Dual structures on Coxeter and Artin groups of rank three

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We extend the theory of dual Coxeter and Artin groups to all rank-three Coxeter systems, beyond the previously studied spherical and affine cases. Using geometric, combinatorial, and topological techniques, we show that rank-three noncrossing partition posets are EL–shellable lattices and give rise to Garside groups isomorphic to the associated standard Artin groups. Within this framework, we prove the $K(\pi, 1)$ conjecture, the triviality of the center, and the solubility of the word problem for rank-three Artin groups. Some of our constructions apply to general Artin groups; we hope they will help develop complete solutions to the $K(\pi, 1)$ conjecture and other open problems in the area.

1 Introduction

The famous $K(\pi, 1)$ conjecture, dating back to the 1960s and due to Arnold, Pham, and Thom, states that the orbit configuration space of Artin groups is an Eilenberg–Mac Lane space (or $K(\pi, 1)$ space). This conjecture was proved fifty years ago by Deligne [1972] for spherical Artin groups. Recently, the second and third authors proved this conjecture for the next important class of Artin groups, namely those of affine type [Paolini and Salvetti 2021]. Between these two results, Dehornoy, Paris, and others developed the theory of Garside structures [Bestvina 1999; Dehornoy and Paris 1999; Bessis 2003; Charney et al. 2004; Dehornoy et al. 2015], generalizing properties of the standard presentation of spherical Artin groups. Applying Garside theory to *dual* presentations (arising from noncrossing partition posets) was a key ingredient to solving the most important open problems on affine Artin groups: not only the $K(\pi, 1)$ conjecture [Paolini and Salvetti 2021], but also the word problem and the triviality of the center [McCammond and Sulway 2017]. An outline of this "dual approach", inspired by [Paolini and Salvetti 2021] and articulated in [Paolini 2021], is given in Section 1.1.

We begin the extension of the dual approach beyond the affine case by tackling Artin groups associated with hyperbolic Coxeter systems of rank three (see Figure 1). While these Artin groups were already partially understood with different techniques [Hendriks 1985; Charney and Davis 1995; Chermak 1998], we are able to solve all important problems about them (including the ones previously solved) independently of prior work. We also answer all questions posed in [Paolini 2021] on the dual structure of rank-three Artin groups. Our main results are summarized in Section 1.2.
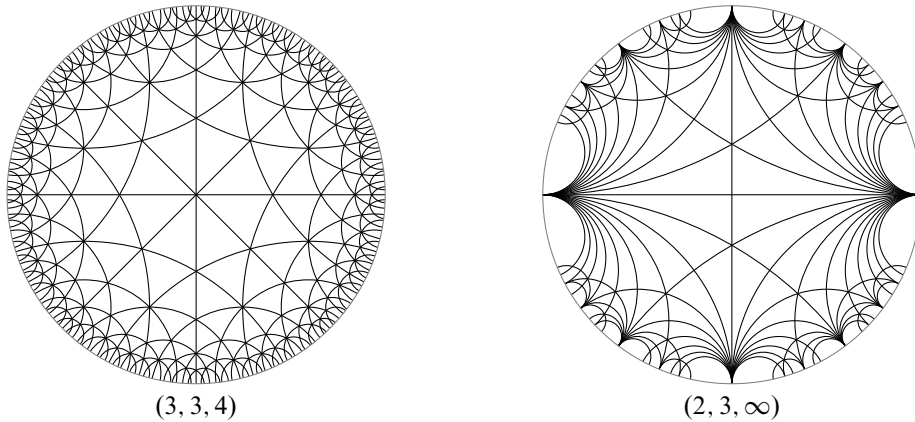
$(3, 3, 4)$ $(2, 3, \infty)$

Figure 1: Examples of arrangements of rank-three hyperbolic Coxeter groups in the Poincaré model. Each picture is captioned with the labels (m_1, m_2, m_3) of the corresponding Coxeter diagram. The hyperbolic plane is tiled by triangles with angles π/m_1, π/m_2, and π/m_3.

1.1 The dual approach

Let W be a finitely generated Coxeter group and G_W the associated Artin group. In the affine case, the proof of the $K(\pi, 1)$ conjecture given in [Paolini and Salvetti 2021] roughly goes as follows. First, the so-called *interval complex* K_W (see below) is shown to be a $K(\pi, 1)$ space. The next step is to identify a finite subcomplex $X'_W \subseteq K_W$ and to prove that it is homotopy equivalent to the orbit configuration space of G_W. Finally, using combinatorial methods (discrete Morse theory and lexicographic shellability), the complex K_W is shown to deformation retract onto X'_W.

The noncrossing partition poset $[1, w]$ is the interval between 1 and w in the (right) Cayley graph of W, using the set R of all reflections as generators. The interval complex K_W is a quotient of the order complex of $[1, w]$; it is a Δ–complex whose d–simplices correspond to sequences $\sigma = [x_1 | \cdots | x_d]$ which are part of a reduced factorization of w. One also introduces a *dual* Artin group W_w associated with W and the chosen Coxeter element w, defined as the fundamental group of K_W. Combinatorial properties of $[1, w]$ reflect topological properties of K_W: most significantly, if $[1, w]$ is a lattice then K_W is a $K(\pi, 1)$.

In addition to the $K(\pi, 1)$ conjecture, several interesting questions can be asked in general and have been answered in the affine case. For example: Is the dual Artin group W_w always naturally isomorphic to the standard one? Is it a Garside group? Is the word problem solvable? And, from the combinatorial point of view: Is the noncrossing partition poset $[1, w]$ a lattice? Is it EL–shellable? We refer to [Paolini 2021] for a more detailed discussion on the dual approach and the several questions related to it.

1.2 Contributions

By studying the geometry and combinatorics of the noncrossing partition poset $[1, w]$, we answer all previous questions (and more) when W is a Coxeter group of rank three, ie generated by three reflections. These groups are also known as *triangle groups*. They are all of hyperbolic type in the sense of
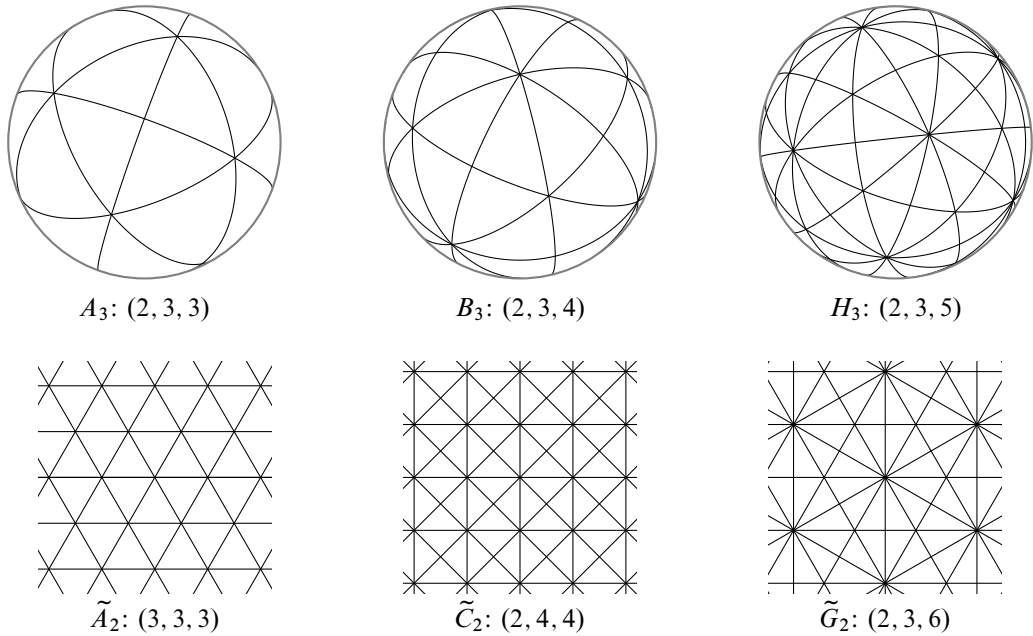
Figure 2: Arrangements of irreducible spherical and affine Coxeter groups of rank three. Each picture is captioned with the standard name (following the classification of spherical and affine Coxeter groups) and the labels (m_1, m_2, m_3) of the corresponding Coxeter diagram. In each case, the sphere or plane is tiled by triangles with angles π/m_1, π/m_2, and π/m_3. The Coxeter groups of the spherical cases (top three) are the symmetry groups of the five platonic solids (tetrahedron, cube/octahedron, and dodecahedron/icosahedron, respectively).

[Humphreys 1990] (Figure 1), except for a finite list of spherical or affine groups that are already well understood (Figure 2).

Going from affine to hyperbolic groups introduces new challenges. For example, a geometric characterization of the elements of $[1, w]$ seems difficult to achieve (in the affine case, such a characterization was obtained by McCammond [2015]). The rank-three case is also peculiar because $[1, w]$ turns out to be a lattice, thus giving rise to a Garside structure on the corresponding dual Artin group. The lattice property does not hold in general, even in the affine case [McCammond 2015, Theorem A]. Aside from the lattice property, we expect many of the results and techniques we develop to extend to hyperbolic Coxeter groups of arbitrary rank. The following are our main results:

Main theorem 1 (dual structure) *Let (W, S) be a Coxeter system of rank three and G_W the associated Artin group. Let $w \in W$ be any Coxeter element, and consider the associated noncrossing partition poset $[1, w]$.*

(i) *$[1, w]$ is a lattice.*

(ii) *$[1, w]$ is EL–shellable.*

(iii) *Every element $u \in [1, w]$ is a Coxeter element for the subgroup generated by the reflections $\leq u$.*

(iv) *The dual Artin group associated with $[1, w]$ is naturally isomorphic to G_W.*

Main theorem 2 (Artin groups) *Let G_W be an Artin group of rank three.*

(i) *G_W is a Garside group.*

(ii) *The $K(\pi, 1)$ conjecture holds for G_W.*

(iii) *The word problem for G_W is solvable.*

(iv) *The center of G_W is trivial unless W is finite.*

Some of the claims in Main theorem 2 have been proved elsewhere by completely different methods, but our setup allows us to obtain particularly succinct proofs for all of them (see Section 7). Specifically, the $K(\pi, 1)$ conjecture [Hendriks 1985; Charney and Davis 1995] and the word problem [Chermak 1998] were already known for rank-three Artin groups. During the preparation of this paper, a preprint appeared showing that the $K(\pi, 1)$ conjecture implies the triviality of the center [Jankiewicz and Schreve 2023] for general Artin groups. All other results are completely novel.

In order to prove the theorems above, we further develop the dual approach by providing new constructions that hold for general Artin groups. In particular, we introduce new subcomplexes of the interval complex K_W and propose a general strategy to deformation retract K_W onto X'_W (among other things, this would imply the isomorphism between standard and dual Artin groups).

Structure In Section 2, we give all definitions and constructions needed later: classical models of the hyperbolic plane; some standard tools of combinatorial topology; interval groups and their relation with Garside structures; Coxeter groups, (standard and dual) Artin groups, their orbit configuration space and the statement of the $K(\pi, 1)$ conjecture; and the definition of the subcomplex X'_W, homotopy equivalent to the orbit configuration space.

In Section 3, we introduce Coxeter elements of rank-three hyperbolic groups and their Coxeter axes. The main result there is that, if w is a Coxeter element, every element u in the interval $[1, w]$ is a Coxeter element for the subgroup of W generated by all reflections below u (Theorem 3.10).

In Section 4, we prove the lattice property for $[1, w]$, which implies that the corresponding dual Artin group is a Garside group. Then we introduce the *axial ordering* on the set of reflections in $[1, w]$. The main result (Theorem 4.15) is that this ordering induces an EL–labeling of $[1, w]$.

In Section 5, we introduce a sequence of subcomplexes of the interval complex K_W, for an arbitrary Coxeter group W. These include the already mentioned subcomplex X'_W (homotopy equivalent to the orbit configuration space) as well as new intermediate subcomplexes between X'_W and K_W. We hypothesize that a deformation retraction $K_W \searrow X'_W$ can be constructed more easily by collapsing each intermediate subcomplex to the next one, as outlined in Figure 10.

In Section 6, we implement the general program of Section 5 in the rank-three case, proving that K_W deformation retracts onto X'_W. In Section 7, we deduce the following consequences: rank-three dual and standard Artin groups are isomorphic (Theorem 7.1), they satisfy the $K(\pi, 1)$ conjecture (Theorem 7.2), they are Garside groups (Theorem 7.3), they have a solvable word problem (Theorem 7.4), and the nonspherical ones have a trivial center (Theorem 7.5).

2 Preliminaries

2.1 The hyperbolic plane

Denote by $\mathbb{H}^2$ the abstract hyperbolic plane and by $\partial\mathbb{H}^2$ the space at infinity. We refer to points in $\partial\mathbb{H}^2$ as *points at infinity* or *ideal points*. Throughout this paper, we are going to use different models for the hyperbolic plane: the hyperboloid model, the half-plane model, the Poincaré model, and the Klein model. We refer to [Cannon et al. 1997; Benedetti and Petronio 1992] for their definitions and the relations between them. The hyperboloid and Klein models are directly linked to the geometry of hyperbolic Coxeter groups acting on the Tits cone (see Section 2.4). However, we often find the half-plane model well-suited for explicit computations with isometries (as we see already in Section 2.1.3). Unless otherwise stated, the figures are drawn in the Poincaré model; both the half-plane and the Poincaré models are conformal (angles are preserved).

2.1.1 Isometries of the hyperbolic plane Every isometry of the hyperbolic plane $\mathbb{H}^2$ can be written as a product of three or fewer reflections and the *reflection length* of an isometry is the minimal number of reflections needed. Isometries can be classified as follows:

- the identity (reflection length 0),
- reflections with respect to a line (reflection length 1),
- rotations around a point in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ (reflection length 2),
- translations along a line (reflection length 2) called the *axis* of the translation (or the *translation axis*),
- glide reflections, ie products of a reflection and a translation along the same line (reflection length 3) called the *axis* of the glide reflection.

The identity, reflections, and rotations around points in $\mathbb{H}^2$ are *elliptic* isometries (they fix at least one point in $\mathbb{H}^2$). Rotations around points in $\partial\mathbb{H}^2$ are *parabolic* isometries (they fix exactly one point in $\partial\mathbb{H}^2$

and no point in $\mathbb{H}^2$). Translations and glide reflections are *hyperbolic* isometries (they fix exactly two points in $\partial\mathbb{H}^2$, namely the endpoints of their axes, and fix no point in $\mathbb{H}^2$).

For an elliptic or parabolic isometry u, denote by $\mathrm{Fix}(u) \subseteq \mathbb{H}^2 \cup \partial\mathbb{H}^2$ the set of its fixed points (also called its *fixed set*). If u is a hyperbolic isometry, then its axis is the set of points $x \in \mathbb{H}^2$ that minimize the distance $d(x, u(x))$. For this reason, the axis of u is often also called its *min-set* and we denote it by $\mathrm{Min}(u)$. The distance $d(x, u(x))$ for any $x \in \mathrm{Min}(u)$ is called the *translation length* of u. The axis of a hyperbolic isometry is naturally oriented according to the translation direction.

2.1.2 Isometries in the hyperboloid model

Denote by

$$L = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 - x_3^2 = -1 \text{ and } x_3 > 0\} \subseteq \mathbb{R}^3$$

the hyperboloid model. Isometries in the hyperboloid model are restrictions of isometries of $\mathbb{R}^3$ with respect to the quadratic form $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$. Conversely, every isometry of the quadratic space $(\mathbb{R}^3, Q)$ that preserves L (as a set) restricts to an isometry of L. In particular, reflections of L are restrictions of reflections of $(\mathbb{R}^3, Q)$ with respect to linear (2–dimensional) planes that intersect L.

Given an isometry in the hyperboloid model (or an isometry of the abstract hyperbolic plane $\mathbb{H}^2$), we refer to its *spectral radius* as the spectral radius of its extension to an isometry of $(\mathbb{R}^3, Q)$. The spectral radius of an elliptic or parabolic isometry is always 1, whereas the spectral radius of a hyperbolic isometry is e^μ where μ is the translation length of u [McMullen 2002, Corollary 3.5].

The *moved set* of an isometry u of $(\mathbb{R}^3, Q)$ is defined as $\mathrm{Mov}(u) = \mathrm{im}(u - \mathrm{id}) \subseteq \mathbb{R}^3$. We also define the moved set of an isometry of the hyperbolic plane (viewed as an isometry in the hyperboloid model) as the moved set of its extension to an isometry of $(\mathbb{R}^3, Q)$. The reflection length of an isometry of $\mathbb{H}^2$ is equal to the dimension of its moved set [McCammond and Paolini 2022, Theorem 5.2].

