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Holomorphic anomaly equations for the Hilbert scheme of points of a K3 surface

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We conjecture that the generating series of Gromov–Witten invariants of the Hilbert schemes of n points on a K3 surface are quasi-Jacobi forms and satisfy a holomorphic anomaly equation. We prove the conjecture in genus 0 and for at most three markings — for all Hilbert schemes and for arbitrary curve classes. In particular, for fixed n, the reduced quantum cohomologies of all hyperkähler varieties of $\mathrm{K3}^{[n]}$–type are determined up to finitely many coefficients.

As an application we show that the generating series of 2–point Gromov–Witten classes are vector-valued Jacobi forms of weight -10, and that the fiberwise Donaldson–Thomas partition functions of an order-2 CHL Calabi–Yau threefold are Jacobi forms.

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1 Introduction

1.1 Overview

An irreducible hyperkähler variety is a simply connected smooth projective variety X such that $H^0(X, \Omega_X^2)$ is generated by a holomorphic–symplectic form [Beauville 1983]. A topological classification of these varieties is unknown so far. However, among the four families of examples which are known, the most studied case is the Hilbert schemes of points on a K3 surface and their deformations, which are called $\mathrm{K3}^{[n]}$–type. We study the Gromov–Witten theory (the intersection theory of the moduli space of stable maps) with target a hyperkähler variety of $\mathrm{K3}^{[n]}$–type. We state two new fundamental conjectures: finite generation by quasi-Jacobi forms, and holomorphic anomaly equations. We prove this conjecture for the case of most interest: in genus 0 and for up to three markings, with no restriction on the curve class. As a corollary, the reduced quantum cohomology of a $\mathrm{K3}^{[n]}$–hyperkähler variety is determined up to finitely many coefficients. We also find that the series of 2–point Gromov–Witten classes define vector-valued Jacobi forms of weight -10. This implies that the Donaldson–Thomas partition functions of CHL Calabi–Yau threefolds are Jacobi forms, proving conjectures of Bryan and Oberdieck [2020].

Together with the multiple cover conjecture [Oberdieck 2022; 2024b] we obtain a complete conjectural picture of the Gromov–Witten theory of $\mathrm{K3}^{[n]}$–hyperkähler varieties, which is proven for genus 0 and up to three markings.

1.2 Gromov–Witten theory

Let $S^{[n]}$ be the Hilbert scheme of n points on a smooth projective K3 surface S. Let

$$\overline{M}_{g,N}(S^{[n]}, \beta + rA)$$

be the moduli space of N–marked genus-g stable maps to $S^{[n]}$ of nonzero degree

$$\beta + rA \in H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z}A,$$

where A is the exceptional curve class. Because $S^{[n]}$ is irreducible hyperkähler, the virtual fundamental class of the moduli space of stable maps in the sense of [Li and Tian 1998; Behrend and Fantechi 1997] vanishes. Instead Gromov–Witten theory is defined by the reduced virtual class [Maulik and Pandharipande 2013; Bryan and Leung 2000; Kool and Thomas 2014; Kiem and Li 2013]:

$$[\overline{M}_{g,N}(S^{[n]}, \beta + rA)]^{\mathrm{vir}} \in A_{\mathrm{vd}}(\overline{M}_{g,N}(S^{[n]}, \beta + rA)), \quad \text{where } \mathrm{vd} = 2n(1-g) + N + 1.$$

The first values of the virtual dimension vd are listed in Table 1.

If $2g - 2 + N > 0$, let $\tau\colon \overline{M}_{g,N}(S^{[n]}, \beta + rA) \to \overline{M}_{g,N}$ be the forgetful morphism to the moduli space of stable curves. Consider the pullback of a *tautological class* [Faber and Pandharipande 2005]

$$\mathrm{taut} := \tau^*(\alpha) \quad \text{for } \alpha \in R^*(\overline{M}_{g,N}).$$

In the unstable cases $2g - 2 + N \leq 0$ we always set $\mathrm{taut} := 1$. Given cohomology classes $\gamma_i \in H^*(S^{[n]})$, the reduced Gromov–Witten invariants of $S^{[n]}$ are defined by

$$\langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,\beta+rA} = \int_{[\overline{M}_{g,n}(S^{[n]}, \beta+rA)]^{\mathrm{vir}}} \mathrm{taut} \cup \prod_{i=1}^{N} \mathrm{ev}_i^*(\gamma_i).$$

genus g	0	1	2	3	4	5	6
$S^{[1]}$	0	1	2	3	4	5	6
$S^{[2]}$	2	1	0				
$S^{[3]}$	4	1					
$S^{[4]}$	6	1					
$S^{[5]}$	8	1					
$S^{[6]}$	10	1					

Table 1: The first nonnegative values of the (reduced) virtual dimension of $\overline{M}_{g,0}(S^{[n]}, \beta + rA)$. If a field is empty, all Gromov–Witten invariants in this genus vanish. Hence for $S^{[n]}$ with $n > 1$ the most interesting case is genus 0.

1.3 Generating series

Consider an elliptic K3 surface $\pi\colon S \to \mathbb{P}^1$ with a section, and let

$$B, F \in H_2(S, \mathbb{Z})$$

be the class of the section and a fiber of π, respectively.

By [Oberdieck 2022, Corollary 2] (based on the global Torelli theorem [Verbitsky 2013; Huybrechts 2012]), for any hyperkähler variety of $\mathrm{K3}^{[n]}$–type X and for any effective curve class $\gamma \in H_2(X, \mathbb{Z})$, there exists an $l \geq 1$ and a deformation

$$(X, \gamma) \rightsquigarrow (S^{[n]}, lB + dF + rA) \quad \text{for } d \geq 0 \text{ and } r \in \mathbb{Z}$$

such that γ is kept of Hodge type along the deformation. If γ is primitive, we can choose $l = 1$. By deformation invariance, it follows that all Gromov–Witten invariants of $\mathrm{K3}^{[n]}$–type hyperkähler varieties are determined by the generating series:

$$F_{g,l}^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d=-l}^{\infty} \sum_{r \in \mathbb{Z}} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle_{g, l(B+F)+dF+rA}^{S^{[n]}} q^d (-p)^r. \tag{1}$$

By convention we assume here that $r = 0$ when $n = 1$. We will always assume that $n \geq 1$.

The series (1) and in particular its modular properties are our main topic. To state our main conjectures and results we require the Looijenga–Lunts–Verbitsky (LLV) Lie algebra and quasi-Jacobi forms.

1.4 Looijenga–Lunts–Verbitsky Lie algebra

The LLV algebra [Looijenga and Lunts 1997; Verbitsky 1996] of the hyperkähler variety $S^{[n]}$ is the Lie subalgebra

$$\mathrm{act}\colon \mathfrak{g}(S^{[n]}) \hookrightarrow \mathrm{End}\, H^*(S^{[n]})$$

generated by the operators of cup product with classes in $H^2(S^{[n]},\mathbb{Q})$ as well as their Lefschetz duals (if they exist); see Section 3.4. Concretely, we have an isomorphism

$$\mathfrak{g}(S^{[n]}) \cong \bigwedge\nolimits^2 (V \oplus U_{\mathbb{Q}}),$$

where $U_{\mathbb{Q}}$ is the hyperbolic lattice with basis $\{e, f\}$ and intersection form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and

$$V = H^2(S^{[n]},\mathbb{Q}) \cong H^2(S,\mathbb{Q}) \oplus^{\perp} \mathbb{Q}\delta, \quad (\delta,\delta) = 2-2n,$$

is endowed with the Beauville–Bogomolov–Fujiki quadratic form.

We require the operators

(2) $\quad U = \mathrm{act}(F \wedge f), \quad T_\alpha = \mathrm{act}(\alpha \wedge F), \quad \alpha \in \{B, F\}^{\perp} \subset V \quad \text{and} \quad \mathrm{Wt} = \mathrm{act}(e \wedge f + B \wedge F).$

The *weight operator* $\mathrm{Wt} \in \mathrm{End}\, H^*(S^{[n]})$ is semisimple and defines a grading:

$$\mathrm{Wt}(\gamma) = \mathrm{wt}(\gamma)\gamma \quad \text{for} \ \ \mathrm{wt}(\gamma) \in \{-n, \dots, n\}.$$

For a class $\gamma \in H^{2i}(S^{[n]})$, the complex cohomological degree of γ is denoted by $\deg(\gamma) = i$.

1.5 Quasi-Jacobi forms

Jacobi forms are holomorphic functions $f : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ which satisfy a transformation law under the Jacobi group $\Gamma \ltimes \mathbb{Z}^2$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup [Eichler and Zagier 1985]. Quasi-Jacobi forms are constant terms of almost-holomorphic Jacobi forms; see Section 2. The algebra of quasi-Jacobi forms is bigraded by weight k and index m:

$$\mathrm{QJac}(\Gamma) = \bigoplus_{m \geq 0} \bigoplus_{k \in \mathbb{Z}} \mathrm{QJac}(\Gamma)_{k,m}.$$

The graded summands $\mathrm{QJac}(\Gamma)_{k,m}$ are finite-dimensional. We usually identify a quasi-Jacobi form with its Fourier expansion in the variables

$$p = e^{2\pi i x} \quad \text{and} \quad q = e^{2\pi i \tau} \quad \text{for} \ \ (x,\tau) \in \mathbb{C} \times \mathbb{H}.$$

Recall that the algebra of quasimodular forms $\mathrm{QMod}(\Gamma)$ is a free polynomial ring over the subalgebra of its modular forms,

$$\mathrm{QMod}(\Gamma) = \mathrm{Mod}(\Gamma)[G_2],$$

where we used the second Eisenstein series

$$G_2(\tau) = -\tfrac{1}{24} + \sum_{n \geq 1} \sum_{d \mid n} d q^n.$$

Similarly, for quasi-Jacobi forms we always have an embedding

$$\mathrm{QJac}(\Gamma) \subset \mathrm{Jac}(\Gamma)[G_2, \mathsf{A}],$$

where $\mathrm{Jac}(\Gamma)$ is the algebra of weak Jacobi forms and A is the logarithmic derivative

$$\mathsf{A}(p,q) = p \frac{d}{dp} \log \Theta(p,q)$$

of the classical Jacobi theta function

$$\Theta(p,q) = (p^{1/2} - p^{-1/2}) \prod_{m \geq 1} \frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}.$$

Since the generators G_2 and A are free over Jac, one obtains *anomaly operators*

$$\frac{d}{dG_2}\colon \mathrm{QJac}(\Gamma)_{k,m} \to \mathrm{QJac}(\Gamma)_{k-2,m}, \quad \frac{d}{d\mathsf{A}}\colon \mathrm{QJac}(\Gamma)_{k,m} \to \mathrm{QJac}(\Gamma)_{k-1,m}$$

which control the transformation behavior of any quasi-Jacobi form under the Jacobi group.

1.6 Main conjectures

We state three fundamental conjectural properties of the series $F_{g,l}$. The first expresses $F_{g,l}$ in terms of the series of primitive invariants $F_{g,1}$. Consider the l^{th} formal Hecke operator of weight k, which acts on power series $f = \sum_{d,r} c(d,r) q^d p^r$ by

$$T_{k,l} f = \sum_{n,r} \Bigg(\sum_{a \mid (l,n,r)} a^{k-1} c\left(\frac{ln}{a^2}, \frac{r}{a}\right)\Bigg) q^n p^r.$$

For $i \in \{1, \dots, N\}$ let $\gamma_i \in H^*(S^{[n]})$ be (wt, deg)–bihomogeneous classes.

Conjecture A (multiple cover conjecture, [Oberdieck 2022, Section 2.6]) *For all $l > 0$ we have*

$$F_{g,l}^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = l^{\sum_i (\deg(\gamma_i) - n - \mathrm{wt}(\gamma_i))} T_{k,l} F_{g,1}^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N), \tag{3}$$

where $k = n(2g - 2 + N) + \sum_i \mathrm{wt}(\gamma_i)$.

The second conjecture concerns the modular behavior. Define the modular discriminant

$$\Delta(q) = q \prod_{n \geq 1} (1-q^n)^{24},$$

which is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight 12, and the congruence subgroup

$$\Gamma_0(l) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod l \right\}.$$

Conjecture B (quasi-Jacobi form property) *For all $l > 0$*

$$F_{g,l}^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(q)^l} \mathrm{QJac}_{k+12l,\, l(n-1)}(\Gamma_0(l)),$$

where $k = n(2g - 2 + N) + \sum_i \mathrm{wt}(\gamma_i) - 10$.

The difference in the values of k in Conjectures A and B was explained in [Oberdieck and Pixton 2019, Section 7.3]. It is responsible for the appearance of the congruence subgroup $\Gamma_0(l)$, and also leads to the unusual fourth term in the holomorphic anomaly equation for d/dG_2 below.

Conjecture B would determine any $F_{g,l}(\cdots)$ up to finitely many coefficients. However, in order to know their transformation property under the Jacobi group and also to make them depend on substantially fewer coefficients, we will conjecture their dependence on the quasi-Jacobi generators G_2 and A:

For $2g-2+N>0$ define the degree 0 Gromov–Witten invariants

$$F_g^{S^{[n]},\mathrm{std}}(\mathrm{taut};\gamma_1,\dots,\gamma_N) := \int_{[\overline{M}_{g,N}(S^{[n]},0)]^{\mathrm{std}}} \tau^*(\mathrm{taut}) \prod_{i=1}^N \mathrm{ev}_i^*(\gamma_i),$$

where we let $[\cdots]^{\mathrm{std}}$ denote the standard (nonreduced!) virtual class in the sense of [Li and Tian 1998; Behrend and Fantechi 1997]. Explicit formulas are given in (67).

Conjecture C (holomorphic anomaly equation) *Assume Conjecture B. We have*

$$\begin{aligned}
&\frac{d}{dG_2} F_{g,l}^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N)\\
&= F_{g-1,l}^{S^{[n]}}(\mathrm{taut}';\gamma_1,\dots,\gamma_N,U) + 2\sum_{\substack{g=g_1+g_2\\ \{1,\dots,N\}=A\sqcup B}} F_{g_1,l}^{S^{[n]}}(\mathrm{taut}_1;\gamma_A,U_1) F_{g_2}^{S^{[n]},\mathrm{std}}(\mathrm{taut}_2;\gamma_B,U_2)\\
&\quad -2\sum_{i=1}^N F_{g,l}^{S^{[n]}}(\psi_i\,\mathrm{taut};\gamma_1,\dots,\gamma_{i-1},U\gamma_i,\gamma_{i+1},\dots,\gamma_N) - \frac{1}{l}\sum_{a,b}(g^{-1})_{ab} T_{e_a} T_{e_b} F_{g,l}^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N)
\end{aligned}$$

and

$$\frac{d}{d\mathsf{A}} F_{g,l}^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N) = T_\delta F_{g,l}^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N),$$

where:

- *We have identified the operator* $U \in \mathrm{End}\, H^*(S^{[n]})$ *with the class*

$$U \in H^*(S^{[n]} \otimes S^{[n]})$$

  using Poincaré duality and the conventions of Section 1.13.
- U_1 *and* U_2 *stand for summing over the Künneth decomposition of* $U \in H^*((S^{[n]})^2)$.
- *The* e_a *form a basis of* $\{F,B\}^\perp \subset H^2(S,\mathbb{Q})$ *and* $g_{ab} = \langle e_a, e_b\rangle$ *is the pairing matrix.*
- *For any* $\alpha \in \{B,F\}^\perp \subset V$ *we set*

$$T_\alpha F_{g,l}^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N) := \sum_{i=1}^N F_{g,l}^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_{i-1},T_\alpha\gamma_i,\gamma_{i+1},\dots,\gamma_N). \tag{4}$$

- *In the stable case, where* $\mathrm{taut}=\tau^*(\alpha)$, *we let* $\mathrm{taut}' := \tau^*\iota^*(\alpha)$ *where* $\iota\colon \overline{M}_{g-1,N+2} \to \overline{M}_{g,N}$ *is the gluing map; in the unstable case, where* $\mathrm{taut}=1$, *we set* $\mathrm{taut}' := 1$.
- taut_1 *and* taut_2 *stand for summing over the Künneth decomposition of* $\xi^*(\mathrm{taut})$, *where* ξ *is the gluing map*

$$\xi\colon \overline{M}_{g_1,|A|+1}(S^{[n]},\beta+rA) \times \overline{M}_{g_2,|B|+1} \to \overline{M}_{g,N}(S^{[n]},\beta+rA).$$

- *We let* $\psi_i \in H^2(\overline{M}_{g,N}(S^{[n]},\beta+rA))$ *be the cotangent line class at the* i^{th} *marking.*

Conjecture C determines any $F_{g,l}$ up to a finite list of coefficients, where the list is sufficiently short for this to be actually useful in applications. For example, the conjecture determines all Gromov–Witten invariants of $S^{[2]}$ from seven elementary computations; see [Cao et al. 2024] where this leads to a Yau–Zaslow type formula for the counts of genus-2 curves on hyperkähler fourfolds of $\mathrm{K3}^{[2]}$–type.

For K3 surfaces (the case of the Hilbert scheme of $n = 1$ points) the above conjectures are well known. In this case, Conjecture A was made in [Oberdieck and Pandharipande 2016], and Conjecture B reduces to the prediction of Maulik, Pandharipande and Thomas that the series $F_{g,l}$ are quasimodular forms for $\Gamma_0(l)$; see [Maulik et al. 2010]. The holomorphic anomaly equation (Conjecture C) was proven in [Oberdieck and Pixton 2018] for $l = 1$ and then conjectured in [Bae and Buelles 2021] for arbitrary l. There is also sufficient evidence for the following:

Theorem 1.1 [Maulik et al. 2010; Oberdieck and Pixton 2018; Bae and Buelles 2021] *For $S^{[1]} \cong S$, the above conjectures hold for all g, N and $l \in \{1, 2\}$.*

For Hilbert schemes of points $S^{[n]}$ with $n > 1$, Conjecture A was proposed in [Oberdieck 2022] based on computations using Noether–Lefschetz theory. Since then the following strong evidence for all $n \geq 1$ was given:

Theorem 1.2 [Oberdieck 2024b, Theorem 1.4] *Conjecture A holds for $g = 0$ and $N \leq 3$ markings.*

The quasi-Jacobi form property (Conjecture B) appeared in an early form already in [Oberdieck 2018a, Conjecture J], where it was stated in genus 0 for primitive classes. On the other hand, the holomorphic anomaly equation (Conjecture C) is new, and one of our main results.

Holomorphic anomaly equations are predicted for the Gromov–Witten theory of Calabi–Yau manifolds by string theory [Bershadsky et al. 1993]. In recent years, this structure was proven in various geometries, such as for elliptic orbifold projective lines [Milanov et al. 2018], elliptic curves [Oberdieck and Pixton 2018], formal elliptic curves [Wang 2019], local $\mathbb{P}^2$ [Lho and Pandharipande 2018; Coates and Iritani 2021], local $\mathbb{P}^1 \times \mathbb{P}^1$ [Lho 2021; Wang 2023] relative $(\mathbb{P}^2, E)$ [Bousseau et al. 2021], $\mathbb{C}^3/\mathbb{Z}_3$ [Lho and Pandharipande 2019a; Coates and Iritani 2021], toric Calabi–Yau 3–folds [Eynard et al. 2007; Eynard and Orantin 2015; Fang et al. 2020; 2019], the formal quintic 3–fold [Lho and Pandharipande 2019b], the quintic 3–fold [Guo et al. 2018; Chang et al. 2018], (partially) elliptic fibrations [Oberdieck and Pixton 2019] and K3 fibrations [Lho 2019]. Conjecture C is maybe the first instance where a general holomorphic anomaly equation is considered in higher dimensions. The interaction here with the LLV Lie algebra is a new phenomenon that needs further exploration. Eg are there connections with the Lie algebra which appears in [Alim et al. 2016]?

1.7 Main results

Theorem 1.3 *For all Hilbert schemes of points $S^{[n]}$ (ie any $n \geq 1$), Conjectures B and C hold for $g = 0$ and $N \leq 3$ markings.*

In particular, this result shows that for fixed n, computing finitely many Gromov–Witten invariants of $S^{[n]}$, where S is the elliptic K3 surface, determines all 3–pointed genus-0 invariants of all Hilbert schemes of n points on K3 surfaces. This shows the following qualitative result (see [Oberdieck 2018a] for the definition of reduced quantum cohomology):

Corollary 1.4 *For any $n \geq 1$, the reduced quantum cohomologies of $\mathrm{QH}^*(X)$ of all hyperkähler varieties of $\mathrm{K3}^{[n]}$–type X can be effectively reconstructed from finitely many Gromov–Witten invariants of $S^{[n]}$, where $S \to \mathbb{P}^1$ is the elliptic K3 surface with section.*

Example 1.5 Let $\mathcal{L} \in H^{2n}(S^{[n]})$ be the class of a fiber of the Lagrangian fibration $S^{[n]} \to \mathbb{P}^n$. An easy computation[1] shows $\mathrm{wt}(\mathcal{L}) = -n$. Hence by the theorem we find

$$F^{S^{[n]}}_{g=0,1}(1; \mathcal{L}, \mathcal{L}) \in \frac{1}{\Delta(q)} \mathrm{QJac}_{2-2n,n-1}. \tag{5}$$

The space $\mathrm{QJac}_{2-2n,n-1}$ is 1–dimensional spanned by $\Theta(p,q)^{2n-2}$, so (5) is determined up to a single constant. The class of a line in the section $\mathbb{P}^n \subset S^{[n]}$ is $B-(n-1)A$. Since there is a unique line through any two points in $\mathbb{P}^n$, we have

$$\langle 1; \mathcal{L}, \mathcal{L} \rangle^{S^{[n]}}_{0,B-(n-1)A} = 1.$$

This yields the explicit evaluation

$$F^{S^{[n]}}_{0,1}(1; \mathcal{L}, \mathcal{L}) = (-1)^{n-1} \frac{\Theta(p,q)^{2n-2}}{\Delta(q)}. \tag{6}$$

This evaluation was previously obtained (with hard work) in [Oberdieck 2018a, Theorem 1].

We give a more fundamental example, where the holomorphic anomaly equation determines the transformation law of the quasi-Jacobi form. Consider the generating series of 2–point Gromov–Witten classes

$$\widetilde{Z}^{S^{[n]}}(p,q) = \sum_{d=-1}^{\infty} \sum_{r \in \mathbb{Z}} q^d (-p)^r (\mathrm{ev}_1 \times \mathrm{ev}_2)_*([\overline{M}_{0,2}(S^{[n]}, B+(d+1)F+rA)]^{\mathrm{vir}})$$

which is an element of $H^*(S^{[n]})^{\otimes 2} \otimes q^{-1}\mathbb{C}((p))[[q]]$. Add a quasi-Jacobi correction term

$$Z^{S^{[n]}}(p,q) := \widetilde{Z}^{S^{[n]}}(p,q) - \frac{\boldsymbol{G}(p,q)^n}{\Theta(p,q)^2 \Delta(q)} \Delta_{S^{[n]}}, \tag{7}$$

where $\Delta_{S^{[n]}}$ is the class of the diagonal in $(S^{[n]})^2$, and

$$\boldsymbol{G}(p,q) = -\Theta(p,q)^2 \Big(p \frac{d}{dp}\Big)^2 \log(\Theta(p,q)).$$

We have the following corollary:

[1] In the Nakajima basis of Section 3.2 we have $\mathcal{L} = \mathfrak{q}_1(F)^n v_{\varnothing}$, which implies the claim.

Corollary 1.6 *Under the variable change $p = e^{2\pi i x}$ and $q = e^{2\pi i \tau}$, the function*

$$Z^{S^{[n]}} : \mathbb{C} \times \mathbb{H} \to H^*(S^{[n]} \times S^{[n]}, \mathbb{C}), \quad (x,\tau) \mapsto Z^{S^{[n]}}(x,\tau)$$

is a vector-valued Jacobi form of weight -10 and index $n-1$ with double poles at lattice points. In particular, we have the transformation laws

$$Z^{S^{[n]}}\left(\frac{x}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-10-\mathrm{Wt}} e\left(\frac{c(n-1)x^2}{c\tau+d}\right) \cdot \exp\left(-\frac{c}{c\tau+d}\left[\frac{1}{4\pi i}\sum_{\alpha,\beta}(\tilde{g}^{-1})_{\alpha\beta} T_\alpha T_\beta + x T_\delta\right]\right) Z^{S^{[n]}}(x,\tau),$$

$$Z^{S^{[n]}}(x+\lambda\tau+\mu,\tau) = e(-(n-1)\lambda^2\tau - 2\lambda(n-1)x)\exp(\lambda T_\delta) Z^{S^{[n]}}(x,\tau),$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$, where we have written $e(x) = e^{2\pi i x}$ for $x \in \mathbb{C}$.

We refer to Section 11.1 for the precise definitions and conventions that we use here. A formula for the series $Z^{S^{[n]}}(p,q)$ was conjectured in [Oberdieck 2018a] and then refined to an explicit conjecture in [van Ittersum et al. 2021]. The above corollary yields strong evidence for this conjecture.

The cycle $Z^{S^{[n]}}(p,q)$ also appears naturally in the Pandharipande–Thomas theory of the relative threefold $(S \times \mathbb{P}^1, S_{0,\infty})$. Indeed, by Denis Nesterov's quasimap wall crossing [2021; 2024] and the computation of the wall-crossing term in [Oberdieck 2024b], one has

$$Z^{S^{[n]}}(p,q) = \sum_{d,r} q^d (-p)^r (\mathrm{ev}_0 \times \mathrm{ev}_\infty)_* [P^{\sim}_{r,(B+(d+1)F,n)}(S \times \mathbb{P}^1, S_{0,\infty})]^{\mathrm{vir}},$$

where the moduli space on the right parametrizes stable pairs (F,s) on the relative rubber target $(S \times \mathbb{P}^1, S_{0,\infty})^\sim$ with Chern character $\mathrm{ch}_3(F) = r$. Consider the Pandharipande–Thomas theory of $S \times E$, where E is an elliptic curve. By using the evaluation in [Oberdieck and Pixton 2018] and by degenerating the elliptic curve [Oberdieck and Pandharipande 2016], one obtains the closed formula

$$\sum_{n=0}^{\infty} \tilde{q}^{n-1} \int_{S^{[n]} \times S^{[n]}} Z^{S^{[n]}}(p,q) \cup \Delta_{S^{[n]}} = -\frac{1}{\chi_{10}(p,q,\tilde{q})},$$

where χ_{10} is the weight-10 Igusa cusp form (as in [Oberdieck and Pandharipande 2016]). Because Fourier coefficients of Siegel modular forms are Jacobi forms, this matches nicely with Corollary 1.6.

1.8 An application: CHL Calabi–Yau threefolds

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section B and fiber class F, and let $g : S \to S$ be a symplectic involution such that

$$\mathrm{Pic}(S) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8(-2),$$

where the first summand is generated by B and F, and the second summand is the anti-invariant part.[2] Let E be an elliptic curve and let $\tau\colon E\to E$ be translation by a 2–torsion point. The Chaudhuri–Hockney–Lykken (CHL) Calabi–Yau threefold associated to (g,τ) is the quotient

$$X = (S\times E)/\langle g\times\tau\rangle.$$

The group of algebraic 1–cycles on X is

$$N_1(X)\cong \mathrm{Span}_{\mathbb{Z}}(B,F)\oplus\mathbb{Z}[E'],$$

where the second summand records the degree over the elliptic curve $E' = E/\langle\tau\rangle$.

Define the Donaldson–Thomas partition function

$$\mathrm{DT}_n(X) = \sum_{d\geq -1}\sum_{r\in\mathbb{Z}} \mathrm{DT}_{r,(B+dF,n)}\, q^{d-1}, (-p)^r$$

where we used the reduced Donaldson–Thomas invariants (see [Bryan and Oberdieck 2020])

$$\mathrm{DT}_{r,\beta} = \int_{[\mathrm{Hilb}_{r,\beta}(X)/E]^{\mathrm{vir}}} 1.$$

Theorem 1.7 *Every $\mathrm{DT}_n(X)$ is a Jacobi form of weight -6 and index n, that is*

$$\mathrm{DT}_n(X)\in\frac{1}{\Theta(p,q)^2\Delta(\tau)}\mathrm{Jac}_{4,n}(\Gamma_0(2)).$$

The rank-1 Donaldson–Thomas invariants of X in arbitrary curve classes are determined from the series DT_n by the multiple cover formula of [Oberdieck 2024b] and a degeneration argument [Bryan and Oberdieck 2020]. Hence Theorem 1.7 puts strong constraints on the full rank-1 Donaldson–Thomas theory of X. For an explicit conjectural formula for the DT_n, see [Bryan and Oberdieck 2020].

Our methods can apply also to arbitrary CHL Calabi–Yau threefolds which are associated to symplectic automorphism of K3 surfaces of any finite order. The above is just the simplest case notationwise, and chosen here to illustrate the method. The Donaldson–Thomas theory of general CHL Calabi–Yau threefolds will be studied at a later time.

1.9 Fiber classes and Lagrangian fibrations

Assume that we are in the stable case $2g-2+N>0$. Consider the generating series of Gromov–Witten invariants in fiber classes of the Lagrangian fibration $S^{[n]}\to\mathbb{P}^n$:

$$F^{S^{[n]}}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N) := \sum_{d\geq 0}\sum_{\substack{k\in\mathbb{Z}\\(d,k)\neq 0}} \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,dF+rA} q^d(-p)^r.$$

[2]These K3 surfaces arise as follows: Let $R\to\mathbb{P}^1$ be a generic rational elliptic surface, and let $\mathbb{P}^1\to\mathbb{P}^1$ be a double cover, branched away from the discriminant. Then consider the K3 surface $S = R\times_{\mathbb{P}^1}\mathbb{P}^1$ and let g be the composition of the covering involutions with the fiberwise multiplication by (-1). This involution is symplectic and has the desired properties; see [Bryan and Oberdieck 2020, Section 5.1].

We have to exclude here the term $(d,r)=(0,0)$, because reduced Gromov–Witten invariants are not defined for a vanishing curve class. The price that we pay for this unnatural definition is that we work modulo the constant term below. Given power series $f,g\in\mathbb{C}((p))[\![q]\!]$ we write $f\equiv g$ if they are equal in $\mathbb{C}((p))[\![q]\!]/\mathbb{C}$, or equivalently if $f=g+c$ for a constant $c\in\mathbb{C}$. In the unstable cases $2g-2+N\le 0$ we define

$$F^{S^{[n]}}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N)=0.$$

We first state the conjectural quasi-Jacobi property and holomorphic anomaly equation:

Conjecture D *Assume that $2g-2+N>0$. We have the following*:

(i) ***Quasi-Jacobi form property*** *Up to a constant term, $F^{S^{[n]}}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N)$ is a meromorphic quasi-Jacobi form of weight $k=n(2g-2+N)+\sum_i \mathrm{wt}(\gamma_i)$ and index 0 with poles at torsion points $z=a\tau+b$, $a,b\in\mathbb{Q}$.*

(ii) ***Holomorphic anomaly equations*** *Modulo constants, ie in $\mathbb{C}((p))[\![q]\!]/\mathbb{C}$, we have*

$$\begin{aligned}\frac{d}{dG_2}F^{S^{[n]}}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N)&\\ \equiv F^{S^{[n]}}_{g-1,0}(\mathrm{taut}';\gamma_1,\dots,\gamma_N,U)&+2\sum_{\substack{g=g_1+g_2\\ \{1,\dots,N\}=A\sqcup B}}F^{S^{[n]}}_{g_1,0}(\mathrm{taut}_1;\gamma_A,U_1)F^{S^{[n]},\mathrm{std}}_{g_2}(\mathrm{taut}_2;\gamma_B,U_2)\\ &-2\sum_{i=1}^{N}F^{S^{[n]}}_{g,0}(\tau^*(\psi_i)\mathrm{taut};\gamma_1,\dots,\gamma_{i-1},U\gamma_i,\gamma_{i+1},\dots,\gamma_N),\end{aligned}$$

where $\psi_i\in H^2(\overline{M}_{g,N})$ is the cotangent line class, and

$$\frac{d}{d\mathsf{A}}F^{S^{[n]}}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N)\equiv\sum_{i=1}^{N}F^{S^{[n]}}_{g,0}(\mathrm{taut};\gamma_1,\dots,T_\delta\gamma_i,\dots,\gamma_N).$$

Theorem 1.8 *Conjecture D holds for*

(i) *the K3 surface S (ie if $n=1$) and for all g and N,*

(ii) *all Hilbert schemes $S^{[n]}$ (that is for arbitrary n), if $(g,N)=(0,3)$.*

We refer to Theorem 10.2 for the precise form which the quasi-Jacobi forms described in (i) have. The multiple cover conjecture (Conjecture A) was proven for the K3 surface S in fiber classes dF in [Bae and Buelles 2021]. The observation that the corresponding generating series is quasimodular and satisfies a holomorphic anomaly equation appears to be new (but follows easily from the known methods). The case of the Hilbert scheme of points also follows from the multiple cover conjecture, together with some subtle vanishing arguments.

Deformation invariance and similar methods as in our proof should show that for any Lagrangian fibration $\pi:X\to\mathbb{P}^n$ of a $\mathrm{K3}^{[n]}$–hyperkähler with a section, the generating series of Gromov–Witten invariants in fiber classes is a (lattice index) quasi-Jacobi form and satisfies a holomorphic anomaly equation. This raises the following question:

Question 1.9 *Consider any Lagrangian fibration $X \to B$ with section of a holomorphic–symplectic variety X. Are the generating series of Gromov–Witten invariants in fiber classes quasi-Jacobi forms, and do they satisfy a holomorphic anomaly equation?*

The answer is very likely "yes". More interestingly, we can ask this for cases where X is quasiprojective hyperkähler. A prototypical example to consider is the Hitchin map $\mathcal{M}_{C,n} \to \bigoplus_i H^0(C, K_C^i)$ from the moduli space of rank-n Higgs bundles on a curve C. Evidence for a positive answer will be given in the genus-1 case (more precisely, for the Hilbert scheme of points on $E \times \mathbb{C}$) in [Oberdieck and Pixton 2023].

1.10 Strategy of the proof

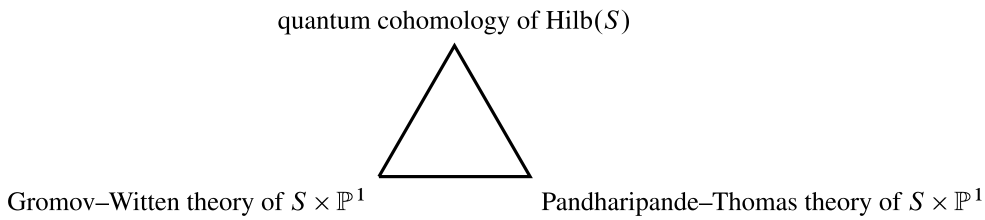
Hilbert schemes of points on K3 surfaces lie in the intersection of two very special classes of varieties: (irreducible) hyperkähler varieties and Hilbert schemes of points on surfaces. The geometry of both of these classes will imply a modular constraint on the generating series of Gromov–Witten invariants. We will show that these two constraints are precisely the two modular transformation equations that a Jacobi form has to satisfy.

From hyperkähler geometry we use the global Torelli theorem [Verbitsky 2013; Huybrechts 2012] and the description of the monodromy in [Markman 2008]. The locus parametrizing Hilbert schemes of points $S^{[n]}$ on K3 surfaces is a divisor in the moduli space of all hyperkähler varieties of $\mathrm{K3}^{[n]}$–type. In particular, there are deformations of $S^{[n]}$ which do not arise from deformations of the underlying K3 surface S (these deformation may be thought of as deforming the K3 surface S in a noncommutative way). Utilizing these extra deformations yields precisely one of the transformation properties that we need.

The other ingredient follows from the Hilbert scheme side. Given a surface S there is a correspondence between three different counting theories:

(i) quantum cohomology (ie $(g, N) = (0, 3)$ Gromov–Witten theory) of $S^{[n]}$,
(ii) Pandharipande–Thomas theory of the relative threefold $(S \times \mathbb{P}^1, S_{0,1,\infty})$,
(iii) Gromov–Witten theory of the relative threefold $(S \times \mathbb{P}^1, S_{0,1,\infty})$.

This correspondence is often represented in the triangle



The GW/PT correspondence (meaning the correspondence between (ii) and (iii)) was proposed in [Maulik et al. 2006a; 2006b] and has since been proven in many instances in [Maulik et al. 2011; Pandharipande and Pixton 2014; 2017]. For $\mathrm{K3} \times \mathbb{P}^1$ it was recently established in [Oberdieck 2024a] for curve classes which are *primitive* over the surface. The Hilb/PT correspondence (between (i) and (ii)) was recently established in full generality by Nesterov [2021]. For $\mathbb{C}^2$ and resolutions of A_n singularities, the triangle of correspondences was worked out previously in [Okounkov and Pandharipande 2010b; Bryan and Pandharipande 2008; Okounkov and Pandharipande 2010a; Maulik and Oblomkov 2009a; 2009b; Maulik 2009; Liu 2021].

