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We systematically introduce and study a new type of singularity, namely, exceptionally noncanonical (enc) singularities. This class of singularities plays an important role in the study of many questions in birational geometry, and has tight connections with local K-stability theory, Calabi–Yau varieties, and mirror symmetry.

We reduce the termination of flips to the termination of terminal flips and the ACC conjecture for minimal log discrepancies (mlds) of enc pairs. As a consequence, the ACC conjecture for mlds of enc pairs implies the termination of flips in dimension 4.

We show that, in any fixed dimension, the termination of flips follows from the lower-semicontinuity for mlds of terminal pairs, and the ACC for mlds of terminal and enc pairs. Moreover, in dimension 3, we give a rough classification of enc singularities, and prove the ACC for mlds of enc pairs. These two results provide a second proof of the termination of flips in dimension 3 which does not rely on any difficulty function.

Finally, we propose and prove the special cases of several conjectures on enc singularities and local K-stability theory. We also discuss the relationship between enc singularities, exceptional Fano varieties, and Calabi–Yau varieties with small mlds or large indices via mirror symmetry.

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1 Introduction

We work over the field of complex numbers $\mathbb{C}$.

The minimal model program (MMP) aims to provide a birational classification of algebraic varieties. The termination of flips in the MMP is one of the major remaining open problems in birational geometry. The goal of this paper is to introduce and study a class of singularities, namely, *exceptionally noncanonical* (*enc*) singularities, and utilize it on the termination of flips and other topics in birational geometry. A pair (X, B) is called *enc* if (X, B) is not canonical, and all but one exceptional prime divisors over X have log discrepancies greater than or equal to 1 (see Definition 3.8).

Termination of flips and the ACC for mlds of enc pairs Shokurov established a relation between the termination of flips and minimal log discrepancies (mlds), a basic but important local invariant in birational geometry. To be specific, Shokurov [2004, Theorem] proved that his ACC conjecture for (local) mlds [1988, Problem 5] together with the lower-semicontinuity (LSC) conjecture for mlds [Ambro 1999, Conjecture 0.2] imply the termination of flips.

On the one hand, the ACC conjecture for mlds remains unknown even in dimension 3, while the termination of flips is proved for threefolds [Kawamata 1992b; Shokurov 1996]. On the other hand, the ACC conjecture for mlds aims to reveal the structure of mlds of *all* singularities, while the singularities appearing in any given sequence of flips should be very special (even of finitely many types as we conjecture that the sequence of flips terminates). This indicates that the ACC conjecture for mlds might be much more difficult than the termination of flips, and we may not need the full power of this conjecture to show the termination of flips.

In this paper, we try to resolve this issue. Our first main result reduces the termination of flips to the ACC for (global) mlds of a very special class of singularities, enc singularities, and the termination of terminal flips. The latter is known up to dimension 4.

Theorem 1.1 *Let d be a positive integer. Assume that*

(1) *the ACC for (global) mlds of enc pairs with finite coefficients of dimension d holds, ie*

$$\{\operatorname{mld}(X, B) \mid (X, B) \text{ is enc}, \dim X = d, \operatorname{coeff}(B) \subseteq \Gamma\}$$

satisfies the ACC, where $\Gamma \subset [0, 1]$ is a finite set (Conjecture 1.9(2′)), and

(2) *any sequence of $\mathbb{Q}$-factorial terminal flips in dimension d terminates.*

Then any sequence of lc flips in dimension $\le d$ terminates.

We note that our proof of Theorem 1.1 does not rely on [Shokurov 2004]. As a consequence of Theorem 1.1, the ACC for (global) mlds of enc pairs implies the termination of flips in dimension 4.

Theorem 1.2 *Assume the ACC for (global) mlds of enc pairs with finite coefficients of dimension* 4. *Then any sequence of lc flips in dimension* 4 *terminates.*

As another application of Theorem 1.1, we may refine Shokurov's approach towards the termination of flips. To be more specific, assuming the ACC for (global) mlds of enc pairs, in order to prove the termination of flips, we only need the ACC and LSC for mlds of terminal pairs instead of lc pairs. We note that the set of terminal singularities is the smallest class for the purpose to run the MMP for smooth varieties, while the set of lc singularities is rather complicated and hard to work with (see [Kollár and Mori 1998, page 57]). Moreover, terminal surface (resp. threefold) singularities are smooth (resp. hyperquotient singularities), and the LSC conjecture for mlds is proven for the smooth varieties and hyperquotient singularities in any dimension [Ein et al. 2003; Nakamura and Shibata 2021].

Theorem 1.3 (Theorem 4.8) *Let d be a positive integer. Assume*

(1) *the ACC for (local) mlds of terminal pairs with finite coefficients*,

(2) *the LSC for mlds of terminal pairs*, *and*

(3) *the ACC for (global) mlds of enc pairs with finite coefficients (Conjecture 1.9*$(2')$)

hold in dimension d. Then any sequence of lc flips in dimension $\leq d$ terminates.

The set of enc singularities is expected to be a really small class of singularities that should possess some nice properties. The local Cartier indices of enc singularities are expected to be bounded from above (see Conjecture 1.8), hence we predict that the set of their mlds is discrete away from zero and should be much smaller than the set of all mlds. We also remark that in Theorems 1.1, 1.2, and 1.3, we only need the ACC for (global) mlds (of enc pairs), which is considered to be much simpler than the ACC for (local) mlds. For instance, consider any normal variety X. The log discrepancy of the exceptional divisor, obtained through the blow-up of X at any smooth codimension 2 point, is 2. Therefore, the global mld of any normal variety is always ≤ 2. However, the boundedness conjecture for (local) mlds remains open in dimension ≥ 4.

ACC for mlds of enc threefolds Recall that the ACC conjecture for mlds is only known in full generality for surfaces [Alexeev 1993] (see [Shokurov 1994; Han and Luo 2023; 2024] for other proofs), toric pairs [Ambro 2006], and exceptional singularities [Han et al. 2024]. It is still open for threefolds in general and only some partial results are known (see [Kawamata 1992a; Markushevich 1996; Kawakita 2015a; Nakamura 2016; Nakamura and Shibata 2022; Jiang 2021; Liu and Xiao 2021; Han et al. 2022; Liu and Luo 2022]). The second main result of this paper is the ACC for mlds of enc pairs in dimension 3. This result suggests that the ACC conjecture for mlds of enc pairs should be much easier than Shokurov's ACC conjecture for mlds.

Theorem 1.4 (cf Theorem E) *Let $\Gamma \subset [0,1]$ be a DCC set. Then*

$$\big\{\mathrm{mld}(X,B) \mid (X,B) \textit{ is enc}, \dim X = 3, \mathrm{coeff}(B) \subseteq \Gamma\big\}$$

satisfies the ACC.