2.1.3 Isometries in the half-plane model

Writing the half-plane model as $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, the group of isometries is given by $\mathrm{PGL}(2, \mathbb{R})$ [Bridson and Haefliger 1999, I.6.14]. An element of $\mathrm{PGL}(2, \mathbb{R})$ acts on H as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \begin{cases} (az + b)/(cz + d) & \text{if the determinant is positive,} \\ (a\bar{z} + b)/(c\bar{z} + d) & \text{if the determinant is negative.} \end{cases}$$

If we denote by $i\mathbb{R}_+$ the positive imaginary line in H, the matrices

$$(1) \qquad\qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } \lambda > 1$$

correspond to the reflection with respect to $i\mathbb{R}_+$ and to the translations in the positive direction of the axis $i\mathbb{R}_+$, respectively. Every reflection and every translation in H is conjugate in $\mathrm{PGL}(2, \mathbb{R})$ to one of the matrices above. In particular, if we consider the line l in H that is represented by a Euclidean semicircle centered in $\alpha \in \mathbb{R}$ and with radius ρ, the reflection with respect to l is given by the matrix

$$(2) \qquad\qquad \begin{pmatrix} \alpha & \rho^2 - \alpha^2 \\ 1 & -\alpha \end{pmatrix}$$

(this can be computed by conjugating the reflection with respect to $i\mathbb{R}_+$ with an isometry that sends $i\mathbb{R}_+$ to l). By multiplying the two matrices in (1), we see that every glide reflection is conjugate to

$$(3) \qquad \begin{pmatrix} \lambda & 0 \\ 0 & -1 \end{pmatrix}$$

for some $\lambda > 1$. If u is a translation as in (1) or a glide reflection as in (3), then its translation length is $\log \lambda$ and its spectral radius is λ. More generally, the spectral radius of an isometry given as a matrix $M \in \mathrm{PGL}(2, \mathbb{R})$ is equal to the absolute value of the ratio between the higher and the lower eigenvalue of M.

For later reference we now prove a general lemma about the composition of glide reflections and reflections:

Lemma 2.1 *Let w be a glide reflection and let r be a reflection whose fixed line does not meet the axis of w (not even at infinity). Then wr is a translation and its (oriented) translation axis meets the (oriented) axis of w with an angle less than $\frac{1}{2}\pi$. In addition, the (oriented) translation axis of wr intersects $\mathrm{Fix}(r)$ before the axis of w. The same applies to rw, except that the (oriented) translation axis of rw intersects $\mathrm{Fix}(r)$ after the axis of w.*

Proof Assume without loss of generality that w is of the form (3) and write r in the form (2). Since $\mathrm{Fix}(r)$ does not intersect the imaginary axis, we have $0 < \rho < |\alpha|$.

The matrix associated with the composition wr is

$$\begin{pmatrix} \lambda\alpha & \lambda(\rho^2 - \alpha^2) \\ -1 & \alpha \end{pmatrix}.$$

Every point $z \in \mathrm{Fix}(wr)$ satisfies $z^2 + (\lambda - 1)\alpha z + \lambda(\rho^2 - \alpha^2) = 0$. This equation has two distinct real roots z_1 and z_2 because $(\lambda + 1)|\alpha| > 2\sqrt{\lambda}\rho$ (recall that $|\alpha| > \rho$, and $\lambda + 1 > 2\sqrt{\lambda}$ for all $\lambda > 1$). Thus wr is a translation whose axis is represented by a Euclidean circle γ that meets the real line at z_1 and z_2. Since $z_1 z_2 < 0$, we have that γ meets the (upwardly oriented) imaginary axis. After relabeling, we can assume that the translation axis is oriented from z_1 to z_2. The situation is depicted in Figure 3.
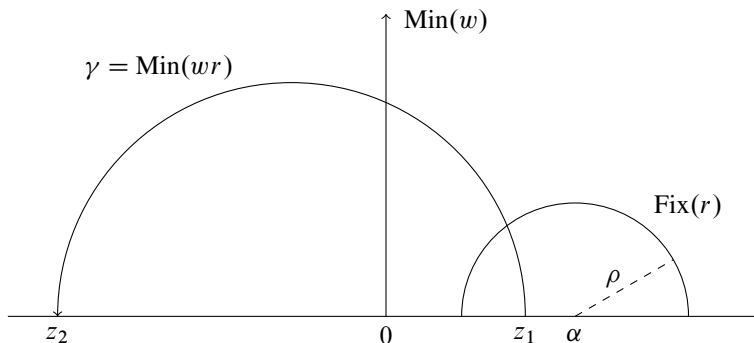

Figure 3: Proof of Lemma 2.1 (in the half-plane model).

Now $\frac{1}{2}(z_1 + z_2)$ (the center of γ) has the same sign as $-\alpha$, which has the same sign as $\lambda(\rho^2 - \alpha^2)/\alpha$ (the image of 0 under wr). This has two consequences. First, 0 is between z_1 and the center of γ. Thus, the (oriented) axis of w and the (oriented) axis of wr intersect at an angle less than $\frac{1}{2}\pi$. Second, the sign of z_1 is the same as the sign of α. Thus, if the axis of wr intersects Fix(r), it does so before intersecting the axis of w. It remains to show that γ intersects Fix(r). For this, we show that exactly one of z_1 and z_2 is between $\alpha - \rho$ and $\alpha + \rho$:

$$(z_1 - (\alpha + \rho))(z_1 - (\alpha - \rho))(z_2 - (\alpha + \rho))(z_2 - (\alpha - \rho)) = (\rho^2 - \alpha^2)(\lambda + 1)^2 \rho^2 < 0.$$

The claim about rw follows with an analogous computation, the difference being that $z_1 + z_2$ has the same sign as α.   □

Remark 2.2 In the setup of Lemma 2.1, suppose that Fix(r) meets the axis of w in 0 (a point at infinity). A similar analysis as in the previous proof (with $|\alpha| = \rho$) yields that wr is again a translation, but the translation axis of wr meets the axis of w in 0 (with an angle of 0). Both axes are oriented outwards from 0.

2.2 Combinatorial topology

A recurring theme of this paper is the relationship between the combinatorial properties of partially ordered sets (in turn related to groups via Garside theory, see Section 2.3) and the topological features of certain canonically associated spaces.

2.2.1 Generalities on posets We review some basic terminology, mainly following [Stanley 1986]. Let $(P, \leq)$ be a partially ordered set (a *poset*). We call P *bounded* if it has a unique maximal element and a unique minimal element. Any two elements $p, q \in P$ define an *interval* $[p, q] := \{u \in P \mid p \leq u \leq q\}$. Write $p \lessdot q$ if $[p, q]$ has cardinality 2; in this case we say that $[p, q]$ is a poset cover. The *Hasse diagram* of P is the graph whose vertices are the elements of P and whose edges $E(P)$ are all poset covers in P.

A *chain* in P is any totally ordered subset, ie any $C \subseteq P$ such that $c \leq c'$ or $c \geq c'$ for all $c, c' \in C$. The length of a chain is defined to be one less than its cardinality. The length of the poset P is the supremum of the lengths of all chains in P. The poset P is *chain-finite* if every chain in P has a finite length. A chain-finite poset P is called *graded* if any two maximal chains within the same interval have equal length. Equivalently, P admits a *rank function* rk: $P \to \mathbb{Z}$ such that $p \lessdot q$ implies rk(q) − rk$(p) = 1$.

A poset P is a *lattice* if every pair of elements $p_1, p_2 \in P$ has a unique maximal lower bound and a unique minimal upper bound. Note that every chain-finite lattice is bounded.

The *order complex* of a partially ordered set P is the abstract simplicial complex of all chains in P. Topological properties of posets are defined by referring to the topology of its order complex. (Recall that every abstract simplicial complex has a geometric realization as a geometric simplicial complex that is unique up to homeomorphism.) Every chain determines a simplex of dimension equal to the chain's length. Thus, the order complex of a chain-finite bounded poset has a finite dimension.

2.2.2 EL–labelings An edge-labeling of a poset P is a function $\lambda\colon E(P) \to \Lambda$, where Λ is any totally ordered set. Given such a labeling, every finite saturated chain $p_1 \lessdot \cdots \lessdot p_k$ in P is associated with a Λ–word

$$\lambda(p_1, p_2)\lambda(p_2, p_3)\cdots\lambda(p_{k-1}, p_k).$$

Using the ordering of Λ, chains of P can be compared using the lexicographical order of the associated words. A chain is called *increasing* if the associated word is strictly increasing.

An edge-labeling of a bounded poset P is called an *EL–labeling* if every interval of P has a unique increasing maximal chain and this chain lexicographically precedes all other maximal chains of the interval. This notion was introduced by Björner [1980; Björner and Wachs 1983] in view of its strong topological implications for the order complex of P. However, we use EL–labelings mainly as a combinatorial tool.

2.2.3 Acyclic matchings of posets and discrete Morse theory We briefly review Forman's [1998] discrete Morse theory for CW complexes in the poset-theoretical formulation introduced by Chari [2000] and later extended by Batzies [2002] to the case of infinite complexes.

A matching of a poset P is a matching of the associated Hasse diagram, ie a subset $\mathcal{M} \subseteq E(P)$ such that $m \cap m' = \varnothing$ whenever $m, m' \in \mathcal{M}$. An element of P is *critical* with respect to a matching $\mathcal{M}$ if it is not contained in any $m \in \mathcal{M}$. A matching defines an orientation of the edges of the Hasse diagram of P: an edge $[p, q] \in E(P)$ is oriented from p to q if $[p, q] \in \mathcal{M}$ and from q to p otherwise. The matching $\mathcal{M}$ is *acyclic* if the resulting oriented graph $H_\mathcal{M}$ has no directed cycles.

If P is a graded poset, then any directed cycle of $H_\mathcal{M}$ must alternate between edges in $\mathcal{M}$ and outside of $\mathcal{M}$. Indeed, if $p \to q$ is an oriented edge and rk is a rank function for P, then $\mathrm{rk}(q) - \mathrm{rk}(p)$ is equal to 1 if the edge is in $\mathcal{M}$ and -1 otherwise. Thus a matching $\mathcal{M}$ of a graded poset P is acyclic if and only if the Hasse diagram of P has no closed cycles that alternate between edges in $\mathcal{M}$ and outside of $\mathcal{M}$.

An acyclic matching $\mathcal{M}$ is called *proper* if, for every $p \in P$, there are only finitely many $q \in P$ that can be reached from p by a directed path in $H_\mathcal{M}$.

Let X be a finite-dimensional CW–complex. The *poset of cells* $\mathcal{F}(X)$ is the set of all (open) cells of X with the partial order given by inclusion of closure: $\tau \leq \sigma$ if $\bar{\tau} \subseteq \bar{\sigma}$. Recall that every cell $\sigma \subseteq X$ has a characteristic map $\Phi_\sigma\colon D^n \to X$, where D^n is the closed n–ball and $n = \dim(\sigma)$. The poset $\mathcal{F}(X)$ is chain-finite and graded with rank function given by $\mathrm{rk}(\sigma) := \dim(\sigma)$.

Let $\sigma \in \mathcal{F}(X)$ be a cell of dimension n. A *regular face* of σ is any cell $\tau \lessdot \sigma$ such that Φ_σ restricts to a homeomorphism $\Phi^{-1}(\tau) \to \tau$, and $\overline{\Phi_\sigma^{-1}(\tau)}$ is homeomorphic to an $(n-1)$–ball in D^n. A matching $\mathcal{M}$ of $\mathcal{F}(X)$ is called *regular* if $[\tau, \sigma] \in \mathcal{M}$ implies that τ is a regular face of σ.

Theorem 2.3 [Forman 1998; Chari 2000; Batzies 2002] *Let X be a finite-dimensional CW complex and let $\mathcal{M}$ be a proper and regular acyclic matching of $\mathcal{F}(X)$. Suppose that the set of critical elements of $\mathcal{M}$ forms a subcomplex Y of X. Then X deformation retracts onto Y (we write $X \searrow Y$). In particular, the inclusion $Y \hookrightarrow X$ is a homotopy equivalence.*

A useful tool for constructing acyclic matchings is given by the following well-known theorem:

Theorem 2.4 (patchwork theorem [Kozlov 2008, Theorem 11.10]) *Let $\eta \colon P \to Q$ be a poset map. For all $q \in Q$, assume there is an acyclic matching $\mathcal{M}_q \subseteq E(P)$ that involves only elements of the fiber $\eta^{-1}(q) \subseteq P$. Then the union of these matchings is an acyclic matching on P.*

2.3 Interval groups and Garside structures

Let G be a group with a generating set R not containing the identity of G and such that $R = R^{-1}$. Then G is partially ordered by assigning $x \leq y$ whenever $l(x) + l(x^{-1}y) = l(y)$, where $l \colon G \to \mathbb{N}$ denotes the length function induced by R. In other words, the relation $x \leq y$ holds if and only if there is a geodesic path between 1 and y passing through x, inside the right Cayley graph of G (with respect to R).

Fix an element $g \in G$ and consider the interval $[1, g] \subseteq G$, consisting (by definition) of all elements $x \in G$ such that $l(x) + l(x^{-1}g) = l(g)$. The interval $[1, g]$ is *balanced* if the set of elements $x \in G$ satisfying $l(x) + l(x^{-1}g) = l(g)$ coincides with the set of elements $x \in G$ satisfying $l(gx^{-1}) + l(x) = l(g)$. In other words, this condition requires that the interval $[1, g]$ inside the right Cayley graph (as we have defined it above) contains the same elements as the interval $[1, g]$ inside the left Cayley graph.

Assuming we have a balanced interval $[1, g]$, construct a new group G_g (called an *interval group*) and a CW complex K (called an *interval complex*) as follows. The interval complex K is a Δ–complex (in the sense of [Hatcher 2002]) having one d–dimensional simplex σ denoted by $[x_1|x_2|\cdots|x_d]$ for every d–tuple of elements $x_1, x_2, \ldots, x_d \in [1, g] \setminus \{1\}$ such that $l(x_1) + l(x_2) + \cdots + l(x_d) = l(x_1 x_2 \cdots x_d)$ and $x_1 x_2 \cdots x_d \in [1, g]$. The faces of $[x_1|x_2|\cdots|x_d]$ are given by $\partial_0(\sigma) = [x_2|\cdots|x_d]$, $\partial_i(\sigma) = [x_1|\cdots|x_i x_{i+1}|\cdots|x_d]$ for $i = 1, \ldots, d-1$, and $\partial_d(\sigma) = [x_1|\cdots|x_{d-1}]$. See [Paolini and Salvetti 2021, Definition 2.8] for more details. Note that K is a quotient of the order complex of $[1, g]$. The 0–cell $[\,]$ of K is not a regular face of the 1–cells; all other faces are regular.

Define the interval group G_g as the fundamental group of K. Then G_g has a presentation with a generator for each 1–cell $[x]$ of K and a relation $[x][y] = [xy]$ for each 2–cell $[x|y]$ of K. The main reason why we consider interval groups and interval complexes is the following result:

Theorem 2.5 [Bestvina 1999; Dehornoy and Paris 1999; Charney et al. 2004; McCammond 2005; Dehornoy et al. 2015] *If the interval $[1, g]$ is a balanced lattice, then G_g is a **Garside group** and the interval complex K is a classifying space for G_g. In addition, the word problem for G_g is solvable, provided that one can algorithmically check equality and compute meets and joins in $[1, g]$.*

Garside groups were introduced by Dehornoy and Paris [1999], building on previous work of Garside [1969]. We will not need the actual definition of a Garside group.

2.4 Coxeter groups

In this section, we outline some basics of the general theory of Coxeter groups. We refer to [Bourbaki 1972; Humphreys 1990] for a more thorough treatment.

A *Coxeter system* is a group W with a distinguished set of generators S such that

(4) $$W = \langle S \mid (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \text{ such that } m(s,t) \neq \infty \rangle,$$

for some function $m\colon S \times S \to \mathbb{N} \cup \{\infty\}$ satisfying $m(s,s') = 1$ if $s = s'$ and $m(s,s') \geq 2$ otherwise. The function m can be encoded in a *Coxeter diagram*, ie a graph on the vertex set S where two vertices s and s' are joined by an edge exactly when $m(s,s') \geq 3$. The edge is labeled with $m(s,s')$ when $m(s,s') \geq 4$. The group W is called a *Coxeter group*. For us, a Coxeter group W always implicitly carries with it a fixed generating set S which makes (W,S) a Coxeter system. For this reason, we often speak of properties of a Coxeter group W which depend on the Coxeter system (and not only on the group structure). For example, a Coxeter group W is called *irreducible* if the corresponding Coxeter diagram is connected.

