Geometry & Topology

Volume 29 (2025)

The smooth classification of 4-dimensional complete intersections

Diarmuid Crowley

Csaba Nagy

msp

msp

Geometry & Topology **29**:1 (2025) 269–311
DOI: 10.2140/gt.2025.29.269
Published: 22 January 2025

The smooth classification of 4-dimensional complete intersections

DIARMUID CROWLEY
CSABA NAGY

We prove the "Sullivan conjecture" on the classification of 4-dimensional complete intersections up to diffeomorphism. Here an n-dimensional complete intersection is a smooth complex variety formed by the transverse intersection of k hypersurfaces in $\mathbb{C}P^{n+k}$.

Previously Kreck and Traving proved the 4-dimensional Sullivan conjecture when 64 divides the total degree (the product of the degrees of the defining hypersurfaces) and Fang and Klaus proved that the conjecture holds up to the action of the group of homotopy 8-spheres $\Theta_8 \cong \mathbb{Z}/2$.

Our proof involves several new ideas, including the use of the Hambleton–Madsen theory of degree-d normal maps, which provide a fresh perspective on the Sullivan conjecture in all dimensions. This leads to an unexpected connection between the Segal conjecture for S^1 and the Sullivan conjecture.

57R55; 32J18

1 Introduction

1.1 Complete intersections and the Sullivan conjecture

A *complete intersection* $X_n(\underline{d}) \subset \mathbb{C}P^{n+k}$ is the transverse intersection of k complex hypersurfaces of degrees $\underline{d} = \{d_1, \dots, d_k\}$. We regard $X_n(\underline{d})$ as an oriented smooth manifold of real dimension $2n$ and consider the problem of classifying complete intersections up to orientation-preserving diffeomorphism. Hence throughout this paper, all manifolds are oriented and all diffeomorphisms and homeomorphisms are assumed to preserve orientations. By an observation of Thom, the diffeomorphism type of $X_n(\underline{d})$ depends only on the *multidegree* $\underline{d}$.

The main conjecture organising the classification of complete intersections for $n \geq 3$ is the "Sullivan conjecture". The statement of the conjecture relies on the following fact (see Remark 2.6): There are integers $p_i(n, \underline{d})$ such that the Pontryagin classes of $X_n(\underline{d})$ satisfy $p_i(X_n(\underline{d})) = p_i(n, \underline{d})x^{2i}$, where $x \in H^2(X_n(\underline{d}))$ is the pullback of a generator of $H^2(\mathbb{C}P^{n+k})$. Let $d := d_1 \cdots d_k$ denote the *total degree of* $X_n(\underline{d})$, which is the product of the individual degrees.

Definition 1.1 The *Sullivan data* associated to the complete intersection $X_n(\underline{d})$ is the tuple

$$\mathrm{SD}_n(\underline{d}) := \big(d, (p_i(n, \underline{d}))_{i=1}^{\lfloor n/2 \rfloor}, \chi(X_n(\underline{d}))\big) \in \mathbb{Z}^+ \times \mathbb{Z}^{\lfloor n/2 \rfloor} \times \mathbb{Z},$$

which consists of the total degree d, the Pontryagin classes of $X_n(\underline{d})$ regarded as integers and the Euler characteristic of $X_n(\underline{d})$. For a fixed n, each of these integers is a polynomial function of the individual degrees; see Section 2.2.

Conjecture 1.2 (the Sullivan conjecture) *Suppose that $n \geq 3$ and $X_n(\underline{d})$ and $X_n(\underline{d}')$ are complete intersections. If $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$, then $X_n(\underline{d})$ is diffeomorphic to $X_n(\underline{d}')$.*

The main result of this paper is that the Sullivan conjecture holds in complex dimension 4.

Theorem 1.3 *Suppose that $X_4(\underline{d})$ and $X_4(\underline{d}')$ are complete intersections with $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$. Then $X_4(\underline{d})$ is diffeomorphic to $X_4(\underline{d}')$.*

1.2 Background and an application

We first list some existing results about the Sullivan conjecture, its analogue in dimensions $n < 3$ and its converse.

When $n = 1$, $X_1(\underline{d})$ is an oriented surface and the classification is classical (in particular the Sullivan conjecture holds but its converse does not).

When $n = 2$, $X_2(\underline{d})$ is a simply connected smooth manifold and smooth classification results are currently out of reach. However, the topological classification can be deduced from results of Freedman [1982]: Two complete intersections are homeomorphic if and only if they have the same Pontryagin class p_1 and the same Euler characteristic. The converse fails, because the total degree is not even a diffeomorphism invariant (eg $X_2(4)$, $X_2(3,2)$ and $X_2(2,2,2)$ are all K3-surfaces.)

When $n \geq 3$, the converse of the Sullivan conjecture holds; see Proposition 2.10.

If $n = 3$, the Sullivan conjecture follows from classification theorems of Wall [1966] or Jupp [1973].

If $n = 4$, Fang and Klaus [1996, Remark 2] proved that the Sullivan conjecture holds up to connected sum with homotopy 8-spheres:

Theorem 1.4 [Fang and Klaus 1996] *Suppose that $X_4(\underline{d})$ and $X_4(\underline{d}')$ are complete intersections with $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$. Then there is a homotopy 8-sphere Σ such that $X_4(\underline{d}')$ and $X_4(\underline{d}) \sharp \Sigma$ are diffeomorphic.*

If $5 \leq n \leq 7$, then Fang and Wang [2010] proved that the Sullivan conjecture holds up to homeomorphism.

For $n \geq 3$, Kreck and Traving proved the following general statement. Let $\nu_p(d)$ be the largest integer such that $p^{\nu_p(d)} \mid d$. If $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$ and $\nu_p(d) \geq (2n+1)/2(p-1) + 1$ for every prime p with $p(p-1) \leq n+1$, then $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic [Kreck 1999, Theorem A]. If $n = 4$, then the condition says that $64 \mid d$.

A motivation for the diffeomorphism classification of complete intersections is [Libgober and Wood 1982, Corollary 8.3], which says that if $n \geq 3$ and diffeomorphic complete intersections have different multidegrees, then their complex structures lie in different connected components of the moduli space of complex structures on the underlying smooth manifold. Here and in general, multidegrees are regarded as equal if one can be obtained from the other by adding or removing 1s, because then the corresponding complete intersections have a common representative. Libgober and Wood used this result to show that for all odd $n \geq 3$ there are complete intersections having a complex moduli space with arbitrarily many connected components. Their proof relied on a counting argument, valid in all dimensions, which shows that the sets $\{\underline{d}' \mid \mathrm{SD}_n(\underline{d}') = \mathrm{SD}_n(\underline{d})\}$ of multidegrees with the same Sullivan data can be arbitrarily large. In future work we give an effective algorithm for finding pairs of multidegrees with the same Sullivan data. The Sullivan conjecture then allows us to construct explicit examples of complete intersections in different components of the complex moduli space and we obtain the following application of Theorem 1.3.

Example 1.5 The complete intersections $X_4(3^{(150)}, 7^{(89)}, 9^{(65)}, 15, 25^{(130)})$ and $X_4(5^{(261)}, 21^{(89)}, 27^{(64)})$ (where $3^{(150)}$ stands for 150 copies of 3, etc) are diffeomorphic by Theorem 1.3 and the formulae in Section 2.2. Hence the corresponding complex structures lie in different components of the complex moduli space.

1.3 The outline of the proof of Theorem 1.3

If $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$, then by Theorem 1.4 of Fang and Klaus there is a diffeomorphism $X_4(\underline{d}) \to X_4(\underline{d}') \sharp \Sigma$ for some homotopy sphere Σ. The group of homotopy 8-spheres, $\Theta_8 \cong \mathbb{Z}/2$, is known from [Kervaire and Milnor 1963] and so we let Σ^8_{ex} denote the unique diffeomorphism class of the exotic 8-sphere and introduce the following terminology.

Definition 1.6
- An 8-manifold M is *Θ-rigid* if $M \sharp \Sigma^8_{\mathrm{ex}}$ is diffeomorphic to M.
- An 8-manifold M is *Θ-flexible* if $M \sharp \Sigma^8_{\mathrm{ex}}$ is not diffeomorphic to M.
- A complete intersection $X_4(\underline{d})$ is *strongly Θ-flexible* if $X_4(\underline{d}) \sharp \Sigma^8_{\mathrm{ex}}$ is not diffeomorphic to a complete intersection.

As our proof of Theorem 1.3 involves treating several cases separately, we shall say that *the Sullivan conjecture holds for a fixed complete intersection $X_n(\underline{d})$* if, for every $\underline{d}'$, $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$ implies that $X_n(\underline{d}')$ is diffeomorphic to $X_n(\underline{d})$. By Theorem 1.4 and Remark 2.11, the Sullivan conjecture holds for $X_4(\underline{d})$ if and only if $X_4(\underline{d})$ is either Θ-rigid or strongly Θ-flexible. To prove the 4-dimensional Sullivan conjecture we consider four cases, which are indexed by the Wu classes of $X_4(\underline{d})$ and the parity of the total degree:

$v_2(X_4(\underline{d}))$	$v_4(X_4(\underline{d}))$	$d \bmod 2$	Θ-rigidity	treated in
0	–	–	strongly Θ-flexible	Theorem 1.7
1	0	–	Θ-rigid	Theorem 1.12
1	1	0	unknown in general	Theorem 1.14 (a)
1	1	1	unknown in general	Theorem 1.14 (b)

Here $v_i(X_4(\underline{d})) \in H^i(X_4(\underline{d}); \mathbb{Z}/2)$ is the i^{th} Wu class of $X_n(\underline{d})$, which can be regarded as an element of $\mathbb{Z}/2$ by Remark 2.6, a "–" indicates the value of the invariant is not relevant in that case, and in the cases when the Θ-rigidity of $X_4(\underline{d})$ is unknown, we conjecture that it depends on $p_1(4, \underline{d})$ mod 8; see Conjecture 1.15.

Now we discuss the proof in each of the four cases.

For a spin complete intersection $X_4(\underline{d})$ (equivalently, by Proposition 2.8, when $v_2(X_4(\underline{d})) = 0$) we find a diffeomorphism invariant property of complete intersections not shared by $X_4(\underline{d}) \sharp \Sigma^8_{ex}$; see Section 3. Namely, if $S(X, \alpha)$ denotes the total space of the circle bundle over a space X with first Chern class α, then $S(X_n(\underline{d}), \pm x)$ admits a framing, making it a null-cobordant framed $(2n+1)$-manifold (for any $X_n(\underline{d})$), whereas $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}; \pm x)$ does not (for a spin $X_4(\underline{d})$). Hence (see Theorem 3.10), we have:

Theorem 1.7 *If $X_4(\underline{d})$ is spin, then $X_4(\underline{d})$ is strongly Θ-flexible. In particular, the Sullivan conjecture holds for $X_4(\underline{d})$.*

Remark 1.8 For the 5-dimensional Sullivan conjecture, the group of homotopy 10-spheres $\Theta_{10} \cong \mathbb{Z}/2 \times \mathbb{Z}/3$ will play a central role. We believe that "transfer" arguments similar to those we use in the 4-dimensional spin case will control the $(\mathbb{Z}/3)$-factor of Θ_{10}. The $(\mathbb{Z}/2)$-factor of Θ_{10} is detected by the α-invariant, and Baraglia [2020] has recently computed the α-invariant of spin complete intersections, verifying its values are consistent with the Sullivan conjecture. We anticipate that these ideas will lead to a proof of the 5-dimensional Sullivan conjecture in future work.

In the nonspin cases we apply Kreck's modified surgery theory [1999]. Consider $B_n := \mathbb{C}P^\infty \times BO\langle n+1\rangle$, with the stable bundle $\xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle}$ over it; for the notation see Definition 2.4 and Section 2.3. Recall from [Kreck 1999, Section 8] that a *normal $(n-1)$-smoothing in* $(B_n, \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$ is a pair $(f, \bar{f})$, where $f\colon M \to B_n$ is an n-connected map from a closed smooth manifold M and $\bar{f}\colon \nu_M \to \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle}$ is a map of stable bundles from the normal bundle of M, which covers f. Recall also that the normal $(n-1)$-type of $X_n(\underline{d})$ is $(B_n, \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$; in particular $X_n(\underline{d})$ admits a normal $(n-1)$-smoothing in $(B_n, \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$. In this setting [Kreck 1999, Proposition 10] reduces the Sullivan conjecture to a statement about bordism classes over $(B_n, \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$. For our purposes, it is useful to state an altered version of [loc. cit., Proposition 10], which compares a complete intersection $X_n(\underline{d})$ to a somewhat more general closed $2n$-manifold X'. The proof of Proposition 1.9 is identical to the proof of the sufficient condition of [loc. cit., Proposition 10].

Proposition 1.9 *Let $n \geq 3$, $X_n(\underline{d})$ be a complete intersection and X' be a closed $2n$-manifold such that $\chi(X_n(\underline{d})) = \chi(X')$, and $X_n(\underline{d})$ and X' admit bordant normal $(n-1)$-smoothings over $(B_n, \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$. If $\underline{d} \neq \{1\}, \{2\}$ or $\{2, 2\}$, then $X_n(\underline{d})$ and X' are diffeomorphic.* $\square$

Remark 1.10 In fact, the assumption that $\underline{d} \neq \{1\}$ can be removed by applying [Kreck 1999, Proposition 8 (i)]. We do not know the situation for $\underline{d} = \{2\}, \{2, 2\}$. However, for all three of these exceptional multidegrees $\underline{d}$, it is elementary that $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$ implies $\underline{d} = \underline{d}'$ and so the Sullivan conjecture holds for these complete intersections.

The main challenge when applying Proposition 1.9 is showing that the bordism condition holds; see the discussion in Section 2.3. Note that the bordism group of 8-manifolds over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 5\rangle})$ is canonically isomorphic to the twisted string bordism group $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$, since $BO\langle 5\rangle = BO\langle 8\rangle = B(O\langle 7\rangle)$.

In the case of a nonspin complete intersection $X_4(\underline{d})$ with $v_4(X_4(\underline{d})) = 0$, we will use Proposition 1.9 to compare $X_4(\underline{d})$ with $X' = X_4(\underline{d}) \sharp \Sigma^8_{ex}$. They admit normal 3-smoothings over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$, whose bordism classes differ by the image of Σ^8_{ex} under the canonical homomorphism $i_0\colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$. The map i_0 factors through $\operatorname{Tors}\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1; \xi_4(\underline{d})|_{\mathbb{C}P^1})$, and (see Lemma 4.2) we prove:

Proposition 1.11 *If $X_4(\underline{d})$ is nonspin, then* $\operatorname{Tors}\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1; \xi_4(\underline{d})|_{\mathbb{C}P^1}) \cong \mathbb{Z}/4$.

When $v_4(X_4(\underline{d})) = 0$, we combine Proposition 1.11 with the computations of [Fang and Klaus 1996, Section 2.2] to show that the map $\Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$ vanishes (Proposition 4.3), which gives (see Theorem 4.4):

Theorem 1.12 *Suppose that $X_4(\underline{d})$ is a nonspin complete intersection with $v_4(X_4(\underline{d})) = 0$. If $\underline{d} \neq \{2,2\}$, then $X_4(\underline{d})$ is Θ-rigid and so the Sullivan conjecture holds for $X_4(\underline{d})$.*

Remark 1.13 In fact $X_4(2,2)$ is Θ-rigid too. This follows from results in Nagy's PhD thesis [2021, Theorem 4.6.1] but will not be proven here.

If $X_4(\underline{d})$ is nonspin, $v_4(X_4(\underline{d})) = 1$ and the total degree d is even, then $16|d$ (see Remark 2.9). We add Proposition 1.11 to the Adams filtration argument of Kreck and Traving [Kreck 1999, Section 8] and the calculations of [Fang and Klaus 1996, Section 2.4] to prove (see Proposition 4.6) part (a) of the following theorem.

Theorem 1.14 *Let $X_4(\underline{d})$ and $X_4(\underline{d}')$ be nonspin complete intersections with* $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$ *and suppose that either*

(a) *$v_4(X_4(\underline{d})) \neq 0$ and the total degree d is even, or*

(b) *the total degree d is odd.*

Then $X_4(\underline{d})$ and $X_4(\underline{d}')$ admit bordant normal 3-smoothings over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$. Consequently, $X_4(\underline{d})$ and $X_4(\underline{d}')$ are diffeomorphic and the Sullivan conjecture holds for $X_4(\underline{d})$.

Note that the cases discussed so far (ie those prior to Theorem 1.14 (b)) have a significant overlap with, but are not implied by, the theorem of Kreck and Traving [Kreck 1999, Theorem A]. However, the case of odd total degree covered in Theorem 1.14 (b) is completely new. Note also that the total degree can be odd only if $v_2(X_4(\underline{d})) \neq 0$ and $v_4(X_4(\underline{d})) \neq 0$; see Proposition 2.8.

To prove Theorem 1.14 (b) we use the Hambleton–Madsen theory of degree-d normal maps [1986]. A complete intersection $X_n(\underline{d})$ (with a canonical choice of normal data) represents an element in the set $\mathcal{N}_d^+(\mathbb{C}P^n)$ of normal bordism classes of degree-d normal maps over $\mathbb{C}P^n$. As explained in Section 5.1, an oriented version of the Hambleton–Madsen theory gives a bijective normal invariant map

$$\eta\colon \mathcal{N}_d^+(\mathbb{C}P^n) \equiv [\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_d],$$

which is the usual normal invariant in the familiar case when $d = 1$ and where $(\mathrm{QS}^0/\mathrm{SO})_d$ is the oriented version of the classifying space for isomorphism classes of stable fibrewise degree-d maps between sphere bundles of vector bundles, which was identified by Brumfiel and Madsen [1976, Section 4]. We establish a relationship between certain "relative divisors" of a vector bundle and degree-d normal maps over the vector bundle (Lemma 5.17) and then use this to give a formula for the canonical degree-d normal invariant of $X_n(\underline{d})$ (Theorem 5.19).

The surgery argument of Proposition 1.9 also works if we have bordant representatives in $\mathcal{N}_d^+(\mathbb{C}P^n)$ (Lemma 5.18). This and the formula of Theorem 5.19 leads to a new perspective on the stable homotopy-theoretic input needed to prove the Sullivan conjecture (see Theorem 5.20). This new perspective allows us to prove the 4-dimensional Sullivan conjecture when the total degree is odd and we anticipate that it will lead to other new results in higher dimensions; eg see Remark 5.34.

Notice that Fang and Klaus (Theorem 1.4) reduced the 4-dimensional Sullivan conjecture to a 2-local problem. When d is odd, [Brumfiel and Madsen 1976] showed that there is an equivalence of 2-localisations $((\mathrm{QS}^0/\mathrm{SO})_d)_{(2)} \simeq (G/O)_{(2)}$, where G/O is the familiar classifying space from classical surgery theory [Browder 1972; Wall 1970]. We can then exploit Sullivan's 2-local splitting (see [Madsen and Milgram 1979, Theorem 5.18]),

$$(G/O)_{(2)} \simeq (B\mathrm{SO})_{(2)} \times \operatorname{coker} J_{(2)},$$

where $\operatorname{coker} J_{(2)}$ is a 2-local space whose homotopy groups are certain large summands of the 2-primary component of the cokernel of the J-homomorphism (see [Madsen and Milgram 1979, Definition 5.16]). It follows that we have a sequence of maps

$$\begin{aligned}[] [\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_d] &\to [\mathbb{C}P^n, ((\mathrm{QS}^0/\mathrm{SO})_d)_{(2)}] \\ &\xrightarrow{\equiv} [\mathbb{C}P^n, (G/O)_{(2)}] \xrightarrow{\equiv} [\mathbb{C}P^n, (B\mathrm{SO})_{(2)}] \times [\mathbb{C}P^n, \operatorname{coker} J_{(2)}]. \end{aligned}$$

The formula for the degree-d normal invariant of $X_n(\underline{d})$ shows that it is the restriction of a map $\mathbb{C}P^\infty \to (\mathrm{QS}^0/\mathrm{SO})_d$. Now the proof of [Feshbach 1986, Theorem 6], which is based on the Segal conjecture for the Lie group S^1, implies that any map $\mathbb{C}P^\infty \to \operatorname{coker} J_{(2)}$ is null-homotopic and this is enough to prove that the $[\mathbb{C}P^n, \operatorname{coker} J_{(2)}]$-factor of the 2-localised normal invariant is trivial (Corollary 5.29). The $[\mathbb{C}P^n, (B\mathrm{SO})_{(2)}]$-factor is controlled by the Sullivan data; hence in dimension 4 the degree-d normal invariant is completely determined by the Sullivan data (Theorem 5.30). The 4-dimensional Sullivan conjecture for complete intersections with odd total degree follows (Theorem 5.31).

1.4 Inertia groups of 4-dimensional complete intersections

Recall that the inertia group of a closed connected m-manifold M is the subgroup

$$I(M) := \{\Sigma \in \Theta_m \mid M \text{ and } M \sharp \Sigma \text{ are diffeomorphic}\} \subseteq \Theta_m$$

of the group of homotopy m-spheres Θ_m [Kervaire and Milnor 1963]. For example, an 8-manifold M is Θ-rigid if and only if $I(M) = \Theta_8$. The results in Section 1.3 determine the inertia groups of a 4-dimensional complete intersection when $X_4(\underline{d})$ is spin, or when $X_4(\underline{d})$ is nonspin and $v_4(X_4(\underline{d})) = 0$. When $X_4(\underline{d})$ is nonspin and $v_4(X_4(\underline{d})) \neq 0$, we have $p_1(4, \underline{d}) \equiv 1 \bmod 4$ (see Proposition 2.8 and the calculations in Section 2.2) and we offer the third and fourth rows of the table in the following conjecture.

Conjecture 1.15 *The inertia groups $I(X_4(\underline{d})) \subseteq \Theta_8 \cong \mathbb{Z}/2$ of 4-dimensional complete intersections are given by the table below*:

$v_2(X_4(\underline{d}))$	$v_4(X_4(\underline{d}))$	$p_1(4, \underline{d}) \bmod 8$	$I(X_4(\underline{d}))$
0	–	–	0
1	0	–	Θ_8
1	1	5	0
1	1	1	Θ_8

Here a "–" indicates the value of the invariant is not relevant for $I(X_4(\underline{d}))$ in that case.

Remark 1.16 The first the line of the table follows from Theorem 1.7 and the second line follows from Theorem 1.12 and Remark 1.13. By [Kasilingam 2016, Remark 2.6 (1)], $I(X_4(1)) = I(\mathbb{C}P^4) = 0$, which is consistent with the third line of the table. The conjecture is based on analysing the homotopy type of the Thom spectrum of $\xi_4(\underline{d})|_{\mathbb{C}P^4}$ and using this to determine the map $\Theta_8 \cong \operatorname{Tors} \Omega_8^{O\langle 7\rangle} \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$.

In the spin case, we identified a diffeomorphism invariant property which distinguishes the manifolds $X_4(\underline{d})$ and $X_4(\underline{d}) \sharp \Sigma^8_{ex}$. In the Θ-flexible nonspin cases, besides the bordism class in $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$, we do not know of such a property.

The rest of this paper is organised as follows. Section 2 covers necessary preliminaries. Section 3 treats the spin case. Section 4 treats the two nonspin cases whose solutions rely on Proposition 1.11, which is the case with $v_4 = 0$ and the case with $v_4 \neq 0$ and even total degree (together comprising all nonspin complete intersections with even total degree). Section 5 treats the case of odd total degree. Finally, we have an appendix about Toda brackets and extensions, which are needed in Section 4 and specifically for the proof of Proposition 1.11.

Acknowledgements

It is a pleasure to thank Dennis Sullivan for stimulating conversations about the origins of the Sullivan conjecture and his work on the classification of manifolds. We thank the anonymous referee for many useful comments. Nagy was supported by the Melbourne Research Scholarship and the EPSRC New Investigator grant EP/T028335/2.

