Geometry & Topology

Volume 28 (2024)

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Geometry & Topology 28:9 (2024) 4233–4255
DOI: 10.2140/gt.2024.28.4233
Published: 27 December 2024

Small-energy isotopies of loose Legendrian submanifolds

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We prove that for a closed Legendrian submanifold L of dimension $n \geq 2$ with a loose chart of size η, any Legendrian isotopy starting at L can be C^0–approximated by a Legendrian isotopy with energy arbitrarily close to $\frac{1}{2}\eta$. This in particular implies that the displacement energy of loose displaceable Legendrians is bounded by half the size of its smallest loose chart, which proves a conjecture of Dimitroglou Rizell and Sullivan (2020).

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1 Introduction

In many situations it requires a positive amount of energy to connect two different Legendrian submanifolds of a contact manifold via a contact isotopy. On the other hand, Murphy [6] showed that for loose Legendrians the existence of a contact isotopy is a purely topological question. The goal of this paper is to give an upper bound for the minimal energy that is required for Legendrian isotopies of loose Legendrians.

Let (M, α) be a strict contact manifold of dimension $2n+1$, and let L be a closed Legendrian submanifold. This means that α is a 1–form on M such that $\alpha \wedge (d\alpha)^n$ defines a volume form, and L is everywhere tangent to $\xi := \ker \alpha$ and of dimension n. The Reeb vector field R_α is the unique vector field on M defined by $i_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$. A Reeb chord of L is a flow line $\gamma\colon [0, l] \to M$ of R_α with endpoints on L, and $l > 0$ is called the *action* of the Reeb chord.

We consider isotopies L_t, $t \in [0, 1]$, of L through Legendrian submanifolds. It is a general fact that such isotopies are always induced by an ambient contact isotopy ϕ_t of M, ie an isotopy of M that preserves ξ (in fact, this even holds for parametrized Legendrians, see Geiges [5, Theorem 2.6.2]). We can associate to ϕ_t its contact Hamiltonian $H\colon M \times [0, 1] \to \mathbb{R}$, which is defined by the formula

$$(1) \qquad H(\phi_t(x), t) = \alpha(\dot{\phi}_t(x)),$$

where $\dot{\phi}_t(x)$ denotes the time-derivative of $\phi_t(x)$. Conversely, given a function $H\colon M \times [0, 1] \to \mathbb{R}$, which we may also view as a time-dependent function $H_t\colon M \to \mathbb{R}$ with $t \in [0, 1]$, we can define its time-dependent contact vector field X_{H_t} via the equations

$$(2) \qquad H_t = \alpha(X_{H_t}) \quad \text{and} \quad -dH_t|_\xi = i_{X_{H_t}} d\alpha|_\xi.$$

The condition $\alpha \wedge (d\alpha)^n \neq 0$ ensures that this defines a unique vector field. The flow of X_{H_t} preserves ξ, and thus gives a contact isotopy ϕ_t^H. It is straightforward to check that these two correspondences between functions on M and contact isotopies are inverse to each other.

To a contact isotopy ϕ_t and its associated Hamiltonian H_t we associate the energy

$$\|\phi_t\|_\alpha = \|H_t\| := \int_0^1 \max_{x \in M} |H_s(x)|\, ds, \tag{3}$$

which induces a nondegenerate metric on the space of contactomorphisms as was shown by Shelukhin [9, Theorem A].

Unless stated otherwise, all manifolds here and below are assumed to be connected, and isotopies always start at the identity.

Assume that L_0 and L_1 are two distinct closed Legendrian submanifolds of M Legendrian isotopic to each other. We are interested in the infimum of the energies of contact isotopies that move L_0 to L_1. Denote this infimum by $d(L_0, L_1)$. In [8], Rosen and Zhang showed that either $d(L_0, L_1) = 0$ or $d(L_0, L_1) > 0$ always holds for fixed L_0 independent of L_1. It is expected that the latter holds under quite general assumptions on M; see [8, Conjecture 1.10]. For example, the following theorem, which combines results obtained by Dimitroglou Rizell and Sullivan [2; 3] and Oh [7], implies that this is indeed the case for displaceable Legendrians in contact manifolds which are either closed[1] or of the form $(P \times \mathbb{R}, \lambda + dz)$, where $(P, d\lambda)$ is an exact geometrically bounded symplectic manifold and z denotes the coordinate on $\mathbb{R}$.

Theorem 1.1 *Let (M, α) be either compact or equal to $(P \times \mathbb{R}, \lambda + dz)$ as above, and let L_0 and L_1 be two distinct closed Legendrian submanifolds that can be connected by a Legendrian isotopy. If there are no Reeb chords between L_0 and L_1, then[2] $2d(L_0, L_1)$ is bounded from below by the minimal action of Reeb chords of L_0 (and by symmetry also of L_1).*

In a strict contact manifold (M, α), a subset A is said to be *displaced* from a subset B if there are no Reeb chords between A and B. The *displacement energy* of a Legendrian L_0 is the infimum of $d(L_0, L_1)$ over all Legendrians L_1 such that there are no Reeb chords between L_0 and L_1, ie Theorem 1.1 states that the displacement energy of L_0 is bounded from below by half of the minimal action of Reeb chords of L_0.

We are concerned with the converse question. Can we give an upper bound on $d(L_0, L_1)$ depending on L_0 and L_1? As a first step, it was proven by Dimitroglou Rizell and Sullivan [2, Theorem 1.8] that the displacement energy of the standard Legendrian 2–sphere in $\mathbb{R}^5$ can be made arbitrarily small by adding

[1] The results of [3] and [7] also hold for more general classes of contact manifolds which may not be closed.

[2] The additional factor of 2 in Theorem 1.2 when compared to the formulation of the results in [2], [3] or [7] appears because in this paper the energy is measured using the maximum norm of a Hamiltonian, and not the oscillatory norm. By Remark 2.7, the displacement energy defined in terms of the oscillatory norm is twice as large as the one used here, as long as the Reeb vector field is complete.

a stabilization contained in a sufficiently small neighborhood of a point $x \in L$. They conjectured that the same should hold for any closed Legendrian in a contact manifold. We will use their techniques to prove this conjecture if $\dim L \geq 2$, and, in fact, give an explicit bound of the displacement energy in terms of the size of the stabilization (Corollary 1.11). It turns out that this upper bound coincides with the lower bound from Theorem 1.1 for "nice" stabilizations and therefore is optimal.

These results follow from the following more general theorem about loose Legendrians, which states that we can guarantee the existence of an isotopy of small energy, and even C^0–approximate any given isotopy.

Theorem 1.2 *Let (M^{2n+1}, α), with $n \geq 2$, be a strict contact manifold, and let $U_0, U_1 \subseteq M$ be open subsets with the property that there exist $\varepsilon_0, \varepsilon_1 > 0$ such that the energy (as a contactomorphism of U_i) of the time-1 map of any compactly supported contact isotopy $\psi_t^i \colon U_i \to U_i$ is smaller than ε_i for $i \in \{0,1\}$. Let $f_t \colon L \to M$, with $t \in [0,1]$, be a homotopy of closed, connected Legendrian embeddings such that $f_i(L) \cap U_i$ is a loose Legendrian submanifold of U_i for $i \in \{0,1\}$. Then, for any given $\eta > 0$, there exist compactly supported contact isotopies ϕ_t and ψ_t^i for $i \in \{0,1\}$, with $\|\phi_t\|_\alpha < \eta$, $\operatorname{supp}(\psi_t^i) \subseteq U_i$, $\|\psi_t^i\|_\alpha < \varepsilon_i$ and $\psi_1^1 \circ \phi_1 \circ \psi_1^0 \circ f_0 = f_1$. Furthermore, given any $\delta > 0$, these isotopies can be chosen in such a way that $\phi_t \circ \psi_1^0 \circ f_0$ is δ–close[3] to f_t for all $t \in [0,1]$; see Figure 1. In particular, the energy of the concatenation $(\psi^0 * \phi * \psi^1)_t$ is smaller than $\varepsilon_0 + \varepsilon_1 + \eta$.*

Remark 1.3 Proposition 2.8 gives an explicit class of examples of sets that satisfy the property of U_0 and U_1 in the statement of Theorem 1.2. In particular, any closed Darboux ball and thus also loose charts in the sense of Murphy [6, Definition 4.3] satisfy this property for some $\varepsilon > 0$. To be more precise, any open subset of a closed Darboux ball can be compressed into any arbitrarily small neighborhood of the origin via contact isotopies with a bound on their energies depending only on the Darboux ball — to see this, consider the contact isotopy $(x, y, z) \mapsto (e^{-\lambda t} x, e^{-\lambda t} y, e^{-2\lambda t} z)$, with $t \in [0,1]$, on $(\mathbb{R}^{2n+1}, f(dz - y\,dx))$ for some function $f \colon \mathbb{R}^{2n+1} \to (0, \infty)$, and proceed as in the proof of Proposition 2.8, using that f is bounded when restricted to the Darboux ball. Following the proof of Proposition 2.8, we may thus assume that the Darboux ball is strict after possibly shrinking it, and then Proposition 2.8 applies.

The main ingredients in the proof of Theorem 1.2 are the following four facts:

(i) $d(\widetilde{L}_0, \widetilde{L}_1) = 0$ whenever $\widetilde{L}_0$ and $\widetilde{L}_1$ are two n–dimensional non-Legendrian submanifolds that can be connected via a contact isotopy, as was shown by Rosen and Zhang [8, Theorem 1.10, Proposition 8.6] and the refinement of this result to parametrized non-Legendrian submanifolds by Dimitroglou Rizell and Sullivan [4, Theorem B]; see Theorems 2.3, 2.4 and Corollary 2.6 below.

(ii) Any formally Legendrian submanifold can be C^0–approximated by loose Legendrians (see [6, Corollary 5.1]) (Lemma 3.4).

[3] Throughout this paper, closeness refers to strict C^0–closeness, ie two functions f and g are δ–close if and only if $\|f - g\|_{C^0} < \delta$.

(iii) Murphy's h-principle for loose Legendrians [6, Theorem 1.2] (Theorem 3.1).

(iv) Upper bounds on the energy of contact isotopies in Weinstein neighborhoods [2] (Proposition 2.8).

In outline, the proof goes as follows. Let Φ_t be a contact isotopy so that $f_t = \Phi_t \circ f_0$. First C^0–perturb f_0 to a formal Legendrian embedding g which is non-Legendrian. By (i), we can find a contact isotopy $\widetilde{\phi}_t$ with arbitrarily small energy so that $\Phi_1 \circ g = \widetilde{\phi}_1 \circ g$. Then C^0–approximate g by a loose Legendrian embedding χ using (ii). Let h_i for $i \in \{0,1\}$, be loose Legendrian embeddings obtained by stabilizing f_i inside a of a small neighborhood of a point in $U_i \cap f_i(L)$. We can perform these steps in such a way that h_0 is formally isotopic to χ inside of a small Weinstein neighborhood of $f_0(L)$ and formally isotopic to f_0 via an isotopy with compact support in U_0, and h_1 is formally isotopic to $\widetilde{\phi}_1 \circ \chi$ inside of a small Weinstein neighborhood of $f_1(L)$ and formally isotopic to f_1 via an isotopy with compact support in U_1. By Murphy's h-principle we can find a contact isotopies θ_t^i and ψ_t^i, with $i \in \{0,1\}$, so that ψ_t^i has compact support in U_i, $\psi_1^0 \circ f_0 = h_0$, $\psi_1^1 \circ h_1 = f_1$, $\theta_1^0 \circ h_0 = \chi$ and $\theta_1^1 \circ \widetilde{\phi}_1 \circ \chi = h_1$. The isotopy ψ_t^i can be chosen to satisfy $\|\psi_t^i\|_\alpha < \varepsilon_i$ by assumption, and $\|\theta_t^i\|_\alpha$ can be assumed to be arbitrarily small by (iv). Then the isotopies ψ_t^i for $i \in \{0,1\}$ and $\phi_t := (\theta^0 * \widetilde{\phi} * \theta^1)_t$ have the desired properties and can, in fact, be chosen so that $\phi_t \circ \psi_1^0 \circ f_0$ is C^0–close to f_t. The details of the proof are explained in Section 4.

As a consequence of Theorem 1.2, we obtain an upper bound on the displacement energy of a loose Legendrian.

