} : \tilde{\lambda} \to \mathbb{R}$ determined by the formula

$$f_{\tau,\lambda,b}(x) = |h'_\gamma|_\tau,$$

where γ is a path in λ which lifts to a path $\tilde{\gamma}$ in $\tilde{\lambda}$ from b to x. By the homotopy invariance property observed in (1), this definition is independent of the choice of the path γ. One should think of $f_{\tau,\lambda,b}$ as the transverse distance to a nearby leaf as measured by τ. Different choices of basepoint yield the same function $f_{\tau,\lambda,b}$ up to a constant factor. To be more precise, if b_1 and b_2 are two choices of basepoint on $\tilde{\lambda}$ and γ is a path in λ which lifts to a path in $\tilde{\lambda}$ from b_1 to b_2, then

$$f_{\tau,\lambda,b_1} = |h'_\gamma|_\tau f_{\tau,\lambda,b_2}.$$

When there is no danger of ambiguity, we abbreviate $f_{\tau,\lambda,b}$ to f_τ.

Remark 10 In a foliation chart with coordinates x, y, z and leaves $z = $ const, one may write any $C^{\infty,1}$ transverse measure τ as $g\,dz$ for some $C^{\infty,1}$ function g. The restriction of g to any given leaf λ agrees with $f_{\tau,\lambda,b}$ up to multiplication by a constant depending only on the choice of basepoint. It follows that $f_{\tau,\lambda,b}$ is C^∞ on each leaf.

A theme in what follows is that for the best transverse measures, $f_{\tau,\lambda,b}$ has no local minima. Fix a Riemannian metric on M. Each leaf inherits a Riemannian metric from the Riemannian metric we chose on M. If $f_{\tau,\lambda,b}$ is a harmonic function on $\tilde{\lambda}$ for every choice of leaf λ, we say that τ is a *harmonic transverse measure*. By Theorem 1c of [Garnett 1983], this is equivalent to more traditional notions of harmonic transverse measure using the leafwise Laplacian operator, at least for differentiable transverse measures like τ. See also [Candel 2003; Deroin and Kleptsyn 2007] for more discussion of harmonic transverse measures. When $\log f_{\tau,\lambda,b}$ is superharmonic (resp strictly superharmonic) on each leaf $\tilde{\lambda}$, we say that τ is *log superharmonic* (resp *strictly log superharmonic*). Since log is a concave function, every harmonic transverse measure is also log superharmonic.

While f_τ is defined on $\tilde\lambda$ and makes sense only up to constant factors for each leaf, several related objects descend to M. The 1–form $d \log f_\tau$ is a well-defined section of $T^*\mathcal{F}$. It measures the infinitesimal rate of contraction or expansion of leaves in directions tangent to $\mathcal{F}$. The function $\Delta \log f_\tau$, where Δ denotes the leafwise Laplace–Beltrami operator, is consequently also defined on M. Finally, the leafwise level sets of f_τ descend to a singular codimension–2 foliation on M.

Proposition 11 *Suppose τ is a $C^{\infty,1}$ transverse measure of full support. Choose a vector field v transverse to $\mathcal{F}$ with $\tau(v) = 1$. Then $\iota_v d\tau = -d \log f_\tau(v)$ as sections of $T^*\mathcal{F}$.*

Proof Choose a foliation chart with coordinates x, y, z, where $\mathcal{F} = \ker(dz)$. We may further arrange that v is parallel to the z–axis. In this coordinate system, $\tau = g\, dz$ for some $C^{\infty,1}$ function g.

Now

$$
\begin{aligned}
(2) \qquad \iota_v d\tau = \iota_v(dg \wedge dz) &= dg(v)\, dz - dz(v)\, dg \\
&= dg(v)\, dz - \frac{1}{g}\, dg \quad \text{(since } \tau(v) = 1\text{)} \\
&= dg(v)\, dz - d \log g.
\end{aligned}
$$

As noted in Remark 10, g agrees with f_τ on each leaf up to a constant factor, so $d \log g = d \log f_\tau$ on each leaf. When we restrict (2) to any given leaf λ, the term $dg(v)\, dz$ vanishes and we are left with $-d \log f_\tau$ as desired. $\qquad\square$

3.7 Leafwise Brownian motion and diffusion

We provide here a summary of the main properties of Brownian motion that we will use. We direct the reader to [Mörters and Peres 2010] for a comprehensive introduction to Brownian motion on $\mathbb{R}^n$. Chapters 3 and 4 of [Hsu 2002] provide a treatment of the Riemannian case. One may also consult [Deroin and Kleptsyn 2007], which specializes to the leafwise Brownian motion that we will consider in this paper.

Let λ be a complete Riemannian manifold with bounded geometry and let x be a point on λ. As suggested by the notation, the Riemannian manifold we consider will usually be a leaf of $\mathcal{F}$ with a Riemannian metric inherited from one on M. Such a surface always has bounded geometry because M is compact. Let Γ_x be the space of all continuous paths $\gamma\colon [0,\infty) \to \lambda$ satisfying $\gamma(0) = x$. We equip Γ_x with the uniform topology on compact sets of $[0,\infty)$ and will use the induced Borel σ–algebra.

On Γ_x there is a probability measure W_x called the Wiener measure. We will often refer to a path drawn from W_x as a Brownian path. For any function f on λ, we define its time t diffusion $D^t(f)$ by

$$D^t(f)(x) = \mathbb{E}_{\gamma \sim W_x}[f(\gamma(t))].$$

Let Δ be the Laplace–Beltrami operator on λ. The Wiener measure satisfies the following properties:

(1) **Markov property** Suppose γ is drawn from W_x. Fix some $t_0 \geq 0$ and any $y \in \lambda$. Then after conditioning on $\gamma(t_0) = y$, $\gamma|_{[t_0,\infty)}$ is distributed like W_y. This implies that, for any $t, t' \geq 0$,

$$D^{t+t'} = D^t \circ D^{t'}.$$

(2) **Adapted to the metric** For a real-valued C^2 function f_0 on λ,

$$\frac{\partial}{\partial t} D^t f_0 \Big|_{t=0} = -\Delta f_0.$$

In other words, $D^t(f)$ is the time–t solution to the heat equation

$$\frac{\partial f}{\partial t} = -\Delta f,$$

with initial condition f_0.

Combining these two properties with linearity of expectation, we find that Δ commutes with diffusion when applied to a C^2 function f:

$$\Delta D^{t'} f = -\frac{\partial}{\partial t} D^t D^{t'} f \Big|_{t=0} \tag{3}$$

$$= -D^{t'} \frac{\partial}{\partial t} D^t f \Big|_{t=0} \tag{4}$$

$$= D^{t'} \Delta f. \tag{5}$$

Remark 12 The hypothesis of bounded geometry allows us to disregard paths that leave the manifold in finite time.

In the course of the paper, we will define another diffusion operator $\mathcal{D}_{T,R,S}$ acting on transverse measures. It should not be confused with D^t.

3.8 Forms and currents

In the rest of the paper, we will use the language of currents carried by foliations. We refer the reader to [Sullivan 1976; Dinh and Sibony 2005] for more background on currents and foliations. Informally, an i–current on a manifold M is an oriented i–dimensional submanifold of M (possibly disconnected, possibly with boundary), or a weak limit of such submanifolds. Similarly, a 2–current carried by a codimension–1 foliation $\mathcal{F}$ on a 3–manifold is informally a weak limit of surfaces contained in leaves of $\mathcal{F}$.

Let us define the relevant spaces more carefully and with some attention to regularity. Let U be a foliation chart. Let x, y, t be local coordinates for U, where x and y are coordinates for tangential directions and t is the transverse coordinate. Let A be a subset of $\{dx, dy, dt\}$, and let f be a $C^{k,l}$ function on U. Let χ be the indicator function for the subset A. Then we say that the differential form $f \bigwedge_{\beta \in A} \beta$ has *adjusted regularity* $(k + |A|, l + \chi(dt))$. For example, if f is $C^{k,l}$, then $f\, dx \wedge dt$ has adjusted regularity $(k + 2, l + 1)$. The adjusted regularity of a sum of such differential forms is the minimum of the adjusted regularities of its summands. The adjusted regularity of a differential form on M is (k, l) if has adjusted regularity (k, l) in each foliation chart. With this definition, the exterior derivative preserves adjusted regularity.

In Section 5, it will be more convenient to work in a Sobolev space. If the function f is in the Sobolev space of functions with L^2 norms on any partial derivatives with order at most l in the transverse direction and total order at most k, then we say that f has *Sobolev adjusted regularity* (k, l). Similarly, we say that $f \bigwedge_{\beta \in A} \beta$ has *Sobolev adjusted regularity* $(k + |A|, l + \chi(dt))$.

Now let $\Omega^i(M)$ denote the space of i–forms of adjusted regularity $C^{\infty,2}$. Note that $\bigoplus_i \Omega^i(M)$ is a chain complex. We define the i–currents to be elements of the topological dual space to $\Omega^i(M)$. We usually use ∂ to denote the differential on the dual complex. A 2–current *carried by* $\mathcal{F}$ is a 2–current which lies in the closure of the set of 2–currents represented by subsurfaces of leaves of $\mathcal{F}$.

Note that i–currents behave like i–dimensional submanifolds; one can evaluate i–forms on them, their boundaries are $(i-1)$–currents, and Stokes' theorem holds.

3.9 Smoothness of heat kernels

In this section, we prove the following proposition which will be useful in the proof of Lemma 23.

Proposition 13 *Let Ω be the unit disk in $\mathbb{R}^d$. Let $\{g_\varepsilon\}_{\varepsilon \in (-1,1)}$ be a 1–parameter family of metrics on a neighborhood of Ω. Let $k_\varepsilon(x,y,t)\colon \Omega \times \Omega \times (0,\infty) \to \mathbb{R}$ be the heat kernel for the metric g_ε with Dirichlet boundary condition 0. Then for any fixed $t > 0$, the derivatives of $k_\varepsilon(x,y,t)$ with respect to ε exist and are continuous as long as the corresponding derivatives of g_ε exist and are continuous.*

Before beginning with the proof of Proposition 13, we set up some notation.

Let $\mathcal{X}$ be the space of smooth functions on $[0,\infty) \times \Omega$ which vanish on $\partial\Omega$. Let $\mathcal{Y}$ be the space of smooth functions on $\Omega \times \Omega \times [0,\infty)$ which extend smoothly to a neighborhood of Ω. The $[0,\infty)$–factor represents time. We will write g for g_ε and use the shorthand $\sqrt{g} = \sqrt{\det g_{ij}}$. We will use Einstein summation notation throughout this section. The coordinates of vectors will be written with upper indices. Given a vector x, we use $x^{\otimes n, i \cdots j}$ to denote the nth tensor power of x with indices $i, \ldots, j$. For example, if x is a vector, then $x^{\otimes 2, ij}$ is the tensor square of x with indices i and j. We will use $|\cdot|$ to denote distance in the standard Euclidean metric.