2 Preliminaries

In this section we recall and establish some basic facts about complete intersections and Sullivan data. We then recall Kreck's modified surgery setting for the classification of complete intersections.

2.1 Complete intersections

Given a finite multiset $\underline{d} = \{d_1, d_2, \dots, d_k\}$ of positive integers, consider homogeneous polynomials $f_1, f_2, \dots, f_k \in \mathbb{C}[x_0, x_1, \dots, x_{n+k}]$ with these degrees. If the zero set $\{[\underline{x}] \in \mathbb{C}P^{n+k} \mid f_i(\underline{x}) = 0\}$ of f_i is a smooth submanifold of $\mathbb{C}P^{n+k}$ for every i and these submanifolds are transverse, then their intersection is a representative of the complete intersection $X_n(\underline{d})$. Any two representatives are diffeomorphic, due to an argument generally attributed to Thom (see eg [Browder 1979]), which we outline below.

Let $P_n(\underline{d})$ denote the space of tuples $(f_1, f_2, \dots, f_k)$ of homogeneous polynomials in $n+k+1$ variables of degrees $d_1, d_2, \dots, d_k$, and let $P_n(\underline{d})^{\mathrm{ns}} \subseteq P_n(\underline{d})$ be the subspace of tuples that define complete intersections. The restriction of the tautological map

$$\{([\underline{x}], (f_1, f_2, \dots, f_k)) \in \mathbb{C}P^{n+k} \times P_n(\underline{d}) \mid f_i(\underline{x}) = 0 \text{ for all } i\} \to P_n(\underline{d})$$

to $P_n(\underline{d})^{\mathrm{ns}}$ is a locally trivial bundle, and its fibres are the representatives of $X_n(\underline{d})$. Since $P_n(\underline{d}) \setminus P_n(\underline{d})^{\mathrm{ns}} \subset P_n(\underline{d})$ is a subvariety of positive complex codimension, $P_n(\underline{d})^{\mathrm{ns}}$ is a generic (ie open and everywhere dense) subset in $P_n(\underline{d})$ and it is path-connected.

This implies that every tuple in $P_n(\underline{d})$ can be approximated by one in $P_n(\underline{d})^{\mathrm{ns}}$. We also get that any two tuples in $P_n(\underline{d})^{\mathrm{ns}}$ can be joined by a path in $P_n(\underline{d})^{\mathrm{ns}}$, which determines a diffeomorphism (up to isotopy) between the fibres over them. So if we take $X_n(\underline{d})$ to mean any of its representatives, then it is well-defined up to diffeomorphism. Moreover, if two representatives are identified via a path as above, then their natural embeddings in $\mathbb{C}P^{n+k}$ are isotopic; hence $X_n(\underline{d})$ comes equipped with an embedding $i\colon X_n(\underline{d}) \to \mathbb{C}P^{n+k}$, well-defined up to isotopy. The embedding i is n-connected (this follows from the Lefschetz hyperplane theorem, or see [Dimca 1992, Chapter 5 (2.6)]).

2.2 Computation of Sullivan data and the converse of the Sullivan conjecture

Definition 2.1 Let $x \in H^2(\mathbb{C}P^\infty)$ denote the standard generator (satisfying $\langle x, [\mathbb{C}P^1]\rangle = 1$). The pullbacks of x (by the standard embeddings) in $H^2(\mathbb{C}P^m)$ and $H^2(X_n(\underline{d}))$ will also be denoted by x.

Definition 2.2 For a (complex) bundle ξ and a positive integer r let $r\xi = \xi \oplus \cdots \oplus \xi$ denote the r-fold Whitney sum of ξ with itself and let $-r\xi$ denote the stable bundle which is the inverse of $r\xi$. Let $\xi^r = \xi \otimes \cdots \otimes \xi$ be the r-fold tensor product (over $\mathbb{C}$) of ξ with itself. For a tuple $\underline{r} = (r_1, r_2, \ldots, r_k)$ let $\xi^{\underline{r}} = \xi^{r_1} \oplus \xi^{r_2} \oplus \cdots \oplus \xi^{r_k}$.

Definition 2.3 Let γ be the conjugate of the tautological complex line bundle over $\mathbb{C}P^\infty$.

With this notation, the tautological bundle is $\bar{\gamma}$, and since $c_1(\bar{\gamma}) = -x$, we have $c_1(\gamma) = x$. It is well known that the normal bundle of $\mathbb{C}P^m$ in $\mathbb{C}P^{m+1}$ is $\nu(\mathbb{C}P^m \to \mathbb{C}P^{m+1}) \cong \gamma|_{\mathbb{C}P^m}$ and that the stable normal bundle of $\mathbb{C}P^m$ is $\nu_{\mathbb{C}P^m} \cong -(m+1)\gamma|_{\mathbb{C}P^m}$ (see eg [Milnor and Stasheff 1974, Section 14]).

Definition 2.4 The stable vector bundle $\xi_n(\underline{d})$ over $\mathbb{C}P^\infty$ is defined to be

$$\xi_n(\underline{d}) := -(n+k+1)\gamma \oplus \gamma^{d_1} \oplus \cdots \oplus \gamma^{d_k}.$$

Since the normal bundle of a degree-r hypersurface in $\mathbb{C}P^m$ is the restriction of γ^r (cf Construction 5.22 and Remark 5.16), we have:

Proposition 2.5 *The stable normal bundle $\nu_{X_n(\underline{d})}$ of $X_n(\underline{d})$ is isomorphic to $i^*(\xi_n(\underline{d})|_{\mathbb{C}P^{n+k}})$.* □

Remark 2.6 Since $\nu_{X_n(\underline{d})}$ is the pullback of a bundle over $\mathbb{C}P^\infty$, all of the stable characteristic classes of $X_n(\underline{d})$ lie in the subring $i^*(H^*(\mathbb{C}P^\infty)) \subseteq H^*(X_n(\underline{d}))$, which is generated by $x \in H^2(X_n(\underline{d}))$. In particular, $p_j(X_n(\underline{d})) \in \langle x^{2j} \rangle \cong \mathbb{Z}$, $c_j(X_n(\underline{d})) \in \langle x^j \rangle \cong \mathbb{Z}$ and if $2j \le n$, then $w_{2j}(X_n(\underline{d})), v_{2j}(X_n(\underline{d})) \in \langle \varrho_2(x^j) \rangle \cong \mathbb{Z}/2$, where $\varrho_2 \colon H^*(X_n(\underline{d})) \to H^*(X_n(\underline{d}); \mathbb{Z}/2)$ is reduction mod 2. (If $2j > n$ and d is even, then $\varrho_2(x^j) = 0$.)

Proposition 2.5 allows us to compute the characteristic classes of $X_n(\underline{d})$ in terms of the degrees $d_1, \ldots, d_k$. Since $c(\gamma^r) = 1 + rx$, the total Chern class of $\xi_n(\underline{d})$ is $c(\xi_n(\underline{d})) = (1+x)^{-(n+k+1)} \prod_{i=1}^k (1 + d_i x)$. The same formula holds for the normal bundle $\nu_{X_n(\underline{d})}$, because it is the pullback of $\xi_n(\underline{d})$. This implies that $c(X_n(\underline{d})) = (1+x)^{n+k+1} \prod_{i=1}^k (1 + d_i x)^{-1}$. For the Pontryagin classes we have $p(\gamma^r) = 1 + r^2 x^2$; hence $p(X_n(\underline{d})) = (1+x^2)^{n+k+1} \prod_{i=1}^k (1 + d_i^2 x^2)^{-1}$.

The Euler characteristic of $X_n(\underline{d})$ can also be determined, namely

$$\begin{aligned}\chi(X_n(\underline{d})) &= \langle c_n(X_n(\underline{d})), [X_n(\underline{d})] \rangle = \langle c_n(-\nu_{X_n(\underline{d})}), [X_n(\underline{d})] \rangle \\ &= \langle c_n(-i^*(\xi_n(\underline{d}))), [X_n(\underline{d})] \rangle = \langle c_n(-\xi_n(\underline{d})), i_*([X_n(\underline{d})]) \rangle,\end{aligned}$$

where $i_*([X_n(\underline{d})]) \in H_{2n}(\mathbb{C}P^{n+k})$ is d times the generator.

It will be useful to explicitly compute the Stiefel–Whitney classes w_2 and w_4 and Wu classes v_2 and v_4 of a 4-dimensional complete intersection $X_4(\underline{d})$.

Definition 2.7 For a multidegree $\underline{d}$ let $p(\underline{d})$ denote the number of even degrees in $\underline{d}$.

Proposition 2.8 *The Stiefel–Whitney classes w_2 and w_4 of $\nu_{X_4(\underline{d})}$ and $X_4(\underline{d})$ and Wu classes v_2 and v_4 of $X_4(\underline{d})$ are determined by $p(\underline{d}) \bmod 4$ as follows (by Remark 2.6 these Stiefel–Whitney classes and Wu classes can be regarded as elements of $\mathbb{Z}/2$):*

$p(\underline{d}) \bmod 4$	0	1	2	3
$w_2(\nu_{X_4(\underline{d})}) = w_2(X_4(\underline{d})) = v_2(X_4(\underline{d}))$	1	0	1	0
$w_4(\nu_{X_4(\underline{d})}) = v_4(X_4(\underline{d}))$	1	1	0	0
$w_4(X_4(\underline{d}))$	0	1	1	0

Proof The total Chern class of $\xi_n(\underline{d})$ is given by the formula

$$\begin{aligned} c(\xi_n(\underline{d})) &= (1+x)^{-(n+k+1)} \prod_{i=1}^{k} (1+d_i x) \\ &= 1+\Big(-(n+1)+\sum_{i=1}^{k}(d_i-1)\Big)x+\Big(\binom{n+2}{2}-(n+2)\sum_{i=1}^{k}(d_i-1)+\sum_{1\le i<j\le k}(d_i-1)(d_j-1)\Big)x^2+\cdots. \end{aligned}$$

We have $w_{2i} = \varrho_2(c_i)$. Therefore

$$w_2(\xi_4(\underline{d})) = \varrho_2\Big(\Big(-5+\sum_{i=1}^{k}(d_i-1)\Big)x\Big) = \varrho_2((1+p(\underline{d}))x),$$

$$w_4(\xi_4(\underline{d})) = \varrho_2\Big(\Big(15-6\sum_{i=1}^{k}(d_i-1)+\sum_{1\le i<j\le k}(d_i-1)(d_j-1)\Big)x^2\Big) = \varrho_2\Big(\Big(1+\binom{p(\underline{d})}{2}\Big)x^2\Big).$$

We have the same formulas for the Stiefel–Whitney classes of $\nu_{X_4(\underline{d})}$, because $\nu_{X_4(\underline{d})}$ is the pullback of $\xi_4(\underline{d})$. Since $H^1(X_4(\underline{d});\mathbb{Z}/2) \cong H^3(X_4(\underline{d});\mathbb{Z}/2) \cong 0$, the Stiefel–Whitney classes $w_2(X_4(\underline{d}))$ and $w_4(X_4(\underline{d}))$ are determined by $w_2(\nu_{X_4(\underline{d})})$ and $w_4(\nu_{X_4(\underline{d})})$ via the Cartan formula. We get that $w_2(X_4(\underline{d})) = w_2(\nu_{X_4(\underline{d})})$ and $w_4(X_4(\underline{d})) = w_2(\nu_{X_4(\underline{d})})^2 + w_4(\nu_{X_4(\underline{d})})$. By applying the Wu formula we get that $v_2(X_4(\underline{d})) = w_2(X_4(\underline{d}))$ and $v_4(X_4(\underline{d})) = w_2(X_4(\underline{d}))^2 + w_4(X_4(\underline{d})) = w_4(\nu_{X_4(\underline{d})})$. □

Remark 2.9 Notice that if $v_2(X_4(\underline{d})) \neq 0$ and $v_4(X_4(\underline{d})) \neq 0$, then $p(\underline{d})$ is divisible by 4. This means that either $p(\underline{d}) = 0$, and hence all degrees are odd, so the total degree is odd; or $p(\underline{d}) \geq 4$, so there are at least four even degrees and then the total degree is divisible by 16.

The following proposition implies that the converse of the Sullivan conjecture holds.

Proposition 2.10 *Let $n \geq 3$ and let $\underline{d}$ and $\underline{d}'$ be two multidegrees. If there is a homotopy equivalence $f: X_n(\underline{d}) \to X_n(\underline{d}')$ such that $f^*(\nu_{X_n(\underline{d}')}) \cong \nu_{X_n(\underline{d})}$ (eg if f is a diffeomorphism), then $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$.*

Proof If $n \geq 3$, then $H^2(X_n(\underline{d})) \cong H^2(X_n(\underline{d}')) \cong \mathbb{Z}$, so any homotopy equivalence $X_n(\underline{d}) \to X_n(\underline{d}')$ preserves x up to sign. If f sends x to $-x$, then we can replace it with another homotopy equivalence that preserves x, by composing it with a self-diffeomorphism of $X_n(\underline{d})$ (or $X_n(\underline{d}')$) that changes the sign of x. (Consider the conjugation map of the ambient $\mathbb{C}P^{n+k}$; it sends x to $-x$. If a representative of $X_n(\underline{d})$ is given by polynomials $f_1, f_2, \dots, f_k$, then its image is another representative of the same complete intersection, given by the conjugate polynomials $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_k$. By Thom's argument there is a diffeomorphism between the two representatives such that after identifying them their embeddings into $\mathbb{C}P^{n+k}$ are isotopic. By composing this diffeomorphism with the restriction of the conjugation map, we get a self-diffeomorphism of either representative that changes the sign of x.) Since $\langle x^n, [X_n(\underline{d})]\rangle = d$ and $\langle x^n, [X_n(\underline{d}')]\rangle = d'$, this means that $d = d'$. The Euler characteristic is a homotopy invariant. The Pontryagin classes are preserved by f because of the assumption on the normal bundles, and since the elements x^{2i} are preserved, the Pontryagin classes are also invariant when regarded as integers. $\square$

Remark 2.11 If $\Sigma \in \Theta_{2n}$ is a homotopy sphere, then there is a homeomorphism between $X_n(\underline{d})$ and $X_n(\underline{d}) \sharp \Sigma$ which preserves normal bundles. Thus if $n \geq 3$ and $X_n(\underline{d}) \sharp \Sigma$ is diffeomorphic to a complete intersection $X_n(\underline{d}')$, then $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$.

2.3 The setting for modified surgery

We recall the setup for the modified surgery arguments of [Kreck 1999, Section 8; Fang and Klaus 1996], which will be used in Sections 4 and 5.

Recall that the inclusion $i: X_n(\underline{d}) \to \mathbb{C}P^\infty$ is n-connected. It is covered by a bundle map $\bar{i}: \nu_{X_n(\underline{d})} \to \xi_n(\underline{d})$ (Proposition 2.5) and therefore $(i, \bar{i})$ is a normal $(n-1)$-smoothing over $(\mathbb{C}P^\infty, \xi_n(\underline{d}))$.

Let γ_{BO} denote the universal stable vector bundle over BO and $\gamma_{BO\langle j\rangle}$ its pullback to $BO\langle j\rangle$, the $(j-1)$-connected cover of BO. Let $B_n := \mathbb{C}P^\infty \times BO\langle n+1\rangle$. Then $(i, \bar{i})$ can be regarded as a normal map over $(B_n, \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$ (and it is still n-connected). Moreover, the map $B_n \to BO$ inducing $\xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle}$ from γ_{BO} is n-coconnected; therefore $(B_n, \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$ is the normal $(n-1)$-type of $X_n(\underline{d})$. When $n = 4$, we have that $BO\langle 5\rangle = BO\langle 8\rangle = B\mathrm{String}$ by Bott periodicity, and thus $(i, \bar{i})$ represents an element in the bordism group of closed 8-manifolds with normal maps to $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$. We denote this bordism group by $\Omega_8^{\mathrm{fr}}(B_4; \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$; it is canonically isomorphic to the twisted string bordism group $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$.

First we will want to apply Proposition 1.9 when $X' = X_4(\underline{d}) \sharp \Sigma^8_{ex}$. There is a canonical homeomorphism $h\colon X_4(\underline{d}) \sharp \Sigma^8_{ex} \to X_4(\underline{d})$, and since homotopy spheres are stably parallelisable [Kervaire and Milnor 1963, Theorem 3.1], h is covered by a bundle map $\bar{h}$ of stable normal bundles. Then $(i \circ h, \bar{i} \circ \bar{h})$ is also a normal 3-smoothing over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$, and in the bordism group $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$ it represents $[i, \bar{i}] + [\Sigma^8_{ex}]$, where $[\Sigma^8_{ex}]$ is the image of Σ^8_{ex} under the canonical homomorphism $i_0\colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$. So to apply Proposition 1.9 in this setting we need to show that this homomorphism is trivial and we do this in the nonspin case with $v_4(X_4(\underline{d})) = 0$; see Proposition 4.3.

Now suppose that $X_4(\underline{d}')$ is another complete intersection with an analogous normal 3-smoothing $(i', \bar{i}')$ over $(\mathbb{C}P^\infty, \xi_4(\underline{d}'))$. If the Pontryagin classes of $X_4(\underline{d})$ and $X_4(\underline{d}')$ agree, in particular if $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$, then the Pontryagin classes p_1 and p_2 of $\xi_4(\underline{d})$ and $\xi_4(\underline{d}')$ also agree. This implies that $\xi_4(\underline{d})|_{\mathbb{C}P^4} \cong \xi_4(\underline{d}')|_{\mathbb{C}P^4}$ (by [Sanderson 1964, Theorem (3.9)] every stable bundle over $\mathbb{C}P^4$ is isomorphic to $\xi_{a,b} := a\gamma \oplus b(\gamma \otimes_{\mathbb{R}} \gamma)$ for some $a, b \in \mathbb{Z}$, and the function $(a,b) \mapsto (p_1(\xi_{a,b}), p_2(\xi_{a,b}))$ is injective). Thus $\xi_4(\underline{d}') \ominus \xi_4(\underline{d})$ is trivial over $\mathbb{C}P^4$, so it has an $O\langle 7\rangle$-structure. Therefore $\mathrm{Id}_{\mathbb{C}P^\infty}$ has a lift $g\colon \mathbb{C}P^\infty \to B_4$ which induces $\xi_4(\underline{d}')$ from $\xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle}$. Hence if $\bar{g}\colon \xi_4(\underline{d}') \to \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle}$ is a bundle map over g, then $(g \circ i', \bar{g} \circ \bar{i}')$ is a normal 3-smoothing of $X_4(\underline{d}')$ over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$.

If $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$, then the discussion in the paragraph above shows that $X_4(\underline{d})$ and $X_4(\underline{d}')$ admit normal 3-smoothings over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$ and $\chi(X_4(\underline{d})) = \chi(X_4(\underline{d}'))$; therefore to apply Proposition 1.9 it is enough to prove that these normal 3-smoothings represent the same bordism class in $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$. Fang and Klaus obtained Theorem 1.4 by showing that the difference of these bordism classes is in the image of the canonical homomorphism $i_0\colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$. In the nonspin cases with $v_4(X_4(\underline{d})) \neq 0$, we are able to show in Sections 4.3 and 5 that the bordism classes agree.

3 The spin case

In this section we prove that 4-dimensional spin complete intersections are strongly Θ-flexible; hence the Sullivan conjecture holds for them.

Definition 3.1 For a smooth manifold X and a cohomology class $\alpha \in H^2(X)$, let $E(X,\alpha)$ denote the total space of the complex line bundle over X with first Chern class α. Let $D(X,\alpha)$ denote its disc bundle and $S(X,\alpha)$ denote its sphere bundle.

Recall that $x \in H^2(X_n(\underline{d}))$ is the pullback of the standard generator of $H^2(\mathbb{C}P^\infty)$. First we will prove that for every complete intersection $X_n(\underline{d})$ the total space $S(X_n(\underline{d}), x)$ admits a framing such that it is framed null-cobordant (where by a framing of a manifold we mean a trivialisation of its stable normal bundle, equivalently, of its stable tangent bundle); see Theorem 3.4.

Recall that (a representative of) the complete intersection $X_{n+1}(\underline{d}) \subset \mathbb{C}P^{n+k+1}$ is the set of common zeros of some homogeneous polynomials $f_1, f_2, \dots, f_k \in \mathbb{C}[x_0, x_1, \dots, x_{n+k+1}]$. If $f_{k+1} \in \mathbb{C}[x_0, x_1, \dots, x_{n+k+1}]$ is linear and its zero set L is transverse to $X_{n+1}(\underline{d})$, then $X_n(\underline{d}) = X_{n+1}(\underline{d}) \cap L$.

Proposition 3.2 *The complement $X_{n+1}(\underline{d}) \setminus X_n(\underline{d})$ is stably parallelisable.*

Proof We have the following commutative diagram of embeddings:

$$\begin{tikzcd} \mathbb{C}P^{n+k+1} \setminus L \arrow[r] & \mathbb{C}P^{n+k+1} \\ X_{n+1}(\underline{d}) \setminus X_n(\underline{d}) \arrow[u, "i"] \arrow[r] & X_{n+1}(\underline{d}) \arrow[u, "i"'] \end{tikzcd}$$

So

$$\nu_{X_{n+1}(\underline{d}) \setminus X_n(\underline{d})} \cong \nu_{X_{n+1}(\underline{d})}|_{X_{n+1}(\underline{d}) \setminus X_n(\underline{d})} \cong i^*(\xi_{n+1}(\underline{d}))|_{X_{n+1}(\underline{d}) \setminus X_n(\underline{d})} \cong i^*(\xi_{n+1}(\underline{d})|_{\mathbb{C}P^{n+k+1} \setminus L})$$

(using Proposition 2.5), and this is trivial, because $\mathbb{C}P^{n+k+1} \setminus L$ is contractible (recall that L is a hyperplane). ◻

Proposition 3.3 *We have* $\nu(X_n(\underline{d}) \to X_{n+1}(\underline{d})) \cong i^*(\gamma)$ (*see* Definition 2.3).

Proof Since L is transverse to $X_{n+1}(\underline{d})$ and $X_n(\underline{d}) = X_{n+1}(\underline{d}) \cap L$, the normal bundle $\nu(X_n(\underline{d}) \to X_{n+1}(\underline{d}))$ is the restriction of $\nu(L \to \mathbb{C}P^{n+k+1})$; hence

$$\nu(X_n(\underline{d}) \to X_{n+1}(\underline{d})) \cong \nu(L \to \mathbb{C}P^{n+k+1})|_{X_n(\underline{d})} \cong \gamma|_{X_n(\underline{d})}. \qquad \square$$

Theorem 3.4 *For any complete intersection* $X_n(\underline{d})$, (*the total space of*) *the* S^1*-bundle* $S(X_n(\underline{d}), x)$ *admits a framing* F_0 *such that* $[S(X_n(\underline{d}), x), F_0] = 0 \in \Omega^{\mathrm{fr}}_{2n+1}$.

