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Renaud Detcherry

Ramanujan Santharoubane

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An embedding of skein algebras of surfaces into localized quantum tori from Dehn–Thurston coordinates

Renaud Detcherry
Ramanujan Santharoubane

We construct embeddings of Kauffman bracket skein algebras of surfaces (either closed or with boundary) into localized quantum tori using the action of the skein algebra on the skein module of the handlebody. We use those embeddings to study representations of Kauffman skein algebras at roots of unity and get a new proof of Bonahon and Wong's unicity conjecture. Our method allows one to explicitly reconstruct the unique representation with fixed classical shadow, as long as the classical shadow is irreducible with image not conjugate to the quaternion group.

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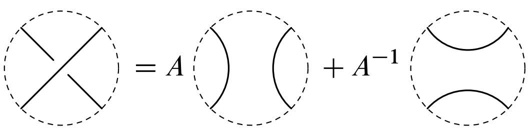
1 Introduction

For Σ a compact connected oriented surface and G an algebraic group, its character variety is

$$X(\Sigma, G) = \{\rho \colon \pi_1(\Sigma) \to G\} /\!\!/ G,$$

where G acts by conjugation. One of the simplest and most intriguing character varieties is obtained when $G = \mathrm{SL}_2(\mathbb{C})$, as it is then well-understood algebraically while being connected to hyperbolic geometry, knot theory and 3-dimensional topology, with many beautiful applications. Concretely speaking, $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ is just an algebraic variety of dimension $6g-6+3n$ when Σ is a surface with negative Euler characteristic of genus g with n boundary components, and the ring of regular functions on the character variety $\mathbb{C}[X(\Sigma, \mathrm{SL}_2(\mathbb{C}))]$ is just a commutative algebra.

Skein algebras and skein modules, introduced independently by Przytycki [1991] and by Turaev [1988], give a quantization of character varieties. The skein module $S(M)$ of a compact oriented 3-manifold M is a $\mathbb{Z}[A^{\pm 1}]$-module which is the quotient of the free module spanned by isotopy classes of framed links in the interior of M, modulo the famous Kauffman relations



and $\quad L \cup U = (-A^2 - A^{-2})L,$

where the first relates three framed links in M that are identical except in a small ball, and in the second U is the unframed unknot. When $M = \Sigma \times [0,1]$, with Σ a compact oriented surface, the skein module has a natural structure of an algebra given by the stacking operation. We will write $S(\Sigma)$ for the skein algebra of a surface (ie the skein module of $\Sigma \times [0,1]$, with its natural algebra structure). We recall also that when M is a compact oriented surface, the skein module $S(M)$ is a module over the skein algebra $S(\partial M)$, again for the natural stacking operation.

A result of Bullock [1997] and Przytycki and Sikora [2000] asserts that, setting the parameter A to -1, the skein algebra $S(\Sigma) \otimes_{A=-1} \mathbb{C}$ is isomorphic to the commutative algebra $\mathbb{C}[X(\Sigma, \mathrm{SL}_2(\mathbb{C}))]$, and a further result of Turaev [1991] states that $S(\Sigma)$ is actually a deformation quantization of that algebra in the direction of the Atiyah–Bott–Goldman Poisson bracket.

The skein algebras of surfaces have deep ties with much of quantum topology, in particular with the Jones polynomials, the Witten–Reshetikhin–Turaev invariants of 3-manifolds and their associated TQFTs. A better understanding of skein algebras (or their 3-dimensional counterparts skein modules) seems to be key for working on many of the open conjectures in quantum topology [Lê and Zhang 2017; Marché and Santharoubane 2021; Bonahon et al. 2021]. Contrary to functions on the $\mathrm{SL}_2(\mathbb{C})$-character variety, skein algebras are a difficult object to tackle, in part due to being noncommutative algebras. For instance, except for a few small surfaces, there is at the moment no known presentations of $S(\Sigma)$ besides the definition.

However, pioneering work of Bonahon and Wong [2011; 2016; 2017; 2019] led to a breakthrough in our understanding of skein algebras. They constructed an embedding, the *quantum trace map*, from the skein algebras of a surface Σ with $n \geq 1$ punctures to a quantum torus. A quantum torus is a noncommutative algebra of the form $\mathbb{Z}[A^{\pm 1}]\langle X_i \mid i \in I\rangle / \{X_i X_j = A^{\sigma_{ij}} X_j X_i\}$; in a way, quantum tori are the simplest possible noncommutative algebras. Bonahon and Wong used the Chekhov–Fock quantization of the Teichmüller space as their target space to define their quantum trace map, then they managed to quantize the map from character variety to shear coordinates. Some difficult computations are required to check that their formulas indeed yield an algebra morphism. Their method of defining a quantum trace map has since been simplified by Lê [2019], and recently extended by Lê and Yu to SL_n character variety.

Our first result is to propose an alternative way of defining an embedding of the skein algebra into a quantum torus, or rather in our case a *localized* quantum torus. Let $\mathcal{P}$ be a pants decomposition of Σ consisting of n curves nonparallel to the boundary and b curves parallel to the boundary. We write $\hat{\Sigma}$ for the closed surface obtained from Σ by filling the boundary components by disks. We consider the quantum torus $\mathcal{T}(\mathcal{P})$ over $\mathbb{Z}[A^{\pm 1}]$ with $2n+b$ variables $E_1, \dots, E_n, Q_1, \dots, Q_n, C_1, \dots, C_b$, where all variables commute except Q_i and E_i (for all $i \in \{1, \dots, n\}$) that satisfy $Q_i E_i = A E_i Q_i$. Viewed as a ring, the quantum torus $\mathcal{T}(\mathcal{P})$ is an integral domain, so we can define $\mathcal{A}(\mathcal{P})$ to be a $\mathbb{Z}[A^{\pm 1}]$-algebra containing $\mathcal{T}(\mathcal{P})$ where $A^k Q_i^2 - A^{-k} Q_i^{-2}$ is invertible for all $1 \leq i \leq n$ and $k \in \mathbb{Z}$. The algebra $\mathcal{A}(\mathcal{P})$ will be called a localized quantum torus.

Theorem 1.1 *There is an injective $\mathbb{Z}[A^{\pm 1}]$-algebra homomorphism*

$$\sigma : S(\Sigma) \to \mathcal{A}(\mathcal{P})$$

that factors through the natural action $S(\Sigma) \to \mathrm{End}(S(H, \mathbb{Q}(A)))$, *where* H *is a handlebody with boundary* $\widehat{\Sigma}$, *such that curves in* $\mathcal{P}$ *bound a disk in* H. *Moreover for any curve* $\alpha \in \mathcal{P}$ *there exists* $Q \in \{Q_1, \dots, Q_n, C_1, \dots, C_b\}$ *such that*

$$\sigma(\alpha) = -(A^2 Q^2 + A^{-2} Q^{-2}).$$

We remark that when Σ has boundary, $S(H, \mathbb{Q}(A))$ is the sum of relative skein modules of H where we add a colored point in each filling disk of $\widehat{\Sigma}$.

Note that in particular, we recover a theorem of Lê [2022] about the faithfulness of the natural action of $S(\Sigma)$ on $S(H)$. The precise definition of the embedding will be given in Definition 2.5; it is based on the study of coefficients of *curve operators* on $S(H, \mathbb{Q}(A))$ where H is a handlebody with boundary Σ, in some basis given by trivalent colored graphs (see Lemma 2.1), which are the skein module version of the basis of WRT-TQFTs given by Blanchet, Habegger, Masbaum and Vogel [Blanchet et al. 1995].

Morally speaking our embedding could be thought as the quantization of Fenchel–Nielsen coordinates associated to a pair of pants decomposition, while Bonahon and Wong's quantum trace map is based on a quantization of shear coordinates on the Teichmüller space.

Compared with Bonahon and Wong's result, our embedding has the drawback of landing in a localized quantum torus instead of just a quantum torus, but in exchange we get several nice features. First, our result applies to closed surfaces as well as surfaces with boundary, whereas Bonahon and Wong's embedding needs punctures to be defined, since they have to start with ideal triangulations of the surface Σ. Second, our embedding arises in a more natural way, by studying the action of $S(\Sigma)$ on the skein module $S(H)$ of a handlebody with boundary Σ, in the graph basis of $S(H)$ associated to the pair of pants decomposition $\mathcal{P}$. The proof that we get an embedding does not require checking difficult formulas, and will be a simple byproduct of the fact that $S(H)$ is a module over $S(\Sigma)$.

Finally, a nice feature of this new embedding is that it is in a way almost surjective. Indeed we see in Theorem 1.1 that the elements $Q_1, \dots, Q_n, C_1, \dots, C_b$ satisfy a degree-4 polynomial equation with coefficients in $\sigma(S(\Sigma))$. A similar property holds for the elements $E_1, \dots, E_n$:

Proposition 1.2 *There exists a finite-index subgroup* Λ *of* $\mathbb{Z}^n$ *such that for all* $k = (k_1, \dots, k_n) \in \Lambda$,

$$E_1^{k_1} \cdots E_n^{k_n} = \sum_{j \in I} \sigma(\gamma_j) G_j,$$

where $\{\gamma_j \mid j \in I\}$ *is a finite set of multicurves on* Σ *and* $\{G_j \mid j \in I\}$ *are rational fractions in* $Q_1, \dots, Q_n, C_1, \dots, C_b$.

Another important aspect of Bonahon and Wong's work is the study of irreducible finite-dimensional complex representations of $S_\xi(\Sigma) = S(\Sigma) \otimes_{A=\xi} \mathbb{C}$ when ξ is root of unity. More precisely, when ξ is root of unity of order twice an odd number, Bonahon and Wong [2016] associate to any such representation ρ of $S_\xi(\Sigma)$ a canonical point $r_\rho \in X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ called the classical shadow of ρ. A natural and important question is whether points in $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ completely classify irreducible finite-dimensional representations of $S_\xi(\Sigma)$. Bonahon and Wong [2017] proved that if Σ has at least one boundary component, any point in $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ satisfying a geometric condition is the classical shadow of an irreducible representation of $S_\xi(\Sigma)$. When Σ has no boundary component they removed this geometric condition to prove that $\rho \mapsto r_\rho$ is surjective in [Bonahon and Wong 2019]. The surjectivity for any surface also follows from [Frohman et al. 2019], where it is derived from the theory of Azumaya algebras. They asked which points in $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ have a single preimage by the map r_ρ; the *unicity conjecture* that they formulated was that for an open dense subset of $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$. The classical shadow determines the representation. We prove:

Theorem 1.3 *Let Σ be a closed compact oriented surface of genus $g \geq 2$, let ξ be a $2p^{th}$ primitive root of unity with $p \geq 3$ an odd number and $\rho\colon S_\xi(\Sigma) \to \mathrm{End}(V)$ be an irreducible representation with classical shadow r. Assume that r is an irreducible representation whose image is not isomorphic to the quaternion group with 8 elements. Then there exists a unique irreducible representation of $S_\xi(\Sigma)$ with classical shadow r, up to isomorphism.*

Remark 1.4 While we restrict to ξ a $2p^{\text{th}}$ root of unity with p odd, our methods should apply for other roots of unity with minimal changes. The restriction to closed surfaces is more fundamental and comes from our use of a certain pants decompositions of surfaces to simplify the computations (see Section 3), Theorem 1.1 of [Santharoubane 2024] that gives generators of skein algebras $S_\xi(\Sigma)$, and the main theorem of [Detcherry et al. 2024].

Theorem 1.3 is a consequence of the following stronger statement:

Theorem 1.5 *Suppose that Σ has at most one boundary component. Let ξ be a $2p^{th}$ primitive root of unity with $p \geq 3$ an odd number and $\rho\colon S_\xi(\Sigma) \to \mathrm{End}(V)$ be an irreducible representation with classical shadow r. Let $\mathcal{P}$ be a pants decomposition of Σ in the same orbit, under the action of the mapping class group of Σ, as the one shown in Figure 4. Suppose that for $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{P}$ bounding a pair of pants*

$$2 + \sum_{k=1}^{3} \mathrm{Tr}(r(\alpha_k^2)) - \prod_{k=1}^{3} \mathrm{Tr}(r(\alpha_k)) \neq 0, \tag{1}$$

and for all $\alpha \in \mathcal{P}$

$$\mathrm{Tr}(r(\alpha)) \neq \pm 2. \tag{2}$$

Then there is a representation $\tilde{\rho}\colon \mathcal{A}_\xi(\Gamma)^0 \to \mathrm{End}(V)$ such that $\tilde{\rho} \circ \sigma_\xi = \rho$.

In this theorem $\mathcal{A}(\Gamma)^0$ is a $\mathbb{Z}[A^{\pm 1}]$-subalgebra of $\mathcal{A}(\mathcal{P})$ containing $\sigma(S(\Sigma))$. The definition of $\mathcal{A}(\Gamma)^0$ is given in Definition 3.1; it is isomorphic to a localized quantum torus (see Lemma 3.2). The notation $\mathcal{A}_\xi(\Gamma)^0$ stands for $\mathcal{A}(\Gamma)^0 \otimes_{A=\xi} \mathbb{C}$ and σ_ξ is the induced map $S_\xi(\Sigma) \to \mathcal{A}_\xi(\Gamma)^0$. As representations of $\mathcal{A}_\xi(\Gamma)^0$ are well understood, we have the following corollary:

Corollary 1.6 *Let ρ_1 and ρ_2 be two irreducible complex finite-dimensional representations of $S_\xi(\Sigma)$ with the same classical shadow r satisfying the hypothesis of Theorem 1.5. Moreover if Σ has a boundary component, we suppose that ρ_1 and ρ_2 have the same scalar value on any simple closed curve parallel to the boundary. Then ρ_1 and ρ_2 are isomorphic with dimension p^{3g-2} when Σ has a boundary component and p^{3g-3} when Σ has no boundary (here g is the genus of Σ).*

Note that the dimensions of representations of $S_\xi(\Sigma)$ in the Azumaya locus have been computed in [Frohman et al. 2021], but our proof is independent.

The connection between conditions (1) and (2) in Theorem 1.5 and nonquaternionic representations is established by the two authors and Thomas Le Fils in [Detcherry et al. 2024], where we prove:

Theorem 1.7 [Detcherry et al. 2024] *Let Σ be a closed compact oriented surface. The set of conjugacy classes of representations $r:\pi_1(\Sigma)\to \mathrm{SL}_2(\mathbb{C})$ satisfying the hypothesis of Theorem 1.5 is equal to the set of irreducible representations minus representations whose image is isomorphic to the quaternion group with 8 elements.*

The fact that we have to exclude the representations with quaternionic image is due to the specific type of pair of pants decomposition we use in Theorem 1.5. Indeed a representation with quaternionic image has trace ± 2 on any separating closed curve on Σ.

Notice that the conditions satisfied by the classical shadow r in Theorem 1.5 defined a Zariski open dense subset of $X(\Sigma,\mathrm{SL}_2(\mathbb{C}))$. Therefore Corollary 1.6 recovers a theorem of Frohman, Kania-Bartoszynska and Lê [Frohman et al. 2019] that showed that there is a Zariski open dense subset of $X(\Sigma,\mathrm{SL}_2(\mathbb{C}))$ on which classical shadows have a single preimage. Our proof thus gives an alternative proof of the unicity conjecture of Bonahon and Wong. We note that the proof in [Frohman et al. 2019] used abstract arguments about Azumaya algebras and facts about the center of skein algebras at roots of unity, and produced a nonexplicit Zariski open subset of $X(\Sigma,\mathrm{SL}_2(\mathbb{C}))$ where the unicity conjecture holds.

An alternative (and more constructive) approach to the unicity conjecture was initiated by Takenov [2015] when Σ is the one-holed torus and the 4-holed sphere. It involved explicitly extending the representations of $S_\xi(\Sigma)$ to a quantum torus in which it embeds. We take advantage of the embedding defined by Theorem 1.1 to extend Takenov's strategy for compact oriented surfaces that are closed or have one boundary component. We show that a representation of classical shadow r can be extended to a representation of a localized quantum torus and use an argument of Bonahon and Liu about the uniqueness of representations of quantum tori to deduce the unicity of the representation of $S_\xi(\Sigma)$ with classical shadow r. This construction also allows one to explicitly reconstruct the representation from the classical shadow, recovering the surjectivity proved in [Bonahon and Wong 2019], on the open dense subset of $X(\Sigma,\mathrm{SL}_2(\mathbb{C}))$ described above.

The theorem of [Frohman et al. 2019] has since been improved by Ganev, Jordan and Safronov [Ganev et al. 2024]: building upon Frohman, Kania-Bartoszynska and Lê's result, they prove that the unicity conjecture actually holds over the set of all irreducible representations.

We end this introduction with the following question:

Question 1.8 *Can one define an embedding similar to that of Theorem 1.1, but with values in a quantum torus instead of a localized quantum torus?*

A possible approach towards this question would be to use a basis for the skein module modeled on the integral basis of SO(3)-TQFTs given by Gilmer and Masbaum [2007]. Note that it is however unlikely that one could both get an integral version of the embedding of Theorem 1.1, while keeping the nice "almost surjectivity" feature described in Proposition 1.2. This last property is key for lifting representations as in Theorem 1.5.

The paper is organized in two largely independent parts. Section 2 is devoted to the definition of the map σ of Theorem 1.1, and the proof that it is an embedding. The short Section 3 introduces a special kind of pair of pants decompositions that are used in the next section, and serves as a transition between the two parts of the paper. Section 4 is devoted to the proof of Theorem 1.5 and Corollary 1.6, and could in principle be read independently of Section 2, although some formulas are deeply inspired by it.

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2 Embeddings of skein algebras of surfaces into localized quantum tori

2.1 Localized quantum tori

Consider the $\mathbb{Z}[A^{\pm 1}]$-module

$$\widetilde{\mathcal{A}}_{n,b} = \mathbb{Z}[A^{\pm 1}][E_1^{\pm 1}, \dots, E_n^{\pm 1}] \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathbb{Q}(A)(Q_1, \dots, Q_n, C_1, \dots, C_b).$$

For convenience, we write an element of $\widetilde{\mathcal{A}}_{n,b}$ as

$$\sum_{k \in \mathbb{Z}^n} E_1^{k_1} \cdots E_n^{k_n} R_k,$$

where $R_k \in \mathbb{Q}(A)(Q_1, \dots, Q_n, C_1, \dots, C_b)$ for all $k \in \mathbb{Z}^n$, all but finitely many R_k are zero, and E^k denotes $E_1^{k_1} \cdots E_n^{k_n}$.

