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**On boundedness of singularities
and minimal log discrepancies of Kollár components, II**

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We show that a set of K -semistable log Fano cone singularities is bounded if and only if their local volumes are bounded away from zero, and their minimal log discrepancies of Kollár components are bounded from above. As corollaries, we confirm the boundedness conjecture for K -semistable log Fano cone singularities in dimension three, and show that local volumes of 3-dimensional klt singularities only accumulate at zero.

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1 Introduction

Following the recent study on K -stability of Fano varieties (see Xu [38] for a comprehensive survey), there has been growing interest in establishing a parallel stability theory for klt singularities, which are the local analog of Fano varieties. In the global theory, boundedness of Fano varieties plays an important role. Building on the seminal work of Birkar [1; 2], Jiang [18] proved that (in any fixed dimension) K -semistable Fano varieties with anticanonical volumes bounded away from zero form a bounded family. (Several different proofs were later found in Li, Liu and Xu [25] and Xu and Zhuang [39].) This is the first step in the general construction of the K -moduli space of Fano varieties; it is also a key ingredient in the proof of the global version of the Higher Rank Finite Generation Conjecture; see Liu, Xu and Zhuang [31].

To advance the local stability theory, it is therefore natural to investigate the boundedness of klt singularities. Several years ago, Li [24] introduced an interesting invariant of klt singularities called the local volume. It has become clear that the stability theory of klt singularities should be built around this invariant. In particular, it is speculated that (in any fixed dimension) klt singularities with local volumes bounded away from zero are specially bounded, i.e. they isotrivially degenerate to a bounded family. This is known in some special cases, see for example Han, Liu and Qi [16], Moraga and Süß [34; 33] and Zhuang [42], but the general situation is still quite mysterious.

By the recent solution of the Stable Degeneration Conjecture — see Blum [4], Li and Xu [27; 28], Xu [37], Xu and Zhuang [39; 40], Li, Wang and Xu [26], as well as Blum, Liu and Qi [8] — every klt singularity

has a canonical “stable degeneration” to a K -semistable log Fano cone singularity; see Section 2.4 for the precise definition, but roughly speaking, K -semistable log Fano cone singularities are generalizations of cones over K -semistable Fano varieties. This suggests a more precise boundedness conjecture:

Conjecture 1.1 (see [40, Conjecture 1.7], also [35, Problem 6.9]) *Fix $n \in \mathbb{N}$, $\varepsilon > 0$ and a finite set $I \subseteq [0, 1]$. Then the set*

$$\{(X, \Delta) \mid x \in (X, \Delta) \text{ is a } K\text{-semistable log Fano cone singularity, } \dim X = n, \text{Coef}(\Delta) \subseteq I \text{ and } \widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon\}$$

is bounded.

Here $\widehat{\text{vol}}(x, X, \Delta)$ denotes the local volume of the singularity $x \in (X, \Delta)$. This conjecture has been verified for toric singularities by Moraga and Süß [34] and Zhuang [42], for hypersurface singularities, and for singularities with torus actions of complexity one also by Moraga and Süß [33].

In this paper, we study Conjecture 1.1 using the minimal log discrepancies of Kollár components (or simply mld^K); see Section 2.2. Our main result gives a boundedness criterion in terms of mld^K and the local volume.

Theorem 1.2 (Corollary 3.10) *Fix $n \in \mathbb{N}$ and consider a set \mathcal{S} of n -dimensional K -semistable log Fano cone singularities with coefficients in a fixed finite set $I \subseteq [0, 1]$. Then \mathcal{S} is bounded if and only if there exist some $\varepsilon, A > 0$ such that*

$$\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon \quad \text{and} \quad \text{mld}^K(x, X, \Delta) \leq A$$

for all $x \in (X, \Delta)$ in \mathcal{S} .

This upgrades the special boundedness result from our previous work [42] to actual boundedness. We also prove boundedness results for more general log Fano cone singularities, replacing the K -semistability requirement by lower bounds on stability thresholds (as introduced by Huang in [17]); see Corollary 3.14.

Comparing Conjecture 1.1 and Theorem 1.2, we are naturally led to the following conjecture, already raised in [42]:

Conjecture 1.3 [42, Conjecture 1.7] *Let $n \in \mathbb{N}$, $\varepsilon > 0$ and let $I \subseteq [0, 1]$ be a finite set. Then there exists some constant A depending only on n , ε and I such that*

$$\text{mld}^K(x, X, \Delta) \leq A$$

for any n -dimensional klt singularity $x \in (X, \Delta)$ with $\text{Coef}(\Delta) \subseteq I$ and $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$.

This is known in dimension up to three [42]. By Theorem 1.2, we then get a complete solution of Conjecture 1.1 in dimension three (the surface case was already treated by Han, Liu and Qi [16]). The same result is also independently proved by Liu, Moraga and Süß [29] using a different method.

Corollary 1.4 (Corollaries 3.16 and 3.17) *Conjecture 1.1 holds in dimension 3. Moreover, for any fixed finite coefficient set $I \subseteq [0, 1]$, the set of possible local volumes of 3-dimensional klt singularities is discrete away from zero.*

1.1 Strategy of the proof

In addition to our previous work [42], the proof of Theorem 1.2 relies on several new ingredients. For simplicity, we assume $\Delta = 0$, so that we are dealing with Fano cone singularities. Every such singularity $x \in X$ is an orbifold cone over some Fano variety V , so a natural idea is to prove Theorem 1.2 by showing the boundedness of the associated Fano variety V .

There are two main reasons why this naïve approach does not directly work. First, the orbifold base is highly nonunique; in fact, for a fixed Fano cone singularity the possible orbifold bases can be unbounded. For example, the simplest Fano cone singularity $0 \in \mathbb{A}^n$ can be realized as an orbifold cone over any weighted projective space of dimension $n - 1$, but without further constraint, weighted projective spaces do not form a bounded family.

Moreover, even when there is a canonical choice of the orbifold base (e.g. when the singularity has a unique \mathbb{G}_m -action), the anticanonical volume of V is only a fraction of the local volume of the singularity, where the factor is related to the Weil index of V (defined as the largest number q such that $-K_V \sim_{\mathbb{Q}} qL$ for some Weil divisor L). In particular, the volume of V can a priori be arbitrarily small, which needs to be ruled out if we want any boundedness of this sort.

Our solution is to turn the local boundedness question into a global one by considering the projective orbifold cone \bar{X} over V . A key observation is that by choosing the appropriate orbifold base V , the anticanonical volume of \bar{X} approximates the local volume of $x \in X$ and in particular is bounded from both above and below. This takes care of the second issue mentioned above.

Still, as we make different choices of V , the corresponding projective orbifold cones \bar{X} can be unbounded. To circumvent this issue, we prove an effective birationality result for \bar{X} . More precisely, we show that regardless of the choice of V , there is some positive integer m depending only on $\widehat{\text{vol}}(x, X)$ and $\text{mld}^K(x, X)$ such that $|-mK_{\bar{X}}|$ induces a birational map that restricts to an embedding on X . This is the main technical part of the proof, and ultimately reduces to the construction of certain isolated non-klt centers and some careful analysis of certain Izumi-type constants; see Section 3.2. Once the effective birationality is established, it is fairly straightforward to conclude the boundedness of X .

1.2 Structure of the article

In Sections 2.2–2.5, we give the necessary background on mld^K , local volumes, log Fano cone singularities and their boundedness. In Sections 2.6–2.7, we collect some useful results in previous works; these include results from Han, Liu and Shokurov [15] tackling pairs with real coefficients, as well as modifications of some results from the prequel [42] of the present work. In Section 3, we present and prove a more general boundedness statement for polarized log Fano cone singularities, Theorem 3.1. Our main Theorem 1.2 will follow as an application of Theorem 3.1.

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2 Preliminaries

2.1 Notation and conventions

We work over an algebraically closed field \mathbb{k} of characteristic 0. We follow the standard terminology from [23; 21].

A pair (X, Δ) consists of a normal variety X together with an effective \mathbb{R} -divisor Δ on X (a priori, we do not require that $K_X + \Delta$ is \mathbb{R} -Cartier). A singularity $x \in (X, \Delta)$ consists of a pair (X, Δ) and a closed point $x \in X$. We will always assume that X is affine and $x \in \text{Supp}(\Delta)$ (whenever $\Delta \neq 0$).

Suppose that X is a normal variety. A prime divisor F on some birational model $\pi: Y \rightarrow X$ (where Y is normal and π is proper) of X is called a *divisor over X* . Given an \mathbb{R} -divisor Δ on X , we denote its strict transform on the birational model Y by Δ_Y . If $K_X + \Delta$ is \mathbb{R} -Cartier, the log discrepancy $A_{X,\Delta}(F)$ is defined to be

$$A_{X,\Delta}(F) := 1 + \text{ord}_F(K_Y - \pi^*(K_X + \Delta)).$$

A *valuation* over a singularity $x \in X$ is an \mathbb{R} -valued valuation $v: K(X)^* \rightarrow \mathbb{R}$ (where $K(X)$ denotes the function field of X) such that v is centered at x (i.e. $v(f) > 0$ if and only if $f \in \mathfrak{m}_x$) and $v|_{\mathbb{k}^*} = 0$. The set of such valuations is denoted as $\text{Val}_{X,x}$. Given $\lambda \in \mathbb{R}$, the corresponding *valuation ideal* $\mathfrak{a}_\lambda(v)$ is

$$\mathfrak{a}_\lambda(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq \lambda\}.$$

When we refer to a constant C as $C = C(n, \varepsilon, \dots)$ it means C only depends on n, ε, \dots

2.2 Kollár components

We first recall some definitions related to klt singularities and Kollár components.

Definition 2.1 [23, Definition 2.34] We say a pair (X, Δ) is klt if $K_X + \Delta$ is \mathbb{R} -Cartier, and for any prime divisor F over X we have $A_{X,\Delta}(F) > 0$. We say $x \in (X, \Delta)$ is a klt singularity if (X, Δ) is klt.

Definition 2.2 [36] Let $x \in (X, \Delta)$ be a klt singularity and let E be a prime divisor over X . If there exists a proper birational morphism $\pi: Y \rightarrow X$ such that $E = \pi^{-1}(x)$ is the unique exceptional divisor, $(Y, E + \Delta_Y)$ is plt and $-(K_Y + \Delta_Y + E)$ is π -ample, we call E a *Kollár component* over $x \in (X, \Delta)$ and $\pi: Y \rightarrow X$ the plt blowup of E .