Proof Let U be a tubular neighbourhood of $X_n(\underline{d})$ in $X_{n+1}(\underline{d})$. By Proposition 3.3 it is diffeomorphic to the disc bundle of $i^*(\gamma)$, whose first Chern class is x, therefore $\partial U \approx S(X_n(\underline{d}), x)$. Its complement, $X_{n+1}(\underline{d}) \setminus \operatorname{int} U$, is a codimension-0 submanifold in $X_{n+1}(\underline{d}) \setminus X_n(\underline{d})$. The latter is stably parallelisable by Proposition 3.2, so $X_{n+1}(\underline{d}) \setminus \operatorname{int} U$ is stably parallelisable too. If we choose F_0 to be the restriction of a framing of $X_{n+1}(\underline{d}) \setminus \operatorname{int} U$ to the boundary $\partial(X_{n+1}(\underline{d}) \setminus \operatorname{int} U) \approx \partial U \approx S(X_n(\underline{d}), x)$, then $(S(X_n(\underline{d}), x), F_0)$ is framed null-cobordant. $\square$

The goal of the rest of this section is to prove that $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ is not framed nullcobordant (with any framing) if $X_4(\underline{d})$ is spin; see Theorem 3.9. First we show that, when an m-manifold X is replaced by $X \sharp \Sigma$ for a homotopy m-sphere Σ, the framed cobordism class of $S(X, \alpha)$ changes by $\Sigma \times S^1$ (with a certain choice of framings); see Lemma 3.5. In Lemma 3.6 we give a formula to compute the framing of the S^1 component. By applying this formula we prove that if $X_4(\underline{d})$ is spin, then $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ has a framing such that it is not framed nullcobordant (Theorem 3.8). Finally we show that we cannot make the framed cobordism class vanish by changing the framing.

Lemma 3.5 *Suppose that* $m \geq 3$, X *is an* m*-manifold*, $\alpha \in H^2(X)$ *and* F_0 *is a framing of* $S(X, \alpha)$. *Then there exists a framing* F_2 *of* S^1 *such that for every* $\Sigma \in \Theta_m$ *and framing* F_1 *of* Σ *there is a framing* F *of* $S(X \sharp \Sigma, \alpha)$ *such that*

$$[S(X, \alpha), F_0] + [\Sigma \times S^1, F_1 \times F_2] = [S(X \sharp \Sigma, \alpha), F] \in \Omega^{\mathrm{fr}}_{m+1}.$$

Proof Fix an embedding $D^m \to X$ where the connected sum is done. There is a homotopically unique homeomorphism between X and $X \sharp \Sigma$ that is the identity on $X \setminus \operatorname{int} D^m$, so there is a canonical isomorphism $H^2(X) \cong H^2(X \sharp \Sigma)$. Thus α can be regarded as an element of $H^2(X \sharp \Sigma)$, and $S(X \sharp \Sigma, \alpha)$ makes sense. The homomorphisms $H^2(X) \to H^2(X \setminus \operatorname{int} D^m) \leftarrow H^2(X \sharp \Sigma)$ are injective (in fact they are isomorphisms); therefore $S(X \sharp \Sigma, \alpha)$ is (the total space of) the unique S^1-bundle over $X \sharp \Sigma$ whose restriction to $X \setminus \operatorname{int} D^m$ is isomorphic to that of $S(X, \alpha)$.

Let $W = (S(X, \alpha) \sqcup \Sigma \times S^1) \times I \cup_f (D^m \times S^1 \times I)$, where the gluing map

$$f \colon D^m \times S^1 \times \partial I \to (S(X, \alpha) \sqcup \Sigma \times S^1) \times \{1\}$$

is the disjoint union of the (homotopically unique) local trivialisation $f_0 \colon D^m \times S^1 \times \{0\} \to S(X, \alpha) \times \{1\}$ of $S(X, \alpha)$ over the fixed D^m and the product map $f_1 \colon D^m \times S^1 \times \{1\} \to \Sigma \times S^1 \times \{1\}$, where $D^m \to \Sigma$

is the embedding used to construct the connected sum $X \sharp \Sigma$. Then $\partial W = \partial_- W \sqcup \partial_+ W$, where $\partial_- W = (S(X,\alpha) \sqcup \Sigma \times S^1) \times \{0\}$ and $\partial_+ W = (S(X,\alpha) \setminus (\operatorname{int} D^m \times S^1) \sqcup (\Sigma \setminus \operatorname{int} D^m) \times S^1) \times \{1\} \cup_f S^{m-1} \times S^1 \times I$. Thus $\partial_+ W$ is an S^1-bundle over $(X \setminus \operatorname{int} D^m) \cup S^{m-1} \times I \cup (\Sigma \setminus \operatorname{int} D^m) \approx X \sharp \Sigma$ and it coincides with $S(X,\alpha)$ over $X \setminus \operatorname{int} D^m$; therefore $\partial_+ W \approx S(X \sharp \Sigma, \alpha)$.

The inclusion $S(X,\alpha) \times \{0\} \to S(X,\alpha) \times I \cup_{f_0} D^m \times S^1 \times I$ is a homotopy equivalence, covered by a bundle map between the stable normal bundles; therefore the framing F_0 can be extended to a framing of $S(X,\alpha) \times I \cup_{f_0} D^m \times S^1 \times I$. The restriction of this framing to $D^m \times S^1 \times \{1\}$ is $E_m \times F_2$, where E_m is the homotopically unique framing of D^m and F_2 is some framing of S^1 (because every framing of $D^m \times S^1$ is of this form). Similarly, we can take the framing $F_1 \times F_2$ of $\Sigma \times S^1$ and extend it to $\Sigma \times S^1 \times I$. The restriction of this framing to $D^m \times S^1 \times \{1\}$ is again $E_m \times F_2$ (up to homotopy); therefore the framings of $S(X,\alpha) \times I \cup_{f_0} D^m \times S^1 \times I$ and $\Sigma \times S^1 \times I$ together determine a framing of W. Let F denote its restriction to $\partial_+ W \approx S(X \sharp \Sigma, \alpha)$. Then W is a framed cobordism between the framed manifolds $(S(X,\alpha), F_0) \sqcup (\Sigma \times S^1, F_1 \times F_2)$ and $(S(X \sharp \Sigma, \alpha), F)$. □

Lemma 3.6 *Suppose that, in addition to the assumptions of Lemma 3.5, there is an $[a] \in \pi_2(X)$ such that $\langle \alpha, \rho([a]) \rangle = 1$, where $\rho\colon \pi_2(X) \to H_2(X)$ is the Hurewicz homomorphism. Then for any such $[a] \in \pi_2(X)$ and the framing F_2 constructed in the proof of Lemma 3.5 we have*

$$[S^1, F_2] = \langle w_2(X), \rho([a]) \rangle + 1,$$

where both sides are regarded as elements of $\mathbb{Z}/2$ (using that $\Omega_1^{\mathrm{fr}} \cong \mathbb{Z}/2$).

Proof Fix a local trivialisation $f_0\colon D^m \times S^1 \to S(X,\alpha)$, as in the proof of Lemma 3.5. The framing F_2 is defined by the property that the restriction of F_0 to $f_0(D^m \times S^1)$ is $E_m \times F_2$ (throughout this proof we will identify the framings of $f_0(D^m \times S^1)$ with the framings of $D^m \times S^1$ via (the derivative of) f_0). First we will give another characterisation of $E_m \times F_2$.

If $\partial\colon \pi_2(X) \to \pi_1(S^1) \cong \mathbb{Z}$ denotes the boundary map in the homotopy long exact sequence of the fibration $S^1 \to S(X,\alpha) \to X$, then $\partial([a]) = \langle \alpha, \rho([a]) \rangle$ (this holds if $X = S^2$ and $a = \mathrm{Id}_{S^2}$, because α is the Euler class of $E(S^2,\alpha)$, and in general $a\colon S^2 \to X$ induces a commutative diagram between the exact sequences). Moreover, ∂ is the composition of the isomorphism $\pi_2(X) \cong \pi_2(S(X,\alpha), S^1)$ and the boundary map $\pi_2(S(X,\alpha), S^1) \to \pi_1(S^1)$. Therefore for any $[a]$ with $\langle \alpha, \rho([a]) \rangle = 1$ there is a map $\tilde{a}\colon D^2 \to S(X,\alpha)$ (well-defined up to homotopy) such that $\tilde{a}|_{S^1}$ is the inclusion of a fibre and $(\tilde{a}, \tilde{a}|_{S^1})$ represents the element in $\pi_2(S(X,\alpha), S^1)$ corresponding to $[a] \in \pi_2(X)$. We can lift $\tilde{a}$ to a map $\bar{a}\colon D^2 \to S(X,\alpha) \times \mathbb{R}_0^+$ (where $\mathbb{R}_0^+$ denotes $[0,\infty)$) such that $\bar{a}$ is an embedding, it is transverse to $S(X,\alpha) \times \{0\}$, $\bar{a}^{-1}(S(X,\alpha) \times \{0\}) = S^1$, $\bar{a}|_{S^1}\colon S^1 \to S(X,\alpha) \times \{0\}$ is the inclusion of a fibre and $[\bar{a}, \bar{a}|_{S^1}] = [\tilde{a}, \tilde{a}|_{S^1}] \in \pi_2(S(X,\alpha) \times \mathbb{R}_0^+, S^1 \times \mathbb{R}_0^+) \cong \pi_2(S(X,\alpha), S^1)$.

Let U be a tubular neighbourhood of $\bar{a}(D^2)$ in $S(X,\alpha) \times \mathbb{R}_0^+$. We can assume that $U \cap S(X,\alpha) \times \{0\} = f_0(D^m \times S^1)$. (Note that U is the total space of a D^m-bundle over D^2, so it has a homotopically unique trivialisation $D^2 \times D^m \to U$, but the restriction of this trivialisation to S^1 may differ from f_0, the difference is given by an element of $\pi_1(\mathrm{SO}_m) \cong \mathbb{Z}/2$.) The framing F_0 can be extended to a framing of $S(X,\alpha) \times \mathbb{R}_0^+$ and then restricted to a framing of U. As mentioned above, if we further restrict this framing to $f_0(D^m \times S^1)$, we get $E_m \times F_2$. Since U is contractible, it has a homotopically unique framing, so this means that $E_m \times F_2$ is the restriction of the homotopically unique framing of U.

The local trivialisation f_0 is the restriction of a local trivialisation $\bar{f_0}\colon D^m \times D^2 \to D(X,\alpha)$. The homotopically unique framing of $\bar{f_0}(D^m \times D^2)$ is $E_m \times E_2$; its restriction to $f_0(D^m \times S^1)$ is $E_m \times (E_2|_{S^1})$. Since $(S^1, E_2|_{S^1})$ is the framed boundary of (D^2, E_2), we have $[S^1, E_2|_{S^1}] = 0 \in \Omega_1^{\mathrm{fr}} \cong \mathbb{Z}/2$. So if $g \in \pi_1(\mathrm{SO}) \cong \mathbb{Z}/2$ denotes the difference of the framings F_2 and $E_2|_{S^1}$ of S^1, then $[S^1, F_2] = g \in \mathbb{Z}/2$.

We have $D(X,\alpha) \cup_{S(X,\alpha)} S(X,\alpha) \times \mathbb{R}_0^+ \approx E(X,\alpha)$ (in each fibre $D^2 \cup_{S^1} S^1 \times \mathbb{R}_0^+ \approx \mathbb{R}^2$) and $\bar{f_0}(D^m \times D^2) \cup U$ is a tubular neighbourhood of $\bar{f_0}(\{0\} \times D^2) \cup \bar{a}(D^2) \approx S^2$ in $E(X,\alpha)$. As a D^m-bundle over S^2 it is classified by an element of $\pi_2(B\mathrm{SO}_m) \cong \mathbb{Z}/2$. Under the isomorphism $\pi_2(B\mathrm{SO}_m) \cong \pi_2(B\mathrm{SO}) \cong \pi_1(\mathrm{SO})$ this element corresponds to g (because it is equal to the difference of the restrictions of the unique framings of $\bar{f_0}(D^m \times D^2)$ and U, which are $E_m \times (E_2|_{S^1})$ and $E_m \times F_2$ respectively).

So we need to determine the normal bundle of the embedding $S^2 \to E(X,\alpha)$ as an element of $\pi_2(B\mathrm{SO})$. Since S^2 is stably parallelisable, it is the same as the restriction of the stable tangent bundle $\tau_{E(X,\alpha)}$ to S^2. The embedding $S^2 \to E(X,\alpha)$ is homotopic to its projection to the zero section (X). Since $\bar{f_0}(\{0\} \times D^2)$ is a fibre of $D(X,\alpha)$, its projection to X is one point. The map $\bar{a}\colon D^2 \to S(X,\alpha) \times \mathbb{R}_0^+$ is a lift of $\tilde{a}\colon D^2 \to S(X,\alpha)$, which is a lift of a map $a\colon S^2 \to X$ representing $[a] \in \pi_2(X)$. Therefore the composition of the embedding $S^2 \to E(X,\alpha)$ and the projection to X is a. The restriction of $\tau_{E(X,\alpha)}$ to X is $E(X,\alpha) \oplus \tau_X$. So the bundle we are interested in is the pullback of $E(X,\alpha) \oplus \tau_X$ by a.

The second Stiefel–Whitney class detects $\pi_2(B\mathrm{SO})$, so

$$\begin{aligned} g &= \langle w_2(a^*(E(X,\alpha) \oplus \tau_X)), [S^2] \rangle = \langle w_2(E(X,\alpha) \oplus \tau_X), a_*([S^2]) \rangle \\ &= \langle w_2(E(X,\alpha)) + w_2(\tau_X), \rho([a]) \rangle = \langle \varrho_2(\alpha), \rho([a]) \rangle + \langle w_2(X), \rho([a]) \rangle = 1 + \langle w_2(X), \rho([a]) \rangle, \end{aligned}$$

where $\varrho_2\colon H^2(X) \to H^2(X;\mathbb{Z}/2)$ denotes reduction mod 2 and we used that $E(X,\alpha)$ is a complex line bundle, so $w_1(E(X,\alpha)) = 0$ and $w_2(E(X,\alpha)) = \varrho_2(c_1(E(X,\alpha))) = \varrho_2(\alpha)$ and that $\langle \alpha, \rho([a]) \rangle = 1$.

We already saw that g corresponds to $[S^1, F_2]$, so the statement follows. □

Proposition 3.7 *If F_2 is the (homotopically unique) framing of S^1 such that $[S^1, F_2]$ is the nontrivial element in $\Omega_1^{\mathrm{fr}} \cong \mathbb{Z}/2$, and F_1 is any framing of Σ_{ex}^8, then $[\Sigma_{\mathrm{ex}}^8 \times S^1, F_1 \times F_2] \neq 0 \in \Omega_9^{\mathrm{fr}}$. Moreover, $[\Sigma_{\mathrm{ex}}^8 \times S^1, F_1 \times F_2]$ is not contained in the image of the J-homomorphism $J_9\colon \pi_9(\mathrm{SO}) \to \Omega_9^{\mathrm{fr}}$.*

Proof It follows from [Kervaire and Milnor 1963, Section 4 and Theorem 5.1] that $[\Sigma_{\mathrm{ex}}^8, F_1] \notin \mathrm{Im}\, J_8$. Under the Pontryagin–Thom isomorphism the map $\times[S^1, F_2]\colon \Omega_8^{\mathrm{fr}} \to \Omega_9^{\mathrm{fr}}$ corresponds to $\cdot\eta\colon \pi_8^s \to \pi_9^s$, which is injective by [Toda 1962, page 189 and Theorem 14.1 i)]. By [Adams 1966, Proof of Example 12.15], we have $\operatorname{Im} J_9 = (\operatorname{Im} J_8)\eta$. Therefore

$$\begin{aligned}[\Sigma^8_{ex}, F_1]\times[S^1, F_2] \in (\Omega^{fr}_8 \setminus \operatorname{Im} J_8)\times[S^1, F_2] &= (\Omega^{fr}_8 \times [S^1, F_2]) \setminus (\operatorname{Im} J_8 \times [S^1, F_2]) \\ &= (\Omega^{fr}_8 \times [S^1, F_2]) \setminus \operatorname{Im} J_9 \subseteq \Omega^{fr}_9 \setminus \operatorname{Im} J_9.\end{aligned}$$

In particular $[\Sigma^8_{ex}, F_1]\times[S^1, F_2] \neq 0$. $\square$

Theorem 3.8 *If $X_4(\underline{d})$ is spin, then there is a framing F such that $[S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x), F] \neq 0 \in \Omega^{fr}_9$.*

Proof Let F_0 be a framing of $S(X_4(\underline{d}), x)$ such that $[S(X_4(\underline{d}), x), F_0] = 0$ (see Theorem 3.4). Let F_1 be any framing of Σ^8_{ex}. By Lemma 3.5 there are framings F_2 and F such that

$$[S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x), F] = [S(X_4(\underline{d}), x), F_0] + [\Sigma^8_{ex} \times S^1, F_1 \times F_2] = [\Sigma^8_{ex} \times S^1, F_1 \times F_2].$$

Since x is a generator of $H^2(X_4(\underline{d}))$, there is a generator $[a]$ of $\pi_2(X_4(\underline{d}))$ such that $\langle x, \rho([a])\rangle = 1$, so we can apply Lemma 3.6, and since $X_4(\underline{d})$ is spin, we get that $[S^1, F_2] = 1$. By Proposition 3.7 $[\Sigma^8_{ex} \times S^1, F_1 \times F_2] \neq 0$ and this implies that $[S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x), F] \neq 0$. $\square$

Theorem 3.9 *If $X_4(\underline{d})$ is spin, then, for every framing F, $[S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x), F] \neq 0 \in \Omega^{fr}_9$.*

Proof First we show that $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ is 3-connected. Recall that the embedding $X_4(\underline{d}) \to \mathbb{C}P^{4+k}$ is 4-connected. Therefore we have $\pi_1(X_4(\underline{d})) \cong \pi_3(X_4(\underline{d})) \cong 0$ and $\pi_2(X_4(\underline{d})) \cong \mathbb{Z}$. From the homotopy long exact sequence of the fibration $S^1 \to S(X_4(\underline{d}), x) \to X_4(\underline{d})$, we obtain that $S(X_4(\underline{d}), x)$ is 3-connected. Since $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ is homeomorphic to $S(X_4(\underline{d}), x)$, it is also a 3-connected 9-manifold. This implies that $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ is homotopy equivalent to a CW-complex with cells only in dimensions 0, 4, 5 and 9 (see [Smale 1962, Theorem 6.1]).

Any two framings of $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ differ by a map $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x) \to \mathrm{SO}$. Since $\pi_4(\mathrm{SO}) \cong \pi_5(\mathrm{SO}) \cong 0$, this difference is in fact an element of $\pi_9(\mathrm{SO})$. Changing the framing of the 9-cell by an element of $\pi_9(\mathrm{SO})$ has the same effect on the framed cobordism class as taking connected sum with S^9 with the corresponding framing, which is given by the J-homomorphism $J_9\colon \pi_9(\mathrm{SO}) \to \Omega^{fr}_9$. Therefore the set of cobordism classes in Ω^{fr}_9 represented by $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ (with any framing) is a coset of $\operatorname{Im} J_9$. By Proposition 3.7 and the proof of Theorem 3.8 this coset has an element which is not in $\operatorname{Im} J_9$; therefore it is not the trivial coset. So it does not contain 0; therefore $0 \in \Omega^{fr}_9$ is not represented by $S(X_4(\underline{d}) \sharp \Sigma^8_{ex}, x)$ with any framing. $\square$

Now we can conclude that 4-dimensional spin complete intersections are strongly Θ-flexible.

Theorem 3.10 *If $X_4(\underline{d})$ is spin, then $X_4(\underline{d}) \sharp \Sigma^8_{ex}$ is not diffeomorphic to a complete intersection.*

Proof Suppose that $X_4(\underline{d}) \sharp \Sigma^8_{ex}$ is diffeomorphic to some complete intersection $X_4(\underline{d}')$. The diffeomorphism induces an isomorphism between $H^2(X_4(\underline{d}'))$ and $H^2(X_4(\underline{d}) \sharp \Sigma^8_{ex})$. We may assume that the generator $x \in H^2(X_4(\underline{d}'))$ goes into the generator of $H^2(X_4(\underline{d}) \sharp \Sigma^8_{ex})$ corresponding to x under the isomorphism $H^2(X_4(\underline{d})\sharp\Sigma^8_{ex})\cong H^2(X_4(\underline{d})\setminus\operatorname{int}D^8)\cong H^2(X_4(\underline{d}))$ (see the proof of Lemma 3.5). This is because $X_4(\underline{d}')$ has a self-diffeomorphism which sends x to $-x$ (see the proof of Proposition 2.10). This implies that $S(X_4(\underline{d})\sharp\Sigma^8_{ex},x)$ is diffeomorphic to $S(X_4(\underline{d}'),x)$.

By Theorem 3.4, $S(X_4(\underline{d}'),x)$ has a framing F_0 such that $(S(X_4(\underline{d}'),x),F_0)$ is framed nullcobordant, but by Theorem 3.9 $S(X_4(\underline{d})\sharp\Sigma^8_{ex},x)$ does not have such a framing, so they are not diffeomorphic. This contradiction shows that $X_4(\underline{d})\sharp\Sigma^8_{ex}$ is not diffeomorphic to any complete intersection $X_4(\underline{d}')$. □

4 The nonspin cases with even total degree

In this section we prove Theorems 1.12 and 1.14 (a). Both of these results rely on the computation of $\operatorname{Tors}\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)\cong\mathbb{Z}/4$ in Lemma 4.2 below, where ξ^1 denotes the (unique up to isomorphism) nontrivial stable bundle over $\mathbb{C}P^1=S^2$. Note that if ξ is a stable bundle over $\mathbb{C}P^\infty$ with $w_2(\xi)\neq 0$, then its restriction to $\mathbb{C}P^1$ is isomorphic to ξ^1.

4.1 The computation of $\operatorname{Tors}\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)$

We first establish the necessary background to state and prove Lemma 4.2. Let $\mathbb{S}^0$ denote the sphere spectrum and write $\mathbb{S}^k$ for the k-fold suspension of $\mathbb{S}^0$. We let $\eta\colon\mathbb{S}^1\to\mathbb{S}^0$ denote the generator of the 1-stem $\pi_1^s\cong\mathbb{Z}/2$, and C_η the cofibre of η. Since ξ^1 is the nontrivial stable bundle over $\mathbb{C}P^1$, the Thom spectrum of ξ^1 is given by $\operatorname{Th}(\xi^1)\simeq C_\eta$, and the Pontryagin–Thom map for $\Omega_*^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)$ is an isomorphism

$$\mathrm{PT}\colon\Omega_*^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)\to\pi_*(MO\langle 8\rangle\wedge\operatorname{Th}(\xi^1))\cong\pi_*(MO\langle 8\rangle\wedge C_\eta),$$

where $\wedge$ denotes the smash product. Smashing the cofibration $\mathbb{S}^0\to C_\eta\to\mathbb{S}^2$ with $MO\langle 8\rangle$ and taking homotopy groups, we obtain the long exact sequence

$$(1)\quad \cdots\to\pi_7(MO\langle 8\rangle)\xrightarrow{\eta_*}\pi_8(MO\langle 8\rangle)\to\pi_8(MO\langle 8\rangle\wedge C_\eta)\to\pi_6(MO\langle 8\rangle)\xrightarrow{\eta_*}\pi_7(MO\langle 8\rangle)\to\cdots.$$

We shall need some basic facts about the low-dimensional string bordism groups $\Omega_*^{O\langle 7\rangle}\cong\pi_*(MO\langle 8\rangle)$ and the natural forgetful map $F\colon\Omega_*^{\mathrm{fr}}\to\Omega_*^{O\langle 7\rangle}$. These facts can be deduced from results of [Giambalvo 1971], and we also give a direct proof below.

Lemma 4.1 (cf [Giambalvo 1971]) *The natural map $F\colon\Omega_*^{\mathrm{fr}}\to\Omega_*^{O\langle 7\rangle}$ satisfies:*

- (a) $\Omega_6^{O\langle 7\rangle}\cong\mathbb{Z}/2$ *and* $F\colon\Omega_6^{\mathrm{fr}}\to\Omega_6^{O\langle 7\rangle}$ *is an isomorphism.*
- (b) $\Omega_7^{O\langle 7\rangle}\cong 0$.
- (c) $\Omega_8^{O\langle 7\rangle}\cong\mathbb{Z}/2\oplus\mathbb{Z}$ *and* $F\colon\Omega_8^{\mathrm{fr}}\to\Omega_8^{O\langle 7\rangle}$ *has image* $\mathbb{Z}/2$ *and kernel the image of J-homomorphism* $J_8\colon\pi_8(\mathrm{SO})\to\pi_8^s\cong\Omega_8^{\mathrm{fr}}\cong(\mathbb{Z}/2)^2$.