By Theorems 1.3 and 1.4, we may reprove the termination of flips in dimension 3.

Corollary 1.5 (Corollary 7.4) *Any sequence of lc flips terminates in dimension* 3.

We remark that our proof of Corollary 1.5 only depends on the ACC and the LSC for mlds, and does not rely on any other auxiliary methods. In particular, our proof does not rely on difficulty functions, which played a key role in the previous proofs on the termination of flips in dimension 3 (see [Shokurov 1985; Kawamata et al. 1987; Kawamata 1992b; Kollár 1992; Shokurov 1996]) but are difficult to be applied in higher dimensions, especially in dimension >4 (we refer the reader to [Alexeev et al. 2007; Shokurov 2004; Fujino 2004] for some progress in dimension 4). The proof of Corollary 1.5, which is based on Theorem 1.3, may shed light on another approach towards the termination of flips in high dimensions. Note that Corollary 1.5 does not follow from Shokurov's approach [2004] directly, as the ACC conjecture for mlds is still open in dimension 3.

Now we turn our attention away from the termination of flips and focus on the ACC conjecture for mlds itself. We may show the following more technical but much stronger result on the ACC conjecture for mlds.

Theorem 1.6 (cf Theorem N) *Let N be a nonnegative integer, and $\Gamma\subset[0,1]$ a DCC set. Then there exists an ACC set Γ' depending only on N and Γ satisfying the following. Assume that (X,B) is a klt pair of dimension* 3 *such that*

(1) $\operatorname{coeff}(B)\subseteq\Gamma$, *and*

(2) *there are at most N different (exceptional) log discrepancies of (X,B) that are ≤ 1, ie*

$$\#(\{a(E,X,B)\mid E \text{ is exceptional over } X\}\cap[0,1])\leq N,$$

then $\operatorname{mld}(X,B)\in\Gamma'$.

Theorem 1.6 is considered to be much stronger than Theorem 1.4 as a result on the ACC conjecture for mlds, and the class of singularities in Theorem 1.6 is much larger than the class of enc singularities. It is clear that the local Cartier indices of these singularities in Theorem 1.6 are unbounded when their mlds have a positive lower bound, while they are expected to be bounded for enc pairs (see Conjecture 1.8). Moreover, when $N=0$, Theorem 1.6 implies the ACC for mlds of terminal threefolds [Han et al. 2022, Theorem 1.1] which is beyond Theorem 1.4, and when $N=1$, Theorem 1.6 implies Theorem 1.4.

Nevertheless, in order to prove Theorem 1.4, we have to prove Theorem 1.6, and the proofs of these two theorems are intertwined with each other (see Sections 2 and 6 for details).

Further remarks and conjectures We remark that, in the proof of the ACC for mlds in dimension 2 [Alexeev 1993], there are two cases:

Case 1 The dual graph of the minimal resolution is bounded.

Case 2 The dual graph of the minimal resolution is unbounded.

These two cases are treated in different ways in [Alexeev 1993]. Note that the minimal resolution of a surface is nothing but its terminalization. Therefore, if we regard "terminalization" as a kind of "minimal resolution" in high dimensions, then Theorem 1.6 implies that the ACC for mlds holds in dimension 3 whenever the dual graph of the terminalization is bounded. In other words, Case 1 in dimension 3 is proved. Moreover, one may show a stronger result for Case 1: any log discrepancy which is not larger than 1 belongs to a finite set (see [Han and Luo 2023, Lemma A.2; 2024]). In the same fashion, we can also show this holds in dimension 3.

Corollary 1.7 *Let N be a nonnegative integer, and $\Gamma \subset [0,1]$ a DCC set. Then there exists an ACC set Γ' depending only on N and Γ satisfying the following. Assume that (X,B) is a klt pair of dimension 3, such that*

1. $\mathrm{coeff}(B) \subseteq \Gamma$, *and*
2. *there are at most N different (exceptional) log discrepancies of (X,B) that are ≤ 1, ie*
$$\#(\{a(E,X,B) \mid E \text{ is exceptional over } X\} \cap [0,1]) \leq N.$$

Then $\{a(E,X,B) \mid E \text{ is exceptional over } X\} \cap [0,1] \subset \Gamma'$.

We propose two conjectures for enc pairs. Conjecture 1.8 is related to the local K-stability theory (see [Han et al. 2023, Theorem 1.5, Conjecture 1.6]). Roughly speaking, we expect that enc singularities have bounded local volumes (see [Li et al. 2020]):

Conjecture 1.8 (local boundedness for enc pairs) *Let d be a positive integer, ϵ a positive real number, and $\Gamma \subset [0,1]$ a DCC set. Then there exists a positive real number δ depending only on d and Γ satisfying the following. Assume that $(X \ni x, B)$ is an enc germ of dimension d such that $\mathrm{coeff}(B) \subseteq \Gamma$. Then*

1. *$(X \ni x, B)$ admits a δ-plt blow-up,*
2. *if $\mathrm{mld}(X \ni x, B) \geq \epsilon$, then the local volume $\widehat{\mathrm{vol}}(X \ni x, B)$ is bounded away from 0.*

Conjecture 1.9 (ACC for mlds of enc pairs) *Let d be a positive integer, and $\Gamma \subset [0,1]$ a set of real numbers. Let*
$$\mathrm{eMLD}_d(\Gamma) := \{\mathrm{mld}(X,B) \mid (X,B) \text{ is enc}, \dim X = d, \mathrm{coeff}(B) \subseteq \Gamma\}.$$

(1) *If Γ satisfies the DCC, then $\mathrm{eMLD}_d(\Gamma)$ satisfies the ACC.*

(2) *If Γ is a finite set, then $\mathrm{eMLD}_d(\Gamma)$ is a discrete set away from 0.*

(2′) *If Γ is a finite set, then $\mathrm{eMLD}_d(\Gamma)$ satisfies the ACC.*

We refer the reader to [Zhuang 2024] on some related works on local K-stability theory and mlds.

Conjecture 1.9(2′) is a weak form of Conjecture 1.9(2). We list it separately as we only need to assume it instead of Conjecture 1.9(2) in many results of this paper.

By [Han et al. 2024, Theorem 1.3], Conjecture 1.8 implies Conjecture 1.9. In particular, in dimension 4, either Conjecture 1.8 or 1.9 implies the termination of flips by Theorem 1.2.

There is some evidence towards Conjectures 1.8 and 1.9. We may prove both conjectures for surfaces. In dimension 3, Conjecture 1.9(1) is nothing but Theorem 1.4, and we also have the following evidence.

