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On the high-dimensional geography problem

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In 1962, Wall showed that smooth, closed, oriented, $(n-1)$ -connected $2n$ -manifolds of dimension at least 6 are classified up to connected sum with an exotic sphere by an algebraic refinement of the intersection form, which he called an n -space.

We complete the determination of which n -spaces are realizable by smooth, closed, oriented, $(n-1)$ -connected $2n$ -manifolds for all $n \neq 63$. In dimension 126, the Kervaire invariant one problem remains open. Along the way, we completely resolve conjectures of Galatius and Randal-Williams and Bowden, Crowley and Stipsicz, showing that they are true outside of the exceptional dimension 23, where we provide a counterexample. This counterexample is related to the Witten genus and its refinement to a map of \mathbb{E}_∞ -ring spectra by Ando, Hopkins and Rezk.

By previous work of many authors, including Wall, Schultz, Stolz, and Hill, Hopkins and Ravenel, as well as recent joint work of Hahn with the authors, these questions have been resolved for all but finitely many dimensions, and the contribution of this paper is to fill in these gaps.

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1 Introduction

A classical problem in differential topology is the following:

Problem 1.1 Classify, or enumerate, all smooth, closed, oriented, $(n-1)$ -connected manifolds of dimension $2n$.

Early progress includes both Adams's solution [1960] to the Hopf invariant one problem and Milnor's discovery [2000] of exotic spheres. A major advance was made by Wall [1962], who showed that the diffeomorphism type of a smooth, closed, oriented, $(n-1)$ -connected $2n$ -manifold of dimension at least 6 is determined, up to connected sum with a homotopy sphere, by the middle homology group, the intersection pairing, and the so-called normal bundle data.¹ Wall refers to such a collection of algebraic invariants as an n -space. The precise definition of an n -space will be given in Section 2. To enumerate all smooth, closed, oriented, $(n-1)$ -connected $2n$ -manifolds in terms of n -spaces, it therefore suffices to answer the following two questions:

- (1) Which n -spaces may be realized by smooth, closed, oriented, $(n-1)$ -connected $2n$ -manifolds?
- (2) Given an n -space which is realized by a manifold M , for which homotopy spheres Σ are $\Sigma \# M$ and M diffeomorphic?

In analogy with the study of smooth structures on simply connected four-dimensional manifolds, we refer to these as the high-dimensional *geography* and *botany* problems. Following a great deal of work over the past half-century, both the high-dimensional geography and botany problems have been resolved in all but finitely many dimensions [Wall 1962; 1967; Kervaire and Milnor 1963; Brown and Peterson 1966; Kosiński 1967; Mahowald and Tangora 1967; Browder 1969; Schultz 1972; Lampe 1981; Barratt et al. 1984; Stolz 1985; 1987; Hill et al. 2016; Burklund et al. 2023].

Developments after 1987 include the work of Hill, Hopkins and Ravenel on the Kervaire invariant one problem, which, when combined with work of Stolz, settled the geography problem for all odd $n > 135$; and work of Hahn and the authors settling the geography problem when $n > 124$ is a multiple of 4, as well as the botany problem when $n > 232$ is congruent to 1 modulo 8.

Here we complete the solution to the high-dimensional geography problem outside of dimension 126, where the answer is contingent on the resolution of the Kervaire invariant one problem. Our answer is phrased in terms of certain n -space invariants studied by Wall. Although some of these invariants, such as the signature and the Kervaire invariant Φ , are likely to be familiar to the reader, others, such as the middle homology class χ , might not be. In Section 2 we recall the definitions for all of the invariants we use.

¹The restriction to dimension at least 6 is inherited from the use of the Whitney trick in Smale's study of handlebody decompositions.

Theorem 1.2 (proven as [Theorem 2.10](#)) *Suppose that $n \geq 3$. With the exception of finitely many n , an n -space $(H, H \otimes H \rightarrow \mathbb{Z}, \alpha)$ is realized by a smooth, closed, oriented, $(n-1)$ -connected $2n$ -manifold if and only if the following conditions hold:*

- (1) *If $n \equiv 0 \pmod{4}$, then $\text{sig} + 4s(Q)_{n/2}\chi^2 \equiv 0 \pmod{\sigma_{n/2}}$.*
- (2) *If $n \equiv 2 \pmod{4}$, then $\text{sig} \equiv 0 \pmod{\sigma_{n/2}}$.*
- (3) *If $n \equiv 1 \pmod{2}$, then $\Phi = 0$.*

The full list of exceptions is as follows:

- *If $n = 3, 7, 15, 31$, then every n -space is realizable.*
- *If $n = 63$, then every n -space is realizable if there exists a closed smooth manifold of Kervaire invariant one in dimension 126. Otherwise, an n -space is realizable if and only if $\Phi = 0$.*
- *If $n = 4$ or 8 , then instead of condition (1) we require $\text{sig} - \chi^2 \equiv 0 \pmod{\sigma_{n/2}}$.*
- *If $n = 9$, we require condition (3) and demand further that $\varphi(\chi) = 0$.*
- *If $n = 12$, we require condition (1) and demand further that $\chi^2 \equiv 0 \pmod{4}$.*

In his 1962 paper, Wall showed for $n \geq 3$ that n -spaces lie in bijection with diffeomorphism classes of oriented, $(n-1)$ -connected, smooth $2n$ -manifolds with boundary a homotopy sphere. An n -space may be realized precisely when this homotopy sphere may be filled in, ie when it is diffeomorphic to the standard $(2n-1)$ -sphere. Therefore, the high-dimensional geography problem is intimately related to the following question:

Question 1.3 *Given an integer $n > 2$, which $(2n-1)$ -dimensional homotopy spheres arise as the boundary of an $(n-1)$ -connected $2n$ -manifold?*

The answer to [Question 1.3](#) invokes knowledge of the Kervaire–Milnor group [[1963](#)] of homotopy spheres, so we begin by recalling its basic structure. Let Θ_m denote the group of h -cobordism classes of oriented, smooth, closed manifolds Σ that are homotopy equivalent to the m -sphere, where the group operation is the connected sum. The Kervaire–Milnor exact sequence

$$0 \rightarrow \text{bP}_{m+1} \rightarrow \Theta_m \rightarrow \text{coker}(J)_m$$

expresses Θ_m in terms of the finite cyclic group bP_{m+1} and the much more complicated group $\text{coker}(J)_m$. Geometrically, the group bP_{m+1} is the subgroup of Θ_m consisting of those homotopy spheres which bound parallelizable manifolds.

Theorem 1.4 *Suppose that $n > 2$ and $n \neq 9, 12$. Then a $(2n-1)$ -dimensional homotopy sphere is the boundary of an $(n-1)$ -connected smooth $2n$ -manifold if and only if it also bounds a parallelizable manifold.*

A homotopy 17–sphere Σ is the boundary of an 8–connected 18–manifold if and only if

$$[\Sigma] \in \{0, \eta\eta_4\} \subset \text{coker}(J)_{17}.$$

A homotopy 23–sphere Σ is the boundary of an 11–connected 24–manifold if and only if

$$[\Sigma] \in \{0, \eta^3\bar{\kappa}\} \subset \text{coker}(J)_{23}.$$

Remark 1.5 [Theorem 1.4](#) is new when

- $n \equiv 0 \pmod{4}$ and $12 \leq n \leq 124$;
- $n \equiv 1 \pmod{8}$ and $9 \leq n \leq 121$.

In the remaining cases the attribution is as follows:

- When $3 \leq n \leq 8$, it is due to [\[Wall 1962\]](#).
- When $n \equiv 3, 5, 6, 7 \pmod{8}$, it follows from [\[Wall 1962, Theorem 2\]](#) and the existence of almost closed parallelizable manifolds of signature 8 (n even) and Kervaire invariant one (n odd). Examples of such manifolds are given by Milnor’s E_8 –plumbing and the Kervaire plumbing, respectively. See eg [\[Browder 1972, Section V.2\]](#).
- When $n \equiv 2 \pmod{8}$, it is due to [\[Schultz 1972, Corollary 3.2 and Theorem 3.4\(i\)\]](#).
- When $n \equiv 1 \pmod{8}$ and $n \geq 129$, it is due to [\[Stolz 1985, Theorem B\]](#).²
- When $n \equiv 0 \pmod{4}$ and $n \geq 128$ it is due to Hahn and the authors [\[Burklund et al. 2023, Theorem 8.6\]](#).

[Theorem 1.4](#) resolves the following conjecture, which is equivalent to [\[Galatius and Randal-Williams 2016, Conjectures A and B\]](#):

Conjecture 1.6 (Galatius and Randal-Williams) *Suppose $n \equiv 0 \pmod{4}$. Then a $(2n-1)$ –dimensional homotopy sphere is the boundary of an $(n-1)$ –connected $2n$ –manifold if and only if it also bounds a parallelizable manifold.*

In particular, we learn that the conjecture is false for $n = 12$ and true otherwise. The interest of Galatius and Randal-Williams in this conjecture was spurred on by their work on mapping class groups of highly connected manifolds. In [Section 9.2](#), we will briefly record some applications of [Theorem 1.4](#) to the computation of mapping class groups.

As we shall see in [Section 9.1](#), [Theorem 1.4](#) also resolves the following conjecture of Bowden, Crowley and Stipsicz:

Conjecture 1.7 [\[Bowden et al. 2014, Conjecture 5.9\]](#) *An odd-dimensional homotopy sphere admits a Stein-fillable contact structure if and only if it also bounds a parallelizable manifold.*

²Stolz [\[1985, Theorem B\]](#) claims this result for all $n \geq 113$, and in this case Stolz’s proof is in fact valid for $n \geq 105$. However, Stolz made crucial use of a theorem announced by Mahowald, whose statement appears in his work as [\[Stolz 1985, Satz 12.9\]](#), of which no proof has appeared in the literature. A similar theorem was proven in [\[Burklund et al. 2023, Section 15\]](#), which when plugged into Stolz’s argument gives the result for $n \geq 129$.

More precisely, we show in [Theorem 9.1](#) that the conjecture is true for all dimensions other than 23, and that a 23–dimensional homotopy sphere Σ admits a Stein-fillable contact structure if and only if

$$[\Sigma] \in \{0, \eta^3 \bar{\kappa}\} \subset \text{coker}(J)_{23}.$$

Bowden et al. [[2014, Proposition 5.3](#)] show how to equip an odd-dimensional homotopy sphere which bounds a parallelizable manifold with an explicit, geometrically defined, Stein-fillable contact structure. This leads us to ask the following question:

Question 1.8 Given an exotic 23–sphere Σ with $[\Sigma] = \eta^3 \bar{\kappa} \in \text{coker}(J)_{23}$, can one construct an explicit Stein-fillable contact structure on Σ in a geometric way? Can this be done in such a way as to shed light on what is special about dimension 23 and the class $\eta^3 \bar{\kappa} \in \text{coker}(J)_{23}$?

Let us take a moment to discuss why the dimension 23 is exceptional, providing counterexamples to both the conjectures of Galatius and Randal-Williams and Bowden, Crowley and Stipsicz. We proceed by contradiction: Supposing that [Conjecture 1.6](#) held for $n = 12$, the work of Wall [[1962](#)] implies the existence of a closed, oriented, 11–connected, smooth 24–manifold with certain Pontryagin numbers. This manifold would have to admit a string structure, and so we may consider its Witten genus, which can be computed in terms of the Pontryagin numbers.

The Witten genus of a closed string manifold is an integral modular form. However, not every integral modular form is the Witten genus of a closed string manifold: a nontrivial restriction on the image of the Witten genus is provided by the Ando–Hopkins–Rezk string orientation [[Ando et al. 2010](#)], which implies that the Witten genus factors through the homotopy groups of the connective spectrum tmf of topological modular forms.³ For example, it follows from computations of Hopkins and Mahowald that the weight 12 modular form Δ does not lie in the image of the Witten genus; instead only multiples of 24Δ lie in the image. Using this restriction, we are able to show that the putative manifold constructed above cannot exist.

Experts are aware that this restriction on the value of the Witten genus gives rise to divisibility constraints on the Pontryagin numbers of string 24–manifolds; see for example [[Teichner 2007, Corollary 90](#)]. Nevertheless, as far as the authors are aware, thus far there have been few concrete geometric applications of the Witten genus and the Ando–Hopkins–Rezk string orientation.⁴ We were therefore pleased to find an application for this beautiful theory in this work.

Remark 1.9 The argument sketched above, whose details are the subject of [Section 3](#), is modeled on a classical argument making use of the \hat{A} –genus, which shows that there is no closed, simply connected, smooth 4–manifold whose intersection form is isomorphic to the E_8 –form, though Freedman [[1982](#)] has

³In fact, a folk theorem of Hopkins and Mahowald, which has now been written up by Devalapurkar [[2019](#)], shows that this is the only restriction on the image of the Witten genus: the map $\Omega_n^{\text{String}} = \pi_n M\text{String} \rightarrow \pi_n \text{tmf}$ is surjective for all n .