Proof Under the Pontryagin–Thom isomorphism, the map $F\colon\Omega_*^{\mathrm{fr}}\to\Omega_*^{O\langle 7\rangle}$ corresponds to the map on homotopy groups induced by the inclusion of the Thom cell $\mathbb{S}^0\to MO\langle 8\rangle$. To compute this map, we first replace $MO\langle 8\rangle$ with a simpler spectrum. Let $f_H\colon S^8 \to BO\langle 8\rangle$ represent a generator of $\pi_8(BO\langle 8\rangle) \cong \mathbb{Z}$ and let ζ_H be the stable vector bundle over S^8 classified by f_H.

The Thom spectrum of any vector bundle over an m-sphere is the cofibre of a map $\mathbb{S}^{m-1} \to \mathbb{S}^0$, and it was Milnor [1958, Lemma 1] who first observed that this map is obtained by applying the stable J-homomorphism to the clutching function of the bundle. Hence the Thom spectrum of ζ_H is $\mathrm{Th}(\zeta_H) \simeq C_{\bar\sigma}$, where $\bar\sigma\colon \mathbb{S}^7 \to \mathbb{S}^0$ is given by applying the J-homomorphism to the clutching function of ζ_H, which generates $\pi_7(\mathrm{SO}\langle 7\rangle) = \pi_7(\mathrm{SO})$.

By construction, f_H induces an isomorphism on π_8. We have $\pi_9(S^8) \cong \pi_1^s \cong \mathbb{Z}_2$ and $\pi_{10}(S^8) \cong \pi_2^s \cong \mathbb{Z}_2$, generated by η and η^2 respectively, so by Bott periodicity and [Adams 1966, Proof of Example 12.5], f_H also induces isomorphisms on π_9 and π_{10}. Hence f_H is 10-connected, and so the induced map of Thom spectra

$$\mathrm{Th}(\zeta_H) \simeq C_{\bar\sigma} \to MO\langle 8\rangle$$

is also 10-connected. Hence in dimensions $* \le 9$ the map $F\colon \Omega_*^{\mathrm{fr}} \to \Omega_*^{O\langle 7\rangle}$ is isomorphic to the map on homotopy groups induced by the inclusion $\mathbb{S}^0 \to C_{\bar\sigma}$.

The cofibration $\mathbb{S}^0 \to C_{\bar\sigma} \to \mathbb{S}^8$ leads to a long exact sequence

$$\cdots \to \pi_1^s \xrightarrow{\bar\sigma_*} \pi_8^s \to \pi_8(C_{\bar\sigma}) \to \pi_0^s \xrightarrow{\bar\sigma_*} \pi_7^s \to \pi_7^s(C_{\bar\sigma}) \to 0 \to \pi_6^s \to \pi_6(C_{\bar\sigma}) \to 0 \to \cdots.$$

We see immediately that $\pi_6^s \to \pi_6(C_{\bar\sigma})$ is an isomorphism, and so $F\colon \Omega_6^{\mathrm{fr}} \to \Omega_6^{O\langle 7\rangle}$ is an isomorphism. Since $\Omega_6^{\mathrm{fr}} \cong \pi_6^s \cong \mathbb{Z}/2$ (where the last isomorphism is given in [Toda 1962, Chapter XIV]), this proves part (a). For part (b), we use that $J_7\colon \pi_7(\mathrm{SO}) \to \pi_7^s$ is onto by [Adams 1966, Example 7.17], and so $\bar\sigma$ generates π_7^s. Hence $\bar\sigma_*\colon \pi_0^s \to \pi_7^s$ is surjective, which proves part (b). For part (c), since $\pi_0^s \cong \mathbb{Z}$ and π_7^s is finite, $\mathrm{Ker}(\bar\sigma_*\colon \pi_0^s \to \pi_7^s) \cong \mathbb{Z}$. We also have $\mathrm{Im}(\bar\sigma_*\colon \pi_1^s \to \pi_8^s) = \langle \eta\bar\sigma\rangle = \mathrm{Im}(J_8)$ (using [Adams 1966, Proof of Example 12.15]). By Toda's calculations [1962, Chapter XIV], $\pi_8^s \cong (\mathbb{Z}/2)^2$ with $\eta\bar\sigma \ne 0$, and this finishes the proof of part (c). □

From the exact sequence (1) and Lemma 4.1 (b) we deduce that there is a short exact sequence

$$0 \to \Omega_8^{O\langle 7\rangle} \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \to \Omega_6^{O\langle 7\rangle} \to 0. \tag{2}$$

Noting that $\Omega_6^{\mathrm{fr}} \cong \Omega_6^{O\langle 7\rangle} \cong \mathbb{Z}/2$ is detected by the Arf invariant, it is easy to see that the homomorphism $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \to \Omega_6^{O\langle 7\rangle}$ can be identified with the codimension-2 Arf invariant

$$A_{\mathbb{C}P^1}\colon \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \to \mathbb{Z}/2,$$

which is defined by making a normal map $(g,\bar g)\colon M \to S^2$ transverse to a point $* \in S^2$ and taking the Arf invariant of the resulting 6-manifold $g^{-1}(*)$, which is canonically framed.

Lemma 4.2 *There is a nonsplit short exact sequence of abelian groups*

$$0 \to \Theta_8 \to \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \xrightarrow{A_{\mathbb{C}P^1}} \mathbb{Z}/2 \to 0.$$

In particular $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \cong \mathbb{Z}/4$.

Proof There is a natural forgetful map $F^1\colon \Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1) \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)$ and the exact sequence of (2) forms part of the following commutative diagram:

$$
\begin{tikzcd}
\Omega_7^{\mathrm{fr}} \arrow[r, "\eta_*"] & \Omega_8^{\mathrm{fr}} \arrow[r] \arrow[d, "F"] & \Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1) \arrow[r, "A^{\mathrm{fr}}_{\mathbb{C}P^1}"] \arrow[d, "F^1"] & \mathbb{Z}/2 \arrow[r, "\eta_*"] \arrow[d, equal] & \Omega_7^{\mathrm{fr}} \\
0 \arrow[r] & \Omega_8^{O\langle 7\rangle} \arrow[r] & \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \arrow[r, "A_{\mathbb{C}P^1}"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}
\tag{3}
$$

Here $A^{\mathrm{fr}}_{\mathbb{C}P^1}\colon \Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1) \to \mathbb{Z}/2$ is a codimension-2 Arf invariant, which is defined analogously to the codimension-2 Arf invariant on $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)$. We shall first compute $\Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1)$ and we do this via the Pontryagin–Thom isomorphism

$$\Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1) \cong \pi_8^s(C_\eta).$$

The cofibration $\mathbb{S}^0 \to C_\eta \to \mathbb{S}^2$ leads to the following long exact sequence (showing in particular that the top row of diagram (3) is also exact):

$$\cdots \to \pi_7^s \xrightarrow{\eta_*} \pi_8^s \to \pi_8(C_\eta) \to \pi_6^s \xrightarrow{\eta_*} \pi_7^s \to \cdots .$$

From Toda's calculations [1962, Chapter XIV], we have $\pi_6^s \cong \mathbb{Z}/2(\nu^2)$, $\pi_7^s \cong \mathbb{Z}/240(\sigma)$, $\pi_8^s \cong \mathbb{Z}/2(\eta\sigma) \oplus \mathbb{Z}/2(\epsilon)$, where $\nu \in \pi_3^s$ is a generator and $\eta\nu \in \pi_4^s = \{0\}$. It follows that $\eta_*\colon \pi_6^s \to \pi_7^s$ is the zero map and that there is a short exact sequence

$$0 \to \mathbb{Z}/2([\epsilon]) \to \pi_8(C_\eta) \to \mathbb{Z}/2 \to 0, \tag{4}$$

where $[\epsilon] \in \pi_8^s/\eta_*(\pi_7^s)$ denotes the equivalence class of ϵ. By Lemma A.1 from the appendix, the extension (4) is determined by the Toda bracket

$$\langle \eta, \nu^2, 2\rangle \subset \pi_8^s.$$

By [loc. cit., Proposition 3.4 ii], there is a Jacobi identity for Toda brackets,

$$0 \in \langle \eta, \nu^2, 2\rangle + \langle 2, \eta, \nu^2\rangle + \langle \nu^2, 2, \eta\rangle,$$

where we have ignored signs since all the Toda brackets consist of elements of order 2 or 1. Now by [loc. cit., Proposition 1.2], $\langle 2, \eta, \nu^2\rangle \subseteq \langle 2, \eta, \nu\rangle\nu$. Since $\langle 2, \eta, \nu\rangle \subset \pi_5^s = \{0\}$, we have $\langle 2, \eta, \nu\rangle\nu = \{0\}$ and so $\langle 2, \eta, \nu^2\rangle = \{0\}$. By [loc. cit., page 189], $\langle \nu^2, 2, \eta\rangle = \{\epsilon, \epsilon + \eta\sigma\}$. It follows that $\langle \eta, \nu^2, 2\rangle = \{\epsilon, \epsilon+\eta\sigma\}$ is nontrivial and maps to the generator $[\epsilon] \in \pi_8^s/\eta_*(\pi_7^s)$. Applying Lemma A.1, we deduce that the extension (4) is nontrivial and hence is isomorphic to the extension

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

The above shows that $\Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1) \cong \mathbb{Z}/4$.

In diagram (3) we can replace the top row with the short exact sequence (4) (noting that, as we saw in the proof of Lemma 4.1, $\mathrm{Im}(J_8) = \eta_*(\pi_7^s)$ and $\mathrm{coker}(J_8) = \Theta_8$) and restrict the bottom row to the torsion subgroups, to get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Theta_8 \arrow[r] \arrow[d, "F"] & \Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1) \arrow[r, "A^{\mathrm{fr}}_{\mathbb{C}P^1}"] \arrow[d, "F^1"] & \mathbb{Z}/2 \arrow[r, "\eta_*"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle} \arrow[r] & \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \arrow[r, "A_{\mathbb{C}P^1}"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

We check that the bottom row is exact. The map $A_{\mathbb{C}P^1}$ is surjective by the commutativity of the diagram, while exactness at $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}$ and $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)$ follows from the exactness of the original sequence. Now by Lemma 4.1 (c) the map $F\colon \Theta_8 \to \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}$ is an isomorphism. So, by the five lemma, the homomorphism $F^1\colon \Omega_8^{\mathrm{fr}}(\mathbb{C}P^1;\xi^1) \to \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1)$ is also an isomorphism, which completes the proof. $\square$

4.2 The nonspin case with $v_4(X_4(\underline{d})) = 0$

Let $X_4(\underline{d})$ be a nonspin complete intersection with $v_4(X_4(\underline{d})) = 0$. We will prove that $X_4(\underline{d})$ is Θ-rigid. As explained in Section 2.3, it is enough to show that the canonical homomorphism $i_0\colon \Theta_8 \cong \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle} \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi_4(\underline{d}))$ is trivial. We will exploit the fact that i_0 factors through the group $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi_4(\underline{d})|_{\mathbb{C}P^1})$.

Proposition 4.3 *Let ξ be a stable bundle over $\mathbb{C}P^\infty$ such that $w_2(\xi) \neq 0$ and $w_4(\xi) = 0$. Then the natural map $i_0\colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$ is trivial.*

Proof By [Fang and Klaus 1996, Section 2.2] we have $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty,*;\xi) \cong \mathbb{Z}$. From the exactness of the sequence

$$\cdots \to \Omega_8^{O\langle 7\rangle} \xrightarrow{j_0} \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi) \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty,*;\xi) \to \cdots$$

we deduce that the image of j_0 contains the torsion subgroup of $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$. The signature defines nontrivial homomorphisms $\Omega_8^{O\langle 7\rangle} \to \mathbb{Z}$ and $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi) \to \mathbb{Z}$ which commute with j_0. By Lemma 4.1 (c), $\Omega_8^{O\langle 7\rangle} \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and so j_0 is rationally injective. Therefore its restriction to $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle} \cong \Theta_8$ is surjective onto $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$. Thus if $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$ has a nontrivial torsion element, then it has order 2.

Let $i_{\mathbb{C}P^1*}\colon \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \to \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$ denote the homomorphism induced by the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$. By Lemma 4.2, $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \cong \mathbb{Z}/4$ and if a denotes a generator, then Σ^8_{ex} represents $2a$. Therefore $i_0(\Sigma^8_{ex}) = i_{\mathbb{C}P^1*}(2a) = 2i_{\mathbb{C}P^1*}(a) = 0$. $\square$

Theorem 4.4 *If $X_4(\underline{d})$ is a nonspin complete intersection with $v_4(X_4(\underline{d})) = 0$ and $\underline{d} \neq \{2,2\}$, then $X_4(\underline{d})$ and $X_4(\underline{d}) \sharp \Sigma^8_{ex}$ are diffeomorphic.*

Proof In this case $w_2(\xi_4(\underline{d})) \neq 0$ and $w_4(\xi_4(\underline{d})) = 0$ (see Proposition 2.8), so by Proposition 4.3 the canonical homomorphism $i_0\colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi_4(\underline{d}))$ is trivial. Therefore $X_4(\underline{d})$ and $X_4(\underline{d}) \sharp \Sigma^8_{ex}$ admit bordant normal 3-smoothings over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$. By Proposition 1.9, $X_4(\underline{d})$ and $X_4(\underline{d}) \sharp \Sigma^8_{ex}$ are diffeomorphic. $\square$

4.3 The nonspin case with $v_4(X_4(\underline{d})) \neq 0$ and even total degree

Now we suppose that $v_4(X_4(\underline{d})) \neq 0$ and the total degree d is even. By Remark 2.9 this implies that $\nu_2(d) \geq 4$ (where $\nu_2(d)$ denotes the exponent of 2 in the prime factorisation of d). We will apply the Adams filtration argument of Kreck and Traving [Kreck 1999, Section 8]. If ξ is a stable vector bundle over $\mathbb{C}P^\infty$, we use the Pontryagin–Thom isomorphism to identify the groups $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi) = \pi_8(MO\langle 8\rangle \wedge \mathrm{Th}(\xi))$ and hence their torsion subgroups. In this way we obtain an Adams filtration on $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$. Recall that a map $\mathbb{S}^i \to \mathbb{E}$ representing a torsion class in $\pi_i(\mathbb{E})$, the i^{th} homotopy group of a spectrum $\mathbb{E}$, has Adams filtration $\geq k$ if it can be factored as a composition of k maps, each of which is trivial on homology with $\mathbb{Z}/2$ coefficients.

We will need the following improvement of Kreck and Traving's vanishing result in dimension 8.

Lemma 4.5 *Let ξ be a stable bundle over $\mathbb{C}P^\infty$ such that $w_2(\xi) \neq 0$, $w_4(\xi) \neq 0$ and the homomorphism $i_0\colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$ is injective. Then the only element of $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$ with Adams filtration 4 or higher is the trivial element.*

Proof Consider the exact sequence

$$\cdots \to \Omega_8^{O\langle 7\rangle} \xrightarrow{j_0} \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi) \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty,*;\xi) \to \cdots.$$

Fang and Klaus [1996, Section 2.4] proved that $\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty,*;\xi) \cong \mathbb{Z}\oplus\mathbb{Z}/2$, where the $\mathbb{Z}/2$ summand is detected by the codimension-2 Arf invariant. Hence we have the following commutative diagram between exact sequences (the bottom sequence is exact, because j_0 is rationally injective, as in the proof of Proposition 4.3):

$$\begin{tikzcd}
\Theta_8 \arrow[r] \arrow[d, "\cong"] & \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \arrow[r, "A_{\mathbb{C}P^1}"] \arrow[d, "i_{\mathbb{C}P^1*}"] & \mathbb{Z}/2 \arrow[d, "="] \\
\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle} \arrow[r, "i_0"] & \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi) \arrow[r, "A"] & \mathbb{Z}/2
\end{tikzcd}$$

Moreover, the top sequence is short exact by Lemma 4.2. The bottom sequence is also short exact (the surjectivity of A follows from the commutativity of the diagram and the injectivity of i_0 was assumed). It follows that $i_{\mathbb{C}P^1*}\colon \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^1;\xi^1) \to \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi)$ is an isomorphism.

Now we choose a generator $a \in \mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi) \cong \mathbb{Z}/4$ and let $[f_a] \in \pi_8(MO\langle 8\rangle \wedge \mathrm{Th}(\xi))$ represent the image of a under the Pontryagin–Thom isomorphism. By [Fang and Klaus 1996, page 144], the image of $[f_a]$ in the group $\pi_8(MO\langle 8\rangle \wedge (\mathrm{Th}(\xi)/\mathbb{S}^0))$ has Adams filtration 2. Since the Adams filtration cannot decrease under composition, $[f_a]$ has Adams filtration ≤ 2. Therefore $2a$, corresponding to $2[f_a]$ under the Pontryagin–Thom isomorphism, has Adams filtration ≤ 3. Since $2a$ is a multiple of every nonzero element of $\mathrm{Tors}\,\Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty;\xi) \cong \mathbb{Z}/4$, the only element with Adams filtration 4 or higher is the trivial element. □

Proposition 4.6 *Let $X_4(\underline{d})$ and $X_4(\underline{d}')$ be nonspin complete intersections with $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$ and $v_4(X_4(\underline{d})) \neq 0$. If the total degree d is even, then $X_4(\underline{d})$ and $X_4(\underline{d}')$ are diffeomorphic.*

Proof By Theorem 1.4 there is a homotopy sphere $\Sigma \in \Theta_8$ such that $X_4(\underline{d}) \approx X_4(\underline{d}') \sharp \Sigma$. By Proposition 2.8 we have $w_2(\xi_4(\underline{d})) \neq 0$ and $w_4(\xi_4(\underline{d})) \neq 0$. Again we consider the natural map $i_0 \colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$.

If i_0 is zero, then $X_4(\underline{d}')$ and $X_4(\underline{d}') \sharp \Sigma$ are diffeomorphic by the same argument as in the proof of Theorem 4.4. Hence $X_4(\underline{d})$ and $X_4(\underline{d}')$ are diffeomorphic.

Now suppose that $i_0 \colon \Theta_8 \to \Omega_8^{O\langle 7\rangle}(\mathbb{C}P^\infty; \xi_4(\underline{d}))$ is nonzero (hence injective). The arguments of Kreck and Traving [Kreck 1999, Section 8] show that $X_4(\underline{d})$ and $X_4(\underline{d}')$ admit normal 3-smoothings over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$ whose bordism classes differ by a torsion element of Adams filtration $\nu_2(d)$ or higher. Since d is even, we have $\nu_2(d) \geq 4$ (see Remark 2.9), so by Lemma 4.5 any such torsion element is trivial. Hence $X_4(\underline{d})$ and $X_4(\underline{d}')$ admit bordant normal 3-smoothings over $(B_4, \xi_4(\underline{d}) \times \gamma_{BO\langle 8\rangle})$ and so by Proposition 1.9, $X_4(\underline{d})$ and $X_4(\underline{d}')$ are diffeomorphic. □

5 The case of odd total degree

It remains then to consider the case where the total degree d is odd. Note that in general this case is not Θ-rigid as the following theorem of Kasilingam shows.

Theorem 5.1 [Kasilingam 2016, Remark 2.6 (1)] *$\mathbb{C}P^4$ is not diffeomorphic to $\mathbb{C}P^4 \sharp \Sigma^8_{ex}$.*

To prove the Sullivan conjecture for $X_4(\underline{d})$ when d is odd, we find a new way to compare normal bordism classes for $X_4(\underline{d})$ and $X_4(\underline{d}')$, which is one of the main achievements of this paper. In particular, we believe that introducing the Hambleton–Madsen theory [1986] of degree-d normal invariants will provide a new perspective on the Sullivan conjecture in all dimensions.

5.1 Degree-r normal maps and their normal invariants

In this subsection we review the surgery classification of bordism classes of degree-r normal maps for any integer r. Our treatment follows [Hambleton and Madsen 1986] but with minor modifications to suit our setting. We will assume that all manifolds and all bundles are oriented and that all bundle maps are orientation-preserving. We also choose to work with stable normal bundles in the source of normal maps, as opposed to stable tangent bundles, and for simplicity, we only formulate the statements in the special case when the target space of a degree-r normal map is a closed smooth connected oriented m-manifold P.

Definition 5.2 Let M and P be closed smooth oriented m-manifolds and assume that P is connected. For $r \in \mathbb{Z}$, a degree-r normal map $(f, \bar{f}) \colon M \to P$ is a map of stable vector bundles

$$\begin{array}{ccc} \nu_M & \xrightarrow{\bar{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & P \end{array}$$

from the stable normal bundle of M to some stable vector bundle over P such that $f \colon M \to P$ has degree r.

When $r = \pm 1$, then ξ is a vector bundle reduction of the Spivak normal fibration of P, but in general this only holds away from r. Normal bordism of degree-r normal maps is defined analogously to normal bordism of degree-1 normal maps [Wall 1970, Proposition 10.2]: the normal maps $(f, \bar{f})\colon (M, \nu_M) \to (P, \xi)$ and $(f', \bar{f}')\colon (M', \nu_{M'}) \to (P, \xi')$ are normally bordant if there is an isomorphism $\alpha\colon \xi' \to \xi$ and a bordism between $(f, \bar{f})$ and $(f', \alpha \circ \bar{f}')$ over (P, ξ).

Definition 5.3 We denote the set of normal bordism classes of degree-r normal maps to P by $\mathcal{N}_r^+(P)$, where the superscript "+" indicates that we are working in the oriented setting.

For a fixed ξ, let $\Omega_m^{\mathrm{fr}}(P;\xi)_r$ denote the subset of $\Omega_m^{\mathrm{fr}}(P;\xi)$ consisting of bordism classes whose representatives have degree r. The group of stable bundle automorphisms of ξ, $\mathrm{Aut}(\xi)$, acts on $\Omega_m^{\mathrm{fr}}(P;\xi)_r$ by postcomposition. Let $\mathcal{N}_r^+(P,\xi) \subseteq \mathcal{N}_r^+(P)$ denote the subset of normal bordism classes that are representable by normal maps to (P,ξ). Then we have a canonical bijection

$$\mathcal{N}_r^+(P,\xi) \equiv \Omega_m^{\mathrm{fr}}(P;\xi)_r/\mathrm{Aut}(\xi).$$

Moreover,

$$\mathcal{N}_r^+(P) = \bigsqcup_{[\xi]} \mathcal{N}_r^+(P,\xi),$$

where we take the union over the isomorphism classes of stable bundles over P which admit degree-r normal maps. To distinguish degree-r normal bordism classes from usual bordism classes we use

Notation 5.4 We denote the bordism class of $(f, \bar{f})$ in $\Omega_m^{\mathrm{fr}}(P;\xi)_r$ by $[f, \bar{f}]_\xi$ and in $\mathcal{N}_r^+(P)$ by $[f, \bar{f}]$.

