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Curvature tensor of smoothable Alexandrov spaces

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We prove weak convergence of curvature tensors of Riemannian manifolds for converging noncollapsing sequences with a lower bound on sectional curvature.

53C20, 53C23, 53C45; 30L99

1 Introduction

The weak convergence and measure-valued tensor used in the following theorem are defined in the next section; a more precise formulation is given in Theorem 2.6.

Main theorem 1.1 *Let $M_1, M_2, \dots$ be a sequence of complete m–dimensional Riemannian manifolds with sectional curvature bounded below by κ. Assume that the sequence M_n Gromov–Hausdorff converges to an Alexandrov space A of the same dimension. Then the curvature tensors of M_n weakly converge to a measure-valued tensor on A.*

Note that from the theorem we get that the limit tensor of the sequence depends only on A and does not depend on the choice of the sequence M_n. Indeed, suppose another sequence M_n' satisfies the assumptions of the theorem. If the limit tensor is different, then a contradiction would occur for the alternated sequence $M_1, M_1', M_2, M_2', \dots$. In particular, if the limit space is Riemannian, then the limit curvature tensor is the curvature tensor of the limit space. The latter statement was announced by the second author [21].

Analogous statements about metric tensor and Levi-Civita connection were essentially proved by Perelman [18], we only had to tie his argument with an appropriate convergence. This part is discussed in Section 7. It provides a technique that could be useful elsewhere as well. For curvature tensor (which has a higher order of derivative), this argument cannot be extended directly; we found a way around applying Bochner-type formulas as in [23].

The following statement looks like a direct corollary of the main theorem, and indeed, it follows from its proof but strictly speaking, it cannot be deduced directly from the main theorem alone. We will denote by Sc the scalar curvature and vol^m the m–dimensional volume; that is, m–dimensional Hausdorff measure calibrated so that the unit m–dimensional cube has unit measure.

Corollary 1.2 *In the assumption of the main theorem, the measures* $\mathrm{Sc}\cdot\mathrm{vol}^m$ *on* M_n *weakly converge to a locally finite signed measure* $\mathfrak{m}$ *on* A.

The following subcorollary requires no new definitions.

Subcorollary 1.3 *In the assumption of the main theorem, suppose* A *is compact. Then the sequence*

$$s_n = \int_{M_n} \mathrm{Sc}\cdot\mathrm{vol}^m$$

converges.

The main theorem in [23] implies that if a sequence of complete m–dimensional Riemannian manifolds M_n has uniformly bounded diameter and uniform lower curvature bound, then the corresponding sequence s_n is bounded; in particular, it has a converging subsequence. However, if M_n is collapsing, then this sequence may not converge. For example, an alternating sequence of flat 2–tori and round 2–spheres might collapse to the one-point space; in this case, the sequence s_n is $0, 4\cdot\pi, 0, 4\cdot\pi, \dots$.

From the main theorem (and the definition of weak convergence) we get the following.

Corollary 1.4 *Let* $\mathfrak{K}$ *be a convex closed subset of curvature tensors on* $\mathbb{R}^m$ *such that all sectional curvatures of tensors in* $\mathfrak{K}$ *are at least* -1. *Assume that* $\mathfrak{K}$ *is invariant with respect to the rotations of* $\mathbb{R}^m$. (*For example, one can take as* $\mathfrak{K}$ *the set of all curvature tensors with nonnegative curvature operator.*)

Suppose M_n *is a sequence of complete* m*–dimensional Riemannian manifolds that converges to a Riemannian manifold* M *of the same dimension. Assume that for any* n, *all curvature tensors of* M_n *belong to* $\mathfrak{K}$. *Then the same holds for the curvature tensors of* M.

Remarks The limit measure $\mathfrak{m}$ in Corollary 1.2 has some specific properties; let us describe a couple of them:

◇ The measure $\mathfrak{m}$ vanishes on any subset of A with a vanishing $(m-2)$–dimensional Hausdorff measure. In particular, $\mathfrak{m}$ vanishes on the set of singularities of codimension 3. This is an easy corollary of [23].

◇ The measure can be explicitly described on the set of singularities of codimension 2. Namely, suppose $A' \subset A$ denotes the set of all points x with tangent space $\mathrm{T}_x A = \mathbb{R}^{m-2} \times \mathrm{Cone}(\theta)$, where $\mathrm{Cone}(\theta)$ is a 2–dimensional cone with the total angle $\theta = \theta(x) < 2 \cdot \pi$. Then

$$\mathfrak{m}|_{A'} = (2 \cdot \pi - \theta) \cdot \mathrm{vol}^{m-2}.$$

This statement follows from Proposition 4.2.

The geometric meaning of our curvature tensor is not quite clear. In particular, we do not see a solution to the following problem; compare to Gigli's conjecture [7, 1.1].

Problem 1.5 *Suppose that the limit curvature tensor of Alexandrov space A as in the main theorem has sectional curvature bounded below by $K > \kappa$. Show that A is an Alexandrov space with curvature bounded below by K.*

The theorem makes it possible to define a curvature tensor for every *smoothable* Alexandrov space. It is expected that the same can be done for general Alexandrov space; so the following problem has to have a solution:

Problem 1.6 *Extend the definition of the measure-valued curvature tensor to general Alexandrov spaces.*

If this is the case, then one may expect to have a generalization of the Gauss formula for the curvature of a convex hypersurface, which in turn might lead to a solution of the following open problems in Alexandrov geometry. This conjecture is open even for convex sets in *smoothable* Alexandrov space.

Conjecture 1.7 *The boundary of an Alexandrov space equipped with its intrinsic metric is an Alexandrov space with the same lower curvature bound.*

More importantly, a solution to Problem 1.6 might provide nontrivial ways to deform Alexandrov space; see [22, Section 9].

Related results The result of the main theorem in dimension 2 is well known; see the book of Alexandr Alexandov and Viktor Zalgaller [1, VII Section 13].

The construction of harmonic coordinates at regular points of RCD space (in particular, Alexandrov space) given by Elia Bruè, Aaron Naber, and Daniele Semola [4] might help to solve Problem 1.6.

The problem of introducing Ricci tensor was studied in far more general settings; see works of Gigli [6], Han [9], Lott [14] and Sturm [27]. Curvature tensor for RCD spaces was defined by Nicola Gigli [7]; it works for a more general class of spaces, but this approach does not see the curvature of singularities. It is expected that our definitions agree on the regular locus.

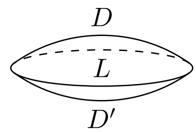
About the proof As it was stated, the 2–dimensional case is proved in [1, VII Section 13]. The 3–dimensional case is the main step in the proof; the higher-dimensional case requires only minor modifications.

We subdivide the limit space A into three subsets: A° — the subset of regular points, A' — points with singularities of codimension 2, A'' — singularities of higher codimension. These sets are treated independently.

First, we show that limit curvature vanishes on A''; this part is an easy application of the main result in [23].

The A'–case is reduced to its partial case when the limit is isometric to the product of the real line and a two-dimensional cone. The proof uses a Bochner-type formula (Theorem 6.1) and Theorem 4.3, which is a more exact version of the following problem from [24].

Problem 1.8 (convex-lens) *Let D and D' be two smooth discs with a common boundary that bound a convex set (a lens) L in a positively curved 3–dimensional Riemannian manifold M. Assume that the discs meet at a small angle. Show that the integral $\int_D k_1 \cdot k_2$ is small; here k_1 and k_2 denote the principal curvatures of D.*



The A°–case is proved by induction. The base is the 2–dimensional case. Further, we apply the induction hypothesis to level sets of special concave functions. By the Gauss formula, these level sets have the same lower curvature bound. In the proof, we use the Bochner-type formula together with the DC-calculus developed in [18]. The first step in the induction is slightly simpler.

As a rule, the calculus is done in the approximating sequence of Riemannian manifolds.

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2 Formulations

In this section, we give the necessary definitions for a precise formulation of the main theorem. For simplicity we will always assume that the lower curvature bound is -1; applying rescaling, we can get the general case.

We denote by $\mathcal{A}lex^m$ the class of m–dimensional Alexandrov's spaces with curvature $\geqslant -1$.

Suppose $A, A_1, A_2, \dots \in \mathcal{A}lex^m$ and $A_n \xrightarrow[\mathrm{GH}]{} A$. That is, A_n converges to A in the sense of Gromov–Hausdorff; since $A \in \mathcal{A}lex^m$, we have no collapse. Denote by $a_n\colon A_n \to A$ the corresponding Hausdorff approximations. If A is compact, then by Perelman's stability theorem [10; 16] we can (and will) assume that a_n is a homeomorphism for every sufficiently large n. In the case of noncompact limit, we assume that for any R, the restriction of a_n to an R–neighborhood of the marked point is a homeomorphism to its image for every sufficiently large n.

We say that $A \in \mathcal{A}lex^m$ is *smoothable* if it can be presented as a Gromov–Hausdorff limit of a noncollapsing sequence of Riemannian manifolds M_n with $\sec M_n \geqslant -1$; here sec stands for sectional curvature. Given a smoothable Alexandrov space A, a sequence of complete Riemannian manifolds M_n as above together with a sequence of approximations $a_n\colon M_n \to A$ will be called *smoothing* of A (briefly, $M_n \xrightarrow[\sim]{} A$, or $M_n \xrightarrow[\sim]{a_n} A$). By Perelman's stability theorem, any smoothable Alexandrov space is a topological manifold without boundary.

2A Weak convergence of measures

In this subsection, we define weak convergence of measures. For more detailed definitions and terminology, we refer to [8].

Let X be a Hausdorff topological space. Denote by $\mathfrak{M}(X)$ the space of signed Radon measures on X. Further, denote by $C_c(X)$ the space of continuous functions on X with a compact support.

We denote by $\langle \mathfrak{m} | f \rangle$ the value of $\mathfrak{m} \in \mathfrak{M}(X)$ on $f \in C_c(X)$. We say that measures $\mathfrak{m}_n \in \mathfrak{M}(X)$ *weakly converge* to $\mathfrak{m} \in \mathfrak{M}(X)$ (briefly $\mathfrak{m}_n \rightharpoonup \mathfrak{m}$) if $\langle \mathfrak{m}_n | f \rangle \to \langle \mathfrak{m} | f \rangle$ for any $f \in C_c(X)$.

Suppose $A_n \xrightarrow[\mathrm{GH}]{} A$ with Hausdorff approximations $a_n\colon A_n \to A$ and $\mathfrak{m}_n$ is a measure on A_n. We say that $\mathfrak{m}_n$ *weakly converges* to a measure $\mathfrak{m}$ on A (briefly $\mathfrak{m}_n \rightharpoonup \mathfrak{m}$) if the pushforwards $\mathfrak{m}_n'$ of $\mathfrak{m}_n$ to A by the Hausdorff approximations $a_n\colon A_n \to A$ weakly converge to $\mathfrak{m}$. If the condition $\langle \mathfrak{m}_n' | f \rangle \to \langle \mathfrak{m} | f \rangle$ holds only for functions f with support in an open subset $\Omega \subset A$, then we say that $\mathfrak{m}_n$ *weakly converges* to $\mathfrak{m}$ in Ω.

Equivalently, the weak convergence can be defined using the uniform convergence of functions. We say that a sequence $f_n \in C_c(A_n)$ *uniformly converges* to $f \in C_c(A)$ if their supports are uniformly bounded and

$$\sup_{x \in A_n} \{\, |f_n(x) - f \circ a_n(x)| \,\} \to 0.$$

Then $\mathfrak{m}_n \rightharpoonup \mathfrak{m}$ if for any sequence $f_n \in C_c(A_n)$ with uniformly bounded supports and uniformly converging to $f \in C_c(A)$ we have $\langle \mathfrak{m}_n | f_n \rangle \to \langle \mathfrak{m} | f \rangle$.

2B Test functions

In this subsection, we introduce a class of test functions and define their convergence.

Test functions form a narrow class of functions defined via a formula. It is just one possible choice of a class containing sufficiently smooth DC functions; see the remarks in the next section.

Recall that the distance between points x, y in a metric space is denoted by $|x - y|$; we will denote by dist_x the distance function $\operatorname{dist}_x : y \mapsto |x - y|$.

Suppose $A_n, A \in \mathcal{A}lex^m$ and $A_n \xrightarrow[\text{GH}]{} A$. Then any distance function $\operatorname{dist}_p : A \to \mathbb{R}$ can be *lifted* to A_n; it means that we can choose a convergent sequence $p_n \to p$ and take the sequence $\operatorname{dist}_{p_n}$.

Choose $r > 0$ and $p \in A$. Let us define *smoothed distance function* as the average:

$$\widetilde{\operatorname{dist}}_{p,r} = \fint_{B(p,r)} \operatorname{dist}_x \, dx.$$

We can lift this function to $\widetilde{\operatorname{dist}}_{p_n,r} : A_n \to [0, \infty)$ by choosing some sequence $A_n \ni p_n \to p \in A$.

We say that f is a *test function* if it can be expressed by the formula

$$f = \varphi(\widetilde{\operatorname{dist}}_{p_1,r_1}, \dots, \widetilde{\operatorname{dist}}_{p_N,r_N}),$$

where $\varphi : (0, \infty)^N \to \mathbb{R}$ is a C^2–smooth function with compact support. If for some sequences of points $A_n \ni p_{i,n} \to p_i \in A$ and C^2–smooth functions φ_n that C^2–converge to φ with compact support we have

$$f_n = \varphi_n(\widetilde{\operatorname{dist}}_{p_{1,n},r_1}, \dots, \widetilde{\operatorname{dist}}_{p_{N,n},r_N}),$$

then we say that f_n is *test-converging* to f (briefly, $f_n \xrightarrow{\text{test}} f$).

Remarks Note that test functions form an algebra.

Let M be a Riemannian manifold. Note that for any open cover of M, there is a subordinate partition of unity of test functions. Further, around any point of M one can take a smoothed distance coordinate chart. One can express any C^2–smooth function in these coordinates, and then apply partition of unity for a covering by charts. This way, we get:

Claim 2.1 *On a smooth complete Riemannian manifold, test functions include all C^2–smooth functions with compact support.*

2C C^1–delta convergence

Here we introduce C^1–delta convergence. It will be necessary to formulate the main theorem in an invariant way, but, except for Section 5B, everywhere in the proofs, we will use test convergence and occasionally DC convergence instead. (As claimed in Claim 2.3 test convergence implies C^1–delta convergence.) By that reason, *it would be wise to skip this section for the first reading*.

The C^1–delta convergence will be used together with other delta convergences introduced in Section 5A.

Convergence of vectors Let A be an Alexandrov space, we denote by $\mathrm{T}A$ the set of all tangent vectors at all points. So far $\mathrm{T}A$ is a disjoint union of all tangent cones; let us define a convergence on it.

We will use gradient exponent gexp: $\mathrm{T}A \to A$ which is defined in [3]. Given a vector $V \in \mathrm{T}A$, it defines its radial curve $\gamma_V : t \mapsto \operatorname{gexp}(t \cdot V)$. We say that a sequence of vectors $V_n \in \mathrm{T}A$ *converges* to $V \in \mathrm{T}A$ (briefly, $V_n \to V$) if γ_{V_n} converges to γ_V pointwise. Since the radial curve γ_V is $|V|$–Lipschitz, we get that any bounded sequence of vectors with base points in a bounded set has a converging subsequence of γ_{V_n}. Further, the pointwise limit of such curves is a radial curve as well. Therefore, any bounded sequence of tangent vectors with base points in a bounded set has a converging sequence.