⁴However, see [[Krannich 2021](#)] for a recent application of the Ando–Hopkins–Rezk orientation to the question of how taking the connected sum with an exotic sphere affects the mapping class group of a highly connected manifold.

famously shown the existence of such a topological 4–manifold. Indeed, if such a smooth 4–manifold existed then it would have to admit a spin structure and its \widehat{A} –genus would be equal to 1. But the fact that the \widehat{A} –genus factors as the composite

$$\Omega_4^{\text{Spin}} \cong \pi_4 M\text{Spin} \rightarrow \pi_4 \text{ko} \rightarrow \pi_4 \text{ku} \cong \mathbb{Z},$$

where the first map is induced by the Atiyah–Bott–Shapiro orientation [Atiyah et al. 1964], and the second map is induced by tensoring up a real vector bundle to the complex numbers, implies that the \widehat{A} –genus of a 4–manifold is even, since the image of the second map is equal to $2\mathbb{Z} \subset \mathbb{Z}$. Since the signature of a simply connected spin 4–manifold is equal to $-\frac{1}{8}$ times its \widehat{A} –genus, this argument also proves Rokhlin’s theorem, which states that the signature of such a manifold must be divisible by 16.

It is interesting to note that the extra integrality for the \widehat{A} –genus used above is apparent from the interpretation of the \widehat{A} –genus as the index of the Dirac operator on the spinor bundle, as the rank 4 real Clifford algebra contains the quaternions. We hope that one day it will be possible to see a similar geometric origin for the restriction on the Witten genus of a closed string manifold used above. At the moment the appropriate replacement for Clifford algebras is not yet clear, but see [Stolz and Teichner 2004; Douglas and Henriques 2011].

An outline of the paper

In Section 2, we deduce Theorem 1.2 from Theorem 1.4, solving the high-dimensional geography problem for manifolds of dimension other than 126. In Section 3, we exploit the integrality properties of the Witten genus to prove the existence of an exceptional 11–connected 24–manifold M_{24} whose boundary ∂M_{24} is a homotopy sphere which does not bound a parallelizable manifold.

In Section 4, we lay the groundwork for the proof of Theorem 1.4, and we divide the proof into four parts. In Section 5, we improve upon the key argument from [Burklund et al. 2023, Section 10] to prove several cases of Theorem 1.4. As a consequence, we are also able to show that $[\partial M_{24}] = \eta^3 \bar{\kappa} \in (\text{coker}(J)_{23})_{(2)}$. In Section 6, we use power operations to prove the existence of the exceptional almost closed manifolds in dimension 18. In Section 7, we again exploit the Ando–Hopkins–Rezk string orientation, this time in order to resolve the 3–primary aspects of the 24–dimensional case. In Section 8, we make a short homological algebra argument necessary to finish the 128–dimensional case of Theorem 1.4.

Finally, in Section 9, we briefly discuss the applications of our work to Stein-fillable homotopy spheres and mapping class groups of highly connected manifolds.

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2 Classification of $(n-1)$ -connected $2n$ -manifolds

In this section, we will reduce the high-dimensional geography problem to [Theorem 1.4](#) and state the answer as [Theorem 2.10](#). While many cases of [Theorem 2.10](#) were previously known, we hope that the reader will find it useful to have a precise answer collected in a single omnibus theorem.

Convention 2.1 All manifolds in this section and [Section 3](#) will be assumed compact, smooth and oriented, and all diffeomorphisms will be assumed orientation-preserving. We will let n denote an integer greater than 2.

2.1 The work of Wall

Let us begin by recalling Wall’s work [\[1962\]](#) on the classification of closed, $(n-1)$ -connected $2n$ -manifolds. Given such a manifold M , Wall associates the data of

- the middle homology group $H = H_n(M; \mathbb{Z})$, which is a finite-dimensional free abelian group;
- the intersection pairing $H \otimes H \rightarrow \mathbb{Z}$, which is a unimodular bilinear form, symmetric if n is even and skew-symmetric if n is odd; and
- the normal bundle data, which is a map of sets $\alpha: H \rightarrow \pi_n BSO(n)$ assigning to $x \in H$ the normal bundle of an embedded sphere representing x . We will recall the values of the groups $\pi_n BSO(n)$ below.

Let $\tau_{S^n} \in \pi_n BSO(n)$ correspond to the tangent bundle of S^n . Moreover, let $J: \pi_n BSO(n) \rightarrow \pi_{2n-1} S^n$ denote the unstable J -homomorphism and let $H: \pi_{2n-1} S^n \rightarrow \mathbb{Z}$ denote the Hopf invariant. Then the above data satisfy the following compatibility conditions: given any $x, y \in H$, we have

$$(1) \quad x^2 = HJ\alpha(x)$$

and

$$(2) \quad \alpha(x + y) = \alpha(x) + \alpha(y) + (xy) \cdot \tau_{S^n},$$

where in both cases we have used multiplication to denote the intersection product.

Definition 2.2 An n -space is a triple $(H, H \otimes H \rightarrow \mathbb{Z}, \alpha)$ which satisfies [\(1\)](#) and [\(2\)](#). Two n -spaces $(H_1, H_1 \otimes H_1 \rightarrow \mathbb{Z}, \alpha_1)$ and $(H_2, H_2 \otimes H_2 \rightarrow \mathbb{Z}, \alpha_2)$ are said to be isomorphic if there is an isomorphism of abelian groups $H_1 \cong H_2$ preserving the intersection forms and normal bundle data.

Wall proved that the n -space of a closed, $(n-1)$ -connected $2n$ -manifold M determines M up to connected sum with a homotopy sphere:

Theorem 2.3 [Wall 1962, pages 169–170] *Two closed, $(n-1)$ -connected $2n$ -manifolds M and N have isomorphic n -spaces if and only if $M \cong N \# \Sigma$ for some homotopy sphere $\Sigma \in \Theta_{2n}$.*

Moreover, call a manifold almost closed if its boundary is a homotopy sphere. Then one may equally well associate an n -space to an almost closed, $(n-1)$ -connected $2n$ -manifold. Wall's invariant is even more powerful in this case.

Theorem 2.4 [Wall 1962, page 170] *The map associating an n -space to an almost closed, $(n-1)$ -connected $2n$ -manifold induces a bijection between the set of almost closed, $(n-1)$ -connected $2n$ -manifolds up to diffeomorphism and the set of n -spaces up to isomorphism.*

Together, these two theorems reduce the classification of $(n-1)$ -connected $2n$ -manifolds to the following two questions, which we have named in analogy with the classification of simply connected smooth 4-manifolds:

- (1) **High-dimensional geography problem** Given an n -space $(H, H \otimes H \rightarrow \mathbb{Z}, \alpha)$, when is it realized by a closed, $(n-1)$ -connected $2n$ -manifold M ? This is equivalent to asking when the boundary of the associated almost closed manifold is diffeomorphic to the standard $(2n-1)$ -sphere.
- (2) **High-dimensional botany problem** Given an n -space $(H, H \otimes H \rightarrow \mathbb{Z}, \alpha)$ which is realized by a $(n-1)$ -connected $2n$ -manifold M , what is the subgroup $I(M) \subset \Theta_{2n}$ of Σ such that $M \# \Sigma \cong M$?

Later, Wall gave a cobordism interpretation of the remaining aspects of the high-dimensional geography problem. This allows one to sum up the problem in an exact sequence of cobordism groups

Definition 2.5 Let $\Omega_{2n}^{(n-1)}$ denote the group of closed, oriented, $(n-1)$ -connected $2n$ -manifolds, modulo $(n-1)$ -connected oriented cobordisms.

Furthermore, let $A_{2n}^{(n-1)}$ denote the group of oriented, almost closed, $(n-1)$ -connected $2n$ -manifolds, modulo $(n-1)$ -connected, oriented cobordisms restricting to h -cobordisms on the boundary.

Proposition 2.6 [Wall 1967, Lemma 32] *There is an exact sequence*

$$\Theta_{2n} \rightarrow \Omega_{2n}^{(n-1)} \rightarrow A_{2n}^{(n-1)} \xrightarrow{\partial} \Theta_{2n-1},$$

where the first map sends a homotopy sphere to its cobordism class, the second map cuts out the interior of a smoothly embedded $2n$ -disk, and the last map sends an almost closed manifold to its boundary.

The high-dimensional geography problem is thus equivalent to the computation of the kernel of the map

$$\partial: A_{2n}^{(n-1)} \rightarrow \Theta_{2n-1}$$

in terms of the associated n -spaces. The results of this paper, building upon a great deal of work in the literature, compute the map ∂ and thereby answer the high-dimensional geography problem for all $n \neq 63$. The remaining case $n = 63$ is equivalent to the Kervaire invariant one problem in dimension 126.

Before proceeding, we find it helpful to unpack the information present in an n -space. First, we note that, since a complete classification of unimodular lattices is not known, the possible bilinear forms are

not completely enumerated. This issue will not affect us, but it is worth mentioning. The definition of an n -space depended on the classification of rank n vector bundles on the n -sphere and the class of the tangent bundle in that group. We recall Kervaire’s work [1960] on this subject. For n at least 8, we have the following table of values:

$n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_n BSO(n)$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2$	\mathbb{Z}	$\mathbb{Z}/2$

Furthermore, the stabilization map to $\pi_n BSO$ is surjective with kernel generated by τ_{S^n} . In the $n \equiv 1 \pmod 8$ case, $\pi_n BSO(n)$ has a basis given by τ_{S^n} and $\eta\iota$, where $\iota \in \pi_{n-1} BSO(n) \cong \mathbb{Z}$ denotes a generator.⁵ Finally, we set up several invariants of n -spaces and definitions which will be useful later (here we assume $n \geq 8$):

- If n is even, let sig denote the signature of the symmetric bilinear form on H .
- If $n \equiv 0 \pmod 4$, the composition $H \xrightarrow{\alpha} \pi_n BSO(n) \rightarrow \pi_n BSO \cong \mathbb{Z}$ is a linear map by (2) and the fact that the image of τ_{S^n} in $\pi_n BSO$ is trivial. This composition therefore corresponds to some element $\chi \in H$ via the unimodular bilinear form.
- If $n \equiv 1, 2 \pmod 8$, the same procedure determines an element $\chi \in H/2$.
- If $n \equiv 0, 2, 4 \pmod 8$, the self-intersection number of χ is an integer χ^2 (well defined modulo 4 in the $n \equiv 2 \pmod 8$ case).
- If $n \equiv 1 \pmod 8$, we let φ denote the map $\pi_n BSO(n) \rightarrow \mathbb{Z}/2$ with kernel $\eta\iota$.⁶
- If $n \equiv 1 \pmod 8$, the element $\varphi(\alpha(\chi))$ gives an element in $\mathbb{Z}/2$, which we denote by $\varphi(\chi)$.
- If n is odd, then, since τ_{S^n} is sent to a generator under the map α ($\alpha \circ \varphi$ in the $n \equiv 1 \pmod 8$ case), this map determines a quadratic refinement of the mod 2 reduction of the intersection pairing. We let Φ denote the Arf–Kervaire invariant of this quadratic form.

Our notation for these invariants follows that in [Wall 1962], with the exception of writing sig for the signature instead of τ . Wall shows that each of the invariants sig, χ^2 , Φ and $\varphi(\chi)$ descends to a linear map from $A_{2n}^{(n-1)}$.⁷ He has further computed the groups $A_{2n}^{(n-1)}$ in terms of these invariants.

Proposition 2.7 [Wall 1962, Theorem 2; 1967, Theorem 11] *Assume $n \geq 9$. The values of $A_{2n}^{(n-1)}$ are given in the following table, along with a choice of basis in terms of the above invariants:*

$n \pmod 8$	0	1	2	3	4	5	6	7
$A_{2n}^{(n-1)}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}/2$	\mathbb{Z}	$\mathbb{Z}/2$
basis	$(\frac{1}{8} \text{ sig}, \frac{1}{2} \chi^2)$	$(\Phi, \varphi(\chi))$	$(\frac{1}{8} \text{ sig}, \frac{1}{2} \chi^2)$	Φ	$(\frac{1}{8} \text{ sig}, \frac{1}{2} \chi^2)$	Φ	$\frac{1}{8} \text{ sig}$	Φ

⁵This follows from the preceding sentence and the fact that $\pi_{8n+1}ko \cong \mathbb{Z}/2\mathbb{Z}$ is generated by η times a generator of $\pi_{8n}ko \cong \mathbb{Z}$.

⁶Wall [1962] did not fix a specific choice of φ , merely asking that the direct sum of φ with the stabilization map be an isomorphism. This leads to an ambiguity in his definition of Φ when $n \equiv 1 \pmod 8$. Here we are careful to fix this specific choice so as to make Theorem 2.10 correct and unambiguous when $n \equiv 1 \pmod 8$.

⁷This follows in a straightforward way from [Wall 1962, Lemma 4] and the definitions.

2.2 The main theorem

We are now ready to state and prove the main theorem of this paper, assuming [Theorem 1.4](#) as input. We first recall some useful quantities computed by Krannich and Reinhold.

Definition 2.8 Let $n > 2$ denote a positive integer. Following [\[Krannich and Reinhold 2020\]](#), we let

- B_{2n} denote the $(2n)^{\text{th}}$ Bernoulli number;
- j_n and k_n denote the denominator and numerator, respectively, of the absolute value of $B_{2n}/4n$ when written in lowest terms;
- a_n denote 1 if n is even and 2 if n is odd;
- σ_n denote the integer $\sigma_n = a_n 2^{2n+1} (2^{2n-1} - 1) k_n$;
- c_n and d_n denote integers such that $c_n k_n + d_n j_n = 1$.

Finally, we let $s(Q)_{2n}$ denote the integer

$$s(Q)_{2n} = \frac{-1}{8j_n^2} (\sigma_n^2 + a_n^2 \sigma_{2n} k_n (c_{2n} k_n + 2(-1)^n d_{2n} j_n)).$$

Remark 2.9 The integers c_n and d_n , and therefore $s(Q)_{2n}$, are not well defined. Nevertheless, $s(Q)_{2n}$ is well defined modulo $\frac{1}{8}\sigma_{2n}$. Since we will only use the value of $s(Q)_{2n}$ modulo $\frac{1}{8}\sigma_{2n}$ in this section, this will not present a problem for us.