As in the degree-1 case, the computation of $\mathcal{N}_r^+(P)$ proceeds via fibrewise degree-r maps between vector bundles. Recall that for a bundle ζ, the total space is denoted by $E\zeta$, the disc bundle is $D\zeta$, the sphere bundle is $S\zeta$ and the projection is π_ζ. For a space Y with oriented vector bundles ζ and θ of the same rank over Y, we consider fibrewise maps

$$\begin{tikzcd} S\zeta \arrow[rr, "g"] \arrow[dr] & & S\theta \arrow[dl] \\ & Y & \end{tikzcd}$$

between the associated sphere bundles, where the restriction of g to each fibre has degree r. Given a fibrewise degree-r_1 map $g_1\colon S\zeta_1 \to S\theta_1$ and a fibrewise degree-r_2 map $g_2\colon S\zeta_2 \to S\theta_2$, their fibrewise join is a fibrewise degree-$r_1 r_2$ map $g_1 * g_2\colon S(\zeta_1 \oplus \zeta_2) \to S(\theta_1 \oplus \theta_2)$ between the spheres bundles of the Whitney sums of the original bundles. An isomorphism between two fibrewise degree-r maps, $g_i\colon S\zeta_i \to S\theta_i$, $i = 0, 1$, is a pair of vector bundle isomorphisms $\alpha\colon \zeta_0 \to \zeta_1$ and $\beta\colon \theta_0 \to \theta_1$ such that the following diagram commutes up to fibre homotopy over Y:

$$\begin{tikzcd} S\zeta_0 \arrow[r, "g_0"] \arrow[d, "S\alpha"'] & S\theta_0 \arrow[d, "S\beta"] \\ S\zeta_1 \arrow[r, "g_1"] & S\theta_1 \end{tikzcd}$$

where $S\alpha$ and $S\beta$ are the induced maps of sphere bundles.

Definition 5.5 Two fibrewise degree-r maps are equivalent if they become isomorphic after fibrewise join with the restriction of a vector bundle isomorphism (ie stabilisation), and we define

$$\mathcal{F}_r^+(Y) := \{g \colon S\zeta \to S\theta\}/\sim$$

to be the set of equivalence classes of fibrewise degree-r maps of vector bundles over Y. The equivalence class of g is denoted by $[g]$.

We now review how taking the transverse inverse image of the zero section is used to define a map $T \colon \mathcal{F}_r^+(P) \to \mathcal{N}_r^+(P)$. If $g \colon S\zeta \to S\theta$ represents an element $[g] \in \mathcal{F}_r^+(P)$, then we can extend it to a fibre-preserving map $f \colon D\zeta \to D\theta$ that is transverse to the zero section $P \subset D\theta$. We set $M := f^{-1}(P)$ and $f_M := f|_M$. Since g has degree r, the map $f_M \colon M \to P$ has degree r too. The map f determines a bundle map $\bar{f}_0 \colon \nu(M \to D\zeta) \to \nu(P \to D\theta) \cong \theta$ over f_M. We have

$$\nu_M \cong \nu(M \to D\zeta) \oplus \nu_{D\zeta}|_M \cong \nu(M \to D\zeta) \oplus (\pi_\zeta|_M)^*(\nu_P \ominus \zeta) = \nu(M \to D\zeta) \oplus f_M^*(\nu_P \ominus \zeta).$$

By adding the canonical map $f_M^*(\nu_P \ominus \zeta) \to \nu_P \ominus \zeta$ to $\bar{f}_0$, we get a bundle map

$$\bar{f}_M \colon \nu_M \to \theta \oplus \nu_P \ominus \zeta$$

over f_M. Then we define $T([g]) := [f_M, \bar{f}_M]$. For the case when P is a smooth manifold (which we have assumed for simplicity), the following theorem is the oriented version of a foundational result of Hambleton and Madsen on degree-r normal maps (their proof applies verbatim in the oriented setting).

Theorem 5.6 (cf [Hambleton and Madsen 1986, Theorem 2.2]) *The map $T \colon \mathcal{F}_r^+(P) \to \mathcal{N}_r^+(P)$ is a well-defined bijection.* □

Remark 5.7 In [Hambleton and Madsen 1986] the source manifold M is defined as the inverse image under g of a section of $S\theta$. The construction above can be seen as a special case of this via stabilisation, as we now explain. Let $\underline{\mathbb{R}} := (\mathbb{R} \times P \to P)$ denote the trivial rank-1 bundle over P and let $\underline{S}^0 := S\,\underline{\mathbb{R}}$. If $\zeta = \zeta' \oplus \underline{\mathbb{R}}$, $\theta = \theta' \oplus \underline{\mathbb{R}}$ and $g = g' * \mathrm{Id}_{\underline{S}^0} \colon S\zeta = S\zeta' * \underline{S}^0 \to S\theta = S\theta' * \underline{S}^0$ for some ζ', θ' and $g' \colon S\zeta' \to S\theta'$, then $S\zeta = D_+\zeta' \cup_{S\zeta'} D_-\zeta'$ (where $D_\pm\zeta'$ are two copies of the disc bundle $D\zeta'$), $S\theta = D_+\theta' \cup_{S\theta'} D_-\theta'$, $f' := g|_{D_+\zeta'} \colon D_+\zeta' \to D_+\theta'$ is an extension of g', and the zero section of $D_+\theta'$ coincides with the section $1 \times P \subset S^0 \times P = \underline{S}^0 \subset S\theta' * \underline{S}^0 = S\theta$ of $S\theta$.

In order to apply Theorem 5.6 we need to be able to compute $\mathcal{F}_r^+(P)$. The assignment $Y \mapsto \mathcal{F}_r^+(Y)$ is a homotopy functor from the category of spaces to the category of sets. By Brown representability [1962], this functor (restricted to CW-complexes) is represented by a classifying space.

Definition 5.8 The classifying space of the functor $\mathcal{F}_r^+$ is denoted by $(\mathrm{QS}^0/\mathrm{SO})_r$, and the canonical bijection from $\mathcal{F}_r^+(Y)$ to $[Y, (\mathrm{QS}^0/\mathrm{SO})_r]$ is denoted by

$$\mathrm{Br} \colon \mathcal{F}_r^+(Y) \to [Y, (\mathrm{QS}^0/\mathrm{SO})_r].$$

Remark 5.9 Hambleton and Madsen [1986, Section 1] and Brumfiel and Madsen [1976, Section 4] allow orientation-reversing bundle isomorphisms when they define the equivalence relation on fibrewise degree-r maps. They denote the corresponding classifying space by $(\mathrm{QS}^0/O)_r$ for $r \geq 0$. The forgetful map induces a map $(\mathrm{QS}^0/\mathrm{SO})_r \to (\mathrm{QS}^0/O)_{|r|}$ of classifying spaces, which is a homotopy equivalence when $r \neq 0$ and a nontrivial $(\mathbb{Z}/2)$-covering when $r = 0$ (see [Brumfiel and Madsen 1976, Proposition 4.3]).

When $r = 1$, we may identify $(\mathrm{QS}^0/\mathrm{SO})_1 = G/O$, where G/O is the homotopy fibre of the canonical map $B\mathrm{SO} \to B\mathrm{SG}$, the forgetful map from the classifying space of stable vector bundles to the classifying space of stable spherical fibrations.

The equivalence Br and Theorem 5.6 combine to give the following important definition.

Definition 5.10 Let $\eta\colon \mathcal{N}_r^+(P) \to [P, (\mathrm{QS}^0/\mathrm{SO})_r]$ denote the composition $\mathrm{Br} \circ T^{-1}$. For a degree-r normal map $(f, \bar{f})\colon M \to P$, the homotopy class $\eta([f, \bar{f}]) \in [P, (\mathrm{QS}^0/\mathrm{SO})_r]$ is called the *normal invariant* of $(f, \bar{f})$.

We shall need two classes of examples of fibrewise degree-r maps. The *trivial degree-r map* of rank k (well-defined up to fibre homotopy) is $h \times \mathrm{Id}_Y : S^{k-1} \times Y \to S^{k-1} \times Y$ for some degree-r map $h\colon S^{k-1} \to S^{k-1}$. For the second class of degree-r maps, ζ has real rank 2, and we regard ζ as a complex line bundle over Y. Setting $\theta := \zeta^r$ to be the r-fold complex tensor product of ζ with itself, we have the *canonical degree-r map*

$$t_r(\zeta)\colon S\zeta \to S\zeta^r, \quad v \mapsto v^r = v \otimes v \otimes \cdots \otimes v.$$

For the classification of complete intersections the universal examples of such maps, where $Y = \mathbb{C}P^n$ or $\mathbb{C}P^\infty$ and $\zeta = \gamma|_{\mathbb{C}P^n}$ or γ, will play a central role.

Definition 5.11 For a k-tuple of integers $\underline{r} = (r_1, \dots, r_k)$ with $r = r_1 r_2 \cdots r_k$ set

$$\eta_n(\underline{r}) := \mathrm{Br}([t_{r_1}(\gamma|_{\mathbb{C}P^n}) * \cdots * t_{r_k}(\gamma|_{\mathbb{C}P^n})]) \in [\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_r],$$
$$\eta_\infty(\underline{r}) := \mathrm{Br}([t_{r_1}(\gamma) * \cdots * t_{r_k}(\gamma)]) \in [\mathbb{C}P^\infty, (\mathrm{QS}^0/\mathrm{SO})_r].$$

The notation in Definition 5.11 is designed to match Theorem 5.19, which states that the complete intersection $X_n(\underline{d})$ admits a degree-d normal map $(f_n(\underline{d}), \bar{f}_n(\underline{d}))\colon X_n(\underline{d}) \to \mathbb{C}P^n$ such that $\eta([f_n(\underline{d}), \bar{f}_n(\underline{d})]) = \eta_n(\underline{d})$.

5.2 The space $(\mathrm{QS}^0/\mathrm{SO})_r$ and connected sums

In order to apply Theorem 5.6 we will need to make computations with the set of normal invariants $[P, (\mathrm{QS}^0/\mathrm{SO})_r]$. For this we need information about the space $(\mathrm{QS}^0/\mathrm{SO})_r$, and we first adapt the discussion of the related space $(\mathrm{QS}^0/O)_r$ from [Brumfiel and Madsen 1976, Section 4] to the oriented setting. When $r = 1$, the space $(\mathrm{QS}^0/\mathrm{SO})_1 = G/O$ has been extensively studied. In general, Brumfiel and Madsen [1976, Proposition 4.3] showed that there is a fibration sequence

$$\mathrm{QS}^0_r \xrightarrow{i_r} (\mathrm{QS}^0/\mathrm{SO})_r \xrightarrow{\delta_r} B\mathrm{SO}, \tag{5}$$

where the map $\delta_r\colon (\mathrm{QS}^0/\mathrm{SO})_r \to B\mathrm{SO}$ classifies taking the formal difference of the source and target vector bundles of a fibrewise degree-r map, QS^0_r is the space of stable degree-r self maps of the sphere, which classifies fibrewise degree-r self-maps of trivialised vector bundles and $i_r\colon \mathrm{QS}^0_r \to (\mathrm{QS}^0/\mathrm{SO})_r$ classifies forgetting that the bundles are trivialised. The space QS^0_1 is often denoted by SG. There is also a map $\cdot r\colon G/O \to (\mathrm{QS}^0/\mathrm{SO})_r$, which classifies taking fibrewise join with the trivial degree-r map and which fits into the following map of fibration sequences:

$$
\begin{tikzcd}
\mathrm{SG} \arrow[r, "i_1"] \arrow[d, "\cdot r"'] & G/O \arrow[r, "\delta"] \arrow[d, "\cdot r"'] & B\mathrm{SO} \arrow[d, "\mathrm{Id}_{B\mathrm{SO}}"] \\
\mathrm{QS}^0_r \arrow[r, "i_r"] & (\mathrm{QS}^0/\mathrm{SO})_r \arrow[r, "\delta_r"] & B\mathrm{SO}
\end{tikzcd}
\tag{6}
$$

where $\cdot r\colon \mathrm{SG} = \mathrm{QS}^0_1 \to \mathrm{QS}^0_r$ is the map obtained by composition with a fixed map of degree r and $\delta := \delta_1$ is the canonical map.

Since $\mathrm{QS}^0 := \bigsqcup_{r\in\mathbb{Z}} \mathrm{QS}^0_r$, the space of stable self maps of the sphere is a grouplike H-space (with the "loop sum" operation, which induces the addition on π^s_0), its connected components are all homotopy equivalent. So $\pi_i(\mathrm{QS}^0_r) \cong \pi_i(\mathrm{SG}) = \pi^s_i$ for all r and i, and under this identification $\pi_i(\cdot r)\colon \pi_i(\mathrm{SG}) \to \pi_i(\mathrm{QS}^0_r)$ is multiplication by r. Therefore when we invert the primes dividing r, the map $\cdot r$ becomes a weak homotopy equivalence and hence a homotopy equivalence (see [Brumfiel and Madsen 1976, Proposition 4.6]). Combining this with the commutative diagram of (6), we get:

Proposition 5.12 (cf [Brumfiel and Madsen 1976, Proposition 4.6]) *If $r \neq 0$, the map $\cdot r\colon G/O \to (\mathrm{QS}^0/\mathrm{SO})_r$ induces a homotopy equivalence*

$$(\cdot r)[1/r]\colon (G/O)[1/r] \simeq (\mathrm{QS}^0/\mathrm{SO})_r[1/r]$$

such that $\delta_r[1/r] \circ (\cdot r)[1/r] \simeq \delta[1/r]$. □

Here $X[1/r]$ denotes the localisation of a space X obtained by inverting the primes dividing r (see [Sullivan 1970, Chapter 2]), and for a map $f\colon X \to Y$, $f[1/r]\colon X[1/r] \to Y[1/r]$ denotes the induced map.

We now consider the effect of taking the connected sum with a framed manifold in the source of a normal map. For this, we will tacitly assume that all manifolds have basepoints, the bundles considered over these basepoints have been trivialised (and for a fibrewise degree-r map, the map over the basepoint is identified with some fixed degree-r map between spheres) and that the connected sum operation is carried out at discs which have the basepoints on their boundaries so that the connected sum is itself based.

Suppose that $(f, \bar{f})\colon M \to P$ is an m-dimensional degree-r normal map and $[N, F] \in \Omega^{\mathrm{fr}}_m$. Let $f_N\colon N \to P$ be the constant map and $\bar{f}_N$ be the bundle map over f_N corresponding to the framing F. We can assume that $(f, \bar{f})$ is constant over a small m-disc $D^m \subset M$, and by taking connected sum in the source and extending with the constant map, we obtain a degree-r normal map $(f \sharp f_N, \bar f \sharp \bar f_N)\colon M \sharp N \to P$. Connected sum defines a natural operation

$$\sharp\colon \mathcal{N}_r^+(P) \times \Omega_m^{\mathrm{fr}} \to \mathcal{N}_r^+(P), \quad ([f, \bar f], [N, F]) \mapsto [f \sharp f_N, \bar f \sharp \bar f_N],$$

and we will explain how to determine $\eta([f \sharp f_N, \bar f \sharp \bar f_N])$ in terms of $\eta([f, \bar f])$ and $[N, F]$. To do this, we need to define an appropriate normal invariant of $[N, F]$. Define the homomorphism

$$\eta_r^{\mathrm{fr}}\colon \Omega_m^{\mathrm{fr}} \to \pi_m(\mathrm{QS}_r^0)$$

to be the composition $\Omega_m^{\mathrm{fr}} \xrightarrow{\mathrm{PT}} \pi_m^s \xrightarrow{\mathrm{ad}} \pi_m(\mathrm{QS}_0^0) \xrightarrow{\mathrm{ls}_{r*}} \pi_m(\mathrm{QS}_r^0)$, where PT denotes the Pontryagin–Thom isomorphism, ad is defined via the adjoint map, ls_r is given by taking the loop sum with a fixed degree-r map and ls_{r*} is the induced map on π_m.

For any connected, oriented m-manifold P and topological space X, connected sum of maps defines an action $\sharp\colon [P, X] \times \pi_m(X) \to [P, X]$ of $\pi_m(X)$ on $[P, X]$, where we can and do assume that maps are constant at the same value in a neighbourhood of the basepoints. This action is natural in X, ie a map $f\colon X \to Y$ determines a commutative diagram

$$\begin{tikzcd}
{[P, X] \times \pi_m(X)} \arrow[r, "\sharp"] \arrow[d, "f_* \times f_*"'] & {[P, X]} \arrow[d, "f_*"] \\
{[P, Y] \times \pi_m(Y)} \arrow[r, "\sharp"] & {[P, Y]}
\end{tikzcd} \tag{7}$$

Lemma 5.13 *Let $(f, \bar f)\colon M \to P$ be an m-dimensional degree-r normal map and $[N, F] \in \Omega_m^{\mathrm{fr}}$. The normal invariant of $(f \sharp f_N, \bar f \sharp \bar f_N)\colon M \sharp N \to P$ is given by*

$$\eta([f \sharp f_N, \bar f \sharp \bar f_N]) = \eta([f, \bar f]) \sharp i_{r*}(\eta_r^{\mathrm{fr}}([N, F])) \in [P, (\mathrm{QS}^0/\mathrm{SO})_r],$$

where $i_r\colon \mathrm{QS}_r^0 \to (\mathrm{QS}^0/\mathrm{SO})_r$ is the inclusion of the fibre appearing in (5).

Proof We will prove that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{N}_r^+(P) \times \Omega_m^{\mathrm{fr}} \arrow[d, "\sharp"'] & \mathcal{F}_r^+(P) \times \pi_m^s \arrow[l, "T \times \mathrm{PT}^{-1}"'] \arrow[r, "\mathrm{Id} \times \alpha"] \arrow[d] & \mathcal{F}_r^+(P) \times \mathcal{F}_r^+(S^m) \arrow[r, "\mathrm{Br} \times \mathrm{Br}"] \arrow[d, "\sharp"] & {[P, (\mathrm{QS}^0/\mathrm{SO})_r] \times \pi_m((\mathrm{QS}^0/\mathrm{SO})_r)} \arrow[d, "\sharp"] \\
\mathcal{N}_r^+(P) & \mathcal{F}_r^+(P) \arrow[l, "T"'] \arrow[r, equal] & \mathcal{F}_r^+(P) \arrow[r, "\mathrm{Br}"] & {[P, (\mathrm{QS}^0/\mathrm{SO})_r]}
\end{tikzcd}$$

where α is the composition of $\mathrm{ad}\colon \pi_m^s \cong \pi_m(\mathrm{QS}_0^0)$, $\mathrm{ls}_{r*}\colon \pi_m(\mathrm{QS}_0^0) \cong \pi_m(\mathrm{QS}_r^0)$ and the canonical map $\pi_m(\mathrm{QS}_r^0) \to \mathcal{F}_r^+(S^m)$ which sends a homotopy class to the adjoint fibrewise degree-r map between trivialised bundles over S^m. The second and third vertical maps will be defined in the course of the proof below.

The commutativity of the diagram above suffices to prove the lemma, because i_{r*} is the composition of the canonical map $\pi_m(\mathrm{QS}_r^0) \to \mathcal{F}_r^+(S^m)$ and Br (recall that QS_r^0 is the classifying space of fibrewise degree-r maps between trivialised bundles and i_r classifies forgetting that the bundles are trivialised).

First we define the map $\mathcal{F}_r^+(P)\times\pi_m^s\to\mathcal{F}_r^+(P)$ and show that the first square commutes. Let $g\colon S\zeta\to S\theta$ represent an element of $\mathcal{F}_r^+(P)$, where ζ and θ are bundles of rank $k+1\gg m$ of the form $\zeta=\zeta'\oplus\underline{\mathbb{R}}$ and $\theta=\theta'\oplus\underline{\mathbb{R}}$. We define the sections $s_\pm\colon P\to S\theta$ by $s_\pm(x)=(\pm1,x)\in S^0\times P\subset S(\theta'\oplus\underline{\mathbb{R}})$, and assume that g is transverse to $s_+(P)$, so that $T([g])$ is represented by $g^{-1}(s_+(P))\to s_+(P)\approx P$ (covered by the appropriate bundle map); see Remark 5.7.

Let $D^m\subset P$ be a small embedded disc, then $\zeta|_{D^m}=\mathbb{R}^{k+1}\times D^m$ and $\theta_{D^m}=\mathbb{R}^{k+1}\times D^m$ are uniquely trivialised (up to homotopy). After a fibre homotopy of g we may assume that $g|_{S\zeta|_{D^m}}$ is trivial, ie $g|_{S\zeta|_{D^m}}=h\times\mathrm{Id}_{D^m}$ for some degree-r map $h\colon S^k\to S^k$, and that for some small disc $D^k\subset S^k$ the map $h|_{D^k}$ is constant with value $-1\in S^0\subset S^{k-1}*S^0\approx S^k$. Then $g(D^k\times D^m)=s_-(D^m)$, so $(g|_{D^k\times D^m})^{-1}(s_+(P))$ is empty.

The adjoint of $g|_{D^k\times D^m}$ is the constant map in $\mathrm{Map}\big((D^m,S^{m-1}),(\mathrm{Map}((D^k,S^{k-1}),(S^k,-1)),c_{-1})\big)$, where c_{-1} denotes the constant map with value -1. Identifying $\mathrm{Map}((D^k,S^{k-1}),(S^k,-1))=\Omega^kS^k$ we have the isomorphism

$$\Phi\colon\pi_0\big(\mathrm{Map}((D^m,S^{m-1}),(\mathrm{Map}((D^k,S^{k-1}),(S^k,-1)),c_{-1}))\big)\cong\pi_m(\Omega^kS^k)\cong\pi_{m+k}(S^k)=\pi_m^s.$$

For $a\in\pi_m^s$, let $u_a\colon D^k\times D^m\to S^k\times D^m$ be the adjoint of a representative of $\Phi^{-1}(a)$. We define the fibrewise degree-r map $g_a\colon S\zeta\to S\theta$ by

$$g_a(v)=\begin{cases}u_a(v) & \text{if } v\in D^k\times D^m,\\ g(v) & \text{if } v\in S\zeta\setminus\mathrm{int}(D^k\times D^m).\end{cases}$$

The map g_a is well-defined and continuous, because g and u_a agree on $\partial(D^k\times D^m)$. We define the map $\mathcal{F}_r^+(P)\times\pi_m^s\to\mathcal{F}_r^+(P)$ by $([g],a)\mapsto[g_a]$.

We may assume that u_a is transverse to $1\times D^m$. Then $u_a^{-1}(1\times D^m)$ is the image of a under the Pontryagin construction. Moreover, g_a is transverse to $s_+(P)$ and $g_a^{-1}(s_+(P))=g^{-1}(s_+(P))\sqcup u_a^{-1}(1\times D^m)$. This is bordant to $g^{-1}(s_+(P))\,\sharp\,u_a^{-1}(1\times D^m)$, showing that $T([g_a])=T([g])\,\sharp\,\mathrm{PT}^{-1}(a)$. Thus the first square commutes.

Next we define the map $\sharp\colon\mathcal{F}_r^+(P)\times\mathcal{F}_r^+(S^m)\to\mathcal{F}_r^+(P)$ and consider the second square. Given fibrewise degree-r maps over P and S^m, we fix embeddings $D^m\to P$ and $D^m\to S^m$ at the basepoints and identify the restrictions of both maps with $h\times\mathrm{Id}_{D^m}$. Then we can take their connected sum over $P\,\sharp\,S^m\approx P$.

Now let $[g]\in\mathcal{F}_r^+(P)$ and $a\in\pi_m^s$ as before. We let $g^0:=h\times\mathrm{Id}_{S^m}\colon S^k\times S^m\to S^k\times S^m$ be a trivial degree-r map and construct $g_a^0\colon S^k\times S^m\to S^k\times S^m$ from u_a and g^0, analogously to g_a. Then $[g_a]=[g]\,\sharp\,[g_a^0]$ (the connected sum $P\,\sharp\,S^m$ is formed using the embedding $D^m\to P$ that was previously used to construct g_a and an embedding $D^m\to S^m$ whose image is the complement of the one used to construct g_a^0). We can extend u_a with the projection $D^k\times(S^m\setminus D^m)\to\{-1\}\times(S^m\setminus D^m)\subset S^k\times(S^m\setminus D^m)$ to get $\tilde{u}_a\colon D^k\times S^m\to S^k\times S^m$. This is again the adjoint of a, and also of the corresponding element

$a' \in \pi_m(\mathrm{QS}^0_0)$. Since $g^0(y,x) = (h(y),x)$ for every $(y,x) \in S^k \times S^m$ and $h(y) = -1$ if $y \in D^k$, we have

$$\begin{aligned} g^0_a(v) &= \begin{cases} u_a(v) & \text{if } v \in D^k \times D^m, \\ g_0(v) & \text{if } v \in S^k \times S^m \setminus \operatorname{int}(D^k \times D^m) \end{cases} \\ &= \begin{cases} \tilde{u}_a(v) & \text{if } v \in D^k \times S^m, \\ g_0(v) = (h(y),x) & \text{if } v = (y,x) \in (S^k \setminus \operatorname{int} D^k) \times S^m. \end{cases} \end{aligned}$$

Since $\mathrm{ls}_r \colon \mathrm{QS}^0 \to \mathrm{QS}^0_r$ is defined by taking loop sum with h, the map g^0_a is the adjoint of $\mathrm{ls}_{r*}(a')$. Therefore $[g_a] = [g] \sharp \alpha(a)$, proving that the second square commutes.