In the case of K3 surfaces the above correspondences take the simplest form: they are straight equalities, without wallcrossing corrections; see Theorem 7.6 and [Nesterov 2024]. By expressing invariants of the Hilbert schemes in terms of invariants of $S \times \mathbb{P}^1$ and then applying the product formula in Gromov–Witten theory, we hence have expressed the Gromov–Witten invariants of the Hilbert scheme in terms of those of the K3 surface. This allows us to lift modular properties which are known for K3 surfaces to the Hilbert scheme of points. Altogether, this provides precisely the other half of the modularity that we were missing.

This leads to the proof of Theorem 1.3 for primitive classes ($l = 1$). To deduce the arbitrary case we use the proven case of the multiple cover conjecture [Oberdieck 2024b] and check the compatibility of our conjectures under the formal Hecke operator. Except for working out the required compatibility on quasi-Jacobi forms, this last step is not difficult.

1.11 History

The Gromov–Witten theory of the Hilbert schemes of points of K3 surfaces was first studied by the author in his PhD thesis [Oberdieck 2015]. Many ideas behind the current work were already anticipated then. For example, the potential role of the monodromy was discussed in [loc. cit., Section 6.3], and the quasi-Jacobi form property was conjectured in a simple case in [loc. cit., Section 5.1.3]. Interestingly, the simplest evaluation on the Hilbert scheme from a weight point of view, given in (6), is precisely also the case where the moduli space of stable maps is the simplest to describe, and indeed this case was the first to be computed back then.

1.12 Outline

In Section 2 we review the definition of quasi-Jacobi forms and prove basic properties regarding their z–expansions, their anomaly operators and how they interact with Hecke operators. In Section 3 we introduce the LLV algebra on the cohomology of the Hilbert scheme and then describe explicitly the two monodromy operators that we need for constraints of the Gromov–Witten generating series (see Sections 3.6.3 and 3.6.3). In Section 4 we use these two monodromies and obtain our first structure result for the generating series of the Hilbert scheme in Proposition 4.1, essentially proving the elliptic transformation law.

Then we turn to the part on GW/PT/Hilb correspondences: In Section 5 we discuss several basic structures in relative Gromov–Witten theory. The main new technical result here is a formula for the restriction of relative Gromov–Witten classes to the nonseparating boundary divisor in the moduli space of curves, which is of independent interest. In Section 6 we specialize to $(K3\times C, K3_z)$ for a curve C, state the GW/Hilb correspondence (Theorem 6.2) and the reduced degeneration formula, and make some preliminary explicit computations of invariants. The goal of Section 7 is to use the product formula and results about the K3 surface to show that the Gromov–Witten invariants of $(K3\times C, K3_z)$ are quasimodular forms and satisfy a holomorphic anomaly equation (Theorem 7.6). This is our second main structure result.

Section 8 is the heart of the paper. Here we combine the two structure results we obtained before (Proposition 4.1 and Theorem 7.6) and match the holomorphic anomaly equation on the Hilbert scheme with the holomorphic anomaly equation for $(K3\times C, K3_z)$ under the GW/Hilb correspondence. This proves Theorem 1.3 when $l=1$. The case $l>1$ follows then in Section 9 by a formal argument using Hecke operators. Section 10 deals with the fiber classes, proving Theorem 1.8 by a combination of the GW/Hilb correspondence and known cases of the multiple cover conjecture. Section 11 discusses the applications to the 2–point function and the CHL Calabi–Yau threefolds.

1.13 Conventions

Let X be a smooth projective variety. Given a cohomology class $\gamma\in H^k(X)$ we let $\deg(\gamma)=\frac{1}{2}k$ denote its complex degree. We will use the identification $H^*(X\times X)\cong \operatorname{End} H^*(X)$ which is given by sending a class $\Gamma\in H^*(X\times X)$ to the operator

$$\Gamma\colon H^*(X)\to H^*(X),\quad \gamma\mapsto \pi_{2*}(\pi_1^*(\gamma)\Gamma),$$

where π_1 and π_2 are the projections of X^2 to the factors. Given a function $Z\colon H^*(X)\to\mathbb{Q}$ we will often write $Z(\Gamma_1)Z(\Gamma_2)$ and say that Γ_1 and Γ_2 stand for summing over the Künneth decomposition of the class $\Gamma\in H^*(X\times X)$. By this we mean

$$Z(\Gamma_1)Z(\Gamma_2):=\sum_i Z(\phi_i)Z(\phi_i^\vee,)$$

where $\Gamma=\sum_i \phi_i\otimes\phi_i^\vee\in H^*(X\times X)$ is a Künneth decomposition. A *curve class* on X is any homology class $\beta\in H_2(X,\mathbb{Z})$. It is effective if there exists a nonempty algebraic curve $C\subset X$ with $[C]=\beta$. In particular, any effective class β is nonzero. An effective class β is primitive if it is not divisible in $H_2(X,\mathbb{Z})$.

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2 Quasi-Jacobi forms

2.1 Overview

Jacobi forms are two-variable generalizations of classical modular forms. Quasi-Jacobi forms are constant terms of almost-holomorphic Jacobi forms. We introduce here the basic facts we need on quasi-Jacobi forms and refer to [Libgober 2011; Oberdieck and Pixton 2019, Section 1; van Ittersum et al. 2021] for more detailed discussions. The topics we cover are the generators of the ring of quasi-Jacobi forms, differential and anomaly operators, and the Fourier and Taylor expansion of quasi-Jacobi forms. Conversely, we give criteria on two-variable generating series to be Taylor or Fourier expansions of quasi-Jacobi forms. In Section 2.8 we discuss Hecke operators on quasi-Jacobi forms, and in Section 2.9 we consider their action on forms of the wrong weight. In Section 2.10 we discuss a classical series of meromorphic quasi-Jacobi forms which will appear for fiber classes of Lagrangian fibrations in Section 10.

2.2 Definition

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the upper half-plane, $q = e^{2\pi i\tau}$, $x \in \mathbb{C}$ and $p = e^{2\pi i x}$. We will also frequently use the variable

$$z = 2\pi i x.$$

We often write $f(p)$ or $f(z)$ for a function $f(x)$ under the above variable change. Consider the real-analytic functions

$$\nu = \frac{1}{8\pi\Im(\tau)} \quad \text{and} \quad \alpha = \frac{\Im(x)}{\Im(\tau)}.$$

An almost-holomorphic function on $\mathbb{C} \times \mathbb{H}$ is a function of the form

$$\Phi = \sum_{i,j\geq 0} \phi_{i,j}(x,\tau)\nu^i\alpha^j \tag{8}$$

such that each of the finitely many nonzero functions $\phi_{i,j}$ is holomorphic and admits a Fourier expansion of the form $\sum_{n\geq 0}\sum_{r\in\mathbb{Z}} c(n,r)q^n p^r$ in the region $|q| < 1$.

Consider a congruence subgroup

$$\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$$

and write $e(x) = e^{2\pi i x}$ for $x \in \mathbb{C}$.

Definition 2.1 An *almost-holomorphic weak Jacobi form* of weight k and index m for the group Γ is a function $\Phi(x,\tau)\colon \mathbb{C}\times\mathbb{H} \to \mathbb{C}$ which

(i) satisfies the transformation laws

$$\begin{gathered}\Phi\left(\frac{x}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e\left(\frac{cmx^2}{c\tau+d}\right)\Phi(x,\tau), \\ \Phi(x+\lambda\tau+\mu,\tau) = e(-m\lambda^2\tau - 2\lambda m x)\Phi(x,\tau),\end{gathered} \tag{9}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$ and $\lambda,\mu \in \mathbb{Z}$, and

(ii) such that

$$(c\tau+d)^{-k} e\left(-\frac{cmx^2}{c\tau+d}\right)\Phi\left(\frac{x}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$$

is an almost-holomorphic function for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Remark 2.2 By taking $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to be the identity in (ii), we see that any almost-holomorphic weak Jacobi form is an almost-holomorphic function, and hence has an expansion (8). Condition (i) implies that (ii) only needs to be checked for a set of representatives of $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$. In particular, if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ the condition (ii) simply says that Φ is an almost-holomorphic function.

An almost-holomorphic weak Jacobi form Φ, which is as a function $\Phi\colon \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ holomorphic, is called a *weak Jacobi form*. More generally, we can consider the holomorphic part of an almost-holomorphic weak Jacobi form:

Definition 2.3 A *quasi-Jacobi form* of weight k and index m for Γ is a function $\phi(x,\tau)$ on $\mathbb{C} \times \mathbb{H}$ such that there exists an almost-holomorphic weak Jacobi form $\sum_{i,j} \phi_{i,j} \nu^i \alpha^j$ of weight k and index m with $\phi_{0,0} = \phi$.

We let $\mathrm{AHJac}_{k,m}(\Gamma)$ (resp. $\mathrm{QJac}_{k,m}(\Gamma)$, resp. $\mathrm{Jac}_{k,m}(\Gamma)$) be the vector space of almost-holomorphic weak (resp. quasi-, resp. weak) Jacobi forms of weight k and index m for the group Γ. We write

$$\mathrm{QJac}(\Gamma) = \bigoplus_{m \geq 0} \bigoplus_{k \in \mathbb{Z}} \mathrm{QJac}(\Gamma)_{k,m}$$

for the bigraded $\mathbb{C}$–algebra of quasi-Jacobi forms, and similar for $\mathrm{AHJac}(\Gamma)$ and $\mathrm{Jac}(\Gamma)$.

Lemma 2.4 *The constant term map*

$$\mathrm{AHJac}(\Gamma)_{k,m} \to \mathrm{QJac}(\Gamma)_{k,m}, \quad \sum_{i,j} \phi_{i,j} \nu^i \alpha^j \mapsto \phi_{0,0}$$

is well-defined and an isomorphism.

Proof This is proven in [Libgober 2011]. □

A quasimodular form of weight k for the congruence subgroup Γ is a quasi-Jacobi form of weight k and index 0 for Γ. The algebra of quasimodular forms is denoted by

$$\mathrm{QMod}(\Gamma) = \bigoplus_k \mathrm{QMod}(\Gamma)_k, \quad \mathrm{QMod}(\Gamma)_k = \mathrm{QJac}(\Gamma)_{k,0}.$$

Remark 2.5 (i) If Γ is the full modular group $\mathrm{SL}_2(\mathbb{Z})$, we will usually omit Γ from our notation, eg

$$\mathrm{QJac} = \mathrm{QJac}(\mathrm{SL}_2(\mathbb{Z})).$$

(ii) In what follows, we will often identify a quasi-Jacobi form $f(x,\tau) \in \mathrm{QJac}_{k,m}$ with its power series in p and q. We will also often write $f(p,q)$ instead of $f(x,\tau)$.

2.3 Presentation by generators: quasimodular forms

For all even $k > 0$ consider the Eisenstein series

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n\geq 1}\sum_{d\,|\,n} d^{k-1}q^n.$$

Set also $G_k = 0$ for all odd $k > 0$. Then G_k is a modular form of weight k for $k > 2$, and G_2 is quasimodular. By [Kaneko and Zagier 1995; Bloch and Okounkov 2000] the algebra of quasimodular forms is a free polynomial ring in G_2 over $\mathrm{Mod}(\Gamma)$, ie the ring of modular forms for the group Γ:

$$\mathrm{QMod}(\Gamma) = \mathrm{Mod}(\Gamma)[G_2].$$

For the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we have

$$\mathrm{QMod} = \mathbb{C}[G_2, G_4, G_6].$$

2.4 Presentation by generators: quasi-Jacobi forms

Consider the odd (renormalized) Jacobi theta function[3]

$$\Theta(x,\tau) = (p^{1/2} - p^{-1/2}) \prod_{m\geq 1} \frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}.$$

Consider the derivative operator $p(d/dp) = 1/(2\pi i)(d/dx) = d/dz$ and consider also the series

$$\mathsf{A}(x,\tau) = \frac{p(d/dp)\Theta(x,\tau)}{\Theta(x,\tau)} = -\frac{1}{2} - \sum_{m\neq 0} \frac{p^m}{1-q^m}.$$

By the same argument as in [Kaneko and Zagier 1995; Bloch and Okounkov 2000], G_2 and A are free generators:

Lemma 2.6 $$\mathrm{QJac}(\Gamma) \subset \mathrm{Jac}(\Gamma)[G_2, \mathsf{A}].$$

As in the case of quasimodular forms, for the full modular group, the algebra of quasi-Jacobi forms can be embedded in a polynomial algebra. Consider the classical Weierstrass elliptic function

$$\wp(x,\tau) = \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d\geq 1}\sum_{k\,|\,d} k(p^k - 2 + p^{-k})q^d.$$

We write $\wp'(x,\tau) = p(d/dp)\wp(x,\tau)$ for its derivative with respect to the first variable. Consider the polynomial algebra

$$\mathrm{MQJac} = \mathbb{C}[\Theta, \mathsf{A}, G_2, \wp, \wp', G_4].$$

Proposition 2.7 [van Ittersum et al. 2021] MQJac *is a free polynomial ring on its generators, and* QJac *is equal to the subring of all polynomials which define holomorphic functions* $\mathbb{C}\times\mathbb{H}\to\mathbb{H}$.

[3]We have $\Theta(x,\tau) = \vartheta_1(x,\tau)/\eta^3(\tau)$, where $\vartheta_1(x,\tau) = \sum_{\nu\in\mathbb{Z}+1/2}(-1)^{\lfloor\nu\rfloor}p^\nu q^{\nu^2/2}$ is the odd Jacobi theta function, ie the unique section on the elliptic curve $\mathbb{C}_x/(\mathbb{Z}+\tau\mathbb{Z})$ which vanishes at the origin, and $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta function.

The generators of MQJac are quasi-Jacobi forms (with poles and character [loc. cit.]) of weight and index given in the following table. The algebra QJac is a graded subring of MQJac.

generator	weight	index
Θ	-1	$\frac{1}{2}$
A	1	0
G_2	2	0
$\wp$	2	0
$\wp'$	3	0
G_4	4	0

Remark 2.8 By the well-known equation

$$\wp'(x)^2 - 4\wp(x)^3 + 20\wp(x)G_4(\tau) + \tfrac{7}{3}G_6(\tau) = 0,$$

the generator G_6 is not needed as a generator of MQJac.

2.5 Differential and anomaly operators

As explained in [Oberdieck and Pixton 2019, Section 2] the algebra $\mathrm{QJac}(\Gamma)$ is closed under the derivative operators

$$D_\tau = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq} \quad \text{and} \quad D_x = \frac{1}{2\pi i}\frac{d}{dx} = \frac{d}{dz} = p\frac{d}{dp}.$$

More precisely, these operators act by

$$D_\tau : \mathrm{QJac}_{k,m}(\Gamma) \to \mathrm{QJac}_{k+2,m}(\Gamma) \quad \text{and} \quad D_x : \mathrm{QJac}_{k,m}(\Gamma) \to \mathrm{QJac}_{k+1,m}(\Gamma).$$

Similarly, we have *anomaly operators*. These can be defined most directly as follows. By Lemma 2.6 every quasi-Jacobi form $f(x,\tau)$ can be uniquely written as a polynomial in A and G_2 with coefficients weak Jacobi-forms. We hence can take the formal derivative at these generators, giving functions $(d/dG_2)f$ and $(d/d\mathsf{A})f$. If $F = \sum_{i,j} f_{i,j}\nu^i\alpha^j$ is the almost-holomorphic function with $f_{0,0} = f$, then by [loc. cit., Section 2] one has

$$\frac{d}{dG_2}f = f_{1,0} \quad \text{and} \quad \frac{d}{d\mathsf{A}}f = f_{0,1}.$$

This can be used to show that d/dG_2 and $d/d\mathsf{A}$ preserve the algebra of quasi-Jacobi forms. Precisely:

Lemma 2.9 [loc. cit., Section 2] *The formal derivation with respect to* A *and* G_2 *defines operators*

$$\frac{d}{dG_2} : \mathrm{QJac}_{k,m}(\Gamma) \to \mathrm{QJac}_{k-2,m}(\Gamma) \quad \textit{and} \quad \frac{d}{d\mathsf{A}} : \mathrm{QJac}_{k,m}(\Gamma) \to \mathrm{QJac}_{k-1,m}(\Gamma).$$

Then we have the commutative diagrams

$$\begin{array}{ccc} \mathrm{QJac}_{k,m} & \xleftarrow{\cong} & \mathrm{AHJac}_{k,m} \\ {\scriptstyle d/dG_2}\downarrow & & \downarrow{\scriptstyle d/d\nu} \\ \mathrm{QJac}_{k-2,m} & \xleftarrow{\cong} & \mathrm{AHJac}_{k-2,m} \end{array} \qquad \begin{array}{ccc} \mathrm{QJac}_{k,m} & \xleftarrow{\cong} & \mathrm{AHJac}_{k,m} \\ {\scriptstyle d/d\mathsf{A}}\downarrow & & \downarrow{\scriptstyle d/d\alpha} \\ \mathrm{QJac}_{k-1,m} & \xleftarrow{\cong} & \mathrm{AHJac}_{k-1,m} \end{array}$$

where the horizontal maps are the "constant term" maps of Lemma 2.4.

Let wt and ind be the operators which act on $\mathrm{QJac}_{k,m}(\Gamma)$ by multiplication by the weight k and the index m, respectively. By [loc. cit., (12)] we have the commutation relations

$$\left[\frac{d}{dG_2}, D_\tau\right] = -2\,\mathrm{wt}, \quad \left[\frac{d}{d\mathsf{A}}, D_x\right] = 2\,\mathrm{ind} \quad \left[\frac{d}{dG_2}, D_x\right] = -2\frac{d}{dA} \quad \text{and} \quad \left[\frac{d}{dA}, D_\tau\right] = D_x. \tag{10}$$

Remark 2.10 These commutation relations are proven by checking them for almost-holomorphic Jacobi forms, where they follow by a straightforward computation of commutators between derivative operators and operators of multiplication by variables. In particular, the argument is not sensitive to the precise holomorphicity conditions we put on Jacobi forms; for example, the commutation relations (10) hold also for MQJac or any other ring of meromorphic quasi-Jacobi forms.

As explained in [loc. cit.], knowing the holomorphic-anomaly equations of a quasi-Jacobi form is equivalent to knowing their transformation properties under the Jacobi group. Concretely:

Lemma 2.11 [loc. cit.] *For any $\phi(x,\tau) \in \mathrm{QJac}_{k,m}(\Gamma)$ we have*

$$\phi\left(\frac{x}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e\left(\frac{cmx^2}{c\tau+d}\right) \exp\left(-\frac{c(d/dG_2)}{4\pi i(c\tau+d)} + \frac{cx(d/d\mathsf{A})}{c\tau+d}\right)\phi(x,\tau),$$

$$\phi(x+\lambda\tau+\mu,\tau) = e(-m\lambda^2\tau - 2\lambda m x)\exp\left(-\lambda\frac{d}{d\mathsf{A}}\right)\phi(x,\tau),$$

for all $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma$ and $\lambda, \mu \in \mathbb{Z}$.

2.6 Elliptic transformation law

Recall from Lemma 2.11 the elliptic transformation law of quasi-Jacobi forms:

Lemma 2.12 *For any $f(p,q) \in \mathrm{QJac}_{k,m}$ and $\lambda \in \mathbb{Z}$ we have*

$$f(pq^\lambda, q) = q^{-\lambda^2 m} p^{-2\lambda m} e^{-\lambda(d/dA)} f(p,q).$$

In particular, if we are given $f(p,q) \in \mathrm{QJac}_{k,m}$ such that $(d/dA)f = 0$, and we let

$$f(p,q) = \sum_{d\geq 0}\sum_{k\in\mathbb{Z}} c(d,k) q^d p^k$$

be its Fourier expansion, then

$$c(d - \lambda k + m\lambda^2, k - 2\lambda m) = c(d,k).$$

Moreover, since $f(p^{-1},q) = (-1)^k f(p,q)$ where k is the weight of f, we have

$$c(d,k) = (-1)^k c(d,-k).$$

We prove the following two useful lemmas, which serve as a partial converse:

Lemma 2.13 *Let $m \geq 0$ and let $f(p,q) = \sum_{n\geq 0}\sum_{k\in\mathbb{Z}} c(d,k)q^d p^k$ be a formal power series such that the following holds for all d, k and $\lambda \in \mathbb{Z}$:*

$$c(d-\lambda k + m\lambda^2, k-2\lambda m) = c(d,k), \tag{11}$$

$$c(d,k) = c(d,-k). \tag{12}$$

Then there exists power series $f_i(q) \in \mathbb{C}[\![q]\!]$ such that

$$f(p,q) = \Theta^{2m}(p,q)\sum_{i=0}^{m} f_i(q)\wp(p,q)^{m-i}.$$

Proof A similar argument has appeared in [Oberdieck and Shen 2020, Section 4.2] but we recall it here for completeness. The vector space of Laurent polynomials $g(p)$ such that $g(p^{-1}) = g(p)$ has a basis given by the set of polynomials

$$(p^{1/2}-p^{-1/2})^{2k} \quad \text{for } k \geq 0.$$

Moreover, by the expansions of Θ and $\wp$ for every $i \in \{0,\dots,m\}$, there exist α_j (all 0 except for finitely many) such that

$$\wp(p,q)^{m-i}\Theta(p,q)^{2m} = (p^{1/2}-p^{-1/2})^{2i} + \sum_{j>i}\alpha_j(p^{1/2}-p^{-1/2})^{2j} + O(q).$$

By an induction argument we can hence find $f_i(q) \in \mathbb{C}[\![q]\!]$ such that the function

$$F(p,q) := f(p,q) - \Theta^{2m}(p,q)\sum_{i=0}^{m} f_i(q)\wp(p,q)^{m-i}$$

has the following property: for all $d \geq 0$ the q^d coefficient of F satisfies

$$F_d(p) := [F(p,q)]_{q^d} = \sum_{l>m} b_{d,l}(p^{1/2}-p^{-1/2})^{2l}. \tag{13}$$

Let $a(d,k)$ be the coefficient of $q^d p^k$ in $F(p,q)$. Since $\Theta^{2m}\wp^{m-i}$ is a (quasi-) Jacobi form of index m, its Fourier-coefficients satisfy (11). Moreover, if the Fourier coefficients of a power series $h(p,q)$ satisfy (11), then the same holds for the Fourier coefficients of $h(p,q)r(q)$ for any power series in q. This implies

$$a(d,k) = a(d-\lambda k + m\lambda^2, k-2\lambda m) \tag{14}$$

for all $d,k,\lambda \in \mathbb{Z}$. Assume $F(p,q)$ is nonzero and let d be the smallest integer such that $F_d(p)$ is nonzero. Since the sum in (13) starts at $l = m+1$, we have

$$a(d,k) \neq 0$$

for some $k \geq m+1 \geq 0$. But then by (14) with $\lambda = 1$, we obtain

$$a(d,k) = a(d-k+m, k-2m) \neq 0.$$

Since $d-k+m < d$ this contradicts the choice of d. □

Lemma 2.14 *Let $m \geq 0$ and let $f(p,q) = \sum_{n \geq 0} \sum_{k \in \mathbb{Z}} c(d,k) q^d p^k$ be a formal power series such that the following holds for all d, k and $\lambda \in \mathbb{Z}$:*

$$c(d - \lambda k + m\lambda^2, k - 2\lambda m) = c(d,k) \quad \text{and} \quad c(d,k) = -c(d,-k).$$

Then there exists power series $f_i(q) \in \mathbb{C}[\![q]\!]$ such that

$$f(p,q) = \Theta^{2m}(p,q) \wp'(p,q) \sum_{i=2}^{m} f_i(q) \wp(p,q)^{m-i}.$$

Proof The vector space of Laurent polynomials $g(p)$ such that $g(p^{-1}) = g(p)$ has the basis

$$(p - p^{-1})(p^{1/2} - p^{-1/2})^{2k} \quad \text{for } k \geq 0.$$

Moreover, for $i \leq m$ we have the expansions

$$\Theta^{2m} \wp' \wp^{m-i} = (p - p^{-1}) \Bigl((p^{1/2} - p^{-1/2})^{2i-4} + \sum_{j > i-2} \alpha_j (p^{1/2} - p^{-1/2})^{2j} \Bigr) + O(q)$$

for some α_j, of which all but finitely many are 0. By induction we conclude that there exists $f_i(q)$ such that

$$F(p,q) = f(p,q) - \Theta^{2m}(p,q) \wp'(p,q) \sum_{i=2}^{m} f_i(q) \wp(p,q)^{m-i}$$

for all $d \geq 0$ satisfies

$$F_d(p) := [F(p,q)]_{q^d} = (p - p^{-1}) \sum_{l > m-2} b_{d,l} (p^{1/2} - p^{-1/2})^{2l}. \tag{15}$$

We argue now as before: Let $a(d,k)$ be the coefficient of $q^d p^k$ in $F(p,q)$. We then still have (14) as well as

$$a(d,k) = -a(d,-k).$$

Assume $F(p,q)$ is nonzero and let d be the smallest integer such that $F_d(p)$ is nonzero. Since the sum in (15) starts at $l = m-1$, we have $a(d,k) \neq 0$ for some $k \geq m \geq 0$. But then by (14) with $\lambda = 1$, we obtain

$$a(d,k) = a(d - k + m, k - 2m) \neq 0.$$

If $k > m$ this yields a contradiction as before, and if $k = m$ we obtain $a(d,k) = a(d,-k)$, but since we also have $a(d,k) = -a(d,k)$ this gives the contradiction $a(d,k) = 0$. □

2.7 The expansion in z

Recall that we have set $z = 2\pi i x$, where $x \in \mathbb{C}$ is the elliptic parameter. To stress the dependence on z, we usually write $f(z)$ for a function $f(x)$ under this variable change. We study here the z–expansions of quasi-Jacobi forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. For that purpose, recall the well-known expansion of the generators of MQJac in z; see eg [van Ittersum et al. 2021]:

$$\Theta(z) = z \exp\biggl(-2\sum_{k\geq 1} G_k(\tau)\frac{z^k}{k!}\biggr), \quad \mathsf{A}(z) = \frac{1}{z} - 2\sum_{k\geq 1} G_k(\tau)\frac{z^{k-1}}{(k-1)!},$$
$$\wp(z) = \frac{1}{z^2} + 2\sum_{k\geq 4} G_k(\tau)\frac{z^{k-2}}{(k-2)!}.$$

Consider the operator that takes the formal derivative with respect to G_2 factorwise,

$$\Bigl(\frac{d}{dG_2}\Bigr)_z : \mathrm{QMod}((z)) \to \mathrm{QMod}((z)).$$

That is, for $f = \sum_r f_r(\tau) z^r$ with $f_r \in \mathrm{QMod}$, we let

$$\Bigl(\frac{d}{dG_2}\Bigr)_z f = \sum_r \frac{df_r}{dG_2} z^r.$$

Consider the decomposition of MQJac according to weight k and index m,

$$\mathrm{MQJac} = \bigoplus_{k,m} \mathrm{MQJac}_{k,m}.$$

Then the following is immediate from the expansions above:

Lemma 2.15 *The coefficient of z^r of any series $f \in \mathrm{MQJac}_{k,m}$ is a quasimodular form of weight $r+k$. Moreover,*

$$\Bigl(\frac{d}{dG_2}\Bigr)_z f = \frac{d}{dG_2} f - 2z\frac{d}{dA} f - 2z^2 m f. \tag{16}$$

We prove the following partial converses:

Lemma 2.16 *Let $f_i(q) \in \mathbb{C}[\![q]\!]$ be power series such that every z^r–coefficient of*

$$f(p,q) = \Theta^{2m}(p,q)\sum_{i=0}^{m} f_i(q)\wp(p,q)^{m-i}$$

is a quasimodular form of weight z^{r+s}. Then every $f_i(q)$ is quasimodular of weight $s+2i$.

Proof We have $\Theta^{2m}\wp^{m-i} = z^{2i} + O(z^{2i+2})$, so we can write $f_i(q)$ as a linear combination of the z^r–coefficients of $f(p,q)$ with coefficients quasimodular forms (of the correct weight). □

Lemma 2.17 *Let $f_i(q) \in \mathbb{C}[\![q]\!]$ be power series such that every z^r–coefficient of*

$$f(p,q) = \Theta^{2m}(p,q)\wp'(p,q)\sum_{i=2}^{m} f_i(q)\wp(p,q)^{m-i}$$

is a quasimodular form of weight z^{r+s}. Then every $f_i(q)$ is quasimodular of weight $s+2i-3$.

Proof The proof is similar. □

2.8 Hecke operators

Let $m \geq 1$ and recall that the m^{th} Hecke operator acts on Jacobi forms $\phi(x,\tau)$ of weight k and index m by

$$(T_{(k,m),l} f)(x,\tau) = l^{k-1} \sum_{A=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \backslash M_l} (c\tau+d)^{-k} e\left(ml \frac{-cx^2}{c\tau+d}\right) f\left(\frac{lx}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right), \tag{17}$$

where A runs over a set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$–left cosets of the set

$$M_l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\, ad-bc = l \right\}.$$

As shown in [Eichler and Zagier 1985, I.4], the action of $T_{(k,m),l}$ is well defined (ie independent of a set of representatives) and defines an operator[4]

$$T_{(k,m),l} \colon \mathrm{Jac}_{k,m} \to \mathrm{Jac}_{k,ml}\,.$$

Since the argument in [loc. cit.] only involves the compatibilities of the slash-operators of the Jacobi forms, the proof carries over identically to almost-holomorphic weak Jacobi forms. Hence using (17) we also obtain a well-defined operator:

$$T_{(k,m),l} \colon \mathrm{AHJac}_{k,m} \to \mathrm{AHJac}_{k,ml^2}, \quad F \mapsto T_{(k,m),l} F.$$

Transporting to quasi-Jacobi forms using the "constant term" map of Lemma 2.4 we hence obtain a Hecke operator on quasi-Jacobi forms

$$T_{(k,m),l} \colon \mathrm{QJac}_{k,m} \to \mathrm{QJac}_{k,ml},$$

defined by the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{QJac}_{k,m} & \xleftarrow{\ \cong\ } & \mathrm{AHJac}_{k,m} \\ {\scriptstyle T_{(k,m),l}}\downarrow & & \downarrow{\scriptstyle T_{(k,m),l}} \\ \mathrm{QJac}_{k,ml} & \xleftarrow{\ \cong\ } & \mathrm{AHJac}_{k,ml} \end{array}$$

The Hecke operator on quasi-Jacobi forms satisfies the following:

Proposition 2.18 *If $f = \sum_{n,r} c(n,r) q^n p^r$ is the Fourier expansion of a quasi-Jacobi form of weight k and index m, then*

$$T_{(k,m),l} f = \sum_{n,r} \Bigg(\sum_{a \,|\, (l,n,r)} a^{k-1} c\left(\frac{ln}{a^2}, \frac{r}{a}\right) \Bigg) q^n p^r. \tag{18}$$

Moreover,

$$\frac{d}{dG_2} T_{k,l} f = l\, T_{k-2,m} \frac{d}{dG_2} f \quad \text{and} \quad \frac{d}{d\mathsf{A}} T_{k,l} f = l\, T_{k-1,m} \frac{d}{d\mathsf{A}} f, \tag{19}$$

where we write $T_{k,l} := T_{(k,m),l}$ since $T_{(k,m),l}$ does not depend on m.

[4] We only require Hecke operators for the full modular group, so we restrict to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ here, ie omit Γ from the notation. This section generalizes to arbitrary congruence subgroups.

Proof For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_l$ we have the transformation properties

$$\nu\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{1}{l}\nu(\tau)|c\tau+d|^2 = \frac{1}{l}\left[(c\tau+d)^2\nu(\tau) + \frac{c(c\tau+d)}{4\pi i}\right],$$
$$\alpha\left(\frac{lx}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\alpha(x,\tau) - cx.$$

Consider the weight-k index-m almost-holomorphic weak Jacobi form

$$F = \sum_{i,j} f_{i,j}\nu^i\alpha^j$$

with $f_{0,0} = f$. With $J = c\tau+d$ and $\tilde{c} = c/4\pi i$ we obtain

$$(T_{(k,m),l}F)(x,\tau) = l^{k-1}\sum_{A,r,s} J^{-k} e\left(ml\frac{-cx^2}{c\tau+d}\right) f_{r,s}\left(\frac{lx}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)\left(\frac{J(J\nu+\tilde{c})}{l}\right)^r (J\alpha-cz)^s. \tag{20}$$

We specialize A now to run over the set of representatives of $\mathrm{SL}_2(\mathbb{Z})\backslash M_l$ given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{for } l = ad \text{ and } b = 0, \dots, d-1.$$

Then (20) becomes

$$(T_{(k,m),l}F)(x,\tau) = l^{k-1}\sum_{r,s\geq 0}\nu^r\alpha^s\left[\frac{1}{l^r}\sum_{\substack{l=ad \\ b=0,\dots,d-1}} d^{-k+2r+s} f_{r,s}\left(az, \frac{a\tau+b}{d}\right)\right].$$

Taking the $\nu^0\alpha^0$ coefficient and inserting $f = \sum_{n,r} c(n,r)q^n p^r$ yields

$$\begin{aligned} T_{(k,m),l}f &= \mathrm{Coeff}_{\nu^0\alpha^0}(T_{(k,m),l}F) = l^{k-1}\sum_{l=ad} d^{-k}\sum_{b=0}^{d-1} f(az, (a\tau+b)/d) \\ &= \sum_{l=ad} a^{k-1}\sum_{\substack{n,r \\ n\equiv 0 \bmod d}} c(n,r)p^{ar}q^{na/d}. \end{aligned}$$

This gives the first claim. The compatibility with the anomaly operators follows from

$$\begin{aligned} \frac{d}{dG_2}T_{k,l}f &= \mathrm{Coeff}_{\nu^1\alpha^0}(T_{(k,m),l}F) = l\sum_{l=ad} a^{k-3}\sum_{\substack{n,r \\ n\equiv 0 \bmod d}} c'(n,r)p^{ar}q^{na/d}, \\ \frac{d}{d\mathsf{A}}T_{k,l}f &= \mathrm{Coeff}_{\nu^0\alpha^1}(T_{(k,m),l}F) = l\sum_{l=ad} a^{k-2}\sum_{\substack{n,r \\ n\equiv 0 \bmod d}} c''(n,r)p^{ar}q^{na/d}, \end{aligned}$$

where c' and c'' are the Fourier coefficients of $f_{1,0}$ and $f_{0,1}$, respectively. □

By a straightforward computation using (18), one finds that for $f \in \mathrm{QJac}_{k,m}$,

$$T_{k+2,l}D_\tau f = lD_\tau T_{k,l}f \quad \text{and} \quad T_{k+1,l}D_z f = D_z T_{k,l}f. \tag{21}$$

Then (19) and (21) are compatible with the commutation relations (10).

2.9 Wrong-weight Hecke operators

For a formal power series $f = \sum_{d,r} c(d,r) q^d p^r$ we can formally define the l^{th} Hecke operator of weight k by

$$T_{k,l} f = \sum_{n,r} \Bigg(\sum_{a \mid (l,n,r)} a^{k-1} c\left(\frac{ln}{a^2}, \frac{r}{a}\right)\Bigg) q^n p^r. \tag{22}$$

In Proposition 2.18 we have seen that $T_{k,l}$ defines an operator

$$T_{k,l} \colon \mathrm{QJac}_{k,m} \to \mathrm{QJac}_{k,ml}\,.$$

More generally, we can ask: what happens if we apply $T_{k,l}$ to quasi-Jacobi forms f of a weight k' different from k? This is answered by the next proposition.