Theorem 2.10 Suppose that $n \geq 3$. With the exception of finitely many n , an n -space $(H, H \otimes H \rightarrow \mathbb{Z}, \alpha)$ is realized by a smooth, closed, oriented, $(n-1)$ -connected $2n$ -manifold if and only if the following conditions hold:

- (1) If $n \equiv 0 \pmod{4}$, then $\text{sig} + 4s(Q)_{n/2} \chi^2 \equiv 0 \pmod{\sigma_{n/2}}$.
- (2) If $n \equiv 2 \pmod{4}$, then $\text{sig} \equiv 0 \pmod{\sigma_{n/2}}$.
- (3) If $n \equiv 1 \pmod{2}$, then $\Phi = 0$.

The full list of exceptions is as follows:

- If $n = 3, 7, 15, 31$, then every n -space is realizable.
- If $n = 63$, then every n -space is realizable if there exists a closed smooth manifold of Kervaire invariant one in dimension 126. Otherwise, an n -space is realizable if and only if $\Phi = 0$.
- If $n = 4$ or 8 , then instead of condition (1) we require $\text{sig} - \chi^2 \equiv 0 \pmod{\sigma_{n/2}}$.
- If $n = 9$, we require condition (3) and demand further that $\varphi(\chi) = 0$.
- If $n = 12$, we require condition (1) and demand further that $\chi^2 \equiv 0 \pmod{4}$.

Remark 2.11 The cases $3 \leq n \leq 8$ were proven in [\[Wall 1962, Theorem 4\]](#), so we will assume $n \geq 9$ in the following.

Almost all of the ingredients in this theorem are already in the literature, and much of what we do here consists merely of their collation. What is original to this paper are the new cases of [Theorem 1.4](#), as well as the special care given to the $n \equiv 1 \pmod 8$ case, which we have not seen spelled out elsewhere.

The proof of [Theorem 2.10](#) will follow immediately from [Proposition 2.7](#) and [Lemma 2.12](#) below, which computes the boundaries of several specific classes. Recall the Kervaire–Milnor exact sequence

$$0 \rightarrow \mathrm{bP}_{2n} \rightarrow \Theta_{2n-1} \rightarrow \mathrm{coker}(J)_{2n-1}.$$

Given an element $\Sigma \in \Theta_{2n-1}$, we will let $[\Sigma] \in \mathrm{coker}(J)_{2n-1}$ denote its image under the map in the above exact sequence.

When n is even, this sequence is short exact, and Brumfiel [\[1968\]](#) constructed a preferred splitting

$$\Theta_{2n-1} \cong \mathrm{bP}_{2n} \oplus \mathrm{coker}(J)_{2n-1}.$$

Lemma 2.12 Assume $n \geq 9$.

- (1) For n even, let $P \in A_{2n}^{(n-1)}$ denote the element with $\frac{1}{8} \mathrm{sig} = 1$ and $\frac{1}{2} \chi^2 = 0$ if $n \equiv 0, 2, 4 \pmod 8$. As noted in [\[Krannich 2020, Section 3.2.2\]](#), we may choose P to be Milnor’s E_8 –plumbing. The homotopy sphere $\partial(P) \in \Theta_{2n-1}$ is a generator of bP_{2n} , which is a cyclic group of order $\frac{1}{8} \sigma_{n/2}$.
- (2) For $n \equiv 0 \pmod 4$, let $Q \in A_{2n}^{(n-1)}$ denote the element with $(\frac{1}{8} \mathrm{sig}, \frac{1}{2} \chi^2) = (0, 1)$. For $n \neq 12$, $\partial(Q)$ is $s(Q)_{n/2} \cdot \partial(P)$. For $n = 12$, $\partial(Q)$ is $s(Q)_6 \cdot \partial(P) + \eta^3 \bar{\kappa}$, where $\eta^3 \bar{\kappa} \in \mathrm{coker}(J)_{23}$ is viewed as an element of Θ_{23} via Brumfiel’s splitting. The class $\eta^3 \bar{\kappa}$ is simple 2–torsion.
- (3) For $n \equiv 2 \pmod 4$, let $L \in A_{2n}^{(n-1)}$ denote the element with $(\frac{1}{8} \mathrm{sig}, \frac{1}{2} \chi^2) = (0, 1)$. Then $\partial(L) = 0$.
- (4) For n odd, let $K \in A_{2n}^{(n-1)}$ denote the element with $\Phi(K) = 1$ (and $\varphi(\chi) = 0$ if $n \equiv 1 \pmod 8$). We may choose K to be the Kervaire plumbing. The homotopy sphere $\partial(K) \in \Theta_{2n-1}$ is a generator of bP_{2n} . This group is zero for $n = 1, 3, 7, 15, 31$ and possibly 63. (It is zero for $n = 63$ precisely if there exists a 126–dimensional manifold of Kervaire invariant one.) It is $\mathbb{Z}/2$ otherwise.
- (5) For $n \equiv 1 \pmod 8$, let $R \in A_{2n}^{(n-1)}$ denote the element with $(\Phi, \varphi(\chi)) = (0, 1)$. For $n \neq 9$, we have $\partial(R) = 0$. For $n = 9$, we have $[\partial(R)] = \eta \eta_4 \in \mathrm{coker}(J)_{17}$, which is simple 2–torsion

Proof of Lemma 2.12(1)–(4) The element P The boundary of Milnor’s E_8 –plumbing is well known to be a generator of bP_{2n} when n is even (use eg [\[Levine 1985, Lemma 3.5\(2\)\]](#) and the fact the signature of the Milnor plumbing is equal to 8). The fact that for even n the group bP_{2n} is a cyclic of order $\frac{1}{8} \sigma_{n/2}$ follows from [\[Levine 1985, Corollary 3.20\]](#).

The element Q We will describe $\partial(Q)$ in terms of Brumfiel’s splitting. On the one hand, Krannich and Reinhold [\[2020, Lemma 2.7\]](#), building on work of Stolz [\[1987\]](#), have computed the bP_{2n} –component of $\partial(Q)$ to be $s(Q)_{n/2} \cdot \partial(P)$ for the explicit quantity $s(Q)_{n/2}$ defined in [Definition 2.8](#). On the other hand, the computation of the $\mathrm{coker}(J)_{2n-1}$ –component of $\partial(Q)$ follows immediately from [Theorem 1.4](#) and the fact that $\partial(P)$ lies in bP_{2n} . In particular, the $\mathrm{coker}(J)_{2n-1}$ –component of $\partial(Q)$ is zero if $n \neq 12$ and is $\eta^3 \bar{\kappa}$ for $n = 12$.

The element L It is a result of Schultz [1972, Corollary 3.2 and Theorem 3.4(iii)] that $\partial(L) = 0 \in \Theta_{2n-2}$.

The element K The boundary of the Kervaire plumbing, $\partial(K)$, is the Kervaire sphere, which generates bP_{2n} by [Kervaire and Milnor 1963, Theorem 8.5]. This case of Theorem 2.10 now follows from the fact that, if n is odd, $\text{bP}_{2n} \cong 0$ if a smooth closed $2n$ -manifold of Kervaire invariant one exists, and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ otherwise. By [Brown and Peterson 1966; Mahowald and Tangora 1967; Browder 1969; Barratt et al. 1984; Hill et al. 2016], such a manifold exists if $n = 1, 3, 7, 15, 31$ and possibly if $n = 63$.

When $n \equiv 1 \pmod 8$, we need to show that $\varphi(\chi)$ of K is zero. To prove this, we note that, since the Kervaire plumbing K is obtained by plumbing together the disk bundles of τ_{S^n} over S^n , the (nonlinear) map $\mathbb{Z} \oplus \mathbb{Z} \cong H_n(K) \xrightarrow{\alpha} \pi_n BSO(n)$ sends both generators to τ_{S^n} , and hence the composite (linear) map $\mathbb{Z} \oplus \mathbb{Z} \cong H_n(K) \xrightarrow{\alpha} \pi_n BSO(n) \rightarrow (\pi_n BSO)/2$ is zero. It follows that $\chi = 0$, and hence $\varphi(\chi) = 0$. \square

Computing $\partial(R)$ takes more work. It follows from Theorem 1.4 that, for $n \neq 9$, $[\partial(R)] = 0 \in \text{coker}(J)_{2n-1}$, but, to show that the condition for realizability of an n -space is $\Phi = 0$ and not $\Phi + \varphi(\chi) = 0$, it is necessary to prove that $\partial(R) = 0 \in \Theta_{2n-1}$. To do this, we recall an argument of Schultz [1972] which reduces it to the case of $n \equiv 0 \pmod 8$.

Recall Bredon’s pairing

$$-\cdot -: \Theta_n \times \pi_{n+k}(S^n) \rightarrow \Theta_{n+k}.$$

Roitberg [1972, Theorem B] has shown that this pairing is that induced by the composition action on $\Theta_n \cong \pi_n \text{PL}/O$, and used this to show that the restriction of this pairing to bP_{n+1} is zero for $k \geq 1$.

Lemma 2.13 *Suppose that $n > 9$ is congruent to 1 modulo 8. Then $\partial(R) = \partial(Q) \cdot \eta^2$, where we have used \cdot for Bredon’s pairing.*

Proof Let $\alpha \in \pi_{n-1} BSO(n-1)$ denote a generator of the image of $\pi_{n-1} BSO(n-2)$ in $\pi_{n-1} BSO(n-1)$. It maps under the stabilization map $i_*: \pi_{n-1} BSO(n-1) \rightarrow \pi_{n-1} BSO(n) \cong \mathbb{Z}$ to a generator.

As in [Krannich 2020, Section 3.2.2], Q may be chosen to be the plumbing of two copies of the $(n-1)$ -dimensional linear disk bundle over S^{n-1} corresponding to α . Similarly, it is not hard to see that R may be constructed by plumbing together two copies of the n -dimensional linear disk bundle over S^n corresponding to $i_*\eta\alpha$. Indeed, for this plumbing we have $H \cong \mathbb{Z}\{x, y\}$ with intersection $xy = 1$, and the map

$$\mathbb{Z}\{x, y\} \cong H \xrightarrow{\alpha} \pi_n BSO(n) \xrightarrow{\varphi} \mathbb{Z}/2$$

sends x and y to zero, and hence $x + y$ to $xy = 1$, from which it follows that the Kervaire invariant is 0. On the other hand, the map

$$\mathbb{Z}\{x, y\} \cong H \xrightarrow{\alpha} \pi_n BSO(n) \rightarrow \pi_n BSO \cong \mathbb{Z}/2$$

sends x and y to 1, so that $\chi = x + y$ and so $\varphi(\chi) = 1$.

It then follows from [Schultz 1972, Theorem 2.5; Lawson 1973, Diagram (B)] and the fact that η lies in the image of J that

$$\partial(R) = \partial(Q) \cdot \eta^2,$$

as desired. □

Remark 2.14 This is essentially [Schultz 1972, Theorem 3.1], except that Schultz does not specify the definition of Φ that he is using.

Proof of Lemma 2.12(5) Suppose that $n > 9$. We have already seen above that $\partial(Q) \in \text{bP}_{2n-2}$. (Note that $n - 1 \equiv 0 \pmod{8}$ and in particular cannot be equal to 12.) Then Lemma 2.13 asserts that $\partial(R) = \partial(Q) \cdot \eta^2$, which is equal to 0 because Bredon’s pairing restricts to zero on bP .

On the other hand, at $n = 9$, Theorem 1.4, Proposition 2.7 and the computation of $\partial(K)$ imply that we must have $[\partial R] = \eta\eta_4 \in \text{coker}(J)_{17}$. □

3 The Witten genus of 11–connected 24–manifolds

In this section, we will show that the 23–dimensional case of Theorem 1.4 is exceptional, disproving the conjecture of Galatius and Randal-Williams. In particular, we prove the following theorem:

Theorem 3.1 *The image of the composition*

$$A_{24}^{(11)} \xrightarrow{\partial} \Theta_{23} \rightarrow \text{coker}(J)_{23} \rightarrow (\text{coker}(J)_{23})_{(2)}$$

is not 0. In particular, there is an 11–connected 24–manifold whose boundary is a homotopy sphere which does not bound a parallelizable manifold.

Remark 3.2 In Sections 5 and 7, we refine this result, showing that the image of the composition

$$A_{24}^{(11)} \xrightarrow{\partial} \Theta_{23} \rightarrow \text{coker}(J)_{23}$$

is $\{0, \eta^3\bar{\kappa}\} \subset \text{coker}(J)_{23}$.

The proof of Theorem 3.1 is a relatively straightforward application of the Ando–Hopkins–Rezk refinement [Ando et al. 2010] of the Witten genus and will be accomplished in two steps. First, we will collect results of Hirzebruch, Hopkins and Mahowald, and Ando, Hopkins and Rezk that provide a divisibility condition on the Pontryagin numbers of a closed 11–connected 24–manifold. Second, we will work through the implications of this restriction for the relevant examples.

3.1 A condition on Pontryagin numbers

In order to motivate the condition on the Pontryagin numbers of an 11–connected 24–manifold which we need, we will begin by working through an analogous restriction for 4–dimensional spin manifolds, which is equivalent to Rokhlin’s theorem.

Example 3.3 The Hirzebruch signature formula tells us that the signature of the intersection form on H_2 of an oriented 4-manifold is given by $\frac{1}{3}p_1$, where p_1 is the first Pontryagin number. In the case where the 4-manifold is spin (ie $w_2 = 0$), the intersection form on H_2 is unimodular and even. Any even unimodular quadratic form has signature divisible by 8, so we may conclude that p_1 is divisible by 24.