Finally we consider the third square. For any space X, the action $\sharp \colon [P,X] \times \pi_m(X) \to [P,X]$ is equal to the composition

$$[P,X] \times \pi_m(X) \xrightarrow{\vee} [P \vee S^m, X] \xrightarrow{p^*} [P,X],$$

where $p \colon P \to P/S^{m-1} \approx P \vee S^m$ is the pinch map, collapsing the boundary of a small embedded m-disc $D^m \subset P$. Similarly, $\sharp \colon \mathcal{F}^+_r(P) \times \mathcal{F}^+_r(S^m) \to \mathcal{F}^+_r(P)$ is the composition of $\vee \colon \mathcal{F}^+_r(P) \times \mathcal{F}^+_r(S^m) \to \mathcal{F}^+_r(P \vee S^m)$ and $p^* \colon \mathcal{F}^+_r(P \vee S^m) \to \mathcal{F}^+_r(P)$. So the commutativity of the third square follows from the naturality of Br. □

5.3 Relative divisors

In this subsection we give another description of the bijection $T \colon \mathcal{F}^+_r(P) \to \mathcal{N}^+_r(P)$ from Theorem 5.6 in terms of sections and divisors. In order to relate fibre-preserving maps to sections we introduce the following notation.

Definition 5.14 Suppose that ζ and θ are vector bundles over some base space Y.

(a) For a fibre-preserving map $g \colon S\zeta \to S\theta$ we define a section $s_g \colon S\zeta \to (\pi_\zeta|_{S\zeta})^*(S\theta) \subseteq S\zeta \times S\theta$ of the sphere bundle $(\pi_\zeta|_{S\zeta})^*(S\theta)$, the pull-back of $S\theta$ via the projection $\pi_\zeta|_{S\zeta} \colon S\zeta \to Y$, by $s_g(x) = (x, g(x))$. The assignment $g \mapsto s_g$ is a bijection between fibre-preserving maps $S\zeta \to S\theta$ and sections of $(\pi_\zeta|_{S\zeta})^*(S\theta)$.

(b) For a fibre-preserving map $f \colon D\zeta \to D\theta$ we define a section $s_f \colon D\zeta \to (\pi_\zeta|_{D\zeta})^*(D\theta) \subseteq D\zeta \times D\theta$ by $s_f(x) = (x, f(x))$. The assignment $f \mapsto s_f$ is a bijection between fibre-preserving maps $D\zeta \to D\theta$ and sections of $(\pi_\zeta|_{D\zeta})^*(D\theta)$. Moreover, f is transverse to the zero section of $D\theta$ if and only if s_f is transverse to the zero section of $(\pi_\zeta|_{D\zeta})^*(D\theta)$.

These two bijections are compatible in the sense that if g is the restriction of some f, then s_g is the restriction of s_f.

Now suppose that $\tilde{\theta}$ is a rank-k smooth vector bundle over a smooth manifold V with boundary ∂V and $s_\partial \colon \partial V \to S\tilde{\theta}|_{\partial V}$ is a section of $S\tilde{\theta}|_{\partial V}$ (hence a nowhere-zero section of $E\tilde{\theta}|_{\partial V}$).

Definition 5.15 If $s\colon V \to E\tilde{\theta}$ is a smooth section of $\tilde{\theta}$, which extends s_∂, and which is transverse to the zero section, s_0, then we call

$$Z(s) := s(V) \cap s_0(V) \subset s_0(V) \approx V$$

a *divisor of* $\tilde{\theta}$ *relative to* s_∂.

Remark 5.16 We have $\nu_{Z(s)} \cong (\tilde{\theta} \oplus \nu_V)|_{Z(s)}$, because the normal bundle of the embedding $Z(s) \hookrightarrow V$ is given by $\nu(Z(s) \to V) \cong \tilde{\theta}|_{Z(s)}$ (in a tubular neighbourhood $U \approx D\nu(Z(s) \to V)$ of $Z(s)$ the section $s|_U$ corresponds to a fibre-preserving map $D\nu(Z(s) \to V) \to E\tilde{\theta}|_{Z(s)}$, and by transversality the restriction of its derivative is an isomorphism $\nu(Z(s) \to V) \cong \tilde{\theta}|_{Z(s)}$).

Since the fibre of $E\tilde{\theta}$ is contractible, s_∂ can always be extended to a (transverse) section s and the extension is unique up to homotopy (rel ∂V). This also implies that the normal bordism class of the normal map

$$\begin{tikzcd} \nu_{Z(s)} \arrow[r] \arrow[d] & \tilde{\theta} \oplus \nu_V \arrow[d] \\ Z(s) \arrow[r] & V \end{tikzcd}$$

is independent of the choice of s (and it only depends on the homotopy class of s_∂ as a nowhere-zero section).

Suppose in addition that $V = D\zeta$ itself is the disc bundle of a rank-k smooth vector bundle ζ over a closed smooth manifold P. Let $\theta = \tilde{\theta}|_P$ be the restriction of $\tilde{\theta}$. Then $\tilde{\theta}$ can be identified with $(\pi_\zeta|_{D\zeta})^*(\theta)$.

Let $g\colon S\zeta \to S\theta$ be a fibrewise degree-r map and s_g the corresponding section (see Definition 5.14). There exists a section $s\colon D\zeta \to (\pi_\zeta|_{D\zeta})^*(D\theta)$ that extends s_g and is transverse to the zero section. Let $p = \pi_\zeta|_{Z(s)}\colon Z(s) \to P$.

Lemma 5.17 *The map $p\colon Z(s) \to P$ has degree r and it is covered by a bundle map $\bar{p}\colon \nu_{Z(s)} \to \theta \oplus \nu_P \ominus \zeta$ such that*

$$T([g]) = [p, \bar{p}] \in \mathcal{N}_r^+(P).$$

Proof Using the bijection from Definition 5.14 (b) there is a fibre-preserving map $f\colon D\zeta \to D\theta$ such that $s = s_f$. This f extends g and it is transverse to the zero section, so it satisfies the conditions in the definition of T (given before Theorem 5.6). The manifold $M = f^{-1}(P)$ is then equal to $Z(s)$ and $f|_M = p$ (and it has degree r). We can choose $\bar{p} = \bar{f}_M$ and then $T([g]) = [p, \bar{p}]$. □

5.4 The canonical degree-d normal invariant of a complete intersection

Consider a complete intersection $X_n(\underline{d})$. By cellular approximation the canonical embedding $i\colon X_n(\underline{d}) \to \mathbb{C}P^{n+k}$ is homotopic to a map $f_n(\underline{d})\colon X_n(\underline{d}) \to \mathbb{C}P^n$ and since $\mathbb{C}P^{n+k}$ has no $(2n+1)$-cells, $f_n(\underline{d})$ is well-defined up to homotopy. Since $i_*([X_n(\underline{d})])$ is d times the preferred generator of $H_{2n}(\mathbb{C}P^n)$, $f_n(\underline{d})$ is a degree-d map.

The main result of this section is the computation of the normal invariant of a certain degree-d normal map covering $f_n(\underline{d})$ in Theorem 5.19 below. The importance of this calculation comes from the next lemma (which can be regarded as a variation of [Kreck 1999, Proposition 10]) and its application, Theorem 5.20.

Lemma 5.18 *Let $X_n(\underline{d})$ and $X_n(\underline{d}')$ be complete intersections with $\chi(X_n(\underline{d})) = \chi(X_n(\underline{d}'))$ and the same total degree d. Suppose that there are degree-d normal maps $(f, \bar{f})\colon (X_n(\underline{d}), \nu_{X_n(\underline{d})}) \to (\mathbb{C}P^n, \xi_n(\underline{d})|_{\mathbb{C}P^n})$ and $(f', \bar{f}')\colon (X_n(\underline{d}'), \nu_{X_n(\underline{d}')}) \to (\mathbb{C}P^n, \xi_n(\underline{d}')|_{\mathbb{C}P^n})$ such that*

$$[f, \bar{f}] = [f', \bar{f}'] \in \mathcal{N}_d^+(\mathbb{C}P^n).$$

If $n \geq 3$, then $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic.

Proof Let $\xi = \xi_n(\underline{d})|_{\mathbb{C}P^n}$ and recall (see Notation 5.4) that $[g, \bar{g}]_\xi \in \Omega_{2n}^{\mathrm{fr}}(\mathbb{C}P^n; \xi)_d$ denotes the element represented by a degree-d normal map $(g, \bar{g})$ and the image of $[g, \bar{g}]_\xi$ in $\mathcal{N}_d^+(\mathbb{C}P^n)$ is $[g, \bar{g}]$. By definition, the condition $[f, \bar{f}] = [f', \bar{f}']$ means that there is an isomorphism $\alpha\colon \xi_n(\underline{d}')|_{\mathbb{C}P^n} \to \xi_n(\underline{d})|_{\mathbb{C}P^n} = \xi$ (which in particular implies that $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$) such that

$$[f, \bar{f}]_\xi = [f', \alpha \circ \bar{f}']_\xi \in \Omega_{2n}^{\mathrm{fr}}(\mathbb{C}P^n; \xi)_d.$$

Now consider the composition

$$\Omega_{2n}^{\mathrm{fr}}(\mathbb{C}P^n; \xi_n(\underline{d})|_{\mathbb{C}P^n})_d \to \Omega_{2n}^{\mathrm{fr}}(\mathbb{C}P^n; \xi_n(\underline{d})|_{\mathbb{C}P^n}) \to \Omega_{2n}^{\mathrm{fr}}(\mathbb{C}P^\infty; \xi_n(\underline{d})) \to \Omega_{2n}^{O\langle n\rangle}(\mathbb{C}P^\infty; \xi_n(\underline{d})).$$

We see that $X_n(\underline{d})$ and $X_n(\underline{d}')$ admit bordant normal $(n-1)$-smoothings over $(B_n; \xi_n(\underline{d}) \times \gamma_{BO\langle n+1\rangle})$ and if $\underline{d} \neq \{1\}, \{2\}$ or $\{2, 2\}$, then the lemma follows from Proposition 1.9. If $\underline{d} = \{1\}, \{2\}$ or $\{2, 2\}$, then $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$ implies that $\underline{d}' = \underline{d}$. □

Theorem 5.19 *There is a bundle map $\bar{f}_n(\underline{d})\colon \nu_{X_n(\underline{d})} \to \xi_n(\underline{d})|_{\mathbb{C}P^n}$ over $f_n(\underline{d})$ such that*

$$\eta([f_n(\underline{d}), \bar{f}_n(\underline{d})]) = \eta_n(\underline{d}) \in [\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_d].$$

(*For the definitions of η and $\eta_n(\underline{d})$ see Definitions 5.10 and 5.11.*)

An immediate consequence of Theorem 5.19, the fact that η is a bijection and Lemma 5.18 is the following:

Theorem 5.20 *Let $X_n(\underline{d})$ and $X_n(\underline{d}')$ be complete intersections with the same total degree d and the same Euler characteristic. If $n \geq 3$ and $\eta_n(\underline{d}) = \eta_n(\underline{d}') \in [\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_d]$, then $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic.* □

Proof of Theorem 5.19 Let $f^0\colon D(k\gamma|_{\mathbb{C}P^n}) \to D(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ denote the Whitney sum of the tensor power maps $D(\gamma|_{\mathbb{C}P^n}) \to D(\gamma^{d_i}|_{\mathbb{C}P^n})$ and let $g^0\colon S(k\gamma|_{\mathbb{C}P^n}) \to S(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ be its restriction to the sphere bundle. Hence, in the notation of Definition 5.11, $g^0 = t_{d_1}(\gamma|_{\mathbb{C}P^n}) * \cdots * t_{d_r}(\gamma|_{\mathbb{C}P^n})$, so $\mathrm{Br}([g^0]) = \eta_n(\underline{d})$ and we must prove the following: there is a map of stable vector bundles $\bar{f}_n(\underline{d})\colon \nu_{X_n(\underline{d})} \to \xi_n(\underline{d})|_{\mathbb{C}P^n}$ over $f_n(\underline{d})$ such that

$$T([g^0]) = [f_n(\underline{d}), \bar{f}_n(\underline{d})] \in \mathcal{N}_d^+(\mathbb{C}P^n).$$

First we describe a way of constructing a representative of the complete intersection $X_n(\underline{d})$ in an arbitrarily small neighbourhood of the subspace $\mathbb{C}P^n \subset \mathbb{C}P^{n+k}$. Let $[x_0, x_1, \dots, x_{n+k}]$ be homogeneous coordinates on the ambient $\mathbb{C}P^{n+k}$. For $i = 1, 2, \dots, k$, define $p_i^0 \in \mathbb{C}[x_0, x_1, \dots, x_{n+k}]$ by $p_i^0(\underline{x}) = x_{n+i}^{d_i}$, where $\underline{x} = (x_0, x_1, \dots, x_{n+k})$. Then

$$\begin{aligned}\{[\underline{x}] \in \mathbb{C}P^{n+k} \mid p_1^0(\underline{x}) = p_2^0(\underline{x}) = \dots = p_k^0(\underline{x}) = 0\} &= \{[\underline{x}] \in \mathbb{C}P^{n+k} \mid x_{n+1} = x_{n+2} = \dots = x_{n+k} = 0\} \\ &= \mathbb{C}P^n.\end{aligned}$$

Note that if $d_i > 1$, then p_i^0 is singular at its zeros, so $\mathbb{C}P^n$ is not a representative of $X_n(\underline{d})$ unless $\underline{d} = \{1, 1, \dots, 1\}$. However, by applying an arbitrarily small perturbation to the p_i^0 we can obtain new polynomials p_i such that

$$\{[\underline{x}] \in \mathbb{C}P^{n+k} \mid p_1(\underline{x}) = p_2(\underline{x}) = \dots = p_k(\underline{x}) = 0\} = X_n(\underline{d})$$

is a complete intersection and it is contained in the interior of a closed tubular neighbourhood U of $\mathbb{C}P^n$ (we will fix a U in Lemma 5.23 below).

By Construction 5.22 the polynomials p_i^0 and p_i define sections of $\gamma^{d_i}|_{\mathbb{C}P^{n+k}}$. Therefore the tuples $(p_1^0, p_2^0, \dots, p_k^0)$ and $(p_1, p_2, \dots, p_k)$ define some sections s^0 and s of $\gamma^{\underline{d}}|_{\mathbb{C}P^{n+k}}$ (so the zero sets of s^0 and s are $\mathbb{C}P^n$ and $X_n(\underline{d})$ respectively). Then we can assume that there is a homotopy $\mathbb{C}P^{n+k} \times I \to \gamma^{\underline{d}}|_{\mathbb{C}P^{n+k}}$ of sections between s^0 and s that is nonzero on $(\mathbb{C}P^{n+k} \setminus \operatorname{int} U) \times I$. In particular, the restrictions of s^0 and s are homotopic as nonzero sections over ∂U.

The normal bundle of $\mathbb{C}P^n$ in $\mathbb{C}P^{n+k}$ is $k\gamma|_{\mathbb{C}P^n}$, so U can be identified with $D(k\gamma|_{\mathbb{C}P^n})$. Moreover, the projection $\pi_U : U \to \mathbb{C}P^n$ of U is a deformation retraction; hence the bundle $\pi_U^*(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ is isomorphic to $\gamma^{\underline{d}}|_U$. We will fix an identification and an isomorphism in Lemma 5.23. With these identifications, the sections $s^0|_U$ and $s|_U$ correspond to fibre-preserving maps $D(k\gamma|_{\mathbb{C}P^n}) \to D(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ under the bijection of Definition 5.14. Let $f : D(k\gamma|_{\mathbb{C}P^n}) \to D(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ be the map such that $s_f = s|_U$. In Lemma 5.23 we prove that $s_{f^0} = s^0|_U$. Let $g : S(k\gamma|_{\mathbb{C}P^n}) \to S(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ be the restriction of f (we can assume that it has values in the sphere bundle, because $f|_{S(k\gamma|_{\mathbb{C}P^n})}$ is nowhere zero since $s|_{\partial U}$ is nowhere zero). Then $s_g = s|_{\partial U}$. The restriction of f^0 is g^0, so $s_{g^0} = s^0|_{\partial U}$. Since $s_{g^0} = s^0|_{\partial U}$ and $s_g = s|_{\partial U}$ are homotopic as nonzero sections, g^0 and g are fibre homotopic. The bijection T is well-defined on fibre homotopy classes, so $T([g^0]) = T([g])$.

By construction $X_n(\underline{d}) = Z(s)$ is a divisor of $\gamma^{\underline{d}}|_{\mathbb{C}P^{n+k}}$ relative to $s|_{\partial U} = s_g$. Since π_U is homotopic to the identity, we have $f_n(\underline{d}) = \pi_U|_{X_n(\underline{d})}$ (up to homotopy). By Lemma 5.17 there is a bundle map $\bar{f}_n(\underline{d}) : \nu_{X_n(\underline{d})} \to \gamma^{\underline{d}} \oplus -(n+1)\gamma \ominus k\gamma|_{\mathbb{C}P^n} \cong \xi_n(\underline{d})|_{\mathbb{C}P^n}$ such that $T([g]) = [f_n(\underline{d}), \bar{f}_n(\underline{d})] \in \mathcal{N}_{\underline{d}}^+(\mathbb{C}P^n)$. Therefore $T([g^0]) = [f_n(\underline{d}), \bar{f}_n(\underline{d})]$. □

Remark 5.21 There is a canonical bundle map $\nu_{X_n(\underline{d})} \to \xi_n(\underline{d})|_{\mathbb{C}P^{n+k}}$ over i, and hence over $f_n(\underline{d})$ (cf Proposition 2.5), because there is a canonical isomorphism $\nu_{X_n(\underline{d})} \cong \nu(X_n(\underline{d}) \to \mathbb{C}P^{n+k}) \oplus \nu_{\mathbb{C}P^{n+k}}$ and the normal bundle of a degree-r hypersurface in $\mathbb{C}P^{n+k}$ is canonically isomorphic to the restriction of γ^r (see Construction 5.22 and Remark 5.16). By following the definitions, we can see that the bundle map $\bar{f}_n(\underline{d})$ constructed in the proof of Theorem 5.19 is equal to this canonical map (up to homotopy).

We used the following (well-known) construction and the lemma below.

Construction 5.22 A homogeneous polynomial q of degree r in variables $x_0, x_1, \dots, x_m$ determines a section of the bundle $\gamma^r|_{\mathbb{C}P^m}$ as follows.

If $r = 1$, then the assignment $[x_0, x_1, \dots, x_m] \mapsto [x_0, x_1, \dots, x_m, q(x_0, x_1, \dots, x_m)]$ is a well-defined map $\mathbb{C}P^m \to \mathbb{C}P^{m+1} \setminus [0, 0, \dots, 0, 1]$. Since the map $\mathbb{C}P^{m+1} \setminus [0, 0, \dots, 0, 1] \to \mathbb{C}P^m$, $[x_0, x_1, \dots, x_{m+1}] \mapsto [x_0, x_1, \dots, x_m]$, can be identified with the projection of the normal bundle of $\mathbb{C}P^m$ in $\mathbb{C}P^{m+1}$, which is isomorphic to $\gamma|_{\mathbb{C}P^m}$, we get that q determines a section of $\gamma|_{\mathbb{C}P^m}$. So every linear monomial x_i determines a section of $\gamma|_{\mathbb{C}P^m}$. If we have sections $s_1, s_2, \dots, s_r$ of some vector bundle ξ, then their symmetric product $s_1 s_2 \cdots s_r$ is a section of the symmetric power $\mathrm{Sym}^r(\xi)$ and if ξ is a line bundle, then $\mathrm{Sym}^r(\xi) = \xi^r$. Therefore every degree-r monomial, and hence every degree-r homogeneous polynomial, determines a section of $\gamma^r|_{\mathbb{C}P^m}$.

Lemma 5.23 *We can identify $D(k\gamma|_{\mathbb{C}P^n})$ with a tubular neighbourhood U of $\mathbb{C}P^n$ in $\mathbb{C}P^{n+k}$ and the bundle $\pi_U^*(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ with $\gamma^{\underline{d}}|_U$ such that after these identifications the section s_{f^0} corresponding to f^0 under the bijection of Definition 5.14(b) is equal to $s^0|_U$.*

Proof First we will introduce "coordinates" on the total space of $\gamma^r|_{\mathbb{C}P^m}$. Then we will define U and describe the necessary identifications. Finally we will show that s_{f^0} (regarded as a section of $\gamma^{\underline{d}}|_U$) is equal to $s^0|_U$.

By Construction 5.22 a pair $([\underline{a}], q)$ (where $[\underline{a}] = [a_0, a_1, \dots, a_m] \in \mathbb{C}P^m$ and q is a homogeneous polynomial of degree r in variables $x_0, x_1, \dots, x_m$) determines a point in $E(\gamma^r|_{\mathbb{C}P^m})$ (namely, the value of the section determined by q over the point $[\underline{a}]$). Every point in $E(\gamma^r|_{\mathbb{C}P^m})$ can be described by such a pair and two pairs, $([\underline{a}], q)$ and $([\underline{a}], q')$, determine the same point if and only if $q(\underline{a}) = q'(\underline{a})$. Similarly, if q_i is a homogeneous polynomial of degree d_i, then a pair $([\underline{a}], (q_1, q_2, \dots, q_k))$ determines a point in $E(\gamma^{\underline{d}}|_{\mathbb{C}P^m})$.

To simplify notation we use the abbreviations $\underline{a} = (a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, $\underline{b} = (b_1, b_2, \dots, b_k) \in \mathbb{C}^k$ and $\underline{c} = (c_0, c_1, \dots, c_{n+k}) \in \mathbb{C}^{n+k+1} \setminus \{0\}$. Also, q_i and r_i will always denote some homogeneous polynomials in variables $x_0, x_1, \dots, x_n$ such that q_i has degree d_i and r_i is linear.

The map $([\underline{a}], (r_1, r_2, \dots, r_k)) \mapsto [\underline{a}, r_1(\underline{a}), r_2(\underline{a}), \dots, r_k(\underline{a})]$ is a homeomorphism between $E(k\gamma|_{\mathbb{C}P^n})$ and an open tubular neighbourhood of $\mathbb{C}P^n$ in $\mathbb{C}P^{n+k}$ (which is diffeomorphic to $\mathbb{C}P^{n+k} \setminus \mathbb{C}P^{k-1}$). We define U to be the image of the disc bundle $D(k\gamma|_{\mathbb{C}P^n})$ under this map. Then this map identifies $D(k\gamma|_{\mathbb{C}P^n})$ with U.