Let h be a positive number strictly smaller than the smallest eigenvalue of $g_{ij}(x)$ for all $x \in \Omega$. For any $x \in \Omega$ at Euclidean distance less than $\sqrt{h}$ to $\partial\Omega$, there is a unique closest point to x on $\partial\Omega$. Call the reflection of x through this closest point $\bar{x}$. Here, "closest" and "reflection" are taken not with respect to the Euclidean metric, nor with respect to the Riemannian metric g_{ij}, but with respect to the flat metric induced by the inner product $g_{ij}(x)$.

Given a smooth function $k\colon \Omega \times \Omega \times (0,\infty) \to \mathbb{R}$, we say that k is *Gaussian-type* if k can be written as

$$(6) \qquad k(x,y,t) = c(x,y,t)\frac{(x-y)^{\otimes a}}{t^{d/2+b}} \exp\left(-\frac{g_{ij}(x)(x-y)^{\otimes 2,ij}}{4t}\right)$$

or

$$(7) \qquad k(x,y,t) = c(x,y,t)\frac{(\bar{x}-y)^{\otimes a}}{t^{d/2+b}} \exp\left(-\frac{g_{ij}(x)(\bar{x}-y)^{\otimes 2,ij}}{4t}\right)$$

for some nonnegative integers a and b and a smooth function $c(x,y,t)$ in $\mathcal{Y}$.

We say that the *order* of k is $a - 2b + 1$. A Gaussian-type function of high order has a relatively mild singularity at $x = y$, $t = 0$.

Proposition 14 *If f is a Gaussian-type function of order k, then $\partial_t f$ is Gaussian-type of order $k-2$ and $\partial_x f$ and $\partial_y f$ are Gaussian-type of order $k-1$. The derivative of f with respect to g is of order k.*

Proposition 15 *If $k(x, y, t)$ is Gaussian-type of nonnegative order and f is a bounded function, then*

$$\left| \int_\Omega k(x, y, t) f(t, x) \, dx \right| \leq \frac{C}{\sqrt{t}}$$

for some constant C.

Proof Suppose k has the form of (6). Let C be an upper bound for $|c(x, y, t)|$ over $\Omega \times \Omega \times [0, t]$. Recall that h is a small enough constant that $g_{ij} - hI$ is positive definite everywhere. Let F be an upper bound for $|f|$. We will make the substitution $r = |x - y|$ in the integral below:

$$\left| \int_\Omega k(x, y, t) f(t, x) \, dx \right| \leq \int_\Omega CF \frac{|x-y|^a}{t^{d/2+b}} \exp\left(-\frac{h|x-y|^2}{4t} \right) dx$$

$$< \int_0^\infty CF \frac{r^{a+d-1}}{t^{d/2+b}} \exp\left(-\frac{hr^2}{4t} \right) dr$$

$$\propto t^{a/2-b}.$$

If the order of k is nonnegative, then $\frac{a}{2} - b \geq -\frac{1}{2}$ as desired. In performing the integral over r, we used the Gaussian integral

$$\int_0^\infty r^k \exp\left(-\frac{r^2}{t} \right) dr \propto t^{(k+1)/2},$$

where the constant of proportionality depends only on k. The same argument works for Gaussian-type functions of the second kind. $\square$

Proposition 16 *Let*

$$Z(n) = \int_{0 \leq t_0 \leq \cdots \leq t_n = 1} \prod_{\alpha=0}^{n-1} \left(\frac{1}{\sqrt{t_{\alpha+1} - t_\alpha}} \right) dt_0 \cdots dt_{n-1}.$$

The integral converges and the explicit bound

$$Z(n) < \frac{10^n}{\sqrt{n!}}$$

holds.

Proof We work by induction. The case $n = 0$ holds trivially. Note that $Z(n)$ has the following scaling property:

$$\int_{0 \leq t_0 \leq \cdots \leq t_n = T} \prod_{\alpha=0}^{n-1} \left(\frac{1}{\sqrt{t_{\alpha+1} - t_\alpha}} \right) dt_0 \cdots dt_{n-1} = \sqrt{T^n} \, Z(n).$$

This is because the measure scales like T^n and the integrand scales like $1/\sqrt{T^n}$.

Now assume that the result holds for $Z(n)$. We now separate integration over the final variable t_n from integration over the remaining variables in the expression for $Z(n+1)$:

$$
\begin{aligned}
Z(n+1) &= \int_0^1 \frac{1}{\sqrt{1-t_n}} \left(\int_{0 \le t_0 \le \cdots \le t_n} \prod_{\alpha=0}^{n-1} \left(\frac{1}{\sqrt{t_{\alpha+1}-t_\alpha}} \right) dt_0 \cdots dt_{n-1} \right) dt_n \\
&= \int_0^1 \frac{1}{\sqrt{1-t_n}} \cdot (\sqrt{t_n})^n \, Z(n) \, dt_n \\
&< \frac{10^n}{\sqrt{n!}} \int_0^1 \frac{t^{n/2}}{\sqrt{(1-t)}} \, dt \quad \text{(by the induction hypothesis)} \\
&= \frac{10^n}{\sqrt{n!}} \left(\int_0^{1-1/n} \frac{t^{n/2}}{\sqrt{(1-t)}} \, dt + \int_{1-1/n}^1 \frac{t^{n/2}}{\sqrt{(1-t)}} \, dt \right) \\
&< \frac{10^n}{\sqrt{n!}} \left(\int_0^{1-1/n} \frac{t^{n/2}}{\sqrt{1/n}} \, dt + \int_{1-1/n}^1 \frac{1}{\sqrt{(1-t)}} \, dt \right) \\
&< \frac{10^n}{\sqrt{n!}} \left(\int_0^1 \sqrt{n} \, t^{n/2} \, dt + \int_{1-1/n}^1 \frac{1}{\sqrt{(1-t)}} \, dt \right) \\
&< \frac{10^n}{\sqrt{n!}} \left(\frac{10}{\sqrt{n+1}} \right) \\
&< \frac{10^{n+1}}{\sqrt{(n+1)!}}.
\end{aligned}
$$

This completes the induction. □

Proof of Proposition 13   Our first step is to construct a reasonable approximation to the heat kernel. To do this, we use the method of images.

Given a source at a point $x \in \Omega$, we add a sink at $\bar{x}$. The solution to the heat equation on $\mathbb{R}^d$ with this source and sink will be approximately zero on $\partial\Omega$, and therefore is approximately a solution to the heat equation on Ω with Dirichlet boundary condition 0. Let us make this more quantitative. Let $\upsilon\colon \Omega \to \mathbb{R}$ be a smooth function satisfying $\upsilon(x) = 0$ when x is at Euclidean distance $\ge \sqrt{h}$ from $\partial\Omega$ and $\upsilon(x) = 1$ when x is at Euclidean distance $\le \sqrt{h}/2$ from $\partial\Omega$. Thus, $\bar{x}$ is well-defined whenever $\upsilon(x) \ne 0$. Let

$$
k_0(x, y, t) = \frac{\sqrt{g(x)}}{(4\pi t)^{d/2}} \exp\left(-\frac{g_{ij}(x)(x-y)^{\otimes 2,ij}}{4t} \right)
$$

and

$$
k_1(x, y, t) = \upsilon(x) \frac{\sqrt{g(x)}}{(4\pi t)^{d/2}} \exp\left(-\frac{g_{ij}(x)(\bar{x}-y)^{\otimes 2,ij}}{4t} \right).
$$

Thanks to the cutoff function υ, we know k_1 is smooth and well-defined at all $x \in \Omega$, $y \in \mathbb{R}^d$, and $t > 0$. Our first approximation to the heat kernel is $k_0 - k_1$. By symmetry, $k_0 - k_1$ vanishes along a hyperplane tangent to $\partial\Omega$. Although $k_0 - k_1$ does not vanish for $y \in \partial\Omega$ as required by the Dirichlet boundary condition, we have the following bound:

Lemma 17 *For $x \in \Omega$ and $y \in \partial\Omega$,*

$$k_0(x, y, t) - k_1(x, y, t) = c(x, y, t)_{ijk} \frac{(x - y)^{\otimes 3, ijk}}{t^{d/2+1}} \exp\left(-\frac{g_{ij}(x)(x - y)^{\otimes 2, ij}}{4t}\right),$$

where c is some smooth tensor-valued function each of whose components is in $\mathcal{Y}$.

Proof Start by comparing $k_0(x, y, t) - k_1(x, y, t)$ to the desired form

(8) $c(x, y, t)_{ijk}(x - y)^{\otimes 3, ijkl}$

$$= \frac{k_0(x, y, t) - k_1(x, y, t)}{t^{-(d/2+1)} \exp(-g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij}/(4t))}$$

$$= \frac{t\sqrt{g(x)}}{(4\pi)^{d/2}} - \upsilon(x)\frac{t\sqrt{g(x)}}{(4\pi)^{d/2}} \exp\left(-\frac{g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij} - g_{ij}(x)(x - y)^{\otimes 2, ij}}{4t}\right).$$

Since $\bar{x}$ is a reflection of x in $\partial\Omega$ and Ω is convex (with respect to the flat metric $g_{ij}(x)$), we have

$$g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij} > g_{ij}(x)(x - y)^{\otimes 2, ij}.$$

Therefore, the exponential term on the right side of (8) is bounded. In the regime in which $|x - y|$ is large, say $\geq \sqrt{h}/2$, we may use (8) to choose a smooth candidate for $c(x, y, t)$.