In a similar fashion, we can define the convergence of tangent vectors to sequences of Alexandrov spaces A_n that converge to A. That is, if $V_n \in \mathrm{T}A_n$ is a bounded sequence of tangent vectors at points on a bounded distance to the base points, then it has a subsequence that converges to some vector $V \in \mathrm{T}A$.

Note that

$$|V| \leqslant \liminf_{n\to\infty} |V_n|$$

and the inequality might be strict.

Recall that if $V \in \mathrm{T}_p$ is the unit vector in the direction of $[pq]$, then γ_V is a unit-speed parametrization of $[pq]$. Using this we get the following observation; it provides a way to apply the convergence.

Observation 2.2 *Let $M_n \underset{\sim}{\longrightarrow} A$ be a smoothing, $p_n, q_n \in M_n$, and $p_n \to p$, $q_n \to q$ as $n \to \infty$. Denote by $V_n \in \mathrm{T}_{p_n}$ and $V \in \mathrm{T}_p$ the directions of geodesics $[p_n q_n]$ and $[pq]$. Suppose that there is a unique geodesic $[pq]$ in A. Then $V_n \to V$.*

C^1–delta smoothness Given a function $f : A \to \mathbb{R}$ and a vector $V \in \mathrm{T}A$, set

$$Vf = (f \circ \gamma_V(t))'|_{t=0}.$$

Note that Vf is defined for all DC functions and, in particular, all test functions.

Two vectors $V, W \in \mathrm{T}_p A$ will be called *δ–opposite* if

$$1-\delta < |V| \leqslant 1, \quad 1-\delta < |W| \leqslant 1, \quad \text{and} \quad |\langle X, V\rangle + \langle X, W\rangle| < \delta$$

for any unit vector $X \in \mathrm{T}_p A$. We say that $V, W \in \mathrm{T}_p A$ are opposite if they are δ–opposite for any $\delta > 0$; in this case, they are both unit vectors and make angle π to each other.

A function $f : A \to \mathbb{R}$ is called C^1–delta smooth if for any compact set $K \subset A$ and $\varepsilon > 0$ there is $\delta > 0$ such that any sequence of points $p_n \to p \in K$ and unit vectors $V_n \in \mathrm{T}_{p_n} A$ that converges to a vector $V \in \mathrm{T}_p A$ that has a δ–opposite vector we have

$$|Vf - \acute{\lim_{n\to\infty}} V_n f| < \varepsilon,$$

where "$\acute{\lim}$" stands for an arbitrary partial limit.

Suppose $M_n \underset{\sim}{\longrightarrow} A$. A sequence of C^1–smooth functions $f_n : M_n \to \mathbb{R}$ is called *C^1–delta converging* to $f : A \to \mathbb{R}$ (briefly, $f_n \xrightarrow{C^1_\delta} f$) if f_n converges to f pointwise and for any compact set $K \subset A$ and any

$\varepsilon > 0$ there is $\delta > 0$ such that if a sequence of unit vectors $V_n \in \mathrm{T}_{p_n} M_n$ converges to a vector $V \in \mathrm{T}_p A$ such that $p \in K$ and V has a δ–opposite vector, then we have

$$|Vf - \acute{\lim_{n\to\infty}} V_n f_n| < \varepsilon.$$

Claim 2.3 *Any test function is C^1–delta smooth. Moreover, for any smoothing $M_n \underset{\sim}{\longrightarrow} A$, sequence of test functions $f_n\colon M_n \to \mathbb{R}$, and test function $f\colon A \to \mathbb{R}$, we have*

$$f_n \xrightarrow{\text{test}} f \implies f_n \xrightarrow{C^1_\delta} f.$$

Proof Let V and W be δ–opposite vectors in $\mathrm{T}_p A$. Note that for almost all points $q \in A$, we have

$$|V \operatorname{dist}_q + W \operatorname{dist}_q| < \delta.$$

It follows that

$$|V\widetilde{\operatorname{dist}}_{q,r} + W\widetilde{\operatorname{dist}}_{q,r}| < \delta \tag{2-1}$$

for any $q \in A$ and $r > 0$.

Suppose V_n is a sequence of unit tangent vectors on M_n such that $V_n \to V$; that is, $\gamma_{V_n} \to \gamma_V$ as $n \to \infty$. By monotonicity of radial curves [3, 16.32], we get

$$V \operatorname{dist}_q \leqslant \liminf_{n\to\infty} V_n \operatorname{dist}_{q_n}$$

if $q_n \to q$. Integrating, we get

$$V\widetilde{\operatorname{dist}}_{q,r} \leqslant \liminf_{n\to\infty} V_n\widetilde{\operatorname{dist}}_{q_n,r}.$$

Suppose V has a δ–opposite vector W. We can assume that W is a unit geodesic vector; that is, there is a geodesic $[ps]$ in the direction of W. Moreover, we can assume that $[ps]$ is a unique geodesic from p to s. Choose points s_n and p_n that converge to s and p respectively. By Observation 2.2, the directions W_n of $[p_n s_n]$ converge to W. Note that W_n is δ–opposite to V_n for all large n.

Repeating the above argument, we get

$$W\widetilde{\operatorname{dist}}_{q,r} \leqslant \liminf_{n\to\infty} W_n\widetilde{\operatorname{dist}}_{q_n,r}.$$

Applying (2-1) we get C^1–delta convergence of $\widetilde{\operatorname{dist}}_{q_n,r}$ and, in particular, C^1–delta smoothness of $\widetilde{\operatorname{dist}}_{q,r}$. Applying the definition of test function, we get the result. □

Recall (Section 2B) that for any smoothing $M_n \underset{\sim}{\longrightarrow} A$ and test function $f\colon A \to \mathbb{R}$ there are test functions $f_n\colon M_n \to \mathbb{R}$ such that $f_n \xrightarrow{\text{test}} f$.

Corollary 2.4 *Given a smoothing $M_n \underset{\sim}{\longrightarrow} A$ and a test function $f\colon A \to \mathbb{R}$, there is a sequence of C^1–smooth functions $f_n\colon M_n \to \mathbb{R}$ such that $f_n \xrightarrow{C^1_\delta} f$.*

Remarks In the next section, we define measure-valued tensor as a functional on an array of test functions. Note that, one test function might have very different presentations that lead to different test convergences. Thus to prove the invariance of measure-valued curvature tensor we need to use the C^1–delta convergence which is more general than test convergence. We could use other classes of functions as well. For example, a subclass of DC_0 functions (see Section 7) or a subclass of C^1–delta function (see Section 2C). Of course, we have to have an analog of Corollary 2.4 for the chosen class. We hope a more natural setting will be found eventually.

2D Tensors

In this subsection, we define measure-valued tensors on Alexandrov spaces. Basically, we reuse the derivation approach to vector fields in classical differential geometry. This definition will be used in Claim 2.9 that reduces the main theorem to Proposition 2.10 and will not show up ever after.

Let $A \in \mathcal{A}lex^m$. Recall that $\mathfrak{M}(A)$ denotes the space of signed Radon measures on A. A *measure-valued vector field* $\mathfrak{v}$ on A is a linear map that takes a test function, spits a measure in $\mathfrak{M}(A)$, and satisfies the *chain rule*: for any collection of test functions $f_1, \dots, f_k$ and a C^2–smooth function $\varphi\colon \mathbb{R}^k \to \mathbb{R}$, we have

$$\mathfrak{v}(\varphi(f_1, \dots, f_n)) = \sum_{i=1}^{n} (\partial_i \varphi)(f_1, \dots, f_n) \cdot \mathfrak{v}(f_i).$$

In the same way, we define (contravariant) measure-valued tensor fields. Namely, a *measure-valued tensor field* $\mathfrak{t}$ of valence k on A is a multilinear map that takes a k–array of test functions, spits a measure in $\mathfrak{M}(A)$, and satisfies the chain rule in each of its arguments.

Suppose that $x_1, \dots, x_m$ are local coordinates in an m–dimensional Riemannian manifold M. Then a measure-valued vector field $\mathfrak{v}$ on M can be described by m components, $(\mathfrak{v}(x_1), \dots, \mathfrak{v}(x_m))$, which are measures. These components transform by contravariant rule under change of coordinates.

By the definition of a measure-valued vector field, we get

$$\mathfrak{v}(f) = \sum_i \partial_i f \cdot \mathfrak{v}(x_i).$$

Similarly, for arbitrary k, a measure-valued tensor field of valence k is defined by m^k components $\mathfrak{t}(x_{i_1}, \dots, x_{i_k})$; namely,

$$\mathfrak{t}(f_1, \dots, f_k) = \sum_{i_1, \dots, i_k} \partial_{i_1} f_1 \cdots \partial_{i_k} f_k \cdot \mathfrak{t}(x_{i_1}, \dots, x_{i_k}).$$

Note that if T is a smooth contravariant tensor field then $\mathfrak{t} = T \cdot \mathrm{vol}$ is a measure-valued tensor field. In other words, usual tensor fields might be considered as a subspace of measure-valued tensor fields.

Definition 2.5 Let $M_n \xrightarrow{\sim} A$ be a smoothing. Assume that $\mathfrak{t}_n$ is a sequence of measure-valued tensor fields on M_n and $\mathfrak{t}$ is a measure-valued tensor field on A, all of the same valence k. We say that $\mathfrak{t}_n$ *weakly* *converges* to $\mathfrak{t}$ (briefly $\mathfrak{t}_n \rightharpoonup \mathfrak{t}$) if

$$f_{i,n} \xrightarrow{C^1_\delta} f_i \quad \text{for all } i \implies \mathfrak{t}_n(f_{1,n},\dots,f_{k,n}) \rightharpoonup \mathfrak{t}(f_1,\dots,f_k)$$

for arbitrary k sequences $f_{1,n},\dots,f_{k,n}$ of C^1–smooth functions and test functions $f_1,\dots,f_k\colon A\to\mathbb{R}$.

2E Dual curvature tensor

The curvature of Riemannian manifold M is usually described by a tensor of valence 4 that will be denoted by Rm. We will use a *dual curvature tensor* — a curvature tensor written in a dual form that will be denoted by Qm; it is a tensor field of valence $2\cdot(m-2)$ defined the by

$$\mathrm{Qm}(X_1,\dots,X_{m-2},Y_1,\dots,Y_{m-2}) = \mathrm{Rm}\big({*}(X_1\wedge\dots\wedge X_{m-2}), {*}(Y_1\wedge\dots\wedge Y_{m-2})\big),$$

where X_i, Y_i are vector fields on M and $*\colon \big(\bigwedge^{m-2}\mathrm{T}\big)M \to \big(\bigwedge^2\mathrm{T}\big)M$ is the Hodge star operator. This definition will be used further mostly for gradient vector fields of semiconcave functions.

In addition, we will need a measure-valued version of Qm denoted by $\mathfrak{qm}$; it will be called *dual measure-valued curvature tensor*. Namely, we define

$$\mathfrak{qm}(f_1,\dots,f_{m-2},g_1,\dots,g_{m-2})$$

as the measure with density

$$\mathrm{Qm}(\nabla f_1,\dots,\nabla f_{m-2},\nabla g_1,\dots,\nabla g_{m-2})\colon M\to\mathbb{R}.$$

Remarks Note that

$$\mathrm{Qm}(X_1,\dots,X_{m-2},X_1,\dots,X_{m-2}) = |X_1\wedge\dots\wedge X_{m-2}|^2\cdot K_\sigma,$$

where K_σ is the sectional curvature of M on a plane σ orthogonal to $(m-2)$–vector $X_1\wedge\dots\wedge X_{m-2}$. Hence, the sectional curvatures of M and therefore its curvature tensor Rm can be computed from $\mathfrak{qm}$. By the symmetry

$$\mathfrak{qm}(f_1,\dots,f_{m-2},g_1,\dots,g_{m-2}) = \mathfrak{qm}(g_1,\dots,g_{m-2},f_1,\dots,f_{m-2}),$$

the density of $\mathfrak{qm}$ is defined by the sectional curvature. Therefore measure-valued tensor $\mathfrak{qm}$ gives an equivalent description of the curvature of Riemannian manifolds.

As you will see further, the described dual form of curvature tensor behaves better in the limit; in particular, it makes it possible to formulate Proposition 4.2.

In the 2–dimensional case, the valence of $\mathfrak{qm}$ is 0; in this case, $\mathfrak{qm}$ coincides with the curvature measure — the standard way to describe the curvature of surfaces [1; 25]. For a smooth surface, the density of this curvature measure with respect to the area is its Gauss curvature. In this case, it is known that *curvature measures are stable under smoothing* [1, VII §13]; in other words, our main theorem is known in the two-dimensional case.

2F Formulation and plan

Theorem 2.6 (main theorem) *Consider a smoothing $M_n \underset{\sim}{\longrightarrow} A$. Denote by $\mathfrak{qm}_n$ the dual measure-valued curvature tensor on M_n. Then there is a measure-valued tensor $\mathfrak{qm}$ on A such that $\mathfrak{qm}_n \rightharpoonup \mathfrak{qm}$.*

Let A be an m–dimensional Alexandrov space without boundary. Let us partition A into three subsets A°, A', and A'':

◇ A° is the set of regular points in A; that is, the set of points with tangent cone isometric to the Euclidean space.

◇ A' — the set of points in $A\backslash A^\circ$ with an isometric copy of $\mathbb{R}^{m-2}$ in their tangent space; in other words, for any $p \in A'$, the tangent space T_p is isometric to the product $\mathrm{Cone}(\theta)\times\mathbb{R}^{m-2}$ where $\mathrm{Cone}(\theta)$ denotes a two-dimensional cone with the total angle $\theta = \theta(p) < 2\cdot\pi$.

◇ A'' — the remaining set; this is the set of points with tangent space that does *not* contain an isometric copy of $\mathbb{R}^{m-2}$.

According to [13], A' is countably $(m-2)$–rectifiable, and A'' is countably $(m-3)$–rectifiable.

Observe that the set of regular points A° can be presented as

$$A^\circ = \bigcap_{\delta>0} A^\delta,$$

where A^δ denotes the set of δ–strained points of A.

