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Helly graphs are graphs in which every family of pairwise-intersecting balls has a nonempty intersection. This is a classical and widely studied class of graphs. We focus on groups acting geometrically on Helly graphs — *Helly groups*. We provide numerous examples of such groups: all (Gromov) hyperbolic groups, CAT(0) cubical groups, finitely presented graphical C(4)–T(4) small cancellation groups and type-preserving uniform lattices in Euclidean buildings of type C_n are Helly; free products of Helly groups with amalgamation over finite subgroups, graph products of Helly groups, some diagram products of Helly groups, some right-angled graphs of Helly groups and quotients of Helly groups by finite normal subgroups are Helly. We show many properties of Helly groups: biautomaticity, existence of finite-dimensional models for classifying spaces for proper actions, contractibility of asymptotic cones, existence of EZ-boundaries, satisfiability of the Farrell–Jones conjecture and satisfiability of the coarse Baum–Connes conjecture. This leads to new results for some classical families of groups (eg for FC-type Artin groups) and to a unified approach to results obtained earlier.

20F06, 20F65, 20F67

1 Introduction

1.1 Motivation and main results

A geodesic metric space is *injective* if any family of pairwise-intersecting balls has a nonempty intersection; see Aronszajn and Panitchpakdi [2]. Injective metric spaces appear independently in various fields of mathematics and computer science: in topology and metric geometry — also known as *hyperconvex spaces* or *absolute retracts* (in the category of metric spaces with 1-Lipschitz maps); in combinatorics — also known as *fully spread spaces*; in functional analysis and fixed-point theory — also known as *spaces with binary intersection property*; in the theory of algorithms — known as *convex hulls*, and elsewhere. They form a very natural and important class of spaces and have been studied thoroughly. The distinguishing feature of injective spaces is that any metric space admits an *injective hull*, ie the smallest injective space into which the input space isometrically embeds; this important result was rediscovered several times in the past; see Chrobak and Larmore [31], Dress [37] and Isbell [59].

A discrete counterpart of injective metric spaces are *Helly graphs* — graphs in which any family of pairwise-intersecting (combinatorial) balls has a nonempty intersection. Again, there are many equivalent definitions of such graphs, and hence they are also known as eg *absolute retracts* (in the category of graphs with nonexpansive maps); see Bandelt and Pesch [8], Bandelt and Prisner [9], Jawhari, Pouzet and Misane [62], Pesch [75; 76] and Quilliot [80].

As the similarities in the definitions suggest, injective metric spaces and Helly graphs exhibit a plethora of analogous features. A simple but important example of an injective metric space is $(\mathbb{R}^n, d_\infty)$, that is, the n-dimensional real vector space with the metric coming from the supremum norm. The discrete analog is $\boxtimes_1^n L$, the direct product of n infinite lines L, which embeds isometrically into $(\mathbb{R}^n, d_\infty)$ with vertices being the points with integral coordinates. The space $(\mathbb{R}^n, d_\infty)$ is quite different from the "usual" Euclidean n-space $\mathbb{E}^n = (\mathbb{R}^n, d_2)$. For example, the geodesics between two points in $(\mathbb{R}^n, d_\infty)$ are not unique, whereas such uniqueness is satisfied in the "nonpositively curved" $\mathbb{E}^n$. However, there is a natural "combing" on $(\mathbb{R}^n, d_\infty)$ — between any two points there is a unique "straight" geodesic line. More generally, every injective metric space admits a unique geodesic bicombing of a particular type (see Section 3.4 for details). The existence of such a bicombing allows us to conclude many properties typical for nonpositively curved — more precisely, for CAT(0) — spaces. Therefore, injective metric spaces can be seen as metric spaces satisfying some version of "nonpositive curvature". Analogously, Helly graphs and the associated *Helly complexes* (that is, flag completions of Helly graphs), enjoy many nonpositive-curvature-like features. Some of them were exhibited in our earlier work: in [23] we prove, for example, a version of the Cartan–Hadamard theorem for Helly complexes. Moreover, the construction of the injective hull associates with every Helly graph an injective metric space into which the graph embeds isometrically and coarsely surjectively. For the example presented above, the injective hull of $\boxtimes_1^n L$ is $(\mathbb{R}^n, d_\infty)$.

Exploration of groups acting nicely on nonpositively curved complexes is one of the main activities in geometric group theory. Here we initiate the study of groups acting geometrically (that is, properly and cocompactly by automorphisms) on Helly graphs. We call them *Helly groups*. We show that the class is vast — it contains many large classical families of groups (see Theorem 1.1), and is closed under various group-theoretic operations (see Theorem 1.3). In some instances, the Helly group structure is the only known nonpositive-curvature-like structure. Furthermore, we show in Theorem 1.5 that Helly groups satisfy some strong algorithmic, group-theoretic and coarse geometric properties. This allows us to derive new results for some classical groups and present a unified approach to results obtained earlier.

Theorem 1.1 *Groups from the following classes are Helly*:

1. *groups acting geometrically on graphs with "near" injective metric hulls*, *in particular*, (*Gromov*) *hyperbolic groups*,
2. CAT(0) *cubical groups*, *that is*, *groups acting geometrically on* CAT(0) *cube complexes*,
3. *finitely presented graphical* C(4)–T(4) *small cancellation groups*,
4. *groups acting geometrically on swm-graphs*, *in particular*, *type-preserving uniform lattices in Euclidean buildings of type* C_n.

As a result of its own interest, as well as a potentially very useful tool for establishing Hellyness of groups (in particular, used successfully here), we prove the following theorem. The *coarse Helly* property is a natural "coarsification" of the Helly property. The property of *β-stable intervals* was introduced by Lang [65] in the context of injective metric spaces and is related to Cannon's property of having finitely many cone types (see Section 1.4 for further explanation).

Theorem 1.2 *A group acting geometrically on a coarse Helly graph with β-stable intervals is Helly.*

Furthermore, it has been shown recently by Huang and Osajda [57] that FC-type Artin groups and weak Garside groups of finite type are Helly. The latter class contains eg fundamental groups of the complements of complexified finite simplicial arrangements of hyperplanes, braid groups of well-generated complex reflection groups, structure groups of nondegenerate, involutive and braided set-theoretical solutions of the quantum Yang–Baxter equation, one-relator groups with nontrivial center and, more generally, tree products of cyclic groups. Conjecturally, there are many more Helly groups — see the discussion in Section 9.

Theorem 1.3 *Let $\Gamma, \Gamma_1, \Gamma_2, \dots, \Gamma_n$ be Helly groups. Then*:

1. *A free product $\Gamma_1 *_F \Gamma_2$ of Γ_1 and Γ_2 with amalgamation over a finite subgroup F and the HNN-extension $\Gamma_1 *_F$ over F are Helly.*
2. *Every graph product of $\Gamma_1, \dots, \Gamma_n$ is Helly. In particular, the direct product $\Gamma_1 \times \dots \times \Gamma_n$ is Helly.*
3. *The $\square$-product of Γ_1 and Γ_2, that is $\Gamma_1 \square \Gamma_2 = \langle \Gamma_1, \Gamma_2, t : [g,h] = [g, tht^{-1}] = 1,\ g \in \Gamma_1,\ h \in \Gamma_2 \rangle$, is Helly.*
4. *The **$\rtimes$-power** of Γ, that is $\Gamma^{\rtimes} = \langle \Gamma, t : [g, tgt^{-1}] = 1,\ g \in \Gamma \rangle$, is Helly.*
5. *The quotient Γ/N by a finite normal subgroup $N \lhd \Gamma$ is Helly.*

Observe also that, by definition, finite-index subgroups of Helly groups are Helly. Again, we conjecture that Hellyness is closed under other group-theoretic constructions — see the discussion in Section 9. Theorem 1.3(2)–(4) are consequences of the following combination theorem for actions on quasimedian graphs with Helly stabilizers. Further consequences of the same result are presented in Section 6.7.

Theorem 1.4 *Let Γ be a group acting topically transitively on a quasimedian graph G. Suppose that*

- *any vertex of G belongs to finitely many cliques*,
- *any vertex stabilizer is finite*,
- *the cubical dimension of G is finite*,
- *G contains finitely many Γ-orbits of prisms*, and
- *for every maximal prism $P = C_1 \times \dots \times C_n$, we have $\mathrm{stab}(P) = \mathrm{stab}(C_1) \times \dots \times \mathrm{stab}(C_n)$.*

If clique stabilizers are Helly, *then Γ is a Helly group.*

The results above show that the class of Helly groups is vast. Nevertheless, we may prove a number of strong properties of such groups. One very interesting and significant aspect of the theory is that the Helly group structure equips the group not only with a specific combinatorial structure that is the source of important algorithmic and algebraic features (eg (1) in the theorem below), but also — via the Helly hull construction — provides a more concrete "nonpositively curved" object acted upon by the group: a metric space with convex geodesic bicombing (see (5) below). Such spaces might be approached using methods typical for the CAT(0) setting, and are responsible for many "CAT(0)-like" results on Helly groups, such as (6)–(9) in the following theorem:

Theorem 1.5 *If Γ is a Helly group, then:*

(1) *Γ is biautomatic.*

(2) *Γ has finitely many conjugacy classes of finite subgroups.*

(3) *Γ is (Gromov) hyperbolic if and only if G does not contain an isometrically embedded infinite ℓ_∞-grid.*

(4) *The clique complex $X(G)$ of G is a finite-dimensional cocompact model for the classifying space $\underline{E}\Gamma$ for proper actions. As a particular case, Γ is always of type F_∞* (see eg Geoghegan [46, Theorem 7.3.1]), *and is of type F when it is torsion-free.*

(5) *Γ acts geometrically on a proper injective metric space of finite combinatorial dimension, and hence on a metric space with a convex geodesic bicombing.*

(6) *Γ admits an EZ-boundary ∂G.*

(7) *Γ satisfies the Farrell–Jones conjecture with finite wreath products.*

(8) *Γ satisfies the coarse Baum–Connes conjecture.*

(9) *The asymptotic cones of Γ are contractible.*

As immediate consequences, we obtain new results on some classical group classes. For example it follows that FC-type Artin groups and finitely presented graphical C(4)–T(4) small cancellation groups are biautomatic. Further discussion of important consequences is presented in Section 1.3. Note also that by Theorem 1.5(5), further properties of Helly groups can be deduced from eg Descombes [32] and Descombes and Lang [33; 34]; see also the discussion in Huang and Osajda [57, Introduction].

Theorems 1.1–1.5 are proved by the use of more general results on Helly graphs. A fundamental property that we use is the following local-to-global characterization of Helly graphs from Chalopin, Chepoi, Hirai and Osajda [23]: a graph G is Helly if and only if G is clique-Helly (ie any family of pairwise-intersecting maximal cliques of G has a nonempty intersection) and its clique complex $X(G)$ is simply connected. Here we present some of the results we obtained about Helly graphs (or complexes) in a simplified form (see Section 1.4 for further explanation).

Theorem 1.6 *The following constructions give rise to Helly graphs*:

(1) *A union of graph-products* (*UGP*) *of clique-Helly graphs satisfying the* 3*-piece condition is clique-Helly. If its clique complex is simply connected then it is Helly.*

(2) *Thickenings of simply connected* C(4)–T(4) *graphical small cancellation complexes are Helly.*

(3) *Rips complexes and face complexes of Helly graphs are Helly.*

(4) *Nerve complexes of the cover of a Helly graph by maximal cliques are Helly.*

1.2 Historical note and general context

As already mentioned, injective metric spaces were introduced by Aronszajn and Panitchpakdi [2], and they show the equivalence between injective metric spaces and hyperconvex spaces. Isbell [59] proves that for any metric space (X,d) there exists a smallest injective space which contains (X,d) as an isometric subspace. This smallest injective space is called the injective hull of (X,d). Later, this result was independently rediscovered by Dress [37] and also established for finite metric spaces by Chrobak and Larmore [31]. Dress provided other characterizations of injective hulls and developed the theory of combinatorial dimension of injective hulls viewed as cell complexes. This concept of dimension was further developed by Lang [65], who was also the first to use injective metric spaces in the context of geometric group theory. Lang also introduced the important concept of β-stable intervals [65] and showed that the injective hulls of locally finite graphs with β-stable intervals are proper and have the structure of a locally finite polyhedral complex with finitely many isometry types of cells of each dimension. This result of Lang is particularly important in the proof of Theorem 1.1(1) and Theorem 1.2. In these proofs, we also use his concept of the bounded distance property [65], which we show to be equivalent to the coarse Helly property introduced by Chepoi and Estellon [29]. As a matter of fact, δ-hyperbolic geodesic spaces and graphs satisfy the bounded distance property [65] and the coarse Helly property [29].

The fact that CAT(0) cubical groups are Helly (Theorem 1.1(2)) follows from the bijection between CAT(0) cube complexes and median graphs (see Chepoi [27] and Roller [81]) and the result of Bandelt and Van de Vel [10] establishing that the thickenings of median graphs are Helly graphs. This result was generalized by Chalopin, Chepoi, Hirai and Osajda [23] to swm-graphs, thus yielding Theorem 1.1(4).

The Helly property is ubiquitous in combinatorics, and is captured by the concept of Helly hypergraphs; see Berge [13]. Berge and Duchet [14] presented a simple "local" characterization of Helly hypergraphs that is useful in showing that the maximal cliques of a graph satisfy the Helly property. This result and the local-to-global characterization of Helly graphs of [23] provide a useful tool to establish the Hellyness of a graph. This method is used in the proof of Theorems 1.1 and 1.6.

Besides the local-to-global characterization of Helly graphs, other characterizations of Helly graphs have been obtained earlier by Bandelt and Pesch [8], Bandelt and Prisner [9] and Hell and Rival [52]. The proof of Theorem 1.5(2)–(9) uses other properties of Helly graphs and injective spaces. Theorem 1.5(2) follows from Polat's [77] fixed-point result for Helly graphs. Theorem 1.5(4) uses the fact that Helly graphs are dismantlable [8] and that fixed-point sets in dismantlable graphs are contractible; see Barmak and Minian [11]. The proof of Theorem 1.5(3) relies on the characterization of (Gromov) hyperbolic weakly modular graphs of [23] and Chepoi, Dragan, Estellon, Habib and Vaxès [28].

Theorem 1.5(5) follows from the fact that a geometric action on a Helly graph extends to a geometric action on its injective hull. The second assertion then follows since injective spaces of finite combinatorial dimension admit a convex geodesic bicombing; see Descombes and Lang [33]. Theorem 1.5(6)–(9) follow from the existence of this geodesic bicombing and results established by [33], Fukaya and Oguni [44] and Kasprowski and Rüping [64].

To establish the biautomaticity of Helly groups (Theorem 1.5(5)), we use the technique introduced by Świątkowski [85] of locally recognized path systems in a graph. In this setting, one can design a canonical path system satisfying a combinatorial bicombing property (this bicombing is different from the convex geodesic bicombing of [33]). That groups acting geometrically on Helly graphs are different from groups acting on injective spaces follows from the recent result of Hughes and Valiunas [58] showing that there exist groups acting geometrically on injective spaces that are neither Helly nor biautomatic.

1.3 Discussion of consequences of main results

Biautomaticity is an important algorithmic property of a group. It implies, among other things, that the Dehn function is at most quadratic and that the word problem and the conjugacy problem are solvable; see eg Epstein, Cannon, Holt, Levy, Paterson and Thurston [40]. Biautomaticity of classical C(4)–T(4) small cancellation groups was proved by Gersten and Short [47]. Our results (Theorem 1.1(3) and Theorem 1.5(1)) imply biautomaticity in the more general graphical small cancellation case.

Biautomaticity of all FC-type Artin groups is a new result of this paper together with work of Huang and Osajda [57]. Also new are the solution to the conjugacy problem and the quadratic bound on the Dehn function. Altobelli [1] showed that FC-type Artin groups are asynchronously automatic, and hence have solvable word problem. Biautomaticity for few classes of Artin groups was shown before by Brady and McCammond [20], Charney [24], Gersten and Short [47], Huang and Osajda [56], Peifer [74] and Pride [79]; (see [57, Subsection 1.3] for a more detailed account).

Although the classical C(4)–T(4) small cancellation groups have been thoroughly investigated and quite well understood (see eg [47] and Lyndon and Schupp [67]), there was no nonpositive curvature structure similar to CAT(0) known for them. Wise [88] equipped groups satisfying the stronger B(4)–T(4) small cancellation condition with a structure of a CAT(0) cubical group, but the question of a similar cubulation of C(4)–T(4) groups is open [88, Problem 1.4]. Theorems 1.5 and 1.1(3) equip such groups with a structure of a group acting geometrically on an injective metric space. This allows us to conclude that the Farrell–Jones and coarse Baum–Connes conjectures hold for them. These results are new; moreover, we prove them in the much more general setting of graphical small cancellation. Note that — although quite similar in definition and basic tools — the graphical small cancellation theories provide examples of groups not achievable in the classical setting (see eg Osajda [71; 70] and Osajda and Prytuła [72] for details and references).

Important examples to which our theory applies are presented in [57]. These — besides the FC-type Artin groups mentioned above — are the weak Garside groups of finite type. This class includes among others: fundamental groups of the complements of complexified finite simplicial arrangements of hyperplanes, spherical Artin groups, braid groups of well-generated complex reflection groups, structure groups of nondegenerate, involutive and braided set-theoretical solutions of the quantum Yang–Baxter equation, one-relator groups with nontrivial center and, more generally, tree products of cyclic groups. To our best knowledge there were no other "CAT(0)-like" structures known for these groups before. Consequently, such results as the existence of an EZ-structure, the validity of the Farrell–Jones conjecture and of the coarse Baum–Connes conjecture obtained by using our approach are new in these settings.

Yet another class to which our theory applies and provides new results are quadric groups introduced and investigated by Hoda [54]. See eg [54, Example 1.4] for a class of quadric groups that are a priori neither CAT(0) cubical nor C(4)–T(4) small cancellation groups.

Finally, we believe that many other groups are Helly — see the discussion in Section 9. Proving Hellyness of those groups would equip them with very rich discrete and continuous structures, and would immediately imply a plethora of strong features, described above. On the other hand, there are still many other properties to be discovered, with the hope that most CAT(0) results can be shown in this setting.

1.4 Organization of the article and further results

The proofs of Theorem 1.1(1)–(4) are provided as follows. Item (1) follows from Proposition 6.7 and Corollary 6.9. Items (2) and (4) follow from Proposition 6.1 and Corollary 6.2. Item (3) is Corollary 6.19.

The coarse Helly property is discussed in Section 3.3, and the proof of Theorem 1.2 (later appearing as Proposition 6.8) is presented in Section 6.3.

The proofs of Theorem 1.3(1)–(5) are provided as follows. Item (1) is proved in Section 6.5. Items (2)–(4) are consequences of Theorem 1.4 (later appearing as Theorem 6.24) and are shown in Section 6.7. There, we also show more general results: Theorem 6.27 on diagram products of Helly groups, and Theorem 6.31 on right-angled graphs of Helly groups. Item (5) follows directly from Theorem 6.21.

Theorem 1.4 is discussed and proved in Section 6.7.

The proofs of Theorem 1.5(1)–(9) are provided as explained below. The proof of (1) is presented in Section 8. Item (2) follows from the fixed point theorem (Theorem 7.1), and is proved in Section 7.1. The proof of (3) is presented in Section 7.2. Item (4) follows from Corollary 7.4 in Section 7.3, (5) follows from Theorems 3.13 and 6.3, and (6)–(9) are proved in Sections 7.4, 7.5, 7.6 and 7.7, respectively.

The proofs of Theorem 1.6(1)–(4) are provided as follows. A union of graph-products (UGP) is defined and studied in Section 5.1, and (1) is a part of Theorem 5.4. Graphical small cancellation complexes are studied in Section 6.4, and (2) is proved there as Theorem 6.18. Rips complexes and face complexes are discussed in Sections 5.5 and 5.6, respectively, and (3) is shown there. We discuss nerve complexes and prove (4) in Section 5.4.

Due to its relevance to our work here, in Section 2.5 we present in detail the Helly property in the general setting of hypergraphs (set systems). We also discuss the conformality property for hypergraphs, which is dual to the Helly property and which is an analog of flagness for simplicial complexes. For the same reason, in Section 3.2 we present the main ideas of Isbell's proof of the existence of injective hulls. Some further notions and additional results can be found in the arXiv version of the paper.

2 Preliminaries

2.1 Graphs

A *graph* $G=(V,E)$ consists of a set of vertices $V:=V(G)$ and a set of edges $E:=E(G)\subseteq V\times V$. All graphs we consider are undirected, connected and locally finite but not necessarily finite, and contain no multiple edges and no loops. (With the exception of the quasimedian graphs considered in Section 6.7.) That is, they are *locally finite one-dimensional simplicial complexes*. For two distinct vertices $v,w\in V$ we write $v\sim w$ (resp. $v\nsim w$) when there is an (resp. there is no) edge connecting v with w, that is, when $vw:=\{v,w\}\in E$. For vertices $v,w_1,\dots,w_k$, we write $v\sim w_1,\dots,w_k$ (resp. $v\nsim w_1,\dots,w_k$) or $v\sim A$ (resp. $v\nsim A$) when $v\sim w_i$ (resp. $v\nsim w_i$), for each $i=1,\dots,k$, where $A=\{w_1,\dots,w_k\}$. As maps between graphs $G=(V,E)$ and $G'=(V',E')$ we always consider *simplicial maps*, that is, functions of the form $f\colon V\to V'$ such that if $v\sim w$ in G then $f(v)=f(w)$ or $f(v)\sim f(w)$ in G'. A (u,w)-path $(v_0=u,v_1,\dots,v_k=w)$ of *length* k is a sequence of vertices with $v_i\sim v_{i+1}$. If $k=2$, then we call P a 2-*path* of G. If $x_i\neq x_j$ for $|i-j|\geq 1$, then P is called a *simple* (a,b)*-path*. A k*-cycle* $(v_0,v_1,\dots,v_{k-1})$ is a path $(v_0,v_1,\dots,v_{k-1},v_0)$. For $A\subseteq V$, the subgraph of $G=(V,E)$ *induced by* A is the graph $G(A)=(A,E')$ such that $uv\in E'$ if and only if $uv\in E$ ($G(A)$ is sometimes called a *full subgraph* of G). A *square* $uvwz$ (resp. *triangle* uvw) is an induced 4-cycle (u,v,w,z) (resp. 3-cycle (u,v,w)). The *wheel* W_k is the graph obtained by connecting a single vertex — the *central vertex* c — to all vertices of the k-cycle $(x_1,x_2,\dots,x_k)$.

The *distance* $d(u,v)=d_G(u,v)$ between two vertices u and v of a graph G is the length of a shortest (u,v)-path. For a vertex v of G and an integer $r\geq 1$, we denote by $B_r(v,G)$ (or by $B_r(v)$) the *ball* in G (and the subgraph induced by this ball) of radius r centered at v, that is, $B_r(v,G)=\{x\in V:d(v,x)\leq r\}$. More generally, the r*-ball around a set* $A\subseteq V$ is the set (or the subgraph induced by) $B_r(A,G)=\{v\in V:d(v,A)\leq r\}$, where $d(v,A)=\min\{d(v,x):x\in A\}$. As usual, $N(v)=B_1(v,G)\setminus\{v\}$ denotes the set of neighbors of a vertex v in G. A graph $G=(V,E)$ is *isometrically embeddable* into a graph $H=(W,F)$ if there exists a mapping $\varphi\colon V\to W$ such that $d_H(\varphi(u),\varphi(v))=d_G(u,v)$ for all vertices $u,v\in V$. A *retraction* φ of a graph G is an idempotent nonexpansive mapping of G into itself, that is, $\varphi^2=\varphi\colon V(G)\to V(G)$ with $d(\varphi(x),\varphi(y))\leq d(x,y)$ for all $x,y\in W$. The subgraph of G induced by the image of G under φ is referred to as a *retract* of G.

The *interval* $I(u,v)$ between u and v consists of all vertices on shortest (u,v)-paths, that is, of all vertices (metrically) *between* u and v: $I(u,v)=\{x\in V:d(u,x)+d(x,v)=d(u,v)\}$. An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any pair of its vertices. The smallest convex subgraph containing a given subgraph S is called the *convex hull* of S and is denoted by $\mathrm{conv}(S)$. An induced subgraph H (or the corresponding vertex set of H) of a graph G is *gated* [39] if for every vertex x outside H there exists a vertex x' in H (the *gate* of x) such that $x' \in I(x,y)$ for any y of H. Gated sets are convex, and the intersection of two gated sets is gated. By Zorn's lemma there exists a smallest gated subgraph $\langle\langle S \rangle\rangle$ containing a given subgraph S, called the *gated hull* of S.

Let G_i for $i \in \Lambda$ be an arbitrary family of graphs. The *Cartesian product* $\prod_{i\in\Lambda} G_i$ is a graph whose vertices are all functions $x\colon i \mapsto x_i$ for $x_i \in V(G_i)$, and where two vertices x and y are adjacent if there exists an index $j \in \Lambda$ such that $x_j y_j \in E(G_j)$ and $x_i = y_i$ for all $i \neq j$. Note that a Cartesian product of infinitely many nontrivial graphs is disconnected. Therefore in this case the connected components of the Cartesian product are called *weak Cartesian products*. The *direct product* $\boxtimes_{i\in\Lambda} G_i$ of graphs G_i for $i \in \Lambda$ is a graph having the same set of vertices as the Cartesian product, and two vertices x and y are adjacent if $x_i y_i \in E(G_i)$ or $x_i = y_i$ for all $i \in \Lambda$.

We continue with definitions of weakly modular graphs and their subclasses. We follow [23; 4]. Recall that a graph is *weakly modular* if it satisfies the following two distance conditions (for every $k > 0$):

- **Triangle condition (TC)** For any vertex u and any two adjacent vertices v and w at distance k to u, there exists a common neighbor x of v and w at distance $k-1$ to u.
- **Quadrangle condition (QC)** For any vertices u and z at distance k and any two neighbors v and w of z at distance $k-1$ to u, there exists a common neighbor x of v and w at distance $k-2$ from u.

Vertices v_1, v_2 and v_3 form a *metric triangle* $v_1v_2v_3$ if $I(v_i,v_j) \cap I(v_i,v_k) = \{v_i\}$ for any distinct $1 \le i,j,k \le 3$. If $d(v_1,v_2) = d(v_2,v_3) = d(v_3,v_1) = k$, then this metric triangle is called *equilateral* of *size* k. All metric triangles of a weakly modular graph are equilateral [25].

We use some classes of weakly modular graphs defined either by forbidden isometric or induced subgraphs, or by restricting the size of the metric triangles of G. A graph is called *median* if every triplet of vertices has a unique median, that is, $|I(u,v) \cap I(v,w) \cap I(w,v)| = 1$ for every triplet of vertices (u,v,w). By a result of [27; 81], median graphs are exactly the 1-skeletons of CAT(0) cube complexes (see below). For other properties and characterizations of median graphs, see the survey [4]; for some other results on CAT(0) cube complexes, see [82]. A graph is called *modular* if $I(u,v) \cap I(v,w) \cap I(w,v) \neq \varnothing$ for every triplet (u,v,w) of vertices, that is, every triplet of vertices admits a median. Clearly, median graphs are modular and modular graphs are weakly modular. A modular graph is called *strongly modular* if it does not contain $K_{3,3}^-$ as an isometric subgraph. We will also consider a nonbipartite generalization of strongly modular graphs, called *sweakly modular graphs* or *swm-graphs*, which are defined as weakly modular graphs without induced K_4^- and isometric $K_{3,3}^-$ (K_4^- is K_4 minus one edge and $K_{3,3}^-$ is $K_{3,3}$ minus one edge). The swm-graphs have been introduced and studied in depth in [23]. The cell complexes of swm-graphs can be viewed as a far-reaching generalization of CAT(0) cube complexes in which the cubes are replaced by cells arising from dual polar graphs, introduced and characterized by Cameron [22]. By [23, Theorem 5.2], dual polar graphs are exactly the thick weakly modular graphs not containing any induced K_4^- or isometric $K_{3,3}^-$ (a graph is *thick* if the interval between two vertices at distance 2 contains at least two other vertices). A set X of vertices of an swm-graph G is *Boolean-gated* if X induces a gated and thick subgraph of G. By [23, Section 6.3] a set X of vertices of an swm-graph G is Boolean-gated if and only if X is a gated set of G that induces a dual-polar graph.

A graph G is called *pseudomodular* if any three pairwise-intersecting balls of G have a nonempty intersection [6]. This condition easily implies both the triangle and quadrangle conditions, and thus pseudomodular graphs are weakly modular. An important subclass of pseudomodular graphs is constituted by Helly graphs, the main subject of our paper, which will be defined below. The *quasimedian graphs* are the K_4^-- and $K_{2,3}$-free weakly modular graphs; equivalently, they are exactly the retracts of Hamming graphs (weak Cartesian products of complete graphs). From the definition it follows that quasimedian graphs are pseudomodular and swm-graphs. For many results about quasimedian graphs see [7; 45], and for a theory of groups acting on quasimedian graphs see [45].

A graph G is called *bridged* [42; 84] if it does not contain any isometric cycle of length greater than 3. Alternatively, a graph G is bridged if and only if the balls $B_r(A, G)$ around convex sets A of G are convex. Bridged graphs are exactly the weakly modular graphs that do not contain induced 4- and 5-cycles [25]. A graph G (or its clique-complex $X(G)$) is called *locally systolic* if the neighborhoods of vertices do not induce 4- and 5-cycles. If additionally the clique complex $X(G)$ of G is simply connected, then the graph G (or its clique-complex $X(G)$) is called *systolic*. If the neighborhoods of vertices of a (locally) systolic graph G do not induce 6-cycles, then G is called (*locally*) 7-*systolic*. It was shown in [27] that bridged graphs are exactly the 1-skeletons of the *systolic complexes* of [61]. In the following, we will use the name *systolic graphs* instead of *bridged graphs*.

A graph $G = (V, E)$ is called *hypercellular* [30] if G can be isometrically embedded into a hypercube and G does not contain Q_3^- as a partial cube minor (Q_3^- is the 3-cube Q_3 minus one vertex). A graph H is called a *partial cube minor* of G if G contains a finite convex subgraph G' which can be transformed into H by successively contracting some classes of parallel edges of G'. Hypercellular graphs are not weakly modular but they generalize median graphs [30].

2.2 Complexes

All complexes we consider are locally finite CW complexes. Following [51, Chapter 0], we call them *cell complexes* or just *complexes*. If all cells are simplices (resp. unit solid cubes) and the nonempty intersection of two cells is their common face, then X is called a *simplicial* (resp. *cube*) *complex*. For a cell complex X, by $X^{(k)}$ we denote its k-*skeleton*. All cell complexes in this paper will have graphs as their 1-skeletons. Therefore we use the notation $G(X) := X^{(1)}$. The *star* of a vertex v in a complex X, denoted by $\mathrm{St}(v, X)$, is the set of all cells containing v.

An *abstract simplicial complex* Δ on a set V is a set of nonempty subsets of V such that each member of Δ, called a *simplex*, is a finite set, and any nonempty subset of a simplex is also a simplex. A simplicial complex X naturally gives rise to an abstract simplicial complex Δ on the set of vertices (0-dimensional cells) of X by setting $U \in \Delta$ if and only if there is a simplex in X having U as its vertices. Combinatorial and topological structures of X are recovered from Δ. Hence we sometimes identify simplicial complexes and abstract simplicial complexes.

The *clique complex* of a graph G is the abstract simplicial complex $X(G)$ having the cliques (ie complete subgraphs) of G as simplices. A simplicial complex X is a *flag simplicial complex* if X is the clique complex of its 1-skeleton. Given a simplicial complex X, the *flag-completion* $\widehat{X}$ of X is the clique complex of the 1-skeleton $G(X)$ of X.

Let C be a cycle in the 1-skeleton of a complex X. Then a cell complex D is called a *singular disk diagram* (or Van Kampen diagram) for C if the 1-skeleton of D is a plane graph whose inner faces are exactly the 2-cells of D and there exists a cellular map $\varphi\colon D \to X$ such that $\varphi|_{\partial D} = C$ (for more details see [67, Chapter V]). According to Van Kampen's lemma [67, pages 150–151], a cell complex X is simply connected if and only if, for every cycle C of X, one can construct a singular disk diagram. A singular disk diagram with no cut vertices (that is, its 1-skeleton is 2-connected) is called a *disk diagram.* A *minimal* (*singular*) *disk* for C is a (singular) disk diagram D for C with a minimum number of 2-faces. This number is called the (*combinatorial*) *area* of C and is denoted by $\mathrm{Area}(C)$. If X is a simply connected triangle (resp. square, triangle-square) complex, then for each cycle C, all inner faces in a singular disk diagram D of C are triangles (resp. squares, triangles or squares).