Now consider the remaining regime where $|x - y|$ is less than $\sqrt{h}/2$. Since $y \in \partial\Omega$, the Euclidean distance between x and $\partial\Omega$ is less than $\sqrt{h}/2$. Therefore, $\upsilon = 1$ and the expression in (8) reduces to

(9) $c(x, y, t)_{ijk}(x - y)^{\otimes 3, ijkl} = \frac{t\sqrt{g(x)}}{(4\pi)^{d/2}}\left(1 - \exp\left(-\frac{g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij} - g_{ij}(x)(x - y)^{\otimes 2, ij}}{4t}\right)\right).$

It will now be convenient to make a linear change of coordinates so that the midpoint of x and $\bar{x}$ is at the origin, x lies on the first coordinate axis, and the hyperplane tangent to $\partial\Omega$ at the origin is spanned by the remaining coordinate axes. In this coordinate system, $g_{1i} = 0$ for $i \neq 1$. By symmetry, $k_0 - k_1$ vanishes whenever y lies on this hyperplane. In this coordinate system, we have

$$\begin{aligned} g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij} - g_{ij}(x)(x - y)^{\otimes 2, ij} &= g_{11}(\bar{x}^1 - y^1)^2 - g_{11}(x^1 - y^1)^2 \\ &= g_{11}(-x^1 - y^1)^2 - g_{11}(x^1 - y^1)^2 \\ &= 4g_{11}x^1y^1 \\ &= O(|x - y|^3). \end{aligned}$$

In the last line we used the constraint that $y \in \partial\Omega$, which implies the geometrical facts that $x^1 = O(|x - y|)$ and $y^1 = O(|y - 0|^2) = O(|x - y|^2)$. See Figure 3. Substituting back into (9), we find

(10) $\frac{k_0(x, y, t) - k_1(x, y, t)}{t^{-(d/2+1)} \exp(-g_{ij}(x)(x - y)^{\otimes 2, ij}/(4t))} = \frac{t\sqrt{g(x)}}{(4\pi)^{d/2}}\left(1 - \exp\left(-\frac{O(|x - y|^3)}{4t}\right)\right)$

(11) $= \frac{\sqrt{g(x)}}{(4\pi)^{d/2}} O(|x - y|^3).$

Thus, this equation defines the desired smooth function $c(x, y, t)$ when $y \in \partial\Omega$. □
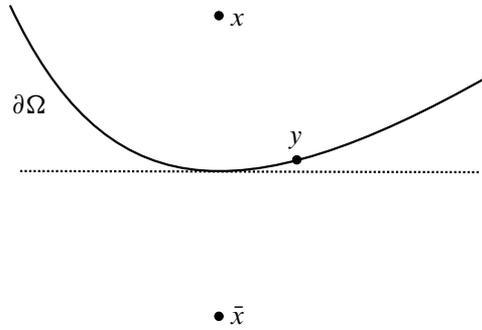
Figure 3: The dotted line is the hyperplane tangent to $\partial\Omega$ at the origin.

By the Whitney extension theorem applied to the function $c(x, y, t)$ constructed in Lemma 17, we can find a Gaussian-type function $k_2(x, y, t)$ of order ≥ 2 such that $k_2(x, y, t) = k_0(x, y, t) - k_1(x, y, t)$ whenever $y \in \partial\Omega$. Now define

$$k_{\text{approx}}(x, y, t) = k_0(x, y, t) - k_1(x, y, t) - k_2(x, y, t).$$

By construction, $k_{\text{approx}}(x, y, t) = 0$ whenever $y \in \partial\Omega$. We will use k_{approx} as our first approximation to the heat kernel.

Now we will write down a formal series for the heat kernel. We will then show that the series converges, as do its derivatives with respect to ε. Finally we will check that the series satisfies the Dirichlet boundary condition.

Define the operator $K_{\text{approx}} \colon \mathcal{X} \to \mathcal{X}$ by

$$K_{\text{approx}} f(t, y) = \int_{\substack{0 \leq t_0 \leq t \\ x \in \Omega}} k_{\text{approx}}(x, y, t_0) \, dt_0 \, dx.$$

K_{approx} should be thought of as a first approximation to the heat kernel operator. Let Δ_g be the Laplace–Beltrami operator on Ω for the metric g. In coordinates, the Laplace–Beltrami operator has the form

$$\Delta_g u = -\frac{1}{\sqrt{g}} \, \partial_i \sqrt{g} g^{ij} \, \partial_j.$$

Now we define the operator $\mathcal{E}$ which measures the failure of K_{approx} to be the true heat kernel operator:

(12) $$\mathcal{E} = \text{Id} - (\partial_t - \Delta_g) K_{\text{approx}}.$$

Rearranging (12), we have

$$(\partial_t - \Delta_g) K_{\text{approx}} (\text{Id} - \mathcal{E})^{-1} = \text{Id}.$$

Thus,

(13) $$K := K_{\text{approx}} (\text{Id} + \mathcal{E} + \mathcal{E}^2 + \cdots)$$

is a formal series for the heat kernel operator. We now show that the terms $K_{\text{approx}} \mathcal{E}^n$ are small enough that the sum converges. In what follows, we will use the notation poly($\dots$) to represent an unspecified

polynomial in its arguments. When the arguments are tensors, poly($\dots$) may be tensor-valued. This polynomial will be allowed to change from line to line as it absorbs constants and the like. The degree and coefficients of this polynomial will crucially not depend on n. We have

$$\partial_t k_0(x,y,t) = \frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \left(\frac{g(x)_{ij}(x-y)^{\otimes 2,ij}}{4t^{d/2+2}} - \frac{d}{2t^{d/2+1}} \right) \exp\left(-\frac{g_{ij}(x)(x-y)^{\otimes 2,ij}}{4t} \right) + \delta(y-x,t),$$

$$\partial_t k_1(x,y,t) = \upsilon(x) \left(\frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \left(\frac{g(x)_{ij}(\bar{x}-y)^{\otimes 2,ij}}{4t^{d/2+2}} - \frac{d}{2t^{d/2+1}} \right) \exp\left(-\frac{g_{ij}(x)(\bar{x}-y)^{\otimes 2,ij}}{4t} \right) + \delta(y-\bar{x},t) \right).$$

Here we are taking distributional derivatives, and $\delta(y-x,t)$ and $\delta(y-\bar{x},t)$ are delta distributions.

Now we apply the Laplace–Beltrami operator in the y–coordinate:

$$\Delta_g k_0(x,y,t) = \frac{1}{\sqrt{g(y)}} \partial_k \sqrt{g(y)} g(y)^{kl} \partial_l \frac{\sqrt{g(x)}}{(4\pi t)^{d/2}} \exp\left(-\frac{g_{ij}(x)(x-y)^{\otimes 2,ij}}{4t} \right)$$

$$= \left(\frac{\sqrt{g(y)}}{(4\pi)^{d/2}} \left(\frac{g(y)^{kl}g(x)_{kl}}{2t^{d/2+1}} - \frac{g(x)_{ij}(x-y)^{\otimes ij}}{t^{d/2+2}} \right) \right.$$

$$+ \frac{\text{poly}(g, \partial g, \partial\partial g)}{t^{d/2}} + \frac{(x^i - y^i)\text{poly}(g, \partial g, \partial\partial g, x-y)_i}{t^{d/2+1}}$$

$$\left. + \frac{(x-y)^{\otimes ijk}\text{poly}(g, \partial g, \partial\partial g, x-y)_{ijk}}{t^{d/2+2}} \right) \exp\left(-\frac{g_{ij}(x)(x-y)^{\otimes 2,ij}}{4t} \right),$$

$$\Delta_g k_1(x,y,t) = \frac{1}{\sqrt{g(y)}} \partial_k \sqrt{g(y)} g(y)^{kl} \partial_l \upsilon(x) \frac{\sqrt{g(x)}}{(4\pi t)^{d/2}} \exp\left(-\frac{g(x)_{ij}(\bar{x}-y)^{\otimes ij}}{4t} \right)$$

$$= \upsilon(x) \left(\frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \left(\frac{g(y)^{kl}g(x)_{kl}}{2t^{d/2+1}} - \frac{g(x)_{ij}(\bar{x}-y^i)^{\otimes ij}}{t^{d/2+2}} \right) \right.$$

$$+ \frac{\text{poly}(g, \partial g, \partial\partial g)}{t^{d/2}} + \frac{(\bar{x}^i - y^i)\text{poly}(g, \partial g, \partial\partial g, \bar{x}-y)_i}{t^{d/2+1}}$$

$$\left. + \frac{(\bar{x}-y)^{\otimes ijk}\text{poly}(g, \partial g, \partial\partial g, \bar{x}-y)_{ijk}}{t^{d/2+2}} \right) \exp\left(-\frac{g_{ij}(x)(\bar{x}-y)^{\otimes 2,ij}}{4t} \right).$$

The terms

$$\frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \left(\frac{g(y)^{kl}g(x)_{kl}}{2t^{d/2+1}} - \frac{g(x)_{ij}(x-y)^{\otimes 2,ij}}{t^{d/2+2}} \right)$$

and

$$\left(\frac{g(y)^{kl}g(x)_{kl}}{2t^{d/2+1}} - \frac{g(x)_{ij}(\bar{x}-y^i)^{\otimes ij}}{t^{d/2+2}} \right)$$

come from the terms where the derivatives hit only $x-y$ and not $g(y)$. Note that all of the terms are Gaussian-type, and all have nonnegative order.

Combining the above, we have

$$\mathcal{E}f(t,y)$$

$$= f(t,y) - (\partial_t - \Delta_g) \int_{\substack{0 \le t_0 \le t \\ x \in \Omega}} (k_0(x,y,t) - k_1(x,y,t) - k_2(x,y,t)) f(t_0,x) \, dt_0 \, dx$$

$$= \int_{\substack{0 \le t_0 \le t \\ x \in \Omega}} \left(\frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \frac{g(x)_{ij} g(y)^{ij} - d}{2(t - t_0)^{d/2}} + \frac{\text{poly}(g, \partial g, \partial \partial g)}{(t - t_0)^{d/2}} \right.$$

$$\left. + \frac{(x^i - y^i)\text{poly}(g, \partial g, \partial \partial g, x - y)_i}{(t - t_0)^{d/2+1}} + \frac{(x - y)^{\otimes 3}\text{poly}(g, \partial g, \partial \partial g, x - y)}{(t - t_0)^{d/2+2}} \right)$$

$$\cdot \exp\left(-\frac{g(x)_{ij}(x - y)^{\otimes 2, ij}}{4(t - t_0)} \right) f(t_0, x)$$

$$+ \upsilon(d(x, \partial\Omega))\left(\frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \frac{g(x)_{ij} g(y)^{ij} - d}{2(t - t_0)^{d/2}} + \frac{\text{poly}(g, \partial g, \partial \partial g)}{(t - t_0)^{d/2}} \right.$$

$$\left. + \frac{(\bar{x}^i - y^i)\text{poly}(g, \partial g, \partial \partial g, \bar{x} - y)_i}{(t - t_0)^{d/2+1}} + \frac{(\bar{x} - y)^{\otimes 3}\text{poly}(g, \partial g, \partial \partial g, \bar{x} - y)}{(t - t_0)^{d/2+2}} \right)$$

$$\cdot \exp\left(-\frac{g(x)_{ij}(x - y)^{\otimes 2, ij}}{4(t - t_0)} \right) f(t_0, x) + (\partial_t - \Delta_g)k_2(x, y, t) f(t_0, x) \, dt_0 \, dx.$$

In the line above, we used that

$$\int_{\substack{0 \le t_0 \le t \\ x \in \Omega}} \delta(x - y, t - t_0) f(t_0, x) \, dx \, dt_0 = f(t, y)$$

and

$$\int_{\substack{0 \le t_0 \le t \\ x \in \Omega}} \delta(\bar{x} - y, t - t_0) f(t_0, x) = 0.$$

The second equality holds because the delta function can only be nonzero when x and y lie on $\partial\Omega$, but f vanishes on $\partial\Omega$.