Further divisibility conditions can be obtained by a more sophisticated analysis. Given a spin manifold, we can consider its \hat{A} -genus, which is an integer (being the index of the Dirac operator on the spinor bundle). In this case, the \hat{A} -genus is equal to $-\frac{1}{24}p_1$, from which we immediately recover our earlier divisibility criterion. But we may go further. Indeed, since the real Clifford algebra $\text{Cl}_4 \cong M_2(\mathbb{H})$ contains the quaternions, the spinor representation inherits a quaternionic structure. The Dirac operator is then quaternion-linear, so its index must be even. The same conclusion can be obtained using the Atiyah–Bott–Shapiro orientation

$$M\text{Spin} \rightarrow \text{ko},$$

which refines the \hat{A} -genus [Atiyah et al. 1964]. Indeed, the composite

$$\pi_4 M\text{Spin} \rightarrow \pi_4 \text{ko} \rightarrow \pi_4 \text{ku} \cong \mathbb{Z}$$

is equal to the \hat{A} -genus, where the first map is induced by the Atiyah–Bott–Shapiro orientation and the second map is induced by tensoring up a real vector bundle to the complex numbers. Since the second map sends a generator to twice a generator of \mathbb{Z} , we learn that the \hat{A} -genus of a spin 4-manifold is divisible by 2, ie that p_1 is divisible by 48.

In the case of 11-connected 24-manifolds, we will use the Ando–Hopkins–Rezk refinement of the Witten genus to prove the following:

Proposition 3.4 *If M is an 11-connected 24-manifold, then $-1177p_3^2 - 311p_6$ is divisible by*

$$237758976000 = 2^{12} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 13.$$

The Witten genus is a cobordism invariant of string manifolds which takes values in the ring of integral modular forms; see Definition 3.5. However, not every integral modular form is the Witten genus of a string manifold. Indeed, the Witten genus factors through the coefficient ring of the spectrum of topological modular forms and it is this restriction which will provide the leverage we need to prove Proposition 3.4.

Definition 3.5 Let MF_n denote the group of integral modular forms of weight n . The direct sum of these groups, $\text{MF}_* = \bigoplus_{n \geq 0} \text{MF}_n$, is a graded ring, and has explicit generators and relations

$$\text{MF}_* \cong \mathbb{Z}[E_4, E_6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta),$$

where E_4 and E_6 are the weight 4 and 6 normalized Eisenstein series, respectively, and Δ is the discriminant. (See [Deligne 1975, proposition 6.1].)

Let Ω_*^{String} denote the cobordism ring of string manifolds. The Witten genus is a ring map

$$\phi_W: \Omega_{2*}^{\text{String}} \rightarrow \text{MF}_*.$$

In fact, by [Ando et al. 2010], the Witten genus can be refined to a map of \mathbb{E}_∞ -ring spectra as follows. Let $M\text{String}$ denote the Thom spectrum of the canonical map $B\text{String} = \tau_{\geq 8}BO \rightarrow BO$. Then there is a canonical isomorphism $\pi_*M\text{String} \cong \Omega_*^{\text{String}}$. Let tmf denote the connective spectrum of topological modular forms [Hopkins and Miller 2014; Behrens 2014; Lurie 2018]. This is an \mathbb{E}_∞ -ring spectrum which comes equipped with a ring map $\pi_{2*}\text{tmf} \rightarrow \text{MF}_*$. Ando et al. [2010] proved the Witten genus lifts to a map of \mathbb{E}_∞ -ring spectra.

Theorem 3.6 [Ando et al. 2010] *There is a map of \mathbb{E}_∞ -rings $M\text{String} \rightarrow \text{tmf}$ such that the induced map*

$$\Omega_{2*}^{\text{String}} \cong \pi_{2*}M\text{String} \rightarrow \pi_{2*}\text{tmf} \rightarrow \text{MF}_*$$

is the Witten genus ϕ_W .

Hopkins and Mahowald have determined the image of the map $\pi_{2*}\text{tmf} \rightarrow \text{MF}_*$:

Theorem 3.7 [Hopkins 2002, Proposition 4.6] *The image of the map $\pi_{2*}\text{tmf} \rightarrow \text{MF}_*$ has a basis given by the monomials*

$$a_{i,j,k} E_4^i E_6^j \Delta^k, \quad i, k \geq 0, j = 0, 1,$$

where

$$a_{i,j,k} = \begin{cases} 1 & \text{if } i > 0, j = 0, \\ 2 & \text{if } j = 1, \\ 24/\text{gcd}(24, k) & \text{if } i, j = 0. \end{cases}$$

Some additional references for this result are [Bauer 2008; Konter 2012, Theorem 1.2; Bruner and Rognes 2021, Section 9.3]. Specializing to the case of 24-dimensional manifolds, we obtain the following result:

Corollary 3.8 *The image of the Witten genus $\Omega_{24}^{\text{String}} \rightarrow \text{MF}_{12}$ lies in the subspace of $\text{MF}_{12} = \mathbb{Z}\{E_4^3, \Delta\}$ spanned by E_4^3 and 24Δ .*

Remark 3.9 At this point one could obtain the desired divisibility criterion on the Pontryagin numbers as follows. The Witten genus is determined by the characteristic series

$$\exp\left(\sum_{k \geq 1} 2G_{2k} \frac{z^{2k}}{(2k)!}\right),$$

where G_{2k} is the weight $2k$ Eisenstein series. From this, one can extract a polynomial in the Pontryagin numbers which computes the Witten genus for 24-manifolds. Plugging in the fact that all terms except p_3^2 and p_6 are zero due to the 11-connectedness assumption would prove the proposition.

Instead of doing this, we will cite results of Hirzebruch and Hopkins and Mahowald, since they have already analyzed the 24-dimensional case in connection with the Hirzebruch prize manifold.

Lemma 3.10 [Hirzebruch et al. 1992, Example, pages 85–86] *Let M denote a 24–dimensional string manifold. Then*

$$\phi_W(M) = \widehat{A}(M)\bar{\Delta} + \widehat{A}(M, T_{\mathbb{C}})\Delta,$$

where $\bar{\Delta} = E_4^3 - 744\Delta$, $\widehat{A}(M)$ is the \widehat{A} –genus of M , and $\widehat{A}(M, T_{\mathbb{C}})$ is the twisted \widehat{A} –genus of M , where the twisting is by the complexified tangent bundle $T_{\mathbb{C}}$.

Since 744 is divisible by 24, combining this lemma with Corollary 3.8 yields the following corollary:

Corollary 3.11 *Let M denote a 24–dimensional string manifold. Then $\widehat{A}(M, T_{\mathbb{C}})$ is divisible by 24.*

We thank the referee for pointing out that this corollary has been noted earlier by Teichner [2007, Corollary 90].

Finally, we recall the formula for $\widehat{A}(M, T_{\mathbb{C}})$ in terms of Pontryagin numbers, conveniently provided by [Mahowald and Hopkins 2002, page 98], specialized to the case of 11–connected 24–manifolds:

Proposition 3.12 [Mahowald and Hopkins 2002, page 98] *Let M denote a smooth, closed, oriented 11–connected 24–manifold. Then*

$$\widehat{A}(M, T_{\mathbb{C}}) = \frac{-1177p_3^2 - 311p_6}{9906624000}.$$

Combining this with Corollary 3.11, we obtain Proposition 3.4.

3.2 Application

To prove Theorem 3.1, we begin by using the results of Section 2 to construct an element of $\Omega_{24}^{(11)}$. Let $P \in A_{24}^{(11)}$ and $Q \in A_{24}^{(11)}$ represent the classes with $(\frac{1}{8} \text{sig}, \frac{1}{2}\chi^2) = (1, 0)$ and $(\frac{1}{8} \text{sig}, \frac{1}{2}\chi^2) = (0, 1)$, respectively. Moreover, let $[\Sigma_Q]$ equal the image of Q in $\text{coker}(J)_{23}$ under the composite

$$A_{24}^{(11)} \xrightarrow{\partial} \Theta_{23} \rightarrow \text{coker}(J)_{23},$$

and let $\text{ord}([\Sigma_Q])$ denote its order. Then, for any choice of $s(Q)_6 \in \mathbb{Z}$ as in Definition 2.8, the class

$$\text{ord}([\Sigma_Q])(Q - s(Q)_6 P) \in A_{24}^{(11)}$$

lifts to $\Omega_{24}^{(11)}$ (see [Krannich and Reinhold 2020, Theorem 2.9]).⁸

Assuming the class $N(Q - s(Q)_6 P)$ lifts to $\Omega_{24}^{(11)}$ for some integer N , we can compute its Pontryagin numbers and check whether they are compatible with Proposition 3.4. This will allow us to conclude that $\text{ord}([\Sigma_Q])$ is even. In order to compute the Pontryagin numbers of $N(Q - s(Q)_6 P)$, we make use of the formulas provided by Krannich and Reinhold.

⁸This statement depends on a choice of integer $s(Q)_6$ and is true for all possible such choices.

Proposition 3.13 [Krannich and Reinhold 2020, Proposition 2.13] *Given $n \geq 3$ and a choice of $s(Q)_{2n} \in \mathbb{Z}$, the Pontryagin numbers of a lift of $N(Q - s(Q)_{2n}P)$ to $\Omega_{8n}^{(4n-1)}$ are given by*

$$p_n^2 = 2Na_n^2(2n-1)!^2, \quad p_{2n} = Na_n^2 \left((2n-1)!^2 + (4n-1)!j_{2n} \frac{|B_{2n}|}{4n} \left(c_{2n} \frac{|B_{2n}|}{4n} + 2d_{2n}(-1)^n \right) \right)$$

for the choices of c_{2n} and d_{2n} corresponding to the choice of $s(Q)_{2n}$.

Evaluating these formulas for $n = 3$ (see Definition 2.8 for the constants which appear) we obtain:

- $B_6 = \frac{1}{42}$ and $B_{12} = -\frac{691}{2730}$.
- $j_6 = \text{denom}(\frac{1}{4 \cdot 6} |B_{12}|) = 65520$.
- $k_6 = \text{num}(\frac{1}{4 \cdot 6} |B_{12}|) = 691$.
- $a_3 = 2$ and $a_6 = 1$.
- We choose $c_6 = -18869$ and $d_6 = 199$.

From this we compute that

$$\begin{aligned} p_3^2 &= 2Na_3^2(2 \cdot 3 - 1)!^2 = 115200 \cdot N, \\ p_6 &= Na_3^2 \left((2 \cdot 3 - 1)!^2 + (4 \cdot 3 - 1)!j_6 \frac{|B_6|}{4 \cdot 3} \left(c_6 \frac{|B_6|}{4 \cdot 3} + 2d_6(-1)^n \right) \right) \\ &= N \cdot 2^2 \cdot \left(5!^2 + 11! \cdot 65520 \cdot \frac{|\frac{1}{42}|}{12} \cdot \left(-18869 \cdot \frac{|\frac{1}{42}|}{12} + 2 \cdot 199 \cdot (-1)^3 \right) \right) \\ &= -9038281766400 \cdot N. \end{aligned}$$

Corollary 3.14 *The Pontryagin numbers of any lift of $N(Q - s(Q)_6 \cdot P)$ to $\Omega_{24}^{(11)}$ are*

$$p_3^2 = 115200 \cdot N \quad \text{and} \quad p_6 = -9038281766400 \cdot N.$$

Finally, plugging these values into Proposition 3.4, we learn that

$$2810905493760000 \cdot N = 2^{11} \cdot 3^6 \cdot 5^4 \cdot 7^2 \cdot 13 \cdot 4729 \cdot N$$

is divisible by

$$237758976000 = 2^{12} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 13,$$

and in particular that N must be even.

4 Reduction to homotopy theory

In this section, we use standard arguments to reduce the proof of Theorem 1.4 to Theorem 4.1, a statement in stable homotopy theory. We then further divide the proof of Theorem 4.1 into several cases, which are the subjects of Sections 5–8. In the statement of Theorem 4.1, we have only included the cases not already in the literature.

Theorem 4.1 Let $MO\langle n \rangle$ denote the Thom spectrum of the canonical vector bundle on $\tau_{\geq n}BO$. Then the kernel of the unit map

$$\pi_{2n-1} \mathbb{S} \rightarrow \pi_{2n-1} MO\langle n \rangle$$

may be described as follows:

- (1) When $n = 9$, it is generated by the image of the J -homomorphism and $\eta\eta_4$.
- (2) When $n = 12$, it is generated by the image of the J -homomorphism and $\eta^3\bar{\kappa}$.
- (3) When $n \equiv 0 \pmod{4}$ and $16 \leq n \leq 124$, it is equal to the image of the J -homomorphism.

Proof of Theorem 1.4 from Theorem 4.1 By Remark 1.5, the only new cases of Theorem 1.4 that we need to establish are when

- $n \equiv 0 \pmod{4}$ and $12 \leq n \leq 124$;
- $n \equiv 1 \pmod{8}$ and $9 \leq n \leq 121$.