Let M be an m–dimensional Riemannian manifold. Denote by $K_{\max}(x)$ the maximal sectional curvature at $x \in M$. The following statement is a direct corollary of the main result in [23]:

Corollary 2.7 *Given an integer $m \geqslant 0$, there is a constant $\mathrm{const}(m)$ such that the following holds:*

Let M be an m–dimensional Riemannian manifold (possibly noncomplete) with sectional curvature bounded below by -1. If for some $r < 1$ the closed ball $\bar{B}(p, 2\cdot r)_M$ is compact, then

$$\int\limits_{B(p,r)_M} K_{\max} \leqslant \mathrm{const}(m)\cdot r^{m-2}.$$

Observation 2.8 *There is another constant $\mathrm{const}'(m)$ such that*

$$|\mathrm{Qm}(X_1,\dots,X_{m-2},Y_1,\dots,Y_{m-2})| \leqslant \mathrm{const}'(m)\cdot K_{\max}\cdot|X_1\wedge\dots\wedge X_{m-2}|\cdot|Y_1\wedge\dots\wedge Y_{m-2}|.$$

Note that Corollary 2.7 and Observation 2.8 imply:

Claim 2.9 *Given a smoothing $M_n \underset{\sim}{\longrightarrow} A$, test functions $f_i\colon A \to \mathbb{R}$, and sequences of C^1–smooth functions $f_{i,n}\colon M_n \to \mathbb{R}$ such that $f_{i,n} \xrightarrow{C^1_\delta} f_i$, the sequence of measures $\mathfrak{qm}_n(f_{1,n},\dots,f_{2\cdot m-4,n})$ has a weakly converging subsequence.*

Moreover, the subsequence can be chosen simultaneously for several choices of function arrays so that it meets the chain rule. More precisely, choose i; *fix all functions* $f_{1,n},\dots,f_{2\cdot m-4,n}$ *except* $f_{i,n}$; *suppose*

$$\mathfrak{q}\hat{\mathfrak{m}}_{i,n}(f_{i,n})=\mathfrak{q}\mathfrak{m}_n(f_{1,n},\dots,f_{2\cdot m-4,n}).$$

Assume $h_{j,n}\colon M_n\to\mathbb{R}$ *are* C^1*–smooth functions such that* $h_{j,n}\xrightarrow{C^1_\delta}h_j$, *each* h_j *is a test function, and*

$$h_{0,n}=\varphi(h_{1,n},\dots,h_{k,n})$$

for a fixed C^2*–function* $\varphi\colon\mathbb{R}^k\to\mathbb{R}$. *Then the sequence of measure arrays* $\mathfrak{q}\hat{\mathfrak{m}}_{i,n}(h_{0,n}),\dots,\mathfrak{q}\hat{\mathfrak{m}}_{i,n}(h_{k,n})$ *has a partial limit* $\mathfrak{q}\hat{\mathfrak{m}}_i(h_0),\dots,\mathfrak{q}\hat{\mathfrak{m}}_i(h_k)$, *and*

$$\mathfrak{q}\hat{\mathfrak{m}}_i(h_0)=\sum_{j=1}^{k}(\partial_j\varphi)(h_1,\dots,h_k)\cdot\mathfrak{q}\hat{\mathfrak{m}}_i(h_j).$$

By Perelman's stability theorem, the space A in the claim is a topological manifold. In particular, A has no boundary; in other words, the singular set in A has codimension at least 2. Together with the claim, it implies that Theorem 2.6 follows from the next statement.

Proposition 2.10 *Let* $M_n \underset{\sim}{\longrightarrow} A$ *and* $\dim A=m$. *Suppose* $h_1,\dots,h_{m-2}$ *are test functions on* A *and* $h_{1,n},\dots,h_{m-2,n}$ *are* C^1*–smooth functions on* M_n *such that* $h_{i,n}\xrightarrow{C^1_\delta}h_i$ *for each* i. *Let* $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *be two measures on* A *that are weak partial limits of the sequence of measures* $\mathfrak{q}\mathfrak{m}_n(h_{1,n},\dots,h_{m,n},h_{1,n},\dots,h_{m,n})$ *on* M_n. *Then the following statements hold*:

(i) $\mathfrak{m}_1|_{A''}=\mathfrak{m}_2|_{A''}=0$.

(ii) $\mathfrak{m}_1|_{A'}=\mathfrak{m}_2|_{A'}$.

(iii) $\mathfrak{m}_1|_{A^\circ}=\mathfrak{m}_2|_{A^\circ}$.

The three parts of the proposition will be proved below in Sections 3, 4, and 5, respectively.

Proofs

3 Singularities of codimension 3

Proof of Proposition 2.10(i) According to [5, 10.6], A'' has a vanishing $(m-2)$–dimensional Hausdorff measure; that is, A'' can be covered by a countable family of balls $B(x_i,r_i)$ such that $\sum r_i^{m-2}$ is arbitrarily small. Therefore, Observation 2.8 and Corollary 2.7 imply the statement. □

4 Singularities of codimension 2

The following lemma will be proved in Section 8.

Lemma 4.1 *Let* A *be an* m*–dimensional Alexandrov space without boundary. Then the subset* $A'\subset A$ *can be covered by a countable set of compact sets* Q_i *that each admit a bi-Lipschitz embedding into* $\mathbb{R}^{m-2}$.

Let $\boldsymbol{h}\colon A\to\mathbb{R}^k$ be a Lipschitz map defined on an m–dimensional Alexandrov space without boundary. Suppose $Q\subset A$ is a closed subset such that there is a bi-Lipschitz embedding $s\colon Q\to\mathbb{R}^k$. By the generalized Rademacher theorem, the metric differential of s^{-1} is defined almost everywhere in the domain of definition of s^{-1}. Moreover, the metric differential is defined by a bilinear form; its determinant is the Jacobian of s^{-1}, briefly $\operatorname{jac} s^{-1}$. The same way we can define $\operatorname{jac}(\boldsymbol{h}\circ s^{-1})$ (we can apply the standard Rademacher theorem this time). Further, set $\operatorname{jac}(\boldsymbol{h}|_Q)=\operatorname{jac}(\boldsymbol{h}\circ s^{-1})/\operatorname{jac} s^{-1}$. It is straightforward to check that this definition is vol^k–almost-everywhere independent of the choice of s.

Consider the function

$$\theta(p)=2\cdot\pi\cdot\frac{\operatorname{vol}^{m-1}\Sigma_p}{\operatorname{vol}^{m-1}\mathbb{S}^{m-1}},\tag{4-1}$$

where Σ_p denotes the space of directions at p. According to [5, 7.14], $\theta\colon A\to\mathbb{R}$ is lower-semicontinuous.

Note that θ is identically $2\cdot\pi$ on A°. Further note that for any point $p\in A'$, its tangent cone is isometric to the product space $\operatorname{Cone}(\theta)\times\mathbb{R}^{m-2}$, where $\theta=\theta(p)<2\cdot\pi$. Since $\operatorname{vol}^{m-2}(A'')=0$, the measure $(2\pi-\theta)\cdot\operatorname{vol}^{m-2}$ vanishes on A''.

Note that Proposition 2.10(ii) follows from Lemma 4.1 and the following statement; it will be proved in Sections 4C–4D.

Proposition 4.2 *Let $\mathfrak{m}$ be one of two limit measures $\mathfrak{m}_i$ in Proposition 2.10 and*

$$\boldsymbol{h}=(h_1,\dots,h_{m-2})\colon A\to\mathbb{R}^{m-2}$$

be an array of test functions. Suppose that $Q\subset A$ is a compact subset that admits a bi-Lipschitz embedding into $\mathbb{R}^{m-2}$. Then

$$\mathfrak{m}|_Q=(2\cdot\pi-\theta)\cdot(\operatorname{jac}(\boldsymbol{h}|_Q))^2\cdot\operatorname{vol}^{m-2}.$$

4A Gauss and mean curvature estimates

Theorem 4.3 *Let f,h be a pair of strongly convex smooth 1–Lipschitz functions defined on an open set of a 3–dimensional Riemannian manifold. Suppose that*

(i) *$|\nabla f|\geqslant 1$ and*

$$|\nabla(f+h)|<\varepsilon\cdot|\nabla f|$$

for some fixed positive $\varepsilon<\frac{1}{2}$;

(ii) *for some $a,b\in\mathbb{R}$, the set*

$$W_{a,b}=\{p\in M\mid f(p)=a,\ h(p)\leqslant b\}$$

is compact.

Denote by $k_1(p) \leqslant k_2(p)$, $H(p) = k_1(p) + k_2(p)$ *and* $G(p) = k_1(p) \cdot k_2(p)$, *the principal, mean, and Gauss curvatures of* $W_{a,b}$ *at* p. *Then*

$$\int\limits_{W_{a,b}} G \leqslant 100 \cdot \varepsilon \tag{4-2}$$

and

$$\int\limits_{W_{a,b}} H \leqslant 10 \cdot \sqrt{\varepsilon} \cdot \operatorname{length}(\partial W_{a,b}). \tag{4-3}$$

The proof is based on the 2–dimensional case of the following statement, which is the integral Bochner formula with Dirichlet boundary condition.

Proposition 4.4 *Assume* Ω *is a compact domain with smooth boundary* $\partial\Omega$ *in a Riemannian manifold and* f *is a smooth function that vanishes on* $\partial\Omega$. *Then*

$$\int\limits_{\Omega} \left(|\Delta f|^2 - |\operatorname{Hess} f|^2 - \langle \operatorname{Ric}(\nabla f), \nabla f\rangle\right) = \int\limits_{\partial\Omega} H \cdot |\nabla f|^2,$$

where H *denotes the mean curvature of* $\partial\Omega$.

Proof of Theorem 4.3 Equip $W_{a,b}$ with unit normal vector field $n = \dfrac{\nabla f}{|\nabla f|}$. Let

$$S_p \colon \mathrm{T}_p W_{a,b} \to \mathrm{T}_p W_{a,b}$$

be the corresponding shape operator, so $S_p \colon v \mapsto \nabla_v n$. Since f is strongly convex, we have that

$$\langle S_p(v), v\rangle \geqslant \delta \cdot |v|^2$$

for a fixed value $\delta > 0$ and any tangent vector $v \in \mathrm{T}_p W_{a,b}$.

Note that the restriction $u = h|_{W_{a,b}}$ is strongly convex. Moreover,

$$\operatorname{Hess}_p u(v, v) \geqslant (1 - \varepsilon) \cdot \langle S_p(v), v\rangle \tag{4-4}$$

for any $p \in W_{a,b}$ and $v \in \mathrm{T}_p W_{a,b}$. Indeed, consider the geodesic γ in $W_{a,b}$ such $\gamma(0) = p$ and $\gamma'(0) = v$. Set $w = \gamma''(t)$. Note that

$$w = -\langle S_p(v), v\rangle \cdot n,$$

Since h is strongly convex, $\operatorname{Hess}_p h \geqslant 0$; therefore

$$(\operatorname{Hess}_p u)(v, v) = (\operatorname{Hess}_p h)(v, v) + \langle \nabla_p h, w\rangle \geqslant -\frac{\langle \nabla_p h, \nabla_p f\rangle}{|\nabla_p f|} \cdot \langle S_p(v), v\rangle \geqslant (1 - \varepsilon) \cdot |\nabla_p f| \cdot \langle S_p(v), v\rangle.$$

Since $|\nabla f| \geqslant 1$, (4-4) follows.

Since $\langle S_p(v), v\rangle \geqslant 0$ and $\varepsilon < \frac{1}{2}$, the inequality (4-4) implies that

$$4\cdot\det(\operatorname{Hess}_p u) \geqslant G(p) \tag{4-5}$$

and

$$-2\cdot\Delta u \geqslant H(p) \tag{4-6}$$

for any $p \in W_{a,b}$.

Denote by $\lambda_1(p), \lambda_2(p)$ the eigenvalues of $\operatorname{Hess}_p u$, so

$$\operatorname{trace}(\operatorname{Hess} u) = \Delta u = \lambda_1 + \lambda_2, \quad |\operatorname{Hess} u|^2 = \lambda_1^2 + \lambda_2^2, \quad \det(\operatorname{Hess} u) = \lambda_1\cdot\lambda_2,$$

and hence

$$2\cdot\det(\operatorname{Hess} u) = |\Delta u|^2 - |\operatorname{Hess} u|^2.$$

Since $W_{a,b}$ is two-dimensional, by Proposition 4.4 we get that

$$\int\limits_{W_{a,b}} 2\cdot\det(\operatorname{Hess} u) = \int\limits_{W_{a,b}} K\cdot|\nabla u|^2 + \int\limits_{\partial W_{a,b}} \kappa\cdot|\nabla u|^2,$$

where $\kappa \geqslant 0$ is the geodesic curvature of $\partial W_{a,b}$ and K is the curvature of $W_{a,b}$.

Since u is a convex function that vanishes on the boundary of $W_{a,b}$, it has a unique critical point, which is its minimum. By the Morse lemma, $W_{a,b}$ is a disc. Therefore, by the Gauss–Bonnet formula, we get that

$$\int\limits_{W_{a,b}} K + \int\limits_{\partial W_{a,b}} \kappa = 2\cdot\pi.$$

Whence,

$$\int\limits_{W_{a,b}} \det(\operatorname{Hess} u) \leqslant \pi\cdot \sup_{p\in W_{a,b}} |\nabla_p u|^2.$$

Note that $\nabla_p u$ is the projection of $\nabla_p h$ to $\mathrm{T}_p W_{a,b}$. Therefore,

$$|\nabla_p u|^2 = |\nabla_p h|^2 - \langle\nabla_p h, n\rangle^2 \leqslant 1-(1-\varepsilon)^2 < 2\cdot\varepsilon.$$

It follows that

$$\int\limits_{W_{a,b}} \det(\operatorname{Hess} u) \leqslant 2\cdot\pi\cdot\varepsilon.$$

Applying (4-5), we obtain (4-2).

Similarly, by the divergence theorem, we get that

$$-\int\limits_{W_{a,b}} \Delta u = \int\limits_{\partial W_{a,b}} |\nabla u|.$$

Whence (4-6) implies

$$\int\limits_{W_{a,b}} H \leqslant 10\cdot\sqrt{\varepsilon}\cdot\operatorname{length}(\partial W_{a,b}). \qquad \square$$

4B Curvature of level sets

Let M be a 3–dimensional Riemannian manifold. Choose a smooth function $f\colon M\to\mathbb{R}$. Consider its level sets

$$L_c=\{x\in M \mid f(x)=c\}.$$

If the level set L_c is a smooth surface in a neighborhood of $x\in L_c$, then denote by $k_1(x)\leqslant k_2(x)$ the principal curvatures of L_c at x. In this case, set

$$G(x)=k_1(x)\cdot k_2(x),\qquad H(x)=k_1(x)+k_2(x);$$

that is, $G(x)$ and $H(x)$ are Gauss and mean curvature of L_c at x.

Recall that $\mathrm{Cone}(\theta)$ denotes a 2–dimensional cone with the total angle θ.

Theorem 4.5 *Let $M_n \underset{\sim}{\longrightarrow} \mathrm{Cone}(\theta)\times\mathbb{R}$ and $f_n\colon M_n\to\mathbb{R}$ be a sequence of strongly concave smooth 1–Lipschitz functions. Suppose that $\sec M_n\geqslant-\frac{1}{n}$ for each n, and f_n converges as $n\to\infty$ to the $\mathbb{R}$–coordinate $f\colon(x,t)\mapsto t$ on $\mathrm{Cone}(\theta)\times\mathbb{R}$. Then G_n and H_n (the Gauss and mean curvatures of the level sets of f_n) weakly converge to zero.*

Proof Choose $p\in\mathrm{Cone}(\theta)\times\mathbb{R}$; set $a=f(p)$.

By the theorem of Artem Nepechiy [15], there is a (-2)–concave function ϱ defined in an r–neighborhood of p such that $\varrho(x)=-|p-x|^2+o(|p-x|^2)$. Moreover, the function ϱ is *liftable*; that is, there is a sequence of (-2)–concave $\varrho_n\colon M_n\to\mathbb{R}$ that converges to ϱ.

Consider a point $q\in\mathrm{Cone}(\theta)\times\mathbb{R}$ *above* p; that is, its $\mathbb{R}$–coordinate is larger, and its $\mathrm{Cone}(\theta)$–coordinate is the same. If the $\mathbb{R}$–coordinate of q is large, then dist_q+f is λ–concave for small $\lambda>0$ and it has a nonstrict minimum at p. Therefore, given $\lambda>0$, we can find q so that the sum $s=f+\operatorname{dist}_q+\lambda\cdot\varrho$ is $(-\lambda)$–concave and has a strict maximum at p. Moreover

$$-\tfrac{1}{2}\cdot\lambda\cdot|p-x|^2_{\mathrm{Cone}(\theta)\times\mathbb{R}}\geqslant s(x)-s(p)\geqslant-\tfrac{3}{2}\cdot\lambda\cdot|p-x|^2_{\mathrm{Cone}(\theta)\times\mathbb{R}}$$

and therefore

$$|\nabla_x s|\leqslant 10\cdot\lambda\cdot|p-x|_{\mathrm{Cone}(\theta)\times\mathbb{R}} \tag{4-7}$$

if $|p-x|_{\mathrm{Cone}(\theta)\times\mathbb{R}}$ is sufficiently small; say if $|p-x|_{\mathrm{Cone}(\theta)\times\mathbb{R}}\leqslant\frac{r}{10}$.