By adjunction, we may write

$$(K_Y + \Delta_Y + E)|_E = K_E + \Delta_E$$

for some effective divisor $\Delta_E = \text{Diff}_E(\Delta_Y)$ (called the different) on E , and (E, Δ_E) is a klt log Fano pair.

Definition 2.3 [42] Let $x \in (X, \Delta)$ be a klt singularity. The minimal log discrepancy of Kollár components, denoted by $\text{mld}^k(x, X, \Delta)$, is the infimum of the log discrepancies $A_{X,\Delta}(E)$ as E varies among all Kollár components over $x \in (X, \Delta)$.

If $x \in (X, \Delta)$ is a klt singularity, then we can write $\Delta = \sum_{i=1}^r a_i \Delta_i$ as a convex combination of \mathbb{Q} -divisors Δ_i such that each (X, Δ_i) is klt. Let $r > 0$ be an integer such that $r(K_X + \Delta_i)$ is Cartier for all i . Then as $A_{X,\Delta}(F) = \sum_{i=1}^r a_i A_{X,\Delta_i}(F)$, the possible values of log discrepancies $A_{X,\Delta}(F)$ belong to the discrete set

$$\left\{ \frac{1}{r} \sum_{i=1}^r a_i m_i \mid m_i \in \mathbb{N} \right\}.$$

This implies that the infimum in the above definition is also a minimum.

The following result will be useful later. Recall that the log canonical threshold $\text{lct}(X, \Delta; D)$ of an effective \mathbb{R} -Cartier divisor D with respect to a klt pair (X, Δ) is the largest number $t \geq 0$ such that $(X, \Delta + tD)$ is klt, and the α -invariant of a log Fano pair (X, Δ) is defined as the infimum of the log canonical thresholds $\text{lct}(X, \Delta; D)$, where $0 \leq D \sim_{\mathbb{R}} -(K_X + \Delta)$. The log canonical threshold $\text{lct}_x(X, \Delta; D)$ at a closed point $x \in X$ is defined analogously.

Lemma 2.4 Let E be a Kollár component over a klt singularity $x \in (X, \Delta)$. Then for any effective \mathbb{R} -Cartier divisor D on X , we have

$$\text{lct}_x(X, \Delta; D) \geq \min\{1, \alpha(E, \Delta_E)\} \cdot \frac{A_{X,\Delta}(E)}{\text{ord}_E(D)}.$$

Proof Let $\alpha = \min\{1, \alpha(E, \Delta_E)\}$, let $t = A_{X,\Delta}(E)/\text{ord}_E(D)$ and let $\pi: Y \rightarrow X$ be the plt blowup of E . Then we have

$$\pi^*(K_X + \Delta + tD) = K_Y + \Delta_Y + tD_Y + E,$$

which gives $tD_Y|_E \sim_{\mathbb{R}} -(K_Y + \Delta_Y + E)|_E \sim_{\mathbb{R}} -(K_E + \Delta_E)$. By the definition of alpha invariants, the pair $(E, \Delta_E + \alpha tD_Y|_E)$ is log canonical, hence by inversion of adjunction [23, Theorem 5.50] we know that $(Y, \Delta_Y + \alpha tD_Y + E)$ is lc around E . As $\alpha \leq 1$, we also have

$$\pi^*(K_X + \Delta + \alpha tD) = K_Y + \Delta_Y + \alpha tD_Y + sE$$

for some $s \leq 1$, thus by the above discussion we deduce that $(Y, \Delta_Y + \alpha tD_Y + sE)$ is sub-lc around E , and hence $(X, \Delta + \alpha tD)$ is lc at x . In other words, $\text{lct}_x(X, \Delta; D) \geq \alpha t$, as desired. \square

2.3 Local volumes

We next briefly recall the definition of the local volumes of klt singularities [24]. Let $x \in (X, \Delta)$ be a klt singularity and let $n = \dim X$. The *log discrepancy* function

$$A_{X,\Delta}: \text{Val}_{X,x} \rightarrow \mathbb{R} \cup \{+\infty\}$$

is defined as in [19] and [9, Theorem 3.1]. It generalizes the usual log discrepancies of divisors; in particular, for divisorial valuations, i.e. valuations of the form $\lambda \cdot \text{ord}_F$, where $\lambda > 0$ and F is a divisor over X , we have

$$A_{X,\Delta}(\lambda \cdot \text{ord}_F) = \lambda \cdot A_{X,\Delta}(F).$$

We denote by $\text{Val}_{X,x}^*$ the set of valuations $v \in \text{Val}_X$ with center x and $A_{X,\Delta}(v) < +\infty$. The *volume* of a valuation $v \in \text{Val}_{X,x}$ is defined as

$$\text{vol}(v) = \text{vol}_{X,x}(v) = \limsup_{m \rightarrow \infty} \frac{\ell(\mathbb{C}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!}.$$

Definition 2.5 Let $x \in (X, \Delta)$ be an n -dimensional klt singularity. For any $v \in \text{Val}_{X,x}^*$, we define the *normalized volume* of v as

$$\widehat{\text{vol}}_{X,\Delta}(v) := A_{X,\Delta}(v)^n \cdot \text{vol}_{X,x}(v).$$

The *local volume* of $x \in (X, \Delta)$ is defined as

$$\widehat{\text{vol}}(x, X, \Delta) := \inf_{v \in \text{Val}_{X,x}^*} \widehat{\text{vol}}_{X,\Delta}(v).$$

By [24, Theorem 1.2], the local volume of a klt singularity is always positive. We will frequently use the following properties of local volumes.

Lemma 2.6 [30, Theorem 1.6] *If $x \in (X, \Delta)$ is a klt singularity of dimension n , then $\widehat{\text{vol}}(x, X, \Delta) \leq n^n$.*

Lemma 2.7 [39, Corollary 1.4] *Let $x \in (X, \Delta)$ be a klt singularity of dimension n and let D be a \mathbb{Q} -Cartier Weil divisor on X . Then the Cartier index of D is at most $n^n/\widehat{\text{vol}}(x, X, \Delta)$.*

2.4 Log Fano cone singularities

In this subsection we recall the definition of log Fano cone singularities and their \mathbb{K} -semistability. These notions originally appear in the study of Sasaki–Einstein metrics [10; 11] and are further explored in works related to the Stable Degeneration Conjecture [27; 26].

Definition 2.8 Let $X = \text{Spec}(R)$ be a normal affine variety and $\mathbb{T} = \mathbb{G}_m^r$ ($r > 0$) an algebraic torus. We say that a \mathbb{T} -action on X is *good* if it is effective and there is a unique closed point $x \in X$ that is in the orbit closure of any \mathbb{T} -orbit. We call x the vertex of the \mathbb{T} -variety X , and call the corresponding singularity $x \in X$ a \mathbb{T} -singularity.

Let $N := N(\mathbb{T}) = \text{Hom}(\mathbb{G}_m, \mathbb{T})$ be the co-weight lattice and $M = N^*$ the weight lattice. We have a weight decomposition

$$R = \bigoplus_{\alpha \in M} R_\alpha,$$

and the action being good implies that $R_0 = \mathbb{k}$ and every R_α is finite-dimensional. For $f \in R$, we denote by f_α the corresponding component in the above weight decomposition.

Definition 2.9 A *Reeb vector* on X is a vector $\xi \in N_{\mathbb{R}}$ such that $\langle \xi, \alpha \rangle > 0$ for all $0 \neq \alpha \in M$ with $R_\alpha \neq 0$. The set $\mathfrak{t}_{\mathbb{R}}^+$ of Reeb vectors is called the *Reeb cone*.

For any $\xi \in \mathfrak{t}_{\mathbb{R}}^+$, we can define a valuation wt_ξ , called a toric valuation, by setting

$$\text{wt}_\xi(f) := \min\{\langle \xi, \alpha \rangle \mid \alpha \in M, f_\alpha \neq 0\},$$

where $f \in R$. It is not hard to verify that $\text{wt}_\xi \in \text{Val}_{X,x}$.

Definition 2.10 A log Fano cone singularity is a klt singularity that admits a nontrivial good torus action. A polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ consists of a log Fano cone singularity $x \in (X, \Delta)$ together with a Reeb vector ξ , called a *polarization*.

By abuse of convention, a good \mathbb{T} -action on a klt singularity $x \in (X, \Delta)$ means a good \mathbb{T} -action on X such that x is the vertex and Δ is \mathbb{T} -invariant. Using terminology from Sasakian geometry, we say a polarized log Fano cone $x \in (X, \Delta; \xi)$ is *quasiregular* if ξ generates a \mathbb{G}_m -action (i.e. ξ is a real multiple of some element of N); otherwise, we say that $x \in (X, \Delta; \xi)$ is *irregular*.

We will often use the following result to perturb an irregular polarization to a quasiregular one.

Lemma 2.11 *The function $\xi \mapsto \widehat{\text{vol}}_{X,\Delta}(\text{wt}_\xi)$ defined on the Reeb cone is continuous and has a minimum.*

Proof This follows from [27, Theorem 2.15(3) and Proposition 2.39]. □

Definition 2.12 We say a polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ is *K-semistable* if

$$\widehat{\text{vol}}(x, X, \Delta) = \widehat{\text{vol}}_{X, \Delta}(\text{wt}_\xi).$$

This definition differs from the original ones from [10; 11], but they are equivalent by [27, Theorem 2.34]. For our purpose, the above definition is more convenient. Since the minimizer of the normalized volume function is unique up to rescaling [39], the polarization ξ is essentially determined by the K-semistability condition and hence we often omit the polarization and simply say $x \in (X, \Delta)$ is K-semistable.

Definition 2.13 The *volume ratio* of a polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ is defined to be

$$\Theta(X, \Delta; \xi) := \frac{\widehat{\text{vol}}(x, X, \Delta)}{\widehat{\text{vol}}_{X, \Delta}(\text{wt}_\xi)}.$$

The *volume ratio* of a log Fano cone singularity $x \in (X, \Delta)$ is defined to be

$$\Theta(x, X, \Delta) := \sup_{\xi} \Theta(X, \Delta; \xi),$$

where the supremum runs over all polarizations on X .

By definition, $0 < \Theta(X, \Delta; \xi) \leq 1$ and $\Theta(X, \Delta; \xi) = 1$ if and only if $x \in (X, \Delta; \xi)$ is K-semistable. By Lemma 2.11, a similar statement holds in the unpolarized case.

