Geometry & Topology

Volume 28 (2024)

Lee filtration structure of torus links

QIUYU REN

msp

Geometry & Topology **28**:8 (2024) 3935–3960
DOI: 10.2140/gt.2024.28.3935
Published: 20 December 2024

Lee filtration structure of torus links

QIUYU REN

We determine the quantum filtration structure of the Lee homology of all torus links. In particular, this determines the s–invariant of a torus link equipped with any orientation. In the special case $T(n,n)$, our result confirms a conjecture of Pardon, as well as a conjecture of Manolescu, Marengon, Sarkar and Willis which establishes an adjunction-type inequality of the s–invariant for cobordisms in $k\overline{\mathbb{CP}}^2$. We also give a few applications of this adjunction inequality.

57K18; 57K10, 57K40

1 Introduction

Khovanov homology is a link invariant introduced by Khovanov [2000], which categorifies the Jones polynomial. Despite its computability for every given link diagram, few families of links have had their Khovanov homology fully determined. Notable examples where the complete answer is known include alternating links [Lee 2005] (more generally, quasialternating links [Manolescu and Ozsváth 2008]), and the torus links $T(2,m)$ [Khovanov 2000] and $T(3,m)$ [Turner 2008; Stošić 2009; Gillam 2012; Benheddi 2017].

Specifically, in the case of positive torus links $T(n,m)$, although the Jones polynomials have a well-known closed form, this is far from true for Khovanov homology. The investigation of the Khovanov homology of torus links dates back at least to Stošić [2007; 2009], where he calculated, for example, some low ($h \leq 4$) homological degree parts of $\mathrm{Kh}(T(n,m))$ and the highest ($h = 2kn^2$) homological degree part of $\mathrm{Kh}(T(2n,2kn))$. More interestingly, he showed the existence of the "stable Khovanov homology groups" of $T(n,m)$ as $m \to \infty$, which have since been extensively investigated; see for example [Gorsky et al. 2013]. It is also worth remarking that the Khovanov–Rozansky triply graded homology of $T(n,m)$ was completely determined [Hogancamp and Mellit 2019]. However, a comprehensive understanding of the ordinary Khovanov homology for torus links remains elusive.

Nevertheless, many useful knot or link invariants that are more computable and possess desirable properties can be derived from Khovanov homology or its variants. A notable example is Rasmussen's s–invariant [2010] for knots, extracted from Lee homology [2005], whose values on torus knots were computed and played a crucial role in providing the first gauge-theory-free proof of Milnor's conjecture on the slice genus of torus knots. In the case of links, the quantum filtration structure of the Lee homology can be considered as a natural generalization of the s–invariant for knots, encompassing the generalization proposed by Beliakova and Wehrli [2008] and Pardon [2012] as special cases. We completely determine the quantum filtration structure of the Lee homology of all torus links.

For ease of exposition, we only state our result in terms of Beliakova and Wehrli's s–invariant for oriented links and Pardon's invariants as the bigraded dimension of the associated graded vector space of the Lee homology. The actual quantum filtration structure, ie the quantum filtration degree function $q\colon \mathrm{Kh}_{\mathrm{Lee}} \to \mathbb{Z} \sqcup \{+\infty\}$, will become apparent during the proof of Theorem 1.2.

In the statements below, let n and m be two positive integers, $d = \gcd(n, m)$, $n_1 = n/d$ and $m_1 = m/d$. For $p, q \geq 0$ with $p + q = d$, let $T(n, m)_{p,q}$ denote the torus link $T(n, m)$ equipped with an orientation in which p of the components are oriented oppositely to the other q components.

Theorem 1.1 *The s–invariant of $T(n, m)_{p,q}$, over any coefficient field, is given by*

$$s(T(n, m)_{p,q}) = (n_1|p-q|-1)(m_1|p-q|-1) - 2\min(p, q).$$

Here the s–invariant of an oriented link over any field of characteristic not equal to 2 is defined in the same way as in [Beliakova and Wehrli 2008]. Over characteristic 2, one should use the Bar-Natan deformation (see [Bar-Natan 2005]) of Khovanov homology instead of the Lee deformation.

By construction of the Khovanov/Lee complex, the Lee homology of $T(n, m)_{p,q}$, as a homologically graded and quantum filtered vector space, equals that of the positive torus link $T(n, m) := T(n, m)_{d,0}$, up to a bidegree shift. Thus we will only state the structure of $\mathrm{Kh}_{\mathrm{Lee}}(T(n, m))$. By Lee [2005, Proposition 4.3], $\mathrm{Kh}_{\mathrm{Lee}}(T(n, m))$ as a homologically graded vector space is determined by the linking matrix of $T(n, m)$. Explicitly,

$$\dim \mathrm{Kh}_{\mathrm{Lee}}^{2n_1m_1pq}(T(n, m)) = \begin{cases} 2\binom{d}{q} & p \neq q, \\ \binom{d}{q} & p = q, \end{cases}$$

for every pair of nonnegative integers (p, q) with $p + q = d$, with other graded components being zero. The following theorem determines (the isomorphism type of) its quantum filtration structure:

Theorem 1.2 *The associated graded vector space of the Lee homology (over $\mathbb{Q}$) of the positive torus link $T(n, m)$ is determined by*

$$\dim \mathrm{gr}(\mathrm{Kh}_{\mathrm{Lee}}(T(n, m)))^{2n_1m_1pq,\,6n_1m_1pq+s(T(n,m)_{p,q})+2r-1}$$
$$= \begin{cases} \dim(d-r, r) & r = 0 \text{ and } p \neq q, \\ \dim(d-r, r) + \dim(d-r+1, r-1) & 0 < r < \min(p, q)+1 \text{ and } p \neq q, \\ \dim(d-r+1, r-1) & r = \min(p, q)+1 \text{ and } p \neq q, \\ \dim(d-r, r) & 0 \leq r \leq \min(p, q) \text{ and } p = q, \end{cases}$$

for every pair of nonnegative integers (p, q) with $p + q = d$, with all other bigraded components being zero. Here (a, b) denotes the irreducible representation of the symmetric group S_{a+b} given by the two-row Young diagram (a, b). Thus

$$\dim(d-r, r) = \binom{d}{r} - \binom{d}{r-1} \quad \text{for } 0 \leq r \leq \tfrac{1}{2}d.$$

The appearance of S_d–representations in Theorem 1.2 is no coincidence. Indeed, we will show that $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$ carries a filtered S_d–action, and determine its structure as a filtered S_d–representation.

In the special case $m = n$, Theorem 1.2 confirms a conjecture of Pardon [2012, Section 5.2].

Since taking the mirror image of a link has the effect of taking the dual on the Lee homology, the quantum filtration structure of $T(n,-m)$ is determined by that of $T(n,m)$. We can also read off its s–invariants as follows (see the second paragraph in the proof of Theorem 1.1).

Corollary 1.3 *The s–invariant, over $\mathbb{Q}$, of $T(n,-m)_{p,q}$ is given by*

$$s(T(n,-m)_{p,q}) = \begin{cases} -(n_1|p-q|-1)(m_1|p-q|-1) & p \neq q, \\ 1 & p = q. \end{cases} \qquad \square$$

As observed by Manolescu, Marengon, Sarkar and Willis [Manolescu et al. 2023], Theorem 1.1 in the special case $m = n$ implies the following corollary, which is an adjunction-type inequality for s–invariants of nullhomologous oriented links in a connected sum of $(S^1 \times S^2)$'s, as defined in their paper. We remark that, however, one cannot deduce an adjunction-type inequality from Corollary 1.3 using the same proof.

Corollary 1.4 *If $\Sigma \subset Z = (I \times l(S^1 \times S^2)) \# k\overline{\mathbb{CP}}^2$ is an oriented cobordism between two nullhomologous oriented links $L_0, L_1 \subset l(S^1 \times S^2)$ with $\pi_0(L_1) \to \pi_0(\Sigma)$ surjective, then*

$$s(L_1) \leq s(L_0) - \chi(\Sigma) - [\Sigma]^2 - \|[\Sigma]\|'.$$

Here $\|[\Sigma]\|'$ is defined as the sum $\sum_{i=1}^{k} |[\Sigma] \cdot z_i|$, where $z_1, \dots, z_k \in H_2(Z) \cong \mathbb{Z}^{l+k}$ are the generators coming from the $\overline{\mathbb{CP}}^2$ factors.

Corollary 1.4 holds over any coefficient field as long as $l = 0$. For $l > 0$, in [Manolescu et al. 2023] the s–invariant is only defined over fields with characteristic not equal to 2, although we expect everything to hold in characteristic 2 as well.

The term $[\Sigma]^2$ above is well defined, and the term $\|[\Sigma]\|'$ is independent of the choice of the decomposition $Z = (I \times l(S^1 \times S^2)) \# k\overline{\mathbb{CP}}^2$, both thanks to the links L_i being nullhomologous. Of course, one may also dualize and obtain a similar adjunction inequality in $(I \times l(S^1 \times S^2)) \# k\mathbb{CP}^2$ (see Section 2.3).

In practice, the special case $l = 0$ might be the most useful, where s reduces to Beliakova and Wehrli's generalization of the classical Rasmussen s–invariant (defined in Section 2.2). In particular, this opens a new approach to detect exotic $k\overline{\mathbb{CP}}^2$ whose existence is not yet known, for example by modifying the constructions in [Manolescu and Piccirillo 2023]. When $l = 0$, $L_0 = \varnothing$ and $L_1 = K$ is a knot, Corollary 1.4 takes the following form:

Corollary 1.5 [Manolescu et al. 2023, Conjecture 9.8] *If $(\Sigma, K) \subset ((k\overline{\mathbb{CP}}^2) \backslash B^4, S^3)$ is a connected orientable properly embedded surface with boundary a knot K, then*

$$s(K) \leq 1 - \chi(\Sigma) - [\Sigma]^2 - \|[\Sigma]\|.$$

Here $\|[\Sigma]\|$ is the L^1–norm of $[\Sigma]$ (denoted by $\|[\Sigma]\|'$ in Corollary 1.4). $\square$

The adjunction-type inequality parallel to Corollary 1.5 for the τ–invariant in knot Floer homology was established by Ozsváth and Szabó [2003] two decades ago. The inequality parallel to the more general Corollary 1.4 appeared recently (for knots, and in some special cases for links) in the work of Hedden and Raoux [2023]. In fact, their inequalities were stated for more general smooth 4–manifolds. This naturally leads us to the question of whether Corollaries 1.4 and 1.5 can be generalized to those settings. Of course, this (in its full generality) entails generalizing the s–invariant to rationally nullhomologous links in arbitrary closed oriented 3–manifolds. It is worth remarking, however, that in order to successfully construct exotic $k\overline{\mathbb{CP}}^2$'s detectable by the s–invariant using a modified version of the construction in [Manolescu and Piccirillo 2023], one should hope that Corollary 1.5 does not have a generalization applicable to arbitrary simply connected negative-definite 4–manifolds.

Two applications of the adjunction inequality for the s–invariant will be given in Section 3.

We summarize the paper's structure and briefly describe the proofs of the main results:

In Section 2.1, we state a "graphical lower bound", Theorem 2.1 (see also Figures 1 and 2), for the Khovanov homology of the torus links $T(n,n)$ and the torus knots $T(n+1,n)$. In the case of $T(n,n)$, in homological degrees $2pq$ with $p+q=n$, the bound is sharp, and the nonvanishing groups with the lowest quantum degrees are $\mathbb{Z}$, which also give rises to Lee homology generators. As we shall see in Section 2.2, this implies Theorem 1.1 in the special case $m=n$. The general case is then proved in Section 2.3 inductively, using the adjunction inequality (Corollary 1.4). The proof of Theorem 2.1, which follows the induction scheme set up by Stošić [2007; 2009], is a cumbersome verification that is not illuminating and is deferred to Section 5.

As further illustrations of the power of the adjunction inequality, we give two applications. In Section 3.1, we provide an optimal bound on the change of the s–invariant when full twists are applied to an oriented link. In particular, the s–invariant grows linearly when the twist number is sufficiently large, answering positively a question in [Manolescu et al. 2023]. In Section 3.2, we show there exist knots in S^3 with simultaneously large $\mathbb{CP}^2$–genus and $\overline{\mathbb{CP}}^2$–genus.