From [Stolz 1985, Satz 1.7], we know that the image of the composite

$$A_{2n}^{\langle n-1 \rangle} \xrightarrow{\partial} \Theta_{2n-1} \rightarrow \text{coker}(J)_{2n-1}$$

is equal to the image of the composite $\pi_{2n-1}(\Sigma^{-1}MO\langle n \rangle/\mathbb{S}) \rightarrow \pi_{2n-1}(\mathbb{S}) \rightarrow \text{coker}(J)_{2n-1}$. The image of $\pi_{2n-1}(\Sigma^{-1}MO\langle n \rangle/\mathbb{S}) \rightarrow \pi_{2n-1}(\mathbb{S})$ is, by the long exact sequence on homotopy groups, the kernel of the unit map $\pi_{2n-1} \mathbb{S} \rightarrow \pi_{2n-1} MO\langle n \rangle$. As a consequence, Theorem 4.1 implies Theorem 1.4 when $n = 9$ or $n = 4m$ and $3 \leq m \leq 31$.

On the other hand, as discussed in Section 2, when $n \equiv 1 \pmod{8}$, $n > 9$ we have $A_{2n}^{\langle n \rangle} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, with basis elements K and R . Here K is the Kervaire plumbing and R has Kervaire invariant 1. The Kervaire sphere $\partial(K)$ bounds a parallelizable manifold and therefore this class maps to zero in $\text{coker}(J)_{2n-1}$. Schultz [1972, Theorem 3.1] (see also Lemma 2.13) shows that $\partial(R) = \partial(Q) \cdot \eta^2$. As we already know that the image of $\partial(Q)$ in $\text{coker}(J)_{2n-1}$ is zero from the $n \equiv 0 \pmod{8}$, $n > 8$ case of Theorem 1.4, this is enough to conclude. □

To prove Theorem 4.1, we first note that the kernel of the unit map

$$\pi_* \mathbb{S} \rightarrow \pi_* MO\langle n \rangle$$

is well known to contain the image of J in degrees $n - 1$ and above, so the question is mostly reduced to finding an effective upper bound on the size of this kernel. Our proof of this upper bound in the $n \equiv 0 \pmod{4}$ case extends the methods and results of [Burklund et al. 2023], so we begin by reviewing the main points of the arguments therein.

Theorem 4.2 [Burklund et al. 2023, Lemma 6.9 and Theorem 7.1] *The kernel of the p -localized unit map*

$$u_{8m-1}: \pi_{8m-1} \mathbb{S}_{(p)} \rightarrow \pi_{8m-1} MO\langle 4m \rangle_{(p)}$$

is generated by the image of J if any of the following conditions are met:

- $p \geq 5$.
- $m \geq 32$ and $p = 3$.
- $m \geq 17$ and $p = 2$.

In [Burklund et al. 2023], Theorem 4.2 is proven in two steps:

- (1) A lower bound on the \mathbb{F}_p -Adams filtration of the classes in the kernel of u_{8m-1} is established.
- (2) This lower bound is compared to an upper bound on the \mathbb{F}_p -Adams filtration of elements in $\pi_{8m-1} \mathbb{S}_{(p)}$ which do not lie in the image of the J -homomorphism.

More precisely, (1) and (2) are accomplished via the following proposition and definition:

Proposition 4.3 [Burklund et al. 2023, Lemma 6.9 and Theorem 10.8] *Suppose that $m \geq 3$. Then the kernel of the p -localized unit map*

$$u_{8m-1}: \pi_{8m-1} \mathbb{S}_{(p)} \rightarrow \pi_{8m-1} MO\langle 4m \rangle_{(p)}$$

is generated by image of J and a single element $w \in \pi_{8m-1} \mathbb{S}_{(p)}$ which lies in \mathbb{F}_p -Adams filtration at least $2N_p - 1$.⁹ Relevant values of $2N_p - 1$ are summarized in Table 1.

Definition 4.4 Let $\Gamma_p(k)$ denote the minimal m such that every $\alpha \in \pi_k \mathbb{S}_{(p)}$ with \mathbb{F}_p -Adams filtration strictly greater than m is in the subgroup generated by the image of J together with Adams’s μ -family (at the prime 2).¹⁰

In Table 1, we have recorded known bounds on Γ_2 and Γ_3 .

- The sharp value of Γ_2 through 87 and the bound in 95 can be extracted from the extensive Adams spectral sequence computations of Isaksen, Wang and Xu [Isaksen et al. 2023].
- The bounds on Γ_2 above 95 are obtained from [Davis and Mahowald 1989, Corollary 1.3].
- The sharp value of Γ_3 through 103 can be extracted from Adams spectral sequence computations of [Oka 1972; Nakamura 1975].
- The bounds on Γ_3 above 103 are obtained from [Burklund 2022, Proposition 6.3.20].

Comparing the values of $2N_p - 1$ and $\Gamma_p(8m - 1)$ in Table 1 and using Theorem 4.2, we see that $w \in \pi_{8m-1} \mathbb{S}_{(p)}$ must lie in the image of J with the possible exception of the six cases

$$(p, 8m - 1) = (2, 23), (3, 23), (2, 31), (3, 39), (2, 47), (2, 127).$$

In order to handle these cases, as well as the $n = 9$ case of Theorem 4.1, we make four essentially different arguments. We outline each of these arguments here.

(1) **Dimensions 23 (prime 2), 31, 39 and 47** In Section 5, we make a mild improvement on the lower bound for the \mathbb{F}_p -Adams filtration of the classes in the kernel of u_{8m-1} established in [Burklund et al.

⁹Note that $2N_p - 1$ depends on m , though this is omitted from the notation. The general definition of N_p is given in [Burklund et al. 2023, Definition 7.5].

¹⁰However, note that the μ -family elements only appear in stems congruent to 1 and 2 mod 8, and in particular are absent from $\pi_{8m-1} \mathbb{S}_{(2)}$.

m	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$8m-1$	23	31	39	47	55	63	71	79	87	95	103	111	119	127
$2N_2-1$	5	5	11	13	19	19	25	27	33	35	41	43	49	49
$\Gamma_2(8m-1)$	9	5	9	13	0	9	15	14	15	≤ 19	≤ 37.9	≤ 41.3	≤ 42.7	≤ 49.1
$2N_3-1$	1	3	5	7	7	9	11	13	15	17	19	21	23	25
$\Gamma_3(8m-1)$	5	0	5	5	5	0	0	0	0	8	0	≤ 14.2	≤ 14.7	≤ 15.2

m	17	18	19	20	21	22	23	24
$8m-1$	135	143	151	159	167	175	183	191
$2N_3-1$	27	29	31	33	33	35	37	39
$\Gamma_3(8n-1)$	≤ 15.7	≤ 16.2	≤ 16.7	≤ 17.2	≤ 17.7	≤ 18.2	≤ 18.7	≤ 19.2

m	25	26	27	28	29	30	31
$8m-1$	199	207	215	223	231	239	247
$2N_3-1$	41	43	45	47	49	51	53
$\Gamma_3(8n-1)$	≤ 19.7	≤ 20.2	≤ 20.7	≤ 21.2	≤ 21.7	≤ 22.2	≤ 22.7

Table 1

2023]. This improvement resolves the cases in dimensions 31, 39 and 47. It further implies that, modulo the image of J , the only element which can lie in the kernel of the 2-localized map u_{23} is $\eta^3\bar{\kappa}$. By Theorem 3.1 and [Stolz 1985, Satz 1.7], it follows that $\eta^3\bar{\kappa}$ does indeed lie in the kernel.

(2) **Dimension 17, ie $n = 9$** In Section 6, we analyze the exceptional situation in dimension 17. We first obtain an upper bound on the kernel of u_{17} by considering the composition

$$\pi_{17}\mathbb{S} \rightarrow \pi_{17}MO\langle 9 \rangle \rightarrow \pi_{17}MO\langle 8 \rangle \rightarrow \pi_{17}\text{tmf},$$

noting that the kernel is generated by $\eta\eta_4$ and the image of J . In order to show $\eta\eta_4$ lies in the kernel of u_{17} , we examine the homotopy power operation P^9 . Using the Steenrod squares on the E_2 -page of the \mathbb{F}_2 -Adams spectral sequence, we find that P^9 sends $\eta\sigma$ to an element of $\pi_{17}\mathbb{S}$ which does not lie in the image of J . Since $\eta\sigma$ is in the image of J , we learn that it maps to zero in $\pi_8MO\langle 9 \rangle$ and therefore $P^9(\eta\sigma)$ must also go to zero in $\pi_{17}MO\langle 9 \rangle$, which concludes the argument.

(3) **Dimension 23 (prime 3)** In Section 7, we study the map of \mathbb{F}_3 -Adams spectral sequences induced by the composition

$$MO\langle 12 \rangle \rightarrow MO\langle 8 \rangle \rightarrow \text{tmf}$$

of the canonical map with the Ando–Hopkins–Rezk string orientation of tmf [Ando et al. 2010]. The string orientation provides the leverage necessary to conclude that the kernel of the 3-localized map u_{23} is equal to the image of J . This appears in Section 7.

(4) **Dimension 127** In Section 8 we make a short homological argument which shows there are no elements of \mathbb{F}_2 -Adams filtration 49 in the 127-stem. It follows that $w \in \pi_{127}\mathbb{S}_{(2)}$ must be of \mathbb{F}_2 -Adams filtration at least 50. Consulting Table 1, we find that it must lie in the image of J .

5 A homotopy argument

In order to finish the proof of [Theorem 4.1](#) in dimensions 31, 39 and 47, as well as the prime 2 case of dimension 23, we will need to improve the \mathbb{F}_p -Adams filtration bound from [\[Burklund et al. 2023, Theorem 10.8\]](#). As in the proof of that bound, this proof relies on the existence and properties of the category of synthetic spectra. We suggest the reader consult [\[Pstragowski 2023\]](#) for a general introduction to synthetic spectra and [\[Burklund et al. 2023, Section 9\]](#) or [\[Burklund 2022\]](#) for a more computational viewpoint.¹¹ Since the proof we give is essentially a refinement of the argument in [\[Burklund et al. 2023, Section 10\]](#), we will assume the reader is generally familiar with that argument.

[Burklund et al. \[2023, Lemma 6.9\]](#) give a Toda bracket expression for an element w which generates the kernel of the unit map

$$\pi_{8m-1} \mathbb{S}_{(p)} \rightarrow \pi_{8m-1} MO\langle 4m \rangle_{(p)}$$

(modulo the image of J). We begin by recalling this expression.

Definition 5.1 Let

$$M \rightarrow \Sigma^\infty O\langle 4m - 1 \rangle$$

denote the inclusion of an $(8m-1)$ -skeleton of $\Sigma^\infty O\langle 4m - 1 \rangle$. By the inclusion of an $(8m-1)$ -skeleton, we mean in particular that the induced map

$$(\mathbb{F}_p)_*(M) \rightarrow (\mathbb{F}_p)_*(\Sigma^\infty O\langle 4m - 1 \rangle)$$

is an isomorphism for $* \leq 8m - 1$ and that $(\mathbb{F}_p)_*(M) \cong 0$ for $* > 8m - 1$. The generator $x \in \pi_{4m-1}(\Sigma^\infty O\langle 4m - 1 \rangle) \cong \mathbb{Z}$ is the image of some class in $\pi_{4m-1} M$, which by abuse of notation we also denote by x . We additionally abuse notation by using J to denote the composite map

$$M \rightarrow \Sigma^\infty O\langle 4m - 1 \rangle \xrightarrow{J} \mathbb{S},$$

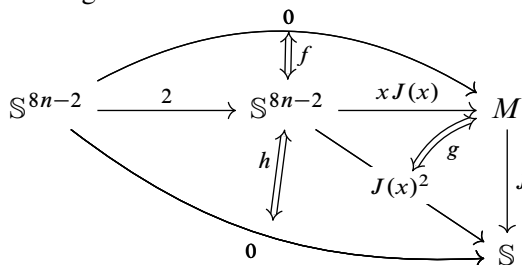
where

$$\Sigma^\infty O\langle 4m - 1 \rangle \xrightarrow{J} \mathbb{S}$$

is adjoint to the map

$$O\langle 4m - 1 \rangle \rightarrow O \xrightarrow{J} GL_1(\mathbb{S}).$$

Construction 5.2 Consider the diagram



where the homotopies f , g and h are chosen as follows:

¹¹In this section all synthetic spectra will be \mathbb{F}_p -synthetic spectra.

- f is an arbitrary nullhomotopy.
- g is the canonical homotopy associated to the fact that J is a map of \mathbb{S} -modules.
- h is the canonical nullhomotopy given by the \mathbb{E}_∞ -ring structure on \mathbb{S} .

This provides a homotopy from 0 to itself which defines the map $w: \mathbb{S}^{8n-1} \rightarrow \mathbb{S}^0$, well defined modulo the image of J . By [Burklund et al. 2023, Lemma 6.9], w generates the kernel of the unit map

$$u_{8m-1}: \pi_{8m-1} \mathbb{S} \rightarrow \pi_{8m-1} MO\langle 4m \rangle.$$

The Adams filtration bound in [Burklund et al. 2023, Theorem 10.8] arises via the construction of a synthetic lift of the diagram in Construction 5.2. Our improvement in Adams filtration will come from producing a slightly better synthetic lift.