We define $e(x, y, t)$ to be the integral kernel for $\mathcal{E}$ appearing above, so that

$$\mathcal{E}f(y, t) = \int_{\substack{0 \le t_0 \le t \\ x \in \Omega}} e(x, y, t - t_0) f(t_0, x) \, dt_0 \, dx.$$

Now we claim that each term appearing in $e(x, y, t)$ is bounded in absolute value by a Gaussian type of nonnegative order. For the term

$$\frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \frac{g(x)_{ij} g(y)^{ij} - d}{2(t - t_0)^{d/2}} \exp\left(-\frac{g_{ij}(x)(x - y)^{\otimes 2, ij}}{4t} \right),$$

observe that $g(x)_{ij} g(y)^{ij} - d$ vanishes at $x = y$. Since the partial derivatives of g are bounded, $g(x)_{ij} g(y)^{ij} - d = O(x - y)$ so the term has nonnegative order. We can use the same trick to show that the other term involving $g(x)_{ij} g(y)^{ij}$ has nonnegative order:

$$\left| \frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \frac{g(x)_{ij} g(y)^{ij} - d}{2(t - t_0)^{d/2}} \exp\left(-\frac{g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij}}{4t} \right) \right|$$

$$= \left| \frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \frac{O(x - y)}{2(t - t_0)^{d/2}} \exp\left(-\frac{g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij}}{4t} \right) \right|$$

$$\le \left| \frac{\sqrt{g(x)}}{(4\pi)^{d/2}} \frac{O(\bar{x} - y)}{2(t - t_0)^{d/2}} \exp\left(-\frac{g_{ij}(x)(\bar{x} - y)^{\otimes 2, ij}}{4t} \right) \right|.$$

In the last line, we used that $|x-y| < c|\bar{x}-y|$ for some constant c independent of x and y. Finally, we apply Proposition 14 to the term involving k_2. Since we constructed k_2 to have order 2, $(\partial_t - \Delta_g)k_2(x, y, t)$ is Gaussian-type of order 0 as desired. Let F be an upper bound for f on $\Omega \times [0, t]$. By Proposition 15, we have

$$|\mathcal{E} f(t, y)| \leq \frac{CF}{\sqrt{t}}$$

for some constant C independent of y and t.

Now we are equipped to handle the more complicated analysis of $K_{\text{approx}} \mathcal{E}^n$. For notational simplicity, set $x_{n+1} = y$. We have

$$|K_{\text{approx}} \mathcal{E}^n f(t, y)| = \left| \int_{\substack{0 \leq t_0 \leq \cdots \leq t_n \leq t \\ x_0, \ldots, x_n \in \Omega}} k_{\text{approx}}(x_n, y, t - t_n) \left(\prod_{\alpha=0}^{n-1} e(x_\alpha, x_{\alpha+1}, t_{\alpha+1} - t_\alpha) \right) \right.$$
$$\left. \cdot f(t_0, x_0) \, dx_0 \cdots dx_n \, dt_0 \cdots dt_n \right|$$

$$(14) \qquad\qquad \leq \int_{0 \leq t_0 \leq \cdots \leq t_n \leq t} C_1 \left(\prod_{\alpha=0}^{n} \frac{C_2}{\sqrt{t_{\alpha+1} - t_\alpha}} \right) F \, dt_0 \cdots dt_n$$

$$(15) \qquad\qquad \leq \frac{C^n}{\sqrt{n!}}$$

for a large enough constant C possibly depending on d, g, f, and t, but not on n. In (14), we used Proposition 15 on both k_{approx} and e. In (15), we used Proposition 16. It follows that the formal series (13) converges uniformly and is indeed the heat kernel operator. The Dirichlet boundary condition is satisfied since $k_{\text{approx}}(x, y, t)$ vanishes whenever $y \in \partial\Omega$.

Finally, we need to check that the derivatives of $K_{\text{approx}}(1 + \mathcal{E} + \mathcal{E}^2 + \cdots)$ with respect to variations of g exist. Recall from Proposition 15 that differentiating a Gaussian-type function with respect to g does not change its order. Therefore, the derivatives of $K_{\text{approx}} \mathcal{E}^n f$ with respect to g have the same form as in (14) except that k_{approx} and e are replaced with new Gaussian-type functions of the same order. Therefore, a similar bound applies and the series for the derivatives of $K_{\text{approx}}(1 + \mathcal{E} + \mathcal{E}^2 + \cdots)$ converges. The corresponding integral kernels are the desired derivatives of the heat kernel with respect to variations of g. $\qquad\square$

3.10 Assorted notation

Given points x, y on the universal cover of a leaf λ, let $d(x, y)$ denote the leafwise Riemannian distance between x and y in $\tilde{\lambda}$. We use $\star_2$ to mean the leafwise Hodge star operator acting on $\Lambda^*(T^*F)$.

4 Existence of transverse Reeb flows

In this section, we prove Theorem 1. For the rest of this section, we fix a closed, oriented 3–manifold M with a cooriented C^2 foliation $\mathcal{F}$. As discussed in Section 3.1, we can assume that $\mathcal{F}$ is $C^{\infty,2}$. We also fix a Riemannian metric on M.

First we explain the claim from the Introduction that an invariant transverse measure is an obstruction to a transverse Reeb flow. Suppose $\mathcal{F}$ supports an invariant transverse measure. As discussed in [Ruelle and Sullivan 1975], M then has a nontrivial closed 2–current Σ carried by $\mathcal{F}$. Suppose that α is a contact form on M with Reeb flow transverse to $\mathcal{F}$. Then $d\alpha > 0$ on $T\mathcal{F}$. Using Stokes' theorem for 2–currents (see for example [Dinh and Sibony 2005] or [Sullivan 1976]), we then have

$$0 > \int_{\Sigma} d\alpha = \int_{\partial\Sigma} \alpha = 0.$$

This is the desired contradiction.

For the rest of the section, we consider the case that $\mathcal{F}$ supports no transverse invariant measure. In particular, this implies that $\mathcal{F}$ is taut. Our plan is to start with a choice of $C^{\infty,1}$ transverse measure τ, and then use a diffusion operator to convert τ into a log superharmonic transverse measure. The diffusion operator depends on our choice of background Riemannian metric. Finally, we will give a recipe to write down a linear perturbation of a log superharmonic transverse measure into a contact structure with Reeb flow transverse to $\mathcal{F}$.

4.1 Structure theorem for C^2 foliations

Deroin and Kleptsyn gave a precise picture of the long-term dynamics of leafwise Brownian motion and the holonomy along such paths. We state their main theorem here for reference:

Theorem 18 [Deroin and Kleptsyn 2007] *Let $\mathcal{F}$ be a C^1 foliation of a closed 3–manifold M. Then either $\mathcal{F}$ supports an invariant transverse measure, or $\mathcal{F}$ has a finite number of minimal sets $\mathcal{M}_1, \ldots, \mathcal{M}_k$ equipped with probability measures $\mu_1, \ldots, \mu_k$, and there exists a real $\kappa > 0$ such that*:

(1) **Contraction** *For every point $x \in M$ and almost every leafwise Brownian path γ starting at x, there is an orthogonal transversal I_γ at x and a constant $C_\lambda > 0$ such that, for every $t > 0$, the holonomy map $h_{\gamma|_{[0,t]}}$ is defined on I_γ and*

$$|h_{\gamma|_{[0,t]}}(I_\gamma)| \leq C_\gamma \exp(-\kappa t).$$

(2) **Distribution** *For every point $x \in M$ and almost every leafwise Brownian path γ starting at x, the path γ tends to one of the minimal sets, $\mathcal{M}_j$, and is distributed with respect to μ_j, in the sense that*

$$\lim_{t\to\infty} \frac{1}{t} \gamma_* \operatorname{leb}_{[0,t]} = \mu_j,$$

where $\operatorname{leb}_{[0,t]}$ is the standard Lebesgue measure on $[0,t]$.

(3) **Attraction** *The probability $p_j(x)$ that a leafwise Brownian path starting at a point x of M tends to $\mathcal{M}_j$ is a continuous leafwise harmonic function (which equals 1 on $\mathcal{M}_j$).*

(4) **Diffusion** *When t goes to infinity, the diffusions D^t of a continuous function $f : M \to \mathbb{R}$ converge uniformly to the function $\sum_j c_j p_j$, where $c_j = \int f \, d\mu_j$. In particular, the functions p_j form a base in the space of continuous leafwise harmonic functions.*

Remark 19 It might be surprising that holonomy is contracting in almost every direction. It is instructive to verify Theorem 18 for the foliation in Example 7. A point undertaking Brownian motion in a pair of pants in $\mathcal{F}$ has three options for a cuff through which to exit. Two of these options have contracting holonomy and one has expanding holonomy. Therefore, with overwhelming probability, holonomy along a leafwise Brownian path is exponentially contracting. In this case, there is only one minimal set $\mathcal{M}_1 = M$. The ergodicity statement reduces to the ergodicity of the associated dynamical system on the S^1–factor transverse to the leaves of M° defined by

$$z \mapsto \begin{cases} \sqrt{z}, & \text{with probability } \tfrac{1}{2}, \\ -\sqrt{z}, & \text{with probability } \tfrac{1}{2}, \end{cases}$$

where S^1 is identified with the unit circle in $\mathbb{C}$.

Remark 20 The nonexistence of an invariant transverse measure is equivalent to an isoperimetric inequality for subsurfaces of leaves, ie the existence of a Cheeger constant $h > 0$ such that $|\partial S|/|S| \geq h$ for any compact subsurface S of the leaves of $\mathcal{F}$ [Goodman and Plante 1979; Calegari 2007, Example 7.6]. Therefore, the theorem above may be regarded as an analogue of the Cheeger inequality for foliations: if $\mathcal{F}$ has a nonzero Cheeger constant, then leafwise random walks converge quickly to a stationary distribution.

Our foliations are C^2, so we can upgrade the contraction result to an infinitesimal version:

Proposition 21 *Let $\mathcal{F}$ be a C^2 foliation of a closed 3–manifold M. Suppose further that $\mathcal{F}$ does not support an invariant transverse measure. Then there exists a real number $\kappa > 0$ such that, for every point $x \in M$ and almost every leafwise Brownian path γ starting at x, there is a constant $C_\gamma > 0$ such that, for every time $t > 0$, the holonomy map $h_{\gamma|_{[0,t]}}$ satisfies*

(16) $$|h'_{\gamma|_{[0,t]}}| \leq C_\gamma \exp(-\kappa t).$$

Moreover, C_γ can be chosen to be a Wiener measurable function of γ.

Remark 22 We don't need to specify a transverse measure for the norm $|\cdot|$ since C_γ can absorb constant factors.