(1) Let $\epsilon_0 := 1 - \sup\{t \mid t \in \mathrm{CT}(3, \Gamma, \mathbb{Z}_{\geq 1})\}$, where $\mathrm{CT}(3, \Gamma, \mathbb{Z}_{\geq 1})$ is a set of threefold canonical thresholds (see Theorem 3.7). Then $(X \ni x, B)$ admits a canonical blow-up which extracts the unique exceptional prime divisor computing $\mathrm{mld}(X \ni x, B)$ if $\mathrm{mld}(X \ni x, B) < \epsilon_0$. This proves a special case of Conjecture 1.8.

(2) To prove Conjecture 1.9(2), we are only left to prove the following two cases:

(a) The case when the index 1 cover of X is strictly canonical.

(b) The case when X is terminal.

All other cases follow from our proofs in this paper.

(3) When $\Gamma = \{0\}$, since the mlds under case (2)(a) belong to the set $\left\{\frac{1}{n} \mid n \in \mathbb{Z}_{\geq 1}\right\}$ while (2)(b) can never happen, we could get Conjecture 1.9(2). In addition, if we assume the index conjecture of Shokurov (see [Chen and Han 2020, Conjecture 6.3]), then we can get Conjecture 1.8(2) as well.

Finally, we remark that enc singularities are deeply related to exceptional Fano varieties and Calabi–Yau varieties with small mlds or large indices via mirror symmetry. They are also tightly connected to the boundedness of log Calabi–Yau varieties. See Section 8 for details.

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2 Sketch of the proofs of theorems 1.4 and 1.6

Since the proofs of Theorems 1.4 and 1.6 are quite complicated, for the reader's convenience, we sketch a proof of them in this section. To prove Theorems 1.4 and 1.6, we need to apply induction on the lower bound of mlds. More precisely, we need to prove the following theorems for positive integers l:

Theorem $\mathbf{E}_l$ (cf Theorem 1.4) *Let l be a positive integer, $\Gamma \subset [0,1]$ a DCC set, and*

$$\mathcal{E}(l,\Gamma) := \Big\{(X,B) \;\Big|\; (X,B) \text{ is } \mathbb{Q}\text{-factorial enc}, \dim X = 3, \operatorname{coeff}(B) \subseteq \Gamma, \operatorname{mld}(X,B) > \frac{1}{l}\Big\}.$$

Then $\{\operatorname{mld}(X,B) \mid (X,B) \in \mathcal{E}(l,\Gamma)\}$ satisfies the ACC.

Theorem $\mathbf{N}_l$ (cf Theorem 1.6) *Let l and N be positive integers, $\Gamma \subset [0,1]$ a DCC set, and*

$$\mathcal{N}(l,N,\Gamma) := \Big\{(X,B) \;\Big|\; (X,B) \text{ is a threefold klt pair}, \operatorname{coeff}(B) \subseteq \Gamma, \frac{1}{l} < \operatorname{mld}(X,B) < 1, \\ \#(\{a(E,X,B) \mid E \text{ is exceptional over } X\} \cap [0,1]) \le N\Big\}.$$

Then $\{\operatorname{mld}(X,B) \mid (X,B) \in \mathcal{N}(l,N,\Gamma)\}$ satisfies the ACC.

To make our proof more clear, we introduce the following auxiliary theorem:

Theorem $\mathbf{C}_l$ *Let l be a positive integer, $\Gamma \subset [0,1]$ a DCC set, and*

$$\mathcal{C}(l,\Gamma) := \Big\{(X,B) \;\Big|\; (X \ni x, B) \text{ is a } \mathbb{Q}\text{-factorial threefold enc germ}, \operatorname{coeff}(B) \subseteq \Gamma, \\ \operatorname{mld}(X) < 1, \operatorname{mld}(X,B) > \frac{1}{l}, \ \tilde{X} \ni \tilde{x} \text{ is strictly canonical}, \\ \text{where } \pi : (\tilde{X} \ni \tilde{x}) \to (X \ni x) \text{ is the index 1 cover of } X \ni x\Big\}.$$

Then $\{\operatorname{mld}(X,B) \mid (X,B) \in \mathcal{C}(l,\Gamma)\}$ satisfies the ACC. (Here "strictly canonical" means canonical but not terminal.)

We will prove Theorems E, N, and C,[1] that is, Theorems E_l, N_l, and C_l for any positive integer l, in the following way:

(1) Theorem E_l implies Theorem N_l; see Lemma 7.1.

(2) Theorem C_l implies Theorem E_l; see Lemma 7.2, and Theorems 5.2, 5.6, 5.7, 6.3, and 6.8.

(3) Theorem N_{l-1} imply Theorem C_l; see Lemma 7.3.

The proof of (1) relies on the following observation: when proving by using contradiction, we may construct a DCC set Γ' depending only on N and Γ, such that for any $(X,B) \in \mathcal{N}(l,N,\Gamma)$, there always exists $(X',B') \in \mathcal{E}(l,\Gamma')$ such that $\operatorname{mld}(X',B')$ belongs to an ACC set if and only if $\operatorname{mld}(X,B)$ belongs to an ACC set. For such construction, one needs to look into different birational models of (X,B) which only extracts noncanonical places of (X,B). Indeed, the proof also works in higher dimensions and its idea is applied to prove Theorems 1.1, 1.2, and 1.3 as well. We refer to Lemma 4.3 for more details.

To prove (2), for any $(X,B) \in \mathcal{E}(l,\Gamma)$, let E be the unique exceptional prime divisor over X such that $a(E,X,B) = \operatorname{mld}(X,B)$, and x the generic point of $\operatorname{center}_X(E)$. We may assume that x is a closed

[1] Regarding the labels of the theorems: E stands for the initial of "enc", N stands for the additional restriction "$\le N$", and C stands for the initial of "canonical" as in "strictly canonical". Coincidentally, these labels together also form the word enc.

point (see Lemma 5.1). There are three cases:

(2.1) $X \ni x$ is strictly canonical,

(2.2) $X \ni x$ is not canonical, and

(2.3) $X \ni x$ is terminal.

We prove (2.1) by applying the cone theorem and the boundedness of complements (Theorem 5.2).

To prove (2.2), let $(\widetilde{X} \ni \tilde{x}) \to (X \ni x)$ be the index 1 cover of $X \ni x$. Then there are three subcases:

(2.2.1) $\widetilde{X}$ is strictly canonical,

(2.2.2) $\widetilde{X}$ is smooth, and

(2.2.3) $\widetilde{X}$ is terminal and has an isolated cDV singularity.

The case (2.2.1) is nothing but Theorem C_l.

