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We show that the mapping torus of a hyperbolic group by a hyperbolic automorphism is cubulable. Along the way, we give an alternate proof of Hagen and Wise’s theorem that hyperbolic free-by-cyclic groups are cubulable, and extend to the case with torsion Brinkmann’s thesis that a torsion-free hyperbolic-by-cyclic group is hyperbolic if and only if it does not contain \mathbb{Z}^2 -subgroups.

20E08, 20E36, 20F65, 20F67

1 Introduction

We prove the following:

Corollary 5.4 *Hyperbolic hyperbolic-by-cyclic groups are cubulable.*

A *hyperbolic-by-cyclic* group is a semidirect product $G \rtimes \mathbb{Z}$ of a hyperbolic group G with the integers \mathbb{Z} . A group is *cubulable* if it admits an isometric action on a CAT(0) cube complex that is cubical, proper, and cocompact. The repetition in the statement is intended: we assume that both G and $G \rtimes \mathbb{Z}$ are hyperbolic (equivalently, G is hyperbolic and $G \rtimes \mathbb{Z}$ does not contain \mathbb{Z}^2 ; see Corollary 5.3). This restricts what G can be.

Emblematic cases of our theorem are known by outstanding works. First and foremost, if G is a closed surface group, then any hyperbolic extension $G \rtimes \mathbb{Z}$ is a closed hyperbolic 3-manifold group [Thurston 1982]. Its cubulation is due to independent works of Bergeron and Wise [2012]—using Kahn and Markovic’s [2012] surface subgroup theorem—and Dufour [2012]—using the immersed quasiconvex surfaces of Cooper, Long, and Reid [Cooper et al. 1994]. Second, when G is free, Hagen and Wise [2016] cubulated the mapping torus $G \rtimes \mathbb{Z}$ of a fully irreducible hyperbolic automorphism.

Hagen and Wise [2015] also treat extensions of free groups by arbitrary hyperbolic automorphisms, a notoriously difficult analysis. We do not rely on, nor follow, that work. Instead, our proof uses the emblematic cases above in a telescopic argument that encompasses the case when G is a torsion-free hyperbolic group (see Theorem 4.2). It provides a hopefully appreciated alternative.

We adopt a relative viewpoint and bootstrap the relative cubulation of certain free-product-by-cyclic groups of [Dahmani and Meda Satish 2022]; this uses recent work of Groves and Manning [2023] on

improper actions on CAT(0) cube complexes along with the malnormal combination theorem of Hsu and Wise [2015]. The need for the theory of train tracks (of free groups or free product automorphisms; see [Bestvina and Handel 1992; Francaviglia and Martino 2015]) is limited to absolute train tracks for the fully irreducible case; it is encapsulated in the relative cubulation of free-product-by-cyclic groups [Dahmani and Meda Satish 2022].

For a hyperbolic group G possibly with torsion, if there exists a hyperbolic extension $G \rtimes \mathbb{Z}$, then G is virtually torsion-free (and residually finite) by Proposition 5.2. In particular, $G \rtimes \mathbb{Z}$ is virtually cubulable hyperbolic, and hence cubulable [Wise 2021, Lemma 7.14]. As a consequence, we have:

Corollary *If a hyperbolic-by-cyclic group Γ is hyperbolic, then*

- (1) Γ is virtually (compact) special [Agol 2013],
- (2) Γ is \mathbb{Z} -linear and its quasiconvex subgroups are separable [Haglund and Wise 2008],
- (3) Γ virtually surjects onto F_2 [Antolín and Minasyan 2015],
- (4) Γ is conjugacy separable [Minasyan and Zalesskii 2016], and
- (5) Γ admits Anosov representations [Douba et al. 2023].

We end this introduction with a question. Proposition 5.2 states that a hyperbolic group is virtually a free product of free and surface groups whenever it admits a hyperbolic automorphism. However, the converse is false as can be seen from a hyperbolic triangle group or the free product of two finite groups — these have finite outer automorphism groups.

Question *Can one algebraically characterise hyperbolic groups that admit hyperbolic automorphisms?*

Note that Pettet [1997] characterised virtually free groups with finite outer automorphism groups.