Section 4 is independent of Section 3. We exploit an S_d–symmetry and decompose $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$ into irreducible S_d–representations. By working equivariantly, we are able to deduce Theorem 1.2 inductively from Theorem 1.1.

Finally, in Section 6, we state a numerical observation as an open question. In particular, we propose a conjectural (recursive) formula for the rational Khovanov homology of $T(n,n)$.

Acknowledgements I would like to thank my advisor Ian Agol for insights, discussions and encouragement. I thank Qianhe Qin for introducing me to this question, Marco Marengon for kindly providing me with the proof of Proposition 3.5, Michael Willis for feedback on a first draft of this paper, and Ciprian Manolescu and Melissa Zhang for many helpful discussions. I also thank the referee for a careful read of the paper and many valuable suggestions.

2 The s–invariants of $T(n,m)$

2.1 A graphical lower bound for $\mathrm{Kh}(T(n,n))$ and $\mathrm{Kh}(T(n+1,n))$

In this section we state our core technical result, which gives a "graphical lower bound" on the Khovanov homology of $T(n,n)$ and $T(n+1,n)$, whose proof is postponed to Section 5. We assume the reader is familiar with Khovanov homology and Lee homology, and refer to the literature mentioned in the introduction if otherwise. See also [Bar-Natan 2002] for a short introduction.

Before stating the theorem, we introduce two families of auxiliary functions $q_{n,n}, q_{n+1,n} : \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$, which serve as quantum lower bounds for the Khovanov homology of $T(n,n)$ and $T(n+1,n)$ in various homological degrees, respectively. Let

$$h_{\max}(T(n,n)) := \lfloor \tfrac{1}{2}n^2 \rfloor \quad \text{and} \quad h_{\max}(T(n+1,n)) := \lfloor \tfrac{1}{2}n^2 \rfloor + \lfloor \tfrac{1}{2}n \rfloor.$$

The functions $q_{n,n}$ are defined by

(a) $q_{n,n}(h) = +\infty$ for $h < 0$ or $h > h_{\max}(T(n,n))$,

(b) $q_{n,n}(0) = n^2 - 2n$,

(c) if $p + q = n$ for $p \geq q > 0$, then for $2(p+1)(q-1) < h \leq 2pq$,

$$q_{n,n}(h) = n^2 + 2\lceil \tfrac{1}{2}h \rceil - 2p.$$

The functions $q_{n+1,n}$ are defined by

(a) $q_{n+1,n}(h) = +\infty$ for $h < 0$ or $h > h_{\max}(T(n+1,n))$,

(b) if $p + q = n$ for $p \geq q > 0$, then

$$q_{n+1,n}(2pq+1) = q_{n,n}(2pq+1) + n - 3,$$

(c) for other $0 \leq h \leq h_{\max}(T(n,n))$,

$$q_{n+1,n}(h) = q_{n,n}(h) + n - 1,$$

(d) for $h_{\max}(T(n,n)) \leq h \leq h_{\max}(T(n+1,n))$,

$$q_{n+1,n}(h) = \lfloor \tfrac{1}{2}n^2 \rfloor + 2h - 1.$$

Note the two definitions of $q_{n+1,n}(h_{\max}(T(n,n)))$ via (c) and (d) agree.

We now state the main technical theorem. See Figures 1 and 2 for an illustration of the case $n = 6$. The reader is warned that the same letter q is (unfortunately) used for two different purposes: the quantum degree and an integer between 0 to n that is complementary to p. It should be clear from context which of these is referred to.

Theorem 2.1 (i) $\mathrm{Kh}^{h,q}(T(n,n)) = 0$ *for* $q < q_{n,n}(h)$*. Moreover, for every* $p + q = n$ *with* $p, q > 0$*, the saddle cobordism* $T(n-2,n-2) \sqcup U \to T(n,n)$ *induces an isomorphism*

$$\mathrm{Kh}^{2(p-1)(q-1),\, q_{n-2,n-2}(2(p-1)(q-1))-1}(T(n-2,n-2) \sqcup U) \xrightarrow{\cong} \mathrm{Kh}^{2pq,\, q_{n,n}(2pq)}(T(n,n)).$$

(ii) $\mathrm{Kh}^{h,q}(T(n+1,n)) = 0$ *for* $q < q_{n+1,n}(h)$*. Moreover,* $\mathrm{Kh}^{2n-1,\, q_{n+1,n}(2n-1)}(T(n+1,n))$ *is torsion.*

q \ h	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
54																			$\mathbb{Z}^5$
52																			$\mathbb{Z}^9$
50																$\mathbb{Z}^5$	$\mathbb{Z}^9$		$\mathbb{Z}^5$
48															$\mathbb{Z}_3$	$\mathbb{Z}^6$	$\mathbb{Z}^{14}$		$\boxed{\mathbb{Z}}$
46													$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}^5$	$\mathbb{Z}$	$\mathbb{Z}^6$		
44													$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}^2 \oplus (\mathbb{Z}_2)^5$	$\mathbb{Z}^6$		$\boxed{\mathbb{Z}}$		
42											$\mathbb{Z}_2$	$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}_2$	$\mathbb{Z}$				
40										$\mathbb{Z}^2$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5$	$\mathbb{Z}^2$						
38								$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^7$								
36						$\mathbb{Z}$		$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^2$		$\boxed{\mathbb{Z}}$								
34						$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$										
32				$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$												
30				$\mathbb{Z}_2$	$\mathbb{Z}$														
28			$\mathbb{Z}$																
26	$\mathbb{Z}$																		
24	$\boxed{\mathbb{Z}}$																		

Figure 1: The Khovanov homology of $T(6,6)$. The lower bound given by $q_{6,6}$ is shown by the thick bars. The homology groups $\mathrm{Kh}^{2pq,q_{6,6}(2pq)}$ are circled.

Corollary 2.2 *For any $p+q=n$ with $p,q \geq 0$, we have* $\mathrm{Kh}^{2pq,q_{n,n}(2pq)}(T(n,n)) = \mathbb{Z}$.

Proof For $pq=0$, this follows from Stošić's calculation [2007, Theorem 3.4] that $\mathrm{Kh}^{0,*}(T(n,n)) = \mathbb{Z}$ for $* = (n-1)^2 \pm 1$ and 0 otherwise. The general case then follows by induction using the last statement in Theorem 2.1(i). □

The case $p = n-1$ and $q = 1$ of Corollary 2.2 confirms a conjecture of Stošić [2007, Conjecture 3.8].

2.2 Proof of Theorem 1.1 for $m = n$

We prove Theorem 1.1 in the special case $m=n$, assuming Theorem 2.1. By symmetry we may assume $p \geq q$. The statement then takes the form $s(T(n,n)_{p,q}) = (p-q-1)^2 - 2q$.

We induct on n. The base cases $n = 1, 2$ are easily checked. Assume $n \geq 3$. If $q = 0$, the result follows either from the sharpness of Kawamura and Lobb's inequality on s–invariants for nonsplit positive links [Abe and Tagami 2017, Corollary 2.4], or Stošić's calculation of $\mathrm{Kh}^{0,*}(T(n,n))$ mentioned above. Assume from now on $q > 0$.

The linking matrix of $T(n,n)$ is $(l_{ij})_{1 \leq i,j \leq n}$, where $l_{ij} = 0$ if $i = j$ and 1 otherwise. Thus by Lee [2005, Proposition 4.3], $\mathrm{Kh}^{2pq}_{\mathrm{Lee}}(T(n,n))$ is the vector space spanned by the canonical generators $[s_{\mathfrak{o}}]$, one for each orientation $\mathfrak{o}$ of $T(n,n)$ realizing $T(n,n)_{p,q}$; see also [Rasmussen 2010]. Moreover, the Lee homology of $T(n,n)$ is that of $T(n,n)_{p,q}$ with a bidegree shift $[2pq]\{6pq\}$ [Khovanov 2000, Proposition 28, typo].

$q \backslash h$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
61																					$\mathbb{Z}_3$	$\mathbb{Z}_2$
59																					$\mathbb{Z}_2\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$
57																	$\mathbb{Z}$		$\mathbb{Z}_4$	$\mathbb{Z}$	$\mathbb{Z}_2$	
55																	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_2\oplus\mathbb{Z}_3$		
53															$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$			
51													$\mathbb{Z}$	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}^2\oplus\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$				
49													$\mathbb{Z}_2\oplus\mathbb{Z}_5$	$\mathbb{Z}^3\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$					
47											$\mathbb{Z}_2$	$\mathbb{Z}^3$	$\mathbb{Z}$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}$							
45										$\mathbb{Z}^2$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus(\mathbb{Z}_2)^2$	$\mathbb{Z}^2$									
43								$\mathbb{Z}$		$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}^2$	$\boxed{\mathbb{Z}_2}$										
41						$\mathbb{Z}$		$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}^2$		$\mathbb{Z}$											
39						$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$													
37				$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$															
35				$\mathbb{Z}_2$	$\mathbb{Z}$																	
33			$\mathbb{Z}$																			
31	$\mathbb{Z}$																					
29	$\mathbb{Z}$																					

Figure 2: The Khovanov homology of $T(7,6)$. The lower bound given by $q_{7,6}$ is shown by the thick bars. The homology group $\mathrm{Kh}^{2\cdot 6-1,q_{7,6}(2\cdot 6-1)}$ is boxed.

According to Beliakova and Wehrli's definition of the s–invariant [2008, Definition 7.1], $s(T(n,n)_{p,q})$ over $\mathbb{Q}$ equals the quantum filtration degree of $[s_{\mathfrak{o}}] \in \mathrm{Kh}_{\mathrm{Lee}}(T(n,n)_{p,q})$ plus 1, where $\mathfrak{o}$ is any orientation that realizes $T(n,n)_{p,q}$. Taking into account the degree shift $6pq$, we need to prove $[s_{\mathfrak{o}}] \in \mathrm{Kh}_{\mathrm{Lee}}(T(n,n))$ has quantum filtration degree $(p-q-1)^2 - 2q - 1 + 6pq = q_{n,n}(2pq)$.

Every element of $\mathrm{Kh}^{2pq}_{\mathrm{Lee}}(T(n,n))$ is a linear combination of these $[s_{\mathfrak{o}}]$, and thus has quantum filtration degree no less than that of these $[s_{\mathfrak{o}}]$. Thus it remains to prove the lowest filtration level of $\mathrm{Kh}^{2pq}_{\mathrm{Lee}}(T(n,n))$ is $q_{n,n}(2pq)$.

This is equivalent to showing that $\mathrm{Kh}^{2pq,q_{n,n}(2pq)}(T(n,n)) \otimes \mathbb{Q} \cong \mathbb{Q}$ (see Corollary 2.2) survives to the E_∞ page in the Lee spectral sequence from $E_1 = \mathrm{Kh}(T(n,n)) \otimes \mathbb{Q}$ to $\mathrm{Kh}_{\mathrm{Lee}}(T(n,n))$. The differential on the E_r page has bidegree $(1,4r)$. By Theorem 2.1, $\mathrm{Kh}^{2pq-1,*}(T(n,n)) = 0$ for $* < q_{n,n}(2pq-1) = q_{n,n}(2pq)$, so $\mathrm{Kh}(T(n,n)) \otimes \mathbb{Q}$ cannot be annihilated by differentials mapping into it. To see that all differentials out of it are zero, we observe that the naturality of the Lee spectral sequence applied to the cobordism $T(n-2,n-2) \to T(n,n)$ gives the following commutative diagram on page E_r:

$$\begin{array}{ccc}
\mathrm{Kh}^{2(p-1)(q-1),q_{n-2,n-2}(2(p-1)(q-1))}(T(n-2,n-2)) \otimes \mathbb{Q} & \xrightarrow{\cong} & \mathrm{Kh}^{2pq,q_{n,n}(2pq)}(T(n,n)) \otimes \mathbb{Q} \\
\big\downarrow d_r = 0 & & \big\downarrow d_r \\
E_r^{2(p-1)(q-1)+1,q_{n-2,n-2}(2(p-1)(q-1))+4r}(T(n-2,n-2)) & \longrightarrow & E_r^{2pq+1,q_{n,n}(2pq)+4r}(T(n,n)).
\end{array}$$

Here the vertical map on the left is zero by the induction hypothesis and the horizontal map on the top is an isomorphism by Theorem 2.1(i). Consequently, the vertical map on the right is also zero.