In the case of (2.2.2), X has toric singularities. Since X is enc, one can show that the degree of the cover $\widetilde{X} \to X$ is bounded from above, and the ACC for $\mathrm{mld}(X, B)$ immediately follows (see Theorem 6.3).

We are left to prove (2.2.3), which is the most tedious part of the whole paper. Here we need to apply the classification of cDV singularities [Mori 1985] to classify enc cDV (cyclic) quotient singularities $X \ni x$. Our ideas are inspired by the ones in the classification of threefold terminal singularities [Mori 1985; Reid 1987] and the (rough) classification of threefold "nearly terminal" singularities [Jiang 2021; Liu and Xiao 2021; Liu and Luo 2022]. The major difference between our classification and the previous ones is that the mlds of enc singularities cannot be assumed to be close to 1: in fact, they can be arbitrarily small. This makes most computations in [Reid 1987; Jiang 2021; Liu and Xiao 2021; Liu and Luo 2022] no longer work. A key observation here is that enc cDV quotient singularities with $\mathrm{mld}(X \ni x) \in \left(\frac{1}{l}, \frac{1}{l-1}\right]$ share similar properties and will be much easier to classify. Indeed, we will show that the local Cartier indices of these $X \ni x$ are (almost) bounded. On the other hand, enc cDV quotient singularities with $\mathrm{mld}(X \ni x) \in \left(\frac{1}{l-1}, 1\right]$ can be classified by induction on l. This will imply (2.2.3), and we conclude the proof of (2.2). See Section 6.2 for more details.

The proof of (2.3) is very tricky. When Γ is a finite set, we can apply the theory of functional pairs introduced in [Han et al. 2024; 2021] and carefully construct some weighted blow-ups to prove this case (Theorem 5.6). The key point is that the unique divisor E over $X \ni x$ which computes $\mathrm{mld}(X, B)$ must also compute the canonical threshold $\mathrm{ct}(X, 0; B)$, and the latter satisfies the ACC [Han et al. 2022, Theorem 1.7; Chen 2022, Theorem 1.1]. However, for the arbitrary DCC coefficient case, we are unable to prove it directly. Nevertheless, with some clever arguments, we can reduce it to the finite coefficient case (although possibly losing the condition that X is terminal). See Theorem 5.7 for more details. Since (2.3) is the only case left to prove in (2), the DCC coefficient case is also automatically resolved. This concludes the proof of (2.3), and hence of (2).

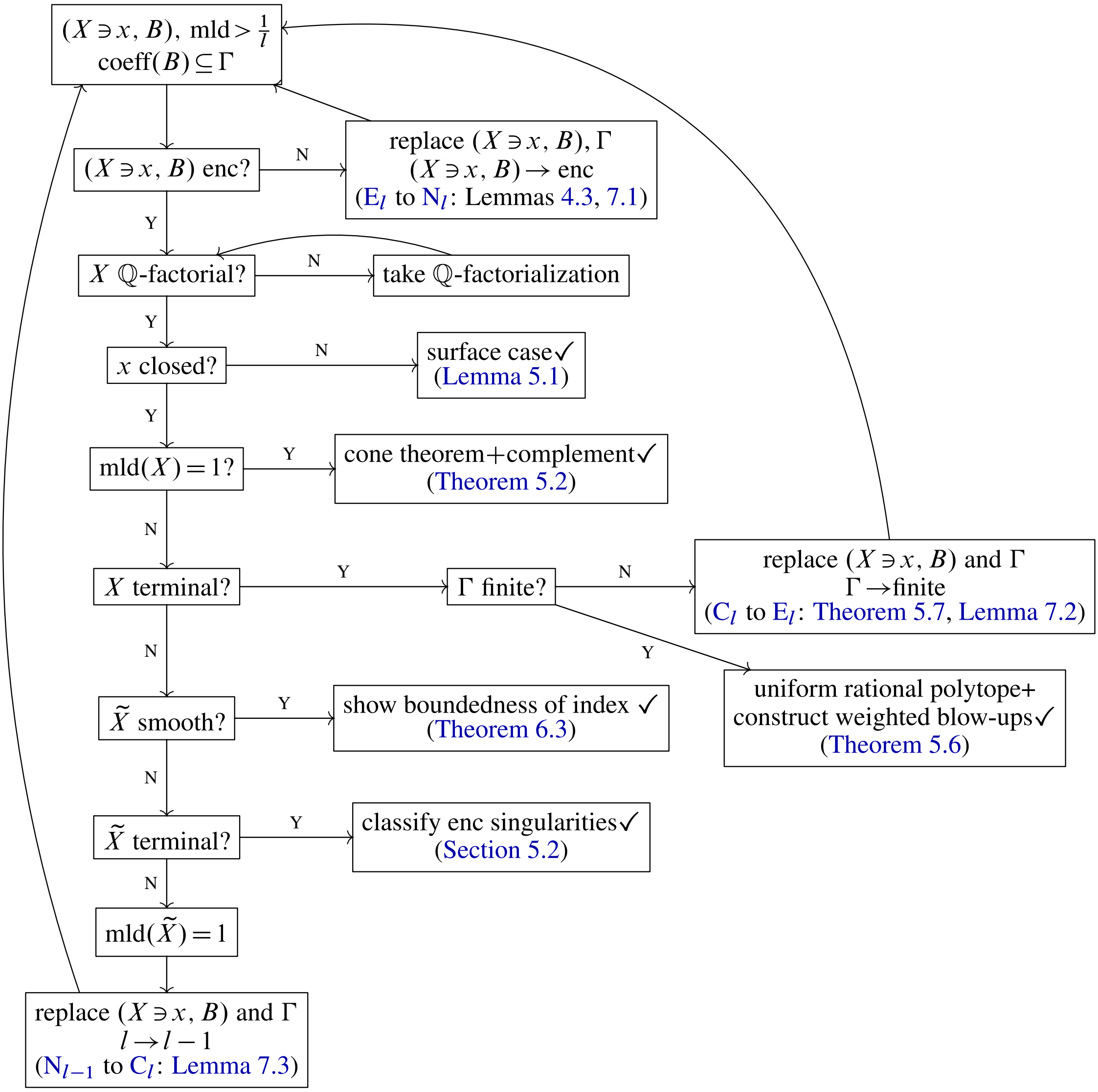

Figure 1: Flowchart of the proofs of Theorems 1.4 and 1.6.