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2 Free factor systems

A *free decomposition* of a group G is an isomorphism $G \cong A_1 * \cdots * A_k * F_r$, where $k \geq 0$, $r \geq 0$, each *peripheral free factor* A_i is not trivial, and F_r is free with rank r . We call $\mathcal{A} = (A_1, \dots, A_k)$ a *free factor system* of G ; it is *proper* unless $k \leq 1$ and $r = 0$. The integer $k + r$ is the *Kurosh corank* of the free factor system \mathcal{A} . A nontrivial group is *freely indecomposable* if its free factor systems have Kurosh corank 1.

Assume G is finitely generated for the rest of this section. A *Grushko decomposition* of G is a free decomposition whose free factor system \mathcal{A} has maximal Kurosh corank and peripheral free factors A_i are not \mathbb{Z} ; here we call \mathcal{A} a *Grushko free factor system* and its Kurosh corank is the *Kurosh–Grushko rank* of G .

Recall the preorder of free factor systems of G : a free factor system $\mathcal{B} = (B_1, \dots, B_l)$ is lower than \mathcal{A} if each B_j is conjugate in G to a subgroup of some A_i . In this case, a free decomposition with peripherals \mathcal{A} refines to one with peripherals \mathcal{B} (as seen by the actions of A_i on $T_{\mathcal{B}}$, a Serre tree whose nontrivial vertex stabilisers are exactly the conjugates of all B_j), and the Kurosh corank of \mathcal{B} is at least that of \mathcal{A} (see [Dahmani and Li 2022, Lemma 1.1] for a similar argument); if it is equal, then \mathcal{A} is also lower than \mathcal{B} .

Let $\mathcal{B} = (B_1, \dots, B_l)$ be a free factor system of G . A *proper (G, \mathcal{B}) -free factor* is a nontrivial point stabiliser of a nontrivial action of G on a tree, for which edge stabilisers are trivial, and in which each B_j is elliptic. In other words, it is a peripheral free factor A_i in a free factor system \mathcal{A} that is higher than \mathcal{B} in the preorder.

A minimal free factor system in this preorder is a Grushko free factor system; it is unique up to the preorder's equivalence relation. So any automorphism preserves the Grushko free factor system (A_1, \dots, A_k) , ie it sends each A_i to a conjugate of some A_j . A free factor system is *periodic* with respect to $\phi \in \text{Aut}(G)$ if some (positive) power of ϕ preserves it.

Lemma 2.1 *Suppose G is a finitely generated group. If $\mathcal{B} = (B_1, \dots, B_l)$ is a proper free factor system, then each B_i has Kurosh–Grushko rank strictly lower than the Kurosh–Grushko rank of G .*

If G has Kurosh–Grushko rank ≥ 2 , then any automorphism $\phi: G \rightarrow G$ has a free factor system that is maximal among ϕ -periodic proper free factor systems.

Proof Since \mathcal{B} is proper, $G \cong B_i * H$ for some nontrivial group H . By uniqueness of the Grushko decomposition, the Kurosh–Grushko rank of G is the sum of those of B_i and H .

For the second assertion, as the Kurosh–Grushko rank is at least 2, the Grushko free factor system is proper and ϕ -periodic. Restricting to ϕ -periodic proper free factor systems, any one with the lowest Kurosh corank is maximal in the preorder. \square

3 Ingredients

Let G be a torsion-free group. For this section, we assume

- a free factor system $\mathcal{B} = (B_1, \dots, B_l)$ has Kurosh corank ≥ 3 ,
- an automorphism $\psi: G \rightarrow G$ preserves \mathcal{B} , denoted by $\psi \in \text{Aut}(G, \mathcal{B})$,
- $\psi \in \text{Aut}(G, \mathcal{B})$ is *relatively fully irreducible*, ie any ψ -periodic (up to conjugacy) proper (G, \mathcal{B}) -free factor must be conjugate to some B_i , and
- $\psi \in \text{Aut}(G, \mathcal{B})$ is *relatively atoroidal*, ie any ψ -periodic conjugacy class of nontrivial elements in G intersects some B_j .

Here is an equivalent definition of relatively fully irreducible:

Lemma 3.1 *An automorphism $\psi \in \text{Aut}(G, \mathcal{B})$ is relatively fully irreducible if and only if \mathcal{B} is a maximal ψ -periodic proper free factor system.*

Proof If some ψ -periodic proper free factor system (A_1, \dots, A_k) is strictly higher than $\mathcal{B} = (B_1, \dots, B_l)$ in the preorder, then some A_i is a ψ -periodic proper (G, \mathcal{B}) -free factor that is not conjugate to any B_j .