For $P \in \mathbb{Q}(A)(Q_1, \dots, Q_n, C_1, \dots, C_b)$ and $k \in \mathbb{Z}^n$, we define

$$\widehat{P}^{(k)} = P(A^{k_1} Q_1, \dots, A^{k_n} Q_n, C_1, \dots, C_b).$$

Now we set the following multiplication on $\widetilde{\mathcal{A}}_{n,b}$:

$$\Big(\sum_{k \in \mathbb{Z}^n} E^k R_k\Big)\Big(\sum_{l \in \mathbb{Z}^n} E^l S_l\Big) = \sum_{k,l \in \mathbb{Z}^n} E^{k+l} \widehat{R}_k^{(l)} S_l.$$

This multiplication makes $\widetilde{\mathcal{A}}_{n,b}$ a noncommutative algebra over the ring $\mathbb{Z}[A^{\pm 1}]$.

Let $\mathcal{R}$ be the subring of $\mathbb{Q}(A)(Q_1, \dots, Q_n, C_1, \dots, C_b)$ consisting of all elements of the form U/V where $U \in \mathbb{Z}[A^{\pm 1}][Q_1^{\pm 1}, \dots, Q_n^{\pm 1}, C_1^{\pm 1}, \dots, C_b^{\pm 1}]$ and V is a finite product (possibly empty) of elements of the form $A^m Q_j^2 - A^{-m} Q_j^{-2}$ for $m \in \mathbb{Z}$ and $1 \le j \le n$.

Let $\mathcal{A}_{n,b}$ be the submodule of $\widetilde{\mathcal{A}}_{n,b}$ generated by elements of the form $E^k R$ where $k \in \mathbb{Z}^n$ and $R \in \mathcal{R}$. It is clear from the multiplicative structure of $\widetilde{\mathcal{A}}_{n,b}$ that $\mathcal{A}_{n,b}$ is a $\mathbb{Z}[A^{\pm 1}]$-subalgebra of $\widetilde{\mathcal{A}}_{n,b}$. The noncommutative algebra $\mathcal{A}_{n,b}$ will be called the *localized quantum torus*.

2.2 An embedding of skein algebras through curve operators

We call a graph unitrivalent if all its vertices have degree 1 or 3. We call an edge of such a graph univalent if at least one of its vertices has degree 1.

Let $\Gamma \subset \mathbb{S}^3$ be a planar banded unitrivalent graph; we denote by $\mathcal{E}$ its set of edges. Let $\mathcal{U} \subset \mathcal{E}$ be the set of univalent edges of $\mathcal{E}$ and $\mathcal{E}' = \mathcal{E} \backslash \mathcal{U}$. Let n be the total number of edges of Γ joining two trivalent vertices and b be the number of univalent edges of Γ. We number edges in $\mathcal{E}'$ from 1 to n and edges in $\mathcal{U}$ from $n+1$ to $n+b$. We will also write $\mathcal{A}(\Gamma)$ for the localized quantum torus $\mathcal{A}_{n,b}$.

Finally, let P be the set of triples $(e, f, g) \in \mathcal{E}^3$ such that the corresponding edges are adjacent to the same trivalent vertex.

Let $H \subset \mathbb{S}^3$ be a tubular neighborhood of Γ. We denote by $\hat{\Sigma}$ the boundary of H and suppose that the genus of $\hat{\Sigma}$ is at least 1. The univalent vertices of Γ define banded points (that is, embedded intervals) on $\hat{\Sigma}$ $x_1, \dots, x_b$. Let Σ be the surface obtained from $\hat{\Sigma}$ by removing small open disks around each x_j. We suppose that Σ has negative Euler characteristic.

A coloring of a banded point x will be a choice of an integer $c \ge 0$. For $c_1, \dots, c_b$ a coloring of the banded points $x_1, \dots, x_b$, the relative skein module $S(H, \mathbb{Q}(A), c)$ will be the module generated by tangles with c_i boundary points on the banded point x_i and with the Jones–Wenzl idempotent f_{c_i} inserted, modulo the Kauffman relations. For the definition of Jones–Wenzl idempotents, which are specific elements of the Temperley–Lieb algebra, we refer for instance to [Masbaum and Vogel 1994].

The relative skein module $S(H, \mathbb{Q}(A))$ will be the direct sum of all $S(H, \mathbb{Q}(A), c)$ over all possible colorings c of the banded points $x_1, \dots, x_b$. We denote by $S(\Sigma)$ the skein algebra of Σ over $\mathbb{Z}[A^{\pm 1}]$. The algebra $S(\Sigma)$ acts on $S(H, \mathbb{Q}(A))$ by the stacking operation. For γ a multicurve, we denote by T^γ the action of γ on $S(H, \mathbb{Q}(A))$.

A map $c\colon \mathcal{E} \to \mathbb{N}$ is called an admissible coloring if, for all $(e, f, g) \in P$, we have triangular inequalities $c(e) \le c(f) + c(g)$ and $c(e) + c(f) + c(g)$ is even. We also introduce a lattice $\Lambda \subset \mathbb{Z}^{\mathcal{E}}$ by

$$\Lambda = \{k \in \mathbb{Z}^{\mathcal{E}} \mid \text{for all } (e, f, g) \in P,\ k(e) + k(f) + k(g) \in 2\mathbb{Z}\}.$$

Given an admissible coloring $c\colon \mathcal{E} \to \mathbb{N}$ of Γ, we denote by $\varphi_c \in S(H, \mathbb{Q}(A))$ the vector obtained by cabling each edge e of Γ using the Jones–Wenzl idempotents $f_{c(e)}$, and at trivalent vertices joining the strands in the unique way that avoids crossing. (Note that the conditions $c(e) \leq c(f) + c(g)$ and $c(e) + c(f) + c(g)$ even imply that this is possible.) It is well known that:

Lemma 2.1 *The set $\{\varphi_c \mid c\colon \mathcal{E} \to \mathbb{N}$ is admissible$\}$ is a basis of $S(H, \mathbb{Q}(A))$.*

Proof The handlebody H with banded points $x_1, \dots, x_b$ is homeomorphic to the thickened surface $\Gamma \times [0, 1]$, with banded points corresponding to univalent vertices. Fix a coloring $\hat{c}$ of the banded points. For thickened surfaces, the skein module is generated by disjoint unions of arcs and nontrivial simple closed curves with boundary $\hat{c}_i$ points on the i^{th} banded point and the $\hat{c}_i^{\text{th}}$ Jones–Wenzl idempotent inserted at that banded point. Such tangles are completely determined by their intersection number with the cocore of each internal edge of Γ. Let ψ_c be the basis element which has c_e intersections with the cocore of the edge e. Now consider the vectors φ_c corresponding to admissible colorings of Γ that coincide with $\hat{c}_i$ on the boundary. The recursive formula for the Jones–Wenzl idempotents shows that φ_c is a linear combination of the vectors ψ_d with $d_e \leq c_e$ for all $e \in \mathcal{E}'$. Moreover, φ_c has nonzero coefficient along ψ_c. This implies that the φ_c are linearly independent, and moreover an easy induction shows that any ψ_c is a linear combination of the φ_c. Therefore the φ_c are also a basis of $S(\Sigma, \mathbb{Q}(A))$. □

The action of curve operators T^γ in the basis φ_c can be computed using the so-called fusion rules derived in [Masbaum and Vogel 1994]. A complete set of fusion rules is described in Figure 1, where coefficients are expressed in terms of A and quantum integers $\{n\} = A^{2n} - A^{-2n}$.

Before studying the form of curve operators in the basis φ_c, we need the following lemma and definition:

Lemma 2.2 *Let Δ be the set of admissible colorings of Γ. Then for all $v_1, \dots, v_n \in \Lambda$ we have*

$$\bigcap_{j=1}^{n} \Delta + v_j \neq \varnothing.$$

We call any subset of $\mathbb{N}^{\mathcal{E}}$ containing a subset of the form $\bigcap_{j=1}^{n} \Delta + v_j$, where $v_j \in \Lambda$, a *large subset* of $\mathbb{N}^{\mathcal{E}}$.

Proof For any $w \in \mathbb{Z}^{\mathcal{E}}$ and $r \in \mathbb{N}$ let $B(w, r)$ be the ball around w for $\|\cdot\|_\infty$. Notice that for any $k \in \mathbb{N}$, the set Δ contains $B\big((2k, \dots, 2k), 2\lfloor \frac{1}{3}k \rfloor\big) \cap \Lambda$. Letting $r \geq 1$, if $2\lfloor \frac{1}{3}k \rfloor > \max(\|v_1\|_\infty, \dots, \|v_n\|_\infty) + r$, then $\bigcap_{j=1}^{n} \Delta + v_j$ contains $B((2k, \dots, 2k), r) \cap \Lambda$. As r can be chosen arbitrarily large, this shows $\bigcap_{j=1}^{n} \Delta + v_j \neq \varnothing$. □

We will define a pants decomposition of Σ to be a collection of simple closed curves on Σ that cuts Σ into a union of pairs of pants. This decomposition is dual to the graph Γ if each curve of the decomposition bounds a disk in H that intersects an edge of Γ transversely in a single interval.

For γ and δ simple closed curves on Σ, we denote by $i(\gamma, \delta)$ the geometric intersection number of γ and δ.

We can now describe the structure of curve operators T^γ associated to multicurves on Σ:

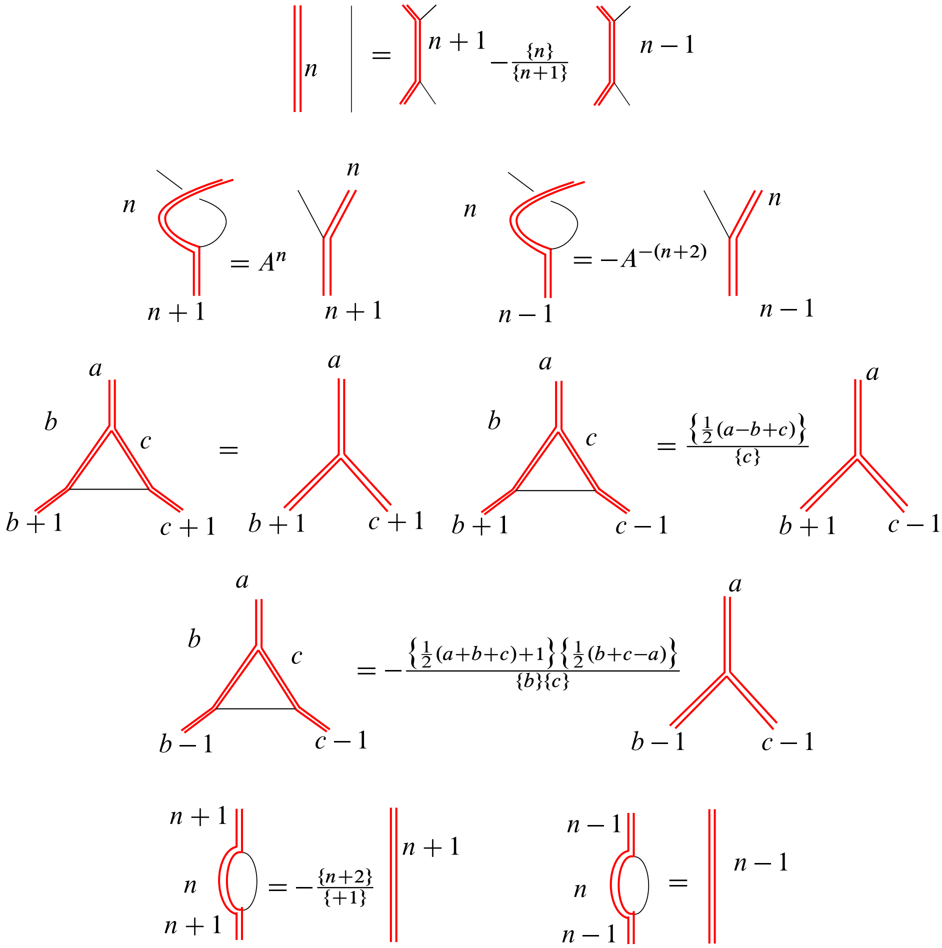


Figure 1: Fusion rules for computing curve operators in the basis φ_c. Thick edges represent edges of the trivalent graph Γ, which are colored by integers, while slim black arcs are colored by 1. We let $\{n\} = A^{2n} - A^{-2n}$.

Proposition 2.3 *Let γ be a multicurve on Σ. There exists a large subset $V_\gamma \subset \mathbb{N}^{\mathcal{E}}$ and $F_k^\gamma \in \mathcal{R}$ (for $k : \mathcal{E}' \to \mathbb{Z}$) such that for all $c \in V_\gamma$,*

$$T^\gamma \varphi_c = \sum_{k : \mathcal{E}' \to \mathbb{Z}} F_k^\gamma (A^{c(1)}, \dots, A^{c(n)}, A^{c(n+1)}, \dots, A^{c(n+b)}) \varphi_{c+k}.$$

Moreover if $\mathcal{P} = \{\alpha_1, \dots, \alpha_n\}$ is a pants decomposition of Σ dual to Γ (with numbering corresponding to that of $\mathcal{E}$) then

(i) *$F_k^\gamma = 0$ when $|k(j)| > i(\gamma, \alpha_j)$ or $k(j) \not\equiv i(\gamma, \alpha_j) \bmod 2$ for some $1 \le j \le n$,*

(ii) *if $k = (\pm i(\gamma, \alpha_1), \dots, \pm i(\gamma, \alpha_n))$ then $F_k^\gamma \neq 0$.*

The proof of Proposition 2.3 involves fusion calculations to compute coefficients of curve operators and will be done in Section 2.4.

Lemma 2.4 *Assume that for some multicurve γ on Σ there exist two large subsets V and V' of $\mathbb{N}^{\mathcal{E}}$ and coefficients F_k and G_k in $\mathcal{R}$ such that for any c in V or V', $T^\gamma\varphi_c$ admits a decomposition as in Proposition 2.3 with coefficients F_k or G_k, respectively. Then $F_k = G_k$ for all $k\colon \mathcal{E}' \to \mathbb{Z}$.*

Proof Indeed, the intersection $V \cap V'$ will be also be a large subset of $\mathbb{N}^{\mathcal{E}}$ and thus will contain, by the proof of Lemma 2.2, subsets of the form $B(v,r)\cap\Lambda$ where $v \in \mathbb{N}^{\mathcal{E}}$ and r can be arbitrarily large. Notice that Λ contains the lattice $2\mathbb{Z}^{\mathcal{E}}$. Assume that r is strictly larger than d, the maximum of the degrees of the rational fractions $F_k - G_k$ (which we define as the maximum of the degrees of their numerator and denominator). By Proposition 2.3 the rational fractions F_k and G_k coincide on the set of all $(A^{c(1)},\dots,A^{c(n+b)})$ where $c \in B(v,r)\cap 2\mathbb{Z}^{\mathcal{E}}$, which is a product of sets that contains more than d elements. By an easy induction on the number of variables n of F_k and G_k, we deduce that $F_k = G_k$. □

Thanks to Lemma 2.4, we can make the following definition:

Definition 2.5 For γ a multicurve,

$$\sigma(\gamma) = \sum_{k\colon \mathcal{E}'\to\mathbb{Z}} E^k F_k^\gamma(Q_1,\dots,Q_n,C_1,\dots,C_b) \in \mathcal{A}(\Gamma),$$

and we linearly extend this definition to a $\mathbb{Z}[A^{\pm1}]$-module morphism

$$\sigma\colon S(\Sigma) \to \mathcal{A}(\Gamma).$$

Lemma 2.6 *The map $\sigma\colon S(\Sigma)\to\mathcal{A}(\Gamma)$ is a $\mathbb{Z}[A^{\pm1}]$-algebra morphism.*

Proof Note that the map $\gamma \in S(\Sigma,\mathbb{Z}[A^{\pm1}]) \mapsto T^\gamma \in \mathrm{End}(S(H,\mathbb{Q}(A)))$ is a morphism of algebras. The lemma will follow from the fact that $\sigma(\gamma)$ encodes the action of $T^\gamma \in \mathrm{End}(S(H,\mathbb{Q}(A)))$ in the basis φ_c, and that the multiplication in $\mathcal{A}(\Gamma)$ corresponds to the composition of operators.

Indeed, let γ and δ be two multicurves, and assume that

$$T^\gamma\varphi_c = \sum_{k\colon \mathcal{E}'\to\mathbb{Z}} F_k^\gamma(A^{c(1)},\dots,A^{c(n+b)})\varphi_{c+k}$$

for any $c \in V_\gamma$ and

$$T^\delta\varphi_c = \sum_{k\colon \mathcal{E}'\to\mathbb{Z}} F_k^\delta(A^{c(1)},\dots,A^{c(n+b)})\varphi_{c+k}$$

for any $c \in V_\delta$. Let $V = \bigcap_{k\in\Lambda,|k_i|\le i(\gamma,\alpha_i)} V_\delta + k$. Then V is a large subset of $\mathbb{N}^{\mathcal{E}'}$ and for any $c \in V$ and any $k\in\Lambda$ such that $F_k^\gamma \neq 0$, we have $c+k \in V_\delta$. Hence

$$\begin{aligned} T^{\delta\cdot\gamma}&\varphi_c \\ &= T^\delta\Big(\sum_{k\colon \mathcal{E}'\to\mathbb{Z}} F_k^\gamma(A^{c(1)},\dots,A^{c(n+b)})\varphi_{c+k}\Big) \\ &= \sum_{k,l\colon \mathcal{E}'\to\mathbb{Z}} F_l^\delta(A^{c(1)+k(1)},\dots,A^{c(n)+k(n)},A^{c(n+1)},\dots,A^{c(n+b)})F_k^\gamma(A^{c(1)},\dots,A^{c(n+b)})\varphi_{c+k+l}.\end{aligned}$$

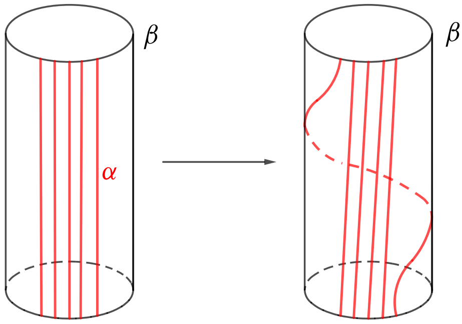
Therefore, comparing with the formula for the product in $\mathcal{A}(\Gamma)$ in Section 2.1, $\sigma(\delta\cdot\gamma) = \sigma(\delta)\sigma(\gamma)$.