The *parabolic subgroup* of W associated with a subset $I \subseteq S$ is the subgroup W_I of W that is generated by the elements of I. Following [McMullen 2002], we call an element $u \in W$ *essential* if it is not conjugated into any parabolic subgroup W_I with $I \neq S$. Let $n := |S|$. This quantity is the *rank* of the given Coxeter group.

2.4.1 Geometric representation and reflections The function m determines a symmetric bilinear form B on the vector space $\mathbb{R}^S$ defined on basis vectors as

$$B(e_s, e_{s'}) := -\cos(\pi/m(s,s')),$$

where we set $B(e_s, e_{s'}) = -1$ if $m(s,s') = \infty$. Now, to every $s \in S$ is naturally associated the *reflection*

$$r_s\colon x \mapsto x - 2B(x, e_s)e_s.$$

The assignment $s \mapsto r_s$ extends to a linear action of W on $\mathbb{R}^S$ that preserves the bilinear form B; see [Humphreys 1990, 5.3]. Denote by R the set of all elements of W that act as reflections (ie that fix a hyperplane and send some nonzero vector to its opposite). This is called the *set of reflections* of the Coxeter group and coincides with the set of all conjugates of elements of S. In particular, R generates W. The *absolute length* (also called *reflection length*) of an element $u \in W$ is the minimum length $l_R(u)$ of a reduced expression of u as a word in the generators R. The elements of S are called *simple reflections*.

2.4.2 Coxeter elements and dual Coxeter systems A *Coxeter element* of a Coxeter system (W,S) is any product of all elements of S in some order. Every Coxeter element has reflection length equal to $n = |S|$ (see [Ingalls and Thomas 2009, Lemma 3.8; Paolini and Salvetti 2021, Lemma 5.1]) and is essential [Paris 2007]. For any choice of a Coxeter element w of (W,S), the triple (W,R,w) is often called a *dual Coxeter system*. Associated with any dual Coxeter system is the corresponding poset of *noncrossing partitions*,[1] namely

$$[1, w] := \{u \in W \mid l_R(u) + l_R(u^{-1}w) = l_R(w)\},$$

partially ordered by

$$u \leq v \quad \text{if and only if} \quad l_R(u) + l_R(u^{-1}v) = l_R(v).$$

---

[1]The name refers to a well-known combinatorial interpretation of such elements in the case of the symmetric group; see for example [Armstrong 2009].

As the notation suggests, the poset $[1, w]$ coincides with the interval between the identity 1 and the Coxeter element w in the right Cayley graph of W with respect to the generating set R. In particular, it contains 1 and w as its unique minimal and maximal elements. Note that $[1, w]$ is balanced because the generating set R is closed under conjugation. The noncrossing partition posets $[1, w]$ are strictly related to "dual Garside structures"; see Section 2.3.

2.4.3 Spherical, affine, and hyperbolic Coxeter groups

A Coxeter group W is called *spherical* if the bilinear form B (defined in Section 2.4.1) is positive definite; it is called *affine* if B is positive semidefinite but not positive definite. The spherical case occurs precisely when W is finite.

For simplicity, suppose now that B is nondegenerate. Then B induces a pairing, and hence an identification, between $\mathbb{R}^S$ and its dual V. In particular, we have an induced action of W on V. The generators $s \in S$ act as reflections with respect to the linear hyperplanes $\hat{H}_s := \{p \in V \mid B(p, e_s) = 0\}$ that bound a closed polyhedron $D \subseteq V$ (a simplicial cone). This polyhedron is a fundamental region for the action of W on V and the union of all W–translates of D is a convex cone I in V, the *Tits cone* [Humphreys 1990, Section 5.13]. The cone I is tiled by copies of D. This tiling corresponds to the subdivision of I determined by the collection of all reflection hyperplanes $\hat{H}_r$ for $r \in R$.

The Coxeter group W is called *hyperbolic* if the bilinear form B is nondegenerate of signature $(n-1, 1)$ and every vector p in the Tits cone satisfies $B(p, p) < 0$ [Humphreys 1990, Section 6.8]. If W is hyperbolic, then the action of W on V is determined by its restriction to the hyperboloid model of the hyperbolic space $\mathbb{H}^{n-1}$ given by all p with $B(p, p) = -1$, with the metric induced by B. Since the action of W preserves the metric, W acts by isometries of the hyperbolic space. Thus, hyperbolic Coxeter groups of rank n are a subclass of all discrete groups of isometries of $\mathbb{H}^{n-1}$ generated by reflections. Note that this class is strictly larger; see [Vinberg 1971].

Let W be a hyperbolic Coxeter group and, for all $r \in R$, let H_r be the intersection of $\hat{H}_r$ with the hyperboloid. The set $\{H_r\}_{r \in R}$ is the *reflection arrangement* of W. It is a locally finite set of hyperplanes of $\mathbb{H}^{n-1}$. The open cells of the induced subdivision of the hyperbolic space are the (open) *chambers* of the reflection arrangement. Chambers are naturally and bijectively labeled by elements of W, once we label the interior of D with the identity element.

The Coxeter graphs corresponding to spherical, affine, and hyperbolic Coxeter systems have been completely classified; see [Humphreys 1990, Chapter 6]. The construction of the reflection arrangement can be extended to all cases (see eg Figures 1 and 2).

2.5 Standard and dual Artin groups

For any Coxeter system (W, S), where W is presented as in (4), there is an associated Artin group defined as

$$(5) \qquad G_W = \langle S \mid \underbrace{stst\cdots}_{m(s,t) \text{ terms}} = \underbrace{tsts\cdots}_{m(s,t) \text{ terms}} \text{ for all } s, t \in S \text{ such that } m(s, t) \neq \infty \rangle.$$

An Artin group is called irreducible, spherical, affine, or hyperbolic if the corresponding Coxeter group is respectively irreducible, spherical (ie finite), affine, or hyperbolic.

Define the *configuration space* associated with W as

$$Y = (I \times I) \setminus \bigcup_{r \in R} \hat{H}_r \times \hat{H}_r.$$

Then W acts freely and properly discontinuously on Y, and the quotient space $Y_W = Y/W$ is the *orbit configuration space* associated with W. The fundamental group of Y_W is isomorphic to the Artin group G_W [Van der Lek 1983]. The orbit configuration space Y_W has the homotopy type of a CW complex (called the *Salvetti complex*) having a k–cell for each subset $T \subseteq S$ of cardinality k that generates a finite subgroup $W_T \subseteq W$ [Salvetti 1987, 1994]; see also [Paris 2014].

In full generality, Artin groups are not well understood, in the sense that there are very few known results that apply to all of them. The following are among the most important open problems on Artin groups: the word problem, determining the center, and solving the $K(\pi, 1)$ conjecture, due to Brieskorn, Arnold, Pham, and Thom.

Conjecture 2.6 ($K(\pi, 1)$ conjecture) The orbit configuration space Y_W is a classifying space for the corresponding Artin group G_W.

Among other things, the $K(\pi, 1)$ conjecture gives a way to compute the homology and cohomology of Artin groups and implies that Artin groups are torsion-free (a property that is also unknown in general). So far, the $K(\pi, 1)$ conjecture has been proved for spherical Artin groups (by Deligne [1972]), affine Artin groups (by the second and third authors [Paolini and Salvetti 2021]), 2–dimensional and FC–type Artin groups (by Charney and Davis [1995]). Special cases of these were previously proved by Fox and Neuwirth [1962], Brieskorn [1973], Okonek [1979], Hendriks [1985], and Callegaro, Moroni and Salvetti [Callegaro et al. 2010]. See [Paris 2014] for a survey on this problem (written before the full solution of the affine case) and [Paolini 2021] for an exposition of the "dual approach" introduced to solve the affine case, which we use and extend here. See also [Charney and Davis 1995; Godelle and Paris 2012] for an overview of open problems on Artin groups (mostly up to date, except for the affine case).

If W is a finite Coxeter group, then G_W is the interval group associated with the interval $[1, \delta] \subseteq W$, where we use S as the generating set of W and define δ as the longest element of W. The interval $[1, \delta]$ is a balanced lattice, and thus makes G_W a Garside group (by Theorem 2.5). This is known as the *standard Garside structure* on spherical Artin groups, introduced by Garside [1969] for braid groups and developed and studied by Brieskorn and Saito [1972], Deligne [1972], and others. The interval complex associated with $[1, \delta]$ is homotopy equivalent to the orbit configuration space Y_W, in accordance with the $K(\pi, 1)$ conjecture (this was explicitly proved in [Delucchi 2010]).

An alternative way to realize spherical Artin groups as Garside groups was introduced by Bessis [2003], following prior work of Birman, Ko, and Lee on braid groups [Birman et al. 1998]. Consider a noncrossing partition poset $[1, w]$ in a finite Coxeter group W. It turns out that $[1, w]$ is a lattice, and the corresponding

interval group is naturally isomorphic to the Artin group G_W. This is known as the *dual Garside structure* on spherical Artin groups. The interval complex associated with $[1, w]$ is therefore another model for the classifying space of the spherical Artin group G_W, with a combinatorial structure that is substantially different from the Salvetti complex X_W and the interval complex arising from the standard Garside structure.

Noncrossing partition posets in general Coxeter groups are not always lattices. Indeed, McCammond [2015] showed that the lattice property fails in most affine cases. Nevertheless, the dual structure (which is not necessarily Garside) proved useful to answer the most important open questions on affine Artin groups, such as the word problem, the center, and the $K(\pi, 1)$ conjecture [McCammond and Sulway 2017; Paolini and Salvetti 2021]. For all affine Coxeter groups W, McCammond and Sulway [2017] proved (among other things) that the interval group W_w associated with $[1, w]$ is isomorphic to the usual Artin group G_W; the second and third author proved that the interval complex K_W associated with $[1, w]$ is a classifying space and deformation retracts onto a finite subcomplex $X'_W \subseteq K_W$ which is homotopy equivalent to the orbit configuration space Y_W [Paolini and Salvetti 2021].

The subcomplex X'_W can be defined for an arbitrary Coxeter system and is always homotopy equivalent to Y_W. We review the definition of X'_W in Section 5. Therefore, proving that K_W deformation retracts onto X'_W implies the isomorphism $G_W \cong W_w$ between the standard and dual Artin groups. In addition, if the interval $[1, w]$ is a lattice, then a deformation retraction $K_W \cong X'_W$ yields a Garside structure on G_W and implies the $K(\pi, 1)$ conjecture. These implications are discussed more thoroughly in [Paolini 2021].

3   Coxeter elements in rank-three hyperbolic groups

Let (W, S) be a rank-three Coxeter system. Recall from [Humphreys 1990, Section 6.7] that, if the three labels $m(s, t)$ for $s \neq t$ are denoted by m_1, m_2, and m_3, then W is hyperbolic if and only if $1/m_1 + 1/m_2 + 1/m_3 < 1$ (set $1/m_i = 0$ if $m_i = \infty$). In particular, all irreducible rank-three Coxeter systems are hyperbolic except when the triple (m_1, m_2, m_3) takes the values $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, $(2, 3, 6)$, $(2, 4, 4)$, or $(3, 3, 3)$, up to permutations. These special cases are either spherical or affine and the corresponding arrangements are shown in Figure 2. All questions we consider have been solved already in the spherical and affine cases, so from now on, *assume that the Coxeter system (W, S) is hyperbolic*, thus acting by isometries on the hyperbolic plane $\mathbb{H}^2$. Two examples are shown in Figure 1.

3.1   Coxeter elements and their axes

Let $S = \{a, b, c\}$ and fix a Coxeter element $w = abc$. Note that w has reflection length equal to 3 (both in W and in the group of isometries of $\mathbb{H}^2$), so it is a glide reflection (see Section 2.1).

Definition 3.1   The (oriented) axis of w is called the *Coxeter axis* and denoted by ℓ. An *axial chamber* is a chamber whose interior intersects the Coxeter axis. We think of the orientation of the Coxeter axis
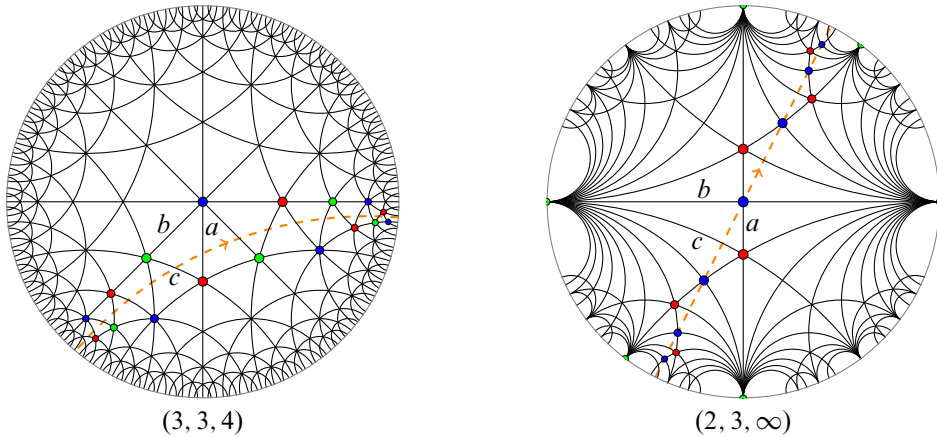
Figure 4: Reflection arrangements of Figure 1 with the axis of a Coxeter element $w = abc$ (orange dashed line). The axial vertices are colored based on their orbit under the action of the infinite cyclic group generated by w (see Lemma 3.5).

as defining the "positive" or "upward" direction. Accordingly, a point $p \in \ell$ (or an axial chamber C) is *above* another point $p' \in \ell$ (or another axial chamber C') if it is further along the Coxeter axis in the positive direction. Any vertex of an axial chamber is called an *axial vertex*; see Figure 4.

In the hyperboloid model $L \subseteq \mathbb{R}^3$, the Coxeter axis ℓ is the intersection between L and a linear plane $P \subseteq \mathbb{R}^3$ which we call the *Coxeter plane*. As noted in [Paolini 2021, Section 2], P is indeed the analog of the classical Coxeter plane of spherical Coxeter groups.

Let C be any chamber of the reflection arrangement. In particular, the closure $\overline{C}$ intersects every W–orbit in exactly one point. Denote by $\pi_C : \mathbb{H}^2 \to \overline{C}$ the projection defined by $\pi_C(x) := Wx \cap \overline{C}$ for every $x \in \mathbb{H}^2$. The following lemma describes the location of the Coxeter axis ℓ:

Lemma 3.2 *For every point $x \in \ell$, the image of the segment $[x, w(x)] \subseteq \ell$ under π_C is the shortest loop that touches the three walls of C and coincides with the orthic triangle of C (see Figure 5).*

Proof The orthic triangle of a triangle $C \subseteq \mathbb{H}^2$ is the (only) shortest loop that touches all three walls of C. This can be proved with the same argument as the one used in [Rademacher and Toeplitz 1957, Chapter 6] for the Euclidean case.

By [McMullen 2002, Theorem 4.1 and Proposition 4.2], there exists a Coxeter element $w' \in W$ such that $\pi_C([x', w'(x')])$ is the shortest loop that touches all three walls of C, where x' is any point on the axis of w'. Note that the length of any such loop is equal to the translation length μ of w'.

Every Coxeter element is conjugate in W to one of abc, bca, cab, cba, bac, or acb (where $\{a, b, c\}$ is a fixed set of simple reflections). The first three (abc, bca, and cab) can be obtained from each other via source-sink flips [McCammond 2015, Lemma 7.4], and hence they are geometrically equivalent Coxeter
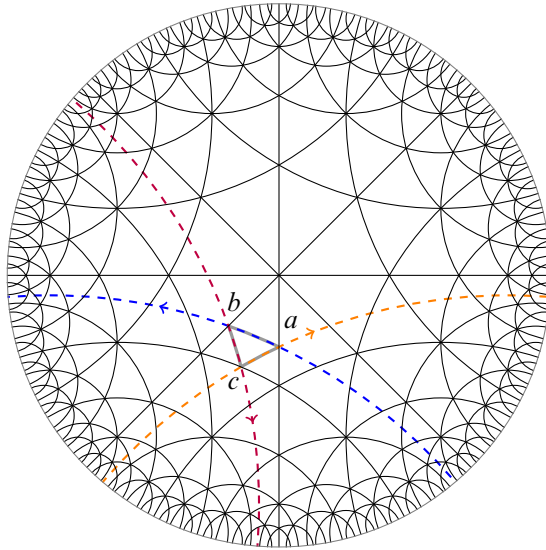
Figure 5: Axes of the Coxeter elements abc (in orange), bca (in blue), and cab (in purple). The orthic triangle of the chamber delimited by $\mathrm{Fix}(a)$, $\mathrm{Fix}(b)$, and $\mathrm{Fix}(c)$ is highlighted in gray.

elements and thus have the same translation length. The same holds for the last three (cba, bac, and acb). In addition, each of the first three is the inverse of one of the last three, so all Coxeter elements have translation lengths equal to μ. In particular, the loop $\gamma = \pi_C([x, w(x)])$ has length μ. Since w is essential, [McMullen 2002, Proposition 4.3] ensures that γ touches all three walls of C. Therefore γ is the shortest loop that touches the three walls of C. □

We now derive two useful corollaries: first, the Coxeter axis is not a reflection line (so axial chambers exist); second, every axial chamber induces a factorization of w.