Conversely, if some ψ -periodic proper (G, \mathcal{B}) -free factor A_1 is not conjugate to any B_i , then the ψ -periodic free factor system (A_1) can be extended to a ψ -periodic proper free factor system (A_1, \dots, A_k) that is strictly higher than \mathcal{B} by including some (conjugates of) B_i . \square

For $h \in G$, $\text{ad}_h: G \rightarrow G$ denotes the inner automorphism $g \mapsto hgh^{-1}$. For a peripheral free factor B_i , let $k_i \geq 1$ be the smallest integer such that $\psi^{k_i}(B_i) = g_i^{-1}B_i g_i$ for some $g_i \in G$. The *peripheral suspension* $B_i \rtimes \mathbb{Z}$ is the suspension of B_i by $\text{ad}_{g_i} \circ \psi^{k_i}|_{B_i}: B_i \rightarrow B_i$; this group naturally embeds in $G \rtimes_{\psi} \mathbb{Z}$ — one can verify using normal forms that the natural homomorphism $B_i \rtimes \langle s \rangle \rightarrow G \rtimes_{\psi} \langle t \rangle$ given by $s \mapsto g_i t^{k_i}$ is injective.

The first two authors recently gave a *relative cubulation* (introduced in [Einstein and Groves 2020]) of the mapping torus of a relatively fully irreducible relatively atoroidal automorphism. Their proof is adapted from Hagen and Wise's [2016] cubulation of hyperbolic irreducible free-by-cyclic groups.

Theorem 3.2 (see [Dahmani and Meda Satish 2022, Theorem 1.1]) *Under this section's assumptions, the mapping torus $G \rtimes_{\psi} \mathbb{Z}$ acts cocompactly on a CAT(0) cube complex, where each cell stabiliser is either trivial or conjugate to a finite-index subgroup of some peripheral suspension $B_i \rtimes \mathbb{Z}$.*

The cited theorem has an additional assumption: absence of twinned subgroups. Two subgroups $H_1 \neq H_2$ of G are *twinned* in \mathcal{B} if they are conjugates of some B_j and B_k , and $\text{ad}_g \circ \psi^n(H_i) = H_i$ (for $i = 1, 2$) for some $n \geq 1$ and $g \in G$. This assumption ensures the family of peripheral suspensions is malnormal (for relative hyperbolicity [Dahmani and Li 2022, Theorem 0.1]), but Guirardel remarked that it is redundant:

Lemma 3.3 (Guirardel) *As \mathcal{B} has Kurosh corank ≥ 3 and $\psi \in \text{Aut}(G, \mathcal{B})$ is relatively fully irreducible, there are no twinned subgroups in \mathcal{B} .*

Our proof of the lemma uses objects (expanding train tracks, limit trees, and geometric trees of surface type) that we do not define here for the sake of brevity; we refer the reader to the cited literature for each.

Proof The automorphism ψ is represented by an expanding irreducible train track; see [Dahmani and Li 2022, Section 1.3]. Projectively iterating the train track produces the limit (G, \mathcal{B}) -tree T and a ψ -equivariant expanding homothety $h: T \rightarrow T$; see [Bestvina et al. 1997, page 232]. Note that nontrivial point stabilisers of T are ψ -periodic (up to conjugacy) by the finiteness of G -orbits of branch points in T [Horbez 2017, Corollary 5.5] and the ψ -equivariance of h .

Let $H \leq G$ be a nontrivial nonperipheral point stabiliser of T — nonperipheral means the subgroup is not conjugate to some B_i . Then no proper (G, \mathcal{B}) -free factor contains H — otherwise, the smallest such factor would be nonperipheral and ψ -periodic, yet $\psi \in \text{Aut}(G, \mathcal{B})$ is relatively fully irreducible. Thus T is geometric of surface type [Horbez 2017, Section 6.2 and Lemma 6.8] and the point stabiliser H is cyclic [Horbez 2017, Proposition 6.10]. As H was arbitrary, all nonperipheral point stabilisers of T are cyclic; therefore there are no twinned subgroups in \mathcal{B} because they would generate a noncyclic nonperipheral T -elliptic subgroup by the ψ -equivariance of h . \square

We use the following theorem of Groves and Manning to upgrade relative cubulations in the next section.