The general case of $\gamma,\delta \in S(\Sigma,\mathbb{Z}[A^{\pm1}])$ follows by linearity. □

2.3 Injectivity of σ

Let γ be a multicurve. From now on, for any $k\colon \mathcal{E}' \to \mathbb{Z}$, the element $F_k^\gamma(Q_1,\dots,Q_n,C_1,\dots,C_b) \in \mathcal{R}$ will be simply denoted by F_k^γ.

For α and β two curves on Σ, the *positive fractional Dehn twist* of α along β is the simple closed curve obtained in the following way: isotope α and β so that they are in minimally intersecting position, then in a neighborhood of β we change the curve α by



The *negative fractional Dehn twist* of α along β is obtained in a similar way.

Lemma 2.7 *Let γ be a multicurve, and let $k\colon \mathcal{E}' \to \mathbb{Z}$. We assume that $|k_j| = \varepsilon k_j = i(\gamma,\alpha_j) \neq 0$ with $\varepsilon = \pm 1$. Let γ_+ be the curve obtained from γ by applying a positive fractional twist along α_j. We have*

$$F_k^{\gamma+} = -A^{2\varepsilon+|k_j|} Q_j^{2\varepsilon} F_k^\gamma.$$

Proof We will treat only the case $\varepsilon = +1$, the case $\varepsilon = -1$ being completely similar. Let γ_- be the curves obtained from γ by applying a negative fractional twist along α_j. We have $\alpha_j\gamma = A^{k_j}\gamma_+ + A^{-k_j}\gamma_- +$ lower order curves, where by lower order we mean less geometric intersection with α_j. Hence by identifying the terms in E^k in $\sigma(\alpha_j\gamma)$, we have $-(A^2Q_j^2 + A^{-2}Q_j^{-2})E^kF_k^\gamma = A^{k_j}E^kF^{\gamma+} + A^{-k_j}E^kF_k^{\gamma-}$. Using $Q_j^2E^k = A^{2k_j}E^kQ_j^2$ and simplify by E^k, we get

$$-(A^{2+2k_j}Q_j^2 + A^{-2-2k_j}Q_j^{-2})F_k^\gamma = A^{k_j}F_k^{\gamma+} + A^{-k_j}F_k^{\gamma-}.$$

Similarly, if we expand $\gamma\alpha_j$, we get

$$-(A^2Q_j^2 + A^{-2}Q_j^{-2})F_k^\gamma = A^{-k_j}F_k^{\gamma+} + A^{k_j}F_k^{\gamma-}.$$

We conclude by solving the system of two equations. □

Proposition 2.8 *The map $\sigma\colon S(\Sigma) \to \mathcal{A}(\Gamma)$ is injective.*

Proof We recall that the set of multicurves is a basis of $S(\Sigma)$. In this proof we will use Dehn–Thurston coordinates for multicurves; we use the convention of [Charles and Marché 2012, Section 2]. This same convention will be used whenever these coordinates are needed.

Let C be a finite set of multicurves and $x = \sum_{\gamma\in C}\lambda_\gamma\gamma \in S(\Sigma)$ such that $\sigma(x) = 0$.

Let us consider an element k of the set of n-tuples of the form $(i(\gamma,\alpha_1),\dots,i(\gamma,\alpha_n))$ where $\gamma \in C$ which is maximal for the lexicographical order. Notice that by Proposition 2.3, only the multicurves γ such that $(i(\gamma,\alpha_1),\dots,i(\gamma,\alpha_n))=k$ contribute to the coefficient in E^k. Let C' be the subset of C of those maximal multicurves, and let us prove that $\lambda_\gamma=0$ for all $\gamma\in C'$. Note that in the Dehn–Thurston coordinates associated to the pair of pants decomposition $(\alpha_1,\dots,\alpha_n)$, the multicurves in C' differ only by their twist coordinates. If we identify the terms in E^k in $\sigma(x)$, we get

$$\sum_{\gamma\in C'}\lambda_\gamma F_k^\gamma=0.$$

Let δ be a multicurve satisfying

(i) $i(\delta,\alpha_j)=k_j$ for all $1\le j\le n$,

(ii) if $k_j=0$ then the j^{th} twist coordinate of δ is trivial.

By Lemma 2.7, for each $\gamma\in C$ there exists $R_\gamma\in\mathbb{Q}(A)[Q_e^{\pm1}]$ such that $F_k^\gamma=R_\gamma F_k^\delta$. We claim that the $\{R_\gamma\mid\gamma\in C'\}$ are linearly independent. Indeed, if the j^{th} intersection coordinate does not vanish, by Lemma 2.7, shifting the j^{th} twist coordinates up by 1 multiplies R_γ by $-A^{2+k_j}Q_j^2$. If the j^{th} intersection coordinate vanishes, shifting the j^{th} twist coordinate up by 1 multiplies R_γ by $(-A^2Q_j^2-A^{-2}Q_j^{-2})$ instead. In all cases, R_γ is a Laurent polynomial in the variables Q_j, whose monomial of highest degree has degree in Q_j equal to twice the j^{th} twist coordinate of γ.

Now, from the equality

$$\sum_{\gamma\in C'}\lambda_\gamma R_\gamma F_k^\delta=0,$$

we can conclude that $\lambda_\gamma=0$ for all $\gamma\in C'$. An easy induction then proves that $\lambda_\gamma=0$ for all $\gamma\in C$. Therefore $x=0$. □

We note that the embedding σ is not surjective onto $\mathcal{A}(\Gamma)$. Indeed, as a consequence of Proposition 2.3(i), the image of the morphism σ is included in the $\mathbb{Q}(A)(Q_1,\dots,Q_n)[C_1^{\pm1},\dots,C_b^{\pm1}]$-subalgebra generated by elements of the form E^k, where $k\in\Lambda$. In the following proposition, we show that the image of σ is a kind of lattice in this subalgebra. The next proposition implies Proposition 1.2 in the introduction:

Proposition 2.9 *Let $\mathcal{F}=\mathbb{Q}(A)(Q_1,\dots,Q_n)[C_1^{\pm1},\dots,C_b^{\pm1}]$. Then the $\mathbb{Q}(A)$-subalgebra of $\widetilde{A}_{n,b}$ generated by $\mathcal{F}$ and $\mathrm{Im}(\sigma)$ is*

$$\bigoplus_{k\in\Lambda}\mathcal{F}E^k.$$

Proof For $k\in\Lambda$ let $|k|=(|k_1|,\dots,|k_n|)$. We will prove, by induction on $|k|$ in the lexicographical order, that E^k is a linear combination over $\mathcal{F}$ of symbols $\sigma(\gamma)$ of multicurves. The image of the empty multicurve settles the case $|k|=0$. Next we note that since $k\in\Lambda$, there is a multicurve γ on Σ such that $i(\gamma,\alpha_i)=|k_i|$ for any $1\le i\le n$. For $\varepsilon\in\{0,1\}^n$, let γ_ε be the multicurve obtained from γ by shifting its i^{th} twist coordinate by ε_i if γ has nonzero intersection with α_i. The curves γ_ε all have $|k|$ geometric intersections with the curve $\{\alpha_1,\dots,\alpha_n\}$.

For any $\mu \in \{\pm 1\}^n$, and any $\varepsilon \in \{0,1\}^n$, the coefficient $F^{\gamma_\varepsilon}_{\mu|k|}$ is nonzero by Proposition 2.3(ii). Let us assume for simplicity that γ has nonzero intersection with the curves $\alpha_1, \dots, \alpha_d$ and is disjoint from the curves $\alpha_{d+1}, \dots, \alpha_n$. By Lemma 2.7, we have

$$F^{\gamma_\varepsilon}_{\mu|k|} = \prod_{1 \le i \le d} (-A^{2\mu_i + |k_i|} Q_i^{2\mu_i})^{\varepsilon_i} F^{\gamma}_{|k|}$$

for any μ and ε. The matrix

$$M = \Big(\prod_{1 \le i \le d} (-A^{2\mu_i + |k_i|} Q_i^{2\mu_i})^{\varepsilon_i} \Big)_{\varepsilon \in \{0,1\}^d, \mu \in \{\pm 1\}^d}$$

is the tensor product of matrices

$$\begin{pmatrix} 1 & 1 \\ -A^{2+|k_i|} Q_i^2 & -A^{-2+|k_i|} Q_i^{-2} \end{pmatrix},$$

and therefore is invertible. Hence for any μ, there is an $\mathcal{F}$-linear combination of the multicurves γ_ε such that its image by σ has coefficient 1 along $E^{\mu|k|}$ and zero coefficient along each other $E^{\mu'|k|}$, where $\mu' \neq \mu \in \{\pm 1\}^d$. As a result, we get an $\mathcal{F}$-linear combination of multicurves $x = \sum \lambda_\varepsilon \gamma_\varepsilon$ such that $\sigma(x) = E^{\mu|k|}$ up to lower-order terms, and by the induction hypothesis we can add another linear combination of multicurves to eliminate those lower-order terms. □

Remark 2.10 In some sense, the embedding is analogous to the Frohman–Gelca embedding [2000] of the skein algebra of the closed torus into the quantum torus $\mathbb{Z}[A^{\pm 1}]\langle Q, E\rangle /_{QE=AEQ}$. The image of the Frohman–Gelca embedding is the symmetric part of the quantum torus, that is, elements invariant under the action of the $\mathbb{Z}[A^{\pm 1}]$-algebra automorphism $\theta\colon Q, E \mapsto Q^{-1}, E^{-1}$. Here, since $\sigma(\alpha_e) = -A^2 Q_e^2 - A^{-2} Q_e^{-2}$, we have that the localized quantum torus is a kind of "finite extension" of the skein algebra by Proposition 2.9.

2.4 Proof of Proposition 2.3

In this section, we will prove Proposition 2.3. Let Σ be a compact oriented surface of genus g and with b boundary components with negative Euler characteristic, $\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3+b}\}$ be a pair of pants decomposition of Σ, and Γ be a trivalent banded graph dual to $\mathcal{P}$. We also view Σ as the boundary of a handlebody H, and thus vectors φ_c associated to colorings of Γ give a basis of $S(H, \mathbb{Q}(A))$ by Lemma 2.1.

Taking two parallel copies of each curve in $\mathcal{P}$, we get a decomposition of Σ into pairs of pants and annuli. We will call the two parallel copies α_i and α_i', with no particular convention for the choice of α_i. Note that the pairs of pants in the decomposition have boundary either the curves α_i and α_i', or the boundary curves of Σ.

If γ is a multicurve on Σ, then up to isotopy γ can be put into Dehn–Thurston position. By this we mean that the geometric intersection number $i(\gamma, \alpha_i)$ of γ with each curve in $\mathcal{P}$ is exactly the number of intersection points of γ with α_i and also with α_i', and that in each pair of pants or annulus of the decomposition, the curve γ looks like one of the patterns described in Figure 2.

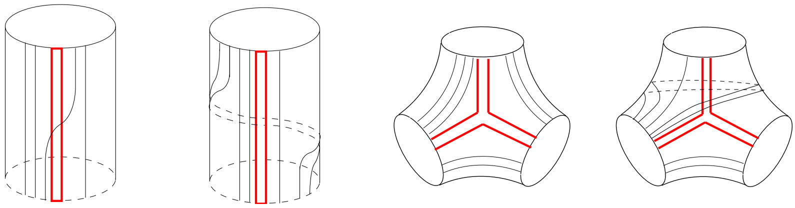


Figure 2: A multicurve in Dehn–Thurston position follows one of the above patterns in the pants and annuli of the decomposition.

The computation of the coefficients of the operator T^γ in the basis φ_c can be done as follows. First we can remove any component β in γ that is parallel to a curve in $\mathcal{P}$ or a boundary curve, at the price of multiplying φ_c by the scalar $-A^{2c_e+2}-A^{-2c_e-2}=-A^2Q_e^2-A^{-2}Q_e^{-2}$ where e is the edge encircled by β.

Second we apply the first fusion rule for each intersection point of γ with a curve α_i or α_i'. Each fusion shifts the color of the corresponding edge by ± 1. After fusion, it only remains to simplify one of the patterns described in Figure 3 to express the coefficients of $T^\gamma\varphi_c$ on the basis $\{\varphi_c\}$. Note that as the skein module of the sphere with two colored points is 1-dimensional if the two colors agree and zero otherwise, to obtain a nonzero vector we need that the sums of shifts at intersection points with α_i and α_i' coincide. Furthermore, the skein module of a sphere with three points colored by c_1, c_2 and c_3 is 1-dimensional if the colors satisfy the admissibility conditions $c_i \le c_j + c_k$ and $c_1+c_2+c_3$ even, and zero-dimensional

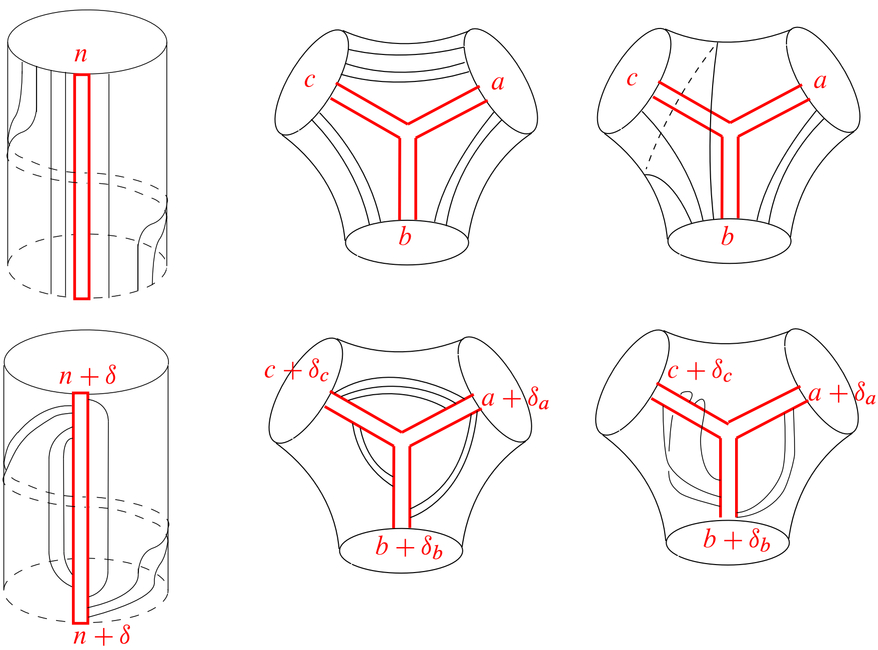


Figure 3: Top: the different patterns of the intersection of a multicurve (in black) with an annulus or pants piece of the decomposition. The trivalent graph Γ is shown in red. Bottom: the remaining patterns after fusion.

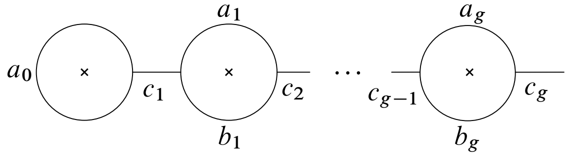
otherwise. By the assumption that c is in the set V_γ, that is always the case after fusion. Therefore each of the terms we obtain after fusion at the intersection points of γ with α_i and α_i' is just a scalar multiple of a vector φ_{c+k}. The color shift k_e at edge e is the common sum of the ± 1 shifts at either intersection points in $\gamma \cap \alpha_i$ or at intersection points in $\gamma \cap \alpha_i'$. This shows that $T^\gamma \varphi_c$ has nonzero coefficient along φ_{c+k} only when $k \in \Lambda$ and $|k_e| \le i(\gamma, c_e)$ for each edge e.

Next we claim that the coefficients are in the ring $\mathcal{R}$. Notice that the remaining patterns after fusion at intersection points $\gamma \cap (\alpha_e \cup \alpha_e')$ shown in Figure 3 can be reduced to remove all black arcs using the fusion rules in Figure 1. Furthermore, all of the fusion rules in Figure 1 involve only rational functions of A^{c_e} and A, where c_e are the colors of edges $e \in \mathcal{E}$. Moreover, the denominators appearing in the fusion rules are all of the form $\{c_e + k\} = A^{2c_e+2k} - A^{-2c_e-2k}$ where $k \in \mathbb{Z}$. (We remark that the first fusion rules will shift colors of Γ, but only by a fixed amount.) We also claim that if e is an external edge, we will never need to use any rule involving a denominator $\{c_e + k\}$, since the geometric intersection of γ and α_e is zero. Those rules correspond to the rules that create or erase a black arc with an endpoint on the edge e. Therefore the coefficients are in the ring $\mathcal{R}$.

It remains to be seen that the extremal coefficients are nonzero. This is a consequence of [Detcherry 2016, Theorem 1.3; Charles and Marché 2012, Lemma 4.3]. The former studies the matrix coefficients of the action of curve operators on SU_2-TQFT spaces of surfaces. A subset of the basis φ_c of the skein module of the handlebody, corresponding to colors c satisfying the additional "r-admissibility conditions", gives a basis of the SU_2-TQFT space of the surface (Σ, c_i) with colored points. The curve operators on S induce curve operators T_r^γ on the TQFT spaces at level r, and their matrix coefficients $F_k^{\gamma,\mathrm{SU}_2}(c/r, 1/r)$ are obtained from the coefficients of T^γ in the E^k by sending A to a $2r^{\text{th}}$ root of unity and applying some renormalization. Theorem 1.3 of [Detcherry 2016] then shows that $F_k^{\gamma,\mathrm{SU}_2}(x,0)$ is the k^{th} Fourier coefficient of the trace function $f_\gamma \colon \rho \mapsto \prod_{i\in I}(-\mathrm{Tr}(\rho(\gamma_i))$ defined on the subset $\{e \in \mathcal{E}' \mid \mathrm{Tr}(\rho(\alpha_e)) = 2\cos(\pi x_e))\}$ of the SU(2) moduli space of Σ. Lemma 4.3 of [Charles and Marché 2012] shows that the extremal Fourier coefficients are nonzero as long as x is taken in the interior of the image of the momentum map $\rho \mapsto \big(\arccos\big(\tfrac{1}{2}\,\mathrm{Tr}(\rho(\alpha_e))\big)\big)_{e\in\mathcal{E}'}$. This implies that the coefficients $F_k^{\gamma,\mathrm{SU}_2}$ are nonzero when $k_e = \pm i(\gamma, \alpha_e)$ for all $e \in \mathcal{E}'$, and the same is true for the coefficients F_k^γ.