As morphisms between cell complexes we always consider *cellular maps*, that is, maps sending the k-skeleton into the k-skeleton. An *isomorphism* is a bijective cellular map that is a linear isomorphism (isometry) on each cell. A *covering* (*map*) of a cell complex X is a cellular surjection $p\colon \widetilde{X} \to X$ such that $p|_{\mathrm{St}(\tilde{v},\widetilde{X})}\colon \mathrm{St}(\tilde{v},\widetilde{X}) \to \mathrm{St}(p(\tilde{v}),X)$ is an isomorphism for every vertex $\tilde{v}$ in $\widetilde{X}$; compare [51, Section 1.3]. The space $\widetilde{X}$ is then called a *covering space*.

2.3 CAT(0) spaces and Gromov hyperbolicity

Let (X,d) be a metric space. A *geodesic segment* joining two points $x, y \in X$ is an isometric embedding $\rho\colon \mathbb{R}^1 \supset [0,l] \to X$ such that $\rho(0) = x$, $\rho(l) = y$ and $d(\rho(t),\rho(t')) = |t-t'|$ for any $t, t' \in [0,l]$ ($d(x,y) = l$ is the *length* of the geodesic ρ). A metric space (X,d) is *geodesic* if every pair of points in X can be joined by a geodesic segment. Every graph $G = (V,E)$ can be transformed into a geodesic space (X_G,d) by replacing every edge $e = uv$ by a segment $\gamma_{uv} = [u,v]$ of length 1; the segments may intersect only at common ends. Then (V,d_G) is isometrically embedded in a natural way into (X_G,d).

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X and a geodesic between each pair of vertices. A *comparison triangle* for $\Delta(x_1,x_2,x_3)$ is a triangle $\Delta(x_1',x_2',x_3')$ in the Euclidean plane $\mathbb{E}^2 = (\mathbb{R}^2,d_2)$ such that $d_2(x_i',x_j') = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. A geodesic metric space (X,d) is a CAT(0) *space* [49] if all geodesic triangles $\Delta(x_1,x_2,x_3)$ of X satisfy the comparison axiom of Cartan, Alexandrov and Toponogov:

> If y is a point on the (x_1,x_2)-geodesic of $\Delta(x_1,x_2,x_3)$ and y' is the unique point on the line segment $[x_1',x_2']$ of the comparison triangle $\Delta(x_1',x_2',x_3')$ such that $d_2(x_i',y')=d(x_i,y)$ for $i=1,2$, then $d(x_3,y)\le d_2(x_3',y')$.

The CAT(0) property is equivalent to the convexity of the function $f\colon[0,1]\to X$ given by $f(t)=d(\alpha(tl_\alpha),\beta(tl_\beta))$, for any two geodesics α and β of lengths l_α and l_β (which is further equivalent to the convexity of the neighborhoods of convex sets). This implies that CAT(0) spaces are contractible. Any two points of a CAT(0) space can be joined by a unique geodesic. See [21] for a detailed account on CAT(0) spaces and their isometry groups.

A cube complex X is CAT(0) if X, endowed with the intrinsic ℓ_2 metric, is a CAT(0) metric space. Gromov [49] characterized CAT(0) cube complexes as the simply connected cube complexes such that the following *cube condition* holds: if three $(k+2)$-dimensional cubes intersect in a k-dimensional cube and pairwise intersect in $(k+1)$-dimensional cubes, then all three are contained in a $(k+3)$-dimensional cube. The cube condition is equivalent to the *flagness condition*, which states that the geometric link of any vertex is a flag simplicial complex. The 1-skeletons of CAT(0) cube complexes are precisely the median graphs [27; 81].

A metric space (X,d) is δ*-hyperbolic* [49; 21] if, for any points u, v, x and y of X, the two larger of the sums $d(u,v)+d(x,y)$, $d(u,x)+d(v,y)$ and $d(u,y)+d(v,x)$ differ by at most $2\delta\ge 0$. A graph $G=(V,E)$ is δ*-hyperbolic* if (V,d_G) is δ-hyperbolic. A metric space or a graph has *bounded hyperbolicity* if it is δ-hyperbolic for some finite δ. For geodesic metric spaces and graphs, δ-hyperbolicity can be defined as spaces in which all geodesic triangles are δ-slim. Recall that a geodesic triangle $\Delta(x,y,z)$ is called δ*-slim* if for any point u on the side $[x,y]$ the distance from u to $[x,z]\cup[z,y]$ is at most δ.

2.4 Group actions

For a set X and a group Γ, a Γ*-action on* X is a group homomorphism $\Gamma\to\mathrm{Aut}(X)$. If X is equipped with an additional structure, then $\mathrm{Aut}(X)$ refers to the automorphism group of this structure. We say then that Γ *acts on* X *by automorphisms*, and $x\mapsto gx$ denotes the automorphism that is the image of g. Here X will be a graph or a cell complex, and thus $\mathrm{Aut}(X)$ will denote graph automorphisms or cellular automorphisms. Let Γ be a group acting by automorphisms on a cell complex X. Recall that the action is *cocompact* if the orbit space X/G is compact. The action of Γ on a locally finite cell complex X is *proper* if stabilizers of cells are finite. Finally, the action is *geometric* (or Γ *acts geometrically* on X) if it is cocompact and proper. If a group Γ acts geometrically on a graph G or on a cell complex X, then G and X are locally finite. This explains why we consider locally finite graphs, complexes and hypergraphs.

2.5 Hypergraphs (set families)

In this subsection, we recall the main notions in hypergraph theory. We closely follow the book by Berge [13] on hypergraphs (with the single difference that our hypergraphs may be infinite). A *hypergraph* is a pair $\mathcal{H}=(V,\mathcal{E})$, where V is a set and $\mathcal{E}=\{H_i\}_{i\in I}$ is a family of nonempty subsets of V; V is called the set of vertices and $\mathcal{E}$ is called the set of edges (or hyperedges) of $\mathcal{H}$. Abstract simplicial complexes are examples of hypergraphs. The *degree* of a vertex v is the number of edges of $\mathcal{H}$ containing v. A hypergraph $\mathcal{H}$ is called *edge-finite* if all edges of $\mathcal{H}$ are finite and *vertex-finite* if the degrees of all vertices are finite. $\mathcal{H}$ is called a *locally finite hypergraph* if $\mathcal{H}$ is edge-finite and vertex-finite. A hypergraph $\mathcal{H}$ is *simple* if no edge of $\mathcal{H}$ is contained in another edge of $\mathcal{H}$. The *simplification* of a hypergraph $\mathcal{H}=(V,\mathcal{E})$ is the hypergraph $\breve{\mathcal{H}}=(V,\breve{\mathcal{E}})$ whose edges are the maximal by inclusion edges of $\mathcal{H}$.

The *dual* of a hypergraph $\mathcal{H}=(V,\mathcal{E})$ is the hypergraph $\mathcal{H}^*=(V^*,\mathcal{E}^*)$ whose vertex set V^* is in bijection with the edge-set $\mathcal{E}$ of $\mathcal{H}$ and whose edge-set $\mathcal{E}^*$ is in bijection with the vertex set V; namely $\mathcal{E}^*$ consists of all $S_v=\{H_j\in\mathcal{E}: v\in H_j\}$ for $v\in V$. By definition, $(\mathcal{H}^*)^*=\mathcal{H}$. The dual of a locally finite hypergraph is also locally finite. The *hereditary closure* $\widehat{\mathcal{H}}$ of a hypergraph $\mathcal{H}$ is the hypergraph whose edge set is the set of all nonempty subsets $F\subset V$ such that $F\subseteq H_i$ for at least one index i. Clearly the hereditary closure $\widehat{\mathcal{H}}$ of a hypergraph $\mathcal{H}$ is a simplicial complex and $\widehat{\mathcal{H}}=\widehat{\widehat{\mathcal{H}}}$. The 2*-section* $[\mathcal{H}]_2$ of a hypergraph $\mathcal{H}$ is the graph having V as its vertex set, and two vertices are adjacent in $[\mathcal{H}]_2$ if they belong to a common edge of $\mathcal{H}$. By definition the 2-section $[\mathcal{H}]_2$ is exactly the 1-skeleton $\widehat{\mathcal{H}}^{(1)}$ of the simplicial complex $\widehat{\mathcal{H}}$, and the 2-section of $\mathcal{H}$ coincides with the 2-section of its simplification $\breve{\mathcal{H}}$. The *line graph* $L(\mathcal{H})$ of $\mathcal{H}$ has $\mathcal{E}$ as its vertex set, and H_i and H_j are adjacent in $L(\mathcal{H})$ if and only if $H_i\cap H_j\neq\varnothing$. By definition (see also [13, Proposition 1, page 32]), the line graph $L(\mathcal{H})$ of $\mathcal{H}$ is precisely the 2-section $[\mathcal{H}^*]_2$ of its dual $\mathcal{H}^*$. A *cycle of length* k of a hypergraph $\mathcal{H}$ is a sequence $(v_1,H_1,v_2,H_2,v_3,\dots,H_k,v_1)$ such that $H_1,\dots,H_k$ are distinct edges of $\mathcal{H}$, $v_1,v_2,\dots,v_k$ are distinct vertices of V, $v_i,v_{i+1}\in H_i$ for $i=1,\dots,k-1$, and $v_k,v_1\in H_k$. A *copair hypergraph* is a hypergraph $\mathcal{H}$ in which $V\setminus H_i\in\mathcal{E}$ for each edge $H_i\in\mathcal{E}$. The *nerve complex* of a hypergraph $\mathcal{H}=(V,\mathcal{E})$ is the simplicial complex $N(\mathcal{H})$ having $\mathcal{E}$ as its vertex set and such that a finite subset $\sigma\subseteq\mathcal{E}$ is a simplex of $N(\mathcal{H})$ if $\bigcap_{H_i\in\sigma}H_i\neq\varnothing$; see [17]. The *nerve graph* $NG(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the 1-skeleton of the nerve complex $N(\mathcal{H})$. The following result is straightforward:

Lemma 2.1 *$N(\mathcal{H})=\widehat{\mathcal{H}}^*$ and $NG(\mathcal{H})=[\mathcal{H}^*]_2=(\widehat{\mathcal{H}}^*)^{(1)}$.*

A family of subsets $\mathcal{F}$ of a set V satisfies the (*finite*) *Helly property* if, for any (finite) subfamily $\mathcal{F}'$ of $\mathcal{F}$, the intersection $\bigcap\mathcal{F}'=\bigcap\{F: F\in\mathcal{F}'\}$ is nonempty if and only if $F\cap F'\neq\varnothing$ for any pair $F,F'\in\mathcal{F}'$. A hypergraph $\mathcal{H}=(V,\mathcal{E})$ is called (*finitely*) *Helly* if its family of edges $\mathcal{E}$ satisfies the (finite) Helly property. We continue with a characterization of Helly hypergraphs. In the finite case this result is due to Berge and Duchet [13; 14]. The case of edge-finite hypergraphs follows from a more general result [5, Proposition 1].

Proposition 2.2 [13; 14] *An edge-finite hypergraph $\mathcal{H}=(V,\mathcal{E})$ is Helly if and only if for any $x,y,z\in V$ the intersection of all edges containing at least two of x, y and z is nonempty.*

We call the condition in Proposition 2.2 the *Berge–Duchet* condition.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *conformal* if all maximal cliques of the 2-section $[\mathcal{H}]_2$ are edges of $\mathcal{H}$. In other words, $\mathcal{H}$ is conformal if and only if its hereditary closure $\widehat{\mathcal{H}}$ is a flag simplicial complex. The following result establishes the duality between conformal and Helly hypergraphs:

Proposition 2.3 [13, page 30] *$\mathcal{H}$ is conformal if and only if its dual $\mathcal{H}^*$ is Helly.*

Analogously to the Helly property, the conformality can be characterized in a local way via the following *Gilmore condition* (the proof follows from Propositions 2.2 and 2.3):

Proposition 2.4 [13, page 31] *A vertex-finite hypergraph $\mathcal{H}$ is conformal if and only if for any three edges H_1, H_2 and H_3 of $\mathcal{H}$ there exists an edge H of $\mathcal{H}$ containing $(H_1 \cap H_2) \cup (H_1 \cap H_3) \cup (H_2 \cap H_3)$.*

A hypergraph $\mathcal{H}$ is *balanced* [13] if any cycle of $\mathcal{H}$ of odd length has an edge containing three vertices of the cycle. Balanced hypergraphs represent an important class of hypergraphs with strong combinatorial properties (the König property) [13; 15]. It was noticed [13, page 179] that the finite balanced hypergraphs are at the same time Helly and conformal; the duals of balanced hypergraphs are also balanced. In fact, those three fundamental properties still hold for a larger class of hypergraphs: we call a hypergraph $\mathcal{H}$ *triangle-free* if any cycle of $\mathcal{H}$ of length 3 has an edge containing the three vertices of the cycle. That is, for any three distinct vertices x, y and z and any three distinct edges H_1, H_2 and H_3 such that $x, y \in H_1$, $y, z \in H_2$ and $z, x \in H_3$, one of the edges H_1, H_2 and H_3 contains the three vertices x, y and z. Equivalently, a hypergraph $\mathcal{H}$ is triangle-free if and only if it satisfies a stronger version of the Gilmore condition: for any three edges H_1, H_2 and H_3 of $\mathcal{H}$ there exists an edge H_i in $\{H_1, H_2, H_3\}$ that contains $(H_1 \cap H_2) \cup (H_1 \cap H_3) \cup (H_2 \cap H_3)$. Since the dual of a triangle-free hypergraph is also triangle-free, locally finite triangle-free hypergraphs are conformal and Helly [15; 13].

Another important class of Helly hypergraphs, extending the class of balanced hypergraphs, is the class of normal hypergraphs. A hypergraph $\mathcal{H}$ is called *normal* [13; 66] if it satisfied the Helly property and its line graph $L(\mathcal{H})$ is perfect (ie by the strong perfect graph theorem $L(\mathcal{H})$ does not contain odd cycles of length > 3 and their complements as induced subgraphs).

With any graph $G = (V, E)$ one can associate several hypergraphs, depending on the studied problem and of the studied class of graphs. In the context of our current work, we consider the following combinatorial and geometric hypergraphs: the *clique-hypergraph* $\mathcal{X}(G)$ of all maximal cliques of G, the *ball-hypergraph* $\mathcal{B}(G)$ of all balls of G, and the *r-ball-hypergraph* $\mathcal{B}_r(G)$ of all balls of a given radius r of G. The ball-hypergraph can be considered for an arbitrary metric space (X, d). The clique-hypergraph $\mathcal{X}(G)$ of any graph G is simple and conformal, and its hereditary closure $\widehat{\mathcal{X}}(G)$ coincides with the clique complex $X(G)$ of G. In the case of median graphs G (and CAT(0) cube complexes), together with the cube complex (cube hypergraph), an important role is played by the copair hypergraph $\mathcal{H}(G)$ of all halfspaces of G (convex sets with convex complements). Since convex sets of median graphs are gated [60, Theorem 1.22] and gated sets satisfy the finite Helly property, the hypergraph $\mathcal{H}(G)$ is finitely Helly. For a graph G we will also consider the nerve complex $N(\mathcal{X}(G))$ of the clique-hypergraph $\mathcal{X}(G)$, as well as the nerve complex $N(\mathcal{B}_r(G))$ of the r-ball-hypergraph $\mathcal{B}_r(G)$ for $r \in \mathbb{N}$.

2.6 Abstract cell complexes

An *abstract cell complex* X (also called a *convexity space* or *closure space*) is a locally finite hypergraph $\mathcal{H}(X) = (V, \mathcal{E})$ with $\varnothing \in \mathcal{E}$ and whose edges are closed under intersections, ie if H_i for $i \in I$ are edges of $\mathcal{H}$, then $\bigcap_{i \in I} H_i$ is also an edge of $\mathcal{H}(X)$. We call the edges of $\mathcal{H}(X)$ the *cells* of X and $\mathcal{H}(X)$ the *cell-hypergraph* of X. The cells of X contained in a given cell C are called the *faces* of C. The faces of a cell C ordered by inclusion define the face lattice $F(C)$ of C. $C' \subsetneq C$ is a *facet* of C if C' is a maximal by inclusion proper face of C; in other words, C' is a coatom of the face lattice $F(C)$. The *dimension* $\dim(C)$ of a cell C is the length of the longest chain in the face lattice of C. Locally finite abstract simplicial complexes are abstract cell complexes. In fact, they are the cell complexes in which the face lattices are Boolean lattices. The dimension of a simplex with $d+1$ vertices is d. Cube complexes are also abstract cell complexes. It suffices to consider the vertex set of each cube as an edge of the cell-hypergraph; the dimension of a cube is the standard dimension.

Abstract cell complexes also arise from swm-graphs and hypercellular graphs. The cells of an swm-graph are its Boolean-gated sets and the dimension of a Boolean-gated set is its diameter. Observe that in an swm-graph, any maximal clique is boolean-gated. In the corresponding abstract cell complex, each such clique is a 1-dimensional cell whose 0-cells are the vertices of the clique. It was shown in [23] that one can also associate a contractible geometric cell complex to any swm-graph G, in which the cells are the orthoscheme complexes of the Boolean-gated sets of G. Note that the geometric dimension of this geometric complex is larger than the dimension of the abstract cell complex. The cells of a hypercellular graph G are the gated subgraphs of G which are the convex hulls of the isometric cycles of G. It was shown in [30] that those cells are Cartesian products of edges and even cycles. It was established in [30] that the geometric realization of the abstract cell complex of a hypercellular graph is contractible. The dimension of such a cell is the number of edge factors plus twice the number of cycle factors. Notice that swm-graphs and hypercellular graphs represent two far-reaching and quite different generalizations of median graphs. Swm-graphs no longer have hyperplanes (ie classes of parallel edges) and halfspaces, and their cells (Boolean-gated sets) have a complex combinatorial structure; nevertheless, they are still weakly modular and admit a local-to-global characterization. On the other hand, hypercellular graphs are no longer weakly modular but they still admit hyperplanes (whose carriers are gated) and halfspaces, and each triplet of vertices admits a unique median cell.

2.7 Helly graphs and Helly groups

We continue with the definitions of our main objects: Helly and clique-Helly graphs and complexes, and Helly groups.

Definition 2.5 A graph G is a *Helly graph* if the ball-hypergraph $\mathcal{B}(G)$ is Helly. A graph G is a 1-*Helly graph* if the 1-ball-hypergraph $\mathcal{B}_1(G)$ is Helly. A *clique-Helly graph* is a graph G in which the hypergraph $\mathcal{X}(G)$ of maximal cliques is Helly.

Observe that a Helly graph is 1-Helly and a 1-Helly graph is clique-Helly, but that the reverse implications do not hold: a cycle of length at least 7 is 1-Helly but not Helly and a cycle of length 4 is clique-Helly but not 1-Helly. Notice also that Helly graphs are pseudomodular and thus weakly modular. For arbitrary graphs not containing infinite cliques, Polat and Pouzet [78] proved that the Helly property and the finite Helly property are equivalent.

Definition 2.6 A *Helly complex* is the clique complex of some Helly graph. A *clique-Helly complex* is the clique complex of some clique-Helly graph.

Remark 2.7 If in Definitions 2.5 and 2.6 instead of a Helly property we consider the corresponding finite Helly property, then the graphs satisfying it are called finitely Helly. For example, finitely clique-Helly graphs are graphs G in which the hypergraph $\mathcal{X}(G)$ has the finite Helly property. For locally finite graphs, the finite Helly properties for balls and cliques implies the Helly property, and thus finitely Helly (resp. clique-Helly) graphs and complexes are Helly (resp. clique-Helly). By [78], the same implication holds for arbitrary graphs not containing infinite cliques.

We continue with the definition of Helly groups:

Definition 2.8 A group Γ is *Helly* if it acts geometrically on a Helly complex X.

If a group Γ acts geometrically on a Helly complex X, then X is locally finite. Moreover X has uniformly bounded degrees.

In case of the clique-Helly property, Proposition 2.2 can be specified in the following way:

Proposition 2.9 [35; 86] *A graph G with finite cliques is clique-Helly if and only if for any triangle T of G the set T^* of all vertices of G adjacent with at least two vertices of T contains a vertex adjacent to all remaining vertices of T^*.*

Remark 2.10 Proposition 2.9 does not hold for graphs containing infinite cliques. For example, consider the graph G defined as follows. First, consider an infinite clique $K = \{v_0, v_1, v_2, \dots, v_k, \dots\}$ whose vertex set is indexed by $\mathbb{N}$. For each $i \in \mathbb{N}$, we add a vertex u_i that is adjacent to all v_j such that $j \geq i$. Observe that any two maximal cliques of G have a nonempty intersection but there is no universal vertex in G. Consequently, G is not clique-Helly. On the other hand, one can easily check that G satisfies the criterion of Proposition 2.9.

For any locally finite graph G, the clique-hypergraph $\mathcal{X}(G)$ is conformal and G is isomorphic to the 2-section of $\mathcal{X}(G)$. Moreover, if G is clique-Helly, then $\mathcal{X}(G)$ is Helly. We conclude this subsection with the following simple but useful converse result (see eg [9]):

Proposition 2.11 *For a locally finite hypergraph $\mathcal{H} = (V, \mathcal{E})$ the following conditions are equivalent*:

(i) *The* 2*-section* $[\mathcal{H}]_2$ *of* $\mathcal{H}$ *is a clique-Helly graph and* $\mathcal{H}$ *is conformal* (*each maximal clique of* $[\mathcal{H}]_2$ *is an edge of* $\mathcal{H}$).

(ii) *The simplification* $\breve{\mathcal{H}}$ *of* $\mathcal{H}$ *is conformal and Helly.*

(iii) $\breve{\mathcal{H}}$ *satisfies the Berge–Duchet and Gilmore conditions.*

In particular, *the* 2*-section of any locally finite triangle-free hypergraph is clique-Helly.*

Proof Since $[\mathcal{H}]_2 = [\breve{\mathcal{H}}]_2$, we can suppose that $\mathcal{H}$ is simple. The equivalence (ii) $\iff$ (iii) follows from Propositions 2.2 and 2.4. If (i) holds, then $\mathcal{H}$ coincides with the hypergraph of maximal cliques of $[\mathcal{H}]_2$, and thus $\mathcal{H}$ is Helly. Also $\mathcal{H}$ is conformal as the clique-hypergraph of a graph. This establishes (i) $\Longrightarrow$ (ii). Conversely, if (ii) holds, since $\mathcal{H}$ is conformal, each clique of $[\mathcal{H}]_2$ is included in an edge of $\mathcal{H}$. Thus the maximal cliques of $[\mathcal{H}]_2$ are in bijection with the edges of $\mathcal{H}$. This shows that $[\mathcal{H}]_2$ is clique-Helly. □

2.8 Hellyfication

There is a canonical way to extend any hypergraph $\mathcal{H} = (V, \mathcal{E})$ to a conformal hypergraph $\mathrm{conf}(\mathcal{H}) = (V, \mathcal{E}')$: $\mathcal{E}'$ consists of $\mathcal{E}$ and all maximal by inclusion cliques C in the 2-section $[\mathcal{H}]$ of $\mathcal{H}$. Any conformal hypergraph $\mathcal{H}''$ extending $\mathcal{H}$ and having the same 2-section $[\mathcal{H}''] = [\mathcal{H}]$ as $\mathcal{H}$ also contains $\mathrm{conf}(\mathcal{H})$ as a subhypergraph, thus $\mathrm{conf}(\mathcal{H})$ can be called the *conformal closure* of $\mathcal{H}$. Since the Helly property and conformality are dual, any hypergraph $\mathcal{H} = (V, \mathcal{E})$ can be extended to a Helly hypergraph $\mathrm{Helly}(\mathcal{H}) = (V', \mathcal{E}')$: for every maximal pairwise-intersecting set $\mathcal{F}$ of edges of $\mathcal{H}$ with empty intersection, add a new vertex $v_{\mathcal{F}}$ to V and to each member of $\mathcal{F}$. In the thus extended hypergraph $\mathrm{Helly}(\mathcal{H})$ any two edges intersect exactly when their traces on V intersect. Hence $\mathrm{Helly}(\mathcal{H})$ satisfies the Helly property and we call $\mathrm{Helly}(\mathcal{H})$ the *Hellyfication* of $\mathcal{H}$. Again, $\mathrm{Helly}(\mathcal{H})$ is contained in any hypergraph satisfying the Helly property, extending $\mathcal{H}$ and having the same line graph as H. This kind of Hellyfication approach was used in [5] to Hellyfy discrete copair hypergraphs and to relate this Hellyfication procedure with the cubulation (median hull) of the associated wall space; see [5, Proposition 3].

3 Injective spaces and injective hulls

In this section we discuss injective metric spaces and Isbell's construction of injective hulls. Those notions are strongly related to Helly graphs: roughly, Helly graphs and ball-Hellyfication can be seen as discrete analogs of (continuous) injective metric spaces and injective hulls.

3.1 Injective spaces

A metric space (X, d) is called *hyperconvex* if every family of closed balls $B_{r_i}(x_i)$ of radii $r_i \in \mathbb{R}^+$ with centers x_i satisfying $d(x_i, x_j) \le r_i + r_j$ has a nonempty intersection. Rephrasing the definition, (X, d) is hyperconvex if it is *Menger-convex* (that is, $B_r(x) \cap B_{d(x,y)-r}(y) \neq \varnothing$ for all $x, y \in X$ and $r \in [0, d(x, y)]$) and the family of closed balls in (X, d) satisfies the Helly property. A metric space

(X,d) is called *integer-valued* if $d(x,y)$ is an integer for any $x,y \in X$. An integer-valued metric space (X,d) is *discretely geodesic* if for any two points $x,y \in X$ with $d(x,y) = n$ there exists a sequence of points $x_0 := x, x_1, x_2, \dots, x_n := y$ such that $d(x_i, x_{i+1}) = 1$. The set of vertices of a connected graph equipped with a graph distance is an example of an integer-valued and discretely geodesic metric space.

Let (Y,d') and (X,d) be two metric spaces. For $A \subset Y$, a map $f: A \to X$ is 1*-Lipschitz* if $d(f(x), f(y)) \le d'(x,y)$ for all $x,y \in A$. The pair (Y,X) has the *extension property* if for any $A \subset Y$, any 1-Lipschitz map $f: A \to X$ admits a 1-Lipschitz extension, ie a 1-Lipschitz map $\tilde{f}: Y \to X$ such that $\tilde{f}|_A = f$. A metric space (X,d) is *injective* if for any metric space (Y,d'), the pair (Y,X) has the extension property. For $Y \subset X$, the map $f: X \to Y$ is a (*nonexpansive*) *retraction* if f is 1-Lipschitz and $f(y) = y$ for any $y \in Y$. A metric space (Y,d') is an *absolute retract* if, whenever (Y,d') is isometrically embedded in a metric space (X,d), there exists a retraction f from X to Y.

In 1956, Aronszajn and Panitchpakdi established the following equivalence between hyperconvex spaces, injective spaces, and absolute retracts:

Theorem 3.1 [2] *A metric space (X,d) is injective if and only if (X,d) is hyperconvex if and only if (X,d) is an absolute* (1*-Lipschitz*) *retract.*

3.2 Injective hulls

By a construction of Isbell [59] (rediscovered twenty years later by Dress [37] and yet another ten years later by Chrobak and Larmore [31] in computer science), for every metric space (X,d) there exists a smallest (with respect to inclusion) injective metric space containing X. More precisely, an *injective hull* (or *tight span*, or *injective envelope*, or *hyperconvex hull*) of (X,d) is a pair $(e, E(X))$ where $e: X \to E(X)$ is an isometric embedding into an injective metric space $E(X)$, and such that no injective proper subspace of $E(X)$ contains $e(X)$. Two injective hulls $e: X \to E(X)$ and $f: X \to E'(X)$ are *equivalent* if they are related by an isometry $i: E(X) \to E'(X)$. Below we describe Isbell's construction in some details and we recall a few important features of injective hulls — all this will be of use in Section 6.

Theorem 3.2 [59] *Every metric space (X,d) has an injective hull and all its injective hulls are equivalent.*

We continue with the main steps in the proof of Theorem 3.2. We follow the Isbell's proof [59], but also use some notations and results from Dress [37] and Lang [65]; see these three papers for a full proof. Let (X,d) be a metric space. A *metric form* on X is a real-valued function f on X such that $f(x) + f(y) \ge d(x,y)$ for all $x,y \in X$. Denote by $\Delta(X)$ the set of all metric forms on X, ie $\Delta(X) = \{f \in \mathbb{R}^X : f(x) + f(y) \ge d(x,y) \text{ for all } x,y \in X\}$. For $f, g \in \Delta(X)$, set $f \le g$ if $f(x) \le g(x)$ for each $x \in X$. A metric form is called *extremal* on X if there is no $g \in \Delta(X)$ such that $g \ne f$ and $g \le f$. Let $E(X) = \{f \in \Delta(X) : f \text{ is extremal}\}$.

Claim 3.3 *If $f \in E(X)$, then $f(x) + d(x,y) \ge f(y)$ for any $x,y \in X$, that is, f is* 1*-Lipschitz.*

If this was false for $x, y \in X$, then defining g to coincide with f everywhere except at y, where $g(y) = f(x) + d(x, y)$, we conclude that $g \in \Delta(X)$. Since $g \le f$, we obtain $g = f$.

The difference $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$ between any two extremal forms f and g is bounded; any number $f(x) + g(x)$ is a bound. Thus $(E(X), d_\infty)$ is a metric space. For a point $x \in X$, let d_x be defined by setting $d_x(y) = d(x, y)$ for any $y \in X$.

Claim 3.4 *For any $x \in X$, the map $d_x \colon y \mapsto d(x, y)$ is extremal on X and the map $e \colon X \to E(X)$ sending x to d_x is an isometric embedding of (X, d) into $(E(X), d_\infty)$.*

The map e is often called the *Kuratowski embedding*.

From the definition of extremal metric forms, the following useful property of $E(X)$ easily follows (this explains why extremal maps are called *tight extensions* in [37]):

Claim 3.5 *If (X, d) is compact then for any $f \in E(X)$ and $x \in X$ there exists y in X such that $f(x) + f(y) = d(x, y)$. In general metric spaces, for any $x \in X$ and any $\epsilon > 0$, there exists y in X such that $f(x) + f(y) < d(x, y) + \epsilon$.*

The inequalities $f(x) + f(y) \ge d(x, y)$ and $f(x) + d(x, y) \ge f(y)$ together are equivalent to:

Claim 3.6 *If $f \in E(X)$, then $f(x) = d_\infty(f, e(x))$ for all $x \in X$.*

The following claim is the main tool in Isbell's proof. Let $\Delta(E(X))$ denote the set of all metric forms on $E(X)$ and let $E(E(X))$ denote the set of all extremal metric forms on $E(X)$.

Claim 3.7 *If s is extremal on $E(X)$, then se is extremal on X.*

First notice that $se \in \Delta(X)$. To prove Claim 3.7, we suppose by way of contradiction that se is not extremal and we obtain a contradiction with the assumption that s is extremal on $E(X)$. Then there exists $h \in E(X)$ such that $h \le se$ and $h(x) < se(x)$ for some $x \in X$. Define the map $t \colon E(X) \to \mathbb{R}$ by setting $t(f) = s(f)$ for all $f \in E(X)$ different from $e(x)$. Set $t(e(x)) = h(x) < s(e(x))$. Since $t < s$, to contradict the extremality of s on $E(X)$ it remains to show that $t \in \Delta(E(X))$, ie $t(f) + t(g) \ge d_\infty(f, g)$ for any $f, g \in E(X)$. Since $s \in \Delta(E(X))$, from the definition of t it suffices to establish the previous inequality for any $f \in E(X)$ and $g = e(x)$ with $f \ne e(x)$, that is, to show that $te(x) + t(f) \ge d_\infty(f, e(x))$. This is done using the definition of $e(x)$ and Claims 3.3 and 3.6. For any $\epsilon > 0$, pick $y \in X$ such that $f(x) + f(y) < d(x, y) + \epsilon$. Then $te(x) + t(f) = te(x) + se(y) - se(y) + s(f) \ge h(x) + h(y) - d_\infty(e(y), f) \ge d(x, y) - f(y) > f(x) - \epsilon = d_\infty(e(x), f) - \epsilon$. Since $\epsilon > 0$ is arbitrary, $te(x) + t(f) \ge d_\infty(f, e(x))$, as required.

Claim 3.8 *The metric space $(E(X), d_\infty)$ is injective.*

To prove Claim 3.8, in view of Theorem 3.1 it suffices to show that $(E(X), d_\infty)$ is hyperconvex: if $f_i \in E(X)$, $r_i \in \mathbb{R}^+$ and $i \in I$ such that $d_\infty(f_i, f_j) \le r_i + r_j$, then $\bigcap_{i \in I} B(f_i, r_i) \ne \varnothing$. We may suppose that $r \colon E(X) \to \Delta(E(X))$ is a metric form on $E(X)$ extending the radius function $r_i \colon r(f_i) = r_i$ (this extension exists by Zorn's lemma). Let $s \in E(E(X))$ such that $s \le r$. By Claim 3.7, se belongs to

$E(X)$. We assert that se belongs to any $r(f)$-ball centered at $f \in E(X)$. Indeed, for any $x \in X$, we have $se(x) - f(x) = se(x) - d_\infty(f, e(x)) \le s(f) \le r(f)$, where the equality follows from Claim 3.6 and the first inequality follows from Claim 3.3 (applied to $E(X)$ and $E(E(X))$ instead of X and $E(X)$). On the other hand, $f(x) - se(x) = d_\infty(f, e(x)) - se(x) \le s(f) \le r(f)$, where the equality follows from Claim 3.6 and the inequality follows by the choice of s in $\Delta(E(X))$. This establishes Claim 3.8.