The proof above works with $\mathbb{Q}$ replaced by any coefficient field with characteristic not equal to 2. For characteristic 2, one should replace the Lee homology with Bar-Natan homology, and the Lee spectral sequences with Bar-Natan–Turner spectral sequences [Bar-Natan 2005; Turner 2006] (whose r^{th} differential has bidegree $(1,2r)$), but everything else goes through. Thus the theorem is proved in the special case $m = n$. □

2.3 Adjunction inequality and proof of Theorem 1.1

Manolescu, Marengon, Sarkar and Willis [Manolescu et al. 2023, Theorem 6.10] proved the adjunction inequality (Corollary 1.4) in the special case of nullhomologous cobordisms, using the calculation of $s(T(2p,2p)_{p,p})$. Having calculated all $s(T(n,n)_{p,q})$, Corollary 1.4 in its full generality is proved in exactly the same way. For completeness, we sketch their proof here. Then we apply this inequality to prove Theorem 1.1 in its full generality.

Proof of Corollary 1.4 Turning the cobordism upside down and reversing the ambient orientation, we obtain a cobordism Σ^t in $\overline{Z}^t = (I \times l(S^1 \times S^2)) \# k\mathbb{CP}^2$ from L_1 to L_0 with $\pi_0(L_1) \to \pi_0(\Sigma^t)$ surjective. Choose embedded 2–spheres $S_1, \dots, S_k$ representing generators $\bar{z}_1, \dots, \bar{z}_k \in H_2(\overline{Z}^t)$ coming from the $\mathbb{CP}^2$ factors. We may assume Σ^t intersects each S_i transversely, in some p_i points positively and q_i points negatively. Since S_i has self-intersection 1, a tubular neighborhood $\nu(S_i)$ has boundary S^3, and the projection to the core $\partial\nu(S_i) \to S_i$ is the Hopf fibration. Therefore, removing all $\nu(S_i)$ and tubing each $\partial\nu(S_i)$ to $\{1\}\times(S^1\times S^2)\subset\bar{Z}^t$ gives a cobordism Σ^t_0 from L_1 to $L_0\sqcup\big(\bigsqcup_i T(n_i,n_i)_{p_i,q_i}\big)$ in $I\times l(S^1\times S^2)$, where $n_i=p_i+q_i$. Topologically, Σ^t_0 is obtained by deleting $\sum_i(p_i+q_i)$ disks in the interior of $\Sigma^t\cong\Sigma$. Now by [Manolescu et al. 2023, Theorem 1.5,Proposition 3.7], we have

$$\left(s(L_0)+\sum_{i=1}^k s(T(n_i,n_i)_{p_i,q_i})-k\right)-s(L_1)\ge\chi(\Sigma^t_0)=\chi(\Sigma)-\sum_{i=1}^k(p_i+q_i).$$

By Theorem 1.1 with $m=n=n_i$, this simplifies to

$$\begin{aligned}s(L_0)-s(L_1)&\ge\chi(\Sigma)-\sum_{i=1}^k((|p_i-q_i|-1)^2-2\min(p_i,q_i)-1+p_i+q_i)\\&=\chi(\Sigma)-\sum_{i=1}^k|p_i-q_i|^2+\sum_{i=1}^k|p_i-q_i|=\chi(\Sigma)+[\Sigma]^2+\|[\Sigma]\|'.\end{aligned}$$ □

Proof of Theorem 1.1 We induct on $m+n$. The base cases are $m=n$, which we have already addressed. To perform the induction step, by symmetry we may assume $n<m$. If $pq=0$ we conclude as before. Assume $p,q>0$, thus $d\ge 2$. There is a cobordism Σ from $T(n,m-n)_{p,q}$ to $T(n,m)_{p,q}$ in $\overline{\mathbb{CP}}^2$ obtained by adding a positive full twist. The surface Σ is a disjoint union of d annuli, which intersects a copy of $\overline{\mathbb{CP}}^1\subset\overline{\mathbb{CP}}^2$ transversely at n_1p points positively and n_1q points negatively. Applying Corollary 1.4 to Σ, we obtain

$$s(T(n,m)_{p,q})\le s(T(n,m-n)_{p,q})+n_1^2|p-q|^2-n_1|p-q|=(n_1|p-q|-1)(m_1|p-q|-1)-2\min(p,q).$$

On the other hand, since $T(n,m)$ is a d–cable on $T(n_1,m_1)$, there is a saddle cobordism

$$T(n-2n_1,m-2m_1)_{p-1,q-1}\sqcup U\to T(n,m)_{p,q}.$$

Thus

$$s(T(n,m)_{p,q})\ge s(T(n-2n_1,m-2m_1)_{p-1,q-1}\sqcup U)-1=(n_1|p-q|-1)(m_1|p-q|-1)-2\min(p,q).$$ □

3 Applications of the adjunction inequality

3.1 Eventual linearity of the s–invariant under full twists

Let L be an oriented link in S^3. A full twist can be performed to L along any (unoriented) 2–disk in S^3 that intersects L transversely in the interior. More generally, if $D_1,\dots,D_l\subset S^3$ are l disjoint such 2–disks, we can independently perform any number of full twists along these disks. Given $\vec{n}=(n_1,\dots,n_l)$, let $L(D_1,D_2,\dots,D_l;\vec{n})\subset S^3$ denote the oriented link obtained by performing n_i full twists (n_i positive ones if $n_i\ge 0$; $-n_i$ negative ones if $n_i<0$) to L along D_i. Let d_i denote the algebraic intersection number of D_i and L (which is well defined up to sign).

Proposition 3.1 *For $n_1,\dots,n_l$ large, the number*

$$s(L;D_1,\dots,D_l;\vec{n}):=s(L(D_1,\dots,D_l;\vec{n}))-\sum_{i=1}^l n_i|d_i|(|d_i|-1)$$

is independent of $n_1,\dots,n_l$.

Therefore the stable number, denoted by $s(L; D_1, \dots, D_l)$, is an isotopy invariant of the oriented link L together with 2–disks $D_1, \dots, D_l$ in S^3. In the special case $n_1 = n_2 = \cdots = n_l$, this answers positively a question of Manolescu, Marengon, Sarkar and Willis [Manolescu et al. 2023, Question 9.1].

Remark 3.2 (i) By performing 0–surgeries on each $\partial D_i \subset S^3$, L can be regarded as an oriented link in $l(S^1 \times S^2)$. Manolescu, Marengon, Sarkar and Willis [loc. cit., Theorem 1.4] proved Proposition 3.1 in the special case $d_1 = \cdots = d_l = 0$ and $n_1 = \cdots = n_l$, and further showed that in this case the stable number $s(L; D_1, \dots, D_l)$ is an invariant of the nullhomologous oriented link L in $l(S^1 \times S^2)$, which can then be defined as the s–invariant of $L \subset l(S^1 \times S^2)$. However, as followed from their Remark 9.6, the stable number in the general case does not define an invariant of $L \subset l(S^1 \times S^2)$. In fact, sliding the strands intersecting D_i over ∂D_i changes the stable number by $\pm 2|d_i|(|d_i| - 1)$. It would be interesting if one could use Proposition 3.1 to define s–invariants of links in some other 3–manifolds, or to define s–invariants for links in $l(S^1 \times S^2)$ valued in $\mathbb{Z}/\gcd(d_1, \dots, d_l)$.

(ii) Manolescu, Marengon, Sarkar and Willis [loc. cit., Conjecture 8.31] conjectured that when $d_1 = \cdots = d_l = 0$, if there is a generic projection of $L, D_1, \dots, D_l$ to $\mathbb{R}^2$ which consists of k–disjoint 2–disks corresponding to the D_i and a positive link diagram corresponding to L, then the number $s(L; D_1, \dots, D_l; \vec{n})$ already stabilizes when $\vec{n} = \vec{0}$, ie it is independent of $\vec{n}$ for $n_1, \dots, n_l \geq 0$. This is not true, because adding a positive full twist appropriately to two oppositely oriented strands in the right-handed trefoil unknots it, but the right-handed trefoil and the unknot have s–invariants 2 and 0, respectively.

The proof of Proposition 3.1 is an easy consequence of the following bound on the behavior of s–invariants under twists.

Proposition 3.3 *Let $L \subset S^3$ be an oriented link, and $D \subset S^3$ a 2–disk intersecting it transversely in the interior, in p points positively and q points negatively, where $p \geq q$. Then for $m' > m$,*

$$s(L; D; m') - s(L; D; m) \in \begin{cases} [-2p+2, 0] & p > q, \\ [-2p, 0] & p = q. \end{cases}$$

Remark 3.4 By considering $T(n, -2n)_{p,q}$, $T(n, -n)_{p,q}$ and a disjoint union of n unknots, we see the bounds in Proposition 3.3 are sharp (see Corollary 1.3).

Proof The lower bounds are exactly those in [Roberts 2011, Theorem 1.2]. We prove the upper bounds. Adding $m' - m$ full twists along D gives a cobordism $L(D; m) \to L(D; m')$ in $(m' - m)\overline{\mathbb{CP}}^2$ with Euler characteristic 0 and homology class $(p - q, \dots, p - q)$. Thus Corollary 1.4 gives

$$s(L(D; m')) \leq s(L(D; m)) + (m - m')((p - q)^2 - (p - q)),$$

or equivalently $s(L; D; m') \leq s(L; D; m)$. □

Proof of Proposition 3.1 Proposition 3.3 implies $s(L; D_1, \dots, D_l; \vec{n})$ is nonincreasing in the coordinates of $\vec{n}$, and has a lower bound independent of $\vec{n}$; consequently, it is constant for large $\vec{n}$. □

3.2 Knots with large $\mathbb{CP}^2$– and $\overline{\mathbb{CP}}^2$–genus

For a closed oriented smooth 4–manifold M and a knot K in S^3, the M–genus of K is the minimal genus of a smooth orientable surface in $M \setminus B^4$ that bounds K. Marco Marengon informed us of the following consequence of Corollary 1.5; see also [Marengon et al. 2024, Proposition 2.1].

Proposition 3.5 *There exist knots with simultaneously arbitrarily large $\mathbb{CP}^2$–genus and $\overline{\mathbb{CP}}^2$–genus.*

Proof Following the argument in [Marengon et al. 2024, Proposition 2.1], after Corollary 1.5, it suffices to construct a knot with large s–invariant, small τ–invariant and vanishing Levine–Tristram signature function. For example, such a knot can be taken to be a connected sum of some copies of the untwisted negative Whitehead double of $T(2,-3)$ — which has vanishing Levine–Tristram signature [Litherland 1979, Theorem 2] and $\tau = -1$ [Hedden and Ording 2008, Theorem 1.2] — with some copies of Piccirillo's companion to the Conway knot, denoted by K' in [Piccirillo 2020] — which has vanishing Levine–Tristram signature and $\tau = 0$ as it is topologically slice, and $s = 2$. □

This is in contrast to the fact that the $(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)$–genus (and the $(S^2 \times S^2)$–genus) of any knot is 0 [Norman 1969, Corollary 3 and Remark], and the fact that every knot has topological $\mathbb{CP}^2$–genus and $\overline{\mathbb{CP}}^2$–genus at most 1 [Kasprowski et al. 2024, Corollary 1.15]. We remark that the proof does not carry to $k\mathbb{CP}^2$ for $k > 1$, and to our knowledge there is currently no knot known to be nonslice in $2\mathbb{CP}^2$. In fact all knots are topologically slice in $2\mathbb{CP}^2$ [Kasprowski et al. 2024, Corollary 1.15].

4 The Lee filtration structure of $T(n,m)$

In this section we prove Theorem 1.2.

By Lee [2005] and Rasmussen [2010], the Lee homology of $T(n,m)$ as a vector space is spanned by the canonical generators $[s_{\mathfrak{o}}]$, one for each orientation $\mathfrak{o}$ of $T(n,m)$. For our purpose in the next subsection, it will be convenient to use the rescaled canonical generators $[\tilde{s}_{\mathfrak{o}}]$ as defined by Rasmussen [2005].