3 Localized quantum torus associated to a sausage graph

In this section, as well as the remainder of the paper, we write $U(x) = x - x^{-1}$ when x is an invertible element in a ring. Let Σ be a surface with at most one boundary component with negative Euler characteristic. Consider the pants decomposition of Σ as in Figure 4. Let Γ be the graph dual to this pants decomposition:



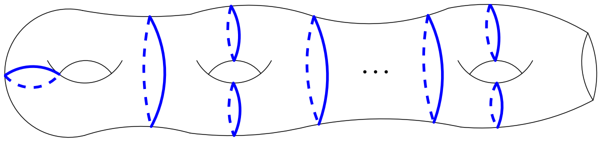


Figure 4: When $\partial\Sigma = \varnothing$, the two rightmost curves coincide.

When $\partial\Sigma = \varnothing$, $a_{g-1} = b_{g-1}$ and c_g does not exist. Finally let $\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}$ be the curves shown in Figure 5.

For $k\colon \mathcal{E}' \to \mathbb{Z}$, we define E^k to be $\prod_{e\in\mathcal{E}'} E_e^{k(e)}$ and Λ to be the set of maps $k\colon \mathcal{E}' \to \mathbb{Z}$ such that if $e_1, e_2, e_3 \in E(\Gamma)$ meet at a vertex then $k(e_1) + k(e_2) + k(e_3)$ is even.

Definition 3.1 Let $\mathcal{R}^0$ be the set of Laurent polynomial with coefficients in $Z[A^{\pm 1}]$ in the variables $Q_{a_0}^2, Q_{a_1}Q_{b_1}, Q_{a_1}Q_{b_1}^{-1}, \dots, Q_{a_{g-1}}Q_{b_{g-1}}, Q_{a_{g-1}}Q_{b_{g-1}}^{-1}, Q_{c_1}, \dots, Q_{c_g}$. We define $\mathcal{A}(\Gamma)^0$ to be the subalgebra of $\mathcal{A}(\Gamma)$ defined by the set of

$$\sum_{k\in\Lambda} E^k F_k,$$

where $F_k = V/W$ with $V \in \mathcal{R}^0$ and W a finite (possibly empty) product of $A^n Q_\alpha^2 - A^{-n} Q_\alpha^{-2}$ for $n \in \mathbb{Z}$ and $\alpha \in \mathcal{P}$.

We define also the *nonlocalized* version $\mathcal{A}(\Gamma)^0$:

$$\mathcal{T}(\Gamma)^0 = \left\{ \sum_{k\in\Lambda} E^k F_k \;\middle|\; F_k \in \mathcal{R}^0 \right\} \subset \mathcal{A}(\Gamma)^0.$$

Lemma 3.2 *$\mathcal{T}(\Gamma)^0$ is generated by the sets*

$$\mathcal{X} = \{Q_{a_0}^2, Q_{a_1}Q_{b_1}, Q_{a_1}Q_{b_1}^{-1}, \dots, Q_{a_{g-1}}Q_{b_{g-1}}, Q_{a_{g-1}}Q_{b_{g-1}}^{-1}, Q_{c_1}, \dots, Q_{c_{g-1}}\},$$
$$\mathcal{Y} = \{E_{a_0}, E_{a_1}E_{b_1}, E_{a_1}E_{b_1}^{-1}, \dots, E_{a_{g-1}}E_{b_{g-1}}, E_{a_{g-1}}E_{b_{g-1}}^{-1}, E_{c_1}^2, \dots, E_{c_{g-1}}^2\},$$
$$\mathcal{Z} = \{Q_{c_g}\},$$

and the inverses of the elements of these sets.

Convention From now on we will use the same symbol for an edge of Γ and the unique curve in $\mathcal{P}$ encircling it.

Recall that $\sigma\colon S(\Sigma) \to \mathcal{A}(\Gamma)$ is an embedding. We see that for $e \in \mathcal{P}$,

$$\sigma(e) = -(A^2 Q_e^2 + A^{-2} Q_e^{-2}).$$

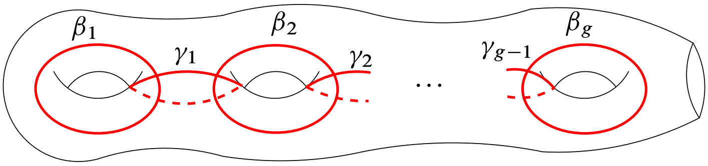


Figure 5: The β and γ curves.

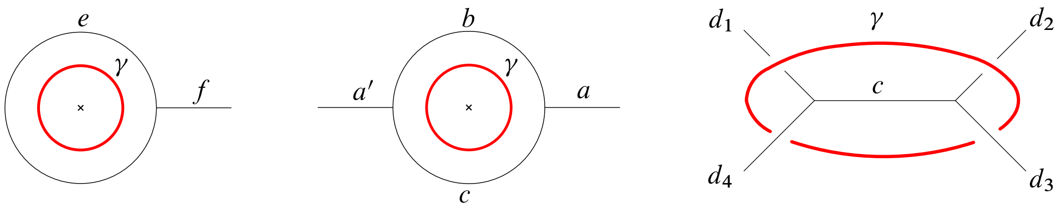


Figure 6: Three configurations for the curve γ: the one-cycle (left), two-cycle (center) and separating edge curve (right).

Let us give an explicit expression of $\sigma(\gamma)$ for $\gamma \in \{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$. There are three local configurations for γ as shown in Figure 6.

Suppose that γ is a one-cycle as in Figure 6, left. The fusion rules say that

$$\sigma(\gamma) = E_e + E_e^{-1} F, \tag{3}$$

where

$$F = U(A^2 Q_e^2 Q_f) U(Q_e^2 Q_f^{-1}) U(A^2 Q_e^2)^{-1} U(Q_e^2)^{-1},$$

and we recall the notation $U(x) = x - x^{-1}$.

Suppose that γ is a two-cycle as in Figure 6, center. A straightforward computation using fusion rules gives that

$$\sigma(\gamma) = E_b E_b + E_b E_c^{-1} F_{1,-1} + E_b^{-1} E_c F_{-1,1} + E_b^{-1} E_c^{-1} F_{-1,-1}, \tag{4}$$

where

$$F_{-1,1} = -\frac{U(Q_{a'} Q_b Q_c^{-1}) U(Q_a Q_b Q_c^{-1})}{U(A^2 Q_b^2) U(Q_b^2)}, \quad F_{1,-1} = -\frac{U(Q_{a'} Q_c Q_b^{-1}) U(Q_a Q_c Q_b^{-1})}{U(A^2 Q_c^2) U(Q_c^2)},$$

$$F_{-1,-1} = \frac{U(A^2 Q_{a'} Q_c Q_b) U(A^2 Q_a Q_c Q_b) U(Q_b Q_c Q_{a'}^{-1}) U(Q_b Q_c Q_a^{-1})}{U(A^2 Q_c^2) U(Q_c^2) U(A^2 Q_b^2) U(Q_b^2)}.$$

Finally suppose that γ is a separating edge curve as in Figure 6, right. We have

$$\sigma(\gamma) = E_c^2 G_2 + G_0 + E_c^{-2} G_{-2}, \tag{5}$$

where

$$G_0 = \frac{(d_1 d_3 + d_2 d_4) c + (A^2 + A^{-2})(d_1 d_2 + d_3 d_4)}{U(Q_c^2) U(A^4 Q_c^2)},$$

$$G_{-2} = -\frac{U(A^2 Q_{d_1} Q_{d_4} Q_c) U(Q_{d_1} Q_c Q_{d_4}^{-1}) U(Q_{d_4} Q_c Q_{d_1}^{-1})}{U(A^{-2} Q_c^2) U(Q_c^2)^2 U(A^2 Q_c^2)} \cdot U(A^2 Q_{d_2} Q_{d_3} Q_c) U(Q_{d_2} Q_c Q_{d_3}^{-1}) U(Q_{d_3} Q_c Q_{d_2}^{-1}),$$

$$G_2 = -U(Q_{d_1} Q_{d_4} Q_c^{-1}) U(Q_{d_2} Q_{d_3} Q_c^{-1}),$$

with $d_j = -A^2 Q_{d_j}^2 - A^{-2} Q_{d_j}^{-2}$ for $j \in \{1, 2, 3, 4\}$.

Proposition 3.3 *If ξ is a nonzero complex number such that $\xi^4 \neq 1$, then the image $\sigma_\xi : S_\xi(\Sigma) \to \mathcal{A}_\xi(\Gamma)$ lies in $\mathcal{A}_\xi(\Gamma)^0$.*

Proof Let $\alpha \in \mathcal{P}$. Then $\sigma_\xi(\alpha) = -(\xi^2 Q_e^2 + \xi^{-2} Q_e^{-2})$ for some edge $e \in E(\Gamma)$, and therefore $\sigma_\xi(\alpha) \in \mathcal{A}_\xi(\Gamma)^0$. Also $\sigma_\xi(\gamma) \in \mathcal{A}_\xi(\Gamma)^0$ for any $\gamma \in \{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$ according to (3)–(5). Since the set of Dehn twists associated to the curves in $\mathcal{P} \cup \{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$ generates the mapping class group of Σ, the set $\mathcal{P} \cup \{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$ generates $S_\xi(\Sigma)$ by [Santharoubane 2024, Theorem 1.1]. □

We give an explicit version of Proposition 1.2:

Proposition 3.4 *Let $\gamma \in \{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$. If γ is a one-cycle as in Figure 6, left, then*

$$E_e = -(t_e(\gamma) + A^{-1}\gamma Q_e^{-2})A^{-1}U(A^2Q_e^2)^{-1}, \quad E_e^{-1} = (A^3\gamma Q_e^2 + t_e(\gamma))A^{-1}U(A^2Q_e^2)^{-1}F^{-1}, \tag{6}$$

where F was defined in (3). If γ is a two-cycle as in Figure 6, center, then

$$E_bE_c = \big(\sigma(\gamma)A^{-2}Q_b^{-2}Q_c^{-2} + \sigma(t_b(\gamma))A^{-1}Q_c^{-2} + \sigma(t_c(\gamma))A^{-1}Q_b^{-2} + \sigma(t_bt_c(\gamma))\big)D^{-1}, \tag{7}$$

$$E_bE_c^{-1}F_{1,-1} = -(\sigma(\gamma)A^2Q_b^{-2}Q_c^2 + \sigma(t_b(\gamma))A^3Q_c^2 + \sigma(t_c(\gamma))A^{-1}Q_b^{-2} + \sigma(t_bt_c(\gamma)))D^{-1}, \tag{8}$$

$$E_b^{-1}E_cF_{-1,1} = -\big(\sigma(\gamma)A^2Q_b^2Q_c^{-2} + \sigma(t_b(\gamma))A^{-1}Q_c^{-2} + \sigma(t_c(\gamma))A^3Q_b^2 + \sigma(t_bt_c(\gamma))\big)D^{-1}, \tag{9}$$

$$E_b^{-1}E_c^{-1}F_{-1,-1} = \big(\sigma(\gamma)A^6Q_b^2Q_c^2 + \sigma(t_b(\gamma))A^3Q_c^2 + \sigma(t_c(\gamma))A^3Q_b^2 + \sigma(t_bt_c(\gamma))\big)D^{-1}, \tag{10}$$

where $D = A^2U(A^2Q_c^2)U(A^2Q_b^2)$ and where $F_{-1,1}$, $F_{1,-1}$ and $F_{-1,-1}$ were defined in (4). Finally if γ is a separating edge curve as in Figure 6, right, then

$$E_c^2G_2 = -\left[\gamma Q_c^{-2} + \tau - \frac{A^{-2}\delta_1 Q_c^{-2} - A^2\delta_2}{U(A^4Q_c^2)}\right]A^{-2}U(A^2Q_c^2)^{-1}, \tag{11}$$

$$E_c^{-2}G_{-2} = \left[\gamma A^2 Q_c^2 + A^{-2}\tau + \frac{\delta_1 Q_c^2 - \delta_2}{U(Q_c^2)}\right]U(A^2Q_c^2)^{-1}, \tag{12}$$

where $\delta_1 = d_1d_3 + d_2d_4$, $\delta_2 = d_1d_2 + d_3d_4$ and τ is as in Figure 7.

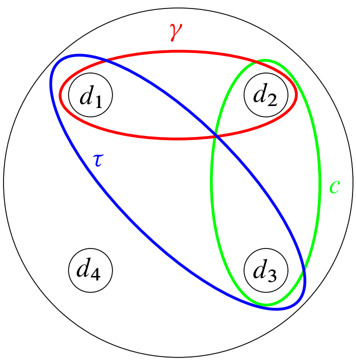


Figure 7: The curve τ.

Proof Suppose that γ is a one-cycle. Recall that $\sigma(e) = -(A^2 Q_e^2 + A^{-2} Q_e^{-2})$ and $Q_e E_e = A E_e Q_e$. Moreover, in the skein algebra of Σ, we have $Ae\gamma - A^{-1}\gamma e = (A^2 - A^{-2}) t_e(\gamma)$. Using (3) we obtain $\sigma(t_e(\gamma)) = -A^3 E_e Q_e^2 - A^{-1} E_e^{-1} Q_e^{-2} F$, and therefore

$$\sigma(\gamma) = E_e + E_e^{-1} F \quad \text{and} \quad \sigma(t_e(\gamma)) = -A^3 E_e Q_e^2 - A^{-1} E_e^{-1} Q_e^{-2} F.$$

Solving this system gives (6).

In the case where γ is a two-cycle, the strategy is very similar. We apply t_b, t_c and $t_b t_c$ to the curve γ to get a system of four equations whose resolution gives (7)–(10).

Suppose now that γ is a separating edge curve. In the skein algebra of Σ we have

$$A^2 c\gamma - A^{-2}\gamma c = (A^4 - A^{-4})\tau + (A^2 - A^{-2})(d_1 d_3 + d_2 d_4).$$

Applying this to (5), one gets

$$\sigma(\gamma) - G_0 = E_c^2 G_2 + E_c^{-2} G_{-2}, \quad \sigma(\tau) + \frac{\sigma(d_1 d_3 + d_2 d_4) - G_0 \sigma(c)}{A^2 + A^{-2}} = -E_c^2 G_2 A^4 Q_c^2 - E_c^{-2} G_{-2} Q_c^{-2}.$$

Solving this system gives (11) and (12). □

We finish this section by analyzing irreducible representations of $\mathcal{A}(\Gamma)^0$ at roots of unity. Let ξ be a $2p^{\text{th}}$ primitive root of unity with p odd, and let $\rho\colon \mathcal{A}_\xi(\Gamma)^0 \to \mathrm{End}(V)$ be a complex irreducible finite-dimensional representation. Let $x \in \mathcal{X} \cup \mathcal{Y}$, and notice that x^p is a central element of $\mathcal{A}_\xi(\Gamma)^0$. Hence $\rho(x^p)$ is a scalar times the identity of V (because ρ is irreducible). Let us denote this scalar by $\lambda_{x,\rho}$. Note that Q_{c_g} is also central, so $\rho(Q_{c_g}) = \lambda_\rho \,\mathrm{Id}_V$ for some $\lambda_\rho \in \mathbb{C}$.

Proposition 3.5 *Let ρ_1 and ρ_2 be two complex irreducible finite-dimensional representations of $\mathcal{A}_\xi(\Gamma)^0$. If $\lambda_{\rho_1} = \lambda_{\rho_2}$ and $\lambda_{x,\rho_1} = \lambda_{x,\rho_2}$ for all $x \in \mathcal{X} \cup \mathcal{Y}$ then ρ_1 and ρ_2 are isomorphic. Any irreducible representation of $\mathcal{A}_\xi(\Gamma)^0$ has dimension p^{3g-2} when Γ has a univalent vertex and p^{3g-3} otherwise.*

Proof It is enough to prove that the restrictions of ρ_1 and ρ_2 to $\mathcal{T}_\xi(\Gamma)^0$ are isomorphic. Let $\mathcal{W}$ be the complex algebra defined by the generators $U^{\pm 1}$ and $V^{\pm 1}$, and by the relation $UV = \xi^2 VU$. From the generators of Lemma 3.2 it is easy to see that $\mathcal{T}_\xi(\Gamma)^0$ is isomorphic to $\mathcal{W}^{\otimes 3g-3}$ when Γ does not have a univalent vertex (which is when $\partial\Sigma = \varnothing$) and to $\mathcal{W}^{\otimes 3g-2} \otimes \mathbb{C}[Z^{\pm 1}]$ (where Z is a formal independent variable) when Γ has one univalent vertex. The result follows directly from [Bonahon and Liu 2007, Lemmas 17 and 18]. □

4 Representation of the skein algebra

In this section Σ is still a surface with at most one boundary component with negative Euler characteristic. The goal of this section is to prove Theorem 1.5 and Corollary 1.6.

Let $p \geq 3$ be an odd number; we warn the reader that from now on A will not be a formal variable but a $2p^{\text{th}}$ primitive root of unity. We recall the Bonahon–Wong theory. Let

$$\rho\colon S_A(\Sigma) \to \mathrm{End}(V)$$

be a finite-dimensional irreducible representation. By the work of Bonahon and Wong, there exists $r\colon \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ such that for any simple close curve γ we have

$$T_p(\rho(\gamma)) = -\mathrm{Tr}(r(\gamma))\,\mathrm{Id}_V. \tag{13}$$

Here T_k is the k^{th} Chebyshev polynomial of the first kind; the important thing to remember is that

$$T_k(u+u^{-1}) = u^k + u^{-k} \quad \text{for all } u \in \mathbb{C}-\{0\}.$$

Moreover r is called the classical shadow of ρ.

The proof of Theorem 1.5 requires several steps; it starts at Section 4.1 and ends at Section 4.7. The goal is to define $\tilde{\rho}$ on the generators of $\mathcal{A}_A(\Gamma)^0$ (given in Lemma 3.2), prove the relations satisfied by these generators are preserved by $\tilde{\rho}$, and show that $\tilde{\rho}$ agrees with ρ on $\sigma_A(\Sigma_A(\Sigma))$.

The proof of Corollary 1.6 is done in Section 4.8. From now on, we fix a irreducible representation $\rho\colon S_A(\Sigma) \to \mathrm{End}(V)$ with classical shadow r satisfying the hypothesis of Theorem 1.5.