Points of the subspace $E(\pi_U^*(\gamma^{\underline{d}}|_{\mathbb{C}P^n})) \subset U \times E(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ are of the form $([\underline{a}, \underline{b}], ([\underline{a}], (q_1, q_2, \dots, q_k)))$. The map $\big([\underline{a}, \underline{b}], ([\underline{a}], (q_1, q_2, \dots, q_k))\big) \mapsto ([\underline{a}, \underline{b}], (\bar{q}_1, \bar{q}_2, \dots, \bar{q}_k)) \in E(\gamma^{\underline{d}}|_{\mathbb{C}P^{n+k}})$ (where $\bar{q}_i$ is equal to q_i, but is regarded as a polynomial in the variables $x_0, x_1, \dots, x_{n+k}$) is an isomorphism between the bundles $\pi_U^*(\gamma^{\underline{d}}|_{\mathbb{C}P^n})$ and $\gamma^{\underline{d}}|_U = (\gamma^{\underline{d}}|_{\mathbb{C}P^{n+k}})|_U$.

By definition the section s^0 is the map $[\underline{c}] \mapsto ([\underline{c}], (p_1^0, p_2^0, \dots, p_k^0))$.

The map f^0 is given by the formula $([\underline{a}], (r_1, r_2, \dots, r_k)) \mapsto ([\underline{a}], (r_1^{d_1}, r_2^{d_2}, \dots, r_k^{d_k}))$. After identifying $D(k\gamma|_{\mathbb{C}P^n})$ with U the formula becomes $[\underline{a}, \underline{b}] \mapsto ([\underline{a}], (r_1^{d_1}, r_2^{d_2}, \dots, r_k^{d_k}))$, where r_i is chosen such that $r_i(\underline{a}) = b_i$. Therefore $s_{f^0}([\underline{a}, \underline{b}]) = ([\underline{a}, \underline{b}], ([\underline{a}], (r_1^{d_1}, r_2^{d_2}, \dots, r_k^{d_k})))$ and this point is identified with $([\underline{a}, \underline{b}], (\bar{r}_1^{d_1}, \bar{r}_2^{d_2}, \dots, \bar{r}_k^{d_k}))$. We have

$$\bar{r}_i^{d_i}(\underline{a}, \underline{b}) = r_i^{d_i}(\underline{a}) = b_i^{d_i} = p_i^0(\underline{a}, \underline{b}),$$

so $([\underline{a}, \underline{b}], (\bar{r}_1^{d_1}, \bar{r}_2^{d_2}, \dots, \bar{r}_k^{d_k})) = ([\underline{a}, \underline{b}], (p_1^0, p_2^0, \dots, p_k^0))$. Therefore $s_{f^0} = s^0|_U$. □

We conclude this section with a discussion of the bundle data $\bar{f}_n(\underline{d})$ in the canonical normal invariant of $X_n(\underline{d})$. Although the degree-d normal map $(f_n(\underline{d}), \bar{f}_n(\underline{d}))\colon X_n(\underline{d}) \to \mathbb{C}P^n$ is canonically constructed, so far we have not been able to characterise its homotopy class amongst all such degree-d normal maps. In particular, if there is a diffeomorphism $h\colon X_n(\underline{d}) \to X_n(\underline{d}')$, then up to homotopy it induces a unique bundle map $\bar{h}\colon \nu_{X_n(\underline{d})} \to \nu_{X_n(\underline{d}')}$ covering h and in general we do not know whether $\bar{f}_n(\underline{d}') \circ \bar{h}$ and $\bar{f}_n(\underline{d})$ are homotopic stable bundle maps. In this paper, we shall only need to address this question when $n = 4$ and $X_4(\underline{d})$ is nonspin. In this case, the problem is solved via the following:

Lemma 5.24 *Let X be a closed, connected nonspin 8-manifold which is homotopy equivalent to a CW-complex with only even-dimensional cells, ξ a stable vector bundle over X and $\bar{g}\colon \xi \to \xi$ an orientation-preserving stable bundle automorphism. Then $\bar{g}$ is fibre homotopic to the identity.*

Proof By standard K-theoretic arguments (given for automorphisms of stable spherical fibrations in [Browder 1972, Lemma I.4.6]), it is sufficient to prove that $[X, \mathrm{SO}] = 0$. By [Hatcher 2002, Proposition 4C.1], we may assume that there is a homotopy equivalence $X \simeq K \cup_f D^8$, where K is a 6-dimensional CW-complex with only even-dimensional cells and $f\colon S^7 \to K$ attaches a single 8-cell. Consider the Puppe sequence of the cofibration $K \to K \cup_f D^8 \to S^8$:

$$[\Sigma K, \mathrm{SO}] \to \pi_8(\mathrm{SO}) \to [K \cup_f D^8, \mathrm{SO}] \to [K, \mathrm{SO}].$$

Obstruction theory [Hatcher 2002, Corollary 4.73] gives that $[K, \mathrm{SO}] = 0$, since $\pi_{2i}(\mathrm{SO}) = 0$ for $0 \le 2i \le 6$. So to prove that $[X, \mathrm{SO}] \cong [K \cup_f D^8, \mathrm{SO}]$ is trivial, it is enough to show that the map $[\Sigma K, \mathrm{SO}] \to \pi_8(\mathrm{SO})$ is surjective.

The map $[\Sigma K, \mathrm{SO}] \to \pi_8(\mathrm{SO})$ sends a homotopy class $[g] \in [\Sigma K, \mathrm{SO}]$ to $[g \circ \Sigma f]$, where $\Sigma f\colon S^8 \to \Sigma K$ is the suspension of f [Whitehead 1978, 6.18 Chapter III]. Since $K \cup_f D^8$ has no odd-dimensional cells, $H^1(X; \mathbb{Z}/2) = 0$, and so X is orientable. Since X is nonspin, $v_2(X) = w_2(X) \neq 0$ by [Milnor and Stasheff 1974, Theorem 11.15], and so $\mathrm{Sq}^2\colon H^6(X; \mathbb{Z}/2) \to H^8(X; \mathbb{Z}/2)$ is nonzero. Let $K^{(4)} \subset K$

denote the 4-skeleton of K, and let $c\colon K \to K/K^{(4)}$ be the collapse map; then $K/K^{(4)} = \bigvee_{i=1}^{b} S^6$ is a wedge of 6-spheres. Since $H^6(S^7) \cong H^5(S^7) \cong H^8\bigl(\bigvee_{i=1}^{b} S^6\bigr) \cong H^7\bigl(\bigvee_{i=1}^{b} S^6\bigr) \cong 0$, we deduce that the functional Steenrod square of Sq^2 applied to $c \circ f\colon S^7 \to \bigvee_{i=1}^{b} S^6$ is unambiguously defined and nonzero on $H^6(K/K^{(4)};\mathbb{Z}/2)$; cf [Mosher and Tangora 1968, Chapter 16]. Since Sq^2 is a stable operation, the functional Steenrod square of Sq^2 applied to the suspension $\Sigma c \circ \Sigma f\colon S^8 \to \bigvee_{i=1}^{b} S^7$ is also nonzero on $H^7\bigl(\bigvee_{i=1}^{b} S^7;\mathbb{Z}/2\bigr)$, showing that $\Sigma c \circ \Sigma f$ is essential (see [Mosher and Tangora 1968, Chapter 16, Proposition 1]). By Hilton's theorem [1955, Theorem A] and the computation of the 1-stem [Toda 1962, Chapter XIV], $\pi_8\bigl(\bigvee_{i=1}^{b} S^7\bigr) \cong \bigoplus_{i=1}^{b} \pi_8(S^7) \cong (\mathbb{Z}/2)^b$, and it follows that $\mathrm{pr}_j \circ (\Sigma c \circ \Sigma f)\colon S^8 \to S^7$ is essential for some $j \in \{1,\dots,b\}$, where $\mathrm{pr}_j\colon \bigvee_{i=1}^{b} S^7 \to S^7$ splits off the j^{th} sphere in the wedge. Using [Adams 1966, Example 12.15], we see that precomposition with $\eta_7\colon S^8 \to S^7$ induces a surjection $\pi_7(\mathrm{SO}) \to \pi_8(\mathrm{SO})$, and so the composition

$$\pi_7(\mathrm{SO}) \xrightarrow{\mathrm{pr}_j^*} [\Sigma\bigl(K/K^{(4)}\bigr),\mathrm{SO}] \xrightarrow{\Sigma c^*} [\Sigma K,\mathrm{SO}] \xrightarrow{\Sigma f^*} \pi_8(\mathrm{SO})$$

is onto. It follows that $[\Sigma K,\mathrm{SO}] \to \pi_8(\mathrm{SO})$ is onto, completing the proof. □

Corollary 5.25 *Let $(f_0,\bar f_0),(f_1,\bar f_1)\colon X_4(\underline{d}) \to (\mathbb{C}P^4, \xi_4(\underline{d})|_{\mathbb{C}P^4})$ be a pair of normal maps from a nonspin complete intersection $X_4(\underline{d})$ such that $f_0^*(x) = f_1^*(x)$. Then $(f_0,\bar f_0)$ and $(f_1,\bar f_1)$ are homotopic.*

Proof It follows from the assumption $f_0^*(x) = f_1^*(x)$ that f_0 and f_1 are homotopic as maps into $\mathbb{C}P^\infty \simeq K(\mathbb{Z},2)$. By cellular approximation they are also homotopic as maps into $\mathbb{C}P^4$, so we may assume that $f_0 = f_1$. Then the bundle maps $\bar f_0,\bar f_1\colon \nu_{X_4(\underline{d})} \to \xi_4(\underline{d})|_{\mathbb{C}P^4}$ differ by precomposition with a bundle automorphism $\bar g\colon \nu_{X_4(\underline{d})} \to \nu_{X_4(\underline{d})}$. By Lemma 5.24, $\bar g$ is homotopic to the identity and so $\bar f_0$ and $\bar f_1$ are homotopic. □

5.5 The Sullivan conjecture in the case of odd total degree

Let $X_n(\underline{d})$ be a complete intersection. By Theorem 5.19 there is a degree-d normal map $(f_n(\underline{d}),\bar f_n(\underline{d}))\colon X_n(\underline{d}) \to \mathbb{C}P^n$ such that

$$\eta(f_n(\underline{d}),\bar f_n(\underline{d})) = \eta_n(\underline{d}) \in [\mathbb{C}P^n,(\mathrm{QS}^0/\mathrm{SO})_d].$$

Our goal is to show that if d is odd and $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$, then $\eta_4(\underline{d}) = \eta_4(\underline{d}')$. Then Theorem 5.20 allows us to deduce the Sullivan conjecture when $n = 4$ and d is odd. To compare the normal invariants $\eta_4(\underline{d})$ and $\eta_4(\underline{d}')$, we will apply results of Feshbach on the Segal conjecture, using the fact that $\eta_n(\underline{d})$ is the restriction of $\eta_\infty(\underline{d})\colon \mathbb{C}P^\infty \to (\mathrm{QS}^0/\mathrm{SO})_d$.

Due to a combination of Theorem 1.4 of Fang and Klaus and Proposition 5.12 of Brumfiel and Madsen, localising at the prime 2 will prove to be an effective strategy when the total degree d is odd. For a simple space Z and a prime p we shall write $Z_{(p)}$ (and even $(Z)_{(p)}$ where necessary) for the p-localisation of Z. Similarly, we write $A_{(p)}$ for the p-localisation of an abelian group A. If $\varphi \in [Y,Z]$ is a homotopy class of maps from some other space Y to Z, we write $\varphi_{(p)} \in [Y,Z_{(p)}]$ for the homotopy class of the composition $Y \xrightarrow{\varphi} Z \to Z_{(p)}$, where $Z \to Z_{(p)}$ is the natural map. Similarly, for $Z[1/p]$, the space obtained from Z by inverting p, we write $\varphi_{[1/p]} \in [Y, Z[1/p]]$ for the homotopy class of the composition $Y \xrightarrow{\varphi} Z \to Z[1/p]$, where $Z \to Z[1/p]$ is the natural map.

Lemma 5.26 *Let n and d be positive integers. Suppose that $\varphi, \psi \in [\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_d]$ are homotopy classes such that $\varphi_{(p)} = \psi_{(p)}$ and $\varphi_{[1/p]} = \psi_{[1/p]}$ for some prime p. Then $\varphi = \psi$.*

Proof We partition the set of all primes into the sets $l := \{p\}$ and $l' := \{q \mid q \neq p\}$. By [Sullivan 1970, (4), page 41], for any simple space Z the natural maps $Z \to Z[1/p]$ and $Z \to Z_{(p)}$ fit into a fibre square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z[1/p] \arrow[d] \\ Z_{(p)} \arrow[r] & Z_{\mathbb{Q}} \end{tikzcd}$$

where $Z_{\mathbb{Q}}$ denotes the rationalisation of Z. Hence there is a homotopy fibration sequence $Z \to Z_{(p)} \times Z[1/p] \to Z_{\mathbb{Q}}$, and for $Z = (\mathrm{QS}^0/\mathrm{SO})_d$ we have a homotopy fibration sequence

$$(\mathrm{QS}^0/\mathrm{SO})_d \xrightarrow{\ell_p} ((\mathrm{QS}^0/\mathrm{SO})_d)_{(p)} \times ((\mathrm{QS}^0/\mathrm{SO})_d)[1/p] \to ((\mathrm{QS}^0/\mathrm{SO})_d)_{\mathbb{Q}}.$$

The Puppe sequence for homotopy classes of maps from $\mathbb{C}P^n$ into this fibration contains the exact sequence

$$[\mathbb{C}P^n, \Omega((\mathrm{QS}^0/\mathrm{SO})_d)_{\mathbb{Q}}] \to [\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_d] \xrightarrow{\ell_{p*}} [\mathbb{C}P^n, ((\mathrm{QS}^0/\mathrm{SO})_d)_{(p)}] \times [\mathbb{C}P^n, ((\mathrm{QS}^0/\mathrm{SO})_d)[1/p]],$$

where $\ell_{p*}(\varphi) = (\varphi_{(p)}, \varphi_{[1/p]})$ and $[\mathbb{C}P^n, \Omega((\mathrm{QS}^0/\mathrm{SO})_d)_{\mathbb{Q}}]$ acts transitively on the fibres of ℓ_{p*}. We will show that $[\mathbb{C}P^n, \Omega((\mathrm{QS}^0/\mathrm{SO})_d)_{\mathbb{Q}}] = 0$, which implies that ℓ_{p*} is injective and proves the lemma.

Since QS^0_d is connected with finite homotopy groups (see Section 5.2), its rationalisation is contractible. So, by the rationalisation of the fibration sequence (5), $((\mathrm{QS}^0/\mathrm{SO})_d)_{\mathbb{Q}} \simeq (B\mathrm{SO})_{\mathbb{Q}}$. It is well known that there is an equivalence $(B\mathrm{SO})_{\mathbb{Q}} \simeq \prod_{i=1}^{\infty} K(\mathbb{Q}, 4i)$ (see eg [Sullivan 1970, (12), pages 42–43]), and so we have a chain of isomorphisms $[\mathbb{C}P^n, \Omega((\mathrm{QS}^0/\mathrm{SO})_d)_{\mathbb{Q}}] \cong [\Sigma\mathbb{C}P^n, ((\mathrm{QS}^0/\mathrm{SO})_d)_{\mathbb{Q}}] \cong [\Sigma\mathbb{C}P^n, (B\mathrm{SO})_{\mathbb{Q}}] \cong \bigoplus_{i=1}^{\infty} H^{4i}(\Sigma\mathbb{C}P^n; \mathbb{Q}) \cong 0$. □

Lemma 5.27 *Let $X_4(\underline{d})$ and $X_4(\underline{d}')$ be nonspin complete intersections such that $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$. If $\eta_4(\underline{d})_{(2)} = \eta_4(\underline{d}')_{(2)} \in [\mathbb{C}P^4, ((\mathrm{QS}^0/\mathrm{SO})_d)_{(2)}]$, then $\eta_4(\underline{d}) = \eta_4(\underline{d}')$.*

Proof By Theorem 1.4, there is a homotopy 8-sphere Σ and a diffeomorphism $h\colon X_4(\underline{d}) \approx X_4(\underline{d}') \sharp \Sigma$. We may assume that h preserves the cohomology class x (see the proof of Proposition 2.10) and hence the maps $f_4(\underline{d})$ and $f_4(\underline{d}') \circ h$ are homotopic (as in the proof of Corollary 5.25). Let $\bar{h}\colon \nu_{X_4(\underline{d})} \to \nu_{X_4(\underline{d}') \sharp \Sigma}$ be the stable bundle map covering h, which is uniquely determined up to homotopy by the derivative of h. By Theorem 5.19, there are bundle maps $\bar{f}_4(\underline{d})$ and $\bar{f}_4(\underline{d}')$ such that $\eta_4(\underline{d}) = \eta([f_4(\underline{d}), \bar{f}_4(\underline{d})])$ and $\eta_4(\underline{d}') = \eta([f_4(\underline{d}'), \bar{f}_4(\underline{d}')])$. As in Section 5.2, let $f_\Sigma\colon \Sigma \to \mathbb{C}P^4$ be the constant map. Then the choice of an arbitrary framing of Σ determines a bundle map $\bar{f}_\Sigma$ over f_Σ and we get a normal map $(f_4(\underline{d}') \sharp f_\Sigma, \bar{f}_4(\underline{d}') \sharp \bar{f}_\Sigma)\colon X_4(\underline{d}') \sharp \Sigma \to \mathbb{C}P^4$. As explained in Section 2.3, since $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$, there is a stable bundle isomorphism $\alpha\colon \xi_4(\underline{d}')|_{\mathbb{C}P^4} \to \xi_4(\underline{d})|_{\mathbb{C}P^4}$. We have the following diagram of stable bundle maps, which commutes by Corollary 5.25:

$$\begin{tikzcd}
\nu_{X_4(\underline{d})} \arrow[r, "\bar{f}_4(\underline{d})"] \arrow[d, "\bar{h}"'] & \xi_4(\underline{d})|_{\mathbb{C}P^4} \\
\nu_{X_4(\underline{d}')\sharp\Sigma} \arrow[r, "\bar{f}_4(\underline{d}')\sharp\bar{f}_\Sigma"] & \xi_4(\underline{d}')|_{\mathbb{C}P^4} \arrow[u, "\alpha"']
\end{tikzcd}$$

It follows that the degree-d normal maps

$$(f_4(\underline{d}), \bar{f}_4(\underline{d}))\colon X_4(\underline{d}) \to \mathbb{C}P^4 \quad \text{and} \quad (f_4(\underline{d}') \sharp f_\Sigma, \bar{f}_4(\underline{d}') \sharp \bar{f}_\Sigma)\colon X_4(\underline{d}') \sharp \Sigma \to \mathbb{C}P^4$$

represent the same element in $\mathcal{N}_d^+(\mathbb{C}P^4)$, so $\eta([f_4(\underline{d}), \bar{f}_4(\underline{d})]) = \eta([f_4(\underline{d}')\sharp f_\Sigma, \bar{f}_4(\underline{d}')\sharp \bar{f}_\Sigma])$. Therefore by Lemma 5.13 we have

$$\eta_4(\underline{d}) = \eta_4(\underline{d}') \sharp [\psi]$$

for some $[\psi] \in (i_d)_*(\pi_8(\mathrm{QS}^0_d))$. By the naturality of $\sharp$ (see (7)) we have $\eta_4(\underline{d})_{[1/2]} = \eta_4(\underline{d}')_{[1/2]} \sharp [\psi_{[1/2]}]$. Since $\pi_8(\mathrm{QS}^0_d) \cong \pi_8^s$ and $\pi_8^s \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ by [Toda 1962, Theorem 7.1], $[\psi] \in \pi_8((\mathrm{QS}^0/\mathrm{SO})_d)$ is 2-torsion. This implies that $[\psi_{[1/2]}] = 0$, and hence $\eta_4(\underline{d})_{[1/2]} = \eta_4(\underline{d}')_{[1/2]}$. We assumed that $\eta_4(\underline{d})_{(2)} = \eta_4(\underline{d}')_{(2)}$ and so by Lemma 5.26, $\eta_4(\underline{d}) = \eta_4(\underline{d}')$. □

From now on we assume that d is odd. Then $(Z[1/d])_{(2)} \simeq Z_{(2)}$ for any simple space Z, so from the Brumfiel–Madsen equivalence $(G/O)[1/d] \simeq (\mathrm{QS}^0/\mathrm{SO})_d[1/d]$ of Proposition 5.12, we deduce the existence of a homotopy equivalence

$$\chi\colon ((\mathrm{QS}^0/\mathrm{SO})_d)_{(2)} \simeq (G/O)_{(2)}$$

such that $\delta_{(2)} \circ \chi = (\delta_d)_{(2)}$, where $\delta_{(2)}$ and $(\delta_d)_{(2)}$ are the 2-localisations the canonical maps δ and δ_d from (6). Moreover, by Sullivan's 2-primary splitting theorem for G/O [Madsen and Milgram 1979, Theorem 5.18], there is a homotopy equivalence

$$\phi\colon (G/O)_{(2)} \to (B\mathrm{SO})_{(2)} \times \operatorname{coker} J_{(2)}, \tag{8}$$

where the space $\operatorname{coker} J_{(2)}$ is defined in [loc. cit., Definition 5.16] and the map ϕ is constructed in the proof of [loc. cit., Theorem 5.18]. From the splitting of $(G/O)_{(2)}$ in (8) we obtain a projection map

$$\pi\colon (G/O)_{(2)} \to \operatorname{coker} J_{(2)}.$$

The following result is contained in the proof of [Feshbach 1986, Theorem 6], where the arguments rely on work of Feshbach [1987] and Ravenel [1984] on the Segal conjecture.

Theorem 5.28 (cf [Feshbach 1986, Proof of Theorem 6]) *For any prime p, $[\mathbb{C}P^\infty, \operatorname{coker} J_{(p)}] \cong 0$.*

Proof The proof of [Feshbach 1986, Theorem 6] states that the stable cohomotopy group $\pi_s^0(\mathbb{C}P^\infty)$ is trivial, where $\pi_s^0(\mathbb{C}P^\infty) = [\mathbb{C}P^\infty, \mathrm{QS}^0_0]$. The natural map $\mathrm{QS}^0_0 \to \prod_p (\mathrm{QS}^0_0)_{(p)}$ from QS^0_0 to the product of its p-localisations, taken over all primes p, is a weak equivalence, because QS^0_0 is connected with finite homotopy groups $\pi_i(\mathrm{QS}^0_0) \cong \pi_i^s$ (hence $\pi_i(\mathrm{QS}^0_0) \cong \prod_p \pi_i(\mathrm{QS}^0_0)_{(p)})$. Now by Sullivan's splitting of $\mathrm{QS}^0_1 \simeq \mathrm{QS}^0_0$ [Madsen and Milgram 1979, Theorem 5.18], $(\mathrm{QS}^0_0)_{(p)} \simeq \mathrm{im} J_{(p)} \times \mathrm{coker}\, J_{(p)}$ for a certain p-local space $\mathrm{im} J_{(p)}$. Therefore

$$0 \cong [\mathbb{C}P^\infty, \mathrm{QS}^0_0] \cong \prod_p \big([\mathbb{C}P^\infty, \mathrm{im} J_{(p)}] \times [\mathbb{C}P^\infty, \mathrm{coker}\, J_{(p)}]\big)$$

and the theorem follows. □

As a consequence of Theorem 5.28 we have:

Corollary 5.29 *If d is odd, then $\pi_*(\chi_*(\eta_n(\underline{d})_{(2)})) = 0 \in [\mathbb{C}P^n, \mathrm{coker}\, J_{(2)}]$ for all n.*

Proof Let $i\colon \mathbb{C}P^n \to \mathbb{C}P^\infty$ be the inclusion and consider the following commutative diagram:

$$\begin{tikzcd}
{[\mathbb{C}P^\infty, (\mathrm{QS}^0/\mathrm{SO})_d]} \arrow[r] \arrow[d, "i^*"'] & {[\mathbb{C}P^\infty, ((\mathrm{QS}^0/\mathrm{SO})_d)_{(2)}]} \arrow[r, "\chi_*"] \arrow[d, "i^*"'] & {[\mathbb{C}P^\infty, (G/O)_{(2)}]} \arrow[r, "\pi_*"] \arrow[d, "i^*"] & {[\mathbb{C}P^\infty, \mathrm{coker}\, J_{(2)}]} \arrow[d, "i^*"] \\
{[\mathbb{C}P^n, (\mathrm{QS}^0/\mathrm{SO})_d]} \arrow[r] & {[\mathbb{C}P^n, ((\mathrm{QS}^0/\mathrm{SO})_d)_{(2)}]} \arrow[r, "\chi_*"] & {[\mathbb{C}P^n, (G/O)_{(2)}]} \arrow[r, "\pi_*"] & {[\mathbb{C}P^n, \mathrm{coker}\, J_{(2)}]}
\end{tikzcd}$$

Now $\eta_n(\underline{d}) = i^*(\eta_\infty(\underline{d}))$ by Definition 5.11 and $[\mathbb{C}P^\infty, \mathrm{coker}\, J_{(2)}] \cong 0$ by Theorem 5.28, so the corollary follows from the commutativity of the diagram. □

Theorem 5.30 *Let $X_4(\underline{d})$ and $X_4(\underline{d}')$ be complete intersections with $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$ and odd total degree. Then $\eta_4(\underline{d}) = \eta_4(\underline{d}') \in [\mathbb{C}P^4, (\mathrm{QS}^0/\mathrm{SO})_d]$.*

The Sullivan conjecture for $n = 4$ and odd total degree follows directly from Theorems 5.20 and 5.30.