To prove (3), we may assume that $l \geq 2$. Let $K_{\widetilde{X}} + \widetilde{B}$ be the pullback of $K_X + B$. Then we may show that

$$\mathrm{mld}(\widetilde{X}, \widetilde{B}) \in \{a(\widetilde{E}, \widetilde{X}, \widetilde{B}) \leq 1 \mid \widetilde{E} \text{ is exceptional over } \widetilde{X}\} \subseteq \{2\,\mathrm{mld}(X, B), \dots, (l-1)\,\mathrm{mld}(X, B)\}$$

and the latter is a finite set with cardinality $l-2$. Thus we only need to show that $\mathrm{mld}(\widetilde{X}, \widetilde{B})$ belongs to an ACC set, which follows from Theorem N_{l-1} as $\mathrm{mld}(\widetilde{X}, \widetilde{B}) \geq 2\,\mathrm{mld}(X, B) > \frac{2}{l} \geq \frac{1}{l-1}$. This finishes the proof of (3).

To summarize, we may show Theorems E, N, and C by induction on l. Now Theorems 1.4 and 1.6 follow from Theorem N and [Han et al. 2022, Theorem 1.1].

The flowchart of Figure 1 may also help the reader.

3 Preliminaries

We will freely use the notation and definitions from [Kollár and Mori 1998; Birkar et al. 2010].

3.1 Pairs and singularities

Definition 3.1 A *contraction* is a projective morphism $f\colon Y \to X$ such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. In particular, f is surjective and has connected fibers.

Definition 3.2 Let $f\colon Y \to X$ be a birational morphism, and $\operatorname{Exc}(f)$ the exceptional locus of f. We say that f is a *divisorial contraction* of a prime divisor E if $\operatorname{Exc}(f) = E$ and $-E$ is f-ample.

Definition 3.3 (pairs, see [Chen and Han 2020, Definition 3.2]) A *pair* $(X/Z \ni z, B)$ consists of a contraction $\pi\colon X \to Z$, a (not necessarily closed) point $z \in Z$, and an $\mathbb{R}$-divisor $B \geq 0$ on X, such that $K_X + B$ is $\mathbb{R}$-Cartier over a neighborhood of z and $\dim z < \dim X$. If π is the identity map and $z = x$, then we may use $(X \ni x, B)$ instead of $(X/Z \ni z, B)$. In addition, if $B = 0$, then we use $X \ni x$ instead of $(X \ni x, 0)$. When we consider a pair $\left(X \ni x, \sum_i b_i B_i\right)$, where B_i are distinct prime divisors and $b_i > 0$, we always assume that $x \in \operatorname{Supp} B_i$ for each i.

If $(X \ni x, B)$ is a pair for any codimension ≥ 1 point $x \in X$, then we call (X, B) a pair. A pair $(X \ni x, B)$ is called a *germ* if x is a closed point. We also say $X \ni x$ is a *singularity* if $X \ni x$ is a germ.

Definition 3.4 (singularities of pairs) Let $(X/Z \ni z, B)$ be a pair associated with the contraction $\pi\colon X \to Z$, and let E be a prime divisor over X such that $z \in \pi(\operatorname{center}_X E)$. Let $f\colon Y \to X$ be a log resolution of (X, B) such that $\operatorname{center}_Y E$ is a divisor, and suppose that $K_Y + B_Y = f^*(K_X + B)$ over a neighborhood of z. We define $a(E, X, B) := 1 - \operatorname{mult}_E B_Y$ to be the *log discrepancy* of E with respect to (X, B).

For any prime divisor E over X, we say that E is *over* $X/Z \ni z$ if $\pi(\operatorname{center}_X E) = \bar{z}$. If π is the identity map and $z = x$, then we say that E is *over* $X \ni x$. We define

$$\operatorname{mld}(X/Z \ni z, B) := \inf\{a(E, X, B) \mid E \text{ is over } Z \ni z\}$$

to be the *minimal log discrepancy* (*mld*) of $(X/Z \ni z, B)$.

Let ϵ be a nonnegative real number. We say that

$$(X/Z \ni z, B) \text{ is lc (resp. klt, } \epsilon\text{-lc, } \epsilon\text{-klt)} \quad \text{if} \quad \operatorname{mld}(X/Z \ni z, B) \geq 0 \text{ (resp. } > 0,\ \geq \epsilon,\ > \epsilon).$$

We say that (X, B) is lc (resp. klt, ϵ-lc, ϵ-klt) if $(X \ni x, B)$ is lc (resp. klt, ϵ-lc, ϵ-klt) for any codimension ≥ 1 point $x \in X$.

We say that (X, B) is *canonical* (resp. *terminal*, *plt*) if $(X \ni x, B)$ is 1-lc (resp. 1-klt, klt) for any codimension ≥ 2 point $x \in X$.

For any (not necessarily closed) point $x \in X$, we say that (X, B) is lc (resp. klt, ϵ-lc, canonical, terminal) near x if (X, B) is lc (resp. klt, ϵ-lc, canonical, terminal) in a neighborhood of x. If X is lc (resp. klt,

ϵ-lc, canonical, terminal) near a closed point x, then we say that $X \ni x$ is an lc (resp. klt, ϵ-lc, canonical, terminal) singularity. We remark that if $(X \ni x, B)$ is lc, then (X, B) is lc near x.

We say that $(X \ni x, B)$ (resp. (X, B)) is *strictly canonical* if $\mathrm{mld}(X \ni x, B) = 1$ (resp. $\mathrm{mld}(X, B) = 1$).

Definition 3.5 Let a be a nonnegative real number, $(X \ni x, B)$ (resp. (X, B)) a pair, and $D \geq 0$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We define

$$a\text{-lct}(X \ni x, B; D) := \sup\{-\infty, t \mid t \geq 0, (X \ni x, B + tD) \text{ is } a\text{-lc}\}$$
$$\big(\text{resp. } a\text{-lct}(X, B; D) := \sup\{-\infty, t \mid t \geq 0, (X, B + tD) \text{ is } a\text{-lc}\}\big)$$

to be the *a-lc threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)). We define

$$\mathrm{ct}(X \ni x, B; D) := 1\text{-lct}(X \ni x, B; D)$$
$$\big(\text{resp. } \mathrm{ct}(X, B; D) := \sup\{-\infty, t \mid t \geq 0, (X, B + tD) \text{ is canonical}\}\big)$$

to be the *canonical threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)). We define

$$\mathrm{lct}(X \ni x, B; D) := 0\text{-lct}(X \ni x, B; D)$$
$$(\text{resp. } \mathrm{lct}(X, B; D) := 0\text{-lct}(X, B; D))$$

to be the *lc threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)).

Theorem 3.6 [Han et al. 2022, Theorem 1.6] *Let $a \geq 1$ be a positive real number, and $\Gamma \subset [0, 1]$, $\Gamma' \subset [0, +\infty)$ two DCC sets. Then the set of a-lc thresholds*

$$\big\{a\text{-lct}(X \ni x, B; D) \mid \dim X = 3,\ X \textit{ is terminal}, \mathrm{coeff}(B) \subseteq \Gamma, \mathrm{coeff}(D) \subseteq \Gamma'\big\},$$

satisfies the ACC.