4.1 Action of the Q operators

For $\alpha \in \mathcal{P}$, let us chose $x_\alpha \neq 0$ such that $\mathrm{Tr}(r(\alpha)) = x_\alpha^{2p} + x_\alpha^{-2p}$. A known fact is that (1) implies that $\rho(\alpha)$ is diagonalizable with eigenvalues

$$-(x_\alpha^2 A^{2k+2} + x_\alpha^{-2} A^{-2k-2})$$

for $k = 0, \dots, p-1$. We define, for $k \in \mathbb{Z}$,

$$V_{\alpha,k} = \mathrm{Ker}(\rho(\alpha) + (x_\alpha^2 A^{2k+2} + x_\alpha^{-2} A^{-2k-2})\,\mathrm{Id}_V).$$

We also define $\tilde{\rho}(Q_\alpha) \in \mathrm{GL}(V)$ by

$$\tilde{\rho}(Q_\alpha)v = x_\alpha(-A)^k v \quad \text{for all } v \in V_{\alpha,k}$$

so that

$$\rho(\alpha) = -(A^2\tilde{\rho}(Q_\alpha)^2 + A^{-2}\tilde{\rho}(Q_\alpha)^{-2}). \tag{14}$$

As the matrices $\{\rho(\alpha) \mid \alpha \in \mathcal{P}\}$ pairwise commute, it is clear that $\{\tilde{\rho}(Q_\alpha) \mid \alpha \in \mathcal{P}\}$ pairwise commute. Hence the set $\{\tilde{\rho}(Q_\alpha) \mid \alpha \in \mathcal{P}\}$ defines a morphism $\tilde{\rho}\colon \mathbb{C}[Q_e^{\pm 1} \mid e \in \mathcal{E}(\Gamma)] \to \mathrm{End}(V)$ — here we use that $\mathcal{P}$ is canonically in bijection with $\mathcal{E}(\Gamma)$.

The following lemma will help us extend $\tilde{\rho}$ further. Recall that $U(W) = W - W^{-1}$ for an invertible element W.

Lemma 4.1 *For all $\alpha \in \mathcal{P}$, we have $U(A^k \tilde{\rho}(Q_\alpha)^2) \in \mathrm{GL}(V)$ for any $k \in \mathbb{Z}$.*

Proof Letting $k \in \mathbb{Z}$ and $\alpha \in \mathcal{P}$, the eigenvalues of $U(A^k \tilde{\rho}(Q_\alpha)^2)$ are $U(A^{k+2l} x_\alpha^2)$ for $l \in \mathbb{Z}$. Suppose that one of these is zero, which is to say that $U(A^k \tilde{\rho}(Q_\alpha)^2) \notin \mathrm{GL}(V)$. This would imply $A^{k+2l} x_\alpha^2 = A^{-k-2l} x_\alpha^{-2}$ for some $l \in \mathbb{Z}$. Taking the p^{th} power of this equality gives $x_\alpha^{4p} = 1$, which implies $r(\alpha) = x_\alpha^{2p} + x_\alpha^{-2p} = \pm 2$. This would contradict (2). □

From this we see that $\tilde{\rho}$ is defined for elements in $\mathcal{A}_A(\Gamma)$ of the form X/Y where $X, Y \in \mathbb{C}[Q_e^{\pm 1} \mid e \in \mathcal{E}(\Gamma)]$ and Y is a finite product of elements in the set $\{U(A^k Q_\gamma) \mid \gamma \in \mathcal{P}, k \in \mathbb{Z}\}$.

Lemma 4.2 *Let $e_1, e_2, e_3 \in E(\Gamma)$ meet at a vertex. For all $k \in \mathbb{Z}$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$ we have*

$$\tilde{\rho}(U(A^k Q_{e_1}^{\epsilon_1} Q_{e_2}^{\epsilon_2} Q_{e_3}^{\epsilon_3})) \in \mathrm{GL}(V).$$

Proof Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{P}$ be dual to e_1, e_2 and e_3, respectively. Suppose there exists $k \in \mathbb{Z}$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$ such that $\tilde{\rho}(U(A^k Q_{e_1}^{\epsilon_1} Q_{e_2}^{\epsilon_2} Q_{e_3}^{\epsilon_3}))$ is not invertible. This would mean that for some $l \in \mathbb{Z}$ we have $U(\pm A^l x_{\alpha_1}^{\epsilon_1} x_{\alpha_2}^{\epsilon_2} x_{\alpha_3}^{\epsilon_3}) = 0$. This implies $A^l x_{\alpha_1}^{\epsilon_1} x_{\alpha_2}^{\epsilon_2} x_{\alpha_3}^{\epsilon_3} = A^{-l} x_{\alpha_1}^{-\epsilon_1} x_{\alpha_2}^{-\epsilon_2} x_{\alpha_3}^{-\epsilon_3}$. Taking the p^{th} power of this equality (and remembering that $A^p = A^{-p}$), one gets

$$x_{\alpha_1}^{p\epsilon_1} x_{\alpha_2}^{p\epsilon_2} x_{\alpha_3}^{p\epsilon_3} = x_{\alpha_1}^{-p\epsilon_1} x_{\alpha_2}^{-p\epsilon_2} x_{\alpha_3}^{-p\epsilon_3}.$$

Let $\lambda_1 = x_{\alpha_1}^{p\epsilon_1}$, $\lambda_2 = x_{\alpha_2}^{p\epsilon_2}$, $\lambda_3 = x_{\alpha_3}^{p\epsilon_3}$ and remember that $\lambda_1\lambda_2\lambda_3 - \lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1} = 0$. Now notice that

$$\prod_{\delta_2, \delta_3 \in \{-1,1\}} (\lambda_1 \lambda_2^{\delta_2} \lambda_3^{\delta_3} - \lambda_1^{-1} \lambda_2^{-\delta_2} \lambda_3^{-\delta_3}) = 2 + \sum_{k=1}^{3} (\lambda_k^4 + \lambda_k^{-4}) - \prod_{k=1}^{3} (\lambda_k^2 + \lambda_k^{-2}),$$

with the left hand side being zero by assumption and the right hand side being

$$2 + \sum_{k=1}^{3} \mathrm{Tr}(r(\alpha_k^2)) - \prod_{k=1}^{3} \mathrm{Tr}(r(\alpha_k)).$$

This would contradict (1). □

4.2 Action of a one-cycle edge shift

Recall that the set of curves $\{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$ was defined in Figure 5, and let γ be one of these curves. Suppose that γ is a one-cycle as in Figure 6, left. In light of (6), let us define

$$\tilde{\rho}(E_e) = -\tilde{\rho}[(t_e(\gamma) + A^{-1}\gamma Q_e^{-2}) A^{-1} U(A^2 Q_e^2)^{-1}],$$
$$\tilde{\rho}(E_e^{-1}) = \tilde{\rho}[(A^3 \gamma Q_e^2 + t_e(\gamma)) A^{-1} U(A^2 Q_e^2)^{-1} F^{-1}],$$

where F is defined in (3). Notice that $\tilde{\rho}(F)$ is invertible by Lemma 4.2, and hence the second formula makes sense.

Convention In the coming proofs, we sometimes drop the symbol $\tilde{\rho}$ when the notation is too cluttered.

Lemma 4.3 *The following statements hold*:

(a) $\tilde{\rho}(Q_e)\tilde{\rho}(E_e) = -A\tilde{\rho}(E_e)\tilde{\rho}(Q_e)$ and $\tilde{\rho}(Q_e)\tilde{\rho}(E_e^{-1}) = -A^{-1}\tilde{\rho}(E_e^{-1})\tilde{\rho}(Q_e)$.

(b) $\tilde{\rho}(E_e)\tilde{\rho}(E_e^{-1}) = \mathrm{Id}_V$.

(c) $\rho(\gamma) = \tilde{\rho}(E_e) + \tilde{\rho}(E_e^{-1})\tilde{\rho}(F)$.

Proof (a) Recall that $V_{e,k} = \mathrm{Ker}(\rho(e) + (x_e^2 A^{2k+2} + x_e^{-2}A^{-2k-2})\,\mathrm{Id}_V)$ for $k \in \mathbb{Z}$. Moreover, it follows from $e\gamma = A^{-2}\gamma e + A^{-1}(A^2 - A^{-2})t_e(\gamma)$ and $et_e(\gamma) = A^2 t_e(\gamma)e - A(A^2 - A^{-2})\gamma$ that

$$\rho(e)\tilde{\rho}(E_e) = \tilde{\rho}(E_e)\tilde{\rho}(-A^4 Q_e^2 - A^{-4}Q_e^{-2}),$$

which implies that $\tilde{\rho}(E_e)V_{e,k} \subseteq V_{e,k+1}$ and in particular $\tilde{\rho}(Q_e)\tilde{\rho}(E_e) = -A\tilde{\rho}(E_e)\tilde{\rho}(Q_e)$. The second part of (a) can be proved with the exact same strategy.

(b) From (a), $U(A^2Q_e^2)^{-1}\tilde{\rho}(E_e^{-1}) = \tilde{\rho}(E_e^{-1})U(Q_e^2)$, so

$$\tilde{\rho}(E_e)\tilde{\rho}(E_e^{-1}) = -(t_e(\gamma) + A^{-1}\beta Q_e^{-2})(A^3\gamma Q_e^2 + t_e(\gamma))A^{-2}U(Q_e^2)^{-1}U(A^2Q_e^2)^{-1}F^{-1}.$$

Note that $-A^2U(Q_e^2)U(A^2Q_e^2)F = -A^2(A^2Q_e^4 + A^{-2}Q_e^{-4} + f)$. Thus it is enough to prove

$$(t_e(\gamma) + A^{-1}\beta Q_e^{-2})(A^3\gamma Q_e^2 + t_e(\gamma)) = -A^2(A^2Q_e^4 + A^{-2}Q_e^{-4} + f). \tag{15}$$

Still from (a),

$$(t_e(\gamma) + A^{-1}\gamma Q_e^{-2})(A^3\gamma Q_e^2 + t_e(\gamma)) = A^3 t_e(\gamma)\gamma Q_e^2 + t_e(\gamma)^2 + A^4\gamma^2 + A\gamma t_e(\gamma)Q_e^{-2}.$$

Now we use $t_e(\gamma)\gamma = A^{-1}e + At_\gamma^{-2}(e)$ to get

$$\begin{aligned}(t_e(\gamma) + A^{-1}\gamma Q_e^{-2})(A^3\gamma Q_e^2 + t_e(\gamma)) &= A^2(Q_e^2 + Q_e^{-2})e + t_e(\gamma)^2 + A^4\gamma^2 - A^2t_\gamma^{-2}(e)e \\ &= -A^2(A^2Q_e^4 + A^{-2}Q_e^{-4} + f).\end{aligned}$$

The last equality follows by $t_\gamma^{-2}(e)e = A^2\gamma^2 + f - A^2 - A^{-2} + A^{-2}t_e(\gamma)^2$ and $e = -(A^2Q_e^2 + A^{-2}Q_e^{-2})$.

(c) This follows from direct computation. □

4.3 Action of a two-cycle edge shift

Let $\gamma \in \{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$ be a two-cycle as in Figure 6, center. In light of (7)–(10), let us define

$$\begin{aligned}\tilde{\rho}(E_bE_c) &= \tilde{\rho}[(\gamma A^{-2}Q_b^{-2}Q_c^{-2} + t_b(\gamma)A^{-1}Q_c^{-2} + t_c(\gamma)A^{-1}Q_b^{-2} + t_bt_c(\gamma))D^{-1}],\\ \tilde{\rho}(E_bE_c^{-1}) &= \tilde{\rho}[-(\gamma A^2Q_b^{-2}Q_c^2 + t_b(\gamma)A^3Q_c^2 + t_c(\gamma)A^{-1}Q_b^{-2} + t_bt_c(\gamma))D^{-1}F_{1,-1}^{-1}],\\ \tilde{\rho}(E_b^{-1}E_c) &= \tilde{\rho}[-(\gamma A^2Q_b^2Q_c^{-2} + t_b(\gamma)A^{-1}Q_c^{-2} + t_c(\gamma)A^3Q_b^2 + t_bt_c(\gamma))D^{-1}F_{-1,1}^{-1}],\\ \tilde{\rho}(E_b^{-1}E_c^{-1}) &= \tilde{\rho}[(\gamma A^6Q_b^2Q_c^2 + t_b(\gamma)A^3Q_c^2 + t_c(\gamma)A^3Q_b^2 + t_bt_c(\gamma))D^{-1}F_{-1,-1}^{-1}],\end{aligned}$$

where $D = A^2U(A^2Q_c^2)U(A^2Q_b^2)$, and where $F_{1,-1}$, $F_{-1,1}$ and $F_{-1,-1}$ were defined in (4).

Lemma 4.4 *The following statements hold*:

(a) $\tilde{\rho}(Q_b)\tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2}) = (-A)^{\epsilon_1}\tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2})\tilde{\rho}(Q_b)$ *for all* $\epsilon_1,\epsilon_2 \in \{-1,1\}$.

(b) $\tilde{\rho}(Q_c)\tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2}) = (-A)^{\epsilon_2}\tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2})\tilde{\rho}(Q_c)$ *for all* $\epsilon_1,\epsilon_2 \in \{-1,1\}$.

(c) $\tilde{\rho}(E_bE_c)\tilde{\rho}(E_b^{-1}E_c^{-1}) = \mathrm{Id}_V$ *and* $\tilde{\rho}(E_bE_c^{-1})\tilde{\rho}(E_b^{-1}E_c) = \mathrm{Id}_V$.

(d) $\tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2})\tilde{\rho}(E_b^{\epsilon_3}E_c^{\epsilon_4}) = \tilde{\rho}(E_b^{\epsilon_3}E_c^{\epsilon_4})\tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2})$ *for all* $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4 \in \{-1,1\}$.

(e) $\rho(\gamma) = \tilde{\rho}(E_bE_c) + \tilde{\rho}(E_bE_c)\tilde{\rho}(F_{1,-1}) + \tilde{\rho}(E_b^{-1}E_c)\tilde{\rho}(F_{-1,1}) + \tilde{\rho}(E_b^{-1}E_c^{-1})\tilde{\rho}(F_{-1,-1})$.

Proof For this proof let us define, for $\epsilon_1,\epsilon_2 \in \{-1,1\}$,

$$X_{\epsilon_1,\epsilon_2} = \tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2})\tilde{\rho}(DF_{\epsilon_1,\epsilon_2}),$$

where we set $F_{1,1} = 1$.

(a) This is very similar to the proof of Lemma 4.3(a). This time

$$b\gamma = A^{-2}\gamma b + A^{-1}(A^2 - A^{-2})t_b(\gamma), \qquad bt_c(\gamma) = A^{-2}t_c(\gamma)b + A^{-1}(A^2-A^{-2})t_b(t_c(\gamma)),$$
$$bt_b(\gamma) = A^2t_b(\gamma)b - A(A^2-A^{-2})\gamma, \qquad bt_bt_c(\gamma) = A^2t_bt_c(\gamma)b - A(A^2-A^{-2})t_c(\gamma),$$

imply that $\tilde{\rho}(b)\tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2}) = \tilde{\rho}(E_b^{\epsilon_1}E_c^{\epsilon_2})\tilde{\rho}(-A^{2+2\epsilon_1}Q_b^2 - A^{2-2\epsilon_1}Q_b^{-2})$ for $\epsilon,\epsilon_2 = \pm 1$. The proof of (b) is exactly the same as (a).

(c) Note that

$$\tilde{\rho}(E_bE_c^{\epsilon})\tilde{\rho}(E_b^{-1}E_c^{-\epsilon}) = X_{1,\epsilon}X_{-1,-\epsilon}(A^4\hat{F}_{1,\epsilon}F_{-1,-\epsilon}\hat{D}D)^{-1},$$

with

$$A^4\hat{F}_{1,\epsilon}F_{-1,-\epsilon}\hat{D}D = A^4(A^{2\epsilon}Q_b^2Q_c^{2\epsilon} + A^{-2\epsilon}Q_b^{-2}Q_c^{-2\epsilon} + a)(A^{2\epsilon}Q_b^2Q_c^{2\epsilon} + A^{-2\epsilon}Q_b^{-2}Q_c^{-2\epsilon} + a').$$

Let us prove that

$$X_{1,\epsilon}X_{-1,-\epsilon} = A^4\hat{F}_{1,\epsilon}F_{-1,-\epsilon}\hat{D}D.$$

We fix $\epsilon \in \{-1,1\}$. Let $x = A^{-1}\gamma Q_b^{-2} + t_b(\gamma)$ and $y = A^3\gamma Q_b^2 + t_b(\gamma)$. Then

$$X_{1,\epsilon} = \epsilon(xA^{1-2\epsilon}Q_c^{-2\epsilon} + t_c(x)), \quad X_{-1,-\epsilon} = \epsilon(yA^{1+2\epsilon}Q_c^{2\epsilon} + t_c(y)),$$
$$X_{1,\epsilon}X_{-1,-\epsilon} = A^4xy + t_c(xy) + A^2(A^{2\epsilon-1}t_c(x)yQ_c^{2\epsilon} + A^{-2\epsilon+1}xt_c(y)Q_c^{-2\epsilon}).$$

Notice that the computations done in Lemma 4.3 (and more precisely, for (15)) can be repeated and give

$$xy = -A^2(A^2Q_b^4 + A^{-2}Q_b^{-4} + \delta_b), \tag{16}$$
$$t_c(xy) = -A^2(A^2Q_b^4 + A^{-2}Q_b^{-4} + t_c(\delta_b)). \tag{17}$$

We compute

$$A^{2\epsilon-1}t_c(x)yQ_c^{2\epsilon} + A^{-2\epsilon+1}xt_c(y)Q_c^{-2\epsilon} = A^4z + t_b(z) + A^2z',$$

where $z = A^{2\epsilon-1}t_c(\gamma)\gamma Q_c^{2\epsilon} + A^{1-2\epsilon}\gamma t_c(\gamma)Q_c^{-2\epsilon}$ and

$z' = A^{2\epsilon}t_c t_b(\gamma)\gamma Q_b^2 Q_c^{2\epsilon} + A^{-2\epsilon}\gamma t_c t_b(\gamma)Q_b^{-2}Q_c^{-2\epsilon} + t_b(\gamma)t_c(\gamma)[A^{2\epsilon-2}Q_b^{-2}Q_c^{2\epsilon} + A^{-2\epsilon+2}Q_b^2 Q_c^{-2\epsilon}]$.