Proof Given an orientation-preserving homeomorphism g between two intervals I_0 and I_1, we define the distortion of g by

$$\text{distortion}(g) = \sup_{a,b,c,d \in I_0} \left(\frac{a-b}{c-d} \right) \left(\frac{g(a)-g(b)}{g(c)-g(d)} \right)^{-1}.$$

The distortion is equal to 1 if and only if g is linear. It is submultiplicative with respect to composition of functions. If g is C^2, then there is a bound on the distortion in terms of g'':

(17) $$\text{distortion}(g) = \frac{\sup_{x \in I_0} g'(x)}{\inf_{x \in I_0} g'(x)}$$

(18) $$\leq 1 + \frac{|I_0| \sup_{x \in I_0} |g''(x)|}{\inf_{x \in I_0} |g'(x)|}.$$

Let ε be small enough that holonomy maps over leafwise paths of length ≤ 1 are defined on all orthogonal transversals of length $\leq \varepsilon$. Let A be an upper bound for the quotient

$$\frac{\sup_{x \in I_0}|g''(x)|}{\inf_{x \in I_0}|g'(x)|}$$

over all holonomy maps g for orthogonal transversals of length $\leq \varepsilon$ along paths of length ≤ 1.

Let us now consider a leafwise Brownian path γ starting at a point $x \in M$. By Theorem 18, it is almost surely possible to choose a short orthogonal transversal I_γ through x such that holonomy of I_γ along γ exists for all time, and moreover that there exists a constant E_γ permitting the inequality

$$(19) \qquad |h_{\gamma|_{[0,t]}}(I_\gamma)| < E_\gamma \cdot \exp(-\kappa t)$$

for all $t > 0$. Shortening I_γ if necessary, we may further assume that

$$(20) \qquad |h_{\gamma|_{[0,t]}}(I_\gamma)| < \varepsilon$$

for all $t > 0$.

Break the holonomy $h_{\gamma|_{[0,t]}}$ into a composition

$$(21) \qquad h_{\gamma|_{[0,t]}} = h_{\gamma|_{[\lfloor t \rfloor, t]}} \circ \cdots \circ h_{\gamma|_{[1,2]}} \circ h_{\gamma|_{[0,1]}}.$$

Let V_i be the Riemannian leafwise distance between $\gamma(i)$ and $\gamma(i+1)$. Since V_i has bounded moments (in particular, bounded second moment), there exists a constant c depending only on the foliation such that $\Pr[V_i > i] \leq c/i^2$. Thus,

$$(22) \qquad \sum_i \Pr[V_i > i] < \sum_i \frac{c}{i^2} < \infty.$$

It follows from the Borel–Cantelli lemma that with probability 1, all but finitely many of the V_i satisfy $V_i < i$. So there exists i_0 depending on γ so that $V_i < i$ for all $i > i_0$.

The following sequence of inequalities holds with probability 1:

$$(23) \qquad \text{distortion}(h_{\gamma|_{[0,t]}}) < \prod_{i=0}^{\lfloor t \rfloor} \text{distortion}(h_{\gamma|_{[i,\min(i+1,t)]}})$$

$$(24) \qquad\qquad < \prod_{i=0}^{\lfloor t \rfloor}(1 + A|h_{\gamma|_{[0,i]}}(I_\gamma)|)^{\lceil V_i \rceil}$$

$$(25) \qquad\qquad < \prod_{i=0}^{\lfloor t \rfloor}(1 + A \cdot E_\gamma \exp(-\kappa i))^{\lceil V_i \rceil}$$

$$(26) \qquad\qquad < \left(\prod_{i=0}^{i_0}(1 + A \cdot E_\gamma \exp(-\kappa i))^{\lceil V_i \rceil}\right)\left(\prod_{i=i_0+1}^{\lfloor t \rfloor}(1 + A \cdot E_\gamma \exp(-\kappa i))^i\right)$$

$$(27) \qquad\qquad < B_\gamma,$$

where B_γ is a constant which can be chosen independent of t. In (23) we invoked submultiplicativity of distortion. In (24), we used the fact that $\gamma|_{[i,i+1]}$ is homotopic rel endpoints to the concatenation of at most $\lceil V_i \rceil$ paths of length at most 1 and invoked (18). We also used (20) to guarantee that the bound involving A applies.

Now we can estimate $h'_{\gamma|_{[0,t]}}$ using our bound on the distortion of $h_{\gamma|_{[0,t]}}$ combined with the macroscopic bound given by Theorem 18:

$$|h'_{\gamma|_{[0,t]}}| < B_\gamma |h_{\gamma|_{[0,t]}}(I_\gamma)| < B_\gamma \cdot E_\gamma \exp(-\kappa t). \tag{28}$$

So we may take the constant in (16) to be $B_\gamma E_\gamma$.

It remains to check that we can make the choice of C_γ a Wiener measurable function of γ. Define the δ neighborhood of γ, denoted by $N_\delta(\gamma)$, to be the set of paths

$$N_\delta(\gamma) = \{\gamma_1 \mid d(\gamma_1(t), \gamma(t)) < \delta \text{ for all } t\}.$$

Set $\delta = \min(1, R/10)$ where R is a lower bound for the injectivity radius of leaves of $\mathcal{F}$. Then, for any $\gamma_1 \in N_\delta(\gamma)$, γ_1 is homotopic rel endpoints to a concatenation $\gamma_2 \circ \gamma$, where γ_2 has length less than 1. For any $t > 0$, if

$$|h'_{\gamma|_{[0,t]}}| \le C_\gamma \exp(-\kappa t),$$

then for any $\gamma_1 \in N_\delta(\gamma)$ we have

$$|h'_{\gamma_1|_{[0,t]}}| \le |h'_{\gamma_2}||h'_{\gamma|_{[0,t]}}| \tag{29}$$

$$\le A B_\gamma E_\gamma \exp(-\kappa t). \tag{30}$$

Therefore, $A B_\gamma E_\gamma$ is a uniform choice for C_{γ_1} that makes (16) work for any $\gamma_1 \in N_\delta(\gamma)$.

Let Γ be a countable set of paths such that the sets $N_\delta(\gamma)$ for $\gamma \in \Gamma$ constitute a cover of the path space. Observe that $N_\delta(\gamma)$ is a Wiener measurable set. Define

$$C_{\gamma_1} = \sup_{\substack{\gamma \in \Gamma \\ \gamma_1 \in N_\delta(\gamma)}} A B_\gamma E_\gamma.$$

This choice of C_{γ_1} satisfies (16) and is Wiener measurable as a countable supremum of Wiener measurable functions. $\qquad \square$

4.2 Logarithmic diffusion

In this subsection, x will be a point in M and λ will be the leaf containing x. Since we assumed $\mathcal{F}$ to be taut and not equal to the foliation of $S^1 \times S^2$ by spheres, the universal cover $\tilde{\lambda}$ is a properly embedded plane in $\tilde{M}$. We will abbreviate $f_{\tau,\lambda,b}$ to f_τ when there is no danger of ambiguity. Recall that W_x is defined to be the distribution of leafwise Brownian paths starting at a point x. Given a time $T > 0$, we define a diffusion operator $\mathcal{D}_T$ acting on $C^{\infty,1}$ transverse measures of full support by

$$\mathcal{D}_T(\tau)|_x = \exp\big(\mathbb{E}_{\gamma \sim W_x}[\log|h'_{\gamma|_{[0,T]}}|\tau]\big)\tau\big|_x.$$

Here, $\mathbb{E}_{\gamma \sim W_x}$ means that we are taking the expectation over paths γ sampled from W_x. Also recall that $|h'_{\gamma|_{[0,T]}}|$ is the infinitesimal holonomy along γ from $\gamma(0)$ to $\gamma(T)$.

In other words, the $\mathcal{D}_T(\tau)$ length of an infinitesimal transverse arc through x is the geometric mean of the τ lengths obtained by holonomy transport along time–T leafwise random walks. For an individual leaf λ, the function $\log f_{\tau, \lambda, b}$ on $\tilde{\lambda}$ evolves according to the standard heat flow. If the geometric mean were to be replaced with an arithmetic mean, we would obtain the leafwise heat flow defined in [Garnett 1983]. Unlike the heat flow operator, our diffusion operator does not conserve mass. Its advantage is that it gives greater weight to the well-behaved smaller holonomies, and therefore allows us to prove a quantitative convergence result.

The main result of this subsection is Proposition 30, which asserts that at some large but finite time T, the Radon–Nikodym derivatives of the diffused transverse measure are exponentials of strictly superharmonic functions. This implies, for example, that the distance to a nearby leaf, as measured by the diffused transverse measure, never has local minima.

As written, it is hard to prove any transverse regularity for the diffused measure. One cannot compare the diffused transverse measure at two nearby points on distinct leaves because holonomy of a transversal connecting these two points typically blows up in finite time along some long paths. We resolve this by introducing a cutoff function $\varphi_{R,S}$. We set

$$\mathcal{D}_{T,R,S}(\tau)|_x = \exp\big(\mathbb{E}_{\gamma \sim W_x}[\log|h'_{\gamma|_{[0,T]}}|_\tau \varphi_{R,S}(x, \gamma(t_{\text{far}}))]\big)\tau\big|_x,$$

where $t_{\text{far}} \in [0, T]$ is the time minimizing $\varphi_{R,S}(x, \gamma(t_{\text{far}}))$ and $\varphi_{R,S}(x, -)$ is a smooth cutoff function defined on $\tilde{\lambda}$ satisfying the following conditions:

- $\varphi_{R,S}(x, y) = 1$ when $d(x, y) < R$.
- $\varphi_{R,S}(x, y) = 0$ when $d(x, y) > SR$.
- φ has all derivatives bounded in absolute value by $10/S$.
- The superlevel sets of $\varphi(x, -)$ on $\tilde{\lambda}$ are topological disks.

The reader may now proceed to the proof of Proposition 30 and refer to the technical lemmas below as needed.

Lemma 23 *If τ is a $C^{\infty,1}$ transverse measure, then $\mathcal{D}_{T,R,S}(\tau)$ is a $C^{\infty,1}$ transverse measure for all T, R, S.*

Proof This is where the cutoff comes in handy. Let x be a point on a leaf λ. In $\tilde{M}$, choose a neighborhood U of the radius SR disk in $\tilde{\lambda}$ centered at x which is skinny enough in the transverse direction that it is foliated as a product. In a neighborhood of x, the values of $\mathcal{D}_{T,R,S}(\tau)$ depend only on information in U. If we were to ignore the dependence of t_{far} on γ, then the smoothness of $\mathcal{D}_{T,R,S}(\tau)$ would follow

from standard results regarding the smoothness of heat kernels with respect to compact variations of the metric. The dependence of t_{far} on γ can indeed be trivialized, as we now detail.