Theorem 1.4 *Let (M^{2n+1}, α), with $n \geq 2$, be a strict contact manifold, and let $U \subseteq M$ be an open subset with the property that there exists an $\varepsilon > 0$ such that the energy (as a contactomorphism of U) of the time-1 map of any compactly supported contact isotopy $\psi_t\colon U \to U$ is smaller than ε. Let $L \subseteq M$ be a closed, displaceable Legendrian submanifold such that $L \cap U$ is loose in U. Then, for any given $\eta > 0$ and $E > 0$, there exist compactly supported contact isotopies ϕ_t and ψ_t with $\|\phi_t\|_\alpha < \eta$, $\operatorname{supp}(\psi_t) \subseteq U$ and $\|\psi_t\|_\alpha < \varepsilon$ such that all Reeb chords between L and $\phi_1(\psi_1(L))$ have action larger than E.*

If in addition the image of L under the Reeb flow is closed as a subset of M, we may choose ϕ_t and ψ_t as above such that there are no Reeb chords between L and $\phi_1(\psi_1(L))$. In particular, the displacement energy of L is not larger than ε.

In the case that L may not be displaceable, we have the following more general statement.

Theorem 1.5 *Let M and U be as in Theorem 1.4, and let $E_1, E_2 \in \mathbb{R}$. Let $L \subseteq M$ be a closed Legendrian submanifold such that $L \cap U$ is loose in U, and assume that there exists a Legendrian submanifold L_1 that is Legendrian isotopic to L such that all Reeb chords from L to L_1 have action larger than E_1, and all Reeb chords from L_1 to L have action larger than E_2. Then, for any given $\eta > 0$, there exist compactly supported contact isotopies ϕ_t and ψ_t with $\|\phi_t\|_\alpha < \eta$, $\operatorname{supp}(\psi_t) \subseteq U$, and $\|\psi_t\|_\alpha < \varepsilon$ such that all Reeb chords from L to $\phi_1(\psi_1(L))$ have action larger than E_1, and all Reeb chords from $\phi_1(\psi_1(L))$ to L have action larger than E_2.*

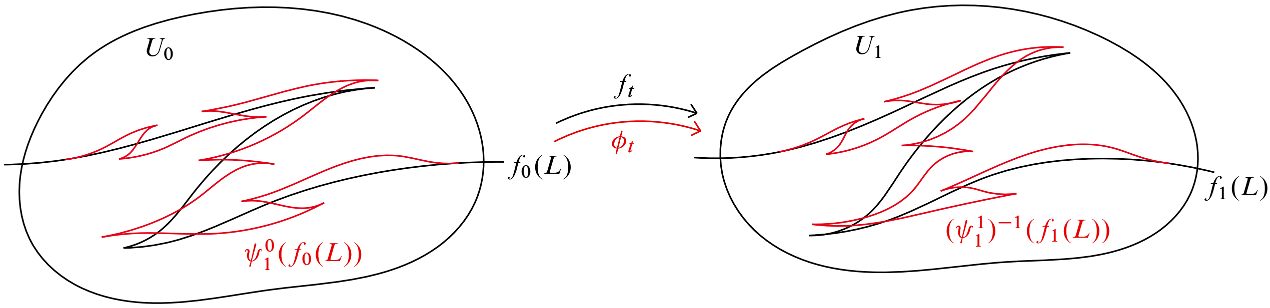

Figure 1: After perturbing the Legendrians there exists an isotopy ϕ_t of small energy.

If, in addition, the image of L under the nonnegative (resp. nonpositive) Reeb flow is closed as a subset of M, the above statement still holds if we allow $E_1 = \infty$ (resp. $E_2 = \infty$), meaning that there are no Reeb chords from L to L_1 (resp. from L_1 to L).

We show in Section 5 how these results follow from Theorem 1.2.

Remark 1.6 In a contactization $(M \times \mathbb{R}, dz + \lambda)$ of an exact symplectic manifold $(M, d\lambda)$, the image of any compact set under the Reeb flow is closed.

Remark 1.7 If $f_t \colon L \to (M, \alpha)$ is a homotopy of Legendrian embeddings of a compact, connected manifold L of dimension ≥ 2 with nonempty boundary, then for any $\varepsilon > 0$ there exists a contact isotopy $\phi_t \colon M \to M$ such that $\phi_1 \circ f_0 = f_1$ and $\|\phi_t\|_\alpha < \varepsilon$ (for unparametrized Legendrians, see [8]). Furthermore, $\phi_t \circ f_0$ can be chosen to C^0–approximate f_t. This follows from the same techniques as Theorem 1.2 since L_0 and L_1 have arbitrarily small loose charts near their respective boundaries (where "small" refers both to the diameter and the size of the loose chart as defined at the end of this section). Indeed, if $f \colon L \to M$ is a Legendrian embedding with nonempty boundary and $\xi_t \colon L \to L$ with $t \in [0,1]$ is a homotopy of embeddings C^0–close to the identity starting at the identity so that $\xi_1(L)$ is contained in the interior of L, then for any stabilization $Sf \colon L \to M$ of f sufficiently close the boundary of L, we have $f \circ \xi_1 = Sf \circ \xi_1$. Now both $f \circ \xi_t$ and $Sf \circ \xi_t$ are induced by ambient contact isotopies ψ_t and $S\psi_t$, respectively, whose contact Hamiltonians vanish along the Legendrians. Thus, we may assume the energies of ψ_t and $S\psi_t$ to be arbitrarily small. Therefore, if $(U, Sf(L) \cap U)$ is a small loose chart for Sf, then $(\psi_1^{-1} \circ S\psi_1)(U, Sf(L) \cap U)$ is a small loose chart for f.

As the following corollary shows, we can also approximate an arbitrary Hamiltonian function instead of requiring the energy of the isotopy to be very small.

Corollary 1.8 *Assume that the assumptions of Theorem 1.2 hold, and let $H \colon M \times [0,1] \to \mathbb{R}$ be a compactly supported Hamiltonian. Then the conclusion of Theorem 1.2 remains true if we replace the condition $\|\phi_t\|_\alpha < \eta$ by $\|H_t - F_t\| < \eta$, where F_t denotes the contact Hamiltonian associated to ϕ_t.*

Corollary 1.8 is proven in Section 5.

Proposition 2.8 shows that local Weinstein neighborhoods of L_0 and L_1 of height 2η satisfy the assumption on the sets U_i with $\varepsilon = 2\eta$. Furthermore, recall that stabilized Legendrians are always loose. Applying the above theorems to this case gives the following results.

Corollary 1.9 *Let $L_0, L_1 \subseteq (M^{2n+1}, \alpha)$, for some $n \geq 2$, be closed Legendrian submanifolds. For $i \in \{0,1\}$, stabilize L_i inside of a local Weinstein neighborhood U_i of L_i of height $2\varepsilon_i > 0$ to obtain a new Legendrian SL_i. Let V_i be an open neighborhood of L_i such that $U_i \subseteq V_i$. If there exists a family of Legendrian embeddings $f_t\colon L \to M$ for $t \in [0,1]$, with $f_i(L) = SL_i$ then, for any given $\eta > 0$, there exist contact isotopies ϕ_t, and ψ_t^i for $i \in \{0,1\}$, with*

$$\operatorname{supp}(\psi_t^i) \subseteq V_i, \quad \|\psi_t^i\|_\alpha < \varepsilon_i, \quad \|\phi_t\|_\alpha < \min\{\eta, (\varepsilon_0 - \|\psi_t^0\|_\alpha), (\varepsilon_1 - \|\psi_t^1\|_\alpha)\}, \quad \psi_1^1 \circ \phi_1 \circ \psi_1^0 \circ f_0 = f_1.$$

Furthermore, given any $\delta > 0$, these isotopies can be chosen in such a way that $\phi_t \circ \psi_1^0 \circ f_0$ is δ–close to f_t for all $t \in [0,1]$. In particular, the energy of the concatenation $(\psi^0 * \phi * \psi^1)_t$ is smaller than $\varepsilon_0 + \varepsilon_1$.

Remark 1.10 Again, it is possible to approximate arbitrary Hamiltonians as in Corollary 1.8 instead of the condition on $\|\phi_t\|_\alpha$.

Corollary 1.11 *Let $L^n \subseteq (M, \alpha)$, for some $n \geq 2$, be a closed Legendrian submanifold, and let U_ε be a local Weinstein neighborhood of L of height 2ε. Let SL denote a Legendrian submanifold of M obtained by stabilizing L inside of U_ε, and assume that SL is displaceable. Then for any $E > 0$, there exists a compactly supported contact isotopy ϕ_t with $\|\phi_t\|_\alpha < \varepsilon$ such that all Reeb chords between SL and $\phi_1(SL)$ have action larger than E.*

If, in addition, the image of SL under the Reeb flow is closed as a subset of M, we may assume that there are no Reeb chords between SL and $\phi_1(SL)$. In particular, the displacement energy of SL is not larger than ε.

Corollaries 1.9 and 1.11 are proven in Section 5.

Remark 1.12 The supremum of the actions of Reeb chords in U_ε is bounded by 2ε. This means that Corollary 1.11 states the reverse of the energy capacity inequality in Theorem 1.1 for stabilized Legendrians.

Remark 1.13 The proof of Proposition 2.8 motivates the following coordinate-independent definition of the size of a loose chart. Namely, let (M, α) be a strict contact manifold, $L \subseteq M$ a Legendrian, and $U \subseteq M$ a connected open set so that $U \cap L \subseteq U$ is loose. Then we say that L has a loose chart of size

$$2 \sup_V \inf_{\phi_t} \|\phi_t\|_\alpha \tag{4}$$

in U, where the supremum is taken over all open sets $V \subseteq U$, and the infimum is taken over all contact isotopies ϕ_t with support in U such that $(V, \phi_1(L) \cap V)$ contains a loose chart. In other words, the size of a loose chart is (up to a factor of 2) the minimal energy that is required to produce an arbitrarily small loose chart. This follows from the observations that

(i) if $V' \subseteq V$ and $(V', \phi_1(L) \cap V')$ contains a loose chart then also $(V, \phi_1(L) \cap V)$ contains a loose chart, and that

(ii) the position of a small open set $V \subseteq U$ does not matter, as any two small open sets can be moved into each other via a contact isotopy that has small energy.

In particular, if the loose chart is contained in a local Weinstein neighborhood of height ε, then its size will be smaller than ε by the arguments in the proof of Proposition 2.8.

Note that the size of a loose chart depends on the chosen contact form. For example, if we replace α by $\lambda\alpha$ for some $\lambda > 0$, then it follows from the definition of the contact Hamiltonian associated to a contact isotopy that the size of any loose chart changes by a factor of λ as well.

With this definition, the above results can be restated vaguely as "If L_0 and L_1 have loose charts of size ε_0 and ε_1, respectively, then, for any $\eta > 0$, one can approximate any Legendrian isotopy from L_0 to L_1 by a Legendrian isotopy of energy less than $\frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_1 + \eta$". Combining this with Murphy's h-principle for loose Legendrians yields the following quantitative version of the h-principle.

Corollary 1.14 *Let L_0 and L_1 be two closed loose Legendrians that are formally isotopic and admit loose charts of size ε_0 and ε_1 (in M), respectively. Then, for any $\eta > 0$, there exists a Legendrian isotopy from L_0 to L_1 of energy less than $\frac{1}{2}\varepsilon_0 + \frac{1}{2}\varepsilon_1 + \eta$.*

Corollary 1.14 is proven in Section 5.

Outline of the paper In Section 2, we prove a C^0–close refinement of results of Rosen and Zhang [8] and Dimitroglou Rizell and Sullivan [4], which allow us to find contact isotopies of arbitrarily small energy between non-Legendrian embeddings, and we discuss energy bounds in local Weinstein neighborhoods of Legendrians.

Section 3 deals with loose Legendrians. We show how to obtain C^0–bounds for the h-principle for loose Legendrians by adapting Murphy's arguments in [6].

Section 4 contains the proof of Theorem 1.2.

In Section 5, we explain how Theorems 1.4 and 1.5, and Corollaries 1.8, 1.9, 1.11 and 1.14 follow from Theorem 1.2.