Theorem 3.7 [Han et al. 2022, Theorem 1.7; Chen 2022, Theorem 1.2] *Let $\Gamma \subset [0, 1]$ and $\Gamma' \subset [0, +\infty)$ be two DCC sets. Then the set*

$$\mathrm{CT}(3, \Gamma, \Gamma') := \big\{\mathrm{ct}(X, B; D) \mid \dim X = 3,\ \mathrm{coeff}(B) \subseteq \Gamma, \mathrm{coeff}(D) \subseteq \Gamma'\big\}$$

satisfies the ACC.

Definition 3.8 (1) Let (X, B) be a pair. We say that (X, B) is *exceptionally noncanonical* (*enc* for short) if $\mathrm{mld}(X, B) < 1$, and the set

$$\{E \mid E \text{ is exceptional over } X, a(E, X, B) \leq 1\}$$

contains a unique element.

(2) Let $(X \ni x, B)$ be a germ. We say that $(X \ni x, B)$ is *exceptionally noncanonical* (*enc* for short) if (X, B) is enc in a neighborhood of x and $\mathrm{mld}(X \ni x, B) = \mathrm{mld}(X, B)$.

It is easy to see that any enc pair is automatically klt.

Lemma 3.9 *Let (X, B) be an enc pair. Then (X, B) is klt.*

Proof Since (X, B) is an enc pair, there exists an exceptional divisor over X. In particular, $\dim X \geq 2$. Let $f: Y \to X$ be a log resolution of $(X, \operatorname{Supp} B)$ and write $K_Y + B_Y := f^*(K_X + B)$. If (X, B) is not klt, then there exists a component D of $\operatorname{Supp} B_Y$ such that $\operatorname{mult}_D B_Y \geq 1$. Let H_1, H_2 be two general hyperplane sections on Y. For each $i \in \{1, 2\}$, let y_i be the generic point of $H_i \cap D$ and let E_i be the exceptional divisor obtained by the blow-up of Y at y_i. Then for each $i \in \{1, 2\}$, E_i is exceptional over X and

$$1 \geq 2 - \operatorname{mult}_D B_Y \geq a(E_i, Y, B_Y) = a(E_i, X, B),$$

which contradicts our assumption. □

Theorem 3.10 [Kollár 1992, Theorem 18.22] *Let $\left(X \ni x, \sum_{i=1}^m b_i B_i\right)$ be an lc germ such that $B_i \geq 0$ are $\mathbb{Q}$-Cartier near x, $b_i \geq 0$, and $x \in \operatorname{Supp} B_i$ for each i. Then $\sum_{i=1}^m b_i \leq \dim X$.*

Definition 3.11 Let S be a set. We define $\#S$ or $|S|$ to be the cardinality of S.

3.2 Complements

Definition 3.12 Let n be a positive integer, $\Gamma_0 \subset (0, 1]$ a finite set, and $(X/Z \ni z, B)$ and $(X/Z \ni z, B^+)$ two pairs. We say that $(X/Z \ni z, B^+)$ is an $\mathbb{R}$-*complement* of $(X/Z \ni z, B)$ if

- $(X/Z \ni z, B^+)$ is lc,
- $B^+ \geq B$, and
- $K_X + B^+ \sim_{\mathbb{R}} 0$ over a neighborhood of z.

We say that $(X/Z \ni z, B^+)$ is an n-*complement* of $(X/Z \ni z, B)$ if

- $(X/Z \ni z, B^+)$ is lc,
- $nB^+ \geq \lfloor (n+1)\{B\} \rfloor + n\lfloor B \rfloor$, and
- $n(K_X + B^+) \sim 0$ over a neighborhood of z.

We say that $(X/Z \ni z, B)$ is $\mathbb{R}$-complementary (resp. n-complementary) if $(X/Z \ni z, B)$ has an $\mathbb{R}$-complement (resp. n-complement).

We say that $(X/Z \ni z, B^+)$ is a *monotonic n-complement* of $(X/Z \ni z, B)$ if $(X/Z \ni z, B^+)$ is an n-complement of $(X/Z \ni z, B)$ and $B^+ \geq B$.

We say that $(X/Z \ni z, B^+)$ is an (n, Γ_0)-*decomposable* $\mathbb{R}$-*complement* of $(X/Z \ni z, B)$ if there exists a positive integer k, $a_1, \dots, a_k \in \Gamma_0$, and $\mathbb{Q}$-divisors $B_1^+, \dots, B_k^+$ on X, such that

- $\sum_{i=1}^k a_i = 1$ and $\sum_{i=1}^k a_i B_i^+ = B^+$,
- $(X/Z \ni z, B^+)$ is an $\mathbb{R}$-complement of $(X/Z \ni z, B)$, and
- $(X/Z \ni z, B_i^+)$ is an n-complement of itself for each i.

Theorem 3.13 [Han et al. 2024, Theorem 1.10] *Let d be a positive integer and $\Gamma \subset [0,1]$ a DCC set. Then there exists a positive integer n and a finite set $\Gamma_0 \subset (0,1]$ depending only on d and Γ and satisfying the following.*

Assume that $(X/Z \ni z, B)$ is a pair of dimension d and $\operatorname{coeff}(B) \subseteq \Gamma$, such that X is of Fano type over Z and $(X/Z \ni z, B)$ is $\mathbb{R}$-complementary. Then $(X/Z \ni z, B)$ has an (n, Γ_0)-decomposable $\mathbb{R}$-complement.

3.3 Threefold singularities

Lemma 3.14 [Kawamata 1988, Lemma 5.1] *Let $X \ni x$ be a terminal threefold singularity and I a positive integer such that IK_X is Cartier near x. Then ID is Cartier near x for any $\mathbb{Q}$-Cartier Weil divisor D on X.*

Theorem 3.15 [Liu and Xiao 2021, Theorem 1.4; Jiang 2021, Theorem 1.3] *Let X be a $\mathbb{Q}$-Gorenstein threefold. If $\operatorname{mld}(X) < 1$, then $\operatorname{mld}(X) \le \frac{12}{13}$.*

Definition 3.16 A *weight vector* is a vector $w \in \mathbb{Q}^d_{>0}$ for some positive integer d.