Consider the congruence subgroup

$$\Gamma_0(l) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod l \right\}.$$

Proposition 2.19 *For any k, k' and m, the l^{th} formal Hecke operator defines a morphism*

$$T_{k,l} \colon \mathrm{QJac}_{k',m} \to \mathrm{QJac}_{k',ml}(\Gamma_0(l)).$$

Moreover, for any $f \in \mathrm{QJac}_{k',m}(\mathrm{SL}_2(\mathbb{Z}))$ we have

$$\frac{d}{dG_2} T_{k,l} f = l T_{k-2,m} \frac{d}{dG_2} f \quad \text{and} \quad \frac{d}{d\mathsf{A}} T_{k,l} f = l T_{k-1,m} \frac{d}{d\mathsf{A}} f. \tag{23}$$

For the proof we will decompose the "wrong-weight Hecke operator" into ordinary Hecke operators and the scaling operators B_N for $N \geq 1$ defined on functions $f \colon \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ by

$$(B_N f)(x,\tau) = f(Nx, N\tau).$$

Lemma 2.20 *If $f \in \mathrm{QJac}_{k,m}$ then $B_N f \in \mathrm{QJac}_{k,mN}(\Gamma_0(N))$, and moreover*

$$\frac{d}{dG_2} B_N f = \frac{1}{N} B_N \frac{d}{dG_2} f \quad \text{and} \quad \frac{d}{d\mathsf{A}} B_N f = \frac{1}{N} B_N \frac{d}{d\mathsf{A}} f.$$

Proof Let $F(x,\tau)$ be a almost-holomorphic weak Jacobi form of weight k and index m. Set

$$\widehat{F}(x,\tau) = (B_N F)(x,\tau) = F(Nx, N\tau).$$

Then for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and with $c = c'N$, we have

$$\begin{aligned}\widehat{F}\left(\frac{x}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= F\left(\frac{Nx}{c\tau+d}, \frac{aN\tau+Nb}{c\tau+d}\right) = F\left(\frac{Nx}{c'(N\tau)+d}, \frac{aN\tau+Nb}{c'(N\tau)+d}\right) \\ &= (c'(N\tau)+d)^k e\left(\frac{mc'(Nx)^2}{c'(N\tau)+d}\right) F(Nx, N\tau) = (c\tau+d)^k e\left(\frac{(mN)cx^2}{c\tau+d}\right) \widehat{F}(x,\tau),\end{aligned}$$

where we have used that $\left(\begin{smallmatrix} a & Nb \\ c' & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Similarly, one proves that

$$\hat{F}(x+\lambda\tau,\tau) = e(-mN(\lambda^2\tau+2\lambda x))\hat{f}(x,\tau).$$

So $\hat{F} \in \mathrm{AHJac}_{k,mN}(\Gamma_0(N))$, and by taking the constant coefficient also $B_N f \in \mathrm{QJac}_{k,mN}(\Gamma_0(N))$. To show the compatibility with the anomaly operators write

$$F(x,\tau) = \sum_{i,j} f_{i,j}\nu^i\alpha^j.$$

Since $\nu(N\tau) = \nu(\tau)/N$, we get

$$B_N F(x,\tau) = \sum_{i,j} \frac{1}{N^i} f_{i,j}(Nx,N\tau)\nu^i\alpha^j.$$

Hence if $f = f_{0,0}$,

$$\frac{d}{dG_2} B_N f = \mathrm{Coeff}_{\nu^1\alpha^0}(B_N F(x,\tau)) = \frac{1}{N} f_{1,0}(Nx,N_\tau) = \frac{1}{N} B_N \frac{d}{dG_2} f.$$

The case for $d/d\mathsf{A}$ is similar. □

Proof of Proposition 2.19 We follow ideas of [Bae and Buelles 2021, Lemma 12]. Given a power series $f = \sum_{d,r} c(d,r)q^d p^r$ define the formal operator

$$U_b f = \sum_{n,r} c(bn,r)q^n p^r.$$

A direct calculation starting from (22) shows that

$$T_{k,l} = \sum_{ab=l} a^{k-1} B_a U_b.$$

Recall the Möbius function

$$\mu(n) = \begin{cases} (-1)^g & \text{if } n = p_1\cdots p_g \text{ for distinct primes } p_i, \\ 0 & \text{else,} \end{cases}$$

which satisfies $\sum_{d\,|\,n,d>0}\mu(d) = \delta_{n1}$. For $s \in \mathbb{Z}$ let Id_s be the function $\mathrm{Id}_s(a) = a^s$. For functions g and h define the Dirichlet convolution $(g*h)(l) = \sum_{l=ab} g(a)h(b)$ and the pointwise product $(gh)(a) = g(a)h(a)$. Both of these are associative operations. We then have

$$(\mu\,\mathrm{Id}_{k'-1}) * \mathrm{Id}_{k'-1}(a) = \delta_{a1},$$

and thus

$$(\mathrm{Id}_{k-1} * (\mu\,\mathrm{Id}_{k'-1}) * \mathrm{Id}_{k'-1})(a) = \mathrm{Id}_{k-1}.$$

After setting

$$c_{k,k'}(e) = (\mathrm{Id}_{k-1} * (\mu\,\mathrm{Id}_{k'-1}))(e)$$

this yields

$$\begin{aligned} T_{k,l} &= \sum_{ab=l} (\mathrm{Id}_{k-1} * (\mu\,\mathrm{Id}_{k'-1}) * \mathrm{Id}_{k'-1})(a) B_a U_b = \sum_{ab=l}\sum_{e\,|\,a} c_{k,k'}(e)\Big(\frac{a}{e}\Big)^{k'-1} B_a U_b \\ &= \sum_{ed=l} c_{k,k'}(e) B_e \sum_{d=bb'} (b')^{k'-1} B_{b'} U_b = \sum_{ed=l} c_{k,k'}(e) B_e T_{k',d}, \end{aligned} \tag{24}$$

where we used $B_a = B_e B_{a/e}$.

Given $f \in \mathrm{QJac}_{k',m}$ we have $T_{k',d} f \in \mathrm{QJac}_{k',md}$ by Proposition 2.18, and hence

$$B_e T_{k',d} f \in \mathrm{QJac}_{k',md} \in \mathrm{QJac}_{k',mde}(\Gamma_0(e)) \tag{25}$$

by Lemma 2.20. Since for $e \mid l$ we have

$$\mathrm{QJac}(\Gamma_0(e)) \subset \mathrm{QJac}(\Gamma_0(l)),$$

we obtain that

$$T_{k,l} f = \sum_{ed=l} c_{k,k'}(e) B_e T_{k',d} f \in \mathrm{QJac}(\Gamma_0(l)).$$

For the second part, observe that

$$c_{k,k'}(e) = \sum_{ab=e} a^{k-1} b^{k'-1} \mu(b) = e^2 c_{k-2,k'-2}(e).$$

Hence by the second parts of Proposition 2.18 and Lemma 2.20, we have

$$\frac{d}{dG_2} T_{k,l} f = \sum_{ed=l} c_{k,k'}(e) \frac{d}{e} B_e T_{k',d} \frac{d}{dG_2} f = l \sum_{ed=l} c_{k-2,k'-2}(e) B_e T_{k',d} \frac{d}{dG_2} f = T_{k-2,l} \frac{d}{dG_2} f. \qquad \square$$

Example 2.21 Recall that $\mathrm{Mod}_2(\Gamma_0(2))$ is 1–dimensional and is generated by

$$F_2(\tau) = 1 + 24 \sum_{\substack{d \mid n \\ d \text{ odd}}} d q^n.$$

Hence $\mathrm{QMod}_2(\Gamma_0(2))$ has the basis given by F_2 and G_2. One computes that

$$T_{k,2} G_2(\tau) = 2^{k-1} B_2 G_2 + U_2 G_2 = 2^{k-1}\big(-\tfrac{1}{48} F_2 + \tfrac{1}{2} G_2\big) + \big(\tfrac{1}{24} F_2 + 2G_2\big).$$

Hence as predicted by Proposition 2.19 we get

$$\frac{d}{dG_2} T_{k,2} G_2(\tau) = 2(1 + 2^{k-3}) = 2T_{k-2,2}(1).$$

In applications below we will consider quasi-Jacobi forms with a pole at $\tau = i\infty$, ie which are of the form

$$f(x,\tau) = \frac{\phi(x,\tau)}{\Delta(\tau)^r}$$

for a quasi-Jacobi form ϕ and some $m \geq 1$. Since the argument used to prove Proposition 2.19 also works when there are poles, the results of Proposition 2.19 remain valid for these quasi-Jacobi forms as well. The only modification concerns the order of poles:

Proposition 2.22 *For any k, k' and m the l^{th} formal Hecke operator acts by*

$$T_{k,l} \colon \frac{1}{\Delta(\tau)} \mathrm{QJac}_{k'+12,m} \to \frac{1}{\Delta(\tau)^l} \mathrm{QJac}_{k'+12l,ml}(\Gamma_0(l)).$$

The relations (23) *hold identically.*

Proof If $f(x,\tau)=\phi(x,\tau)/\Delta(\tau)$ is a weight-k index-m quasi-Jacobi for the group $\mathrm{SL}_2(\mathbb{Z})$, then the "correct weight" Hecke transform $T_{k,l} f$ is also quasi-Jacobi for the full group $\mathrm{SL}_2(\mathbb{Z})$. The poles of $T_{k,l} f$ are located at the single cusp $\tau=i\infty$, and here (22) shows that the pole order is increased by l. So $\Delta(\tau)^l T_{k,l} f$ is holomorphic quasi-Jacobi, ie it lies in $\mathrm{QJac}_{k+12l,ml}$. Hence the claim holds if $k=k'$. In the general case we use again the decomposition (24), the fact that B_N is a ring homomorphism and that for any $N\geq 1$ (see eg [Koblitz 1993, Proposition 17(a)])

$$B_N\Big(\frac{1}{\Delta(\tau)}\Big)\in \frac{1}{\Delta(\tau)^N}\,\mathrm{Mod}_{12(N-1)}(\Gamma_0(N)). \qquad \square$$

2.10 Index-0 meromorphic Jacobi forms

Consider the algebra of index-0 Jacobi forms,

$$\mathrm{MQJac}_0 := \bigoplus_{k\geq 0}\mathrm{MQJac}_{k,0}=\mathbb{C}[\mathsf{A},G_2,\wp,\wp',G_4].$$

The algebra MQJac_0 is precisely the ring of index-0 meromorphic Jacobi forms with poles only at lattice points $x=a\tau+b$ for $a,b\in\mathbb{Z}$; see [Libgober 2011].

Consider once more the Jacobi theta function

$$\Theta(z)=(p^{1/2}-p^{-1/2})\prod_{m\geq 1}\frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2},$$

which we view in this section as a function of $z=2\pi i x$ (and drop τ from notation). Define functions $\mathsf{A}_n(z,\tau)$ for all $n\in\mathbb{Z}$ by the expansion

$$\frac{\Theta(z+w)}{\Theta(z)\Theta(w)}=\sum_{n\geq 0}\frac{\mathsf{A}_n(z,\tau)}{n!}w^{n-1}. \tag{26}$$

In particular $\mathsf{A}_0=1$ and $\mathsf{A}_1=\mathsf{A}$. The function $\Theta(z+w)/(\Theta(z)\Theta(w))$ is a meromorphic Jacobi of lattice index $\left(\begin{smallmatrix}0 & 1/2\\ 1/2 & 0\end{smallmatrix}\right)$, which leads to the proof of the following:

Theorem 2.23 [Zagier 1991; Libgober 2011] (*a*) *For all n we have $\mathsf{A}_n\in\mathrm{MQJac}_{0,n}$ and*

$$\frac{d}{dG_2}\mathsf{A}_n=0,\qquad \frac{d}{d\mathsf{A}}\mathsf{A}_n=n\mathsf{A}_{n-1}.$$

(*b*) *For all $n\geq 0$ we have the expansion*

$$\mathsf{A}_n(z,\tau)=B_n+\delta_{n,1}\frac{1}{2}\frac{p^{1/2}+p^{-1/2}}{p^{1/2}-p^{-1/2}}-n\sum_{k,d\geq 1}d^{n-1}(p^k+(-1)^n p^{-k})q^{kd},$$

where the Bernoulli numbers B_n are defined by $\frac{1}{2}\coth\big(\frac{1}{2}z\big)=\sum_{n\geq 0}(B_n/n!)z^{n-1}$.

Proof The first part follows immediately from the transformation properties given in the theorem of [Zagier 1991, Section 3], but see also [Libgober 2011] for why the A_n lie in MQJac, and [Oberdieck

2012, Lemmata 5 and 6] for the holomorphic anomaly equation (the functions A_n were called J_n in [loc. cit.]). Part (b) follows from the expansion proven in [Zagier 1991, Section 3]

$$\frac{\Theta(z+w)}{\Theta(z)\Theta(w)} = \tfrac{1}{2}\big(\coth\tfrac{1}{2}w + \coth\tfrac{1}{2}z\big) - 2\sum_{n=1}^{\infty}\bigg(\sum_{d\,|\,n}\sinh\Big(dw + \frac{n}{d}z\Big)\bigg)q^n. \qquad \square$$

Remark 2.24 (historical remark) The function (26) already centrally appeared in work of Eisenstein on elliptic functions in the 1850s; see [Weil 1976] for a historical account,

3 Cohomology and monodromy of the Hilbert scheme

3.1 Overview

Let S be a K3 surface and let $S^{[n]}$ be the Hilbert scheme of n points on S. There are two basic structures on the cohomology of the Hilbert scheme. The first is the Nakajima Heisenberg action (Section 3.2), which gives a natural additive basis of the cohomology and allows us to identify the curve classes on the Hilbert scheme (Section 3.3). The second is the Looijenga–Lunts–Verbitsky (LLV) Lie algebra (Section 3.4), which will appear in the statement of the holomorphic anomaly equations. In Section 3.5 we use the LLV algebra to define several gradings on the cohomology. In Section 3.6 we recall work of Markman on how the LLV algebra controls the monodromy. Two particular monodromy operators are of special importance to us because they lead to the elliptic transformation property of the generating series. These are discussed in detail in Sections 3.6.3 and 3.6.4. In particular, we describe how they act on the Nakajima basis.

3.2 Nakajima operators

We follow the work [Nakajima 1997]; see also [Grojnowski 1996]. For any $n, k \in \mathbb{N}$, consider the closed subscheme

$$S^{[n,n+k]} = \{(I \supset I') \mid I/I' \text{ is supported at a single } x \in S\} \subset S^{[n]} \times S^{[n+k]}$$

endowed with projection maps

(27)
$$\begin{array}{ccccc} & & S^{[n,n+k]} & & \\ & \swarrow^{p_-} & \downarrow{\scriptstyle p_S} & \searrow^{p_+} & \\ S^{[n]} & & S & & S^{[n+k]} \end{array}$$

which remember I, x and I', respectively. For $\alpha \in H^*(S)$ and $k > 0$ we define the k^{th} Nakajima operator by letting $S^{[n,n+k]}$ act as a correspondence; that is we define

$$\mathfrak{q}_k(\alpha)\colon H^*(S^{[n]}) \to H^*(S^{[n+k]}), \quad \mathfrak{q}_k(\alpha)\gamma = p_{+*}(p_-^*(\gamma)p_S^*(\alpha)).$$

Similarly, we can go the other way and define $\mathfrak{q}_{-k}(\alpha)\colon H^*(S^{[n+k]}) \to H^*(S^{[n]})$ by

$$\mathfrak{q}_{-k}(\alpha)\gamma = (-1)^k p_{-*}(p_+^*(\gamma)p_S^*(\alpha)).$$

We also set $\mathfrak{q}_0(\gamma) = 0$ for all γ.

Consider the direct sum

$$H^*(\mathrm{Hilb}) = \bigoplus_{n\geq 0} H^*(S^{[n]}).$$

Because the correspondences above are defined for all n, we obtain operators

$$\mathfrak{q}_i(\alpha)\colon H^*(\mathrm{Hilb}) \to H^*(\mathrm{Hilb}).$$

By the main result of [Nakajima 1997] we have the commutation relations of the Heisenberg algebra

$$[\mathfrak{q}_k(\alpha), \mathfrak{q}_l(\beta)] = k(\alpha,\beta)\,\mathrm{Id}_{\mathrm{Hilb}}\,. \tag{28}$$

Moreover, $H^*(\mathrm{Hilb})$ is generated by the operators $\mathfrak{q}_k(\alpha)$ for $k > 0$ from the vacuum vector

$$v_\varnothing \in H^*(S^{[0]}) = \mathbb{Q}.$$

In particular, the set of classes

$$\mathfrak{q}_{\lambda_1}(\gamma_{i_1}) \cdots \mathfrak{q}_{\lambda_{\ell(\lambda)}}(\gamma_{i_{\ell(\lambda)}}) v_\varnothing,$$

where $\lambda = (\lambda_j, \gamma_{i_j})$ runs over all partitions of size n weighted by cohomology classes from a fixed basis $\{\gamma_i\}_{i=1}^{24}$ of $H^*(S)$, forms a basis of $H^*(S^{[n]},\mathbb{Q})$.

For homogeneous $\alpha_i \in H^*(S)$, the degree of a Nakajima cycle is

$$\deg(\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing) = n - l + \sum_i \deg(\alpha_i). \tag{29}$$

The *length* of a Nakajima cycle is defined to be the number of Nakajima factors:

$$l(\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing) = l. \tag{30}$$

3.3 Curve classes

For $n \geq 2$, the fiber of the Hilbert–Chow morphism $S^{[n]} \to \mathrm{Sym}^n(S)$ over a generic point in the discriminant is isomorphic to $\mathbb{P}^1$ and has (co)homology class

$$A = \mathfrak{q}_2(\mathsf{p})\mathfrak{q}_1(\mathsf{p})^{n-2}v_\varnothing \in H_2(S^{[n]},\mathbb{Z}),$$

where $\mathsf{p} \in H^4(S,\mathbb{Z})$ is the class of a point. Similarly, given a class $\beta \in H_2(S,\mathbb{Z})$ we have an associated class on the Hilbert scheme given by

$$\beta_{[n]} := \mathfrak{q}_1(\beta)\mathfrak{q}_1(\mathsf{p})^{n-1}v_\varnothing \in H_2(S^{[n]},\mathbb{Z}).$$

If β is the class of a curve $C \subset S$, then $\beta_{[n]}$ is the class of the curve parametrizing subschemes consisting of $n-1$ distinct fixed points away from C and a single free point on C.

By Nakajima's theorem [1997] (discussed in the last section), we have an isomorphism:

$$H_2(S^{[n]},\mathbb{Z}) \cong H^2(S,\mathbb{Z}) \oplus \mathbb{Z}A, \quad \beta_{[n]} + rA \leftarrow\!\shortmid (\beta, r). \tag{31}$$

Usually we simply write $\beta + rA$ for the class associated to (β, r) on the Hilbert scheme. If $n \leq 1$, we set $A = 0$ and always assume that $r = 0$; if $n = 0$ we also assume that $\beta = 0$.

3.4 The Looijenga–Lunts–Verbitsky algebra

Let X be an (irreducible) hyperkähler variety of dimension $2n$. The lattice $H^2(X,\mathbb{Z})$ is equipped with an integral and nondegenerate quadratic form, called the Beauville–Bogomolov–Fujiki form [Fujiki 1987]. We will also view $H^*(X,\mathbb{Z})$ as a lattice using the Poincaré pairing. Both pairings are extended to the $\mathbb{C}$–valued cohomology groups by linearity.

The Looijenga–Lunts–Verbitsky Lie algebra of X is defined as follows; see [Looijenga and Lunts 1997; Verbitsky 1996]. For any $a \in H^2(X,\mathbb{Q})$ such that $(a,a)\neq 0$, consider the operator on cohomology which takes the cup product with a,

$$e_a\colon H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{Q}), \quad x \mapsto a \cup x.$$

Let h be the Lefschetz grading operator which acts on $H^{2i}(X,\mathbb{Z})$ by multiplication by $i-n$. Then there exists a unique operator

$$f_a\colon H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$$

such that the $\mathfrak{sl}_2$ commutation relations are satisfied:

$$[e_a, f_a] = h, \quad [h, e_a] = e_a, \quad [h, f_a] = -f_a.$$

The LLV Lie algebra $\mathfrak{g}(X)$ is defined as the Lie subalgebra of $\operatorname{End} H^*(X,\mathbb{Q})$ generated by e_a, f_a and h for all $a \in H^2(X,\mathbb{Q})$ as above. By the central result of [Verbitsky 1996] one has

$$\mathfrak{g}(X) = \mathfrak{so}(H^2(X,\mathbb{Q}) \oplus U_{\mathbb{Q}}),$$

where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic plane.

The degree-0 part of $\mathfrak{g}(X)$ decomposes as

$$\mathfrak{g}(X)_0 = \mathfrak{so}(H^2(X,\mathbb{Q})) \oplus \mathbb{Q}h.$$

The summand $\mathfrak{so}(H^2(X,\mathbb{Q}))$ is also called the reduced LLV algebra. Base changing to $\mathbb{C}$ and integrating this yields the *LLV representation*:

$$\rho_{\mathrm{LLV}}\colon \mathrm{SO}(H^2(X,\mathbb{C})) \to \mathrm{GL}(H^*(X,\mathbb{C})). \tag{32}$$

The LLV representation acts by degree-preserving orthogonal ring isomorphisms [Looijenga and Lunts 1997, Proposition 4.4(ii)], where orthogonal means with respect to the Poincaré pairing.

The Hilbert scheme of points $S^{[n]}$ on a K3 surface are irreducible hyperkähler varieties [Beauville 1983]. The LLV algebra we use here was described explicitly in the Nakajima basis in [Oberdieck 2021]. We recall the explicit formulas, using the conventions of [Neguţ et al. 2021]. First recall the isomorphism

$$V = H^2(S^{[n]}) \cong H^2(S) \oplus \mathbb{Q}\delta, \tag{33}$$

which can be obtained by dualizing (31). In particular, δ is $-\frac{1}{2}$ times the class of the locus of nonreduced subschemes and satisfies $\delta A = 1$. Moreover, for $\alpha \in H^2(S,\mathbb{Q})$ the associated divisor on the Hilbert scheme is $(1/(n-1)!)\mathfrak{q}_1(\alpha)\mathfrak{q}_1(1)^{n-1}v_\varnothing$. The Beauville–Bogomolov–Fujiki form is then the form on V which extends the intersection form on $H^2(S)$ and satisfies

$$(\delta,\delta)=2-2n \quad\text{and}\quad (\delta, A^1(S))=0.$$

The LLV algebra is given by

$$\mathfrak{g}(S^{[n]})={\textstyle\bigwedge}^2(V\oplus U_{\mathbb{Q}}),$$

where the Lie bracket is defined for all $a,b,c,d\in V\oplus U_{\mathbb{Q}}$ by

$$[a\wedge b, c\wedge d]=(a,d)b\wedge c-(a,c)b\wedge d-(b,d)a\wedge c+(b,c)a\wedge d.$$

Consider for all $\alpha\in H^2(S,\mathbb{Q})$ the following operators:

$$\begin{aligned}&e_\alpha=-\sum_{n>0}\mathfrak{q}_n\mathfrak{q}_{-n}(\Delta_*\alpha),\quad e_\delta=-\frac{1}{6}\sum_{i+j+k=0}{:}\mathfrak{q}_i\mathfrak{q}_j\mathfrak{q}_k(\Delta_{123}){:},\quad \tilde{f}_\alpha=-\sum_{n>0}\frac{1}{n^2}\mathfrak{q}_n\mathfrak{q}_{-n}(\alpha_1+\alpha_2),\\ &\tilde{f}_\delta=-\frac{1}{6}\sum_{i+j+k=0}{:}\mathfrak{q}_i\mathfrak{q}_j\mathfrak{q}_k\Big(\frac{1}{k^2}\Delta_{12}+\frac{1}{j^2}\Delta_{13}+\frac{1}{i^2}\Delta_{23}+\frac{2}{jk}c_1+\frac{2}{ik}c_2+\frac{2}{ij}c_3\Big){:}.\end{aligned}\tag{34}$$

Here :−: is the normal ordered product defined by

$${:}\mathfrak{q}_{i_1}\cdots\mathfrak{q}_{i_k}{:}=\mathfrak{q}_{i_{\sigma(1)}}\cdots\mathfrak{q}_{i_{\sigma(k)}},$$

where σ is any permutation such that $i_{\sigma(1)}\geq\cdots\geq i_{\sigma(k)}$. We define operators e_α and $\tilde{f}_\alpha$ for general $\alpha\in V$ by linearity in α. By [Lehn 1999], e_α is precisely the operator of the cup product with α. By [Oberdieck 2021], if $(\alpha,\alpha)\neq 0$, the multiple $\tilde{f}_\alpha/(\alpha,\alpha)$ acts on cohomology as the Lefschetz dual of e_α. Then, as shown in [loc. cit.], the assignment

$$\text{act}\colon \mathfrak{g}(S^{[n]})\to \text{End}\, H^*(S^{[n]}),\quad \forall\alpha\in V,\ \text{act}(e\wedge\alpha)=e_\alpha \text{ and } \text{act}(\alpha\wedge f)=\tilde{f}_\alpha\tag{35}$$

induces a Lie algebra homomorphism, which is precisely the action of the LLV algebra. The element $e\wedge f$ acts by the Lefschetz grading operator

$$h=\text{act}(e\wedge f)=\sum_{k>0}\frac{1}{k}\mathfrak{q}_k\mathfrak{q}_{-k}(\mathsf{p}_2-\mathsf{p}_1).\tag{36}$$

3.5 Weight grading

With the notation of the previous section, consider vectors $W,F\in H^2(S,\mathbb{Z})$ which span a hyperbolic lattice, that is, which have intersection form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We associate three operators on $H^*(S^{[n]})$ to this pair:

(i) the Lefschetz dual operator (which will appear in the holomorphic anomaly equation for d/dG_2),

$$U=\tilde{f}_F=\text{act}(F\wedge f)=-\sum_{n>0}\frac{1}{n^2}\mathfrak{q}_n\mathfrak{q}_{-n}(F_1+F_2)$$

(ii) for any $\alpha\in V$ with $\alpha\perp\{W,F\}$, the degree-preserving operator

$$T_\alpha=[e_\alpha,U]=\text{act}(\alpha\wedge F),$$

where for the class $\delta\in V$ we have explicitly

$$T_\delta=\frac{1}{2}\sum_{i+j+k=0}\frac{1}{i}{:}\mathfrak{q}_i\mathfrak{q}_j\mathfrak{q}_k((F_1+F_2)\Delta_{23}){:},\tag{37}$$

(iii) the weight grading operator

$$\mathrm{Wt} = [e_W, U] = \mathrm{act}(e \wedge f + W \wedge F) = \sum_{k>0} \frac{1}{k} \mathfrak{q}_k \mathfrak{q}_{-k} (\mathsf{p}_2 - \mathsf{p}_1 + W_2 F_1 - W_1 F_2). \tag{38}$$

The action of $x = e \wedge f + W \wedge F$ on $H^2(X, \mathbb{Q}) \oplus U_{\mathbb{Q}}$ is semisimple, so x is a semisimple element of the LLV algebra. Hence $H^*(S^{[n]})$ decomposes into eigenspaces under Wt. We can describe the eigenspaces quite explicitly: Define a weight grading on $H^*(S)$ by

$$\mathrm{wt}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \{W, \mathsf{p}\}, \\ -1 & \text{if } \alpha \in \{F, 1\}, \\ 0 & \text{if } \alpha \in \{F, W, 1, \mathsf{p}\}^{\perp}. \end{cases}$$

This induces a grading of $H^*(S^{[n]})$ by setting

$$\mathrm{wt}(\gamma) = \sum_i \mathrm{wt}(\alpha_i) \quad \text{for all } \gamma = \prod_i \mathfrak{q}_{k_i}(\alpha_i) v_{\varnothing}, \tag{39}$$

so that all α_i are wt–homogeneous. By the explicit formula (38), a direct check shows that

$$\mathrm{Wt}(\gamma) = \mathrm{wt}(\gamma)\gamma$$

for a wt–homogeneous element $\gamma \in H^*(S^{[n]})$.

Lemma 3.1 *The action of* Wt *on* $H^*(S^{[n]})$ *is semisimple with eigenspace decomposition*

$$H^*(S^{[n]}) = \bigoplus_{\substack{d=-n \\ d \in \mathbb{Z}}}^{n} V_d, \quad \mathrm{Wt}|_{V_d} = d\, \mathrm{id}_{V_d}.$$

The operators T_α *(for* $\alpha \perp \{W, F\}$*) and* U *act with respect to this grading with weights* -1 *and* -2*, respectively; that is,*

$$T_\alpha \colon V_d \to V_{d-1} \quad \text{and} \quad U \colon V_d \to V_{d-2}.$$

Proof The first claim follows since $\mathrm{wt}(\gamma)$ takes values in $\{-n, \dots, n\}$. The second claim follows from

$$\begin{aligned} [\mathrm{Wt}, T_\alpha] &= \mathrm{act}([e \wedge f + W \wedge F, \alpha \wedge F]) = \mathrm{act}(F \wedge \alpha) = -T_\alpha, \\ [\mathrm{Wt}, U] &= \mathrm{act}([e \wedge f + W \wedge F, F \wedge f]) = \mathrm{act}(f \wedge F - F \wedge f) = -2U. \end{aligned}$$ ∎

We have the following weight computation for the class

$$U \in H^*(S^{[n]} \times S^{[n]})$$

associated to the operator U according to the conventions of Section 1.13:

Lemma 3.2 *Consider a Künneth decomposition* $U = \sum_i a_i \otimes b_i \in H^*(S^{[n]})^{\otimes 2}$ *with* a_i *and* b_i *homogeneous with respect to* wt. *Then for all* i *we have*

$$\mathrm{wt}(a_i) + \mathrm{wt}(b_i) = -2.$$

Proof This follows from

$$(\mathrm{id} \otimes \mathrm{Wt} + \mathrm{Wt} \otimes \mathrm{id})(U) = \mathrm{Wt} \circ U + U \circ \mathrm{Wt}^t = \mathrm{Wt} \circ U - U \circ \mathrm{Wt} = [\mathrm{Wt}, U] = -2U. \qquad \square$$

The weight grading also interacts nicely with the cup product:

Lemma 3.3 *The product $\gamma_1 \cdots \gamma_k$ of any* wt*–homogeneous classes γ_i is again* wt*–homogeneous, and has weight*

$$\mathrm{wt}(\gamma_1 \cdots \gamma_k) = (k-1)n + \sum_i \mathrm{wt}(\gamma_i).$$

Proof The grading operator $\tilde{h} = h + n\,\mathrm{id}$ is multiplicative, ie $\tilde{h}(xy) = \tilde{h}(x)y + x\tilde{h}(y)$. Moreover, since the LLV representation (32) acts by ring isomorphisms,

$$h_{WF} := \mathrm{act}(W \wedge F) = \frac{d}{dt}\Big|_{t=0} \rho_{\mathrm{LLV}}(e^{t(W \wedge F)})$$

is multiplicative. Hence $\widetilde{\mathrm{Wt}} := \mathrm{Wt} + n\,\mathrm{id} = \tilde{h} + h_{WF}$ is multiplicative. If we use this to compute $\mathrm{Wt}(\gamma_1 \cdots \gamma_k)$, we obtain the claim. $\square$

Remark 3.4 For $\gamma \in H^*(S^{[n]})$, the modified degree function $\underline{\deg}(\gamma)$ of [Oberdieck 2022, Section 2.6] is related to the weight $\mathrm{wt}(\gamma)$ defined above by $\underline{\deg}(\gamma) = n + \mathrm{wt}(\gamma)$.

3.6 Monodromy

3.6.1 Monodromy group Let $X = S^{[n]}$. Let $\mathrm{Mon}(X)$ be the subgroup of $O(H^*(X,\mathbb{Z}))$ generated by all monodromy operators, and let $\mathrm{Mon}^2(X)$ be its image in $O(H^2(X,\mathbb{Z}))$. We let

$$\mathrm{mon}\colon \mathrm{Mon}(X) \to O(H^*(X,\mathbb{Z}))$$

denote the monodromy representation.

By results of Markman [2011, Theorem 1.3; 2021, Lemma 2.1]

$$\mathrm{Mon}(X) \cong \mathrm{Mon}^2(X) = \widetilde{O}^+(H^2(X,\mathbb{Z})), \tag{40}$$

where the first isomorphism is the restriction map and $\widetilde{O}^+(H^2(X,\mathbb{Z}))$ is the subgroup of $O(H^2(X,\mathbb{Z}))$ of orientation-preserving lattice automorphisms which act by ± 1 on the discriminant.[5] If $g \in \mathrm{Mon}^2(X)$, we let $\tau(g) \in \{\pm 1\}$ be the sign by which g acts on the discriminant lattice. This defines a character

$$\tau\colon \mathrm{Mon}^2(X) \to \mathbb{Z}_2.$$

3.6.2 Zariski closure By [Markman 2008, Lemma 4.11], if $n \geq 3$ the Zariski closure of the subgroup $\mathrm{Mon}(X) \subset O(H^*(X,\mathbb{C}))$ is $O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2$. The inclusion yields the representation

$$\rho\colon O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2 \to O(H^*(X,\mathbb{C})), \tag{41}$$

[5] Let $\mathcal{C} = \{x \in H^2(X,\mathbb{R}) \mid \langle x, x\rangle > 0\}$ be the positive cone. Then $\mathcal{C}$ is homotopy equivalent to S^2. An automorphism is orientation preserving if it acts by $+1$ on $H^2(\mathcal{C}) = \mathbb{Z}$.

which acts by degree-preserving orthogonal ring isomorphism. There is a natural embedding

$$\widetilde{O}^+(H^2(X,\mathbb{Z})) \to O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2, \quad g \mapsto (g, \tau(g))$$

under which ρ restricts to the monodromy representation; that is,

$$\text{mon}(g) = \rho(g, \tau(g)) \quad \text{for all } g \in \text{Mon}(X). \tag{42}$$

If $n \in \{1, 2\}$ the Zariski closure of $\text{Mon}(X)$ is $O(H^2(X,\mathbb{C}))$. In this case, we define the representation (41) by projection to $O(H^2(X,\mathbb{C}))$ followed by the natural inclusion.

The representation ρ is determined by and has the following properties:

Property 0 For any $(g, \tau) \in O(H^2(X,\mathbb{C})) \times \mathbb{Z}_2$ we have

$$\rho(g, \tau)|_{H^2(X,\mathbb{C})} = g.$$

Property 1 The restriction of ρ to $\text{SO}(H^2(X,\mathbb{C})) \times \{1\}$ is the integrated action of the Looijenga–Lunts–Verbitsky algebra [Looijenga and Lunts 1997; Verbitsky 1996],

$$\rho|_{\text{SO}(H^2(X,\mathbb{C}))\times\{0\}} = \rho_{\text{LLV}}.$$

Property 2 We have

$$\rho(1, -1) = D \circ \rho(-\text{id}_{H^2(X,\mathbb{C})}, 1),$$

where D acts on $H^{2i}(X,\mathbb{C})$ by multiplication by $(-1)^i$.

Property 3 The action is equivariant with respect to the Nakajima operators: For any $g \in O(H^2(X,\mathbb{C}))$ such that $g(\delta) = \delta$, let $\tilde{g} = g|_{H^2(S,\mathbb{C})} \oplus \text{id}_{H^0(S,\mathbb{Z})\oplus H^4(S,\mathbb{Z})}$. Then

$$\rho(g, 1)\biggl(\prod_i \mathfrak{q}_{k_i}(\alpha_i)1\biggr) = \prod_i \mathfrak{q}_{k_i}(\tilde{g}\alpha_i)1.$$

Property 1 follows by [Markman 2008, Lemma 4.13]. Property 3 follows since the Nakajima operator is naturally equivariant with respect to the action of the monodromy group $\text{Mon}(S) = O(H^2(S,\mathbb{Z}))^+$ (of deformations of the K3 surfaces), and this group is Zariski dense in $O(H^2(X,\mathbb{C}))_\delta$. Property 0 follows by construction. Property 2 is implicit in [loc. cit., Section 4]; compare also with [loc. cit., Section 1.1.2].

3.6.3 Example 1: involution The element $g \in \widetilde{O}^+(H^2(X,\mathbb{Z}))$ given under the isomorphism (33) by

$$g|_{H^2(S,\mathbb{Z})} = \text{id}, \quad g(\delta) = -\delta$$

is orientation preserving (it fixes a slice of the positive cone) and acts by -1 on the discriminant lattice. We want to describe the action of the corresponding monodromy operator of X defined by (40).

By Property 2,

$$\text{mon}(g) = D \circ \rho(-g, 1).$$

Since $-g$ fixes δ, we obtain the equivariance with respect to the Nakajima operators in the sense of Property 3; that is, if we let

$$\tilde{g} = \text{id}_{H^0\oplus H^4} \oplus -\text{id}_{H^2(S,\mathbb{Z})}$$

then

$$\rho(-g,1)\biggl(\prod_i \mathfrak{q}_{k_i}(\alpha_i)1\biggr) = \prod_i \mathfrak{q}_{k_i}(\tilde{g}\alpha_i)1.$$

In particular, if all α_i are homogeneous, we see that

$$\rho(-g,1)(\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing) = (-1)^{\tilde{l}}\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing,$$

where $\tilde{l} = |\{i \mid \alpha_i \in H^2(S,\mathbb{Q})\}|$. Using (29), we conclude that

$$\mathrm{mon}(g)(\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing) = (-1)^{n+l}\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing.$$

3.6.4 Example 2: shift The element $\delta\wedge F$ acts on $H^2(X,\mathbb{Z})$ by

$$W\mapsto\delta,\quad \delta\mapsto(2n-2)F,\quad F\mapsto 0 \quad\text{and}\quad (\delta\wedge F)|_{\{W,F,\delta\}^\perp}=0.$$

Let $T_\delta = \mathrm{act}(\delta\wedge F)$ as before, and for any $\lambda\in\mathbb{Z}$ consider the operator

$$e^{\lambda T_\delta}\colon H^*(X,\mathbb{Z})\to H^*(X,\mathbb{Z}).$$

By a direct check, the operator $e^{\lambda(\delta\wedge F)}\colon H^2(X,\mathbb{Z})\to H^2(X,\mathbb{Z})$ is an isometry which is orientation preserving, acts with $+1$ on the discriminant and has determinant $+1$. By (40) it hence defines a monodromy operator of X. Moreover, by (42) and Property 1 we have

$$\mathrm{mon}(e^{\lambda(\delta\wedge F)}) = \rho_{\mathrm{LLV}}(e^{\lambda(\delta\wedge F)}) = e^{\lambda T_\delta}.$$

In particular, $e^{\lambda T_\delta}$ is a monodromy operator.