Acknowledgements The idea for this paper originated in discussions with Georgios Dimitroglou Rizell. I am very grateful to him for many helpful comments and for suggesting the definition of the size of a loose chart in Remark 1.13 (personal communication, December 3, 2020). I am also grateful to the referee for pointing out gaps in the proofs of the main result and Theorem 1.4 in the original version of this paper, and for many comments that helped to greatly improve this article. Also, I would like to thank my adviser Tobias Ekholm for interesting discussions about the contents of this paper.

This work is partially supported by the Knut and Alice Wallenberg Foundation, grant KAW2020.0307.

2 The energy of a contactomorphism

Let (M,α) be a strict contact manifold, and let $\mathrm{Cont}_0(M)$ denote the set of all compactly supported contactomorphisms on M that are isotopic to the identity through compactly supported contactomorphisms. For any function $H\colon M\times[0,1]\to\mathbb{R}$ with compact support define

$$\|H\| := \int_0^1 \max_{x\in M} |H(x,s)|\,ds. \tag{5}$$

$\|\phi_t^H\|_\alpha := \|H\|$ will also be called the *energy* of ϕ_t^H, where ϕ_t^H denotes the contact isotopy associated to H.

For any contactomorphism $\phi\in\mathrm{Cont}_0(M)$ define[4]

$$\|\phi\|_\alpha := \inf_{\psi_t}\|\psi_t\|_\alpha, \tag{6}$$

where the infimum is taken over all compactly supported contact isotopies ψ_t on M that satisfy $\psi_1=\phi$. Recall that in this paper all isotopies start at the identity.

Shelukhin proved the following theorem.

Theorem 2.1 [9, Theorem A] *Let $\phi,\psi\in\mathrm{Cont}_0(M)$ be two contactomorphisms. Then*:

(i) ***Nondegeneracy*** $\|\phi\|_\alpha\ge 0$, *and* $\|\phi\|_\alpha=0$ *if and only if* $\phi=\mathrm{id}_M$.

(ii) ***Triangle inequality*** $\|\phi\psi\|_\alpha\le\|\phi\|_\alpha+\|\psi\|_\alpha$.

(iii) ***Symmetry*** $\|\phi^{-1}\|_\alpha=\|\phi\|_\alpha$.

(iv) ***Naturality*** $\|\psi\phi\psi^{-1}\|_\alpha=\|\phi\|_{\psi^*\alpha}$.

In fact, the following lemma is an easy consequence of the definition of a contact Hamiltonian isotopy associated to a contact isotopy.

Lemma 2.2 *Let $H\colon M\times[0,1]\to\mathbb{R}$ be a compactly supported Hamiltonian with associated contact isotopy ϕ_t^H. Then $-H_{1-t}$ is the contact Hamiltonian associated to $\phi_{1-t}^H\circ(\phi_1^H)^{-1}$.*

Let $\psi: M\to M$ be a contactomorphism and denote by $f: M\to\mathbb{R}_{>0}$ the function defined by $\psi^*\alpha=f\alpha$. Then $(fH_t)\circ\psi^{-1}$ is the Hamiltonian associated to the contact isotopy $\psi\phi_t^H\psi^{-1}$.

Rosen and Zhang [8] analyzed how $\|\cdot\|_\alpha$ behaves with respect to orbits of certain subsets of M under the contactomorphism group. The following result follows immediately from [8, Theorem 1.10 and Proposition 8.6] and will be essential in our argument.

[4] By abuse of notation, we will use the same symbol to denote the energy of a contact isotopy and a contactomorphism. When the contactomorphism is written with the subscript t, we will always mean the energy of the contact isotopy.

Theorem 2.3 *Let $L^n \subseteq M^{2n+1}$ be a closed, connected non-Legendrian submanifold, and $\Phi_t\colon M \to M$ a compactly supported contact isotopy. Then there exist compactly supported contact isotopies $\phi_t\colon M \to M$ of arbitrarily small energies that satisfy $\phi_1(L) = \Phi_1(L)$.*

Recently, Dimitroglou Rizell and Sullivan [4] refined Rosen and Zhang's result to (parametrized) proper non-Legendrian embeddings. The following is a weaker version of [4, Theorem B].

Theorem 2.4 *Let $f\colon L^n \to M^{2n+1}$ be a proper, connected non-Legendrian embedding of a manifold L, and let $\Phi_t\colon M \to M$ be a compactly supported contact isotopy. Then there exist compactly supported contact isotopies $\phi_t\colon M \to M$ of arbitrarily small energies that satisfy $\phi_1 \circ f = \Phi_1 \circ f$.*

Remark 2.5 From the proof of [4, Theorem B] it is clear that Theorem 2.4 also holds in the case that L is disconnected with finitely many connected components as long as we assume that every component is non-Legendrian, since the heart of the argument is purely local around the image of any connected component of L. Furthermore, even if L has countably infinitely many connected components, only finitely many components of $f(L)$ can intersect the (compact!) support of Φ_t by properness of f. Thus, the conclusion of Theorem 2.4 also holds in this case (still under the assumption that f is non-Legendrian on every connected component of L).

We will need the following C^0–close version of Theorem 2.4.

Corollary 2.6 *Let $f\colon L^n \to M^{2n+1}$ be a proper embedding which is non-Legendrian almost everywhere (ie $D_x f(T_x L) \nsubseteq \ker\alpha$ for a.e. $x \in L$), and let $\Phi_t\colon M \to M$ be a compactly supported contact isotopy. Then for any $\delta > 0$, there exist compactly supported contact isotopies $\phi_t\colon M \to M$ of arbitrarily small energies that satisfy $\phi_1 \circ f = \Phi_1 \circ f$ so that $\phi_t \circ f$ is δ–close to $\Phi_t \circ f$ for all $t \in [0,1]$.*

Proof Let $\{U_j^k\}$, with $k \in \{1, \dots, K_j\}$ and $j \in \{1, \dots, J\}$, be an open cover of the support of Φ_t by relatively compact subsets such that

(i) the diameter of U_j^k is smaller than $\delta/(J+1)$ for all j and k, and
(ii) $U_j^k \cap U_j^{k'} = \varnothing$ for all $j \in \{1, \dots, J\}$ and $k, k' \in \{1, \dots, K_j\}$ with $k \neq k'$.

The existence of such sets can be seen as follows. After choosing a proper embedding of M into some $\mathbb{R}^H$, with $H \in \mathbb{N}$, we may assume that M is a proper submanifold of $\mathbb{R}^H$. For any $\lambda > 0$, $\mathbb{R}^H$ is covered by the cubes

$$C_{\lambda,j}^k := \Pi_{i=1}^H \big(\lambda\big(2k_i + j_i - \tfrac{1}{3}\big), \lambda\big(2k_i + j_i + \tfrac{4}{3}\big)\big), \tag{7}$$

where $k = (k_1, \dots, k_H) \in \mathbb{Z}^H$ and $j = (j_1, \dots, j_H) \in \{0,1\}^H$. We set $U_{\lambda,j}^k := C_{\lambda,j}^k \cap M$. Then (ii) is clearly satisfied. We define

$$\mathcal{K}_\lambda := \{k \in \mathbb{Z}^H \mid U_{\lambda,j}^k \cap \operatorname{supp}\Phi_t \neq \varnothing \text{ for some } j \in \{0,1\}^H\}. \tag{8}$$

Then the sets $\{U_{\lambda,j}^k\}$, for $j \in \{0,1\}^H$ and $k \in \mathcal{K}_\lambda$, cover the support of Φ_t, and it is straightforward to see that (i) is satisfied if λ is sufficiently small.

By the proof of the fragmentation lemma in [1], there exist a subdivision $t_0 = 0 < t_1 < \cdots < t_N = 1$, for some $N \in \mathbb{N}$, of the interval $[0,1]$, and contact isotopies $\phi_t^{i,j}$, for $j \in \{1,\ldots,J\}$ and $t \in [t_i, t_{i+1}]$ for $i \in \{0,\ldots,N-1\}$, such that each $\phi_t^{i,j}$ is supported in $\bigcup_{1\le k\le K_j} U_j^k$, and for every $i \in \{0,\ldots,N-1\}$, $\Phi_t \circ (\Phi_{t_i})^{-1} = \phi_t^{i,J} \circ \cdots \circ \phi_t^{i,2} \circ \phi_t^{i,1}$ for $t \in [t_i, t_{i+1}]$. Furthermore, we may assume that $\Phi_t \circ (\Phi_{t_i})^{-1}$, for $t \in [t_i, t_{i+1}]$, is $\delta/(J+1)$–close to the identity for every $i \in \{0,\ldots,N-1\}$.

First, we assume that $N = 1$, and we drop i from the notation. Let $\varepsilon > 0$. We will define contact isotopies $\widetilde{\phi}_t^j$ inductively over j. Since f is non-Legendrian almost everywhere, $f|_{f^{-1}(\bigcup_{1\le k\le K_1} U_1^k \cap f(L))}$ is non-Legendrian on each connected component of $f^{-1}\big(\bigcup_{1\le k\le K_1} U_1^k \cap f(L)\big)$. By Theorem 2.4 applied to $\phi_t^1|_{\bigcup_{1\le k\le K_1} U_1^k}$ and $f|_{f^{-1}(\bigcup_{1\le k\le K_1} U_1^k \cap f(L))}$, there exists a compactly supported contact isotopy $\widetilde{\phi}_t^1$ on $\bigcup_{1\le k\le K_1} U_1^k$ such that $\|\widetilde{\phi}_t^1\|_\alpha < \varepsilon/J$ and $\widetilde{\phi}_1^1 \circ f = \phi_1^1 \circ f$. In the following we will view $\widetilde{\phi}_t^1$ as a contact isotopy of M with compact support in $\bigcup_{1\le k\le K_1} U_1^k$. Assume now that $\widetilde{\phi}_t^1, \ldots, \widetilde{\phi}_t^j$ have already been defined. As before, it follows from Theorem 2.4 applied to

$$\phi_t^{j+1}|_{\bigcup_{1\le k\le K_{j+1}} U_{j+1}^k} \quad \text{and} \quad \phi_1^j \circ \cdots \circ \phi_1^1 \circ f|_{(\phi_1^j\circ\cdots\circ\phi_1^1\circ f)^{-1}\left(\bigcup_{1\le k\le K_{j+1}} U_{j+1}^k \cap (\phi_1^j\circ\cdots\circ\phi_1^1\circ f)(L)\right)}$$

that we can find a contact isotopy $\widetilde{\phi}_t^{j+1}$ with compact support in $\bigcup_{1\le k\le K_{j+1}} U_{j+1}^k$ such that

$$\|\widetilde{\phi}_t^{j+1}\|_\alpha < \frac{\varepsilon}{J} \quad \text{and} \quad \widetilde{\phi}_1^{j+1} \circ \phi_1^j \circ \cdots \circ \phi_1^1 \circ f = \phi_1^{j+1} \circ \phi_1^j \circ \cdots \circ \phi_1^1 \circ f.$$

After J steps we have defined isotopies $\widetilde{\phi}_t^1, \ldots, \widetilde{\phi}_t^J$ such that $\phi_t := \widetilde{\phi}_t^1 * \cdots * \widetilde{\phi}_t^J$ satisfies $\phi_1 \circ f = \Phi_1 \circ f$, where $*$ denotes the concatenation of isotopies.

Furthermore, $\|\phi_t\|_\alpha < \varepsilon$ and each $\widetilde{\phi}_t^j$ is $\delta/(J+1)$–close to the identity since it is supported in a disjoint union of sets of diameter less than $\delta/(J+1)$. In particular, ϕ_t is $J\delta/(J+1)$–close to the identity. Also, Φ_t is $\delta/(J+1)$–close to the identity by assumption. All in all, we see that $\phi_t \circ f$ is δ–close to $\Phi_t \circ f$ for all $t \in [0,1]$. This finishes the proof for $N = 1$.

In the case that $N > 1$, we perform the above constructions on each time interval $[t_i, t_{i+1}]$ separately (with ε replaced by ε/N) and concatenate the obtained contact isotopies to find the desired isotopy ϕ_t. □

Remark 2.7 Similarly to the Hofer metric in symplectic manifolds, one can also consider the oscillatory seminorm

$$\|H\|_{\mathrm{osc}} := \int_0^1 \Big(\max_{x\in M} H(x,s) - \min_{x\in M} H(x,s) \Big)\, ds \tag{9}$$

of a compactly supported function $H\colon M \times [0,1] \to \mathbb{R}$. If M is noncompact,

$$2 \max_{x\in M} |H(x,t)| \ge \max_{x\in M} H(x,t) - \min_{x\in M} H(x,t) \ge \max_{x\in M} |H(x,t)| \tag{10}$$

holds for all times $t \in [0,1]$. This implies that the same inequalities also hold for the energies of contact isotopies that are defined using $\|\cdot\|$ and $\|\cdot\|_{\mathrm{osc}}$.