Claim 3.9 *The embedding $e\colon X \to E(X)$ is an injective hull and is equivalent to every injective hull of X.*

Let $\alpha\colon E(X) \to E(X)$ be 1-Lipschitz such that $\alpha(e(x)) = e(x)$ for any $x \in X$. Let $f \in E(X)$ and let $g = \alpha(f)$. By Claim 3.6, for any $x \in X$ we have $g(x) = d_\infty(g, e(x)) = d_\infty\big(\alpha(f), \alpha(e(x))\big) \le d_\infty(f, e(x)) = f(x)$. Hence $g \le f$, whence α is the identity map. Thus $E(X)$ cannot be retracted to any subset $S \subsetneq E(X)$ containing $e(X)$, and hence S is not injective.

Finally, consider any injective hull $e'\colon X \to E'(X)$ of (X, d). Let f be an isometry from $e(X)$ to $e'(X)$ and let f' be its inverse. Since both $E(X)$ and $E'(X)$ are injective, there exist 1-Lipschitz maps $\tilde{f}\colon E(X) \to E'(X)$ and $\tilde{f}'\colon E'(X) \to E(X)$ extending f and f', respectively. Note that the composition $\tilde{f}'\tilde{f}$ is a 1-Lipschitz map from $E(X)$ to $E(X)$ that is the identity on $e(X)$. Therefore $\tilde{f}'\tilde{f}$ is the identity map by what has been shown above, and thus $\tilde{f}$ is injective and $\tilde{f}'$ is surjective. Since $\tilde{f}$ and $\tilde{f}'$ are 1-Lipschitz and $\tilde{f}'\tilde{f}$ is the identity on $E(X)$, necessarily $\tilde{f}$ is an isometric embedding of $E(X)$ in $E'(X)$. Then since $E(X)$ is injective, the image of $\tilde{f}$ contains $e'(X) = \tilde{f}(e(X))$ and $E'(X)$ is an injective hull, so $\tilde{f}$ must be surjective and thus $\tilde{f}$ is an isometry. Then $\tilde{f}'$ is injective, otherwise $\tilde{f}'\tilde{f}$ cannot be the identity map on $E(X)$. Thus both $\tilde{f}$ and $\tilde{f}'$ are isometries. This concludes the proof of Theorem 3.2.

Dress [37] defined $E(X)$ as the set of all maps $f \in \mathbb{R}^X$ such that $f(x) = \sup\{d(x, y) - f(y) : y \in X\}$ for all $x \in X$. He established the following nice property of $E(X)$ (which in fact characterizes $E(X)$; see [37, Theorem 1]):

Claim 3.10 *If $f, g \in E(X)$, then*

$$d_\infty(f, g) = \sup\{d_\infty(e(x), e(y)) - d_\infty(e(y), f) - d_\infty(e(x), g) : x, y \in X\}.$$

For simplicity, we prove Claim 3.10 for compact metric spaces, for which the supremum can be replaced by maximum. The claim asserts that any pair of extremal functions f and g lies on a geodesic between the images $e(x)$ and $e(y)$ in $E(X)$ of two points x and y of X. Let x be a point of X such that $d_\infty(f, g) = f(x) - g(x)$. By Claim 3.5 there exists $y \in X$ such that $f(x) = d(x, y) - f(y)$. Hence $d_\infty(f, g) = f(x) - g(x) = d(x, y) - f(y) - g(x) = d_\infty(e(x), e(y)) - f(y) - g(x)$. By Claim 3.6, $f(y) = d_\infty(f, e(y))$ and $g(x) = d_\infty(g, e(x))$. Consequently $d_\infty(f, g) = d_\infty(e(x), e(y)) - f(y) - g(x) = d_\infty(e(x), e(y)) - d_\infty(f, e(y)) - d_\infty(g, e(x))$ and we are done.

One interesting property of injective hulls is their monotonicity:

Corollary 3.11 *If (X, d) is isometrically embeddable into (X', d'), then $E(X)$ is isometrically embeddable into $E(X')$.*

Proof Since (X,d) is isometrically embeddable into (X',d') and $E(X)$, and (X',d') is isometrically embeddable into $E(X')$, there exists an isometric embedding of $e(X) \subset E(X)$ into $E(X')$. Since $E(X')$ is injective, this isometric embedding extends to a 1-Lipschitz map α from $E(X)$ to $E(X')$. If $d_\infty(\alpha(f),\alpha(g)) < d_\infty(f,g)$ for $f,g \in E(X)$, then $d_\infty\big(\alpha(e(x)),\alpha(e(y))\big) < d_\infty(e(x),e(y))$ for points $x,y \in X$ occurring in Claim 3.10, contrary to the assumption that α isometrically embeds $e(X)$. □

As shown by Dress [37], the injective hull of a finite metric space is a finite polyhedral complex. Using this, he defined the *combinatorial dimension* of a general metric space X as the supremum of the dimensions of the polyhedral complexes $E(Y)$ for all finite subspaces Y of X. Any $f \in E(X)$ belongs to the interior of a unique cell of the polyhedral complex. Dress combinatorially characterized the cells of $E(X)$. Goodman and Moulton gave a presentation of Dress's result in the finite case [48]. Lang presented Dress's result in the case of general metric spaces, and formulated conditions under which the injective hull is finite-dimensional or has a finite number of types of cells for each dimension [65]. In the following, we continue with this combinatorial description following the presentation of [65; 33].

For any $f \in \Delta(X)$, consider the graph $(X, A(f))$ where $A(f)$ is the set of all pairs $\{x,y\}$ of points in X such that $f(x)+f(y)=d(x,y)$. If X is finite (or compact), then f belongs to $E(X)$ if and only if $(X,A(f))$ has no isolated vertices. This is no longer true when X is not compact (see Claim 3.5). For this, Dress and Lang introduced the subset $E'(X)=\big\{f \in \Delta(X) : \bigcup A(f)=X\big\}$ of $E(X)$. They show that $E'(x)$ is dense in $E(X)$ if the metric on X is integer-valued.

A set A of unordered pairs of points in X is called *admissible* if there exists $f \in E'(X)$ with $A(f)=A$. Denote by $\mathcal{A}(X)$ the set of all such admissible sets. The family of polyhedral faces of $E(X)$ is then given by $\{P(A)\}_{a\in\mathcal{A}(X)}$ where $P(A)=\{f \in \Delta(X) : A \subseteq A(f)\}$. As noticed in [65], $P(A)=P(A)\cap E(X)=P(A)\cap E'(X)$. The *rank* $\operatorname{rk}(A)$ of an admissible set A is the dimension of $P(A)$. The rank $\operatorname{rk}(A)$ can be characterized as follows. If $f,g \in P(A)$, then $f(x)+f(y)=d(x,y)=g(x)+g(y)$ for $\{x,y\}\in A$ and thus $f(y)-g(y)=-(f(x)-g(x))$. So the difference $f-g$ has alternating sign along all paths in the graph (X,A). Consequently, for each connected component of (X,A), there is at most one degree of freedom for the values of $f \in P(A)$. If the connected component C contains an odd cycle, then f and g coincide on all vertices of C. Alternatively, if the connected component C is bipartite, then the restrictions of all functions $f \in P(A)$ on the vertices of C form a 1-parameter family: given the value $f(x)$ on one vertex of C, one can deduce all the other values of f on C. Then the rank $\operatorname{rk}(A)=\dim(P(A))$ is precisely the number of bipartite components of the graph (X,A).

Dress [37] characterized spaces of combinatorial dimension at most n by a $2(n+1)$-point inequality. These notions are important to state and establish some results of Lang [65] that we present and use in Section 6.3.

3.3 Coarse Helly property

A metric space (X,d) is *coarsely hyperconvex* if there exists some $\delta \geq 0$ such that, for any set of centers $\{x_i\}_{i\in I}$ in X and any set of radii $\{r_i\}_{i\in I}$ in $\mathbb{R}^+$ satisfying $d(x_i,x_j) \leq r_i+r_j$, there exists

$x \in X$ such that $d(x, x_i) \leq r_i + \delta$ for all $i \in I$, ie the intersection $\bigcap_{i \in I} B_{r_i+\delta}(x_i)$ is not empty. A metric space (X, d) has the *coarse Helly property* if there exists some $\delta \geq 0$ such that, for any family $\mathcal{B} = \{B_{r_i}(x_i) : i \in I\}$ of pairwise-intersecting closed balls of X, the intersection $\bigcap_{i \in I} B_{r_i+\delta}(x_i)$ is not empty. If the space (X, d) is Menger-convex (in particular, if (X, d) is geodesic), both properties are equivalent. In a discretely geodesic metric space (in particular, in a graph), if $d(x_i, x_j) \leq r_i + r_j$, then the balls $B_{\lceil r_i \rceil}(x_i)$ and $B_{\lceil r_j \rceil}(x_j)$ intersect. In particular if the $\{r_i\}_{i \in I}$ are integers, then $d(x_i, x_j) \leq r_i + r_j$ if and only if $B_{r_i}(x_i)$ and $B_{r_j}(x_j)$ intersect. Consequently, a discretely geodesic metric space (X, d) is coarsely hyperconvex with some constant δ if and only if it satisfies the coarse Helly property with some constant δ', where δ and δ' differ by at most 1. The injective hull $E(X)$ of a metric space (X, d) has the *bounded distance property* if there exists $\delta \geq 0$ such that for any $f \in E(X)$ there exists a point $x \in X$ such that $d_\infty(f, e(x)) \leq \delta$. The coarse Helly property was introduced in [29] and the bounded distance property in [65], in both cases for δ-hyperbolic spaces and graphs.

We show that the coarse hyperconvexity of a metric space is equivalent to the fact that its injective hull satisfies the bounded distance property.[1]

Proposition 3.12 *A metric space (X, d) is coarsely hyperconvex if and only if its injective hull $E(X)$ satisfies the bounded distance property. Consequently, if (X, d) is a geodesic or discretely geodesic metric space, then the coarse hyperconvexity of (X, d), the coarse Helly property for (X, d) and the bounded distance property for $E(X)$ are all equivalent.*

Proof First suppose that (X, d) is coarsely hyperconvex with some constant $\delta \geq 0$. Let $f \in E(X)$. Then $f(x) + f(y) \geq d(x, y)$ for any x and y. By the coarse hyperconvexity of (X, d) applied to the radius function f, there exists a point $z \in X$ such that $d(z, x) \leq f(x) + \delta$ for any $x \in X$. We assert that $d_\infty(f, e(z)) \leq \delta$. Indeed, $d_\infty(f, e(z)) = \sup_{x \in X} |f(x) - d(x, z)|$. By the choice of z in $B_{f(x)+\delta}(x)$, $d(x, z) - f(x) \leq \delta$. It remains to show the other inequality, $f(x) - d(x, z) \leq \delta$. Assume by contradiction that $f(x) - d(x, z) > \delta$. Let $\epsilon = \frac{1}{2}(f(x) - d(x, z) - \delta)$ and observe that $f(x) > d(x, z) + \delta + \epsilon$. By Claim 3.5, there exists $y \in X$ such that $f(x) + f(y) < d(x, y) + \epsilon$. But since $z \in B_{f(y)+\delta}(y)$, we have $f(y) \geq d(y, z) - \delta$, and so $f(x) + f(y) > d(x, z) + \delta + \epsilon + d(y, z) - \delta = d(x, z) + d(y, z) + \epsilon \geq d(x, y) + \epsilon$ (the last inequality follows from the triangle inequality), a contradiction.

Conversely, let $E(X)$ satisfy the bounded distance property with $\delta \geq 0$. We will show that (X, d) is coarsely hyperconvex. Let $B(x_i, r_i)$ for $i \in I$ be a collection of closed balls of (X, d) such that $r_i + r_j \geq d(x_i, x_j)$ for all $i, j \in I$. Let $r \in \Delta(X)$ be a metric form on X extending the radius function r_i for $i \in I$ (its existence follows from Zorn's lemma). Let $f \in E(X)$ such that $f(x) \leq r(x)$ for any $x \in X$. By the bounded distance property, X contains a point z such that $d_\infty(f, e(z)) \leq \delta$. This implies that $|f(x) - e(z)(x)| = |f(x) - d(x, z)| \leq \delta$ for any $x \in X$. In particular, $d(x, z) \leq f(x) + \delta \leq r(x) + \delta$, and thus z belongs to all closed balls $B_{r(x)+\delta}(x)$, $x \in X$. □

[1] Independently, this was also observed by Urs Lang (personal communication, 2019).

3.4 Geodesic bicombings

One important feature of injective metric spaces is the existence of a nice (bi)combing. Recall that a *geodesic bicombing* on a metric space (X, d) is a map $\sigma\colon X \times X \times [0,1] \to X$ such that for every pair $(x, y) \in X \times X$ the function $\sigma_{xy} := \sigma(x, y, \cdot)$ is a constant-speed geodesic from x to y. We call σ *convex* if the function $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex for all $x, y, x', y' \in X$. The bicombing σ is *consistent* if $\sigma_{pq}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t)$ for all $x, y \in X$, $0 \le s \le t \le 1$, $p := \sigma_{xy}(s)$, $q := \sigma_{xy}(t)$ and $\lambda \in [0,1]$. It is called *reversible* if $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $x, y \in X$ and $t \in [0,1]$. From the definition of injective hulls and [33, Lemma 2.1 and Theorems 1.1–1.2] we have the following:

Theorem 3.13 *A proper injective metric space of finite combinatorial dimension admits a unique convex consistent reversible geodesic bicombing.*

4 Helly graphs and complexes

In this section, we recall the basic properties and characterizations of Helly graphs. We also show that any graph admits a Hellyfication, a discrete counterpart of Isbell's construction.

4.1 Characterizations

Helly graphs are the discrete analogs of hyperconvex spaces: namely, the requirement that radii of balls are nonnegative reals is modified by replacing the reals by the integers. A vertex x of a graph G is *dominated* by another vertex y if the unit ball $B_1(y)$ includes $B_1(x)$. A graph G is *dismantlable* if its vertices can be well ordered (denoted by $\prec$) so that, for each v there is a neighbor w of v with $w \prec v$ which dominates v in the subgraph of G induced by the vertices $u \preceq v$.

The following result presents a local-to-global and a topological characterization of all (not necessarily finite or locally finite) Helly graphs.

Theorem 4.1 [23] *For a graph G, the following conditions are equivalent*:

(i) *G is Helly.*

(ii) *G is a weakly modular 1-Helly graph.*

(iii) *G is a dismantlable clique-Helly graph.*

(iv) *G is clique-Helly with a simply connected clique complex.*

Moreover, if the clique complex $X(G)$ of G is finite-dimensional, then (i)–(iv) *are equivalent to*:

(v) *G is clique-Helly with a contractible clique complex.*

Let G be a (finitely) clique-Helly graph and let $\widetilde{G}$ be the 1-skeleton of the universal cover $\widetilde{X} := \widetilde{X}(G)$ of the clique complex $X := X(G)$ of G. Then $\widetilde{G}$ is a (finitely) Helly graph. In particular, G is a (finitely) Helly graph if and only if G is (finitely) clique-Helly and its clique complex is simply connected.

Conditions (ii) and (iii) of Theorem 4.1 refine and generalize the characterizations of finite Helly graphs given in [9; 8]. The second part of Theorem 4.1 and its proof lead to two conclusions. First, if a simplicial complex X is clique-Helly (for arbitrary families of maximal cliques), then its universal cover $\widetilde{X}$ is Helly (for arbitrary families of balls of its 1-skeleton). Second, if X is finitely clique-Helly, then its universal cover is finitely Helly (this holds even if X contains infinite cliques). From [23, Theorem 9.1] it follows that Helly graphs satisfy a quadratic isoperimetric inequality. It was shown in [80] that any finite Helly graph G has the stabilized clique property: there exists a complete subgraph of G invariant under the action of the automorphism group of G. Other properties of Helly graphs will be presented below.

4.2 Injective hulls and Hellyfication

We will show that for any graph G there exists a smallest Helly graph $\mathrm{Helly}(G)$ comprising G as an isometric subgraph; we call $\mathrm{Helly}(G)$ the *Hellyfication* of G (analogously, we will denote by $\mathrm{Helly}(X(G))$ the clique complex of $\mathrm{Helly}(G)$ and refer to it as to the *Hellyfication* of $X(G)$).

Let (X,d) be an integer-valued metric space. An *integer metric form* on X is a function $f\colon X\to\mathbb{Z}$ such that $f(v)+f(w)\geq d(v,w)$ for all $v,w\in X$. Let $\Delta^0(X)$ denote the set of all integer metric forms on X. An integer metric form is *extremal* if it is minimal pointwise. We define the metric space $E^0(X)\subset\Delta^0(X)$ as the set of all extremal integer metric forms on (X,d) endowed with the sup-metric d_∞. The embedding $e\colon X\to E^0(X)$ is defined as $v\mapsto d(v,\cdot)$. The pair $(e,E^0(X))$ is the *discrete injective hull* of X. We define a graph structure on $E^0(X)$ by putting an edge between two extremal forms $f,g\in E^0(X)$ if $d_\infty(f,g)=1$. With some abuse of notation, we also denote this graph by $E^0(X)$. If $G=(V,E)$ is a graph with the path metric d, we will denote by $E^0(G)$ and $E(G)$ the discrete injective hull $E^0(V(G))$ and the injective hull of the metric space $(V(G),d)$, respectively. Similarly, we write $e(G)$ instead of $e(V(G))$.

The following result is well known (see [62; 75; 76]), and is the discrete counterpart of Isbell's Theorem 3.2.

Theorem 4.2 *If (X,d) is an integer-valued metric space, then $E^0(X)=E(X)\cap\mathbb{Z}^X$ is the smallest Helly graph into which (X,d) is isometrically embedded. In particular, the discrete injective hull $E^0(G)$ of a graph G is contained as an isometric subgraph in any Helly graph G' containing G as an isometric subgraph and is the Hellyfication* $\mathrm{Helly}(G)$ *of G.*

Proof First we show that the sets $E^0(X)$ and $E(X)\cap\mathbb{Z}^X$ coincide. Observe that by the definitions of $E^0(X)$ and $E(X)\cap\mathbb{Z}^X$, we have $E(X)\cap\mathbb{Z}^X\subseteq E^0(X)$. To show the converse inclusion, first note that $E^0(X)$ satisfies the discrete analog of Claim 3.5: if $f\in E^0(X)$, then for any x in X there exists y in X such that $f(x)+f(y)=d(x,y)$. By way of contradiction, suppose there exist $f\in E^0(X)$ and $g\in E(X)$ such that $g\neq f$ and $g\leq f$. Then $g(x)<f(x)$ for some point x of X. By the discrete analog of Claim 3.5, there exists y in X such that $f(x)+f(y)=d(x,y)$. But since $g(x)<f(x)$ and $g(y)\leq f(y)$, we obtain $g(x)+g(y)<d(x,y)$, contrary to the assumption $g\in E(X)$. Therefore $E^0(X)\subseteq E(X)\cap\mathbb{Z}^X$ and thus $E^0(X)=E(X)\cap\mathbb{Z}^X$. Consequently, $(E^0(X),d_\infty)$ is also an integer-valued metric space.

Next we show that the balls of $(E^0(X), d_\infty)$ satisfy the Helly property. Let $f_i \in E^0(X)$ and $r_i \in \mathbb{Z}^+$ for $i \in I$ such that $d_\infty(f_i, f_j) \le r_i + r_j$. We may suppose that $r \in \Delta^0(E^0(X))$ is an integer metric form on $E^0(X)$ extending the radius function r_i ($r(f_i) = r_i$ for $i \in I$) and $t \in E^0(E^0(X)) = E(E^0(X)) \cap \mathbb{Z}^{E^0(X)}$ is an integer metric form on $E^0(X)$ such that $t \le r$. Let $t' \in \Delta(E(X))$ be a metric form on $E(X)$ extending t, ie for any $f \in E^0(X)$ we have $t'(f) = t(f)$ — its existence follows by Zorn's lemma. Let $s \in E(E(X))$ such that $s \le t'$. By the discrete analog of Claim 3.5, for any $f \in E^0(X)$ there exists $g \in E^0(X)$ such that $t(f) + t(g) = d_\infty(f, g)$. Since $s(f) + s(g) \le t'(f) + t'(g) = t(f) + t(g) = d_\infty(f, g) \le s(f) + s(g)$, we have that $s(f) = t'(f) = t(f)$ and $s(g) = t'(g) = t(g)$ since $s(h) \le t'(h) = t(h)$ for any $h \in E^0(X)$. Consequently, $s|_{E^0(X)} = t$. By Claim 3.7 and the proof of Claim 3.8, se belongs to $E(X)$ and is a common point of all balls $B_{r_i}(f_i)$. Since $e(x) \in E^0(X)$ for any $x \in X$, and since s and t coincide on $E^0(X)$, we have $se = te$. Therefore te belongs to $E^0(X)$ and is a common point of all balls $B_{r_i}(f_i)$. This shows that the balls of $(E^0(X), d_\infty)$ satisfy the Helly property.

We show by induction on $k = d_\infty(f, g)$ that any two vertices $f, g \in E^0(X)$ are connected in the graph $E^0(X)$ by a path of length k. Indeed, pick a ball of radius 1 centered at f and a ball of radius $k-1$ centered at g. By the Helly property, there exists $h \in E^0(X)$ such that $d_\infty(f, h) \le 1$ and $d_\infty(h, g) \le k-1$. By the triangle inequality, these two inequalities are equalities. Thus $E^0(X)$ is a Helly graph isometrically embedded in $E(X)$. The proof that $E^0(X)$ does not contain any Helly subgraph containing X and that all discrete injective hulls are isometric is identical to the proof of Claim 3.9. The proof that $E^0(X)$ is an isometric subgraph of any Helly graph G' containing G as an isometric subgraph is similar to the proof of Corollary 3.11. □

Remark 4.3 A direct consequence of the second assertion of Theorem 4.2 is that if G is Helly, then $\mathrm{Helly}(G)$ coincides with G.

Remark 4.4 For an integer-valued metric space (X, d), the injective hull $E(E^0(X))$ of the discrete injective hull $E^0(X)$ of X coincides with the injective hull $E(X)$ of X.

4.3 Hyperbolicity and Helly graphs

In Helly graphs, hyperbolicity can be characterized by forbidding isometric square-grids.

Proposition 4.5 *For a Helly graph G, the following are equivalent*:

(1) *G has bounded hyperbolicity.*

(2) *The size of isometric ℓ_1-square-grids of G is bounded.*

(3) *The size of isometric ℓ_∞-square-grids of G is bounded.*

Proof Since any Helly graph G is weakly modular, by [23, Theorem 9.6] G has bounded hyperbolicity if and only if the metric triangles and the isometric square-grids are of bounded size. Since G is Helly, all metric triangles of G are of size at most one. Therefore G has bounded hyperbolicity if and only if the size of the isometric ℓ_1-square-grids of G are bounded. We now show that in a Helly graph G, the size of the isometric ℓ_1-square-grids is bounded if and only if the size of the isometric ℓ_∞-square-grids is bounded.

Let G contain an isometric $2k \times 2k$ ℓ_1-grid H_1, where

$$V(H_1) = \{(i,j) \in \mathbb{Z}^2 : |i| + |j| \le 2k \text{ and } i+j \text{ is even}\}$$

and $(i,j)(i',j') \in E(H_1)$ if and only if $|i-i'| = |j-j'| = 1$, ie if and only if $d_\infty((i,j),(i',j')) = 1$. Since G is Helly, the Hellyfication H_1' of H_1 is an isometric subgraph of G and H_1' can then be described as follows: $V(H_1') = \{(i,j) \in \mathbb{Z}^2 : |i| + |j| \le 2k\}$ and $(i,j)(i',j') \in E(H_1')$ if and only if $d_\infty((i,j),(i',j')) = 1$. But then the set $\{(i,j) \in V(H_1') : |i| \le k \text{ and } |j| \le k\}$ induces a $2k \times 2k$ ℓ_∞-grid in H_1', and thus in G. Suppose now that G contains an isometric $2k \times 2k$ ℓ_∞-grid H_2, where $V(H_2) = \{(i,j) \in \mathbb{Z}^2 : |i| \le k \text{ and } |j| \le k\}$ and $(i,j)(i',j') \in E(H_1')$ if and only if $d_\infty((i,j),(i',j')) = 1$. Let H_2' be the graph induced by $V(H_2') = \{(i,j) \in \mathbb{Z}^2 : |i| + |j| < k \text{ and } i+j \text{ is even}\}$. Note that H_2' is isomorphic to a $k \times k$ ℓ_1-grid. Since H_2' is an isometric subgraph of H_2, G contains an isometric $k \times k$ ℓ_1-grid. □

Dragan and Guarnera [36] precisely characterize the hyperbolicity of a Helly graph by three families of isometric subgraphs of the ℓ_∞-grid.

5 Helly graph constructions

In the previous section, with any connected graph G we associated in a canonical way a Helly graph Helly(G). However, not every group acting geometrically on G also acts geometrically on Helly(G). In this section, we prove or recall that several standard graph-theoretical operations preserve Hellyness and that other operations applied to some non-Helly graphs lead to Helly graphs. As we will show in the next section, those constructions also preserve the geometric action of the group, allowing us to prove that some classes of groups are Helly.

5.1 Direct products and amalgams

We start with the following well-known result:

Proposition 5.1 *The classes of Helly and clique-Helly graphs are closed under direct products of finitely many factors and retracts.*

The first assertion follows from the fact that the balls in a direct product are direct products of balls in the factors and that the maximal cliques of a direct product are direct products of maximal cliques. The second assertion follows from the fact that retractions are 1-Lipschitz maps and therefore preserve the Helly property.

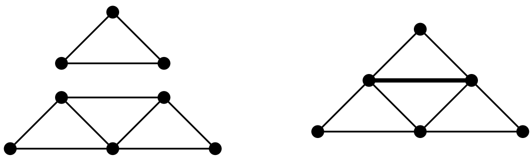

Figure 1: The 3-sun can be obtained from the amalgam of a triangle and a 3-fan over an edge.

The amalgam of two Helly graphs along a Helly graph is not necessarily Helly: the 3-sun (which is not Helly) can be obtained as an amalgam over an edge of a triangle and a 3-fan (which are both Helly); see Figure 1. Consider now amalgams of direct products of (clique-)Helly graphs and, more generally, of graphs obtained by amalgamating together a collection of direct products of (clique-)Helly graphs along common subproducts. We provide sufficient conditions for these amalgams to be (clique-)Helly.

Given a family $\mathcal{H} = \{H_j\}_{j\in J}$ of locally finite graphs, a *finite subproduct* of the direct product $\boxtimes\,\mathcal{H} = \boxtimes_{j\in J} H_j$ is a subgraph $G = \boxtimes_{j\in J} G^j$ of $\boxtimes\,\mathcal{H}$ such that $G^j = H_j$ for finitely many indices and $G^j = \{v_j\}$ where $v_j \in V(H_j)$ for all other indices. For each vertex v of $\boxtimes\, H$ (or any of its subgraphs), we denote by v_j the coordinate of v in H_j.

A locally finite connected graph G is a *union of graph products* (*UGP*) over a family $\mathcal{H} = \{H_j\}_{j\in J}$ of locally finite graphs if there exists a family $\{G_i\}_{i\in I}$ of distinct finite subproducts of $\boxtimes\,\mathcal{H}$ such that $G = \bigcup_i G_i$. The graphs G_i are called the *pieces* of G. Since each $H_j \in \mathcal{H}$ is locally finite and each piece of G is a finite subproduct of $\boxtimes\,\mathcal{H}$, each piece of G is also locally finite. Observe that G is a subgraph of $\boxtimes\,\mathcal{H}$ but not necessarily an induced subgraph. However, each piece G_i of G is an induced subgraph of $\boxtimes\,\mathcal{H}$. We say that the pieces of a collection $\{G_{i_k}\}_{k\in K}$ of pieces of a UGP $G = \bigcup_{i\in I} G_i \subseteq \boxtimes\,\mathcal{H}$ over $\mathcal{H} = \{H_j\}_{j\in J}$ *agree* on a factor H_j if there exists $v_j \in V(H_j)$ such that for each $k \in K$, either $G^j_{i_k} = H_j$ or $G^j_{i_k} = \{v_j\}$.

Lemma 5.2 *Two pieces G_1 and G_2 of a UGP $G \subseteq \boxtimes\,\mathcal{H}$ have a nonempty intersection if and only if G_1 and G_2 agree on all factors $H_j \in \mathcal{H}$.*

The set of pieces $\{G_i\}_{i\in I}$ satisfies the Helly property: any collection $\{G_{i_k}\}_{k\in K}$ of pairwise-intersecting pieces has a nonempty intersection, ie there exists a vertex w of G such that for each $k \in K$ and each factor $H_j \in \mathcal{H}$, either $G^j_{i_k} = \{w_j\}$ or $G^j_{i_k} = H_j$.

Proof First, if G_1 and G_2 agree on all factors H_j, then for each j there exists $w'_j \in V(G^j_1) \cap V(G^j_2)$. Let w be a vertex of $\boxtimes\,\mathcal{H}$ such that $w_j = w'_j$ for all j. Since for each j, $G^j_1 = \{w_j\}$ or $G^j_1 = H_j$, the vertex w belongs to G_1. Similarly, w belongs to G_2 and thus G_1 and G_2 have a nonempty intersection. Conversely, let $u \in V(G_1) \cap V(G_2)$ and note that $u_j \in V(G^j_1)$ for every j. Consequently, either $G^j_1 = H_j$ or $G^j_1 = \{u_j\}$. Similarly, either $G^j_2 = H_j$ or $G^j_2 = \{u_j\}$. In both cases, G_1 and G_2 agree on H_j.

Let $\{G_{i_k}\}_{k\in K}$ be a collection of pairwise-intersecting pieces. By the first statement, any two pieces of this collection agree on all factors $H_j \in \mathcal{H}$. Consequently, for any factor H_j, there exists $w'_j \in V(H_j)$ such that for any $k \in K$, $G^j_{i_k} = \{w_j\}$ or $G^j_{i_k} = H_j$. Consider the vertex w of $\boxtimes\,\mathcal{H}$ such that $w_j = w'_j$ for all j and observe that w belongs to every piece of the collection. □

We say that a UGP satisfies the 3-*piece condition* if, for any three pairwise-intersecting pieces G_1, G_2 and G_3, there exists a piece G_4 intersecting G_1, G_2 and G_3 such that for every factor $H_j \in \mathcal{H}$, if for two pieces G_{i_1} and G_{i_2} among G_1, G_2 and G_3 we have $G_{i_1}^j = G_{i_2}^j = H_j$, then $G_4^j = H_j$.