We identify a rescaled canonical generator $[s_{\mathfrak{o}}]$ with the orientation $\mathfrak{o}$. Upon labeling the d components of $T(n,m)$ by $1, 2, \dots, d$ and choosing a preferred direction, we further identify $\mathfrak{o}$ with a subset of $[d] := \{1, 2, \dots, d\}$. Thus $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$ is identified with $\mathbb{Q}\{2^{[d]}\}$, where $2^{[d]}$ denotes the power set of $[d]$. Since the linking number between any two different components of $T(n,m)$ is $n_1 m_1$, under this identification, the span of subsets of $[d]$ with cardinality k or $d-k$ is identified with the homological degree $2n_1 m_1 k(d-k)$ part of the Lee homology [Lee 2005, Proposition 4.3; Rasmussen 2010].

Hence, the Lee homology of $T(n,m)$ is zero for homological degrees not equal to $2n_1m_1pq$ for any $p+q=d$. For each $p+q=d$ we have

$$\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m)) \cong \mathbb{Q}\{X\subset[d] : \#X=p \text{ or } q\}. \tag{1}$$

It remains to prove for each $p+q=d$ that $\mathrm{gr}(\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m)))$ has the desired graded dimension. By symmetry we assume throughout that $p\geq q$.

4.1 Halve the dimensions

Let $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m))$ (resp. $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},1}(T(n,m))$) denote the subspace of $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m))$ spanned by subsets of $[d]$ with cardinality q (resp. p), namely,

$$\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m)) \cong \mathbb{Q}\{X\subset[d] : \#X=q\}. \tag{2}$$

Thus when $p=q$, we simply have $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m))=\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},1}(T(n,m))=\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m))$.

Lemma 4.1 *For $p>q$, as graded vector spaces, we have*

$$\mathrm{gr}(\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m))) \cong \mathrm{gr}(\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m)))\otimes(\mathbb{Q}\oplus\mathbb{Q}\{2\}).$$

Proof The Lee homology $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m))$ has a quantum $\mathbb{Z}/4$–grading, and the elements are supported in odd gradings if d is even, and even gradings if d is odd. In either case, we can write $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m))=V_1\oplus V_2$ as a direct sum of two $\mathbb{Z}/4$–homogeneous components. Let ι be an involution on $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$ which is 1 on V_1 and -1 on V_2. Then ι maps every $[\tilde{s}_{\mathfrak{o}}]$ to $\pm[\tilde{s}_{\bar{\mathfrak{o}}}]$ [Rasmussen 2010, Lemma 3.5], where $\bar{\mathfrak{o}}$ denotes the reverse orientation of $\mathfrak{o}$. Thus it interchanges the two subspaces $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0/1}(T(n,m))$.

Let $q\colon \mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee}}(T(n,m))\to\mathbb{Z}\sqcup\{+\infty\}$ denote the quantum filtration degree function. Then $q(x)=q(\iota x)=\min(q(x+\iota x),q(x-\iota x))$ for nonzero x. We claim that $q(x)=\max(q(x+\iota x),q(x-\iota x))-2$ if $x\in\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m))$. This would imply the desired statement, for example by performing an induction from the top filtration degree.

Let $X_i\colon \mathrm{Kh}_{\mathrm{Lee}}(T(n,m))\to\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$ be the map induced by putting a dot on the i^{th} component of $T(n,m)$ (see [Bar-Natan 2005, Section 11.2]) for $1\leq i\leq d$. Then X_i has quantum filtration degree -2. For a suitable choice of sign for X_i, $X_i[\tilde{s}_{\mathfrak{o}}]=\epsilon[\tilde{s}_{\mathfrak{o}}]$ where $\epsilon=1$ if $i\in\mathfrak{o}$ and -1 otherwise. It follows that for $x\in\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m))$ we have $\bigl(\sum_{i=1}^d X_i\bigr)x=(q-p)x$ and $\bigl(\sum_{i=1}^d X_i\bigr)\iota x=(p-q)\iota x$. Hence, $\bigl(\sum_{i=1}^d X_i\bigr)/(q-p)$ is a map of quantum filtration degree -2 that interchanges $x\pm\iota x$. □

4.2 An S_d–symmetry

The torus link $T(n,m)$ can be seen as a d–cable of the torus knot $T(n_1,m_1)$. Thus every element in the braid group B_d lifts to an isotopy from $T(n,m)$ to itself. This induces a B_d–action on $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$

up to sign, which respects the homological grading, the quantum $\mathbb{Z}/4$–grading and the quantum filtration. Our choice of using the rescaled canonical generators $[\tilde{s}_{\mathfrak{o}}]$ has the advantage that an element $\alpha \in B_d$ acts by $\alpha \cdot [\tilde{s}_{\mathfrak{o}}] = \pm[\tilde{s}_{\bar{\alpha}\cdot\mathfrak{o}}]$ [Rasmussen 2005, Proposition 3.2], where $\bar{\alpha}$ denotes the image of α in S_d, which acts on the power set $2^{[d]}$ in the natural way. In particular, the B_d–action descends to an S_d–action, up to sign. In fact, following Grigsby, Licata and Wehrli [Grigsby et al. 2018, Theorem 2], we can fix a sign convention (ie a choice of signs for the action of each $\alpha \in B_d$ on $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$) to remove the sign ambiguity (see the proof of Proposition 4.4).

Proposition 4.2 *As S_d–representations over $\mathbb{Q}$,*

$$\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m)) \cong \bigoplus_{r=0}^{q}(d-r,r). \tag{3}$$

Proof By the proceeding paragraph, upon changing the signs of some of the $[\tilde{s}_{\mathfrak{o}}]$, the identification $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m)) \cong \mathbb{Q}\{2^{[d]}\}$ is S_d–equivariant. Now the statement follows by restricting to (2) and the standard fact that $\mathbb{Q}\{X \subset [d] : \#X = q\} \cong \bigoplus_{r=0}^{q}(d-r,r)$ as S_d–representations; see eg [Pasechnik 2013]. □

Remark 4.3 Grigsby, Licata and Wehrli [Grigsby et al. 2018, Theorem 2] actually showed that the S_d–action on $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$ descends to an action of the Temperley–Lieb algebra $TL_d(1)$. In fact, the irreducible S_d–representations $(d-r,r)$ are exactly the ones pulled back from irreducible $TL_d(1)$–representations.

The Lee homology of the unknot U is generated by the two rescaled canonical generators, denoted by A and B. In standard notation of the Lee homology defined via the Frobenius algebra $\mathbb{Q}[X]/(X^2-1)$, we can take $A = \frac{1}{2}(X+1)$ and $B = \frac{1}{2}(X-1)$. Define an involution ι on $\mathrm{Kh}_{\mathrm{Lee}}(U)$ by $A \mapsto B$, which equips $\mathrm{Kh}_{\mathrm{Lee}}(U)$ with a $\mathbb{Z}/2$–action. Then $\mathrm{Kh}_{\mathrm{Lee}}(U) \cong 1 \oplus \epsilon$ as $\mathbb{Z}/2$–representations. Abuse the notation and use 1 and ϵ to also denote the corresponding subrepresentations of $\mathrm{Kh}_{\mathrm{Lee}}(U)$. The quantum filtration structure of $\mathrm{Kh}_{\mathrm{Lee}}(U)$ is determined by

$$q(\epsilon) = 1, \quad q(1) = -1. \tag{4}$$

From the description of $T(n,m)$ as a d–cable on $T(n_1,m_1)$, when $d \geq 2$ there is a saddle cobordism

$$T(n-2n_1, m-2m_1) \sqcup U \to T(n,m).$$

Proposition 4.4 *The induced map*

$$\mathrm{Kh}_{\mathrm{Lee}}(T(n-2n_1, m-2m_1) \sqcup U) \to \mathrm{Kh}_{\mathrm{Lee}}(T(n,m)) \tag{5}$$

by the saddle cobordism is $(S_{d-2}\times\mathbb{Z}/2)$–equivariant. Here $S_{d-2}\times\mathbb{Z}/2 = S_{d-2}\times S_2 \subset S_d$ via the natural inclusion. Moreover, upon negating the involution on $\mathrm{Kh}_{\mathrm{Lee}}(U)$, the induced map

$$\mathrm{Kh}_{\mathrm{Lee}}(T(n,m)) \to \mathrm{Kh}_{\mathrm{Lee}}(T(n-2n_1, m-2m_1) \sqcup U) \tag{6}$$

by the (backward) saddle cobordism is $(S_{d-2}\times\mathbb{Z}/2)$–equivariant.

Proof We only prove the equivariance of (5). The equivariance of (6) is proved similarly.

Let Σ denote the saddle cobordism $T(n-2n_1, m-2m_1) \sqcup U \to T(n,m)$. For a braid $\alpha \in B_d$, let Σ_α denote the corresponding self-isotopy of $T(n,m)$. By an explicit calculation, one can show the involution ι on $\mathrm{Kh}_{\mathrm{Lee}}(U)$ is the map induced by the self-isotopy Σ_ι of U flipping itself around. Now the following pairs of cobordisms that are isotopic rel boundary show the equivariance of (5) up to sign:

$$\Sigma \circ \Sigma_{\sigma_i} \sim \Sigma_{\sigma_i} \circ \Sigma \quad \text{for } i = 1, 2, \dots, d-3, \qquad \Sigma \circ \Sigma_\iota \sim \Sigma_{\sigma_{d-1}} \circ \Sigma.$$

Here $\sigma_1, \dots, \sigma_{d-1}$ are the usual braid group generators.

To remove the sign ambiguity, we have to look into the sign convention in [Grigsby et al. 2018, Section 7.2] which we adopted. Let $\mathfrak{o}_p$ denote the parallel orientation $\varnothing \subset [d]$. The sign of $\varphi_i := \mathrm{Kh}_{\mathrm{Lee}}(\Sigma_{\sigma_i})$ is chosen so that $\varphi_i([\tilde{s}_{\mathfrak{o}_p}]) = [\tilde{s}_{\mathfrak{o}_p}]$. This is the sign convention we adopted to define $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$ as an S_d–representation.

Grigsby, Licata and Wehrli [Grigsby et al. 2018] also considered maps ψ_i on $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m))$, each induced by the annular cobordism $T(n,m) \to T(n-2n_1, m-2m_1)$ that annihilates the components labeled i and $i+1$, followed by the annular cobordism $T(n-2n_1, m-2m_1) \to T(n,m)$ that recreates two components with labels i and $i+1$. This defines ψ_i up to sign. Let $\mathfrak{o}_a$ denote the alternating orientation $\{2, 4, \dots, 2\lfloor \frac{1}{2}d \rfloor\} \subset [d]$, and $\mathfrak{o}_{a,i}$ denote the symmetric difference between $\mathfrak{o}_a$ and $\{i, i+1\}$. Then $\psi_i([\tilde{s}_{\mathfrak{o}_a}]) = \epsilon_1[\tilde{s}_{\mathfrak{o}_a}] + \epsilon_2[\tilde{s}_{\mathfrak{o}_{a,i}}]$ for some signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ [Rasmussen 2005, Proposition 3.2]. The sign of ψ_i is fixed by demanding

$$\psi_i([\tilde{s}_{\mathfrak{o}_a}]) = -[\tilde{s}_{\mathfrak{o}_a}] \pm [\tilde{s}_{\mathfrak{o}_{a,i}}]. \tag{7}$$

Under these two sign conventions, they showed that $\varphi_i = \mathrm{id} + \psi_i$ [Grigsby et al. 2018, Proposition 9].

The above sign fixes do not depend on the signs of the rescaled canonical generators $[\tilde{s}_{\mathfrak{o}}]$, but our assumption (in the proof of Proposition 4.2) that $\mathrm{Kh}_{\mathrm{Lee}}(T(n,m)) \cong \mathbb{Q}\{2^{[d]}\}$ is S_d–equivariant does. Since $\varphi_i([\tilde{s}_{\mathfrak{o}_a}]) = [\tilde{s}_{\mathfrak{o}_a}] - [\tilde{s}_{\mathfrak{o}_a}] \pm [\tilde{s}_{\mathfrak{o}_{a,i}}] = \pm[\tilde{s}_{\mathfrak{o}_{a,i}}]$, the sign in (7) is $+$ by our convention.

