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We study the antipodal subsets of the full flag manifolds $\mathscr{F}(\mathbb{R}^d)$. As a consequence, for natural numbers $d \geq 2$ such that $d \neq 5$ and $d \not\equiv 0, \pm 1 \mod 8$, we show that Borel Anosov subgroups of $\mathrm{SL}(d,\mathbb{R})$ are virtually isomorphic to either a free group or the fundamental group of a closed hyperbolic surface. This gives a partial answer to a question asked by Andrés Sambarino. Furthermore, we show restrictions on the hyperbolic spaces admitting uniformly regular quasi-isometric embeddings into the symmetric space X_d of $\mathrm{SL}(d,\mathbb{R})$.

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Dedicated to Misha Kapovich on the occasion of his 60th birthday

1 Introduction

In the past decade, Anosov subgroups of higher-rank Lie groups have emerged as a well-regarded higher-rank extension of the classical convex-cocompact Kleinian groups. The notion of Anosov representations was introduced by Labourie [16] from a dynamical perspective in his pioneering work on Hitchin representations of surface groups, and then extended by Guichard and Wienhard [10] for any hyperbolic groups. Afterward, Kapovich, Leeb and Porti [14] gave several geometrical and dynamical characterizations of Anosov subgroups; see the article by Kapovich and Leeb [13] giving an overview of their characterizations. A main feature of Anosov subgroups is that they have a well-defined limit set in suitable generalized flag varieties, and any two distinct points in these limit sets are in general position.

We are motivated by a question asked by Andrés Sambarino, namely whether Borel Anosov subgroups of $\mathrm{SL}(d,\mathbb{R})$ are necessarily virtually free or surface groups. Combined works of Canary and Tsouvalas [4] and Tsouvalas [22] have affirmatively answered this question for $d = 3, 4$ and $d \equiv 2 \mod 4$ (note that $d = 2$ case is classical). Using a different approach, we give an affirmative answer to this question for all $d \in \mathbb{N}$ satisfying

$$d \neq 5 \qquad \text{and} \qquad d \equiv 2, 3, 4, 5 \text{ or } 6 \mod 8. \tag{1}$$

See Corollary D.

We summarize our main objectives:

(i) We study the subsets of full flag manifolds $\mathscr{F}(\mathbb{R}^d)$ where all pairs points are *antipodal*, ie are in general position. As noted above, the limit sets of Anosov subgroups share this property. We are specifically interested in understanding when antipodal subsets of $\mathscr{F}(\mathbb{R}^d)$ are *maximally* antipodal; see Section 1.1 for discussions related to this matter.

(ii) We aim to understand which hyperbolic groups can be realized as *Borel Anosov subgroups* of $\mathrm{SL}(d,\mathbb{R})$; see Section 1.2 for the discussion related to this.

(iii) Finally, we aim to understand which geodesic metric spaces may admit *coarsely uniformly regular quasi-isometric embeddings*, a notion introduced by Kapovich, Leeb and Porti [13] strengthening the classical notion of *quasi-isometric embeddings*, into the symmetric space X_d of $\mathrm{SL}(d,\mathbb{R})$. Notably, the *orbit maps* of Anosov subgroups are such embeddings; see Section 1.3 for further discussions.

1.1 Antipodal subsets

For $d \geq 2$, let $\mathscr{F}_d := \mathscr{F}(\mathbb{R}^d)$ denote the manifold consisting of all complete flags in $\mathbb{R}^d$. A pair of points $\sigma_\pm \in \mathscr{F}_d$ is called *antipodal* (or *transverse*) if

$$\sigma_-^{(k)} + \sigma_+^{(d-k)} = \mathbb{R}^d \quad \text{for all } k \in \{1,\dots,d-1\}.$$

Here, for $\sigma \in \mathscr{F}_d$, we use the notation $\sigma^{(k)}$ to denote the k-dimensional vector subspace of $\mathbb{R}^d$ appearing in the complete flag σ. We denote by $\mathscr{E}_\sigma$, $\sigma \in \mathscr{F}_d$ the set of all points in $\mathscr{F}_d$ which are *not* antipodal to σ. The complementary subset of $\mathscr{E}_\sigma$ in $\mathscr{F}_d$, which we denote by $\mathscr{C}_\sigma$, is an open dense subset of $\mathscr{F}_d$ homeomorphic to a cell. The subset $\mathscr{C}_\sigma$ is called a *maximal Schubert cell* or *big cell*, whereas $\mathscr{E}_\sigma$ is the closure of the union of all codimension-1 Schubert cells in the Schubert cell decomposition of $\mathscr{F}_d$ corresponding to σ.

Theorem A *Let d be any natural number satisfying* (1). *Let $\sigma_\pm \in \mathscr{F}_d$ be any pair of antipodal points, and let Ω be any connected component of $\mathscr{F}_d \setminus (\mathscr{E}_{\sigma_-} \cup \mathscr{E}_{\sigma_+}) = \mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$. If $c\colon [-1,1] \to \mathscr{F}_d$ is any continuous map such that*

$$c(\pm 1) = \sigma_\pm \quad \textit{and} \quad c((-1,1)) \subset \Omega,$$

then, for all $\sigma \in \Omega$, the image of c intersects $\mathscr{E}_\sigma$.

Although the following example is not covered in the setting of the theorem, we believe that it would still serve as a simple illustration of the statement: In the case corresponding to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and its minimal parabolic subgroup, the "full flag manifold" is realized as a torus. Let D denote the unit square in $\mathbb{R}^2$ from which we obtained the torus by identifying the opposite edges. We identify the four corners of D with σ_-, and $\mathscr{E}_{\sigma_-}$ with its edges. Given any point $\hat\sigma$ in the interior of D, the subset $\mathscr{E}_{\hat\sigma}$ can be realized as the union of the horizontal and vertical line segments passing through $\hat\sigma$. Therefore, for any $\sigma_+ \in \mathscr{C}_{\sigma_-} = \operatorname{int} D$, the intersection $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ can be seen as the disjoint union of four open rectangles. For any path c connecting $\sigma_\pm$ lying in (except for the endpoints) one such rectangles Ω, and for any point $\sigma \in \Omega$, it can be checked that $\mathscr{E}_\sigma$ intersects c.

We prove Theorem A in Section 3. The main technical ingredient in the proof is Theorem 2.4, which states that, for natural numbers d satisfying (1), an involution ι defined on $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ does not leave invariant any connected components; see Section 2.

Remark 1.1 Theorem A and the other key results below have the restriction (1) on d because Theorem 2.4 may fail when d is of the form $8k-1$, $8k$ or $8k+1$, for $k\in\mathbb{N}$. Specifically, we show that Theorem 2.4 indeed fails when $d=8k\pm 1$, yet the status of its validity remains unclear for $d=8k$. However, with help from Su Ji Hong, we could computationally verify the validity of Theorem 2.4 when $d=5$. Further discussions on this matter are detailed in Remark 2.5. Consequently, for $d=5$, Theorems A, C and F, and Corollaries D and G (as well as Corollary E(i) for $n=2$), remain valid.

Nevertheless, it is an intriguing prospect to investigate whether Theorem A holds true for these remaining natural numbers d, provided the hypothesis is strengthened by requiring the map $c\colon[-1,1]\to\mathscr{F}_d$ to also be antipodal.

We apply Theorem A to get information about (*locally*) *maximally antipodal subsets* of $\mathscr{F}_d$, defined as follows:

Definition 1.2 (antipodal subsets and maps)

(i) A subset $\Lambda\subset\mathscr{F}_d$ is called *antipodal* if all distinct pairs of points in Λ are antipodal.

(ii) An antipodal subset $\Lambda\subset\mathscr{F}_d$ is called *maximally* antipodal if it is not contained in a strictly larger antipodal subset of $\mathscr{F}_d$.

(iii) We call an antipodal subset $\Lambda\subset\mathscr{F}_d$ *locally* maximally antipodal if there exists an open neighborhood N of Λ in $\mathscr{F}_d$ such that Λ is not contained in any strictly larger antipodal subset of N; equivalently, every point of N is *not* antipodal to some point of Λ.

(iv) A continuous map $\phi\colon Z\to\mathscr{F}_d$ is called *antipodal* if, for all distinct points $z_\pm\in Z$, $\phi(z_+)$ and $\phi(z_-)$ are antipodal.

Note that antipodal subsets of $\mathscr{F}_d$ form a poset, partially ordered by inclusions, and the maximally antipodal subsets are precisely the maximal elements.

As an application of Theorem A, we get the following result:

Corollary B *Let d be any natural number satisfying* (1). *If $c\colon S^1\to\mathscr{F}_d$ is an antipodal embedding, then $\Lambda:=c(S^1)$ is a locally maximally antipodal subset of $\mathscr{F}_d$.*

Proof Let $x_1,x_2,x_3\in S^1$ be any distinct triple. For distinct indices $i,j,k\in\{1,2,3\}$, let Ω_{ijk} denote the connected component of $\mathscr{F}_d\setminus(\mathscr{E}_{c(x_i)}\cup\mathscr{E}_{c(x_k)})$ containing $c(x_j)$. Then $Y=\Omega_{123}\cup\Omega_{231}\cup\Omega_{312}$ is an open neighborhood of Λ. Applying Theorem A, one can verify that every point in Y is nonantipodal to some point in Λ. □

It is unclear whether one can omit the word "locally" in the conclusion of the above result. However, if the image of $c\colon S^1\to\mathscr{F}_d$ is the limit set of a Borel Anosov subgroup of $\mathrm{SL}(d,\mathbb{R})$, then $\Lambda:=c(S^1)$ is a maximally antipodal subset of $\mathscr{F}_d$; see Proposition 5.1.