The action of T_δ is compatible with the identification of $H^2(X,\mathbb{Q})$ and $H_2(X,\mathbb{Q})$ under the Beauville–Bogomolov form. So using $\delta=(2-2n)A$ under this identification, one finds that T_δ acts on $H_2(X,\mathbb{Z})$ by

$$W\mapsto(2-2n)A,\quad A\mapsto -F\quad\text{and}\quad F\mapsto 0.$$

We conclude that

$$e^{\lambda T_\delta}(W+dF+rA) = W+(d-r\lambda+\lambda^2(n-1))F+(r-2\lambda(n-1))A.$$

3.7 Monodromies preserving the Hodge type of a curve class

The Gromov–Witten invariants of $S^{[n]}$ in an effective curve class $\alpha\in H_2(S^{[n]})$ are invariant under deformations which preserve the Hodge type of α. Consider two classes $\alpha,\alpha'\in H_2(S^{[n]})$ which are of Hodge type $(2n-1,2n-1)$ and which *pair positively with a Kähler class*. If there is a monodromy operator $\varphi\in\mathrm{Mon}(S^{[n]})$ such that $h\alpha=\alpha'$, then by the global Torelli theorem [Verbitsky 2013; Huybrechts 2012] there exists a monodromy of $S^{[n]}$ which induces φ and which preserves the Hodge type of α along the deformation. In this case we conclude that

$$\langle\mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,\alpha} = \langle\mathrm{taut};\varphi(\gamma_1),\dots,\varphi(\gamma_N)\rangle^{S^{[n]}}_{g,\varphi(\alpha)}.$$

Remark 3.5 The condition that α and $\varphi(\alpha)$ both pair positively with a Kähler class is necessary. For example, the monodromy operator of Section 3.6.3 sends A to $-A$, but obviously does not preserve the Gromov–Witten invariants (since $-A$ is not effective).

4 Constraints from the monodromy

4.1 Overview

Let S be an elliptic K3 surface with section B and fiber class F, and define the class

$$W = B + F.$$

Let $n \geq 2$ and consider the generating series of Gromov–Witten invariants of $S^{[n]}$:

$$F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d \geq -1} \sum_{r \in \mathbb{Z}} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle_{g, W+dF+rA}^{S^{[n]}} q^d (-p)^r. \tag{43}$$

Our goal in this section is to prove the following:

Proposition 4.1 *There exist unique power series $f_{i,j,s}(q) \in \mathbb{Q}[\![q]\!]$ such that*

$$F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \frac{\Theta(p,q)^{2n-2}}{\Delta(q)} \sum_{i=0}^{2n} \sum_{j=0}^{n-1} \sum_{s \in \{0,1\}} f_{i,j,s}(q) \mathsf{A}(p,q)^i \wp(p,q)^j \wp'(p,q)^s.$$

Moreover, we have the following properties:

(a) *In the ring $(1/\Delta(q))\mathbb{Q}[\![q]\!][\mathsf{A}, \wp, \wp', \Theta]$ we have*

$$\frac{d}{d\mathsf{A}} F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = T_\delta F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N),$$

where the right-hand side is defined as in (4).

(b) *The series $F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N)$ is a power series in q with coefficients which are Laurent polynomials in p.*

(c) *If the γ_i are written in the Nakajima basis (of length $l(\gamma_i)$ as defined in* (30)), *then*

$$F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N)(p^{-1}) = (-1)^{Nn + \sum_i l(\gamma_i)} F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N).$$

The idea of the proof of Sections 4.1 is not difficult: The monodromy operators described in Section 3.6.3 and 3.6.4 together with the invariance of Gromov–Witten invariants under deformations (which preserve the Hodge type of the curve class) yield two basic identities on the generating series $F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N)$. Up to correction terms coming from insertions of lower weight, these identities are precisely the conditions given in Lemmata 2.13 and 2.14, and hence up to the correction term force an expression of the series as a certain polynomial in $\wp$, Θ and $\wp'$. To control the correction term, we argue by an induction on the order of the weight. The correction term is then controlled by the A–holomorphic anomaly equation, and the claim follows by a formal argument.

4.2 Proof of Proposition 4.1

We split the proof in two parts:

Step 1 (the $p \mapsto p^{-1}$ symmetry) We first prove (b) and (c). By Section 3.6.3 there exists a monodromy $\mathrm{mon}(g)$ of $S^{[n]}$ which acts on cohomology by

$$\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing \mapsto (-1)^{n+l}\mathfrak{q}_{k_1}(\alpha_1)\cdots\mathfrak{q}_{k_l}(\alpha_l)v_\varnothing.$$

In particular, it acts on $H_2(S^{[n]},\mathbb{Z})$ by the identity on $H_2(S,\mathbb{Z})$ and sends A to $-A$. By deformation invariance of the Gromov–Witten invariants we obtain that

$$\begin{aligned}\langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,W+dF+rA} &= \langle \mathrm{taut};\mathrm{mon}(g)\gamma_1,\dots,\mathrm{mon}(g)\gamma_N\rangle^{S^{[n]}}_{g,\mathrm{mon}(g)(W+dF+rA)}\\ &= (-1)^{Nn+l_1+\cdots+l_N}\langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,W+dF-rA}.\end{aligned}$$

For any curve class $\beta \in H_2(S,\mathbb{Z})$ there exists an integer r_β such that for all $r \geq r_\beta$ there are no curves in $S^{[n]}$ of class $\beta - rA \in H_2(S^{[n]})$. Hence this equality proves (b) and (c).

Step 2 (the $p \mapsto pq^\lambda$ symmetry) We apply the deformation invariance with respect to the monodromy considered in Section 3.6.4. It yields

$$\langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,W+dF+rA} = \langle \mathrm{taut};e^{\lambda T_\delta}\gamma_1,\dots,e^{\lambda T_\delta}\gamma_N\rangle^{S^{[n]}}_{g,W+(d-r\lambda+\lambda^2(n-1))F+(r-2\lambda(n-1))A}.$$

By multiplying with $(-p)^{r-2\lambda m}q^{d-r\lambda+\lambda^2 m}$, summing over r and d and replacing λ by $-\lambda$, we obtain

$$F_g^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N)(pq^\lambda,q) = p^{-2\lambda m}q^{-\lambda^2 m}F_g^{S^{[n]}}(\mathrm{taut};e^{-\lambda T_\delta}\gamma_1,\dots,e^{-\lambda T_\delta}\gamma_N). \tag{44}$$

We proceed by induction on the total weight of the insertions

$$\sum_i \mathrm{wt}(\gamma_i) = L.$$

Assume that the claim of the proposition holds for all insertions γ_i' with $\sum_i \mathrm{wt}(\gamma_i') < L$. (Since we always have $\mathrm{wt}(\gamma_i) \geq -n$, the statement is true for $L < -nN$. This provides the base of the induction.) Since T_δ decreases the weight by one (see Lemma 3.1), the series

$$\sum_{i=1}^N F_g^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_{i-1},T_\delta\gamma_i,\gamma_{i+1},\dots,\gamma_N) \tag{45}$$

satisfies the induction hypothesis, and hence has all the desired properties. In particular, it is equal to $\Theta^{2n-2}\Delta(q)^{-1}$ times a polynomial in A, $\wp$ and $\wp'$ with coefficients power series in q. Consider the integral with respect to A

$$\widetilde{F} = \sum_{i=1}^N \int F_g^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_{i-1},T_\delta\gamma_i,\gamma_{i+1},\dots,\gamma_N)\,dA,$$

which is defined here formally as the right inverse to d/dA with constant term in A to be 0. (In other words, $\int A^i\,dA = A^{i+1}/(i+1)$.) By Lemma 2.12 and using the induction hypothesis to calculate d/dA, we obtain the transformation property:

$$\begin{aligned}p^{2\lambda m}q^{\lambda^2 m}\widetilde{F}(pq^\lambda,q) &= e^{-\lambda(d/dA)}\widetilde{F}(p,q) = \widetilde{F}(p,q) - \lambda\frac{d}{dA}\widetilde{F} + \tfrac{1}{2}\lambda^2\Big(\frac{d}{dA}\Big)^2\widetilde{F} + \cdots\\ &= \widetilde{F}(p,q) - F_g^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N) + F_g^{S^{[n]}}(\mathrm{taut};e^{-\lambda T_\delta}\gamma_1,\dots,e^{-\lambda T_\delta}\gamma_N).\end{aligned}$$

Using this equation and (44) we conclude that

$$F(p,q) = F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) - \widetilde{F}(p,q)$$

satisfies

$$p^{2\lambda m} q^{\lambda^2 m} F(pq^\lambda, q) = F(p,q).$$

Since T_δ is a cubic in Nakajima operators (see (37)) its action on a cohomology class changes the parity of the number of Nakajima factors in which it is written. In particular, if $r = Nn + \sum_i l(\gamma_i)$ is even, then the function (45) is odd in p by Step 1, and, since $A(p,q)$ is odd in p, its integration with respect to A is again even in p. Similar arguments apply if r is odd. We obtain that

$$F(p^{-1}, q) = (-1)^{Nn+\sum_i l(\gamma_i)} F(p,q).$$

Using Lemmata 2.13 and 2.14 (depending on the parity of $Nn + \sum_i l_i$) we conclude that

$$F(p,q) = \begin{cases} \Delta(q)^{-1}\Theta^{2m}(p,q)\wp'(p,q)\sum_{i=2}^m f_i(q)\wp(p,q)^{m-i} & \text{if } Nn + \sum_i l(\gamma_i) \text{ is even,} \\ \Delta(q)^{-1}\Theta^{2m}(p,q)\sum_{i=0}^m f_i(q)\wp(p,q)^{m-i} & \text{if } Nn + \sum_i l(\gamma_i) \text{ is odd,} \end{cases} \tag{46}$$

for some power series $f_i(q) \in \mathbb{C}[\![q]\!]$. This proves the main claim.

Since $F(p,q)$ is written without any A, we have

$$\begin{aligned} 0 = \frac{d}{d\mathsf{A}} F(p,q) &= \frac{d}{d\mathsf{A}} F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) - \frac{d}{d\mathsf{A}} \widetilde{F}(p,q) \\ &= \frac{d}{d\mathsf{A}} F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) - \sum_{i=1}^N F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_{i-1}, T_\delta \gamma_i, \gamma_{i+1}, \dots, \gamma_N), \end{aligned}$$

that is, we also have the holomorphic anomaly equation (a) with respect to A. □

The argument in Step 1 of the proof more generally shows the following:

Lemma 4.2 *For any K3 surface S and effective curve class $\beta \in H_2(S, \mathbb{Z})$, the series*

$$Z_{g,\beta}^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) := \sum_{r \in \mathbb{Z}} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle_{g,\beta+rA}^{S^{[n]}} (-p)^r$$

is a Laurent polynomial in p, and if the γ_i are in the Nakajima basis, then

$$Z_{g,\beta}^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N)(p^{-1}) = (-1)^{Nn+\sum_i l(\gamma_i)} Z_{g,\beta}^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N).$$

5 Relative Gromov–Witten theory

5.1 Overview

Let X be a smooth projective divisor and let $D \subset X$ be a smooth divisor with connected components D_i for $i = 1, \dots, N$. In this section we consider the relative Gromov–Witten theory of the pair (X, D) introduced by Li [2001; 2002], see also [Argüz et al. 2023; Oberdieck 2024a] for introductions. In the first part we introduce the basic structures of the theory: moduli spaces, evaluation maps, psi classes, and rubber moduli spaces. Then, we recall three basic equations that will be needed later on: a splitting formula for the relative diagonal, proven recently in [Argüz et al. 2023] (Proposition 5.2); a splitting formula for relative psi-classes (Proposition 5.3), and finally we prove a new formula for the restriction of relative Gromov–Witten classes to the nonseparating boundary divisor in the moduli space of curves (Proposition 5.4).

5.2 Moduli space

Let $\beta \in H_2(X,\mathbb{Z})$ be a curve class and let $\vec{\lambda} = (\vec{\lambda}_1, \dots, \vec{\lambda}_N)$ be a tuple of ordered partitions $\lambda_i = (\lambda_{i,j})_{j=1}^{l}$ of size and length

$$|\vec{\lambda}_i| = \sum_j \lambda_{i,j} = D_i \beta \quad \text{and} \quad \ell(\lambda_i) = l.$$

Consider the moduli space of r–pointed genus-g degree-β relative stable maps from connected curves to the pair (X, D) with ordered ramification profile $\vec{\lambda}_i$ along the divisor D_i,

$$\overline{M}_{g,r,\beta}((X,D),\vec{\lambda}).$$

By definition, an element of the moduli space is a map $f : C \to X[k]$ where $X[k]$ is a target degeneration of X along D which satisfies a list of conditions (finite automorphism, predeformability, no components mapping entirely mapped to the singular fibers, relative multiplicities as specified). The degree of the map is $\pi_* f_*[C] = \beta$ where $\pi : X[k] \to X$ is the canonical map that contracts the expansion.

5.3 Evaluation maps

For every boundary divisor D_i we have relative evaluation maps

$$\mathrm{ev}^{\mathrm{rel}}_{i,j} : \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) \to D_i \quad \text{for } j = 1, \dots, \ell(\vec{\lambda}_i)$$

which send a stable map to the j^{th} intersection point with the divisor D_i.

We also have an interior evaluation map

$$\mathrm{ev} : \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) \to (X,D)^r$$

which takes values in the (smooth projective) moduli space $(X,D)^r$ of (ordered) tuples of r points on the relative geometry (X,D); see [Kim and Sato 2009] for a construction. For example, as a variety $(X,D)^1$ is isomorphic to X, and $(X,D)^2$ is the blowup $\mathrm{Bl}_{\bigsqcup_i D_i \times D_i}(X \times X)$. We refer to [Pandharipande and Pixton 2017; Argüz et al. 2023, Section 3.4] for beautiful self-explaining figures illustrating the situation. By forgetting points we have, for any $I \subset \{1, \dots, r\}$, contraction maps $p_I : (X,D)^r \to (X,D)^{|I|}$. We can hence view classes on $\prod_i (X,D)^{a_i}$ with $\sum_i a_i = r$ (such as X^r) as defining cohomology classes on $(X,D)^r$ via pullback by the projections. We write $\mathrm{ev}_I = p_I \circ \mathrm{ev}$.

The class of the locus in $(X,D)^2$ of incident points (the relative diagonal) is denoted by

$$\Delta^{\mathrm{rel}}_{(X,D)} \subset H^*((X,D)^2).$$

5.4 Psi-classes

There are cotangent line bundles at both interior and relative markings. We let their first Chern classes be denoted, respectively, by

$$\psi_i \quad \text{for } i = 1, \dots, r \qquad \text{and} \qquad \psi^{\mathrm{rel}}_{i,j} \quad \text{for } i = 1, \dots, N \text{ and } j = 1, \dots, \ell(\lambda_i).$$

Let also $\mathbb{L}_{D_i}$ be the cotangent line bundle associated to D_i on the stack of target expansions $\mathcal{T}$ as defined in [Maulik and Pandharipande 2006, 1.5.2]. The line bundle $\mathbb{L}_{D_i}$ has a section which vanishes precisely at expansions corresponding to bubbling at D_i. Let $\Psi_{D_i} = c_1(\mathbb{L}_{D_i})$ and let

$$\mathsf{q}\colon \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) \to \mathcal{T}$$

be the classifying map corresponding to the universal target over the moduli space. The relative ψ–classes then satisfy the following well-known lemma:

Lemma 5.1 $$\lambda_{i,j}\psi^{\mathrm{rel}}_{i,j} = \mathsf{q}^*(\Psi_i) - \mathrm{ev}^{\mathrm{rel}*}_{i,j}(c_1(N_{D_i/X})).$$

Proof See for example [Oberdieck and Pixton 2019, Proof of Lemma 12]. □

5.5 Cohomology-weighted partitions

Consider a $H^*(D_i)$–weighted partition μ

$$((\mu_1,\delta_1),\dots,(\mu_l,\delta_l)) \quad \text{for } \delta_j \in H^*(D_i) \text{ and } \mu_1 \geq \dots \geq \mu_l \geq 1. \tag{47}$$

We write $l = \ell(\mu)$ for the length and $|\mu| = \sum_i \mu_i$ for the size of the partition. The *partition underlying* μ is the ordered partition

$$\vec{\mu} = (\mu_1,\dots,\mu_l).$$

While the δ_i are arbitrary cohomology classes on D_i, we often take them to be elements of a fixed basis $\mathcal{B}$ of $H^*(D_i)$. In this case we say μ is $\mathcal{B}$–weighted. Given a $\mathcal{B}$–weighted partition μ, the automorphism group $\mathrm{Aut}(\mu)$ consists of the permutation symmetries of μ.

5.6 Gromov–Witten invariants

For $i \in \{1,\dots,N\}$ consider $H^*(D_i)$–weighted partitions

$$\lambda_i = ((\lambda_{i,j},\delta_{i,j}))_{j=1}^{\ell(\lambda_i)}$$

and let $\vec{\lambda}_i$ be the partition underlying λ_i. Fix also a class

$$\gamma \in H^*((X,D)^n).$$

We define relative Gromov–Witten invariants by integration over the virtual fundamental class [Li 2002] of the moduli space:

$$\langle \lambda_1,\dots,\lambda_N \mid \gamma\rangle^{(X,D)}_{g,\beta} := \int_{[\overline{M}_{g,r,\beta}((X,D),\vec{\lambda})]^{\mathrm{vir}}} \mathrm{ev}^*(\gamma) \prod_{i=1}^{N}\prod_{j=1}^{\ell(\lambda_i)} \mathrm{ev}^{\mathrm{rel}}_{i,j}(\delta_{i,j}).$$

We will also sometimes need to include ψ–classes in the integral. A more general definition is hence the following. Let $a_{i,j}$ and b_i be arbitrary nonnegative integers. Then

$$\Big\langle \Big(\lambda_i \prod_{j=1}^{\ell(\lambda_i)} (\psi^{\mathrm{rel}}_{i,j})^{a_{ij}}\Big)_{i=1}^{N} \,\Big|\, (\tau_{b_1}\cdots\tau_{b_r})(\gamma)\Big\rangle^{(X,D)}_{g,\beta} := \int_{[\overline{M}_{g,r,\beta}((X,D),\vec{\lambda})]^{\mathrm{vir}}} \prod_{i=1}^{r} \psi_i^{b_i}\, \mathrm{ev}^*(\gamma) \prod_{i=1}^{N} \prod_{j=1}^{\ell(\lambda_i)} (\psi^{\mathrm{rel}}_{i,j})^{a_{ij}}\, \mathrm{ev}^{\mathrm{rel}}_{i,j}(\delta_{i,j}). \tag{48}$$

If all $b_i = 0$ we will simply write γ instead of $\tau_{b_1}\cdots\tau_{b_r}(\gamma)$.

The discussion above also works when we allow the source curve of our relative stable map to be disconnected. More precisely, we let

$$\overline{M}^{\bullet}_{g,r,\beta}((X,D),\vec{\lambda})$$

denote the moduli space of relative stable maps to (X, D) as above except that we allow disconnected domain curves and require the following condition:

($\bullet$) For any stable map $f : \Sigma \to (S \times C)[l]$ to a target expansion of the pair $(S \times C, S_z)$, the stable map f has nonzero degree on each of its connected components.

We define Gromov–Witten invariants in the disconnected case completely parallel to (48). The brackets on the left-hand side will be denoted by a superscript $\bullet$, as in $\langle - \rangle^{(X,D),\bullet}$.

5.7 Rubber moduli space

For any of the divisors $E \in \{D_1, \dots, D_N\}$ consider the projective bundle

$$\mathbb{P} = \mathbb{P}(N_{E/X} \oplus \mathcal{O}_E) \to E.$$

The projection has two canonical sections $E_0, E_\infty \subset \mathbb{P}$, called the zero and infinite sections, with normal bundles $N_{E/\mathbb{P}} \cong N^{\vee}_{E/X}$ and $N_{E/X}$, respectively. Let

$$\overline{M}^{\sim}_{g,r,\alpha}((\mathbb{P}, E_0 \sqcup E_\infty), \vec{\lambda}) \tag{49}$$

be the moduli space of genus-g degree-$\alpha \in H_2(E,\mathbb{Z})$ rubber stable maps with target $(\mathbb{P}, E_0 \sqcup E_\infty)$. Elements of the moduli space are maps $f : C \to \mathbb{P}_l$, where $\mathbb{P}_l$ is a chain of l copies of $\mathbb{P}$ with zero sections glued along infinite section of the next components, satisfying a list of conditions. The degree of a rubber stable map is fixed here to be $\pi_{E*} f_*[C] = \alpha$, where $\pi_E : \mathbb{P}_l \to E$ is the natural projection. In the definition of (49) we let the source curve be connected. If we allow disconnected domains and require condition ($\bullet$), we decorate the moduli space (and the invariants below) with the superscript $\bullet$. As before, we have evaluation maps at the relative markings denoted by $\mathrm{ev}^{\mathrm{rel}}_{i,j}$. By evaluating the composition $\pi_E \circ f$ at the interior marked points we also have a well-defined interior evaluation map:

$$\mathrm{ev} : \overline{M}^{\sim}_{g,r,\alpha}((\mathbb{P}, E_0 \sqcup E_\infty), \vec{\lambda}) \to E^r.$$

Given $H^*(E)$–weighted partitions λ and μ, and $\gamma \in H^*(E^r)$, we define:

$$\langle \lambda, \mu \mid \gamma \rangle_{g,\alpha}^{(\mathbb{P}, E_0 \sqcup E_\infty),\sim} = \int_{[\overline{M}^{\sim}_{g,r,\alpha}((\mathbb{P}, E_0 \sqcup E_\infty),\vec{\lambda})]^{\mathrm{vir}}} \mathrm{ev}^*(\gamma) \prod_{i=1}^{N} \prod_{j=1}^{\ell(\lambda_i)} \mathrm{ev}_{i,j}^{\mathrm{rel}}(\delta_{i,j}).$$

5.8 Splitting formulas

We state two splitting formulas that we will need later on. Let $\iota\colon D \to X$ denote the inclusion. We begin with the splitting of the relative diagonal:

Proposition 5.2 *We have*

$$\begin{aligned}
&\langle \lambda_1, \dots, \lambda_N \mid \Delta^{\mathrm{rel}}_{(X,D)} \rangle_{g,\beta}^{(X,D),\bullet} \\
&\quad = \langle \lambda_1, \dots, \lambda_N \mid \Delta_X \rangle_{g,\beta}^{(X,D),\bullet} \\
&\qquad - \sum_{i=1}^{N} \sum_{\mu} \sum_{\substack{g_1+g_2=g+1-\ell(\mu) \\ \iota_*\alpha+\beta'=\beta}} \frac{\prod_i \mu_i}{|\mathrm{Aut}(\mu)|} \langle \lambda_1, \dots, \underbrace{\mu}_{i^{\mathrm{th}}}, \dots, \lambda_N \rangle_{g_1,\beta'}^{(X,D),\bullet} \langle \lambda_i, \mu^\vee \mid \Delta_D \rangle_{g_2,\alpha}^{(\mathbb{P}, D_{i,0} \sqcup D_{i,\infty}),\bullet,\sim}.
\end{aligned}$$

In the above formula, μ runs over all cohomology weighted partitions $\mu = \{(\mu_i, \gamma_{s_i})\}$ of size βD_i, with weights from a fixed basis $\{\gamma_i\}$ of $H^*(D_i)$. Moreover, we let $\mu^\vee = \{(\eta_i, \gamma_{s_i}^\vee)\}$ be the dual partition, with weights from the basis $\{\gamma_i^\vee\}$ which is dual to $\{\gamma_i\}$.

Proof This is a special case of [Argüz et al. 2023, Theorem 3.10]. □

Next we explain how to remove the relative ψ–classes. Again we only need a special case (the general case is similar), and without loss of generality we can consider relative ψ–classes for the first component D_1.

Proposition 5.3 *For any $j \in \{1, \dots, \ell(\lambda_1)\}$,*

$$\begin{aligned}
&\lambda_{1,j} \langle \psi^{\mathrm{rel}}_{1,j} \lambda_1, \dots, \lambda_N \rangle_{g,\beta}^{(X,D),\bullet} \\
&\quad = -\langle \hat{\lambda}_1, \lambda_2, \dots, \lambda_N \rangle_{g,\beta}^{(X,D),\bullet} \\
&\qquad + \sum_{\mu} \sum_{\substack{g_1+g_2=g+1-\ell(\mu) \\ \iota_*\alpha+\beta'=\beta}} \frac{\prod_i \mu_i}{|\mathrm{Aut}(\mu)|} \langle \lambda_1, \dots, \underbrace{\mu}_{i^{\mathrm{th}}}, \dots, \lambda_N \rangle_{g_1,\beta'}^{(X,D),\bullet} \langle \lambda_i, \mu^\vee \rangle_{g_2,\alpha}^{(\mathbb{P}, D_{1,0} \sqcup D_{1,\infty}),\bullet,\sim},
\end{aligned}$$

where $\hat{\lambda}_1$ is the weighted partition λ_1 but with j^{th} cohomology weight δ_{1j} replaced by $\delta_{1j} \cup c_1(N_{D_1/X})$. Moreover, μ runs over the same data as in Proposition 5.2.

Proof This follows from Lemma 5.1 and [Li 2002]; compare [Oberdieck and Pixton 2019, Lemma 12]. □

5.9 Boundary restriction

We will also require the restriction of relative Gromov–Witten classes to the boundary. Consider the class in $H_*(\overline{M}_{g,r,\beta}((X,D)))$ defined by

$$J_{g,\beta}^{(X,D)}(\lambda \mid \gamma) = \mathrm{ev}^*(\gamma) \prod_{i=1}^{N} \prod_{j=1}^{\ell(\lambda_i)} \mathrm{ev}_{i,j}^{\mathrm{rel}}(\delta_{i,j})[\overline{M}_{g,r,\beta}((X,D),\vec{\lambda})]^{\mathrm{vir}}. \tag{50}$$

If there exists a forgetful morphism

$$\tau\colon \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) \to \overline{M}_{g,n},$$

where $n = r + \sum_i \ell(\vec{\lambda}_i)$, consider also the pushforward

$$I_{g,\beta}^{(X,D)}(\lambda \mid \gamma) = \tau_*\biggl(\mathrm{ev}^*(\gamma) \prod_{i=1}^{N} \prod_{j=1}^{\ell(\lambda_i)} \mathrm{ev}_{i,j}^{\mathrm{rel}}(\delta_{i,j})[\overline{M}_{g,r,\beta}((X,D),\vec{\lambda})]^{\mathrm{vir}}\biggr). \tag{51}$$

Let $u\colon \overline{M}_{g-1,n+1} \to \overline{M}_{g,n}$ be the natural gluing morphism.

Proposition 5.4 *We have*

$$\begin{aligned}
&u^* I_{g,\beta}^{(X,D)}(\lambda_1,\dots,\lambda_N) \\
&\quad= I_{g-1,\beta}^{(X,D)}(\lambda_1,\dots,\lambda_N \mid \Delta_{(X,D)}^{\mathrm{rel}}) + \sum_{i=1}^{N} \sum_{\substack{m\ge 0\\ g=g_1+g_2+m\\ \beta=\beta'+\iota_*\alpha}} \sum_{\substack{b,b_1,\dots,b_m\\ l,l_1,\dots,l_m}} \frac{\prod_{i=1}^m b_i}{m!} \\
&\qquad \Bigl\{\xi_* j^*\Bigl[J_{g_1,\beta'}^{(X,D),\bullet}\bigl(\lambda_1,\dots,\lambda_{i-1},(\underbrace{(b,\Delta_{D_i,l})}_{(n+1)^{th}},(b_j,\Delta_{D_i,l_j})_{j=1}^m),\lambda_{i+1},\dots,\lambda_N\bigr) \\
&\qquad\qquad \boxtimes J_{g_2,\alpha}^{(\mathbb{P},D_{i,0}\sqcup D_{i,\infty}),\bullet,\sim}\bigl((\underbrace{(b,\Delta_{D_i,l}^\vee)}_{(n+2)^{th}},(b_j,\Delta_{D_i,l_j}^\vee)_{j=1}^m),\lambda_i\bigr)\Bigr] \\
&\qquad + \xi_* j^*\Bigl[J_{g_1,\beta'}^{(X,D),\bullet}\bigl(\lambda_1,\dots,\lambda_{i-1},(\underbrace{(b,\Delta_{D_i,l})}_{(n+2)^{th}},(b_j,\Delta_{D_i,l_j})_{j=1}^m),\lambda_{i+1},\dots,\lambda_N\bigr) \\
&\qquad\qquad \boxtimes J_{g_2,\alpha}^{(\mathbb{P},D_{i,0}\sqcup D_{i,\infty}),\bullet,\sim}\bigl((\underbrace{(b,\Delta_{D_i,l}^\vee)}_{(n+1)^{th}},(b_j,\Delta_{D_i,l_j}^\vee)_{j=1}^m),\lambda_i\bigr)\Bigr]\Bigr\},
\end{aligned}$$

where

- $(n+1)^{th}$ *stands for labeling the corresponding marked points by* $n+1$,
- $b, b_1, \dots, b_m$ *run over all positive integers such that* $b + \sum_j b_j = \beta D_i$,
- $\Delta_D = \sum_l \Delta_{D,l} \otimes \Delta_{D,l}^\vee$ *is a Künneth decomposition of the diagonal of* D.

Moreover, j *is the embedding of the (closed and open) component*

$$U \subset \overline{M}_{g_1,\beta'}((X,D),(\vec{\lambda}\setminus\vec{\lambda}_i,(b,b_1,\dots,b_m))) \times \overline{M}_{g_2,\alpha}^{\bullet,\sim}((\mathbb{P},D_{i,0}\sqcup D_{i,\infty}),\vec{\lambda}_i,(b,b_1,\dots,b_m))$$

parametrizing pairs $(f_1\colon C_1 \to X[k], p_i)$ *and* $(f_2\colon C_2 \to \mathbb{P}_l, p_i')$ *such that the curve, which is obtained by gluing* C_1 *to* C_2 *pairwise along the* m *markings labeled by* b_i*, is connected. And we let*

$$\xi\colon U \to \overline{M}_{g-1,n+2}$$

be the map that forgets the maps f_1 *and* f_2*, glues together the curves* C_1 *and* C_2 *pairwise along the markings labeled by* b_i*, and then contracts unstable components.*

A related formula for the restriction of the double ramification cycle to the divisor $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ was given (only with a sketch) by Zvonkine [2015].

Proof Let $\mathfrak{M}_{g,n}$ be the Artin stack of prestable curves, where $n = \sum_i \ell(\lambda_i)$. We refer to [Bae and Schmitt 2022] for an introduction to the stack $\mathfrak{M}_{g,n}$. The map τ factors as a morphism $\tilde{\tau}$ to $\mathfrak{M}_{g,n}$ followed by the stabilization map $\mathrm{st}\colon \mathfrak{M}_{g,n} \to \overline{M}_{g,n}$. Form the fiber diagram

$$\begin{array}{ccccc}
M_1 & \xrightarrow{\;q\;} & M_2 & \longrightarrow & \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) \\
\big\downarrow{\scriptstyle\rho} & & \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\tilde{\tau}} \\
\mathfrak{M}_{g-1,n+2} & \xrightarrow{\;\tilde{q}\;} & W & \xrightarrow{\;u'\;} & \mathfrak{M}_{g,n} \\
 & & \big\downarrow{\scriptstyle\mathrm{st}} & & \big\downarrow{\scriptstyle\mathrm{st}} \\
 & & \overline{M}_{g-1,n+2} & \xrightarrow{\;u\;} & \overline{M}_{g,n}
\end{array}$$

Consider also the gluing map on prestable curves

$$\tilde{u} = u' \circ \tilde{q}\colon \mathfrak{M}_{g-1,n+2} \to \mathfrak{M}_{g,n}.$$

We want to apply Proposition 5.6. Observe the following:

- $\mathfrak{M}_{g,n}$ is smooth, and by [Bae and Schmitt 2022, Example 4] has a good filtration by quotient stacks.
- Since $u'\colon W \to \mathfrak{M}_{g,n}$ is representable and $\mathfrak{M}_{g,n}$ has affine stabilizers at geometric points [loc. cit., Proposition 3.1], by [Kresch 1999, Propositions 3.5.5 and 3.5.9] W has affine stabilizers at geometric points.
- The gluing maps $u\colon \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ and $\tilde{u}\colon \mathfrak{M}_{g-1,n+2} \to \mathfrak{M}_{g,n}$ are both representable [Bae and Schmitt 2022, Lemma 2.2].
- By [loc. cit., Proposition 3.13] the map
$$\tilde{q}\colon \mathfrak{M}_{g-1,n+2} \to W = \mathfrak{M}_{g,n} \times_{\overline{M}_{g,n}} \overline{M}_{g-1,n+2}$$
is proper and birational. Since $\tilde{u}$ is representable, $\tilde{q}$ is representable.
- Since the domain and target of $\tilde{u}$ are smooth, $\tilde{u}$ is lci.
- By [Behrend 1997, Proposition 3], $\mathrm{st}\colon \mathfrak{M}_{g,n} \to \overline{M}_{g,n}$ is flat. Since $u\colon \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ is lci, and this is preserved by flat base change (see [Stacks 2005–, Tag 069I]), u' is also lci.
- The map $a\colon \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) \to \mathfrak{M}_{g,n}$ is representable, since it is injective on stabilizers: the group of automorphisms of $(C \to X[k], p_i)$ is a subgroup of the group of automorphisms of (C, p_i).

By the above, $\tilde{u}$ and u are proper representable, so M_1 and M_2 are proper DM stacks.

By Proposition 5.6 we obtain that

$$(\iota')^! = q_* \tilde{\iota}^! \colon A_*(\overline{M}_{g,r,\beta}((X,D),\vec{\lambda})) \to A_{*-1}(M_2).$$

Consider the class

$$J := J^{(X,D)}_{g,\beta}(\lambda).$$

We obtain

$$u^* I^{(X,D)}_{g,\beta}(\lambda) = (\mathrm{st}\circ\sigma)_* u^! J = (\mathrm{st}\circ\sigma)_*(u')^! J = (\mathrm{st}\circ\sigma)_* q_* \tilde{u}^! J) = (q\circ\mathrm{st}\circ\sigma)_* \tilde{u}^! J = \pi_* \tilde{u}^! J,$$

where

$$\pi = q\circ\mathrm{st}\circ\sigma \colon \mathfrak{M}_{g-1,n+2} \times_{\mathfrak{M}_{g,n}} \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) \to \overline{M}_{g,n+2}.$$

Hence we need to compute the refined pullback $\tilde{u}^! J$.