Proposition 5.3 *If a UGP G over $\mathcal{H}$ satisfies the* 3*-piece condition, then every clique K of G is contained in a piece of G.*

Proof Since G is locally finite, the cliques of G are finite and we can proceed by induction on the size k of K. Suppose that the assertion holds for all cliques of size at most $k-1$. By definition of G, each edge belongs to a piece of G. Let u, v and w be three vertices of K. Since $K \setminus \{w\}$ is a clique of size $k-1$, there exists a piece G_1 containing all vertices of $K \setminus \{w\}$. If $w \in V(G_1)$, we are done. Assume now that $w \notin V(G_1)$. Similarly, we can assume that there exist pieces G_2 and G_3 such that $K \cap V(G_2) = K \setminus \{u\}$ and $K \cap V(G_3) = K \setminus \{v\}$. Since $u \in V(G_1) \cap V(G_3)$, the pieces G_1 and G_3 agree on every factor $H_j \in \mathcal{H}$. Similarly, G_1 and G_2 as well as G_2 and G_3 agree on every factor $H_j \in \mathcal{H}$. Since $u \notin V(G_2)$, necessarily there exists a factor H_{j_2} such that $G_2^{j_2}$ does not contain u_{j_2}. Thus $G_2^{j_2}$ consists of a single vertex $v_2 \neq u_{j_2}$. Since both G_1 and G_3 agree with G_2 on H_{j_2} and since they both contain u_{j_2}, necessarily $G_1^{j_2} = G_3^{j_2} = H_{j_2}$. Similarly, there exist $H_{j_1}, H_{j_3} \in \mathcal{H}$ and vertices $v_1 \in H_{j_1}$ and $v_3 \in H_{j_3}$ such that $G_1^{j_1} = \{v_1\}$, $G_2^{j_1} = G_3^{j_1} = H_{j_1}$, $G_3^{j_3} = \{v_3\}$ and $G_1^{j_3} = G_2^{j_3} = H_{j_3}$. By the 3-piece condition, there exists G_4 intersecting G_1, G_2, and G_3 such that for every factor $H_j \in \mathcal{H}$, if for two pieces G_{i_1} and G_{i_2} among G_1, G_2 and G_3 we have $G_{i_1}^j = G_{i_2}^j = H_j$, then $G_4^j = H_j$. We assert that K is a clique of G_4. Pick any vertex $x \in K$ and note that x belongs to at least two pieces among G_1, G_2 and G_3, say to G_1 and G_2. For each factor $H_j \in \mathcal{H}$, if $G_4^j \neq H_j$, since G_4 agrees with G_1 and G_2 and by the definition of G_4, either $G_4^j = G_1^j = \{x_j\}$ or $G_4^j = G_2^j = \{x_j\}$. Consequently, x is a vertex of G_4 and thus K is a clique of G_4. Therefore all vertices of K belong to a piece of G, and since any piece is an induced subgraph of $\boxtimes \mathcal{H}$ we conclude that K is a clique of this piece. □

Theorem 5.4 *If a UGP G over $\mathcal{H}$ satisfies the* 3*-piece condition and every piece of G is clique-Helly, then G is a clique-Helly graph. Furthermore, if the clique complex $X(G)$ of G is simply connected then G is a Helly graph.*

Proof Since G has finite cliques, we can use Proposition 2.9 to establish the clique-Helly property for G. Pick any triangle $T = u_1u_2u_3$ of G and let T^* be the set of vertices of G adjacent to at least two vertices of T. For any $v \in T^*$, by Proposition 5.3 there exists a piece containing a triangle vu_iu_j; let P^* be the set of all pieces containing such triangles. Since the pieces of P^* piecewise intersect, by the first assertion of Lemma 5.2 they pairwise agree on every factor $H_j \in \mathcal{H}$. By the second assertion of Lemma 5.2, there exists a vertex $w \in G$ such that either $G_i^j = \{w_j\}$ or $G_i^j = H_j$ for any piece G_i of P^*. Thus w belongs to every piece of P^*.

For each factor $H_j \in \mathcal{H}$, let $T_j = \{u_j : u \in T\}$ and $T_j^* = \{v_j : v \in T_j^*\}$. Note that T_j is either a vertex, an edge or a triangle in H_j. Moreover, in the first two cases, there exists $u_j \in T_j$ that belongs to the 1-ball of every vertex $v_j \in T_j^*$. If T_j is a triangle, then every vertex $v_j \in T_j^*$ is in the 1-ball of at least two vertices of T_j. Since H_j is clique-Helly, in all three cases there exists a vertex $w_j \in V(H_j)$ belonging to the 1-ball of each vertex $v_j \in T_j^*$. Observe that if there exists a piece G_i of P^* such that G_i^j contains only one vertex, then T_j is a vertex or an edge and we can choose $w_j \in V(H_j)$ such that $G_i^j = \{w_j\}$.

Let w^* be the vertex of G such that $w_j^* = w_j$ for every factor $H_j \in \mathcal{H}$. By our choice of w_j, for any piece G_i of P^* such that G_i^j contains only one vertex, $G_i^j = \{w_j\}$ and for any other piece G_i of P^*, w_j is a vertex of $G_i^j = H_j$. Therefore w^* is a vertex that belongs to all pieces of P^*. For any vertex $v \in T^*$ and any factor $H_j \in \mathcal{H}$, v_j is in the 1-ball of w_j in H_j by our choice of w_j. Since each piece G_i of G is an induced subgraph of $\boxtimes \mathcal{H}$, w^* is in the 1-ball in G of all vertices v of T^*, establishing that G is clique-Helly.

The second assertion follows from Theorem 4.1. □

Given a family $\mathcal{H} = \{H_j\}_{j \in J}$ of locally finite graphs, an *abstract graph of subproducts* (*GSP*) $(\mathcal{H}, \mathcal{G}, \ell)$ is given by a connected graph $\mathcal{G}$ without infinite clique and a map $\ell\colon V(\mathcal{G}) \to 2^{\mathcal{H}}$ satisfying the following conditions:

(A1) $\ell(v)$ is a finite subset of $\mathcal{H}$ for each $v \in V(\mathcal{G})$.

(A2) For each edge $uv \in E(\mathcal{G})$, $\ell(u) \neq \ell(v)$.

A *realization* of an abstract GSP $(\mathcal{H}, \mathcal{G}, \ell)$ is a set of maps

$$\Big\{ p_v : \mathcal{H} \setminus \ell(v) \to \bigcup_{j \in J} V(H_j) \Big\}_{v \in V(\mathcal{G})}$$

satisfying the following conditions:

(A3) For each $v \in V(\mathcal{G})$, $p_v(H_j) \in V(H_j)$ for every factor $H_j \in \mathcal{H} \setminus \ell(v)$.

(A4) For any vertices $u, v \in V(\mathcal{G})$, there is an edge $uv \in E(\mathcal{G})$ if and only if $p_u(H_j) = p_v(H_j)$ for every factor $H_j \in \mathcal{H} \setminus (\ell(u) \cup \ell(v))$.

A GSP admitting a realization is called a *realizable GSP*.

Proposition 5.5 *For any realizable GSP $(\mathcal{H}, \mathcal{G}, \ell)$ and any of its realizations $\{p_v\}_{v \in V(\mathcal{G})}$, we can define a UGP $G(\mathcal{G}) = \bigcup_{v \in V(\mathcal{G})} G_v$ where there is a piece $G_v = \boxtimes_{j \in J} G_v^j$ for each $v \in V(\mathcal{G})$ such that $G_v^j = H_j$ if $H_j \in \ell(v)$ and $G_v^j = \{p_v(H_j)\}$ otherwise. Conversely, any UGP $G \subseteq \boxtimes \mathcal{H}$ is the realization of a realizable GSP over $\boxtimes \mathcal{H}$.*

Proof First note that (A4) is equivalent to the following condition on the pieces of $G(\mathcal{G})$:

(A4′) For any vertices $u, v \in V(\mathcal{G})$, there is an edge $uv \in E(\mathcal{G})$ if and only if $V(G_u) \cap V(G_v) \neq \varnothing$.

In order to show that $G(\mathcal{G})$ is a UGP, we must show that it is locally finite. Consider a vertex $u \in G(\mathcal{G})$ that has an infinite number of neighbors. Since each piece containing u is locally finite, there are an infinite number of pieces containing u. By (A4′), these pieces form an infinite clique in $\mathcal{G}$, a contradiction. Moreover, if there exist two vertices $u, v \in V(\mathcal{G})$ such that the pieces G_u and G_v coincide, then $\ell(u) = \ell(v)$ and $p_u(H_j) = p_v(H_j)$ for any $H_j \in \mathcal{H} \setminus \{\ell(u)\}$. So $uv \in E(\mathcal{G})$ and $\ell(u) = \ell(v)$, contradicting (A2).

Conversely, given a UGP G over $\mathcal{H}$, we construct a realizable GSP as follows. In $\mathcal{G}$, there is a vertex v_i for each piece G_i of G, and we set $\ell(v_i) = \{H_j \in \mathcal{H} : G_i^j = H_j\}$. For each $H_j \notin \ell(v_i)$, there exists $w_j \in V(H_j)$ such that $G_i^j = \{w_j\}$, and we set $p_{v_i}(H_j) = w_j$. For any vertices $v_i, v_{i'} \in V(\mathcal{G})$, there is an edge $v_i v_{i'} \in E(\mathcal{G})$ if and only if $p_{v_i}(H_j) = p_{v_{i'}}(H_j)$ for every factor $H_j \notin \ell(v_i) \cup \ell(v_{i'})$.

Since each piece G_i is a finite subproduct of $\boxtimes H$, $\ell(v_i)$ is finite for each $v_i \in V(\mathcal{G})$ and thus (A1) holds. By definition of p_{v_i} and of the edges of $E(\mathcal{G})$, (A2) and (A4) also hold. Observe also that $G(\mathcal{G})$ and G are isomorphic and thus G is the realization of $\mathcal{G}$. It remains to show that $\mathcal{G}$ does not contain infinite cliques. By (A4′), if there exists an infinite clique in $\mathcal{G}$, then there exists an infinite collection $\{G_{i_k}\}_{k \in K}$ of pairwise-intersecting pieces. By Lemma 5.2, this implies that there exists a vertex w that belongs to every piece G_{i_k}. Since all pieces of G are distinct and since w belongs to an infinite number of pieces, there exists an infinite collection of factors $\{H_{j'}\}_{j' \in J'}$ such that for each $H_{j'}$ there exists a piece G_{i_k} with $w \in G_{i_k}$ and $G_{i_k}^{j'} = H_{j'}$. Consequently, for each $j' \in J'$, one can find a vertex $w^{j'} \in \boxtimes \mathcal{H}$ in G obtained from w by replacing the coordinate $w_{j'}$ by one of its neighbors in $H_{j'}$. All the $w^{j'}$ constructed in this way are distinct and they are all neighbors of w in G. Consequently, w has infinitely many neighbors in G and thus G is not locally finite, a contradiction. □

We say that a GSP $(\mathcal{H}, \mathcal{G}, \ell)$ satisfies the *product-Gilmore condition* if for every triangle $\mathcal{T} = x_1x_2x_3$ of $\mathcal{G}$ there exists $y \in V(\mathcal{G})$ such that $y = x_i$ or $y \sim x_i$ for $1 \le i \le 3$ and

$$(\ell(x_1) \cap \ell(x_2)) \cup (\ell(x_2) \cap \ell(x_3)) \cup (\ell(x_1) \cap \ell(x_3)) \subseteq \ell(y).$$

Proposition 5.6 *For a realizable GSP $(\mathcal{H}, \mathcal{G}, \ell)$ and any of its realizations $\{p_v\}_{v \in V(\mathcal{G})}$, the UGP $G(\mathcal{G})$ obtained from $\mathcal{G}$ and $\{p_v\}_{v \in V(\mathcal{G})}$ satisfies the 3-piece condition if and only if $(\mathcal{H}, \mathcal{G}, \ell)$ satisfies the product-Gilmore condition.*

Proof Assume $(\mathcal{H}, \mathcal{G}, \ell)$ satisfies the product-Gilmore condition. By (A4′), two pieces in the UGP $G(\mathcal{G})$ obtained from a realization of a GSP $\mathcal{G}$ intersect if and only if there is an edge between the corresponding vertices of $\mathcal{G}$. Thus it is enough to consider three pieces G_{x_1}, G_{x_2} and G_{x_3} corresponding to three vertices x_1, x_2 and x_3 that are pairwise adjacent in $\mathcal{G}$. By our assumption, there exists a vertex $y \in V(\mathcal{G})$ such that $y = x_i$ or $y \sim x_i$ for any $1 \le i \le 3$ and $(\ell(x_1) \cap \ell(x_2)) \cup (\ell(x_2) \cap \ell(x_3)) \cup (\ell(x_1) \cap \ell(x_3)) \subseteq \ell(y)$. Consider the piece G_y in $G(\mathcal{G})$. By (A4′), G_y intersects G_{x_1}, G_{x_2}, and G_{x_3}. Moreover, for any factor $H_j \in \mathcal{H}$, if $G_{x_1}^j = G_{x_2}^j = H_j$, by the definition of $G(\mathcal{G})$ we obtain $H_j \in \ell(x_1) \cap \ell(x_2) \subseteq \ell(y)$. Similarly, for any $H_j \in \mathcal{H}$ such that $G_{x_2}^j = G_{x_3}^j = H_j$ or $G_{x_1}^j = G_{x_3}^j = H_j$, we have $H_j \in \ell(y)$. This establishes the 3-piece condition for $G(\mathcal{G})$.

Conversely, suppose that $G(\mathcal{G})$ satisfies the 3-piece condition and consider a triangle $x_1x_2x_3$ of $\mathcal{G}$ and the three corresponding pieces G_{x_1}, G_{x_2} and G_{x_3} of $G(\mathcal{G})$. By (A4′), $V(G_{x_1})$, $V(G_{x_2})$ and $V(G_{x_3})$ pairwise intersect. By the 3-piece condition, there exists $x_4 \in V(\mathcal{G})$ such that $V(G_{x_4})$ intersects $V(G_{x_1})$, $V(G_{x_2})$, and $V(G_{x_3})$, ie x_4 either coincides with or is adjacent to each x_i for $1 \le i \le 3$. Moreover, for each $H_j \in \ell(x_1) \cap \ell(x_2)$ we have $G^j_{x_1} = G^j_{x_2} = H_j$, and the definition of G_{x_4} implies that $G^j_{x_4} = H_j$, ie $H_j \in \ell(x_4)$. Consequently $\ell(x_1) \cap \ell(x_2) \subseteq \ell(x_4)$, and similarly $(\ell(x_2) \cap \ell(x_3)) \cup (\ell(x_1) \cap \ell(x_3)) \subseteq \ell(x_4)$. This proves the product-Gilmore condition for $(\mathcal{H}, \mathcal{G}, \ell)$. □

From Propositions 5.1 and 5.6 and Theorem 5.4 we obtain the following corollary:

Corollary 5.7 *Consider a realizable GSP $(\mathcal{H}, \mathcal{G}, \ell)$ and any of its realizations $\{p_v\}_{v \in V(\mathcal{G})}$. If $(\mathcal{H}, \mathcal{G}, \ell)$ satisfies the product-Gilmore condition and if each factor $H \in \mathcal{H}$ is clique-Helly, then $G(\mathcal{G})$ is a clique-Helly graph. Furthermore, if the clique complex $X(G(\mathcal{G}))$ is simply connected, then $G(\mathcal{G})$ is a Helly graph.*

Thickenings of locally finite median graphs (ie of CAT(0) cube complexes) is an instructive example of clique-Helly graphs that can be obtained via Theorem 5.4 or Corollary 5.7. The pieces of a median graph G seen as a UGP are the thickenings of the maximal cubes of G. The fact that it satisfies the product-Gilmore condition follows from the fact that the cell hypergraph is conformal, which can be derived from the cube condition of the CAT(0) cube complex $X_{\text{cube}}(G)$.

5.2 Thickening

The direct product of graphs considered above is the l_∞ version of the Cartesian product. Thus, when we turn all k-cubes of the Cartesian product of k paths into simplices, we have the corresponding direct product of k paths. More generally, a similar operator transforms median graphs into Helly graphs: let G^Δ be the graph having the same vertex set as G, where two vertices are adjacent if and only if they belong to a common cube of G. The graph G^Δ is called the *thickening* of G (for l_∞-metrization of cube complexes, of median graphs and, more generally, of median spaces; see [19; 87]).

Proposition 5.8 [10] *If G is a locally finite median graph, then G^Δ is a Helly graph and each maximal clique of G^Δ is a cube of G.*

The *thickening* X^Δ of an abstract cell complex X is a graph obtained from X by making adjacent all pairs of vertices of X belonging to a common cell of X. Equivalently, the thickening of X is the 2-section $[\mathcal{H}(X)]_2$ of the hypergraph $\mathcal{H}(X)$. We say that an abstract cell complex X is *simply connected* if the clique complex of its thickening X^Δ is simply connected.

Proposition 5.8 of Bandelt and Van de Vel was extended to the thickenings of the abstract cell complexes arising from swm-graphs and from hypercellular graphs.

Proposition 5.9 [23; 30] *The thickening $G^\Delta := X(G)^\Delta$ of the abstract cell complex $X(G)$ associated to any locally finite swm-graph or any hypercellular graph G is a Helly graph. Each maximal clique of G^Δ is a cell of $X(G)$.*

The existing proofs of Propositions 5.8 and 5.9 are based on the following global property of G^Δ: each ball of G^Δ defines a gated subgraph of G; thus G^Δ is Helly since gated sets satisfy the finite Helly property.

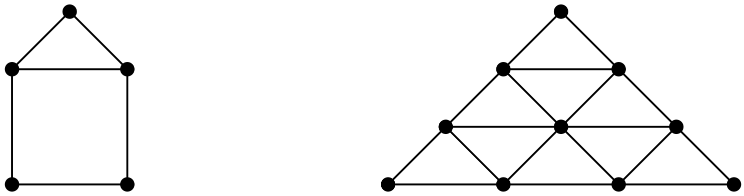

Figure 2: A house (left) and a 3-deltoid (right).

5.3 Coarse Helly graphs

The coarse Helly property of a graph G can be used to show via Hellyfication that a group acting on G geometrically is Helly. In this subsection, we recall the result of [29] that δ-hyperbolic graphs are coarse Helly and we deduce from a result of [26] that several subclasses of weakly modular graphs (in particular, cube-free median graphs, hereditary modular graphs and 7-systolic graphs) are coarse Helly.

Proposition 5.10 [29] *If G is a δ-hyperbolic graph, then G is coarse Helly for 2δ.*

Proposition 5.11 [26] *A weakly modular graph not containing isometric cycles of length > 5, houses or 3-deltoids (see Figure 2) is coarse Helly with constant 1. In particular, cube-free median graphs, hereditary modular graphs and 7-systolic graphs are coarse Helly.*

It is known that the systolic (bridged) graphs satisfying the conditions of Proposition 5.11 are all hyperbolic [23; 28]. Cube-free median graphs and, more generally, hereditary modular graphs (which by a result of [3] are exactly the graphs in which all isometric cycles have length 4) in general are not hyperbolic. On the other hand, general median graphs are not coarse Helly: already the cubic grid $\mathbb{Z}^3$ is not coarse Helly as shown by the following example:

Example 5.12 In $\mathbb{Z}^3$, for any integer n, consider four balls of radius $2n$ centered at $x_1 = (-2n, 2n, -2n)$, $x_2 = (2n, 2n, 2n)$, $x_3 = (-2n, -2n, 2n)$ and $x_4 = (2n, -2n, -2n)$. Observe first that for any two such nodes x_l and $x_{l'}$, $d(x_l, x_{l'}) = 4n$ and thus the four balls pairwise intersect. We show that for any node $y = (i, j, k) \in \mathbb{Z}^3$, we have $\max\{d(y, x_l) : 1 \le l \le 4\} \ge 6n$. Assume that y minimizes this maximum. Observe that if $y \notin [-2n, 2n]^3$, then its gate y' in the box $[-2n, 2n]^3$ is strictly closer to each x_l, contrary to our choice of y. Consequently $i, j, k \in [-2n, 2n]$, $d(y, x_1) = i + 2n + 2n - j + k + 2n = 6n + i - j + k$, $d(y, x_2) = 6n - i - j - k$, $d(y, x_3) = 6n + i + j - k$ and $d(y, x_4) = 6n - i + j + k$, and thus $\sum_{i=1}^{4} d(x_i, y) = 24n$. Therefore $\max\{d(y, x_l) : 1 \le l \le 4\} \ge 6n$.

Analogously, the triangular grid (alias, the systolic plane) is also not coarse Helly:

Example 5.13 $\mathbb{T}_3$ is the graph of the tiling of the plane into equilateral triangles with side length 1. $\mathbb{T}_3$ is a bridged graph. Pick three vertices x_1, x_2 and $z = x_3$ of $\mathbb{T}_3$ which define an equilateral triangle $\Delta(x_1, x_2, x_3)$ of $\mathbb{T}_3$ with side length $6n$. Consider the three balls $B_{3n}(x_1)$, $B_{3n}(x_2)$ and $B_{3n}(x_3)$. We assert that $\max\{d(y,x_i) : 1 \le i \le 3\} \ge 4n$ for any vertex y of $V(\mathbb{T}_3)$. If $y \notin \Delta(x_1,x_2,x_3)$, then y is in one of the half-planes defined by the sides of $\Delta(x_1,x_2,x_3)$ and not containing $\Delta(x_1,x_2,x_3)$, say in the halfspace defined by x_1 and x_2. But then $d(x_3,y) \ge 6n$ because x_3 has distance $\ge 6n$ to any vertex of $\mathbb{T}_3$ defined by the line between x_1 and x_2. Now let $y \in \Delta(x_1,x_2,x_3)$. It can be shown easily by induction on k that if $\Delta(x_1,x_2,x_3)$ is a deltoid of size k of $\mathbb{T}_3$, then $d(y,x_1)+d(y,x_2)+d(y,x_3)=2k$ for any $y \in \Delta(x_1,x_2,x_3)$. This shows that in our case $d(y,x_1)+d(y,x_2)+d(y,x_3) \ge 12n$, ie $\max\{d(y,x_i) : 1 \le i \le 3\} \ge 4n$.

5.4 Nerve graphs of clique-hypergraphs

We first show that (clique-)Hellyness is preserved by the nerve complex $N(\mathcal{X}(G))$ of the clique-hypergraph $\mathcal{X}(G)$ of a Helly graph G. Nerve complexes of clique-hypergraphs are also called *clique graphs* in the literature; see eg [9]. In general, the nerve complex $N(\mathcal{X}(G))$ of the clique-hypergraph of a graph G is not a flag simplicial complex. However, $N(\mathcal{X}(G))$ is flag if G is clique-Helly:

Lemma 5.14 *For any locally finite graph G, $N(\mathcal{X}(G))$ is a flag simplicial complex if and only if G is a (finitely) clique-Helly graph.*

Proof By definition, $N(\mathcal{X}(G))$ is a flag simplicial complex if and only if any finite set of pairwise-intersecting cliques $K_1,K_2,\dots,K_p$ of G have a nonempty intersection. This is precisely the definition of a finitely clique-Helly graph. Since G is locally finite, G is finitely clique-Helly if and only if G is clique-Helly. □

The first assertion of the following result was first proved by Escalante [41] (he also proved the converse, that any clique-Helly graph is the clique graph of some graph):

Proposition 5.15 *If G is a locally finite clique-Helly graph, then the nerve graph $NG(\mathcal{X}(G))$ of the clique-hypergraph $\mathcal{X}(G)$ is a clique-Helly graph and its flag-completion is a clique-Helly complex. If G is a locally finite Helly graph, then $NG(\mathcal{X}(G))$ is a Helly graph and its flag-completion is a Helly complex.*

Proof Let G be a locally finite clique-Helly graph. Let G' be the nerve graph of the clique-hypergraph $\mathcal{X}(G)$. Since G is locally finite, G' is also locally finite. We prove that G' is clique-Helly by using the triangle criterion from Proposition 2.9. Let uvw be a triangle in G'. It corresponds to three pairwise intersecting, and thus intersecting, maximal cliques in G, denoted by the same symbols u, v and w. Observe that all vertices of $(u\cap v)\cup(v\cap w)\cup(w\cap u)$ are pairwise adjacent in G, and thus $u\cap v$, $v\cap w$ and $w\cap u$ are all contained in a common maximal clique x in $G(X)$. We claim that every vertex y in G' that is adjacent to u and v in G' is also adjacent to x in G'. This is so because in G, the maximal clique y intersects u and v, and hence intersects $u\cap v$ since G is a clique-Helly graph. Since $u\cap v \subseteq x$, y intersects x in G and thus $x \sim y$ in G'. Similarly, the vertex $x \in G'$ is a universal vertex for triangles containing $\{v,w\}$ and $\{w,u\}$ in G'. Consequently, the nerve graph G' is clique-Helly.

Suppose now that G is Helly. By Theorem 4.1, $X(G)$ is simply connected and G is a clique-Helly graph. By the first part of the theorem, the 1-skeleton $G' = G(Y)$ of the nerve complex Y of the clique-hypergraph $\mathcal{X}(G)$ is clique-Helly. By Borsuk's nerve theorem [18; 17], $X(G)$ and Y have the same homotopy type. Consequently, Y is also simply connected. By Theorem 4.1, this implies that $G' = G(Y)$ is Helly. □

We now show that the clique-Hellyness of the nerve graph is preserved by taking covers:

Theorem 5.16 *Given two locally finite graphs G and G' such that the clique complex $X(G)$ is a cover of the clique complex $X(G')$, the nerve graph $NG(\mathcal{X}(G))$ is clique-Helly if and only if the nerve graph $NG(\mathcal{X}(G'))$ is clique-Helly.*

Theorem 4.1 immediately gives the following corollary since the nerve complex of the maximal simplices of a simply connected simplicial complex is simply connected by Borsuk's nerve theorem [18; 17].

Corollary 5.17 *For a locally finite graph G, the nerve graph $NG(\mathcal{X}(\widetilde{G}))$ of the clique-hypergraph of the 1-skeleton $\widetilde{G}$ of the universal cover $\widetilde{X}(G)$ of $X(G)$ is Helly if and only if the nerve graph $NG(\mathcal{X}(G))$ of the clique-hypergraph $\mathcal{X}(G)$ is clique-Helly.*

Theorem 5.16 is proved via the following lemma, establishing that a covering map between the clique complexes of two graphs extends to a covering map between the nerve complexes of the corresponding clique-hypergraphs.

Lemma 5.18 *Given two locally finite simple graphs G and G', any covering map $\varphi\colon X(G) \to X(G')$ induces a covering map from $N(\mathcal{X}(G))$ to $N(\mathcal{X}(G'))$.*

Proof In $N(\mathcal{X}(G))$, the vertices are the maximal cliques of G and a finite set $\sigma = \{K_1, \dots, K_p\}$ of cliques of G is a simplex of $N(\mathcal{X}(G))$ if $\bigcap_{i=1}^{p} K_i \neq \varnothing$. Since G is locally finite, each maximal clique K of G is finite. We extend the map φ to all cliques of G: for any clique $K = \{u_1, \dots, u_k\}$ of G, we set $\varphi(K) = \{\varphi(u_1), \dots, \varphi(u_k)\}$. Observe that for any clique $K = \{u_1, \dots, u_k\}$ of G we have $u_i \sim u_j$, and thus $\varphi(u_i) \sim \varphi(u_j)$. Since G' is loop-free, $\varphi(K)$ is a clique of G' and $|\varphi(K)| = |K|$.

Consider two cliques K of G and K' of G' such that $K' = \varphi(K)$. For any $u \in K$, φ induces a bijection between the cliques containing u and the cliques containing $\varphi(u)$. So K is a maximal clique of G if and only if K' is a maximal clique of G'. Therefore φ induces a map from $V(N(\mathcal{X}(G)))$ to $V(N(\mathcal{X}(G')))$.

We now prove a useful claim:

Claim 5.19 *For any maximal cliques K_1 and K_2 of G such that $K_1 \cap K_2 \neq \varnothing$, we have $\varphi(K_1) \cap \varphi(K_2) = \varphi(K_1 \cap K_2)$.*

Proof The inclusion $\varphi(K_1 \cap K_2) \subseteq \varphi(K_1) \cap \varphi(K_2)$ is trivial. Suppose now that the reverse inclusion does not hold, ie that there exist $u_1 \in K_1 \setminus K_2$ and $u_2 \in K_2 \setminus K_1$ such that $\varphi(u_1) = \varphi(u_2)$. Pick $u \in K_1 \cap K_2$ and observe that $u \sim u_1$ since K_1 is a clique and $u \sim u_2$ since K_2 is a clique. Consequently the map φ is not locally injective at u, a contradiction. □

Note that if $\sigma = \{K_1, \dots, K_p\}$ is a simplex of $N(\mathcal{X}(G))$, then there exists $u \in \bigcap_{i=1}^p K_i$. Consequently $\varphi(u) \in \bigcap_{i=1}^p \varphi(K_i)$, and thus the image of a simplex of $N(\mathcal{X}(G))$ is a simplex of $N(\mathcal{X}(G'))$. Thus φ is a simplicial map from $N(\mathcal{X}(G))$ to $N(\mathcal{X}(G'))$. Moreover, for any $1 \le i < j \le p$ we have $u \in K_i \cap K_j$, and consequently, by Claim 5.19, $\varphi(K_i) \cap \varphi(K_j) = \varphi(K_i \cap K_j)$. Since $|\varphi(K_i)| = |K_i|$ and $|\varphi(K_j)| = |K_j|$, this implies that if $K_i \neq K_j$ then $\varphi(K_i) \neq \varphi(K_j)$. Consequently, $|\varphi(\sigma)| = |\sigma|$.

We now show that φ is locally surjective. Let $K_0 \in V(N(\mathcal{X}(G)))$ and $K'_0 = \varphi(K) \in V(N(\mathcal{X}(G')))$ and consider a simplex $\sigma' = \{K'_0, K'_1, \dots, K'_p\}$ in $N(\mathcal{X}(G'))$. By definition of $N(\mathcal{X}(G'))$, there exists $u' \in \bigcap_{i=1}^p K'_i$. Since $K'_0 = \varphi(K_0)$, there exists $u \in K_0$ such that $u' = \varphi(u)$. Since φ is a covering map from G to G', for each $1 \le i \le p$, there exists $K_i \in V(N(\mathcal{X}(G)))$ such that $u \in K_i$ and $K'_i = \varphi(K_i)$. Since $u \in \bigcap_{i=1}^p K_i$, $\sigma = \{K_0, K_1, \dots, K_p\}$ is a simplex of $N(\mathcal{X}(G'))$ that is mapped to σ' by φ.

We now show that φ is locally injective. Consider $K_0 \in V(N(\mathcal{X}(G)))$ and assume that there exist two distinct simplices σ_1 and σ_2 in $N(\mathcal{X}(G))$ such that $K_0 \in \sigma_1 \cap \sigma_2$ and $\varphi(\sigma_1) = \varphi(\sigma_2)$. Since $|\varphi(\sigma_1)| = |\sigma_1|$ and $|\varphi(\sigma_2)| = |\sigma_2|$, it implies that there exist $K_1 \in \sigma_1 \setminus \sigma_2$ and $K_2 \in \sigma_2 \setminus \sigma_1$ such that $\varphi(K_1) = \varphi(K_2)$. If $K_1 \cap K_2 \neq \varnothing$, since $|\varphi(K_1)| = |K_1|$ and $|\varphi(K_2)| = |K_2|$, by Claim 5.19 we have $K_1 = K_2$, a contradiction. Consequently, $K_1 \cap K_2 = \varnothing$. Consider two distinct vertices $u_1 \in K_0 \cap K_1$ and $u_2 \in K_0 \cap K_2$. Since $|\varphi(K_0)| = |K_0|$, we have $\varphi(u_1) \neq \varphi(u_2)$. Since $\varphi(K_2) = \varphi(K_1)$, there exists $v_1 \in K_2$ such that $\varphi(v_1) = \varphi(u_1)$. But $u_2 \sim u_1$ since $u_1, u_2 \in K_0$ and $u_2 \sim v_1$ since $u_2, v_1 \in K_2$. This contradicts the local injectivity of φ at u_2.

Consequently, φ defines a simplicial map from $N(\mathcal{X}(G))$ to $N(\mathcal{X}(G'))$ that induces a bijection between the simplices containing a vertex of $N(\mathcal{X}(G))$ and the simplices containing its image, ie φ defines a covering map from $N(\mathcal{X}(G))$ to $N(\mathcal{X}(G'))$. □

Since a covering map is locally bijective, from Lemma 5.18 and Proposition 2.9 we conclude that the nerve graph $NG(\mathcal{X}(G))$ is clique-Helly if and only if the nerve graph $NG(\mathcal{X}(G'))$ is clique-Helly. This concludes the proof of Theorem 5.16.

5.5 Rips complexes and nerve complexes of δ-ball-hypergraphs

The *Rips complex* $R_\delta(M)$ of a metric space (M, d) and positive real δ is an abstract simplicial complex that has a simplex for every finite set of points of M that has diameter at most δ. If (M, d) is a connected unweighted graph G, then for any positive real δ, $R_\delta(G)$ and $R_{\lfloor\delta\rfloor}(G)$ coincide. In this case, we can thus assume that δ is a positive integer, and then the Rips complex $R_\delta(G)$ is just the δ^{th} power G^δ of G. Notice that for any $\delta \in \mathbb{N}$, the nerve complex $N(\mathcal{B}_\delta(G))$ of the δ-ball-hypergraph $\mathcal{B}_\delta(G)$ is isomorphic to the Rips complex $R_{2\delta}(G)$.

Lemma 5.20 *Rips complexes $R_\delta(G)$ of a Helly graph G are Helly.*

Proof As noted above, δ can be assumed to be an integer and thus the Rips complex $R_\delta(G)$ coincides with the δ^{th} power G^δ of G. For any vertex v and any radius r, note that $B_r(v, G^\delta) = B_{r\delta}(v, G)$. Thus the result follows since the family of balls of G satisfies the Helly property. □

5.6 Face complexes

The *face complex* $F(X)$ of a locally finite abstract simplicial complex X is the simplicial complex whose vertex set $V(F(X))$ is the set of nonempty simplices of X and where $\{F_1, F_2, \dots, F_k\}$ is a simplex of $F(X)$ if $\bigcup_{i=1}^{k} F_i$ is contained in a common simplex F of X. If X is the clique complex of a graph G, then the vertices of $F(X)$ are the cliques of G and two cliques K_1 and K_2 of G are adjacent in the 1-skeleton of $F(X)$ if $K_1 \cup K_2$ is a clique.