In fact, when dealing with displacement, these two seminorms differ exactly by a factor of 2 in the following sense. Assume that $A_0, A_1 \subseteq M$ are two compact subsets such that there exists a contact isotopy $\phi_t\colon M \to M$ with $\phi_1(A_0) = A_1$. Assume that the Reeb vector field is complete and denote the Reeb flow by ϕ_t^α. Then a straightforward argument shows that

$$2 \inf_{T\in\mathbb{R}} \inf_{\phi_t} \|\phi_t\|_\alpha = \inf_{\phi_t} \|\phi_t\|_{\mathrm{osc}}, \tag{11}$$

where the infimum on the left-hand (resp. right-hand) side is taken over all compactly supported contact isotopies $\phi_t\colon M \to M$ with $\phi_1^H(A_0) = \phi_T^\alpha(A_1)$ (resp. $\phi_1^H(A_0) = A_1$).

Let N be a manifold, and denote its 1–jet bundle by J^1N. For $\delta > 0$ we define *a local Weinstein neighborhood of the zero section of height* 2δ to be an open set of the form

$$U_\delta := (\{(q,p,z) \in J^1N \mid q \in V,\, p \in W_q,\, |z| < \delta_q\}, \alpha_{\mathrm{std}}), \tag{12}$$

where $V \subseteq N$ is open, $W_q \subseteq T_q^*N$ is a star-shaped neighborhood of 0, and $V \to (0,\infty)$, $q \mapsto \delta_q$, is a function with $\sup_{q\in V} \delta_q \le \delta$. Here, α_{std} is locally defined as $\alpha_{\mathrm{std}} = \sum_i p_i\, dq_i - dz$, where $\{q_i\}$ are local coordinates on N, and $\{p_i\}$ are the conjugate coordinates on the cotangent fibers.

Similarly, for a Legendrian submanifold $N \subseteq (M,\alpha)$ we call $U_\delta \subseteq M$ *a local Weinstein neighborhood of* N *of height* 2δ if it is strictly contactomorphic to a local Weinstein neighborhood $\widetilde{U}_\delta \subseteq J^1N$ of the zero section of height 2δ via a contactomorphism that identifies $N \cap U_\delta$ with $N \cap \widetilde{U}_\delta$. Recall that any closed Legendrian has strict Weinstein neighborhoods [5, Theorem 6.2.2]. Such a neighborhood $U_\delta \approx \widetilde{U}_\delta$ is said to have fibers of diameter $\varepsilon > 0$ (with respect to some metric on M) if the diameter in M of the set of points identified with $\pi^{-1}(\{q\}) \subseteq \widetilde{U}_\delta$ is less than ε for all $q \in N \cap \widetilde{U}_\delta \approx N \cap U_\delta$, where $\pi\colon J^1N \to N$ denotes the projection onto the zero section.

The next proposition gives us bounds on the energies of contact isotopies with support in local Weinstein neighborhoods. The first part is an adaption of the last paragraph of the proof of [2, Theorem 1.8].

Proposition 2.8 *For any manifold N and any local Weinstein neighborhood U_δ of the zero section of height $2\delta > 0$ and every compactly supported contact isotopy $\phi_t\colon U_\delta \to U_\delta$ with $t \in [0,1]$, there exists a compactly supported contact isotopy $\widetilde{\phi}_t\colon U_\delta \to U_\delta$ such that $\widetilde{\phi}_1 = \phi_1$ and $\|\widetilde{\phi}_t\|_\alpha < 2\delta$.*

If, in addition, U_δ has fibers of diameter $\varepsilon > 0$, then for any compact subset $K \subseteq U_\delta$, the isotopy $\widetilde{\phi}_t|_K$ can be chosen to be ε–close to $\phi_t|_K$ for all $t \in [0,1]$.

Proof We assume that N is closed and that $U_\delta = J^1_\delta N := \{(q,p,z) \in J^1N \mid |z| < \delta\}$. The general case is directly analogous by restricting to a compact subset containing the support of ϕ_t. Let $0 < \eta < \delta$ be such that the isotopy ϕ_t is supported in $J^1_\eta N$. For $\lambda > 0$ consider a time-dependent function $H_t\colon M \to \mathbb{R}$, with $t \in [0,1]$, such that $H_t(q,p,z) = \lambda z$ on $J^1_{e^{-\lambda t}\eta} N$, and such that H_t is appropriately cut off outside of $J^1_{e^{-\lambda t}\eta}N$. Then its associated contact isotopy ψ_t^H will map (q,p,z) in $J^1_\eta N$ to $(q,e^{-\lambda t}p,e^{-\lambda t}z)$ in $J^1_{e^{-\lambda t}\eta}N$, and it satisfies

$$\|\psi_t^H\|_\alpha \leq \int_0^1 \lambda\eta e^{-\lambda t}\,dt + \eta e^{-\lambda} = \eta, \tag{13}$$

where the $\eta e^{-\lambda}$–summand is due to the chosen cut-off. Note also that $((\psi_1^H)^*\alpha)|_{J^1_\eta N} = e^{-\lambda}\alpha|_{J^1_\eta N}$.

Now let

$$\widetilde{\phi}_t =: \big((\psi_s^H) * (\psi_1^H\phi_s(\psi_1^H)^{-1}) * (\psi_{1-s}^H \circ (\psi_1^H)^{-1})\big)_t, \tag{14}$$

where $*$ denotes the concatenation of isotopies.

As a consequence of Lemma 2.2,

$$\begin{aligned} \|\widetilde{\phi}_t\|_\alpha &= \|\big((\psi_s^H) * (\psi_1^H\phi_s(\psi_1^H)^{-1}) * (\psi_{1-s}^H \circ (\psi_1^H)^{-1})\big)_t\|_\alpha \\ &= \|\psi_t^H\|_\alpha + \|\psi_1^H\phi_t(\psi_1^H)^{-1}\|_\alpha + \|\psi_{1-t}^H \circ (\psi_1^H)^{-1}\|_\alpha \\ &= 2\|\psi_t^H\|_\alpha + e^{-\lambda}\|\phi_t\|_\alpha \leq 2\eta + e^{-\lambda}\|\phi_t\|_\alpha, \end{aligned} \tag{15}$$

which is smaller than 2δ if λ is sufficiently large.

Now assume that $K \subseteq U_\delta$ is a compact subset. By choosing η sufficiently close to δ, we may assume that $K \subseteq J^1_\eta N$. If, in addition, U_δ has fibers of diameter $\varepsilon > 0$, then $\psi_t^H|_{J^1_\eta N}$ and $\psi_{1-t}^H \circ (\psi_1^H)^{-1}|_{J^1_{e^\lambda \eta}N}$ are ε–close to the identity as they preserves the fibers of the projection onto the zero section. Therefore, $\psi_1^H\phi_t(\psi_1^H)^{-1} \circ \psi_1^H|_{J^1_\eta N} = \psi_1^H\phi_t|_{J^1_\eta N}$ is ε–close to $\phi_t|_{J^1_\eta N}$, and $\widetilde{\phi}_t|_{J^1_\eta N}$ will be ε–close to $\phi_t|_{J^1_\eta N}$ if we perform the concatenation in the definition of $\widetilde{\phi}_t$ is such a way that ψ_t^H and $\psi_{1-t}^H \circ (\psi_1^H)^{-1}$ are traversed very quickly. □

3 Loose Legendrians

Murphy's h-principle for loose Legendrians is an important ingredient in the proof of our main result. Roughly speaking, it states that for two loose Legendrians in the sense of [6, Definition 4.3], the existence of a Legendrian isotopy between them is a purely homotopy-theoretical problem. We recall this result in this section and explain how to refine Murphy's proof to obtain C^0–bounds.

Recall that a formal Legendrian embedding of an n–dimensional manifold L into a contact manifold (M^{2n+1},ξ) is a pair (f,F_s) consisting of an embedding $f\colon L \to M$ and a homotopy of fiberwise injective bundle maps $F_s\colon TL \to f^*(TM)$, with $s \in [0,1]$, covering f, such that F_0 is equal to the differential Df of f and $F_1(TL)$ is a Lagrangian subspace of $f^*\xi$ at every point with respect to the conformal symplectic structure on ξ. We identify a Legendrian embedding $f\colon L \to M$ with the formal Legendrian embedding (f,Df), where Df is viewed as the constant homotopy.

The first result that we need is the following version of Murphy's h-principle.

Theorem 3.1 *Let (M^{2n+1}, ξ), for some $n \geq 2$, be a contact manifold endowed with a Riemannian metric, and let L^n be a connected manifold. Let (f_t, F_s^t) with $t \in [0,1]$ be a homotopy of proper formal Legendrian embeddings $L \to (M, \xi)$, constant (in t) outside of a compact subset of L, such that (f_0, F_s^0) and (f_1, F_s^1) are Legendrian embeddings which admit loose charts $(U_0, U_0 \cap f_0(L))$ and $(U_1, U_1 \cap f_1(L))$, respectively. Then for any $\delta > 0$, there exists a homotopy $\Phi_t : L \to M$ of Legendrian embeddings, constant outside of a compact subset of L, such that $\Phi_i = f_i$ for $i \in \{0,1\}$, and Φ_t is pointwise $(d+\delta)$–close to f_t for all $t \in [0,1]$, where d denotes the maximum of the diameters of U_0 and U_1 (with distances measured in M).*

Proof We will explain how to adjust the arguments in the proof of [6, Theorem 1.2] to prove this theorem. The C^0–close part in the case of a fixed loose chart and the fact that we may choose compactly supported homotopies are consequences of the constructions in Murphy's proof. We will first explain how to obtain these two results and then reduce the proof in the general case to these special cases.

Let $\delta > 0$ be an arbitrary number, and define d as in the statement.

First note that the assumptions of the theorem imply in particular that f_t is Legendrian outside of a compact subset of L for all t. Furthermore, we may assume that $f_0^{-1}(U_0 \cap f_0(L))$ and $f_1^{-1}(U_1 \cap f_1(L))$ are contained in a compact subset of L after possibly replacing U_0 and U_1 by slightly smaller loose charts. Note that this will not increase d.

First, we prove the theorem under the assumptions that $U_0 = U_1$, and that $f_t^{-1}(U_0 \cap f_0(L))$ and $f_t|_{f_t^{-1}(U_0 \cap f_0(L))}$ do not depend on t (in other words, we assume that there is a fixed loose chart for the family f_t).

By [6, Proposition 3.4] and the subsequent paragraph in [6], there exists a compactly supported homotopy $\overline{f_t} : L \to M$ of wrinkled Legendrian embeddings[5] agreeing with f_t outside of a compact subset such that $\overline{f_t}^{-1}(U_0 \cap f_0(L))$ and $\overline{f_t}|_{\overline{f_t}^{-1}(U_0 \cap f_0(L))}$ do not depend on t, $\overline{f_i} = f_i$ for $i \in \{0,1\}$, and $\overline{f_t}$ is $\frac{1}{2}\delta$–close to f_t.

Following [6], there exists a homotopy g_t of wrinkled Legendrian embeddings obtained from $\overline{f_t}$ by replacing loose charts in U_0 by inside-out wrinkles which agrees with $\overline{f_t}$ outside of $f_0^{-1}(U_0 \cap f_0(L))$, satisfies $g_t(f_0^{-1}(U_0 \cap f_0(L))) \subseteq U_0$, and admits a (finite) collection of markings for its wrinkles which agree with the markings of the model inside-out wrinkle near $t \in \{0,1\}$. It follows that g_t is d–close to $\overline{f_t}$ since U_0 has diameter d.

Let $\widetilde{g}_t : L \to M$ denote the homotopy of (smooth) Legendrian embeddings obtained from g_t by resolving the singularities using the collection of markings; see [6, Lemma 4.2]. Note that $\widetilde{g}_t$ agrees with f_t outside of a compact subset of L. Furthermore, we may assume that $\widetilde{g}_t$ is $\frac{1}{2}\delta$–close to g_t.