1.2 Borel Anosov subgroups

First, let us recall the notion of *boundary embedded subgroups* of $\mathrm{SL}(d,\mathbb{R})$ introduced by Kapovich, Leeb and Porti [14].

Definition 1.3 (boundary embedded subgroups) A subgroup Γ of $\mathrm{SL}(d,\mathbb{R})$ is called *B-boundary embedded* if Γ, as an abstract group, is hyperbolic and there exists a Γ-equivariant antipodal embedding $\xi\colon \partial_\infty\Gamma \to \mathscr{F}_d$ of the Gromov boundary $\partial_\infty\Gamma$ of Γ to the complete flag manifold $\mathscr{F}_d$.

Due to the fact that nonelementary hyperbolic groups act as convergence groups on their Gromov boundaries, it can be inferred that nonelementary B-boundary embedded subgroups of $\mathrm{SL}(d,\mathbb{R})$ are discrete. The following result shows that the group-theoretic structures of the B-boundary embedded subgroups of $\mathrm{SL}(d,\mathbb{R})$ are highly restricted:

Theorem C *Let d be any natural number satisfying* (1). *If a subgroup Γ of $\mathrm{SL}(d,\mathbb{R})$ is B-boundary embedded, then Γ is virtually isomorphic to either a free group or the fundamental group of a closed hyperbolic surface.*

This result, which we prove in Section 4, directly applies to the class of Borel Anosov subgroups introduced by Labourie [16], who proved the seminal result that the images of the *Hitchin representations* of surface groups into $\mathrm{SL}(d,\mathbb{R})$ are Borel Anosov subgroups. While Labourie's original definition of Borel Anosov subgroups was intricate, a more straightforward definition has since emerged thanks to the work of Kapovich, Leeb and Porti [14; 15] and Bochi, Potrie and Sambarino [2].

For $g \in \mathrm{SL}(d,\mathbb{R})$, let

$$\sigma_1(g) \ge \cdots \ge \sigma_d(g)$$

denote the singular values of g. For a finitely generated group Γ, let $|\cdot|\colon \Gamma \to \mathbb{N}\cup\{0\}$ denote the word-length function with respect to some symmetric finite generating set of Γ. The following definition does not depend on the choice of such a generating set, although the implied constants may vary.

Definition 1.4 (Borel Anosov subgroups) A finitely generated subgroup Γ of $\mathrm{SL}(d,\mathbb{R})$ is called *Borel Anosov* if there exist constants $L \ge 1$ and $A \ge 0$ such that, for all $k \in \{1,\dots,d-1\}$ and for all $\gamma \in \Gamma$,

$$\log\left(\frac{\sigma_k(\gamma)}{\sigma_{k+1}(\gamma)}\right) \ge L^{-1}|\gamma| - A. \tag{2}$$

The main features of the Borel Anosov subgroups Γ of $\mathrm{SL}(d,\mathbb{R})$ include: Γ, as an abstract group, is hyperbolic, and there exists a Γ-equivariant antipodal embedding, called the *limit map*,

$$\xi\colon \partial_\infty\Gamma \to \mathscr{F}_d,$$

from the Gromov boundary $\partial_\infty\Gamma$ of Γ to the complete flag manifold $\mathscr{F}_d$; see [2; 16]. In particular, Borel Anosov subgroups of $\mathrm{SL}(d,\mathbb{R})$ are B-boundary embedded (Definition 1.3). Therefore, Theorem C has the following direct implication:

Corollary D *Let d be any natural number satisfying* (1). *If Γ is a Borel Anosov subgroup of* $\mathrm{SL}(d,\mathbb{R})$, *then Γ is virtually isomorphic to either a free group or the fundamental group of a closed hyperbolic surface.*

Remark 1.5 This result partially answers a question asked by Sambarino (see Canary and Tsouvalas [4, Section 7]), who asked if the statement is true for all $d \geq 2$. As mentioned above, this question previously has been affirmatively answered for $d = 3$ and $d = 4$ by Canary and Tsouvalas [4], and for all d of the form $4k + 2$ by Tsouvalas [22]. In fact, we give a new (and possibly simpler) proof for the previously known cases from [4; 22]. However, for the remaining integers $d \geq 2$ not covered by Corollary D (except for $d = 5$), we are unable to provide a conclusive answer to this question; see Remark 1.1. We emphasize a connection between the maximal antipodality of limit sets and Sambarino's question, which could be beneficial for further exploration in these remaining cases: Suppose there exists $d \in \mathbb{N}$ and a Borel Anosov subgroup $\Gamma < \mathrm{SL}(d,\mathbb{R})$, isomorphic to a surface group, such that the limit set of Γ in $\mathscr{F}_d$ is not maximally antipodal. In this case, by applying the combination theorem for Anosov subgroups by Dey, Kapovich and Leeb [7] (see also Dey and Kapovich [6]), one can construct a Borel Anosov subgroup of $\mathrm{SL}(d,\mathbb{R})$ isomorphic to $\Gamma' \star \mathbb{Z}$, where Γ' is a finite-index subgroup (and thus a surface subgroup) of Γ. Such a construction could produce a counterexample.

More generally, given a connected noncompact real semisimple Lie group G with finite center and a parabolic subgroup P of G, there is a distinguished class of discrete subgroups of G called *P-Anosov subgroups*; see Guichard and Wienhard [10] and also Kapovich, Leeb and Porti [14]. By a *B-Anosov subgroup* of G, we are referring to a P-Anosov subgroup, where P is assumed to be a minimal parabolic subgroup of G.[1]

In the special case $G = \mathrm{SO}_0(n, n+1)$ (resp. $G = \mathrm{Sp}(2n,\mathbb{R})$), the Borel Anosov subgroups of G map to Borel Anosov subgroups of $\mathrm{SL}(2n+1,\mathbb{R})$ (resp. $\mathrm{SL}(2n,\mathbb{R})$) under the inclusion $\mathrm{SO}_0(n,n+1) \hookrightarrow \mathrm{SL}(2n+1,\mathbb{R})$ (resp. $\mathrm{Sp}(2n,\mathbb{R}) \hookrightarrow \mathrm{SL}(2n,\mathbb{R})$). Thus Corollary D also yields the following:

Corollary E *Let G be one of*

(i) $\mathrm{SO}_0(n,n+1)$, *where* $n \neq 2$ *and* $n \equiv 1$ *or* $2 \bmod 4$,

(ii) $\mathrm{Sp}(2n,\mathbb{R})$, *where* $n \equiv 1, 2$ *or* $3 \bmod 4$.

If Γ is a Borel Anosov subgroup of G, then Γ is virtually isomorphic to either a free group or the fundamental group of a closed hyperbolic surface.

Finally, it is worth remarking that restrictions on Anosov subgroups of the symplectic groups are explored further in the subsequent papers of Dey, Greenberg and Riestenberg [5] and Pozzetti and Tsouvalas [18].

[1] When dealing with a connected algebraic group G defined over an algebraically closed field, the minimal parabolic subgroups are Borel subgroups. This is why we refer this class of subgroups as "B-Anosov". For the same reason, we referred to the class of subgroups defined in Definition 1.3 as "B-boundary embedded". When the minimal parabolic subgroups of G are Borel (eg if G is split), we may also refer to B-Anosov subgroups as *Borel Anosov subgroups* as done in Definition 1.4 for the case $G = \mathrm{SL}(d,\mathbb{R})$.

1.3 Uniformly regular quasi-isometric embeddings

The notion of *uniformly regular quasi-isometric embeddings*, introduced by Kapovich, Leeb and Porti, of geodesic metric spaces into the symmetric space

$$X_d := \mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d,\mathbb{R})$$

is a strengthening of quasi-isometric embeddings. Since the definition of uniformly regular quasi-isometric embeddings requires a lengthier discussion, we refer our reader to Kapovich and Leeb [13, Definition 2.26]. This notion is especially interesting in the context of Anosov subgroups since, by [13, Theorem 3.41], a subgroup $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is Borel Anosov if and only if Γ is finitely generated and the orbit map

$$\Gamma \to X_d, \quad \gamma \mapsto \gamma \cdot x_0, \tag{3}$$

is a uniformly regular quasi-isometric embedding where Γ is equipped with any word metric and $x_0 \in X_d$ is any basepoint; see (2).

Theorem F *Consider any locally compact geodesic Gromov hyperbolic space Z with $\partial_\infty Z$ denoting its Gromov boundary. Suppose there exists a topological embedding $c\colon S^1 \to \partial_\infty Z$ such that the image of c is not an open set. Then Z does not admit any uniformly regular quasi-isometric embeddings into X_d, given d satisfies* (1).

Theorem F is proved in Section 4. This result obstructs uniformly regular quasi-isometric embeddings of certain simply connected complete Riemannian manifolds of nonpositive sectional curvature (also called *Cartan–Hadamard manifolds*) into X_d. More precisely, if Y is a Cartan–Hadamard manifold with sectional curvature bounded below and Y admits a uniformly regular quasi-isometric embedding into X_d, then Y is Gromov hyperbolic as a metric space (by Kapovich, Leeb and Porti [15, Theorem 1.2]), whereas the Gromov boundary of Y is homeomorphic to the sphere of dimension $\dim Y - 1$ (by Kaimanovich [11, Theorem 2.10]). Thus if d satisfies (1), then Theorem F implies that $\dim Y \le 2$. A special case of this is as follows:

Corollary G *The hyperbolic plane is the only symmetric space of noncompact type that admits uniformly regular quasi-isometric embeddings into X_d with d satisfying* (1).