Given a maximal simplex $\sigma = \{F_1, F_2, \dots, F_k\}$ of $F(X)$, $\bigcup_{i=1}^{k} F_i$ is contained in a common simplex F of X. By maximality of σ, $F = \bigcup \sigma = \bigcup_{i=1}^{k} F_i$ and F is a maximal simplex of X. Moreover, $F \in \sigma$, and consequently, since σ is maximal, $\sigma = \mathcal{P}(F) \setminus \{\varnothing\}$ where $\mathcal{P}(F)$ is the set of all subsets of F. Conversely, for any maximal simplex F of X, by definition of $F(X)$, $\sigma = \mathcal{P}(F) \setminus \{\varnothing\}$ is a simplex of $F(X)$. Since F is a maximal simplex of X, σ must be a maximal simplex of $F(X)$. As a result, we obtain the following lemma:

Lemma 5.21 *For any simplicial complex X, the map $\sigma \mapsto \bigcup \sigma$ defines a bijection from the set of maximal simplices of $F(X)$ to the set of maximal simplices of X, with inverse given by $F \mapsto \mathcal{P}(F) \setminus \{\varnothing\}$.*

The face complexes of clique complexes are also clique complexes:

Lemma 5.22 *For any clique complex X, its face complex $F(X)$ is also a clique complex.*

Proof Let $G = G(X)$ be the 1-skeleton of X and let $G' = G(F(X))$ be the 1-skeleton of $F(X)$. For any edge F_1F_2 in G', F_1, F_2 and $F_1 \cup F_2$ are cliques of G. Consequently, for any clique $\sigma = \{F_1, F_2, \dots, F_k\}$ in $G(F(X))$, $F_1 \cup F_2 \cup \dots \cup F_k$ is a clique of G. Since X is the clique complex of G, $F_1 \cup F_2 \cup \dots \cup F_k$ is a clique of X and thus σ is a simplex of $F(X)$. □

Proposition 5.23 *The face complex $F(X)$ of a locally finite clique-Helly (resp. Helly) complex X is a locally finite clique-Helly (resp. Helly) complex.*

Proof By Lemma 5.22, $F(X)$ is a clique complex. Let $G = G(X)$ be the 1-skeleton of X and $G' = G(F(X))$ be the 1-skeleton of $F(X)$. Since G is locally finite, G' is also locally finite and thus $F(X)$ is a locally finite simplicial complex. Consider the bijection $\sigma \mapsto \bigcup \sigma$ between the maximal cliques of $F(X)$ and the maximal cliques of X defined in Lemma 5.21. Observe that if $(\sigma_i)_{i \in I}$ is a family of maximal cliques of $F(X)$, then $\bigcap_{i \in I} \sigma_i \neq \varnothing$ if and only if $\bigcap_{i \in I} \left(\bigcup \sigma_i\right) \neq \varnothing$. Consequently, since X is clique-Helly, $F(X)$ is also clique-Helly.

Suppose now that X is simply connected. Since the nerve complexes of the clique hypergraphs of X and $F(X)$ are isomorphic thanks to the bijection $\sigma \mapsto \bigcup \sigma$, X and $F(X)$ are homotopy equivalent by Borsuk's nerve theorem [18; 17]. Consequently $F(X)$ is simply connected, and thus by Theorem 4.1 $F(X)$ is a Helly complex when X is a Helly complex. □

6 Helly groups

As defined above, a group is *Helly* if it acts geometrically on a (necessarily, locally finite) Helly graph. The main goal of this section is to provide examples of Helly groups. More precisely, in this section we prove Theorems 1.1, 1.2, 1.3 and 1.4, some of their consequences and related results.

6.1 Proving Hellyness of a group

To prove that a group Γ (geometrically) acting on a cell complex X (or on its 1-skeleton $G(X)$) is Helly, we will derive from X a Helly complex X^* and prove that Γ acts geometrically on X^*. The natural (and most canonical) way would be to take as X^* the Hellyfication $\mathrm{Helly}(X)$ of X. By Theorem 4.2, $\mathrm{Helly}(X)$ is well defined and Helly for all complexes X. The group Γ acts on $\mathrm{Helly}(X)$, but the group action is not always geometrical. However, using the results from Sections 4.2 and 5.3, and a result of Lang [65], we will prove that hyperbolic groups act geometrically on the Hellyfication of their Cayley graphs that are hyperbolic, and thus hyperbolic groups are Helly.

In several other cases there are more direct ways to derive X^*. In the case of CAT(0) cubical groups, based on Proposition 5.8 and the bijection between median graphs and 1-skeletons of CAT(0) cube complexes [27; 81], it follows that thickenings along cubes of locally finite CAT(0) cube complexes are Helly. Thus CAT(0) cubical groups are Helly. By Proposition 5.9, the thickenings of locally finite hypercellular complexes and of locally finite swm-complexes are Helly. Consequently, groups acting geometrically on hypercellular graphs or swm-graphs are Helly. We use the same technique by thickening (along cells) to show that classical C(4)–T(4) small-cancellation and graphical C(4)–T(4) small-cancellation groups are Helly. In all these cases, the maximal cliques of the thickenings correspond to cells of the original complex. This allows us to establish that the group Γ acts geometrically on the thickening. By considering face complexes, we show that Helly groups are stable by free products with amalgamation over finite subgroups and by quotients by finite normal subgroups. Using the theory of quasimedian groups of [45], we provide criteria allowing the construction of Helly groups from groups acting on quasimedian graphs. This allows us to show that Helly groups are stable by taking graph products, $\square$-products, $\rtimes$-powers and $\bowtie$-products of groups. We also show that the fundamental groups of right-angled graphs of Helly groups are Helly.

6.2 CAT(0) cubical hypercellular, and swm-groups via thickening

A group Γ is called *cubical* if Γ acts geometrically on a median graph G (or on the CAT(0) cube complex of G). A group Γ is called an *swm-group* if it acts geometrically on an swm-graph G (or on the orthoscheme complex of G). A group Γ is called *hypercellular* if it acts geometrically on a hypercellular graph G (or on the geometric realization of G).

Any group Γ acting geometrically on a median graph, swm-graph or hypercellular graph G also acts geometrically on its thickening G^{Δ}. From Propositions 5.8 and 5.9 we obtain:

Proposition 6.1 *Cubical groups, swm-groups and hypercellular groups are Helly.*

In [23], with every building Δ of type C_n we associated an swm-graph $H(\Delta)$ in such a way that any (proper or geometric) type-preserving group action on Δ induces a (proper or geometric) action on $H(\Delta)$.

Corollary 6.2 *Uniform type-preserving lattices in isometry groups of buildings of type C_n are Helly.*

6.3 Hyperbolic and quadric groups via Hellyfication

If a group Γ acts geometrically on a graph G, it also acts on its Hellyfication $\mathrm{Helly}(G) = E^0(G)$ and on its injective hull $E(G)$. However in general, this action is no longer geometric. This is because the injective hull $E(G)$ is not necessarily proper and because the points of $E(G)$ may be arbitrarily far from $e(G)$. This does not happen if G is a Helly graph:

Theorem 6.3 *Let G be a locally finite Helly graph.*

(1) *The injective hull $E(G)$ of G is proper and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in ℓ^n_∞, for every $n \geq 1$. Moreover, $d_H(E(G), e(G)) \leq 1$. Furthermore, if G has uniformly bounded degrees, then $E(G)$ has finite combinatorial dimension.*

(2) *A group acting cocompactly, properly or geometrically on G acts cocompactly, properly or geometrically, respectively, on its injective hull $E(G)$.*

For $\beta \geq 1$, the graph G has *β-stable intervals* [65] if for every triple of vertices w, v and v' with $v \sim v'$, we have $d_H(I(w,v), I(w,v')) \leq \beta$, where d_H denotes the Hausdorff distance. The proof of the first assertion of Theorem 6.3(1) is based on the following theorem of Lang [65]:

Theorem 6.4 [65, Theorem 1.1] *Let G be a locally finite graph with β-stable intervals. Then the injective hull of G is proper (that is, bounded closed subsets are compact) and has the structure of a locally finite polyhedral complex with only finitely many isometry types of n-cells, isometric to injective polytopes in ℓ^n_∞, for every $n \geq 1$.*

Next we show that weakly modular graphs (and thus Helly graphs) have β-stable intervals:

Lemma 6.5 *Every weakly modular graph has 1-stable intervals.*

Proof We need to show that for every triplet of vertices w, v and v' with $d(v,v') \leq 1$ and every $u \in I(w,v)$, there exists a vertex $u' \in I(w,v')$ with $d(u,u') \leq 1$. If $v = v'$, we are done by taking $u' = u$. Suppose now that $v \sim v'$. We proceed by induction on $k = d(w,v) + d(w,v')$. The case $k = 0$ is obvious. Assume that the statement holds for any $j < k$ and let $d(w,v) + d(w,v') = k$. If $d(w,v') = d(w,v) + 1$, then $I(w,v) \subseteq I(w,v')$ and the result holds. If $d(w,v) = d(w,v')$, then, by the triangle condition (TC) (see Section 2.1) there exists a vertex $v^* \sim v, v'$ such that $v^* \in I(w,v) \cap I(w,v')$. Since $d(w,v) + d(w,v^*) = d(w,v) + d(w,v') - 1 = k - 1$, by the induction hypothesis, for any $u \in I(w,v)$ there exists $u' \in I(w,v^*) \subseteq I(w,v')$ such that $d(u,u') \leq 1$. Let $d(w,v') = d(w,v) - 1$, ie $v' \in I(w,v)$.

For any $u \in I(w,v)$, let $u^* \in N(v) \cap I(u,v)$. By the quadrangle condition, there exists v^* such that $v^* \sim v', u^*$ and $v^* \in I(w,v') \cap I(w,u^*)$. Since $d(w,u^*) + d(w,v^*) = k-2$ and since $u \in I(w,u^*)$, by the induction hypothesis there exists u' such that $d(u,u') \leq 1$ and $u' \in I(w,v^*) \subseteq I(w,v')$. □

To establish the second assertion of Theorem 6.3(1), we use Lang's results relating the combinatorial dimension with the notion of cones. In a graph G, the *cone* [65] determined by the directed pair (x,v) of vertices of G is the set $C(x,v) = \{y \in V(G) : v \in I(x,y)\}$. Given a vertex $v \in V(G)$, we denote by $\mathcal{C}(v)$ the set of all cones $C(x,v)$ for $x \in V(G)$. For a ball B of G, we denote by $\mathcal{C}(B)$ the set of all pointed cones $(v, C(x,v))$ with $v \in B$ and $x \in V(G)$. By [65, Lemma 5.8], the size of $\mathcal{C}(B)$ is finite and bounded by a function of the size of B.

Proposition 6.6 [65, Proposition 5.12] *Let G be a locally finite graph with β-stable intervals. Given a vertex $z \in V(G)$ and $\alpha > 0$, let B be the ball $B_{2\alpha\beta}(z)$. Then for every $f \in E'(G)$ such that $f(z) \leq \alpha$, we have* $\mathrm{rk}(A(f)) \leq \frac{1}{2}|\mathcal{C}(B)|$.

Proof of Theorem 6.3(1) Properness and the structure of a locally finite polyhedral complex follow from Theorem 6.4 and Lemma 6.5.

We now show that $d_H(E(G), e(G)) \leq 1$. Pick any $f \in E(G)$ and consider $f' \in \Delta^0(G)$ defined by setting $f'(x) = \lceil f(x) \rceil$ for any $x \in V(G)$. Let $f'' \in E^0(G)$ such that $f'' \leq f'$, and notice that for any $x \in V(G)$ we have $f''(x) \leq f'(x) < f(x) + 1$. On the other hand, for any $x \in V(G)$, by Claim 3.5, for any $\epsilon > 0$, there exists $y \in V(G)$ such that $f(x) + f(y) < d(x,y) + \epsilon \leq f''(x) + f''(y) + \epsilon \leq f''(x) + f(y) + 1 + \epsilon$. Consequently $f(x) < f''(x) + 1 + \epsilon$ for any $\epsilon > 0$, and thus $f(x) \leq f''(x) + 1$. Since G is a Helly graph, by Theorem 4.2, $E^0(G)$ and G coincide, and thus there exists a vertex $z \in V(G)$ such that $f'' = d_z$, establishing that $d_\infty(f, d_z) \leq 1$.

Now, additionally suppose that G has uniformly bounded degrees. To show that $E(G)$ is finite-dimensional, pick any $f \in E'(G)$ and consider the vertex $z \in V(G)$ such that $f(z) = d_\infty(f, d_z) \leq 1$. By Lemma 6.5, G has 1-stable intervals, and by Proposition 6.6 applied with $\alpha = \beta = 1$, we have that $\mathrm{rk}(A(f)) \leq \frac{1}{2}|\mathcal{C}(B_2(z))|$. Since G has bounded degrees, the size of the balls of radius 2 in G is also bounded, and by [65, Lemma 5.8] the size of $|\mathcal{C}(B_2(z))|$ is uniformly bounded by some constant K. Consequently, by Proposition 6.6 all cells of $E'(X)$ are of dimension at most $\frac{1}{2}K$. By [65, Theorem 4.5], $E'(G) = E(G)$. This proves that $E(G)$ has finite combinatorial dimension. □

Theorem 6.3(2) is an immediate corollary of Theorem 6.3(1) and of the next proposition.

Proposition 6.7 *Let G be a locally finite graph such that the injective hull $E(G)$ is proper and satisfies the bounded distance property, and let Γ be a group acting on G.*

(1) *If Γ acts cocompactly on G, then Γ acts cocompactly on $E(G)$ and $E^0(G)$.*
(2) *If Γ acts properly on G, then Γ acts properly on $E(G)$ and $E^0(G)$.*
(3) *If Γ acts geometrically on G, Γ acts geometrically on $E(G)$ and $E^0(G)$; thus Γ is a Helly group.*

Proof Consider the Helly graph $E^0(G)$. Since the set $E^0(G)$ is an integer-valued subspace of $E(G)$ and $E(G)$ is proper, the balls of $E^0(G)$ are compact. Therefore the graph $E^0(G)$ is a proper metric space and thus is locally finite. In particular, all compact sets of $E^0(G)$ are finite. Since $E(G)$ satisfies the bounded distance property, there exists δ such that, for each $f \in E(G)$, we have $d_\infty(f, e(G)) \le \delta$.

We first assume that Γ acts cocompactly on G, and then show that Γ acts cocompactly on $E^0(G)$ and $E(G)$. The proof is the same in both cases; we provide it for $E^0(G)$. Since Γ acts cocompactly on G, there exists $v \in V(G)$ and $r \in \mathbb{N}$ such that $V(G) = \bigcup_{g\in\Gamma} V(B_r(gv, G))$. Let $R = r+\delta$ and consider $\bigcup_{g\in\Gamma} V\big(B_R(ge(v), E^0(G))\big)$. For any $f \in E^0(G)$, there exists $v' \in V(G)$ such that $d_\infty(f, e(v')) \le \delta$. Since there exists $g \in \Gamma$ such that $d_G(v', gv) \le r$,

$$d_\infty(f, ge(v)) = d_\infty(f, e(gv)) \le d_\infty(f, e(v')) + d_\infty(e(v'), e(gv)) \le \delta + d_G(v', gv) \le \delta + r.$$

This shows that $E^0(G) = \bigcup_{g\in\Gamma} V\big(B_R(ge(v), E^0(G))\big)$, and thus Γ acts cocompactly on $E^0(G)$.

We now assume that Γ acts properly on G, and then show that Γ acts properly on $E^0(G)$ and $E(G)$. Consider a compact set K in $E^0(G)$ or $E(G)$ and let $K' = \{v \in V(G) : \exists f \in K \text{ such that } d_\infty(f, e(v)) \le \delta\}$. Since K' is a bounded subset of $V(G)$, K' is finite, and thus $e(K')$ is also finite. Pick any $g \in \Gamma$ such that $\bar{g}K \cap K \ne \varnothing$ (where $\bar{g}$ is the inverse of g in Γ) and some $f \in K$ such that $\bar{g}f \in K$. Let $v \in K'$ such that $d_\infty(f, e(v)) \le \delta$. Since Γ acts on $E^0(G)$ and $E(G)$, $d_\infty(\bar{g}f, \bar{g}e(v)) = d_\infty(f, e(v)) \le \delta$. Since $\bar{g}e(v) = e(\bar{g}v)$ we have $\bar{g}v \in K'$, and thus $v \in K' \cap gK'$. Hence $\{g \in \Gamma : \bar{g}K \cap K \ne \varnothing\} \subseteq \{g \in \Gamma : gK' \cap K' \ne \varnothing\}$. Since Γ acts properly on G, the second set is finite and thus Γ acts properly on $E^0(G)$ and $E(G)$.

Finally, if Γ acts geometrically on $E^0(G)$, since $E^0(G)$ is Helly, Γ is a Helly group. □

If we consider a group Γ acting on a coarse Helly graph G, then the following holds:

Proposition 6.8 *A group acting geometrically on a coarse Helly graph with β-stable intervals is Helly.*

This result is a particular case of Theorem 6.4 and Propositions 3.12 and 6.7.

From Propositions 5.10 and 6.8, we also get the following corollary:

Corollary 6.9 *Hyperbolic groups are Helly.*

Proof By Proposition 5.10, any δ-hyperbolic graph G is coarse Helly with constant 2δ. Moreover, if G has δ-thin geodesic triangles, then one can easily check that G has $(\delta+1)$-stable intervals. The result then follows from Proposition 6.8. □

A group Γ is *quadric* if it acts geometrically on a quadric complex [54]. *Quadric complexes* are cell complexes that have hereditary modular graphs as 1-skeletons.

Corollary 6.10 *Quadric groups are Helly.*

Proof Since hereditary modular graphs are weakly modular, they have 1-stable intervals by Lemma 6.5 and they are coarse Helly by Proposition 5.11. □

By [54, Theorem B], any group admitting a finite C(4)–T(4) presentation acts geometrically on a quadric complex, leading thus to the following corollary:

Corollary 6.11 *Any group admitting a finite* C(4)–T(4) *presentation is Helly.*

6.4 Graphical small cancellation groups via thickening

Here we prove that finitely presented graphical C(4)–T(4) small cancellation groups are Helly. We closely follow [72, Section 6], where graphical $C(6)$ groups were studied. We begin with general notions concerning complexes, then graphical C(4)–T(4) complexes, and proving the Helly property for a class of graphical C(4)–T(4) complexes. From this we conclude the Hellyness of the corresponding groups.

In this subsection, unless otherwise stated, all complexes are 2-dimensional CW-complexes with combinatorial attaching maps (that is, restriction to an open cell is a homeomorphism onto an open cell) being immersions — see [72, Section 6] for details. A *polygon* is a 2-disk with the cell structure that consists of n vertices, n edges and a single 2-cell. For any 2-cell C of a 2-complex X there exists a map $R \to X$, where R is a polygon and the attaching map for C factors as $S^1 \to \partial R \to X$. By a *cell* we will mean a map $R \to X$ where R is a polygon. An *open cell* is the image in X of the single 2-cell of R. A *path* in X is a combinatorial map $P \to X$ where P is either a subdivision of the interval or a single vertex. In the latter case we call $P \to X$ a *trivial* path. The *interior* of the path is the path minus its endpoints. Given paths $P_1 \to X$ and $P_2 \to X$ such that the terminal point of P_1 is equal to the initial point of P_2, their *concatenation* is the obvious path $P_1 P_2 \to X$ whose domain is the union of P_1 and P_2 along these points. A *cycle* is a map $C \to X$, where C is a subdivision of the circle S^1. The cycle $C \to X$ is *nontrivial* if it does not factor through a map to a tree. A path or cycle is *simple* if it is injective on vertices. Notice that a simple cycle (of length at least 3) is nontrivial. The *length* of a path P or a cycle C, denoted by $|P|$ or $|C|$, respectively, is the number of 1-cells in the domain. A subpath $Q \to X$ of a path $P \to X$ (or a cycle) is a path that factors as $Q \to P \to X$ such that $Q \to P$ is an injective map.

A *disk diagram* is a contractible finite 2-complex D with a specified embedding into the plane. We call D *nonsingular* if it is homeomorphic to the 2-disc, otherwise D is called *singular*. The *area* of D is the number of 2-cells. The boundary cycle ∂D is the attaching map of the 2-cell that contains the point $\{\infty\}$, when we regard $S^2 = \mathbb{R}^2 \cup \{\infty\}$. A *boundary path* is any path $P \to D$ that factors as $P \to \partial D \to D$. An *interior path* is a path such that none of its vertices, except for possibly endpoints, lie on the boundary of D. If X is a 2-complex, then a *disk diagram in* X is a map $D \to X$.

A *piece* in a disk diagram D is a path $P \to D$ for which there exist two different lifts to 2-cells of D, ie there are 2-cells $R_i \to D$ and $R_j \to D$ such that $P \to D$ factors both as $P \to R_i \to D$ and $P \to R_j \to D$, but there does not exist an isomorphism $R_j \to R_i$ making the following diagram commutative:

$$\begin{array}{ccc} P & \longrightarrow & R_i \\ \downarrow & \nearrow & \downarrow \\ R_j & \longrightarrow & D \end{array}$$

Let $\varphi\colon G \to \Theta$ be an immersion of graphs where Θ is connected and G does not have vertices of degree 0 or 1. For convenience we will write G as the union of its connected components $G = \bigsqcup_{i\in I} G_i$, and refer to the connected graphs G_i as *relators*.

A *thickened graphical complex* X is a 2-complex with 1-skeleton Θ and a 2-cell attached along every immersed cycle in G (if a cycle $C \to G$ is immersed, then in X there is a 2-cell attached along the composition $C \to G \to \Theta$). A (nonthickened) *graphical complex* X^* is a 2-complex obtained by gluing a simplicial cone $C(G_i)$ along each $G_i \to \Theta$:

$$X^* = \Theta \cup_\varphi \bigsqcup_{i\in I} C(G_i).$$

For any $G_i \to X$ we have a *thick cell* $\mathrm{Th}(G_i) \to X$, where $\mathrm{Th}(G_i)$ is formed by gluing 2-cells along all immersed cycles in G_i. In X^* a *cone-cell* is the corresponding map $C(G_i) \to X$. Note that the two complexes X and X^* have the same fundamental groups. To be consistent with the approach in [72], in the following material we usually work with the thickened complex X, however the results could also be formulated for X^*.

Let X be a thickened graphical complex. A *piece* in X is a path $P \to X$ for which there exist two different lifts to G, ie there are two relators G_i and G_j such that the path $P \to X$ factors as $P \to G_i \to X$ and $P \to G_j \to X$, but there does not exist an isomorphism $\mathrm{Th}(G_j) \to \mathrm{Th}(G_i)$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Th}(G_i) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Th}(G_j) & \longrightarrow & X \end{array}$$

A disk diagram $D \to X$ is *reduced* if for every piece $P \to D$ the composition $P \to D \to X$ is a piece in X.

Lemma 6.12 (Lyndon–Van Kampen lemma) *Let X be a thickened graphical complex and let $C \to X$ be a closed homotopically trivial path. Then*:

(1) *There exists a disk diagram $D \to X$ such that the path C factors as $C \to \partial D \to X$, and $C \to \partial D$ is an isomorphism.*

(2) *If a diagram $D \to X$ is not reduced, then there exists a diagram $D_1 \to X$ with smaller area and the same boundary cycle in the sense that there is a commutative diagram*

$$\begin{array}{ccc} \partial D_1 & \xrightarrow{\cong} & \partial D \\ & \searrow & \downarrow \\ & & X \end{array}$$

(3) *Any minimal-area diagram $D \to X$ such that C factors as $C \xrightarrow{\cong} \partial D \to X$ is reduced.*

Definition 6.13 We say that a thickened graphical complex X satisfies

- the C(4) *condition* if no immersed cycle $C \to X$ that factors as $C \to G_i \to X$ is the concatenation of fewer than four pieces, and
- the T(4) *condition* if there does not exist a reduced nonsingular disk diagram $D \to X$ with D containing an internal 0-cell v, of valence 3, that is, contained in exactly three corners of 2-cells.

If X satisfies both conditions we call it a C(4)–T(4) *thickened graphical complex*. The corresponding complex X^* is called a C(4)–T(4) *graphical complex*.

If D is a disk diagram, we define small cancellation conditions in a very similar way, except that a *piece* is understood as a piece in a disk diagram.

Proposition 6.14 *If X is a* C(4)–T(4) *thickened graphical complex and $D \to X$ is a reduced disk diagram, then D is a* C(4)–T(4) *diagram.*

Proof The assertion follows immediately from the definitions of a reduced map and a piece. □

The following lemma is a graphical C(4)–T(4) analog of [72, Theorem 6.10] — the graphical $C(6)$ case — and [54, Propositions 3.4, 3.5 and 3.7 and Corollary 3.6] — the classical C(4)–T(4) case.

Lemma 6.15 *Let X be a simply connected* C(4)–T(4) *thickened graphical complex. Then*:

(1) *For every relator G_i, the map $G_i \to X$ is an embedding.*

(2) *The intersection of (the images of) any two relators is either empty or it is a finite tree.*

(3) *If three relators pairwise intersect then they all intersect and the intersection is a finite tree.*

Proof We proceed by contradiction, assuming the statement does not hold and showing that this leads to a forbidden reduced disk diagram in each case.

(1) Suppose there is a relator G_1 that does not embed. Let v and v' be two vertices of G_1 mapped to a common vertex v_{11} in X, and let γ be a geodesic path in G_1 between v and v'. The path γ is mapped to a loop γ_1 in X. By simple connectedness and Lemma 6.12 there exists a reduced disk diagram D for γ_1; see Figure 3, left. We may assume that we choose a counterexample so that the area (the number of 2-cells) of D is minimal among all counterexamples.

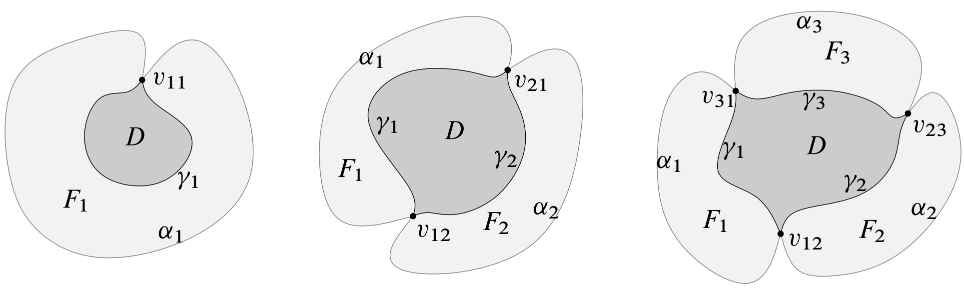

Figure 3: The proof of Lemma 6.15. From left to right: (1), (2) and (3).

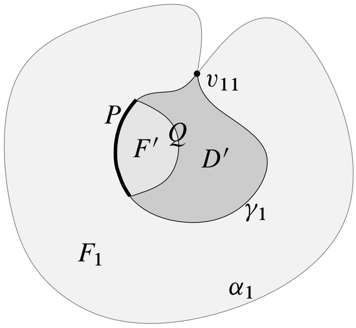

Figure 4: The proof of Lemma 6.15(1).

Now consider a larger disk diagram $D \cup F_1$ where F_1 is a cell whose boundary is the concatenation $\gamma_1\alpha_1$ which is mapped to a loop in G_1, and the only common point of γ_1 and α_1 is v_{11}; see Figure 3, left. The existence of F_1 follows from our assumption that there are no degree-1 vertices in relators. The diagram $D \cup F_1$ cannot be reduced, otherwise it would be a C(4)–T(4) diagram by Proposition 6.14, and this would contradict eg [54, Proposition 3.4]. Hence, by the definition of a reduced diagram, there is a piece P in $D \cup F_1$ that does not lift to a piece in X. Since D is reduced, it follows that P has to lie on γ_1. Since P does not lift to a piece in X, P is a part of the boundary of a cell F' such that its other boundary part Q maps to G_1 as well; see Figure 4. Thus replacing the subpath P of γ_1 by Q and, if necessary, reducing the resulting loop to get an immersed one, we get a new counterexample with a diagram D', such that $D = D' \cup F'$, of smaller area. This is a contradiction and so proves (1).

(2) First we prove that the intersection of two relators is connected. We proceed analogously to the proof of (1). Suppose not, and let G_1 and G_2 intersect in a nonconnected subgraph leading to a reduced disk diagram as in Figure 3, middle, with the boundary of F_i mapping to G_i. Again, we assume that D has minimal area among counterexamples and consider the extended disk diagram $D \cup F_1 \cup F_2$. By [54, Proposition 3.5] the new diagram is not reduced, and hence, as in the proof of (1), we get to a contradiction by finding a new counterexample with a smaller area diagram. This proves the connectedness of the intersection of two relators. The fact that such intersections do not contain cycles follows immediately from the C(4) condition.

(3) By (1) and (2) it is enough to show that the triple intersection is nonempty. Here we proceed analogously to (1) and (2). The corresponding diagrams are depicted in Figure 3, right, and the fact that the extended diagram $D \cup F_1 \cup F_2 \cup F_3$ is not reduced follows from [54, Proposition 3.7]. □

Lemma 6.16 *Let G_1, G_2 and G_3 be three pairwise-intersecting relators in a simply connected* C(4)–T(4) *thickened graphical complex X. Then the intersection $G_i \cap G_j$ of any two relators is contained in the third one.*

Proof Suppose not. Let v_i be a vertex in $G_j \cap G_k$ not in G_i for $\{i, j, k\} = \{1, 2, 3\}$. By Lemma 6.15 there exists a vertex $v \in G_1 \cap G_2 \cap G_3$ and immersed paths $\gamma_i \subseteq G_j \cap G_k$ from v to v_i for all $\{i, j, k\} = \{1, 2, 3\}$.

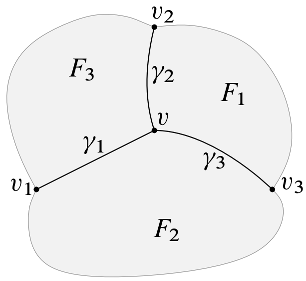

Figure 5: The proof of Lemma 6.16.

By our assumption that there are no degree-1 vertices, we may find a reduced disk diagram consisting of cells F_i mapped to G_i for $i = 1, 2, 3$, as in Figure 5. This contradicts the T(4) condition. □

Lemma 6.17 *Let X be a simply connected* C(4)–T(4) *thickened graphical complex and consider a collection $\{G_i \to X\}_{i \in I}$ of relators. If for every $i, j \in I$ the intersection $G_i \cap G_j$ is nonempty, then the intersection $\bigcap_{i \in I} G_i$ is a nonempty tree.*

Proof This follows directly from Lemmas 6.16 and 6.15(3). □

In view of Lemmas 6.16 and 6.15, for a simply connected C(4)–T(4) graphical complex X^* we may define a flag simplicial complex X^Δ, called its *thickening*, as follows: vertices of X^Δ are the vertices of X^*, and two vertices are connected by an edge if and only if they are contained in a common cone-cell. (Observe that the thickening of a graphical complex is not the corresponding thickened graphical complex.)

Theorem 6.18 *Let X^* be a simply connected* C(4)–T(4) *graphical complex. Then the* 1*-skeleton of the thickening X^Δ of X^* is Helly. Consequently, a group acting geometrically on X^* is Helly.*

Proof Since cone-cells are contractible and, by Lemma 6.17, all their intersections are contractible or empty, by Borsuk's nerve theorem [18; 17] the thickening X^Δ is homotopically equivalent to X^*. By Lemmas 6.16 and 6.15, the hypergraph defined by the thickening is triangle-free, and hence, by Proposition 2.11 the 1-skeleton of X^Δ is clique-Helly. The theorem now follows by Theorem 4.1. □

Examples of groups as in Theorem 6.18 are given by the following construction. A *graphical presentation* $\mathcal{P} = \langle S : \varphi \rangle$ is a graph $G = \bigsqcup_{i \in I} G_i$ and an immersion $\varphi\colon G \to R_S$, where every G_i is finite and connected and R_S is a rose, ie a wedge of circles with edges (cycles) labeled by a set S. Alternatively, the map $\varphi\colon G \to R_S$, called a *labeling*, may be thought of as an assignment: to every edge of G we assign a direction (orientation) and an element of S.

A graphical presentation $\mathcal{P}$ defines a group $\Gamma = \Gamma(\mathcal{P}) = \pi_1(R_S)/\langle\langle \varphi_*(\pi_1(G_i))_{i \in I} \rangle\rangle$. In other words Γ is the quotient of the free group $F(S)$ by the normal closure of the group generated by all words (over $S \cup S^{-1}$) read along cycles in G (where an oriented edge labeled by $s \in S$ is identified with the edge of the opposite orientation and the label s^{-1}). Observe that removing vertices of degree 1 from G does not change the group. Hence we may assume that there are no such vertices in G. A *piece* is a path P labeled by S such that there exist two immersions $p_1\colon P \to G$ and $p_2\colon P \to G$, and there is no automorphism $\Phi\colon G \to G$ such that $p_1 = \Phi \circ p_2$.