Combining the C^0- estimates, we see that $\widetilde{g}_t$ is $(d+\delta)$–close to f_t.

[5] Here and below, we omit the data of the Darboux charts in the definition of a wrinkled Legendrian from the notation.

The explicit construction of an inside-out wrinkle in [6] shows that, in addition, we may assume that for $i \in \{0,1\}$, there exists a homotopy $h_t^i\colon L \to M$ of Legendrian embeddings which agrees with $f_i = \overline{f_i}$ outside of $f_0^{-1}(U_0 \cap f_0(L))$ so that $h_0^i = f_i$, $h_1^i = \widetilde{g}_i$ and $h_t^i(f_0^{-1}(U_0 \cap f_0(L))) \subseteq U_0$. It follows again that h_t^i is d–close to f_i.

Now consider the homotopy $\Phi_t = (h^0 * \widetilde{g} * \overline{h^1})_t$ of Legendrian embeddings, where $\overline{h_t^1} = h_{1-t}^1$. By construction, $\Phi_i = f_i$ for $i \in \{0,1\}$, and Φ_t agrees with f_t outside of a compact subset of L. Since h_t^i, f_t and $\widetilde{g}_t$ are constant (in t) outside of a compact set (in fact, they are equal to each other outside of a compact set), h_t^i is d–close to f_i for $i \in \{0,1\}$, and $\widetilde{g}_t$ is $(d+\delta)$–close to f_t, Φ_t will be $(d+\delta)$–close to f_t for all $t \in [0,1]$ if the concatenation in the definition of Φ_t is performed in such a way that h_t^0 and $\overline{h_t^1}$ are traversed sufficiently fast.

It follows that Φ_t satisfies the required properties.

This finishes the proof in the case that there exists a fixed loose chart.

In the general case, pick a path p_t, with $t \in [0,1]$, in L so that f_i has a loose chart of diameter bounded by d in the complement of some neighborhood of $f_i(p_i)$ for $i \in \{0,1\}$. We may assume that (f_t, F_s^t) is Legendrian in a neighborhood of p_t for all $t \in [0,1]$ by Lemma 3.2 below. Let ξ_t be a compactly supported isotopy of L with $\xi_t(p) = p_t$ for all t. After replacing (f_t, F_s^t) by $(f_t \circ \xi_t, F_s^t \circ D\xi_t)$, we may assume that $p := p_t$ does not depend on t. Then there exists a compactly supported contact isotopy $\psi_t\colon M \to M$ such that $\psi_t \circ f_0 = f_t$ on a neighborhood of p since homotopies of compact Legendrian embeddings can always be extended to contact isotopies. Because ψ_t has compact support, it is uniformly C^0–continuous. Thus, we can find $\varepsilon > 0$ with $\varepsilon < \frac{1}{2}\delta$ such that for any two points $x, y \in M$, the distance between $\psi_t(x)$ and $\psi_t(y)$ is smaller than $\frac{1}{2}\delta$ for all $t \in [0,1]$ whenever the distance between x and y is smaller than ε.

The homotopy $(\psi_t^{-1} \circ f_t, (D\psi_t)^{-1} \circ F_s^t)$ of formal Legendrian embeddings is genuinely Legendrian for $t \in \{0,1\}$ and equal to (f_0, Df_0) on a neighborhood V of p for all $t \in [0,1]$, and does not depend on t outside of a compact subset of L. Let U be a Darboux ball around $f_0(p)$ so that $f_0^{-1}(U) \subseteq V$, $(\psi_t^{-1} \circ f_t)^{-1}(U) = f_0^{-1}(U)$, and $(U, U \cap f_0(L), f_0(p)) \subseteq (M, f_0(L), f_0(p))$ is contactomorphic to $(B_\rho, B_\rho \cap \mathbb{R}^n, \{0\}) \subseteq (\mathbb{R}^{2n+1}, \mathbb{R}^n, \{0\})$ for some $\rho > 0$, with its standard contact structure. Furthermore, we ask that the diameter of U is smaller than $\frac{1}{2}\varepsilon$ and that f_i has a loose chart of diameter bounded by d in the complement of $\psi_i(U)$. Now let (g_t, G_s^t) be a homotopy of formal Legendrian embeddings which agrees with $(\psi_t^{-1} \circ f_t, (D\psi_t)^{-1} \circ F_s^t)$ outside of a compact subset of $f_0^{-1}(U)$ so that $(g_t, G_s^t)|_{f_0^{-1}(U)}$ is Legendrian, has image contained in U, and does not depend on t, and so that there is a loose chart for $g_t(L)$ contained in a compact subset of U. In addition, we assume that there exists a homotopy of formal Legendrian embeddings from $(g_t, G_s^t)|_{f_0^{-1}(U)}$ to $f_0|_{f_0^{-1}(U)}$ with compact support in $f_0^{-1}(U)$ and image in U. For example, we can find such a (g_t, G_s^t) by performing a $(\chi = 0)$–stabilization of $\psi_t^{-1} \circ f_t$ inside of U.

In particular, (g_t, G^t_s) has a fixed loose chart of diameter smaller than $\frac{1}{2}\varepsilon$, does not depend on t outside of a compact subset of L, and is $\frac{1}{2}\varepsilon$–close to $\psi_t^{-1} \circ f_t$ for $t \in \{0,1\}$. By what we have proven above, we may find a homotopy $h_t\colon L \to M$ of Legendrian embeddings which does not depend on t outside of a compact subset of L so that $h_i = g_i$ for $i \in \{0,1\}$ and so that h_t is $\frac{1}{2}\varepsilon$–close to g_t for all $t \in [0,1]$. In particular, h_t is ε–close to $\psi_t^{-1} \circ f_t$ for all t.

By our choice of ε, $\psi_t \circ h_t$ is $\frac{1}{2}\delta$–close to f_t for all t, by our choice of U, $\psi_i \circ h_i$ has a loose chart of diameter bounded by d in the complement of $\psi_i(U)$, and by our choice of g_t, there exists a formal Legendrian isotopy from $\psi_i \circ h_i$ to f_i with support in $\psi_i(U)$ for $i \in \{0,1\}$. It follows that this formal isotopy is $\frac{1}{2}\delta$–close to the constant isotopy since the diameter of $\psi_i(U)$ is bounded by $\frac{1}{2}\delta$.

Now we can apply the special case of Theorem 3.1 which we have already proven to these formal isotopies to find for $i \in \{0,1\}$ a homotopy $\phi^i_t\colon L \to M$ of Legendrian embeddings from $\psi_i \circ h_i$ to f_i which is $(d+\delta)$–close to f_i for all t.

Then $\Phi_t := (\phi^0_{1-s} * (\psi_s \circ h_s) * \phi^1_s)_t$ is a compactly supported homotopy of Legendrian embeddings from f_0 to f_1. If we perform the concatenation again in such a way that $\overline{\phi^0_s}$ and ϕ^1_s are traversed very quickly, then Φ_t is $(d+\delta)$–close to f_t for all $t \in [0,1]$. □

The following lemma is needed to reduce Theorem 3.1 to the case of a fixed loose chart.

Lemma 3.2 *Let $(f_t\colon L \to (M,\xi), F^t_s)$, with $t \in [0,1]$, be a homotopy of formal Legendrian embeddings such that (f_i, F^i_s) is Legendrian for $i \in \{0,1\}$. Let $p_t \in L$, with $t \in [0,1]$, be a smooth path, and let $W \subseteq L \times [0,1]$ be an open neighborhood of $\bigcup_t \{(p_t, t)\}$. Then there exists a C^0–small formal homotopy from (f_t, F^t_s) to a homotopy of formal Legendrian embeddings (g_t, G^t_s) which is fixed for $t \in \{0,1\}$ and outside of a compact subset of W, so that (g_t, G^t_s) is genuinely Legendrian on a neighborhood of p_t for all t.*

Proof **Step 1** *We may assume that p_t does not depend on t.*

Let ξ_t be a compactly supported isotopy of L so that $\xi_t(p) = p_t$ for all t. If the lemma holds for $(f_t \circ \xi_t, F^t_s \circ D\xi_t)$, then it also holds for (f_t, F^t_s).

Step 2 *We may assume that $f_t(p)$ does not depend on t.*

Let Φ_t be a compactly supported contact isotopy of M such that $\Phi_t(f_0(p)) = f_t(p)$. If the lemma holds for $(\Phi_t^{-1} \circ f_t, (D\Phi_t)^{-1} \circ F^t_s)$, then it also holds for (f_t, F^t_s) by uniform C^0–continuity of Φ_t.

As all of the constructions below are inherently performed inside of an arbitrarily small neighborhood of p, the homotopies of formal Legendrian homotopies will be supported in a compact subset of W.

Step 3 *We may assume that $F^t_s(p) = D_p f_t$ for all $s, t \in [0,1]$.*

Let $\Xi_{s,t}\colon T_{f_t(p)}M \to T_{f_t(p)}M$, with $s,t \in [0,1]$, be a family of isomorphisms which is equal to the identity for $(s,t) \in \{0\} \times [0,1] \cup [0,1] \times \{0,1\}$ and satisfies $\Xi_{s,t} \circ D_p f_t = F^t_s$ for all $s,t \in [0,1]$. Such

$\Xi_{s,t}$ exist since the map $\mathrm{GL}_{2n+1}(\mathbb{R}) \to \mathrm{Mono}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^{2n+1})$ given by restriction to the first n coordinate directions is a Serre fibration. Choose a local coordinate chart around p diffeomorphic to $\mathbb{R}^{2n+1}$ in which $f_t(p)$ is identified with the origin. Let $\Phi_{s,t}\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, with $s,t \in [0,1]$, be a family of compactly supported diffeomorphisms that agrees with $\Xi_{s,t}$ near the origin (where we identify $T_{f_t(p)}M$ with $\mathbb{R}^{2n+1}$ using the local coordinates) and is equal to the identity for $(s,t) \in \{0\} \times [0,1] \cup [0,1] \times \{0,1\}$. Such a $\Phi_{s,t}$ can be constructed by cutting off the generating vector field of $\Xi_{s,t}$ for fixed t (viewed as an isotopy with time-parameter s) outside of a compact neighborhood of the origin. Using the chosen coordinates, $\Phi_{s,t}$ may be viewed as a family of diffeomorphisms fixing $f_t(p)$, with compact support near p, which is equal to the identity for $(s,t) \in \{0\} \times [0,1] \cup [0,1] \times \{0,1\}$, and which satisfies $D_{f_t(p)}\Phi_{s,t} = \Xi_{s,t}$. Then $\Phi_{u,t} \circ f_t$, with $t,u \in [0,1]$, defines an isotopy of families of embeddings starting at f_t and ending at $\Phi_{1,t} \circ f_t$. It is well-known and easy to check that $\Phi_{u,t} \circ f_t$ extends to a family of formal embeddings $(\Phi_{u,t} \circ f_t, F_s^{t,u})$, with $s,t,u \in [0,1]$, rel $t \in \{0,1\}$ starting at $(f_t, D_p f_t *_s F_s^t)$, which can be chosen so that $F_s^{t,u}(p) = ((D_{f_t(p)}\Phi_{u(1-s),t} \circ D_{f_t(p)}\Phi_{u,t}^{-1}) \circ (D_p(\Phi_{u,t} \circ f_t)) *_s F_s^t(p) = (D_{f_t(p)}\Phi_{u(1-s),t} \circ D_p f_t)) *_s F_s^t(p) = (\Xi_{u(1-s),t} \circ D_p f_t) *_s F_s^t(p) = F_{u(1-s)}^t *_s F_s^t(p)$. For $u = 1$, this becomes $F_{1-s}^t(p) *_s F_s^t(p)\colon T_pL \to T_{f_t(p)}M$, which is clearly homotopic rel $(s,t) \in \partial([0,1]^2)$ to the constant (in s) family $F_1^t = \Xi_{1,t} \circ D_p f_t = D_p(\Phi_{1,t} \circ f_t)$. From this it follows that $F_s^{t,1}$ is homotopic (with fixed underlying embedding) rel $(s,t) \in \partial([0,1]^2)$ to a family $\widetilde{F}_s^t$, $s,t \in [0,1]$, which satisfies $\widetilde{F}_s^t(p) = D_p(\Phi_{1,t} \circ f_t)$ for all $s,t \in [0,1]$. This finishes the third step as $(f_t, D_p f_t *_s F_s^t)$ is clearly homotopic to (f_t, F_s^t) rel $(s,t) \in \partial([0,1])^2$.