Corollary 3.3 The Coxeter axis ℓ does not coincide with any reflection line of W. In addition, if two reflection lines intersect ℓ in the same point, then they are perpendicular.

Proof The orthic triangle of any chamber C touches every wall at exactly one point. In addition, it touches two walls at the same point if and only if the two walls are perpendicular (in this case, the orthic triangle is degenerate since two vertices coincide). The statement then follows from Lemma 3.2, since $\pi_C(\ell)$ is the orthic triangle of C. □

Corollary 3.4 Suppose that C is an axial chamber, with associated reflections s_1, s_2, and s_3 in the order the sides of C are touched by the loop $\pi_C([x, w(x)])$ for any $x \in \ell \cap C$. Then $w = s_1 s_2 s_3$.

Proof Let $x \in C \cap \ell$. Now $\pi_C(x) = \pi_C(w(x))$, and no other point in $\pi_C(\ell)$ is in the same W–orbit. By Lemma 3.2, the segment of $[x, w(x)] \subseteq \ell$ intersects exactly three reflection lines, and the corresponding reflections are s_1, $s_1 s_2 s_1$, and $s_1 s_2 s_3 s_2 s_1$. Therefore $w = (s_1 s_2 s_3 s_2 s_1)(s_1 s_2 s_1) s_1 = s_1 s_2 s_3$. □

We close this section with a geometric observation on axial vertices which is exemplified in Figure 4.

Lemma 3.5 *Let p be an axial vertex. Every axial chamber has exactly one vertex in $\{w^j(p) \mid j \in \mathbb{Z}\}$.*

Proof Let C be an axial chamber. Since $\overline{C}$ is a fundamental domain for the action of W on $\mathbb{H}^2$, at most one vertex of C belongs to the orbit $\{w^j(p) \mid j \in \mathbb{Z}\}$. In particular, the three vertices of C necessarily belong to three different orbits.

Let C' be another axial chamber. By induction on the number of axial chambers between C and C', we prove that the orbits of the three vertices of C are the same as the orbits of the three vertices of C'. It is enough to consider the case where C' is the axial chamber immediately above C. Denote by s_1, s_2, and s_3 the reflections with respect to the walls of C, ordered as in Corollary 3.4, so that $w = s_1 s_2 s_3$.

Case 1 Suppose that C and C' are separated by a single reflection line, $\mathrm{Fix}(s_1)$. Then $C' = s_1(C)$. The vertex q' of C' opposite to $\mathrm{Fix}(s_1)$ is equal to $s_1(q) = w(q)$ where q is the vertex of C opposite to $\mathrm{Fix}(s_1)$. The other two vertices are in common between C and C'; see for example Figure 4, left.

Case 2 Suppose that C and C' are separated by two reflection lines, $\mathrm{Fix}(s_1)$ and $\mathrm{Fix}(s_2)$, which are orthogonal by Corollary 3.3. Then $C' = s_1 s_2(C)$. The vertices q'_1 and q'_2 of C' opposite to $\mathrm{Fix}(s_1)$ and $\mathrm{Fix}(s_2)$ are equal to $s_1 s_2(q_1) = w(q_1)$ and $s_1 s_2(q_2) = w(q_2)$, where q_1 and q_2 are the vertices of C opposite to $\mathrm{Fix}(s_1)$ and $\mathrm{Fix}(s_2)$. The third vertex is in common between C and C'; see for example Figure 4, right.

In all cases, the vertices of C and the vertices of C' belong to the same three orbits. $\square$

3.2 Reflections

Let r be a reflection in the noncrossing partition poset $[1, w]$. By analogy with the affine case, we say that r is *vertical* if $\mathrm{Fix}(r)$ intersects the Coxeter axis ℓ and *horizontal* if it does not.

Remark 3.6 If r is a horizontal reflection, then $\mathrm{Fix}(r)$ does not intersect ℓ at infinity, because the distance between $\mathrm{Fix}(r)$ and ℓ is bounded away from 0. Indeed, we can fix a point $x \in \ell$ and find an ϵ–neighborhood N of the closed segment $[x, w(x)]$ which intersects no fixed lines of horizontal reflections; then the union of all $w^k(N)$ for $k \in \mathbb{Z}$ is an ϵ–neighborhood of ℓ with the same property.

Differently from the affine case, the roots corresponding to horizontal reflections are not necessarily orthogonal to the Coxeter plane. In fact, $[1, w]$ contains infinitely many horizontal reflections. However, we now show that wr is a rotation if r is vertical and a translation if r is horizontal (as in the affine case).

Lemma 3.7 *If r is vertical, then wr (resp. rw) is a rotation around an axial vertex p (possibly at infinity). Specifically, p is the vertex opposite to $\mathrm{Fix}(r)$ in the axial chamber immediately above (resp. below) $\mathrm{Fix}(r) \cap \ell$. If r is horizontal, then wr (resp. rw) is a translation whose axis meets the Coxeter axis ℓ with an angle $< \frac{1}{2}\pi$.*

Proof If r is horizontal, then the statement follows by Lemma 2.1. Suppose from now on that r is vertical. By Corollary 3.3, the intersection point $\mathrm{Fix}(r) \cap \ell$ is fixed by at most one other reflection $r' \in W$ and, if this happens, the two reflection lines are perpendicular. Therefore, the axial chamber C immediately above $\mathrm{Fix}(r) \cap \ell$ has $\mathrm{Fix}(r)$ as one of its walls. By Corollary 3.4, the walls of C yield a factorization $w = r_1 r_2 r$. Therefore $wr = r_1 r_2$ is a rotation around the vertex $\mathrm{Fix}(r_1) \cap \mathrm{Fix}(r_2)$ of C opposite to $\mathrm{Fix}(r)$. Note that this vertex might be at infinity (if there is no relation between r_1 and r_2) and in such case, wr is a parabolic isometry. □

3.3 Rotations and translations

Next, we give further insights on rank-two elements of the noncrossing partition poset $[1, w]$, namely, rotations and translations.

Lemma 3.8 (rotations) *Let $u \in [1, w]$ be a rotation around a point $p \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$. Then u is a Coxeter element for the parabolic subgroup $W_u \subseteq W$ that fixes p. In addition, a point $p \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$ is the fixed point of a rotation $u \in [1, w]$ if and only if p is an axial vertex.*

Proof Let $r = wu^{-1}$ be the left complement of u. Then r is a vertical reflection by Lemma 3.7 and p is an axial vertex. Let C be the axial chamber immediately below $\mathrm{Fix}(r)$. Note that $\mathrm{Fix}(r)$ is a wall of C by Corollary 3.3. By Corollary 3.4 there is a factorization $w = r r_1 r_2$ associated with the walls of C, and r comes first. Therefore $u = r_1 r_2$. We finish the proof of the first part of the statement by noting that $\{r_1, r_2\}$ is a set of simple reflections for the parabolic subgroup W_u that fixes p.

It remains to show that every axial vertex p is the fixed point of a rotation $u \in [1, w]$. Let C be an axial chamber having p as one of its vertices. By Corollary 3.4 there is a factorization $w = s_1 s_2 s_3$ where s_1, s_2, and s_3 are the reflections with respect to the walls of C. The three rotations $s_1 s_2$, $s_2 s_3$, and $s_1 s_3$ are all in $[1, w]$, and one of them is a rotation around p. □

The case of translations (given by the following lemma) is less trivial. Our proof relies on the fact that Coxeter elements minimize the spectral radius among all essential elements of W.

Lemma 3.9 (translations) *Let $t \in [1, w]$ be a translation. A reflection $r \in W$ is in $[1, t]$ if and only if $\mathrm{Fix}(r)$ is orthogonal to $\mathrm{Min}(t)$. In addition, t is a Coxeter element for the type-$\widetilde{A}_1$ subgroup $W_t \subseteq W$ generated by the reflections below t.*

Proof The product of two reflections r and r' in the hyperbolic plane is a translation if and only if $\mathrm{Fix}(r)$ and $\mathrm{Fix}(r')$ do not meet (not even at infinity); when $t = rr'$ is a translation, the translation axis $\mathrm{Min}(t)$ is orthogonal to both $\mathrm{Fix}(r)$ and $\mathrm{Fix}(r')$. In particular, if $t \in [1, w]$ is a translation as in the statement, $\mathrm{Fix}(r)$ is orthogonal to $\mathrm{Min}(t)$ for any reflection $r \in [1, t]$. Conversely, if $r \in W$ is a reflection such that $\mathrm{Fix}(r)$ is orthogonal to $\mathrm{Min}(t)$, then $r' = rt \in W$ is also a reflection and thus $r, r' \in [1, t]$. This proves the first part of the statement.

The reflections below t form an infinite discrete sequence $\dots, r_{-1}, r_0, r_1, r_2, \dots$ ordered according to the positions of $\mathrm{Fix}(r_i) \cap \mathrm{Min}(t)$ along the oriented translation axis $\mathrm{Min}(t)$. We need to prove that $t = r_{i+1} r_i$ for any (or equivalently all) $i \in \mathbb{Z}$. Suppose that $t = r_j r_i$ for some $j > i$.

Let r be the left complement of t, so that $w = rt$. Let t_λ be the translation with the same translation axis as t but with spectral radius changed to an arbitrary $\lambda > 1$ (ie the translation length is $\log \lambda$). Up to a change of coordinates in the half-plane model, we have

$$t_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

Writing the reflection r as in (2), we can express rt_λ as

$$(6) \qquad\qquad rt_\lambda = \begin{pmatrix} \lambda\alpha & \rho^2 - \alpha^2 \\ \lambda & -\alpha \end{pmatrix}$$

for some $\rho > |\alpha| > 0$. This is an isometry of the hyperbolic plane with an odd reflection length. It is not a reflection because $\mathrm{Fix}(r)$ is not orthogonal to the translation axis of t_λ, so it is a glide reflection. The spectral radius of rt_λ is given by the absolute value of the ratio of the two eigenvalues of the matrix (6). The characteristic polynomial of (6) is $t^2 - (\lambda - 1)\alpha t - \lambda\rho^2$. The absolute value of the ratio between the larger and the smaller eigenvalue is

$$\left(\frac{\sqrt{(\lambda - 1)^2 + 4\lambda\beta} + (\lambda - 1)}{2\sqrt{\lambda\beta}} \right)^2,$$

where $\beta = \rho^2/\alpha^2 > 1$. This quantity is strictly increasing in λ for $\lambda \geq 1$ (and it is equal to 1 for the degenerate case $\lambda = 1$ where t_λ becomes the identity). Thus the spectral radius of rt_λ increases with the translation length of t_λ.

Recall that $t = r_j r_i$ and suppose for the sake of contradiction that $j > i + 1$. Then the translation length of $t' = r_{i+1} r_i$ is strictly smaller than the translation length of t. Therefore, the spectral radius of $w' = rt'$ is strictly smaller than the spectral radius of $w = rt$ (and strictly greater than 1). As noted in the proof of Lemma 3.2, all the Coxeter elements of W have the same translation length and thus the same spectral radius. We reach a contradiction because w' is essential and, by [McMullen 2002, Theorem 4.1], the Coxeter elements minimize the spectral radius among all essential elements of W. $\qquad\square$

The previous two lemmas yield the following analog of [Paolini and Salvetti 2021, Theorem 3.22]:

Theorem 3.10 *For every element $u \in [1, w]$, we have that u is a Coxeter element for the subgroup of W generated by the reflections below u.*

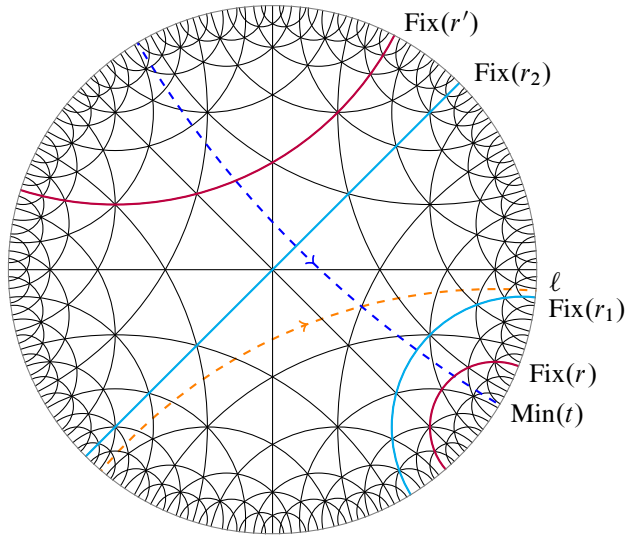We end this section by proving a geometric property of factorizations of translations into horizontal reflections:

Figure 6: The oriented axis $\mathrm{Min}(t)$ of a translation $t = r_1 r_2$ (dashed in blue) together with the other five lines of Lemma 3.11. The Coxeter axis ℓ is the same as in Figure 4, left (dashed in orange).

Lemma 3.11 (five lines) *Let $t \in [1, w]$ be a translation. Suppose that there is a factorization $t = r_1 r_2$ where r_1 and r_2 are horizontal reflections whose fixed lines are on opposite sides of ℓ. Let $r = wt^{-1}$ be the left complement of t and $r' = t^{-1}w$ be the right complement of t. Then the (oriented) translation axis of t intersects the five lines $\mathrm{Fix}(r')$, $\mathrm{Fix}(r_2)$, ℓ, $\mathrm{Fix}(r_1)$, and $\mathrm{Fix}(r)$ in this order, and these five lines are pairwise disjoint.*

Proof The situation is depicted in Figure 6. By construction and by Lemma 3.7, all reflections r, r', r_1, and r_2 are horizontal and therefore their fixed lines do not intersect ℓ. By Lemma 2.1, the translation axis of t intersects $\mathrm{Fix}(r')$ and $\mathrm{Fix}(r_2)$ before ℓ, and it intersects $\mathrm{Fix}(r)$ and $\mathrm{Fix}(r_1)$ after ℓ. Since $w = rt = rr_1r_2$, we have that rr_1 is a translation by Lemma 2.1, and therefore $\mathrm{Fix}(r)$ does not intersect $\mathrm{Fix}(r_1)$. Similarly, $\mathrm{Fix}(r')$ does not intersect $\mathrm{Fix}(r_2)$. Applying Lemma 2.1 to w and r_2, we find that the (oriented) translation axis of rr_1 intersects $\mathrm{Fix}(r_2)$ before ℓ. Therefore $\mathrm{Fix}(r_1)$ is between ℓ and $\mathrm{Fix}(r)$. Similarly, $\mathrm{Fix}(r_2)$ is between ℓ and $\mathrm{Fix}(r')$. $\square$

4 The poset of noncrossing partitions

In this section, we study some order-theoretic properties of the noncrossing partition poset $[1, w]$, where w is a Coxeter element of a rank-three hyperbolic Coxeter group. The following preliminary observation will allow us to link the order relation in $[1, w]$ with the geometric attributes of its elements:

Lemma 4.1 *The map $u \mapsto \mathrm{Mov}(u)$ from $[1, w]$ to the poset of linear subspaces of $\mathbb{R}^3$ (ordered by inclusion) is a poset isomorphism onto its image.*

Proof Denote by $[1, w]_{\text{Isom}(\mathbb{H}^2)}$ the interval between the identity and w in the group $\text{Isom}(\mathbb{H}^2)$ of all isometries of the hyperbolic plane $\mathbb{H}^2$ (using all reflections as generators). Since the reflection length of w is 3 both in W and in $\text{Isom}(\mathbb{H}^2)$, every geodesic between 1 and w in the Cayley graph of W is also a geodesic in the Cayley graph of $\text{Isom}(\mathbb{H}^2)$. Therefore there is a natural order-preserving and rank-preserving inclusion $[1, w] \hookrightarrow [1, w]_{\text{Isom}(\mathbb{H}^2)}$. To show that this inclusion is a poset isomorphism onto its image, we need to check that $u \leq v$ in $[1, w]_{\text{Isom}(\mathbb{H}^2)}$ implies $u \leq v$ in $[1, w]$. The only nontrivial case is when u has reflection length 1 and v has reflection length 2 (in both W and $\text{Isom}(\mathbb{H}^2)$): in this case, $v = ur$ for some reflection $r \in \text{Isom}(\mathbb{H}^2)$ and r is also a reflection of W (because $u, v \in W$), so $u \leq v$ in $[1, w]$. Finally, by [McCammond and Paolini 2022, Theorem 5.3], the map $u \mapsto \text{Mov}(u)$ from $[1, w]_{\text{Isom}(\mathbb{H}^2)}$ to the poset of linear subspaces of $\mathbb{R}^3$ is a poset isomorphism onto its image. $\qquad\square$

4.1 The lattice property

We now prove that $[1, w]$ is a lattice and thus defines a Garside structure on the dual Artin group W_w.