Recollection 5.3 As in [Burklund et al. 2023, Construction 10.5], once we move over to the category of synthetic spectra there is a lift of the map of synthetic spectra

$$\nu J: \nu M \rightarrow \mathbb{S}^{0,0}$$

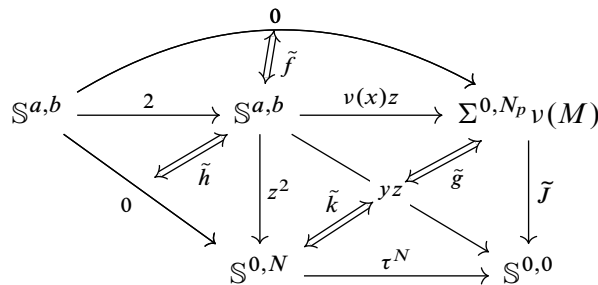
along the map $\tau^{N_p}: \mathbb{S}^{0,-N_p} \rightarrow \mathbb{S}^{0,0}$ to a map $\nu M \rightarrow \mathbb{S}^{0,-N_p}$, which we view as a map

$$\tilde{J}: \Sigma^{0,N_p} \nu M \rightarrow \mathbb{S}^{0,0}.$$

The main result of this section is the following:

Lemma 5.4 Suppose that $\tilde{J}(\nu(x)) = \tau^N z$. Then the Toda bracket w has Adams filtration at least $2N_p + N - 1$.

Proof Let us fix the following notation: we set $y = \tilde{J}(\nu(x))$, $a = 8n - 2$ and $b = 8n - 2 + 2N_p + N$. Then we construct the diagram in synthetic spectra



where the homotopies \tilde{f} , \tilde{g} and \tilde{h} are chosen as follows:

- \tilde{f} is an arbitrary nullhomotopy, which exists as a consequence of the fact that $\pi_{8n-2, 8n-2+\gamma}(\nu M) = 0$ for all $\gamma \geq 2$ [Burklund et al. 2023, Proof of Proposition 10.7].
- \tilde{g} is the canonical homotopy that expresses the fact that \tilde{J} is a map of right $\mathbb{S}^{0,0}$ -modules.

- \tilde{h} is the canonical nullhomotopy that comes from the fact that $\mathbb{S}^{0,0}$ is an \mathbb{E}_∞ -ring in the symmetric monoidal ∞ -category $\text{Syn}_{\mathbb{F}_p}$.
- \tilde{k} is the composite of a homotopy expressing that $z\tau^N = y$ and the natural homotopy expressing that composition with z is homotopic to multiplication by z .

This diagram determines a homotopy of 0 with itself, and hence a map

$$\tilde{w}: \mathbb{S}^{a+1,b} \rightarrow \mathbb{S}^{0,0}.$$

On applying τ^{-1} , the above diagram recovers [Construction 5.2](#), so \tilde{w} maps to w under τ^{-1} . The desired Adams filtration bound now follows from [\[Burklund et al. 2023, Corollary 9.21\]](#). \square

Using this lemma, we are able to finish the proof of [Theorem 4.1](#) in the promised cases.

Proposition 5.5 *When $(p, 8m - 1) = (2, 31), (3, 39)$ and $(2, 47)$, the element w lies in the image of J . On the other hand, when $(p, 8m - 1) = (2, 23)$, the image of the element w in the cokernel of J is $\eta^3\bar{\kappa}$.*

Proof Throughout this proof, we will freely make use of [\[Burklund et al. 2023, Theorem 9.19\]](#) in order to translate between knowledge of the \mathbb{F}_p -Adams spectral sequence and knowledge of \mathbb{F}_p -synthetic homotopy groups.

Dimension 23, prime 2 In this case, $J(x) = \zeta_{11}$ and we can determine that $\tilde{J}(v(x)) = \tau^2\tilde{\zeta}_{11}$ because there is no τ -torsion in this bidegree. Thus, w has \mathbb{F}_2 -Adams filtration at least 7. It follows from [Theorem 3.1](#) and [\[Stolz 1985, Satz 1.7\]](#) that the image of w in the 2-localized cokernel of J must be nonzero. Using our restriction on its \mathbb{F}_2 -Adams filtration, we conclude that w must be equal to $\eta^3\bar{\kappa}$ in the 2-localized cokernel of J .

Dimension 31, prime 2 In this case, $J(x) = \rho_{15}$ and we can determine that $\tilde{J}(v(x)) = \tau\tilde{\rho}_{15}$ because there is no τ -torsion in this bidegree. Thus, w has Adams filtration at least 6 and so it must be in the image of J (see [Table 1](#)).

Dimension 39, prime 3 In this case, $J(x) = \alpha_5$ and we can determine that $\tilde{J}(v(x)) = \tau^2\tilde{\alpha}_5$ because there is no τ -torsion in this bidegree. Thus, w has Adams filtration at least 7 and so it must be in the image of J (see [Table 1](#)).

Dimension 47, prime 2 Once again, $\tilde{J}(v(x))$ lands in a bidegree with no τ -torsion where every element is divisible by τ^2 . Thus, w has Adams filtration at least 15 and so it must be in the image of J (see [Table 1](#)). \square

6 The case of dimension 17

The goal of this section is to prove the following theorem, which shows that the dimension 17 is exceptional:

Theorem 6.1 *The kernel of the unit map*

$$u_{17}: \pi_{17} \mathbb{S} \rightarrow \pi_{17} MO\langle 9 \rangle$$

is generated by the image of J and $\eta\eta_4 \in \pi_{17} \mathbb{S}$.

We begin the proof of [Theorem 6.1](#) with the following bounds on the size of the kernel of u_{17} :

Lemma 6.2 *The kernel of the unit map*

$$\pi_{17} \mathbb{S} \rightarrow \pi_{17} MO\langle 9 \rangle$$

is either equal to the image of J or the subgroup of $\pi_{17} \mathbb{S}$ generated by the image of J and $\eta\eta_4$.

Proof It is well known that in degrees 8 and above the image of J is in the kernel of the unit map for $MO\langle 9 \rangle$. Composing the Ando–Hopkins–Rezk string orientation [[Ando et al. 2010](#)] with the canonical map $MO\langle 9 \rangle \rightarrow MO\langle 8 \rangle$, we obtain an \mathbb{E}_∞ -ring map $MO\langle 9 \rangle \rightarrow \text{tmf}$. One may read off from the computations of [[Bauer 2008](#)] that the kernel of

$$\pi_{17} \mathbb{S} \rightarrow \pi_{17} MO\langle 9 \rangle \rightarrow \pi_{17} \text{tmf}$$

is generated by the image of J and $\eta\eta_4$, from which the proposition follows. \square

It now suffices to show that the kernel of the unit map

$$\pi_* \mathbb{S} \rightarrow \pi_* MO\langle 9 \rangle$$

contains an element not in the image of J in dimension 17. To prove this, we will use the fact that this kernel is closed under spherical power operations. The power operation of interest to us is described in the following proposition:

Proposition 6.3 [[Bruner et al. 1986](#), Table V.1.3] *Let R be an \mathbb{E}_∞ -ring. There is a natural, not necessarily additive, operation P^9 from $\pi_8 R$ to $\pi_{17} R$, with indeterminacy. The indeterminacy of $P^9(x)$ is ηx^2 .*

Moreover, if $x \in \pi_8 R$ is detected by $a \in E_2^{s, 8+s}$ on the E_2 -page of the \mathbb{F}_2 -Adams spectral sequence, then $P^9 x$ lies in \mathbb{F}_2 -Adams filtration at least $2s - 1$ and its image in $E_2^{2s-1, 17+2s-1}$ is $\text{Sq}^9 a$.

Proof of Theorem 6.1 Applying this power operation to $\eta\sigma \in \pi_8 \mathbb{S}$, which lies in the image of J and thus the kernel of

$$\pi_8 \mathbb{S} \rightarrow \pi_8 MO\langle 9 \rangle,$$

we learn that $P^9(\eta\sigma)$ is in the kernel of the unit map for $MO\langle 9 \rangle$. Since $\eta(\eta\sigma)^2 = 0$, the operation $P^9(\eta\sigma) \in \pi_{17} \mathbb{S}$ has no indeterminacy.

The class $P^9(\eta\sigma)$ is detected on the E_2 -page of the \mathbb{F}_2 -Adams spectral sequence by $Sq^9(h_1h_3)$. Using the Cartan formula and the fact that $Sq^{2^i}(h_i) = h_{i+1}$ — see [Bruner et al. 1986, Proposition 1.4(i)] — it follows that

$$Sq^9(h_1h_3) = Sq^1(h_1)Sq^8(h_3) + Sq^2(h_1)Sq^7(h_3) = h_1^2h_4 + h_2h_3^2 = h_1^2h_4.$$

Since no element detected by $h_1^2h_4$ lies in the image of J , we are done. □

7 The case of dimension 23 at the prime 3

In this section, we will show that, although the kernel of the unit map $\pi_{23} \mathbb{S} \rightarrow \pi_{23} MO\langle 12 \rangle$ contains an exceptional element at the prime 2, at the prime 3 the kernel contains only the image of J . This is the final step in the proof of the dimension 23 case of Theorem 4.1.

We will prove this by directly computing of the \mathbb{F}_3 -Adams spectral sequence for $MO\langle 12 \rangle$. As at the prime 2, one of the key techniques in this argument is comparison with tmf via the Ando–Hopkins–Rezk string orientation [Ando et al. 2010]. The first step we take is to compute the homology of $MO\langle 12 \rangle$ as an \mathcal{A}_* -comodule in a range.

As is common in odd primary Adams spectral sequence computations, everything will be implicitly 3-completed and we will make use of the \doteq notation, which means that an equation holds up to multiplication by a 3-adic unit. Similarly, since we do not keep track of constants, all claims in this section should be regarded as true up to multiplication by a 3-adic unit.

Lemma 7.1 *In degrees ≤ 25 , the \mathbb{F}_3 -homology of $MO\langle 12 \rangle$ has the following properties:*

- (1) *It is isomorphic to $\mathbb{F}_3 \oplus (\mathbb{F}_3)_*(\Sigma^{12}ko) \oplus \Sigma^{24} \mathbb{F}_3$ as an \mathcal{A}_* -comodule.*
- (2) *The only nontrivial product is the square of the generator in degree 12, which is equal to a generator of the third summand.*
- (3) *On \mathbb{F}_3 -homology, the composition of the canonical map with the string orientation*

$$MO\langle 12 \rangle \rightarrow MO\langle 8 \rangle \rightarrow \text{tmf}$$

is only nonzero on the unit.

Proof We begin by showing that, in degrees ≤ 25 , the Thom isomorphism

$$\mathbb{F}_3 \otimes MO\langle 12 \rangle \simeq \mathbb{F}_3 \otimes \Sigma_+^\infty BO\langle 12 \rangle,$$

which is an equivalence of \mathbb{E}_∞ -rings, preserves the \mathcal{A}_* -comodule structure. To do this, we just need to show that the action of the Steenrod algebra on the Thom class $u \in (\mathbb{F}_3)^0(MO\langle 12 \rangle)$ is trivial through degree 25. Since the Steenrod algebra is generated by β , P^1 and P^3 in this range, it suffices to show that the action of these operations on u is zero.

To show this, we note that u is the pullback of another class $u \in (\mathbb{F}_3)^0(\text{tmf})$ along the composition $MO\langle 12 \rangle \rightarrow MO\langle 8 \rangle \rightarrow \text{tmf}$. From [Culver 2021, Section 4.1], we can extract that through dimension 12 the cohomology of tmf is given by $\mathbb{F}_3\{u\} \oplus \mathbb{F}_3\{b_4\} \oplus \mathbb{F}_3\{z\}$ where $|u| = 0$, $|b_4| = 8$ and $|z| = 12$ with Steenrod action $z \doteq P^1(b_4) \doteq P^3(u)$. For degree reasons, $\beta(u) = P^1(u) = 0$ and b_4 maps to zero in $(\mathbb{F}_3)^8(MO\langle 12 \rangle) = 0$; therefore, the Steenrod operations β , P^1 and P^3 act trivially on u in $MO\langle 12 \rangle$.

Using the Goodwillie tower of the identity for \mathbb{E}_∞ -rings, as worked out in [Kuhn 2006], we obtain a tower of nonunital \mathbb{E}_∞ -rings

$$\begin{array}{ccccccc} & & D_3(\tau_{\geq 12}ko) & & D_2(\tau_{\geq 12}ko) & & \tau_{\geq 12}ko \\ & & \downarrow & & \downarrow & & \downarrow \simeq \\ \Sigma^\infty BO\langle 12 \rangle & \rightarrow & \cdots & \longrightarrow & Q_3 & \longrightarrow & Q_2 & \longrightarrow & Q_1 \end{array}$$

For connectivity reasons, through degree 25 we only need to work with Q_2 . Since Q_1 is the stabilization of $\Sigma^\infty BO\langle 12 \rangle$, the product on Q_1 is zero.

Note that

$$(\mathbb{F}_3)_*(D_2(\tau_{\geq 12}ko)) \cong \begin{cases} 0 & \text{if } * \leq 23 \text{ or } * = 25, \\ \mathbb{F}_3 & \text{if } * = 24, \end{cases}$$

and let x denote a generator of $(\mathbb{F}_3)_{12}(BO\langle 12 \rangle)$. In order to finish the proof of (1), we only need to show that the vertical map into Q_2 is injective on \mathbb{F}_3 -homology in degree 24. In fact, this would follow from knowing that x^2 is nonzero, which itself would imply (2), given that we know the product on Q_1 is zero. In order to show x^2 is nonzero, we note that x must be primitive for degree reasons and consider the coproduct, in $(\mathbb{F}_3)_*(BO\langle 12 \rangle)$,

$$\Delta(x^2) = x^2 \otimes 1 + 2(x \otimes x) + 1 \otimes x^2,$$

where the middle term is clearly nonzero.