2.5 Bounded family of singularities

In this subsection we define boundedness of singularities and recall some properties of singularity invariants in bounded families. They will be useful in proving the easier direction of Theorem 1.2.

Definition 2.14 We call $B \subseteq (\mathcal{X}, \mathcal{D}) \rightarrow B$ an \mathbb{R} -Gorenstein family of klt singularities (over a normal but possibly disconnected base B) if

- (1) \mathcal{X} is flat over B , and $B \subseteq \mathcal{X}$ is a section of the projection,
- (2) for any closed point $b \in B$, \mathcal{X}_b is connected, normal and is not contained in $\text{Supp}(\mathcal{D})$, and
- (3) $K_{\mathcal{X}/B} + \mathcal{D}$ is \mathbb{R} -Cartier and $b \in (\mathcal{X}_b, \mathcal{D}_b)$ is a klt singularity for any $b \in B$.

Lemma 2.15 Let $B \subseteq (\mathcal{X}, \mathcal{D}) \rightarrow B$ be an \mathbb{R} -Gorenstein family of klt singularities. Then there exist constants $\varepsilon, A > 0$ such that

$$\widehat{\text{vol}}(b, \mathcal{X}_b, \mathcal{D}_b) \geq \varepsilon \quad \text{and} \quad \text{mld}^{\text{K}}(b, \mathcal{X}_b, \mathcal{D}_b) \leq A$$

for all closed points $b \in B$.

Proof The uniform lower bound of the local volumes follows from their lower semicontinuity [6] in families together with Noetherian induction. Note that although the main result of [7] is stated for families with \mathbb{Q} -coefficients, it remains valid for real coefficients. This is because there exist \mathbb{Q} -divisors $\mathcal{D}' \leq \mathcal{D}$ such that $K_{\mathcal{X}/B} + \mathcal{D}'$ is \mathbb{Q} -Cartier and the coefficients of $\mathcal{D} - \mathcal{D}'$ are arbitrarily small; in particular, $\widehat{\text{vol}}(b, \mathcal{X}_b, \mathcal{D}_b) \geq \widehat{\text{vol}}(b, \mathcal{X}_b, \mathcal{D}'_b)$ and their difference can be made arbitrarily small for any fixed $b \in B$. Thus the lower semicontinuity of $\widehat{\text{vol}}(b, \mathcal{X}_b, \mathcal{D}_b)$ in $b \in B$ follows from the lower semicontinuity of $\widehat{\text{vol}}(b, \mathcal{X}_b, \mathcal{D}'_b)$. Alternatively, the local volume lower bound follows from the constructibility of the volume function [37, Theorem 1.3], which also holds for families with real coefficients by the discussions in Theorem 2.19.

The uniform upper bound of mld^{K} is a direct consequence of [16, Theorem 2.34]: after a (quasifinite and surjective) base change, the family admits a flat family of Kollár components (in the sense of [16, Definition 2.32]), whose log discrepancy is locally constant and hence uniformly bounded. \square

We next define the boundedness of log Fano cone singularities. Note that our definition is a priori stronger than their boundedness as klt singularities (it is not clear to us whether they are equivalent).

Definition 2.16 We say that a set \mathcal{S} of polarized log Fano cone singularities is bounded if there exist finitely many \mathbb{R} -Gorenstein families $B_i \subseteq (\mathcal{X}_i, \mathcal{D}_i) \rightarrow B_i$ of klt singularities, each with a fiberwise good \mathbb{T}_i -action for some nontrivial algebraic torus \mathbb{T}_i , such that every $x \in (X, \Delta; \xi)$ in \mathcal{S} is isomorphic to $b \in (\mathcal{X}_{i,b}, \mathcal{D}_{i,b}; \xi_b)$ for some i , some $b \in B_i$ and some $\xi_b \in N(\mathbb{T}_i)_{\mathbb{R}}$. We say \mathcal{S} is log bounded if we only ask $x \in (X; \xi)$ to be isomorphic to $b \in (\mathcal{X}_{i,b}; \xi_b)$ and that $\text{Supp}(\Delta) \subseteq \text{Supp}(\mathcal{D}_{i,b})$ under this isomorphism.

We say that a set \mathcal{C} of log Fano cone singularities is bounded if it is the underlying set of singularities from a bounded set of polarized log Fano cone singularities.

Lemma 2.17 *Let \mathcal{C} be a bounded set of log Fano cone singularities. Then there exists some constant $\theta > 0$ such that $\Theta(x, X, \Delta) \geq \theta$ for all $x \in (X, \Delta)$ in \mathcal{C} .*

Proof By definition, it suffices to show that for any \mathbb{R} -Gorenstein family of log Fano cone singularities $B \subseteq (\mathcal{X}, \mathcal{D}) \rightarrow B$ with a fiberwise good torus \mathbb{T} -action, we have a uniform positive lower bound of $\Theta(b, \mathcal{X}_b, \mathcal{D}_b)$. We may assume that B is connected. Replacing \mathcal{X} by \mathbb{T} -invariant affine subset, we may also assume that $\mathcal{X} = \text{Spec}(\mathcal{R})$ is affine. We have a weight decomposition $\mathcal{R} = \bigoplus_{\alpha \in M} \mathcal{R}_{\alpha}$, which reduces to the weight decomposition on the fibers. By flatness of $\mathcal{X} \rightarrow B$, each \mathcal{R}_{α} is flat over B . It follows that the Reeb cone $\mathfrak{t}_{\mathbb{R},b}^+ \subseteq N_{\mathbb{R}}$ is locally constant in $b \in B$, hence is constant as B is connected. Choose any primitive $\xi \in N$ that lies in the (constant) Reeb cone. It suffices to show that $\Theta(b, \mathcal{X}_b, \mathcal{D}_b; \xi)$ is uniformly bounded from below. Since the \mathbb{T} -action is good, the \mathcal{R}_{α} are also finite over B , hence locally free by flatness. From the definition of volume, this easily implies that $\text{vol}(\text{wt}_{\xi})$ is constant as $b \in B$ varies. Since $\xi \in N$ comes from a \mathbb{G}_m -action, there is a divisor \mathcal{E} over \mathcal{X} such that $\text{ord}_{\mathfrak{e}} = \text{wt}_{\xi}$ as valuations over \mathcal{X} and hence $A_{\mathcal{X}_b, \mathcal{D}_b}(\text{wt}_{\xi}) = A_{\mathcal{X}_b, \mathcal{D}_b}(\text{ord}_{\mathfrak{e}_b}) = A_{\mathcal{X}, \mathcal{D}}(\mathcal{E})$ for general $b \in B$. By Noetherian induction, this

implies that $A_{\mathcal{X}_b, \mathcal{D}_b}(\text{wt}_\xi)$ is constructible in $b \in B$ and thus uniformly bounded from above. Finally, by Lemma 2.15 we know that $\widehat{\text{vol}}(b, \mathcal{X}_b, \mathcal{D}_b) \geq \varepsilon$ for some constant $\varepsilon > 0$. Putting these together we deduce from the definition that $\Theta(b, \mathcal{X}_b, \mathcal{D}_b; \xi)$ is uniformly bounded from below. \square

2.6 Real coefficients

The following result will help us reduce many questions about pairs with real coefficients to ones with rational coefficients. We will often use it without explicit mention when we refer to results that are originally stated for \mathbb{Q} -coefficients.

Lemma 2.18 *Let $n \in \mathbb{N}$ and let $I \subseteq [0, 1]$ be a finite set. Then there exists a finite set $I' \subseteq [0, 1] \cap \mathbb{Q}$ depending only on n and I such that the following holds.*

For any n -dimensional klt singularity $x \in (X, \Delta)$ with coefficients in I , there exists some effective \mathbb{Q} -divisor $\Delta' \geq \Delta$ on X , with coefficients in I' , such that $\text{Supp}(\Delta) = \text{Supp}(\Delta')$, $x \in (X, \Delta')$ is klt, $\widehat{\text{vol}}(x, X, \Delta') \geq 2^{-n}\widehat{\text{vol}}(x, X, \Delta)$ and $\text{mld}^{\text{K}}(x, X, \Delta') \leq 2 \cdot \text{mld}^{\text{K}}(x, X, \Delta)$. If in addition $x \in (X, \Delta; \xi)$ is a log Fano cone singularity, then $\Theta(x, X, \Delta'; \xi) \geq 4^{-n}\Theta(x, X, \Delta; \xi)$.

Proof This is essentially a consequence of [15, Theorem 5.6]. Without loss of generality we may assume $1 \in I$. Write $I = \{a_1, \dots, a_m\}$ and let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m$. Let $V \subseteq \mathbb{R}^m$ be the rational envelope of \mathbf{a} , i.e. the smallest affine subspace defined over \mathbb{Q} that contains \mathbf{a} . By [15, Theorem 5.6], there exists an open neighborhood U of $\mathbf{a} \in V$ depending only on n and I such that for any n -dimensional singularity $x \in X$ and any Weil divisors $\Delta_1, \dots, \Delta_m \geq 0$, if $x \in (X, \sum_{i=1}^m a_i \Delta_i)$ is klt (resp. plt) then $x \in (X, \sum_{i=1}^m a'_i \Delta_i)$ is klt (resp. plt) for any $\mathbf{a}' = (a'_1, \dots, a'_m) \in U$. Note that the plt case implies that if E is a Kollár component over $x \in (X, \sum_{i=1}^m a_i \Delta_i)$ then it is also a Kollár component over $x \in (X, \sum_{i=1}^m a'_i \Delta_i)$.