Consider the graphical complex $X^* = R_S \cup_\varphi \bigsqcup_{i\in I} C(G_i)$. The fundamental group of X^* is isomorphic to Γ. In the universal cover $\widetilde{X}^*$ of X^* there might be multiple copies of cones $C(G_i)$ whose attaching maps differ by lifts of $\mathrm{Aut}(G_i)$. After identifying all such copies, we obtain the complex $\widetilde{X}^+$. The group Γ acts geometrically, but not necessarily freely, on $\widetilde{X}^+$. We call the presentation $\mathcal{P}$ a C(4)–T(4) *graphical small cancellation presentation* when the complex X^* is a C(4)–T(4) graphical complex. The presentation $\mathcal{P}$ is *finite*, and the group Γ is *finitely presented* if the graph G is finite and the set S (of generators) is finite. As an immediate consequence of Theorem 6.18 we obtain the following:

Corollary 6.19 *Finitely presented graphical* C(4)–T(4) *small cancellation groups are Helly.*

6.5 Free products with amalgamation over finite subgroups

Let H be a graph with vertex set $\{w_j\}_{j\in J}$. For a collection $\{H_j\}_{j\in J}$ of graphs indexed by vertices of H, we consider the collection $\mathcal{FH} := \{F(H_j)\}_{j\in J}$ of their face complexes. For every edge $e = \{u_j, u_{j'}\}$ in H we pick vertices $w_j^e \in F(H_j)$ and $w_{j'}^e \in F(H_{j'})$. The *amalgam of* $\mathcal{FH}$ *over* H, denoted by $H(\mathcal{FH})$, is a graph defined as follows: Vertices of $H(\mathcal{FH})$ are equivalence classes of the equivalence relation on $\bigcup_{j\in J} V(F(H_j))$ induced by the relation $w_j^e \sim w_{j'}^e$ for all edges e of H. Edges of $H(\mathcal{FH})$ are induced by edges in the disjoint union $\bigsqcup_{j\in J} F(H_j)$. The part of Theorem 1.3(1) concerning free products with amalgamations over finite subgroups is implied by the following result. The case of HNN-extensions follows analogously.

Theorem 6.20 *For $i = 1, 2$ let Γ_i act geometrically on a Helly graph G_i, and let $\Gamma_i' < \Gamma_i$ be finite subgroups such that Γ_1' and Γ_2' are isomorphic. Then the free product $\Gamma_1 *_{\Gamma_1'\cong\Gamma_2'} \Gamma_2$ of Γ_1 and Γ_2 with amalgamation over $\Gamma_1' \cong \Gamma_2'$ acts geometrically on an amalgam $H(\mathcal{FH})$ of $\mathcal{FH}$ over H, where H is a tree, elements of $\mathcal{H}$ are copies of G_1 and G_2, and such that $H(\mathcal{FH})$ is Helly.*

Proof Let H be the Bass–Serre tree for $\Gamma_1 *_{\Gamma_1'\cong\Gamma_2'} \Gamma_2$. For a vertex w_j of H corresponding to Γ_i we define H_j to be a copy of G_i. For an edge e in H we define w_j^e to be a vertex fixed in H_j by the corresponding conjugate of $\Gamma_1' \cong \Gamma_2'$ (such a vertex exists by Theorem 7.1 and Proposition 5.23). An equivariant choice of vertices w_j^e leads to an amalgam $H(\mathcal{FH})$ acted geometrically upon by $\Gamma_1 *_{\Gamma_1'\cong\Gamma_2'} \Gamma_2$. The graph $H(\mathcal{FH})$ is Helly since it can be obtained by consecutive gluings of two Helly graphs along a common vertex — such a gluing obviously results in a Helly graph (for a more general gluing procedure see [68]). □

6.6 Quotients by finite normal subgroups

Let Γ act (by automorphisms) on a complex X. Then Γ acts on $F(X)$ and we define the *fixed-point complex* $F(X)^\Gamma$ in the face complex as the subcomplex spanned by all vertices of $F(X)$ fixed by Γ (that correspond to the cliques of X stabilized by Γ). The following theorem implies Theorem 1.3(5):

Theorem 6.21 *Let Γ be a group acting by automorphisms on a clique-Helly graph G. Let $N \lhd \Gamma$ be a finite normal subgroup. Then Γ/N acts by automorphisms on the clique-Helly complex $F(X(G))^N$. If G is Helly then $F(X(G))^N$ is Helly as well. If the Γ action on G is proper or cocompact then the induced action of Γ/N on $F(X(G))^N$ is proper or cocompact, respectively.*

Proof The Γ-action on G induces the Γ-action on $F(X(G))$, and consequently the Γ/N-action on $F(X(G))^N$. It is clear that the latter is proper or cocompact if the initial action is so. By Lemma 7.7 and Corollary 7.8 the complex $F(X(G))^N$ is (clique-)Helly if G is so. □

6.7 Actions with Helly stabilizers

Our goal now is to apply the general theory developed in [45] in order to show that the family of Helly groups is stable under several group-theoretic operations. The main theorem in this direction is Theorem 6.24, which shows that if a group acts on a quasimedian graph in a specific way and if clique stabilizers are Helly, then the group must be Helly as well. We emphasize that, contrary to the rest of the article, our quasimedian graphs may not be locally finite; in particular, their cliques will typically be infinite. We begin by giving general definitions and properties related to quasimedian graphs.

6.7.1 Preliminaries on quasimedian graphs Recall that a graph is *quasimedian* if it is weakly modular and does not contain K_4^- or $K_{3,2}$ as induced subgraphs. Several subgraphs are of interest in the study of quasimedian graphs:

- In this subsection, by a *clique*, we mean a maximal complete subgraph.
- A *prism* is an induced subgraph which decomposes as a Cartesian product of cliques. The maximal number of factors of a prism in a quasimedian graph is referred to as its *cubical dimension* (which may be infinite). (Observe that, by maximality of our cliques, a single vertex defines a prism of zero cubical dimension if and only if it is isolated.)
- A *hyperplane* is an equivalence class of edges with respect to the transitive closure of the relation which identifies two edges whenever they belong to a common triangle or they are opposite sides of a square (ie a four-cycle). Two cliques are *parallel* if they belong to the same hyperplane. Two hyperplanes are *transverse* if their union contains two adjacent edges of some square.
- According to [45, Proposition 2.15], a hyperplane *separates* a quasimedian graph, that is, the graph obtained by removing the interiors of the edges of a hyperplane contains at least two connected components. Such a component is a *sector* delimited by the hyperplane.

According to [7; 45, Lemmas 2.16 and 2.80], cliques and prisms are gated subgraphs. For convenience, in the sequel we will refer to the map sending a vertex to its gate in a given gated subgraph as the *projection* onto this subgraph.

6.7.2 Systems of metrics Given a quasimedian graph G, a *system of metrics* is the data of a metric δ_C on each clique C of G. Such a system is *coherent* if for any two parallel cliques C and C' one has

$\delta_C(x,y) = \delta_{C'}(t_{C\to C'}(x), t_{C\to C'}(y))$ for all vertices $x, y \in C$, where $t_{C\to C'}$ denotes the projection of C onto C'. As shown in [45, Section 3.2], it is possible to extend a coherent system of metrics to a global metric on G. Several constructions are possible; we focus on the one which will be relevant for our study of Helly groups. A *chain* R between two vertices $x, y \in V(G)$ is a sequence of vertices $(x_1 = x, x_2, \dots, x_{n-1}, x_n = y)$ such that, for every $1 \le i \le n-1$, the vertices x_i and x_{i+1} belong to a common prism, say P_i. The *length* of R is $\ell(R) = \sum_{i=1}^{n-1} \delta_{P_i}(x_i, x_{i+1})$, where δ_{P_i} denotes the ℓ_∞-metric associated to the local metrics defined on the cliques of P_i. Then the global metric extending our system of metrics is

$$\delta_\infty \colon (x,y) \mapsto \min\{\ell(R) : R \text{ is a chain between } x \text{ and } y\}.$$

Throughout this section, all our local metrics will be graph metrics. It is worth noticing that, in this case, δ_∞ turns out to be a graph metric as well. Consequently, (G, δ_∞) will be considered as a graph. More precisely, this graph has $V(G)$ as its vertex set and its edges link two vertices if they are at δ_∞-distance 1. Notice that if $P = C_1 \times \cdots \times C_n$ is a prism of G, then the graph (P, δ_∞) is isometric to the direct product $(C_1, \delta_{C_1}) \boxtimes \cdots \boxtimes (C_n, \delta_{C_n})$.

The main result of this section is that extending a system of Helly graph metrics produces a global metric which is again Helly. More precisely:

Proposition 6.22 *Let G be a quasimedian graph of finite cubical dimension endowed with a coherent system of graph metrics $\{\delta_C : C$ is a clique of $G\}$. Suppose that (C, δ_C) is a locally finite Helly graph for every clique C of G and that each vertex belongs to only finitely many cliques. Then (G, δ_∞) is a Helly graph.*

We begin by proving the following preliminary lemma:

Lemma 6.23 *Suppose G is a quasimedian graph endowed with a coherent system of graph metrics $\{\delta_C : C$ is a clique of $G\}$ and that the clique complex of (C, δ_C) is simply connected for every clique C of G. Then the clique complex of (G, δ_∞) is simply connected as well.*

Proof Let γ be a cycle in the 1-skeleton of (G, δ_∞). We prove by induction on the number of hyperplanes of G crossed by γ that γ is nullhomotopic in the clique complex of (G, δ_∞). Of course, if γ does not cross any hyperplane then it has to be reduced to a single vertex and there is nothing to prove. So from now on we assume that γ crosses at least one hyperplane.

Let $Y \subseteq V(G)$ denote the gated hull of the vertex set of γ. Notice that the subgraph of (G, δ_∞) spanned by the vertices of Y coincides with (Y, δ_∞). According to [45, Proposition 2.68], the hyperplanes of Y are exactly the hyperplanes of G crossed by γ. If the hyperplanes of Y are pairwise transverse, then it follows from [45, Lemma 2.74] that Y is a single prism. Consequently, (Y, δ_∞) is the direct product of graphs whose clique complexes are simply connected, so γ must be nullhomotopic in the clique complex of (G, δ_∞). From now on assume that Y contains at least two hyperplanes, say J and H, which are not transverse.

Let S denote the sector delimited by H which contains J. Decompose γ as a concatenation of subpaths $\alpha_1\beta_1\cdots\alpha_n\beta_n\alpha_{n+1}$ such that $\alpha_1,\dots,\alpha_{n+1}$ are included in S and $\beta_1,\dots,\beta_n$ intersect S only at their endpoints. For every $1 \le i \le n$, fix a path $\sigma_i \subset (Y,\delta_\infty)$ between the endpoints of β_i which does not cross J (such a path exists as a consequence of [45, Proposition 3.16]). Notice that $\beta_i\sigma_i^{-1}$ is a cycle which does not cross H, so by our induction assumptions we know that β_i and σ_i are homotopic (in the clique complex). Therefore γ is homotopic (in the clique complex) to the cycle $\alpha_1\sigma_1\cdots\alpha_n\sigma_n\alpha_{n+1}$, which does not cross H. We conclude that γ is nullhomotopic (in the clique complex) by our induction assumptions. □

Proof of Proposition 6.22 Fix a set $\mathcal{C}$ of representatives of cliques modulo parallelism. For every $C \in \mathcal{C}$, let $\pi_C : G \to C$ denote the projection onto C. We claim that

$$\pi : (G,\delta_\infty) \to \boxtimes_{C\in\mathcal{C}} (C,\delta_C), \quad x \mapsto (\pi_C(x)),$$

is an injective graph morphism.

Let $x, y \in (G,\delta_\infty)$ be two adjacent vertices, ie $\delta_\infty(x,y) = 1$. So there exists a prism P of G, thought of as a product of cliques $C_1\times\cdots\times C_n$, which contains x and y and such that the projections of x and y onto each C_i are identical or δ_{C_i}-adjacent. For every $1 \le i \le n$, let $C_i' \in \mathcal{C}$ denote the representative of C_i. Because our system of metrics is coherent, we also know that the projections of x and y onto each C_i' are identical or $\delta_{C_i'}$-adjacent. Therefore $\pi(x)$ and $\pi(y)$ are adjacent in the subgraph $\boxtimes_{1\le i\le n}(C_i',\delta_{C_i'})$ of $\boxtimes_{C\in\mathcal{C}}(C,\delta_C)$. Thus π is a graph morphism.

Now, let $x, y \in (G,\delta_\infty)$ be two distinct vertices. By [45, Proposition 2.30], there exists a hyperplane separating x and y. Therefore if $C \in \mathcal{C}$ denotes the representative clique dual to this hyperplane, then $\pi_C(x) \ne \pi_C(y)$. Hence $\pi(x) \ne \pi(y)$, proving that π is indeed injective.

Notice that the image of a prism of G under π is a finite subproduct of $\boxtimes_{C\in\mathcal{C}}(C,\delta_C)$. Moreover, because every vertex of G belongs to only finitely many cliques and because each (C,δ_C) is locally finite, (G,δ_∞) must be locally finite. As a consequence, (G,δ_∞) is a UGP over $\{(C,\delta_C) : C \in \mathcal{C}\}$. We claim that our UGP satisfies the 3-piece condition, so let P_1, P_2 and P_3 be three pairwise-intersecting prisms in G. Because prisms are gated, they satisfy the Helly property, so there exists a vertex $x \in P_1\cap P_2\cap P_3$. Let $\mathcal{J}$ denote the set of all the hyperplanes that have a clique in at least two prisms among P_1, P_2 and P_3. Observe that any two distinct hyperplanes $J_1, J_2 \in \mathcal{J}$ are transverse (ie there exists a prism containing cliques from both J_1 and J_2). For every $J \in \mathcal{J}$, fix a clique $C_J \subset P_1\cup P_2\cup P_3$ in J that contains x and let P denote the gated hull of the union of all the C_J for $J \in \mathcal{J}$. Because the hyperplanes in $\mathcal{J}$ are pairwise transverse, we deduce from [45, Proposition 2.68 and Lemma 2.74] that P is a prism. Our goal is to show that P is the piece of G we are looking for. So let $C \in \mathcal{C}$ be such that at least two prisms among P_1, P_2 and P_3 have projection C on the C-coordinate. It follows from [45, Lemma 2.20] that the hyperplane J containing C intersects at least two prisms among P_1, P_2 and P_3, and hence $J \in \mathcal{J}$. By construction, P contains a clique in J, and hence a clique parallel to C. In other words, C is also the projection of P on the C-coordinate, as desired. Thus we have verified that the 3-piece condition holds. We conclude that (G,δ_∞) is a Helly graph by combining Theorem 5.4 with Lemma 6.23. □

6.7.3 Constructing Helly groups We are now ready to construct new Helly groups from old ones. Recall from [45] that the action of a group Γ on a quasimedian graph G is *topical-transitive* if it satisfies the two following conditions:

(1) For every hyperplane J, every clique $C \subset J$ and every $g \in \mathrm{stab}(J)$, there exists $h \in \mathrm{stab}(C)$ such that g and h induce the same permutation on the set of sectors delimited by J.

(2) For every clique C of G either
- C is finite and $\mathrm{stab}(C) = \mathrm{fix}(C)$, or
- $\mathrm{stab}(C) \curvearrowright C$ is free and transitive on the vertices.

Then the statement we are interested in is:

Theorem 6.24 *Let Γ be a group acting topically transitively on a quasimedian graph G. Suppose that*

- *every vertex of G belongs to finitely many cliques*,
- *every vertex stabilizer is finite*,
- *the cubical dimension of G is finite*,
- *G contains finitely many Γ-orbits of cliques*, *and*
- *for every maximal prism $P = C_1 \times \cdots \times C_n$, we have $\mathrm{stab}(P) = \mathrm{stab}(C_1) \times \cdots \times \mathrm{stab}(C_n)$.*

If clique stabilizers are Helly, *then so is Γ.*

Before turning to the proof of Theorem 6.24, we need the following easy observation (which can be proved by following [45, Lemma 4.34]):

Lemma 6.25 *For every Helly group Γ*, *there exist a Helly graph G and a vertex $x_0 \in G$ such that Γ acts geometrically on G and $\mathrm{stab}(x_0)$ is trivial.* □

Proof of Theorem 6.24 First, observe that G contains only finitely many Γ-orbits of prisms. Indeed, let $\mathcal{C}$ be a finite collection of representatives of cliques modulo the action of Γ. For every $C \in \mathcal{C}$, fix a vertex $x_C \in C$. Let $\mathcal{P}$ denote the set of all the prisms in G that contain x_C for some $C \in \mathcal{C}$. Since each vertex belongs to finitely many cliques by assumption, we know that $\mathcal{P}$ is a finite collection. Now, if P is an arbitrary prism in G, there must exist $g \in \Gamma$ and $C \in \mathcal{C}$ such that gP contains C, and a fortiori x_C, and hence $gP \in \mathcal{P}$. This proves our observation. By combining Lemma 6.25 with [45, Proposition 7.8], there exists a new quasimedian graph Y endowed with a coherent system of metrics $\{\delta_C : C \text{ is a clique of } Y\}$ such that Γ acts geometrically on (Y, δ_∞) and such that (C, δ_C) is a Helly graph for every clique C of Y. Since (Y, δ_∞) is a Helly graph by Proposition 6.22, we conclude that Γ is a Helly group. □

We now record several applications of Theorem 6.24.

6.7.4 Graph products of groups Given a *simplicial graph* G and $\mathcal{G} = \{\Gamma_u : u \in V(G)\}$ a collection of groups indexed by the vertices of G (called *vertex-groups*), the *graph product* $G\mathcal{G}$ is the quotient

$$\Big(\mathop{\ast}_{u \in V(G)} \Gamma_u \Big) / \langle\langle [g,h] = 1,\ g \in \Gamma_u,\ h \in \Gamma_v \text{ if } (u,v) \in E(G) \rangle\rangle.$$

For instance, if G has no edge then $G\mathcal{G}$ is the free product of $\mathcal{G}$, and if G is a complete graph then $G\mathcal{G}$ is the direct sum of $\mathcal{G}$. One often says that graph products interpolate between free products and direct sums.

By combining Theorem 6.24 with [45, Proposition 8.14], one obtains:

Theorem 6.26 *Let G be a finite simplicial graph and $\mathcal{G}$ a collection of groups indexed by $V(G)$. If the vertex-groups are Helly, then so is the graph product $G\mathcal{G}$.*

6.7.5 Diagram products of groups Let $\mathcal{P} = \langle \Sigma : \mathcal{R} \rangle$ be a semigroup presentation. We assume that if $u = v$ is a relation which belongs to $\mathcal{R}$ then $v = u$ does not belong to $\mathcal{R}$; in particular, $\mathcal{R}$ does not contain relations of the form $u = u$. The *Squier complex* $S(\mathcal{P})$ is the square-complex

- vertices are the positive words $w \in \Sigma^+$,
- edges $(a, u = v, b)$ link aub and avb where $(u = v) \in \mathcal{R}$, and
- squares $(a, u = v, b, p = q, c)$ are delimited by the edges $(a, u = v, bpc)$, $(a, u = v, bqc)$, $(aub, p = q, c)$ and $(avb, p = q, c)$.

The connected component of $S(\mathcal{P})$ containing a given word $w \in \Sigma^+$ is denoted by $S(\mathcal{P}, w)$. Given a collection of groups $\mathcal{G} = \{\Gamma_s, s \in \Sigma\}$ labeled by the alphabet Σ, the *diagram product* $D(\mathcal{P}, \mathcal{G}, w)$ is isomorphic to the fundamental group of the following 2-complex of groups:

- The underlying 2-complex is the 2-skeleton of the Squier complex $S(\mathcal{P}, w)$.
- To any vertex $u = s_1 \cdots s_r \in \Sigma^+$ is associated the group $\Gamma_u = \Gamma_{s_1} \times \cdots \times \Gamma_{s_r}$.
- To any edge $e = (a, u \to v, b)$ is associated the group $\Gamma_e = \Gamma_a \times \Gamma_b$.
- To any square is associated the trivial group.
- For every edge $e = (a, u \to v, b)$, the monomorphisms $\Gamma_e \to \Gamma_{aub}$ and $\Gamma_e \to \Gamma_{avb}$ are the canonical maps $\Gamma_a \times \Gamma_b \to \Gamma_a \times \Gamma_u \times \Gamma_b$ and $\Gamma_a \times \Gamma_b \to \Gamma_a \times \Gamma_v \times \Gamma_b$.

We refer to [50; 45, Section 10] for more information about diagram products of groups. By Theorem 6.24 and [45, Proposition 10.33 and Lemma 10.34], one obtains:

Theorem 6.27 *Let $\mathcal{P} = \langle \Sigma : \mathcal{R} \rangle$ be a finite semigroup presentation, $\mathcal{G}$ a collection of groups indexed by the alphabet Σ and $w \in \Sigma^+$ a baseword. If $\{u \in \Sigma^+ : u = w \bmod \mathcal{P}\}$ is finite and if the groups of $\mathcal{G}$ are all Helly, then the diagram product $D(\mathcal{P}, \mathcal{G}, w)$ is a Helly group.*

Explicit examples of diagram products can be found in [45, Section 10.7]. For instance, the $\square$*-product* of two groups Γ_1 and Γ_2, defined by the relative presentation

$$\Gamma_1 \square \Gamma_2 = \langle \Gamma_1, \Gamma_2, t : [g, h] = [g, tht^{-1}] = 1 \text{ for } g \in \Gamma_1 \text{ and } h \in \Gamma_2 \rangle$$

is a diagram product [45, Example 10.65]. As it satisfies the assumptions of Theorem 6.27, it follows that:

Corollary 6.28 *If Γ_1 and Γ_2 are two Helly groups, then so is $\Gamma_1 \square \Gamma_2$.*

6.7.6 Right-angled graphs of groups Roughly speaking, right-angled graphs of groups are fundamental groups of graphs of groups obtained by gluing graph products together along "simple" subgroups. We refer to [83] for more information about graphs of groups.

Definition 6.29 Let G and H be two simplicial graphs, and $\mathcal{G}$ and $\mathcal{H}$ be two families of groups indexed by $V(G)$ and $V(H)$, respectively. A morphism $\Phi\colon G\mathcal{G} \to H\mathcal{H}$ is a *graphical embedding* if there exists an embedding $f\colon G \to H$ and isomorphisms $\varphi_v\colon \Gamma_v \to \Gamma_{f(v)}$ for $v \in V(G)$, such that $f(G)$ is an induced subgraph of H and $\Phi(g) = \varphi_v(g)$ for every $v \in V(G)$ and $g \in \Gamma_v$.

Definition 6.30 A *right-angled graph of groups* is a graph of groups such that each (vertex- and edge-)group has a fixed decomposition as a graph product and such that each monomorphism of an edge-group into a vertex-group is a graphical embedding (with respect to the structures of graph products we fixed).

In the following, a *factor* will refer to a vertex-group of one of these graph products. Let $\mathfrak{G}$ be a right-angled graph of groups. Notice that, if e is an oriented edge from a vertex x to another y, then the two embeddings of Γ_e in Γ_x and Γ_y given by $\mathfrak{G}$ provide an isomorphism φ_e from a subgroup of Γ_x to a subgroup of Γ_y. Moreover, if $\Gamma \subset \Gamma_x$ is a factor, then $\varphi_e(\Gamma) := \{g \in \Gamma_y : \exists h \in \Gamma \text{ such that } \varphi_e(h) = g\}$ is either empty or a factor of Γ_y. Set

$$\Phi(\Gamma) = \{\varphi_{e_k} \circ \cdots \circ \varphi_{e_1} : e_1, \dots, e_k \text{ is an oriented cycle at } x,\ \varphi_{e_k} \circ \cdots \circ \varphi_{e_1}(\Gamma) = \Gamma\},$$

thought of as a subgroup of the automorphism group $\mathrm{Aut}(\Gamma)$.

By combining Theorem 6.24 with [45, Proposition 11.26 and Lemma 11.27], one obtains:

Theorem 6.31 *Let $\mathfrak{G}$ be a right-angled graph of groups such that $\Phi(\Gamma) = \{\mathrm{Id}\}$ for every factor Γ. Suppose that the underlying abstract graph and the simplicial graphs defining the graph products are all finite. If the factors are Helly, then so is the fundamental group of $\mathfrak{G}$.*

For explicit examples of fundamental groups of right-angled graphs of groups see [45, Section 11.4]. For instance, the $\rtimes$-*power* of a group Γ [45, Example 11.38], defined by the relative presentation $\Gamma^{\rtimes} = \langle \Gamma, t : [g, tgt^{-1}] = 1 \text{ for } g \in \Gamma\rangle$, is the fundamental group of a right-angled graph of groups satisfying the assumptions of Theorem 6.31, and hence:

Corollary 6.32 *If Γ is a Helly group, then so is $\Gamma^{\rtimes}$.*

Also, the $\bowtie$-*product* of two groups Γ_1 and Γ_2 [45, Example 11.39], defined by the relative presentation $\Gamma_1 \bowtie \Gamma_2 = \langle \Gamma_1, \Gamma_2, t : [g, h] = [g, tht^{-1}] = [h, tht^{-1}] = 1 \text{ for } g \in \Gamma_1 \text{ and } h \in \Gamma_2\rangle$, is the fundamental group of a right-angled graph of groups satisfying the assumptions of Theorem 6.31, and hence:

Corollary 6.33 *If Γ_1 and Γ_2 are Helly groups, then so is $\Gamma_1 \bowtie \Gamma_2$.*

7 Properties of Helly groups

The main goal of this section is proving Theorem 1.5(2)–(4) and (6)–(9). (Theorem 1.5(1) is proved in the subsequent Section 8 and Theorem 1.5(5) follows from Theorems 3.13 and 6.3.) On the way we show also some immediate consequences of the main results and prove related facts concerning groups acting on Helly graphs.

7.1 Fixed points for finite group actions

In this subsection we prove Theorem 1.5(2), which states that every Helly group has only finitely many conjugacy classes of finite subgroups. It is an immediate consequence of the following result, which is interesting on it own:

Theorem 7.1 (fixed point theorem) *Let Γ be a group acting by automorphisms on a Helly graph G without infinite cliques. If Γ has bounded orbits, then there exists a clique of G stabilized by Γ. In particular, there is a fixed vertex of the induced action of Γ on the face complex $F(G)$.*

Proof Pick a vertex v of G and consider its Γ-orbit Γv. Let N be the diameter of Γv. The intersection $B := \bigcap_{g \in \Gamma} B_N(gv)$ of N-balls centered at vertices of the orbit Γv is a nonempty bounded Γ-invariant Helly graph. Since G does not contain infinite simplices, by [77, Theorem A], the graph B contains a clique stabilized by Γ. □

Proof of Theorem 1.5(2) This follows immediately from the fixed point theorem, Theorem 7.1, as in, for example, the case of CAT(0) groups in [21, Proposition I.8.5]. □

Remark 7.2 Theorem 1.5(2) can be also deduced from [38] or [65, Proposition 1.2] combined with our Theorem 6.3.

7.2 Flats vs hyperbolicity

Proof of Theorem 1.5(3) Suppose that Γ is hyperbolic. Then G is hyperbolic and, clearly, does not contain an isometric ℓ_∞-square-grid. For the converse, recall that if Γ is not hyperbolic then G contains isometric finite ℓ_∞-square-grids of arbitrary size, by Proposition 4.5. Since Γ acts geometrically on G (and, in particular, G is locally finite), by a diagonal argument it follows that G contains an isometric infinite ℓ_∞-grid; see eg [21, Lemma II.9.34 and Theorem II.9.33]. □

7.3 Contractibility and Hellyness of the fixed-point set

The aim of this section is to prove that for a group acting on a Helly complex its fixed-point set is contractible. This leads to a proof of Theorem 1.5(4) showing that the Helly complex is a model for the classifying space for proper actions. Furthermore, we show that the fixed-point subcomplex of the face complex (of the Helly complex on which the group acts) is Helly.

Lemma 7.3 *Let $\Gamma < \mathrm{Aut}(X)$ be a group of automorphisms of a locally finite Helly complex X. The fixed-point set ${X'}^{\Gamma}$ of the barycentric subdivision X' of X is contractible.*

Proof Let σ be a simplex of X stabilized by Γ. For every $N > 0$, the intersection $B_N := \bigcap_{v \in \sigma^{(0)}} B_N(v)$ of N-balls centered at vertices of σ is Helly, and hence dismantlable. It is also Γ-invariant, by construction. The fixed-point set ${B'_N}^{\Gamma}$ in the barycentric subdivision B'_N of B_N is contractible by [11, Theorem 6.5] or [53, Theorem 1.2]. Since the sets B_N exhaust X, it follows that the fixed-point set ${X'}^{\Gamma}$ in the barycentric subdivision X' of X is contractible. □

Theorem 1.5(4) is a part of the following corollary of Theorem 7.1 and Lemma 7.3:

Corollary 7.4 *Let Γ be a group acting properly on a locally finite Helly graph G. Then the Helly complex $X(G)$ is a model for the classifying space $\underline{E}\Gamma$ for proper actions of Γ. If the action is cocompact then the model is finite-dimensional and cocompact.*

In view of Theorem 6.3 and [65, Theorem 1.4] there exists another model for $\underline{E}\Gamma$, defined as follows:

Theorem 7.5 *Let Γ be a group acting properly on a locally finite Helly graph G. The injective hull $E(G)$ of G is a model for the classifying space $\underline{E}\Gamma$ for proper actions of Γ. If the action is cocompact then the model is finite-dimensional and cocompact.*

Remark 7.6 Observe that $X(G)$ can be nonhomeomorphic to $E(G)$. For example, if G is an $(n+1)$-clique then obviously the clique complex $X(G)$ is an n-simplex, whereas the injective hull $E(G)$ is a cone over $n+1$ points, that is, a tree.

Recall that the fixed-point complex $F(X)^{\Gamma}$ in the face complex is the subcomplex spanned by all vertices of $F(X)$ fixed by Γ. We now prove that $F(X)^{\Gamma}$ is Helly.

Lemma 7.7 (clique-Helly fixed-point set) *Let $\Gamma < \mathrm{Aut}(X)$ be a group of automorphisms of a locally finite clique-Helly complex X. Then the fixed-point complex $F(X)^{\Gamma}$ is clique-Helly.*

Proof Let uvw be a triangle in $F(X)^{\Gamma}$. By the clique-Helly property for $F(X)$ (Proposition 5.23) there is a vertex $z \in F(X)$ adjacent to all vertices of $F(X)$ spanning triangles with an edge of uvw (Proposition 2.9). Since uvw belongs to $F(X)^{\Gamma}$, all vertices in the orbit Γz have the same property as z, ie they are adjacent to all vertices of $F(X)$ spanning triangles with an edge of uvw. Thus they span a simplex of $F(X)$. Let σ be the union of the simplices of X corresponding to the vertices of Γz in $F(X)$. By Lemma 5.21, σ is a simplex of X. Let y be the vertex of $F(X)$ corresponding to σ. Notice that y belongs to $F(X)^{\Gamma}$. We now prove that y satisfies the assumption of Proposition 2.9. Pick a vertex x of $F(X)^{\Gamma}$ spanning a triangle with an edge of uvw, say with uv. By the definition of z, x is adjacent to z and to any $z' \in \Gamma z$. Consequently, for any $z' \in \Gamma z$, x and z' correspond to two subsimplices τ_x and $\tau_{z'}$ of a common simplex of X. Therefore all vertices of τ_x are adjacent to all vertices of $\tau_{z'}$. Since $\sigma = \bigcup_{z' \in \Gamma z} \tau_{z'}$, τ_x and σ are also subsimplices of a common simplex of X. Thus x and y are adjacent in $F(X)$. □

Corollary 7.8 (Helly fixed-point set) *Let $\Gamma < \mathrm{Aut}(X)$ be a group of automorphisms of a locally finite Helly complex X. Then the fixed-point complex $F(X)^\Gamma$ is Helly.*

Proof Since every edge in $F(X)^\Gamma$ is homotopic to a path in X'^Γ, every cycle in $F(X)^\Gamma$ is homotopic to a cycle in X'^Γ, and hence $F(X)^\Gamma$ is simply connected by Lemma 7.3. So by Lemma 7.7 and Theorem 4.1, $F(X)^\Gamma$ is Helly. □

7.4 EZ-boundaries

For a group Γ acting geometrically on X, by an *EZ-structure for* Γ we mean a pair $(\overline{X}, \partial X)$, where $\overline{X} = X \cup \partial X$ is a compactification of X that is a Euclidean retract with the following additional properties. The *EZ-boundary* ∂X is a Z-set in $\overline{X}$ such that, for every compact $K \subset X$, the sequence $(gK)_{g \in \Gamma}$ is a null sequence, and the action $\Gamma \curvearrowright X$ extends to an action $\Gamma \curvearrowright \overline{X}$ by homeomorphisms. This notion was first introduced by Bestvina [16] (without the requirement of extending $\Gamma \curvearrowright X$ to $\Gamma \curvearrowright \overline{X}$), then by Farrell and Lafont [43] (for free actions) and finally in [73] (in the form above). Homological invariants of the boundary are related to homological invariants of the group, and the existence of an EZ-structure has some important consequences (eg it implies the Novikov conjecture in the torsion-free case). Conjecturally, all groups with finite classifying spaces admit EZ-structures, but such objects were constructed only for limited classes of groups — notably for hyperbolic groups and for CAT(0) groups. Theorem 1.5(6) is a consequence of the following:

Theorem 7.9 *Let Γ act geometrically on a Helly graph G. Then there exists an EZ-boundary ∂G such that $(X(G) \cup \partial G, \partial G)$ and $(E(G) \cup \partial G, \partial G)$ are EZ-structures for Γ.*

Proof It is shown in [33] that for a complete metric space $E(G)$ with a convex and consistent bicombing, there exists ∂G (a space of equivalence classes of combing rays) such that $(E(G) \cup \partial G, \partial G)$ is a so-called Z-structure. The proof is easily adapted to show that it is an EZ-structure (see eg [73] where a much weaker version of a "coarse bicombing" is used to define an EZ-structure). It follows that $(X(G) \cup \partial G, \partial G)$ is an EZ-structure as well. □

7.5 The Farrell–Jones conjecture

For a discrete group Γ, the *Farrell–Jones conjecture* asserts that the K-theoretic (resp. L-theoretic) *assembly map*

$$H_n^\Gamma(E_{\mathcal{VCY}}(\Gamma); \boldsymbol{K}_R) \to K_n(R\Gamma) \quad \text{(resp. } H_n^\Gamma(E_{\mathcal{VCY}}(\Gamma); \boldsymbol{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(R\Gamma))$$

is an isomorphism. Here R is an associative ring with a unit, $R\Gamma$ is the group ring and $K_n(R\Gamma)$ are the algebraic K-groups of $R\Gamma$. By $E_{\mathcal{VCY}}(\Gamma)$ we denote the classifying space for the family of virtually cyclic subgroups of Γ, and $\boldsymbol{K}_R$ is the spectrum given by algebraic K-theory with coefficients from R (resp. we have the L-theoretic analogs); see eg [12; 64] for more details. We say that Γ satisfies the *Farrell–Jones conjecture with finite wreath products* if for any finite group F the wreath product $\Gamma \wr F$ satisfies the Farrell–Jones conjecture.