In a similar vein, applying Theorem F, a stronger conclusion than Corollary D can be obtained: finitely generated groups Γ, unless Γ is virtually a free group or a surface group, do not even admit uniformly regular quasi-isometric embeddings[2] into X_d when d satisfies (1), since in such cases, if Γ admits a uniformly regular quasi-isometric embedding into X_d, then Γ is a hyperbolic group [15, Theorem 1.2] and there exist such nonisolated circles in $\partial_\infty \Gamma$ (see Bonk and Kleiner [3, Corollary 2]) as required by the hypothesis of Theorem F to get a contradiction.

[2] These need not arise from a group homomorphism $\Gamma \to \mathrm{SL}(d,\mathbb{R})$ as in (3).

Outline In Section 2, we present and prove our main technical result, Theorem 2.4. Using this, we establish Theorem A in Section 3. Subsequently, we apply Corollary B, an immediate consequence of Theorem A, to prove Theorems C and F in Section 4. Finally, in Section 5, we explore some additional applications of the methods we introduce.

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2 An involution on the intersection of two opposite maximal Schubert cells

The goal of this section is to state and prove the main technical result behind the results discussed in the introduction; see Theorem 2.4.

We recall that SL$(d,\mathbb{R})$ acts transitively on the set consisting of all antipodal pairs of points in $\mathscr{F}_d$. From now on, we reserve the notation $\sigma_\pm$ for the *descending/ascending* flags defined as follows: Let $\mathbb{R}^d$ be equipped with the standard basis $\{e_1,\dots,e_d\}$. Define

$$\begin{aligned} &\sigma_+ \quad \text{as } \{0\}\subset \operatorname{span}\{e_d\}\subset \operatorname{span}\{e_d,e_{d-1}\}\subset\cdots\subset\operatorname{span}\{e_d,\dots,e_1\}=\mathbb{R}^d,\\ &\sigma_- \quad \text{as } \{0\}\subset \operatorname{span}\{e_1\}\subset \operatorname{span}\{e_1,e_2\}\subset\cdots\subset\operatorname{span}\{e_1,\dots,e_d\}=\mathbb{R}^d. \end{aligned}$$

It can be seen easily that $\sigma_\pm$ are antipodal.

We also reserve the notation U_d to denote the subgroup of SL$(d,\mathbb{R})$ consisting of all upper-triangular unipotent matrices. It is easy to check that U_d fixes σ_-, and hence preserves the big cell $\mathscr{C}_{\sigma_-}$. Moreover, U_d acts on $\mathscr{C}_{\sigma_-}$ simply transitively, so we have a diffeomorphism

$$F_{\sigma_+}\colon U_d\to\mathscr{C}_{\sigma_-},\quad F(u)=u\sigma_+.$$

For notational convenience, for all $\sigma\in\mathscr{C}_{\sigma_-}$, let us write

$$u_\sigma := F_{\sigma_+}^{-1}(\sigma).$$

We identify U_d with $\mathscr{C}_{\sigma_-}$ under the diffeomorphism F_{σ_+}.

Furthermore, we identify U_d (and hence $\mathscr{C}_{\sigma_-}$) with $\mathbb{R}^{\binom{d}{2}}$ by sending a matrix $u\in U_d$ to the vector $(u_{ij})_{1\le i<j\le d}$. Under this identification, $\sigma\in\mathscr{C}_{\sigma_-}$ lies in $\mathscr{E}_{\sigma_+}=\mathscr{F}_d\setminus\mathscr{C}_{\sigma_+}$ if and only if there exists some $k\in\{1,\dots,d-1\}$ such that $\sigma^{(k)}+\sigma_+^{(d-k)}$ is a proper subspace of $\mathbb{R}^d$ or, equivalently,

$$p_k(u_\sigma) := \frac{(u_\sigma e_{d-k+1})\wedge\cdots\wedge(u_\sigma e_d)\wedge e_{k+1}\wedge\cdots\wedge e_d}{e_1\wedge\cdots\wedge e_d}=0.$$

Thus we can describe the set $\mathscr{E}_{\sigma_+} \cap \mathscr{C}_{\sigma_-}$ algebraically as a subset of $\mathbb{R}^{\binom{d}{2}}$ by

$$\mathscr{E}_{\sigma_+} \cap U_d = \bigcup_{k=1}^{d-1} \mathscr{E}^k_{\sigma_+},$$

where $\mathscr{E}^k_{\sigma_+} := \{u \in U_d \mid p_k(u) = 0\}$.

Example 2.1 If $d = 3$, then

$$p_1\left(\begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix}\right) = y, \quad p_2\left(\begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix}\right) = xz - y.$$

Therefore $\mathscr{E}_{\sigma_+} \cap U_3$ can be written as the union of the hypersurfaces $\mathscr{E}^1_{\sigma_+} = \{(x,y,z) \mid p_1(x,y,z) = y = 0\}$ and $\mathscr{E}^2_{\sigma_+} = \{(x,y,z) \mid p_2(x,y,z) = xz - y = 0\}$ in $\mathbb{R}^3$; see Figure 1.

The following lemma can be verified by linear algebra. We omit the details.

Lemma 2.2 *The polynomial $p_k(u)$ can be expressed as $p_k(u) = \det u^{(k)}$, where $u^{(k)}$ denotes the upper-right $k \times k$ block submatrix of u. In particular,*

$$\mathscr{E}^k_{\sigma_+} = \{u \in U_d \mid \det u^{(k)} = 0\}.$$

We define an involution

$$\iota \colon U_d \to U_d, \quad u \mapsto u^{-1}.$$

This simple involution plays a key role here. Note that ι is a diffeomorphism, $\mathrm{Fix}(\iota) = \{\boldsymbol{I}\}$, where $\boldsymbol{I}$ denotes the identity matrix, and $d\iota|_{T_{\boldsymbol{I}} U_d} = -\mathrm{id}$.

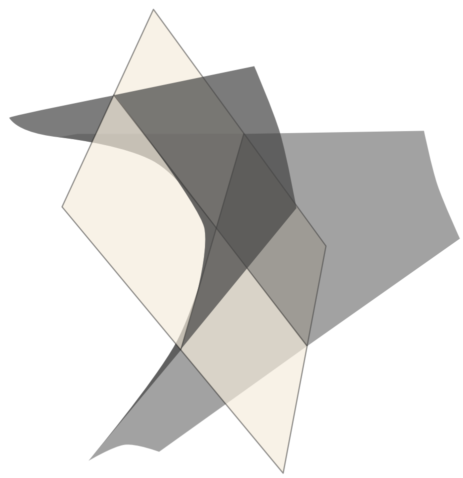

Figure 1: The part of the set $\mathscr{E}_{\sigma_+}$ lying in $\mathbb{R}^3 \cong \mathscr{C}_{\sigma_-} \subset \mathscr{F}_3$. The six components of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ are visible in the complement of this algebraic surface.

Proposition 2.3 *For all* $k \in \{1,\dots,d-1\}$,

$$p_k(u^{-1}) = (-1)^{k(d+1)} p_{d-k}(u).$$

In particular, $\iota(\mathscr{E}^k_{\sigma_+}) = \mathscr{E}^{d-k}_{\sigma_+}$, ι *preserves* $\mathscr{E}_{\sigma_+} \cap U_d$, *and hence preserves* $U_d \setminus \mathscr{E}_{\sigma_+}$.

Proof We apply *Jacobi's complementary minor formula*: if A is an invertible $d \times d$ matrix, then for any subsets $I, J \subset \{1,\dots,d\}$ of size k,

$$\det A_{IJ} = (-1)^{\sum I + \sum J} (\det A) \det((A^{-1})_{J^c I^c}).$$

Here we use the notation A_{IJ} to denote the submatrix of A obtained by its I^{th} rows and J^{th} columns. Since in our case $\det u = 1$, the above formula reduces to

$$\det((u^{-1})_{IJ}) = (-1)^{\sum I + \sum J} \det u_{J^c I^c}.$$

Fix $k \in \{1,\dots,d-1\}$. Notice that when $I = \{1,\dots,k\}$ and $J = \{d-k+1,\dots,d\}$, we have $p_k(\hat{u}) = \det \hat{u}_{IJ}$, $p_{d-k}(\hat{u}) = \det \hat{u}_{J^c I^c}$ and $\sum I + \sum J = k(d+1)$. Hence, by the formula in the previous paragraph, $p_k(u^{-1}) = (-1)^{k(d+1)} p_{d-k}(u)$. □

By the above result, we thus have a well-defined involution ι on $U_d \setminus \mathscr{E}_{\sigma_+}$. Our main result of this section is as follows:

Theorem 2.4 *Suppose that* d *is any natural number such that* $d \neq 5$ *and* $d \equiv 2, 3, 4, 5$ *or* $6 \bmod 8$. *Then the involution* $\iota\colon U_d \to U_d$ *does not leave invariant any connected components of* $U_d \setminus \mathscr{E}_{\sigma_+}$.