Choose some $\mathbf{a}' = (a'_1, \dots, a'_m) \in U \cap \mathbb{Q}^m$ such that $2\mathbf{a}' - \mathbf{a}, 2\mathbf{a} - \mathbf{a}' \in U$. We claim that $I' = \{a'_1, \dots, a'_m\}$ satisfies the required conditions. Indeed, for any n -dimensional klt singularity $x \in (X, \Delta = \sum_{i=1}^m a_i \Delta_i)$ (where the Δ_i are Weil divisors), if we set $\Delta' = \sum_{i=1}^m a'_i \Delta_i$ and let $\Delta_1 = 2\Delta' - \Delta$, then by our choice of U and \mathbf{a}' we know that $\text{Supp}(\Delta') = \text{Supp}(\Delta)$, $x \in (X, \Delta')$ and $x \in (X, \Delta_1)$ are both klt, and any Kollár component over $x \in (X, \Delta)$ is also a Kollár component over $x \in (X, \Delta')$. Moreover, since $\Delta' = \frac{1}{2}(\Delta + \Delta_1)$, we have $A_{X, \Delta'}(v) = \frac{1}{2}(A_{X, \Delta}(v) + A_{X, \Delta_1}(v)) \geq \frac{1}{2}A_{X, \Delta}(v)$ for any valuation $v \in \text{Val}_{X, x}^*$. Similarly, using $2\mathbf{a} - \mathbf{a}' \in U$ we get $A_{X, \Delta'}(v) \leq 2A_{X, \Delta}(v)$. In particular, we see that $2^{-n}\widehat{\text{vol}}_{X, \Delta}(v) \leq \widehat{\text{vol}}_{X, \Delta'}(v) \leq 2^n\widehat{\text{vol}}_{X, \Delta}(v)$. These imply the inequalities about $\widehat{\text{vol}}$, mld^{K} and the volume ratio (all by definition). \square

We also note the Stable Degeneration Conjecture holds for pairs with real coefficients.

Theorem 2.19 [4; 27; 37; 39; 40] *Every klt singularity $x \in (X, \Delta)$ has a special degeneration to a K -semistable log Fano cone singularity $x_0 \in (X_0, \Delta_0)$ with $\widehat{\text{vol}}(x, X, \Delta) = \widehat{\text{vol}}(x_0, X_0, \Delta_0)$*

Proof The arguments in [4; 27; 39; 40] extend directly to real coefficients, since they are not sensitive to the coefficients. The proof in [37] uses the existence of monotonic N -complement which requires \mathbb{Q} -coefficients, but the main results in loc. cit. can be extended to \mathbb{R} -coefficients by [16, Theorems 3.3 and 3.4]. Thus [40, Theorem 1.2] holds for \mathbb{R} -pairs. In particular, the minimizer v of the normalized volume function $\widehat{\text{vol}}_{X,\Delta}$ induces a special degeneration of $x \in (X = \mathbf{Spec}(R), \Delta)$ to a \mathbb{K} -semistable log Fano cone $x_0 \in (X_0 = \mathbf{Spec}(\text{gr}_v R), \Delta_0; \xi_v)$. By [27, Lemma 2.58], we have

$$\widehat{\text{vol}}_{X,\Delta}(v) = \widehat{\text{vol}}_{X_0,\Delta_0}(\text{wt}_{\xi_v}).$$

Since $\widehat{\text{vol}}(x, X, \Delta) = \widehat{\text{vol}}_{X,\Delta}(v)$ (as v is the minimizer of $\widehat{\text{vol}}_{X,\Delta}$) and $\widehat{\text{vol}}(x_0, X_0, \Delta_0) = \widehat{\text{vol}}_{X_0,\Delta_0}(\text{wt}_{\xi_v})$ (as $x_0 \in (X_0, \Delta_0; \xi_v)$ is \mathbb{K} -semistable), we see that the degeneration preserves the local volume. \square

2.7 Results from Part I

In this subsection we collect slight modification of several results from [42] that we need in later proofs. For the first result, recall that we say a singularity $x \in (X, \Delta)$ is of klt type if there exists some effective \mathbb{R} -divisor D on X such that $x \in (X, \Delta + D)$ is klt.

Lemma 2.20 *Let $n \in \mathbb{N}$ and let $\varepsilon > 0$. Then there exists some constant $c = c(n, \varepsilon) > 0$ such that for any n -dimensional klt singularity $x \in (X, \Delta)$ with $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$, we have that $x \in (X, (1 + c)\Delta)$ is of klt type.*

Proof Since $x \in (X, \Delta)$ is klt, by [3] (cf. [42, Lemma 4.7]) there exists a small birational morphism $\pi : Y \rightarrow X$ such that K_Y is \mathbb{Q} -Cartier and relatively ample. Since π is small, we have $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$, hence by [30, Lemma 2.9(2)] (cf. the proof of [42, Lemma 2.10]) we have

$$\widehat{\text{vol}}(y, Y, \Delta_Y) \geq \widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$$

for all $y \in \pi^{-1}(x)$. By [42, Theorem 3.1], there exists some constant $c = c(n, \varepsilon) > 0$ such that the pair $(Y, (1 + c)\Delta_Y)$ is klt for all $y \in \pi^{-1}(x)$. Note that $-(K_Y + (1 + c)\Delta_Y) \sim_{\mathbb{Q},\pi} cK_Y$, hence π is also the ample model of $-(K_X + (1 + c)\Delta)$. By [41, Lemma 2.4], this implies that $x \in (X, (1 + c)\Delta)$ is of klt type. \square

The second result is extracted from the proof of [42, Theorem 4.1].

Lemma 2.21 *Let $n \in \mathbb{N}$, let $\varepsilon, A > 0$, and let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Then there exists some integer $N = N(n, \varepsilon, A, I) > 0$ such that for any n -dimensional klt singularity $x \in (X, \Delta)$ with coefficients in I and $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$, and for any Kollár component E over $x \in (X, \Delta)$ with $A_{X,\Delta}(E) \leq A$, we have that NE and $N(K_Y + \Delta_Y + E)$ are Cartier on the plt blowup $Y \rightarrow X$ of E .*

Proof By [42, (4.1)], the local volumes $\widehat{\text{vol}}(y, Y, \Delta_Y)$ (where $y \in E$) are bounded from below by some constants that only rely on the given constants n, ε and A . Thus by Lemma 2.7 we get the desired Cartier index bound. \square

3 Boundedness

In this section, we prove the following statement on boundedness of log Fano cone singularities, which will imply the main results of this paper.

Theorem 3.1 *Let $n \in \mathbb{N}$ and let $I \subseteq [0, 1]$ be a finite set. Let $\varepsilon, \theta, A > 0$. Let \mathcal{S} be the set of n -dimensional polarized log Fano cone singularities $x \in (X, \Delta; \xi)$ with coefficients in I such that*

$$\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon, \quad \Theta(X, \Delta; \xi) \geq \theta, \quad \text{mld}^{\mathbb{K}}(x, X, \Delta) \leq A.$$

Then \mathcal{S} is bounded.

Recall that $\Theta(X, \Delta; \xi)$ is the volume ratio of the log Fano cone singularity (Definition 2.13). We refer to Sections 2.2–2.5 for the other relevant definitions.

3.1 Orbifold cones

Since the volume ratio is continuous in the Reeb vector ξ (Lemma 2.11), by perturbing the polarization, we see that it suffices to prove Theorem 3.1 when \mathcal{S} consists of quasiregular log Fano cones. Every quasiregular log Fano cone singularity has a natural affine orbifold cone structure induced by the polarization. The proof of Theorem 3.1 relies on the associated projective orbifold cone construction. In this subsection, we first fix some notation and recall some basic properties of orbifold cones from [20].

Definition 3.2 Let V be a normal projective variety. Let L be an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor on V .

- (1) The *affine orbifold cone* $C_a(V, L)$ is defined as

$$C_a(V, L) := \text{Spec} \bigoplus_{m=0}^{\infty} H^0(V, \mathcal{O}_V(\lfloor mL \rfloor)).$$

- (2) The *projective orbifold cone* $C_p(V, L)$ is defined as

$$C_p(V, L) := \text{Proj} \bigoplus_{m=0}^{\infty} \bigoplus_{i=0}^{\infty} H^0(V, \mathcal{O}_V(\lfloor mL \rfloor)) \cdot s^i,$$

where the grading of $H^0(V, \mathcal{O}_V(\lfloor mL \rfloor))$ and s are m and 1, respectively.

For ease of notation, denote $X = C_a(V, L)$, $\bar{X} = C_p(V, L)$ and let $x \in X$ be the vertex of the orbifold cone. On the projective orbifold cone, we also denote by V_{∞} the divisor at infinity, i.e. the divisor corresponding to $(s = 0)$. We have $X \cong \bar{X} \setminus V_{\infty}$. Note that $X \setminus \{x\}$ is a Seifert \mathbb{C}_m -bundle over V (in the sense of [20]). Thus for any effective \mathbb{R} -divisor Δ_V on V , we can define the affine (resp. projective) orbifold cone over the polarized pair $(V, \Delta_V; L)$ as the pair (X, Δ) (resp. $(\bar{X}, \bar{\Delta})$), where Δ (resp. $\bar{\Delta}$) is the closure of the pullback of Δ_V to $X \setminus \{x\}$ (since the projection $X \setminus \{x\} \rightarrow V$ is equidimensional,

the pullback of a Weil divisor is well-defined). Every quasiregular polarized log Fano cone singularity $x \in (X, \Delta; \xi)$ arises in this way, so we can talk about its associated projective orbifold cone $(\bar{X}, \bar{\Delta})$.

Alternatively, the Seifert \mathbb{G}_m -bundle $X \setminus \{x\}$ can be compactified by adding V_∞ and the missing zero section V_0 ; the resulting space \bar{Y} is also the orbifold blowup of \bar{X} at x . Similarly, let Y be the orbifold blowup of X at x , i.e. $Y = \bar{Y} \setminus V_\infty$. Write the fractional part of L as $\{L\} = \sum_{i=1}^k (a_i/b_i) D_i$ for some prime divisors D_i on V and $0 < a_i < b_i$ coprime integers. We denote $\Delta_L := \sum_{i=1}^k ((b_i - 1)/b_i) D_i$. Then the pairs $(V_0, \text{Diff}_{V_0}(0))$ and $(V_\infty, \text{Diff}_{V_\infty}(0))$ obtained by taking adjunction over some big open set of V from the pair $(\bar{X}, V_0 + V_\infty)$ are both isomorphic to (V, Δ_L) (this is a local computation; see [20, Section 4]).

With this notation, we have the following well-known result [20] (cf. [21, Section 3.1] for analogous statement for usual cones).

Lemma 3.3 *The following conditions are equivalent:*

- (1) $x \in (X, \Delta)$ is klt;
- (2) $(\bar{X}, \bar{\Delta} + V_\infty)$ is plt;
- (3) $(V, \Delta_V + \Delta_L)$ is a log Fano pair, and $-(K_V + \Delta_V + \Delta_L) \sim_{\mathbb{R}} rL$ for some $r > 0$.