Step 4 *We may assume that $D_p f_t$ does not depend on t.*

Let $k_t\colon U \to M$ be a family of Legendrian embeddings of a neighborhood $U \subseteq L$ of p such that $k_t(p) = f_t(p)$ and $D_p k_t = D_p f_t$. Then there exists a contact isotopy Ψ_t with compact support near $f_t(p)$ so that $k_0 = \Psi_t \circ k_t$ near p. Now $u \mapsto (\Psi_{ut} \circ f_t, D\Psi_{ut} \circ F_s^t)$ defines a homotopy of formal Legendrian isotopies. For $u = 1$ and at p this becomes $(\Psi_t \circ f_t, D\Psi_t \circ F_s^t)(p) = (k_0(p), D\Psi_t \circ D_p f_t) = (k_0(p), D_p k_0)$ by Step 2 and our choice of Ψ_t and k_t.

Step 5 *We prove the lemma.*

Let $\widetilde{f}\colon U \to M$ be a Legendrian embedding of a neighborhood $U \subseteq L$ of p that satisfies $D_p \widetilde{f} = D_p f_t$. On a sufficiently small neighborhood $V \subseteq U$ of p, $\widetilde{f}$ is C^1–close to f_t and $D\widetilde{f}$ is C^0–close to F_s^t for all $s,t \in [0,1]$ by Step 4. The first observation implies that we can find a family of C^1–small isotopies $\Xi_{s,t}\colon M \to M$, with $s,t \in [0,1]$, with compact support near $f_t(p)$ so that $\widetilde{f} = \Xi_{1,t} \circ f_t$ near p for all $t \in [0,1]$, and $\Xi_{s,t}(p) = p$ and $D_p \Xi_{s,t} = \mathrm{id}$ for all $s,t \in [0,1]$. Then $(\Xi_{1,t} \circ f_t, D(\Xi_{1-s,t} \circ f_t) *_s F_s^t)$ defines a formal Legendrian homotopy which is clearly homotopic to (f_t, F_s^t), where we have again identified[6] the tangent fibers using local coordinates near p so that we can view $D(\Xi_{1-s,t} \circ f_t) *_s F_s^t$ as maps of tangent bundles covering $\Xi_{1,t} \circ f_t$. Near p, $D(\Xi_{1-s,t} \circ f_t) *_s F_s^t$ is C^0–close to $D(\Xi_{1,t} \circ f_t)$.

[6] As in the proof of the Step 3, we can use an identification of the tangent fibers in which the contact structure is constant near p.

Since the space of monomorphisms $T_x L \to T_{\Xi_{1,t}\circ f_t(x)} M$ forms a smooth manifold and the Legendrian monomorphisms (ie those whose image is contained in ξ and Lagrangian) form a smooth submanifold, all depending smoothly on x and t, this implies that we can find a homotopy $\mathscr{F}_s^{u,t}$, with $s, t, u \in [0,1]$, with support near p starting at $D(\Xi_{1-s,t} \circ f_t) *_s F_s^t$ covering $\Xi_{1,t} \circ f_t$ with $\mathscr{F}_0^{u,t} = D(\Xi_{1,t} \circ f_t)$ and $\mathscr{F}_1^{u,t}$ Legendrian for all $t, u \in [0,1]$ so that $\mathscr{F}_s^{1,t} = D(\Xi_{1,t} \circ f_t)$ near p for all $s, t \in [0,1]$.

This finishes the proof since $\Xi_{1,t} \circ f_t$ is Legendrian near p by construction. □

Remark 3.3 As is clear from the proof, Lemma 3.2 also holds if (f_i, F_s^i) is not Legendrian for $i = 0$ or $i = 1$ and we don't demand that the homotopy from (f_i, F_s^i) to (g_i, G_s^i) is constant.

We will also need the following statement that allows us to approximate a formal Legendrian by a loose Legendrian.

Lemma 3.4 *Let $(f\colon L \to M, F_s)$ be a formal Legendrian embedding of a closed and connected manifold L into a contact manifold (M, ξ) with $\dim L \geq 2$. Then (f, F_s) is C^0–closely formally isotopic to a loose Legendrian embedding. This isotopy can be chosen to be constant outside of any nonempty neighborhood of the set where (f, F_s) is not genuinely Legendrian.*

Proof We explain how this is a consequence of the proof of [6, Corollary 5.1] in a way similar to how Theorem 3.1 above is a consequence of the proof of [6, Theorem 1.2]. Let $\varepsilon > 0$. As in [6], we may assume that there exists an open set $U \subseteq M$ of diameter smaller than ε so that f is Legendrian on $f^{-1}(U)$ and that $(U, U \cap f(L))$ is a loose chart; see Remark 3.3 above. By [6, Proposition 3.4], there exists a formal homotopy ε–close to f which fixes the loose chart from (f, F_s) to a wrinkled Legendrian embedding g. We can find pairwise disjoint loose charts inside of U, one for each wrinkle of g. By replacing each of those loose charts by an inside-out wrinkle, we can find a wrinkled Legendrian embedding $\widetilde{g}$ ε–close to g which admits markings for its wrinkles. Using the markings to resolve the wrinkles, we obtain a Legendrian embedding which is 3ε–closely formally isotopic to f. Since [6, Proposition 3.4] holds relatively, and the other constructions in the proof are compactly supported, the lemma follows. □

4 Proof of Theorem 1.2

In this section we present the proof of our main result as outlined in the introduction. The idea of the proof is taken from the proof of [2, Theorem 1.8].

Proof of Theorem 1.2 Let $U_0, U_1 \subseteq M$ be two open subsets and f_t a homotopy of Legendrian embeddings as in the statement of Theorem 1.2. Let $\Phi_t\colon M \to M$ be a contact isotopy with $\Phi_t \circ f_0 = f_t$. Let $\delta, \eta > 0$ be two positive numbers.

Let W_i denote a Weinstein neighborhood of $f_i(L)$ of height $\frac{1}{3}\eta$ with fibers of diameter $\frac{1}{3}\delta$ for $i = 1, 2$. By choosing W_0 close enough to $f_0(L)$, we may assume that $\Phi_1(W_0) \subseteq W_1$.

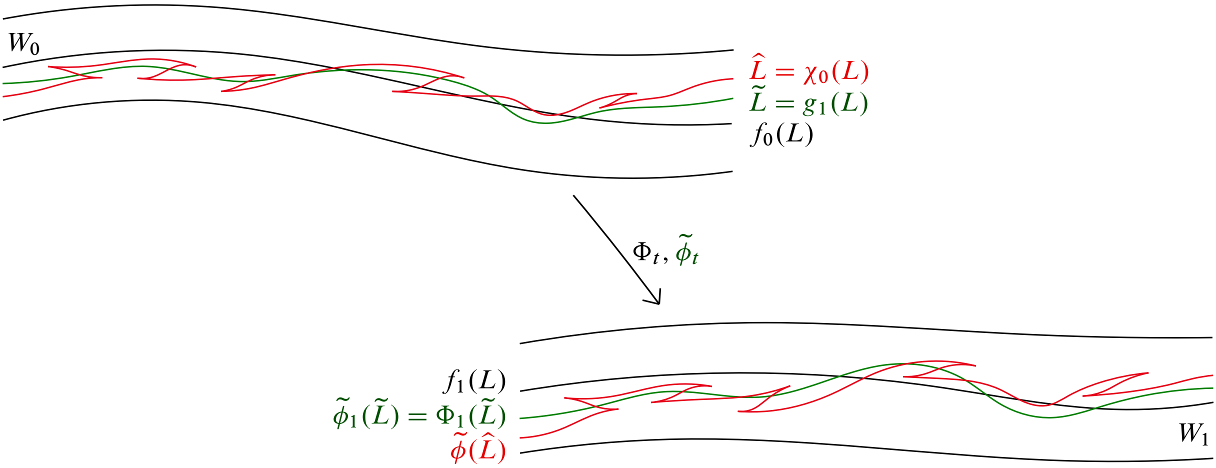

Figure 2: Starting from an isotopy f_t of Legendrian embeddings, we construct a non-Legendrian perturbation g_1, a loose Legendrian approximation χ_0 of g_1 inside of W_0, and contact isotopies Φ_t and $\widetilde{\phi}_t$ that satisfy $\Phi_t \circ f_0 = f_t$ and $\widetilde{\phi}_1(\widetilde{L}) = \Phi_1(\widetilde{L})$.

We perturb f_0 to get a formal Legendrian homotopy $(g_t\colon L \to W_0, G_{s,t})$ from f_0 to a formal Legendrian $(g_1, G_{s,1})$ which is non-Legendrian almost everywhere so that g_t is $\frac{1}{6}\delta$–close to f_0 and $\Phi_t \circ g_1$ is $\frac{1}{3}\delta$–close to f_t for all $t \in [0,1]$. Write $\widetilde{L} := g_1(L)$.

According to Corollary 2.6, there exists a contact isotopy $\widetilde{\phi}_t$ with $\widetilde{\phi}_1 \circ g_1 = \Phi_1 \circ g_1$ and $\|\widetilde{\phi}_t\|_\alpha < \frac{1}{3}\eta$ such that $\widetilde{\phi}_t \circ g_1$ is $\frac{1}{3}\delta$–close to $\Phi_t \circ g_1$ for all $t \in [0,1]$. In particular, $\widetilde{\phi}_t \circ g_1$ is $\frac{2}{3}\delta$–close to f_t for all $t \in [0,1]$.

For any $\hat{\delta} > 0$, we can find a formal Legendrian homotopy $(\chi_t\colon L \to W_0, X_{s,t})$ from a loose Legendrian embedding $\chi_0\colon L \to W_0$ to $(g_1, G_{s,1})$ so that χ_t is $\hat{\delta}$–close to the constant homotopy; see Lemma 3.4. Write $\hat{L} := \chi_0(L)$. We may choose χ_0 in such a way that $\hat{L}$ and $\widetilde{\phi}_t(\hat{L})$ have loose charts of diameter smaller than $\frac{1}{3}\delta$ contained in W_0 and W_1, respectively. We choose $\hat{\delta} < \frac{1}{6}\delta$ so small that $\widetilde{\phi}_t \circ \chi_0$ is $\frac{1}{3}\delta$–close to $\widetilde{\phi}_t \circ g_1$ for all $t \in [0,1]$. In particular, $\widetilde{\phi}_t \circ \chi_0$ is δ–close to f_t for all $t \in [0,1]$.

We claim that we may assume that there exists a formal Legendrian homotopy $(\xi_t\colon L \to W_1, \Xi_{s,t})$ from $\widetilde{\phi}_1 \circ \chi_0$ to f_1 that is $\frac{1}{3}\delta$–close to f_1. In order to see this, let $\widetilde{\delta} > 0$ be so small that for all points $x \in f_t(L)$, with $t \in [0,1]$, and $y \in M$ so that the distance between x and y is smaller than $\widetilde{\delta}$, the distance between $(\Phi_1 \circ \Phi_t^{-1})(x)$ and $(\Phi_1 \circ \Phi_t^{-1})(y)$ is smaller than $\frac{1}{3}\delta$ and $(\Phi_1 \circ \Phi_t^{-1})(y) \in W_1$. For any $\delta > 0$, we have constructed a formal Legendrian homotopy $(g_t, G_{s,t}) * (\chi_{1-t}, X_{s,1-t})$ from f_0 to χ_0 which is $\frac{1}{3}\delta$–close to f_0, and we proved that we can find a Legendrian isotopy $\widetilde{\phi}_t \circ \chi_0$ from χ_0 to $\widetilde{\phi}_1 \circ \chi_0$ which is δ–close to f_t. We can apply the above constructions[7] to some δ smaller than $\widetilde{\delta}$ so that we may assume that $(g_t, G_{s,t}) * (\chi_{1-t}, X_{s,1-t})$ and $\widetilde{\phi}_t \circ \chi_0$ are, in fact, $\widetilde{\delta}$–close to f_0 and f_t, respectively. Then by our

[7] Note that we have not used the properties of W_1 in our constructions yet. This implies that we can apply those constructions to a smaller δ while keeping W_1 fixed.