Theorem 3.4 (see [Groves and Manning 2023, Theorem D]) *If a hyperbolic group Γ acts cocompactly on a CAT(0) cube complex so that cell stabilisers are quasiconvex and cubulable, then Γ is cubulable.*

The cited theorem has “virtually special” in place of “cubulable”. Since virtually cubulable hyperbolic groups are cubulable [Wise 2021, Lemma 7.14], the properties “virtually special” and “cubulable” are equivalent for hyperbolic groups by Agol’s theorem [2013]. In particular, for hyperbolic groups, being cubulable is a commensurability invariant.

Finally, for sporadic cases when the Kurosh corank is 2, we will need a specialisation of Hsu and Wise’s malnormal combination theorem:

Theorem 3.5 (see [Hsu and Wise 2015, Corollary C]) *Suppose $\Gamma = \Gamma_1 *_{\langle c \rangle} \Gamma_2$ or $\Gamma_1 *_{\langle c \rangle}$ is hyperbolic and $\langle c \rangle$ is an infinite cyclic malnormal subgroup of Γ . If each Γ_i is cubulable, then Γ is cubulable.*

The two decompositions can be stated together as “ Γ splits over $\langle c \rangle$ ”.

4 The bootstrap

The following proposition is due to Sela (see Proposition 5.1 for a proof):

Proposition 4.1 (see [Sela 1997, Corollary 1.10]) *Assume G is a torsion-free hyperbolic group and some extension $G \rtimes_{\phi} \mathbb{Z}$ does not contain a copy of \mathbb{Z}^2 . If G is freely indecomposable, then it is the fundamental group of a closed surface.*

We now prove our central result:

Theorem 4.2 *Let G be a torsion-free hyperbolic group. If $G \rtimes_{\phi} \mathbb{Z}$ is hyperbolic, then it is cubulable.*

Proof We proceed by induction on the Kurosh–Grushko rank.

If the Kurosh–Grushko rank of G is 1, then G is freely indecomposable. By Proposition 4.1, G is a closed surface group and, by the classification of its automorphisms, ϕ is pseudo-Anosov [Thurston 1982, Theorem 5.5]. Then $G \rtimes_{\phi} \mathbb{Z}$ is famously the fundamental group of a closed hyperbolic 3-manifold [Thurston 1982, Theorem 5.6] and cubulable, as already mentioned in Section 1. Assume $n \geq 2$ and the theorem holds for torsion-free hyperbolic groups of Kurosh–Grushko rank $< n$.

Let the Kurosh–Grushko rank of G be n . Lemma 2.1 provides a maximal ϕ -periodic proper free factor system $\mathcal{B} = (B_1, \dots, B_l)$, and each B_i has Kurosh–Grushko rank $< n$. As each peripheral free factor B_i is quasiconvex in the hyperbolic group G , a closest point projection $G \rightarrow B_i$ is Lipschitz and extends (cosetwise) to a *peripheral retraction* $G \rtimes_{\phi} \mathbb{Z} \rightarrow B_i \rtimes \mathbb{Z}$ to the peripheral suspension. Since ϕ is a quasi-isometry, the peripheral retractions are Lipschitz by the Morse lemma (in G) — a variation of this idea appears in [Mitra 1998, Section 3]. Thus the peripheral suspensions are quasiconvex and hyperbolic. By the induction hypothesis, each $B_i \rtimes \mathbb{Z}$ is cubulable.

We distinguish two cases. The first is when the Kurosh corank of \mathcal{B} is at least 3. Some positive power ψ of ϕ preserves \mathcal{B} and, by Lemma 3.1, $\psi \in \text{Aut}(G, \mathcal{B})$ is relatively fully irreducible. Since $G \rtimes_{\psi} \mathbb{Z}$ is hyperbolic, it has no \mathbb{Z}^2 -subgroups and there are no ψ -periodic conjugacy classes of nontrivial elements in G . In particular, $\psi \in \text{Aut}(G, \mathcal{B})$ is relatively atoroidal. By Theorem 3.2, $G \rtimes_{\psi} \mathbb{Z}$ acts cocompactly on a CAT(0) cube complex, where each cell stabiliser is either trivial or conjugate to a finite-index subgroup of some quasiconvex cubulable $B_i \rtimes \mathbb{Z}$. Groves and Manning’s Theorem 3.4 thus implies $G \rtimes_{\psi} \mathbb{Z}$ is cubulable. It naturally embeds in $G \rtimes_{\phi} \mathbb{Z}$ with finite index, so the latter is also cubulable by [Wise 2021, Lemma 7.14].