The stack

$$\mathfrak{M}_{g-1,n+2} \times_{\mathfrak{M}_{g,n}} \overline{M}_{g,r,\beta}((X,D),\vec{\lambda})$$

parametrizes relative stable maps $(f\colon C \to X[k], p_1,\dots,p_r)$ together with a chosen nonseparating nodal point $p \in C$ and two markings p_{n_1} and p_{n+2} on the partial normalization $\widetilde{C} \to C$ at p. By [Argüz et al. 2023, Section 1.5] we have a disjoint union (with both components open and closed)

$$\mathfrak{M}_{g-1,n+2} \times_{\mathfrak{M}_{g,n}} \overline{M}_{g,r,\beta}((X,D),\vec{\lambda}) = \mathcal{P}_{g,r,\beta}((X,D),\vec{\lambda}) \sqcup \mathcal{N}_{g,r,\beta}((X,D),\vec{\lambda}).$$

The component $\mathcal{P}_{g,r,\beta}((X,D),\vec{\lambda})$ parametrizes relative stable maps where the marked point p maps to a *nonsingular* point on some expanded degeneration $X[k]$ of (X,D). By [Argüz et al. 2023, Theorem 3.2] we then have

$$\pi_*(\tilde{u}^!(J)|_{\mathcal{P}_{g,r,\beta}((X,D),\vec{\lambda})}) = I^{(X,D)}_{g-1,\beta}(\lambda_1,\dots,\lambda_N \mid \Delta^{\mathrm{rel}}_{(X,D)}).$$

The other component $\mathcal{N}_{g,r,\beta}((X,D),\vec{\lambda})$ parametrizes maps where p maps to the singular locus, and hence forces a splitting of the source curve C,

$$C = C_1 \cup C_2,$$

where $f|_{C_1}\colon C_1 \to X[a]$ is a relative stable map to (X,D) and $f|_{C_2}\colon C_2 \to \mathbb{P}_l$ maps entirely into a bubble of D_i for some i. The marked points p_{n+1} and p_{n+2} have to lie on different components C_i, and hence there are two choices: p_{n+1} can lie on C_1 and p_{n+2} on C_2, or vice versa. The curve C is obtained by gluing C_1 and C_2 along p_{n+1} and p_{n+2}, as well as along "secondary" markings $q_i \in C_1$ and $q_i' \in C_2$ for $i = 1,\dots,m$. These markings are called "secondary" because they will be forgotten by pushforward along π to $\overline{M}_{g-1,n+2}$. Let b be the contact order of f with the divisor at p_{n+1}, and let b_i be the contact order at the q_i.

We consider the local structure of the component $\mathcal{N}_{g,r,\beta}((X,D),\vec{\lambda})$. A local versal family for the gluing nodes of C is given by $xy = s$ and $x_i y_i = s_i$ for $i = 1,\dots,m$. Let t be étale locally the coordinate defining the bubble splitting $X[a] \cup \mathbb{P}_l$. The coordinate t is pulled back from the stack of target degeneration. Then the local analysis of [Li 2002, Section 4.4] shows that $t = s^b$ and $t = s_i^{b_i}$. Hence $\mathcal{N}_{g,r,\beta}((X,D),\vec{\lambda})$, which is cut out by $s=0$, is given by the equations $\{s=0, s_i^{b_i}=0\}$. On the other hand, the image stack of the gluing morphism

$$\overline{M}_{g_1,\beta'}\big((X,D),(\vec{\lambda}\setminus\vec{\lambda}_i,(b,b_1,\dots,b_m))\big)\times_{D^{m+1}}\overline{M}^{\bullet,\sim}_{g_2,\alpha}((\mathbb{P},D_{i,0}\sqcup D_{i,\infty}),\vec{\lambda}_i,(b,b_1,\dots,b_m)) \xrightarrow{\xi} \mathcal{N}_{g,r,\beta}((X,D),\vec{\lambda}) \tag{52}$$

is given by $\{s=0, s_i=0\}$. Since the gluing morphism is finite of degree $|\mathrm{Aut}(\eta)|$, by the arguments in [Li 2002], especially Lemma 3.12, one obtains that the virtual class of $\mathcal{N}_{g,r,\beta}((X,D),\vec{\lambda})$ is $\prod_{i=1}^m b_i/|\mathrm{Aut}(b_1,\dots,b_m)|$ times the pushforward by ξ of the natural virtual class on the domain of the map (52).[6] In total one obtains:

$$\begin{aligned}(\tilde{u}^![\overline{M}_{g,r,\beta}(X,D)]^{\mathrm{vir}})|_{\mathcal{N}_{g,r,\beta}((X,D),\vec{\lambda})} &= \sum_{i=1}^N \sum_{\substack{m\geq 0\\ g=g_1+g_2+m\\ \beta'+\iota_*\alpha}} \sum_{b;b_1,\dots,b_m} \frac{\prod_{i=1}^m b_i}{m!}\xi_*\Delta^!_{D^{m+1}} j^* \\ &\quad\cdot\Big([\overline{M}_{g_1,\beta'}((X,D),\vec{\lambda}\setminus\vec{\lambda}_i,(\underbrace{b}_{n+1},b_1,\dots,b_m))]^{\mathrm{vir}}\times[\overline{M}^{\bullet,\sim}_{g_2,\alpha}((\mathbb{P},D_{i,0}\sqcup D_{i,\infty}),\vec{\lambda}_i,(\underbrace{b}_{n+2},b_1,\dots,b_m))]^{\mathrm{vir}}\Big) \\ &\quad+(\text{same term with the roles of } (n+1) \text{ and } (n+2) \text{ interchanged}).\end{aligned} \tag{53}$$

Here we have viewed $(b_1,\dots,b_m)$ as a list of numbers and not as a partition, so that the factor $1/|\mathrm{Aut}(b_1,\dots,b_m)|$ has to be replaced by $1/m!$ to compensate for overcounting.

Pushing forward (53) by π completes the proof. □

Example 5.5 We adapt a basic example from [Li 2004] which illustrates the local analysis in the last step of the proof above in the case of a universal target $(\mathcal{A},\mathcal{D})=(\mathbb{A}^1/\mathbb{G}_m, 0/\mathbb{G}_m)$. The universal target was introduced in [Abramovich et al. 2017]; see also [Argüz et al. 2023, Proof of Theorem 3.2]. We let w_0 be the coordinate on the chart $\mathbb{A}^1\to\mathcal{A}$. Let $\mathcal{T}^1=\mathbb{A}^1/\mathbb{G}_m$ be the stack of 1–step target expansion of $(\mathcal{A},\mathcal{D})$. The universal family of targets over $\mathcal{T}^1$ is

$$\widetilde{\mathbb{A}^1[1]}=\mathrm{Bl}_0(\mathbb{A}^1\times\mathbb{A}^1)\to\mathbb{A}^1,$$

modulo a quotient by $\mathbb{G}_m^3$. Explicitly, if t is the coordinate on $\mathbb{A}^1$ (the chart of $\mathcal{T}^1$), then

$$\widetilde{\mathbb{A}^1[1]}=\mathrm{Bl}_0(\mathbb{A}^1\times\mathbb{A}^1)=V(w_0z_1=tw_1)\subset\mathbb{A}^1_{w_0}\times\mathbb{P}^1\times\mathbb{A}^1_t,$$

where w_1 and z_1 are the homogeneous coordinates on $\mathbb{P}^1$.

[6] The more modern viewpoint is to work relative to the moduli space of stable maps to the universal target $(\mathbb{A}^1/\mathbb{G}_m, 0/\mathbb{G}_m)$ as proposed in [Abramovich et al. 2017]. The moduli space $\overline{M}_{g,n,d}(\mathbb{A}^1/\mathbb{C}^*, 0/\mathbb{G}_m)$ is pure of expected dimension, and the virtual class on $\overline{M}_{g,r,\beta}(X,D)$ is the virtual pullback of the fundamental class on $\overline{M}_{g,n,d}(\mathbb{A}^1/\mathbb{C}^*, 0/\mathbb{G}_m)$. The local argument above proves an equality of codimension-1 classes in $\overline{M}_{g,n,d}(\mathbb{A}^1/\mathbb{C}^*, 0/\mathbb{G}_m)$. The equality (53) of virtual classes on $\overline{M}_{g,r,\beta}(X,D)$ follows from this by virtual pullback (after matching the relative perfect obstruction theories). See [Argüz et al. 2023, Proof of Theorem 3.2] for a similar case. I thank P Bousseau for discussions related to this point.

Consider a family of degenerating curves $C = \mathbb{A}^2 \to \mathbb{A}^1_s$ given by $(x, y) \mapsto s = xy$, and consider the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathbb{A}^2_{w_0,Z} \subset \widetilde{\mathbb{A}^1[1]} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_s & \longrightarrow & \mathbb{A}^1_t \end{array}$$

where we let $\mathbb{A}^2_{w_0,Z}$ be the affine chart $\operatorname{Spec}(\mathbb{C}[w_0, Z]) \subset \widetilde{\mathbb{A}^1[1]}$ for $Z = z_1/w_1$, and the map f is described by $x \mapsto w^r$ and $y \mapsto Z^r$. Then the lower horizontal map is given by $t \mapsto s^r$, that is the coordinate defining the bubble t corresponds to the r^{th} power of the coordinate defining the node of C.

Proposition 5.6 (Schmitt) *Consider the following data*:

- *Let X, Y and Z be algebraic stacks locally of finite type over $\mathbb{C}$ of pure dimension, and assume that Y has affine stabilizers at geometric points, and that Z is smooth and has a good filtration by finite-type substacks.*[7]
- *Let $g\colon X \to Y$ be proper birational of DM type, let $f\colon Y \to Z$ be representable and lci of relative dimension k, and assume that $h = g \circ f\colon X \to Z$ is representable and lci.*
- *Let W be a finite-type DM stack and let $a\colon W \to Z$ be a representable morphism.*

Consider the fiber diagram

$$\begin{array}{ccccc} U & \xrightarrow{\tilde{g}} & V & \xrightarrow{\tilde{f}} & W \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \end{array}$$
$$h\colon X \to Z$$

Then we have

$$f^! = \tilde{g}_* h^! \colon A_*(W) \to A_{*+k}(V).$$

Proof We work with the Chow groups as introduced in [Bae and Schmitt 2022, Appendix A; Kresch 1999]. In particular, for any locally finite-type algebraic stack $\mathcal{X}$ over $\mathbb{C}$ we define

$$A_*(\mathcal{X}) = \varprojlim_i A_*(\mathcal{U}_i),$$

where $(\mathcal{U}_i)_{i \in I}$ is a directed system of finite-type open substacks of $\mathcal{X}$ whose union is all of $\mathcal{X}$, and the Chow groups $A_*(\mathcal{U}_i)$ are taken with $\mathbb{Q}$–coefficients in the sense of Kresch [1999]. If $\mathcal{X}$ is pure dimensional and admits a good filtration $(\mathcal{U}_m)_{m \in \mathbb{N}}$ by finite-type substacks then

$$A_{\dim(\mathcal{X})-d}(\mathcal{X}) = A_{\dim(\mathcal{X})-d}(\mathcal{U}_m) \quad \text{for all } m > d.$$

In this case all functionalities of Kresch's Chow groups also apply to $A_*(\mathcal{X})$.

[7] In the sense of [Bae and Schmitt 2022, Definition A.2] or [Oesinghaus 2019, Definition 5], ie there exists a collection $(\mathcal{U}_m)_{m \in \mathbb{N}}$ of open substacks of finite type of Z with $\mathcal{U}_m \subset \mathcal{U}_l$ for $m \leq l$ and such that $\dim(Z \setminus \mathcal{U}_m) < \dim(Z) - m$.

Kresch defines only a projective pushforward. A proper pushforward along proper morphisms of DM type has been defined in [Bae and Schmitt 2022, Theorem B.17], assuming that the target has affine stabilizers at geometric points, or equivalently is stratified by quotient stacks [Kresch 1999, Theorem 2.1.12]. In particular, by our assumptions on Y there exists a proper pushforward g_*.

Assume first that W is a smooth finite-type scheme. Then since the source and target of a are smooth, a is lci. By the commutativity of refined pullbacks [Kresch 1999], and the compatibility [Bae and Schmitt 2022, Proposition B.18] of proper pushforward (along the DM-type morphism g) and refined Gysin pullback (along the representable morphism a), we then have

$$f^![W]=f^!a^![Z]=a^!f^![Z]=a^![Y]\overset{(*)}{=}a^!g_*[X]=\tilde{g}_*a^![X]=\tilde{g}_*a^!h^![Z]=\tilde{g}_*h^!a^![Z]=\tilde{g}_*h^![X], \tag{54}$$

where $(*)$ follows since g is birational and hence of degree 1; compare [Bae et al. 2023, Proposition 25].

In the general case, the Chow group of W is generated by $\iota_*[\widetilde{W}]$, where $\widetilde{W}$ are smooth finite-type schemes and $\iota\colon \widetilde{W}\to W$ is proper and representable. Form the fiber diagram

$$\begin{array}{ccccc}
\widetilde{U} & \xrightarrow{\tilde{\tilde{g}}} & \widetilde{V} & \xrightarrow{\tilde{\tilde{f}}} & \widetilde{W} \\
\downarrow{\scriptstyle \iota''} & & \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} \\
U & \xrightarrow{\tilde{g}} & V & \xrightarrow{\tilde{f}} & W \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\
X & \xrightarrow{g} & Y & \xrightarrow{f} & Z
\end{array}
\qquad h = f\circ g\colon X\to Z$$

With (54), and using again the compatibility [Bae and Schmitt 2022, Proposition B.18] of proper pushforward and refined Gysin pullback (along the representable morphisms f, h), we find:

$$f^!\iota_*[\widetilde{W}]=\iota'_*f^![\widetilde{W}]=\iota'_*\tilde{\tilde{g}}_*h^![\widetilde{W}]=\tilde{g}_*\iota''_*h^![\widetilde{W}]=\tilde{g}_*h^!\iota_*[\widetilde{W}]. \qquad \square$$

6 Relative Gromov–Witten theory of $(\mathrm{K3}\times C, \mathrm{K3}_z)$

6.1 Overview

Let S be a smooth projective K3 surface, let C be a smooth curve and let $z=(z_1,\dots,z_N)$ be a tuple of distinct points $z_i\in C$. We specialize here to the relative Gromov–Witten theory of the pair

$$(S\times C, S_z),\qquad S_z=\bigsqcup S\times\{z_i\}. \tag{55}$$

After introducing our notation for the relative Gromov–Witten invariants in Section 6.2, we state in Section 6.3 our main input: the correspondence between relative invariants of $(S\times C, S_z)$ and the invariants of the Hilbert scheme of S (Theorem 6.2).

Then we discuss further preliminaries. In Section 6.4 we state the reduced degeneration formula. In Sections 6.6 and 6.7 we give basic evaluations of nonreduced invariants and reduced rubber invariants.

The curve classes on $S \times C$ will be denoted throughout by

$$(\beta, n) = \iota_* \beta + n[C] \in H_2(S \times C, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z}[C].$$

6.2 Definition

For $i \in \{1, \dots, N\}$, consider $H^*(S)$–weighted partitions

$$\lambda_i = ((\lambda_{i,j}, \delta_{i,j}))_{j=1}^{\ell(\lambda_i)}$$

of size n with underlying partition $\vec{\lambda}_i$. Let $\gamma \in H^*((S \times C, S_z)^r)$ be a cohomology class. If $\beta \neq 0$, define the partition function of reduced Gromov–Witten invariants

$$\begin{aligned}(56)\quad & Z^{(S\times C, S_z)}_{\mathrm{GW},(\beta,n)}(\lambda_1, \dots, \lambda_N \mid (\tau_{k_1} \cdots \tau_{k_r})(\gamma)) \\ &= (-1)^{(1-g(C)-N)n + \sum_i \ell(\lambda_i)} z^{(2-2g(C)-N)n + \sum_i \ell(\lambda_i)} \\ &\quad \cdot \sum_{g \in \mathbb{Z}} (-1)^{g-1} z^{2g-2} \langle \lambda_1, \dots, \lambda_N \mid (\tau_{k_1} \cdots \tau_{k_r})(\gamma) \rangle^{(S \times C, S_z), \bullet}_{g,(\beta,n)},\end{aligned}$$

where the invariants on the right-hand side are defined by integration over the *reduced virtual fundamental class* of the moduli space which is obtained by cosection localization [Kiem and Li 2013] from the surjective cosection constructed in [Maulik and Pandharipande 2013; Maulik et al. 2010]. If all $k_i = 0$, we often just write γ instead of $\tau_{k_1} \cdots \tau_{k_r}(\gamma)$. Sometimes we will also include psi classes $\psi^{\mathrm{rel}}_{i,j}$ at the relative markings where we follow the notation of (48). If $\beta = 0$, the series (56) is defined to vanish.

For any (β, m) the moduli space $\overline{M}^{\bullet}_{g,r,(\beta,n)}((S \times C, S_z), \vec{\lambda})$ also carries the ordinary or standard (ie nonreduced) virtual class. By the existence of the nontrivial cosection it vanishes for all $\beta \neq 0$, so it is only interesting for $\beta = 0$. In case $\beta = 0$ we denote it by $[-]^{\mathrm{std}}$. If we integrate over the "standard" virtual class, we decorate the corresponding Gromov–Witten bracket and the partition function Z with a superscript std. The rest of the notation is unchanged.

We can associate to every $H^*(S)$–weighted partition a class on the Hilbert scheme:

Definition 6.1 The class in $H^*(S^{[n]})$ associated to a $H^*(S)$–weighted partition $\mu = \{(\mu_i, \delta_i)\}$ of size n is defined by

$$\mu = \frac{1}{\prod_i \mu_i} \prod_i \mathfrak{q}_{\mu_i}(\delta_i) v_{\varnothing}. \tag{57}$$

We extend the Gromov–Witten bracket (48) for $(S \times C, S_z)$, and the partition functions $Z(-)$ by multilinearity in the entries λ_i. Since the Gromov–Witten bracket is invariant under permutations of relative markings that preserve the ramification profile (ie under $\mathrm{Aut}(\vec{\lambda}_i)$), the partition function $Z^{(S\times C, S_z)}_{\mathrm{GW},(\beta,n)}(\lambda_1, \dots, \lambda_N \mid \gamma)$ only depends on the associated class $\lambda_i \in H^*(S^{[n]})$. Hence we obtain a morphism:

$$Z^{(S\times C, S_z)}_{\mathrm{GW},(\beta,n)}(-, \dots, - \mid \gamma) \colon H^*(S^{[n]})^{\otimes N} \to \mathbb{Q}((z)).$$

6.3 Hilb/GW correspondence

Assume that $2g(C)-2+N>0$ so that $(C,z_1,\dots,z_N)$ is a marked stable curve,

$$\xi=[(C,z_1,\dots,z_N)]\in \overline{M}_{g,N}.$$

Given classes $\lambda_1,\dots,\lambda_k\in H^*(S^{[n]})$ we define the generating series

$$Z^{(S\times C,S_z)}_{\mathrm{Hilb},(\beta,n)}(\lambda_1,\dots,\lambda_N)=\sum_{r\in\mathbb{Z}}(-p)^r\int_{[\overline{M}_{g(C),N}(S^{[n]},\beta+rA)]^{\mathrm{vir}}}\tau^*([\xi])\prod_i \mathrm{ev}_i^*(\lambda_i). \tag{58}$$

By Lemma 4.2 the series (58) is a Laurent polynomial in p.

Theorem 6.2 [Nesterov 2021; 2024; Oberdieck 2024a] *If $\beta\in H_2(S,\mathbb{Z})$ is primitive, then*

$$Z^{(S\times C,S_z)}_{\mathrm{Hilb},(\beta,n)}(\lambda_1,\dots,\lambda_N)=Z^{(S\times C,S_z)}_{\mathrm{GW},(\beta,n)}(\lambda_1,\dots,\lambda_N)$$

under the variable change $p=e^z$.

Proof Nesterov [2021; 2024] showed that the left-hand side is equal to a partition function of relative Pandharipande–Thomas invariants of $(S\times C,S_z)$; see in particular [Nesterov 2024, Corollary 4.5]. The statement then follows from the GW/PT correspondence for $(S\times C,S_z)$ proven in [Oberdieck 2024a, Theorem 1.2] whenever β is primitive. □

Remark 6.3 If the multiple cover conjecture [Oberdieck and Pandharipande 2016, C2] holds for an effective curve class $\beta\in H_2(S,\mathbb{Z})$ then Theorem 6.2 also holds for β [Oberdieck 2024a, Proposition 1.4].

6.4 Degeneration formula

We recall the reduced degeneration formula for reduced invariants. Let $C\rightsquigarrow C_1\cup_x C_2$ be a degeneration of C. Let

$$\{1,\dots,N\}=A_1\sqcup A_2$$

be a partition of the index set of relative divisors, and write $z(A_i)=\{z_j\mid j\in A_i\}$. We choose that the points in A_i specialize to the curve C_i disjoint from x. Recall also the Künneth decomposition of the diagonal of the Hilbert scheme in the Nakajima basis:

Lemma 6.4 *In $H^*(S^{[n]}\times S^{[n]})$ we have*

$$\Delta_{S^{[n]}}=\sum_{\mu}(-1)^{n-\ell(\mu)}\frac{\prod_i\mu_i}{|\mathrm{Aut}(\mu)|}\mu\boxtimes\mu^\vee, \tag{59}$$

where μ runs over all cohomology-weighted partitions $\mu=\{(\mu_i,\gamma_{s_i})\}$ with weights from a fixed basis $\mathcal{B}=(\gamma_1,\dots,\gamma_{24})$ of $H^*(S)$, and $\mu^\vee=\{(\eta_i,\gamma_{s_i}^\vee)\}$ is the dual partition.

Proof For $\mathcal{B}$–weighted partitions μ and ν one has $\int_{S^{n]}}\mu\nu^\vee=\delta_{\mu\nu}(-1)^{n+\ell(\mu)}|\mathrm{Aut}(\mu)|/\prod_i\mu_i$. □

Proposition 6.5 *For any $\alpha_i \in H^*(S\times C)$ we have*

$$Z^{(S\times C,S_z)}_{\mathrm{GW},(\beta,n)}\Big(\lambda_1,\dots,\lambda_N \,\Big|\, \prod_i \tau_{k_i}(\alpha_i)\Big) = \sum_{\{1,\dots,r\}=B_1\sqcup B_2} \Bigg(Z^{(S\times C_1,S_{z(A_1),x})}_{(\beta,n)}\Big(\prod_{i\in A_1}\lambda_i,\Delta_1 \,\Big|\, \prod_{i\in B_1}\tau_{k_i}(\alpha_i)\Big) Z^{(S\times C_2,S_{z(A_2),x}),\mathrm{std}}_{(0,n)} \cdot \Big(\prod_{i\in A_2}\lambda_i,\Delta_2 \,\Big|\, \prod_{i\in B_2}\tau_{k_i}(\alpha_i)\Big)$$
$$+ Z^{(S\times C_1,S_{z(A_1),x}),\mathrm{std}}_{(0,n)}\Big(\prod_{i\in A_1}\lambda_i,\Delta_1 \,\Big|\, \prod_{i\in B_1}\tau_{k_i}(\alpha_i)\Big) Z^{(S\times C_2,S_{z(A_2),x})}_{(\beta,n)} \cdot \Big(\prod_{i\in A_2}\lambda_i,\Delta_2 \,\Big|\, \prod_{i\in B_2}\tau_{k_i}(\alpha_i)\Big)\Bigg),$$

where (Δ_1,Δ_2) stands for summing over the Künneth decomposition of the diagonal (59).

Proof The required modifications to the usual degeneration formula of Li [2001; 2002] needed in the reduced case are discussed in [Maulik et al. 2010]. We refer also to [Oberdieck 2024a, Section 5.3] for a discussion of the matching of signs and exponents, and to [loc. cit., Section 8.1] for a conceptual explanation for the form of the equation. □

6.5 Rubber invariants

We will also need generating series of rubber invariants. For any $\alpha_i \in H^*(S)$ define

$$Z^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{\mathrm{GW},(\beta,n)}\Big(\lambda,\mu \,\Big|\, \prod_i \tau_{k_i}(\alpha_i)\Big) = (-1)^{-n+\ell(\lambda)+\ell(\mu)} z^{\ell(\lambda)+\ell(\mu)} \sum_{g\in\mathbb{Z}} (-1)^{g-1} z^{2g-2} \Big\langle \lambda,\mu \,\Big|\, \prod_i \tau_{k_i}(\alpha_i)\Big\rangle^{(S\times\mathbb{P}^1,S_{0,\infty}),\bullet,\sim}_{g,(\beta,n)},$$

where the brackets on the right-hand side are defined by integrating over the reduced virtual class of the moduli space of rubber stable maps to $(S\times\mathbb{P}^1, S\times\{0,\infty\})$. The rubber invariants for the standard (nonreduced) virtual class are denoted by std.

6.6 Nonreduced invariants

We state two explicit evaluations of nonreduced relative invariants:

Proposition 6.6 [Bryan and Pandharipande 2008] *For any cohomology-weighted partitions $\lambda_1,\dots,\lambda_N$ of size n,*

$$Z^{(S\times\mathbb{P}^1,S_z),\mathrm{std}}_{\mathrm{GW},(0,n)}(\lambda_1,\dots,\lambda_N) = \int_{S^{[n]}} \lambda_1\cup\dots\cup\lambda_N.$$

Recall the class $\delta \in H^2(S^{[n]})$ from Section 3.

Proposition 6.7
$$Z^{\sim,\mathrm{std}}_{\mathrm{GW},(0,n)}(\lambda,\mu) = z\int_{S^{[n]}} \delta \cup \lambda \cup \mu.$$

Proof Consider first the connected rubber invariants $\langle \lambda,\mu\rangle^{\sim}_{g,(0,n)}$ (the lack of $\bullet$ means it is connected). By the stability of the moduli space $2g-2+\ell(\lambda)+\ell(\mu)>0$. Hence we can apply the product formula, which shows that the invariant vanishes for $g\geq 2$. If $g=1$ all the cohomology weights of λ and μ have to be of degree 0, and hence $\deg(\lambda)+\deg(\mu)\leq 2n-2$. Since the moduli space is of virtual dimension $2n-1$, the integral vanishes. This leaves $g=0$. Let $\lambda=(\lambda_i,\gamma_i)$ and $\mu=(\mu_i,\gamma_i')$. We find
$$\sum_i \deg(\gamma_i) + \sum_i \deg_i(\gamma_i') = 2.$$
On the other hand, by (29) we have
$$\deg(\lambda) = n-\ell(\lambda)+\sum_i \deg(\gamma_i) \quad\text{and}\quad \deg(\mu) = n-\ell(\mu)+\sum_i \deg(\gamma_i'),$$
and moreover we can assume the dimension constraint:
$$\deg(\lambda)+\deg(\mu) = 2n-1.$$
Substituting, we find $\ell(\lambda)+\ell(\mu)=3$. If we assume that $\lambda=((\lambda_a,\gamma_a)(\lambda_b,\gamma_b))$ and $\mu=((\mu_c,\gamma_c'))$, then by the product formula we obtain
$$\langle\lambda,\mu\rangle^{\sim}_{g,(0,n)} = \delta_{g0}\int_{[\overline{M}_{0,3}(S,0)]^{\mathrm{std}}} \pi^*(\mathrm{DR}_0(\lambda_a,\lambda_b,-\mu_c))\,\mathrm{ev}_1^*(\gamma_a)\,\mathrm{ev}_2^*(\gamma_b)\,\mathrm{ev}_3^*(\gamma_c') = \delta_{g0}\int_S \gamma_a\gamma_b\gamma_c',$$
where $\mathrm{DR}_g(a)$ is the double ramification cycle and we used that it is equal to 1 in genus 0.

For the disconnected case, recall that all connected nonrubber invariants of $(S\times\mathbb{P}^1, S_{0,\infty})$ with only relative insertions vanish (see eg [Oberdieck and Pixton 2018, Lemma 2]), except for the tube evaluation
$$\int_{[\overline{M}^{\bullet}_g(S\times\mathbb{P}^1/S_{0,\infty},(0,n),((n),(n)))]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma)\,\mathrm{ev}_2^*(\gamma') = \delta_{0g}\frac{1}{n}\int_S \gamma\gamma'.$$
(This also proves Proposition 6.6 in the case $N=2$.) Moreover, in the disconnected series, we have one rubber term and the remaining terms are nonrubber.

We conclude that we must have $\ell(\lambda)=\ell(\mu)\pm 1$, otherwise all invariants vanish. We assume that $\ell(\lambda)=\ell(\mu)+1$; the other case is parallel. We find that
$$\begin{aligned} Z^{\sim,\mathrm{std}}_{\mathrm{GW},(0,n)}(\lambda,\mu) &= \sum_{g\in\mathbb{Z}} (-1)^{g-1}(-1)^{-n+\ell(\lambda)+\ell(\mu)} z^{2g-2+\ell(\lambda)+\ell(\mu)} \langle\lambda,\mu\rangle^{\bullet,\sim}_{g,(0,n)} \\ &= \sum_{g\in\mathbb{Z}} (-1)^{g-1}(-1)^{-n+\ell(\lambda)+\ell(\mu)} z^{2g-2+\ell(\lambda)+\ell(\mu)} \sum_{\substack{1\leq a,b\leq \ell(\lambda)\\ a\neq b\\ 1\leq c\leq \ell(\mu)}} \left(\delta_{g+\ell(\lambda'),0}\int_S \gamma_a\gamma_b\gamma_c'\right) \\ &\quad\cdot\left((-1)^{|\lambda'|+\ell(\lambda')}\frac{1}{\prod_{i\neq a,b}\lambda_i\prod_{i\neq c}\mu_i}\int_{S^{[|\lambda'|]}} \prod_{i\neq a,b}\mathfrak{q}_{\lambda_i}(\gamma_i)v_\varnothing \cup \prod_{i\neq c}\mathfrak{q}_{\mu_i}(\gamma_i')v_\varnothing\right), \end{aligned}$$

where λ' is the partition λ without the parts (λ_a,γ_a) and (λ_b,γ_b). Since it is of length $\ell(\lambda')=\ell(\lambda)-2$, we obtain

$$Z^{\sim,\mathrm{std}}_{\mathrm{GW},(0,n)}(\lambda,\mu)=\frac{z}{\prod_i\lambda_i\prod_i\mu_i}\sum_{\substack{1\le a,b\le\ell(\lambda)\\ a\neq b\\ 1\le c\le\ell(\mu)}}(-1)^{\lambda_a+\lambda_b}\lambda_a\lambda_b\mu_c\biggl(\int_S\gamma_a\gamma_b\gamma'_c\biggr)\cdot\int_{S^{[n-\lambda_a-\lambda_b]}}\prod_{i\neq a,b}\mathfrak{q}_{\lambda_i}(\gamma_i)v_\varnothing\cup\prod_{i\neq c}\mathfrak{q}_{\mu_i}(\gamma'_i)v_\varnothing.$$

On the other side, recall that the operator of cup product with δ can be explicitly described as a cubic in Nakajima operators (34). For $i=j+k$, one obtains

$$(\mathfrak{q}_i(\gamma_i),e_\delta\mathfrak{q}_j(\gamma_j)\mathfrak{q}_k(\gamma_k)v_\varnothing)=(-1)^{j+k}ijk\int_S\gamma_i\gamma_j\gamma_k,$$

where we write $(-,-)$ for the intersection pairing on $S^{[n]}$. One finds that $\int_{S^{[n]}}\delta\cup\lambda\cup\mu$ vanishes unless $\ell(\lambda)=\ell(\mu)\pm1$. Assuming that $\ell(\lambda)=\ell(\mu)+1$, we compute:

$$\int_{S^{[n]}}\delta\cup\lambda\cup\mu=\frac{1}{\prod_i\lambda_i\prod_i\mu_i}\sum_{\substack{1\le a,b\le\ell(\lambda)\\ a\neq b\\ 1\le c\le\ell(\mu)}}(-1)^{\lambda_a+\lambda_b}\biggl(\int_S\gamma_a\gamma_b\gamma'_c\biggr)\int_{S^{[n-\lambda_a-\lambda_b]}}\prod_{i\neq a,b}\mathfrak{q}_{\lambda_i}(\gamma_i)v_\varnothing\cup\prod_{i\neq c}\mathfrak{q}_{\mu_i}(\gamma'_i).$$

The claim follows by comparison. ∎

6.7 Reduced rubber invariants

The reduced rubber invariants can be expressed in terms of the nonreduced ones by rigidification. This is the K3 surface analogue of [Maulik 2009, Proposition 4.4]:

Proposition 6.8 *For any $D\in H^2(S)$ and $\beta\neq0$ we have*

$$(D\beta)Z^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{\mathrm{GW},(\beta,n)}(\lambda,\mu)=Z^{(S\times\mathbb{P}^1,S_{0,1,\infty})}_{\mathrm{GW},(\beta,n)}(\lambda,\mu,D)+\biggl(\int_{S^{[n]}}\lambda\mu\biggr)Z^{(S\times\mathbb{P}^1,S_0)}_{\mathrm{GW},(\beta,n)}((1,\mathrm{p})^n\mid\tau_0(\omega D)),$$

where $D=(1/(n-1)!)((1,D)(1,1)^{n-1})$.

Proof Rigidification of the rubber as discussed in [Maulik 2009, Proposition 4.3] (or [Maulik and Pandharipande 2006] or [Oberdieck 2024a, Proposition 3.12]) implies

$$(\beta D)\langle\lambda,\mu\rangle^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{g,(\beta,n)}=\langle\tau_0(D)\mid\lambda,\mu\rangle^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{g,(\beta,n)}=\langle\tau_0(\omega D)\mid\lambda,\mu\rangle^{(S\times\mathbb{P}^1,S_{0,\infty})}_{g,(\beta,n)}.$$

For the disconnected rubber invariants we hence obtain that

$$
\begin{aligned}
&(\beta, D) Z^{(S\times\mathbb{P}^1, S_{0,\infty}),\sim}_{\mathrm{GW},(\beta,n)}(\lambda,\mu) \\
&\quad = \sum_{g\in\mathbb{Z}} (-1)^{g-1}(-1)^{-n+\ell(\lambda)+\ell(\mu)} z^{2g-2+\ell(\lambda)+\ell(\mu)} \sum_{\substack{\lambda=\lambda'\sqcup\lambda'' \\ \mu=\mu'\sqcup\mu''}} \langle \tau_0(\omega D) \mid \lambda'', \mu''\rangle^{(S\times\mathbb{P}^1, S_{0,\infty})}_{g+\ell(\lambda'),(\beta,n)} \\
&\qquad\qquad \cdot \biggl((-1)^{|\lambda'|+\ell(\lambda')} \frac{1}{\prod_{\lambda_i\in\lambda'}\lambda_i \prod_{i\in\mu'}\mu_i} \int_{S^{[|\lambda'|]}} \prod_{\lambda_i\in\lambda'} \mathfrak{q}_{\lambda_i}(\gamma_i) v_\varnothing \cup \prod_{\mu_i\in\mu'} \mathfrak{q}_{\mu_i}(\gamma_i') v_\varnothing \biggr) \\
&\quad = Z^{(S\times\mathbb{P}^1, S_{0,\infty})}_{\mathrm{GW},(\beta,n)}(\tau_0(\omega D) \mid \lambda,\mu).
\end{aligned}
$$

We now apply the degeneration formula, which gives

$$
\begin{aligned}
(60)\quad & Z^{(S\times\mathbb{P}^1, S_{0,\infty})}_{\mathrm{GW},(\beta,n)}(\tau_0(\omega D) \mid \lambda,\mu) \\
&= \sum_{\nu} Z^{(S\times\mathbb{P}^1, S_{0,1,\infty})}_{\mathrm{GW},(\beta,n)}(\lambda,\mu,\nu)(-1)^{|\nu|+\ell(\nu)} \frac{\prod_i \nu_i}{|\mathrm{Aut}(\nu)|} Z^{(S\times\mathbb{P}^1, S_0),\mathrm{std}}_{\mathrm{GW},(0,n)}(\tau_0(\omega D) \mid \nu^\vee) \\
&\quad + \sum_{\nu} Z^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\mathrm{std}}_{\mathrm{GW},(0,n)}(\lambda,\mu,\nu)(-1)^{|\nu|+\ell(\nu)} \frac{\prod_i \nu_i}{|\mathrm{Aut}(\nu)|} Z^{(S\times\mathbb{P}^1, S_0)}_{\mathrm{GW},(\beta,n)}(\tau_0(\omega D) \mid \nu^\vee).
\end{aligned}
$$

We have the straightforward evaluation

$$
Z^{(S\times\mathbb{P}^1, S_0),\mathrm{std}}_{\mathrm{GW},(0,n)}(\tau_0(\omega D) \mid \nu^\vee) = \begin{cases} \int_S \gamma D & \text{if } \nu = (1,\gamma)(1,\mathsf{p})^{n-1}, \\ 0 & \text{if } \nu = (2,\mathsf{p})(1,\mathsf{p})^{n-2}, \end{cases}
$$

which gives us

$$
\sum_{\nu} (-1)^{|\nu|+\ell(\nu)} \frac{\prod_i \nu_i}{|\mathrm{Aut}(\nu)|} Z^{(S\times\mathbb{P}^1, S_0),\mathrm{std}}_{\mathrm{GW},(0,n)}(\tau_0(\omega D) \mid \nu^\vee)\nu = \frac{1}{(n-1)!}((1,D)(1,1)^{n-1}) = D.
$$

Moreover, in the second summand on the right of (60) we must have $\nu = (1,1)^n$ for dimension reasons. Using Proposition 6.6 the claim follows. □

For primitive β the second term on the right of the proposition is known:

Proposition 6.9 [Oberdieck 2019] *If $\beta \in H_2(S,\mathbb{Z})$ is primitive, then*

$$
Z^{(S\times\mathbb{P}^1, S_0)}_{\mathrm{GW},(\beta,n)}((1,\mathsf{p})^n \mid \tau_0(\omega D)) = (\beta, D)\, \mathrm{Coeff}_{q^{\beta^2/2}}\biggl(\frac{\boldsymbol{G}^n(z,q)}{\Theta^2(z,q)\Delta(q)} \biggr),
$$

where $\boldsymbol{G}(z,q) = -\Theta(z,\tau)^2 D_z^2 \log(\Theta(z,\tau))$ with $D_z = d/dz$ and $q = e^{2\pi i\tau}$.