The proof of Theorem 2.4 is split into several cases and occupies the rest of this section. Here is our plan: The proof for $d = 3$ is discussed in Section 2.1; see Section 2.2 for the case when d is of the form $4k+2$. These initial cases are approached in an elementary manner. However, as our elementary approach appears to be insufficient for the remaining cases, we rely upon some sophisticated invariants developed by Shapiro, Shapiro and Vainshtein [19; 20], which characterize the connected components of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ in a combinatorial manner. In Section 2.3, we recall some necessary background on these papers. Subsequently, the proof for $d = 4$ is discussed in Section 2.4. Following that, we prove the theorem in the rest of the odd cases of d in Section 2.5 and in the remaining even cases of d in Section 2.6.

Remark 2.5 When d is of the form $8m \pm 1$, then the $\iota\colon U_d \to U_d$ leaves invariant some components of $U_d \setminus \mathscr{E}_{\sigma_+}$; see below. Therefore Theorem 2.4 is false in those cases of d; see Proposition 2.10(ii). When d is of the form $8m$, we are unable to make the conclusion because we could not study some "exceptional" connected components; see Remark 2.12. However, in all these cases, the total number of these components is quite "small" compared to the total number of connected components of $U_d \setminus \mathscr{E}_{\sigma_+}$.

With help from Su Ji Hong, we managed to computationally verify Theorem 2.4 for $d = 5$. Despite this effort, we decided to exclude the $d = 5$ case because we couldn't find a way to present a proof for it.

2.1 Proof of Theorem 2.4 when $d = 3$

When $d = 3$, the connected components of $U_3 \setminus \mathscr{E}_{\sigma+}$ in $\mathbb{R}^3 = U_3$ are

$$\begin{aligned} \Omega_1 &= \{(x,y,z) \mid x>0,\ y>0,\ z>0,\ xz-y>0\}, & \widehat{\Omega}_1 &= \{(x,y,z) \mid x<0,\ y>0,\ z<0,\ xz-y>0\},\\ \Omega_2 &= \{(x,y,z) \mid x<0,\ y<0,\ z>0,\ xz-y<0\}, & \widehat{\Omega}_2 &= \{(x,y,z) \mid x>0,\ y<0,\ z<0,\ xz-y<0\},\\ \Omega_3 &= \{(x,y,z) \mid y>0,\ xz-y<0\}, & \widehat{\Omega}_3 &= \{(x,y,z) \mid y<0,\ xz-y>0\}. \end{aligned}$$

See Figure 1. By picking a representative in each component and applying ι to the representative, it can be checked that $\iota\Omega_k = \widehat{\Omega}_k$ for $k = 1, 2, 3$. We omit the details.

2.2 Proof of Theorem 2.4 when $d \equiv 2 \bmod 4$

Suppose that $d \equiv 2 \bmod 4$. Let $u \in U_d \setminus \mathscr{E}_{\sigma+}$ be any point. By Proposition 2.3, $\iota u \in U_d \setminus \mathscr{E}_{\sigma+}$. Let $c\colon [-1,1] \to U_d$ for $c(\pm 1) = u^{\pm 1}$ be any path. We show that such a path c must intersect $\mathscr{E}_{\sigma+} \cap U_d$. In this case, since $\frac{1}{2}d$ is odd, by Proposition 2.3

$$p_{d/2}(u^{-1}) = -p_{d/2}(u).$$

Thus, by continuity, the image of c must intersect $\mathscr{E}^{d/2}_{\sigma+}$. Therefore u and ιu lie in different connected components of $U_d \setminus \mathscr{E}^{d/2}_{\sigma+}$, and hence of $U_d \setminus \mathscr{E}_{\sigma+}$.

2.3 Some preparation before the proof of Theorem 2.4 in the remaining cases

We recall some notions of Shapiro, Shapiro and Vainshtein [19; 20]. Throughout, we try to be consistent with their papers so that we can freely refer to those for more details.

Let $n := d - 1$. Denote by $T^n = T^n(\mathbb{F}_2)$ the vector space of all $n \times n$ upper-triangular matrices with $\mathbb{F}_2$-valued entries, where $\mathbb{F}_2 = \{0, 1\}$ is the finite field of order 2. There is a certain subgroup $\mathfrak{G}_n < \mathrm{GL}(T^n)$ acting linearly on T^n. This action is called the *first* $\mathfrak{G}_n$-action; see the introduction of [20]. There is also another $\mathfrak{G}_n$-action defined in that paper, which is called the *second* $\mathfrak{G}_n$-action. However we do not need to discuss the second action, and hence will simply call the *first $\mathfrak{G}_n$-action* the *$\mathfrak{G}_n$-action*. For the reader's convenience, we recall this action. For $1 \le i \le j \le n-1$, let $g_{ij} \in \mathrm{GL}(T^n)$ be the element acting linearly on T^n as follows: Let M^{ij} be the 2×2 submatrix of M formed by the rows i and $i+1$, and the columns j and $j+1$ (or its upper triangle when $i = j$). Then $g_{ij} \cdot M$ is the matrix obtained by adding to each entry of M^{ij} its trace, and keeping the rest of the entries of M unchanged. The subgroup $\mathfrak{G}_n < \mathrm{GL}(T^n)$ is generated by all these g_{ij}.

Example 2.6 We revisit the case $d = n + 1 = 3$, where

$$T^2 = \left\{ \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix} \right\}, \tag{4}$$

and $\mathfrak{G}_2$ is the group of order 2, with g_{11} as its nontrivial generating element. We note that $\mathfrak{G}_2$ has precisely four fixed points in T^2, represented by the initial four elements in (4). The remaining four elements in T^2 form two distinct $\mathfrak{G}_2$-orbits. Consequently, there are six $\mathfrak{G}_2$-orbits in total. Remarkably, each of these orbits corresponds to a unique connected component of $U_3 \setminus \mathscr{E}_{\sigma_+}$ (see Section 2.1). We elaborate this further in our discussion below.

When $d = n + 1 = 4$, the $\mathfrak{G}_3$-orbits in T^3 can be found in Table 1.

By [19], the connected components of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ are in one-to-one correspondence with the $\mathfrak{G}_n$-orbits in T^n. The correspondence can be realized as follows (see [19, Sections 2 and 3] for more details): Let S_{n+1} denote the group of all permutations of $\{1,\dots,n+1\}$, let w_0 denote the *longest element* in S_{n+1}, and let s_k for $k = 1,\dots,n$ denote the transposition which swaps k and $k+1$ in $\{1,\dots,n+1\}$. The element w_0 can be written as

$$w_0 = (s_1 s_2 \cdots s_n)(s_1 s_2 \cdots s_{n-1}) \cdots (s_1 s_2 s_3)(s_1 s_2)(s_1). \tag{5}$$

Corresponding to this (fixed) reduced decomposition of w_0, by [1; 17], a generic matrix $u \in U_d = U_{n+1}$ can be uniquely factorized as

$$u = (\boldsymbol{I} + t_{1n}\boldsymbol{E}_{s_1})(\boldsymbol{I} + t_{2(n-1)}\boldsymbol{E}_{s_2}) \cdots (\boldsymbol{I} + t_{n1}\boldsymbol{E}_{s_n})(\boldsymbol{I} + t_{1(n-1)}\boldsymbol{E}_{s_1}) \cdots (\boldsymbol{I} + t_{(n-1)1}\boldsymbol{E}_{s_{n-1}}) \\ \cdots (\boldsymbol{I} + t_{12}\boldsymbol{E}_{s_1})(\boldsymbol{I} + t_{21}\boldsymbol{E}_{s_2})(\boldsymbol{I} + t_{11}\boldsymbol{E}_{s_1}), \tag{6}$$

where t_{ij} represents the coefficient of $\boldsymbol{E}_{s_i}$ when $\boldsymbol{E}_{s_i}$ appears the j^{th} time from right to left in the above expression, the t_{ij} are nonzero real numbers, and $\boldsymbol{E}_{s_i}$ denotes the $(n+1) \times (n+1)$ matrix with only nonzero entry 1 at the place $(i, i+1)$. Using this unique factorization of u, we assign to it the matrix $M_u \in T^n$ given by

$$M_u = \begin{bmatrix} \epsilon_{11} & \epsilon_{21} & \cdots & \epsilon_{(n-1)1} & \epsilon_{n1} \\ & \epsilon_{12} & \ddots & & \epsilon_{(n-1)2} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \epsilon_{2(n-1)} \\ & & & & \epsilon_{1n} \end{bmatrix}, \tag{7}$$

where $\epsilon_{ij} = 0$ if $t_{ij} > 0$ in (6), and $\epsilon_{ij} = 1$ if $t_{ij} < 0$; see [19, Section 2.8].