Moreover, when the above conditions are satisfied, we have $K_{\bar{X}} + \bar{\Delta} \sim_{\mathbb{R}} -(1+r)V_\infty$ and $A_{X,\Delta}(V_0) = r$.

The next result is a key observation in the proof of Theorem 3.1. It expresses the normalized volume of a polarized log Fano cone singularities as the global volume of the associated projective orbifold cone.

Lemma 3.4 *Assume that $x \in (X, \Delta)$ is klt. Then under the notation of Lemma 3.3 we have*

$$\widehat{\text{vol}}_{X,\Delta}(\text{ord}_{V_0}) = \text{vol}(-(K_{\bar{X}} + \bar{\Delta} + V_\infty)) = r \cdot \text{vol}(-(K_V + \Delta_V + \Delta_L)).$$

Proof This follows from a direct calculation. First we have $A_{X,\Delta}(V_0) = r$ by Lemma 3.3. We also have $-V_0|_{V_0} \cong L$, hence

$$\widehat{\text{vol}}_{X,\Delta}(\text{ord}_{V_0}) = A_{X,\Delta}(V_0)^n \cdot \text{vol}_{X,\Delta}(\text{ord}_{V_0}) = r^n \text{vol}(-V_0|_{V_0}) = r^n L^{n-1},$$

where $n = \dim X$. On the other hand, by Lemma 3.3 we get $-(K_{\bar{X}} + \bar{\Delta} + V_\infty) \sim_{\mathbb{R}} rV_\infty$; recall also that $-(K_{\bar{X}} + \bar{\Delta} + V_\infty)|_{V_\infty} \sim_{\mathbb{R}} -(K_V + \Delta_V + \Delta_L) \sim_{\mathbb{Q}} rL$ (where we identify V_∞ with V), thus

$$\begin{aligned} \text{vol}(-(K_{\bar{X}} + \bar{\Delta} + V_\infty)) &= (-(K_{\bar{X}} + \bar{\Delta} + V_\infty))^{n-1} \cdot (rV_\infty) \\ &= r \cdot \text{vol}(-(K_V + \Delta_V + \Delta_L)) = r \cdot (rL)^{n-1} = r^n L^{n-1}, \end{aligned}$$

which proves the desired equality. □

We also need a slight generalization of Lemma 3.3.

Lemma 3.5 *Notation as before. Assume that $x \in (X, \Delta)$ is of klt type. Then $-(K_V + \Delta_V + \Delta_L)$ is big.*

Proof By assumption, there exists some effective \mathbb{R} -divisor D such that $x \in (X, \Delta + D)$ is klt (note that D is not necessarily invariant under the \mathbb{G}_m -action). In particular, $A_{X, \Delta + D}(V_0) > 0$ and we have

$$K_Y + \Delta_Y + D_Y + V_0 \sim_{\mathbb{R}} A_{X, \Delta + D}(V_0) \cdot V_0$$

over X . It follows that $-(K_Y + \Delta_Y + D_Y + V_0)|_{V_0} \sim_{\mathbb{R}} A_{X, \Delta + D}(V_0) \cdot L$ is ample. By adjunction, this implies that

$$-(K_V + \Delta_V + \Delta_L) = -(K_Y + \Delta_Y + D_Y + V_0)|_{V_0} + D_Y|_{V_0}$$

is the sum of an ample divisor and an effective divisor, hence is big. \square

3.2 Effective birationality

Given Lemma 3.4, a naïve idea to prove Theorem 3.1 is to associate to each log Fano cone singularity a projective orbifold cone and show that the corresponding set of projective orbifold cones is bounded. In general, this is too much to hope for, as the projective orbifold cone depends on the (auxiliary) choice of a quasiregular polarization. The following example shows that even for a fixed singularity we can get an unbounded family of projective orbifold cones by choosing different polarizations.

Example 3.6 Let $a_1, \dots, a_n \in \mathbb{N}$ be pairwise coprime positive integers. Then $\xi = (a_1, \dots, a_n) \in \mathbb{N}^n$ gives a polarization of the Fano cone singularity $0 \in \mathbb{A}^n$; it generates the \mathbb{G}_m -action with weights a_1, \dots, a_n on the coordinates. This endows \mathbb{A}^n with an affine orbifold cone structure $C_a(V, L)$, where $V = \mathbb{P}(a_1, \dots, a_n)$ and $L = \mathcal{O}_V(1)$. The associated projective orbifold cone is $\bar{X} = \mathbb{P}(1, a_1, \dots, a_n)$, which is clearly unbounded as the weights a_i vary. By choosing appropriate weights, we can even ensure that the normalized volume $\widehat{\text{vol}}(\text{wt}_\xi) = (a_1 + \dots + a_n)^n / (a_1 \cdots a_n)$ is fixed.

Nonetheless, we observe that in the above example the projective orbifold cones we get satisfy the following interesting property: the linear system $|-K_{\bar{X}}|$ always defines a birational map that is an embedding at the vertex $[1 : 0 : \dots : 0]$. In fact, if $[s : x_1 : \dots : x_n]$ are the weighted homogeneous coordinates of \bar{X} , then for every $i \in \{1, \dots, n\}$ there exists some $k_i \in \mathbb{N}$ such that $s^{k_i} x_i \in H^0(-K_{\bar{X}})$ (this is possible because s has weight 1); it is not hard to see that the sublinear system spanned by $s^{k_i} x_i$ for $i = 0, \dots, n$ is basepoint-free and restricts to an embedding on the affine chart $\mathbb{A}^n = \bar{X} \setminus (s = 0)$.

This motivates us to raise the following question.

Question 3.7 Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Is there some integer $m = m(n, \varepsilon, I) > 0$ such that for any n -dimensional quasiregular polarized log Fano cone $x \in (X, \Delta; \xi)$ with $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$, we have that $|-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)|$ defines a birational map that restricts to an embedding on X ?

As before, $(\bar{X}, \bar{\Delta})$ denotes the associated projective orbifold cone, and $V_\infty = \bar{X} \setminus X$ is the divisor at infinity.

The technical core of the proof of Theorem 3.1 is the answer to this question after imposing upper bounds on the minimal log discrepancies of Kollár components.

Proposition 3.8 *Let $n \in \mathbb{N}$ and $\varepsilon, A > 0$. Let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. Then there is some integer $m = m(n, \varepsilon, A, I) > 0$ such that for any n -dimensional quasiregular polarized log Fano cone $x \in (X, \Delta; \xi)$ with $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$ and $\text{mld}^K(x, X, \Delta) \leq A$, we have that $-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)$ defines a birational map that restricts to an embedding on X .*

We remark that the integer m is independent of the polarization ξ .

Proof We start with some reductions. Let $H := -(K_{\bar{X}} + \bar{\Delta} + V_\infty)$. We view X as an affine orbifold cone $C_a(V, L)$ and keep the notation $(\Delta_V, \Delta_L, \text{etc.})$ from Section 3.1. First note that since the coefficients of $\bar{\Delta}$ belong to a fixed finite set I of rational numbers, we can choose some m_0 depending only on I such that $m_0 H$ has integer coefficients. Since $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$, by Lemma 2.7 we know that the Cartier index of $m_0 H$ at x is bounded from above by n^n/ε . Thus replacing m_0 by a sufficiently large fixed multiple (e.g. $\lfloor n^n/\varepsilon \rfloor!$) we may further assume that $m_0 H$ is Cartier in a neighborhood of x . By \mathbb{G}_m -translation, this implies that $m_0 H$ is Cartier on X .

The next step is to produce enough sections (of some multiple of $m_0 H$) that do not vanish at x . We do this by creating isolated non-klt centers at x and apply Nadel vanishing. Let $\varphi: \bar{Y} \rightarrow \bar{X}$ be the orbifold blowup of the vertex x , and let $\Delta_{\bar{Y}}$ be the strict transform of $\bar{\Delta}$. Let V_0 be the exceptional divisor as before. Recall that \bar{Y} has an orbifold \mathbb{P}^1 -bundle structure $\pi: \bar{Y} \rightarrow V$. Then we have

$$-(K_{\bar{Y}} + \Delta_{\bar{Y}} + V_0 + V_\infty) \sim_{\mathbb{Q}} -\pi^*(K_V + \Delta_V + \Delta_L).$$

(See [20, Proposition 40 and Corollary 41].) The left-hand side is $\sim_{\mathbb{Q}} \varphi^* H - aV_0$, where $a = A_{X, \Delta}(V_0) > 0$, while the right-hand side is semiample since $-(K_V + \Delta_V + \Delta_L)$ is ample on V by Lemma 3.3. From here we deduce that $\varphi^* H - aV_0$ is semiample. For any positive integer m , if we take D to be a general member of the \mathbb{Q} -linear system $\varphi_*|\varphi^* H - aV_0|_{\mathbb{Q}}$, then $D \sim_{\mathbb{Q}} H$ and $(\bar{X}, \bar{\Delta} + mD)$ is klt away from x (since this is the only basepoint of the \mathbb{Q} -linear system). In particular, the multiplier ideal $\mathcal{J}(\bar{X}, \bar{\Delta} + mD)$ is co-supported at x . Furthermore, we have

$$\text{ord}_{V_0} \mathcal{J}(\bar{X}, \bar{\Delta} + mD) > m \cdot \text{ord}_{V_0}(D) - A_{\bar{X}, \bar{\Delta}}(V_0) = (m - 1)a$$

by the definition of multiplier ideals. Thus we obtain

$$(3-1) \quad \mathcal{J}(\bar{X}, \bar{\Delta} + mD) \subseteq \mathfrak{a}_{(m-1)a},$$

where $\mathfrak{a}_{(m-1)a} := \mathfrak{a}_{(m-1)a}(\text{ord}_{V_0})$ denotes the valuation ideals. Recall that $m_0 H$ is Cartier at x , hence as

$$mH - (K_{\bar{X}} + \bar{\Delta} + mD) \sim_{\mathbb{Q}} -(K_{\bar{X}} + \bar{\Delta})$$

is ample, by the following Lemma 3.9 (Nadel vanishing for Weil divisor), we have

$$H^1(\bar{X}, \mathcal{J}(\bar{X}, \bar{\Delta} + mD) \otimes \mathcal{O}_{\bar{X}}(mH)) = 0$$

for all $m \in \mathbb{N}$ that are divisible by m_0 . From the long exact sequence we then deduce that the natural map

$$H^0(\bar{X}, \mathcal{O}_{\bar{X}}(mH)) \rightarrow H^0(\bar{X}, \mathcal{O}_{\bar{X}}/\mathcal{J}(\bar{X}, \bar{\Delta} + mD))$$

is surjective. Combined with (3-1), we get a surjection

$$(3-2) \quad H^0(\bar{X}, \mathcal{O}_{\bar{X}}(mH)) \rightarrow H^0(\bar{X}, \mathcal{O}_{\bar{X}}/\mathfrak{a}_{(m-1)a})$$

for all $m \in \mathbb{N}$ that are divisible by m_0 .