7 Holomorphic anomaly equations: $(\mathrm{K3}\times C, \mathrm{K3}_z)$

7.1 Overview

In this section we prove that the natural generating series of Gromov–Witten invariants of $(S\times C, K3_z)$ for an elliptic K3 surface S in primitive classes are quasimodular forms and satisfy a holomorphic anomaly equation (Theorem 7.6). The idea is straightforward: we apply the product formula in Gromov–Witten theory and use the corresponding results from the Gromov–Witten theory of K3 surfaces which were proven in [Maulik et al. 2010; Oberdieck and Pixton 2018].

The details require some work: First in Section 7.2 we introduce a special set of disconnected invariants labeled by $\sharp$ which is well adapted to the holomorphic anomaly equation. In Sections 7.3 and 7.4 we recall the quasimodularity and holomorphic anomaly equations for K3 surfaces in this convention. In Section 7.5 we then state and prove Theorem 7.6 using the product formula, and by a careful application of the splitting formulas and the new boundary restriction formulas introduced in Section 5.

7.2 Preliminaries

To state the holomorphic anomaly equations we will need another convention for disconnected Gromov–Witten invariants. Let $\pi : X \to B$ be an elliptic fibration and let

$$\overline{M}^{\sharp}_{g,n}(X, \beta)$$

be the moduli space of stable maps $f : C \to X$ from possibly disconnected curves of genus g in class β, with the following requirement:

($\sharp$) For every connected component $C' \subset C$ at least one of the following holds:

(i) $\pi \circ f|_{C'}$ is nonconstant, or

(ii) C' has genus g' and carries n' markings with $2g' - 2 + n' > 0$.

Parallel definitions apply to relative targets (X, D) admitting an elliptic fibration to a pair (B, A), moduli spaces of rubber stable maps, etc. We will denote the invariants defined from moduli satisfying condition ($\sharp$) by a superscript $\sharp$.

7.3 Quasimodularity

Let $\pi : S \to B \cong \mathbb{P}^1$ be an elliptic K3 surface with a section, let B and F denote the class of the section and a fiber, respectively, and set $W = B + F$. For any tautological class $\mathrm{taut} \in \tau^* R^*(\overline{M}_{g,n})$ — or $\mathrm{taut} = 1$ in the unstable cases $2g - 2 + N \leq 0$ — and $\gamma_i \in H^*(S)$, consider (or recall from (43)) the generating series

$$F^S_g(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d \geq -1} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^S_{g, W+dF} q^d.$$

Theorem 7.1 [Maulik et al. 2010; Bryan et al. 2018, Section 4.6] *For* wt*–homogeneous classes $\gamma_i \in H^*(S)$, we have*

$$F^S_g(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(q)} \mathrm{QMod}_s$$

for $s = 2g + N + \sum_i \mathrm{wt}(\gamma_i)$.

Consider the generating series of disconnected invariants (for the $\sharp$–condition)

$$F^{S,\sharp}_g(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d \geq -1} q^d \int_{[\overline{M}^{\sharp}_{g,N}(S, W+dF)]^{\mathrm{vir}}} \pi^*(\mathrm{taut}) \prod_{i=1}^{N} \mathrm{ev}_i^*(\gamma_i).$$

Corollary 7.2 *For* wt*–homogeneous classes* $\gamma_i \in H^*(S)$, *we have*

$$F_g^{S,\sharp}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(q)} \mathrm{QMod}_s$$

for $s = 2g + N + \sum_i \mathrm{wt}(\gamma_i)$.

Proof Recall that the standard virtual class satisfies

$$[\overline{M}_{g,n}(S,0)]^{\mathrm{std}} = \begin{cases} [\overline{M}_{0,n} \times S] & \text{if } g = 0, \\ c_2(S) \cap [\overline{M}_{1,n} \times S] & \text{if } g = 1, \\ 0 & \text{if } g \geq 2. \end{cases}$$

If an invariant

$$\int_{[\overline{M}_{g,n}(S,0)]^{\mathrm{std}}} \pi^*(\mathrm{taut}) \prod_i \mathrm{ev}_i^*(\gamma_i)$$

is to contribute, we must have

- $g = 0$ and $\sum_i \mathrm{wt}(\gamma_i) = 2 - n$,
- $g = 1$ and $\sum_i \mathrm{wt}(\gamma_i) = -n$.

In both cases

$$-2 + 2g + n + \sum_i \mathrm{wt}(\gamma_i) = 0.$$

Now, if a connected components of $\overline{M}^{\sharp}_{g,N}(S,\beta)$ contributes nontrivially to the disconnected Gromov–Witten invariant, then by a second-cosection argument the component must parametrize stable maps $f: C \to S$ which are nonconstant only on one component C'. Let g' and N' be the genus and number of markings on C'. The above computation shows that

$$2g + N + \sum_{i=1}^{N} \mathrm{wt}(\gamma_i) = 2g' + N' + \sum_{j=1}^{N'} \mathrm{wt}(\gamma_{i_j}),$$

where i_j are the indices of marked points on C'. The claim hence follows from Theorem 7.1. □

7.4 Holomorphic anomaly equation

We state the holomorphic anomaly equation for K3 surfaces in primitive classes:

Theorem 7.3 [Oberdieck and Pixton 2018] *We have*

$$\begin{aligned} &\frac{d}{dG_2} F_g^S(\mathrm{taut}; \gamma_1, \dots, \gamma_r) \\ &\quad = F_{g-1}^S(\mathrm{taut}'; \gamma_1, \dots, \gamma_r, \Delta_B) + 2 \sum_{\substack{g = g_1 + g_2 \\ \{1,\dots,r\} = A \sqcup B}} F_{g_1}^S(\mathrm{taut}_1; \gamma_A, \Delta_{B,1}) F_{g_2}^{S,\mathrm{std}}(\mathrm{taut}_2; \gamma_B, \Delta_{B,2}) \\ &\qquad - 2 \sum_{i=1}^{r} F_g^S(\psi_i \, \mathrm{taut}; \gamma_1, \dots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \dots, \gamma_r) - \sum_{a,b} (g^{-1})_{ab} T_{e_a} T_{e_b} F_g^S(\mathrm{taut}; \gamma_1, \dots, \gamma_r), \end{aligned}$$

where we follow the notation of Conjecture C, and moreover:

- $\Delta_{B,1}, \Delta_{B,2}$ *stands for summing over the Künneth decomposition of the diagonal class* $\Delta_B \in H^*(B \times B)$*, and we have suppressed the pullback to* $S \times S$.

This immediately yields the following for the series of disconnected invariants (compare with [Oberdieck and Pixton 2019, Section 3.2] for a similar case):

Corollary 7.4 *We have*

$$\begin{aligned}\frac{d}{dG_2} F_g^{S,\sharp}(\mathrm{taut}; \gamma_1, \dots, \gamma_r) &= F_{g-1}^{S,\sharp}(\mathrm{taut}'; \gamma_1, \dots, \gamma_r, \Delta_B) - 2\sum_{i=1}^{r} F_g^{S,\sharp}(\psi_i\, \mathrm{taut}; \gamma_1, \dots, \gamma_{i-1}, \pi^*\pi_*\gamma_i, \gamma_{i+1}, \dots, \gamma_r) \\ &\quad - \sum_{a,b} (g^{-1})_{ab} T_{e_a} T_{e_b} F_g^{S,\sharp}(\mathrm{taut}; \gamma_1, \dots, \gamma_r).\end{aligned}$$

Example 7.5 Instead of the proof (which is straightforward) let us consider a concrete example that highlights all the main points. Consider the series

$$F_0^S(W, F, F) = F_0^{S,\sharp}(W, F, F) = q\frac{d}{dq}\frac{1}{\Delta(q)}.$$

We compute in three different ways the G_2–derivative. First directly:

$$\frac{d}{dG_2} F_0^S(W, F, F) = \left[\frac{d}{dG_2}, q\frac{d}{dq}\right]\frac{1}{\Delta(q)} = -2(-12)\frac{1}{\Delta(q)} = 24\frac{1}{\Delta(q)}.$$

Second, by the holomorphic anomaly equations for the connected series:

$$\frac{d}{dG_2} F_0^S(W, F, F) = 2 \cdot 2 \cdot F_0^S(F, U_1) F_0^{S,\mathrm{std}}(U_2, W, F) + 20 F_0^S(F, F, F) = 24\frac{1}{\Delta(q)}.$$

Here the extra factor 2 comes from choosing which of the two F's goes to the two factors. Third, by the disconnected holomorphic anomaly equation:

$$\frac{d}{dG_2} F_0^{S,\sharp}(W, F, F) = F_{-1}^{S,\sharp}(W, F, F, \Delta_B) - 2F_0^{S,\sharp}(\psi \cdot 1, F, F) + 20 F_0^{S,\sharp}(F, F, F) = (6-2+20)\frac{1}{\Delta(q)}.$$

Here we have used that

$$F_{-1}^{S,\sharp}(W, F, F, \Delta_B) = 2F_{-1}^{S,\sharp}(W, F, F, F, 1) = 6F_0^S(F, F) F_0^{\mathrm{std}}(W, F, 1) = 6\frac{1}{\Delta(q)},$$

$$-2F_0^{S,\sharp}(\psi \cdot 1, F, F) = -2\left(24\int_{\overline{M}_{1,1}} \psi_1\right) F_0^S(F, F).$$

7.5 Relative geometry $(S \times C, S_z)$

Consider the relative geometry

$$(S \times C, S_z) \quad \text{for } z = (z_1, \dots, z_N) \text{ and } S_z = \bigsqcup_i S \times \{z_i\}, \tag{61}$$

where we assume that the pair $(C, z_1, \dots, z_N)$ is stable, ie $2g-2+N>0$. Define the generating series of relative invariants satisfying the $\sharp$ condition

$$F_g^{(S\times C,S_z),\sharp}(\lambda_1,\dots,\lambda_N \mid \gamma) = \sum_{d\geq -1} q^d \langle \lambda_1,\dots,\lambda_N \mid \gamma\rangle^{(S\times C,S_z),\sharp}_{g,(W+dF,n)},$$

where λ_i are $H^*(S)$–weighted partitions and $\gamma \in H^*((S\times C, S_z)^r)$. Similarly, we have the corresponding series of reduced rubber invariants; see Section 6.

We also require the nonreduced invariants:

$$F_g^{(S\times C,S_z),\sharp,\mathrm{std}}(\lambda_1,\dots,\lambda_N \mid \gamma) = \langle \lambda_1,\dots,\lambda_N \mid \gamma\rangle^{(S\times C,S_z),\sharp,\mathrm{std}}_{g,(0,n)}.$$

Theorem 7.6 (a) *For cohomology-weighted partitions $\lambda_i = (\lambda_{i,j}, \delta_{i,j})$ where $\delta_{i,j} \in H^*(S)$ are* wt–*homogeneous, we have*

$$F_g^{(S\times C,S_z),\sharp}(\lambda_1,\dots,\lambda_N) \in \frac{1}{\Delta(q)}\,\mathrm{QMod}_s,$$

where $s = 2g + \sum_{i=1}^N \ell(\lambda_i) + \sum_{i,j} \mathrm{wt}(\delta_{i,j})$.

(b) *We have the holomorphic anomaly equation*

$$\begin{aligned}
&\frac{d}{dG_2} F_g^{(S\times C,S_z),\sharp}(\lambda_1,\dots,\lambda_N) \\
&\quad = F_{g-1}^{(S\times C,S_z),\sharp}(\lambda_1,\dots,\lambda_N \mid \Delta^{\mathrm{rel}}_{(B\times C,B_z)}) + 2\sum_{i=1}^N \sum_{\substack{m\geq 0\\ g=g_1+g_2+m}} \sum_{\substack{b,b_1,\dots,b_m\\ l,l_1,\dots,l_m}} \frac{\prod_{i=1}^m b_i}{m!} \\
&\qquad\quad \cdot \Big(F_{g_1}^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim,\sharp,\mathrm{std}}\big(\lambda_i, ((b,\Delta_{B,l}), (b_i,\Delta_{S,l_i})_{i=1}^m)\big) \\
&\qquad\qquad \cdot F_{g_2}^{(S\times C,S_z),\sharp}\big(\lambda_1,\dots,\lambda_{i-1}, ((b,\Delta^\vee_{B,l}), (b_i,\Delta^\vee_{S,l_i})_{i=1}^m), \lambda_{i+1},\dots,\lambda_N\big) \\
&\qquad\qquad + F_{g_1}^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim,\sharp}\big(\lambda_i, ((b,\Delta_{B,l}), (b_i,\Delta_{S,l_i})_{i=1}^m)\big) \\
&\qquad\qquad \cdot F_{g_2}^{(S\times C,S_z),\sharp,\mathrm{std}}\big(\lambda_1,\dots,\lambda_{i-1}, ((b,\Delta^\vee_{B,l}), (b_i,\Delta^\vee_{S,l_i})_{i=1}^m), \lambda_{i+1},\dots,\lambda_N\big)\Big) \\
&\qquad - 2\sum_{i=1}^N \sum_{j=1}^{\ell(\lambda_i)} F_g^{(S\times C,S_z),\sharp}(\lambda_1,\dots,\lambda_{i-1}, \psi^{\mathrm{rel}}_{i,j}\lambda_i^{(j)}, \lambda_{i+1},\dots,\lambda_N) \\
&\qquad - \sum_{a,b} (g^{-1})_{ab} T_{e_a} T_{e_b} F_g^{(S\times C,S_z),\sharp}(\lambda_1,\dots,\lambda_N).
\end{aligned}$$

Here the $b, b_1,\dots,b_m$ run over all positive integers such that $b+\sum_i b_i = n$, and the l and l_i run over the splitting of the diagonals of B and S, respectively:

$$\Delta_B = \sum_l \Delta_{B,l} \otimes \Delta^\vee_{B,l}, \quad \forall i,\ \Delta_S = \sum_{l_i} \Delta_{S,l_i} \otimes \Delta^\vee_{S,l_i}.$$

Moreover, $\lambda_i^{(j)}$ is the weighted partition λ_i but with j^{th} weight δ_{ij} replaced by $\pi^*\pi_*(\delta_{ij})$.

Proof Consider a stable map $f\colon \Sigma \to (S\times C)[k]$ parametrized by $\overline{M}^{\sharp}_{g,(W+dF,n)}((S\times C, S_z),\vec{\lambda})$. In order for the connected component of the moduli space containing f to contribute nontrivially to the Gromov–Witten invariant, there must be precisely one connected component $\Sigma' \subset \Sigma$ where f is of nonzero degree over the K3 surface S. Moreover, we claim that $f|_{\Sigma'}$ in this case is also of nonzero degree over C. Indeed if not, then the remaining components yield a factor of

$$\langle \lambda_1,\dots,\lambda_N\rangle^{(S\times C,S_z),\sharp,\mathrm{std}}_{g',(0,n)},$$

which have to vanish for dimension reasons (since the standard virtual class is dimension one less than the reduced virtual class and the degree of the insertions λ_1, λ_2 and λ_3 are chosen to sum up to the degree of the reduced virtual class). Since (C,z) is stable, it follows that Σ' satisfies $2g(\Sigma')-2+n(\Sigma')>0$, so its stabilization is well defined. Similarly, if $\Sigma'\subset\Sigma$ is a connected component whose degree over the K3 surface S is trivial, then either $2g(\Sigma')-2+n(\Sigma)>0$ by assumption of the moduli space, or the degree over C is nontrivial. In the latter case by the stability of (C,z) we have that Σ' has again at least N special points and genus $\geq g(C)$; hence Σ' and its markings defines a stable curve. Note also since we have no interior markings, there are no contributions from contracted genus-$g\geq 2$ components. Let $\overline{M}^{\sharp,\mathrm{contr}}_{g,(W+dF,n)}((S\times C,S_z),\vec{\lambda})$ be the union of connected components which have a nontrivial contribution, where we have written $\vec{\lambda}=(\vec{\lambda}_1,\dots,\vec{\lambda}_N)$. We have shown that there exists a commutative diagram

$$\begin{array}{ccc}
\overline{M}^{\sharp,\mathrm{contr}}_{g,(W+dF,n)}((S\times C,S_z),\vec{\lambda}) & \xrightarrow{\;q\;} & \overline{M}^{\sharp}_{g,\sum_i \ell(\lambda_i)}(S,W+dF) \\
\downarrow & & \downarrow \\
\overline{M}'_{g,n}((C,z),\vec{\lambda}) & \xrightarrow{\;\pi\;} & \overline{M}'_{g,n}
\end{array}$$

where $\overline{M}'_{g,n}(X,D)$ is the moduli space of disconnected relative stable maps where each connected component of the source is stable, and $\overline{M}'_{g,n}$ is simply the moduli space of disconnected stable curves (where each connected component is stable).

Recall from (51) the class

$$I^{(C,z),\prime}_{g,n}(\vec{\lambda}\mid\gamma)=\pi_*(\mathrm{ev}^*(\gamma)[\overline{M}_{g,r,\beta}((C,z),\vec{\lambda})]^{\mathrm{vir}}).$$

Then applying the product formula of [Behrend 1999; Lee and Qu 2018] we conclude that

$$F^{(S\times C,S_z),\sharp}_{g}(\lambda_1,\dots,\lambda_N)=F^{S,\sharp}_{g}\biggl(I^{(C,z),\prime}_{g,n}(\vec{\lambda});\prod_{i=1}^{\ell(\lambda_i)}\prod_{j}\delta_{i,j}\biggr).$$

The first claim hence follows from Corollary 7.2.

For the second claim we apply the holomorphic anomaly equation of Corollary 7.4. Let

$$\iota\colon \overline{M}'_{g-1,n+2}\to\overline{M}'_{g,n}$$

be the morphism that glues the $(n+1)^{\text{th}}$ and $(n+2)^{\text{th}}$ marked points. By an application of Proposition 5.4 we then have

$$\iota^* I_{g,n}^{(C,z),\prime}(\vec{\lambda}) = I_{g-1,\beta}^{(C,z),\prime}(\vec{\lambda} \mid \Delta_{(C,z)}^{\text{rel}}) + \sum_{i=1}^{N} \sum_{\substack{m\geq 0 \\ g=g_1+g_2+m}} \sum_{\substack{b,b_1,\dots,b_m \\ l,l_1,\dots,l_m}} \frac{\prod_{i=1}^m b_i}{m!} \cdot \Big\{ \xi_* j^* \Big[J_{g_1,\beta'}^{(C,z),\bullet}(\lambda_1,\dots,\lambda_{i-1},(b,b_1,\dots,b_m),\lambda_{i+1},\dots,\lambda_N) \boxtimes J_{g_2,\alpha}^{(\mathbb{P}^1,\{0,z_i\}),\bullet,\sim}((b,b_1,\dots,b_m),\lambda_i) \Big] + (\text{reversed}) \Big\},$$

where (reversed) stands for the same term as before but with the role of the markings $(n+1)$ and $(n+2)$ reversed, and the rest of the notation is as in Proposition 5.4 (except that we do not require the glued curve to be connected). Since only the $(n+1,n+2)^{\text{th}}$ marked points are not glued, we exclude precisely those components of the moduli space where there is a totally ramified morphism from a genus-0 component to rubber $(\mathbb{P}^1, 0\sqcup\infty)$ which is ramified over 0 by some relative marking $\lambda_{i,j}$ and over ∞ by b (corresponding to the marking labeled $n+1$ or $n+2$). Applying the product formula in reverse, we hence find that

$$F_{g-1}^{S,\sharp}\Big(\iota^* I_{g,n}^{(C,z),\prime}(\vec{\lambda}); \prod_{i=1}^{\ell(\lambda_i)} \prod_j \delta_{i,j} \Big)$$

accounts for precisely the first two terms on the right of Theorem 7.6(b), except for the components where we have a contribution from a totally ramified map to a bubble attached to the marking b.

The second term on the right of Corollary 7.4 is

$$-2\sum_{i=1}^{r} F_g^{S,\sharp}(I_{g,n}^{(C,z),\bullet}(\vec{\lambda}); \gamma_1,\dots,\gamma_{j-1},\psi_i\pi^*\pi_*\gamma_j,\gamma_{j+1},\dots,\gamma_r),$$

where we write $(\gamma_1,\dots,\gamma_r) = (\delta_{ij})_{i,j}$. Again we apply the product formula in reverse. For that we need to compare the psi-classes ψ_i on the domain and target of the morphism:

$$\overline{M}^{\sharp,\text{contr}}_{g,(W+dF,n)}((S\times C, S_z),\vec{\lambda}) \xrightarrow{q} \overline{M}^{\sharp}_{g,\sum_i \ell(\lambda_i)}(S, W+dF).$$

Precisely,

$$q^*(\psi_{i,j}) = \psi_{i,j}^{\text{rel}} - D,$$

where D is the virtual boundary divisor parametrizing splittings of maps $f : C \to X[k]$ where the relative marking $\lambda_{i,j}$ lies on a genus-0 component mapping entirely into the bubble such that the underlying curve is contracting after forgetting the map to (C,z). We hence obtain precisely the third term in part (b) of the claim, plus the contribution we were missing in the first two terms.

Finally, the third term in Corollary 7.4 yields precisely part (b) in our claim. □

Remark 7.7 The holomorphic anomaly equation of Theorem 7.6 is a version (for reduced virtual classes) of the holomorphic anomaly equation conjectured for the relative Gromov–Witten theory of elliptic fibrations in [Oberdieck and Pixton 2019, Conjecture D]. The form in [loc. cit.] is more natural, but requires more notation (for one thing, it is defined on the cycle level). Theorem 7.6 is then a special case of the following statement: if the holomorphic anomaly equation (in the form of [loc. cit., Conjecture B]) holds for an elliptic fibration $S \to B$, then for any relative pair (X, D) the holomorphic anomaly equation holds for the elliptic fibration $S \times X \to B \times X$ relative to $S \times D \to B \times D$ (in the form of [loc. cit., Conjecture D]).

8 Holomorphic anomaly equations: primitive case

8.1 Overview

Let $S \to B$ be an elliptic K3 surface and recall the generating series

$$F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d \geq -1} \sum_{r \in \mathbb{Z}} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle_{g, W+dF+rA}^{S^{[n]}} q^d (-p)^r,$$

where $W = B + F$, and B and F are the section and fiber class. The following are the conjectural quasi-Jacobi form property and holomorphic anomaly equation in the special case of primitive classes. We follow parallel notation as in Conjecture C.

Conjecture E (*a*) *For* wt*–homogeneous classes* γ_i, *we have*

$$F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(q)} \mathrm{QJac}_{k, n-1},$$

where $k = n(2g - 2 + N) + 2 + \sum_i \mathrm{wt}(\gamma_i)$.

(*b*) *Assuming part* (*a*), *we have*

$$\begin{aligned}
\frac{d}{dG_2} F_g^{S^{[n]}}&(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \\
&= F_{g-1}^{S^{[n]}}(\mathrm{taut}'; \gamma_1, \dots, \gamma_N, U) + 2 \sum_{\substack{g = g_1 + g_2 \\ \{1, \dots, N\} = A \sqcup B}} F_{g_1}^{S^{[n]}}(\mathrm{taut}_1; \gamma_A, U_1) F_{g_2}^{S^{[n]}, \mathrm{std}}(\mathrm{taut}_2; \gamma_B, U_2) \\
&\qquad - 2 \sum_{i=1}^{N} F_g^{S^{[n]}}(\psi_i \mathrm{taut}; \gamma_1, \dots, \gamma_{i-1}, U(\gamma_i), \gamma_{i+1}, \dots, \gamma_N) \\
&\qquad - \sum_{a,b} (g^{-1})_{ab} T_{e_a} T_{e_b} F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N).
\end{aligned}$$

In this section we prove the following:

Theorem 8.1 *Conjecture E holds when* $g = 0$ *and* $N \leq 3$.

The proof below proceeds in three steps. After reducing to $N = 3$ and taut $= 1$, the GW/Hilb correspondence (Theorem 6.2) implies the following basic statement (see (62))

$$F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3) = \sum_{g \in \mathbb{Z}} z^{2g-2-n+\sum_i l(\lambda_i)} (-1)^{g-1+\sum_i l(\lambda_i)} F_g^{(S \times \mathbb{P}^1, S_{0,1,\infty}), \sharp}(\lambda_1, \lambda_2, \lambda_3)$$

under the variable change $p = e^z$. By Theorem 7.6 we know that each $F_g^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}(\lambda_1,\lambda_2,\lambda_3)$ is a quasimodular form satisfying a holomorphic anomaly equation. Moreover, the Hilbert scheme series $F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3)$ on the left satisfies the structure described in Proposition 4.1. Our main work is then to turn these two inputs into the quasi-Jacobi form property and the holomorphic anomaly equation for quasi-Jacobi forms for the Hilbert scheme series. In Section 8.2 we first discuss that the left-hand side is a quasi-Jacobi form. Then in Section 8.3 we reduce the holomorphic anomaly equation for the left-hand side to an identity of the corresponding z–series. This is done by using Lemma 2.15 on the comparison of the G_2–holomorphic anomaly equation for quasi-Jacobi forms with the factorwise G_2–holomorphic anomaly equation on the z–expansion. Finally, the required identity is checked in Section 8.4 in a longer and technical 4–step argument.

8.2 Quasi-Jacobi form property

We start with the quasi-Jacobi form part of Theorem 8.1:

Proposition 8.2 *Assume that $g = 0$ and $N \le 3$. For* wt*–homogeneous classes γ_i we have*

$$F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(q)} \mathrm{QJac}_{s,n-1},$$

where $s = n(2g-2+N) + 2 + \sum_i \mathrm{wt}(\gamma_i)$.

Proof For $g = 0$ and $N \le 3$ we can take $\mathrm{taut} = 1$. By using the divisor equation the claim for $N \in \{0,1,2\}$ reduces to $N = 3$. Consider three $H^*(S)$–weighted partitions,

$$\lambda_i = (\lambda_{ij}, \delta_{ij})_j \quad \text{for } i = 1,2,3.$$

We argue in three steps:

Step 1 Under the variable change $p = e^z$ the z^r coefficient in $\Delta(q) F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3)$ is a quasimodular form of weight $r + n + 2 + \sum_i \mathrm{wt}(\lambda_i)$.

Proof of Step 1 By Theorem 6.2 under the variable change $p = e^z$ we have

$$F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3) = \sum_{d \ge -1} Z_{\mathrm{GW},(W+dF,n)}^{(S\times\mathbb{P}^1, S_{0,1,\infty})}(\lambda_1,\lambda_2,\lambda_3) q^d.$$

Since $(\mathbb{P}^1, 0, 1, \infty)$ is stable and there are no interior markings, we have the inclusion

$$\overline{M}^{\sharp}_{g,(W+dF,n)}((S\times\mathbb{P}^1, S_{0,1,\infty}), \vec{\lambda}) \subset \overline{M}^{\bullet}_{g,(W+dF,n)}((S\times\mathbb{P}^1, S_{0,1,\infty}), \vec{\lambda}),$$

and moreover, every connected component in the complement does not contribute to the Gromov–Witten invariant since the obstruction theory will admit an extra cosection coming from stable maps with two components of the domain curve of nontrivial degree over S. Hence

$$F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3) = \sum_{g\in\mathbb{Z}} z^{2g-2-n+\sum_i l(\lambda_i)} (-1)^{g-1+\sum_i l(\lambda_i)} F_g^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}(\lambda_1,\lambda_2,\lambda_3). \tag{62}$$

By Theorem 7.6(a) the series $\Delta(q) F_g^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}(\lambda_1,\lambda_2,\lambda_3)$ is a quasimodular form of weight

$$2g + \sum_{i=1}^3 \ell(\lambda_i) + \sum_{i,j} \mathrm{wt}(\delta_{i,j}).$$

Hence under $p = e^z$ the z^r coefficient of $\Delta(q) F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3)$ is a quasimodular form of weight $r+s$ where

$$s = \Big(2g + \sum_i \ell(\lambda_i) + \sum_{i,j} \mathrm{wt}(\delta_{ij})\Big) - \Big(2g - 2 - n + \sum_i \ell(\lambda_i)\Big) = n + 2 + \sum_i \mathrm{wt}(\lambda_i). \qquad \square$$

Step 2 $\Delta(q) F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3) \in \mathrm{MQJac}_{s,n-1}$, where $s = n + 2 + \sum_i \mathrm{wt}(\lambda_i)$.

Proof of Step 2 We argue by induction on the total weight of the insertions

$$\sum_i \mathrm{wt}(\lambda_i) = L.$$

We assume that the claim holds for all insertions λ_i' with $\sum_i \mathrm{wt}(\lambda_i') < L$. By induction and Lemma 3.1 we have

$$\sum_{i=1}^3 F_0^{S^{[n]}}(\lambda_1,\dots,\lambda_{i-1}, T_\delta\lambda_i, \lambda_{i+1},\dots,\lambda_3) \in \mathrm{MQJac}_{s-1,n-1}.$$

As in Step 2 of the proof of Proposition 4.1, we consider the integral with respect to A

$$\widetilde{F} = \sum_{i=1}^3 \int F_0^{S^{[n]}}(\lambda_1,\dots,\lambda_{i-1}, T_\delta\lambda_i, \lambda_{i+1},\dots,\lambda_3) dA,$$

which lies in $\mathrm{MQJac}_{s,n-1}$. Consider also the difference

$$F(p,q) = F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3) - \widetilde{F}(p,q).$$

Then as shown in (46) there exists power series $f_i(q) \in \mathbb{Q}[\![q]\!]$ such that

$$F(p,q) = \begin{cases} \Delta^{-1}(q)\Theta^{2m}(p,q)\wp'(p,q)\sum_{i=2}^m f_i(q)\wp(p,q)^{m-i} & \text{if } 3n + \sum_{i=1}^3 \ell(\lambda_i) \text{ is even,} \\ \Delta^{-1}(q)\Theta^{2m}(p,q)\sum_{i=0}^m f_i(q)\wp(p,q)^{m-i} & \text{if } 3n + \sum_{i=1}^3 \ell(\lambda_i) \text{ is odd.} \end{cases}$$

By Step 1 (for the term $F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3)$) and by Lemma 2.15 (for $\widetilde{F} \in \mathrm{MQJac}_{s,n-1}$) every z^r coefficient of $F(p,q)$ is a quasimodular form of weight $r+s$. By Lemma 2.16 or Lemma 2.17 (depending on the parity of $3n + \sum_{i=1}^3 \ell(\lambda_i)$) the claim follows. $\square$

Step 3 $\Delta(q) F_0^{S}(\lambda_1,\lambda_2,\lambda_3) \in \mathrm{QJac}_{s,n-1}$, where $s = n + 2 + \sum_i \mathrm{wt}(\lambda_i)$.

Proof of Step 3 The function $F(z,\tau) = \Delta(q) F_0^S(\lambda_1,\lambda_2,\lambda_3)$ defines a meromorphic function $\mathbb{C}\times\mathbb{H} \to \mathbb{C}$ which is holomorphic away from the lattice points $z/(2\pi i) = \lambda\tau + \mu$ for all $\lambda,\mu \in \mathbb{Z}$.

By Proposition 4.1(b) the expansion of z around $z = 0$ takes the form

$$F(z,\tau) = \sum_{k\geq 0} f_k(\tau) z^k,$$

where $f_k(\tau)$ are quasimodular forms. This shows that $F(z,\tau)$ is holomorphic at $z=0$.

To check the other lattice points we apply Lemma 2.12, which yields the transformation

$$F(z+2\pi i(\lambda\tau+\mu),\tau)=q^{-\lambda^2 m}p^{-2\lambda m}e^{-\lambda(d/dA)}F(z,\tau).$$

By Proposition 4.1(a) (the behavior under d/dA) this equals

$$q^{-\lambda^2 m}p^{-2\lambda m}\Delta(q)F_0^S(\mathrm{taut};e^{-\lambda T_\delta}\lambda_1,e^{-\lambda T_\delta}\lambda_2,e^{-\lambda T_\delta}\lambda_3).$$

Since T is nilpotent there are only finitely many terms on the right-hand side. Hence by Proposition 4.1(b) again, the right-hand side is holomorphic at $z=0$. □

8.3 Reduction

Recall the operator that takes the G_2–derivative of a power series in z with coefficients quasimodular forms factorwise:

$$\left(\frac{d}{dG_2}\right)_z:\mathrm{QMod}((z))\to\mathrm{QMod}((z)).$$

After having shown Conjecture E(a) we now reduce part (b) to a statement about the z–series of the 3–point function:

Proposition 8.3 *Conjecture E(b) holds for $g=0$ and $N\le 3$ if, for any cohomology-weighted partitions λ_1,λ_2 and λ_3, we have*

$$\begin{aligned}(63)\quad &\left(\frac{d}{dG_2}\right)_z F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3)\\ &=2\Big(F_0^{S^{[n]}}(\lambda_1,U(\lambda_2\lambda_3))-F_0^{S^{[n]}}(U\lambda_1,\lambda_2\lambda_3)+F_0^{S^{[n]}}(\lambda_2,U(\lambda_1\lambda_3))-F_0^{S^{[n]}}(U\lambda_2,\lambda_1\lambda_3)\\ &\qquad+F_0^{S^{[n]}}(\lambda_3,U(\lambda_1\lambda_2))-F_0^{S^{[n]}}(U\lambda_3,\lambda_1\lambda_2)\Big)\\ &\quad-\sum_{a,b}(G^{-1})_{ab}T_{e_a}T_{e_b}F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3)\\ &\quad-2z(F_0^{S^{[n]}}(T_\delta\lambda_1,\lambda_2,\lambda_3)+F_0^{S^{[n]}}(\lambda_1,T_\delta\lambda_2,\lambda_3)+F_0^{S^{[n]}}(\lambda_1,\lambda_2,T_\delta\lambda_3))\\ &\quad-2(n-1)z^2F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3).\end{aligned}$$

Proof Part (b) states that Conjecture E is compatible under the divisor equations, string equation and restriction to boundary. This can be proven parallel to [Oberdieck and Pixton 2018, Section 2] or [Bae and Buelles 2021, Section 3]. Hence it suffices to consider the case $\alpha=1$, $g=0$ and $N=3$, ie to prove the holomorphic anomaly equation for $F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3)$.

One has that

$$\begin{aligned}\sum_{\{1,\dots,3\}=A\sqcup B}&F_0^{S^{[n]}}(1;\lambda_A,U_1)F_0^{S^{[n]},\mathrm{std}}(1;\lambda_B,U_2)\\ &=F_0^{S^{[n]}}(\lambda_1,U(\lambda_2\lambda_3))+F_0^{S^{[n]}}(\lambda_2,U(\lambda_1\lambda_3))+F_0^{S^{[n]}}(\lambda_3,U(\lambda_1\lambda_2)),\end{aligned}$$

and by expressing ψ_i as boundary we also get

$$F_0^{S^{[n]}}(\psi_1; U\lambda_1, \lambda_2, \lambda_3) = F_0^{S^{[n]}}(U\lambda_1, \lambda_2\lambda_3).$$

Hence the equation that we need to prove is

$$\begin{aligned}(64)\quad & \frac{d}{dG_2} F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3) \\ &= 2\Big(F_0^{S^{[n]}}(\lambda_1, U(\lambda_2\lambda_3)) - F_0^{S^{[n]}}(U\lambda_1, \lambda_2\lambda_3) + F_0^{S^{[n]}}(\lambda_2, U(\lambda_1\lambda_3)) - F_0^{S^{[n]}}(U\lambda_2, \lambda_1\lambda_3) \\ &\qquad + F_0^{S^{[n]}}(\lambda_3, U(\lambda_1\lambda_2)) - F_0^{S^{[n]}}(U\lambda_3, \lambda_1\lambda_2)\Big) \\ &\quad - \sum_{a,b} (G^{-1})_{ab} T_{e_a} T_{e_b} F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3).\end{aligned}$$

We now apply the variable change $p = e^z$ and view $F_0(\lambda_1, \lambda_2, \lambda_3)$ as a power series in z with coefficients quasimodular forms. Since $F_0(\lambda_1, \lambda_2, \lambda_3)$ are quasi-Jacobi forms of index $n-1$ by Lemma 2.15, we have the following relation of Jacobi and factorwise G_2–derivative:

$$\begin{aligned}&\Big(\frac{d}{dG_2}\Big)_z F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3) \\ &\quad = \frac{d}{dG_2} F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3) - 2z \frac{d}{dA} F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3) - 2z^2(n-1) F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3).\end{aligned}$$

By Proposition 4.1 we have that

$$\frac{d}{d\mathsf{A}} F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3) = F_0^{S^{[n]}}(T_\delta\lambda_1, \lambda_2, \lambda_3) + F_0^{S^{[n]}}(\lambda_1, T_\delta\lambda_2, \lambda_3) + F_0^{S^{[n]}}(\lambda_1, \lambda_2, T_\delta\lambda_3).$$

Expressing the left-hand side in (64) in terms of $(d/dG_2)_z$ then yields the claim. □

8.4 Conclusion

We aim to prove the holomorphic anomaly equation (63), which by Proposition 8.3 gives us the remaining part of Theorem 8.1. We start with the expression given in (62),

$$(65)\qquad F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3) = \sum_{g \in \mathbb{Z}} z^{2g-2-n+\sum_i l(\lambda_i)} (-1)^{g-1+\sum_i l(\lambda_i)} F_g^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}(\lambda_1, \lambda_2, \lambda_3).$$

We will compute the factorwise G_2–derivative $(d/dG_2)_z$ using the holomorphic anomaly equation given in Theorem 7.6, and then match all the terms with the right-hand side of (63).