Lemma 2.7 *For a generic matrix $u \in U_d$, the matrices M_u and $M_{u^{-1}}$ are related by*

$$(M_{u^{-1}})_{ij} = (M_u)_{(n+1-j)(n+1-i)} + 1 \quad \textit{for all } 1 \le i \le j \le n.$$

Proof Suppose that the factorization of u is given by (6) and the corresponding M_u has the expression given by (7). To obtain an expression for $M_{u^{-1}}$, we first notice

$$u^{-1} = (\boldsymbol{I} - t_{11}\boldsymbol{E}_{s_1})(\boldsymbol{I} - t_{21}\boldsymbol{E}_{s_2})(\boldsymbol{I} - t_{12}\boldsymbol{E}_{s_1}) \cdots (\boldsymbol{I} - t_{(n-1)1}\boldsymbol{E}_{s_{n-1}}) \cdots (\boldsymbol{I} - t_{1(n-1)}\boldsymbol{E}_{s_1})(\boldsymbol{I} - t_{n1}\boldsymbol{E}_{s_n}) \\ \cdots (\boldsymbol{I} - t_{2(n-1)}\boldsymbol{E}_{s_2})(\boldsymbol{I} - t_{1n}\boldsymbol{E}_{s_1}).$$

This is not in the order required by the chosen reduced form of w_0 in (5); see (6). However, using the fact that $\boldsymbol{E}_{s_i}$ and $\boldsymbol{E}_{s_j}$ commute if $|i-j| \ge 2$, we can easily put this expression in the desired form (6): If t'_{ij} denote the coefficients involved in this expression for u^{-1}, then

$$t'_{ij} = -t_{i(n+1-j)}.$$

Ω	$\iota\Omega$
$\begin{bmatrix}1&1&1\\&1&1\\&&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\&0&0\\&&0\end{bmatrix}$
$\begin{bmatrix}1&1&0\\&1&1\\&&1\end{bmatrix}$	$\begin{bmatrix}0&0&1\\&0&0\\&&0\end{bmatrix}$
$\begin{bmatrix}1&0&0\\&1&0\\&&1\end{bmatrix}$	$\begin{bmatrix}0&1&1\\&0&1\\&&0\end{bmatrix}$
$\begin{bmatrix}1&0&1\\&1&0\\&&1\end{bmatrix}$	$\begin{bmatrix}0&1&0\\&0&1\\&&0\end{bmatrix}$
$\begin{bmatrix}1&0&0\\&1&1\\&&1\end{bmatrix} \begin{bmatrix}1&1&1\\&0&0\\&&1\end{bmatrix} \begin{bmatrix}1&1&1\\&1&1\\&&0\end{bmatrix} \begin{bmatrix}0&0&1\\&1&0\\&&1\end{bmatrix}$	$\begin{bmatrix}0&0&1\\&0&1\\&&0\end{bmatrix} \begin{bmatrix}0&1&0\\&1&0\\&&0\end{bmatrix} \begin{bmatrix}1&0&0\\&0&0\\&&0\end{bmatrix} \begin{bmatrix}0&1&0\\&0&1\\&&1\end{bmatrix}$
$\begin{bmatrix}0&1&1\\&0&0\\&&0\end{bmatrix} \begin{bmatrix}0&0&0\\&1&1\\&&0\end{bmatrix} \begin{bmatrix}0&0&0\\&0&0\\&&1\end{bmatrix} \begin{bmatrix}1&1&0\\&0&1\\&&0\end{bmatrix}$	$\begin{bmatrix}1&1&0\\&1&0\\&&1\end{bmatrix} \begin{bmatrix}1&0&1\\&0&1\\&&1\end{bmatrix} \begin{bmatrix}0&1&1\\&1&1\\&&1\end{bmatrix} \begin{bmatrix}1&0&1\\&1&0\\&&0\end{bmatrix}$
$\begin{bmatrix}1&0&1\\&1&1\\&&1\end{bmatrix} \begin{bmatrix}1&1&0\\&0&0\\&&1\end{bmatrix} \begin{bmatrix}1&1&0\\&1&1\\&&0\end{bmatrix} \begin{bmatrix}0&0&0\\&1&0\\&&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\&0&1\\&&0\end{bmatrix} \begin{bmatrix}0&1&1\\&1&0\\&&0\end{bmatrix} \begin{bmatrix}1&0&1\\&0&0\\&&0\end{bmatrix} \begin{bmatrix}0&1&1\\&0&1\\&&1\end{bmatrix}$
$\begin{bmatrix}0&1&0\\&0&0\\&&0\end{bmatrix} \begin{bmatrix}0&0&1\\&1&1\\&&0\end{bmatrix} \begin{bmatrix}0&0&1\\&0&0\\&&1\end{bmatrix} \begin{bmatrix}1&1&1\\&0&1\\&&0\end{bmatrix}$	$\begin{bmatrix}1&1&1\\&1&0\\&&1\end{bmatrix} \begin{bmatrix}1&0&0\\&0&1\\&&1\end{bmatrix} \begin{bmatrix}0&1&0\\&1&1\\&&1\end{bmatrix} \begin{bmatrix}1&0&0\\&1&0\\&&0\end{bmatrix}$
$\begin{bmatrix}1&1&0\\&0&0\\&&0\end{bmatrix} \begin{bmatrix}0&0&0\\&1&0\\&&0\end{bmatrix} \begin{bmatrix}0&0&0\\&0&1\\&&1\end{bmatrix}$ $\begin{bmatrix}0&1&1\\&1&0\\&&1\end{bmatrix} \begin{bmatrix}1&0&1\\&1&1\\&&0\end{bmatrix} \begin{bmatrix}1&0&1\\&0&0\\&&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\&1&0\\&&0\end{bmatrix} \begin{bmatrix}1&1&1\\&0&1\\&&1\end{bmatrix} \begin{bmatrix}0&0&1\\&1&1\\&&1\end{bmatrix}$ $\begin{bmatrix}0&1&0\\&0&0\\&&1\end{bmatrix} \begin{bmatrix}1&0&0\\&0&1\\&&0\end{bmatrix} \begin{bmatrix}0&1&0\\&1&1\\&&0\end{bmatrix}$
$\begin{bmatrix}1&1&1\\&0&0\\&&0\end{bmatrix} \begin{bmatrix}1&0&0\\&1&1\\&&0\end{bmatrix} \begin{bmatrix}0&0&1\\&1&0\\&&0\end{bmatrix}$ $\begin{bmatrix}1&0&0\\&0&0\\&&1\end{bmatrix} \begin{bmatrix}0&1&0\\&1&0\\&&1\end{bmatrix} \begin{bmatrix}0&0&1\\&0&1\\&&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\&1&0\\&&0\end{bmatrix} \begin{bmatrix}1&0&1\\&0&1\\&&0\end{bmatrix} \begin{bmatrix}1&1&0\\&0&1\\&&1\end{bmatrix}$ $\begin{bmatrix}0&1&1\\&1&1\\&&0\end{bmatrix} \begin{bmatrix}0&1&1\\&0&0\\&&1\end{bmatrix} \begin{bmatrix}0&0&0\\&1&1\\&&1\end{bmatrix}$

Table 1: All the $\mathfrak{S}_3$-orbits (see Section 2.4). Each table entry represents a single $\mathfrak{S}_3$-orbit, which is a collection of upper-triangular matrices comprising only 0's or 1's.

Hence the matrix $M_{u^{-1}}$ is derived from M_u through a two-step process: first reflecting it across its antidiagonal to adjust the indices, and then adding 1 to each entry in the upper-triangular region to accommodate the sign changes. □

To each connected component Ω of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ we associate the set

$$S_\Omega := \{M_u \mid u \in \Omega \text{ is generic}\} \subset T^n.$$

The vector space T^n can be partitioned into subsets of the form S_Ω, and the correspondence $\Omega \leftrightarrow S_\Omega$ is one-to-one. Moreover, by the main theorem of [19], the subsets S_Ω are precisely the orbits of the $\mathfrak{S}_n$-action. Let us define an involution $\iota\colon T^n \to T^n$ by

(8) $$\iota(M)_{ij} := M_{(n+1-j)(n+1-i)} + 1 \quad \text{for all } 1 \le i \le j \le n.$$

A consequence of Lemma 2.7 is that

(9) $$\iota S_\Omega = S_{\iota\Omega}.$$

Thus Ω is ι-invariant if and only if S_Ω is. The first part of the following lemma records this discussion:

Lemma 2.8 *Let $n \in \mathbb{N}$. The involution $\iota\colon U_{n+1} \to U_{n+1}$ preserves a connected component Ω of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ if and only if the map $\iota\colon T^n \to T^n$ leaves S_Ω invariant.*

Further, the involution ι has no fixed points in T^n. In particular, no singleton $\mathfrak{S}_n$-orbits are preserved by ι.

The "further" part of the lemma above is verified by noticing that the entries in upper-right corner of $M \in T^n$ and $\iota(M)$ are different.

We identify the dual space $(T^n)^*$ with the space of $n \times n$ upper-triangular matrices with $\mathbb{F}_2$-entries so that, for $M \in T^n$ and $M^* \in (T^n)^*$,

(10) $$\langle M, M^* \rangle = \sum_{i \le j} M_{ij} M^*_{ij}.$$

We recall the elements $E_k \in T^n$ and $R_k \in (T^n)^*$ for $k = 1, \dots, n$ from [20, Section 2.1],

$$E_k = \sum_{s-r=k-1} E_{rs} \quad \text{and} \quad R_k = \sum_{1 \le r \le k \le s \le n} E_{rs},$$

where E_{rs} denotes the matrix whose only nontrivial entry is at the position (r,s). The subspace of $(T^n)^*$ (resp. T^n) spanned by the matrices R_k (resp. E_k) is denoted by $\mathscr{D}_n$ (resp. $\mathscr{I}_n$). One checks that $\iota E_k = \sum_{i \ne k} E_i$, and hence

(11) $$\iota \mathscr{I}_n = \mathscr{I}_n.$$

Moreover, note that the matrices E_k are symmetric with respect to the antidiagonal; therefore any element $I \in \mathscr{I}_n = \operatorname{span}\{E_1, \dots, E_n\}$ is also symmetric with respect to the antidiagonal. Hence

(12) $$\iota(I + M) = I + \iota(M) \quad \text{for all } M \in T^n \text{ and } I \in \mathscr{I}_n.$$