Suppose $x(\varepsilon)$ is a smooth path in U with $x(0) = x$. To show that $\mathcal{D}_{T,R,S}(\tau)$ is once differentiable at x, we need to show that

$$\frac{\partial}{\partial \varepsilon} (\mathcal{D}_{T,R,S}(\tau)(v)\,|_{x(\varepsilon)})$$

exists for any smooth vector field v.

Let $\tilde{\lambda}_\varepsilon \subset U$ be the leaf containing $x(\varepsilon)$. Let $\psi_\varepsilon \colon \mathbb{R}^2 \to \tilde{\lambda}_\varepsilon$ be a smooth family of diffeomorphisms parametrized by ε such that $\psi_\varepsilon(0) = x(\varepsilon)$ and that $\psi_\varepsilon^*(\varphi(x(\varepsilon), -))$ is a standard, rotationally symmetric function on $\mathbb{R}^2$ independent of ε. Call this standard cutoff function $\varphi_0 \colon \mathbb{R}^2 \to \mathbb{R}$. We suppress the parameters R and S in this notation since we can take them as constant in the proof of this lemma.

Let

$$g_\varepsilon = \psi_\varepsilon^*(g),$$
$$f_\tau^\varepsilon = \psi_\varepsilon^*(f_{\tau, \lambda_\varepsilon, x(\varepsilon)}),$$

where g is the Riemannian metric on $\widetilde{M}$. With this definition, f_τ^ε is a $C^{\infty,1}$ function on $\mathbb{R}^2 \oplus \mathbb{R}$, where the $\mathbb{R}$–factor is parametrized by ε. Now we may write

$$(31)\quad \mathcal{D}_{T,R,S}(\tau)(v)\,|_{x(\varepsilon)} = \exp\big(\mathbb{E}_{\gamma \sim W_{x(\varepsilon)}}[\log|h'_{\gamma|_{[0,T]}}|\tau\varphi_{R,S}(x, \gamma(t_{\mathrm{far}}))]\big)\tau(v)\,\big|_{x(\varepsilon)}$$

$$(32)\quad = \exp\big(\mathbb{E}_{\gamma \sim W_{x(\varepsilon)}}[\log f_{\tau, \lambda_\varepsilon, x(\varepsilon)}(\gamma(T))\varphi_{R,S}(x, \gamma(t_{\mathrm{far}}))]\big)\tau(v)\,\big|_{x(\varepsilon)}$$

$$(33)\quad = \exp\bigg(\int_0^1 \mathbb{E}_{\gamma \sim W_{x(\varepsilon)}}[\log f_{\tau, \lambda_\varepsilon, x(\varepsilon)}(\gamma(T)) \mid \varphi_{R,S}(x, \gamma(t_{\mathrm{far}})) > a]\,da\bigg)\tau(v)\,\bigg|_{x(\varepsilon)}$$

$$(34)\quad = \exp\bigg(\int_0^1 \int_{\Omega(a)} k_{\varepsilon, \Omega(a)}(0, x, T) \log f_\tau^\varepsilon(y)\,dx\,da\bigg)\tau(v)\,\bigg|_{x(\varepsilon)},$$

where $\Omega(a)$ is the compact disk $\{x \in \mathbb{R}^2 \mid \varphi_0(x) \geq a\}$ and

$$(35)\qquad k_{\varepsilon, \Omega(a)}(x, y, t)\colon \Omega(a) \times \Omega(a) \times (0, \infty) \to \mathbb{R}$$

is the heat kernel on $\Omega(a)$ for the metric g_ε with zero boundary condition. By Proposition 13, the partial derivative of $k_{\varepsilon, \Omega(a)}(0, x, T)$ with respect to ε exists and is continuous because $\frac{\partial g}{\partial \varepsilon}$ exists and is continuous. Moreover, by another application of Proposition 13, $\frac{\partial}{\partial \varepsilon} k_{\varepsilon, \Omega(a)}(0, x, T)$ varies continuously as we vary a. Thus, the partial derivative of (34) with respect to ε exists and is continuous.

A similar argument for higher partial derivatives shows that $\mathcal{D}_{T,R,S}(\tau)$ has as much regularity as do g_ε and f_τ^ε, which is to say $C^{\infty,1}$. $\qquad\square$

Lemma 24 *For any fixed T, S,*

$$(36)\qquad \lim_{R\to\infty} \mathbb{E}_{\gamma\sim W_x}[\log f_\tau(\gamma(T))\varphi_{R,S}(x, \gamma(t_{\mathrm{far}}))] = \lim_{R\to\infty} \mathbb{E}_{\gamma\sim W_x}[\log f_\tau(\gamma(T))\varphi_{R,S}(x, \gamma(T))]$$

$$(37)\qquad = \mathbb{E}_{\gamma\sim W_x}[\log f_\tau(\gamma(T))]$$

and the convergence for either limit is uniform over $x \in M$ and over S. In other words, tail events from abnormally long paths do not contribute much to the expectation. In particular, the right side exists and is continuous.

Proof Say γ is a tail event if $d(x, \gamma(t_{\text{far}})) > R$. In other words, γ is a tail event if γ ever travels a distance R away from x before time T. The expectations in (36)–(37) differ only on tail events. In order to quantify the contribution of tail events to the expectations, we need to bound both how fast $\log f$ grows on $\tilde{\lambda}$ and how fast Brownian motion can travel on $\tilde{\lambda}$. Since $d \log f_\tau$ is continuous on M, it is bounded above in norm. Therefore, $\log f_\tau$ is L–Lipschitz on $\tilde{\lambda}$ for some L. In particular,

$$(38) \qquad\qquad |\log f_\tau(\gamma(T))| < L\, d(x, \gamma(T)) + C$$

for some constant C.

Let K be a global upper bound for the absolute value of the Gaussian curvature of leaves in M. By the heat kernel estimates from [Cheng et al. 1981] or [Candel and Conlon 2003, Theorem B.7.1], we have

$$(39) \qquad\qquad \Pr[d(x, \gamma(t_{\text{far}})) > r] < C_0 \exp\!\left(-\frac{c_0 r}{T\sqrt{K}}\right)$$

for some absolute constants c_0 and C_0.

Note that

$$(40) \qquad\qquad \Pr[d(x, \gamma(T)) > r] < \Pr[d(x, \gamma(t_{\text{far}})) > r],$$

so $\Pr[d(x, \gamma(T)) > r]$ enjoys the same upper bound as in (39).

Combining (38) and (39), the maximum possible contribution of tail events to any of (36)–(37) is bounded above by

$$(41) \qquad\qquad \int_R^\infty (Lr + C)C_0 \exp\!\left(-\frac{c_0 r}{T\sqrt{K}}\right) dr,$$

where L, C, and K are constants depending only on M, $\mathcal{F}$, and the metric. This integral converges to zero as $R \to \infty$. □

Remark 25 Here is a heuristic explanation for the bound (39). Brownian motion travels at the usual square root speed on length scales below $1/\sqrt{K}$, and in negative curvature travels at linear speed at length scales above $1/\sqrt{K}$. This is because a random walk is exponentially unlikely to backtrack in the presence of negative curvature. A concentration result for the linear speed gives the desired bound.

In what follows, Δ denotes the leafwise Laplace operator.

Lemma 26 *For fixed T,*

$$\lim_{R,S \to \infty} \mathbb{E}_{\gamma \sim W_x}[\Delta(\log f_\tau(-)\varphi(x, -))(\gamma(T))] = \mathbb{E}_{\gamma \sim W_x}[\Delta(\log f_\tau)(\gamma(T))]$$

and the convergence is uniform over M. In particular, the right side exists and is continuous.

Proof This follows from an argument parallel to that in the proof of Lemma 24. We just need to check that $\Delta(\log f_\tau(-)\varphi(x,-))$ grows slowly enough on $\tilde{\lambda}$. We have

$$\Delta(\log f_\tau(-)\varphi(x,-)) = (\Delta \log f_\tau)\varphi(x,-) + \langle d \log f_\tau, d\varphi(x,-)\rangle + \log f_\tau \Delta\varphi(x,-).$$

The terms $\Delta \log f_\tau$ and $d \log f_\tau$ are both bounded because they descend to functions on M. The terms $d\varphi(x,-)$ and $\Delta\varphi(x,-)$ are bounded; in fact, they tend to zero as $S \to \infty$ as arranged in the construction of φ. Finally, $\log f_\tau$ is Lipschitz as noted in the proof of Lemma 24. Therefore, we have a bound of the form

$$\Delta(\log f_\tau(-)\varphi(x,-))(\gamma(T)) < L\, d(x,\gamma(T)) + C$$

for some constants L and C depending only on M, $\mathcal{F}$, and the Riemannian metric. The rest of the proof carries through as in Lemma 24. □

It will be necessary to understand how $\log f_\tau(x)$ behaves as x undergoes Brownian motion. Itô's lemma gives the answer:

Lemma 27 (Itô's lemma) *Suppose x evolves according to Brownian motion on a Riemannian manifold λ with diffusion rate σ. Then for a differentiable real-valued function g on λ, $g(x)$ follows a drift-diffusion process with diffusion rate $\sigma|\nabla g|$ and drift rate $\frac{1}{2}\sigma^2\Delta g$.*

For the reader unfamiliar with the language of Itô calculus, we provide a restatement in our setting for the case $\sigma = 1$:

Lemma 28 (Itô's lemma, reformulated) *If g is a real-valued C^2 function on a Riemannian manifold λ and $x \in \lambda$, then, for any $y \in \lambda$ and $t_0 > 0$, we have*

$$\lim_{t \to t_0^+} \frac{\mathrm{Var}_{\gamma \sim W_x}[g(\gamma(t)) \mid \gamma(t_0) = y]}{t - t_0} = |\nabla g(y)|,$$

$$\frac{d}{dt}\, \mathbb{E}_{\gamma \sim W_x}[g(\gamma(t)) \mid \gamma(t_0) = y]\Big|_{t=t_0} = \frac{1}{2}\Delta g(y).$$

For more discussion of Itô's lemma and drift-diffusion process, we point the reader to [Mörters and Peres 2010, Chapter 7].

Lemma 29 *There exists $\kappa > 0$ such that for each minimal set $\mathcal{M}_i$,*

(42)
$$\int_{\mathcal{M}_i} \Delta \log f_\tau \, d\mu_i < -\kappa,$$

where μ_i is the probability measure on $\mathcal{M}_i$.

Proof The basic idea is that, by Itô's lemma, the integral on the left side of (42) dictates the average drift rate of $\log f_\tau$ for long Brownian paths in $\mathcal{M}_i$. This drift rate should be negative in accordance with Theorem 18. We must take a bit more care because, as written, Theorem 18 doesn't give bounds on the expectation of C_γ.