We analyze all four terms appearing in the right-hand side of Theorem 7.6 in a sequence of lemmata:

Lemma 8.4 (term 1) *We have*

$$\begin{aligned}&\sum_{g\in\mathbb{Z}} z^{2g-2-n+\sum_i l(\lambda_i)} (-1)^{g-1+\sum_i l(\lambda_i)} F_{g-1}^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}(\lambda_1, \lambda_2, \lambda_3 \mid \Delta^{\mathrm{rel}}_{(B\times C, B_z)}) \\ &\qquad = (2-2n) z^2 F_0^{S^{[n]}}(\lambda_1, \lambda_2, \lambda_3).\end{aligned}$$

Proof Let $\beta = \beta_d = W + dF$. By the splitting formula of Proposition 5.2 applied in the reduced case we have

$$\begin{aligned}
&\langle \lambda_1, \lambda_2, \lambda_3 \mid \Delta^{\mathrm{rel}}_{(B\times C, B_z)} \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}_{g,(\beta,n)} \\
&\quad = \langle \lambda_1, \lambda_2, \lambda_3 \mid \Delta_{B\times C} \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}_{g,(\beta,n)} - \sum_{\substack{i\in\{1,2,3\},\mu \\ g_1+g_2=g+1-\ell(\mu)}} \frac{\prod_i \mu_i}{|\mathrm{Aut}(\mu)|} \\
&\qquad \cdot \Big(\langle \lambda_1, \dots, \underbrace{\mu}_{i^{\mathrm{th}}}, \dots, \lambda_N \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp,\mathrm{std}}_{g_1,(0,n)} \langle \lambda_i, \mu^\vee \mid \Delta_D \rangle^{(S\times\mathbb{P}^1, S_{0,\infty}),\sharp,\sim}_{g_2,(\beta,n)} \\
&\qquad\qquad + \langle \lambda_1, \dots, \underbrace{\mu}_{i^{\mathrm{th}}}, \dots, \lambda_N \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}_{g_1,(\beta,n)} \langle \lambda_i, \mu^\vee \mid \Delta_D \rangle^{(S\times\mathbb{P}^1, S_{0,\infty}),\sharp,\sim,\mathrm{std}}_{g_2,(0,n)} \Big)
\end{aligned}$$

To analyze the first term above we now use the Künneth decomposition

$$\Delta_{B\times C} = \Delta_B \Delta_C = (\omega_1 + \omega_2)(F_1 + F_2).$$

The moduli space $\overline{M}_{1,1,(0,0)}((S\times\mathbb{P}^1, S_{0,1,\infty}), \varnothing)$ is naturally isomorphic to $\overline{M}_{1,1} \times S \times \mathbb{P}^1$ with virtual class given by

$$e\big(H^1(\Sigma, f^*(T^{\log}_{(S\times\mathbb{P}^1, S_{0,1,\infty})}))\big) = e(T^{\log}_{(S\times\mathbb{P}^1, S_{0,1,\infty})}) - \lambda_1 c_2(T^{\log}_{(S\times\mathbb{P}^1, S_{0,1,\infty})}),$$

where we used the log tangent bundle

$$T^{\log}_{(S\times\mathbb{P}^1, S_{0,1,\infty})} = T_S \oplus T^{\log}_{(\mathbb{P}^1,\{0,1,\infty\})} = T_S \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

It follows that

$$\langle \tau_0(\alpha) \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty})}_{g=1,(0,0)} = \begin{cases} 0 & \text{if } \alpha \in \{1, F\}, \\ -1 & \text{if } \alpha = \omega. \end{cases}$$

Observe that under the ($\sharp$) convention we can have genus-1 components that are contracted, but since we only have two interior markings there can be no contracted genus-0 component. Moreover, genus ≥ 2 contracted component are ruled out since the K3 virtual class vanishes. Hence applying the divisor equation yields

$$\begin{aligned}
\langle \lambda_1, \lambda_2, \lambda_3 \mid \Delta_{B\times C} \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}_{g,(\beta,n)} &= 2\langle \lambda_1, \lambda_2, \lambda_3 \mid F, \omega \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\sharp}_{g,(\beta,n)} \\
&= 2\langle \tau_0(\omega) \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty})}_{g=1,(0,0)} \langle \lambda_1, \lambda_2, \lambda_3 \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\bullet}_{g,(\beta,n)} \\
&\qquad + 2\bigg(\int_{(\beta,n)} \omega \bigg) \langle \lambda_1, \lambda_2, \lambda_3 \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\bullet}_{g,(\beta,n)} \\
&= (-2+2n) \langle \lambda_1, \lambda_2, \lambda_3 \rangle^{(S\times\mathbb{P}^1, S_{0,1,\infty}),\bullet}_{g,(\beta,n)}.
\end{aligned}$$

On the other hand, $\Delta_D = F_1 + F_2$, so we find

$$\langle \lambda_i, \mu^\vee \mid \Delta_D \rangle^{(S\times\mathbb{P}^1, S_{0,\infty}),\sharp,\sim}_{g_2,(\beta,n)} = 2\langle \lambda_i, \mu^\vee \mid \tau_0(1)\tau_0(F) \rangle^{(S\times\mathbb{P}^1, S_{0,\infty}),\sharp,\sim}_{g_2,(\beta,n)}.$$

If the marked point carrying $\tau_0(1)$ lies on a component of a curve which remains stable after forgetting the marking, ie where on the corresponding connected component of the moduli space the morphism forgetting the marking is well defined, then since the integrand is pulled back from the forgetful morphism, the contribution vanishes. Alternatively, $\tau_0(1)$ lies on a contracted genus-1 component, which yields the contribution

$$\langle \tau_0(1)\rangle^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{g=1,(0,0)} \langle \lambda_i, \mu^\vee \mid \tau_0(F)\rangle^{(S\times\mathbb{P}^1,S_{0,\infty}),\bullet}_{g,(\beta,n)},$$

where since $\tau_0(1)$ stabilizes the rubber action, the second factor is nonrubber(!). The first factor is nonzero, but the second factor vanishes by the product formula and the general vanishing (see eg [Oberdieck and Pixton 2018, Lemma 2])

$$\pi_*[\overline{M}_{g,r}(\mathbb{P}^1,\mu,\nu)]^{\mathrm{vir}} = 0$$

for π the forgetful morphism to $\overline{M}_{g,r+\ell(\mu)+\ell(n)}$ whenever $2g-2+r+\ell(\mu)+\ell(\nu)>0$. The case where the rubber carries the standard virtual class is similar.

In summary:

$$\langle \lambda_1,\lambda_2,\lambda_3 \mid \Delta^{\mathrm{rel}}_{(B\times C,B_z)}\rangle^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\sharp}_{g,(\beta,n)} = 2(n-1)\langle \lambda_1,\lambda_2,\lambda_3\rangle^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\bullet}_{g,(\beta,n)}.$$

Replacing g by $g-1$ and summing over the genus then yields

$$\begin{aligned}
\sum_{g\in\mathbb{Z}} & z^{2g-2-n+\sum_i l(\lambda_i)}(-1)^{g-1+\sum_i l(\lambda_i)}\langle \lambda_1,\lambda_2,\lambda_3 \mid \Delta^{\mathrm{rel}}_{(B\times C,B_z)}\rangle^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\sharp}_{g-1,(\beta,n)} \\
&= z^2(-1)\sum_{g\in\mathbb{Z}} z^{2(g-1)-2-n+\sum_i l(\lambda_i)}(-1)^{(g-1)-1+\sum_i l(\lambda_i)} 2(n-1)\langle \lambda_1,\lambda_2,\lambda_3\rangle^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\bullet}_{g,(\beta,n)} \\
&= -2(n-1)z^2 Z^{(S\times\mathbb{P}^1,S_{0,1,\infty})}_{\mathrm{GW},(\beta,n)}(\lambda_1,\lambda_2,\lambda_3) = -2(n-1)z^2 Z^{(S\times\mathbb{P}^1,S_{0,1,\infty})}_{\mathrm{Hilb},(\beta,n)}(\lambda_1,\lambda_2,\lambda_3),
\end{aligned}$$

where we used the triangle of correspondences in the last step. Summing over the curve class β_d completes the lemma. □

For the second term we need first some preparation:

Lemma 8.5 *The class $U \in H^*(S^{[n]})$ has Künneth decomposition*

$$U = \sum_{m\geq 0}\sum_{\substack{b;b_1,\dots,b_m\\ l;l_1,\dots,l_m}} (-1)^{m+n+1}\frac{\prod_{i=1}^m b_i}{m!}((b,\Delta_{B,l}),(b_i,\Delta_{S,l_i})_{i=1}^m)\boxtimes((b,\Delta^\vee_{B,l}),(b_i,\Delta^\vee_{S,l_i})_{i=1}^m),$$

where the $b,b_1,\dots,b_m$ run over all positive integers such that $b+\sum_i b_i = n$, and the l and l_i run over the splitting of the diagonals of B and S, respectively:

$$\Delta_B = \sum_l \Delta_{B,l}\otimes\Delta^\vee_{B,l}, \quad \forall i,\ \Delta_S = \sum_{l_i}\Delta_{S,l_i}\otimes\Delta^\vee_{S,l_i}.$$

Proof Let $\mathfrak{q}_i$ and $\mathfrak{q}'_i$ denote the Nakajima operators acting on the first and second copies of $S^{[j]}\times S^{[j]}$, respectively. Then

$$U = -\sum_{b>0}\frac{1}{b^2}\mathfrak{q}_b\mathfrak{q}_{-b}(F_1+F_2) = -\sum_{b>0}\sum_{|\lambda|=n-b}\frac{1}{b^2}(-1)^b\mathfrak{q}_b\mathfrak{q}'_b(F_1+F_2)\Delta_{S^{[n-b]}}$$

$$= -\sum_{b>0}\sum_{|\lambda|=n-b} \frac{1}{b^2}(-1)^b \mathfrak{q}_b\mathfrak{q}'_b(F_1+F_2)\frac{(-1)^{|l|+\ell(\lambda)}}{|\mathrm{Aut}(\lambda)|\prod_i \lambda_i}\mathfrak{q}_{\lambda_1}\mathfrak{q}'_{\lambda_1}(\Delta_S)\cdots\mathfrak{q}_{\lambda_{\ell(\lambda)}}\mathfrak{q}'_{\lambda_{\ell(\lambda)}}(\Delta_S)$$

$$= \sum_{m\geq 0}\sum_{b;b_1,\dots,b_m} (-1)^{n+m+1}\frac{1}{m!}\frac{1}{b^2}\frac{1}{\prod_i b_i}\mathfrak{q}_b\mathfrak{q}'_b(F_1+F_2)\mathfrak{q}_{b_1}\mathfrak{q}'_{b_1}(\Delta_S)\cdots\mathfrak{q}_{b_m}\mathfrak{q}'_{b_m}(\Delta_S).$$

Using Definition 6.1 to rewrite this in terms of weighted partitions yields the claim. □

Lemma 8.6 (term 2a) *We have*

(66)
$$2\sum_{g\in\mathbb{Z}} z^{2g-2-n+\sum_i l(\lambda_i)}(-1)^{g-1+\sum_i l(\lambda_i)} \sum_{\substack{m\geq 0\\ g=g_1+g_2+m}}\sum_{\substack{b;b_1,\dots,b_m\\ l;l_1,\dots,l_m}} \frac{\prod_{i=j}^m b_j}{m!}$$
$$\cdot F_{g_1}^{\sim,\sharp,\mathrm{std}}\big(\lambda_1,((b,\Delta_{B,l}),(b_i,\Delta_{S,l_i})_{i=1}^m)\big)F_{g_2}^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\sharp}\big(((b,\Delta_{B,l}^\vee),(b_i,\Delta_{S,l_i}^\vee)_{i=1}^m),\lambda_2,\lambda_3\big)$$
$$= 2zF_0^{S^{[n]}}(U(\delta\lambda_1),\lambda_2,\lambda_3).$$

Proof By Lemma 8.5, via a careful matching of the signs and z factors, and observing that since we have no interior markings the ($\sharp$) convention yields the same invariant as the ($\bullet$) convention, the left-hand side in (66) equals

$$2\sum_d q^d Z_{(0,n)}^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim,\mathrm{std}}(\lambda_1,U_1)Z_{(W+dF,n)}^{(S\times\mathbb{P}^1,S_{0,1,\infty})}(U_2,\lambda_2,\lambda_3),$$

where we write U_1 and U_2 for summing over the Künneth factors of the class $U\in H^*(S^{[n]}\times S^{[n]})$. By Proposition 6.7 and Theorem 6.2 the above then becomes

$$2\sum_d q^d z\left(\int_{S^{[n]}}\delta\lambda_1 U_1\right)Z_{(W+dF,n)}^{(S\times\mathbb{P}^1,S_{0,1,\infty})}(U_2,\lambda_2,\lambda_3) = 2\sum_d q^d z Z_{(W+dF,n)}^{(S\times\mathbb{P}^1,S_{0,1,\infty})}(U(\delta\delta_1),\lambda_2,\lambda_3)$$
$$= 2zF_0^{S^{[n]}}(U(\delta\lambda_1),\lambda_2,\lambda_3). \qquad □$$

Lemma 8.7 (term 2b) *We have*

$$2\sum_{g\in\mathbb{Z}} z^{2g-2-n+\sum_i l(\lambda_i)}(-1)^{g-1+\sum_i l(\lambda_i)} \sum_{\substack{m\geq 0\\ g=g_1+g_2+m}}\sum_{\substack{b;b_1,\dots,b_m\\ l;l_1,\dots,l_m}} \frac{\prod_{i=j}^m b_j}{m!}$$
$$\cdot F_{g_1}^{\sim,\sharp}\big(\lambda_1,((b,\Delta_{B,l}),(b_i,\Delta_{S,l_i})_{i=1}^m)\big)F_{g_2}^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\sharp,\mathrm{std}}\big(((b,\Delta_{B,l}^\vee),(b_i,\Delta_{S,l_i}^\vee)_{i=1}^m),\lambda_2,\lambda_3\big)$$
$$= 2F_0^{S^{[n]}}(\lambda_1,U(\lambda_2\lambda_3)) + 2\left(\int_{S^{[n]}}\lambda_1 U(\lambda_2\lambda_3)\right)\frac{\boldsymbol{G}(p,q)^n}{\Theta^2(p,q)\Delta(q)}.$$

Proof With similar reasoning as for term 2a and using Propositions 6.6 and 6.8 this becomes

$$2\sum_d q^d Z_{(W+dF,n)}^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}(\lambda_1,U_1)Z_{(0,n)}^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\mathrm{std}}(U_2,\lambda_2,\lambda_3)$$
$$= 2\sum_d q^d Z_{(W+dF,n)}^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}(\lambda_1,U_1)\int_{S^{[n]}}U_2\lambda_2\lambda_3 = 2\sum_d q^d Z_{(W+dF,n)}^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}(\lambda_1,U(\lambda_2\lambda_3)).$$

Let $D(F)=(1/(n-1)!)((1,F)(1,1)^{n-1})\in H^2(S^{[n]})$. Employing Proposition 6.8 and the evaluation of Proposition 6.9 we get

$$2\sum_d q^d Z^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{(W+dF,n)}(\lambda_1,U(\lambda_2\lambda_3),D(F))+2\biggl(\int_{S^{[n]}}\lambda_1U(\lambda_2\lambda_3)\biggr)\frac{\boldsymbol{G}(z,q)^n}{\Theta^2(z,q)\Delta(q)}$$
$$=2F_0^{S^{[n]}}(\lambda_1,U(\lambda_2\lambda_3))+2\biggl(\int_{S^{[n]}}\lambda_1U(\lambda_2\lambda_3)\biggr)\frac{\boldsymbol{G}(p,q)^n}{\Theta^2(p,q)\Delta(q)},$$

as desired. □

Lemma 8.8 (term 3) *Let $\lambda_i^{(j)}$ be the weighted partition λ_i but with j^{th} weight δ_{ij} replaced by $\pi^*\pi_*(\delta_{ij})$. Then*

$$-2\sum_{g\in\mathbb{Z}} z^{2g-2-n+\sum_i l(\lambda_i)}(-1)^{g-1+\sum_i l(\lambda_i)}\sum_{j=1}^{\ell(\lambda_1)} F_g^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\sharp}(\psi^{\mathrm{rel}}_{1,j}\lambda_1^{(j)},\lambda_2,\lambda_3)$$
$$=-2zF_0^{S^{[n]}}(\delta U(\lambda_1),\lambda_2,\lambda_3)-2F_0^{S^{[n]}}(U(\lambda_1),\lambda_2\lambda_3)-2\biggl(\int_{S^{[n]}}U(\lambda_1)\lambda_2\lambda_3\biggr)\frac{\boldsymbol{G}(p,q)^n}{\Theta^2(p,q)\Delta(q)}.$$

Proof We employ the splitting formula for the relative ψ–class given in Proposition 5.3. The left-hand side term becomes

$$-2\sum_d\sum_{j=1}^{\ell(\lambda_1)}\frac{1}{\lambda_{1,i}}q^dZ^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{(W+dF,n)}(\lambda_1^{(i)},\Delta_1)Z^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\mathrm{std}}_{(0,n)}(\Delta_2,\lambda_2,\lambda_3)$$
$$-2\sum_d\sum_{j=1}^{\ell(\lambda_1)}\frac{1}{\lambda_{1,i}}q^dZ^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim,\mathrm{std}}_{(0,n)}(\lambda_1^{(i)},\Delta_1)Z^{(S\times\mathbb{P}^1,S_{0,1,\infty})}_{(W+dF,n)}(\Delta_2,\lambda_2,\lambda_3),$$

where Δ_1 and Δ_2 stand for summing over the Künneth decomposition of the diagonal in $(S^{[n]})^2$.

Observe that U acts on a $H^*(S)$–weighted partition $\lambda=((\lambda_j,\delta_j))_{j=1}^l$ by

$$U\lambda=\sum_{j=1}^{\ell(\lambda)}\frac{1}{\lambda_j}\bigl((\lambda_1,\delta_1)\cdots\underbrace{(\lambda_i,\pi^*\pi_*(\gamma_i))}_{i^{\mathrm{th}}}\cdots(\lambda_l,\delta_l)\bigr).$$

Hence with Propositions 6.6 and 6.7, the above becomes

$$-2\sum_d q^dZ^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{(W+dF,n)}(U(\lambda_1),\Delta_1)\int_{S^{[n]}}\Delta_2\lambda_2\lambda_3$$
$$-2z\sum_d q^d\biggl(\int_{S^{[n]}}U(\lambda_1)\delta\Delta_1\biggr)Z^{(S\times\mathbb{P}^1,S_{0,1,\infty})}_{(W+dF,n)}(\Delta_2,\lambda_2,\lambda_3)$$
$$=-2\sum_d q^dZ^{(S\times\mathbb{P}^1,S_{0,\infty}),\sim}_{(W+dF,n)}(U(\lambda_1),\lambda_2\lambda_3)-2z\sum_d q^dZ^{(S\times\mathbb{P}^1,S_{0,1,\infty})}_{(W+dF,n)}(\delta U(\lambda_1),\lambda_2,\lambda_3)$$
$$=-2F_0^{S^{[n]}}(U(\lambda_1),\lambda_2\lambda_3)-2\biggl(\int_{S^{[n]}}U(\lambda_1)\lambda_2\lambda_3\biggr)\frac{\boldsymbol{G}(p,q)^n}{\Theta^2(p,q)\Delta(q)}-2zF_0^{S^{[n]}}(\delta U(\lambda_1),\lambda_2,\lambda_3).\quad □$$

Lemma 8.9 (term 4) *We have*

$$-\sum_{a,b}(G^{-1})_{ab}T_{e_a}T_{e_b}F_g^{(S\times\mathbb{P}^1,S_{0,1,\infty}),\sharp}(\lambda_1,\lambda_2,\lambda_3) = -\sum_{a,b}(G^{-1})_{ab}T_{e_a}T_{e_b}F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3).$$

Proof Since there are no interior markings, the ($\sharp$) condition yields the same invariants as the ($\bullet$) condition. Hence the claim is just the application of Theorem 6.2. □

Proof of Theorem 8.1 Part (a) was proven in Proposition 8.2. For Part (b) it suffices to prove the equality in Proposition 8.3. We start with (62), and compute $(d/dG_2)_z$ of the left-hand side of (62) by applying the holomorphic anomaly equation for $(S\times\mathbb{P}^1, S_{0,1,\infty})$ stated in Theorem 7.6. This holomorphic anomaly equation produces four terms. These four terms are precisely the terms labeled 1, 2a, 2b, 3 and 4 in the above lemmata (up to permutation). Summing these four terms together yields

$$\begin{aligned}\left(\frac{d}{dG_2}\right)_z F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3) = (2-2n)z^2F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3) + 2zF_0^{S^{[n]}}(U(\delta\lambda_1),\lambda_2,\lambda_3)\\ + 2F_0^{S^{[n]}}(\lambda_1,U(\lambda_2\lambda_3)) + 2\left(\int_{S^{[n]}}\lambda_1U(\lambda_2\lambda_3)\right)\frac{\boldsymbol{G}(p,q)^n}{\Theta^2(p,q)\Delta(q)}\\ -2zF_0^{S^{[n]}}(\delta U(\lambda_1),\lambda_2,\lambda_3) - 2F_0^{S^{[n]}}(U(\lambda_1),\lambda_2\lambda_3)\\ -2\left(\int_{S^{[n]}}U(\lambda_1)\lambda_2\lambda_3\right)\frac{\boldsymbol{G}(p,q)^n}{\Theta^2(p,q)\Delta(q)}\\ -\sum_{a,b}(G^{-1})_{ab}T_{e_a}T_{e_b}F_0^{S^{[n]}}(\lambda_1,\lambda_2,\lambda_3) + (\cdots),\end{aligned}$$

where $(\cdots)$ stands for the terms where the role of λ_1 is played by λ_2 and λ_3 in the four middle terms. The above is precisely the right-hand side in Proposition 8.3 if we observe two basic facts: First, the operator U is symmetric (since the adjoint of $\mathfrak{q}_n(\alpha)$ is $(-1)^n\mathfrak{q}_{-n}(\alpha)$):

$$\int_{S^{[n]}}U(\lambda)\mu = \int_{S^{[n]}}\lambda U(\mu).$$

Hence the $\boldsymbol{G}^n$ terms cancel. And second,

$$T_\delta = [e_\delta, U], \quad \text{and hence } T_\delta\lambda = \delta U(\lambda) - U(\delta\lambda).$$ □

9 Holomorphic anomaly equations: nonprimitive case

9.1 Overview

Let g and N be fixed. For the elliptic K3 surface $S\to\mathbb{P}^1$ recall the generating series

$$F_{g,l}(\mathrm{taut};\gamma_1,\ldots,\gamma_N) = \sum_{d=-l}^{\infty}\sum_{r\in\mathbb{Z}}\langle\mathrm{taut};\gamma_1,\ldots,\gamma_N\rangle_{g,lW+dF+rA}^{S^{[n]}}q^d(-p)^r,$$

where we have dropped the superscript $S^{[n]}$ on the left.

We show that the quasi-Jacobi form property and the holomorphic anomaly equation for the primitive series $F_{g,1}$ (Conjecture E) together with the multiple cover conjecture (Conjecture A) imply both claims for the general series $F_{g,l}$. More precisely:

Proposition 9.1 *If Conjectures A and E hold for all g' and N' such that either $g' < g$ or ($g' = g$ and $N' < N$), then Conjectures B and C hold for g and N.*

Using Proposition 9.1 we obtain the proof of our main theorem:

Proof of Theorem 1.3 If $g = 0$ and $N \leq 3$, then Conjecture A holds by Theorem 1.2, and Conjecture E was proven in Theorem 8.1. Hence the claim follows from Proposition 9.1. □

The proof of Proposition 9.1 is purely formal: if the multiple cover formula holds, then $F_{g,l}$ is obtained from $F_{g,1}$ by applying the Hecke operator. The statement then follows from results about Hecke operators on quasi-Jacobi forms (Section 2.8) and basic properties of the operators appearing in the holomorphic anomaly equation.

9.2 Proof

Proof of Proposition 9.1 Recall the formal l^{th} weight k Hecke operator $T_{k,l}$ defined in (22). If the multiple cover conjecture holds, then for all $l > 0$ we have

$$F_{g,l}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = l^{\sum_i (\deg(\gamma_i) - n - \mathrm{wt}(\gamma_i))} T_{k,l} F_{g,1}(\mathrm{taut}; \gamma_1, \dots, \gamma_N),$$

where $k = n(2g - 2 + N) + \sum_i \mathrm{wt}(\gamma_i)$. Assuming Conjecture E(a), we have

$$F_{g,1}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(\tau)} \mathrm{QJac}_{k',n-1},$$

where $k' = n(2g - 2 + N) + \sum_i \mathrm{wt}(\gamma_i) - 10$. Hence by Proposition 2.22 (describing the action of Hecke operators of weight k on weight-k' forms) we find that

$$F_{g,l}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(\tau)^l} \mathrm{QJac}_{k'+12l,(n-1)l}(\Gamma_0(l)),$$

that is Conjecture B holds.

To prove Conjecture C, the multiple cover conjecture and (23) give

$$\begin{aligned} \frac{d}{dG_2} F_{g,l}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) &= l^{e(\gamma_1,\dots,\gamma_N)} \frac{d}{dG_2} T_{k,l} F_{g,1}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \\ &= l^{e(\gamma_1,\dots,\gamma_N)+1} T_{k-2,l} \frac{d}{dG_2} F_{g,1}(\mathrm{taut}; \gamma_1, \dots, \gamma_N), \end{aligned}$$

where

$$k = k(g, N, \gamma_1, \dots, \gamma_N) := n(2g - 2 + N) + \sum_i \mathrm{wt}(\gamma_i),$$

$$e(\gamma_1, \dots, \gamma_N) = \sum_i (\deg(\gamma_i) - n - \mathrm{wt}(\gamma_i)).$$

Assuming Conjecture E(b) this equals

$$l^{e(\gamma_1,\dots,\gamma_N)+1} T_{k-2,l} \cdot \Bigg[F_{g-1,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N,U) + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\dots,N\}=A\sqcup B}} F_{g_1,1}(\mathrm{taut}_1;\gamma_A,U_1) F_{g_2}^{\mathrm{std}}(\mathrm{taut}_2;\gamma_B,U_2) \\ -2\sum_{i=1}^{N} F_{g,1}(\psi_i\,\mathrm{taut};\gamma_1,\dots,\gamma_{i-1},U\gamma_i,\gamma_{i+1},\dots,\gamma_N) - \sum_{a,b} (g^{-1})_{ab} T_{e_a} T_{e_b} F_{g,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N) \Bigg].$$

By Lemmata 9.3 and 9.2 we can apply Conjecture A to this term in reverse, eg

$$l^{e(\gamma_1,\dots,\gamma_N)+1} T_{k-2,l} F_{g-1,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N,U) = F_{g-1,l}(\mathrm{taut};\gamma_1,\dots,\gamma_N,U),$$

or the exceptional case

$$l^{e(\gamma_1,\dots,\gamma_N)+1} T_{k-2,l} F_{g,1}(\mathrm{taut};\dots,T_{e_a}\gamma_i,\dots,T_{e_b}\gamma_j,\dots) = \frac{1}{l} F_{g,l}(\mathrm{taut};\dots,T_{e_a}\gamma_i,\dots,T_{e_b}\gamma_j,\dots),$$

etc. As a result we obtain precisely the right-hand side for the (d/dG_2)–holomorphic anomaly equation in Conjecture C.

Similarly, by Proposition 4.1 we have

$$\frac{d}{d\mathsf{A}} F_{g,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N) = T_\delta F_{g,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N).$$

Hence by (23) we have

$$\begin{aligned} \frac{d}{d\mathsf{A}} F_{g,l}(\mathrm{taut};\gamma_1,\dots,\gamma_N) &= l^{\sum_i (\deg(\gamma_i)-n-\mathrm{wt}(\gamma_i))} \frac{d}{d\mathsf{A}} T_{k,l} \frac{d}{d\mathsf{A}} F_{g,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N) \\ &= l\cdot l^{\sum_i (\deg(\gamma_i)-n-\mathrm{wt}(\gamma_i))} T_{k-1,l} \frac{d}{d\mathsf{A}} F_{g,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N) \\ &= l\cdot l^{\sum_i (\deg(\gamma_i)-n-\mathrm{wt}(\gamma_i))} T_{k-1,l} T_\delta F_{g,1}(\mathrm{taut};\gamma_1,\dots,\gamma_N) \\ &= T_\delta F_{g,l}(\mathrm{taut};\gamma_1,\dots,\gamma_N), \end{aligned}$$

where we used that T_δ is of weight -1 (Lemma 3.1). □

Lemma 9.2 *If $U=\sum_i a_i\otimes b_i$ is a* wt*–homogeneous Künneth decomposition of $U\in H^*(S^{[n]})^{\otimes 2}$, then for every i we have*

$$\begin{aligned} k(g-1,N+2,\gamma_1,\dots,\gamma_N,a_i,b_i) &= k(g,N,\gamma_1,\dots,\gamma_N,a_i,b_i)-2, \\ k(g_1,|A|+1,\gamma_A,a_i) &= k(g,N,\gamma_1,\dots,\gamma_N)-2 \quad \text{if } F_{g_2}^{S^{[n]},\mathrm{std}}(\mathrm{taut}_2;\gamma_B,b_i)\neq 0, \\ k(g,N,\gamma_1,\dots,U(\gamma_i),\dots,\gamma_N) &= k(g,N,\gamma_1,\dots,\gamma_N)-2, \\ k(g,N,\gamma_1,\dots,T_{e_a}\gamma_i,\dots,T_{e_b}\gamma_j,\dots,\gamma_N) &= k(g,N,\gamma_1,\dots,\gamma_N)-2. \end{aligned}$$

Proof The first of these equations follows from Lemma 3.2, and the third and fourth follow from Lemma 3.1. For the second, recall that for $X=S^{[n]}$ we have

$$(67) \qquad [\overline{M}_{g,N}(X,0)]^{\mathrm{std}} = \begin{cases} [\overline{M}_{0,N}\times X] & \text{if } g=0,\ N\geq 3, \\ [\overline{M}_{1,N}\times X]\pi_2^*(c_{2n}(X)) & \text{if } g=1,\ N\geq 1, \\ 0 & \text{if } g\geq 2. \end{cases}$$

If $F^{\mathrm{std}}_{g_2}(\mathrm{taut}_2;\gamma_B,b_i)\neq 0$, we hence find

$$\sum_i F^{\mathrm{std}}_{g_2}(\mathrm{taut}_2;\gamma_B,b_i)a_i=\biggl(\int_{\overline{M}_{g_2,|B|+1}}\mathrm{taut}_2\biggr)\begin{cases}U\bigl(\prod_{i\in B}\gamma_i\bigr) & \text{if } g_2=0,\\ U\bigl(c_{2n}(X)\prod_{i\in B}\gamma_i\bigr) & \text{if } g_2=1.\end{cases}$$

Hence using Lemma 3.3 we get

$$\begin{aligned}k(g_1,|A|+1,\gamma_A,a_i)&=\begin{cases}k\bigl(g,|A|+1,\gamma_A,U\bigl(\prod_{i\in B}\gamma_i\bigr)\bigr) & \text{if } g_2=0,\\ k\bigl(g-1,|A|+1,\gamma_A,U\bigl(c_{2n}(X)\prod_{i\in B}\gamma_i\bigr)\bigr) & \text{if } g_2=1,\end{cases}\\ &=k(g,N,\gamma_1,\dots,\gamma_N),\end{aligned}$$

where we used $\mathrm{wt}(c_{2n}(X))=n$ in the last step. □

Lemma 9.3 *If $U=\sum_i a_i\otimes b_i$ is a wt–homogeneous Künneth decomposition of $U\in H^*(S^{[n]})^{\otimes 2}$, then for every i we have*

$$\begin{aligned}e(\gamma_1,\dots,\gamma_N,a_i,b_i)&=e(\gamma_1,\dots,\gamma_N)+1,\\ e(\gamma_A,a_i)&=e(\gamma)+1\quad \textit{if } F^{\mathrm{std}}_{g_2}(\mathrm{taut}_2;\gamma_B,b_i)\neq 0 \textit{ for some } g_2,\\ e(\gamma_1,\dots,U(\gamma_i),\dots,\gamma_N)&=e(\gamma_1,\dots,\gamma_N)+1,\\ e(\gamma_1,\dots,T_{e_a}\gamma_i,\dots,T_{e_b}\gamma_j,\dots,\gamma_N)&=e(\gamma_1,\dots,\gamma_N)+2.\end{aligned}$$

Proof With the notation of Section 3.5 define $h_{FW}:=\mathrm{act}(F\wedge W)$, which acts semisimply on $H^*(S^{[n]})$. For an eigenvector γ, define $\deg_{FW}(\gamma)$ to be the eigenvalue of h_{FW}:

$$h_{FW}(\gamma)=\deg_{FW}(\gamma)\gamma.$$

Then because

$$(\deg(\gamma)-n-\mathrm{wt}(\gamma))\gamma=(h-\mathrm{Wt})\gamma=-\mathrm{act}(W\wedge F)\gamma=h_{FW}(\gamma),$$

we find

$$e(\gamma_1,\dots,\gamma_N)=\sum_i \deg_{FW}(\gamma_i).$$

The claim now follows parallel to Lemma 9.2 (use that $h_{FW}=h-\mathrm{Wt}$, so the corresponding properties for the grading operator h_{FW} are easily derived). □

10 Fiber classes

10.1 Overview

We study the generating series of Gromov–Witten invariants of $S^{[n]}$ in fiber classes of the Lagrangian fibration $S^{[n]}\to\mathbb{P}^n$,

$$F_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N)=\sum_{d\geq 0}\sum_{\substack{r\in\mathbb{Z}\\(d,r)\neq(0,0)}}\langle\mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,dF+rA}q^d(-p)^r.$$

Recall from Theorem 2.23 the weight-n (meromorphic) quasi-Jacobi forms

$$\mathsf{A}_n(z,\tau) = B_n + \delta_{n,1}\frac{1}{2}\frac{p^{1/2}+p^{-1/2}}{p^{1/2}-p^{-1/2}} - n\sum_{k,d\geq 1} d^{n-1}(p^k+(-1)^n p^{-k})q^{kd} \in \mathrm{MQJac}_{0,n}\,.$$

For any (deg, wt)–bihomogeneous class γ, define the modified degree

$$\deg_{WF}(\gamma) = n + \mathrm{wt}(\gamma) - \deg(\gamma).$$

Remark 10.1 Consider the basis of $H^*(S,\mathbb{Q})$ given by $\mathcal{B} = \{1, \mathsf{p}, W, F, e_a\}$, where $\{e_a\}$ is a basis of $\{W,F\}^\perp \subset H^2(S,\mathbb{Q})$. If $\gamma = \prod_i \mathfrak{q}_{n_i}(\delta_i)v_\varnothing$ for $\delta_i \in \mathcal{B}$, we have

$$\deg_{WF}(\gamma) = |\{i \mid \delta_i = W\}| - |\{i \mid \delta_i = F\}|.$$

By Section 3.5, $\deg_{WF}(\gamma)$ is also the eigenvalue of the operator $h_{WF} := \mathrm{act}(W\wedge F)$.

The main result of this section is the following:

Theorem 10.2 *Fix g and N with $2g-2+N>0$ such that*

(i) *the multiple cover conjecture (Conjecture A) holds for this g and N,*

(ii) $\langle \mathrm{taut}; \gamma_1,\dots,\gamma_N\rangle_{g,dF+rA} = 0$ *for all* $(d,r)\neq(0,0)$, *whenever* $\sum_i \deg_{WF}(\gamma_i) < 0$.