Let $\mathscr{D}_n^\perp \subset T^n$ denote the subspace orthogonal to $\mathscr{D}_n$ with respect to the standard pairing $\langle \cdot, \cdot \rangle$ in (10). A translation of $\mathscr{D}_n^\perp$ by a matrix $M \in T^n$ is called a *slice*. If $S \subset T^n$ is a slice, then its *height* h^S is defined to be the vector

$$h^S = (h_1^S, \dots, h_n^S) \in \mathbb{F}_2^n,$$

where $h_k^S := \langle M, R_k \rangle \in \mathbb{F}_2$ and $M \in S$ is an arbitrary matrix. A straightforward computation using (8) shows that for any $M \in T^n$,

$$\langle \iota(M), R_{n+1-k} \rangle \equiv \langle M, R_k \rangle + k(n+1-k) \mod 2.$$

In particular, ι sends slices to slices. The following lemma relates the height vectors of S and ιS, and follows from the formula above:

Lemma 2.9 *For every slice $S \subset T^n$ and $k \in \{1, \dots, n\}$,*

$$h_k^S \equiv h_{n+1-k}^{\iota S} + k(n+1-k) \mod 2.$$

Note that $\mathscr{D}_n^\perp$ is the slice at height zero. Moreover, the correspondence $S \leftrightarrow h^S$ is one-to-one. A slice S is called *symmetric* if its height vector h^S is symmetric with respect to its middle, ie $h_k^S = h_{n+1-k}^S$ for all $k \in \{1, \dots, n\}$. By the $\mathfrak{G}_n$-orbit structure theorem [20, Theorem 2.2], every orbit of $\mathfrak{G}_n \curvearrowright T^n$ lies in some slice $S \subset T^n$.

With the help of Lemmata 2.8 and 2.9, our strategy now is to apply [20, Theorem 2.2] to check if any $\mathfrak{G}_n$-orbit is preserved under the involution ι.

2.4 Proof of Theorem 2.4 when $d = 4$

This case is illustrative, and also does not fit into the discussion of the more general cases below. Here T^3 has 64 elements, and there are twenty $\mathfrak{G}_3$-orbits, each corresponding to one connected component of $U_4 \setminus \mathscr{E}_{\sigma_+}$. The orbits are listed in Table 1; the theorem can be verified directly from the table.

2.5 Proof of Theorem 2.4 when d is odd and $d \geq 7$

More precisely, we prove the following:

Proposition 2.10 *Let $d \geq 7$ be an odd integer.*

(i) *If $d \equiv 3$ or $5 \mod 8$, then the involution $\iota\colon U_d \to U_d$ does not preserve any connected component of $U_d \setminus \mathscr{E}_{\sigma_+}$.*

(ii) *If $d \equiv 1$ or $7 \mod 8$, then ι preserves $2^{(d+1)/2}$ connected components of $U_d \setminus \mathscr{E}_{\sigma_+}$.*

Note that there are $3 \cdot 2^{d-1}$ connected components of $U_d \setminus \mathscr{E}_{\sigma_+}$ [20].

Let $d \geq 7$ be any odd integer. Equivalently, we assume that $n = d - 1 \geq 6$ is even. Applying Lemma 2.9, $h_k^S = h_{n+1-k}^{\iota S}$ for all $k \in \{1, \dots, n\}$. Thus, if $S \subset T^n$ is a nonsymmetric slice, then $h^{\iota S} \neq h^S$. Hence ι does not preserve any orbits lying in the nonsymmetric slices.

However, for every symmetric slice S,

$$h^S = h^{\iota S} \tag{13}$$

or, equivalently, $\iota S = S$. Using [20, Theorem 2.2(ii)], every symmetric slice S decomposes into a number of singleton $\mathfrak{S}_n$-orbits and two nonsingleton $\mathfrak{S}_n$-orbits of equal sizes. By Lemma 2.8, no singleton $\mathfrak{S}_n$-orbit is preserved under ι. So it remains only to check how ι acts on the pair of nonsingleton $\mathfrak{S}_n$-orbits in each symmetric slice.

Any symmetric slice can be sent to any other by the action $\mathscr{I}_n \curvearrowright T^n$ by translations. It has been noted in the proof of [20, Lemma 6.7] that $\mathscr{I}_n \curvearrowright T^n$ maps $\mathfrak{S}_n$-orbits to $\mathfrak{S}_n$-orbits.

Claim 1 *If ι swaps (resp. preserves) the pair of nonsingleton $\mathfrak{S}_n$-orbits in one symmetric slice, then it swaps (resp. preserves) those for all symmetric slices.*

Proof Let S_1 and S_2 be any two symmetric slices, and let $I \in \mathscr{I}_n$ be a matrix such that $I + S_1 = S_2$. Let $S_1^{\pm} \subset S_1$ denote the distinct nonsingleton $\mathfrak{S}_n$-orbits. Then $S_2^{\pm} := I + S_1^{\pm} \subset S_2$ are the distinct nonsingleton $\mathfrak{S}_n$-orbits in S_2. If $\iota(S_1^+) = S_1^-$, then by (12),

$$\iota(S_2^+) = \iota(I + S_1^+) = I + \iota(S_1^+) = I + S_1^- = S_2^-. \qquad \square$$

Thus it is enough to understand how ι acts on the pair of nonsingleton $\mathfrak{S}_n$-orbits in $\mathscr{D}_n^{\perp}$, the symmetric slice at zero height. Consider the matrix $M_n^- \in T^n$ whose only nontrivial entries are the ones contained in the 2×2 submatrix at the upper-right corner, and let $M_n^+ := \iota(M_n^-)$. We note that $M_n^- \in \mathscr{D}_n^{\perp}$, and hence so is M_n^+. Using the description of the $\mathfrak{S}_n$-action above, it is easy to observe that the $\mathfrak{S}_n$-orbits $S_n^{\pm} := \mathfrak{S}_n \cdot M_n^{\pm}$ are both nonsingleton. Finally, since $M_n^+ = \iota(M_n^-)$, we get $S_n^+ = \iota(S_n^-)$.

Claim 2 *For all even numbers $n \geq 6$, $S_n^+ \cap S_n^- = \varnothing$ precisely when $n \equiv 2$ or $4 \bmod 8$.*

Proof Let $\Phi_n : (T^n)^* \to T^{n-1}$ denote the linear map given by sending a matrix $M \in (T^n)^*$ to $N \in T^{n-1}$ such that

$$N_{ij} = M_{ij} + M_{i+1,j} + M_{i,j+1} + M_{i+1,j+1}.$$

It is proven in [20, Lemma 6.6] that the dual map $\Phi_n^* : (T^{n-1})^* \to T^n$ maps $(T^{n-1})^*$ isomorphically onto $\mathscr{D}_n^{\perp}$. The dual map Φ^* can be computed by

$$\Phi^*(E_{ij}^{n-1}) = E_{ij}^n + E_{i,j+1}^n + E_{i+1,j}^n + E_{i+1,j+1}^n, \tag{14}$$

where E_{ij}^k denotes the $k\times k$ matrix with only nontrivial entry at the position (i,j), if $i \leq j$, or the zero $k\times k$ matrix, otherwise. Let $N_{n-1}^- \in (T^{n-1})^*$ be the matrix whose only nontrivial entry is contained in the upper-right corner. Let $N_{n-1}^+ = N_{n-1}^- + P_{n-1}$, where P_{n-1} denotes the $(n-1)\times(n-1)$ matrix whose nontrivial entries are precisely located at the (i,j) such that $i \leq j$, and i and j are both odd numbers. By the description of Φ^* above, it is easy to check that

$$\Phi_n^*(N_{n-1}^{\pm}) = M_n^{\pm}.$$

There is a quadratic function

$$Q\colon (T^{n-1})^* \to \mathbb{F}_2 \tag{15}$$

defined in [20, Section 5.1] which distinguishes the two nonsingleton orbits. Applying [20, Lemma 5.1], we get

$$Q(N_{n-1}^-) = 1 \quad \text{and} \quad Q(N_{n-1}^+) = \tfrac{1}{2}\cdot\tfrac{1}{2}n\big(\tfrac{1}{2}n+1\big) - 1 \mod 2.$$

Note that the quantity $\frac{1}{2}\cdot\frac{1}{2}n\big(\frac{1}{2}n+1\big)$ counts the number of 1's in the matrix P_{n-1}. Therefore $Q(N_{n-1}^+) \neq Q(N_{n-1}^-)$ exactly in the cases when $n \equiv 2$ or $4 \mod 8$. Applying [20, Lemmata 4.3, 5.5 and 6.6], the claim follows. □

Proof of Proposition 2.10 Let $n = d-1$.

(i) By the second claim above, for all even integers $n \geq 6$ satisfying $n \equiv 2$ or $4 \mod 8$, ι swaps the pair of nonsingleton $\mathfrak{S}_n$-orbits in $\mathscr{D}_n^\perp$. Following the discussion before that claim, we conclude that no orbit of $\mathfrak{S}_n \curvearrowright T^n$ is preserved by ι.