Theorem 5.31 *Let $X_4(\underline{d})$ and $X_4(\underline{d}')$ be complete intersections with $\mathrm{SD}_4(\underline{d}) = \mathrm{SD}_4(\underline{d}')$ and odd total degree. Then $X_4(\underline{d})$ is diffeomorphic to $X_4(\underline{d}')$.* □

Proof of Theorem 5.30 By Lemma 5.27, it is enough to prove that $\eta_4(\underline{d})_{(2)} = \eta_4(\underline{d}')_{(2)}$. Since the map $\chi\colon ((\mathrm{QS}^0/\mathrm{SO})_d)_{(2)} \to (G/O)_{(2)}$ is a homotopy equivalence, it suffices to show that $\chi_*(\eta_4(\underline{d})_{(2)}) = \chi_*(\eta_4(\underline{d}')_{(2)}) \in [\mathbb{C}P^4, (G/O)_{(2)}]$. To simplify the notation we set

$$\hat{\eta}(\underline{d}) := \chi_*(\eta_4(\underline{d})_{(2)}) \quad \text{and} \quad \hat{\eta}(\underline{d}') := \chi_*(\eta_4(\underline{d}')_{(2)}).$$

Let $\mu\colon (G/O)_{(2)} \to B\mathrm{SO}_{(2)}$ and $\alpha_{(2)}\colon (B\mathrm{SO})_{(2)} \to (G/O)_{(2)}$ be the projection and inclusion defined by the Sullivan splitting of $(G/O)_{(2)}$ in (8) respectively, so that $\phi = \mu \times \pi\colon (G/O)_{(2)} \to (B\mathrm{SO})_{(2)} \times \mathrm{coker}\, J_{(2)}$, and consider the bijection

$$\mu_* \times \pi_*\colon [\mathbb{C}P^4, (G/O)_{(2)}] \equiv [\mathbb{C}P^4, (B\mathrm{SO})_{(2)}] \times [\mathbb{C}P^4, \mathrm{coker}\, J_{(2)}].$$

Corollary 5.29 states that $\pi_*(\hat{\eta}(\underline{d})) = \pi_*(\hat{\eta}(\underline{d}')) = 0$, hence

(a) it remains to show that $\mu_*(\hat\eta(\underline{d})) = \mu_*(\hat\eta(\underline{d}'))$, and

(b) we have
$$\hat\eta(\underline{d}) = (\alpha_{(2)} \circ \mu)_*(\hat\eta(\underline{d})) \quad \text{and} \quad \hat\eta(\underline{d}') = (\alpha_{(2)} \circ \mu)_*(\hat\eta(\underline{d}')).$$

It follows from Lemma 5.32 that $\delta_{(2)*}(\hat\eta(\underline{d})) = \delta_{(2)*}(\hat\eta(\underline{d}'))$; hence
$$(\delta_{(2)} \circ \alpha_{(2)} \circ \mu)_*(\hat\eta(\underline{d})) = (\delta_{(2)} \circ \alpha_{(2)} \circ \mu)_*(\hat\eta(\underline{d}')).$$

By Lemma 5.33 $(\delta_{(2)} \circ \alpha_{(2)})_* : [\mathbb{C}P^4, (BSO)_{(2)}] \to [\mathbb{C}P^4, (BSO)_{(2)}]$ is injective, so $\mu_*(\hat\eta(\underline{d})) = \mu_*(\hat\eta(\underline{d}'))$, which completes the proof. $\square$

Lemma 5.32 *Suppose that* $n \not\equiv 1 \bmod 4$. *If* $X_n(\underline{d})$ *and* $X_n(\underline{d}')$ *are complete intersections such that* $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$, *then* $(\delta_{(2)} \circ \chi)_*(\eta_n(\underline{d})_{(2)}) = (\delta_{(2)} \circ \chi)_*(\eta_n(\underline{d}')_{(2)})$.

Proof By the assumption $\mathrm{SD}_n(\underline{d}) = \mathrm{SD}_n(\underline{d}')$, the complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$, and hence their normal bundles $\nu_{X_n(\underline{d})}$ and $\nu_{X_n(\underline{d}')}$, have the same Pontryagin classes (regarded as integers). This implies that $p_j(\xi_n(\underline{d})) = p_j(\xi_n(\underline{d}'))$ for $2j \le n$ (see Section 2.2). By [Sanderson 1964, Theorem 3.9], if $n \not\equiv 1 \bmod 4$, then $[\mathbb{C}P^n, BSO] \cong \mathbb{Z}^{\lfloor n/2 \rfloor}$, detected by the total Pontryagin class. Therefore $\xi_n(\underline{d})|_{\mathbb{C}P^n} \cong \xi_n(\underline{d}')|_{\mathbb{C}P^n}$, and hence $(n+1)\gamma \oplus \xi_n(\underline{d})|_{\mathbb{C}P^n} \cong (n+1)\gamma \oplus \xi_n(\underline{d}')|_{\mathbb{C}P^n}$.

Since δ_d classifies taking the formal difference of the source and target vector bundles of a fibrewise degree-d map (see Section 5.2), $(\delta_d)_*(\eta_n(\underline{d})) \in [\mathbb{C}P^n, BSO]$ is the classifying map of $-k\gamma \oplus \gamma^{d_1} \oplus \cdots \oplus \gamma^{d_k}|_{\mathbb{C}P^n} \cong (n+1)\gamma \oplus \xi_n(\underline{d})|_{\mathbb{C}P^n}$. Therefore we have $(\delta_d)_*(\eta_n(\underline{d})) = (\delta_d)_*(\eta_n(\underline{d}'))$.

By localising at 2 we get that $(\delta_d)_{(2)*}(\eta_n(\underline{d})_{(2)}) = (\delta_d)_{(2)*}(\eta_n(\underline{d}')_{(2)})$. We saw that χ satisfies $\delta_{(2)} \circ \chi = (\delta_d)_{(2)}$ (see Proposition 5.12), so this means that $(\delta_{(2)} \circ \chi)_*(\eta_n(\underline{d})_{(2)}) = (\delta_{(2)} \circ \chi)_*(\eta_n(\underline{d}')_{(2)})$. $\square$

Lemma 5.33 *Suppose* $n \not\equiv 1 \bmod 4$. *Then the map* $(\delta_{(2)} \circ \alpha_{(2)})_* : [\mathbb{C}P^n, (BSO)_{(2)}] \to [\mathbb{C}P^n, (BSO)_{(2)}]$ *is injective.*

Proof The proof of [Madsen and Milgram 1979, Theorem 5.18] shows that there is a commutative diagram

$$\begin{tikzcd} & (G/O)_{(2)} \arrow[d, "\delta_{(2)}"] \\ (BSO)_{(2)} \arrow[ur, "\alpha_{(2)}"] \arrow[r, "\psi^3-\mathrm{Id}"] & (BSO)_{(2)} \end{tikzcd}$$

where ψ^3 is the map induced by the third-power Adams operation; see [loc. cit., 5.13 and Theorem 5.18]. The map $\psi^3 - \mathrm{Id}$ is a rational homotopy equivalence (to see this, we note that the homotopy fibre of $\psi^3 - \mathrm{Id}$ is connected and has finite homotopy groups by the second Sullivan splitting in [loc. cit., Theorem 5.18]); hence $\delta_{(2)} \circ \alpha_{(2)}$ is a rational homotopy equivalence.

Now let $j_{\mathbb{Q}}\colon (BSO)_{(2)} \to ((BSO)_{(2)})_{\mathbb{Q}} = BSO_{\mathbb{Q}}$ be the natural map to the rationalisation of BSO. There is a commutative square

$$\begin{tikzcd} {[\mathbb{C}P^n, (BSO)_{(2)}]} \arrow[r, "j_{\mathbb{Q}*}"] \arrow[d, "(\delta_{(2)}\circ\alpha_{(2)})_*"] & {[\mathbb{C}P^n, BSO_{\mathbb{Q}}]} \arrow[d, "(\delta_{\mathbb{Q}}\circ\alpha_{\mathbb{Q}})_*"] \\ {[\mathbb{C}P^n, (BSO)_{(2)}]} \arrow[r, "j_{\mathbb{Q}*}"] & {[\mathbb{C}P^n, BSO_{\mathbb{Q}}]} \end{tikzcd}$$

where $\delta_{\mathbb{Q}}$ and $\alpha_{\mathbb{Q}}$ are the rationalisations of $\delta_{(2)}$ and $\alpha_{(2)}$ respectively. In particular, $\delta_{\mathbb{Q}} \circ \alpha_{\mathbb{Q}}$ is a homotopy equivalence; hence $(\delta_{\mathbb{Q}} \circ \alpha_{\mathbb{Q}})_*$ is a bijection.

Since there are isomorphisms $[\mathbb{C}P^n, BSO_{\mathbb{Q}}] \cong [\mathbb{C}P^n, (BSO)_{(2)}] \otimes \mathbb{Q} \cong (\mathbb{Z}_{(2)})^{\lfloor n/2 \rfloor} \otimes \mathbb{Q} \cong \mathbb{Q}^{\lfloor n/2 \rfloor}$, it follows that $j_{\mathbb{Q}*}$ is injective. Therefore $(\delta_{\mathbb{Q}} \circ \alpha_{\mathbb{Q}})_* \circ j_{\mathbb{Q}*} = j_{\mathbb{Q}*} \circ (\delta_{(2)} \circ \alpha_{(2)})_*$ is injective, which implies that $(\delta_{(2)} \circ \alpha_{(2)})_*$ is injective. $\square$

Remark 5.34 The arguments of this section can be generalised to prove the Sullivan conjecture "prime to the total degree". We plan to take this up in future work.

Appendix. Extensions and Toda brackets

Appendix Extensions and Toda brackets

Recall that $\mathbb{S}^0$ denotes the sphere spectrum and that the i^{th} stable stem, $\pi_i(\mathbb{S}^0)$, is denoted by π_i^s. The k-fold suspension of $\mathbb{S}^0$ is denoted by $\mathbb{S}^k$, and if $f\colon \mathbb{S}^k \to \mathbb{S}^0$ is a map, then C_f denotes the cofibre of f. The aim of this appendix is to prove Lemma A.1, which concerns the role of Toda brackets in computing extensions for homotopy groups of C_f. Lemma A.1 is presumably well known, but we did not find a proof for it in the literature so far.

The stable homotopy groups of C_f lie in the following fragment of the long exact Puppe sequence:

$$\cdots \to \pi_{j-k}^s \xrightarrow{f_*} \pi_j^s \xrightarrow{i_*} \pi_j(C_f) \xrightarrow{c_*} \pi_{j-k-1}^s \to \cdots .$$

Here f_*, i_* and c_* are respectively the homomorphisms induced by composition with f, the inclusion $i\colon \mathbb{S}^0 \subset C_f$, and the collapse map $c\colon C_f \to \mathbb{S}^{k+1}$. We shall be interested in describing the extension

$$0 \to \operatorname{im}(i_*) \to \pi_j(C_f) \to \operatorname{im}(c_*) \to 0. \tag{9}$$

To do this we take an element $g \in \pi_{j-k-1}^s$ of order a for some positive integer a, which lifts to $\bar{g} \in \pi_j(C_f)$. Then $a\bar{g} \in \operatorname{im}(i_*) \cong \operatorname{coker}(f_*)$. The element $a\bar{g} \in \pi_j^s$ will of course depend on the choice of $\bar{g}$ in general.

To describe $a\bar{g}$ we consider the sequence of maps

$$\mathbb{S}^{j-1} \xrightarrow{f} \mathbb{S}^{j-k-1} \xrightarrow{g} \mathbb{S}^0 \xrightarrow{a} \mathbb{S}^0.$$

Since $g \circ f$ and $a \circ g$ are both null-homotopic, the Toda bracket

$$\langle a, g, f \rangle \subseteq \pi_j^s$$

is defined. Representatives for the elements of $\langle a, g, f \rangle$ are defined as unions

$$(a \circ H_1) \cup (C(f) \circ H_2) \colon C(\mathbb{S}^{j-1}) \cup C(\mathbb{S}^{j-1}) \to \mathbb{S}^0,$$

where H_1 is a null-homotopy of $g \circ f$, H_2 is a null-homotopy of $a \circ g$ and $C(-)$ denotes the cone of a spectrum or a map. The indeterminacy of $\langle a, g, f \rangle$ arises from the choice of null-homotopies H_1 and H_2 and is given by

$$I(\langle a, g, f \rangle) = f_*(\pi_{j-k}^s) + a\pi_j^s \subseteq \pi_j^s.$$

We now relate the restriction of the extension (9) to the cyclic subgroup $\langle g \rangle \subset \pi_{j-k-1}^s$ generated by g to the Toda bracket $\langle a, g, f \rangle$.

Lemma A.1 *Suppose that $g \in \pi_{j-k-1}^s$ has order a and that $\bar{g} \colon \mathbb{S}^j \to C_f$ is a map such that $c \circ \bar{g} = g$. Then*

$$a\bar{g} \in i_*(\langle a, g, f \rangle) \subset \pi_j(C_f).$$

In particular, the extension

$$0 \to \operatorname{im}(i_*) \to (c_*)^{-1}(\langle g \rangle) \to \langle g \rangle \to 0$$

is trivial if and only if $0 \in \langle a, g, f \rangle$.

Proof Given $H_1 \colon C(\mathbb{S}^{j-1}) \to \mathbb{S}^0$, a null-homotopy of $g \circ f \colon \mathbb{S}^{j-1} \to \mathbb{S}^0$, we define a choice of $\bar{g} \in \pi_j(C_f)$ by

$$\bar{g} = H_1 \cup C(g) \colon C(\mathbb{S}^{j-1})_1 \cup C(\mathbb{S}^{j-1})_2 \to C_f,$$

where the subscripts label two copies of $C(\mathbb{S}^{j-1})$. There is an a-fold fold map $a_{C_f} \colon (C_f, \mathbb{S}^0) \to (C_f, \mathbb{S}^0)$, which extends $a \colon \mathbb{S}^0 \to \mathbb{S}^0$, and we have $a\bar{g} = a_{C_f} \circ \bar{g}$. On the first copy of $C(\mathbb{S}^{j-1})$ we have $(a_{C_f} \circ \bar{g})|_{C(\mathbb{S}^{j-1})_1} = a \circ H_1$. On the second copy of $C(\mathbb{S}^{j-1})$, the map $(a_{C_f} \circ \bar{g})|_{C(\mathbb{S}^{j-1})_2}$ defines the zero element of $\pi_j^s(C_f, \mathbb{S}^0) \cong \pi_{j-1}^s$. It follows that $(a_{C_f} \circ \bar{g})|_{C(\mathbb{S}^{j-1})_2}$ is homotopic rel $\mathbb{S}^{j-1}$ to $H_2 \circ C(f)$, where $H_2 \colon C(\mathbb{S}^{j-k-1}) \to \mathbb{S}^0$ is a null-homotopy of ag. It follows that $a\bar{g} = a_{C_f} \circ \bar{g}$ is homotopic to $i \circ \big((a \circ H_1) \cup (H_2 \circ C(f))\big)$ and so $a\bar{g} \in i_*(\langle a, g, f \rangle)$ as required.

Finally, the extension $0 \to \operatorname{im}(i_*) \to (c_*)^{-1}(\langle g \rangle) \to \langle g \rangle \to 0$ is trivial if and only if there is $\bar{g} \in \pi_j(C_f)$ such that $a\bar{g} = 0$. Given such a $\bar{g}$, we have $0 \in i_*(\langle a, g, f \rangle)$ by the previous paragraph and so $\langle a, g, f \rangle$ contains an element of $\ker(i_*) = f_*(\pi_{j-k}^s)$. Hence $\langle a, g, f \rangle \cap I(\langle a, g, f \rangle) \neq 0$ and so $0 \in \langle a, g, f \rangle$. Conversely, $0 \in \langle a, g, f \rangle$ if and only if $\langle a, g, f \rangle = f_*(\pi_{j-k}^s) + a\pi_j^s$ and then $a\bar{g} \in i_*(\langle a, g, f \rangle) = ai_*(\pi_j^s)$. Hence we can modify our choice of $\bar{g}$ to achieve $a\bar{g} = 0$. $\square$

References

[Adams 1966] **J F Adams**, *On the groups $J(X)$, IV*, Topology 5 (1966) 21–71 MR Zbl

[Baraglia 2020] **D Baraglia**, *The alpha invariant of complete intersections*, preprint (2020) arXiv 2002.06750

[Browder 1972] **W Browder**, *Surgery on simply-connected manifolds*, Ergebnisse der Math. 65, Springer (1972) MR Zbl

[Browder 1979] **W Browder**, *Complete intersections and the Kervaire invariant*, from "Algebraic topology" (J L Dupont, I H Madsen, editors), Lecture Notes in Math. 763, Springer (1979) 88–108 MR Zbl

[Brown 1962] **E H Brown, Jr**, *Cohomology theories*, Ann. of Math. 75 (1962) 467–484 MR Zbl

[Brumfiel and Madsen 1976] **G Brumfiel**, **I Madsen**, *Evaluation of the transfer and the universal surgery classes*, Invent. Math. 32 (1976) 133–169 MR Zbl

[Dimca 1992] **A Dimca**, *Singularities and topology of hypersurfaces*, Springer (1992) MR Zbl

[Fang and Klaus 1996] **F Fang**, **S Klaus**, *Topological classification of 4-dimensional complete intersections*, Manuscripta Math. 90 (1996) 139–147 MR Zbl

[Fang and Wang 2010] **F Fang**, **J Wang**, *Homeomorphism classification of complex projective complete intersections of dimensions 5, 6 and 7*, Math. Z. 266 (2010) 719–746 MR Zbl

[Feshbach 1986] **M Feshbach**, *Essential maps exist from BU to coker J*, Proc. Amer. Math. Soc. 97 (1986) 539–545 MR Zbl

[Feshbach 1987] **M Feshbach**, *The Segal conjecture for compact Lie groups*, Topology 26 (1987) 1–20 MR Zbl

[Freedman 1982] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR Zbl

[Giambalvo 1971] **V Giambalvo**, *On $\langle 8\rangle$-cobordism*, Illinois J. Math. 15 (1971) 533–541 MR Zbl

[Hambleton and Madsen 1986] **I Hambleton**, **I Madsen**, *Local surgery obstructions and space forms*, Math. Z. 193 (1986) 191–214 MR Zbl

[Hatcher 2002] **A Hatcher**, *Algebraic topology*, Cambridge University Press (2002) MR Zbl

[Hilton 1955] **P J Hilton**, *On the homotopy groups of the union of spheres*, J. Lond. Math. Soc. 30 (1955) 154–172 MR Zbl

[Jupp 1973] **P E Jupp**, *Classification of certain 6-manifolds*, Proc. Cambridge Philos. Soc. 73 (1973) 293–300 MR Zbl

[Kasilingam 2016] **R Kasilingam**, *Classification of smooth structures on a homotopy complex projective space*, Proc. Indian Acad. Sci. Math. Sci. 126 (2016) 277–281 MR Zbl

[Kervaire and Milnor 1963] **M A Kervaire**, **J W Milnor**, *Groups of homotopy spheres, I*, Ann. of Math. 77 (1963) 504–537 MR Zbl

[Kreck 1999] **M Kreck**, *Surgery and duality*, Ann. of Math. 149 (1999) 707–754 MR Zbl

[Libgober and Wood 1982] **A S Libgober**, **J W Wood**, *Differentiable structures on complete intersections, I*, Topology 21 (1982) 469–482 MR Zbl

[Madsen and Milgram 1979] **I Madsen**, **R J Milgram**, *The classifying spaces for surgery and cobordism of manifolds*, Ann. of Math. Stud. 92, Princeton Univ. Press (1979) MR Zbl

[Milnor 1958] **J Milnor**, *On the Whitehead homomorphism J*, Bull. Amer. Math. Soc. 64 (1958) 79–82 MR Zbl

[Milnor and Stasheff 1974] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Ann. of Math. Stud. 76, Princeton Univ. Press (1974) MR Zbl

[Mosher and Tangora 1968] **R E Mosher**, **M C Tangora**, *Cohomology operations and applications in homotopy theory*, Harper & Row, New York (1968) MR Zbl

[Nagy 2021] **Cs Nagy**, *The classification of* 8*-dimensional* E*-manifolds*, PhD thesis, University of Melbourne (2021) Available at `http://hdl.handle.net/11343/279361`

[Ravenel 1984] **D C Ravenel**, *The Segal conjecture for cyclic groups and its consequences*, Amer. J. Math. 106 (1984) 415–446 MR Zbl

[Sanderson 1964] **B J Sanderson**, *Immersions and embeddings of projective spaces*, Proc. Lond. Math. Soc. 14 (1964) 137–153 MR Zbl

[Smale 1962] **S Smale**, *On the structure of manifolds*, Amer. J. Math. 84 (1962) 387–399 MR Zbl

[Sullivan 1970] **D P Sullivan**, *Geometric topology: localization, periodicity and Galois symmetry*, K-Monogr. Math. 8, Springer (1970) MR Zbl

[Toda 1962] **H Toda**, *Composition methods in homotopy groups of spheres*, Ann. of Math. Stud. 49, Princeton Univ. Press (1962) MR Zbl

[Wall 1966] **C T C Wall**, *Classification problems in differential topology, V: On certain 6-manifolds*, Invent. Math. 1 (1966) 355–374 MR Zbl Correction in 2 (1966) 306

[Wall 1970] **C T C Wall**, *Surgery on compact manifolds*, Lond. Math. Soc. Monogr. 1, Academic Press, London (1970) MR Zbl

[Whitehead 1978] **G W Whitehead**, *Elements of homotopy theory*, Graduate Texts in Math. 61, Springer (1978) MR Zbl

School of Mathematics and Statistics, University of Melbourne
Parkville, VIC, Australia

School of Mathematics and Statistics, University of Glasgow
Glasgow, United Kingdom

`dcrowley@unimelb.edu.au`, `csaba.nagy@glasgow.ac.uk`

Proposed: András I Stipsicz — Received: 26 March 2020
Seconded: Ciprian Manolescu, Benson Farb — Revised: 11 December 2023

GEOMETRY & TOPOLOGY

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2025 is US $865/year for the electronic version, and $1210/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 29 Issue 1 (pages 1–548) 2025