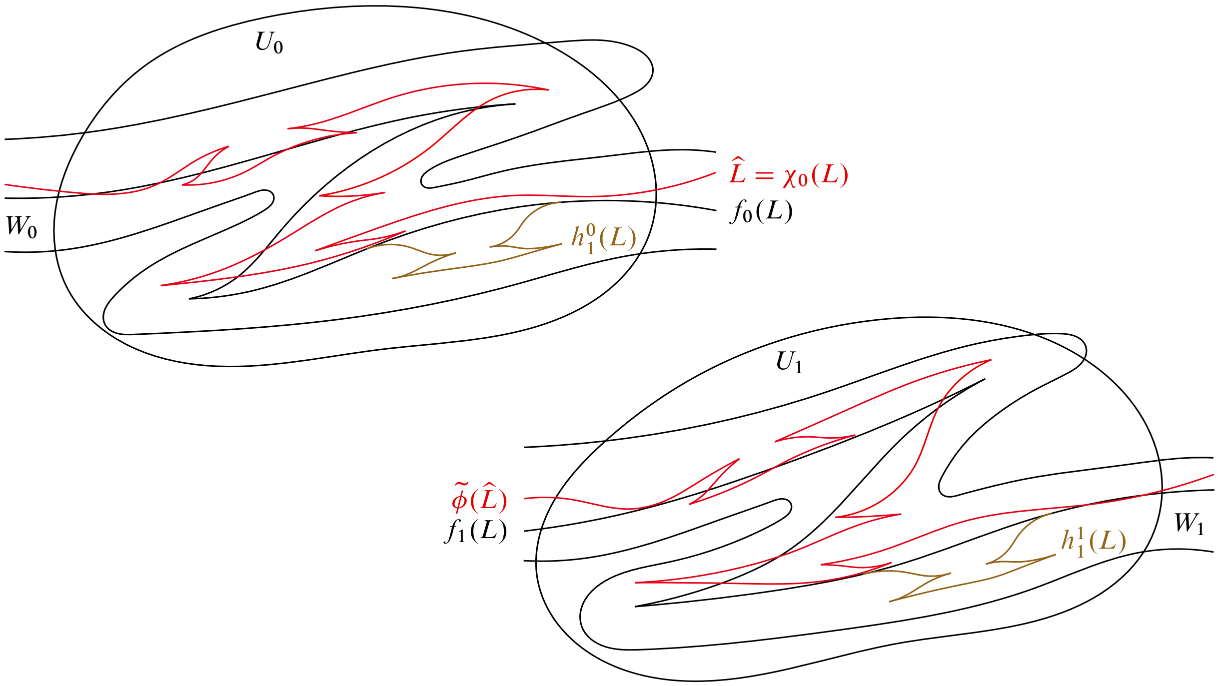

Figure 3: Inside of $V_i \cap W_i$, we find a formal Legendrian homotopy $(h_t^i, H_{s,t}^i)$ from f_i to a loose Legendrian embedding h_1^i which admits a small loose chart.

choice of $\widetilde{\delta}$, the formal Legendrian homotopy

$$\big(\Phi_1 \circ \Phi_{1-t}^{-1} \circ (\widetilde{\phi}_{1-t} \circ \chi_0)\big) * \big((\Phi_1, D\Phi_1) \circ ((\chi_t, X_{s,t}) * (g_{1-t}, G_{s,1-t}))\big)$$

from $\widetilde{\phi}_1 \circ \chi_0$ to f_1 is $\frac{1}{3}\delta$–close to f_1, and the underlying embeddings have image contained in W_1. This proves the claim.

For $i \in \{0, 1\}$, let V_i be a relatively compact subset of U_i so that $(V_i, f_i(L) \cap V_i)$ is a loose chart for $f_i(L)$. Let $(h_t^i \colon L \to W_i, H_{s,t}^i)$ be a formal Legendrian homotopy $\frac{1}{3}\delta$–close to f_i from f_i to a loose Legendrian embedding $h_1^i \colon L \to W_i$ which admits a loose chart contained in $W_i \cap V_i$ whose diameter is smaller than $\frac{1}{3}\delta$, which is constant outside of a compact subset of $f_i^{-1}(f_i(L) \cap W_i \cap V_i)$ and maps $f_i^{-1}(f_i(L) \cap W_i \cap V_i)$ into $W_i \cap V_i$; see Figure 3. Such $(h_t^i, H_{s,t}^i)$ may be found for example by performing a $(\chi = 0)$–stabilization of f_i.

By applying Murphy's h-principle for $i \in \{0, 1\}$ to the formal Legendrian homotopies $(h_t^i, H_{s,t}^i)$ inside of $V_i \subseteq U_i$, we find contact isotopies $\psi_t^i \colon M \to M$ with compact support in U_i such that $\psi_1^0 \circ f_0 = h_1^0$ and $\psi_1^1 \circ h_1^1 = f_1$. By definition of U_i, we may assume that $\|\psi_t^i\|_\alpha < \varepsilon_i$.

As $h_1^i(L)$, $\widehat{L}$ and $\widetilde{\phi}_1(\widehat{L})$ have loose charts of diameter smaller than $\frac{1}{3}\delta$, we can apply the C^0–close version of Murphy's h-principle (Theorem 3.1) to the formal Legendrian homotopies

$$(h_{1-t}^0, H_{s,1-t}^0) * (g_t, G_{s,t}) * (\chi_{1-t}, X_{s,1-t}) \quad \text{and} \quad (\xi_t, \Xi_{s,t}) * (h_t^1, H_{s,t}^1)$$

to find contact isotopies $\theta_t^i\colon M \to M$, for $i \in \{0,1\}$, with compact support in W_i so that $\theta_t^0 \circ h_1^0$ is a homotopy of Legendrian embeddings from h_1^0 to χ_0 and $\theta_t^1 \circ \widetilde{\phi}_1 \circ \chi_0$ is a homotopy of Legendrian embeddings from $\widetilde{\phi}_1 \circ \chi_0$ to h_1^1 which are $\frac{1}{3}\delta$–close to $(h_{1-t}^0, H_{s,1-t}^0) * (g_t, G_{s,t}) * (\chi_{1-t}, X_{s,1-t})$ and $(\xi_t, \Xi_{s,t}) * (h_t^1, H_{s,t}^1)$, respectively.

Note that $(h_{1-t}^0, H_{s,1-t}^0) * (g_t, G_{s,t}) * (\chi_t, X_{s,t})$ and $(\xi_t, \Xi_{s,t}) * (h_t^1, H_{s,t}^1)$ are $\frac{1}{3}\delta$–close to f_0 and f_1, respectively, and thus $\theta_t^0 \circ h_1^0$ is $\frac{2}{3}\delta$–close to f_0, and $\theta_t^1 \circ \widetilde{\phi}_1 \circ \chi_0$ is $\frac{2}{3}\delta$–close to f_1 for all $t \in [0,1]$.

Since W_i has height $\frac{1}{3}\eta$ and fibers of diameter $\frac{1}{3}\delta$, it follows from Proposition 2.8 applied to θ_t^i that we can find contact isotopies $\widetilde{\theta}_t^i\colon M \to M$ that satisfy $\widetilde{\theta}_1^i = \theta_1^i$ and $\|\widetilde{\theta}_t^i\|_\alpha < \frac{1}{3}\eta$ so that $\widetilde{\theta}_t^0|_{h_1^0(L)}$ and $\widetilde{\theta}_t^1|_{\widetilde{\phi}_1 \circ \chi_0(L)}$ are $\frac{1}{3}\delta$–close to $\theta_t^0|_{h_1^0(L)}$ and $\theta_t^1|_{\widetilde{\phi}_1 \circ \chi_0(L)}$, respectively. Note that then $\widetilde{\theta}_t^0 \circ h_1^0 = \widetilde{\theta}_t^0 \circ \psi_1^0 \circ f_0$ and $\widetilde{\theta}_t^1 \circ \widetilde{\phi}_1 \circ \chi_0$ are δ–close to f_0 and f_1, respectively, for all $t \in [0,1]$.

Because $\widetilde{\phi}_t \circ \chi_0$ is δ–close to f_t, $(\widetilde{\theta}^0 * \widetilde{\phi} * \widetilde{\theta}^1)_t \circ \psi_1^0 \circ f_0$ will be δ–close to f_t if the concatenation is performed in such a way that $\widetilde{\theta}_t^0$ and $\widetilde{\theta}_t^1$ are traversed very fast. Since also $\|\widetilde{\phi}_t\|_\alpha < \frac{1}{3}\eta$, we see that $\|(\widetilde{\theta}^0 * \widetilde{\phi} * \widetilde{\theta}^1)_t\|_\alpha < \eta$.

Then the isotopies ψ_t^0, ψ_t^1, and $\phi_t := (\widetilde{\theta}^0 * \widetilde{\phi} * \widetilde{\theta}^1)_t$ have the desired properties. □

5 Proofs of the corollaries

In this section we present the proofs of Theorems 1.4 and 1.5, and Corollaries 1.8, 1.9, 1.11 and 1.14.

Proof of Theorems 1.4 and 1.5 Assume that the assumptions in Theorem 1.4 are satisfied. Let $L_1 \subseteq M$ be a closed Legendrian submanifold which is Legendrian isotopic to L such that there are no Reeb chords between L and L_1. In particular, L_1 is loose. It follows from Murphy's h-principle that L_1 is Legendrian isotopic to some Legendrian L_2 which agrees with L_1 outside of an arbitrarily small strict Darboux ball U_1 around some point in L_1, and so that $U_1 \cap L_2$ is loose in U_1. If U_1 is chosen sufficiently small, all Reeb chords between L and L_2 will have action larger than E, there will be no Reeb chords at all if the image of L under the Reeb flow is closed, and U_1 will satisfy the property in the statement of Theorem 1.2 for $\varepsilon_1 = \frac{1}{2}\eta$ by Proposition 2.8. Now we can apply Theorem 1.2 to find compactly supported contact isotopies ψ_t^0, ϕ_t and ψ_t^1 such that ψ_t^0 has support in U, $\|\psi_t^0\|_\alpha < \varepsilon$, $\|\phi_t\|_\alpha < \frac{1}{2}\eta$, $\|\psi_t^1\|_\alpha < \frac{1}{2}\eta$ and $L_2 = (\psi_1^1 \circ \phi_1 \circ \psi_1^0)(L)$. In particular, the isotopies ψ_t^0 and $(\phi * \psi^1)_t$ have the desired properties.

The proof of Theorem 1.5 works in the same way after we note that if all Reeb chords from L to L_1 have action larger than E_1, and all Reeb chords from L_1 to L have action larger than E_2, then also all Reeb chords from L to L_2 have action larger than E_1, and all Reeb chords from L_2 to L have action larger than E_2, as long as L_2 is sufficiently C^0–close to L_1, since

$$\bigcup_{t \in [-E_2, E_1]} \phi_t^\alpha(L) \tag{16}$$

is closed as a subset of M, where ϕ_t^α denotes the Reeb flow of α. □

Proof of Corollary 1.8 Assume that the assumptions in Corollary 1.8 are satisfied. Let ϕ_t^H denote the contact isotopy associated to the function H_t. Let $f_t\colon L \to M$ be a homotopy of Legendrian embeddings and assume that there exist open sets U_0, U_1 as in the statement of Theorem 1.2 such that $f_i(L) \cap U_i \subseteq U_i$ is loose for $i = 1, 2$. For any $\tilde{\eta} > 0$, we can apply Theorem 1.2 to the family $(\phi_t^H)^{-1} \circ f_t$ to find isotopies ψ_t^0, $\tilde{\psi}_t^1$ and $\tilde{\phi}_t$ with $\|\psi_t^0\|_\alpha < \varepsilon_0$, $\operatorname{supp}(\psi_t^0) \subseteq U_0$, $\operatorname{supp}(\tilde{\psi}_t^1) \subseteq (\phi_1^H)^{-1}(U_1)$, $\|\tilde{\phi}_t\|_\alpha < \tilde{\eta}$ and $\tilde{\psi}_1^1 \circ \tilde{\phi}_1 \circ \psi_1^0 \circ f_0 = (\phi_1^H)^{-1} \circ f_1$. Let $\tilde{F}_t$ denote the contact Hamiltonian associated to $\tilde{\phi}_t$. Then the contact Hamiltonian F_t associated to $\phi_t := \phi_t^H \circ \tilde{\phi}_t$ is given by $F_t(x) = H_t(x) + h_t((\phi_t^H)^{-1}(x))\tilde{F}_t((\phi_t^H)^{-1}(x))$, where h_t denotes the positive function, which is equal to 1 outside of a compact set, defined by $(\phi_t^H)^*\alpha = h_t\alpha$. In particular, $\|H_t - F_t\|$ can be made arbitrarily small by decreasing $\tilde{\eta}$. Define $\psi_t^1 := \phi_1^H \circ \tilde{\psi}_t^1 \circ (\phi_1^H)^{-1}$. Then ψ_t^1 is supported in U_1, and we may assume that $\|\psi_t^1\|_\alpha < \varepsilon_1$. It also follows that $\psi_1^1 \circ \phi_1 \circ \psi_1^0 \circ f_0 = f_1$. Furthermore, if we choose these isotopies in such a way that $\tilde{\phi}_t \circ \psi_1^0 \circ f_0$ is sufficiently C^0–close to $(\phi_t^H)^{-1} \circ f_t$, then $\phi_t \circ \psi_1^0 \circ f_0$ is C^0–close to f_t. □