Let κ and C_γ be the constants from [Proposition 21](). Let $g = \Delta \log f_\tau$. Choose $x \in \mathcal{M}_i$ and let $\gamma \sim W_x$, where W_x is the distribution of leafwise Brownian paths starting at x. Choose C large enough that $\Pr_{\gamma \sim W_x}[C_\gamma < C] > 0$. Let A be the event that $C_\gamma < C$. Let A_t be the event that, for all $s \in [0, t]$,

$$(43) \qquad\qquad f_\tau(\gamma(s)) < C \exp(-\kappa s).$$

By definition, $A = \bigcap_t A_t$. Since $\Pr[A] > 0$, the conditional probabilities satisfy

$$\lim_{\substack{i,j \to \infty \\ i > j}} \Pr[A_i \mid A_j] = 1.$$

For any $\varepsilon > 0$, [Theorem 18]() lets us choose a time dt large enough that the time–dt diffused function, $D^{dt} g$, satisfies

$$\left| D^{dt} g - \int_{\mathcal{M}_i} g \, d\mu_i \right| < \varepsilon$$

at every point in $\mathcal{M}_i$. For any time t, set $t' := \lfloor t - dt \rfloor$. The essential properties of t' are that it is (locally) independent of t, it is a bounded time in the past, and yet is far enough in the past that Brownian motion has had some time to diffuse.

Now taking the logarithm of the defining constraint [(43)](), we have

$$(44) \quad -\kappa t + \log C > \mathbb{E}[\log f_\tau(\gamma(t)) \mid A_t]$$

$$(45) \qquad\qquad = \frac{1}{\Pr[A_t \mid A_{t'}]} \big(\mathbb{E}[\log f_\tau(\gamma(t)) \mid A_{t'}] - \mathbb{E}[\log f_\tau(\gamma(t)) \mid \neg A_t \cap A_{t'}] \Pr[\neg A_t \mid A_{t'}] \big)$$

$$(46) \qquad\qquad > \frac{1}{\Pr[A_t \mid A_{t'}]} \big(\mathbb{E}[\log f_\tau(\gamma(t)) \mid A_{t'}] - \mathbb{E}[\log f_\tau(\gamma(t)) \\ - \log f_\tau(\gamma(t')) + \kappa t' - \log C \mid \neg A_t \cap A_{t'}] \cdot \Pr[\neg A_t \mid A_{t'}] \big)$$

$$(47) \qquad\qquad = \frac{\mathbb{E}[\log f_\tau(\gamma(t)) \mid A_{t'}]}{\Pr[A_t \mid A_{t'}]} + o(t).$$

Since $\lim_{t \to \infty} \Pr[A_t \mid A_{t'}] \to 1$, we conclude that

$$(48) \qquad\qquad \mathbb{E}[\log f_\tau(\gamma(t)) \mid A_{t'}] < -\kappa t + o(t).$$

Equation [(47)]() requires some justification. By [Lemma 24](), conditioned on any value of $\gamma(t')$, the distribution of

$$\log f_\tau(\gamma(t)) - \log f_\tau(\gamma(t'))$$

has a finite expectation. Moreover, [Lemma 24]() gives a uniform bound on the tails of this distribution depending only on $t - t'$. Since $\Pr[\neg A_t \mid A_{t'}] \to 0$, the term dropped in [(47)]() is negligible as claimed.

On the other hand, we can compute the growth rate of the left side of [(48)]() using Itô's lemma:

$$\frac{d \, \mathbb{E}[\log f_\tau(t) \mid A_{t'}]}{dt} = \mathbb{E}[g(\gamma(t)) \mid A_{t'}] = \mathbb{E}[D^{t-t'} g(\gamma(t')) \mid A_{t'}] > \int_{\mathcal{M}_i} g \, d\mu_i - \varepsilon.$$

Here, it was critical for the application of Itô's lemma that $A_{t'}$ is a piecewise constant function of t so that $\gamma(t) \mid A_{t'}$ is a diffusion process. Comparing with (48), we conclude

$$\int_{\mathcal{M}_i} g \, d\mu_i < -\kappa + \varepsilon.$$

Taking $\varepsilon \to 0$ finishes the proof of the lemma. $\qquad\square$

Now we are ready to prove the main result of this section.

Proposition 30 *If T, R and S are chosen sufficiently large, then $\mathcal{D}_{T,R,S}(\tau)$ is log superharmonic. That is, on each leaf $\tilde{\lambda}$ of $\tilde{\mathcal{F}}$, the function $\log f_{\mathcal{D}_{T,R,S}(\tau),\lambda}$ is strictly superharmonic.*

Proof The function $\Delta \log(f_\tau)$ is continuous on M. By Theorem 18, as $T \to \infty$,

$$\mathbb{E}_{\gamma \sim W_x}[\Delta \log(f_\tau)(\gamma(T))]$$

uniformly tends to a linear combination

$$\sum_i p_i \int_{\mathcal{M}_i} \Delta \log f_\tau \, d\mu_i,$$

where μ_i is the probability measure on the ith minimal set, $\mathcal{M}_i$, and p_i are continuous, leafwise harmonic functions on M. Choose κ satisfying (42) as guaranteed by Lemma 29. For large enough T, we can guarantee that, for any point $x \in M$,

$$(49) \qquad \mathbb{E}_{\gamma \sim W_x}[\Delta \log(f_\tau)(\gamma(T))] < -\frac{\kappa}{2}.$$

Now we are ready to estimate $\Delta \log f_{\mathcal{D}_{T,R,S}(\tau)}$:

$$(50) \quad \lim_{R,S\to\infty} \Delta \log f_{\mathcal{D}_{T,R,S}(\tau)} = \lim_{R,S\to\infty} \Delta \, \mathbb{E}_{\gamma \sim W_x}[\log(f_\tau(\gamma(T))\varphi_{R,S}(x,\gamma(t_{\text{far}})))]$$

$$(51) \qquad\qquad\qquad = \lim_{R,S\to\infty} \Delta \, \mathbb{E}_{\gamma \sim W_x}[\log(f_\tau(\gamma(T))\varphi_{R,S}(x,\gamma(T)))] \quad \text{(by Lemma 24)}$$

$$(52) \qquad\qquad\qquad = \lim_{R,S\to\infty} \mathbb{E}_{\gamma \sim W_x}[\Delta(\log(f_\tau(-))\varphi_{R,S}(x,-))(\gamma(T))] \quad \text{(by (5))}$$

$$(53) \qquad\qquad\qquad = \mathbb{E}_{\gamma \sim W_x}[\Delta(\log f_\tau)(\gamma(T))] \quad \text{(by Lemma 26)}$$

$$(54) \qquad\qquad\qquad < -\frac{\kappa}{2} \quad \text{(by (49))}.$$

In (52), the use of (5) to commute Δ with leafwise diffusion was valid because f_τ and φ are leafwise C^∞ functions; see also Remark 10. Lemmas 24 and 26 further say that the limits above converge uniformly over $x \in M$. Therefore, for large enough R and S, $\log f_{\mathcal{D}_{T,R,S}(\tau)}$ is a strictly superharmonic function on each leaf $\tilde{\lambda}$ of $\tilde{\mathcal{F}}$. $\qquad\square$

Proof of Theorem 1 Suppose that $\mathcal{F}$ supports no invariant transverse measure. Then by Proposition 30, we can find a log superharmonic transverse measure τ. Let β be the section of $T^*\mathcal{F}$ defined by $\beta = \star_2 d \log f$, where $\star_2$ is the Hodge star operator on a leaf. The fact that $\log f$ is strictly superharmonic

can be written as

$$\star_2 d \star_2 d \log f > 0.$$

Therefore $d\beta > 0$ on $T\mathcal{F}$. Furthermore, $\beta \geq 0$ on $\ker(d \log f_\tau)$.

Choose a vector field v transverse to $\mathcal{F}$ with $\tau(v) = 1$, and extend β to a 1–form on M by choosing $\beta(v) = 0$. Let σ be the unit norm 2–vector tangent to $T\mathcal{F}$. Now consider the 1–form $\alpha = \tau + \varepsilon\beta$:

$$\alpha \wedge d\alpha = (\tau + \varepsilon\beta) \wedge d(\tau + \varepsilon\beta)$$
$$= \varepsilon(\tau \wedge d\beta + \beta \wedge d\tau) + O(\varepsilon^2).$$

Here, $\tau \wedge d\tau$ vanishes because τ defines a foliation. Evaluating both sides on $v \otimes \sigma$, we get

$$(\alpha \wedge d\alpha)(v \otimes \sigma) = \varepsilon\tau(v)\, d\beta(\sigma) - \varepsilon(\beta \wedge (\iota_v d\tau))(\sigma) + O(\varepsilon^2)$$
$$= \varepsilon\tau(v)\, d\beta(\sigma) + \varepsilon(\beta \wedge d \log f_\tau)(\sigma) + O(\varepsilon^2).$$

In the last line, we made use of [Proposition 11](). By construction, $d\beta(\sigma) > 0$ and $\beta \wedge d \log f_\tau \geq 0$. Therefore, for small enough ε, the right side is positive everywhere in M. So α is a contact perturbation of $\mathcal{F}$.

Moreover, we have

$$d\alpha(\sigma) = \varepsilon d\beta(\sigma) > 0.$$

So the Reeb flow of α is transverse to $\mathcal{F}$. Note that α is only C^1; if preferred, we can approximate α with a C^∞ 1–form having the same properties. □

Remark 31 We have actually shown that with the hypotheses of [Theorem 1](), there are 1–forms τ, β such that $\ker \tau = T\mathcal{F}$ and $\tau + \varepsilon\beta$ is a contact structure for all sufficiently small ε. This is a linear perturbation in the sense of [Eliashberg and Thurston 1998, Chapter 2]. This gives another proof of [loc. cit., Theorem 2.1.2].

5 Obstruction 2–currents and Farkas' lemma

In the proof of [Theorem 1](), we saw that the desired perturbing 1–form $\beta \in \Gamma(T^*\mathcal{F})$ need only satisfy $d\beta > 0$ on $T\mathcal{F}$ and $\beta \geq 0$ on the level sets of f_τ. When f_τ is log superharmonic, the 1–form $\beta = \star_2 d \log f$ immediately satisfies these conditions. In this section, we give a more flexible criterion for the existence of such a β which depends only on the topology of the level sets of f_τ. It has the advantage that, for transverse measures one sees in the wild, one can often directly verify the condition and avoid using logarithmic diffusion. The main observation leading to the criterion is that the constraints on β are linear, and so can be analyzed via linear programming duality. This idea goes back at least to [Sullivan 1976] where several similar alternatives are proven: either a solution to a system of linear inequalities on i–forms exists, or there is an i–current furnishing an obstruction. See for example Sullivan's Theorems II.1 and II.2.

In this section, it will be more convenient to work with reflexive Banach spaces. In [Section 3.8](), we introduced the i–forms of Sobolev adjusted regularity (k, l). In this section, we fix some large $K > 2$.

Our i–forms will be those of Sobolev adjusted regularity $(K, 2)$, and our currents will be elements of the corresponding dual spaces.