The last case is when the Kurosh corank of \mathcal{B} is 2. There are three possibilities: G is $B_1 * B_2$, $B_1 * F_1$, or F_2 . We rule out the third possibility as $F_2 \rtimes \mathbb{Z}$ is never hyperbolic — it is a classical theorem of Nielsen [1917] that any automorphism of F_2 maps the commutator of a basis to a conjugate of itself or its inverse. To conclude, we will prove that $\Gamma = G \rtimes_{\phi} \langle t \rangle$ (virtually) satisfies the hypotheses of Hsu and Wise’s Theorem 3.5, and hence is cubulable. Note that $\langle t \rangle$ is a maximal cyclic subgroup of Γ , and hence malnormal. It remains to show that Γ splits over $\langle t \rangle$ as needed.

In the first possibility, up to taking the square of ϕ , we may assume that ϕ preserves the conjugacy classes of both B_1 and B_2 . After conjugation (which does not change the mapping torus), we may assume it fixes B_1 (setwise) and, being an automorphism, it sends B_2 to a conjugate by an element of B_1 . After further conjugation, it fixes both B_1 and B_2 . Then the mapping torus Γ is

$$(B_1 * B_2) \rtimes_{\phi} \langle t \rangle \cong (B_1 \rtimes \langle t \rangle) *_{\langle t \rangle} (B_2 \rtimes \langle t \rangle).$$

In the second possibility, we write $G = B_1 * \langle s \rangle$. Up to taking the square of ϕ and composing with a conjugation, we may assume that $\phi(B_1) = B_1$ and $\phi(s) = sb$ for some $b \in B_1$. Consider $G \rtimes_{\phi} \langle t \rangle$, where $tst^{-1} = sb$, or written differently $s^{-1}ts = bt$. Then, rewriting the presentation, one has that

$$\Gamma = (B_1 * \langle s \rangle) \rtimes_{\phi} \langle t \rangle \cong (B_1 \rtimes \langle t \rangle) *_{\langle t \rangle s = (bt)},$$

where the last operation is an HNN extension with a stable letter s that (right) conjugates $\langle t \rangle$ to $\langle bt \rangle$ (and actually t to bt). \square

5 Once more, with torsion

Now G is a finitely presented group (possibly with torsion). It has a maximal decomposition as the fundamental group of a finite graph of groups with finite edge groups [Dunwoody 1985]. The infinite vertex groups are thus one-ended [Stallings 1971]. We call this a *Dunwoody–Stallings decomposition*. It is not unique, but the conjugacy classes of infinite vertex groups are uniquely defined: they are conjugacy classes of the maximal one-ended subgroups of G . The following is a generalisation of Proposition 4.1:

Proposition 5.1 *Assume G is a hyperbolic group (possibly with torsion) and some extension $G \rtimes_{\phi} \mathbb{Z}$ does not contain a copy of \mathbb{Z}^2 . Then every maximal one-ended subgroup of G is virtually a closed surface group.*

Proof Let H be a maximal one-ended subgroup of G . Since there are only finitely many conjugacy classes of such subgroups, $\psi = (\text{ad}_g \circ \phi^k)|_H$ is an automorphism of H for some integer $k \geq 1$ and element $g \in G$.

Similar to the discussion in Section 3, the suspension $H \rtimes_{\psi} \mathbb{Z}$ naturally embeds in $G \rtimes_{\phi} \mathbb{Z}$. As H is one-ended, its JSJ decomposition is preserved by ψ [Bowditch 1998, Theorem 0.1]. The lack of \mathbb{Z}^2 in $G \rtimes_{\phi} \mathbb{Z}$ imposes that the JSJ is trivial but not a rigid vertex [Bestvina and Feighn 1995, Corollary 1.3]. It is therefore a vertex of surface type. In particular, H is virtually a closed surface group; see, for instance, [Martino 2007, Section 4]. \square

We are now ready to state the main observation of this section:

Proposition 5.2 *If G is a hyperbolic group (possibly with torsion) and some extension $G \rtimes_{\phi} \mathbb{Z}$ does not contain a copy of \mathbb{Z}^2 , then G has a characteristic finite-index subgroup that is a free product of closed surface groups and free groups. In particular, G is residually finite.*

Proof Let \mathbb{X} be a Dunwoody–Stallings decomposition of G . We need notation for the decomposition: the underlying finite graph is X , for each vertex v in X its vertex group is \mathbb{X}_v , and for each edge e in X its finite edge group is \mathbb{X}_e . For each vertex v , denote by H_v a normal finite-index subgroup of \mathbb{X}_v that is either trivial or a closed surface group, as guaranteed by Proposition 5.1.