At this point, we have produced sections of mH that separate the jets in $\mathcal{O}_{\bar{X}}/\mathfrak{a}_{(m-1)a}$. Our goal is to make $|mH|$ into a birational map that restricts to an embedding on X . Clearly, a necessary condition is that $|mH|$ separates tangent directions at x . Let us first show that this latter condition can be achieved. In view of (3-2), it suffices to show that

$$(3-3) \quad \mathfrak{a}_{(m-1)a} \subseteq \mathfrak{m}_x^2,$$

which becomes a local question. If X is fixed, this is certainly true for $m \gg 0$, so the main point is to make sure that the bound on m depends only on the given data n, ε, A rather than on X . To this end, note that one way to interpret the opposite condition $\mathfrak{a}_{(m-1)a} \not\subseteq \mathfrak{m}_x^2$ is that there exists some $f \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$ such that $\text{ord}_{V_0}(f) \geq (m-1)a$; in particular, $\text{lct}(f) := \text{lct}(X, \Delta; (f=0)) \leq 1/(m-1)$. This suggests to us that we should try to analyze the Izumi type constant

$$(3-4) \quad \inf\{\text{lct}(f) \mid f \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2\}$$

for the singularities in question.

Let E be a Kollár component over $x \in (X, \Delta)$ such that $A_{X,\Delta}(E) \leq A$, whose existence is guaranteed by our assumption that $\text{mld}^K(x, X, \Delta) \leq A$. Let $f: Z \rightarrow X$ be the plt blowup that extract E and let $\Delta_E = \text{Diff}_E(\Delta_Z)$ be the different. By Lemma 2.4, we have

$$(3-5) \quad \text{lct}(f) \geq \min\{1, \alpha(E, \Delta_E)\} \cdot \frac{A_{X,\Delta}(E)}{\text{ord}_E(f)}.$$

In order to give a uniform estimate of (3-4) using (3-5), we need further information about the log discrepancy $A_{X,\Delta}(E)$, the α -invariant $\alpha(E, \Delta_E)$, and the value of

$$d_E := \sup\{\text{ord}_E(f) \mid f \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2\}.$$

(This last value can be thought of as measuring how well the valuation ideals of ord_E approximate the first two powers of the maximal ideal \mathfrak{m}_x .)

For the log discrepancy, recall from the previous discussion that $m_0(K_X + \Delta)$ is Cartier. Since (X, Δ) is klt, this implies that $A_{X,\Delta}(E) \geq 1/m_0$. For the α -invariant, we know by Lemma 2.21 that there exists

some positive integer $N_1 = N_1(n, \varepsilon, A, I)$ divisible by m_0 such that $N_1 E$ and $N_1(K_Z + \Delta_Z + E)$ are Cartier. By adjunction, we see that $N_1(K_E + \Delta_E)$ is also Cartier. By [14, Corollary 1.8], the log Fano pair (E, Δ_E) belongs to a bounded family (which relies only on N_1). This implies that $\alpha(E, \Delta_E)$ is uniformly bounded below. Finally, to give an upper bound for d_E , let $L = -E|_E$ be the (ample) \mathbb{Q} -divisor defined as in [15, Definition A.4] and let

$$b_k := \alpha_k(\text{ord}_E) = f_* \mathbb{O}_Z(-kE),$$

where $k = 1, 2, \dots$. Then $N_1 L$ is Cartier as $N_1 E$ is Cartier. Since $-(K_E + \Delta_E) \sim_{\mathbb{Q}} A_{X,\Delta}(E) \cdot L$ and $A_{X,\Delta}(E) \geq 1/m_0$, we see that the coefficient and degree of L are bounded, hence the triple (E, Δ, L) belongs to a bounded family. Thus there exists an integer $N = N(n, \varepsilon, A, I) > 0$ such that the section ring $\bigoplus_{k \in \mathbb{N}} H^0(E, [kL])$ is generated in degree $\leq N$. We claim that

$$(3-6) \quad b_{N+1} \subseteq m_x^2.$$

Taking this for granted, it follows immediately that $d_E \leq N$. Putting this information together we deduce from (3-5) that (3-4) is bounded below by some positive constant that only relies on n, ε, A, I . From the discussion right above (3-4) we also know that (3-4) is bounded from above by $1/(m-1)$, where m is the largest integer such that $\alpha_{(m-1)a} \not\subseteq m_x^2$. Thus we see that there is some fixed m depending only on n, ε, A, I such that mH is Cartier and (3-3) holds. Combined with (3-2), we deduce that

$$(3-7) \quad H^0(\bar{X}, \mathbb{O}_{\bar{X}}(mH)) \rightarrow H^0(\bar{X}, \mathbb{O}_{\bar{X}}/m_x^2)$$

is surjective, i.e. $|mH|$ separates tangent directions at x .

Before we proceed to show that $|mH|$ also defines a birational map, let us finish the proof of the claim (3-6). In fact, we shall prove by descending induction that $b_k \subseteq m_x^2$ for all $k \geq N + 1$. This is clear when $k \gg 0$. Suppose that $b_{k+1} \subseteq m_x^2$ and $k \geq N + 1$, then since $b_{k+1} \subseteq b_k$ for all k and since $\bigoplus_{k \in \mathbb{N}} b_k/b_{k+1} \cong \bigoplus_{k \in \mathbb{N}} H^0(E, [kL])$ is generated in degree $\leq N$ (see [28, Section 2.4] or [32, Proposition 2.10] for a proof of the isomorphism $b_k/b_{k+1} \cong H^0(E, [kL])$), we see that

$$b_k/b_{k+1} \subseteq \sum_{i=1}^N (b_i/b_{i+1}) \cdot (b_{k-i}/b_{k-i+1}) \subseteq b_1^2/b_{k+1}.$$

In other words, $b_k \subseteq b_{k+1} + b_1^2$. As $b_{k+1} \subseteq m_x^2$ by induction hypothesis and clearly $b_1 \subseteq m_x$, we obtain $b_k \subseteq m_x^2$, as desired.

We are finally in a position to show that $|mH|$ defines a birational map that restricts to an embedding on X . By (3-7), we already know that in a neighborhood of x , the linear system $|mH|$ is basepoint-free and the induced map is unramified. Since the base locus of $|mH|$ is closed and invariant under the \mathbb{G}_m -action (coming from the orbifold cone structure), we see that $|mH|$ has no basepoint in X . Similarly, as the ramification locus of the induced map φ on X is a closed \mathbb{G}_m -invariant subset, we see that φ is unramified on X and thus it is quasifinite. This implies that $\varphi(x') \neq \varphi(x)$ for all $x' \neq x \in X$, otherwise

by \mathbb{G}_m -translation we deduce that φ contracts the closed orbit $\overline{\mathbb{G}_m \cdot x'}$. In particular, $\varphi|_X^{-1}(\varphi(x))$ is supported at x ; as φ is also unramified, the scheme-theoretic preimage $\varphi|_X^{-1}(\varphi(x))$ equals $\{x\}$. By upper semicontinuity and the \mathbb{G}_m -action, this implies that $\varphi|_X^{-1}(\varphi(x'))$ has length 1 and thus consists of a single point for all $x' \in X$. It follows that $\varphi|_X$ is an embedding on X and we finish the proof. \square

We have used the following vanishing result in the above proof. This should be well-known to experts, but we are unable to find a suitable reference.

Lemma 3.9 *Let (X, Δ) be a pair such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let L be a \mathbb{Q} -Cartier Weil divisor such that $L - (K_X + \Delta)$ is nef and big. Assume that (X, Δ) is klt along the non-Cartier locus of L . Then*

$$H^i(X, \mathcal{F}(X, \Delta) \otimes \mathcal{O}_X(L)) = 0 \quad \text{for all } i > 0.$$

Proof If L is a line bundle this is just the usual Nadel vanishing; in the general case we follow the proof of Nadel vanishing. Let $f: Y \rightarrow X$ be a log resolution. We may write

$$K_Y + \Delta_Y = f^*(K_X + \Delta) + \sum a_i E_i \quad \text{and} \quad f^*L = \lfloor f^*L \rfloor + \sum b_i E_i,$$

where the E_i are the exceptional divisors, and $0 \leq b_i < 1$. Let

$$L' = \lfloor f^*L \rfloor - \lfloor \Delta_Y \rfloor + \sum [a_i + b_i] E_i \quad \text{and} \quad D = \{\Delta_Y\} + \sum \{-a_i - b_i\} E_i.$$

Then it's easy to check that (Y, D) is klt (i.e. $\{D\} = 0$ since Y is a log resolution) and $L' - (K_Y + D) \sim_{\mathbb{Q}} f^*(L - K_X - \Delta)$ is nef and big. By Kawamata–Viehweg vanishing we have $H^i(Y, \mathcal{O}_Y(L')) = 0$ and $R^i f_* \mathcal{O}_Y(L') = 0$ for all $i > 0$. It follows that $Rf_* \mathcal{O}_Y(L') = f_* \mathcal{O}_Y(L')$ and hence

$$H^i(X, f_* \mathcal{O}_Y(L')) = H^i(X, Rf_* \mathcal{O}_Y(L')) = H^i(Y, \mathcal{O}_Y(L')) = 0.$$

It remains to show that

$$(3-8) \quad f_* \mathcal{O}_Y(L') = \mathcal{F}(X, \Delta) \otimes \mathcal{O}_X(L).$$

This is a local question, so it suffices to check the equality at any $x \in X$. If (X, Δ) is klt at x , then locally $\mathcal{F}(X, \Delta) = \mathcal{O}_X$, $\lfloor \Delta_Y \rfloor = 0$ and $a_i > -1$, which implies $[a_i + b_i] \geq 0$. It follows that

$$f_* \mathcal{O}_Y(L') = \mathcal{O}_X(L) = \mathcal{F}(X, \Delta) \otimes \mathcal{O}_X(L).$$

If (X, Δ) is not klt at x , then L is Cartier around x , which gives $b_i = 0$ for every E_i whose image contains x , thus locally

$$f_* \mathcal{O}_Y(L') = f_* \mathcal{O}_Y(f^*L + \lceil -\Delta_Y + \sum a_i E_i \rceil) = \mathcal{F}(X, \Delta) \otimes \mathcal{O}_X(L)$$

by the projection formula and the definition of the multiplier ideal. Thus we see that (3-8) always holds. This completes the proof. \square

3.3 Proof of Theorem 3.1

We are now in a position to prove Theorem 3.1.