Hence

$$X_{1,\epsilon}X_{-1,-\epsilon} = A^4 xy + t_c(xy) + A^2(A^4 z + t_b(z) + A^2 z'). \tag{18}$$

We compute using skein relations:

$$z = -A^2\gamma^2 - A^{-2}(t_c(\gamma))^2 - \delta_c + (A^2 + A^{-2}) + c(A^{2-2\epsilon}Q_c^2 + A^{-2+2\epsilon}Q_c^{-2}), \tag{19}$$

$$t_b(z) = -A^2(t_b(\gamma))^2 - A^{-2}(t_b t_c(\gamma))^2 - t_b(\delta_c) + (A^2 + A^{-2}) + c(A^{2-2\epsilon}Q_c^2 + A^{-2+2\epsilon}Q_c^{-2}), \tag{20}$$

$$\begin{aligned} z' = t_c(\gamma)t_b(\gamma)cb + (a + a')(A^{2\epsilon}Q_b^2 Q_c^{2\epsilon} + A^{-2\epsilon}Q_b^{-2}Q_c^{-2\epsilon}) \\ + cb(A^{2\epsilon-2}Q_b^2 Q_c^{2\epsilon} + A^{2-2\epsilon}Q_b^{-2}Q_c^{-2\epsilon}). \end{aligned} \tag{21}$$

The computation of $t_c(\gamma)t_b(\gamma)cb$ gives

$$\begin{aligned} A^4\gamma^2 + A^2(\delta_b + \delta_c) + (b^2 + c^2 - (A^2 + A^{-2})^2 + aa' + (t_b(\gamma))^2 + (t_c(\gamma))^2) \\ + A^{-2}(t_b(\delta_c) + t_c(\delta_b)) + A^{-4}(t_b t_c(\gamma))^2. \end{aligned}$$

Combining (16), (17) and (19)–(21) in (18) we get

$$X_{1,\epsilon}X_{-1,-\epsilon} = A^4(A^{2\epsilon}Q_b^2 Q_c^{2\epsilon} + A^{-2\epsilon}Q_b^{-2}Q_c^{-2\epsilon} + a)(A^{2\epsilon}Q_b^2 Q_c^{2\epsilon} + A^{-2\epsilon}Q_b^{-2}Q_c^{-2\epsilon} + a').$$

(d) By (c) it is enough to prove that

$$\tilde\rho(E_b E_c)\tilde\rho(E_b E_c^{-1}) = \tilde\rho(E_b E_c^{-1})\tilde\rho(E_b E_c).$$

We have

$$\begin{aligned} \tilde\rho(E_b E_c)\tilde\rho(E_b E_c^{-1}) &= X_{1,1}X_{1,-1}(A^4 U(A^2 Q_b^2)U(A^2 Q_c^2)U(A^4 Q_b^2)U(Q_c^2)F_{1,-1})^{-1}, \\ \tilde\rho(E_b E_c^{-1})\tilde\rho(E_b E_c) &= X_{1,-1}X_{1,1}(A^4 U(A^2 Q_b^2)U(A^2 Q_c^2)U(A^4 Q_b^2)U(Q_c^2)F_{1,-1})^{-1}. \end{aligned}$$

Thus it is enough to prove $X_{1,1}X_{1,-1} = X_{1,-1}X_{1,1}$. This reduces to

$$\begin{aligned} A^4x^2 + t_c(x^2) + Axt_c(x)Q_c^{-2} + A^3t_c(x)xQ_c^2 &= A^4x^2 + t_c(x^2) + A^5xt_c(x)Q_c^2 + A^{-1}t_c(x)xQ_c^{-2} \\ \iff Axt_c(x)Q_c^{-2} + A^3t_c(x)xQ_c^2 &= A^5xt_c(x)Q_c^2 + A^{-1}t_c(x)xQ_c^{-2} \\ \iff Axt_c(x)(A^2Q_c^2 - A^{-2}Q_c^{-2}) &= A^{-1}t_c(x)x(A^2Q_c^2 - A^{-2}Q_c^{-2}) \\ \iff \qquad Axt_c(x) &= A^{-1}t_c(x)x. \end{aligned}$$

The last equivalence is obtained because $U(A^2Q_c^2)$ is invertible. Let us expand $Axt_c(x) - A^{-1}t_c(x)x$ remembering that $Q_b x = AxQ_b$ and prove it is zero:

$$\begin{aligned} &Axt_c(x) - A^{-1}t_c(x)x \\ &\quad = A^{-4}[A\gamma t_c(\gamma) - A^{-1}t_c(\gamma)\gamma]Q_b^{-4} + A^{-2}[\gamma t_c(\gamma) - t_c(\gamma)\gamma + (A^2 - A^{-2})t_b(\gamma)t_c(\gamma)]Q_b^{-2} \\ &\qquad + t_b(A\gamma t_c(\gamma) - A^{-1}t_c(\gamma)\gamma). \end{aligned}$$

Expanding $\gamma t_c(\gamma)$ and $t_c(\gamma)\gamma$ using skein relations, this expression reduces to

$$\begin{aligned}(A^2-A^{-2})(A^{-4}cQ_b^{-4}+A^{-2}bcQ_b^{-2}+c) &= (A^2-A^{-2})[c(A^2Q_b^2+A^{-2}Q_b^{-2})+bc]Q_b^{-2}A^{-2}\\ &= (A^2-A^{-2})[-cb+bc]Q_b^{-2}A^{-2}=0. \qquad \square\end{aligned}$$

4.4 Action of the square of a separating edge shift

Let γ be a separating edge curve as shown in Figure 6, right. Again inspired by (11) and (12), let

$$\tilde\rho(E_c^2) = -\tilde\rho\Big[(\gamma Q_c^{-2}+\tau-\frac{A^{-2}\delta_1Q_c^{-2}-A^2\delta_2}{U(A^4Q_c^2)})A^{-2}U(A^2Q_c^2)^{-1}G_2^{-1}\Big],$$
$$\tilde\rho(E_c^{-2}) = \tilde\rho\Big[(\gamma A^2Q_c^2+A^{-2}\tau+\frac{\delta_1Q_c^2-\delta_2}{U(Q_c^2)})U(A^2Q_c^2)^{-1}G_{-2}^{-1}\Big].$$

Lemma 4.5 *The following statements hold*:

(a) $\tilde\rho(Q_c)\tilde\rho(E_c^2)=A^2\tilde\rho(E_c^2)\tilde\rho(Q_c)$ *and* $\tilde\rho(Q_c)\tilde\rho(E_c^{-2})=A^{-2}\tilde\rho(E_c^{-2})\tilde\rho(Q_c)$.

(b) $\tilde\rho(E_c^2)\tilde\rho(E_c^{-2})=\mathrm{Id}_V$.

(c) $\rho(\gamma)=\tilde\rho(E_c^2)\tilde\rho(G_2)+\tilde\rho(G_0)+\tilde\rho(E_c^{-2})\tilde\rho(G_{-2})$.

Proof (a) Using skein relations we have

$$\begin{aligned}c\gamma &= A^{-4}\gamma c+A^{-2}(A^4-A^{-4})\tau+A^{-2}(A^2-A^{-2})(d_1d_3+d_2d_4),\\ c\tau &= A^4\tau c-A^2(A^4-A^{-4})c-A^2(A^2-A^{-2})(d_1d_2+d_3d_4).\end{aligned}$$

Then a computation implies

$$cE_c^{\pm2}=E_c^{\pm2}(-A^{2\pm4}Q_c^2-A^{-(2\pm4)}Q_c^{-2}),$$

which implies the desired equalities.

(b) Let $Y_2=\tilde\rho(E_c^2)A^2U(A^2Q_c^2)^1G_2^1$ and $Y_{-2}=\tilde\rho(E_c^{-2})U(A^2Q_c^2)G_{-2}$. Using (a), $\tilde\rho(E_c^2)\tilde\rho(E_c^{-2})=Y_2Y_{-2}(A^2U(A^{-2}Q_c^2)\widehat G_2U(A^2Q_c^2)G_{-2})^{-1}$, where

$$\widehat G_2=-U(A^2Q_{d_1}Q_{d_4}Q_c^{-1})U(A^2Q_{d_2}Q_{d_3}Q_c^{-1}).$$

Let us prove that $Y_2Y_{-2}=A^2U(A^{-2}Q_c^2)\widehat G_2U(A^2Q_c^2)G_{-2}$. A brute force computation gives

$$\begin{aligned}&A^2U(A^{-2}Q_c^2)\widehat G_2U(A^2Q_c^2)G_{-2}\\ &\qquad=-A^2(-T^4+\delta_3T^3+(8-\Delta)T^2+(\delta_1\delta_2-4\delta_3)T+(4\Delta-16-\delta_1^2-\delta_2^2))U(Q_c^2)^{-2},\end{aligned}$$

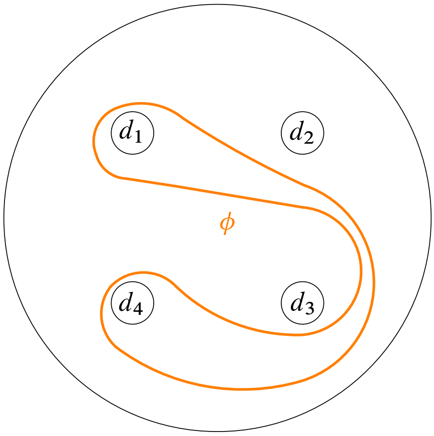
where $\Delta=d_1^2+d_2^2+d_3^2+d_4^2+d_1d_2d_3d_4$ and $T=Q_c^2+Q_c^{-2}$. (Note that this equality is equivalent to an equality of two Laurent polynomials in A and the variables Q_e, and hence can be checked using Sage, for example.) Therefore it is enough to prove that

$$-Y_2Y_{-2}=A^2(-T^4+\delta_3T^3+(8-\Delta)T^2+(\delta_1\delta_2-4\delta_3)T+(4\Delta-16-\delta_1^2-\delta_2^2))U(Q_c^2)^{-2}.$$

Let us prove this:

$$\begin{aligned}
-&Y_2Y_{-2}\\
&= \left(\gamma Q_c^{-2} + \tau - \frac{A^{-2}\delta_1 Q_c^{-2} - A^2\delta_2}{U(A^4 Q_c^2)}\right)\left(\gamma A^2 Q_c^2 + A^{-2}\tau + \frac{\delta_1 Q_c^2 - \delta_2}{U(Q_c^2)}\right)\\
&= A^6\gamma^2 + A^2\gamma\tau Q_c^{-2} + A^4\gamma\delta_1 U(Q_c^2)^{-1} - A^4\gamma\delta_2 Q_c^{-2}U(Q_c^2)^{-1}\\
&\qquad + A^2\tau\gamma Q_c^2 + A^{-2}\tau^2 + \tau\delta_1 Q_c^2 U(Q_c^2)^{-1} - \tau\delta_2 U(Q_c^2)^{-1}\\
&\qquad - A^4\delta_1\gamma U(Q_c^2)^{-1} - \delta_1\tau Q_c^{-2}U(Q_c^2)^{-1} - A^2\delta_1^2 U(Q_c^2)^{-2} + A^2\delta_1\delta_1 Q_c^{-2}U(Q_c^2)^{-2}\\
&\qquad + A^4\delta_2\gamma Q_c^2 + \delta_2\tau U(Q_c^2)^{-1} + A^2\delta_1\delta_2 Q_c^2 U(Q_c^2)^{-2} - A^2\delta_2^2 U(Q_c^2)^{-2}\\
&= A^4\delta_2\gamma + \tau\delta_1 + [A^2\delta_1\delta_2(Q_c^2+Q_c^{-2}) - A^2(\delta_1^2+\delta_2^2)]U(Q_c^2)^{-2} + A^6\gamma^2 + A^2(\gamma\tau Q_c^{-2} + \tau\gamma Q_c^2) + A^{-2}\tau^2.
\end{aligned}$$

The term $\gamma\tau$ can reduced using skein relations to $A^2c + \delta_3 + A^{-2}\varphi$, where $\delta_3 = d_1d_4 + d_2d_3$ and φ is



Hence $\gamma\tau Q_c^{-2} + \tau\gamma Q_c^2 = (A^{-2}Q_c^2 + A^2Q_c^{-2})c + \delta_3(Q_c^2 + Q_c^{-2}) - \varphi c$. The expansion of φc gives

$$\varphi c = \Delta - (A^2 + A^{-2})^2 + A^4\gamma^2 + A^2\delta_2\gamma + A^{-2}\delta_1\tau + A^{-4}\tau^2.$$

We recall that $\Delta = d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_1d_2d_3d_4$. Now plugging this back into the expression for $-Y_2Y_{-2}$, we get, after simplifications,

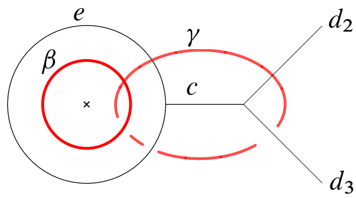
$$\begin{aligned}
-Y_2Y_{-2} &= \left(\gamma Q_c^{-2} + \tau - \frac{A^{-2}\delta_1 Q_c^{-2} - A^2\delta_2}{U(A^4Q_c^2)}\right)\left(\gamma A^2 Q_c^2 + A^{-2}\tau + \frac{\delta_1 Q_c^2 - \delta_2}{U(Q_c^2)}\right)\\
&= A^2(\delta_1\delta_2(Q_c^2+Q_c^{-2})U(Q_c^2)^{-2} - (\delta_1^2+\delta_2^2)U(Q_c^2)^{-2} + \delta_3(Q_c^2+Q_c^{-2}) - \Delta - U(Q_c^2)^2)\\
&= A^2(\delta_1\delta_2(Q_c^2+Q_c^{-2}) - (\delta_1^2+\delta_2^2) + \delta_3(Q_c^2+Q_c^{-2})U(Q_c^2)^2 - \Delta U(Q_c^2)^2 - U(Q_c^2)^4)U(Q_c^2)^{-2}\\
&= A^2(-T^4 + \delta_3T^3 + (8-\Delta)T^2 + (\delta_1\delta_2 - 4\delta_3)T + (4\Delta - 16 - \delta_1^2 - \delta_2^2))U(Q_c^2)^{-2},
\end{aligned}$$

where we recall that $T = Q_c^2 + Q_c^{-2}$. The last equality is obtained from the identity $U(Q_c^2)^2 = T^2 - 4$.

(c) This is derived from a direct computation. □

4.5 Commutation between the square of an edge and a one-cycle

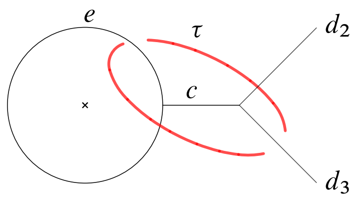
Consider the following portion of the graph



Recall that $\tilde{\rho}(E_c^2) = \tilde{\rho}[Y_2 A^{-2} U(A^2 Q_c^2)^{-1} G_2^{-1}]$ where $G_2 = -U(Q_e^2 Q_c^{-1}) U(Q_{d_2} Q_{d_3} Q_c^{-1})$, and where

$$-Y_2 = \gamma Q_c^{-2} + \tau - \frac{A^{-2}\delta_1 Q_c^{-2} - A^2 \delta_2}{U(A^4 Q_c^2)}$$

with $\delta_1 = ed_3 + ed_2$, $\delta_2 = ed_2 + ed_3$ and τ being the following curve:



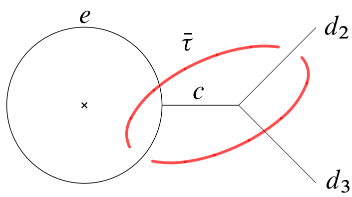
Recall also that $\tilde{\rho}(E_e) = -\tilde{\rho}[(t_e(\beta) + A^{-1}\beta Q_e^{-2}) A^{-1} U(A^2 Q_e^2)^{-1}]$.

Lemma 4.6 $$\tilde{\rho}(E_e)\tilde{\rho}(E_c^2) = \tilde{\rho}(E_c^2)\tilde{\rho}(E_e).$$

Proof Let us first simplify the expression of $\tilde{\rho}(E_c^2)$. A computation shows that

$$\tilde{\rho}(E_c^2) = -[(\gamma - t_c(\gamma) + (\bar{\tau} - \tau) Q_c^{-2}) A^{-2} U(A^4 Q_c^2)^{-1} U(A^2 Q_c^2)^{-1} G_2^{-1}],$$

where $\bar{\tau} = t_c^{-1}(\tau)$ is the following curve:



Let us define $Y_2' = \gamma - t_c(\gamma) + (\bar{\tau} - \tau) Q_c^{-2}$, $\varphi = \gamma - t_c(\gamma)$ and $\psi = \bar{\tau} - \tau$. Using A-commutation, $\tilde{\rho}(E_e)\tilde{\rho}(E_c^2) = \tilde{\rho}(E_c^2)\tilde{\rho}(E_e)$ is equivalent to

$$X_1 Y_2' (A^2 Q_e^2 Q_c^{-1} - A^{-2} Q_e^{-2} Q_c) = Y_2' X_1 (Q_e^2 Q_c^{-1} - Q_e^{-2} Q_c), \tag{22}$$

where $X_1 = t_e(\beta) + A^{-1}\beta Q_e^{-2}$. For two elements x and y we define $[x, y]_A = Axy - A^{-1} yx$. Proving (22) is equivalent to proving that

$$(A[X_1, Y_2']_A Q_e^2 + A^{-1}[Y_2', X_1]_A Q_e^{-2} Q_c^2) Q_c^{-1} = 0.$$

Now let us expand the expression $\mathcal{E} = A[X_1, Y_2']_A Q_e^2 + A^{-1}[Y_2', X_1]_A Q_e^{-2} Q_c^2$ using the fact that X_1 commutes with Q_c and Y_2' commutes with Q_e:

$$\begin{aligned}
&A[X_1, Y_2']_A Q_e^2 + A^{-1}[Y_2', X_1]_A Q_e^{-2} Q_c^2 \\
&\quad = A([t_e(\beta), \varphi]_A + [t_e(\beta), \psi]_A Q_c^{-2} + A^{-1}[\beta, \varphi]_A Q_e^{-2} + A^{-1}[\beta, \psi]_A Q_c^{-2} Q_e^{-2}) Q_e^2 \\
&\qquad\quad + A^{-1}([\varphi, t_e(\beta)]_A + A^{-1}[\varphi, \beta]_A Q_e^{-2} + [\psi, t_e(\beta)]_A Q_c^{-2} + A^{-1}[\psi, \beta]_A Q_c^{-2} Q_e^{-2}) Q_e^{-2} Q_c^2 \\
&\quad = A([t_e(\beta), \varphi]_A Q_e^2 + [t_e(\beta), \psi]_A Q_c^{-2} Q_e^2 + A^{-1}[\beta, \varphi]_A + A^{-1}[\beta, \psi]_A Q_c^{-2}) \\
&\qquad\quad + A^{-1}([\varphi, t_e(\beta)]_A Q_e^{-2} Q_c^2 + A^{-1}[\varphi, \beta]_A Q_e^{-4} Q_c^2 + [\psi, t_e(\beta)]_A Q_e^{-2} + A^{-1}[\psi, \beta]_A Q_e^{-4}).
\end{aligned}$$