Choose a sequence of points $q_n\in M_n$ that converges to q. Let us apply the Green–Wu smoothing procedure to $\operatorname{dist}_{q_n}+\lambda\cdot\varrho_n$; denote by h_n the obtained function; we can assume that $|h_n-\operatorname{dist}_{q_n}-\lambda\cdot\varrho_n|\to 0$. Observe that (4-7) implies that the first condition in Theorem 4.3 is met for all large n in an $\frac{r}{2}$–neighborhood of p_n with $\varepsilon=10\cdot\lambda\cdot r$. Moreover, one can choose b so that the second condition is satisfied and $B_n=B(p_n,r/10)\cap L_a\subset W_{a,b}$. Applying Theorem 4.3, we get that for any $\delta>0$, we have

$$\int\limits_{B_n} G_n<\delta,\quad\text{and}\quad\int\limits_{B_n} H_n<\delta.$$

for all large n. It remains to integrate the obtained inequalities by a and pass to a limit as $n\to\infty$. □

For a product $\mathbb{R}^{m-2}\times\mathrm{Cone}(\theta)$, denote by $\mathcal{V}$ the vol^{m-2}–measure on the vertical line $\mathbb{R}^{m-2}\times\{0\}$. Further, consider the *curvature measure*

$$\omega = (2\cdot\pi-\theta)\cdot\mathcal{V}$$

on $\mathbb{R}^{m-2}\times\mathrm{Cone}(\theta)$.

Corollary 4.6 *Suppose that $M_n \underset{\sim}{\longrightarrow} \mathrm{Cone}(\theta)\times\mathbb{R}$ and $f_n\colon M_n\to\mathbb{R}$ be as in Theorem 4.5. Set $u_n=\dfrac{\nabla f_n}{|\nabla f_n|}$. Then*:

- (i) *$\langle\mathrm{Ric}\, u_n,u_n\rangle$ weakly converges to zero.*
- (ii) *Let v_n and w_n be sequences of uniformly bounded, continuous vector fields on M_n. Suppose that $\langle v_n,u_n\rangle$ and $\langle w_n,u_n\rangle$ converge uniformly as $n\to\infty$ to some constants a and b respectively. Then*

$$\mathrm{Qm}(v_n,w_n)\rightharpoonup a\cdot b\cdot\omega,$$

where ω is the measure on $\mathrm{Cone}(\theta)\times\mathbb{R}$ described above.

Proof (i) Passing to a subsequence if necessary, we can assume weak convergence of $\langle\mathrm{Ric}\,u_n,u_n\rangle\cdot\mathrm{vol}^3$ to a measure $\mathfrak{m}$ on $\mathrm{Cone}(\theta)\times\mathbb{R}$. Since $\sec M_n\geqslant-\frac{1}{n}$, we have that $\mathfrak{m}\geqslant 0$. Therefore it is sufficient to show that $\mathfrak{m}\leqslant 0$.

By Theorem 6.1 we have that

$$\int\limits_{\Omega}\varphi_n\cdot\langle\mathrm{Ric}\,u_n,u_n\rangle=\int\limits_{\Omega}\bigl[\varphi_n\cdot G_n+H_n\cdot\langle u_n,\nabla\varphi_n\rangle-\langle\nabla\varphi_n,\nabla_{u_n}u_n\rangle\bigr]$$

holds for any function φ_n with compact support on M_n, assuming that all expressions in the formula have sense.

It remains to find a sequence of nonnegative functions $\varphi_n\colon M_n\to\mathbb{R}$ with compact support that converges to a $\varphi\colon\mathrm{Cone}(\theta)\times\mathbb{R}\to\mathbb{R}$ such that (1) φ is unit in a neighborhood of a given point $p\in\mathrm{Cone}(\theta)\times\mathbb{R}$ and (2) we have control on the three terms on the right-hand side of the formula; the latter means that we have the weak convergences

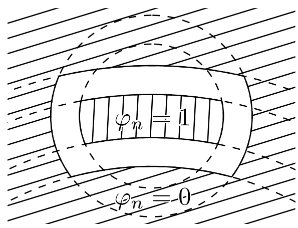
$$\varphi_n\cdot G_n\rightharpoonup 0,\qquad H_n\cdot\langle u_n,\nabla\varphi_n\rangle\rightharpoonup 0,\qquad \langle\nabla\varphi_n,\nabla_{u_n}u_n\rangle\rightharpoonup 0. \tag{4-8}$$

For the first convergence, it is sufficient to choose the sequence φ_n so that in addition it is universally bounded. Indeed, since $|\nabla f_n|\to 1$, we have that Theorem 4.5 implies the first convergence in (4-8).

Similarly, to prove the second convergence in (4-8), it is sufficient to assume in addition that $|\nabla\varphi_n|$ is universally bounded and apply Theorem 4.5.

To prove the last convergence in (4-8), note that $|\nabla_{u_n}u_n|\rightharpoonup 0$ away from the singular locus. The latter follows from Lemmas 5.3 and 5.4. Indeed, $\nabla_{u_n}u_n$ can be written in a common chart away from the singular locus. The lemmas imply that its components converge to the components of $\nabla_u u$ in the limit space. By assumption u is parallel in the limit space; in particular $\nabla_u u = 0$.

This observation will be used to control the term $\langle \nabla\varphi_n, \nabla_{u_n} u_n\rangle$ at the points far from the singular locus of $\mathrm{Cone}(\theta)\times\mathbb{R}$. To do this we only need to assume that $|\nabla\varphi_n|$ is bounded. Next, we describe how to control it near the singularity.



Since $|u_n| = 1$, we have $\nabla_{u_n} u_n \perp \nabla f_n$. Therefore if $\nabla\varphi_n$ is proportional to ∇f_n at some point, then $\langle \nabla\varphi_n, \nabla_{u_n} u_n\rangle = 0$ at this point. This observation makes it possible to choose φ_n so that the term $\langle \nabla\varphi_n, \nabla_{u_n} u_n\rangle$ vanish around the singular locus of $\mathrm{Cone}(\theta)\times\mathbb{R}$. Namely, in addition to the above conditions on φ_n we have to assume that the identity $\varphi_n = \psi\circ f_n$ holds at the points of M_n that are sufficiently close to the singular locus of $\mathrm{Cone}(\theta)\times\mathbb{R}$.

Finally, observe that the needed sequence exists. Indeed, one can take

$$\varphi_n = (\sigma\circ \mathrm{dist}_{p_n})\cdot(\psi\circ f_n)$$

for appropriately chosen fixed mollifiers $\sigma, \psi : \mathbb{R}\to\mathbb{R}$ and $M_n \ni p_n \to p$.

(ii) Since $G_n \rightharpoonup 0$, we get that the curvature measure of level sets of f_n weakly converges to the curvature of $\mathrm{Cone}(\theta)$. It follows that

$$\mathrm{Qm}(u_n, u_n) \rightharpoonup \omega.$$

Suppose $v_n' \perp u_n$ for all n. Part (i) implies that $\mathrm{Qm}(v_n', v_n') \rightharpoonup 0$.

Fix $t\in\mathbb{R}$. Since the lower bound on sectional curvature of M_n converges to 0, any partial weak limit of $\mathrm{Qm}(v_n' + t\cdot w_n, v_n' + t\cdot w_n)$ is nonnegative. It follows that

$$\mathrm{Qm}(v_n', w_n) \rightharpoonup 0$$

for any sequence of fields v_n', w_n such that $v_n' \perp u_n$.

Consider the vector fields v_n', w_n' such that

$$\begin{aligned} v_n' &\perp u_n, & v_n &= a_n\cdot u_n + v_n',\\ w_n' &\perp u_n, & w_n &= b_n\cdot u_n + w_n'. \end{aligned}$$

Since Qm is bilinear, we get that

$$\mathrm{Qm}(v_n, w_n) = \mathrm{Qm}(v_n', w_n) + a_n\cdot[\mathrm{Qm}(u_n, w_n') + b_n\cdot \mathrm{Qm}(u_n, u_n)].$$

By assumption, $a_n = \langle u_n, v_n\rangle$ and $b_n = \langle u_n, w_n\rangle$ uniformly converge to a and b respectively. Whence the statement follows. $\square$

4C Three-dimensional case

Proof of the 3–dimensional case in Proposition 4.2 Suppose $M_n \xrightarrow[\sim]{} A$ and $\dim A = 3$. Choose a set $Q \subset A$ that admits a bi-Lipschitz embedding into $\mathbb{R}$.

Let us split $\mathfrak{m}$ into negative and positive parts $\mathfrak{m} = \mathfrak{m}^+ - \mathfrak{m}^-$; that is,

$$\mathfrak{m}^{\pm}(X) := \sup\{\pm\mathfrak{m}(Y) \mid Y \subset X\}.$$

Since the sectional curvature of M_n is bounded below, we get that $\mathfrak{m}^-$ has bounded density; in other words, $\mathfrak{m}^-$ is a regular measure with respect to vol^3 on A. Since Q has zero volume, we get $\mathfrak{m}^-(Q) = 0$.

Set $\mathfrak{n} = \mathfrak{m}|_Q$; from above we have $\mathfrak{n} \geqslant 0$. By [23], $\mathfrak{n}$ is regular with respect to vol^1 on Q. Therefore it is sufficient to show that

$$(2\cdot\pi - \theta)\cdot(\mathrm{jac}(\boldsymbol{h}|_Q))^2$$

is the vol^1–density of $\mathfrak{n}$ at vol^1–almost all $p \in Q$.

Choose a bi-Lipschitz embedding $s\colon Q \to \mathbb{R}$; set $K = s(Q)$. Since s^{-1} and $h \circ s^{-1}$ are Lipschitz, by Rademacher's theorem, we can assume that s^{-1} and $h \circ s^{-1}$ are differentiable at almost all $x \in K$. Moreover, we can assume that $d_x s^{-1}(y) = (\lambda\cdot y, 0) \in \mathbb{R} \times \mathrm{Cone}(\theta) = \mathrm{T}_p$ and the vol^1–density of $\mathfrak{n}$ at $p = s^{-1}(x)$ is defined.

Shifting and scaling the interval K, we may assume that $x = 0$ and $\lambda = 1$. In this case, $|\mathrm{jac}_p(h|_Q)| = |d_0(h \circ s^{-1})|$.

Note that we can choose a sequence of points $p_n \in M_n$ and a sequence of factors $c_n \to \infty$ such that $(c_n \cdot M_n, p_n)$ converges to the tangent space $\mathrm{T}_p = \mathbb{R} \times \mathrm{Cone}(\theta)$.

Applying Perelman's construction [22, 7.1.1 and 7.2.3] for a horizontal vector in $\mathrm{T}_p = \mathbb{R} \times \mathrm{Cone}(\theta)$, we can choose a sequence of functions $f_n\colon c_n \cdot M_n \to \mathbb{R}$ satisfying the assumptions in Theorem 4.5; let $u_n = \nabla f_n / |\nabla f_n|$. Consider the sequence of functions $\hat{h}_n\colon c_n \cdot M_n \to \mathbb{R}$ defined by

$$\hat{h}_n(x) = c_n \cdot (h_n(x) - h_n(p_n)).$$

Since $\lambda = 1$, we have that $|\langle \nabla \hat{h}_n, u_n\rangle|$ uniformly converges to $|d_0(h \circ s^{-1})|$. By Corollary 4.6(ii), the sequence of measures $\mathfrak{qm}(\hat{h}_n, \hat{h}_n)$ on $c_n \cdot M_n \to \mathbb{R}$ weakly converges to $|d_0(h \circ s^{-1})|^2 \cdot \omega_{\mathrm{Cone}(\theta)\times\mathbb{R}}$. Recall that vol^1–density of $\omega_{\mathrm{Cone}(\theta)\times\mathbb{R}}$ on the singular line is $2\cdot\pi - \theta(p)$.

Observe that

$$\mathfrak{qm}(\hat{h}_n, \hat{h}_n)[B(p_n, 1)_{c_n\cdot M_n}] = c_n \cdot \mathfrak{qm}(h_n, h_n)[B(p_n, 1/c_n)_{M_n}]$$

Whence $(2\cdot\pi - \theta(p))\cdot(\mathrm{jac}_p(h|_Q))^2$ is the vol^1–density of $\mathfrak{n}$ at p as required. □

4D Higher-dimensional case

Suppose $M_n \underset{\sim}{\longrightarrow} \mathrm{Cone}(\theta) \times \mathbb{R}^{m-2}$, where $\mathrm{Cone}(\theta)$ denotes a two-dimensional cone with the total angle $\theta < 2 \cdot \pi$. First, we will show that the curvatures of M_n in the vertical sectional directions of $\mathrm{Cone}(\theta) \times \mathbb{R}^{m-2}$ weakly converge to zero; an exact statement is given in the following proposition. By combining this result with the 3–dimensional case we get Proposition 4.2 in all dimensions.

For $X, Y \in \mathrm{T}_p$, denote by $K(X \wedge Y)$ the curvature in the sectional direction of $X \wedge Y$. A function $f : \mathrm{Cone}(\theta) \times \mathbb{R}^{m-2} \to \mathbb{R}$ will be called a *vertical affine function* if f can be obtained as a composition of the projection to $\mathbb{R}^{m-2}$ and an affine function on $\mathbb{R}^{m-2}$.

Proposition 4.7 *Let $M_n \underset{\sim}{\longrightarrow} \mathrm{Cone}(\theta) \times \mathbb{R}^{m-2}$ and $f_n, h_n : M_n \to \mathbb{R}$ be sequences of strongly concave smooth Lipschitz functions. Suppose that $\sec M_n \geqslant -\frac{1}{n}$ for each n, and we have pointwise convergences $f_n \to f$ and $h_n \to h$, where f and h are vertical affine functions on $\mathbb{R}^{m-2} \times \mathrm{Cone}(\theta)$. Then*

$$K(\nabla f_n \wedge \nabla h_n) \cdot \mathrm{vol}^m \rightharpoonup 0.$$

Let Σ be a convex hypersurface in an m–dimensional Riemannian manifold M. Suppose x is a smooth point of Σ; that is, the tangent hyperplane H_x of Σ is defined at x; denote by $e_1, \dots, e_{m-1}$ an orthonormal basis of H_x. Set

$$\mathrm{Zc}_\Sigma(x) = \sum_{i,j} K(e_i \wedge e_j).$$

In other words,

$$\mathrm{Zc}_\Sigma = \mathrm{Sc} - 2 \cdot \mathrm{Ric}(n_\Sigma, n_\Sigma),$$

where n_Σ is the unit normal vector to Σ.

Since tangent hyperplanes are defined at almost all points of convex hypersurfaces, $\mathrm{Zc}_\Sigma(x)$ is defined almost everywhere on Σ.

Lemma 4.8 *Let Σ be a strongly convex hypersurface in an m–dimensional Riemannian manifold M with curvature $\geqslant -1$. Suppose that for some point $p \in \Sigma$ and $r < 1$ the closed ball $\bar{B}(p, 2 \cdot r)$ in the intrinsic metric of Σ is compact.*

Then,

$$\int\limits_{x \in B(p,r)} \mathrm{Zc}_\Sigma(x) \cdot \mathrm{vol}^{m-1} \leqslant (m-1) \cdot (m-2) \cdot \mathrm{const}(m-1) \cdot r^{m-3},$$

where $\mathrm{const}(m-1)$ is the constant in Corollary 2.7.