Now we turn to (3). Applying \mathbb{F}_3 -cohomology to the map $MO\langle 12 \rangle \rightarrow \text{tmf}$, we obtain a map of \mathcal{A}^* -modules

$$(\mathbb{F}_3)^*\text{tmf} \rightarrow (\mathbb{F}_3)^*(MO\langle 12 \rangle).$$

Since $(\mathbb{F}_3)^*(\text{tmf})$ has a two-stage filtration by cyclic \mathcal{A}^* -modules with generators in degrees 0 and 8, respectively [Rezk 2002, Theorem 21.5(2)], and $(\mathbb{F}_3)^*(\Sigma^{12}ko) \oplus \Sigma^{24}\mathbb{F}_3$ begins in degree 12, it follows that through degree 25 the map

$$(\mathbb{F}_3)^*\text{tmf} \rightarrow (\mathbb{F}_3)^*(MO\langle 12 \rangle)$$

factors through the unit. □

As a consequence of this lemma, we learn that the E_2 -page of the \mathbb{F}_3 -Adams spectral sequence for $MO\langle 12 \rangle$ takes the form shown in Figure 1. Next we determine the easy differentials in this spectral sequence.

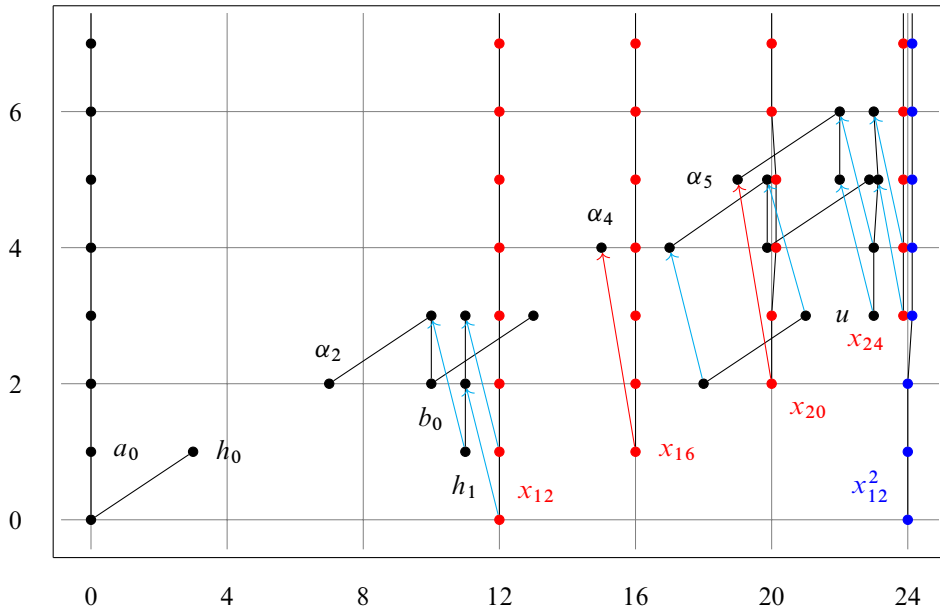


Figure 1: The \mathbb{F}_3 -Adams spectral sequence for $MO\langle 12 \rangle$. Classes in black come from the unit copy of \mathbb{F}_3 . Classes in red come from the copy of $(\mathbb{F}_3)_*(ko)$ in degree 12. Classes in blue come from the copy of \mathbb{F}_3 in degree 24. The nontrivial product in homology from Lemma 7.1 lets us conclude that x_{12}^2 is as labeled.

Lemma 7.2 *Through degree 24, the differentials in the \mathbb{F}_3 -Adams spectral sequence for $MO\langle 12 \rangle$ fit into one of the following two families:*

- (1) *the differentials induced from the sphere, all of which occur; and*
- (2) *the extra differentials*

$$d_2(x_{12}) \doteq a_0 h_1, \quad d_3(x_{16}) \doteq \alpha_4, \quad d_3(x_{20}) \doteq \alpha_5, \quad d_2(x_{24}) \doteq a_0^2 u, \quad d_5(x_{12}^2) \doteq ?.$$

Both families are displayed in Figure 1.

After proving this lemma, the final task of this section will be to show that x_{12}^2 is a permanent cycle.

Proof First we note that, for degree reasons, nothing can interfere with the differentials induced from the sphere. We also know that the elements $\alpha_{3/2}$, α_4 and α_5 in the image of J must each map to zero in $MO\langle 12 \rangle$ as well. For each of these there is a unique possible differential which could enforce that relation. It remains to show that $d_2(x_{24}) \doteq a_0^2 u$.

In the sphere we can use Moss’s theorem [1970] to conclude that $\langle \alpha_5, \alpha_1, 3 \rangle$ is detected by $a_0^2 u$ in the \mathbb{F}_3 -Adams spectral sequence. Since the indeterminacy is just 3 times this same element, we find that this bracket is equal to $\alpha_{6/2}$ up to a unit. The class α_5 maps to zero in $MO\langle 12 \rangle$; therefore, $\alpha_{6/2}$ becomes divisible by 3. Examining Figure 1, this can only happen if $\alpha_{6/2}$ is zero. Thus, we know it gets hit by

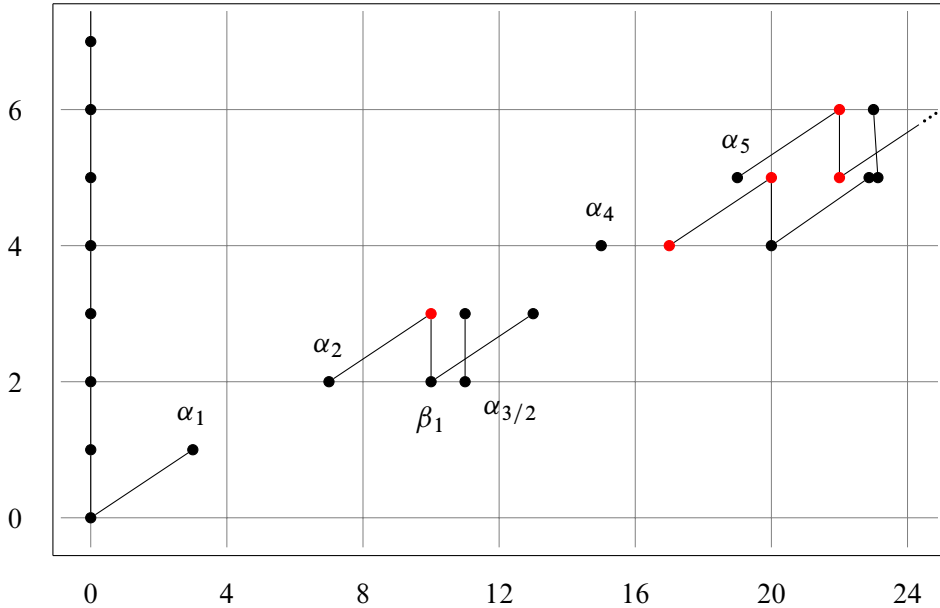


Figure 2: The bigraded homotopy groups of the \mathbb{F}_3 -synthetic sphere displayed in the $(t-s, s)$ -plane. Black dots denote τ -torsion-free classes, red dots denote τ^{-1} -torsion classes. Note that in order to reconstruct the group in a given bidegree one must examine all degrees lying above it.

some differential. We will finish the proof by using synthetic spectra to bound the length of the Adams differential that hits a_0^2u .

The first step is to lift the Toda bracket above to one in the synthetic category. Using [Burklund et al. 2023, Theorem 9.19], we may compute the \mathbb{F}_3 -synthetic homotopy of the sphere through 24 using the known computation of its \mathbb{F}_3 -Adams spectral sequence in this range. The result is displayed in Figure 2. We next fix some names for specific elements of $\pi_{*,*}\mathbb{S}$:

- Let $\tilde{3} \in \pi_{0,1}\mathbb{S}$ denote the unique element in that degree which maps to 3 in $\pi_0\mathbb{S}$ and a_0 in $\pi_{0,1}(C\tau) \cong \text{Ext}_{\mathcal{A}}^{1,4}(\mathbb{F}_3, \mathbb{F}_3)$.
- Let $\alpha_1 \in \pi_{3,4}\mathbb{S}$ denote the unique element in that degree which maps to α_1 in $\pi_3\mathbb{S}$ and h_0 in $\pi_{3,4}(C\tau) \cong \text{Ext}_{\mathcal{A}}^{1,4}(\mathbb{F}_3, \mathbb{F}_3)$.
- Let $\alpha_5 \in \pi_{19,24}\mathbb{S}$ denote the unique element in that degree which maps to α_5 in $\pi_3\mathbb{S}$ and a generator of $\pi_{19,24}(C\tau)$.

Since the image of $\alpha_1\alpha_5$ in $\pi_{22,28}(C\tau)$ is hit by a d_2 differential and there are no classes above it we learn that $\tau\alpha_1\alpha_5 = 0$. Since $\pi_{3,5}\mathbb{S} = 0$ we learn that $\tilde{3}\alpha_1 = 0$. This means we can form the Toda bracket $x := \langle \tau\alpha_5, \alpha_1, \tilde{3} \rangle$ in synthetic spectra. Upon inverting τ , the bracket x goes to the bracket $\langle \alpha_5, \alpha_1, 3 \rangle$, which is detected by a_0^2u on the E_2 -page. We may therefore conclude that x maps to a_0^2u in the homotopy of $C\tau$. If we show that the image of x in $\nu MO\langle 12 \rangle$ is simple τ -torsion, then this will imply that a_0^2u is hit by a d_2 differential in the \mathbb{F}_3 -Adams spectral sequence for $MO\langle 12 \rangle$.

Toda brackets are preserved by \mathbb{E}_1 -ring maps, and $\mathbb{S} \rightarrow \nu MO\langle 12 \rangle$ is a map of \mathbb{E}_∞ -rings, therefore we may make the following manipulations of Toda brackets (now considered in $\nu MO\langle 12 \rangle$):

$$\tau x \in \tau \langle \tau \alpha_5, \alpha, \tilde{3} \rangle \subset \langle \tau^2 \alpha_5, \alpha, \tilde{3} \rangle = \langle 0, \alpha, \tilde{3} \rangle = \tilde{3} \pi_{23,26}(MO\langle 12 \rangle),$$

where the fact that $\tau^2 \alpha_5 = 0$ follows from it getting hit by the d_3 differential off of x_{20} . Finally, since there is no possible nonzero $\tilde{3}$ -division for τx , we conclude that it is zero. \square

In dimension 12 we will need slightly finer information than that provided by Lemma 7.2.

Lemma 7.3 *The element c_6 which generates $\pi_{12} \text{tmf}_3$ lifts to $\pi_{12} MO\langle 12 \rangle$, where it is detected (up to a unit) by $a_0^2 x_{12}$ in the \mathbb{F}_3 -Adams spectral sequence.*

Proof We begin by considering the sequence of maps

$$\mathbb{S}^{11} \xrightarrow{9} \mathbb{S}^{11} \xrightarrow{\alpha_{3/2}} \mathbb{S}^0 \xrightarrow{\iota} \text{tmf}.$$

Since each pairwise composite is nullhomotopic, we can form the Toda bracket $\langle 9, \alpha_{3/2}, \iota \rangle$. This Toda bracket has indeterminacy $9\pi_{12}(\text{tmf}) + \pi_{12}(\mathbb{S})\iota = (9\mathbb{Z}) \cdot c_6$. We will begin by showing that, up to a unit, c_6 is contained in this bracket.

This bracket can be evaluated using the corresponding Massey product $\langle a_0^2, a_0 h_1, \iota \rangle$. In particular, it will suffice to show that this Massey product is equal to c_6 as an element of the E_2 -page for tmf . After consulting this E_2 -term, we note that c_6 is the only element in its bidegree, so it will suffice to simply show that the bracket is nontrivial. In order to do this, we shuffle the bracket with h_0 :

$$h_0 \langle a_0^2, a_0 h_1, \iota \rangle = \langle h_0, a_0^2, a_0 h_1 \rangle \iota \doteq \alpha_4 \iota.$$

Finally, we note that the image of α_4 in the E_2 -page for tmf is nontrivial [Culver 2021]. From this we can read off that $\langle 9, \alpha_{3/2}, \iota \rangle = u c_6$ for some $u \in \mathbb{Z}_3^\times$. The claim that c_6 lifts to $MO\langle 12 \rangle$ now follows from the fact that the bracket $\langle 9, \alpha_{3/2}, \iota' \rangle$ is defined, where ι' is the unit of $MO\langle 12 \rangle$. Since c_6 is a generator of $\pi_{12} \text{tmf}_3$, the only possibility for an \mathbb{F}_3 -Adams representative of its lift to $\pi_{12} MO\langle 12 \rangle$ is (up to a unit) $a_0^2 x_{12}$. \square

Proposition 7.4 *The kernel of the unit map $\pi_{23} \mathbb{S}_3 \rightarrow \pi_{23} MO\langle 12 \rangle_3$ does not contain $\alpha_1 \beta_1^2$, and hence is generated by the image of J . Equivalently, $d_5(x_{12}^2) = 0$.*

Proof We will proceed by contradiction. Suppose that $d_5(x_{12}^2) \doteq h_0 b_0^2$.

- Let y denote an element of $\pi_{24} MO\langle 12 \rangle$ which is detected by $a_0^2 x_{24}$.
- Let z denote an element of $\pi_{24} MO\langle 12 \rangle$ which is detected by $a_0 x_{12}^2$.