Since we have shown that (5) is equivariant up to sign, to prove the full equivariance it now suffices to check on particular Lee generators.

First we check $\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)\varphi_i = \varphi_i \,\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)$ for $1 \le i \le d-3$. Since ψ_i vanishes on any $[\tilde{s}_{\mathfrak{o}}]$ where the i and $i+1$ components are parallel in $\mathfrak{o}$, we see $\varphi_i = \mathrm{id}$ on such generators. Therefore $\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)\varphi_i([\tilde{s}_{\mathfrak{o}}] \otimes A) = \mathrm{Kh}_{\mathrm{Lee}}(\Sigma)([\tilde{s}_{\mathfrak{o}}] \otimes A) = \varphi_i \,\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)([\tilde{s}_{\mathfrak{o}}] \otimes A)$ (which is nonzero) for any such $\mathfrak{o}$.

Next we check $\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)\iota = \varphi_{d-1} \,\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)$. By definition, ψ_{d-1} factors through $\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)$, and we see $\psi_{d-1}([\tilde{s}_{\mathfrak{o}_a}]) = \mathrm{Kh}_{\mathrm{Lee}}(\Sigma)([\tilde{s}_{\mathfrak{o}_a}] \otimes 1)$ up to sign. In view of (7) and noting $1 = A - B$ in $\mathrm{Kh}_{\mathrm{Lee}}(U)$, upon switching A and B we may assume $\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)([\tilde{s}_{\mathfrak{o}_a}] \otimes A) = \pm[\tilde{s}_{\mathfrak{o}_a}]$ and $\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)([\tilde{s}_{\mathfrak{o}_a}] \otimes B) = \pm[\tilde{s}_{\mathfrak{o}_{a,d-1}}]$, where the two signs $\pm$ are equal. It follows that

$$\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)\iota([\tilde{s}_{\mathfrak{o}_a}] \otimes A) = \mathrm{Kh}_{\mathrm{Lee}}(\Sigma)([\tilde{s}_{\mathfrak{o}_a}] \otimes B) = \pm[\tilde{s}_{\mathfrak{o}_{a,d-1}}] = \pm\varphi_{d-1}([\tilde{s}_{\mathfrak{o}_a}]) = \varphi_{d-1} \,\mathrm{Kh}_{\mathrm{Lee}}(\Sigma)([\tilde{s}_{\mathfrak{o}_a}] \otimes A),$$

so we are done. □

4.3 Proof of Theorem 1.2

By Lemma 4.1, Theorem 1.2 reduces to showing that nonzero components of $\mathrm{gr}(\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m)))$ are determined by

$$\text{(8)}\quad \dim \mathrm{gr}(\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m)))^{6n_1m_1pq+s(T(n,m)_{p,q})+2r-1} = \dim(d-r,r) \quad \text{for } r=0,1,\dots,q.$$

Let $V^{2n_1m_1pq}_{d-r,r} \subset \mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m))$ denote the irreducible S_d–subrepresentation that corresponds to $(d-r,r)$ via (3). Since the S_d–action respects the quantum filtration structure, all nonzero elements in $V^{2n_1m_1pq}_{d-r,r}$ have the same filtration degree, denoted by $q(V^{2n_1m_1pq}_{d-r,r})$. Moreover, since every irreducible S_d–representation appears at most once in $\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m))$, we conclude that

$$\mathrm{gr}(\mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m))) \cong \bigoplus_{r=0}^{q} V^{2n_1m_1pq}_{d-r,r}$$

as graded vector spaces. Now (8) reduces to showing that

$$\text{(9)}\qquad q(V^{2n_1m_1pq}_{d-r,r}) = 6n_1m_1pq + s(T(n,m)_{p,q}) + 2r - 1 \quad \text{for } r=0,1,\dots,q.$$

We proceed by induction on d. The case $d=0$ is plain, and the case $d=1$ follows directly from Theorem 1.1. Below we assume $d \geq 2$.

In the case $q>0$, the map (5) restricts to

$$\text{(10)}\qquad \Phi\colon \mathrm{Kh}^{2n_1m_1(p-1)(q-1)}_{\mathrm{Lee},0}(T(n-2n_1,m-2m_1)) \otimes \mathrm{Kh}_{\mathrm{Lee}}(U) \to \mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m)),$$

where the left-hand side is isomorphic to $\bigoplus_{r=0}^{q-1}((d-r-2,r)\otimes(1\oplus\epsilon))$ as $(S_{d-2}\times\mathbb{Z}/2)$–representations. Since

$$\mathrm{Res}^{S_d}_{S_{d-2}\times\mathbb{Z}/2}(d-r,r) = \begin{cases} (d-r,r-2)\otimes 1 \oplus (d-r-1,r-1)\otimes(1\oplus\epsilon) \oplus (d-r-2,r)\otimes 1 & r<\frac{1}{2}d, \\ (d-r,r-2)\otimes 1 \oplus (d-r-1,r-1)\otimes\epsilon & r=\frac{1}{2}d, \end{cases}$$

(here a non-Young-diagram (a,b) is considered to be zero), the right-hand side of (10) contains a unique copy of $(d-r-2,r)\otimes\epsilon$ for every $0\leq r\leq q-1$, denoted by $W^{2n_1m_1pq}_{d-r-1,r+1}$, which is a subspace of $V^{2n_1m_1pq}_{d-r-1,r+1}$.

By an explicit description of cobordism maps on Lee homology in terms of the canonical generators (see [Rasmussen 2005, Proposition 3.2]), (5) is injective. Consequently Φ is injective, and thus by Proposition 4.4 it maps $V^{2n_1m_1(p-1)(q-1)}_{d-r-2,r}\otimes\epsilon$ isomorphically onto $W^{2n_1m_1pq}_{d-r-1,r+1}$. Similarly, the backward map

$$\Psi\colon \mathrm{Kh}^{2n_1m_1pq}_{\mathrm{Lee},0}(T(n,m)) \to \mathrm{Kh}^{2n_1m_1(p-1)(q-1)}_{\mathrm{Lee},0}(T(n-2n_1,m-2m_1)) \otimes \mathrm{Kh}_{\mathrm{Lee}}(U)$$

is surjective and maps $W^{2n_1m_1pq}_{d-r-1,r+1}$ isomorphically onto $V^{2n_1m_1(p-1)(q-1)}_{d-r-2,r}\otimes 1$. Upon shifting

$$\mathrm{Kh}_{\mathrm{Lee},0}(T(n-2n_1,m-2m_1)) \otimes \mathrm{Kh}_{\mathrm{Lee}}(U)$$

by $[2n_1m_1(d-1)]\{6n_1m_1(d-1)\}$ to account for the bidegree differences between $\mathrm{Kh}(T(n,m))$ and $\mathrm{Kh}(T(n,m)_{p,q})$ and between $\mathrm{Kh}(T(n-2n_1,m-2m_1))$ and $\mathrm{Kh}(T(n-2n_1,m-2m_1)_{p-1,q-1})$, the maps Φ and Ψ preserve the homological degree and have quantum filtration degree -1. By (4) and induction hypothesis, we conclude that

$$\begin{aligned} q(V^{2n_1m_1pq}_{d-r-1,r+1}) = q(W^{2n_1m_1pq}_{d-r-1,r+1}) &= q(V^{2n_1m_1(p-1)(q-1)}_{d-r-2,r}) + 6n_1m_1(d-1) \\ &= 6n_1m_1pq + s(T(n,m)_{p,q}) + 2r + 1. \end{aligned}$$

This proves (9) for $r \neq 0$. Finally, Theorem 1.1 implies $q(V^{2n_1m_1pq}_{d-r,r}) = 6n_1m_1pq + s(T(n,m)_{p,q}) - 1$ for some r, which is now necessarily 0. □

5 Proof of Theorem 2.1

We follow the induction scheme set up by Stošić [2007; 2009]. For ease of notation, in this section, we write $T_{n,m}$ for the torus link $T(n,m)$. Define an auxiliary family of links $D^i_{n,m}$ for $m,n \geq 0$ and $0 \leq i \leq n-1$, as the braid closure of the braid $(\sigma_1 \cdots \sigma_{n-1})^m \sigma_1 \cdots \sigma_i \in B_n$. Thus $T_{n,m} = D^0_{n,m} = D^{n-1}_{n,m-1}$. For $i > 0$, performing a 0–resolution to the crossing of $D^i_{n,m}$ corresponding to the last letter σ_i gives the link $D^{i-1}_{n,m}$, while performing a 1–resolution gives another link, which we denote by $E^{i-1}_{n,m}$. The reader is warned this notation does not agree with that of Stošić [2007; 2009].

For our purpose, the cases $m = n, n-1$ will be useful. The following statement is easily checked. The first two items appeared in [Stošić 2009, Proof of Theorem 1]. See Figure 3 for an illustration of the third item.

Lemma 5.1

- $E^{n-2}_{n,n-1} \simeq D^{n-3}_{n-2,n-3} \sqcup U$,
- $E^i_{n,n-1} \simeq D^i_{n-2,n-3}$ *for* $i = 0, 1, \ldots, n-3$,
- $E^i_{n,n} \simeq D^{i-1}_{n-2,n-2}$ *for* $i = 1, 2, \ldots, n-2$,
- $E^0_{n,n} \simeq D^0_{n-2,n-2} \sqcup U$. □

Equip $D^i_{n,m}$ with the orientation where all components are oriented in the same direction. Equip $E^i_{n,m}$ with the orientation coming from the right-hand sides of Lemma 5.1 for $m = n, n-1$. Then all crossings in $D^i_{n,m}$ are positive, while $E^i_{n,n-1}$ has $2n-3$ negative crossings and $E^i_{n,n}$ has $2n-2$ negative crossings.

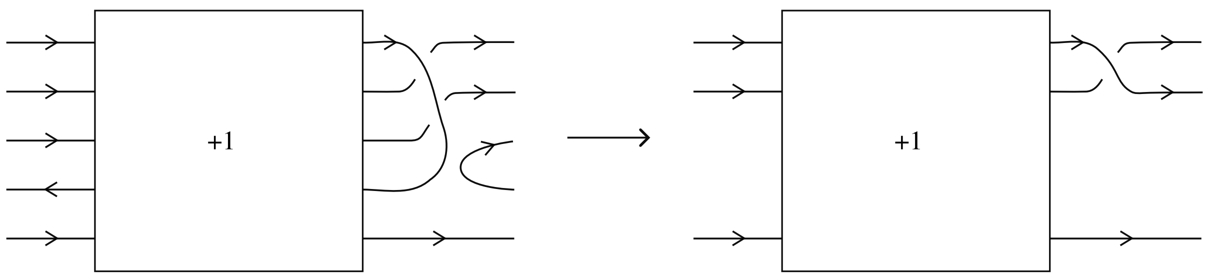


Figure 3: The oriented link $E^3_{5,5} \simeq D^1_{3,3}$.

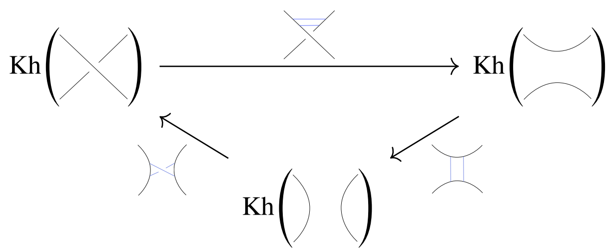
Keeping track of the degree shifts (see eg [Turner 2017, Section 3.1, Case II]), the skein long exact sequences on Khovanov homology corresponding to the resolution at the last letter σ_i in $D^i_{n,m}$ read

$$\cdots \to \mathrm{Kh}^{h-2n+2,q-6n+7}(E^{i-1}_{n,n-1}) \to \mathrm{Kh}^{h,q}(D^i_{n,n-1}) \to \mathrm{Kh}^{h,q-1}(D^{i-1}_{n,n-1}) \to \cdots, \tag{11}$$

$$\cdots \to \mathrm{Kh}^{h-2n+1,q-6n+4}(E^{i-1}_{n,n}) \to \mathrm{Kh}^{h,q}(D^i_{n,n}) \to \mathrm{Kh}^{h,q-1}(D^{i-1}_{n,n}) \to \cdots. \tag{12}$$

Moreover, the maps in the above long exact sequences are the maps induced by the obvious saddle cobordisms between the relevant links. Since the author is not aware of this last claim on cobordisms in the existing literature, we take a short detour:

Lemma 5.2 *At a crossing of a link diagram, we have the following skein exact triangle in Khovanov homology, where the morphisms are induced by the saddles as indicated. (For simplicity we have suppressed all grading shifts.)*



Proof Let CKh denote the Khovanov chain complex. Then by definition

$$\mathrm{CKh}(\;) = \mathrm{Cone}(\mathrm{CKh}(\;) \longrightarrow \mathrm{CKh}(\;)).$$

This gives rise to an exact triangle as stated, except that we still have to check that the top and the left morphisms in the exact triangle agree with the morphisms induced by the saddle cobordisms given in the statement.