Theorem 4.2 *The interval $[1, w]$ is a lattice.*

Proof The poset $[1, w]$ is bounded and of rank three. Thus, if it is not a lattice then there are two reflections $r_1 \neq r_2$ and two rank-two elements $u_1 \neq u_2$ with $r_i < u_j$ for all $i, j \in \{1, 2\}$ (this configuration is usually called a *bowtie*). By Lemma 4.1 $\text{Mov}(r_i) \subseteq \text{Mov}(u_j)$, where each $\text{Mov}(r_i)$ is one-dimensional and each $\text{Mov}(u_j)$ is two-dimensional. Also, $\text{Mov}(r_1) \neq \text{Mov}(r_2)$, and therefore $\text{Mov}(u_1) = \text{Mov}(r_1) + \text{Mov}(r_2) = \text{Mov}(u_2)$, which is a contradiction (again by Lemma 4.1). $\qquad\square$

Corollary 4.3 *The dual Artin group W_w is a Garside group.*

Proof This follows immediately from Theorems 2.5 and 4.2, since W_w is the interval group associated with the poset $[1, w]$, which is balanced (see Section 2.4). $\qquad\square$

4.2 Axial ordering

Denote by R_0 the set of all reflections of $[1, w]$. We now describe a total ordering of R_0 which we call *axial ordering*.

Working in the hyperboloid model, let $P \subseteq \mathbb{R}^3$ be the Coxeter plane, ie the linear span of the Coxeter axis ℓ. Given a reflection r, let $\overline{\text{Fix}}(r) = \ker(\bar{r} - \text{id})$ where $\bar{r}$ is the extension of r to a linear reflection of $\mathbb{R}^3$ (with respect to the quadratic form Q of Section 2.1.2).

Note that $\overline{\text{Fix}}(r)$ is the fixed set of $\bar{r}$ in $\mathbb{R}^3$ and coincides with the linear span of $\text{Fix}(r)$ in the hyperboloid model.

Definition 4.4 (axial ordering of reflections) Let $\prec$ be the total ordering of R_0 defined as follows. Fix an axial chamber C_0 and a point $q \in \ell \cap C_0$. Define $r_1 \prec r_2$ whenever the line $\overline{\text{Fix}}(r_1) \cap P$ comes before the line $\overline{\text{Fix}}(r_2) \cap P$ when traversing the projective line $\mathbb{P}(P)$ starting from the line spanned by q and going in the positive direction of ℓ. If $\overline{\text{Fix}}(r_1) \cap P = \overline{\text{Fix}}(r_2) \cap P$, define the relative order between r_1 and r_2 arbitrarily.
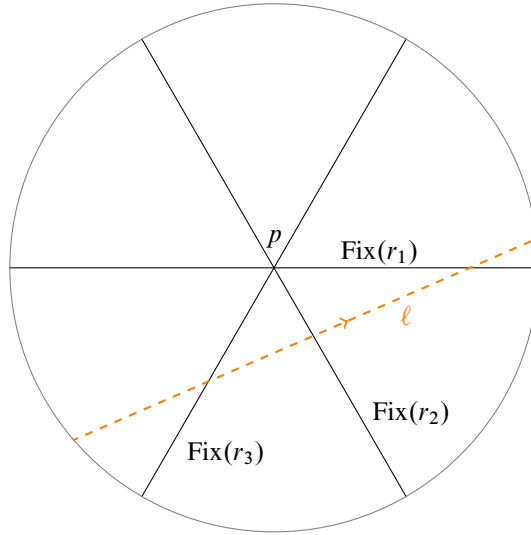
Figure 7: Reflections below a $\frac{2}{3}\pi$–rotation $u = r_1 r_2$ in the Klein model, as in Remark 4.5 and in Case 1 of the proof of Theorem 4.15. The Coxeter axis ℓ does not pass through p, so it defines a reflection ordering of $R_0 \cap [1, u]$, say $r_1 \prec r_3 \prec r_2$ (it could also be $r_2 \prec r_1 \prec r_3$ or $r_3 \prec r_2 \prec r_1$, depending on the location of the chamber C_0 along the axis ℓ). We have that $u = r_1 r_2 = r_2 r_3 = r_3 r_1$, so any of the three possible reflection orderings above is compatible with u.

Remark 4.5 (dihedral subgroups) Let $u \in [1, w]$ be a rotation, so that $[1, u]$ generates a dihedral subgroup W_u for which u is a Coxeter element (Lemma 3.8). By [Paolini and Salvetti 2021, Proposition 4.4], the restriction $\prec_u$ of the total order $\prec$ to $R_0 \cap [1, u]$ is a *reflection ordering* for W_u: whenever α, α_1, and α_2 are distinct positive roots and α is a positive linear combination of α_1 and α_2, we have either

$$r_{\alpha_1} \prec r_\alpha \prec r_{\alpha_2} \quad \text{or} \quad r_{\alpha_2} \prec r_\alpha \prec r_{\alpha_1}$$

(see [Bourbaki 1972; Dyer 1993; Björner and Brenti 2005] for more background on reflection orderings). The situation is depicted in Figure 7. Reflection orderings play an important role in the proof that finite noncrossing partition lattices are EL–shellable [Athanasiadis et al. 2007].

Next we give a more intrinsic reformulation of the definition of axial ordering:

Definition 4.6 Given a reflection $r \in R_0$, define $\xi(r) \in \ell$ to be the point of the Coxeter axis ℓ which is closest to $\mathrm{Fix}(r)$. Note that $\mathrm{Fix}(r)$ does not meet ℓ at infinity by Remark 3.6, so $\xi(r)$ is not at infinity. An example of the construction of $\xi(r)$ in the Poincaré model is shown in Figure 8.

Lemma 4.7 *Let $r_1, r_2 \in R_0$ be horizontal reflections. We have that $\overline{\mathrm{Fix}}(r_1) \cap P = \overline{\mathrm{Fix}}(r_2) \cap P$ if and only if $\xi(r_1) = \xi(r_2)$. In addition, $r_1 \prec r_2$ if and only if $\xi(r_1)$ is above $\xi(r_2)$.*

Proof Working in the Poincaré model, assume without loss of generality that the Coxeter axis ℓ is a diameter of the disk, as in Figure 8. Let r be a horizontal reflection and let l be its fixed line. Then l
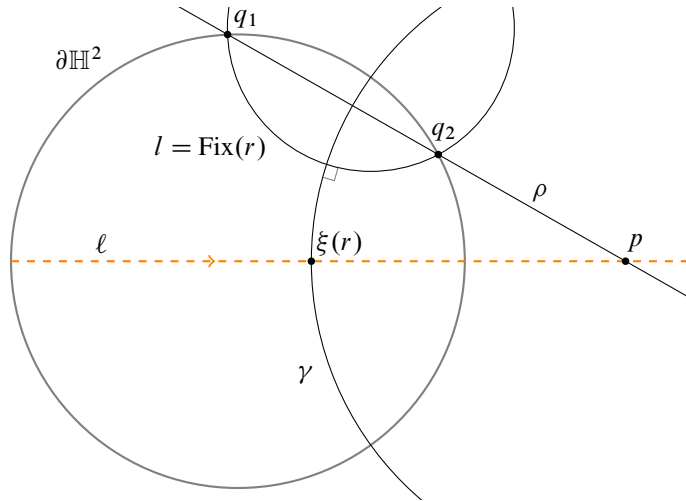
Figure 8: Proof of Lemma 4.7.

is represented by a circle that intersects $\partial\mathbb{H}^2$ perpendicularly in two points q_1 and q_2 on the same side of ℓ. Let p be the point where the radical axis ρ of l and $\partial\mathbb{H}^2$ meets the extension of the Coxeter axis ℓ outside the disk (ie the intersection of the Coxeter plane P with the plane in which the Poincaré disk is placed when compared to the hyperboloid model). Now, in order to identify the point $\xi(r)$, let γ be the circle centered at p and perpendicular to l; if ρ and ℓ are parallel, take γ to be the axis of the segment connecting q_1 and q_2. Since the power of p with respect to l is the same as with respect to $\partial\mathbb{H}^2$, the circle γ is perpendicular to $\partial\mathbb{H}^2$ as well (and of course to ℓ). Thus, conformality of the Poincaré model implies that $\gamma \cap \ell = \xi(r)$.

We now pass to the Klein model, noting that the canonical transformation from the Poincaré to the Klein disk maps the boundary points (such as q_1 and q_2) identically and preserves the axis ℓ. However, now ρ is the transform of the line l, and so p represents $\overline{\mathrm{Fix}}(r) \cap P$. Thus the position of p along ℓ determines the position of r in the axial ordering of Definition 4.4. Since $\xi(r) = \gamma \cap \ell$ only depends on the power of p with respect to the boundary of the model disk, the first claim follows immediately.

For the second claim, observe that if we move p further away from the center of the disk, its power with respect to the boundary circle increases. Accordingly, the point $\xi(r)$ will move towards the center of the disk, ie in the opposite direction with respect to p. □

Remark 4.8 The previous lemma gives a reformulation of the ordering $\prec$ of Definition 4.4: First come the vertical reflections r such that $\mathrm{Fix}(r) \cap \ell$ is above C_0, ordered by $\mathrm{Fix}(r) \cap \ell$ using the orientation of ℓ. Then come the horizontal reflections r, ordered by $\xi(r)$ using the *reverse* orientation of ℓ. Finally come the vertical reflections such that $\mathrm{Fix}(r) \cap \ell$ is below C_0, ordered by $\mathrm{Fix}(r) \cap \ell$ using the orientation of ℓ.

Definition 4.9 Once a Coxeter element w has been fixed, denote by $\varphi(u) := w^{-1}uw$ the conjugation by w.

The following lemmas describe how φ interacts with the axial ordering $\prec$:

Lemma 4.10 *Let $r_1, r_2 \in R_0$ be two horizontal reflections. We have $r_1 \prec r_2$ if and only if $\varphi(r_1) \prec \varphi(r_2)$.*

Proof Conjugating a horizontal reflection r by w has the effect of translating $\xi(r)$ in the negative direction of ℓ by an amount equal to the translation length of w: $\xi(\varphi(r)) = w^{-1}(\xi(r))$. The statement then follows from Remark 4.8. □

Lemma 4.11 *If $r \in R_0$ is a vertical reflection, then $\varphi(r) \prec r$ unless r is among the three $\prec$–smallest reflections. If r is horizontal, then $\varphi(r) \succ r$.*

Proof If r is vertical, then φ moves $\mathrm{Fix}(r) \cap \ell$ in the negative direction of ℓ by the translation length of w. Therefore $\varphi(r) \succ r$ if and only if $\mathrm{Fix}(r) \cap \ell$ is between C_0 and $w(C_0)$. By Lemma 3.2, there are exactly three reflection hyperplanes separating C_0 and $w(C_0)$, so the result follows. If r is horizontal, we immediately have $\varphi(r) \succ r$ by Remark 4.8. □

Lemma 4.12 *For every vertical reflection $r \in R_0$, there exists a unique $j \in \mathbb{Z}$ such that $\varphi^j(r)$ is one of the three $\prec$–smallest reflections. In addition, $j \geq 0$ if and only if $\mathrm{Fix}(r)$ intersects ℓ above C_0.*

Proof As already noted in the proof of the previous lemma, exactly three reflection hyperplanes separate C_0 and $w(C_0)$, and they are the fixed lines of the three $\prec$–smallest reflections. The statement immediately follows. □

Lemma 4.13 *Let $r \in R_0$ be a vertical reflection. Its right complement $u = rw$ fixes a vertex of C_0 if and only if r is among the three $\prec$–smallest reflections.*

Proof Let C be the axial chamber immediately below $\mathrm{Fix}(r) \cap \ell$. By Lemma 3.7, u is a rotation around the vertex p of C opposite to the wall $\mathrm{Fix}(r)$.

Suppose that p is a vertex of C_0. In particular, $\mathrm{Fix}(r)$ intersects ℓ above C_0, because otherwise p and C_0 would be on opposite sides of $\mathrm{Fix}(r)$. By Lemma 3.5, C_0 and $w(C_0)$ have no vertices in common, so C is between C_0 (included) and $w(C_0)$ (excluded). Therefore $\mathrm{Fix}(r)$ intersects ℓ below $w(C_0)$. We conclude that r is one of the three $\prec$–smallest reflections.

Conversely, suppose that r is one of the three $\prec$–smallest reflections. Then $\mathrm{Fix}(r)$ intersects ℓ between C_0 and $w(C_0)$, so C is between C_0 (included) and $w(C_0)$ (excluded). Therefore p and $w(C_0)$ lie on opposite sides of $\mathrm{Fix}(r)$, so p is not a vertex of $w(C_0)$. By Lemma 3.5, p is a vertex of C_0. □

4.3 EL–labeling

Next, we prove that the interval $[1, w]$ admits an EL–labeling (see Section 2.2.2), so in particular it is EL–shellable. Note that noncrossing partition posets are known to be EL–shellable in the spherical case [Athanasiadis et al. 2007] and in the affine case [Paolini and Salvetti 2021].

Definition 4.14 The natural labeling of $[1, w]$ by reflections is the labeling $E([1, w]) \to R_0$ which maps a poset cover $u \lessdot ur$ to the reflection $r \in R_0$. The set of reflections R_0 is totally ordered by the axial ordering $\prec$ constructed in Section 4.2.

Theorem 4.15 *The natural labeling of $[1, w]$ by reflections ordered by $\prec$ is an EL–labeling. In other words, every element $u \in [1, w]$ has a unique $\prec$–increasing factorization into reflections, and this factorization is lexicographically minimal.*

Proof **Case 1** (*u is a rotation*) Then u is a rotation around a point $p \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$ and it is a Coxeter element for the (parabolic) dihedral subgroup W_u that fixes p by Lemma 3.8. If u is a rotation through an angle of π, then there are exactly two factorizations $u = r_1 r_2 = r_2 r_1$; exactly one of them is $\prec$–increasing and lexicographically smaller than the other one.

Suppose from now on that u is a rotation through an angle less than π. The restriction of $\prec$ to $R_0 \cap [1, u]$ is a reflection ordering (see Remark 4.5 and Figure 7). By [Athanasiadis et al. 2007, Theorem 3.5], it is enough to show that this ordering is compatible with u, ie that there is at least one factorization $u = r_1 r_2$ such that r_1 comes immediately after r_2 in the cyclic order $\prec'$ where the last reflection comes immediately before the first (see the caption of Figure 7). This is true if r_1 and r_2 are as in the proof of Lemma 3.8, because $\text{Fix}(r_2)$ intersects ℓ immediately below the axial chamber C.

Case 2 (*u is a translation*) By Lemma 3.9, u is a product of any two consecutive reflections below u. Recall from Lemma 2.1 that the translation axis of u and the Coxeter axis ℓ intersect with an angle less than $\frac{1}{2}\pi$. Then, if $u = r_1 r_2$, the reflection r_1 comes immediately after r_2 in the cyclic order $\prec'$ on $R_0 \cap [1, u]$. (This is easily seen in the Klein model; see Figure 9.) Therefore $r_1 \succ r_2$, unless r_1 is the $\prec$–first reflection and r_2 is the last.

Case 3 (*u = w*) The lexicographically minimal factorization $w = abc$ is increasing by construction (if a and b commute, or b and c commute, we may need to swap a and b, or b and c, for this to become true). Therefore it is enough to show that every increasing factorization $w = r_1 r_2 r_3$ coincides with the factorization $w = abc$.