Let γ_i be (wt, deg)*–bihomogeneous classes and let*

$$a = 3g-3+N-\deg(\mathrm{taut}) \quad \textit{and} \quad b = \sum_{i=1}^N \deg_{WF}(\gamma_i).$$

If $a,b\geq 0$, then in $\mathbb{C}((p))[[q]]/\mathbb{C}$ we have

$$F_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N) \equiv \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,F}\sum_{d,k\geq 1} k^a d^b q^{kd} + \sum_{r\geq 1}(-1)^r \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,F+rA}\left(\frac{-1}{b+1}\left(p\frac{d}{dp}\right)^a \mathsf{A}_{b+1}(p,q)\right)\bigg|_{p\mapsto p^r}. \tag{68}$$

In particular, $F_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N)$ is a meromorphic quasi-Jacobi form of weight

$$k = n(2g-2+N) + \sum_i \mathrm{wt}(\gamma_i)$$

and index 0, *with poles at torsion points.*

Here for two power series $f(p,q), g(p,q) \in \mathbb{C}((p))[[q]]$, we write $f\equiv g$ if they are equal in $\mathbb{C}((p))[[q]]/\mathbb{C}$, that is if there exists a constant $c\in\mathbb{C}$ such that $f(p,q) = g(p,q)+c$.

In (68) the sum over r is finite by Lemma 4.2, and hence the statement of the theorem is well defined. If $a<0$ in Theorem 10.2, then $\mathrm{taut}=0$, so all Gromov–Witten invariants would vanish. Theorem 10.2(ii) would follow from a family version of the GW/Hilb correspondence (Section 6.3), where one does not fix the complex structure of the source curve. Hence (ii) is expected to hold for all g and N with $2g-2+N>0$. We prove (ii) for $(g,N)=(0,3)$ below and obtain the following:

Theorem 10.3 *For any $\gamma_1, \gamma_2, \gamma_3 \in H^*(S^{[n]})$ the series $F_{g=0,0}(\mathrm{taut}; \gamma_1, \dots, \gamma_N)$ is a meromorphic quasi-Jacobi form of weight $n + \sum_i \mathrm{wt}(\gamma_i)$ and index 0 with poles at torsion points (of the form given in (68)). Moreover, in $\mathbb{C}((p))[[q]]/\mathbb{C}$ we have*

$$\frac{d}{dG_2} F_{0,0}(\mathrm{taut}; \gamma_1, \gamma_2, \gamma_3) \equiv 0 \quad \text{and} \quad \frac{d}{d\mathsf{A}} F_{0,0}(\mathrm{taut}; \gamma_1, \gamma_2, \gamma_3) \equiv T_\delta F_{0,0}(\mathrm{taut}; \gamma_1, \gamma_2, \gamma_3). \tag{69}$$

10.2 Multiple cover conjecture

We first recall an equivalent form of the multiple cover conjecture (Conjecture A). Let S be any K3 surface with an effective curve class $\beta \in H_2(S, \mathbb{Z})$. For every divisor $k \mid \beta$ let S_k be some K3 surface and consider any *real* isometry

$$\varphi_k \colon H^2(S, \mathbb{R}) \to H^2(S_k, \mathbb{R})$$

such that $\varphi_k(\beta/k) \in H_2(S_k, \mathbb{Z})$ is a primitive effective curve class. We extend φ_k to the full cohomology lattice by $\varphi_k(\mathsf{p}) = \mathsf{p}$ and $\varphi_k(1) = 1$. Define an extension to the Hilbert scheme by acting factorwise in the Nakajima operators:

$$\varphi_k \colon H^*(S^{[n]}) \to H^*(S_k^{[n]}), \quad \prod_i \mathfrak{q}_{n_i}(\delta_i) v_\varnothing \mapsto \prod_i \mathfrak{q}_{n_i}(\varphi_k(\delta_i)) v_\varnothing.$$

Conjecture F *We have*

$$\langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g, \beta + rA} = \sum_{k \mid (\beta, r)} k^{3g-3+N-\deg(\mathrm{taut})} (-1)^{r + r/k} \langle \mathrm{taut}; \varphi_k(\gamma_1), \dots, \varphi_k(\gamma_N) \rangle^{S^{[n]}}_{g, \varphi_k(\beta/k) + (r/k)A}.$$

This conjecture is equivalent to the one we have given in the introduction:

Lemma 10.4 [Oberdieck 2022, Lemma 3] *Conjecture F is equivalent to Conjecture A.*

10.3 Proof of Theorem 10.2

Step 1 (positive part) We apply the multiple cover conjecture (in the form of Conjecture F) to the following series, where we sum only over curve classes which have positive fiber degree:

$$F^+_{g,0}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d \geq 1} \sum_{r \in \mathbb{Z}} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g, dF + rA} q^d (-p)^r.$$

For any $k \mid (d, r)$ let $\varphi_k \colon H^2(S, \mathbb{Q}) \to H^2(S, \mathbb{Q})$ be the isometry defined by

$$F \mapsto \frac{k}{d} F, \quad W \mapsto \frac{d}{k} W \quad \text{and} \quad \varphi_k|_{\{W,F\}^\perp} = \mathrm{id}.$$

Assuming that all γ_i are written in the Nakajima basis with weightings from the fixed basis $\mathcal{B}$ (defined in Remark 10.1), we obtain

$$F^+_{g,0}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d \geq 1} \sum_{r \in \mathbb{Z}} \sum_{k \mid (d,r)} k^b \left(\frac{d}{k}\right)^a (-1)^{r/k} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g, F + rA} p^r q^d.$$

Using the monodromy of Section 3.6.3 we have

$$\langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,F+rA} = (-1)^{nN+\sum_i l(\gamma_i)} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,F-rA}.$$

Hence we conclude that

$$\begin{aligned} &F^{+}_{g,0}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \\ &\quad = \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,F} \sum_{d,k\geq 1} k^a d^b q^{kd} \\ &\qquad + \sum_{r\geq 1} (-1)^r \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,F+rA} \biggl(\sum_{k,d\geq 1} k^a d^b (p^k + (-1)^{nN+\sum_i l(\gamma_i)} p^{-k}) q^{kd} \biggr) \bigg|_{p\mapsto p^r}. \end{aligned}$$

We now analyze the second term on the right. Since otherwise all invariants vanish, we can assume the dimension constraint

$$\mathrm{vd}\overline{M}_{g,N}(S^{[n]}, \beta) = (2n-3)(1-g) + N + 1 = \deg(\mathrm{taut}) + \sum_i \deg(\gamma_i),$$

or equivalently,

$$a = 3g - 3 + N - \deg(\mathrm{taut}) = 2n(g-1) - 1 + \sum_i \deg(\gamma_i). \tag{70}$$

Furthermore, let $\gamma_{i,j} \in H^*(S)$ be the cohomology weights of γ_i in the Nakajima basis. Let $V = \{W, F\}^{\perp} \subset H^2(S, \mathbb{Z})$. Since $\mathrm{ev}_*[\overline{M}_{g,N}(S^{[n]}, dF + rA)]^{\mathrm{vir}}$ is invariant under the monodromy group $O(V, \mathbb{Z})$, by standard invariant theory for the orthogonal group (eg [Oberdieck 2024a, Section 6.1]) we can assume that there are an even number of γ_{ij} such that $\gamma_{ij} \in V$. Indeed, otherwise all the invariants $\langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,F+rA}$ vanish and there is nothing to prove. We obtain the following parity result:

Lemma 10.5

$$a + nN + \sum_i l(\gamma_i) \equiv b - 1 \mod 2.$$

Proof Using (29) we have

$$\sum_i \deg(\gamma_i) = nN - \sum_i l(\gamma_i) + \sum_{i,j} \deg(\gamma_{ij}).$$

Hence by the dimension constraint (70) and modulo 2,

$$\begin{aligned} a + nN + \sum_i l(\gamma_i) &\equiv -1 + \deg(\gamma_i) + nN + \sum_i l(\gamma_i) \equiv -1 + \sum_{i,j} \deg(\gamma_{ij}) \\ &\equiv -1 + \sum_i |\{j \mid \gamma_{ij} \in H^2(S)\}| \overset{(*)}{\equiv} -1 + \sum_i |\{j \mid \gamma_{ij} \in \{W, F\}\}| \equiv b - 1, \end{aligned}$$

where in $(*)$ we used that there are an even number of γ_{ij} in $\{W, F\}^{\perp}$. □

So

$$F^{+}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N) \tag{71}$$
$$= \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,F} \sum_{d,k\geq 1} k^a d^b q^{kd}$$
$$+ \sum_{r\geq 1} (-1)^r \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,F+rA} \biggl(\Bigl(p\frac{d}{dp}\Bigr)^a \sum_{k,d\geq 1} d^b (p^k + (-1)^{b+1} p^{-k}) q^{kd} \biggr)\bigg|_{p\mapsto p^r}.$$

Step 2 (fiber degree-0 part) It remains to compute the degree-0 part

$$F^{(0)}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N) = \sum_{r\geq 1} \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,rA} (-p)^r.$$

Lemma 10.6 *If $\sum_i \deg_{WF}(\gamma_i) \neq 0$, then $F^{(0)}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N) = 0$.*

Proof By monodromy invariance, the class

$$\mathrm{ev}_*(\mathrm{taut}[\overline{M}_{g,N}(S^{[n]}, rA)]^{\mathrm{vir}}) \in H^*(S^{[n]})^{\otimes N}$$

has weight 0 with respect to the grading operator $h_{WF} = \mathrm{act}(W \wedge F)$. On the other hand,

$$h_{WF}(\gamma_1\otimes\cdots\otimes\gamma_N) = \sum_i \gamma_1\otimes\cdots\otimes h_{WF}(\gamma_i)\otimes\cdots\otimes\gamma_N = \Bigl(\sum_i \deg_{WF}(\gamma_i)\Bigr)\gamma_1\otimes\cdots\otimes\gamma_N.$$

Hence if $\sum_i \deg_{WF}(\gamma_i) \neq 0$, the pairing between these two classes vanishes. □

Lemma 10.7 *If $\sum_i \deg_{WF}(\gamma_i) = 0$ and under the assumptions of Theorem 10.2, we have*

$$F^{(0)}_{g,0}(\mathrm{taut};\gamma_1,\dots,\gamma_N) = \sum_{r\geq 1} (-1)^r \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,F+rA} \sum_{k\geq 1} k^a p^{kr}.$$

Proof Recall the monodromy e^{-T_δ} from Section 3.6.4 which satisfies $e^{-T_\delta}A = A+F$. We conclude that

$$\langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,rA} = \langle \mathrm{taut}; e^{-T_\delta}\gamma_1,\dots,e^{-T_\delta}\gamma_N\rangle^{S^{[n]}}_{g,rF+rA}. \tag{72}$$

The operator T_δ satisfies the commutation relation

$$[h_{WF}, T_\delta] = [\mathrm{act}(W\wedge F), \mathrm{act}(\delta\wedge F)] = -T_\delta,$$

and hence $\deg_{WF}(T_\delta\gamma) = \deg_{WF}(\gamma) - 1$. Because we assumed $\sum_i \deg_{WF}(\gamma_i) = 0$ and Theorem 10.2(ii), only the leading term in $e^{-T_\delta}\gamma_i$ can contribute:

$$(\text{term in (72)}) = \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,rF+rA}.$$

Using the multiple cover formula (Conjecture F) and $b = \sum_i \deg_{WF}(\gamma_i) = 0$ this becomes

$$\sum_{k\mid r} k^a (-1)^{r+r/k} \langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle^{S^{[n]}}_{g,F+(r/k)A}.$$

The lemma follows by rearranging the sums. □

Step 3 (proof of (68)) If $b = \sum_i \deg_{WF}(\gamma_i) > 0$, then by Lemma 10.6 the series $F_{g,0}(\mathrm{taut}; \gamma_1, \dots, \gamma_N)$ is given by (71), and since $[\mathsf{A}_{b+1}]_{q^0}$ is a constant in p, the right-hand side of (71) is precisely as claimed in (68). If $b = 0$, we add the evaluation of Lemma 10.7 to (71) and use the straightforward identity

$$\sum_{k \geq 1} k^a p^{kr} = \text{constant} + \left(-\left(p \frac{d}{dp}\right)^a \frac{1}{2} \frac{p^{1/2} + p^{-1/2}}{p^{1/2} - p^{-1/2}} \right)_{p \mapsto p^r}.$$

Step 4 (quasi-Jacobi form property) Since $\mathsf{A}_{b+1} \in \mathrm{MQJac}_{b+1,0}$, the derivative $p(d/dp)$ increases the weight by 1, and if the operator $f(p,q) \mapsto f(p^r, q)$ sends quasi-Jacobi forms of weight k and index m to quasi-Jacobi forms of weight k and index mr^2 (see [Eichler and Zagier 1985, Theorem I.4.1]), the second term on the right in (68) is a quasi-Jacobi form of weight

$$a + b + 1 = 2n(g-1) + \sum_i \deg(\gamma_i) + \sum_i \deg_{WF}(\gamma_i) = n(2g - 2 + N) + \sum_i \mathrm{wt}(\gamma_i).$$

By the monodromy of Section 3.6.3 we have

$$\langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,F} = (-1)^{nN + l(\gamma_1) + \cdots + l(\gamma_N)} \langle \mathrm{taut}; \gamma_1, \dots, \gamma_N \rangle^{S^{[n]}}_{g,F}.$$

Hence the first term in (68) is even unless $nN + l(\gamma_1) + \cdots + l(\gamma_N)$, in which case $a \equiv b + 1$ modulo 2 by Lemma 10.5. If $a > b$ we find in $\mathbb{C}[\![q]\!]/\mathbb{C}$ the equality

$$\sum_{d,k \geq 1} k^a d^b q^{kd} \equiv \left(q \frac{d}{dq}\right)^b \sum_{m \geq 1} \sum_{k \mid m} k^{a-b} q^m,$$

and since this is the q–derivative of an Eisenstein series we get

$$\text{constant} + \sum_{d,k \geq 1} k^a d^b q^{kd} \in \mathrm{QMod}_{a+b+1}.$$

The case $b > a$ is parallel. □

10.4 Conclusion

We prove Theorem 10.3, and Theorem 1.8 of the introduction.

Proof of Theorem 1.8(i) If there is an index i (let us say $i = 1$) with $\gamma_1 = F\tilde{\gamma}_1$, then by using $F = [E]$ for a smooth elliptic fiber $\iota \colon E \hookrightarrow S$, a straightforward computation gives

$$F^S_{g,0}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d=1}^{\infty} \langle \mathrm{taut}(-1)^{g-1} \lambda_{g-1}; \iota^*(\tilde{\gamma}_1), \iota^*(\gamma_2), \dots, \iota^*(\gamma_N) \rangle^E_{g,d[E]} q^d, \tag{73}$$

where we used the standard notation for the (ordinary nonreduced) Gromov–Witten invariants of the elliptic curve E. In this case Conjecture D(i) follows from [Okounkov and Pandharipande 2006a], and one checks that the holomorphic anomaly equation of [Oberdieck and Pixton 2018] implies the one stated in Conjecture D(ii).

If there is no such i, by expressing taut as boundary classes and the splitting formula, as well as using the divisor equation, we can reduce the claim to the case $(g,N)=(0,3)$. This base case holds by inspection from the explicit evaluation

$$F^S_{g,0}(1;W,W,W)=\langle W,W,W\rangle^S_{0,F}\sum_{k,d\geq 1} d^3q^{kd}=\text{constant}+24G_4(q). \qquad \square$$

We prove a basic vanishing for the Gromov–Witten theory of the elliptic K3 surface S:

Lemma 10.8 *If* $\sum_i \deg_{WF}(\gamma_i)<0$, *then* $\langle \text{taut};\gamma_1,\dots,\gamma_N\rangle^S_{g,dF}=0$ *for all* $d>0$.

Proof We assume $\gamma_i\in\mathcal{B}$ for all i. By Remark 10.1, if $\sum_i \deg_{WF}(\gamma_i)<0$ there exists at least one cohomology class with $\gamma_i=F$. Hence by expressing the invariants of S in terms of the invariants of the elliptic fiber E as in (73), we see that if $\gamma_j=F$ for some $j\neq i$ then the invariant vanishes, and if there are no other cohomology classes with $\gamma_i=W$ then the integrand on $\overline{M}_{g,N}(E,d)$ is invariant under translation by E and hence the integral vanishes; see eg [Okounkov and Pandharipande 2006b, Section 5.4]. Since we are always in at least one of these cases, this proves the claim. $\square$

Proof of Theorem 10.3 If $(g,N)=(0,3)$ we can take taut $=1$, so $a=0$. By Theorem 1.2 the multiple cover conjecture holds for this (g,N). Moreover, using the GW/Hilb correspondence (Theorem 6.2), the product formula for the relative Gromov–Witten theory of $(S\times\mathbb{P}^1,S_{0,1,\infty})$ and Lemma 10.8, Theorem 10.2(ii) also holds. Hence the first two claims follow directly from Theorem 10.2 and the (d/dG_2)–holomorphic anomaly equation for A_n proven in Theorem 2.23. It remains to prove (69). This follows by either using the monodromy of Section 3.6.4 to derive the elliptic transformation law in the meromorphic case, or by applying the GW/Hilb correspondence (this is possible since the multiple cover conjecture is proven for fiber classes [Bae and Buelles 2021]; see Remark 6.3) and then using (16) to calculate the $d/d\mathsf{A}$ derivative in terms of the z–expansion (similarly to what was done in Section 8). We leave the details to the reader. $\square$

11 Applications

In this section we prove two applications of the holomorphic anomaly equation for the Hilbert scheme stated in the introduction. The first considers the 2–point function on the Hilbert scheme (Corollary 1.6) which is implied by Proposition 11.1. The second concerns the Jacobi form property for CHL Calabi–Yau threefolds (Theorem 1.7). Here we first prove, by a deformation argument, a version of the holomorphic anomaly equation for generating series which keep track of curve classes of the form $W+dF+\alpha$ where α runs over a lattice $E_8(-2)\subset \mathrm{Pic}(S)$ orthogonal to W and F (Proposition 11.2). Then Theorem 1.7 follows formally by the degeneration formula and the GW/Hilb correspondence (Theorem 6.2).

11.1 The 2–point function

Recall the notation of Section 3.5, in particular the LLV algebra

$$\mathfrak{g}(S^{[n]}) = \textstyle\bigwedge^2(V \oplus U_{\mathbb{R}}) \quad \text{for } V = H^2(S^{[n]}).$$

Extend the definition of the operator T_α by defining

$$T_\alpha := \mathrm{act}(\alpha \wedge F)$$

for all $\alpha \in V \oplus U_{\mathbb{R}}$ with $\alpha \perp \{W, F\}$. In particular,

(74) $$T_e = \mathrm{act}(e \wedge F) = e_F \quad \text{and} \quad T_f = \mathrm{act}(f \wedge F) = -U.$$

For any operator $a \in \mathfrak{g}(S^{[n]})$ which is homogeneous of degree $\deg(a)$ — ie if $\deg(a\gamma) = \deg(\gamma) + \deg(a)$ for all homogeneous γ — define the induced operator

(75) $$a\colon H^*(S^{[n]})^{\otimes N} \to H^*(S^{[n]})^{\otimes N},$$
$$a(\gamma_1 \otimes \cdots \otimes \gamma_N) = \sum_{i=1}^{N} \gamma_1 \otimes \cdots \otimes \gamma_{i-1} \otimes ((-1)^{i \deg(a)} a\gamma_i) \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_N.$$

By the quasi-Jacobi form part of Theorem 1.3, the generating series $Z^{S^{[n]}}(p,q)$ defined in (7) can be identified with a vector with entries quasi-Jacobi forms. We prove the following anomaly equation, which combined with Lemma 2.11 (and using that Wt is antisymmetric) immediately implies Corollary 1.6:

Proposition 11.1 *We have*

$$\frac{d}{dG_2} Z^{S^{[n]}}(p,q) = -\sum_{\alpha,\beta} (\tilde{g}^{-1})_{\alpha\beta} T_\alpha T_\beta Z^{S^{[n]}}(p,q) \quad \text{and} \quad \frac{d}{d\mathsf{A}} Z^{S^{[n]}}(p,q) = -T_\delta Z^{S^{[n]}}(p,q),$$

where α and β run over a basis of $\{W, F\}^\perp \subset V \oplus U_{\mathbb{Q}}$ with intersection matrix $\tilde{g}_{ab} = \langle \alpha, \beta \rangle$.

Proof By Theorem 1.3, for any $\gamma_1, \gamma_2 \in H^*(S^{[n]})$ we have

$$\frac{d}{dG_2} F_{0,1}^{S^{[n]}}(\gamma_1,\gamma_2)$$
$$= 2F_{0,1}^{S^{[n]}}(U(\gamma_1 \cup \gamma_2)) - 2F_{0,1}^{S^{[n]}}(\psi_1; U\gamma_1, \gamma_2) - 2F_{0,1}(\psi_2; \gamma_1, U\gamma_2) - \sum_{a,b} (g^{-1})_{ab} F_{0,1}^{S^{[n]}}(T_{e_a} T_{e_b}(\gamma_1 \otimes \gamma_2)).$$

Let $\mathsf{p}_{S^{[n]}} = \mathfrak{q}_1(\mathsf{p})^n v_\varnothing$ be the class of a point on $S^{[n]}$. By $U(\mathsf{p}_{S^{[n]}}) = n\mathfrak{q}_1(F)\mathfrak{q}_1(\mathsf{p})^{n-1} v_\varnothing$ and the evaluation [Oberdieck 2018a, Theorem 2], we have

$$2F_0^{S^{[n]}}(U(\gamma_1 \cup \gamma_2)) = 2F_0^{S^{[n]}}(U(\mathsf{p})) \int_{S^{[n]}} \gamma_1 \cup \gamma_2 = 2n \frac{\boldsymbol{G}(p,q)^{n-1}}{\Delta(q)} \int_{S^{[n]}} \gamma_1 \cup \gamma_2.$$

Similarly, using the divisor equation with respect to $(1/(n-1)!)\mathfrak{q}_1(F)\mathfrak{q}_1(1)^{n-1}v_\varnothing$ to add a marking, rewriting the ψ–class in terms of boundary and applying the splitting axiom of Gromov–Witten theory (see for example [Cao et al. 2024, Section 1.2] for a similar case) yields

$$F_{0,1}^{S^{[n]}}(\psi_1; U\gamma_1, \gamma_2) = F_{0,1}^{S^{[n]}}(U\gamma_1, e_F\gamma_2) - F_{0,1}^{S^{[n]}}(e_F U\gamma_1, \gamma_2).$$

Rewriting this using (74) and using convention (75) we get

$$-2F_{0,1}^{S^{[n]}}(\psi_1; U\gamma_1, \gamma_2) - 2F_{0,1}^{S^{[n]}}(\psi_2; \gamma_1, U\gamma_2) = 2F_{0,1}^{S^{[n]}}(Ue_F(\gamma_1\otimes\gamma_2)) = -2F_{0,1}^{S^{[n]}}(T_eT_f(\gamma_1\otimes\gamma_2)).$$

Finally, by the commutation relations (10) we have

$$\frac{d}{dG_2}\boldsymbol{G}(p,q) = 2\Theta(p,q)^2.$$

Putting all this together we obtain

$$\begin{aligned}\frac{d}{dG_2}\int_{S^{[n]}\times S^{[n]}} Z^{S^{[n]}}(p,q)\cup(\gamma_1\otimes\gamma_2) &= \frac{d}{dG_2}F_{0,1}^{S^{[n]}}(\gamma_1\otimes\gamma_2) - \left(\int_{S^{[n]}}\gamma_1\cup\gamma_2\right)\frac{d}{dG_2}\frac{\boldsymbol{G}^n}{\Theta^2\Delta(q)} \\ &= -\sum_{\alpha,\beta}(G^{-1})_{\alpha\beta}T_\alpha T_\beta Z^{S^{[n]}}(p,q).\end{aligned}$$

The first claim now follows since T_α is antisymmetric if $\alpha\in V$, and symmetric if $\alpha\in U_{\mathbb{Q}}$ (both orthogonal to W and F). The second claim follows from $(d/d\mathsf{A})\boldsymbol{G}=0$, the holomorphic anomaly equation for $d/d\mathsf{A}$ (proven in Theorem 1.3), and since T_δ is antisymmetric. □

11.2 CHL Calabi–Yau threefolds

We work in the setting introduced in Section 1.8. For a general element $\alpha\in E_8(-2)$ — where $E_8(-2)\subset \operatorname{Pic}(S)$ is the anti-invariant part of the symplectic involution $g: S\to S$ — and with $W=B+F$ as usual, consider the curve class

$$W + dF + \alpha \in H_2(S,\mathbb{Z}).$$

Let $b_1,\dots,b_8$ be a fixed integral basis of $E_8(-2)$, and identify $w=(w_1,\dots,w_8)\in\mathbb{C}^8$ with $\sum_i w_ib_i\in E_8(-2)\otimes\mathbb{C}$. Given a class $\alpha\in E_8(-2)$, we write

$$\zeta^\alpha = \exp(\langle w,\alpha\rangle) = \prod_{i=1}^{8} e(\langle b_i,\alpha\rangle w_i). \tag{76}$$

We also refer to [Oberdieck and Pixton 2019, Section 2.1.4] for parallel definitions.

Form the extended generating series

$$\widetilde{F}_g^{S^{[n]}}(\mathrm{taut};\gamma_1,\dots,\gamma_N) = \sum_{d=-1}^{\infty}\sum_{r\in\mathbb{Z}}\sum_{\alpha\in E_8(-2)}\langle \mathrm{taut};\gamma_1,\dots,\gamma_N\rangle_{g,W+dF+\alpha+rA}^{S^{[n]}} q^d(-p)^r\zeta^\alpha.$$

Usually we drop the superscript $S^{[n]}$. The first step is to prove the following:

Proposition 11.2 *If Conjectures B and C hold for (g, N), then*

$$\widetilde{F}_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) \in \frac{1}{\Delta(q)} \mathrm{QJac}_{k+12,(n-1)\oplus(1/2)E_8(-2)}(\Gamma_0(2) \ltimes (2\mathbb{Z} \oplus \mathbb{Z})),$$

where $k = n(2g-2+N) + \sum_i \mathrm{wt}(\gamma_i) - 6$ and $\mathrm{QJac}_{k,L}$ is the vector space of weight-k multivariable quasi-Jacobi forms of lattice index L as defined in [Oberdieck and Pixton 2019, Section 1], *except that here we work with respect to the Jacobi group $\Gamma_0(2) \ltimes (2\mathbb{Z} \oplus \mathbb{Z})$.*[8] *Moreover,*

$$(77) \quad \frac{d}{dG_2} \widetilde{F}_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \widetilde{F}_{g-1}^{S^{[n]}}(\mathrm{taut}'; \gamma_1, \dots, \gamma_N, U) + 2 \sum_{\substack{g=g_1+g_2 \\ \{1,\dots,N\}=A \sqcup B}} \widetilde{F}_{g_1}^{S^{[n]}}(\mathrm{taut}_1; \gamma_A, U_1) F_{g_2}^{S^{[n]},\mathrm{std}}(\mathrm{taut}_2; \gamma_B, U_2)$$
$$- 2 \sum_{i=1}^N \widetilde{F}_g^{S^{[n]}}(\psi_i\, \mathrm{taut}; \gamma_1, \dots, \gamma_{i-1}, U\gamma_i, \gamma_{i+1}, \dots, \gamma_N)$$
$$- \sum_{a,b} (\hat{g}^{-1})_{ab} T_{e_a} T_{e_b} F_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N),$$

where the e_a form a basis of $(\mathrm{Span}_{\mathbb{Z}}(B, F) \oplus E_8(-2))^{\perp} \subset H^2(S, \mathbb{Q})$ with intersection matrix $\hat{g}_{ab} = \langle e_a, e_b \rangle$, and

$$\frac{d}{d\mathsf{A}} \widetilde{F}_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = T_\delta \widetilde{F}_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N).$$

Proof For $\alpha \in E_8(-2)$ the operator $T_\alpha = \mathrm{act}(\alpha \wedge F)$ satisfies

$$e^{-T_\alpha}(W + dF + rA + \alpha) = W + \big(d + \tfrac{1}{2}\langle \alpha, \alpha \rangle\big) F + rA.$$

Moreover, e^{-T_α} can either be viewed as a monodromy operator (as in Section 3.6) or identified with the induced action on the Hilbert schemes coming from the automorphism $t_{-\alpha} \colon S \to S$ given by translation by the section labeled by $-\alpha$; compare [Oberdieck and Pixton 2019, Section 3.4]. In either case, we have invariance of Gromov–Witten invariants, so

$$\widetilde{F}_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = \sum_{d,r} \sum_{\alpha \in E_8(-2)} \langle \mathrm{taut}; e^{-T_\alpha}\gamma_1, \dots, e^{-T_\alpha}\gamma_N \rangle^{S^{[n]}}_{g, W+(d+(1/2)\langle\alpha,\alpha\rangle)F+rA} q^d (-p)^r \zeta^\alpha$$
$$= \sum_{\tilde{d},r} \langle \mathrm{taut}; e^{-T_\alpha}\gamma_1, \dots, e^{-T_\alpha}\gamma_N \rangle^{S^{[n]}}_{g, W+\tilde{d}F+rA} q^{\tilde{d}} q^{-(1/2)\langle\alpha,\alpha\rangle} (-p)^r \zeta^\alpha$$
$$= \sum_{\alpha \in E_8(-2)} F_g(\mathrm{taut}; e^{-T_\alpha}\gamma_1, \dots, e^{-T_\alpha}\gamma_N) q^{-(1/2)\langle\alpha,\alpha\rangle} \zeta^\alpha.$$

Let h^{ij} be the inverse matrix of the intersection matrix $\langle b_i, b_j \rangle$. Then

$$T_\alpha = \sum_{i,j} h^{ij} \langle \alpha, b_i \rangle T_{b_j}.$$

[8] More explicitly, the quasi-Jacobi forms we consider will simply be linear combinations of derivatives of the theta function of the $E_8(2)$–lattice; see the proof.

Moreover, let

$$\Theta_{E_8(2)}(\zeta, q) = \sum_{\alpha \in E_8(-2)} q^{-(1/2)\langle \alpha, \alpha \rangle} \zeta^{\alpha} \tag{78}$$

be the theta functions of the $E_8(2)$ lattice, which is a Jacobi form of weight $\frac{1}{2}\operatorname{rk} E_8(-2) = 4$ and lattice index $\frac{1}{2}E_8(2)$ for the Jacobi group $\Gamma_0(2) \ltimes (2\mathbb{Z} \times \mathbb{Z})$; see [Ziegler 1989, Section 3].[9] Similarly, if we multiply the summand in (78) with products of $\langle \alpha, b_i \rangle$, the function becomes derivatives of the theta functions by the differential operators

$$D_{b_i} = \frac{1}{2\pi i} \frac{d}{dw_i}.$$

For example,

$$\sum_{\alpha \in E_8(-2)} \langle \alpha, b_i \rangle q^{-(1/2)\langle \alpha, \alpha \rangle} \zeta^{\alpha} = D_{b_i} \Theta_{E_8(2)}(\zeta, q).$$

Putting this together, we find that

$$\widetilde{F}_g^{S^{[n]}}(\mathrm{taut}; \gamma_1, \dots, \gamma_N) = F_g(\mathrm{taut}; e^{-\sum_{i,j} h^{ij} D_{b_i} T_{b_j}} \gamma_1, \dots, e^{-\sum_{i,j} h^{ij} D_{b_i} T_{b_j}} \gamma_N) \Theta_{E_8(2)}(\zeta, q), \tag{79}$$

which is understood as expanding all the exponentials and then applying the derivatives D_{b_i} to the theta function. The operator D_{b_i} preserves the algebra of quasi-Jacobi forms; see [Oberdieck and Pixton 2019]. Moreover, since D_{b_i} increases the weight by 1, and T_{b_i} is of degree -1 with respect to the weight grading wt on cohomology, we conclude that (79) is a quasi-Jacobi form of weight equal to the weight of $F_g(\mathrm{taut}; \gamma_1, \dots, \gamma_N)$ plus 4. Finally, the claimed holomorphic anomaly equations also follow from (79) by a straightforward computation: The terms where d/dG_2 does not interact with the derivatives D_{b_i} are evaluated by Conjecture C. For any $\alpha \in E_8(-2)$ one has $(e^{-T_\alpha} \otimes e^{-T_\alpha})(U) = U$ (proven by differentiating with respect to α and then as in Lemma 3.2). Hence one sees that these terms give precisely the four terms in (77) up to the extra term coming from summing over the basis of $E_8(-2)$ in the last term. This extra term cancels with the terms coming from interactions of d/dG_2 with the D_{b_i}. These are calculated using the commutation relations [Oberdieck and Pixton 2019, (12)]. Since the E_8–theta function does not depend on p, the $d/d\mathsf{A}$ derivative follows directly from the one in Conjecture C. □

Proof of Theorem 1.7 By the arguments of [Oberdieck 2018b] we can work with stable pairs invariants of X. We then use the degeneration formula for the degeneration

$$(S \times E)/\mathbb{Z}_2 \rightsquigarrow (S \times \mathbb{P}^1)/((s, 0) \sim (gs, \infty)),$$

which was worked out explicitly in [Bryan and Oberdieck 2020, Section 1.6]. This reduces us to invariants of $(S \times \mathbb{P}^1, S_{0,\infty})$ with relative condition specified with the graph of the automorphism of $S^{[n]}$ induced by the involution $g\colon S \to S$,

$$\Gamma_g \in H^*(S^{[n]} \times S^{[n]}).$$

[9] Concretely, the theta function $\Theta_{E_8}(\tau, z)$ for the unimodular lattice E_8 is a Jacobi form for the full Jacobi group $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, and we replace τ by 2τ, which introduces the congruence subgroup.

We then apply Nesterov's wall-crossing [2021; 2024; Oberdieck 2024b]. Putting all this together yields

$$
\begin{aligned}
\text{(80)}\quad \mathrm{DT}_n(p,q) &= \frac{1}{2}\widetilde{F}_0(\Gamma_g)|_{\zeta^\alpha=1} \\
&\qquad - \frac{1}{2}\sum_{\alpha,d,r} q^d p^r \operatorname{Coeff}_{q^{d+(1/2)\langle\alpha,\alpha\rangle}p^r}\left(\frac{\boldsymbol{G}(p,q)^n}{\Theta(p,q)^2\Delta(q)}\right)\int_{S^{[n]}\times S^{[n]}}\Delta_{S^{[n]}}\Gamma_g \\
&= \frac{1}{2}\widetilde{F}_0(\Gamma_g)|_{\zeta^\alpha=1} - \frac{1}{2}\frac{\boldsymbol{G}(p,q)^n}{\Theta^2(p,q)\Delta(q)}\Theta_{E_8(2)}(q)\operatorname{Tr}(g \mid H^*(S^{[n]})),
\end{aligned}
$$

where $\Theta_{E_8(2)}(q)=\sum_{\alpha\in E_8(-2)} q^{-(1/2)\langle\alpha,\alpha\rangle}=E_4(q^2)$ is the theta function of the E_8–lattice. This shows that $\mathrm{DT}_n(p,q)$ is a quasi-Jacobi form of weight -6 and index $n-1$ for $\Gamma_0(2)$.

It remains to compute the derivative with respect to G_2 and A of the first term (the second is clearly Jacobi). Since the anomaly operators d/dG_2 and $d/d\mathsf{A}$ commute with specializing of the variable ζ (compare [Oberdieck and Pixton 2019, Section 1.3.5]) we have

$$\frac{d}{dG_2}(\widetilde{F}_0(\Gamma_g)|_{\zeta^\alpha=1}) = \left(\frac{d}{dG_2}\widetilde{F}_0(\Gamma_g)\right)\Big|_{\zeta^\alpha=1}.$$

By Proposition 11.2, arguing then as in the proof of Proposition 11.1, and using $[T_{e_a},g]=0$ for $e_a\in E_8(-2)^\perp$ and $[U,g]=0$, one finds

$$\frac{d}{dG_2}\widetilde{F}_0(\Gamma_g)|_{\zeta^\alpha=1} = 2n\frac{\boldsymbol{G}(p,q)^{n-1}}{\Delta(q)}\Theta_{E_8(2)}(q)\operatorname{Tr}(g\mid H^*(S^{[n]})).$$

Since this cancels precisely with the G_2–derivative of the second term in (80), we get

$$\frac{d}{dG_2}\mathrm{DT}_n(p,q)=0.$$

The claim $(d/d\mathsf{A})\,\mathrm{DT}_n(p,q)=0$ follows from $T_\delta(\Gamma_g)=[T_\delta,g]=0$. □

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