For any vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\ge 0}$, we define $\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, and $w(\boldsymbol{x}^{\boldsymbol{\alpha}}) := \sum_{i=1}^d w_i\alpha_i$ to be *the weight of $\boldsymbol{x}^{\boldsymbol{\alpha}}$ with respect to w*. For any analytic function $0 \ne h := \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^d_{\ge 0}} a_{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}$, we define $w(h) := \min\{w(\boldsymbol{x}^{\boldsymbol{\alpha}}) \mid a_{\boldsymbol{\alpha}} \ne 0\}$ to be *the weight of h with respect to w*. If $h = 0$, then we define $w(h) := +\infty$.

Definition 3.17 Let $\left(X \ni x, B := \sum_{i=1}^k b_iB_i\right)$ be a threefold germ such that X is terminal, $b_i \ge 0$, and $B_i \ge 0$ are $\mathbb{Q}$-Cartier Weil divisors. Let d, n, and $m < d$ be positive integers, such that

$$(X \ni x) \cong (\phi_1 = \cdots = \phi_m = 0) \subset (\mathbb{C}^d \ni o)/\frac{1}{n}(a_1, \dots, a_d)$$

for some nonnegative integers $a_1, \dots, a_d$ and some semi-invariant irreducible analytic function $\phi_1 \dots, \phi_m \in \mathbb{C}\{x_1, \dots, x_d\}$ such that $\operatorname{mult}_o \phi_i > 1$ for each i, and the group action on $\mathbb{C}^d$ corresponding to $\frac{1}{n}(a_1, \dots, a_d)$ is free outside o. By [Kawamata 1988, Lemma 5.1], B_i can be identified with

$$\left((h_i = 0) \subset (\mathbb{C}^d \ni o) \,/\, \frac{1}{n}(a_1, \dots, a_d)\right)\Big|_X$$

for some semi-invariant analytic function $h_i \in \mathbb{C}\{x_1, \dots, x_d\}$ near $x \in X$. We say that B_i is locally defined by $(h_i = 0)$ for simplicity. The set of *admissible weight vectors* of $X \ni x$ is defined by

$$\left\{\frac{1}{n}(w_1, \dots, w_d) \in \frac{1}{n}\mathbb{Z}^d_{>0} \;\middle|\; \text{there exists } b \in \mathbb{Z} \text{ such that } w_i \equiv ba_i \bmod n,\ 1 \le i \le d\right\}.$$

For any admissible weight vector $w = \frac{1}{n}(w_1, \dots, w_d)$, we define

$$w(X \ni x) := \frac{1}{n}\sum_{i=1}^d w_i - \sum_{i=1}^m w(\phi_i) - 1, \quad \text{and} \quad w(B) := \sum_{i=1}^k b_iw(h_i).$$

By construction, $w(B)$ is independent of the choices of b_i and B_i.

Let $f'\colon W\to(\mathbb{C}^d\ni o)/\frac{1}{n}(a_1,\dots,a_d)$ be the weighted blow-up at o with an admissible weight vector $w:=\frac{1}{n}(w_1,\dots,w_d)$, Y the strict transform of X on W, and E' the exceptional locus of f'. Let $f:=f'|_Y$ and $E:=E'|_Y$. We say that $f\colon Y\to X\ni x$ is a *weighted blow-up* at $x\in X$ and E is the exceptional divisor of this weighted blow-up.

We will use the following well-known lemma frequently.

Lemma 3.18 (see [Mori 1985, the proof of Theorem 2; Hayakawa 1999, Section 3.9]) *Settings as in Definition 3.17. For any admissible weight vector w of $X\ni x$, let E be the exceptional divisor of the corresponding weighted blow-up $f\colon Y\to X$ at x (see Definition 3.17). If E is a prime divisor, then*

$$K_Y=f^*K_X+w(X\ni x)E,\quad\text{and}\quad f^*B=B_Y+w(B)E,$$

where B_Y is the strict transform of B on Y. In particular, $a(E,X,B)=1+w(X\ni x)-w(B)$.

4 On termination of flips

4.1 Proofs of Theorems 1.1 and 1.2

We need the following auxiliary lemma for induction purposes.

Lemma 4.1 *Let $d\geq 2$ be a positive integer. We have the following.*

(1) *The ACC for (global) mlds of enc pairs with finite coefficients in dimension d implies the ACC for (global) mlds of enc pairs with finite coefficients in dimension $\leq d$.*

(2) *The termination of $\mathbb{Q}$-factorial terminal (resp. klt, lc) flips in dimension d implies the termination of $\mathbb{Q}$-factorial terminal (resp. klt, lc) flips in dimension $\leq d$.*

Proof (1) It suffices to show that for any enc pair (X,B), $(X',B'):=(X\times\mathbb{C},B\times\mathbb{C})$ is also an enc pair, and $\mathrm{mld}(X',B')=\mathrm{mld}(X,B)$. Let E be the unique exceptional prime divisor over X, such that $a(E,X,B)=\mathrm{mld}(X,B)<1$. By [Kollár and Mori 1998, Proposition 2.36], there exists a log resolution $f\colon Y\to X$ of (X,B), such that $\operatorname{Supp}B_Y^{\geq 0}$ is log smooth, where $K_Y+B_Y:=f^*(K_X+B)$. Then $\mathrm{mult}_E B_Y=1-a(E,X,B)>0$. Let $Y':=Y\times\mathbb{C}$, $B_{Y'}:=B_Y\times\mathbb{C}$, $E'=E\times\mathbb{C}$, and $f':=f\times\mathrm{id}_{\mathbb{C}}$. Then $f'\colon Y'\to X'$ is a log resolution of (X',B'),

$$K_{Y'}+B_{Y'}=f'^*(K_{X'}+B'),$$

$\operatorname{Supp}B_{Y'}^{\geq 0}$ is log smooth, and $\mathrm{mult}_{E'}B_{Y'}=\mathrm{mult}_E B_Y>0$.

Since (X,B) is enc, for any point y' on Y' such that $\overline{y'}\neq E'$ is exceptional over X', we have $\operatorname{codim}y'>\mathrm{mult}_{y'}B_{Y'}+1$. Thus by [Chen and Han 2020, Lemma 3.3],

$$\mathrm{mld}(Y'\ni y',B_{Y'})=\operatorname{codim}y'-\mathrm{mult}_{y'}B_{Y'}>1.$$

In other words, E' is the unique exceptional prime divisor over X' such that $a(E',X',B')\leq 1$. Hence (X',B') is an enc pair, and $\mathrm{mld}(X',B')=a(E',X',B')=a(E,X,B)=\mathrm{mld}(X,B)$.