Suppose that r_1 is vertical and let C be the axial chamber immediately below $\text{Fix}(r_1)$. By Lemma 3.7, $v = r_2 r_3$ is a rotation around the vertex p of C opposite to $\text{Fix}(r_1)$. In addition, we can write $v = r_2' r_3'$ where $\text{Fix}(r_2')$ and $\text{Fix}(r_3')$ are the two walls of C containing p. By Corollary 3.4, $\text{Fix}(r_3')$ intersects the Coxeter axis ℓ below $\text{Fix}(r_1)$, so $r_3' \prec r_1$ unless $C = C_0$. By Case 1, since $r_2 \prec r_3$, we have that r_2 is the $\prec$–smallest reflection below v; in particular, $r_2 \preceq r_3'$. Putting this all together, if $C \neq C_0$, we get $r_2 \preceq r_3' \prec r_1$ which is a contradiction. If $C = C_0$, the factorization $r_1 r_2 r_3$ coincides with abc. A similar argument can be carried out if r_3 is vertical.

Suppose now that r_1 and r_3 are both horizontal. Since $r_1 \prec r_2 \prec r_3$, we have that r_2 is also horizontal. By Case 2, if $t = r r'$ is the increasing factorization of a translation and both r and r' are horizontal,
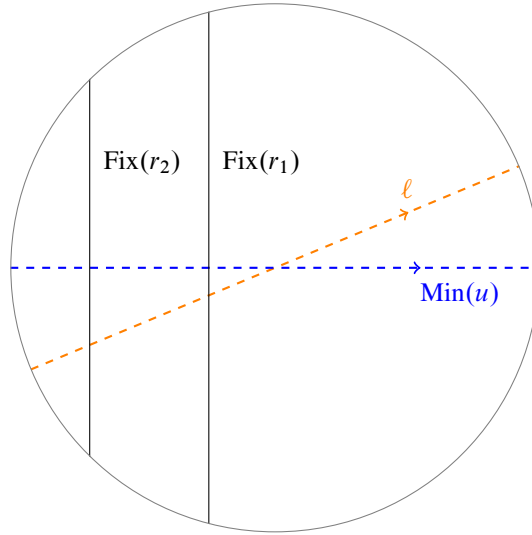
Figure 9: Case 2 of the proof of Theorem 4.15, depicted in the Klein model. Both the Coxeter axis ℓ and the translation axis $\mathrm{Min}(u)$ are taken as diameters of the disk. Then any reflection $r \leq u$ fixes a line perpendicular to $\mathrm{Min}(u)$. In addition, the oriented axes ℓ and $\mathrm{Min}(u)$ form an angle smaller than $\frac{1}{2}\pi$. Therefore, if $u = r_1 r_2$, then $\mathrm{Fix}(r_1)$ intersects (the extension of) ℓ above $\mathrm{Fix}(r_2)$.

then $\mathrm{Fix}(r)$ and $\mathrm{Fix}(r')$ are on opposite sides of the Coxeter axis. Applying this observation to the three factorizations $r_1 r_2$ (translation), $r_2 r_3$ (translation), and $r_1 r_3$ (translation or rotation), we find that $r_1 r_3$ has to be a rotation. Then its left complement $r_1 r_2 r_1$ is a vertical reflection. However, $\mathrm{Fix}(r_1)$ and $\mathrm{Fix}(r_2)$ are on opposite sides of the Coxeter axis ℓ, so $\mathrm{Fix}(r_1 r_2 r_1) = r_1(\mathrm{Fix}(r_2))$ and ℓ are on opposite sides of $\mathrm{Fix}(r_1)$, and in particular they cannot intersect. This is a contradiction.   □

5 The interval complex K_W and its subcomplexes

Throughout this section, let (W, S) be a nonspherical irreducible Coxeter system of arbitrary rank n, where $S = \{s_1, s_2, \ldots, s_n\}$ is the set of reflections with respect to some chamber C_0. Fix a Coxeter element $w = s_1 s_2 \cdots s_n$. As usual, $[1, w]$ denotes the interval between 1 and w in the right Cayley graph of W with respect to the set of all reflections $R \subseteq W$. Let K_W be the interval complex associated with $[1, w]$ (see Section 2.3).

For every $T \subseteq S$, consider the Coxeter element w_T of the standard parabolic subgroup W_T consisting of the product of the elements of T in the same relative order as in the sequence $s_1, s_2, \ldots, s_n$. As shown in [Paolini and Salvetti 2021, Section 5], the interval $[1, w_T]$ is the same in W_T and in W (using all reflections as generators) and the interval complex K_{W_T} is naturally a subcomplex of K_W. Let $X'_W \subseteq K_W$ be the subcomplex defined as

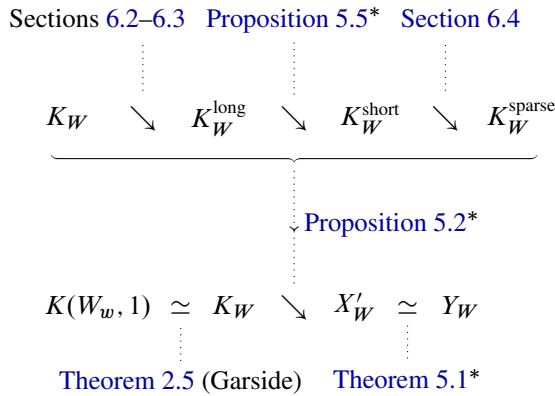$$X'_W = \bigcup_{\substack{T \subseteq S \\ W_T \text{ finite}}} K_{W_T}.$$

Sections 6.2–6.3   Proposition 5.5*   Section 6.4

$$K_W \quad \searrow \quad K_W^{\text{long}} \quad \searrow \quad K_W^{\text{short}} \quad \searrow \quad K_W^{\text{sparse}}$$

Proposition 5.2*

$$K(W_w, 1) \quad \simeq \quad K_W \quad \searrow \quad X_W' \quad \simeq \quad Y_W$$

Theorem 2.5 (Garside)   Theorem 5.1*

Figure 10: Relationships between the various subcomplexes of K_W and strategy of the proof of Theorem 6.1. The statements marked with an asterisk are valid for a general Coxeter group W. Note that Proposition 5.2 requires the collapses on the top row of the picture to hold for every infinite standard parabolic subgroup of W.

Theorem 5.1 [Paolini and Salvetti 2021, Theorem 5.5] *For any Coxeter system (W, S) and Coxeter element w, the subcomplex $X_W' \subseteq K_W$ is homotopy equivalent to the orbit configuration space Y_W.*

A deformation retraction $K_W \searrow X_W'$ was a crucial step in the proof of the $K(\pi, 1)$ conjecture in the affine case [loc. cit., Theorem 8.14] and it is natural to ask whether such a deformation retraction can be constructed for a general W (see [Paolini 2021, Question 5.5]). In this section, we introduce three additional subcomplexes of K_W that will be useful in answering the previous question when W has rank three (Section 6). The overall structure of the retraction argument is summarized in Figure 10. The reader might find it helpful to refer to Figure 10 when reading the definitions of the three new complexes below. We define the subcomplexes in full generality, with the hope that they will also prove useful when dealing with arbitrary Coxeter groups.

5.1 The subcomplex K_W^{sparse} and inductive collapses

Consider the following subcomplex of K_W:

$$K_W^{\text{sparse}} = \bigcup_{T \subsetneq S} K_{W_T}.$$

Since W is infinite, every finite standard parabolic subgroup W_T is a proper subgroup of W and therefore $X_W' \subseteq K_W^{\text{sparse}}$. Note that $X_W' = K_W^{\text{sparse}}$ whenever W is affine or compact hyperbolic because every proper parabolic subgroup of W is finite. Also, note that a simplex $[x_1|\cdots|x_d]$ of K_W belongs to K_W^{sparse} if and only if the product $x_1 \cdots x_d$ fixes a vertex of C_0. The nickname "sparse" is motivated in Figure 12.

The reason for introducing K_W^{sparse} is given by the following criterion, which allows us to deformation retract K_W onto X_W' inductively:

Proposition 5.2 *Suppose that $K_{W_T} \searrow K_{W_T}^{\text{sparse}}$ for every infinite standard parabolic subgroup W_T of W. Then $K_W \searrow X'_W$.*

Proof Let $T_1, T_2, \ldots, T_k$ be the list of all subsets of S such that W_T is infinite. Order this list in such a way that $T_i \subseteq T_j$ implies $i > j$. In particular, $T_1 = S$. Note that any deformation retraction of K_{W_T} onto $K_{W_T}^{\text{sparse}}$ must delete all the simplices of K_{W_T} that are not contained in any $K_{W_{T'}}$ with $T' \subsetneq T$ while fixing all $K_{W_{T'}}$ with $T' \subsetneq T$. Therefore it is enough to start from K_W and perform the deformation retractions $K_{W_T} \searrow K_{W_T}^{\text{sparse}}$ one at a time for $T = T_1, \ldots, T_k$. The final subcomplex is precisely X'_W. □

Hence, in order to deformation retract K_W onto X'_W it is enough to be able to deformation retract K_W onto the larger subcomplex K_W^{sparse} (and to be able to do so for all infinite $W_T \subseteq W$).

5.2 Fiber components and the subcomplexes K_W^{short} and K_W^{long}

As in the affine case [Paolini and Salvetti 2021, Section 7], consider the poset map $\eta \colon \mathcal{F}(K_W) \to \mathbb{N}$ defined as

$$(7) \qquad \eta([x_1|x_2|\cdots|x_d]) = \begin{cases} d & \text{if } x_1 x_2 \cdots x_d = w, \\ d+1 & \text{otherwise.} \end{cases}$$

The connected components of any fiber $\eta^{-1}(d)$ in the Hasse diagram of $\mathcal{F}(K_W)$ are called the *d–fiber components*. A d–fiber component has the form

$$[x_0|\cdots|x_{d-1}] \qquad [x_1|\cdots|x_d] \qquad [x_2|\cdots|x_{d+1}]$$

$$[x_0|\cdots|x_{d-2}] \qquad [x_1|\cdots|x_{d-1}] \qquad [x_2|\cdots|x_d] \qquad [x_3|\cdots|x_{d+1}]$$
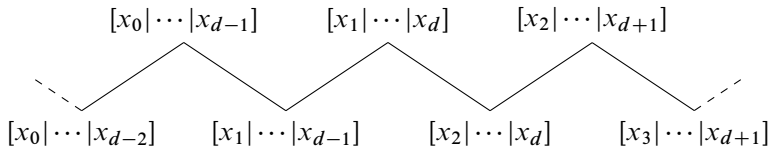
where $x_i x_{i+1} \cdots x_{i+d-1} = w$ for all i. The bi-infinite sequence $(x_i)_{i \in \mathbb{Z}}$ satisfies $x_{i+d} = \varphi(x_i)$ for all i, where φ is the conjugation by w introduced in Definition 4.9. The collection of all d–fiber components for $d = 1, \ldots, n$ yields a partition of $\mathcal{F}(K_W)$. Note that a fiber component can be finite (so its Hasse diagram is a closed loop) or infinite.

We now use the notion of fiber components to introduce two further subcomplexes of K_W that interpolate between K_W and its subcomplex K_W^{sparse} defined in Section 5.1.

Definition 5.3 (subcomplexes K_W^{short} and K_W^{long}) (a) Define K_W^{short} as the union of all simplices σ of K_W such that there is a simplex of K_W^{sparse} both (weakly) to the left and (weakly) to the right of σ in the fiber component $\mathcal{C}$ containing σ. Roughly speaking, K_W^{short} is constructed by taking the "convex hull" of the simplices of K_W^{sparse} in the Hasse diagram of each fiber component.

(b) Define K_W^{long} as the union of all fiber components that intersect K_W^{sparse}.

The nicknames "short" and "long" are motivated in Figure 12. The following lemma shows that K_W^{short} and K_W^{long} indeed are subcomplexes of K_W:
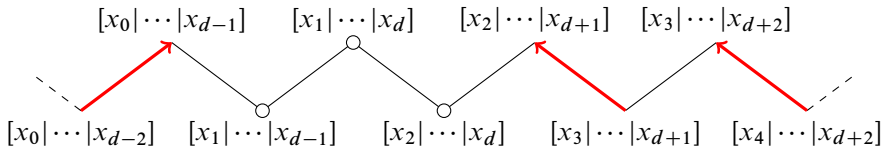
Figure 11: A matching on an infinite d–fiber component leaving the critical simplices $[x_1|\cdots|x_{d-1}]$, $[x_1|\cdots|x_d]$, and $[x_2|\cdots|x_d]$ unmatched. More generally, one can construct a similar matching that leaves an arbitrary finite set of contiguous critical simplices unmatched, starting and ending with $(d-1)$–dimensional simplices. This matching is acyclic and proper.

Lemma 5.4 *Let $\sigma \in \mathcal{F}(K_W)$ and let τ be a face of σ. Suppose that the fiber component of σ contains a simplex of K_W^{sparse} weakly to the right (resp. to the left) of σ. Then the fiber component of τ contains a simplex of K_W^{sparse} weakly to the right (resp. to the left) of τ.*

Proof Let $d = \eta(\sigma)$, so that σ is of the form $[x_1|\cdots|x_d]$ or $[x_1|\cdots|x_{d-1}]$ where $x_1 \cdots x_d = w$. Denote by $\mathcal{C}$ the d–fiber component of σ. Let σ' be a simplex of K_W^{sparse} weakly to the right of σ in $\mathcal{C}$. Then σ' is a minimal element of $\mathcal{C}$ and takes the form

$$\sigma' = [\varphi^{-k}(x_{j+1})|\cdots|\varphi^{-k}(x_d)|\varphi^{-k+1}(x_1)|\cdots|\varphi^{-k+1}(x_{j-1})]$$

for some $j = 1, \ldots, d$ and some $k > 0$.

Recall from Section 2.3 that the faces of an m–simplex $[y_1|\cdots|y_m]$ are denoted by $\partial_i([y_1|\cdots|y_m])$ for $i = 0, \ldots, m$. Let $\tau = \partial_i(\sigma)$. If σ has dimension d and $i = 0$ or $i = d$, then $\tau \in \mathcal{C}$ and σ' is weakly to the right of τ. In all other cases, the fiber component of τ contains the simplex $\partial_k(\sigma')$ weakly to the right of τ, where $k = i - j \mod d$. □

Recall from [Paolini and Salvetti 2021, Section 7] that, given an infinite d–fiber component, for any connected subgraph of its Hasse diagram starting and ending with a $(d-1)$–simplex, there exists a proper acyclic matching with critical simplices given by this subgraph (see Figure 11). This immediately gives us the following result:

Proposition 5.5 *The complex K_W^{long} deformation retracts onto K_W^{short}.*

Proof Every d–fiber component $\mathcal{C}$ of K_W^{long} intersects K_W^{sparse} (and thus K_W^{short}), so it admits a proper acyclic matching with $\mathcal{C} \cap \mathcal{F}(K_W^{\text{short}})$ as the set of critical simplices (if $\mathcal{C}$ is finite, then $\mathcal{C} \subseteq \mathcal{F}(K_W^{\text{short}})$, so the empty matching works). Let $\mathcal{M}$ be the union of all these matchings. By the patchwork theorem (Theorem 2.4), $\mathcal{M}$ is acyclic. Let $H_{\mathcal{M}}$ be the acyclic graph defined in Section 2.2.3, where H is the Hasse diagram of $\mathcal{F}(K_W^{\text{long}})$. If σ is any simplex in a d–fiber component, then every oriented path starting from σ in the graph $H_{\mathcal{M}}$ can change the fiber component at most $d - 1$ times (each time the fiber component changes, the value of η decreases by 1; the initial value is d and the value on any simplex is ≥ 1). By induction on d, for any σ in a d–fiber component, there are only a finite number of simplices reachable from σ. Therefore $\mathcal{M}$ is proper. By Theorem 2.3, we conclude that $K_W^{\text{long}} \searrow K_W^{\text{short}}$. □
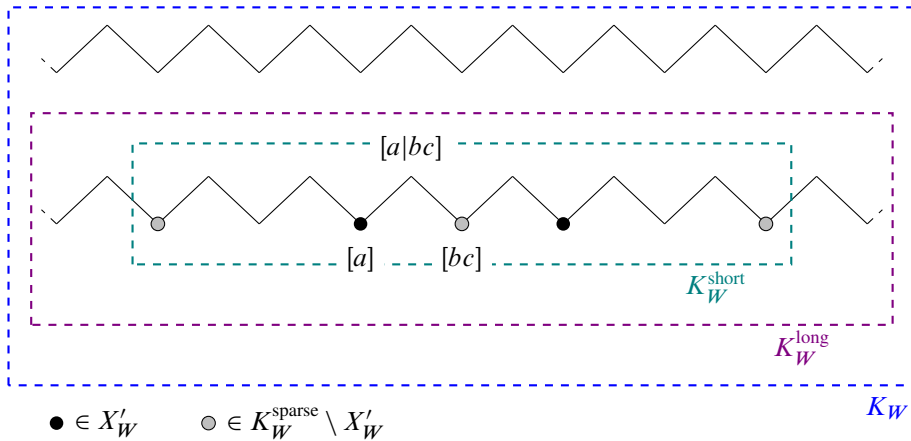
Figure 12: Example of how the subcomplexes $X'_W \subseteq K_W^{\text{sparse}} \subseteq K_W^{\text{short}} \subseteq K_W^{\text{long}} \subseteq K_W$ can intersect two different fiber components. The bottom fiber component intersects X'_W (and thus also K_W^{sparse}, K_W^{short}, and K_W^{long}), whereas the top one does not. This figure visually motivates the names "sparse", "short", and "long" given to the three subcomplexes. The depicted situation can be realized eg if W is the $(2, 3, \infty)$ triangle group (Figure 4, right), the bottom component contains the simplices $\dots, [a], [a|bc], [bc], \dots$, and on top is any component that does not intersect K_W^{sparse} (see Remark 6.4).