As the subgroups H_v are torsion-free, the surjections $q_v: \mathbb{X}_v \rightarrow \mathbb{X}_v/H_v$ are injective on finite subgroups. Thus we define a graph of finite groups \mathbb{Y} with underlying graph X , vertex groups \mathbb{X}_v/H_v , and edge groups \mathbb{X}_e ; the surjections q_v induce a surjection $q: G \rightarrow \pi_1(\mathbb{Y})$ with a torsion-free kernel. The quotient $\pi_1(\mathbb{Y})$ is virtually free by Karrass, Pietrowski, and Solitar’s characterisation [Karrass et al. 1973, Theorem 1].

Let $J \leq \pi_1(\mathbb{Y})$ be a free finite-index subgroup. Since J and the kernel of q are torsion-free, the preimage $q^{-1}(J) \leq G$ is a torsion-free finite-index subgroup. The intersection H of subgroups of G with index $[G : q^{-1}(J)]$ is a characteristic torsion-free finite-index subgroup. The decomposition \mathbb{X} of G induces a Grushko decomposition of H whose freely indecomposable free factors are closed surface groups. \square

We may extend Brinkmann's thesis [2000] to the case with torsion:

Corollary 5.3 *Suppose G is a hyperbolic group. Then $G \rtimes_{\phi} \mathbb{Z}$ is hyperbolic if and only if it does not contain a copy of \mathbb{Z}^2 .*

The forward implication is standard. Conversely, if $G \rtimes_{\phi} \mathbb{Z}$ does not contain a copy of \mathbb{Z}^2 , then the same holds for the finite-index subgroup $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$, where G_0 is the torsion-free subgroup given by Proposition 5.2. As $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$ is hyperbolic [Brinkmann 2000], so is $G \rtimes_{\phi} \mathbb{Z}$.

Corollary 5.4 *If G and $G \rtimes_{\phi} \mathbb{Z}$ are hyperbolic groups, then $G \rtimes_{\phi} \mathbb{Z}$ is cubulable.*

Again, consider the finite-index subgroup $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$ of $G \rtimes_{\phi} \mathbb{Z}$, where G_0 is given by Proposition 5.2. $G_0 \rtimes_{\phi|_{G_0}} \mathbb{Z}$ is cubulable by Theorem 4.2, and hence, by [Wise 2021, Lemma 7.14], so is $G \rtimes_{\phi} \mathbb{Z}$.

References

- [Agol 2013] **I Agol**, *The virtual Haken conjecture*, Doc. Math. 18 (2013) 1045–1087 MR Zbl
- [Antolín and Minasyan 2015] **Y Antolín**, **A Minasyan**, *Tits alternatives for graph products*, J. Reine Angew. Math. 704 (2015) 55–83 MR Zbl
- [Bergeron and Wise 2012] **N Bergeron**, **D T Wise**, *A boundary criterion for cubulation*, Amer. J. Math. 134 (2012) 843–859 MR Zbl
- [Bestvina and Feighn 1995] **M Bestvina**, **M Feighn**, *Stable actions of groups on real trees*, Invent. Math. 121 (1995) 287–321 MR Zbl
- [Bestvina and Handel 1992] **M Bestvina**, **M Handel**, *Train tracks and automorphisms of free groups*, Ann. of Math. 135 (1992) 1–51 MR Zbl
- [Bestvina et al. 1997] **M Bestvina**, **M Feighn**, **M Handel**, *Laminations, trees, and irreducible automorphisms of free groups*, Geom. Funct. Anal. 7 (1997) 215–244 MR Zbl
- [Bowditch 1998] **B H Bowditch**, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180 (1998) 145–186 MR Zbl
- [Brinkmann 2000] **P Brinkmann**, *Mapping tori of automorphisms of hyperbolic groups*, PhD thesis, University of Utah (2000) Available at <https://www.proquest.com/docview/304625866>
- [Cooper et al. 1994] **D Cooper**, **D D Long**, **A W Reid**, *Bundles and finite foliations*, Invent. Math. 118 (1994) 255–283 MR Zbl
- [Dahmani and Li 2022] **F Dahmani**, **R Li**, *Relative hyperbolicity for automorphisms of free products and free groups*, J. Topol. Anal. 14 (2022) 55–92 MR Zbl