First we check the top morphism. Identify CKh( ) = CKh( ) ⊕ CKh( ) as modules; the top morphism in the exact triangle is induced by the projection of CKh( ) onto CKh( ). On the other hand, the morphism given in the statement is induced by the chain map

$$\mathrm{CKh}(\;) \longrightarrow \mathrm{CKh}(\;) \xrightarrow{(R1^+)^{-1}} \mathrm{CKh}(\;), \tag{13}$$

where $(R1^+)^{-1}$ denotes the chain homotopy equivalence induced by undoing the positive twist. See [Hayden and Sundberg 2024, Tables 1 and 3] for succinct descriptions of the induced maps by saddle cobordisms and Reidemeister I moves. On the direct summand CKh( ), (13) equals

$$\mathrm{CKh}(\;) \to \mathrm{CKh}(\;) \to \mathrm{CKh}(\;), \tag{14}$$

where the first map is induced by the splitting saddle cobordism $\Delta\colon 1 \mapsto 1 \otimes X + X \otimes 1$, $X \mapsto X \otimes X$ on the top strand, and the second map is induced by the death cobordism $\epsilon\colon 1 \mapsto 0$, $X \mapsto 1$ on the middle circle. Since $(1 \otimes \epsilon) \circ \Delta = 1$, the composition (14) is the identity. On the direct summand CKh( ), (13) equals

$$\mathrm{CKh}(\;) \to \mathrm{CKh}(\;) \to \mathrm{CKh}(\;),$$

where the second map is identically zero. We have thus shown that the top morphism in the exact triangle equals the stated morphism.

Next we check the left cobordism. The morphism in the exact triangle is induced by the inclusion of CKh( ) into CKh( ) as a direct summand. On the other hand, the morphism given in the statement is induced by the chain map

(15) $$\mathrm{CKh}(\ ) \xrightarrow{R1^-} \mathrm{CKh}(\ ) \longrightarrow \mathrm{CKh}(\ ),$$

where $R1^-$ denotes the chain homotopy equivalence induced by creating the negative twist, which is equal to the map $\mathrm{CKh}(\ ) \to \mathrm{CKh}(\circ)$ induced by the birth cobordism $\iota\colon 1 \mapsto 1$ creating the middle circle, followed by the inclusion $\mathrm{CKh}(\circ) \to \mathrm{CKh}(\ )$ as a direct summand. On the direct summand $\mathrm{CKh}(\circ)$, the second map in (15) equals the merging saddle cobordism $m\colon 1 \otimes 1 \mapsto 1$, $1 \otimes X \mapsto X$, $X \otimes 1 \mapsto X$, $X \otimes X \mapsto 0$ on the right two components. Since $m \circ (\iota \otimes 1) = 1$, the composition (15) equals the inclusion map, and the proof is complete. □

Next we introduce some notation that will be convenient. For two functions $f, g\colon \mathbb{Z} \to \mathbb{Z} \sqcup \{+\infty\}$, define $\min(f, g)$ to be the function $\mathbb{Z} \to \mathbb{Z} \sqcup \{+\infty\}$ with $\min(f, g)(h) = \min(f(h), g(h))$. We write $f > g$ if $f(h) > g(h)$ for all h with $f(h) < +\infty$, and write $f < g$, $f \geq g$ and $f \leq g$ analogously. For $h, q \in \mathbb{Z}$, we define $f[h]\{q\}\colon \mathbb{Z} \to \mathbb{Z} \sqcup \{+\infty\}$ by $(f[h]\{q\})(h') = f(h' - h) + q$. For a function defined on a subset of $\mathbb{Z}$ with values in $\mathbb{Z} \sqcup \{+\infty\}$, we abuse the notation and use the same expression to denote its extension by $+\infty$ to all of $\mathbb{Z}$. Finally, define $t_{n,n}(h) := \inf\{q : \mathrm{Kh}^{h,q}(T_{n,n}) \neq 0\}$ to be the quantum infimum function for $\mathrm{Kh}(T_{n,n})$, and similarly $t_{n+1,n}$ and $d^i_{n,m}, e^i_{n,m}$ to be the quantum infimum functions for $\mathrm{Kh}(T_{n+1,n})$, $\mathrm{Kh}(D^i_{n,m})$ and $\mathrm{Kh}(E^i_{n,m})$, respectively.

Thus, for example, the lower bound of Theorem 2.1(i) says $t_{n,n} \geq q_{n,n}$.

Before launching into the proof of Theorem 2.1, we note the following relations among the functions $q_{n,n}$ and $q_{n+1,n}$ defined in Section 2.1:

Lemma 5.3 *We have*

(16a) $$q_{n-2,n-2}[2n-2]\{6n-8\} = q_{n,n}|_{h \geq 2n-2} \geq q_{n,n},$$

(16b) $$q_{n-1,n-2}[2n-2]\{6n-6\} \geq q_{n+1,n}|_{h \leq h_{\max}(T_{n+1,n})-1},$$

(16c) $$q_{n,n}\{n-1\} \geq q_{n+1,n},$$

(16d) $$q_{n,n-1}\{n-1\} \geq q_{n,n},$$

(16e) $$q_{n,n-1}[1]\{2\}|_{1 \neq h \leq h_{\max}(T_{n,n-1})} \geq q_{n,n-1},$$

(16f) $$q_{n,n-1}[2]\{4\}|_{h \leq h_{\max}(T_{n,n-1})} \geq q_{n,n-1}.$$

Moreover, (16c) *is strict at* $h = 2n-1$. *Also,* (16d) *is strict at* $h = 2pq$ *for any* $p + q = n$ *with* $p \geq q > 0$; *it is strict at* $h = 2pq - 1$ *for any* $p + q = n$ *with* $p \geq q > 1$.

These are elementary. We sketch mostly geometric proofs:

Proof The functions $q_{n,n}$ are thought of as staircases as indicated by the thick segments in Figure 1. The height of a step is usually 2, but is 4 right after homological degrees $2pq$. Thus to check the equality in (16a), note that the two sets $\{2pq : p+q = n \text{ with } p \geq q > 0\}$ and $\{2pq : p+q = n-2 \text{ with } p \geq q \geq 0\}$ are identical up to a shift of $2n-2$, and that $q_{n,n}(2n-2) = q_{n-2,n-2}(0) + 6n - 8$.

The function $q_{n+1,n}$ is thought of as a shift of $q_{n,n}$ by $\{n-1\}$ with all but the first big step flattened (by decreasing the height by 2 at $2pq+1$ for $p \geq q > 0$), plus $\lfloor \frac{1}{2}n \rfloor$ short small steps as tail; see Figure 2. This description immediately gives (16c), (16e) and (16f). Together with (16a), this also proves (16b) (we can apply the truncation in the end because the tail of $q_{n+1,n}$ is one longer than that of $q_{n-1,n-2}$). Since $2n-1 = 2(n-1)\cdot 1 + 1$, (16c) is strict at $h = 2n-1$.

Finally, (16d) is more contrived, so we first calculate algebraically. At $h = 0$, (16d) is an equality. For $0 < h \leq h_{\max}(T_{n-1,n-1})$, choose $p+q = n$ for $p \geq q > 0$ and $p'+q' = n-1$ for $p' \geq q' > 0$ with $h \in (2(p+1)(q-1), 2pq] \cap (2(p'+1)(q'-1), 2p'q']$. Then

$$\text{(17)}\quad q_{n-1,n-1}(h)+2n-3-q_{n,n}(h) = (n-1)^2+2\lceil \tfrac{1}{2}h\rceil-2p'+2n-3-n^2-2\lceil \tfrac{1}{2}h\rceil+2p = 2(q'-q).$$

If $h \neq 2(p'+1)(q'-1)+1$ or $q'=1$, (17) gives $q_{n,n-1}\{n-1\}(h) = q_{n-1,n-1}(h)+2n-3 \geq q_{n,n}(h)$ since $q' \geq q$. If $h = 2(p'+1)(q'-1)+1$ and $q' \neq 1$, (17) gives $q_{n,n-1}\{n-1\}(h) = q_{n-1,n-1}(h)+2n-5 = q_{n,n}(h)$ since $q' > q$.

To prove (16d) for $h_{\max}(T_{n-1,n-1}) \leq h \leq h_{\max}(T_{n,n-1})$, we note that in this range $q_{n,n-1}\{n-1\}$ is a tail of short small steps while $q_{n,n}$ is a combination of long small and long big steps. Thus (16d) follows from its validity at $h = h_{\max}(T_{n-1,n-1})$, unless $q_{n,n}$ has a big step right after $h = h_{\max}(T_{n-1,n-1})$. In this exceptional case, $h_{\max}(T_{n-1,n-1}) = 2pq$ for some $p+q = n$ with $p \geq q > 0$. This implies $q' > q$ in (17), so (16d) is strict at $h = h_{\max}(T_{n-1,n-1})$, and (16d) also follows.

Finally we prove the addendum about the strictness of (16d). For $0 < h \leq h_{\max}(T_{n-1,n-1})$ we have $h \in (2(p+1)(q-1), 2pq] \cap (2(p'+1)(q'-1), 2p'q']$, (16d) is strict if and only if $q' > q+1$ or $q' = q+1$ and $h \neq 2(p'+1)(q'-1)+1$. This is the case for $h = 2pq$ if $p \geq q > 0$, and for $h = 2pq-1$ if $p \geq q > 1$.

Now assume $h = 2pq, 2pq-1$ for $h > h_{\max}(T_{n-1,n-1})$. The case for $h = 2pq$ is trivial because

$$q_{n,n-1}\{n-1\}(2pq) = q_{n,n-1}\{n-1\}(2pq-1)+2 \geq q_{n,n}(2pq-1)+2 = q_{n,n-1}(2pq)+2.$$

For $h = 2pq-1$, we divide into three cases:

Case 1 If $2pq-1 > \max(2(p+1)(q-1)+1, h_{\max}(T_{n-1,n-1})+1)$, then

$$q_{n,n-1}\{n-1\}(2pq-1) = q_{n,n-1}\{n-1\}(2pq-3)+4 \geq q_{n,n}(2pq-3)+4 = q_{n,n}(2pq-1)+2.$$

Case 2 If $2pq-1 = 2(p+1)(q-1)+1$, then $p = q+1$, so $h_{\max}(T_{n,n-1}) = 2q^2+q$ is less than $2pq-1$ unless $q = 1$, which is excluded in our hypothesis.

Case 3 If $2pq-1 = h_{\max}(T_{n-1,n-1})+1 > 2(p+1)(q-1)+1$, then the strictness of (16d) at $2pq-1$ is equivalent to that at $h_{\max}(T_{n-1,n-1})$. The latter is true because $q' > q$ in (17) unless $q = 1$. □

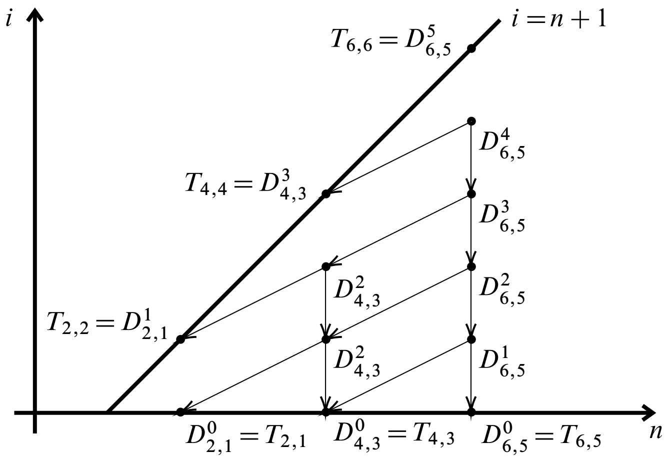


Figure 4: Flowchart of applying (18) to $D^4_{6,5}$.