Using that $Q_e^{-2} = -A^2 e - A^4 Q_e^2$, $Q_c^{-2} = -A^2 c - A^4 Q_c^2$ and $Q_e^{-4} = A^4 e^2 + A^6 e Q_e^2 - A^4$, we get

$$\mathcal{E} = C_1 Q_c^2 Q_e^2 + C_2 Q_e^2 + C_3 Q_c^2 + C_4,$$

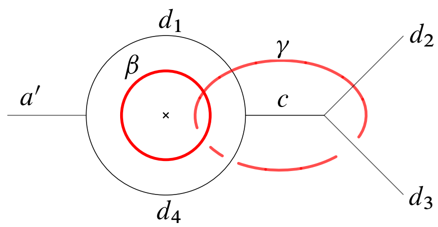
where

$$\begin{aligned}
C_1 &= -A^5[t_e(\beta), \psi]_A - A^3[\varphi, t_e(\beta)]_A + A^4[\varphi, \beta]_A e, \\
C_2 &= A[t_e(\beta), \varphi]_A - A^3[t_e(\beta), \psi]_A c - A^3[\psi, t_e(\beta)]_A + A^4[\psi, \beta]_A e, \\
C_3 &= -A^4[\beta, \psi]_A - A[\varphi, t_e(\beta)]_A e + A^2[\varphi, \beta]_A e^2 - A^2[\varphi, \beta]_A, \\
C_4 &= [\beta, \varphi]_A - A^2[\beta, \psi]_A c - A[\psi, t_e(\beta)]_A e + A^2[\psi, \beta]_A e^2 - A^2[\psi, \beta]_A.
\end{aligned}$$

The elements C_1, C_2, C_3 and C_4 are skein elements that can be computed using skein relations. After a straightforward computation we get $C_1 = C_2 = C_3 = C_4 = 0$, which shows that $\mathcal{E} = 0$. More details on those computations are given in the appendix. □

4.6 Commutation between the square of an edge and a two-cycle

Consider the following portion of the graph



Using the proof of Lemma 4.6, we have

$$\tilde{\rho}(E_c^2) = -\tilde{\rho}[Y_2' A^{-2} U(A^4 Q_c^2)^{-1} U(A^2 Q_c^2)^{-1} G_2^{-1}],$$

with $Y_2' = \gamma - t_c(\gamma) + (\bar{\tau} - \tau) Q_c^{-2}$. Recall also that we set $\varphi = \gamma - t_c(\gamma)$ and $\psi = \bar{\tau} - \tau$. For the two-cycle β we have

$$\begin{aligned}
\tilde{\rho}(E_{d_1} E_{d_4}) &= \tilde{\rho}[(\beta A^{-2} Q_{d_1}^{-2} Q_{d_4}^{-2} + t_{d_1}(\beta) A^{-1} Q_{d_4}^{-2} + t_{d_4}(\beta) A^{-1} Q_{d_1}^{-2} + t_{d_1} t_{d_4}(\beta)) D^{-1}], \\
\tilde{\rho}(E_{d_1} E_{d_4}^{-1}) \tilde{\rho}(F_{1,-1}) &= \tilde{\rho}[-(\beta A^2 Q_{d_1}^{-2} Q_{d_4}^2 + t_{d_1}(\beta) A^3 Q_{d_4}^2 + t_{d_4}(\beta) A^{-1} Q_{d_1}^{-2} + t_{d_1} t_{d_4}(\beta)) D^{-1}],
\end{aligned}$$

where $D = A^2 U(A^2 Q_{d_1}^2) U(A^2 Q_{d_4}^2)$ and

$$F_{1,-1} = -\frac{U(Q_{a'} Q_{d_4} Q_{d_1}^{-1}) U(Q_c Q_{d_4} Q_{d_1}^{-1})}{U(A^2 Q_{d_4}^2) U(Q_{d_4}^2)}.$$

Lemma 4.7 *The following statements hold*:

(a) $\tilde\rho(E_{d_1} E_{d_4}) \tilde\rho(E_c^2) = \tilde\rho(E_c^2) \tilde\rho(E_{d_1} E_{d_4})$.

(b) $\tilde\rho(E_{d_1} E_{d_4}^{-1}) \tilde\rho(E_c^2) = \tilde\rho(E_c^2) \tilde\rho(E_{d_1} E_{d_4}^{-1})$.

Proof To make the notation less cluttered, we write Q_4 for Q_{d_4} and Q_1 for Q_{d_1}.

(a) First let us set $X = A^{-1} \beta Q_1^{-2} + t_{d_1}(\beta)$ so that

$$\tilde\rho(E_{d_1} E_{d_4}) = (A^{-1} X Q_4^{-2} + t_{d_4}(X)) D^{-1}.$$

Using known commutation relations, $\tilde\rho(E_{d_1} E_{d_4}) \tilde\rho(E_c^2) = \tilde\rho(E_c^2) \tilde\rho(E_{d_1} E_{d_4})$ is equivalent to

$$\mathcal{E} = A^{-1}(A[X, Y_2']_A Q_1^2 Q_4^2 + A^{-1}[Y_2', X] Q_c^2) Q_4^{-2} + t_{d_4}(A[X, Y_2']_A Q_1^2 Q_4^2 + A^{-1}[Y_2', X] Q_c^2) = 0.$$

If we set $\mathcal{U} = A[X, Y_2']_A Q_1^2 Q_4^2 + A^{-1}[Y_2', X] Q_c^2$, the expression of $\mathcal{E}$ is simply $A^{-1} \mathcal{U} Q_4^{-2} + t_{d_4}(\mathcal{U})$. Let us first compute $\mathcal{U}$ by expanding the expression and remembering that $Q_1^{-2} = -A^4 Q_1^2 - A^2 d_1$ and $Q_c^{-2} = -A^4 Q_c^2 - A^2 c$:

$$\begin{aligned}\mathcal{U} &= A[X, Y_2']_A Q_1^2 Q_4^2 + A^{-1}[Y_2', X] Q_c^2 \\ &= -A^5 [t_{d_1}(\beta), \psi]_A Q_c^2 Q_1^2 Q_4^2 - (A^3 [t_{d_1}(\beta), \psi]_A c + A[t_{d_1}(\beta), \varphi]_A) Q_1^2 Q_4^2 - A^4 [\beta, \psi]_A Q_c^2 Q_4^2 \\ &\qquad - A^2 [\varphi, \beta]_A Q_1^2 Q_c^2 + ([\beta, \varphi]_A - A^2 [\beta, \psi]_A c) Q_4^2 - A^2 [\psi, \beta]_A Q_1^2 \\ &\qquad + (A^{-1} [\varphi, t_{d_1}(\beta)]_A - [\varphi, \beta]_A d_1) Q_c^2 + (A^{-1} [\psi, t_{d_1}(\beta)]_A - [\psi, \beta]_A d_1).\end{aligned}$$

Using the same technique we get

$$A^{-1} \mathcal{U} Q_4^{-2} + t_{d_4}(\mathcal{U}) = \sum_{\epsilon_1, \epsilon_2, \epsilon_3 = 0,1} C_{\epsilon_1, \epsilon_2, \epsilon_3} Q_1^{2\epsilon_1} Q_4^{2\epsilon_2} Q_c^{2\epsilon_3},$$

where

$$\begin{aligned}
C_{1,1,1} &= -A^5 [t_{d_4} t_{d_1}(\beta), \psi]_A + A^5 [\varphi, \beta]_A, \\
C_{1,1,0} &= A^5 [\psi, \beta]_A - A^3 [t_{d_4} t_{d_1}(\beta), \psi]_A c + A[t_{d_4} t_{d_1}(\beta), \varphi]_A, \\
C_{1,0,1} &= -A^2 [\varphi, t_{d_4}(\beta)]_A + A^3 [\varphi, \beta]_A d_4 - A^4 [t_{d_1}(\beta), \psi]_A, \\
C_{0,1,1} &= -A^2 [\varphi, t_{d_1}(\beta)]_A + A^3 [\varphi, \beta]_A d_1 - A^4 [t_{d_4}(\beta), \psi]_A, \\
C_{0,0,1} &= -A^3 [\beta, \psi]_A - [\varphi, t_{d_1}(\beta)]_A d_4 + A[\varphi, \beta]_A d_1 d_4 + A^{-1} [\varphi, t_{d_4} t_{d_1}(\beta)]_A - [\varphi, t_{d_4}(\beta)]_A d_1, \\
C_{0,1,0} &= A^3 [\psi, \beta]_A d_1 + [t_{d_4}(\beta), \varphi]_A - A^2 [t_{d_4}(\beta), \psi]_A c - A^2 [\psi, t_{d_1}(\beta)]_A, \\
C_{1,0,0} &= -A^2 [t_{d_1}(\beta), \psi]_A c + [t_{d_1}(\beta), \varphi]_A - A^2 [\psi, t_{d_4}(\beta)]_A + A^3 [\psi, \beta]_A d_4, \\
C_{0,0,0} &= -[\psi, t_{d_4}(\beta)]_A d_1 + A^{-1} [\psi, t_{d_4} t_{d_1}(\beta)]_A + A^{-1} [\beta, \varphi]_A - A[\beta, \psi]_A c + A[\psi, \beta]_A d_1 d_4 \\ &\qquad - [\psi, t_{d_1}(\beta)]_A d_4.
\end{aligned}$$

Here again, these elements are in the skein algebra of the surface, and hence can be computed using skein relations. A long but straightforward computation shows that $C_{\epsilon_1,\epsilon_2,\epsilon_3}=0$ for all $\epsilon_1,\epsilon_2,\epsilon_3\in\{0,1\}$. We refer to the appendix for more details. Thus $\mathcal{E}=A^{-1}\mathcal{U}Q_4^{-2}+t_{d_4}(\mathcal{U})=0$.

(b) The proof goes as in (a) and we use the same notation. Here $\tilde\rho(E_{d_1}E_{d_4}^{-1})\tilde\rho(E_c^2)=\tilde\rho(E_c^2)\tilde\rho(E_{d_1}E_{d_4}^{-1})$ is equivalent to

$$\mathcal{E}=A^3(A[Y_2',X]_AQ_c^2Q_4^2+A^{-1}[X,Y_2']Q_1^2)Q_4^2+t_{d_4}(A[Y_2',X]_AQ_c^2Q_4^2+A^{-1}[X,Y_2']Q_1^2)=0.$$

Expanding this using $Q_1^{-2}=-A^4Q_1^2-A^2d_1$, $Q_c^{-2}=-A^4Q_c^2-A^2c$ and $Q_4^4=-A^{-2}Q_4^2d_4-A^{-4}$ we get

$$\mathcal{E}=\sum_{\epsilon_1,\epsilon_2,\epsilon_3=0,1}D_{\epsilon_1,\epsilon_2,\epsilon_3}Q_1^{2\epsilon_1}Q_4^{2\epsilon_2}Q_c^{2\epsilon_3},$$

where

$$D_{1,1,1}=A^5[\varphi,\beta]_Ad_4-A^6[t_{d_1}(\beta),\psi]_A-A^4[\varphi,t_{d_4}(\beta)]_A,$$
$$D_{1,1,0}=A^5[\psi,\beta]_Ad_4+A^2[t_{d_1}(\beta),\varphi]_A-A^4[t_{d_1}(\beta),\psi]_Ac-A^4[\psi,t_{d_4}(\beta)]_A,$$
$$D_{1,0,1}=A^3[\varphi,\beta]_A-A^3[t_{d_1}t_{d_4}(\beta),\psi]_A,$$
$$D_{0,1,1}=-A^2[\varphi,t_{d_1}(\beta)]_Ad_4+A^3[\varphi,\beta]_Ad_1d_4-A^5[\beta,\psi]_A+A[\varphi,t_{d_4}t_{d_1}(\beta)]_A-A^2[\varphi,t_{d_4}(\beta)]_Ad_1,$$
$$D_{0,0,1}=-[\varphi,t_{d_1}(\beta)]_A+A[\varphi,\beta]_Ad_1-A^2[t_{d_4}(\beta),\psi]_A,$$
$$D_{0,1,0}=-A^2[\psi,t_{d_1}(\beta)]_Ad_4-A^3[\beta,\psi]_Ac+A[\beta,\varphi]_A+A^3[\psi,\beta]_Ad_1d_4+A[\psi,t_{d_1}t_{d_4}(\beta)]_A-A^2[\psi,t_{d_4}(\beta)]_Ad_1,$$
$$D_{1,0,0}=A^3[\psi,\beta]_A+A^{-1}[t_{d_1}t_{d_4}(\beta),\varphi]_A-A[t_{d_1}t_{d_4}(\beta),\psi]_Ac,$$
$$D_{0,0,0}=A[\psi,\beta]_Ad_1-[\psi,t_{d_1}(\beta)]_A-[t_{d_4}(\beta),\psi]_Ac+A^{-2}[t_{d_4}(\beta),\varphi]_A.$$

Notice that $D_{1,1,1}=A^2C_{1,0,1}$, $D_{1,1,0}=A^2C_{1,0,0}$, $D_{1,0,1}=A^{-2}C_{1,1,1}$, $D_{0,1,1}=A^2C_{0,0,1}$, $D_{0,0,1}=A^{-2}C_{0,1,1}$, $D_{0,1,0}=A^2C_{0,0,0}$, $D_{1,0,0}=A^{-2}C_{1,1,0}$, and $D_{0,0,0}=A^{-2}C_{0,1,0}$. Hence these elements are again all vanishing. □

4.7 Proof of Theorem 1.5

Lemma 3.2 gives us a list of generators for $\mathcal{A}_A(\Gamma)^0$. Subsections 4.1–4.4 define $\tilde\rho$ on these generators. Lemmas 4.3(a)–(b), 4.4(a)–(d), 4.5(a)–(b), 4.6 and 4.7 insure that $\tilde\rho$ preserve the relations between the generators and therefore defines a representation $\tilde\rho\colon\mathcal{A}_A(\Gamma)^0\to\mathrm{End}(V)$. Now Lemmas 4.3(c), 4.4(e) and 4.5(c) and (14) tell us that $\tilde\rho\circ\sigma_A$ coincides with ρ on $\mathcal{P}\cup\{\beta_1,\dots,\beta_g,\gamma_1,\dots,\gamma_{g-1}\}$. As A is a $2p^{\text{th}}$ primitive root of unity with $p\geq 3$, $\mathcal{P}\cup\{\beta_1,\dots,\beta_g,\gamma_1,\dots,\gamma_{g-1}\}$ generates $S_A(\Sigma)$ by [Santharoubane 2024, Theorem 1.1], and hence $\tilde\rho\circ\sigma_A=\rho$.

4.8 Classical shadow

We still work with an irreducible representation $\rho\colon S_A(\Sigma)\to\mathrm{End}(V)$ with classical shadow r satisfying the hypothesis of Theorem 1.5. Let $\tilde\rho\colon\mathcal{A}_A(\Gamma)^0\to\mathrm{End}(V)$ be the lift of ρ built in the previous subsections. For γ a simple a closed curve on Σ, let $r_\gamma=-\mathrm{Tr}(r(\gamma))$.

Proposition 4.8 *Let $\gamma \in \{\beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{g-1}\}$.*

(a) *If γ is a one-cycle as in Figure 6, left, then $\tilde{\rho}(Q_e^{2p}) = x_e^{2p} \operatorname{Id}_V$ and*

$$\tilde{\rho}(E_e^p) = \frac{r_\gamma x_e^{-2p} + r_{t_e(\gamma)}}{x_e^{2p} - x_e^{-2p}} \operatorname{Id}_V .$$

(b) *If γ is a two-cycle as in Figure 6, center, then $\tilde{\rho}((Q_b Q_c^{\pm 1})^p) = x_b^p x_c^{\pm p} \operatorname{Id}_V$,*

$$\tilde{\rho}((E_b E_c)^p) = \frac{r_\gamma x_b^{-2p} x_c^{-2p} - r_{t_b(\gamma)} x_c^{-2p} - r_{t_c(\gamma)} x_b^{-2p} + r_{t_b t_c(\gamma)}}{(x_b^{2p} - x_b^{-2p})(x_c^{2p} - x_c^{-2p})} \operatorname{Id}_V$$

and

$$\tilde{\rho}((E_b E_c^{-1})^p) = \frac{-r_\gamma x_b^{-2p} x_c^{2p} + r_{t_b(\gamma)} x_c^{2p} + r_{t_c(\gamma)} x_b^{-2p} - r_{t_b t_c(\gamma)}}{(x_b^{2p} - x_b^{-2p})(x_c^{2p} - x_c^{-2p})\omega} \operatorname{Id}_V,$$

where

$$\omega = -\prod_{k=0}^{p-1} \frac{U((-A)^k x_{a'} x_c x_b^{-1}) U((-A)^k x_a x_c x_b^{-1})}{U((-A)^k x_c^2)^2}.$$

(c) *If γ is a separating edge curve as in Figure 6, right, then $\tilde{\rho}(Q_c^p) = x_c^p \operatorname{Id}_V$ and*

$$\tilde{\rho}(E_c^{2p}) = \frac{r_{t_c^{-1}(\gamma)} x_c^{-2p} + r_\gamma r_c + r_{t_c(\gamma)} x_c^{2p}}{(x_c^{2p} - x^{-2p})^2 r_c \omega'} \operatorname{Id}_V,$$

where $\omega' = \prod_{k=0}^{p-1} U((-A)^k x_{d_1} x_{d_4} x_c^{-1}) U((-A)^k x_{d_2} x_{d_3} x_c^{-1})$.

Proof Before starting the proof, let us recall some important facts from Lemma 4.3. Given e an edge, we can decompose V as the direct sum of the subspaces $V_{e,k} = \operatorname{Ker}(\rho(e) + (x_e^2 A^{2k+2} + x_e^{-2} A^{-2k-2}) \operatorname{Id}_V)$.