(ii) If $n \geq 6$ and $n \equiv 0$ or $6 \mod 8$, then by the above claim it follows that $\iota S_n^+ = S_n^- = S_n^+$. Hence ι preserves the nonsingleton orbits of $\mathfrak{S}_n \curvearrowright T^n$ lying in the symmetric slices. Finally, by the first item of [20, Theorem 2.2(ii)], there are exactly $2^{n/2+1}$ such orbits. □

2.6 Proof of Theorem 2.4 when $d \equiv 4 \mod 8$ and $d \geq 12$

The only remaining case of Theorem 2.4 is as follows:

Proposition 2.11 *Let $d \geq 12$ be an integer such that $d \equiv 4 \mod 8$. The involution $\iota\colon U_d \to U_d$ does not preserve any connected components of $U_d \setminus \mathscr{E}_{\sigma+}$.*

Proof Suppose that $n = d-1 \geq 11$ is an odd integer such that $n \equiv 3 \mod 8$. Applying Lemma 2.9, we observe that the $\mathfrak{S}_n$-orbits in the *symmetric* slices are not preserved, since, for every symmetric slice S, $h_1^S = 1 + h_1^{\iota S}$. Furthermore, by a similar application of Lemma 2.9 to the nonsymmetric slices S, we observe that $h^S \neq h^{\iota S}$ unless h^S satisfies

$$h_k^S = h_{n+1-k}^S \text{ for all even } k \quad \text{and} \quad h_k^S = h_{n+1-k}^S + 1 \text{ for all odd } k. \tag{16}$$

Therefore our discussion reduces to the case of $\mathfrak{S}_n$-orbits contained in the *nonsymmetric* slices whose height vectors h^S satisfy (16); we call such nonsymmetric slices *special*. By definition, it follows that a slice S is special if and only if $\iota(S) = S$. By [20, Theorem 2.2(i)], every nonsymmetric (in particular, special) slice decomposes into a pair of orbits of equal sizes.

Claim *Any special slice can be brought to any other by the action $\mathscr{I}_n \curvearrowright T^n$ by translations.*

Proof If S and S' are any two special slices, then the difference vector $h^S - h^{S'}$ is symmetric with respect to the middle. We only need to remark that the image of the map $h\colon \mathscr{I}_n \to \mathbb{F}_2^n$ which sends a matrix $M \in \mathscr{I}_n$ to the vector $(h_1^M,\dots,h_n^M) \in \mathbb{F}_2^n$, where $h_k^M := (M, R_k) \in \mathbb{F}_2$, consists of all vectors $h \in \mathbb{F}_2^n$ which are symmetric with respect to the middle. □

Since ι preserves the orbit structure of the action $\mathscr{I}_n \curvearrowright T^n$ (by (12)), and the action $\mathscr{I}_n \curvearrowright T^n$ preserves the $\mathfrak{S}_n$-orbit structure of the slices, by the above claim, it is enough to understand the involution $\iota\colon S \to S$ on *only* one special slice S. Let $\bar{S}_n$ denote the special slice at height

$$\bar{h}_n = (\underbrace{1,0,1,0,\dots,1}_{\text{first } (n-1)/2 \text{ entries}},0,\dots,0) \in \mathbb{F}_2^n.$$

Note that $\bar{h}_n$ satisfies (16). Let $\bar{M}_n^- \in \bar{S}_n \subset T^n$ denote the diagonal matrix whose diagonal entries are given by the vector $\bar{h}_n$, and let $\bar{M}_n^+ := \iota(\bar{M}_n^-)$. Let $\bar{S}_n^\pm := \mathfrak{S}_n \cdot \bar{M}_n^\pm \subset \bar{S}_n$.

Define a map $f\colon T^n \to T^{n+1}$ by sending a matrix M to the matrix $f(M) \in T^{n+1}$ obtained by appending the transpose of the vector

$$(\underbrace{1,\dots,1}_{(n+1)/2},0,\dots,0) \in \mathbb{F}_2^{n+1}$$

to M as the last column. By a direct calculation of the height, we observe that $f(\bar{S}_n) \subset \mathscr{D}_{n+1}^\perp$. Moreover, by definition of the $\mathfrak{S}_n$-action, it follows that $f(\bar{S}_n^-)$ (and similarly $f(\bar{S}_n^+)$) are contained in a nonsingleton $\mathfrak{S}_{n+1}$-orbit in $\mathscr{D}_{n+1}^\perp$. Therefore it is enough to show that $f(\bar{S}_n^-)$ and $f(\bar{S}_n^+)$ lie in two different $\mathfrak{S}_{n+1}$-orbits. Recall that $Q \circ (\Phi_{n+1}^*)^{-1}$, where $Q\colon (T^n)^* \to \mathbb{F}_2^n$ is the quadratic function in (15), distinguishes between the pair of nonsingleton $\mathfrak{S}_{n+1}$-orbits in $\mathscr{D}_{n+1}^\perp$. Let $\bar{N}_n^\pm \in (T^n)^*$ denote the $n\times n$ matrices given by

$$(\bar{N}_n^-)_{ij} = \begin{cases} 1 & i \le j,\ i \text{ is odd and } i \le \tfrac{1}{2}(n+1), \\ 0 & \text{otherwise,} \end{cases}$$

$$(\bar{N}_n^+)_{ij} = \begin{cases} 1 & i \le j,\ i \text{ and } j \text{ are both odd and } i \le \tfrac{1}{2}(n+1), \\ 1 & i \le j,\ i \text{ is odd, } j \text{ is even and } i \ge \tfrac{1}{2}(n+1), \\ 0 & \text{otherwise.} \end{cases}$$

Using the description of the dual map Φ_n^* in (14), one checks that $f(\bar{M}_n^\pm) = \Phi_n^*(\bar{N}_n^\pm)$. With the help of [20, Lemma 5.1], we obtain that (modulo 2) the quantity $Q(\bar{N}_n^-)$ counts the number of nontrivial rows in $\bar{N}_n^-$, whereas $Q(\bar{N}_n^+)$ counts the number of 1's in $Q(\bar{N}_n^+)$; since n is of the form $8m+3$,

$$Q(\bar{N}_n^-) = 1 \quad \text{and} \quad Q(\bar{N}_n^+) = 0.$$ □

Remark 2.12 Most of the discussion in the above proof applies to the case when d is divisible by 8, except that in this case $Q(\bar{N}_{d-1}^\pm)$ are both zero.

3 Proof of Theorem A

We first need the following lemma:

Lemma 3.1 *Let d be any natural number satisfying* (1), *and let Ω be any connected component of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$. Then $\sigma_+ \notin u_\sigma \Omega$ for every $\sigma \in \Omega$.*

Proof The equivalent statement that $u_\sigma^{-1}\sigma_+ \notin \Omega$ for every $\sigma \in \Omega$ follows directly from Theorem 2.4. $\square$

Proof of Theorem A Suppose that $d \in \mathbb{N}$ is any number satisfying (1), and let Ω be a connected component of $\mathscr{F}_d \setminus (\mathscr{E}_{\sigma_+} \cup \mathscr{E}_{\sigma_-})$. Let $\sigma \in \Omega$ be any point. Pick a continuous path u_t, for $t \in [0,1]$, in U_d from the identity element to u_σ. The set $\mathscr{E} = \bigcup_{t\in[0,1]} u_t \mathscr{E}_{\sigma_+} = \bigcup_{t\in[0,1]} \mathscr{E}_{u_t\sigma_+}$ is compact and does not contain σ_-. Let $B_r(\sigma_-)$ denote the closed ball in $\mathscr{F}_d$ centered at σ_- (with respect to some background metric on $\mathscr{F}_d$ compatible with the manifold topology) of radius $r > 0$ small enough that it does not intersect $\mathscr{E}$. We show that

$$(B_r(\sigma_-) \cap \Omega) \subset u_\sigma \Omega. \tag{17}$$

By our choice of the radius r, any point $\hat\sigma \in B_r(\sigma_-) \cap \Omega$ is antipodal to σ_- and $u_t\sigma_+$, for all $t \in [0,1]$. Equivalently, for all $t \in [0,1]$, $u_t^{-1}\hat\sigma$, is antipodal to σ_- and σ_+. Therefore we obtain a path $u_t^{-1}\hat\sigma$, for $0 \le t \le 1$, from $\hat\sigma$ to $u_1^{-1}\hat\sigma$ which lies completely in a single connected component of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$. Since, by assumption, $\hat\sigma \in \Omega$, we must have $u_1^{-1}\hat\sigma \in \Omega$. Hence $\hat\sigma \in u_1\Omega = u_\sigma\Omega$.

Now we can complete the proof of the theorem. Let $c\colon [-1,1] \to \mathscr{F}_d$ be a continuous path such that

$$c(\pm 1) = \sigma_\pm \quad \text{and} \quad c((-1,1)) \subset \Omega.$$

Then, by (17), there exists $t_0 \in (-1,1)$ such that

$$c(t) \in u_\sigma\Omega \quad \text{whenever } -1 \le t \le t_0. \tag{18}$$

However, by Lemma 3.1, $\sigma_+ \notin u_\sigma\Omega$. Since σ_+ is antipodal to both σ_- and σ, $\sigma_+ \notin u_\sigma\overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω in $\mathscr{F}_d$. Therefore there exists $t_1 \in (t_0, 1)$ such that

$$c(t) \notin u_\sigma\Omega \quad \text{whenever } t_1 \le t \le 1. \tag{19}$$

By (18) and (19), $c([t_0,t_1])$ must intersect the boundary $\partial(u_\sigma\Omega)$ of the subset $u_\sigma\Omega$ in $\mathscr{F}_d$. Note that the boundary of $u_\sigma\Omega$ is contained in $\mathscr{E}_{\sigma_-} \cup \mathscr{E}_\sigma$, because

$$\partial(u_\sigma\Omega) = u_\sigma(\partial\Omega) \subset u_\sigma(\mathscr{E}_{\sigma_-} \cup \mathscr{E}_{\sigma_+}) = \mathscr{E}_{\sigma_-} \cup \mathscr{E}_\sigma.$$

Furthermore, under our hypothesis, $c((-1,1)) \cap \mathscr{E}_{\sigma_-} = \varnothing$, and so

$$c([t_0,t_1]) \cap \mathscr{E}_\sigma \neq \varnothing. \qquad \square$$

4 Proofs of Theorems C and F

We first prove Theorem C. The proof of Theorem F is similar and given afterwards.