Proof of Theorem 2.1 In the notation above, we need to prove $t_{n,n} \geq q_{n,n}$, $t_{n+1,n} \geq q_{n+1,n}$ and the two addenda to (i) and (ii).

We induct on n. The base cases $n = 1, 2$ are easily checked. Below we assume $n \geq 3$.

(i) **The statements for $T_{n,n}$** The long exact sequence (11) gives

$$d^i_{n,n-1} \geq \min(e^{i-1}_{n,n-1}[2n-2]\{6n-7\}, d^{i-1}_{n,n-1}\{1\}). \tag{18}$$

Using Lemma 5.1, we thus have

$$t_{n,n} \geq \min(t_{n-2,n-2}[2n-2]\{6n-8\}, d^{n-2}_{n,n-1}\{1\}) =: \min(A, B),$$

where $A \geq q_{n-2,n-2}[2n-2]\{6n-8\} \geq q_{n,n}$ by the induction hypothesis and (16a). Inductively applying (18) (see Figure 4), we obtain

$$B \geq \min\Bigg(\min_{\substack{n-m=2r\geq 0\\ m\geq 2}} t_{m,m-1}\Big[\sum{}'(2i-2)\Big]\Big\{\sum{}'(6i-8)+n-1\Big\}, \\ \min_{\substack{n-m=2r>0\\ m\geq 1}} t_{m,m}\Big[\sum{}'(2i-2)\Big]\Big\{\sum{}'(6i-8)+n-m\Big\}\Bigg). \tag{19}$$

Here and henceforth, each Σ' is a sum over $i \in (m, n]$ with the same parity as m and n. By the induction hypothesis and (16a), each term in the second sum in (19) is bounded below by

$$q_{m,m}\Big[\sum{}'(2i-2)\Big]\Big\{\sum{}'(6i-8)+n-m\Big\} \geq q_{n,n}\{n-m\} > q_{n,n}.$$

Similarly, by the induction hypothesis and (16b), (16d) and (16f), each term in the first sum in (19) is bounded below by

$$q_{m,m-1}\Big[\sum{}'(2i-4)+2r\Big]\Big\{\sum{}'(6i-12)+4r+n-1\Big\} \geq q_{n,n-1}[2r]\{4r+n-1\} \\ \geq \min(q_{n,n-1}\{n-1\}, \text{tail}) \geq q_{n,n},$$

where tail is the function defined on $[h_{\max}(T_{n,n-1}), h_{\max}(T_{n,n-1})+2r]$ which consists of short small steps (in the sense of the proof of Lemma 5.3) and agrees with $q_{n,n-1}\{n-1\}$ at $h = h_{\max}(T_{n,n-1})$.

Now it remains to prove the statement about the saddle cobordism. The relevant induced map α fits in the long exact sequence (11),

$$\cdots \to \mathrm{Kh}^{h-1,q-1}(D^{n-2}_{n,n-1}) \xrightarrow{\beta} \mathrm{Kh}^{h-2n+2,q-6n+7}(T_{n-2,n-2} \sqcup U) \xrightarrow{\alpha} \mathrm{Kh}^{h,q}(T_{n,n}) \to \mathrm{Kh}^{h,q-1}(D^{n-2}_{n,n-1}) \to \cdots, \tag{20}$$

for $h = 2pq > 0$ and $q = q_{n,n}(h)$. The map α would be an isomorphism if

$$\mathrm{Kh}^{h-1,q-1}(D^{n-2}_{n,n-1}) = \mathrm{Kh}^{h,q-1}(D^{n-2}_{n-1,n-2}) = 0,$$

or equivalently, using the notation above, if $B(h-1), B(h) > q (= q_{n,n}(h) = q_{n,n}(h-1))$.

We reexamine the estimate $B \geq q_{n,n}$ above. The contribution from the second term in (19) is always strict. The contribution from the first term is strict for h, and is strict for $h-1$ if $q > 1$, because (16d) is strict for such homological degrees by the second addendum of Lemma 5.3. If $q = 1$, however, the inequality is strict at $h-1 = 2n-3$ for $r > 0$ (as the left-hand side is $+\infty$), but not for $r = 0$. In this exceptional case, the addendum in the induction hypothesis for $T_{n,n-1}$ states that the relevant homology group $\mathrm{Kh}^{2n-3,q_{n,n-1}(2n-3)}(T_{n,n-1})$ is torsion, and thus so is every $\mathrm{Kh}^{2n-3,q_{n,n-1}(2n-3)+i}(D^i_{n,n-1})$, in view of (12). This group for $i = n-2$ and the group $\mathrm{Kh}^{0,(n-3)^2-2}(T_{n-2,n-2} \sqcup U) = \mathbb{Z}$ appear as the first two terms in (20), which implies β in (20) is zero, and thus α is an isomorphism.

(ii) **The statements for $T_{n+1,n}$** The long exact sequence (12) gives

$$d^i_{n,n} \geq \min(e^{i-1}_{n,n}[2n-1]\{6n-4\}, d^{i-1}_{n,n}\{1\}). \tag{21}$$

Using Lemma 5.1, we obtain

$$t_{n+1,n} \geq \min(t_{n-1,n-2}[2n-1]\{6n-4\}, d^{n-2}_{n,n}\{1\}) =: \min(A, B),$$

where $A \geq q_{n-1,n-2}[2n-1]\{6n-4\} \geq q_{n+1,n}[1]\{2\}|_{2n-1 \leq h \leq h_{\max}(T_{n+1,n})} \geq q_{n+1,n}$ by the induction hypothesis, (16b) and (16e), and

$$B \geq \min_{n-m=2r\geq 0} t_{m,m}\Big[{\textstyle\sum}'(2i-1)\Big]\Big\{{\textstyle\sum}'(6i-6)+n-1\Big\}. \tag{22}$$

By the induction hypothesis, (16a), (16c) and (16e), the $r = 0$ term in the summation is bounded below by $q_{n,n}\{n-1\} \geq q_{n+1,n}$, and each of the $r > 0$ terms is bounded below by

$$\begin{aligned} q_{m,m}\Big[{\textstyle\sum}'(2i-1)\Big]\Big\{{\textstyle\sum}'(6i-6)+n-1\Big\} &\geq q_{n,n}[r]\{2r+n-1\}|_{h\geq 2n-1} \\ &\geq q_{n+1,n}[r]\{2r\}|_{2n-1\leq h\leq h_{\max}(T_{n,n})+r} \geq q_{n+1,n}. \end{aligned}$$

It remains to show $\mathrm{Kh}^{2n-1,q_{n+1,n}(2n-1)}(T_{n+1,n})$ is torsion. This group sits in the long exact sequence (12):

$$\cdots \to \mathrm{Kh}^{2n-2,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n}) \xrightarrow{\gamma} \mathrm{Kh}^{0,q_{n-1,n-2}(0)}(T_{n-1,n-2}) \to \mathrm{Kh}^{2n-1,q_{n+1,n}(2n-1)}(T_{n+1,n}) \to \mathrm{Kh}^{2n-1,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n}) \to \cdots. \tag{23}$$

Thus it suffices to show, upon tensoring with $\mathbb{Q}$, that γ is surjective and $\mathrm{Kh}^{2n-1,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n})=0$. By Lemma 5.2, γ is induced by a saddle cobordism $D^{n-2}_{n,n}\to T_{n-1,n-2}$. Let

$$\theta\colon \mathrm{Kh}^{0,q_{n-1,n-2}(0)+2}(T_{n-1,n-2})\to \mathrm{Kh}^{2n-2,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n})$$

be the map induced by the backward saddle cobordism. Then by the neck-cutting relation [Bar-Natan 2005, Section 11.2] we have $\gamma\circ\theta=2X$, where

$$X\colon \mathrm{Kh}^{0,q_{n-1,n-2}(0)+2}(T_{n-1,n-2})\to \mathrm{Kh}^{0,q_{n-1,n-2}(0)}(T_{n-1,n-2}) \tag{24}$$

is the map induced by the dotted cobordism on $T_{n-1,n-2}$. By Stošić [2007, Theorem 3.4], we have $\mathrm{Kh}^{0,*}(T_{n-1,n-2})=\mathbb{Z}$ for $*=q_{n-1,n-2}(0)+1\pm1$ and 0 otherwise, and $\mathrm{Kh}^{1,*}(T_{n-1,n-2})=0$. Hence the reduced Khovanov homology of $T_{n-1,n-2}$ has $\widetilde{\mathrm{Kh}}^{1,*}(T_{n-1,n-2})=0$, and thus $\widetilde{\mathrm{Kh}}^{0,*}(T_{n-1,n-2})=\mathbb{Z}$ for $*=q_{n-1,n-2}(0)+1$ and 0 otherwise, in view of the long exact sequence

$$\cdots\to\widetilde{\mathrm{Kh}}^{-1,*-1}\to\widetilde{\mathrm{Kh}}^{0,*+1}\to\mathrm{Kh}^{0,*}\to\widetilde{\mathrm{Kh}}^{0,*-1}\to\widetilde{\mathrm{Kh}}^{1,*+1}\to\cdots$$

applied to $T_{n-1,n-2}$. Consequently, (24) is an isomorphism. This proves the surjectivity of $\gamma\otimes\mathbb{Q}$.

Next we show $\mathrm{Kh}^{2n-1,q_{n+1,n}(2n-1)-1}(D^{n-2}_{n,n})\otimes\mathbb{Q}=0$. Let $d^i_{n,m,\mathbb{Q}}$ denote the quantum infimum function for $\mathrm{Kh}(D^i_{n,m})\otimes\mathbb{Q}$. The above estimates apply equally well with $d^i_{n,m}$ replaced by $d^i_{n,m,\mathbb{Q}}$. We now need to show the inequality $d^{n-2}_{n,n,\mathbb{Q}}\{1\}\ge q_{n+1,n}$ is strict at homological degree $2n-1$. We reexamine the estimate for B above. The contribution from the $r=0$ term in (22) is strict at $2n-1$, because (16c) is strict at $2n-1$ by the first addendum of Lemma 5.3. The $r\ge2$ terms are also strict because the left-hand sides are $+\infty$. For the $r=1$ term, the contribution is not necessarily strict, but it would be if we could improve the contribution coming from the first term in (21) at homological degree $2n-1$ by an extra positive quantum shift, for each i. Upon tensoring with $\mathbb{Q}$, we can indeed make this improvement, by showing that the relevant maps $\mathrm{Kh}^{2n-2,q_{n-2,n-2}(0)+i-2+6n-5}(D^{i-1}_{n,n})\otimes\mathbb{Q}\to\mathrm{Kh}^{0,q_{n-2,n-2}(0)+i-2}(E^{i-1}_{n,n})\otimes\mathbb{Q}$ in (12)$\otimes\mathbb{Q}$ are surjective. Note that for $i=n-1$ this map is exactly $\gamma\otimes\mathbb{Q}$, whose surjectivity has just been shown. The $i<n-1$ cases follow from exactly the same argument, using the fact that, by an induction on i using (12), $\mathrm{Kh}^{0,*}(D^i_{n-2,n-2})\otimes\mathbb{Q}=\mathbb{Q}$ for $*=q_{n-2,n-2}(0)+i+1\pm1$ and 0 otherwise, and $\mathrm{Kh}^{1,*}(D^i_{n-2,n-2})=0$ (the base case for $D^0_{n-2,n-2}=T_{n-2,n-2}$ follows again from [Stošić 2007, Theorem 3.4]). □

Remark 5.4 The argument above carries through with $\mathbb{Q}$ replaced by any $\mathbb{F}_p$ for $p\neq2$. This shows the order of every element in $\mathrm{Kh}^{2n-1,q_{n+1,n}(2n-1)}(T_{n+1,n})$ is a power of 2.