Proof If Σ is smooth, then the inequality follows from Corollary 2.7, and the fact that curvature cannot decrease when we pass to a convex hypersurface.

In the general case, the surface Σ can be approximated by a smooth convex surface; this can be done by applying the Green–Wu smoothing procedure; compare to [2].

It remains to pass to the limit. More precisely, suppose Σ_n is a sequence of smooth strongly convex hypersurfaces that converges to Σ. For each n, choose a point $p_n \in \Sigma_n$ such that $p_n \to p$. By [20, Theorem 1.2], (Σ_n, p_n) converges to (Σ, p) as in the pointed Gromov–Hausdorff convergence.

Recall that vol_{m-1} on Σ_n weakly converges to vol_{m-1} on Σ (see [5, 10.8]). Further, since M is smooth, Zc_{Σ_n} is bounded in $B(p_n, r)_{\Sigma_n}$. Therefore,

$$\int\limits_{x\in B(p_n,r)_{\Sigma_n}} \mathrm{Zc}_{\Sigma_n}(x)\cdot \mathrm{vol}^{m-1} \to \int\limits_{x\in B(p,r)_{\Sigma}} \mathrm{Zc}_{\Sigma}(x)\cdot \mathrm{vol}^{m-1} \quad \text{as } n\to\infty$$

follows if for almost all $x \in \Sigma$ we have that *for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if* $x_n \in \Sigma_n$ *and* $|x_n - x| < \delta$ *for large* n*, then*

$$|\mathrm{Zc}_{\Sigma_n}(x_n) - \mathrm{Zc}_{\Sigma}(x)| < \varepsilon.$$

This condition holds if the tangent plane H_x is defined. Whence the nonsmooth case follows. $\square$

Proof of Proposition 4.7 Passing to a subsequence, we can assume that

$$K(\nabla f_n \wedge \nabla h_n)\cdot \mathrm{vol}^m \rightharpoonup \mathfrak{m}$$

for some measure $\mathfrak{m}$ on $\mathbb{R}^{m-2}\times \mathrm{Cone}(\theta)$.

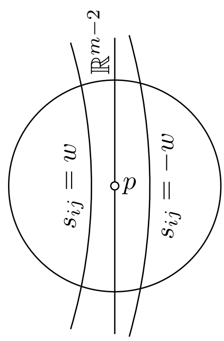
First, let us show that $\mathfrak{m}$ is supported on the singular locus. If p is not singular, then it has a flat neighborhood. Therefore by a local version of Key lemma 5.5 (see also Section 9) we get that $\mathfrak{m}$ vanishes in a neighborhood $U \ni p$. Indeed, we can include copies of U (which is flat) in the approximating sequence $U_n \subset M_n$ of U and argue as in the introduction.

Let p be a singular point on $\mathrm{Cone}(\theta)\times\mathbb{R}^{m-2}$; let us denote its liftings by $p_n \in M_n$. We can assume that p corresponds to the origin of $\mathbb{R}^{m-2}$. Choose points $a_{1,n},\dots,a_{m-2,n}, b_{1,n},\dots,b_{m-2,n}$ in M_n such that the functions $f_{i,n} = \mathrm{dist}_{a_{i,n}} - |a_{i,n} - p|$ and $-h_{i,n} = -\mathrm{dist}_{b_{i,n}} + |b_{i,n} - p|$ converge to i^{th} vertical coordinate function on $\mathrm{Cone}(\theta)\times\mathbb{R}^{m-2}$. Further, choose points $c_{1,n}, c_{2,n}, c_{3,n}$ so that the functions $g_{i,n} = \mathrm{dist}_{c_{i,n}} - |c_{i,n} - p|$ converge to Busemann functions for different horizontal rays in $\mathrm{Cone}(\theta)\times\mathbb{R}^{m-2}$ emerging from p. Note that the latter implies that the angles $\tilde\measuredangle(p_n\,{}^{c_{i,n}}_{c_{j,n}})$ are bounded away from zero for all large n.

By [22, Lemma 7.2.1], there is an increasing concave function φ defined in a neighborhood of zero in $\mathbb{R}$ such that φ' is close to 1 and for any $\varepsilon > 0$ and $i \neq j$ the function

$$s_{ij,n} = \varphi\circ g_{i,n} + \varphi\circ g_{j,n} + \sum_i [\varphi(\varepsilon\cdot f_{i,n}) + \varphi(\varepsilon\cdot h_{i,n})]$$

is strongly concave in $B(p_n, R)$ for fixed $R > 0$ and every large n.



Denote by $s_{ij}:\mathrm{Cone}(\theta)\times\mathbb{R}^{m-2}\to\mathbb{R}$ the limits of $s_{ij,n}$. Note that given $w > 0$, we can take small $\varepsilon > 0$ so that for all $i \neq j$ the set $s_{ij}^{-1}[-w, w]$ covers the singular locus in $B(p, R)$.

Note that we can choose $\varepsilon_0 > 0$ so that for almost all points $x \in B(p_n, R)$ the differential $d_x s_{ij,n}$ is linear and $|d_x s_{ij,n}| > \varepsilon_0 > 0$ for some i and j. Indeed, these differentials are linear outside cutlocuses of $a_{i,n}$, $b_{i,n}$, and $c_{i,n}$; in particular, they are linear at almost any point $x_n \in M_n$. Further, if the differential $d_{x_n} s_{12,n}$ is very close to zero, then the directions of $[x_n, c_{1,n}]$ and $[x_n, c_{2,n}]$ are nearly opposite. Since x_n is close to p_n, we get that for large n the angles $\measuredangle[x_n\,{}^{c_{i,n}}_{c_{j,n}}]$ is bounded away from zero, we get $|d_{x_n} s_{13,n}|$ is bounded away from zero as well.

Since f_n and h_n are converging to vertical affine functions, we get that for large n their gradients are nearly orthogonal to $\nabla_{x_n} g_{i,n}$ at almost all $x_n \in M_n$. Suppose $d_x s_{ij,n}$ is linear and $|d_x s_{ij,n}| > \varepsilon_0 > 0$. Then gradients $\nabla_{x_n} f_n$ and $\nabla_{x_n} h_n$ are nearly orthogonal to $\nabla_x s_{ij,n}$.

Set $\Sigma_{ij,n} = \Sigma_{ij,n}(c) = \{x \in M_n \mid s_{ij,n}(x) = c\}$. The argument above implies that for almost all points $x \in B(p_n, R)$ one of the sectional directions of the tangent directions σ of $\Sigma_{ij,n}$ is close to the sectional direction $\nabla f_n \wedge \nabla h_n$. In particular, given $\delta > 0$, we have

$$K(\nabla f_n \wedge \nabla h_n)(x) \leqslant K(\sigma) + \delta \cdot |K_{\max}(x)|$$

for all large n ($K_{\max}$ is defined in Corollary 2.7).

By Lemma 4.8 and the coarea formula, the sum of integral curvatures of M_n in the directions of $\Sigma_{ij,n}$ at x_n at the subsets where $|d_{x_n} s_{ij,n}| > \varepsilon_0$ is bounded by $\mathrm{const} \cdot w$. By Corollary 2.7, the same holds for the integral of $K(\nabla f_n \wedge \nabla h_n)$ if n is large. The proposition follows since w can be taken arbitrarily small. □

Proof of the general case of Proposition 4.2 Choose $m-2$ sequences of strongly concave functions $f_{1,n}, \dots, f_{m-2,n} \colon M_n \to \mathbb{R}$ that converge to vertical affine functions $f_1, \dots, f_{m-2}$ on $\mathrm{Cone}(\theta) \times \mathbb{R}^{m-2}$ with orthonormal gradients. It is done using Perelman's construction [22, 7.1.1 and 7.2.3] for the corresponding vertical vectors in $\mathrm{Cone}(\theta) \times \mathbb{R}^{m-2}$.

Note that the fields $e_{1,n} = \nabla f_{1,n}, \dots, e_{m-2,n} = \nabla f_{m-2,n}$ are nearly orthonormal; in particular, they are linearly independent for all large n. Let us add two fields $e_{m-1,n}$ and $e_{m,n}$ so that $e_{1,n}, \dots, e_{m,n}$ form a nearly orthonormal frame in M_n; that is, $\langle e_{i,n}, e_{j,n}\rangle \to 0$ for $i \neq j$ and $\langle e_{i,n}, e_{i,n}\rangle \to 1$ for any i as $n \to \infty$.

Observe that Proposition 4.7 implies that $K(e_i \wedge e_j) \cdot \mathrm{vol} \rightharpoonup 0$ if $i, j \leqslant m-2$.

Let us show that $K(e_i \wedge e_j) \cdot \mathrm{vol} \rightharpoonup 0$ for $i \leqslant m-2$ and $j \geqslant m-1$. The 3–dimensional case is done already; it is used as a base of induction. Let us apply the induction hypothesis to the level surfaces of $f_{1,n}$. (Formally speaking, we apply the local version of the induction hypothesis described in Section 9.) Since the curvature of convex hypersurfaces is larger than the curvature of the ambient manifold in the same direction, we get the statement for $i \neq 1$. Applying the same argument for the level surfaces of $f_{2,n}$, we get the claim.

Now let us show that $K(e_{m-1} \wedge e_m) \cdot \mathrm{vol}^m \rightharpoonup \omega_{\mathrm{Cone}(\theta)\times\mathbb{R}^{m-2}}$. Consider the level sets L_n defined by

$$f_{1,n} = c_1, \quad \dots, \quad f_{m-2,n} = c_{m-2}. \tag{4-9}$$

Note that $L_n \xrightarrow{\sim} \mathrm{Cone}(\theta)$. Applying the 2–dimensional case to L_n and the coarea formula, we get that curvatures of L_n weakly converge to $\omega_{\mathrm{Cone}(\theta)\times\mathbb{R}^{m-2}}$. It remains to show that the extra term in the Gauss formula for the curvature of L_n weakly converges to zero; in other words, the difference between the curvature of L_n and sectional curvature of M_n in the direction tangent to L_n weakly converges to zero.

The 3–dimensional case is done already. To prove the general case, we apply the 3–dimensional case to the 3–dimensional level sets defined by $m-3$ equations from the $m-2$ equations in (4-9). (The same argument is used in the proof of Key lemma 5.5, and it is written with more details.)

Note that for $\theta = 0$, the last argument implies:

Claim 4.9 *Let $M_n \xrightarrow{\sim} A$. If A has a flat open set U, then $|K_{\max}| \cdot \mathrm{vol}_n \rightharpoonup 0$ on U.*

In particular, the weak limit of dual curvature tensor has support on the singularity of $\mathrm{Cone}(\theta) \times \mathbb{R}^{m-2}$.

The same argument as in Corollary 4.6 shows that $\langle \mathrm{Rm}(e_i, e_j)e_q, e_r\rangle \cdot \mathrm{vol}^m \rightharpoonup 0$ if at least one of the indices i, j, q, r is at most $m-2$. The latter statement implies the result. □

5 Regular points

5A Common chart and delta-convergence

Choose a smoothing $M_n \xrightarrow{\sim} A$ of an m–dimensional Alexandrov space A. Let $p \in A$ be a point of rank m; that is, there are $m+1$ points $a_0, \dots, a_m \in A$ such that $\tilde\measuredangle(p\,{}^{a_i}_{a_j}) > \frac{\pi}{2}$ for all $i \neq j$.

Recall [22, Sec. 7] that we can choose small $r > 0$, finite set of points $\boldsymbol{a}_i$ near a_i, and a smooth concave increasing real-to-real function φ defined on an open interval such that

$$f_i = \sum_{x \in \boldsymbol{a}_i} \varphi \circ \widetilde{\mathrm{dist}}_{x,r}$$

is strongly a concave function that is defined in a neighborhood $U \ni p$; it will be called *smoothed distance chart*.

Since r is small, and all points in $\boldsymbol{a}_i$ are near a_i we get that the functions $f_0, \dots, f_m$ are tight in U; see the definition in [22]. In particular, the map $U \to \mathbb{R}^m$ defined by $x \mapsto (f_1(x), \dots, f_m(x))$ is a coordinate system in U.

The presented construction can be lifted to M_n. As a result, we obtain a chart of an open set $U_n \subset M_n$. Passing to smaller sets we may assume that U and each U_n is mapped to a fixed open set $\Omega \subset \mathbb{R}^m$ for all large n. Further, we assume that it holds for all n; it could be achieved by cutting off the beginning of the sequence M_n.

The obtained collection of charts $\boldsymbol{x}_n : U_n \to \Omega$ and $\boldsymbol{x} : U \to \Omega$ will be called a *common chart* at p. It will be used to identify points of Ω, M_n, and A; in addition, we will use it to identify the tangent spaces $\mathrm{T}M_n$ and $\mathrm{T}A$ with $\mathbb{R}^m$. For example, we will use the same notation for function $M_n \to \mathbb{R}$ and its composition $\Omega \to \mathbb{R}$ with the inverse of the chart $U_n \to \Omega$. We will use index n or skip it to indicate that the calculations are performed in M_n or A respectively. For example, given a function $f\colon \Omega \to \mathbb{R}$, we denote by $\nabla_n f$ and ∇f the gradients of $f \circ \boldsymbol{x}_n$ in M_n and $f \circ \boldsymbol{x}$ in A respectively.

Recall that A^δ denotes the set of δ–strained points in A. For a fixed common chart $\boldsymbol{x}$ we will use the notation A^δ_Ω for the image $\boldsymbol{x}(A^\delta) \subset \Omega$.

Part (iii) of Proposition 2.10 will follow from certain estimates in one common chart.

Definitions 5.1 Let $M_n \underset{\sim}{\longrightarrow} A$, $\dim A = m$; choose a common chart with range $\Omega \subset \mathbb{R}^m$.

A sequence of measures $\mathfrak{n}_n$ defined on Ω is called *weakly delta-converging* if the following conditions hold:

- ◇ Every subsequence of $\mathfrak{n}_n$ has a weak partial limit.
- ◇ For any $\varepsilon > 0$ there is $\delta > 0$ such that for any two weak partial limits $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $(\mathfrak{n}_n)$ we have

$$|(\mathfrak{m}_1 - \mathfrak{m}_2)(S)| < \varepsilon$$

for any Borel set $S \subset A^\delta_\Omega$.

A sequence of bounded functions f_n on Ω is called *weakly delta-converging* if the measures $f_n \cdot \mathrm{vol}_n$ are weakly delta-converging.

A sequence of functions f_n defined on Ω is called *uniformly delta-converging* if the following conditions hold:

- ◇ For any $\varepsilon > 0$ there is $\delta > 0$ such that such that

$$\limsup_{n\to\infty}\{f_n(x)\} - \liminf_{n\to\infty}\{f_n(x)\} < \varepsilon$$

for any $x \in A^\delta_\Omega$.

Observation 5.2 *If f_n is uniformly delta-converging and $\mathfrak{n}_n$ is weakly delta-converging, then $f_n \cdot \mathfrak{n}_n$ is weakly delta-converging.*

5B Convergences

Lemma 5.3 *Let $M_n \underset{\sim}{\longrightarrow} A$, $\dim A = m$; choose a common chart with range $\Omega \subset \mathbb{R}^m$. Let $f_n\colon M_n \to \mathbb{R}$ be a sequence of C^1–functions such that $f_n \xrightarrow{C^1_\delta} f\colon A \to \mathbb{R}$. Let us denote by $\partial_1, \dots, \partial_m$ the partial derivatives on $\Omega \subset \mathbb{R}^m$. Denote by $g_{ij,n}$ and g^{ij}_n the components of the metric tensors on M_n. Then:*

1. (i) *f_n uniformly converges to f on Ω.*
2. (ii) *$\partial_i f_n$ are uniformly delta-converging.*
3. (iii) *$g_{ij,n}$ and g^{ij}_n are uniformly delta-converging for all i, j; moreover, $\det g_{ij,n}$ is bounded away from zero.*
4. (iv) *$|\nabla_n f_n|$ uniformly delta-converges on Ω.*

Proof Part (i) is trivial.