Proof of Theorem C Suppose that $d \in \mathbb{N}$ is as in the hypothesis. By definition, since Γ is a B-boundary embedded subgroup of $\mathrm{SL}(d,\mathbb{R})$, Γ is a hyperbolic group. Furthermore, since Γ is finitely generated, appealing to the Selberg lemma, we know that Γ is virtually torsion-free. After passing to a subgroup of finite index, we may (and will) assume that Γ is torsion-free. Then, by the Stallings decomposition theorem, Γ is isomorphic to a free product

$$\Gamma = F_k \star \Gamma_1 \star \cdots \star \Gamma_n, \tag{20}$$

where F_k is a free group of rank $k \geq 0$, and the Γ_j are one-ended hyperbolic groups for $n \geq 0$. Supposing Γ is not free gives $n \geq 1$. We show that for $k = 0$ and $n = 1$, Γ_1 is a surface group.

The subgroup Γ_1 is naturally B-boundary embedded in $\mathrm{SL}(d,\mathbb{R})$: Let $\xi\colon \partial_\infty\Gamma \to \mathscr{F}_d$ denote a (fixed) Γ-equivariant antipodal embedding, and let ξ_1 denote the composition

$$\partial_\infty\Gamma_1 \hookrightarrow \partial_\infty\Gamma \xrightarrow{\xi} \mathscr{F}_d.$$

Then $\xi_1\colon \partial_\infty\Gamma_1 \to \mathscr{F}_d$ is a Γ_1-equivariant antipodal embedding.

Lemma 4.1 *If $\widehat{G}$ is a one-ended hyperbolic group then there exists a topological embedding $i\colon S^1 \to \partial_\infty\widehat{G}$.*

Proof Such embedded circles can be constructed either by a direct topological argument, by using the fact that boundaries of one-ended groups are locally connected and without any global cut points (see Swarup [21]) or by using Bonk and Kleiner's [3, Corollary 2]. □

Let $i\colon S^1 \to \partial_\infty\Gamma_1$ be a topological embedding, and define

$$c := \xi_1 \circ i\colon S^1 \to \mathscr{F}_d.$$

Then c is an antipodal map. Define

$$\mathscr{E}_c := \bigcup_{x\in S^1} \mathscr{E}_{c(x)} \subset \mathscr{F}_d.$$

We show that $\partial_\infty\Gamma_1 = i(S^1)$, ie $\partial_\infty\Gamma_1$ is homeomorphic to a circle: Suppose to the contrary that $\partial_\infty\Gamma_1 \supsetneq i(S^1)$. Consider a sequence of points (y_n) in $\partial_\infty\Gamma_1 \setminus i(S^1)$ which converges to some point $y \in i(S^1)$. Then $\xi_1(y_n) \to \xi(y)$ as $n \to \infty$. Since ξ_1 is antipodal, $\xi_1(y_n) \notin \mathscr{E}_c$. However, by Corollary B, the image of c is contained in the interior of $\mathscr{E}_c$, giving a contradiction.

Since $\partial_\infty\Gamma_1$ is homeomorphic to a circle, Γ_1 is isomorphic to a surface group due to the deep work by Tukia, Gabai, Freden, Casson and Jungreis; see the survey by Kapovich and Benakli [12, Theorem 5.4].

Finally, we show that $\Gamma = \Gamma_1$ in (20): Suppose, to the contrary, that $\Gamma\setminus\Gamma_1$ is nonempty. Then $\partial_\infty\Gamma\setminus\partial_\infty\Gamma_1$ is also nonempty (for example, the fixed points in $\partial_\infty\Gamma$ of any element $\gamma \in \Gamma \setminus \Gamma_1$ lie outside $\partial_\infty\Gamma_1$). Let $z \in \partial_\infty\Gamma \setminus \partial_\infty\Gamma_1$ be an arbitrary point, and let $\gamma_1 \in \Gamma_1$ be a nontrivial element. Then $(\gamma_1^n z)_{n\in\mathbb{N}}$ is a sequence in $\partial_\infty\Gamma \setminus \partial_\infty\Gamma_1$ accumulating in $\partial_\infty\Gamma_1 \cong S^1$. By a similar argument as in the previous paragraph, we obtain a contradiction. □

Proof of Theorem F If the image of $c\colon S^1 \to \partial_\infty Z$ is not open, then there exists a sequence (z_n) in $\partial_\infty Z$ outside $c(S^1)$ converging to a point $z \in c(S^1)$. Suppose, to the contrary, that there exists a uniformly regular quasi-isometric embedding $f\colon Z \to X_d$, where d satisfies (1). Since Z is locally compact, by [15, Theorem 1.2], f admits a continuous extension

$$\bar{f}\colon \bar{Z} \to X_d \sqcup \mathscr{F}_d,$$

where $\bar{Z}$ is the compactification of Z by attaching the Gromov boundary $\partial_\infty Z$, such that the restriction $\bar{f}|_{\partial_\infty Z}\colon \partial_\infty Z \to \mathscr{F}_d$ is an antipodal map. As a consequence of continuity, the sequence $(\bar{f}(z_n))$ converges to $\bar{f}(z)$. However, due to the antipodality of the map $\bar{f}|_{\partial_\infty Z}$, $(\bar{f}(z_n))$ must remain antipodal to $\bar{f}(c(S^1))$. This is a contradiction since Corollary B asserts that $\bar{f}(c(S^1))$ is locally maximally antipodal. □

5 Some further remarks

Suppose that d is any natural number satisfying (1). Recall that, by Corollary B, antipodal circles Λ in $\mathscr{F}_d$ are locally maximally antipodal. The following result shows that, if such a circle Λ is the limit set of some Borel Anosov subgroup of $\mathrm{SL}(d,\mathbb{R})$, then Λ is maximally antipodal, ie

$$\bigcup_{\sigma\in\Lambda} \mathscr{E}_\sigma = \mathscr{F}_d.$$

Proposition 5.1 *Let d be any natural number satisfying* (1)*. If $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is a Borel Anosov subgroup which is isomorphic to a surface group, then its flag limit set Λ is a maximally antipodal subset of $\mathscr{F}_d$.*

Proof Suppose, to the contrary, that there exists a point $\hat{\sigma} \in \mathscr{F}_d$ antipodal to every point in Λ. Let $\gamma \in \Gamma$ be any hyperbolic element with *attracting/repelling* points $\sigma_\pm \in \Lambda$. Then $\gamma^k\hat{\sigma} \to \sigma_+$ as $k \to \infty$. However, since γ preserves Λ, $\gamma^k\hat{\sigma}$ remains antipodal to Λ for all $k \in \mathbb{N}$. Since Λ is homeomorphic to a circle, by Corollary B, Λ is locally maximally antipodal in $\mathscr{F}_d$, and so we get a contradiction with the preceding two sentences. □

We prove the following statement, answering a question asked by Hee Oh, which was motivated Oh and Edwards [8, Theorem 5.2], where the authors mention knowing the result for $d = 3$ or when d is even; see Remark 5.4(4) in that paper.

Proposition 5.2 *Let $d \geq 2$ be any natural number. The image in $\mathscr{F}_d$ of the equivariant limit maps corresponding to the Hitchin representations of surface groups into* $\mathrm{PSL}(d,\mathbb{R})$ *for $d \geq 2$ are maximally antipodal subsets.*

Proof By Proposition 5.1, this result is true for all d covered under the hypothesis of Proposition 5.1.

In any case, for all $d \geq 2$ it is enough to verify that Λ is locally maximally antipodal in $\mathscr{F}_d$ (see the proof of Proposition 5.1): By Fock and Goncharov [9], the Hitchin representations are characterized by Γ-equivariant *positive* limit maps $\xi\colon \partial_\infty\Gamma \to \mathscr{F}_d$. Let $x_-, x, x_+ \in \partial_\infty\Gamma$ be any distinct points, and let $\sigma_\pm := \xi(x_\pm)$ and $\sigma := \xi(x)$. Then the configuration of flags $(\sigma_-, \sigma, \sigma_+)$ in $\mathscr{F}_d$ is *positive*, ie with an appropriate identification of U_d with the unipotent radical in the stabilizer of σ_- in $\mathrm{PSL}(d,\mathbb{R})$ there exists a *totally positive* matrix $u \in U_d$ such that $\sigma = u\sigma_+$. Such a matrix u corresponds to the zero matrix $\mathbf{0} \in T^{d-1}(\mathbb{F}_2)$; see (6) and (7). By Lemma 2.8, the involution ι does not preserve the connected component Ω_d^+ of $\mathscr{C}_{\sigma_-} \cap \mathscr{C}_{\sigma_+}$ corresponding to $\mathbf{0}$, since $\mathbf{0}$ is a $\mathfrak{S}_{d-1}$-fixed point for the action $\mathfrak{S}_{d-1} \curvearrowright T^{d-1}$; see Section 2 for these notions. Therefore, for all $d \geq 2$, Lemma 3.1, and hence Theorem A, hold for the specific component Ω_d^+. Following the proof of Corollary B, one verifies that Λ is a locally maximally antipodal subset of $\mathscr{F}_d$. □

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