- [Dahmani and Meda Satish 2022] **F Dahmani, S K Meda Satish**, *Cubulating a free-product-by-cyclic group* (2022) arXiv 2212.09869 To appear in *Algebr. Geom. Topol.*
- [Douba et al. 2023] **S Douba, B Fléchéelles, T Weisman, F Zhu**, *Cubulated hyperbolic groups admit Anosov representations*, preprint (2023) arXiv 2309.03695
- [Dufour 2012] **G Dufour**, *Cubulations de variétés hyperboliques compactes*, PhD thesis, Université Paris-Sud (2012) Available at <https://theses.hal.science/tel-00690334>
- [Dunwoody 1985] **M J Dunwoody**, *The accessibility of finitely presented groups*, *Invent. Math.* 81 (1985) 449–457 MR Zbl
- [Einstein and Groves 2020] **E Einstein, D Groves**, *Relative cubulations and groups with a 2-sphere boundary*, *Compos. Math.* 156 (2020) 862–867 MR Zbl
- [Francaviglia and Martino 2015] **S Francaviglia, A Martino**, *Stretching factors, metrics and train tracks for free products*, *Illinois J. Math.* 59 (2015) 859–899 MR Zbl
- [Groves and Manning 2023] **D Groves, J F Manning**, *Hyperbolic groups acting improperly*, *Geom. Topol.* 27 (2023) 3387–3460 MR Zbl
- [Hagen and Wise 2015] **M F Hagen, D T Wise**, *Cubulating hyperbolic free-by-cyclic groups: the general case*, *Geom. Funct. Anal.* 25 (2015) 134–179 MR Zbl
- [Hagen and Wise 2016] **M F Hagen, D T Wise**, *Cubulating hyperbolic free-by-cyclic groups: the irreducible case*, *Duke Math. J.* 165 (2016) 1753–1813 MR Zbl
- [Haglund and Wise 2008] **F Haglund, D T Wise**, *Special cube complexes*, *Geom. Funct. Anal.* 17 (2008) 1551–1620 MR Zbl
- [Horbez 2017] **C Horbez**, *The boundary of the outer space of a free product*, *Israel J. Math.* 221 (2017) 179–234 MR Zbl
- [Hsu and Wise 2015] **T Hsu, D T Wise**, *Cubulating malnormal amalgams*, *Invent. Math.* 199 (2015) 293–331 MR Zbl
- [Kahn and Markovic 2012] **J Kahn, V Markovic**, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, *Ann. of Math.* 175 (2012) 1127–1190 MR Zbl
- [Karrass et al. 1973] **A Karrass, A Pietrowski, D Solitar**, *Finite and infinite cyclic extensions of free groups*, *J. Austral. Math. Soc.* 16 (1973) 458–466 MR Zbl
- [Martino 2007] **A Martino**, *A proof that all Seifert 3-manifold groups and all virtual surface groups are conjugacy separable*, *J. Algebra* 313 (2007) 773–781 MR Zbl
- [Minasyan and Zalesskii 2016] **A Minasyan, P Zalesskii**, *Virtually compact special hyperbolic groups are conjugacy separable*, *Comment. Math. Helv.* 91 (2016) 609–627 MR Zbl
- [Mitra 1998] **M Mitra**, *Cannon–Thurston maps for hyperbolic group extensions*, *Topology* 37 (1998) 527–538 MR Zbl
- [Nielsen 1917] **J Nielsen**, *Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden*, *Math. Ann.* 78 (1917) 385–397 MR Zbl
- [Pettet 1997] **M R Pettet**, *Virtually free groups with finitely many outer automorphisms*, *Trans. Amer. Math. Soc.* 349 (1997) 4565–4587 MR Zbl
- [Sela 1997] **Z Sela**, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups, II*, *Geom. Funct. Anal.* 7 (1997) 561–593 MR Zbl

- [Stallings 1971] **J Stallings**, *Group theory and three-dimensional manifolds*, Yale Math. Monogr. 4, Yale Univ. Press, New Haven, CT (1971) MR Zbl
- [Thurston 1982] **W P Thurston**, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982) 357–381 MR Zbl
- [Wise 2021] **D T Wise**, *The structure of groups with a quasiconvex hierarchy*, Ann. of Math. Stud. 209, Princeton Univ. Press (2021) MR Zbl

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