Note that any choice of y and z forms a basis for $\pi_{24} MO\langle 12 \rangle$ over \mathbb{Z}_3 . By Moss's theorem [1970], we can choose z such that $z \in \langle 3, \alpha \beta^2, \iota' \rangle$, where again ι' is the unit of $MO\langle 12 \rangle$. Similarly, from the d_4 off

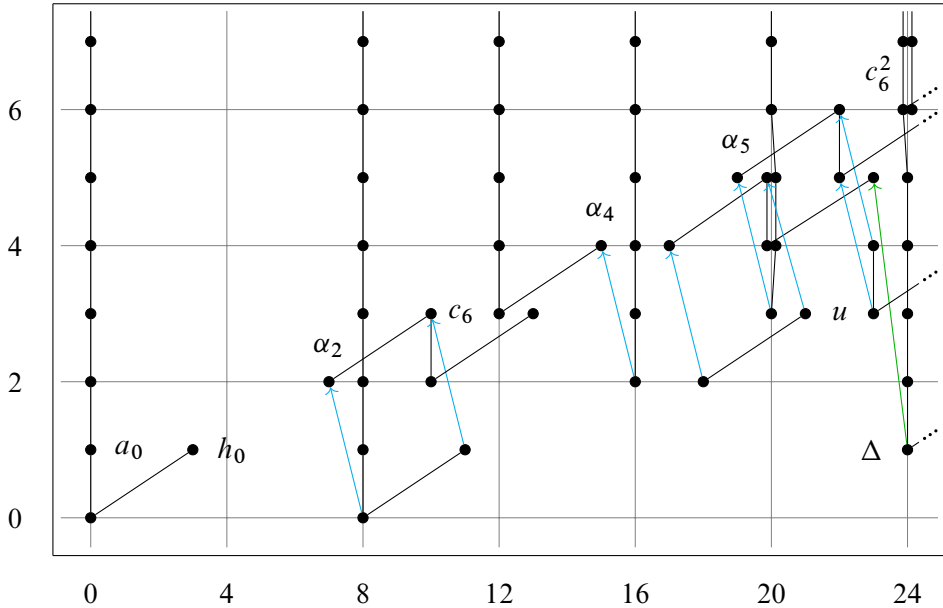


Figure 3: The \mathbb{F}_3 -Adams spectral sequence for tmf as it appears in [Culver 2021].

of Δ in the \mathbb{F}_3 -Adams spectral sequence for tmf , we learn that $[3\Delta] \in \langle 3, \alpha\beta^2, \iota \rangle$. Postcomposing with the string orientation lets us conclude that z maps to $[3\Delta]$ up to higher filtration elements. Then, using the fact that c_6^2 and $[3\Delta]$ are generators for $\pi_{24}tmf$, we may conclude that the map $\pi_{24}MO\langle 12 \rangle \rightarrow \pi_{24}tmf$ is surjective. Now we may choose z such that it maps to $[3\Delta]$ in tmf .

Using the \mathbb{F}_3 -Adams filtration of $MO\langle 12 \rangle$ and the fact that a lift of c_6 is detected by $a_0^2x_{12}$, we can conclude that $c_6^2 = u_1 27z + u_2 y$ for some constants $u_1 \in \mathbb{Z}_3^\times$ and $u_2 \in \mathbb{Z}_3$. Rearranging, we can write

$$c_6^2 - 27u_1 z = u_2 y.$$

Now consider the Adams filtration of the image of each side of this equality in tmf . The left-hand side maps to $c_6^2 - 27u_1[3\Delta]$, which has Adams filtration 5. The element y is detected by $a_0^2x_{24}$ in filtration 5. However, by Lemma 7.1, we know that x_{24} maps to zero under the map of E_2 -pages induced by the string orientation. Thus, the right-hand side maps to an element of Adams filtration at least 6, a contradiction. \square

8 The case of dimension 127 at the prime 2

In this section, we will prove the following proposition, which implies the dimension 127 case of Theorem 4.1:

Proposition 8.1 *There is no element of $\pi_{127}\mathbb{S}_{(2)}$ which is detected in \mathbb{F}_2 -Adams filtration 49. In other words, if $x \in \pi_{127}\mathbb{S}_{(2)}$ has \mathbb{F}_2 -Adams filtration at least 49, then it also has \mathbb{F}_2 -Adams filtration at least 50.*

Proof of Theorem 4.1 in dimension 127, given Proposition 8.1 By Proposition 4.3 and Table 1, the element $w \in \pi_{127} \mathbb{S}_{(2)}$ lies in \mathbb{F}_2 -Adams filtration at least 49. By Proposition 8.1, w in fact lies in \mathbb{F}_2 -Adams filtration at least 50. Consulting Table 1 once more, we find that w must lie in the image of J , as desired. \square

The proof of Proposition 8.1 will be based on two further lemmas.

Lemma 8.2 Let $E_2^{s,t}$ denote the E_2 -page of the \mathbb{F}_2 -Adams spectral sequence. Then

$$E_2^{49,176} = \mathbb{F}_2\{h_0^{48}h_7\}.$$

Proof It may be read off from [Tangora 1970, Figure 5] that $E_2^{17,80} = \mathbb{F}_2\{h_0^{16}h_6\}$. It follows from [Ravenel 1986, Theorem 3.4.6] that Adams periodicity determines an isomorphism $E_2^{17,80} \cong E_2^{49,176}$, so that the latter is a one-dimensional \mathbb{F}_2 -vector space. By [Ravenel 1986, Lemma 3.4.15], $h_0^{63}h_7$ is nonzero, so that $h_0^{48}h_7 \in E_2^{49,176}$ must be nonzero and so a basis for $E_2^{49,176}$. \square

Remark 8.3 The conclusion of this lemma can also be read off directly from Nassau’s computer calculations [2000] of Ext over the Steenrod algebra.

Lemma 8.4 Suppose that $x \in \pi_{2^n-1} \mathbb{S}_{(2)}$ is detected on the E_2 -page of the \mathbb{F}_2 -Adams spectral sequence by the class $h_0^{2^{n-1}-1}h_n$. Then x lies in the image of the J -homomorphism.

Proof By Adams vanishing [1966b, Theorem 2.1], there can be no elements above $h_0^{2^{n-1}-1}h_n$ in the \mathbb{F}_2 -Adams spectral sequence, ie $E^{1+s,2^n+s} = 0$ for all $s \geq 2^{n-1}$. It therefore suffices to establish the existence of an element in the image of J detected by $h_0^{2^{n-1}-1}h_n$. This follows from [Ravenel 1986, Lemma 3.4.15 and Theorem 3.4.16]. \square

Proof of Proposition 8.1 Suppose that $x \in \pi_{127} \mathbb{S}_{(2)}$ were of \mathbb{F}_2 -Adams filtration 49. Then it would have to be detected on the E_2 -page by $h_0^{48}h_7$ by Lemma 8.2, so that $2^{15}x$ is detected by $h_0^{63}h_7$.

By Lemma 8.4, this implies that $2^{15}x$ lies in the image of J . Since the image of J is a summand of $\pi_{127} \mathbb{S}_{(2)}$ [Adams 1966a; Quillen 1971], this implies that the image of J must contain an element of order 2^{16} , which contradicts the fact that it is a cyclic group of order 2^8 [Adams 1966a; Quillen 1971]. \square

9 Further applications

9.1 Stein-fillable homotopy spheres

In this section, we complete the enumeration of odd-dimensional homotopy spheres which admit a Stein-fillable contact structure, answering a question of Eliashberg [2012, 3.8]. Bowden et al. [2014,

Conjecture 5.9] have constructed Stein-fillable contact structures on homotopy spheres which bound parallelizable manifolds and conjectured that these are all of them. We show that their conjecture is true in dimensions other than 23. In dimension 23, we provide a counterexample and analyze the extent to which it fails. This result is new in dimensions $n = 23$ and $39 \leq n \leq 247$ congruent to 7 modulo 8; see [Bowden et al. 2014, Theorem 5.4], Remark 9.2 and [Burklund et al. 2023, Theorem 3.1] for the other cases.

Theorem 9.1 *Let $q \neq 11$ be a positive integer. A homotopy sphere $\Sigma \in \Theta_{2q+1}$ admits a Stein-fillable contact structure if and only if $\Sigma \in bP_{2q+2}$, ie if and only if the class $[\Sigma]$ of Σ in $\text{coker}(J)_{2q+1}$ is zero.*

On the other hand, a homotopy sphere $\Sigma \in \Theta_{23}$ admits a Stein-fillable contact structure if and only if

$$[\Sigma] \in \{0, \eta^3 \bar{\kappa}\} \subset \text{coker}(J)_{23}.$$

Proof As in [Bowden et al. 2014], let $A_{2q+2}^{U(q+1)}$ denote the group of almost closed, q -connected, almost complex $(2q+2)$ -manifolds, modulo q -connected almost complex cobordisms restricting to h -cobordisms on the boundary. We then have the sequence of maps

$$A_{2q+2}^{U(q+1)} \rightarrow A_{2q+2}^{(q+1)} \xrightarrow{\partial} \Theta_{2q+1} \rightarrow \text{coker}(J)_{2q+1},$$

and, as in [Bowden et al. 2014, Proof of Theorem 5.4], we see that an exotic sphere $\Sigma \in \Theta_{2q+1}$ admits a Stein-fillable contact structure if and only if the class $[\Sigma] \in \text{coker}(J)_{2q+1}$ is in the image of the composite map from $A_{2q+2}^{U(q+1)}$.

By [Bowden et al. 2014, Theorem 5.4], it suffices to deal with the case when $q \equiv 3 \pmod{4}$ or $q = 9$. In the case $q = 9$, Schultz [1972, Theorem 3.4(iii)] states that the map $A_{2q+2}^{(q+1)} \rightarrow \text{coker}(J)_{2q+1}$ vanishes. In the case when $q \equiv 3 \pmod{4}$ and $q \neq 11$, this map is zero by Theorem 1.4.

On the other hand, when $q = 11$, we note that, by the argument in [Bowden et al. 2014, page 28], the map $A_{2q+2}^{U(q+1)} \rightarrow A_{2q+2}^{(q+1)}$ is surjective, as $\pi_{11}(U) \rightarrow \pi_{11}(SO)$ is an isomorphism. It follows that the image of the composite $A_{2q+2}^{U(q+1)} \rightarrow \text{coker}(J)_{2q+1}$ is equal to $\{0, \eta^3 \bar{\kappa}\}$ by Theorem 1.4, as desired. \square

Remark 9.2 Case (1) of [Bowden et al. 2014, Theorem 5.4] assumes that $q \neq 9$ in the $q \equiv 1 \pmod{8}$ case. This is because [Schultz 1972, Corollary 3.2] does not cover this case. However, this case is in fact covered in [Schultz 1972, Theorem 3.4(iii)], so this hypothesis may be removed.

In particular, we obtain a counterexample to [Bowden et al. 2014, Conjecture 5.9]:

Corollary 9.3 *There exists a 23-dimensional homotopy sphere which admits a Stein-fillable contact structure but does not bound a parallelizable manifold.*

9.2 Mapping class groups

Our results also have application to the computation of mapping class groups of highly connected manifolds. Indeed, this was the original motivation of Galatius and Randal-Williams [2016] in making their conjecture.

Definition 9.4 Let $W_g^{2n} = \#^g(S^n \times S^n)$ denote the connected sum of g copies of $S^n \times S^n$. We further let

$$\Gamma_g^n = \pi_0 \text{Diff}^+(W_g^{2n})$$

denote the group of isotopy classes of orientation-preserving diffeomorphisms of W_g^{2n} .

Building on work on Kreck [1979], Krannich [2020] determined the group Γ_g^n for $n \geq 3$ odd and $g \geq 1$ in terms of two extensions. In the case $n \equiv 3 \pmod{4}$, his answer is phrased in terms of a certain exotic $(2n+1)$ -sphere Σ_Q , which is the boundary of the manifold Q considered in Section 2. In Section 2, we computed Σ_Q for all n . Therefore, our results completely resolve the identity of the mysterious Σ_Q which appeared in Krannich’s work. We refer the interested reader to [Krannich 2020] for more details.

One consequence of Krannich’s results is a computation of the abelianization of Γ_g^n , extending and reproving an earlier result of Galatius and Randal-Williams [2016].¹² When combined with Theorem 1.4, Krannich [2020, Corollary E(i), page 4] implies the following result, which demonstrates the effect that our 23-dimensional counterexample to [Galatius and Randal-Williams 2016, Conjectures A and B] can have on the abelianization of the mapping class groups of highly connected manifolds:

Theorem 9.5 *Suppose that $n \geq 9$ is odd. Then, if $n \neq 11$ and $g \geq 3$, there is an isomorphism*

$$H_1(\Gamma_g^n) \cong \text{coker}(J)_{2n+1} \oplus \mathbb{Z}/4\mathbb{Z},$$

and if $n \neq 11$ and $g = 2$, we have

$$H_1(\Gamma_g^n) \cong \text{coker}(J)_{2n+1} \oplus (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}).$$

On the other hand, if $n = 11$ and $g \geq 3$, there is an isomorphism

$$H_1(\Gamma_g^n) \cong (\text{coker}(J)_{23}/\eta^3\bar{\kappa}) \oplus \mathbb{Z}/4\mathbb{Z} \cong (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/4\mathbb{Z},$$

and if $n = 11$ and $g = 2$, we have

$$H_1(\Gamma_g^n) \cong \text{coker}(J)_{2n+1}/\eta^3\bar{\kappa} \oplus (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}).$$

In particular, one consequence of the existence of the exceptional 23-dimensional counterexample to [Galatius and Randal-Williams 2016, Conjectures A and B] is to make the abelianization of Γ_g^{11} smaller than would otherwise be expected.

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¹²Krannich [2020] also determines the abelianization of Γ_g^n for even $n \geq 4$ and $g \geq 1$.

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