(ii) Suppose $a_0, a_1, \dots, a_m$ struts p (see the definition in [3]), and the geodesics $[pa_i]$ are uniquely defined. In this case, for any sequence of points $a_{i,n}, p_n \in M_n$ such that $a_{i,n} \to a_i$, and $p_n \to p$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \measuredangle[p_n\,{}^{a_{i,n}}_{a_{j,n}}] \geqslant \measuredangle[p\,{}^{a_i}_{a_j}].$$

If T_p is Euclidean, then $(n+1)$–point comparison implies that equality holds in the last inequality.

Note that the angles $\measuredangle[p_n\,{}^{a_{i,n}}_{a_{j,n}}]$ for all $i, j > 0$ completely describe the metric tensor at p_n in the basis $V_{1,n}, \dots, V_{m,n}$, where $V_{i,n}$ is the unit vector in the direction of $[p_n, a_{i,n}]$.

If $f_n \xrightarrow{C^1_\delta} f$, then $V_{i,n} f_n$ completely describes $\nabla_{p_n} f_n$ in the basis $V_{1,n}, \dots, V_{m,n}$. From above, we can express $|\nabla_{p_n} f_n|$ in terms of $V_{i,n} f_n$ and the angles $\measuredangle[p_n\,{}^{a_{i,n}}_{a_{j,n}}]$. Whence we get convergence $|\nabla_{p_n} f_n| \to |\nabla_p f|$ and therefore

$$\langle \nabla_{p_n} f_n, \nabla_{p_n} h_n \rangle \to \langle \nabla_p f, \nabla_p h \rangle$$

if $h_n \xrightarrow{C^1_\delta} h$; the latter follows by the identity $4 \cdot B(x,y) = B(x+y, x+y) - B(x-y, x-y)$ for any bilinear form B.

Note that the partial derivatives $\partial_i f_n$ at a regular point p can be expressed in terms of $\langle d_p f_n, d_p x_j \rangle_n$ and $\langle d_p x_j, d_p x_k \rangle_n$, where $x_1, \dots, x_m$ are the coordinate functions of the chart. Therefore, we get that all $\partial_i f_n$ converge at any regular point.

Finally, observe that if p is a δ–strained point for sufficiently small $\delta > 0$, then the calculations above go thru with a small error. Whence the statement follows.

(iii) This part follows from the proof of (ii) since $g_n^{ij} = \langle d_p x_i, d_p x_j \rangle_n$ and $g_{ij,n}$ can be expressed thru g_n^{ij}.

(iv) Note that $|\nabla_n f_n|$ can be expressed from g_n^{ij} and $\partial_i f_n$. Since these quantities are delta-converging, so is $|\nabla_n f_n|$. $\square$

The following lemma relies on the DC-calculus which is discussed in Section 7; this section includes the definition of DC and DC_0 functions, as well as DC convergence. Since *test convergence implies DC convergence* (see Observation 7.1), the lemma also holds for test-converging sequences of functions.

Lemma 5.4 *Let $M_n \underset{\sim}{\longrightarrow} A$. Choose a common chart $\boldsymbol{x}_n \colon U_n \subset M_n \to \Omega$ and $\boldsymbol{x} \colon U \subset A \to \Omega$ with range $\Omega \subset \mathbb{R}^m$. Let $f_n \colon M_n \to \mathbb{R}$ be a sequence of smooth functions that DC converges to a DC_0 function $f \colon A \to \mathbb{R}$. Let us denote by $\partial_1, \dots, \partial_m$ the partial derivatives on $\Omega \subset \mathbb{R}^m$. Denote by $g_{ij,n}$ and g_n^{ij} the components of the metric tensors on M_n. Then the partial derivatives $\partial_k g_{ij,n}$, $\partial_k g_n^{ij}$, $\partial_j \partial_i f_n$, as well as their products to uniformly delta-converging functions, are weakly converging.*

Proof The weak convergence of $\partial_k g_{ij,n}$, $\partial_k g_n^{ij}$, and $\partial_j \partial_i f_n$ follows from Theorem 7.4. Products of these partial derivatives with uniformly delta-converging sequences of functions are weakly delta-converging, by Observation 5.2.

Let $h_n\colon M_n\to\mathbb{R}$ be a uniformly delta-converging sequence. Note that its limit is well defined in A°; denote it by h; let us extend it by 0 to the whole A.

Denote by $\mathfrak{m}_n$ one of the measures on Ω with the density $\partial_k g_{ij,n}$, $\partial_k g_n^{ij}$, or $\partial_j\partial_i f_n$. Let $\mathfrak{m}$ be the corresponding limit measure $\partial_k g_{ij}$, $\partial_k g^{ij}$, or $\partial_j\partial_i f$. We need to show that

$$\int\limits_{\Omega}(h_n\circ \boldsymbol{x}_n^{-1})\cdot\varphi\cdot\mathfrak{m}_n\to\int\limits_{\Omega}(h\circ \boldsymbol{x}^{-1})\cdot\varphi\cdot\mathfrak{m}\quad\text{as } n\to\infty \tag{5-1}$$

for any continuous function $\varphi\colon\Omega\to\mathbb{R}$ with compact support.

Choose $\varepsilon>0$; let $\delta>0$ be as in Definitions 5.1 (for h_n). The set $S^\delta_\Omega=\Omega\setminus A^\delta_\Omega$ is a closed subset of Ω. By Proposition 7.3, $|\mathfrak{m}|(S^\delta_\Omega)=0$. Therefore we can choose an open neighborhood $N\subset\Omega$ of S^δ_Ω such that $|\mathfrak{m}|(N)<\varepsilon$. Choose two nonnegative continuous functions φ_0 and φ_1 such that

$$\varphi=\varphi_0+\varphi_1,\quad \operatorname{supp}\varphi_0\subset N,\quad \operatorname{supp}\varphi_1\subset A^\delta_\Omega=\Omega\setminus S^\delta_\Omega.$$

Note that the sequence $a_n=\int_\Omega(h_n\circ \boldsymbol{x}_n^{-1})\cdot\varphi_0\cdot\mathfrak{m}_n$ *converges with error* $\varepsilon_0=\varepsilon\cdot c\cdot\max\{\,|\varphi|\,\}$, where c is a bound on $|h_n|$. In other words, the upper and lower limits of a_n differ by at most ε_0. Similarly, $b_n=\int_\Omega(h_n\circ \boldsymbol{x}_n^{-1})\cdot\varphi_1\cdot\mathfrak{m}_n$ converges with error $\varepsilon_1=\varepsilon\cdot|\mathfrak{m}|\cdot c\cdot\max\{\,|\varphi|\,\}$. Since $\varepsilon>0$ is arbitrary, we get (5-1). □

5C Proof modulo a key lemma

Key lemma 5.5 *Choose a common chart with range $\Omega\subset\mathbb{R}^m$ for a smoothing $M_n\rightsquigarrow A$. Choose a component $\mathrm{Rm}_{ijsr,n}$ of the curvature tensor of M_n in Ω. Then $\mathrm{Rm}_{ijsr,n}\cdot\mathrm{vol}^m_n$ is a weakly delta-converging sequence of measures.*

The proof of the key lemma will take the remaining part of this section; in the current subsection, we show that it implies Proposition 2.10(iii).

Proof of Proposition 2.10(iii) modulo Key lemma 5.5 Recall that components of $\mathfrak{qm}_n$ can be expressed from the components of Rm_n. Therefore, the key lemma implies delta-convergence of components of $\mathfrak{qm}_n$.

Choose sequences of test functions $f_{1,n},\dots,f_{m-2,n}$, $h_{1,n},\dots,h_{m-2,n}$ on M_n that test-converge to $f_1,\dots,f_{m-2},h_1,\dots,h_{m-2}\colon A\to\mathbb{R}$. By Lemma 5.3, we have delta-convergence of the partial derivatives $\partial_i f_{j,n}$ and $\partial_i h_{j,n}$ to $\partial_i f_j$ and $\partial_i h_j$, respectively. The measures $\mathfrak{qm}_n(f_{1,n},\dots,f_{m-2,n},h_{1,n},\dots,h_{m-2,n})$ can be expressed as a linear combination of the components of $\mathfrak{qm}_n$ with coefficients expressed in terms of $\partial_i f_{j,n}$. By Observation 5.2, it follows that the sequence of measures

$$\mathfrak{m}_n=\mathfrak{qm}_n(f_{1,n},\dots,f_{m-2,n},h_{1,n},\dots,h_{m-2,n})$$

is delta-converging.

Finally, recall that

$$A^\circ = \bigcap_{\delta>0} A^\delta.$$

Therefore delta-convergence of $\mathfrak{qm}_n(f_{1,n},\dots,f_{m-2,n},h_{1,n},\dots,h_{m-2,n})$ implies Proposition 2.10(iii). $\square$

5D Strange curvature

Suppose M is a 3–dimensional Riemannian manifold. *Strange curvature tensor* Str on M is a bilinear form that is uniquely defined by

$$\mathrm{Str}(w,w) = \mathrm{Sc}\cdot|w|^2 - \mathrm{Ric}(w,w)$$

for $w \in \mathrm{T}M$. Note that Str completely describes the Ricci curvature tensor Ric. Further, since M is 3–dimensional, Str completely describes the curvature tensor Rm of M.

In Riemannian manifolds, we can (and will) use the metric tensor to identify tangent and cotangent bundles. Therefore the tensor Str can be applied to vector fields and forms; in particular, for any smooth function f we have

$$\mathrm{Str}(df,df) = \mathrm{Str}(\nabla f,\nabla f).$$

Proposition 5.6 *Let $M_n \underset{\sim}{\longrightarrow} A$ and $\dim A = 3$; choose a common chart with range $\Omega \subset \mathbb{R}^3$. Suppose that f is a convex combination of coordinate functions of the chart. Then the measures*

$$\mathfrak{m}_n = \mathrm{Str}_n(df,df)\cdot\mathrm{vol}_n^3$$

are weakly delta-converging in Ω.

The definition of strange curvature tensor is motivated by the following integral expression from Theorem 6.1:

$$\int_\Omega \varphi\cdot\mathrm{Str}(u,u) = \int_\Omega \varphi\cdot\mathrm{Int} + \int_\Omega [H\cdot\langle u,\nabla\varphi\rangle - \langle\nabla\varphi,\nabla_u u\rangle], \tag{5-2}$$

where

- $u = \nabla f/|\nabla f|$,
- $H(x)$ is the mean curvature of the level set $f^{-1}(f(x))$,
- $\mathrm{Int}(x)$ is the scalar curvature of $f^{-1}(f(x))$.

This formula is the main tool in the proof of the proposition. It reduces the proposition to the following two lemmas; each lemma provides the convergence of an integral term in the right-hand side of (5-2).

Lemma 5.7 *In the assumptions of Proposition 5.6, $\mathrm{Int}_n\cdot\mathrm{vol}_n^3$ is a delta-converging sequence of measures on Ω.*

The proof of this lemma uses the convergence of curvature measures $\mathrm{Int}_n\cdot\mathrm{vol}^2$ on the 2–dimensional level sets of concave functions f and the coarea formula. Recall that the sequence $|\nabla_n f|$ is only weakly delta-convergent (see Lemma 5.3(iv)). Since the factor $|\nabla_n f|$ appears in the coarea formula, we get that $\mathrm{Int}_n \cdot \mathrm{vol}_n^3$ is only weakly delta-convergent.

Proof Recall that any point in an Alexandrov space A has a convex neighborhood [22]. This construction can be lifted to the smoothing sequence (M_n). Let $V \subset A$ be an open convex neighborhood of x and $V_n \subset M_n$ be open convex sets such that $\overline{V}_n \xrightarrow[\mathrm{GH}]{} \overline{V}$.

Set

$$\begin{aligned} L_{t,n} &= f^{-1}(t) \cap V_n, & C_{t,n} &= f^{-1}[t,\infty) \cap \overline{V}_n, \\ L_t &= f^{-1}(t) \cap V, & C_t &= f^{-1}[t,\infty) \cap \overline{V}. \end{aligned}$$

For every t and n, the set $C_{t,n}$ is a convex subset in Alexandrov space and hence is an Alexandrov space with curvature $\geqslant -1$. Note that $C_{t,n} \xrightarrow[\mathrm{GH}]{} C_t$. Let us equip the boundaries $\partial C_{t,n}$ and ∂C_t with the induced inner metrics. By [20, Theorem 1.2], $\partial C_{t,n}$ converges to ∂C_t as $n \to \infty$.

By [2], $\partial C_{t,n}$ is an Alexandrov space with curvature $\geqslant -1$; hence, so is the limit ∂C_t.

Note that $L_{t,n}$ with induced inner metric is isometric to its image in $\partial C_{t,n}$. Since ∂C_t is an extremal subset of C_t, the inner metric of ∂C_t is bi-Lipschitz to the metric restricted from A. It follows that we can take r sufficiently small such that for all t and $U_{t,n} = L_{t,n} \cap B(x_n, r)$ we will have

$$h_n \leqslant \tfrac{1}{10} \cdot \mathrm{dist}(U_{t,n}, \partial C_{t,n} \setminus L_{t,n}),$$

where h_n denotes the intrinsic diameter of $U_{t,n}$. Then the local version of the 2–dimensional case of the main theorem can be applied to $U_{t,n}$; it implies weak convergence of measures $\mathrm{Int}_n \cdot \mathrm{vol}_n^2$ on $L_{t,n}$.

Choose a smooth function $\varphi\colon B(x_n, r) \to \mathbb{R}$ with a compact support in A_Ω^δ. Applying the coarea formula, we get

$$\int\limits_{s \in \Omega} \mathrm{Int}_n(s) \cdot \varphi(s) \cdot \mathrm{vol}_n^3 = \int\limits_{-h}^{h} dt \cdot \int\limits_{s \in U_{t,n}} \frac{\varphi_n(s)}{|\nabla_n f(s)|} \cdot \mathrm{Int}_n(s) \cdot \mathrm{vol}^2 . \tag{5-3}$$

Note that $\nabla_n f$ is bounded away from zero. By Lemma 5.3(iv), $1/|\nabla_n f(s)|$ is uniformly delta-converging. Recall that $\mathrm{Int}_n \cdot \mathrm{vol}^2$ are weakly converging measures on L_n [1, VII §13]. Therefore Observation 5.2 implies that $\mathrm{Int}_n \cdot \mathrm{vol}_n^3$ are weakly delta-converging measures on M_n. □

The following lemma is related to the convergence of the second integral in (5-2), the proof uses the DC calculus in a common chart; see Section 7.

Lemma 5.8 *In the assumptions of Proposition 5.6, suppose $\varphi\colon \Omega \to \mathbb{R}$ is a smooth function with compact support. Then*

$$\int\limits_{\Omega} \big[H_n \cdot \langle u_n, \nabla_n \varphi\rangle_n - \langle \nabla_{u_n} u_n, \nabla_n \varphi\rangle_n\big] \cdot \mathrm{vol}_n^3 \tag{5-4}$$

converges, where H_n and u_n as in (5-2).