The following is the chain of subcomplexes we have introduced:

$$X'_W \subseteq K_W^{\text{sparse}} \subseteq K_W^{\text{short}} \subseteq K_W^{\text{long}} \subseteq K_W.$$

The definition of K_W^{short} extends the definition of the *canonical nice subcomplex* introduced in [Paolini and Salvetti 2021, Section 7] for the affine case, where the chain above simplifies to $X'_W = K_W^{\text{sparse}} \subseteq K_W^{\text{short}} \subseteq K_W^{\text{long}} = K_W$. However, differently from the affine case, K_W^{sparse} (and thus K_W^{short}) does not necessarily intersect every fiber component of K_W in general (see Section 6). Figure 12 exemplifies the definition of all subcomplexes; see also [loc. cit., Figure 8] for an affine example.

The reason for introducing the subcomplexes K_W^{short} and K_W^{long} is twofold. First, Proposition 5.5 yields a deformation retraction $K_W^{\text{long}} \searrow K_W^{\text{short}}$. Second, the proof that $K_W^{\text{short}} \searrow K_W^{\text{sparse}}$ in the affine case ([loc. cit., Section 8]) applies more generally, provided that one has an axial ordering of R_0 that induces an EL–labeling of $[1, w]$.

6 Discrete Morse theory for the rank-three case

We apply discrete Morse theory to the complexes introduced in the previous section in the case of a rank-three Coxeter system, which we assume to be irreducible and hyperbolic. The end goal of this section is to prove the following main theorem:

Theorem 6.1 *The interval complex K_W deformation retracts onto X'_W. In particular, K_W is homotopy equivalent to the orbit configuration space Y_W.*

The strategy of the proof is outlined in Figure 10. The only missing ingredients at this point are the deformation retractions $K_W \searrow K_W^{\mathrm{long}}$ and $K_W^{\mathrm{short}} \searrow K_W^{\mathrm{sparse}}$, which we construct in Sections 6.2, 6.3 and 6.4.

We will proceed in steps. First, we will classify the fiber components of the complex K_W. Based on this classification, we will construct a discrete Morse matching $\mathcal{M}$ on K_W that will prove the collapsing of K_W onto K_W^{long}. Then we will explain how the arguments of [Paolini and Salvetti 2021, Section 8] carry over to our context, allowing us to prove the collapse of K_W^{short} onto K_W^{sparse}. Finally, we will summarize all the steps and prove Theorem 6.1.

6.1 Classification of fiber components

We provide a geometric classification of the fiber components of K_W for a rank-three hyperbolic Coxeter group:

Lemma 6.2 *Let $\mathcal{C}$ be a d–fiber component of K_W, encoded by the bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$. Up to translation of the indices, exactly one of the following cases occurs:*

(i) $\mathcal{C} = \{[\,], [w]\}$, $d = 1$, *and* $x_i = w$ *for all i.*

(ii) $d = 2$, x_{2i} *is a vertical reflection, and x_{2i+1} is a rotation (possibly around a point at infinity).*

(iii) $d = 2$, x_{2i} *is a horizontal reflection, and x_{2i+1} is a translation.*

(iv) $d = 3$, *all the x_i are reflections, and at least one of the subsequences $(x_{3i})_{i\in\mathbb{Z}}$, $(x_{3i+1})_{i\in\mathbb{Z}}$, and $(x_{3i+2})_{i\in\mathbb{Z}}$ consists of vertical reflections.*

(v) $d = 3$ *and all the x_i are horizontal reflections.*

Proof The cases $d = 1$ and $d = 3$ are obvious. In the case $d = 2$, we only need to use the fact that the (left or right) complement of a vertical reflection is a rotation, whereas the complement of a horizontal reflection is a translation (Lemma 3.7). □

The classification of Lemma 6.2 has some similarities with the (rank-three) affine case ([Paolini and Salvetti 2021, Section 7]). There, components of type (iii) are finite, type (v) does not occur, and overall there are only a finite number of components. Here on the other hand, all components are infinite except for the single component of type (i). In addition, there are infinitely many components of types (iii) and (v).

Remark 6.3 Even though a reflection can never occur twice in the same minimal factorization of w, the sequence $(x_i)_{i\in\mathbb{Z}}$ defining a fiber component can have repetitions. For example, if t is the translation of Figure 6 and $t = r_1 r_2$ is its $\prec$–increasing factorization, then $r_2 = \varphi^3(r_1)$. Therefore the fiber component of type (v) containing $[r_1|r_2]$ has a repetition of both r_1 and r_2.

Remark 6.4 The fiber components that intersect K_W^{sparse} (and thus constitute the subcomplex K_W^{long}) are all those of types (i), (ii), and (iv), and some of type (iii). Indeed, for every minimal element $[r_i|r_{i+1}]$ of a type (v) fiber component, the product $r_i r_{i+1}$ is a translation by Lemma 3.7 and fixes no vertices of the chamber C_0.

If a fiber component of type (iii) intersects K_W^{sparse}, we say that it is *exceptional*. Exceptional components are characterized in the following lemma:

Lemma 6.5 *Let* $\mathcal{C}$ *be a fiber component of type* (iii). *The following are equivalent*:

(I) $\mathcal{C}$ *intersects* K_W^{sparse} (*ie* $\mathcal{C}$ *is exceptional*).

(II)   *For every* $[r] \in \mathcal{C}$ *where* r *is a* (*horizontal*) *reflection*, r *fixes an axial vertex*.

(III) *For every* $[t] \in \mathcal{C}$ *where* t *is a translation*, *there is at least one vertical reflection below* t *in* $[1, w]$.

Proof   If (I) holds, then there exists a simplex $[r] \in \mathcal{C}$ such that r is a horizontal reflection that fixes a vertex of C_0. Then $\varphi^k(r)$ fixes an axial vertex for every $k \in \mathbb{Z}$, and thus (II) holds. Conversely, if (II) holds, then by Lemma 3.5 there is at least one horizontal reflection r that fixes a vertex of C_0 with $[r] \in \mathcal{C}$, so (I) holds.

If (II) holds, then by Lemma 3.8, for any horizontal reflection r with $[r] \in \mathcal{C}$ we have that $r \leq u$ for some rotation $u \in [1, w]$ (around an axial vertex fixed by r). Therefore $t \geq r'$, where t is the right complement of r (a translation) and r' is the right complement of u (a vertical reflection). Thus (III) holds. Finally, if (III) holds and t is a translation with $[t] \in \mathcal{C}$, there is a vertical reflection $r' \leq t$. Then $u \geq r$, where u is the right complement of r' (a rotation around an axial vertex) and r is the right complement of t (a horizontal reflection). In particular, r fixes an axial vertex. Therefore (II) holds. $\square$

6.2 Componentwise construction of the matching $\mathcal{M}$

We now describe a perfect matching $\mathcal{M}$ (ie a matching with no critical simplices) on $K_W \setminus K_W^{\text{long}}$. Recall that $K_W \setminus K_W^{\text{long}}$ consists of the union of all fiber components of type (v) and all nonexceptional components of type (iii).

Fix any point $p_0 \in \ell$ and consider the semiopen segment $I \subseteq \ell$ between p_0 (included) and $w(p_0)$ (excluded). Then I is a fundamental domain for the action of $\mathbb{Z} = \langle w \rangle$ on ℓ. Recall from Definition 4.6 that the function ξ assigns to every reflection r the point $\xi(r)$ on the Coxeter axis that is closest to $\text{Fix}(r)$.

Definition 6.6   A translation $t \in [1, w]$ is *special* if its left complement r satisfies $\xi(r) \in I$.

We could alternatively define a translation to be special if its *right* complement r satisfies $\xi(r) \in I$, or if the translation axis of t intersects ℓ in a point of I. All these definitions are equivalent up to changing the point $p_0 \in \ell$ in the definition of I.

Remark 6.7   By construction, for every translation $t \in [1, w]$, there is exactly one $j \in \mathbb{Z}$ such that the translation $\varphi^j(t)$ is special. Therefore every fiber component of type (iii) contains exactly one simplex $[t]$ such that t is a special translation. For the same reason, every component of type (v) contains exactly three simplices of the form $[r_i | r_{i+1}]$ such that $r_i r_{i+1}$ is a special translation.
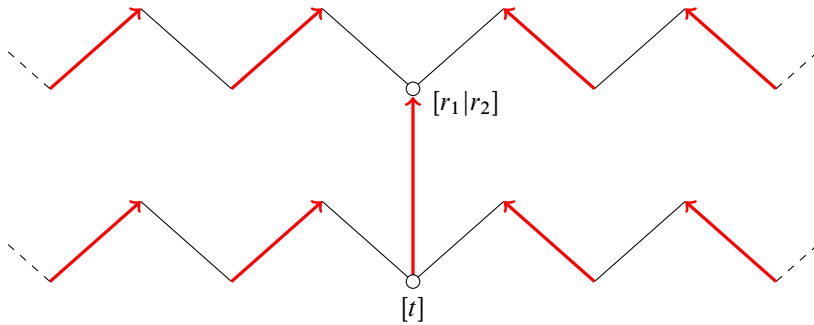
Figure 13: The matching $\mathcal{M}$ on a fiber component of type (v) (top) and the corresponding nonexceptional component of type (iii) (bottom), as described in Sections 6.2.1, 6.2.2 and 6.2.3. The translation t is special and $t = r_1 r_2$ is its increasing factorization.

6.2.1 Matching on fiber components of type (v) Let $\mathcal{C}$ be a fiber component of type (v), defined by a bi-infinite sequence of horizontal reflections $(r_i)_{i \in \mathbb{Z}}$. Consider the sequence of axial order relations ($\prec$ or $\succ$) between consecutive reflections along the sequence $(r_i)_{i \in \mathbb{Z}}$.

Lemma 6.8 *Up to translation of the indices, the sequence of axial order relations among the reflections defining a component of type (v) is*

$$\cdots \prec r_{-1} \succ r_0 \succ r_1 \prec r_2 \succ r_3 \succ \cdots,$$

ie an infinite 3–periodic repetition of one $\prec$ and two $\succ$.

Proof By Lemma 4.10, the sequence of order relations is 3–periodic: $r_i \prec r_{i+1}$ if and only if $r_{i+3} \prec r_{i+4}$. In addition, there cannot be two consecutive $\prec$, because then we would have an increasing factorization of w into three horizontal reflections, contradicting Theorem 4.15 (the only increasing factorization of w is induced by the fundamental chamber C_0 as in Corollary 3.4 and uses at least two vertical reflections). Finally, if we have $\cdots \succ r_{-1} \succ r_0 \succ r_1 \succ r_2 \succ \cdots$ (an all-$\succ$ sequence of order relations), then $r_0 \succ r_3 = \varphi(r_0)$, which is impossible by Lemma 4.11. $\square$

Apply the previous lemma and assume that $r_{3i+1} \prec r_{3i+2}$ for all $i \in \mathbb{Z}$. By Remark 6.7, among all translations $t_i = r_{3i+1} r_{3i+2}$, exactly one is special. Without loss of generality, assume that $t_0 = r_1 r_2$ is special. Then we match simplices of $\mathcal{C}$ as follows:

> On the component $\mathcal{C}$, the matching $\mathcal{M}$ is defined as the unique acyclic matching that has $[r_1 | r_2]$ as the only critical simplex (see Figure 13, top).

6.2.2 Matching on nonexceptional fiber components of type (iii) Let $\mathcal{C}$ be a nonexceptional fiber component of type (iii). By Remark 6.7, $\mathcal{C}$ has a unique simplex $[t]$ such that t is a special translation.

> On the component $\mathcal{C}$, the matching $\mathcal{M}$ is defined as the unique acyclic matching that has $[t]$ as the only critical simplex (see Figure 13, bottom).

6.2.3 Cross-fiber edges Let $\mathcal{C}$ be a nonexceptional fiber component of type (iii) and let $[t]$ be the unique simplex of $\mathcal{C}$ such that t is a special translation. Then t has a unique increasing factorization $t = r_1 r_2$. By Lemma 6.5, the reflections r_1 and r_2 are horizontal and so $[r_1 | r_2]$ belongs to a fiber component of type (v). Notice that, so far, in the construction of $\mathcal{M}$ the simplices $[t]$ and $[r_1 | r_2]$ have not been matched yet.

> Add to $\mathcal{M}$ all edges $[t] \to [r_1 | r_2]$ where t is a special translation in a nonexceptional component of type (iii) and $t = r_1 r_2$ is its increasing factorization (see Figure 13).

Remark 6.9 The matching $\mathcal{M}$ has no critical cells in $K_W \setminus K_W^{\mathrm{long}}$. In fact, let $\mathcal{C}$ be a fiber component of type (v) and consider the unique simplex $[r_1 | r_2] \in \mathcal{C}$ such that $r_1 \prec r_2$ and $r_1 r_2$ is special. Since the increasing factorization $t = r_1 r_2$ has two horizontal reflections, there is no vertical reflection below t, so $[r_1 r_2]$ is in a nonexceptional component of type (iii) by Lemma 6.5 and is matched with $[r_1 | r_2]$.

6.3 Acyclicity and properness of $\mathcal{M}$

The matching $\mathcal{M}$ is regular because it does not involve the 0–cell $[\,]$, which is the only simplex of K_W that is a nonregular face of some other simplex (see Section 2.3). In order to prove the acyclicity and properness of $\mathcal{M}$, we will weight every matched cell by a real number in such a way that the weight (weakly) decreases along directed paths (this will suffice to prove acyclicity) and the weights form a discrete subset of the positive real numbers (which will allow us to prove properness by induction on the weight).

Given a cell σ in a fiber component of type (v) or in a nonexceptional component of type (iii), let $\underline{r}(\sigma)$ be the left complement of the special translation associated with the unique cross-fiber edge exiting the fiber component of σ (see Sections 6.2.1, 6.2.2 and 6.2.3). Then construct a poset map $\omega \colon \mathcal{F}(K_W) \setminus \mathcal{F}(K_W^{\mathrm{long}}) \to \mathbb{R}$ by setting

$$\omega(\sigma) = d(\mathrm{Fix}(\underline{r}(\sigma)), \ell),$$

where $d(\,\cdot\,, \cdot\,)$ denotes the distance between two lines in the hyperbolic plane. Note that ω is constant on fiber components and $\underline{r}(\sigma)$ is always a horizontal reflection.

Lemma 6.10 *The image of ω is a discrete subset of the positive real numbers bounded away from 0.*

Proof By Remark 3.6, the values taken by ω are positive and bounded away from zero. We now show that, for every $\delta > 0$, the intersection $\mathrm{im}(\omega) \cap (0, \delta)$ is finite. Let σ be a cell such that $\omega(\sigma) < \delta$. By definition of special translation, $\xi(\underline{r}(\sigma)) \in I$ (recall that I is introduced at the beginning of Section 6.2). Therefore $\mathrm{Fix}(\underline{r}(\sigma))$ intersects the closed δ–neighborhood N of the segment I. Since N is compact, it intersects only a finite number of reflection hyperplanes. In particular, $\omega(\sigma) = d(\mathrm{Fix}(\underline{r}(\sigma)), \ell)$ can only take a finite number of possible values. $\square$

Lemma 6.11 *The function ω is order-preserving. In addition, $\omega(\sigma) = \omega(\tau)$ whenever $[\tau, \sigma] \in \mathcal{M}$.*

Proof Suppose that τ is a face of σ, with both simplices belonging to $\mathcal{F}(K_W) \setminus \mathcal{F}(K_W^{\text{long}})$. For the first part of the claim, we want to show that $\omega(\tau) \leq \omega(\sigma)$. The only nontrivial case is if τ and σ belong to different components, and this only happens if τ is in a (nonexceptional) component of type (iii) and σ is in a component of type (v).