Proof First assume that $I \subseteq [0, 1] \cap \mathbb{Q}$. Let $x \in (X, \Delta; \xi)$ be a polarized log Fano cone singularity in \mathcal{S} and let \mathbb{T} be the torus generated by ξ . By Lemma 2.11, there exists some quasiregular polarization ξ' that is sufficiently close to ξ in the Reeb cone such that $\Theta(X, \Delta; \xi') \geq \frac{1}{2}\theta$. Using ξ' , we may realize (X, Δ) as an orbifold cone over some polarized pair $(V, \Delta_V; L)$. Moreover, if V_0 is the exceptional divisor of the orbifold vertex blowup as in Section 3.1, then ord_{V_0} is proportional to $\text{wt}_{\xi'}$. By assumption and Lemma 2.6, we obtain

$$(3-9) \quad \widehat{\text{vol}}_{X, \Delta}(\text{ord}_{V_0}) = \widehat{\text{vol}}_{X, \Delta}(\text{wt}_{\xi'}) \leq 2\theta^{-1} \widehat{\text{vol}}(x, X, \Delta) \leq 2\theta^{-1} n^n.$$

Let $(\bar{X}, \bar{\Delta})$ be the projective orbifold cone over $(V, \Delta_V; L)$ and let V_∞ be the divisor at infinity. By Proposition 3.8, there exists some integer m depending only on n, ε, A, I such that $|-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)|$ defines a birational map φ that restricts to an embedding on X . By (3-9) and Lemma 3.4, we see that

$$\text{vol}(-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)) = m^n \cdot \widehat{\text{vol}}_{X, \Delta}(\text{ord}_{V_0}) \leq 2\theta^{-1} (mn)^n$$

is bounded above.

Let us show that $\text{vol}(-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)|_{\bar{\Delta}})$ is also bounded from above, so that the pair $(\bar{X}, \bar{\Delta})$ belongs to a birationally bounded family. By Lemma 2.20, we know that $x \in (X, (1+c)\Delta)$ is of klt type for some constant $c = c(n, \varepsilon) > 0$. By Lemma 3.5, this implies that $-(K_V + (1+c)\Delta_V + \Delta_L)$ is big. By a similar calculation as in Lemma 3.4, it follows that

$$\begin{aligned} \text{vol}(-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)|_{\bar{\Delta}}) &= (-m(K_{\bar{X}} + \bar{\Delta} + V_\infty))^{n-2} \cdot r V_\infty \cdot \bar{\Delta} = r \cdot (-m(K_V + \Delta_V + \Delta_L))^{n-2} \cdot \Delta_V \\ &\leq c^{-1} r \cdot (-m(K_V + \Delta_V + \Delta_L))^{n-1} = c^{-1} \widehat{\text{vol}}_{X, \Delta}(\text{ord}_{V_0}) \leq 2(c\theta)^{-1} n^n, \end{aligned}$$

where the first equality is by Lemma 3.3, the second by adjunction along $V_\infty \cong V$, the next inequality by the bigness of $-(K_V + (1+c)\Delta_V + \Delta_L)$, and the last equality by Lemma 3.4. Thus

$$\text{vol}(-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)|_{\bar{\Delta}})$$

is also bounded from above, as desired. Note that $\varphi|_{\text{Supp}(\bar{\Delta})}$ is also birational since φ is an embedding at x . Therefore, the image (W, Δ_W) of $(\bar{X}, \bar{\Delta})$ under the birational map induced by $|-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)|$ belongs to a fixed bounded family $(\mathcal{W}, \bar{\mathcal{D}}) \rightarrow B$.

By construction, $(\bar{X}, \bar{\Delta})$ carries an effective \mathbb{T} -action and V_∞ is \mathbb{T} -invariant. Thus $|-m(K_{\bar{X}} + \bar{\Delta} + V_\infty)|$ is a \mathbb{T} -invariant linear system and the induced birational map φ is \mathbb{T} -equivariant. In particular, the image (W, Δ_W) also carries an effective \mathbb{T} -action. We claim that there exist finitely many morphisms $B_i \rightarrow B$ (depending only on the family $(\mathcal{W}, \bar{\mathcal{D}}) \rightarrow B$) such that

- (1) after base change, each family $(\mathcal{W}_i, \bar{\mathcal{D}}_i) = (\mathcal{W}, \bar{\mathcal{D}}) \times_B B_i \rightarrow B_i$ admits an effective fiberwise \mathbb{T}_i -action for some torus \mathbb{T}_i , and

- (2) $(W, \Delta_W) \cong (\mathcal{W}_{b_i}, \overline{\mathcal{D}}_{b_i})$ for some $b_i \in B_i$, and under this isomorphism, the \mathbb{T} -action on W is induced by some group homomorphism $\mathbb{T} \rightarrow \mathbb{T}_i$ (in other words, the \mathbb{T} -action on W is induced by the \mathbb{T}_i -action on \mathcal{W}_{b_i}).

To see this, first observe that for any torus action on a projective variety, the induced action on the Picard scheme is trivial. This is because both Pic^0 (an abelian variety) and Pic/Pic^0 (a discrete group) have no \mathbb{G}_m -action. Thus if \mathcal{L} is a relatively ample line bundle on \mathcal{W} , then it is invariant under any fiberwise torus action. Consider the relative automorphism group scheme \mathbb{G} over B , which parametrizes the automorphisms of the polarized fibers $(\mathcal{W}_b, \overline{\mathcal{D}}_b; \mathcal{L}_b)$. Note that \mathbb{G} is affine over B . Possibly after stratifying the base B , we may also assume that \mathbb{G} is smooth over B . By [12, Exposé XII, Théorème 1.7(a)], after a further stratification of B , we may assume that the dimension of the maximal torus of \mathbb{G}_b is locally constant in $b \in B$. We may discard the components of B where this torus dimension is zero. By [12, Exposé XII, Théorème 1.7(b)], it then follows that there exists an (étale) cover $\bigcup B_i \rightarrow B$ of the remaining components of B and a subgroup scheme $\mathbb{G}_i \subseteq \mathbb{G} \times_B B_i$ that reduces to the maximal tori on the fibers. Passing to a further finite cover of the B_i , we may assume that the \mathbb{G}_i are split, i.e. $\mathbb{G}_i = \mathbb{T}_i \times B_i$ for some torus \mathbb{T}_i . In particular, the \mathbb{G} -action on $(\mathcal{W}, \overline{\mathcal{D}})$ induces a fiberwise \mathbb{T}_i -action on $(\mathcal{W}_i, \overline{\mathcal{D}}_i) := (\mathcal{W}, \overline{\mathcal{D}}) \times_B B_i$. This gives the family in (1). Since (W, Δ_W) has a nontrivial torus action, we see that (W, Δ_W) appears as a fiber of $(\mathcal{W}_i, \overline{\mathcal{D}}_i) \rightarrow B_i$ for some i . Since all maximal tori in $\text{Aut}(W, \Delta_W)$ are conjugate to each other, we see that \mathbb{T} is conjugate to a subtorus of the maximal torus \mathbb{T}_i . In other words, there exists an isomorphism $(W, \Delta_W) \cong (\mathcal{W}_{b_i}, \overline{\mathcal{D}}_{b_i})$ such that (2) holds. This proves the claim.

Taking the fiberwise isolated \mathbb{T}_i -fixed points $(\mathcal{W}_i, \overline{\mathcal{D}}_i) \rightarrow B_i$ for all i , we get families $B_i \subseteq (\mathcal{X}_i, \mathcal{D}_i) \rightarrow B_i$ of singularities (possibly after a refinement of the B_i), each with an effective fiberwise torus \mathbb{T}_i -action. Since φ restricts to an embedding on X , by the second part of the above claim we see that $x \in (X, \Delta; \xi)$ is isomorphic to $b_i \in (\mathcal{W}_{b_i}, \overline{\mathcal{D}}_{b_i}; \xi_{b_i})$ for some $b_i \in B_i$ and some $\xi_{b_i} \in N(\mathbb{T}_i)_{\mathbb{R}}$. In particular, this gives log boundedness. Note that when $I \not\subseteq \mathbb{Q}$, we can still apply the above argument to the singularities and the coefficient set constructed from Lemma 2.18, thus the same conclusion holds.

To get boundedness, we need to further stratify the family $(\mathcal{W}_i, \overline{\mathcal{D}}_i) \rightarrow B_i$ so it becomes \mathbb{R} -Gorenstein klt. By [22, Lemma 4.44] and inversion of adjunction, there exists a finite collection of locally closed subsets S_j of $\bigcup_i B_i$ such that the family becomes \mathbb{R} -Gorenstein after base change to $\bigcup_j S_j$ and enumerates exactly all the klt fibers of $\bigcup_i (\mathcal{W}_i, \overline{\mathcal{D}}_i) \rightarrow \bigcup_i B_i$. (Note that [22, Lemma 4.44] requires the family to be proper but the proof applies to our situation, essentially because the section $B_i \subseteq \mathcal{X}_i$ is proper over the base.) Replacing the B_i by the S_j , we obtain the desired family. The proof is now complete. \square

3.4 Applications

We now explain how to deduce the other main results of this paper from Theorem 3.1. First we prove the boundedness criterion for \mathbb{K} -semistable log Fano cone singularities.

Corollary 3.10 *Let \mathcal{S} be a set of n -dimensional K -semistable log Fano cone singularities with coefficients in a fixed finite set $I \subseteq [0, 1]$. Then \mathcal{S} is bounded if and only if there exist some $\varepsilon, A > 0$ such that*

$$\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon \quad \text{and} \quad \text{mld}^K(x, X, \Delta) \leq A$$

for all $x \in (X, \Delta)$ in \mathcal{S} .

Proof One direction follows from Lemma 2.15, while the other direction is implied by Theorem 3.1 as the volume ratio of a K -semistable log Fano cone singularity (with the K -semistable polarization) is equal to 1. □

Similarly, we have an unpolarized version of Theorem 3.1.