Proof Note that $H_n = \operatorname{div} u_n$. Let us rewrite the first term of (5-4) in coordinates:

$$\int\limits_{\Omega}\Big[\sum_i(\partial_i u_n^i + \tfrac{1}{2}u_n^i\cdot\partial_i \log\det g_n)\Big]\cdot\Big[\sum_{i,j} u_n^i\cdot\partial_i\varphi\Big]\cdot\sqrt{\det g_n}\cdot dx^1\,dx^2\,dx^3.$$

We also have

$$u_n^i = \frac{\sum_j g^{ij}\cdot\partial_j f}{\sqrt{\sum_{j,k} g^{jk}\cdot\partial_j f\cdot\partial_k f}}.$$

Taking the derivatives, we see under the integral a sum of products the following two types of expressions: the first a partial derivative $\partial_k g_{ij,n}$, $\partial_k g_n^{ij}$, or $\partial_i\partial_j f$, and the second is an expression made from $g_{ij,n}$, g_n^{ij}, $\partial_i f$, $\partial_i\varphi$. Applying Lemmas 5.3 and 5.4, we get that the integral converges.

Further, for the second term in (5-4) we have

$$\int\limits_{M_n}\langle\nabla\varphi_n,\nabla_{u_n}u_n\rangle\cdot\mathrm{vol}_n^3$$
$$=\int\limits_{\Omega} dx^1\,dx^2\,dx^3\cdot\sum_{i,j,k} u_n^i\cdot\partial_k\varphi\cdot\sqrt{\det g_n}\cdot\Big(\partial_i u_n^k + \tfrac{1}{2}u_n^j\cdot\sum_s(\partial_i g_{js,n}+\partial_j g_{si,n}-\partial_s g_{ij,n})\cdot g_n^{ks}\Big).$$

The convergence follows by the same argument. □

Proof of Proposition 5.6 By (5-2) and Lemmas 5.7, and 5.8 we get that $\mathrm{Str}_n(u_n,u_n)\cdot\mathrm{vol}_n^3$ is a weakly delta-converging sequence of measures. It remains to apply Lemma 5.3(iv) and Observation 5.2. □

5E Three-dimensional case

In this section, we prove Key lemma 5.5 in the 3–dimensional case.

Vectors $w_1,\dots,w_{m(m+1)/2}\in\mathbb{R}^m$ are said to be *in general position* if the vectors $w_i\otimes w_i$ form a basis in $\mathbb{R}^{m\cdot(m+1)/2}$ — the symmetric square of $\mathbb{R}^m$. In this case, any quadratic form Q on $\mathbb{R}^m$ can be computed from the $m(m+1)/2$ values

$$Q(w_1,w_1),\quad\dots,\quad Q(w_{m(m+1)/2},w_{m(m+1)/2}).$$

More precisely, there are rational functions $s_1,\dots,s_{m(m+1)/2}$ that take $m(m+1)/2$ vectors in $\mathbb{R}^m$ and return a quadratic form on $\mathbb{R}^m$ such that

$$Q=\sum_{k=1}^{m(m+1)/2} s_k(w_1,\dots,w_{m(m+1)/2})\cdot Q(w_k,w_k). \tag{5-5}$$

Note that the vectors $w_1,\dots,w_{m(m+1)/2}\in\mathbb{R}^m$ are in general position if and only if $s_k(w_1,\dots,w_{m(m+1)/2})$ are finite for all k. Since s_k are rational functions, we get:

Observation 5.9 *Suppose that vectors $w_1,\dots,w_{m(m+1)/2}\in\mathbb{R}^m$ are in general position. Then the functions $s_1,\dots,s_{m(m+1)/2}$ are Lipschitz in a neighborhood of $(w_1,\dots,w_{m(m+1)/2})\in(\mathbb{R}^m)^{m(m+1)/2}$.*

Proof of the 3–dimensional case of Key lemma 5.5 Choose a common chart

$$M_n \supset U_n \to \Omega, \quad \text{and} \quad A \supset U \to \Omega.$$

Let us use it to identify tangent spaces of M_n and A with $\mathbb{R}^3$.

Choose 6 sequences of convex combinations of coordinate functions $f_1, \dots, f_6$, such that $\nabla f_1, \dots, \nabla f_6$ are in general position at $p \in \Omega$. We can assume that Ω is a small neighborhood of p, so by Proposition 5.6 the measures $\operatorname{Str}_n(\nabla_n f_k, \nabla_n f_k) \cdot \operatorname{vol}_n^3$ weakly delta-converges on A_Ω^δ for $k = 1, \dots, 6$.

By Observation 5.9, the functions s_i are Lipschitz in a neighborhood of $(\nabla f_1, \dots, \nabla f_6) \in (\mathbb{R}^3)^6$. Applying (5-5), we get that

$$\operatorname{Str} = \sum_{k=1}^{6} s_k(\nabla_n f_1, \dots, \nabla_n f_6) \cdot \operatorname{Str}_n(\nabla_n f_k, \nabla_n f_k).$$

Hence the measure $\operatorname{Str}_n(dx_i, dx_j) \cdot \operatorname{vol}_n^3$ are weakly delta-converging for all i and j, where x_1, x_2, x_3 is the standard coordinates in $\mathbb{R}^3$.

By Lemma 5.3, the sequence of metric tensors g_n of M_n on Ω is uniformly delta-converging. Since the following equality

$$\operatorname{Tr}\operatorname{Str}_n = \sum_{i,j} g_{ij,n} \cdot \operatorname{Str}_n(dx_i, dx_j)$$

holds almost everywhere, we get that the sequence of measures $\operatorname{Tr}\operatorname{Str}_n \cdot \operatorname{vol}_n^3$ is weakly delta-converging.

Note that for 3–dimensional manifolds we have

$$\operatorname{Qm}_n(V, V) = \operatorname{Str}_n(V, V) - \tfrac{1}{4} \cdot |V|^2 \cdot \operatorname{Tr}\operatorname{Str}_n. \tag{5-6}$$

Hence the measures $\operatorname{Qm}_n(dx_i, dx_j) \cdot \operatorname{vol}_n^3$ are weakly delta-converging for all i and j.

Finally, according to Lemma 5.3(ii), the components $\alpha_{ik,n}$ of $\nabla_n f_k$ are uniformly delta-converging. The result follows since

$$\mathfrak{qm}(f_k, f_k) = \sum_{i,j} \alpha_{ik,n} \cdot \alpha_{jk,n} \cdot \mathfrak{qm}(x_i, x_j). \qquad \square$$

5F Higher-dimensional case

Observation 5.10 *Choose a common chart with the range $\Omega \subset \mathbb{R}^m$ for a smoothing $M_n \rightsquigarrow A$. Consider the sequence of coordinate level sets $\Omega = L_m \supset L_{m-1} \supset \cdots \supset L_0$, where $L_i = L_i(c_{i+1}, \dots, c_m)$ is defined by setting the last $m - i$ coordinates to be $c_{i+1}, \dots, c_m$, respectively. Then each level set L_i is a smooth convex hypersurface in L_{i+1} in each M_n; in particular, each L_i has sectional curvature bounded below by -1.*

Moreover, there is an open set O in the space of linear transformations of $\mathbb{R}^n$ such that the same holds after applying any linear transformation $T \in O$ to Ω.

Proof of Key lemma 5.5 Let us use notations as in the observation. By Key lemma 5.5 in dimensions 2 and 3, we get weak delta-convergence of curvature tensors on L_2 and L_3. (Again, we apply the local version of these statements as described in Section 9.) In particular, applying the coarea formula, we get convergence of sectional curvatures of L_3 in the directions of L_2 as well as the sectional curvature of L_2 for all values $c_3, \dots, c_m$. The difference between these curvatures is the Gauss curvature G_n of L_2 as a submanifold in L_3. Therefore, G_n is weakly delta-converging as well.

Consider a linear transformation of Ω that preserves the direction of L_2. By the last statement in Observation 5.10, the above argument shows weak delta-convergence of $G_n(w)$, where the direction w of L_3 on L_2 can be chosen in an open set of $\mathbb{R}^{m-2}$ — the space transversal to L_2. In particular, we may choose directions $w_1, \dots, w_{(m-2)\cdot(m-1)/2}$ in $\mathbb{R}^{m-2}$ that form a generic set (see the definition in Section 5E).

Denote by G_n^+ the term in the Gauss formula for L_2 in M_n; that is, G_n^+ is the difference between the curvature of L_2 and the sectional curvature of M_n in the same direction. Denote by g_n the Riemannian metric of M_n in Ω. Note that

$$G_n^+ = \sum \alpha_{k,n} \cdot G_n(w_k),$$

where the coefficients $\alpha_{k,n}$ depend continuously on $w_1, \dots, w_{(m-2)\cdot(m-1)/2}$, and the components of g_n. It follows that weak delta-convergence of $G_n(w_k)$ implies weak delta-convergence of G_n^+ as $n \to \infty$. Since the curvature of L_2 is weakly delta-converging, it implies weak delta-convergence of sectional curvature in the direction of L_2.

By the second statement in the observation, the above argument can be repeated after applying a linear transformation of Ω that changes the direction of L_2 slightly. It follows that sectional curvatures converge for a generic array of simple bivectors in $\mathbb{R}^m$. Note that the curvature tensor can be expressed from these sectional curvatures and the metric tensor. Hence, the weak delta-convergence of components of curvature tensor and therefore dual curvature tensor follows. □

Details

6 Bochner formula

Let M be a Riemannian m–manifold and $f : M \to \mathbb{R}$ be a smooth function without critical points on an open domain $\Omega \subset M$. Set $u = \nabla f / |\nabla f|$. Let us define $\mathrm{Int}_f(x)$ (or just Int) to be scalar curvature of the level set $L_x = f^{-1}(f(x))$ at $x \in L_x \subset M$. Set

(1) $\kappa_1(x) \leqslant \kappa_2(x) \leqslant \dots \leqslant \kappa_{m-1}(x)$, as the principal curvatures of L_x at x;

(2) $H = H_f(x) = \kappa_1 + \kappa_2 + \dots + \kappa_{m-1}$, the mean curvature of L_x at x;

(3) $G = G_f(x) = 2 \cdot \sum_{i<j} \kappa_i \cdot \kappa_j$, the extrinsic term in the Gauss formula for $\mathrm{Int}_f(x)$.

Recall that the strange curvature Str is defined as

$$\operatorname{Str}(u) = \operatorname{Sc} - \langle \operatorname{Ric}(u), u\rangle,$$

where Sc and Ric denote scalar and Ricci curvature respectively.

Theorem 6.1 (Bochner's formula) *Let M be an m–dimensional Riemannian manifold, $f\colon M\to\mathbb{R}$ be a smooth function without critical points on an open domain $\Omega\subset M$, and $u = \nabla f/|\nabla f|$. Assume $\varphi\colon \Omega\to\mathbb{R}$ is a smooth function with compact support. Then*

$$\int\limits_{\Omega} \varphi\cdot\langle \operatorname{Ric} u, u\rangle = \int\limits_{\Omega} [\varphi\cdot G + H\cdot\langle u,\nabla\varphi\rangle - \langle \nabla\varphi, \nabla_u u\rangle] \tag{6-1}$$

and

$$\int\limits_{\Omega} \varphi\cdot\operatorname{Str}(u) = \int\limits_{\Omega} [H\cdot\langle u,\nabla\varphi\rangle - \langle \nabla\varphi, \nabla_u u\rangle] + \int\limits_{\Omega} \varphi\cdot\operatorname{Int}_f\,. \tag{6-2}$$

The following calculations are based on [11, Chapter II]. The Dirac operator will be denoted by D. We use the Riemannian metric to identify differential forms and multivector fields on M. Therefore the statement about differential forms can be also formulated in terms of multivector fields and the other way around.

Proof of Theorem 6.1 Assume $b_1,\dots,b_m$ is an orthonormal frame such that $b_m = u$. Then

$$\operatorname{Sc} - 2\cdot\langle \operatorname{Ric}(u), u\rangle = 2\cdot\sum_{i<j<m}\sec(b_i\wedge b_j).$$

Therefore the Gauss formula can be written as

$$\operatorname{Int} = G + \operatorname{Sc} - 2\cdot\langle \operatorname{Ric}(u), u\rangle = G + \operatorname{Str}(u) - \langle \operatorname{Ric}(u), u\rangle. \tag{6-3}$$

We can assume that $b_i(x)$ points in the principal directions of L_x for $i<m$; so we have $\nabla_{b_i}u = \kappa_i\cdot b_i$ at x. We will denote by "$\bullet$" the Clifford multiplication; recall that $b_i\bullet b_i = -1$. Note that

$$Du = \sum_i b_i\bullet\nabla_{b_i}u = \sum_{i<m}\kappa_i\cdot b_i\bullet b_i + u\bullet\nabla_u u = -H + u\bullet\nabla_u u.$$

Since $\langle\nabla_u u, u\rangle = 0$, we get $H\perp(u\bullet\nabla_u u)$. Therefore

$$\langle Du, Du\rangle = \Big(\sum_{i<m}\kappa_i\Big)^2 + |u\bullet\nabla_u u|^2 = H^2 + |\nabla_u u|^2.$$

On the other hand

$$\nabla u = \sum_{i<m}\kappa_i\cdot b_i\otimes b_i + \nabla_u u\otimes u,$$

hence

$$\langle\nabla u,\nabla u\rangle = \sum_{i<m}\kappa_i^2 + |\nabla_u u|^2.$$

Therefore

$$\langle Du, Du\rangle - \langle\nabla u,\nabla u\rangle = 2\cdot\sum_{i<j}\kappa_i\cdot\kappa_j = G.$$

Following the calculations in [11, II.5.3], we get

$$\int_\Omega \varphi\cdot[\langle Du, Du\rangle - \langle D^2u, u\rangle] = -\int_\Omega \langle\nabla\varphi \bullet u, Du\rangle = -\int_\Omega [H\cdot\langle\nabla\varphi, u\rangle - \langle\nabla\varphi, \nabla_u u\rangle].$$

Since $|u|\equiv 1$, we have $\langle\nabla_{\nabla\varphi}u, u\rangle = 0$. Therefore

$$\int_\Omega \varphi\cdot[\langle\nabla u, \nabla u\rangle - \langle\nabla^*\nabla u, u\rangle] = \int_\Omega \langle\nabla_{\nabla\varphi}u, u\rangle = 0.$$

By the Bochner formula [11, II.8.3],

$$D^2u - \nabla^*\nabla u = \operatorname{Ric}(u);$$

in particular,

$$\varphi\cdot\langle D^2u, u\rangle - \varphi\cdot\langle\nabla^*\nabla u, u\rangle = \varphi\cdot\langle\operatorname{Ric}(u), u\rangle. \tag{6-4}$$

Integrating (6-4) and applying the derived formulas, we get

$$\int_\Omega \varphi\cdot G = \int_\Omega \varphi\cdot[\langle Du, Du\rangle - \langle\nabla u, \nabla u\rangle] = \int_\Omega \varphi\cdot\operatorname{Ric}(u,u) - \int_\Omega [H\cdot\langle u, \nabla\varphi\rangle - \langle\nabla\varphi, \nabla_u u\rangle].$$

It remains to apply the Gauss formula (6-3). □

7 DC-calculus

Let f be a continuous function defined on an open domain of an m–dimensional Alexandrov space A. Recall that f is *DC* if it can be presented locally as a difference between two concave functions. Recall that for any point $p\in A$ there is a (-1)–concave function defined in a neighborhood of p [17, 3.6]. Therefore we can say that *f is DC if and only if it can be presented locally as a difference between two semiconcave functions.*

Suppose that a sequence of Alexandrov spaces A_n converges to Alexandrov space A without collapse. Let f_n and f be DC functions defined on open domains $\operatorname{Dom} f_n\subset A_n$ and $\operatorname{Dom} f\subset A$. Suppose that for any $p\in\operatorname{Dom} f$ there is a sequence $p_n\in\operatorname{Dom} f_n$ and $R>0$ such that $p_n\to p$ and $B(p_n,R)_{A_n}\subset\operatorname{Dom} f_n$, $B(p,R)_A\subset\operatorname{Dom} f$ and for some fixed $\lambda\in\mathbb{R}$, and each large n we have λ–concave functions a_n and b_n defined in $B(p_n,R)_{A_n}$ and λ–concave functions a and b defined in $B(p,R)_A$ such that $f_n = a_n - b_n$ and $f = a - b$ and the sequences a_n and b_n converge to functions a and b respectively. In this case, we say that f_n is *DC-converging* to $f = a-b\colon A\to\mathbb{R}$ as $n\to\infty$; briefly $f_n \xrightarrow[\mathrm{DC}]{} f$.