Moreover, each $V_{e,k}$ is stable by all operators $\tilde{\rho}(Q_e)$ and $\tilde{\rho}(E_f), \tilde{\rho}(Q_f)$ with $f \neq e$, and finally $\tilde{\rho}(E_e)(V_{e,k}) \subset V_{e,k+1}$. When computing $T_p(\gamma)$ for γ a curve on Σ and where T_p is the p^{th} Chebyshev polynomial, since we know that $T_p(\gamma)$ is a multiple of Id_V, it is sufficient to compute the "diagonal part" of $T_p(\gamma)$, ie the contribution that corresponds to maps $V_{e,k} \to V_{e,k}$, since the nondiagonal parts will vanish.

(a) $\tilde{\rho}(Q_e^{2p}) = x_e^{2p} \operatorname{Id}_V$ is immediate from the definition of $\tilde{\rho}(Q_e)$. To compute $\tilde{\rho}(E_e^p)$, notice that taking the p^{th} Chebyshev in the equality $\rho(\gamma) = \tilde{\rho}(E_e) + \tilde{\rho}(E_e^{-1})\tilde{\rho}(F)$ we have

$$T_p(\rho(\gamma)) = \tilde{\rho}(E_e^p) + \tilde{\rho}((E_e^{-1} F)^p).$$

Similarly $\rho(t_e(\gamma)) = -A^3 \tilde{\rho}(E_e)\tilde{\rho}(Q_e^2) - A^{-1}\tilde{\rho}(E_e^{-1} F)\tilde{\rho}(Q_e^{-2})$ gives

$$T_p(\rho(t_e(\gamma))) = -\tilde{\rho}(E_e^p)\tilde{\rho}(Q_e^{2p}) - \tilde{\rho}((E_e^{-1} F)^p)\tilde{\rho}(Q_e^{-2p}).$$

We thus have

$$r_\gamma \operatorname{Id}_V = \tilde{\rho}(E_e^p) + \tilde{\rho}((E_e^{-1} F)^p) \quad \text{and} \quad r_{t_e(\gamma)} \operatorname{Id}_V = -\tilde{\rho}(E_e^p) x_e^{2p} - \tilde{\rho}((E_e^{-1} F)^p) x_e^{-2p}.$$

The conclusion is obtained by solving this system.

(b) $\tilde{\rho}(Q_bQ_c^{\pm 1}) = x_b^p x_c^{\pm p}\,\mathrm{Id}_V$ is immediate. The other part is very similar to (a). This time we start from the system

$$\begin{cases} \sigma(\gamma) = X_{1,1} + X_{1,-1} + X_{-1,1} + X_{-1,-1}, \\ \sigma(t_b(\gamma)) = -A^3X_{1,1}Q_b^2 - A^3X_{1,-1}Q_b^2 - A^{-1}X_{-1,1}Q_b^{-2} - A^{-1}X_{-1,-1}Q_b^{-2}, \\ \sigma(t_c(\gamma)) = -A^3X_{1,1}Q_c^2 - A^{-1}X_{1,-1}Q_c^{-2} - A^3X_{-1,1}Q_c^2 - A^{-1}X_{-1,-1}Q_c^{-2}, \\ \sigma(t_bt_c(\gamma)) = A^6X_{1,1}Q_b^2Q_c^2 + A^2X_{1,-1}Q_b^2Q_c^{-2} + A^2X_{-1,1}Q_b^{-2}Q_c^2 + A^{-2}X_{-1,-1}Q_b^{-2}Q_c^{-2}, \end{cases}$$

where $X_{\epsilon_1,\epsilon_2} = E_b^{\epsilon_1}E_c^{\epsilon_2}F_{\epsilon_1,\epsilon_2}$ with $F_{1,1} = 1$. Applying $\tilde{\rho}$ and taking the p^{th} Chebyshev polynomial for these four equalities, one gets

$$\begin{cases} r_\gamma = X_{1,1}^p + X_{1,-1}^p + X_{-1,1}^p + X_{-1,-1}^p, \\ r_{t_b(\gamma)} = X_{1,1}^p x_b^{2p} + X_{1,-1}^p x_b^{2p} + X_{-1,1}^p x_b^{-2p} + X_{-1,-1}^p x_b^{-2p}, \\ r_{t_c(\gamma)} = X_{1,1}^p x_c^{2p} + X_{1,-1}^p x_c^{-2p} + X_{-1,1}^p x_c^{2p} + X_{-1,-1}^p x_c^{-2p}, \\ r_{t_bt_c(\gamma)} = X_{1,1}^p x_b^{2p}x_c^{2p} + X_{1,-1}^p x_b^{2p}x_c^{-2p} + X_{-1,1}^p x_b^{-2p}x_c^{2p} + X_{-1,-1}^p x_b^{-2p}x_c^{-2p}. \end{cases}$$

In this system, $X_{\pm 1,\pm 1}$ has to be understood as $\tilde{\rho}(X_{\pm 1,\pm 1})$. As $X_{1,1}^p = \tilde{\rho}((E_bE_c)^p)$, the desired expression is immediate by solving the system. For $\tilde{\rho}((E_bE_c^{-1})^p)$, the resolution of the system gives

$$X_{1,-1}^p = \tilde{\rho}((E_bE_c^{-1}F_{1,-1})^p) = \frac{-r_\gamma x_b^{-2p}x_c^{2p} + r_{t_b(\gamma)}x_c^{2p} + r_{t_c(\gamma)}x_b^{-2p} - r_{t_bt_c(\gamma)}}{(x_b^{2p} - x_b^{-2p})(x_c^{2p} - x_c^{-2p})}\,\mathrm{Id}_V\,.$$

Now from Lemma 4.4(a)–(b) and the fact that $\tilde{\rho}(Q_a)$ and $\tilde{\rho}(Q_{a'})$ commute with $\tilde{\rho}(E_bE_c^{-1})$, we get

$$\tilde{\rho}((E_bE_c^{-1}F_{1,-1})^p) = -\tilde{\rho}((E_bE_c^{-1})^p)\prod_{k=0}^{p-1}\frac{U(A^{-2k}Q_{a'}Q_cQ_b^{-1})U(A^{-2k}Q_aQ_cQ_b^{-1})}{U(A^{2-2k}Q_c^2)U(A^{-2k}Q_c^2)}.$$

Letting v be an eigenvector common to Q_a, $Q_{a'}$, Q_b and Q_c, we see that

$$\frac{U(A^{-2k}Q_{a'}Q_cQ_b^{-1})U(A^{-2k}Q_aQ_cQ_b^{-1})}{U(A^{2-2k}Q_c^2)U(A^{-2k}Q_c^2)}v = \prod_{k=0}^{p-1}\frac{U((-A)^kx_{a'}x_cx_b^{-1})U((-A)^kx_ax_cx_b^{-1})}{U((-A)^kx_c^2)^2}v.$$

Thus $\tilde{\rho}((E_bE_c^{-1}F_{1,-1})^p) = \tilde{\rho}((E_bE_c^{-1})^p)\omega$ and we can conclude.

(c) The equality $\tilde{\rho}(Q_c^p) = x_c^p\,\mathrm{Id}_V$ is clear. Let us prove the other one. Applying the p^{th} Chebyshev polynomial to $\rho(\gamma) = \tilde{\rho}(E_c^2)\tilde{\rho}(G_2) + \tilde{\rho}(G_0) + \tilde{\rho}(E_c^{-2})\tilde{\rho}(G_{-2})$ we get

$$T_p(\rho(\gamma)) = \tilde{\rho}((E_c^2G_2)^p) + H + \tilde{\rho}((E_c^{-2}G_{-2})^p),$$

where H is the degree-zero term in E_c. We recall that

$$\sigma(t_c(\gamma)) = E_c^2G_2A^8Q_c^4 + G_0 + E_c^{-2}G_{-2}Q_c^{-4}.$$

To compute $T_p(\rho(t_c(\gamma)))$, let us introduce an algebra automorphism τ_c on $\mathcal{A}_A(\Gamma)$ by the formulas $\tau_c(Q_f) = Q_f$ for all $f \in \mathcal{E}$, $\tau_c(E_f) = E_f$ for $f \neq c$ and $\tau_c(E_c) = (-A)^3E_cQ_c^2$. This indeed defines an automorphism of $\mathcal{A}_A(\Gamma)$. Note that $\tau_c(E_c^k) = (-A)^{(k+1)^2-1} E_c^k$ (for $k \in \mathbb{Z}$) and $\sigma(t_c(\gamma)) = \tau_c(\sigma(\gamma))$. We deduce that

$$T_p(\rho(t_c(\gamma))) = \tilde{\rho}(\tau_c(\sigma(\gamma))) = \tilde{\rho}((E_c^2 G_2 A^8 Q_c^4)^p) + H + \tilde{\rho}((E_c^{-2} G_{-2} Q_c^{-4})^p).$$

Similarly

$$T_p(\rho(t_c^{-1}(\gamma))) = \tilde{\rho}(\tau_c^{-1}(\sigma(\gamma))) = \tilde{\rho}((E_c^2 G_2 A^{-8} Q_c^{-4})^p) + H + \tilde{\rho}((E_c^{-2} G_{-2} Q_c^{4})^p).$$

Now we have

$$(E_c^2 G_2 A^{\pm 8} Q_c^{\pm 4})^p = (E_c^2 G_2)^p x_c^{\pm 4p} \quad \text{and} \quad (E_c^{-2} G_{-2} Q_c^{\pm 4})^p = (E_c^{-2} G_{-2})^p x_c^{\pm 4p}.$$

We deduce the system

$$\begin{cases} T_p(\rho(t_c^{-1}(\gamma))) = \tilde{\rho}((E_c^2 G_2)^p) x_c^{-4p} + H + \tilde{\rho}((E_c^{-2} G_{-2})^p x_c^{4p}), \\ T_p(\rho(\gamma)) = \tilde{\rho}((E_c^2 G_2)^p) + H + \tilde{\rho}((E_c^{-2} G_{-2})^p), \\ T_p(\rho(t_c(\gamma))) = \tilde{\rho}((E_c^2 G_2)^p) x_c^{4p} + H + \tilde{\rho}((E_c^{-2} G_{-2})^p) x_c^{-4p}. \end{cases}$$

Solving this system gives

$$\tilde{\rho}((E_c^2 G_2)^p) = \frac{T_p(t_c^{-1}(\gamma)) x_c^{-2p} - T_p(\gamma)(x_c^{2p} + x_c^{-2p}) + T_p(t_c(\gamma)) x_c^{2p}}{(x_c^{2p} - x^{-2p})^2 (x_c^{2p} + x_c^{-2p})} \operatorname{Id}_V .$$

On the other hand,

$$\tilde{\rho}((E_c^2 G_2)^p) = -\tilde{\rho}(E_c^{2p}) \prod_{k=0}^{p-1} U((-A)^k x_{d_1} x_{d_4} x_c^{-1}) U((-A)^k x_{d_2} x_{d_3} x_c^{-1}). \qquad \square$$

Proof of Corollary 1.6 Let ρ_1 and ρ_2 be two irreducible representations of $S_A(\Sigma)$ with the same classical shadow r satisfying the hypothesis of Theorem 1.5. Let $\tilde{\rho}_1$ and $\tilde{\rho}_2$ be the lifts of ρ_1 and ρ_2 to $\mathcal{A}_A(\Gamma)^0$ built from the previous subsections. We build the lifts $\tilde{\rho}_1$ and $\tilde{\rho}_2$ from the same quantities $\{x_\alpha \mid \alpha \in \mathcal{P}\}$. Proposition 4.8 shows that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ have the same scalar values on the p^{th} powers of the noncentral generators given in Lemma 3.2. Finally Proposition 3.5 allows us to conclude. $\square$

Appendix Skein computations

In this section, we will give more details on some of the skein computations skipped in Sections 4.5 and 4.6, showing that the skein elements C_1 in the proof of Lemma 4.6 and $C_{0,0,0}$ in the proof of Lemma 4.7 both vanish.

Recall that for $x, y \in S(\Sigma)$, we write $[x, y]_A$ for $Axy - A^{-1}yx$.

Lemma A.1 *Let a and b be two simple closed curves on Σ, viewed as elements of $S(\Sigma)$, that intersect once geometrically. Let t_a and t_b be the associated Dehn twists. Then*

$$ab = At_a(b) + A^{-1} t_a^{-1}(b), \quad [a, b]_A = (A^2 - A^{-2}) t_a(b) \quad \text{and} \quad t_a(b) = t_b^{-1}(a).$$

Proof The first two are a direct consequence of Kauffman relations, and the third is a simple isotopy. □

Lemma A.2 *Let γ, τ, $\bar\tau$, β, c and e be the curves described in Section 4.5, let $\varphi = \gamma - t_c(\gamma)$ and $\psi = \bar\tau - \tau = \bar\tau - t_c(\tau)$, and let*

$$C_1 = -A^5[t_e(\beta), \psi]_A - A^3[\varphi, t_e(\beta)]_A + A^4[\varphi, \beta]_A e.$$

Then $C_1 = 0$.

Proof One can check that $C_1 = x_1 - t_c(x_1)$, where $x_1 = -A^5[t_e(\beta), \bar\tau]_A - A^3[\gamma, t_e(\beta)]_A + A^4[\gamma, \beta]_A$. We compute x_1 using Lemma A.1:

$$\begin{aligned} x_1 &= (A^2 - A^{-2})(-A^5 t_e t_\beta t_c^{-1}(\tau) - A^3 t^{-1}_{t_e(\beta)}(\gamma) + A^4 t_\beta^{-1}(\gamma) e) \\ &= (A^2 - A^{-2})(-A^5 t_e t_\beta t_c^{-1}(\tau) - A^3 t^{-1}_{t_e(\beta)}(\gamma) + A^5 t_e^{-1} t_\beta^{-1}(\gamma) + A^3 t_e t_\beta^{-1}(\gamma)). \end{aligned}$$

To conclude, let us remark that $t^{-1}_{t_e(\beta)}(\gamma) = t_e t_\beta^{-1} t_e^{-1}(\gamma) = t_e t_\beta^{-1}(\gamma)$, and that a simple isotopy shows that $t_e^{-1} t_\beta^{-1}(\gamma) \sim t_e t_\beta t_c^{-1}(\tau)$. Hence $x_1 = 0$ and therefore $C_1 = 0$. □

Lemma A.3 *Let γ, $\bar\tau$, β, c, d_1 and d_4 be the curves described in Section 4.6, let $\varphi = \gamma - t_c(\gamma)$ and $\psi = \bar\tau - t_c(\bar\tau)$ and let*

$$C_{0,0,0} = -[\psi, t_{d_4}(\beta)]_A + A^{-1}[\psi, t_{d_4} t_{d_1}(\beta)]_A + A^{-1}[\beta, \varphi]_A - A[\beta, \psi]_A c + A[\psi, \beta]_A d_1 d_4 - [\psi, t_{d_1}(\beta)]_A d_4.$$

Then $C_{0,0,0} = 0$.

Proof Again, $C_{0,0,0} = x_{0,0,0} - t_c(x_{0,0,0})$ where

$$x_{0,0,0} = -[\bar\tau, t_{d_4}(\beta)]_A + A^{-1}[\bar\tau, t_{d_4} t_{d_1}(\beta)]_A + A^{-1}[\beta, \gamma]_A - A[\beta, \bar\tau]_A c + A[\bar\tau, \beta]_A d_1 d_4 - [\bar\tau, t_{d_1}(\beta)]_A d_4.$$

Using Lemma A.1, we get

$$\begin{aligned} &\frac{1}{A^2 - A^{-2}} x_{0,0,0} \\ &\quad = -t_{d_4} t_\beta^{-1}(\bar\tau) d_1 + A^{-1} t_{d_1} t_{d_4} t_\beta^{-1}(\bar\tau) + A^{-1} t_\beta(\gamma) + A t_\beta(\bar\tau) c + A t_\beta^{-1}(\bar\tau) d_1 d_4 - t_{d_1} t_\beta^{-1}(\bar\tau) d_4. \end{aligned}$$

Notice that $t_\beta(\bar\tau)c = t_\beta(\bar\tau c)$. The curves $\bar\tau$ and c have geometric intersection 2, and an easy skein computation shows that $\bar\tau c = A^2 t_c^{-1}(\gamma) + \delta_2 + A^{-2}\gamma$, where $\delta_2 = d_1 d_2 + d_3 d_4$. Using this and the second equation in Lemma A.1, we get

$$\begin{aligned} \frac{1}{A^2 - A^{-2}} x_{0,0,0} &= -A^{-1} t_{d_1} t_{d_4} t_\beta^{-1}(\bar\tau) - A t_{d_1}^{-1} t_{d_4} t_\beta^{-1}(\bar\tau) + A^{-1} t_{d_1} t_{d_4} t_\beta^{-1}(\bar\tau) + A^{-1} t_\beta(\gamma) \\ &\quad - A^3 t_\beta t_c^{-1}(\gamma) - A t_\beta(\delta_2) - A^{-1} t_\beta(\gamma) + A^3 t_{d_1}^{-1} t_{d_4}^{-1} t_\beta^{-1}(\bar\tau) + A t_{d_1} t_{d_4}^{-1} t_\beta^{-1}(\bar\tau) \\ &\quad + A t_{d_1}^{-1} t_{d_4} t_\beta^{-1}(\bar\tau) + A^{-1} t_{d_1} t_{d_4} t_\beta^{-1}(\bar\tau) - A t_{d_1} t_{d_4}^{-1} t_\beta^{-1}(\bar\tau) - A^{-1} t_{d_1} t_{d_4} t_\beta^{-1}(\bar\tau) \\ &= -A t_\beta(\delta_2) - A^3 t_\beta t_c^{-1}(\gamma) + A^3 t_{d_1}^{-1} t_{d_4}^{-1} t_\beta^{-1}(\bar\tau). \end{aligned}$$

A drawing shows that the simple closed curves $t_{d_1}^{-1} t_{d_4}^{-1} t_\beta^{-1}(\bar\tau)$ and $t_\beta t_c^{-1}(\gamma)$ are actually isotopic. Hence $x_{0,0,0} = -A(A^2 - A^{-2}) t_\beta(\delta_2)$. But $t_\beta(\delta_2)$ is a linear combination of multicurves that are disjoint from c, and hence $x_{0,0,0}$ is t_c invariant and $C_{0,0,0} = 0$. □

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Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université de Bourgogne
Dijon, France

Laboratoire Mathématiques d'Orsay, UMR 8628 CNRS, Université Paris-Saclay
Orsay, France

renaud.detcherry@u-bourgogne.fr, ramanujan.santharoubane@universite-paris-saclay.fr

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