Proof of Theorem 1.5(7) Kasprowski and Rüping [64] showed that the Farrell–Jones conjecture with finite wreath products holds for groups acting geometrically on spaces with convex geodesic bicombing. Hence our result follows from Theorems 6.3 and 3.13. □

7.6 The coarse Baum–Connes conjecture

For a metric space X the *coarse assembly map* is a homomorphism from the coarse K-homology of X to the K-theory of the Roe algebra of X. The space X satisfies the *coarse Baum–Connes conjecture* if the coarse assembly map is an isomorphism. A finitely generated group Γ satisfies the coarse Baum–Connes conjecture if the conjecture holds for Γ seen as a metric space with a word metric given by a finite generating set. Equivalently, the conjecture holds for Γ if a metric space (equivalently, every metric space) acted geometrically upon by Γ satisfies the conjecture.

Proof of Theorem 1.5(8) Fukaya and Oguni [44] introduced the notion of *geodesic coarsely convex* space, and proved that the coarse Baum–Connes conjecture holds for such spaces. A geodesic coarsely convex space is a metric space with a coarse version of a bicombing satisfying some coarse convexity condition. In particular, metric spaces with a convex bicombing — and hence all proper injective metric spaces (Theorem 3.13) — are geodesic coarsely convex spaces. Therefore our result follows from Theorem 6.3. □

7.7 Asymptotic cones

In this section, we are interested in asymptotic cones of Helly groups and prove Theorem 1.5(9). Before turning to the proof, let us begin with a few definitions.

An *ultrafilter* ω over a set S is a collection of subsets of S satisfying the following conditions:

- $\varnothing \notin \omega$ and $S \in \omega$.
- For every $A, B \in \omega$, $A \cap B \in \omega$.
- For every $A \subset S$, either $A \in \omega$ or $A^c \in \omega$.

Basically, an ultrafilter may be thought of as a labeling of the subsets of S as "small" (if they do not belong to ω) or "big" (if they belong to ω). More formally, the map

$$\mathfrak{P}(S) \to \{0,1\}, \qquad A \mapsto \begin{cases} 0 & \text{if } A \notin \omega, \\ 1 & \text{if } A \in \omega, \end{cases}$$

defines a finitely additive measure on S. The easiest example of an ultrafilter is the following. Fixing some $s \in S$, set $\omega = \{A \subset S : s \in A\}$. Such an ultrafilter is called *principal*. The existence of nonprincipal ultrafilters is assured by Zorn's lemma; see [63, Section 3.1] for a brief explanation.

Now fix a metric space (X,d), a nonprincipal ultrafilter ω over $\mathbb{N}$, a *scaling sequence* $\epsilon = (\epsilon_n)$ satisfying $\epsilon_n \to 0$ and a sequence of basepoints $o = (o_n) \in X^{\mathbb{N}}$. A sequence $(r_n) \in \mathbb{R}^{\mathbb{N}}$ is *ω-bounded* if there exists some $M \geq 0$ such that $\{n \in \mathbb{N} : |r_n| \leq M\} \in \omega$ (ie if $|r_n| \leq M$ for "ω-almost all n"). Set $B(X,\epsilon,o) = \{(x_n) \in X^{\mathbb{N}} : (\epsilon_n d(x_n,o_n)) \text{ is } \omega\text{-bounded}\}$. We may define a pseudodistance on $B(X,\epsilon,o)$ as follows. First, we say that a sequence $(r_n) \in \mathbb{R}^{\mathbb{N}}$ *ω-converges* to a real $r \in \mathbb{R}$ if, for every $\epsilon > 0$,

$\{n \in \mathbb{N} : |r_n - r| \leq \epsilon\} \in \omega$. If so, we write $r = \lim_\omega r_n$. It is worth noticing that an ω-bounded sequence of $\mathbb{R}^{\mathbb{N}}$ always ω-converges; see [63, Section 3.1] for more details. Then our pseudodistance is

$$B(X,\epsilon,o)^2 \to [0,+\infty), \quad (x,y) \mapsto \lim_\omega \epsilon_n d(x_n,y_n).$$

The previous ω-limit always exists since the sequence under consideration is ω-bounded.

Definition 7.10 The *asymptotic cone* $\mathrm{Cone}_\omega(X,\epsilon,o)$ of X is the metric space obtained by quotienting $B(X,\epsilon,o)$ by the relation $(x_n) \sim (y_n)$ if $d((x_n),(y_n)) = 0$.

The picture to keep in mind is that $(X,\epsilon_n d)$ is a sequence of spaces we get from X by "zooming out", and the asymptotic cone is the "limit" of this sequence. Roughly speaking, the asymptotic cones of a metric space are asymptotic pictures of the space. For instance, any asymptotic cone of $\mathbb{Z}^2$, thought of as the infinite grid in the plane, is isometric to $\mathbb{R}^2$ endowed with the ℓ_1-metric, and the asymptotic cones of a simplicial tree (and more generally of any Gromov-hyperbolic space) are real trees.

One can define asymptotic cones of finitely generated groups up to bi-Lipschitz homeomorphism by looking at word metrics associated to finite generating sets, since quasi-isometric metric spaces have bi-Lipschitz-homeomorphic asymptotic cones [63, Proposition 3.12].

We are now ready to prove Theorem 1.5(9) as a consequence of the following result:

Proposition 7.11 *Let (X,d) be a finite-dimensional proper injective metric space. Then its asymptotic cones are contractible.*

Proof Let $\sigma\colon X \times X \times [0,1]$ denote the combing provided by Theorem 3.13. Fix a nonprincipal ultrafilter ω, a sequence of basepoints $o = (o_n)$ and a sequence of scalings $\epsilon = (\epsilon_n)$. For every point $x = (x_n) \in \mathrm{Cone}_\omega(X,o,\epsilon)$ and every $t \in [0,1]$, let $\rho(t,x)$ denote $(\sigma(o_n,x_n,t))$. Since σ is geodesic, $\rho(t,x)$ defines a point of $\mathrm{Cone}_\omega(X,o,\epsilon)$. Also, since σ is convex, the map

$$\rho\colon [0,1] \times \mathrm{Cone}_\omega(X,o,\epsilon) \to \mathrm{Cone}_\omega(X,o,\epsilon), \quad (t,x) \mapsto \rho(t,x),$$

is continuous. In other words, ρ defines a retraction of $\mathrm{Cone}_\omega(X,o,\epsilon)$ to the point o. □

Proof of Theorem 1.5(9) Let Γ be a group acting geometrically on a Helly graph G. By Theorem 6.3, Γ acts geometrically on the injective hull $E(G)$ of G, which is a finite-dimensional proper injective metric space. As every asymptotic cone of Γ must be bi-Lipschitz-homeomorphic to an asymptotic cone of $E(G)$, the desired conclusion follows from Proposition 7.11. □

8 Biautomaticity of Helly groups

Biautomaticity is a strong property implying numerous algorithmic and geometric features of a group [40; 21]. Sometimes the fact that a group acting on a space is biautomatic may be established from the geometric and combinatorial properties of the space. For example, one of the important and nice results about CAT(0) cube complexes is a theorem by Niblo and Reeves [69] stating that the groups acting geometrically on such complexes are biautomatic. Januszkiewicz and Świątkowski [61] established a similar result for groups acting on systolic complexes. It is also well known that hyperbolic groups are biautomatic [40]. Świątkowski [85] presented a general framework of locally recognized path systems in a graph G under which proving biautomaticity of a group acting geometrically on G is reduced to proving local recognizability and the 2-sided fellow traveler property for some paths.

In this section we use a different meaning of the term "bicombing". Here the bicombing is a combinatorial object that should not be confused with the (continuous) geodesic bicombing from Section 3.4.

8.1 Main results

In this section, similarly to the results of [69] for CAT(0) cube complexes, of [61] for systolic complexes and of [23] for swm-graphs, we define the normal clique-path and prove the existence and uniqueness of normal clique-paths in all Helly graphs G. These clique-paths can be viewed as usual paths in the 1-skeleton of the face complex $F(X(G))$ of $X(G)$ and give rise to paths in the 1-skeleton $\beta(G)$ of the first barycentric subdivision of $X(G)$. From their definition, it follows that the sets of normal clique-paths are locally recognized sensu [85]. Moreover, we prove that they satisfy the 2-sided fellow traveler property. As a consequence, groups acting geometrically on Helly graphs are biautomatic.

Theorem 8.1 *The set of normal clique-paths between all vertices of a Helly graph G defines a regular geodesic bicombing in $\beta(G)$. Consequently, a group acting geometrically on a Helly graph is biautomatic.*

Remark 8.2 A natural generalization of this theorem would be to prove that injective groups (ie groups acting geometrically on injective metric spaces) are biautomatic. Recently, Hugues and Valiunas [58] proved this is not the case: they constructed an injective group that is not biautomatic and thus not Helly.

8.2 Bicombings and biautomaticity

We continue by recalling the definitions of (geodesic) bicombing and biautomatic group [40; 21]. Let $G = (V, E)$ be a graph and suppose that Γ is a group acting geometrically by automorphisms on G. These assumptions imply that the graph G is locally finite and that the degrees of the vertices of G are uniformly bounded. Denote by $\mathcal{P}(G)$ the set of all paths of G. A *path system* $\mathcal{P}$ [85] is any subset of $\mathcal{P}(G)$. The action of Γ on G induces the action of Γ on the set $\mathcal{P}(G)$ of all paths of G. A path system $\mathcal{P} \subseteq \mathcal{P}(G)$ is called Γ*-invariant* if $g \cdot \gamma \in \mathcal{P}$ for all $g \in \Gamma$ and $\gamma \in \mathcal{P}$.

Let $[0,n]^*$ denote the set of integer points from the segment $[0,n]$. Given a path γ of length $n = |\gamma|$ in G, we can parametrize it and denote it by $\gamma\colon [0,n]^* \to V(G)$. It will be convenient to extend γ over $[0,\infty]$ by setting $\gamma(i) = \gamma(n)$ for any $i > n$. A path system $\mathcal{P}$ of a graph G is said to satisfy the 2*-sided fellow traveler property* if there are constants $C > 0$ and $D \geq 0$ such that for any two paths $\gamma_1, \gamma_2 \in \mathcal{P}$, the following inequality holds for all natural i:

$$d_G(\gamma_1(i), \gamma_2(i)) \leq C \max\{d_G(\gamma_1(0), \gamma_2(0)), d_G(\gamma_1(\infty), \gamma_2(\infty))\} + D.$$

A path system $\mathcal{P}$ is *complete* if any two vertices are endpoints of some path in $\mathcal{P}$. A *bicombing* of a graph G is a complete path system $\mathcal{P}$ satisfying the 2-sided fellow traveler property. If all paths in the bicombing $\mathcal{P}$ are shortest paths of G, then $\mathcal{P}$ is called a *geodesic bicombing*.

We quickly recall the definition of a biautomatic structure for a group; for details see [40; 21; 85]. Let Γ be a group generated by a finite set S. A *language* over S is some set of words in $S \cup S^{-1}$ (in the free monoid $(S \cup S^{-1})^*$). A language over S defines a Γ-invariant path system in the Cayley graph $\mathrm{Cay}(\Gamma, S)$. A language is *regular* if it is accepted by some finite-state automaton. A *biautomatic structure* is a pair $(S, \mathcal{L})$, where S is as above, $\mathcal{L}$ is a regular language over S and the associated path system in $\mathrm{Cay}(\Gamma, S)$ is a bicombing. A group is *biautomatic* if it admits a biautomatic structure. In what follows, we use specific conditions implying biautomaticity for groups acting geometrically on graphs. The method, relying on the notion of a locally recognized path system, was developed by Świątkowski [61].

Let G be a graph and let Γ be a group acting geometrically on G. Two paths γ_1 and γ_2 of G are Γ-*congruent* if there is $g \in \Gamma$ such that $g \cdot \gamma_1 = \gamma_2$. Denote by $\mathcal{S}_k$ the set of Γ-congruence classes of paths of length k of G. Since Γ acts geometrically on G, the sets $\mathcal{S}_k$ are finite for any natural k. For any path γ of G, denote by $[\gamma]$ its Γ-congruence class.

For a subset $R \subset \mathcal{S}_k$, let $\mathcal{P}_R$ be the path system in G consisting of all paths γ satisfying the following two conditions:

(1) If $|\gamma| \geq k$, then $[\eta] \in R$ for any subpath η of length k of γ.

(2) If $|\gamma| < k$, then γ is a prefix of some path η such that $[\eta] \in R$.

A path system $\mathcal{P}$ in G is k-*locally recognized* if, for some $R \subset \mathcal{S}_k$, we have $\mathcal{P} = \mathcal{P}_R$, and $\mathcal{P}$ is *locally recognized* if it is k-locally recognized for some k. Świątkowski [85] established the following sufficient conditions for biautomaticity in terms of local recognition and bicombing:

Theorem 8.3 [85, Corollary 7.2] *Let Γ be group acting geometrically on a graph G and let $\mathcal{P}$ be a path system in G satisfying the following conditions*:

(1) *$\mathcal{P}$ is locally recognized.*

(2) *There exists $v_0 \in V(G)$ such that any two vertices from the orbit $\Gamma \cdot v_0$ are connected by a path from $\mathcal{P}$.*

(3) *$\mathcal{P}$ satisfies the 2-sided fellow traveler property.*

Then Γ is biautomatic.

8.3 Normal clique-paths in Helly graphs

For a set S of vertices of a graph $G = (V, E)$ and an integer $k \geq 0$, let $B^*_k(S) := \bigcap_{s \in S} B_k(s)$. In particular, if S is a clique, then $B^*_1(S)$ is the union of S and the set of vertices adjacent to all vertices in S. If $S \subseteq S'$, then $B^*_k(S) \supseteq B^*_k(S')$. For two cliques τ and σ of G, let $\bar{d}(\tau, \sigma) := \max\{d(t, s) : t \in \tau \text{ and } s \in \sigma\}$. We also recall the notation $d(\tau, \sigma) = \min\{d(t, s) : t \in \tau \text{ and } s \in \sigma\}$ for the standard distance between τ and σ.

We say that two cliques σ and τ of a graph G are at *uniform distance* k (denoted by $\sigma \bowtie_k \tau$) if $d(s,t)=k$ for any $s \in \sigma$ and any $t \in \tau$. Equivalently, $\sigma \bowtie_k \tau$ if and only if $\bar{d}(\tau,\sigma)=d(\tau,\sigma)=k$.

Given two cliques σ and τ of G with $\bar{d}(\tau,\sigma)=k \geq 2$, let $\widehat{R}_\tau(\sigma) := B_k^*(\tau) \cap B_1^*(\sigma)$ and let $f_\tau(\sigma) := B_{k-1}^*(\tau) \cap B_1^*(\widehat{R}_\tau(\sigma))$. The following observations can help to understand these notions:

- $\widehat{R}_\tau(\sigma)$ is the union of the maximal cliques of $B_k^*(\tau)$ that contain σ.
- $B_1^*(\widehat{R}_\tau(\sigma))$ is the intersection of the maximal cliques of $B_k^*(\tau)$ that contain σ.
- $f_\tau(\sigma)$ is the intersection of $B_{k-1}^*(\tau)$ and the maximal cliques of $B_k^*(\tau)$ that contain σ.

Since G is a Helly graph, the set $f_\tau(\sigma)$ is nonempty, and we call it the *imprint* of σ with respect to τ. Note that since σ is a clique $\sigma \subseteq \widehat{R}_\tau(\sigma)$, and thus $f_\tau(\sigma) \subseteq \widehat{R}_\tau(\sigma)$. Note also that each vertex in $f_\tau(\sigma)$ is adjacent to all other vertices in $\widehat{R}_\tau(\sigma)$, whence $\widehat{R}_\tau(\sigma) \subseteq B_1^*(f_\tau(\sigma))$ and $f_\tau(\sigma)$ is a clique.

Lemma 8.4 *For any two cliques σ and τ of a Helly graph G such that $\bar{d}(\tau,\sigma)=k \geq 2$, the imprint $f_\tau(\sigma)$ is a nonempty clique such that $\bar{d}(\tau, f_\tau(\sigma)) = k-1 = \bar{d}(\tau,s')$ for any $s' \in f_\tau(\sigma)$. Moreover, if $\sigma \bowtie_k \tau$, then $f_\tau(\sigma) \bowtie_{k-1} \tau$.*

Proof By definition $f_\tau(\sigma) \subseteq B_{k-1}^*(\tau)$. Also, for any $r,r' \in \widehat{R}_\tau(\sigma)$, we have $\sigma \subseteq B_1(r) \cap B_1(r')$. Moreover, for any $r \in \widehat{R}_\tau(\sigma)$ and any $t \in \tau$, $d(r,t) \leq k$ and thus $B_{k-1}(t) \cap B_1(r) \neq \varnothing$. Note also that since τ is a clique and $k \geq 2$, $\tau \subseteq B_{k-1}^*(\tau)$. Consequently, since G is a Helly graph, $f_\tau(\sigma) \neq \varnothing$. Since $f_\tau(\sigma) \cup \sigma \subseteq \widehat{R}_\tau(\sigma)$ and each vertex of $f_\tau(\sigma)$ is adjacent to all other vertices of $\widehat{R}_\tau(\sigma)$, necessarily $f_\tau(\sigma) \cup \sigma$ is a clique. Therefore, for any $t \in \tau$ and $s \in \sigma$ such that $d(t,s)=\bar{d}(\tau,\sigma)=k$, and any $s' \in f_\tau(\sigma)$, we have $d(s',t) \geq d(s,t)-d(s,s')=k-1$. Since $s' \in f_\tau(\sigma) \subseteq B_{k-1}^*(\tau)$, we have $d(s',t)=k-1$. Thus, $\bar{d}(\tau, f_\tau(\sigma))=k-1$ and $f_\sigma(\tau) \bowtie_{k-1} \tau$ when $\sigma \bowtie_k \tau$. □

Lemma 8.5 *Consider three cliques σ, σ' and τ of a Helly graph G such that $\bar{d}(\tau,\sigma)=\bar{d}(\tau,\sigma')=k \geq 2$. If $\sigma' \subseteq \sigma$, then $\widehat{R}_\tau(\sigma) \subseteq \widehat{R}_\tau(\sigma')$ and $f_\tau(\sigma') \subseteq f_\tau(\sigma)$. In particular, if $\sigma \bowtie_k \tau$, then for every $s \in \sigma$ we have $f_\tau(s) \subseteq f_\tau(\sigma)$.*

Proof Recall that $\widehat{R}_\tau(\sigma) := B_k^*(\tau) \cap B_1^*(\sigma)$ and $\widehat{R}_\tau(\sigma') := B_k^*(\tau) \cap B_1^*(\sigma')$. Since $\sigma' \subseteq \sigma$, we have $B_1^*(\sigma) \subseteq B_1^*(\sigma')$ and thus $\widehat{R}_\tau(\sigma) \subseteq \widehat{R}_\tau(\sigma')$. Consequently, $B_1^*(\widehat{R}_\tau(\sigma')) \subseteq B_1^*(\widehat{R}_\tau(\sigma))$ and thus $f_\tau(\sigma') = B_{k-1}^*(\tau) \cap B_1^*(\widehat{R}_\tau(\sigma')) \subseteq B_{k-1}^*(\tau) \cap B_1^*(\widehat{R}_\tau(\sigma)) = f_\tau(\sigma)$. □

A sequence of cliques $(\sigma_0,\sigma_1,\dots,\sigma_k)$ of a Helly graph G is called a *normal clique-path* if the following local conditions hold:

(1) For any $0 \leq i \leq k-1$, σ_i and σ_{i+1} are disjoint and $\sigma_i \cup \sigma_{i+1}$ is a clique of G.

(2) For any $1 \leq i \leq k-1$, σ_{i-1} and σ_{i+1} are at uniform distance 2.

(3) For any $1 \leq i \leq k-1$, $\sigma_i = f_{\sigma_{i-1}}(\sigma_{i+1})$.

Notice that if $k \geq 2$, then (1) follows from (2) and (3).

Theorem 8.6 (normal clique-paths) *For any pair τ and σ of cliques of a Helly graph G such that $\sigma \bowtie_k \tau$, there exists a unique normal clique-path $\gamma_{\tau\sigma} = (\tau = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_k = \sigma)$ such that*

$$\sigma_i = f_\tau(\sigma_{i+1}) \quad \text{for each } i = k-1, \ldots, 2, 1, \tag{8-1}$$

and any sequence of vertices $P = (s_0, s_1, \ldots, s_k)$ with $s_i \in \sigma_i$ for $0 \le i \le k$ is a shortest path from s_0 to s_k. In particular, any two vertices p and q of G are connected by a unique normal clique-path γ_{pq}.

Proof We first prove that $\gamma_{\tau\sigma}$ is a normal clique-path. The proof uses the following result:

Lemma 8.7 *Let σ, σ', σ'' and τ be four cliques of a Helly graph G such that $\sigma \bowtie_k \tau$ with $k \ge 3$, $\sigma' \subseteq f_\tau(\sigma)$ and $\sigma'' \subseteq f_\tau(\sigma')$. Then $f_\tau(\sigma) = f_{\sigma''}(\sigma)$.*

Proof Note that our conditions and Lemma 8.4 imply that $\sigma' \bowtie_{k-1} \tau$, $\sigma'' \bowtie_{k-2} \tau$ and $\sigma \bowtie_2 \sigma''$.

We first show that $\widehat{R}_{\sigma''}(\sigma) = \widehat{R}_\tau(\sigma)$. Recall that $\widehat{R}_\tau(\sigma) = B^*_k(\tau) \cap B^*_1(\sigma)$ and $\widehat{R}_{\sigma''}(\sigma) = B^*_2(\sigma'') \cap B^*_1(\sigma)$. Since $\tau \bowtie_{k-2} \sigma''$, we have $B^*_2(\sigma'') \subseteq B^*_k(\tau)$. Consequently, $\widehat{R}_{\sigma''}(\sigma) \subseteq \widehat{R}_\tau(\sigma)$. Conversely, by the definition of σ'', we have $\sigma' \subseteq B^*_1(\sigma'')$. Since $\sigma'' \subseteq f_\tau(\sigma')$, we have $B^*_1(\sigma'') \supseteq B^*_1(f_\tau(\sigma')) \supseteq \widehat{R}_\tau(\sigma') \supseteq \sigma'$. Since $\sigma' \subseteq f_\tau(\sigma)$, we have $\widehat{R}_\tau(\sigma) \subseteq B^*_1(f_\tau(\sigma)) \subseteq B^*_1(\sigma') \subseteq B^*_2(\sigma'')$ where the last containment follows from $\sigma'' \subseteq f_\tau(\sigma')$. So $\widehat{R}_\tau(\sigma) = B^*_k(\tau) \cap B^*_1(\sigma) \subseteq B^*_2(\sigma'') \cap B^*_1(\sigma) = \widehat{R}_{\sigma''}(\sigma)$, and thus $\widehat{R}_{\sigma''}(\sigma) = \widehat{R}_\tau(\sigma)$.

Set $\widehat{R} := \widehat{R}_{\sigma''}(\sigma) = \widehat{R}_\tau(\sigma)$, $\varrho' := f_{\sigma''}(\sigma)$ and $\nu' := f_\tau(\sigma)$. Recall that $\nu' = f_\tau(\sigma) = B^*_{k-1}(\tau) \cap B^*_1(\widehat{R})$ and $\varrho' = f_{\sigma''}(\sigma) = B^*_1(\sigma'') \cap B^*_1(\widehat{R})$. Since $\tau \bowtie_{k-2} \sigma''$, we have $B^*_1(\sigma'') \subseteq B^*_{k-1}(\tau)$ and thus $\varrho' \subseteq \nu'$. Conversely, since $\nu' \subseteq \widehat{R}_\tau(\nu') = B^*_1(\nu') \cap B_{k-1}(\tau) \subseteq B^*_1(\sigma') \cap B_{k-1}(\tau) = \widehat{R}_\tau(\sigma')$, we have $\nu' \subseteq B^*_1(\sigma'')$ by definition of σ''. Consequently, $\nu' \subseteq B^*_1(\sigma'') \cap B^*_1(\widehat{R}) = \varrho'$. Thus $\nu' = \varrho'$. □

To prove that $\gamma_{\tau\sigma}$ is a normal clique-path, we proceed by induction on k. If $k \le 2$, there is nothing to prove. Assume now that $k \ge 3$. Since $\tau \bowtie_k \sigma_k$, $\sigma_{k-1} = f_\tau(\sigma_k)$ and $\sigma_{k-2} = f_\tau(\sigma_{k-1})$, we have that $\tau \bowtie_{k-1} \sigma_{k-1}$, $\tau \bowtie_{k-2} \sigma_{k-2}$ and $\sigma_{k-2} \bowtie_2 \sigma_k$. By the induction hypothesis, $(\sigma_0 = \tau, \sigma_1, \sigma_2, \ldots, \sigma_{k-1})$ is a normal clique-path. Applying Lemma 8.7 with $\sigma = \sigma_k$, $\sigma' = \sigma_{k-1}$ and $\sigma'' = \sigma_{k-2}$, we have that $\sigma_{k-1} = f_{\sigma_{k-2}}(\sigma_k)$, and thus $\gamma_{\tau\sigma}$ is a normal clique-path as well.

We now prove that an arbitrary normal clique-path $\gamma'_{\tau\sigma} = (\tau = \varrho_0, \varrho_1, \varrho_2, \ldots, \varrho_l = \sigma)$ coincides with $\gamma_{\tau\sigma}$. In fact, we prove this result under a weaker assumption than $\sigma \bowtie_k \tau$.

Proposition 8.8 *Let σ and τ be two cliques of a Helly graph G, and let k be an integer such that $\bar{d}(s, \tau) = k$ for every $s \in \sigma$. Then any normal clique-path $\gamma'_{\tau\sigma} = (\tau = \varrho_0, \varrho_1, \varrho_2, \ldots, \varrho_l = \sigma)$ coincides with $\gamma_{\tau\sigma} = (\tau = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_k = \sigma)$, whose cliques are given by (8-1).*

Proof The proof of the proposition is based on the following result:

Lemma 8.9 *Let ϱ, ϱ', ϱ'' and τ be four cliques of a Helly graph G with $\bar{d}(\tau, \varrho) = 1 + \bar{d}(\tau, \varrho') =: k \ge 3$, $\bar{d}(\varrho, \varrho'') \ge 2$, $\varrho' = f_{\varrho''}(\varrho)$ and $\varrho'' \subseteq f_\tau(\varrho')$. Then $\varrho' = f_\tau(\varrho)$.*

Proof Let $\sigma' = f_\tau(\varrho)$, and note that our conditions and Lemma 8.4 imply that $\bar{d}(\tau,\sigma') = 1 + \bar{d}(\tau,\varrho'') = k-1$ and $\bar{d}(\varrho,\varrho'') = 2$.

We first show that $\widehat{R}_{\varrho''}(\varrho) = \widehat{R}_\tau(\varrho)$. Recall that $\widehat{R}_\tau(\varrho) = B^*_k(\tau)\cap B^*_1(\varrho)$ and $\widehat{R}_{\varrho''}(\varrho) = B^*_2(\varrho'')\cap B^*_1(\varrho)$. Since $\bar{d}(\tau,\varrho'') = k-2$, necessarily $B^*_2(\varrho'')\subseteq B^*_k(\tau)$, and consequently, $\widehat{R}_{\varrho''}(\varrho)\subseteq \widehat{R}_\tau(\varrho)$. In particular, note that $\varrho' \subseteq \widehat{R}_{\varrho''}(\varrho)\subseteq \widehat{R}_\tau(\varrho)$, so $\sigma'\subseteq B^*_1(\widehat{R}_\tau(\varrho))\subseteq B^*_1(\varrho')$. Since $\sigma'\subseteq B^*_{k-1}(\tau)$, we have $\sigma'\subseteq B^*_{k-1}(\tau)\cap B^*_1(\varrho') = \widehat{R}_\tau(\varrho')$. Therefore, by the definition of $\varrho''\subseteq f_\tau(\varrho')$, we have $\sigma'\subseteq B^*_1(\varrho'')$. Hence $B^*_1(\sigma')\subseteq B^*_2(\varrho'')$, and thus $\widehat{R}_\tau(\varrho)\subseteq B^*_1(\sigma')\subseteq B^*_2(\varrho'')$. Therefore $\widehat{R}_\tau(\varrho)\subseteq B^*_2(\varrho'')\cap B^*_1(\varrho) = \widehat{R}_{\varrho''}(\varrho)$ and thus $\widehat{R}_\tau(\varrho) = \widehat{R}_{\varrho''}(\varrho)$.

Let $\widehat{R} = \widehat{R}_\tau(\varrho) = \widehat{R}_{\varrho''}(\varrho)$ and recall that $\varrho' = f_{\varrho''}(\varrho) = B^*_1(\varrho'')\cap B^*_1(\widehat{R})$ and that $\sigma' = f_\tau(\varrho) = B^*_{k-1}(\tau)\cap B^*_1(\widehat{R})$. Since $\sigma'\subseteq B^*_1(\varrho'')$, necessarily $\sigma'\subseteq\varrho'$. Conversely, since $\bar{d}(\tau,\varrho'') = k-2$, necessarily $B^*_1(\varrho'')\subseteq B^*_{k-1}(\tau)$, and consequently, $\varrho'\subseteq\sigma'$. Thus $\varrho' = \sigma'$. □

We prove the proposition by induction on the length l of the normal clique-path $\gamma'_{\tau\sigma}$. If $l\le 2$, there is nothing to prove. Assume now that $l\ge 3$ and let $k = \bar{d}(\tau,\sigma)$.

Suppose first that $\bar{d}(\tau,\varrho_{l-1}) = k-1$. Since $\varrho_{l-1}\cup\sigma$ is a clique and since $\bar{d}(s,\tau) = k$ for every $s\in\sigma$, necessarily $\bar{d}(p',\tau) = k-1$ for every $p'\in\varrho_{l-1}$. By the induction hypothesis, the clique-path $\gamma'_{\tau\varrho_{l-1}} = (\tau = \varrho_0,\varrho_1,\varrho_2,\dots,\varrho_{l-1})$ coincides with $\gamma_{\tau\varrho_{l-1}}$. Consequently, $l = k$ and $\varrho_{l-2} = f_\tau(\varrho_{l-1})$. Applying Lemma 8.9 with $\varrho = \sigma$, $\varrho' = \varrho_{l-1}$ and $\varrho'' = \varrho_{l-2}$, we have that $f_\tau(\sigma) = f_{\varrho_{l-2}}(\sigma) = \varrho_{l-1}$. Hence $\gamma'_{\tau\sigma}$ and $\gamma_{\tau\sigma}$ coincide.