A DC function $f\colon A\to\mathbb{R}$ is called *DC_0* if it is continuously differentiable in A°. More preciously, for any smoothed distance chart $\boldsymbol{x}\colon U\subset A\to\mathbb{R}^m$ (see Section 5A) the restriction $f\circ\boldsymbol{x}^{-1}|_{\boldsymbol{x}(A^\circ)}$ is continuously differentiable.

Observation 7.1 *Any test function is* DC_0*. Moreover, test convergence implies DC-convergence.*

Proof Choose a test function $f=\varphi(\widetilde{\operatorname{dist}}_{p_1,r},\dots,\widetilde{\operatorname{dist}}_{p_n,r})$. Note that the function φ can be presented locally as a difference between C^2–smooth concave functions increasing in each argument; say $\varphi=\psi-\chi$.

For the first part of the observation, it remains to observe that the functions

$$a=\psi(\widetilde{\operatorname{dist}}_{p_1,r},\dots,\widetilde{\operatorname{dist}}_{p_n,r}),\quad b=\chi(\widetilde{\operatorname{dist}}_{p_1,r},\dots,\widetilde{\operatorname{dist}}_{p_n,r})$$

are semiconcave and continuously differentiable in A°.

Suppose that a sequence of functions φ_i is C^2–converging to φ. Choose $x=(x_1,\dots,x_n)$ in the domain of definition of φ. Note that φ_n and its partial derivatives up to order 2 are bounded; fix a bound c. Then in a neighborhood of $(x_1,\dots,x_n)$ we may choose ψ_n that is uniquely defined by $\psi_n(x)=0$, $\partial_i\psi_n(x)=2\cdot c$, $\partial_i\partial_j\psi_n\equiv 0$ for $i\ne j$, and $\partial_i^2\psi_n\equiv -d$ for a large constant d. In this case, $\chi_n=\psi_n-\varphi_n$ is concave. Moreover, C^2–convergence of φ_n implies convergence of ψ_n and χ_n. Hence, the second statement follows. □

The definition of DC-convergence extends naturally to sequences of functions defined on a fixed domain $\Omega\subset\mathbb{R}^m$. The proof of the following statement is a straightforward modification of [18, Section 3]:

Proposition 7.2 *Let* $M_n \underset{\sim}{\longrightarrow} A$; *choose a common chart* $\boldsymbol{x}_n\colon U_n\subset M_n\to\Omega$, $\boldsymbol{x}\colon U\subset A\to\Omega\subset\mathbb{R}^m$.

Consider functions f_n *and* f *defined on* U_n *and* U*, respectively. Then*

$$f_n \xrightarrow[\mathrm{DC}]{} f \quad \text{if and only if} \quad f_n\circ\boldsymbol{x}_n^{-1} \xrightarrow[\mathrm{DC}]{} f\circ\boldsymbol{x}^{-1}.$$

The following statement follows from the lemma in [18, Section 4].

Proposition 7.3 *Let* A *be an* m*–dimensional Alexandrov space and* $\boldsymbol{x}\colon U\to\mathbb{R}^m$ *— a smoothed chart for* $U\subset A$*. Denote by* g_{ij} *components of metric tensors in this chart and by* g^{ij} *components of the inverse matrix. Let* $f\colon U\to\mathbb{R}$ *be a* DC_0 *function.*

Then the partial derivatives $\partial_k g_{ij}$, $\partial_k g^{ij}$, $\partial_i\partial_j f$ *are Radon measures on* A *that vanish on* $\boldsymbol{x}^{-1}(A'\cup A'')$.

Theorem 7.4 *Let* $M_n \underset{\sim}{\longrightarrow} A$, $\dim A=m$; *choose a common chart* $\boldsymbol{x}_n$ *defined on* $U_n\subset M_n$, $\boldsymbol{x}$ *defined on* $U\subset A$ *with a common range* $\Omega\subset\mathbb{R}^m$*. Denote by* $g_{ij,n}$ *components of metric tensors in this chart and by* g_n^{ij} *components of the inverse matrix. Let* $f_n\colon U_n\to\mathbb{R}$ *be a sequence of DC function that DC-converges to a* DC_0 *function* $f\colon U\to\mathbb{R}$*. Then partial derivatives* $\partial_k g_{ij,n}$, $\partial_k g_n^{ij}$, $\partial_i\partial_j f_n$ *weakly converge to the Radon measures* $\partial_k g_{ij}$, $\partial_k g^{ij}$, $\partial_i\partial_j f$ *described in Proposition 7.3.*

By Observation 7.1, the theorem applies to any test-converging sequence $f_n\xrightarrow{\text{test}} f$. In the proof, we will modify the argument in [18, Section 4] slightly.

Proof Let's start with the partial derivatives of metric tensors. In [18, Subsection 4.2], it was shown that components of metric tensors can be expressed as a rational function of partial derivatives of distance functions to a finite collection of points. The distance functions are semiconcave, in particular DC.

The base points $p_{i,n} \in M_n$ of these distance functions can be chosen so that they converge to some point $p_i \in A$. In this case, the distance functions are DC-converging. Now, applying Proposition 7.2, we get the statement.

The case of $\partial_i \partial_j f_n$ is similar. □

8 Bi-Lipschitz covering

In this section we will prove Lemma 4.1. A more general version of the lemma can be proved along the same lines as Lemma 11.1 in [26].

Note that the lemma follows from the next proposition.

Proposition 8.1 *Let A be an m–dimensional Alexandrov space with curvature at least -1 and $p \in A'$. Then there is a compact set Q such that*

(i) *Q admits a bi-Lipschitz embedding into $\mathbb{R}^{m-2}$ and*

(ii) *there is a neighborhood $U \ni p$ and $\varepsilon > 0$ such that $q \in Q$ for any point $q \in U \cap A'$ such that*
$$\theta(q) < \theta(p) + \varepsilon.$$

Let x be a point in an Alexandrov space A with curvature at least -1. Recall that Bishop–Gromov inequality implies that
$$\frac{\mathrm{vol}^m B(x,R)_A}{\mathrm{vol}^m B(\tilde{x},R)_{\mathbb{H}^m}} \leqslant \frac{\mathrm{vol}^{m-1}\Sigma_x}{\mathrm{vol}^{m-1}\mathbb{S}^{m-1}}$$
for any $R > 0$; here $\mathbb{H}^m$ denotes the m–dimensional hyperbolic space. The following lemma makes this inequality more precise.

Lemma 8.2 *Let x be a point in an m–dimensional Alexandrov space A with curvature at least -1. Suppose $y \in A$ is a point such that $|x-y| < R$ and $\measuredangle[y\,{}^{x}_{z}] < \pi - \varepsilon$ for any point z. Then*
$$\frac{\mathrm{vol}^m B(x,R)_A}{\mathrm{vol}^m B(R)_{\mathbb{H}^m}} \leqslant (1-\delta)\cdot\frac{\mathrm{vol}^{m-1}\Sigma_x}{\mathrm{vol}^{m-1}\mathbb{S}^{m-1}},$$
where δ is a positive number that depends on m, $|x-y|$, R and ε.

Proof To simplify the presentation we will assume that A is nonnegatively curved; it is straightforward to adapt the proof to the general case. In this case, we need to show that

$$\frac{\mathrm{vol}^m B(x,R)_A}{\mathrm{vol}^m B(R)_{\mathbb{R}^m}} \leqslant (1-\delta)\cdot\frac{\mathrm{vol}^{m-1}\Sigma_x}{\mathrm{vol}^{m-1}\mathbb{S}^{m-1}},$$

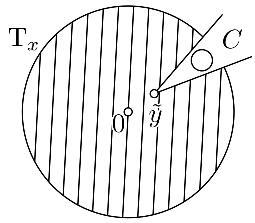
Let us denote by $\tilde p$ a vector in T_x that is tangent to a geodesic path $\gamma\colon[0,1]\to A$ from x to p. By comparison, the map $p\mapsto\tilde p$ is a distance-noncontracting map.

Since $\measuredangle[y\,{}^x_z]<\pi-\varepsilon$ for any z, the image of the map $p\mapsto\tilde p$ does not include points in a cone C behind $\tilde y$ of angle ε. It follows that

$$\mathrm{vol}^m(B(0,R)_{\mathrm{T}_x}\setminus C)\geqslant\mathrm{vol}^m(B(x,R)_A)$$

for any $R>0$.

Since $R>|x-y|$, the intersection $C\cap B(0,R)_{\mathrm{T}_x}$ includes a ball of a certain radius $r>0$ that can be found in terms of $|x-y|$, R and ε. By the Bishop–Gromov inequality, we get $\delta=\delta(m,|x-y|,R,\varepsilon)>0$ such that



$$\frac{\mathrm{vol}^m(C\cap B(0,R)_{\mathrm{T}_x})}{\mathrm{vol}^m(B(0,R)_{\mathrm{T}_x})}>\delta.$$

Further, observe that

$$\frac{\mathrm{vol}^m(B(0,R)_{\mathrm{T}_x})}{\mathrm{vol}^m(B(0,R)_{\mathbb{R}^m})}=\frac{\mathrm{vol}^{m-1}\Sigma_x}{\mathrm{vol}^{m-1}\mathbb{S}^{m-1}},$$

whence the lemma. □

Proof of Proposition 8.1 Since the tangent cone at p has $\mathbb{R}^{m-2}$–factor, we can choose points $a_1,\dots,a_{m-2}$, $b_1,\dots,b_{m-2}$ that are δ–strainers of p for arbitrary $\delta>0$. The corresponding distance map

$$s\colon x\mapsto(|a_1-x|,\dots,|x-a_{m-2}|)$$

is an almost submersion of a neighborhood $U\ni p$ to $\mathbb{R}^{m-2}$. Choose small $\varepsilon>0$ and set

$$Q'=\{x\in U\cap A'\mid\theta(x)<\theta(p)+\varepsilon\}.$$

Let us show that $s|_{Q'}$ is bi-Lipschitz. Once it is done, passing to the closure $Q=\bar Q'$ gives the required set.

Note that for some $R>0$ the ball $B(p,10\cdot R)_A$ is almost isometric to the ball $B(0,10\cdot R)_{\mathrm{T}_p}$ and we can assume that $U\subset B(p,R)_A$. By the volume convergence (see [5, 10.8]) and Bishop–Gromov inequality, we can assume that

$$\mathrm{vol}^m B(x,R)_A>\tfrac{\theta(p)-\varepsilon}{2\cdot\pi}\cdot\mathrm{vol}^m B(0,R)_{\mathbb{H}^m}$$

for any $x\in U$; here $\mathbb{H}^m$ denotes the m–dimensional hyperbolic space.

Assume x and y in Q'. Since ε is small, the lemma implies that there is $z\in A$ such that $\measuredangle[y\,{}^x_z]$ is near π. It follows that $\Uparrow_{[yx]}$ lies very close to the $\mathbb{R}^{m-2}$–factor in T_y. The same way we can show that $\Uparrow_{[xy]}$ lies very close to the $\mathbb{R}^{m-2}$–factor in T_x. In other words $[xy]$ lies nearly horizontally with respect to almost submetry s. In particular,

$$|s(x)-s(y)|_{\mathbb{R}^{m-2}} \lessgtr \lambda^{\pm 1}\cdot|x-y|_A$$

for some constant $\lambda > 1$. (In fact, we can take λ arbitrarily close to 1, but we do not need it.) □

9 Localization

In this section we formulate a local version of the main theorem. This version is more general, but its proof requires just a slight change of language. A couple of times we had to use this local version in the proof. In a perfect world, we had to rewire the whole paper using this language. However, this is not a principle moment, so we decided to keep the paper more readable at the cost of being not fully rigorous. A more systematic discussion of this topic is given in [12].

First, we need to define Alexandrov region; its main example is an open set in Alexandrov space.

Definition 9.1 Let A be a locally compact metric space. We say that a point $p\in A$ is *ε–inner* if the closed ball $\bar{B}(x,2\cdot\varepsilon)$ is compact.

Definition 9.2 We say that a locally compact inner metric space A of finite Hausdorff dimension is an *Alexandrov region* if any point has a neighborhood where the Alexandrov comparison for curvature $\geqslant -1$ holds.

The *comparison radius* $r_c(p)$ for $p\in A$ is defined as the maximal number r such that p is r–inner point and Alexandrov comparison for curvature $\geqslant -1$ holds in $B(x,r)$.

Any point p in an Alexandrov region admits a convex neighborhood. Moreover, its size can be controlled in terms of dimension, $r_c(p)$, and a lower bound on the volume of ball $B(p,r_c)$. The construction is the same as for Alexandrov space [19, 4.3].

By the globalization theorem (see, for example, [3]), a compact convex subset in an Alexandrov region is an Alexandrov space. So the statement above makes it possible to apply most of the arguments and constructions for Alexandrov spaces to Alexandrov regions. Moreover, in the case when an Alexandrov region is a Riemannian manifold (possibly noncomplete) it is possible to take the doubling of a convex neighborhood from the proposition and smooth it with almost the same lower curvature bound. This allows us to apply the main result from [23], where the complete manifold can be replaced by a convex domain in a possibly open manifold.

Further, let us define a local version of smoothing. Let us denote by $\mathcal{M}^m_{\geqslant -1}$ a class of m–dimensional Riemannian manifolds without boundary, but possibly noncomplete, with sectional curvature bounded from below by -1.

Definition 9.3 Let $M_n \in \mathcal{M}^m_{\geqslant -1}$ (with corresponding intrinsic metric) converge in Gromov–Hausdorff sense to some metric space A via approximation. Suppose that $M_n \ni x_n \to x \in A$, $\dim A = m$, and $r_c(x_n) \geqslant R > 0$. Let $U_n = B(x_n, R)_{M_n}$. Then we say that U_n is a local smoothing of $U = B(x, R)_A$ (briefly, $U_n \underset{\sim}{\longrightarrow} U$).

It is straightforward to redefine test functions and weak convergence for local smoothings. Using this language we can make a local version for each statement in this paper; the proofs go without changes. As a result, we get the following local version of the main theorem.

Theorem 9.4 (local version of Theorem 2.6) *Let $M_n \in \mathcal{M}^m_{\geqslant -1}$, $M_n \xrightarrow[\mathrm{GH}]{} A$, $U_n \subset M_m$, $U \subset A$, and $U_n \underset{\sim}{\longrightarrow} U$ be a local smoothing.*

Denote by $\mathfrak{qm}_n$ the dual measure-valued curvature tensor on U_n. Then there is a measure-valued tensor $\mathfrak{qm}$ on U such that $\mathfrak{qm}_n \rightharpoonup \mathfrak{qm}$.

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