Let τ be an arbitrary nonvanishing $C^{\infty,1}$ transverse measure. Such a form has adjusted regularity $(\infty, 2)$; see Section 3.8. Orient the level sets of f_τ so as to agree with the orientation induced as the boundary of a superlevel set. An *obstruction 2–current for τ* is a 2–current in M which is carried by and cooriented with $\mathcal{F}$, and whose boundary consists of level sets for f_τ with the negative orientation. For example, a sublevel set near a local minimum for f_τ is an obstruction 2–current. While an arbitrary sublevel set of f_τ in a leaf $\tilde{\lambda} \subset \tilde{\mathcal{F}}$ might appear to be an obstruction 2–current, it typically does not project to a 2–current in M with finite mass.

Proposition 32 *Either there exists a section β of $T^*\mathcal{F}$ with $d\beta > 0$ on $T\mathcal{F}$ and $\beta \geq 0$ on level sets of f_τ, or there exists an obstruction 2–current for τ.*

Before giving the proof of Proposition 32, we will recall some background from convex optimization. A closed, convex cone in a Banach space is *proper* if it does not contain a line through the origin. We will need a hybrid version of Farkas' lemma with allowances for some strict and some nonstrict inequalities. We include a proof because we couldn't find the form we require in the literature.

Lemma 33 (Farkas' lemma) *Let X, Y_1, Y_2 be Banach spaces with X reflexive and separable. Let $B: Y_1 \oplus Y_2 \to X^*$ be a continuous linear map. Let $\langle \cdot, \cdot \rangle: (Y_1 \oplus Y_2) \otimes X \to \mathbb{R}$ denote the induced pairing. Let C_i be a closed, convex cone in Y_i. Suppose further that C_2 is proper and that the images $B(C_1)$ and $B(C_2)$ are closed. Then exactly one of the following alternatives holds:*

(1) *There exists $x \in X$ satisfying $\langle x, C_1 \rangle \geq 0$ and $\langle x, C_2 \setminus \{0\} \rangle > 0$.*

(2) *There exists $y_1 \in C_1$ and $y_2 \in C_2 \setminus \{0\}$ satisfying $\langle X, (y_1, y_2) \rangle = 0$.*

Proof It is clear that the alternatives are mutually exclusive, so we need only prove that at least one of the alternatives holds. We will use the shorthand $B(C) = B(C_1) + B(C_2)$.

Suppose that $B(C) = X^*$. Choose any nonzero $(a_1, a_2) \in (C_1, C_2)$. Then choose $(b_1, b_2) \in (C_1, C_2)$ satisfying $B((b_1, b_2)) = -B((a_1, a_2))$. The element $(y_1, y_2) = (a_1 + b_1, a_2 + b_2)$ satisfies alternative (2). The fact that $y_2 \neq 0$ follows from properness of C_2.

Suppose instead that $B(C)$ does not contain some point $v \in X^*$. By the Hahn–Banach hyperplane separation theorem, there exists a linear functional $x \in X^{**}$ separating v from $B(C)$, in the sense that $x(v) < 0$ and $x(B^*(C)) \geq 0$. Since X is reflexive, x may be realized as a point in X.

Let H_x be the hyperplane in X^* defined by x. Let $A = B(C) \cap -B(C)$. Points in A are "corners" of $B(C)$. Observe that $A \subset H_x$. We will now show that it can be further arranged that $H_x \cap B(C) = A$.

If $w \in (H_x \cap B(C)) \setminus A$, then another application of the Hahn–Banach theorem gives a linear functional $x' \in X^{**}$ separating $-w$ from $B(C)$. Now $H_{x+x'} \cap B(C)$ is strictly contained in $H_x \cap B(C)$, and does not

include w. In effect, we have tilted H_x away from w. So we have a strategy for removing a single unwanted point from $H_x \cap B(C)$. We will now use Zorn's lemma to show that we can remove all the unwanted points.

Let $\mathcal{S} = \{H_x \cap B(C)\}$, where x varies over linear functionals supporting $B(C)$. The elements of $\mathcal{S}$ are all closed subsets of $B(C)$. The family $\mathcal{S}$ is partially ordered by inclusion. Since X is separable, any chain in $\mathcal{S}$ may be refined to a countable chain. Given a countable chain $\{H_{x_i} \cap B(C)\}_{i \in \mathbb{N}}$, let

$$\bar{x} = \sum_i \frac{1}{2^i \|x_i\|} x_i.$$

With this choice,

$$H_{\bar{x}} \cap B(C) = \bigcap_{i \in \mathbb{N}} H_{x_i} \cap B(C).$$

Therefore, every chain in $\mathcal{S}$ has a lower bound in $\mathcal{S}$. By Zorn's lemma, $\mathcal{S}$ has a minimal element. By the discussion in the previous paragraph, such a minimal element must be equal to A.

Now we have a nonzero candidate $x \in X$ solving the nonstrict versions of the inequalities in alternative (1). This choice of x satisfies $\langle x, (y_1, y_2) \rangle = 0$ if and only if $B((y_1, y_2)) \in A$. If there exists $a_2 \in C_2 \setminus \{0\}$ with $B((0, a_2)) \in A$, then we may find $(b_1, b_2) \in (C_1, C_2)$ satisfying $B((b_1, b_2)) = -B((0, a_2))$. As before, properness of C_2 guarantees that $a_2 + b_2 \neq 0$. Thus, the element $(b_1, a_2 + b_2)$ satisfies alternative (2). Otherwise, alternative (1) holds. $\qquad\square$

Proof of Proposition 32 Let X be the Sobolev space of sections of $T^*\mathcal{F}$ with Sobolev adjusted regularity $(K, 2)$. Since X is a separable Hilbert space, Farkas' lemma will apply. For $i \in \{1, 2\}$, let Y_i be the space of i–currents carried by $\mathcal{F}$. Let $C_1 \subset Y_1$ be the closed cone generated by 1–currents contained in and cooriented with the level sets of f_τ. Let Y_2 be the space of 2–currents tangent to $\mathcal{F}$ and let $C_2 \subset Y_2$ be the closed cone generated by 2–currents cooriented with $\mathcal{F}$.

Although C_1 is not proper, C_2 is proper. The lack of properness for C_1 occurs near critical points of f_τ. At such points, there is a tangent vector v such that both $-v$ and v are limits of oriented subarcs of the level sets of τ. Therefore, v and $-v$ are both elements of C_1. On the other hand, if $y \in C_2 \setminus \{0\}$, then $-y$ is a 2–current negatively tangent to $\mathcal{F}$ and does not lie in C_2.

Define the pairing $\langle \cdot, \cdot \rangle \colon X \otimes (Y_2 \oplus Y_2) \to \mathbb{R}$ by

$$\langle \beta, (y_1, y_2) \rangle = \int_{y_2} d\beta + \int_{y_1} \beta.$$

By Stokes' theorem, an element $(y_1, y_2) \in Y_1 \oplus Y_2$ satisfies $\langle X, (y_1, y_2) \rangle = 0$ if and only if $y_1 = -\partial y_2$. So by Farkas' lemma, either there exists a section of $T^*\mathcal{F}$ satisfying the desired properties, or there exists a nonzero 2–current y_2 positively tangent to $\mathcal{F}$ such that ∂y_2 is negatively tangent to the level sets of f_τ. This is exactly an obstruction 2–current. $\qquad\square$

Proposition 34 *If $\log f_\tau$ is a strictly superharmonic function on each leaf, then there are no obstruction 2–currents for τ.*

Proof This is a generalization of the fact that superharmonic functions have no local minima. Suppose that S is an obstruction 2–current for τ. It is helpful to keep in mind the simplest case of a compact surface S in a leaf λ with boundary a negatively oriented union of level sets of $f_{\tau,\lambda}$.

Strict superharmonicity of $\log f_\tau$ is equivalent to

$$\star_2 d \star_2 d \log(f_\tau) < 0,$$

where $\star_2$ is the Hodge star operator on a leaf. So by Stokes' theorem,

$$0 > \int_S d \star_2 d \log f_\tau = \int_{\partial S} \star_2 d \log f_\tau \geq 0.$$

The last inequality uses the property of obstruction 2–currents that ∂S is a subcurrent of the level sets of f_τ, which are in turn equal to the level sets of $\log f_\tau$. This is a contradiction. $\square$

Example 35 (Example 7 revisited) Adopt the notation from Example 7. Let T be the torus in M along which one can cut to obtain M°. Let $\tau = f\, d\theta$. Let us show that τ has no obstruction 2–current. Call the purported obstruction 2–current Σ. It is possible to modify Σ so that its boundary lies on T. Then Σ is a positive combination of horizontal pairs of pants. A pair of pants has two components which are positively oriented level sets of f and one that is a negatively oriented level set of f. Therefore, the positively oriented boundary of Σ has twice the length of the negatively oriented boundary. Even with cancellation, the positive boundary cannot be empty. Therefore Σ cannot be an obstruction 2–current. By Proposition 32, $f\, d\theta$ may be perturbed to a contact structure with Reeb flow transverse to $\mathcal{F}$.

6 Questions

• Can the results of the present paper be extended to $C^{\infty,1}$ or even to $C^{\infty,0}$ foliations? Many powerful constructions of foliations proceed by iteratively splitting a branched surface [Li 2002]. The resulting foliation is typically only $C^{\infty,0}$. Kazez and Roberts [2017] and Bowden [2016a] independently showed that the Eliashberg–Thurston theorem does extend to $C^{\infty,0}$ foliations. The difficulty in extending our approach is that the direction of expanding holonomy is no longer well-defined for $C^{\infty,0}$ foliations. One possible line of attack would be to use [Ishii et al. 2023], which shows that branched surfaces satisfying a certain handedness condition admit transverse Reeb flows.

• What can be said in higher dimensions? One would like to generalize the Eliashberg–Thurston theorem to codimension–1 leafwise symplectic foliations in arbitrary dimension. Proposition 30 and its dependencies all work in arbitrary dimension, giving a log superharmonic transverse measure in the absence of an invariant transverse measure. When can it be upgraded to a log plurisuperharmonic transverse measure?

• Is a Reeb flow R transverse to a foliation $\mathcal{F}$ product-covered, ie conjugate to the standard flow $\frac{\partial}{\partial z}$ on $\widetilde{M} \cong \mathbb{R}^3$? This is equivalent to the statement that $\widetilde{M}/R$ is topologically an open disk. Since R is transverse to a taut foliation, $\widetilde{M}/R$ is a (possibly non-Hausdorff) 2–manifold. The fact that the flow

of R preserves contact planes prohibits certain types of non-Hausdorff behavior in $\widetilde{M}/R$. In particular, a smooth 1–parameter family of flow lines cannot break into two different families. However, as pointed out to the author by Fenley, there could conceivably still be a sequence of flow lines with more than one limiting flow line.

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