Suppose now that $\bar{d}(\tau,\varrho_{l-1})\ge k$. In this case $l\ge k+1$, and so $\bar{d}(\varrho_l,\tau) = k\le l-1$. Consider the minimal index i for which there exists $p\in\varrho_i$ such that $\bar{d}(p,\tau)\le i-1$. Note that $i\ge 2$, since otherwise $\tau = \varrho_0 = \{p\}$ and $\varrho_0\cap\varrho_1\ne\varnothing$, contradicting the fact that $\gamma'_{\tau\sigma}$ is a normal clique-path. Note also that since $\gamma'_{\tau\sigma}$ is a normal clique-path, $\varrho_0 \bowtie_2 \varrho_2$, and thus $i\ge 3$. By the induction hypothesis, $\gamma'_{\tau\varrho_{i-1}} = (\tau = \varrho_0,\varrho_1,\varrho_2,\dots,\varrho_{i-1})$ and $\gamma_{\tau\varrho_{i-1}}$ coincide. In particular, this implies that $\varrho_{i-2} = f_\tau(\varrho_{i-1})$. Note that $p\in B^*_{i-1}(\tau)$ by our choice of p and that $p\in B^*_1(\varrho_{i-1})$ since $\varrho_{i-1} = f_{\varrho_{i-2}}(\varrho_i)$. Consequently $p\in\widehat{R}_\tau(\varrho_{i-1})\subseteq B^*_1(\varrho_{i-2})$. But then ϱ_i and ϱ_{i-2} are not at uniform distance 2, contradicting the fact that $\gamma'_{\tau\sigma}$ is a normal clique-path. This finishes the proof of Proposition 8.8. □

To conclude the proof of Theorem 8.6, consider any sequence $P = (s_0,s_1,\dots,s_k)$ such that $s_i\in\sigma_i$ for $0\le i\le k$. Note that P is a path since $\sigma_i\cup\sigma_{i+1}$ is a clique for every $0\le i\le k-1$, and that it is a shortest path since $d(s_0,s_k) = \bar{d}(\sigma_0,\sigma_k) = k$. □

8.4 Normal paths in Helly graphs

In this subsection, we define the notion of a normal path between any two vertices t and s of a Helly graph. Analogously to normal clique-paths, normal paths can be characterized in a local-to-global way, and therefore they are locally recognized. Any two vertices t and s of G can be connected by at least one normal path, and all normal (t,s)-paths are hosted by the normal clique-path γ_{ts}.

A path $(t = s_0, s_1, \dots, s_k = s)$ between two vertices t and s of a Helly graph G is called a *normal path* if the following local conditions hold:

(1) For any $1 \le i \le k-1$, $d(s_{i-1}, s_{i+1}) = 2$.

(2) For any $1 \le i \le k-1$, $s_i \in f_{s_{i-1}}(s_{i+1})$.

Proposition 8.10 (normal paths) *A path $P_{ts} = (t = s_0, s_1, \dots, s_k = s)$ between two vertices t and s of a Helly graph G is a normal path if and only if $s_i \in f_t(s_{i+1})$ for any $1 \le i \le k-1$. In particular, this implies that P_{ts} is a shortest path of G. If $\gamma_{ts} = (\{t\} = \sigma_0, \sigma_1, \dots, \sigma_k = \{s\})$ is the unique normal clique-path between t and s, then for any normal path $P'_{ts} = (t = s_0, s_1, \dots, s_k = s)$, we have $s_i \in \sigma_i$ for $0 \le i \le k$.*

Proof The proof of the first statement is similar to the proof of Theorem 8.6. We first prove that P_{ts} is a normal path. Observe that by Lemma 8.4, P_{ts} is a shortest path of G. We proceed by induction on the distance $k = d(t,s)$. If $k \le 2$, there is nothing to prove. Assume now that $k \ge 3$. Since $d(t, s_k) = k$, $s_{k-1} \in f_t(s_k)$ and $s_{k-2} \in f_t(s_{k-1})$, we have $d(t, s_{k-1}) = k-1$ and $d(t, s_{k-2}) = k-2$. By the induction hypothesis $(s_0 = t, s_1, s_2, \dots, s_{k-1})$ is a normal path. Applying Lemma 8.7 with $\sigma = \{s_k\}$, $\sigma' = \{s_{k-1}\}$, $\sigma'' = \{s_{k-2}\}$ and $\tau = \{t\}$, we conclude that $s_{k-1} \in f_t(s_k) = f_{s_{k-2}}(s_k)$ and thus P_{ts} is a normal path as well.

We now prove that any normal path $P'_{ts} = (t = p_0, p_1, \dots, p_l = s)$ is a shortest path of G and that $p_i \in f_t(p_{i+1})$ for every $1 \le i \le l$. To do so, we proceed by induction on the length l of P'_{ts}. If $l \le 2$, there is nothing to prove. Assume now that $l \ge 3$ and let $k = d(t, p_l)$. By the induction hypothesis applied to the normal path $P'_{tp_{l-1}} = (t = p_0, p_1, \dots, p_{l-1})$, $P'_{tp_{l-1}}$ is a shortest path of G and $p_i \in f_t(p_{i+1})$ for every $1 \le i \le l-2$. In particular, $d(t, p_{l-1}) = l-1$.

Suppose first that $d(t, p_{l-1}) = k-1$. Then $l = k$, and therefore P'_{ts} is a shortest path. Since $p_{l-2} \in f_\tau(p_{l-1})$, applying Lemma 8.9 with $\varrho = \{s\}$, $\varrho' = f_{p_{l-2}}(s)$ and $\varrho'' = \{p_{l-2}\}$, we have that $f_t(s) = f_{p_{l-2}}(s)$, and thus $p_{l-1} \in f_{p_{l-2}}(s) = f_t(s)$. Consequently, $p_i \in f_t(p_{i+1})$ for every $1 \le i \le l$ and the proposition holds in this case. Suppose now that $l-1 = d(t, p_{l-1}) \ge k$, ie $l \ge k+1$. By the induction hypothesis applied to the path $P'_{tp_{l-1}}$, we have $p_{l-2} \in f_t(p_{l-1})$. Note that $p_l \in B_{l-1}(t)$ because $d(t, p_l) = k \le l-1$, and that $p_l \in B_1(p_{l-1})$. Consequently, $p_l \in \widehat{R}_t(p_{l-1}) \subseteq B_1(p_{l-2})$. But then $d(p_l, p_{l-2}) \le 1$, contradicting the fact that P'_{ts} is a normal path.

Consider now the normal clique-path $\gamma_{ts} = (\{t\} = \sigma_0, \sigma_1, \dots, \sigma_k = \{s\})$ between two vertices t and s and any normal path $P_{ts} = (t = s_0, s_1, \dots, s_k = s)$. We show by reverse induction on i that $s_i \in \sigma_i$ for $0 \le i \le k$. For $i = k$, there is nothing to prove. Suppose now that $i < k$ and that $s_{i+1} \in \sigma_{i+1}$. Since $s_i \in f_t(s_{i+1})$ by the first assertion of the proposition and since $f_t(s_{i+1}) \subseteq f_t(\sigma_{i+1}) = \sigma_i$ by Lemma 8.5, we have $s_i \in \sigma_i$. □

Remark 8.11 Figure 6 is a Helly graph and contains two vertices s and t such that the cliques of the normal clique-path γ_{ts} contain a vertex not included in any normal (t,s)-path.

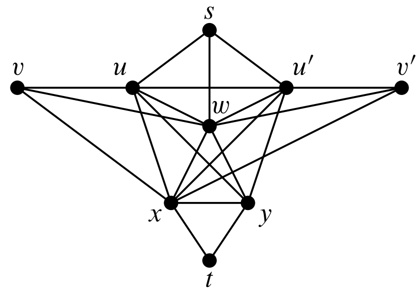

Figure 6: In this graph, y appears in a clique of the normal clique-path $\gamma_{ts} = (t, \{x, y\}, \{u, u', w\}, s)$. However, for any normal path $(t = s_0, s_1, s_2, s_3 = s)$, $\widehat{R}_t(s_2)$ contains either v or v', and thus $y \notin f_t(s_2)$.

8.5 Normal (clique-)paths are fellow travelers

Proposition 8.12 *Let G be a Helly graph. Consider two cliques σ and τ, two vertices p and q of G, and two integers $k' \geq k$ such that $p \bowtie_{k'} \sigma$, $q \bowtie_k \tau$, $d(\sigma, \tau) \leq 1$ and $d(p, q) \leq 1$. For the normal clique-paths $\gamma_{p\sigma} = (p = \sigma_0, \sigma_1, \ldots, \sigma_{k'} = \sigma)$ and $\gamma_{q\tau} = (q = \tau_0, \tau_1, \ldots, \tau_k = \tau)$, we have $d(\sigma_i, \tau_i) \leq 1$ for every $0 \leq i \leq k$ and $d(\sigma_i, \tau_k) \leq 1$ for every $k \leq i \leq k'$.*

Proof We prove the result by induction on k'. If $k' \leq 1$, there is nothing to prove. Assume now that $k' \geq 2$ and that the lemma holds for any cliques σ and τ, any vertices p and q, and any integers $l \leq l' \leq k' - 1$ such that $p \bowtie_{l'} \sigma$, $q \bowtie_l \tau$, $d(\sigma, \tau) \leq 1$ and $d(p, q) \leq 1$.

Suppose first that $k < k'$. Note that $k + 1 \leq k' \leq k + 2$ since $d(p, q) \leq 1$ and $d(\sigma, \tau) \leq 1$. Let $s \in \sigma$ and $t \in \tau$ such that $d(s, t) = d(\sigma, \tau) \leq 1$. Note that $d(p, t) \leq d(q, t) + 1 = k + 1 \leq k'$. Consequently $t \in \widehat{R}_p(s)$, and thus $f_p(s) \subseteq B_1(t)$. So since $f_p(s) \subseteq f_p(\sigma) = \sigma_{k'-1}$ by Lemma 8.5, we have $d(\sigma_{k'-1}, \tau_k) \leq 1$. By Lemma 8.4, $p \bowtie_{k'-1} \sigma_{k'-1}$, and thus we can apply the induction hypothesis to $\sigma_{k'-1}$, τ, p and q. Therefore $d(\sigma_i, \tau_i) \leq 1$ for every $0 \leq i \leq k$ and $d(\sigma_i, \tau_k) \leq 1$ for every $k \leq i \leq k' - 1$. Since, by our assumptions, $d(\sigma_{k'}, \tau_k) \leq 1$, we are done.

Suppose now that $k = k'$. By the induction hypothesis, it is enough to show that $d(f_p(\sigma), f_q(\tau)) \leq 1$. Consider any two vertices $s \in \sigma$ and $t \in \tau$ such that $d(s, t) = d(\sigma, \tau)$. By Lemma 8.5, it is enough to show that $d(f_p(s), f_q(t)) \leq 1$.

Assume first that $d(p, t) \leq k$ (in this case $s = t$ or $p = q$). Note that $t \in B_k(p) \cap B_1(s) = \widehat{R}_p(s)$, and so $f_p(s) \subseteq B_1(t)$. Since $f_p(s) \subseteq B_{k-1}(p) \subseteq B_k(q)$, we have $f_p(s) \subseteq B_k(q) \cap B_1(t) = \widehat{R}_q(t)$. Therefore $f_q(t) \subseteq B_1^*(f_p(s))$ and $d(f_p(s), f_q(t)) \leq 1$. Using symmetric arguments, we have $d(f_p(s), f_q(t)) \leq 1$ when $d(q, s) \leq k$.

Assume now that $d(q, s) = d(p, t) = k + 1$. This implies that $p \neq q$, $s \neq t$, $p \bowtie_k f_q(t)$ and $q \bowtie_k f_p(s)$. Since $d(p, s) = k$ and $p \bowtie_k f_q(t)$, we have $\{s, t\} \cup f_q(t) \subseteq \widehat{R}_p(t)$. Consider a vertex $u \in f_p(t)$. By definition of u, we have $d(p, u) = k$ and $\{s, t\} \cup f_q(t) \subseteq B_1(u)$. Also, $d(q, u) = k$ since $d(q, s) = k + 1$ and since $\bar{d}(q, f_q(t)) = k - 1$. Therefore, by the previous case replacing t by u, we have

$d(f_p(s), f_q(u)) \leq 1$. Note that $\widehat{R}_q(t) = B_1(t) \cap B_k(q) \subseteq B_1(t) \cap B_{k+1}(p) = \widehat{R}_p(t)$. Since $u \in f_p(t)$, we obtain $\widehat{R}_q(t) \subseteq \widehat{R}_p(t) \subseteq B_1^*(f_p(t)) \subseteq B_1(u)$. Consequently, $\widehat{R}_q(t) \subseteq B_1(u) \cap B_k(q) = \widehat{R}_q(u)$ and $f_q(u) \subseteq f_q(t)$. Therefore $d(f_p(s), f_q(t)) \leq d(f_p(s), f_q(u)) \leq 1$, concluding the proof. □

From Propositions 8.10 and 8.12, we immediately get the following result:

Corollary 8.13 *In a Helly graph G, the set of normal paths satisfies the 2-sided fellow traveler property. More precisely, for any four vertices s, t, p and q and two integers $k' \geq k$ such that $d(p,s) = k'$, $d(q,t) = k$, $d(s,t) \leq 1$ and $d(p,q) \leq 1$, and for any normal paths $P = (p = s_0, s_1, \dots, s_{k'} = s)$ and $Q = (q = t_0, t_1, \dots, t_k = t)$ we have $d(s_i, t_i) \leq 3$ for every $0 \leq i \leq k$ and $d(s_i, t_k) \leq 3$ for every $k \leq i \leq k'$.*

We are ready to conclude the proof of biautomaticity from Theorem 8.1:

Proposition 8.14 *Let a group Γ act geometrically on a Helly graph G. Then Γ is biautomatic.*

Proof Let $\mathcal{P}$ denote the set of all normal paths of G. We will prove now that the path system $\mathcal{P}$ satisfies Theorem 8.3(1)–(3). Condition (2) is satisfied because any two vertices of G are connected by a path of $\mathcal{P}$. That $\mathcal{P}$ satisfies the 2-sided fellow traveler property follows from Corollary 8.13. Finally, condition (1), that the set $\mathcal{P}$ can be 2-locally recognized, follows from the definition of normal paths and the fact that conditions (1) and (2) of this definition can be tested within balls of G of radius 2. Since Γ acts geometrically on G, there exists only a constant number of types of such balls. □

Remark 8.15 Proposition 8.14 can be also proved by viewing the set $\mathcal{P}^*$ of normal clique-paths of a Helly graph G as paths of the face complex $F(X(G))$ of the clique complex of G and establishing that $\mathcal{P}^*$ satisfies Theorem 8.3(1)–(3).

The set $\mathcal{P}^*$ in $F(X(G))$ gives rise to a set $\mathcal{P}'$ of paths of the first barycentric subdivision $\beta(G)$ of the clique complex $X(G)$ of G. Combinatorially, $\beta(G)$ can be defined in the following way: The cliques of G are the vertices of $\beta(G)$ and two different cliques σ and σ' are adjacent in $\beta(G)$ if and only if $\sigma \subset \sigma'$ or $\sigma' \subset \sigma$. For each path P in $\mathcal{P}^*$, each edge $\sigma\sigma'$ of P is replaced by the 2-path $(\sigma, \sigma \cup \sigma', \sigma')$ in the path P' of $\mathcal{P}'$ corresponding to P. Again, one can establish that $\mathcal{P}^*$ satisfies Theorem 8.3(1)–(3).

9 Final remarks and questions

We strongly believe that the theory of Helly graphs, injective metric spaces and groups acting on them deserves intensive study on its own. In this article we focused mostly on geometric actions of groups on Helly graphs, but similarly to other nonpositive curvature settings, just proper or cocompact actions should be studied as well.

Below we pose a few arbitrary problems following the overall scheme of our main results; the first two concern examples of Helly groups, and the last one is about their properties.

Problem 9.1 (*When*) *are the following groups* (*virtually*) *Helly*: *mapping class groups*, *cubical small cancellation groups*, *Artin groups and Coxeter groups*?

Confirming a conjecture stated by the authors of the current article, Nima Hoda [55] proved recently that the Coxeter group acting on the Euclidean plane and generated by three reflections in the sides of the equilateral Euclidean triangle is not Helly. This group is CAT(0) and systolic (and hence also biautomatic).

Problem 9.2 (combination theorems for group actions with Helly stabilizers) *Is a free product of two Helly groups with amalgamation over an infinite cyclic subgroup Helly*? *Are groups hyperbolic relative to Helly subgroups Helly*? (*When*) *are small cancellation quotients of Helly groups Helly*?

As for general properties of Helly groups, it is natural to ask which of the properties of CAT(0) groups hold in the Helly setting. For a choice of such properties a standard reference is [21].

Problem 9.3 *Are abelian subgroups of Helly groups finitely generated*? *Is there a solvable subgroup theorem for Helly groups*? *Describe centralizers of infinite-order elements in Helly groups. Construct low-dimensional models for classifying spaces for families of subgroups* (*eg for virtually cyclic subgroups*) *of Helly groups. Describe quasiflats in Helly groups.*

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References

[1] **J A Altobelli**, *The word problem for Artin groups of FC type*, J. Pure Appl. Algebra 129 (1998) 1–22 MR Zbl

[2] **N Aronszajn**, **P Panitchpakdi**, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. 6 (1956) 405–439 MR Zbl

[3] **H-J Bandelt**, *Hereditary modular graphs*, Combinatorica 8 (1988) 149–157 MR Zbl

[4] **H-J Bandelt**, **V Chepoi**, *Metric graph theory and geometry: a survey*, from "Surveys on discrete and computational geometry" (J E Goodman, J Pach, R Pollack, editors), Contemp. Math. 453, Amer. Math. Soc., Providence, RI (2008) 49–86 MR Zbl

[5] **H-J Bandelt**, **V Chepoi**, **D Eppstein**, *Combinatorics and geometry of finite and infinite squaregraphs*, SIAM J. Discrete Math. 24 (2010) 1399–1440 MR Zbl

[6] **H-J Bandelt**, **H M Mulder**, *Pseudomodular graphs*, Discrete Math. 62 (1986) 245–260 MR Zbl

[7] **H-J Bandelt**, **H M Mulder**, **E Wilkeit**, *Quasi-median graphs and algebras*, J. Graph Theory 18 (1994) 681–703 MR Zbl

[8] **H-J Bandelt**, **E Pesch**, *Dismantling absolute retracts of reflexive graphs*, European J. Combin. 10 (1989) 211–220 MR Zbl

[9] **H-J Bandelt**, **E Prisner**, *Clique graphs and Helly graphs*, J. Combin. Theory Ser. B 51 (1991) 34–45 MR Zbl

[10] **H-J Bandelt**, **M van de Vel**, *Superextensions and the depth of median graphs*, J. Combin. Theory Ser. A 57 (1991) 187–202 MR Zbl

[11] **J A Barmak**, **E G Minian**, *Strong homotopy types, nerves and collapses*, Discrete Comput. Geom. 47 (2012) 301–328 MR Zbl

[12] **A Bartels**, **W Lück**, *The Borel conjecture for hyperbolic and* CAT(0)*-groups*, Ann. of Math. 175 (2012) 631–689 MR Zbl

[13] **C Berge**, *Hypergraphs: combinatorics of finite sets*, North-Holland Math. Library 45, North-Holland, Amsterdam (1989) MR Zbl

[14] **C Berge**, **P Duchet**, *A generalization of Gilmore's theorem*, from "Recent advances in graph theory" (M Fiedler, editor), Academia, Prague (1975) 49–55 MR Zbl

[15] **C Berge**, **M Las Vergnas**, *Sur un théorème du type König pour hypergraphes*, Ann. New York Acad. Sci. 175 (1970) 32–40 MR Zbl

[16] **M Bestvina**, *Local homology properties of boundaries of groups*, Michigan Math. J. 43 (1996) 123–139 MR Zbl

[17] **A Björner**, *Topological methods*, from "Handbook of combinatorics, II" (R L Graham, M Grötschel, L Lovász, editors), Elsevier, Amsterdam (1995) 1819–1872 MR Zbl

[18] **K Borsuk**, *On the imbedding of systems of compacta in simplicial complexes*, Fund. Math. 35 (1948) 217–234 MR Zbl

[19] **B H Bowditch**, *Median and injective metric spaces*, Math. Proc. Cambridge Philos. Soc. 168 (2020) 43–55 MR Zbl

[20] **T Brady**, **J P McCammond**, *Three-generator Artin groups of large type are biautomatic*, J. Pure Appl. Algebra 151 (2000) 1–9 MR Zbl

[21] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer (1999) MR Zbl

[22] **P J Cameron**, *Dual polar spaces*, Geom. Dedicata 12 (1982) 75–85 MR Zbl

[23] **J Chalopin**, **V Chepoi**, **H Hirai**, **D Osajda**, *Weakly modular graphs and nonpositive curvature*, Mem. Amer. Math. Soc. 1309, Amer. Math. Soc., Providence, RI (2020) MR Zbl

[24] **R Charney**, *Artin groups of finite type are biautomatic*, Math. Ann. 292 (1992) 671–683 MR Zbl

[25] **V D Chepoi**, *Classification of graphs by means of metric triangles*, Metody Diskret. Analiz. (1989) 75–93 MR Zbl In Russian

[26] **V Chepoi**, *A note on r-dominating cliques*, Discrete Math. 183 (1998) 47–60 MR Zbl

[27] **V Chepoi**, *Graphs of some* CAT(0) *complexes*, Adv. in Appl. Math. 24 (2000) 125–179 MR Zbl

[28] **V Chepoi**, **F F Dragan**, **B Estellon**, **M Habib**, **Y Vaxès**, *Diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs*, from "Symposium on computational geometry", ACM, New York (2008) 59–68 MR Zbl

[29] **V Chepoi**, **B Estellon**, *Packing and covering δ-hyperbolic spaces by balls*, from "Approximation, randomization, and combinatorial optimization: algorithms and techniques" (M Charikar, K Jansen, O Reingold, J D P Rolim, editors), Lecture Notes in Comput. Sci. 4627, Springer (2007) 59–73 Zbl

[30] **V Chepoi**, **K Knauer**, **T Marc**, *Hypercellular graphs: partial cubes without Q_3^- as partial cube minor*, Discrete Math. 343 (2020) art. id. 111678 MR Zbl

[31] **M Chrobak**, **L L Larmore**, *Generosity helps or an 11-competitive algorithm for three servers*, J. Algorithms 16 (1994) 234–263 MR Zbl

[32] **D Descombes**, *Asymptotic rank of spaces with bicombings*, Math. Z. 284 (2016) 947–960 MR Zbl

[33] **D Descombes**, **U Lang**, *Convex geodesic bicombings and hyperbolicity*, Geom. Dedicata 177 (2015) 367–384 MR Zbl

[34] **D Descombes**, **U Lang**, *Flats in spaces with convex geodesic bicombings*, Anal. Geom. Metr. Spaces 4 (2016) 68–84 MR Zbl

[35] **F F Dragan**, Центры в графах и свойство Хелли, PhD thesis, Moldova State University (1989)

[36] **F F Dragan**, **H M Guarnera**, *Obstructions to a small hyperbolicity in Helly graphs*, Discrete Math. 342 (2019) 326–338 MR Zbl

[37] **A W M Dress**, *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces*, Adv. Math. 53 (1984) 321–402 MR Zbl

[38] **A W M Dress**, *A note on compact groups, acting on $\mathbb{R}$-trees*, Beitr. Algebra Geom. 29 (1989) 81–90 MR Zbl

[39] **A W M Dress**, **R Scharlau**, *Gated sets in metric spaces*, Aequationes Math. 34 (1987) 112–120 MR Zbl

[40] **D B A Epstein**, **J W Cannon**, **D F Holt**, **S V F Levy**, **M S Paterson**, **W P Thurston**, *Word processing in groups*, Jones & Bartlett, Boston, MA (1992) MR Zbl

[41] **F Escalante**, *Über iterierte Clique-Graphen*, Abh. Math. Sem. Univ. Hamburg 39 (1973) 59–68 MR Zbl

[42] **M Farber**, **R E Jamison**, *On local convexity in graphs*, Discrete Math. 66 (1987) 231–247 MR Zbl

[43] **F T Farrell**, **J-F Lafont**, *EZ-structures and topological applications*, Comment. Math. Helv. 80 (2005) 103–121 MR Zbl

[44] **T Fukaya**, **S-i Oguni**, *A coarse Cartan–Hadamard theorem with application to the coarse Baum–Connes conjecture*, J. Topol. Anal. 12 (2020) 857–895 MR Zbl

[45] **A Genevois**, *Cubical-like geometry of quasi-median graphs and applications to geometric group theory*, PhD thesis, Aix-Marseille Université (2017) arXiv 1712.01618

[46] **R Geoghegan**, *Topological methods in group theory*, Graduate Texts in Math. 243, Springer (2008) MR Zbl

[47] **S M Gersten**, **H B Short**, *Small cancellation theory and automatic groups*, Invent. Math. 102 (1990) 305–334 MR Zbl

[48] **O Goodman**, **V Moulton**, *On the tight span of an antipodal graph*, Discrete Math. 218 (2000) 73–96 MR Zbl

[49] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR Zbl

[50] **V S Guba**, **M V Sapir**, *On subgroups of the R Thompson group F and other diagram groups*, Mat. Sb. 190 (1999) 3–60 MR Zbl In Russian; translated in Sb. Math. 190 (1999) 1077–1130

[51] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[52] **P Hell**, **I Rival**, *Absolute retracts and varieties of reflexive graphs*, Canad. J. Math. 39 (1987) 544–567 MR Zbl

[53] **S Hensel**, **D Osajda**, **P Przytycki**, *Realisation and dismantlability*, Geom. Topol. 18 (2014) 2079–2126 MR Zbl

[54] **N Hoda**, *Quadric complexes*, Michigan Math. J. 69 (2020) 241–271 MR Zbl

[55] **N Hoda**, *Crystallographic Helly groups*, Bull. Lond. Math. Soc. 55 (2023) 2991–3011 Zbl

[56] **J Huang**, **D Osajda**, *Large-type Artin groups are systolic*, Proc. Lond. Math. Soc. 120 (2020) 95–123 MR Zbl

[57] **J Huang**, **D Osajda**, *Helly meets Garside and Artin*, Invent. Math. 225 (2021) 395–426 MR Zbl

[58] **S Hughes**, **M Valiunas**, *Commensurating HNN-extensions: hierarchical hyperbolicity and biautomaticity* (2022) arXiv 2203.11996 To appear in Comment. Math. Helv.

[59] **J R Isbell**, *Six theorems about injective metric spaces*, Comment. Math. Helv. 39 (1964) 65–76 MR Zbl

[60] **J R Isbell**, *Median algebra*, Trans. Amer. Math. Soc. 260 (1980) 319–362 MR Zbl

[61] **T Januszkiewicz**, **J Świątkowski**, *Simplicial nonpositive curvature*, Publ. Math. Inst. Hautes Études Sci. 104 (2006) 1–85 MR Zbl

[62] **E M Jawhari**, **M Pouzet**, **D Misane**, *Retracts: graphs and ordered sets from the metric point of view*, from "Combinatorics and ordered sets" (I Rival, editor), Contemp. Math. 57, Amer. Math. Soc., Providence, RI (1986) 175–226 MR Zbl

[63] **M Kapovich**, **B Leeb**, *On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds*, Geom. Funct. Anal. 5 (1995) 582–603 MR Zbl

[64] **D Kasprowski**, **H Rüping**, *The Farrell–Jones conjecture for hyperbolic and* CAT(0)*-groups revisited*, J. Topol. Anal. 9 (2017) 551–569 MR Zbl

[65] **U Lang**, *Injective hulls of certain discrete metric spaces and groups*, J. Topol. Anal. 5 (2013) 297–331 MR Zbl

[66] **L Lovász**, *Combinatorial problems and exercises*, North-Holland, Amsterdam (1979) MR Zbl

[67] **R C Lyndon**, **P E Schupp**, *Combinatorial group theory*, Ergebnisse der Math. 89, Springer (1977) MR Zbl

[68] **B Miesch**, *Gluing hyperconvex metric spaces*, Anal. Geom. Metr. Spaces 3 (2015) 102–110 MR Zbl

[69] **G A Niblo**, **L D Reeves**, *The geometry of cube complexes and the complexity of their fundamental groups*, Topology 37 (1998) 621–633 MR Zbl

[70] **D Osajda**, *Residually finite non-exact groups*, Geom. Funct. Anal. 28 (2018) 509–517 MR Zbl

[71] **D Osajda**, *Small cancellation labellings of some infinite graphs and applications*, Acta Math. 225 (2020) 159–191 MR Zbl

[72] **D Osajda**, **T Prytuła**, *Classifying spaces for families of subgroups for systolic groups*, Groups Geom. Dyn. 12 (2018) 1005–1060 MR Zbl

[73] **D Osajda**, **P Przytycki**, *Boundaries of systolic groups*, Geom. Topol. 13 (2009) 2807–2880 MR Zbl

[74] **D Peifer**, *Artin groups of extra-large type are biautomatic*, J. Pure Appl. Algebra 110 (1996) 15–56 MR Zbl

[75] **E Pesch**, *Minimal extensions of graphs to absolute retracts*, J. Graph Theory 11 (1987) 585–598 MR Zbl

[76] **E Pesch**, *Retracts of graphs*, Math. Syst. Econ. 110, Athenäum, Frankfurt (1988) MR Zbl

[77] **N Polat**, *Finite invariant simplices in infinite graphs*, Period. Math. Hungar. 27 (1993) 125–136 MR Zbl

[78] **N Polat**, *Convexity and fixed-point properties in Helly graphs*, Discrete Math. 229 (2001) 197–211 MR Zbl

[79] **S J Pride**, *On Tits' conjecture and other questions concerning Artin and generalized Artin groups*, Invent. Math. 86 (1986) 347–356 MR Zbl

[80] **A Quilliot**, *On the Helly property working as a compactness criterion on graphs*, J. Combin. Theory Ser. A 40 (1985) 186–193 MR Zbl

[81] **M Roller**, *Poc sets, median algebras and group actions*, Habilitationsschrift, Universität Regensburg (1998) arXiv 1607.07747

[82] **M Sageev**, *Ends of group pairs and non-positively curved cube complexes*, Proc. Lond. Math. Soc. 71 (1995) 585–617 MR Zbl

[83] **J-P Serre**, *Trees*, Springer (1980) MR Zbl

[84] **V P Soltan**, **V D Chepoi**, *Conditions for invariance of set diameters under d-convexification in a graph*, Kibernetika (Kiev) (1983) 14–18 MR Zbl In Russian; translated in Cybernetics 19 (1983) 750–756

[85] **J Świątkowski**, *Regular path systems and (bi)automatic groups*, Geom. Dedicata 118 (2006) 23–48 MR Zbl

[86] **J L Szwarcfiter**, *Recognizing clique-Helly graphs*, Ars Combin. 45 (1997) 29–32 MR Zbl

[87] **M van de Vel**, *Collapsible polyhedra and median spaces*, Proc. Amer. Math. Soc. 126 (1998) 2811–2818 MR Zbl

[88] **D T Wise**, *Cubulating small cancellation groups*, Geom. Funct. Anal. 14 (2004) 150–214 MR Zbl

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Helly groups 1

JÉRÉMIE CHALOPIN, VICTOR CHEPOI, ANTHONY GENEVOIS, HIROSHI HIRAI and DAMIAN OSAJDA

Topologically trivial proper 2-knots 71

ROBERT E GOMPF

The stable Adams operations on Hermitian K-theory 127

JEAN FASEL and OLIVIER HAUTION

On Borel Anosov subgroups of $\mathrm{SL}(d,\mathbb{R})$ 171

SUBHADIP DEY

Global Brill–Noether theory over the Hurwitz space 193

ERIC LARSON, HANNAH LARSON and ISABEL VOGT

Hyperbolic hyperbolic-by-cyclic groups are cubulable 259

FRANÇOIS DAHMANI, SURAJ KRISHNA MEDA SATISH and JEAN PIERRE MUTANGUHA

The smooth classification of 4-dimensional complete intersections 269

DIARMUID CROWLEY and CSABA NAGY

An embedding of skein algebras of surfaces into localized quantum tori from Dehn–Thurston coordinates 313

RENAUD DETCHERRY and RAMANUJAN SANTHAROUBANE

Virtual classes via vanishing cycles 349

TASUKI KINJO

On termination of flips and exceptionally noncanonical singularities 399

JINGJUN HAN and JIHAO LIU

Lower Ricci curvature and nonexistence of manifold structure 443

ERIK HUPP, AARON NABER and KAI-HSIANG WANG

Independence of singularity type for numerically effective Kähler–Ricci flows 479

HOSEA WONDO and ZHOU ZHANG

Subgroups of genus-2 quasi-Fuchsian groups and cocompact Kleinian groups 495

ZHENGHAO RAO