Proof of Corollary 1.9 Assume that U_i, L_i, V_i, SL_i, for $i \in \{0, 1\}$, and f_t are as in the statement of Corollary 1.9. Let $\Phi_t\colon M \to M$ be a compactly supported contact isotopy with $\Phi_t \circ f_0 = f_t$. Recall that $SL_i \cap U_i \subseteq U_i$ is loose. Let $\eta, \mu, \delta > 0$. We identify U_i with a subset of the 1–jet bundle J^1L with coordinates (q, p, z), with $q \in L$, $p \in T_q^*L$, $z \in \mathbb{R}$, as in the definition of a local Weinstein neighborhood and write the height explicitly as $U_{\varepsilon_i}^i := U_i$. For $\lambda \in (0, 1]$, we denote by $U_{\lambda\varepsilon_i}^i \subseteq U_{\varepsilon_i}^i$ the image of $U_{\varepsilon_i}^i$ under the contactomorphism $(q, p, z) \mapsto (q, \lambda p, \lambda z)$. We let $\tilde{\varepsilon}_i < \varepsilon_i$ be such that SL_i is still stabilized inside of $U_{\tilde{\varepsilon}_i}^i$. It follows from the proof of Proposition 2.8 that for any $\lambda \in (0, 1]$ there exists a contact isotopy $\tilde{\psi}_t^i$ with compact support[8] in $U_{\varepsilon_i}^i$ that satisfies $\tilde{\psi}_1^i(U_{\tilde{\varepsilon}_i}^i) \subseteq U_{\lambda\tilde{\varepsilon}_i}^i$ and $\|\tilde{\psi}_t^i\|_\alpha < \tilde{\varepsilon}_i + \mu$ for $i \in \{0, 1\}$. Choose μ and λ so small that $\tilde{\varepsilon}_i + \mu + 2\lambda\tilde{\varepsilon}_i < \varepsilon_i$ for $i \in \{0, 1\}$. By Proposition 2.8, $U_{\lambda\tilde{\varepsilon}_i}^i$ satisfies the property of U_i in the statement of Theorem 1.2 with the constant $2\lambda\tilde{\varepsilon}_i$. Hence, we can apply Theorem 1.2 to the homotopy $\big((\tilde{\psi}_{1-t}^0 \circ (\tilde{\psi}_1^0)^{-1}) * \Phi_t * \tilde{\psi}_t^1\big) \circ \tilde{\psi}_1^0 \circ f_0$ to conclude that there exist contact isotopies $\hat{\phi}_t$ and $\hat{\psi}_t^i$ for $i \in \{0, 1\}$, such that

$$\|\hat{\phi}_t\|_\alpha < \min\left\{\eta, \frac{\varepsilon_0 - (\tilde{\varepsilon}_0 + \mu + 2\lambda\tilde{\varepsilon}_0)}{2}, \frac{\varepsilon_1 - (\tilde{\varepsilon}_1 + \mu + 2\lambda\tilde{\varepsilon}_1)}{2}\right\}, \tag{17}$$

$\|\hat{\psi}_t^i\|_\alpha < 2\lambda\tilde{\varepsilon}_i$, $\operatorname{supp}(\hat{\psi}_t^i) \subseteq U_{\lambda\tilde{\varepsilon}_i}^i$ and $\hat{\psi}_1^1 \circ \hat{\phi}_1 \circ \hat{\psi}_1^0 \circ \tilde{\psi}_1^0 \circ f_0 = \tilde{\psi}_1^1 \circ f_1$. Furthermore, we can assume that $\hat{\phi}_t \circ \hat{\psi}_1^0 \circ \tilde{\psi}_1^0 \circ f_0$ is δ–close to

$$\big((\tilde{\psi}_{1-s}^0 \circ (\tilde{\psi}_1^0)^{-1}) * \Phi_s * \tilde{\psi}_s^1\big)_t \circ \tilde{\psi}_1^0 \circ f_0.$$

This concatenation is performed so that $\tilde{\psi}_{1-t}^0 \circ (\tilde{\psi}_1^0)^{-1}$ is traversed during the time interval $\left[0, \frac{1}{3}\right]$, Φ_t during the time interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, and $\tilde{\psi}_t^1$ during the time interval $\left[\frac{2}{3}, 1\right]$. As $\tilde{\psi}_t^i(SL_0)$ is contained in V_i by construction, we may assume after potentially using appropriate cut-offs outside of a compact subset of V_i that $\{\hat{\phi}_{t/3}\}_{t\in[0,1]}$ and $\{\hat{\phi}_{2/3+t/3} \circ (\hat{\phi}_{2/3})^{-1}\}_{t\in[0,1]}$ have compact support contained in V_0 (resp. V_1)

[8]Technically speaking, we may also have to shrink $U_{\tilde{\varepsilon}_i}^i$ a bit in the q–coordinate in order to ensure that $\tilde{\psi}_t^i$ can be chosen to have compact support in $U_{\varepsilon_i}^i$, but this causes no issue and we omit to write this explicitly.

as long as $\widehat{\phi}_t \circ \widehat{\psi}_1^0 \circ \widetilde{\psi}_1^0 \circ f_0$ is sufficiently close to $((\widetilde{\psi}_{1-s}^0 \circ (\widetilde{\psi}_1^0)^{-1})) * \Phi_s * \widetilde{\psi}_s^1)_t \circ \widetilde{\psi}_1^0 \circ f_0$ so that $\widehat{\phi}_{t/3} \circ \widetilde{\psi}_1^0(SL_0) \subseteq V_0$ and $\widehat{\phi}_{2/3+t/3} \circ \widetilde{\psi}_1^0(SL_0) \subseteq V_1$ for all $t \in [0,1]$.

Now, for $t \in [0,1]$, we define

$$\begin{aligned}\psi_t^0 &= (\widetilde{\psi}_s^0 * \widehat{\psi}_s^0 * \widehat{\phi}_{s/3})_t,\\ \psi_t^1 &= \big((\widehat{\phi}_{2/3+s/3} \circ (\widehat{\phi}_{2/3})^{-1}) * \widehat{\psi}_s^1 * (\widetilde{\psi}_{1-s}^1 \circ (\widetilde{\psi}_1^1)^{-1})\big)_t,\\ \phi_t &= \widehat{\phi}_{1/3+t/3} \circ (\widehat{\phi}_{1/3})^{-1}.\end{aligned}$$

Here, $s \in [0,1]$ for all of the isotopies in the concatenations. It is now straightforward to check that these maps have the desired properties. □

Proof of Corollary 1.11 The proof of Corollary 1.11 combines the proofs of Corollary 1.9 and Theorem 1.4.

Let $L^n \subseteq M$, for some $n \geq 2$, be a closed Legendrian submanifold and U_ε a local Weinstein neighborhood of L of height $2\varepsilon > 0$. Let SL be a displaceable stabilized version of L such that the stabilization is performed in U_ε. Again, $SL \cap U_\varepsilon \subseteq U_\varepsilon$ is loose. As above, we denote by $U_{\lambda\varepsilon} \subseteq U_\varepsilon$ the image of U_ε, viewed as a subset of J^1L, under the contactomorphism $(q,p,z) \mapsto (q,\lambda p,\lambda z)$ for $\lambda \in (0,1]$. Let $\widetilde{\varepsilon} < \varepsilon$ be such that SL is stabilized inside of $U_{\widetilde{\varepsilon}}$. Let $\mu > 0$. It follows as before that for any $\lambda \in (0,1]$ there exists a compactly supported contact isotopy ψ_t with $\psi_1(U_{\widetilde{\varepsilon}}) \subseteq U_{\lambda\widetilde{\varepsilon}}$ and $\|\psi_t\|_\alpha < \widetilde{\varepsilon} + \mu$. According to Proposition 2.8, $U_{\lambda\widetilde{\varepsilon}}$ satisfies the property of U in the statement of Theorem 1.2 with the constant $2\lambda\widetilde{\varepsilon}$.

Let $E > 0$, and choose μ and λ so small that $\widetilde{\varepsilon} + \mu + 2\lambda\widetilde{\varepsilon} < \varepsilon$. Let L_1 be a Legendrian which is Legendrian isotopic to SL so that there are no Reeb chords between L_1 and SL. In particular L_1 is loose, and it is Legendrian isotopic to L_2, where L_2 is obtained from L_1 via a stabilization inside of some Darboux ball U_1. We may choose U_1 in such a way that there are no Reeb chords between SL and L_2 of action larger than E and no Reeb chords at all if the image of SL under the Reeb flow is closed, and U_1 satisfies the property in the statement of Theorem 1.2 with $\varepsilon_1 = \frac{1}{2}(\varepsilon - \widetilde{\varepsilon} - \mu - 2\lambda\widetilde{\varepsilon}) > 0$. We now apply Theorem 1.2 to a Legendrian isotopy from $\psi_1(SL)$ to L_2, to conclude that there exist contact isotopies ψ_t^0, ψ_t^1 and ϕ_1 so that $\psi_1^1 \circ \phi_1 \circ \psi_1^0(\psi_1(SL)) = L_2$ and $\|\psi_t^0\|_\alpha < 2\lambda\widetilde{\varepsilon}$, $\|\phi_t\|_\alpha < \frac{1}{2}(\varepsilon - \widetilde{\varepsilon} - \mu - 2\lambda\widetilde{\varepsilon})$ and $\|\psi_t^0\|_\alpha < \frac{1}{2}(\varepsilon - \widetilde{\varepsilon} - \mu - 2\lambda\widetilde{\varepsilon})$. Then the concatenation $\psi_t * \psi_t^0 * \phi_t * \psi_t^1$ has the desired properties. □

Proof of Corollary 1.14 Let L_0 and L_1 be two closed loose Legendrian submanifolds of M that are formally isotopic and admit loose charts of size ε_0 and ε_1, respectively. By Murphy's h-principle for loose Legendrians, L_0 and L_1 are Legendrian isotopic. Let $\eta > 0$, and let $V \subseteq M$ be an open subset which satisfies the property of U_i in the statement of Theorem 1.2 with ε_i (in Theorem 1.2) equal to $\frac{1}{3}\eta$. By definition of the size of a loose chart, there exist contact isotopies ψ_t^i for $i \in \{0,1\}$, with $\|\psi_t^i\|_\alpha \leq \frac{1}{2}\varepsilon_i$ so that $\psi_1^i(L_i)$ has a loose chart contained in V. By applying Theorem 1.2 to $\psi_1^0(L_0)$ and $\psi_1^1(L_1)$, it follows that there exists a contact isotopy ϕ_t with $\phi_1(\psi_1^0(L_0)) = \psi_1^1(L_1)$ and $\|\phi_t\|_\alpha < \eta$. Then the concatenation $\psi_t^0 * \phi_t * ((\psi_{1-t}^1)^{-1} \circ \psi_1^1)$ has the desired properties. □

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Proposed: Yakov Eliashberg Received: 4 August 2021
Seconded: Leonid Polterovich, Dmitri Burago Revised: 30 January 2023

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

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mathematical sciences publishers

nonprofit scientific publishing

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Geometry & Topology

Volume 28 Issue 9 (pages 3973–4381) 2024