Let $(r_i)_{i \in \mathbb{Z}}$ be the sequence of (horizontal) reflections defining the fiber component $\mathcal{C}$ of σ. By Lemma 6.8, without loss of generality,

$$\cdots \prec r_{-1} \succ r_0 = \underline{r}(\sigma) \succ r_1 \prec r_2 \succ r_3 \succ \cdots.$$

By Case 2 in the proof of Theorem 4.15 applied to the (special) translation $t = r_1 r_2$, the fact that $r_1 \prec r_2$ implies that r_1 and r_2 must be the $\prec$–first and $\prec$–last reflections below t, and so the fixed lines of r_1 and r_2 are on opposite sides of ℓ. Since $r_0 = \underline{r}(\sigma)$ is the left complement of the (special) translation $t = r_1 r_2$, it follows immediately from Lemma 3.11 that the distances $d(\text{Fix}(r_1), \ell)$ and $d(\text{Fix}(r_2), \ell)$ are both strictly lower than the distance $d(\text{Fix}(r_0), \ell) = d(\text{Fix}(r_3), \ell)$. Now recall that the distance $d(\text{Fix}(\cdot), \ell)$ is invariant under conjugation by w, whence $\omega(\sigma) = d(\text{Fix}(r_{3k}), \ell)$ and $d(\text{Fix}(r_i), \ell) = d(\text{Fix}(r_{3k+i}), \ell)$ for all k and for $i = 1, 2$. This implies that $\omega(\sigma) = \max\{d(\text{Fix}(r_i), \ell) \mid i \in \mathbb{Z}\}$. In addition, $\omega(\tau) = d(\text{Fix}(r_i), \ell)$ for some $i \in \mathbb{Z}$ and therefore $\omega(\tau) \leq \omega(\sigma)$.

For the second part of the claim, since ω is constant on fiber components, it is enough to check that ω is constant along cross-fiber edges of $\mathcal{M}$. Such edges are of the form $[t] \to [r_1|r_2]$ where $t = r_1 r_2$ is a special translation. By definition of ω, we have $\omega([t]) = d(\text{Fix}(wt^{-1}), \ell) = \omega([r_1|r_2])$. $\square$

Lemma 6.12 *The matching $\mathcal{M}$ is acyclic.*

Proof By the patchwork theorem (Theorem 2.4), it is enough to show that there is no alternating cycle on which ω is constant. Suppose for the sake of contradiction that such an alternating cycle γ exists.

Case 1 Suppose that γ has no cross-fiber edges. Then γ is also an alternating cycle with respect to the smaller matching $\mathcal{M}' \subseteq \mathcal{M}$ where all cross-fiber edges are removed. Note that every edge of $\mathcal{M}'$ matches elements in the same fiber component. By the patchwork theorem applied to the map η of Section 5.2, γ needs to be entirely contained in a single fiber component. However, the restriction of $\mathcal{M}'$ to every fiber component is obviously acyclic.

Case 2 Suppose that γ has at least one cross-fiber edge $[t] \to [r_1|r_2]$, where t is a special translation and $r_1 r_2$ is its increasing factorization. After that edge, γ has to continue either with $[r_1|r_2] \to [r_1]$ or $[r_1|r_2] \to [r_2]$. In both cases, the value of ω strictly decreases by Lemma 3.11, which is a contradiction. $\square$

Lemma 6.13 *The matching $\mathcal{M}$ is proper.*

Proof Without using cross-fiber edges, from any simplex it is possible to reach only a finite number of simplices. Indeed, this is true while staying in the same fiber component, and any path starting from a d–simplex can only change the fiber component $d - 1$ times.

Therefore it is enough to check that there is only a finite number of simplices reachable from $[t]$, where t is any special translation. We prove this by induction on $\omega([t])$, which takes values in a discrete subset of the positive real numbers bounded away from 0 (by Lemma 6.10). A directed path starting from $[t]$ necessarily begins with $[t] \to [r_1|r_2] \to [r_i]$, where $i \in \{1, 2\}$. Then it continues as an alternating path inside the fiber component C_i of $[r_i]$ until it reaches the unique simplex $[t_i] \in C_i$ such that t_i is a special translation. As already noted in the proof of Lemma 6.12, $\omega([t_i]) = \omega([r_i]) < \omega([t])$ by Lemma 3.11. By induction, there are only a finite number of simplices reachable from $[t_1]$ and $[t_2]$, so there are only a finite number of simplices reachable from $[t]$. □

6.4 Construction of the matching $\mathcal{N}$

In this section, we construct an acyclic and proper matching $\mathcal{N}$ on K_W^{short} whose critical cells are exactly the faces of K_W^{sparse}. The construction of $\mathcal{N}$, as well as the proof of acyclicity and properness, closely follow the treatment given in [Paolini and Salvetti 2021, Section 8] for the case of affine Artin groups, but notation differs slightly: our complexes K_W^{sparse} and K_W^{short} respectively take the place of X_W' and K_W' in [loc. cit.] where, accordingly, a matching on $\mathcal{F}(K_W') \setminus \mathcal{F}(X_W')$ is constructed (note that $X_W' = K_W^{\mathrm{sparse}}$ in the affine case).

The matching $\mathcal{N}$ will be defined as the set of orbits of an involution μ on $\mathcal{F}(K_W^{\mathrm{short}}) \setminus \mathcal{F}(K_W^{\mathrm{sparse}})$; see Definition 6.18. For the remainder of this subsection, we consider simplexes of the form

$$\sigma = [x_1|x_2|\cdots|x_k] \in \mathcal{F}(K_W^{\mathrm{short}}) \setminus \mathcal{F}(K_W^{\mathrm{sparse}}),$$

and we set $\pi(\sigma) := x_1 \cdots x_k \in [1, w]$.

Definition 6.14 Let $\sigma = [x_1|\cdots|x_d]$ be such that $\pi(\sigma) = w$. The depth $\delta = \delta(\sigma)$ is the minimum $i \in \{1, 2, \ldots, d\}$ such that either x_i has reflection length at least 2, or else $i \le d - 1$ and x_i precedes in the axial ordering every reflection that is below x_{i+1} in $[1, w]$. If no such i exists, set $\delta(\sigma) = \infty$.

Lemma 6.15 Let $\sigma = [x_1|\cdots|x_d]$ be such that $\pi(\sigma) = w$ and $x_2 \cdots x_d$ fixes some vertex of C_0. Then $\delta(\sigma) < \infty$.

Proof This is the claim of [loc. cit., Lemma 8.4], whose proof relies on [loc. cit., Lemma 8.2] and [loc. cit., Proposition 3.10 and Remark 3.2]. These correspond to our Lemmas 4.13 and 3.5, respectively. □

Definition 6.16 (the involution μ) Recall the map η from (7) and let $d = \eta(\sigma)$, so that $\sigma = [x_1|\cdots|x_k]$ belongs to a d–fiber component C.

(i) If $\pi(\sigma) \ne w$, then $k = d - 1$ and we let $\mu(\sigma)$ be the simplex $[x|x_1|\cdots|x_{d-1}] \in C$ that lies immediately to the left of σ.

(ii) If $\pi(\sigma) = w$ (so $k = d$) and $x_2 \cdots x_d$ does not fix any vertex of C_0, let $\mu(\sigma) = [x_2|\cdots|x_d]$.

Suppose now that $\pi(\sigma) = w$ (so $k = d$) and $x_2 \cdots x_d$ fixes a vertex of C_0. Let $\delta = \delta(\sigma)$ and note that $\delta \neq \infty$ by Lemma 6.15.

(iii)   If x_δ is not a reflection, let $\mu(\sigma) = [x_1| \cdots |x_{\delta-1}|y|z|x_{\delta+1}| \cdots |x_d]$, where $x_\delta = yz$ and y is the $\prec$–smallest reflection below x_δ in $[1, w]$.

(iv)   If x_δ is a reflection, let $\mu(\sigma) = [x_1| \cdots |x_\delta x_{\delta+1}| \cdots |x_d]$.

Lemma 6.17   *The map μ is well defined and involutive on $\mathcal{F}(K_W^{\mathrm{short}}) \setminus \mathcal{F}(K_W^{\mathrm{sparse}})$.*

Proof   The claim can be proved by the same arguments as [loc. cit., Lemma 8.8 and Proposition 8.9], using properties of vertical reflections proved in Lemmas 4.12 and 4.13 (which replace [loc. cit., Lemmas 8.1 and 8.2]), the order-preserving correspondence between elements of $[1, w]$ and linear subspaces of $\mathbb{R}^3$ proved in Lemma 4.1 (which replaces [loc. cit., Lemmas 2.15 and 2.16]), and Lemma 3.8 (which replaces [loc. cit., Lemma 3.18]). We also use the fact that the natural edge-labeling of $[1, w]$ is an EL–labeling with respect to the axial ordering (Theorem 4.15). $\square$

Definition 6.18   The matching $\mathcal{N}$ on $\mathcal{F}(K_W^{\mathrm{short}}) \setminus \mathcal{F}(K_W^{\mathrm{sparse}})$ consists of all pairs of cells $(\mu(\sigma), \sigma)$ such that $\mu(\sigma)$ is a face of σ.

Lemma 6.19   *The matching $\mathcal{N}$ is proper and acyclic.*

Proof   The main tool consists in a special total ordering $\lhd$ on the set of all minimal-length factorizations of w as a product of reflections. This total ordering is induced by the axial ordering $\prec$ as defined in [loc. cit., pages 548–549]. Now, given any simplex $\sigma = [x_1|x_2| \cdots |x_k] \in \mathcal{F}(K_W^{\mathrm{short}}) \setminus \mathcal{F}(K_W^{\mathrm{sparse}})$, we can consider the minimal-length factorization of w obtained by concatenating the $\prec$–increasing factorizations of $y, x_1, x_2, \ldots, x_k$, where y is the left complement of $x_1 x_2 \cdots x_k$ (such increasing factorizations are unique because of the EL property). This defines an order-preserving map from $\mathcal{F}(K_W^{\mathrm{short}}) \setminus \mathcal{F}(K_W^{\mathrm{sparse}})$ to the (totally) $\lhd$–ordered set of minimal-length factorizations of w. Properness and acyclicity of $\mathcal{N}$ follow from the fact that this order-preserving map is strictly increasing along alternating paths of $\mathcal{N}$. This can be proved as in [loc. cit., Lemmas 8.10–8.13] with (extensive) use of the EL property of the axial ordering (Theorem 4.15). $\square$

6.5   Proof of Theorem 6.1

Proof of Theorem 6.1   By Lemmas 6.12 and 6.13, $\mathcal{M}$ is a proper acyclic matching on K_W having K_W^{long} as the subcomplex of critical simplices. Using discrete Morse theory (Theorem 2.3), we deduce that $K_W \searrow K_W^{\mathrm{long}}$. Proposition 5.5 shows that $K_W^{\mathrm{long}} \searrow K_W^{\mathrm{short}}$. Discrete Morse theory applied to the matching $\mathcal{N}$ of Section 6.4 shows that $K_W^{\mathrm{short}} \searrow K_W^{\mathrm{sparse}}$. Combining these collapses we obtain a collapse $K_W \searrow K_W^{\mathrm{sparse}}$. In addition, $K_{W_T} \searrow K_{W_T}^{\mathrm{sparse}}$ for all infinite proper standard parabolic subgroups W_T by [loc. cit., Theorem 8.14] (such subgroups are necessarily affine of type $\widetilde{A}_1$). Finally, Proposition 5.2 implies that $K_W \searrow X'_W$. $\square$

7 Consequences for rank-three Artin groups

In this section, we use the theory developed in the rest of the paper to derive several results about rank-three Artin groups. All results are already known in the spherical and affine cases, so in our proofs below we implicitly restrict ourselves to the hyperbolic case.

Using the deformation retraction $K_W \searrow X'_W$, we are now able to establish the isomorphism between the standard and the dual Artin groups associated with W, as well as the $K(\pi, 1)$ conjecture in the rank-three case. Note that the $K(\pi, 1)$ conjecture was already proved by other means by Charney and Davis [1995] for 2–dimensional Artin groups, which include rank-three Artin groups as a special case.

Theorem 7.1 *If W has rank three, then the natural map from the standard Artin group G_W to the dual Artin group W_w is an isomorphism.*

Proof The natural map $G_W \to W_w$ is induced by the inclusion of the subcomplex X'_W into K_W, which is a homotopy equivalence by Theorem 6.1. □

Theorem 7.2 *The $K(\pi, 1)$ conjecture holds for all rank-three Artin groups G_W.*

Proof By Theorem 6.1, the orbit configuration space Y_W is homotopy equivalent to K_W, which is a classifying space by Theorems 4.2 and 2.5. □

In light of the isomorphism $G_W \cong W_w$, we can now use the Garside structure of W_w to study the standard Artin group G_W:

Theorem 7.3 *Rank-three Artin groups are Garside groups.*

Proof This is an immediate consequence of Theorem 7.1 and Corollary 4.3. □

In particular, using the dual Garside structure, we can solve the word problem and easily prove that the center is trivial (except in the spherical cases):

Theorem 7.4 *Rank-three Artin groups have a solvable word problem.*

Proof By Theorem 2.5, it is enough to be able to check equality and compute meets and joins in $[1, w]$. Equality can be checked using a solution for the word problem in W, expressing elements as a product of simple reflections (see for example [Björner and Brenti 2005, Chapter 4]). Since $[1, w]$ is self-dual, it is enough to be able to compute joins.

The only nontrivial case to consider is the join of two distinct reflections r_1 and r_2. Representing W as a group of isometries of the hyperbolic plane, compute the intersection point of $\mathrm{Fix}(r_1)$ and $\mathrm{Fix}(r_2)$. If such a point p exists (possibly at infinity), then the join $r_1 \vee r_2$ is either the only rotation $u \in [1, w]$ fixing p (if p is an axial vertex) or w otherwise. To check if p is an axial vertex, check whether p coincides with one of the vertices of the axial chambers whose closure contains the projection of p onto the Coxeter axis ℓ.

Suppose now that $\mathrm{Fix}(r_1)$ and $\mathrm{Fix}(r_2)$ do not intersect in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. The join $r_1 \vee r_2$ is either the only translation $t \in [1, w]$ whose axis is the line l orthogonal to both $\mathrm{Fix}(r_1)$ and $\mathrm{Fix}(r_2)$ (if such a translation exists in $[1, w]$) or w otherwise. There are only a finite number of reflections $r \in W$ whose fixed line intersects l between $\mathrm{Fix}(r_1)$ and $\mathrm{Fix}(r_2)$ (included). By Lemma 3.9, the translation t (if it exists) has to be equal to rr_1 or $r_1 r$ for at least one of these reflections r. Then we can consider all elements of this form and check whether one of them is a translation t such that wt^{-1} is a reflection (this is equivalent to t being in $[1, w]$, because t has reflection length 2). If we find such a translation $t \in [1, w]$, then $t = r_1 \vee r_2$ because the axis of t is orthogonal to both $\mathrm{Fix}(r_1)$ and $\mathrm{Fix}(r_2)$ by the first part of Lemma 3.9. $\square$

Theorem 7.5 *Nonspherical rank-three Artin groups have a trivial center.*

Proof Let x be an element of the center of a rank-three Artin group G_W. The dual Garside structure yields a normal form $x = \iota(w)^{-m} \iota(u_1) \iota(u_2) \cdots \iota(u_k)$, where $u_1, \ldots, u_k \in [1, w]$ and $\iota \colon [1, w] \to G_W$ is the natural immersion. A well-known property of the normal form is that x commutes with $\iota(w)$ if and only if all the u_i commute with w in W [McCammond and Sulway 2017, Proposition 2.14]. In particular, this has to hold since x is in the center of G_W. However, for any $u \in [1, w]$ different from 1 and w, no nontrivial power of w commutes with u: if u is a reflection, then $\xi(w^n u w^{-n}) = w^n \xi(u) \neq \xi(u)$; passing to the (left) complement, we reach the same conclusion if u is a rotation or a translation. Therefore x is a power of $\iota(w)$. Thus x does not commute with any $\iota(u)$ where $u \in [1, w] \setminus \{1, w\}$, unless $x = 1$. $\square$

Note that the word problem was already solved by Chermak [1998] for locally nonspherical Artin groups, a class that includes rank-three Artin groups. In a preprint by Jankiewicz and Schreve [2023], posted while this article was in preparation, it is shown that the $K(\pi, 1)$ conjecture implies that the center is trivial (for a nonspherical irreducible Artin group), thus providing another proof that nonspherical rank-three Artin groups have a trivial center.

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