Corollary 3.11 *Let \mathcal{S} be a set of n -dimensional log Fano cone singularities with coefficients in a fixed finite set $I \subseteq [0, 1]$. Then \mathcal{S} is bounded if and only if there exist positive constants $\varepsilon, \theta, A > 0$ such that*

$$\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon, \quad \Theta(x, X, \Delta) \geq \theta, \quad \text{mld}^K(x, X, \Delta) \leq A,$$

for all $x \in (X, \Delta)$ in \mathcal{S} .

Proof One direction follows from Lemmas 2.15 and 2.17, while the other direction is immediate by Theorem 3.1. □

We next prove a version of Theorem 3.1 that replaces the volume ratios with the stability thresholds introduced in [17]. First we recall the definition. Let $x \in (X = \mathbf{Spec}(R), \Delta; \xi)$ be a polarized log Fano cone singularity and let \mathbb{T} be the torus generated by ξ . For this definition it is more convenient to rescale the polarization so that $A_{X, \Delta}(\text{wt}_\xi) = 1$, which we will assume in what follows. Using the weight decomposition $R = \bigoplus_\alpha R_\alpha$, we set

$$R_m := \bigoplus R_\alpha.$$

An m -basis type \mathbb{Q} -divisor of $x \in (X = \mathbf{Spec}(R), \Delta; \xi)$ is defined to be a \mathbb{Q} -divisor of the form

$$D = \frac{1}{mN_m} \sum_{i=1}^{N_m} \{s_i = 0\},$$

where $N_m = \dim R_m$ and s_1, \dots, s_{N_m} form a basis of R_m . Set $\delta_m = \inf_D \text{lct}_x(X, \Delta; D)$, where the infimum runs over all m -basis type \mathbb{Q} -divisors D . The stability threshold of $x \in (X = \mathbf{Spec}(R), \Delta; \xi)$ is defined as

$$\delta(X, \Delta; \xi) := \lim_{m \rightarrow \infty} \delta_m.$$

If $(X, \Delta; \xi)$ is the cone over a log Fano pair (V, Δ_V) , then this definition is closely related to the stability threshold of (V, Δ_V) introduced in [13]; see also [5]. In fact, using inversion of adjunction it is not hard to show that $\delta(X, \Delta; \xi) = \min\{1, \delta(V, \Delta_V)\}$; cf. [39, Theorem 3.6].

The stability threshold version of Theorem 3.1 is a direct consequence of the following result.

Lemma 3.12 *Let $x \in (X = \text{Spec}(R), \Delta; \xi)$ be a polarized log Fano cone singularity of dimension n . Then*

$$\Theta(X, \Delta; \xi) \geq \delta(X, \Delta; \xi)^n.$$

Proof Let $\delta := \delta(X, \Delta; \xi)$. We first recall the valuative interpretation of $\delta(X, \Delta; \xi)$ as explained in [17]. Fix some \mathbb{T} -invariant valuation $v \in \text{Val}_{X,x}^*$. Set $S_m(v) := \sup_D v(D)$, where D varies among m -basis type \mathbb{Q} -divisors. By an Okounkov body argument [17, Section 4], we know that $S(v) := \lim_{m \rightarrow \infty} S_m(v)$ exists and, by [17, Theorem 4.3.5],

$$(3-10) \quad A_{X,\Delta}(v) \geq \delta \cdot S(v).$$

We next relate $S(v)$ to the “relative” S -invariant $S(\text{wt}_\xi; v)$ defined in [39, Section 3.1]. In our notation (and under our assumption that $A_{X,\Delta}(\text{wt}_\xi) = 1$), we have $S(\text{wt}_\xi; v) = \lim_{m \rightarrow \infty} S_m(\text{wt}_\xi; v)$, where

$$S_m(\text{wt}_\xi; v) := \frac{\sum_{k=0}^m k N_k S_k(v)}{\sum_{k=0}^m k N_k}.$$

Intuitively, the previous $S_m(v)$ is defined using basis type divisors for R_m while $S_m(\text{wt}_\xi; v)$ is defined via basis type divisors for $\bigoplus_{k \leq m} R_k$ (the main reason for doing so in [39] is that in the more general situation, the dimension of the analogous space $\bigoplus_{k \leq m} R_k$ has an asymptotic expression, while the individual R_m does not). Note that our $S_m(\text{wt}_\xi; v)$ differs slightly from the one in [39, Section 3.1] by some round-downs, but after taking the $m \rightarrow \infty$ limit we get the same value of $S(\text{wt}_\xi; v)$. Since $S_m(v) \rightarrow S(v)$ as $m \rightarrow \infty$, from the above expression we get $S(v) = S(\text{wt}_\xi; v)$. From the proof of [39, Theorem 3.7], especially the inequality after (3.8) in loc. cit., we then obtain

$$S(v) \geq \left(\frac{\text{vol}(\text{wt}_\xi)}{\text{vol}(v)} \right)^{1/n}.$$

Combined with (3-10) and recalling that $A_{X,\Delta}(\text{wt}_\xi) = 1$, we deduce

$$\widehat{\text{vol}}_{X,\Delta}(v) = A_{X,\Delta}(v)^n \cdot \text{vol}(v) \geq \delta^n S(v)^n \cdot \text{vol}(v) \geq \delta^n \text{vol}(\text{wt}_\xi) = \delta^n \widehat{\text{vol}}_{X,\Delta}(\text{wt}_\xi)$$

for all \mathbb{T} -invariant valuation $v \in \text{Val}_{X,x}^*$. Since the local volume of $x \in (X, \Delta)$ is computed by some \mathbb{T} -invariant valuation (see [4] and [39, Corollary 1.2]), this implies that $\widehat{\text{vol}}(x, X, \Delta) \geq \delta^n \widehat{\text{vol}}_{X,\Delta}(\text{wt}_\xi)$ and hence $\Theta(X, \Delta; \xi) \geq \delta^n$. \square

Remark 3.13 We also have an inequality in the reverse direction, namely, there exists some positive constant $c > 0$, that only depends on the dimension, such that

$$(3-11) \quad c \cdot \delta(X, \Delta; \xi) \geq \Theta(X, \Delta; \xi).$$

To see this, let $c = c_0^{-1}$, where c_0 is the constant from [42, Lemma 3.4], and let D be any m -basis type \mathbb{Q} -divisor of $(X, \Delta; \xi)$. By definition we have $\text{wt}_\xi(D) \leq 1 = A_{X,\Delta}(\text{wt}_\xi)$, hence by loc. cit. we see that

$$\text{lct}_x(X, \Delta; D) \geq c_0 \cdot \Theta(X, \Delta; \xi),$$

which gives (3-11). Note that the upper bound on the volume ratio is at least linear in the stability threshold, as can be seen on the smooth toric singularities $(X, \Delta) = (\mathbb{A}^n, 0)$: if $\xi = (a_1, \dots, a_n) \in \mathfrak{t}_{\mathbb{R}}^+ = \mathbb{R}_{>0}^n$ and $a_1 \ll a_2 = \dots = a_n$, then we have that $\Theta(X, \Delta; \xi) = n^n a_1 \cdots a_n / (a_1 + \dots + a_n)^n$ is roughly linear in $\delta(X, \Delta; \xi) = na_1 / (a_1 + \dots + a_n)$.

Corollary 3.14 *Let $n \in \mathbb{N}$ and let $I \subseteq [0, 1]$ be a finite set. Let $\varepsilon, \delta, A > 0$. Let \mathcal{S} be the set of n -dimensional polarized log Fano cone singularities $x \in (X, \Delta; \xi)$ with coefficients in I such that*

$$\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon, \quad \delta(X, \Delta; \xi) \geq \delta, \quad \text{mld}^K(x, X, \Delta) \leq A.$$

Then \mathcal{S} is bounded.

Proof This is a direct consequence of Theorem 3.1 and Lemma 3.12. □

Finally we specialize our results to dimension three. For this we need the following result from [42].

Proposition 3.15 *Let $\varepsilon > 0$ and let $I \subseteq [0, 1]$ be a finite set. Then there exists some constant $A = A(\varepsilon, I)$ such that*

$$\text{mld}^K(x, X, \Delta) \leq A$$

for all 3-dimensional klt singularities $x \in (X, \Delta)$ with coefficients in I and $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$.

Proof If $I \subseteq \mathbb{Q}$ this is [42, Corollary 6.11]; in general we apply Lemma 2.18 to reduce to the rational coefficient case. □

Corollary 3.16 *For any finite set $I \subseteq [0, 1]$ and any $\varepsilon > 0$, the set of 3-dimensional K -semistable log Fano cone singularities with coefficients in I and with local volume at least ε is bounded.*

Proof Immediate from Corollary 3.10 and Proposition 3.15. □

The last application concerns the distribution of local volumes in dimension 3. For any $n \in \mathbb{N}$ and $I \subseteq [0, 1]$, consider the set $\text{Vol}_{n,I}^{\text{loc}}$ of all possible local volumes of n -dimensional klt singularities $x \in (X, \Delta)$ with coefficients in I .

Corollary 3.17 *Let $I \subseteq [0, 1]$ be a finite set. Then $\text{Vol}_{3,I}^{\text{loc}}$ is discrete away from zero.*

Proof By Theorem 2.19, it suffices to consider local volumes of K -semistable log Fano cone singularities. Let $\varepsilon > 0$. By Corollary 3.16, the set of 3-dimensional K -semistable log Fano cone singularities $x \in (X, \Delta)$ with coefficients in I and $\widehat{\text{vol}}(x, X, \Delta) \geq \varepsilon$ is bounded. On the other hand, the local volume function is constructible in \mathbb{R} -Gorenstein families by [37, Theorem 1.3]; see [16, Theorem 3.5] for the real coefficient case. In particular, the local volumes only take finitely many possible values in a bounded family. This implies $\text{Vol}_{3,I}^{\text{loc}} \cap [\varepsilon, +\infty)$ is a finite set, and we are done. □

Remark 3.18 By combining the ideas in this work with some generalization of the proof of [16, Theorem 1.2(2)], one should be able to show that if the set I in Corollary 3.17 is not finite but satisfies DCC (descending chain condition), then $\text{Vol}_{3,I}^{\text{loc}}$ satisfies ACC (ascending chain condition). We leave the details to the interested readers.

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