6 Questions

In this section we make some comments mostly related to Theorem 2.1. First, we remark that the statements in Theorem 2.1 are almost designed minimally so that the $m=n$ case of Theorem 1.1 can be proven. One may try to prove more about $\mathrm{Kh}(T(n,n))$ and $\mathrm{Kh}(T(n+1,n))$ of their own interests using the same induction scheme. For example, it seems true that the bound $q_{n,n}$ is sharp not only for $h = 2pq$, but for any even h; not only $\mathrm{Kh}^{2n-1,q_{n+1,n}(2n-1)}(T(n+1,n))$ but also any $\mathrm{Kh}^{2pq+1,q_{n+1,n}(2pq+1)}(T(n+1,n))$ is torsion, which equals 0 if n is odd. Moreover, one can similarly expect to obtain a "graphical upper bound" for $\mathrm{Kh}(T(n,n))$ and $\mathrm{Kh}(T(n+1,n))$. This might lead to a proof of Corollary 1.3 (over any coefficient field) without resorting to Theorem 1.2. One may also try to prove similar results on Khovanov homology of more general torus links $T(n,m)$.

By pushing the representation theory techniques further, one may try to prove Theorem 1.2 for more general coefficient fields.

More interestingly, we state the following numerical observation as a conjecture. Let $D^i_{n,m}$ and $E^i_{n,m}$ be defined as in the previous section.

Conjecture 6.1 *The saddle cobordism $D^i_{n,n-1} \to D^{i-1}_{n,n-1}$ at the crossing of $D^i_{n,n-1}$ corresponding to the last letter σ_i induces a surjection in rational Khovanov homology.*

Equivalently, the conjecture states that the exact triangle

$$\begin{array}{ccc} \mathrm{Kh}(E^{i-1}_{n,n-1})[2n-2]\{6n-7\} & \longrightarrow & \mathrm{Kh}(D^i_{n,n-1}) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \mathrm{Kh}(D^{i-1}_{n,n-1})\{1\} & \end{array}$$

splits into a short exact sequence

$$0 \to \mathrm{Kh}(E^{i-1}_{n,n-1})[2n-2]\{6n-7\} \to \mathrm{Kh}(D^i_{n,n-1}) \to \mathrm{Kh}(D^{i-1}_{n,n-1})\{1\} \to 0.$$

An affirmative answer to Conjecture 6.1 enables one to express the rational Khovanov homology of the torus links $T(n,n)$ entirely in terms of that of the torus knots $T(n'+1,n')$. Explicitly, let $K_n(t,q)$ denote the Poincaré polynomial of the Khovanov homology of $T(n+1,n)$ and $L_n(t,q)$ denote that of $T(n,n)$. Then Conjecture 6.1 is equivalent to the following:

Conjecture 6.1′ *The Poincaré polynomial of* $\mathrm{Kh}(T(n,n))$ *is recursively defined by*

$$L_0 = 1, \quad L_1 = q^{-1} + q,$$

$$L_n = t^{2n-2}(q^{6n-8} + q^{6n-6})L_{n-2} + \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} C_{i-1} t^{2i(n-i)} q^{6i(n-i)} L_{n-2i} + \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} \left(\binom{n-2}{i} - \binom{n-2}{i-1} \right) t^{2i(n-i)} q^{6i(n-i)+n-2i-1} K_{n-2i-1} \quad \text{for } n \geq 2.$$

Here $C_n = \binom{2n}{n}/(n+1)$ is the n^{th} Catalan number.

On the other hand, Shumakovitch and Turner have the following conjecture:

Conjecture 6.2 [Gorsky et al. 2013, Conjecture 1.8] *The Poincaré polynomial of* $\mathrm{Kh}(T(n+1,n))$ *is recursively defined by*

$$K_0 = K_1 = q^{-1} + q, \quad K_2 = q + q^3 + t^2q^5 + t^3q^9,$$
$$K_n = q^{2n-2}K_{n-1} + t^{2n-2}q^{6n-6}K_{n-2} + t^{2n-1}q^{8n-8}K_{n-3} \quad \text{for } n \geq 3.$$

Conjectures 6.1′ and 6.2 together establish a recursive formula for the rational Khovanov homology of $T(n,n)$.

With the help of the computer program KnotJob [Schütz 2023], we are able to verify Conjecture 6.1 for the cases $n \leq 8$. The integral version of the conjecture is not true, and the first counterexample is found at $n = 7$ and $i = 6$. We also verified Conjecture 6.2 for $n \leq 8$.

One can also consider the (rational) annular Khovanov homology version of Conjecture 6.1. With the help of the program implemented in [Hunt et al. 2015], we are able to verify the cases $n \leq 4$ for the annular version. Due to the existence of a spectral sequence from annular Khovanov homology to the usual Khovanov homology, an affirmative answer to the annular version implies the version as stated. We remark that, however, the annular version of Conjecture 6.2 is not true.

More generally, if Conjecture 6.1 has an affirmative answer, one can ask for what values of m, n and i does $D^i_{n,m} \to D^{i-1}_{n,m}$ induce a surjection on Khovanov homology, aiming for a complete calculation of rational Khovanov homology of all torus links. This is not true for most cases where $n \mid m$. For $n \nmid m$ with $m+n \leq 7$, all violations for the annular version of this question are $(n,m,i) = (4,2,1), (4,2,3), (5,2,3)$.

References

[Abe and Tagami 2017] **T Abe**, **K Tagami**, *Characterization of positive links and the s–invariant for links*, Canad. J. Math. 69 (2017) 1201–1218 MR Zbl

[Bar-Natan 2002] **D Bar-Natan**, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. 2 (2002) 337–370 MR Zbl

[Bar-Natan 2005] **D Bar-Natan**, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. 9 (2005) 1443–1499 MR Zbl

[Beliakova and Wehrli 2008] **A Beliakova**, **S Wehrli**, *Categorification of the colored Jones polynomial and Rasmussen invariant of links*, Canad. J. Math. 60 (2008) 1240–1266 MR Zbl

[Benheddi 2017] **M Benheddi**, *Khovanov homology of torus links: structure and computations*, PhD thesis, Université de Genève (2017) Available at `https://archive-ouverte.unige.ch/unige:101697`

[Gillam 2012] **W D Gillam**, *Knot homology of* $(3,m)$ *torus knots*, J. Knot Theory Ramifications 21 (2012) art. id. 1250072 MR Zbl

[Gorsky et al. 2013] **E Gorsky**, **A Oblomkov**, **J Rasmussen**, *On stable Khovanov homology of torus knots*, Exp. Math. 22 (2013) 265–281 MR Zbl

[Grigsby et al. 2018] **J E Grigsby**, **A M Licata**, **S M Wehrli**, *Annular Khovanov homology and knotted Schur–Weyl representations*, Compos. Math. 154 (2018) 459–502 MR Zbl

[Hayden and Sundberg 2024] **K Hayden**, **I Sundberg**, *Khovanov homology and exotic surfaces in the* 4*–ball*, J. Reine Angew. Math. 809 (2024) 217–246 MR Zbl

[Hedden and Ording 2008] **M Hedden**, **P Ording**, *The Ozsváth–Szabó and Rasmussen concordance invariants are not equal*, Amer. J. Math. 130 (2008) 441–453 MR Zbl

[Hedden and Raoux 2023] **M Hedden**, **K Raoux**, *Knot Floer homology and relative adjunction inequalities*, Selecta Math. 29 (2023) art. id. 7 MR Zbl

[Hogancamp and Mellit 2019] **M Hogancamp**, **A Mellit**, *Torus link homology*, preprint (2019) arXiv 1909.00418

[Hunt et al. 2015] **H Hunt**, **H Keese**, **A Licata**, **S Morrison**, *Computing annular Khovanov homology*, preprint (2015) arXiv 1505.04484

[Kasprowski et al. 2024] **D Kasprowski**, **M Powell**, **A Ray**, **P Teichner**, *Embedding surfaces in* 4*–manifolds*, Geom. Topol. 28 (2024) 2399–2482 MR Zbl

[Khovanov 2000] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR Zbl

[Lee 2005] **E S Lee**, *An endomorphism of the Khovanov invariant*, Adv. Math. 197 (2005) 554–586 MR Zbl

[Litherland 1979] **R A Litherland**, *Signatures of iterated torus knots*, from "Topology of low-dimensional manifolds" (R A Fenn, editor), Lecture Notes in Math. 722, Springer (1979) 71–84 MR Zbl

[Manolescu and Ozsváth 2008] **C Manolescu**, **P Ozsváth**, *On the Khovanov and knot Floer homologies of quasi-alternating links*, from "Proceedings of Gökova Geometry/Topology Conference" (S Akbulut, T Önder, R J Stern, editors), GGT, Gökova (2008) 60–81 MR Zbl

[Manolescu and Piccirillo 2023] **C Manolescu**, **L Piccirillo**, *From zero surgeries to candidates for exotic definite* 4*–manifolds*, J. Lond. Math. Soc. 108 (2023) 2001–2036 MR Zbl

[Manolescu et al. 2023] **C Manolescu**, **M Marengon**, **S Sarkar**, **M Willis**, *A generalization of Rasmussen's invariant, with applications to surfaces in some four-manifolds*, Duke Math. J. 172 (2023) 231–311 MR Zbl

[Marengon et al. 2024] **M Marengon**, **A N Miller**, **A Ray**, **A I Stipsicz**, *A note on surfaces in* $\mathbb{CP}^2$ *and* $\mathbb{CP}^2\#\mathbb{CP}^2$, Proc. Amer. Math. Soc. Ser. B 11 (2024) 187–199 MR Zbl

[Norman 1969] **R A Norman**, *Dehn's lemma for certain* 4*–manifolds*, Invent. Math. 7 (1969) 143–147 MR Zbl

[Ozsváth and Szabó 2003] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geom. Topol. 7 (2003) 615–639 MR Zbl

[Pardon 2012] **J Pardon**, *The link concordance invariant from Lee homology*, Algebr. Geom. Topol. 12 (2012) 1081–1098 MR Zbl

[Pasechnik 2013] **D Pasechnik**, *Permutation character of the symmetric group on subsets of certain size*, Reply in MathOverflow thread (2013) Available at `https://mathoverflow.net/q/123754`

[Piccirillo 2020] **L Piccirillo**, *The Conway knot is not slice*, Ann. of Math. 191 (2020) 581–591 MR Zbl

[Rasmussen 2005] **J Rasmussen**, *Khovanov's invariant for closed surfaces*, preprint (2005) arXiv math/0502527

[Rasmussen 2010] **J Rasmussen**, *Khovanov homology and the slice genus*, Invent. Math. 182 (2010) 419–447 MR Zbl

[Roberts 2011] **L Roberts**, *Extending Van Cott's bounds for the τ and s–invariants of a satellite knot*, J. Knot Theory Ramifications 20 (2011) 1237–1245 MR Zbl

[Schütz 2023] **D Schütz**, *KnotJob*, Java program (2023) Available at `https://www.maths.dur.ac.uk/users/dirk.schuetz/knotjob.html`

[Stošić 2007] **M Stošić**, *Homological thickness and stability of torus knots*, Algebr. Geom. Topol. 7 (2007) 261–284 MR Zbl

[Stošić 2009] **M Stošić**, *Khovanov homology of torus links*, Topology Appl. 156 (2009) 533–541 MR Zbl

[Turner 2006] **P R Turner**, *Calculating Bar-Natan's characteristic two Khovanov homology*, J. Knot Theory Ramifications 15 (2006) 1335–1356 MR Zbl

[Turner 2008] **P Turner**, *A spectral sequence for Khovanov homology with an application to* $(3, q)$*–torus links*, Algebr. Geom. Topol. 8 (2008) 869–884 MR Zbl

[Turner 2017] **P Turner**, *Five lectures on Khovanov homology*, J. Knot Theory Ramifications 26 (2017) art. id. 1741009 MR Zbl

Department of Mathematics, University of California, Berkeley
Berkeley, CA, United States

qiuyu_ren@berkeley.edu

Proposed: Ciprian Manolescu
Seconded: András I Stipsicz, Peter Ozsváth

Received: 25 May 2023
Revised: 11 December 2023

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2024 is US $805/year for the electronic version, and $1135/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Geometry & Topology

Volume 28 Issue 8 (pages 3511–3972) 2024