(2) Let (X, B) be a $\mathbb{Q}$-factorial terminal (resp. klt, lc) pair, and

$$(X, B) := (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow \cdots (X_i, B_i) \dashrightarrow \cdots$$

a sequence of flips. Let C be an elliptic curve. Then

$$(X \times C, B \times C) := (X_0 \times C, B_0 \times C) \dashrightarrow (X_1 \times C, B_1 \times C) \dashrightarrow \cdots (X_i \times C, B_i \times C) \dashrightarrow \cdots$$

is also a sequence of flips of dimension $\dim X + 1$. Now (2) follows from our assumptions. □

We will use the following notation in the proofs of Lemma 4.3 and Theorem 1.1.

Definition 4.2 Let (X, B) be an lc pair. We define

$$D(X, B)_{\le 1} := \{E \mid E \text{ is exceptional over } X, a(E, X, B) \le 1\}.$$

By [Kollár and Mori 1998, Proposition 2.36], $D(X, B)$ is a finite set when (X, B) is klt.

The following lemma plays a key role in this section, and it will be applied to prove Theorems 1.1, 7.6, and Lemma 7.1.

Lemma 4.3 *Let d, N be two positive integers, and $\Gamma \subset [0, 1]$ a DCC set. Let $\{(X_i, B_i)\}_{i=1}^{\infty}$ be a sequence of klt pairs of dimension d, and*

$$\Gamma_i := \{a(E_i, X_i, B_i) \mid E_i \text{ is exceptional over } X_i\} \cap [0, 1].$$

Suppose that

- $\operatorname{coeff}(B_i) \subseteq \Gamma$ *for each i,*
- $1 \le \#\Gamma_i \le N$ *for each i, and*
- $\bigcup_{i=1}^{\infty} \Gamma_i$ *does not satisfy the ACC.*

Then possibly passing to a subsequence, there exists a DCC set $\Gamma' \subset [0, 1]$, and a sequence $\{(X_i', B_i')\}_{i=1}^{\infty}$ of $\mathbb{Q}$-factorial enc pairs of dimension d, such that

(1) $\operatorname{coeff}(B_i') \subseteq \Gamma'$ *for each i,*

(2) $\{\operatorname{mld}(X_i', B_i')\}_{i=1}^{\infty}$ *is strictly increasing, and*

(3) $\operatorname{mld}(X_i', B_i') \ge \operatorname{mld}(X_i, B_i)$ *for each i.*

Moreover, if we further assume that Γ is a finite set and $\bigcup_{i=1}^{\infty} \Gamma_i$ is a DCC set, then we may choose Γ' to be a finite set.

Proof Possibly passing to a subsequence we may assume that $\#\Gamma_i = k \geq 1$ for some positive integer $k \leq N$. For each i, there exist positive integers $r_{1,i}, \ldots, r_{k,i}$ and real numbers $\{a_{i,j}\}_{i\geq 1, 1\leq j\leq k}$, such that

$$D(X_i, B_i)_{\leq 1} = \left\{E_{i,1,1}, \ldots, E_{i,1,r_{1,i}}; E_{i,2,1}, \ldots, E_{i,2,r_{2,i}}; \ldots; E_{i,k,1}, \ldots, E_{i,k,r_{k,i}}\right\}$$

for some distinct exceptional prime divisors $E_{i,j,l}$ over X_i, $a_{i,j} = a(E_{i,j,l}, X_i, B_i) \leq 1$ for any i, j, l, and $\{a_{i,j}\}_{i\geq 1, 1\leq j\leq k}$ does not satisfy the ACC. Possibly reordering indices and passing to a subsequence, we may assume that there exists $1 \leq j_0 \leq k$, such that

- $a_{i,j}$ is strictly increasing for any $1 \leq j \leq j_0$, and
- $a_{i,j}$ is decreasing for any $j_0 + 1 \leq j \leq k$.

Let $f_i\colon Y_i \to X_i$ be a birational morphism which extracts exactly the set of divisors

$$\mathcal{F}_i := \{E_{i,j,l}\}_{j_0+1\leq j\leq k,\, 1\leq l\leq r_{j,i}}$$

and let $K_{Y_i} + B_{Y_i} := f_i^*(K_{X_i} + B_i)$. Since $a_{i,j}$ is decreasing for any $j_0 + 1 \leq j \leq k$, the coefficients of B_{Y_i} belong to the DCC set $\widetilde{\Gamma} := \Gamma \cup \{1 - a_{i,j}\}_{j_0+1\leq j\leq k}$. By construction, $D(Y_i, B_{Y_i})_{\leq 1} = D(X_i, B_i) \backslash \mathcal{F}_i$, and $D(Y_i, B_{Y_i})_{\leq 1}$ is a nonempty set as $E_{i,1,1} \in D(Y_i, B_{Y_i})_{\leq 1}$.

Let $a_j := \lim_{i\to+\infty} a_{i,j}$ for any $1 \leq j \leq j_0$. By [Liu 2018, Lemma 5.3], for each i, there exist $1 \leq j_i \leq j_0$ and $1 \leq l_i \leq r_{j_i,i}$, and a birational morphism $g_i : X_i' \to Y_i$ which extracts exactly all divisors in $D(Y_i, B_{Y_i})_{\leq 1}$ except E_{i,j_i,l_i}, such that X_i' is $\mathbb{Q}$-factorial and

$$\sum_{1\leq j\leq j_0,\, 1\leq l\leq r_{j,i},\, (j,l)\neq(j_i,l_i)} (a_j - a_{i,j}) \operatorname{mult}_{E_{i,j_i,l_i}} E_{i,j,l} < a_{j_i} - a_{i,j_i}.$$

Let $B_{X_i'} := (g_i^{-1})_* B_{Y_i} + \sum_{1\leq j\leq j_0,\, 1\leq l\leq r_{j,i},\, (j,l)\neq(j_i,l_i)} (1 - a_j) E_{i,j,l}$. Then the coefficients of $B_{X_i'}$ belong to the DCC set $\Gamma' := \widetilde{\Gamma} \cup \{1 - a_j\}_{1\leq j\leq j_0}$. Moreover, if Γ is a finite set and $\bigcup_{i=1}^{\infty} \Gamma_i$ is a DCC set, then Γ' is a finite set.

By construction, $(X_i', B_{X_i'})$ is enc and $\operatorname{mld}(X_i, B_i) \leq a_{i,j_i} \leq a(E_{i,j_i,l_i}, X_i', B_{X_i'}) < a_{j_i} \leq 1$. Thus

$$a_{i,j_i} \leq a(E_{i,j_i,l_i}, X_i', B_{X_i'}) = \operatorname{mld}(X_i', B_{X_i'}) < a_{j_i} \leq 1.$$

Possibly passing to a subsequence, we may assume that $j_i = j_1$ is a constant. Since a_{i,j_1} is strictly increasing for each i, $\{\operatorname{mld}(X_i', B_{X_i'}) \mid i \in \mathbb{Z}_{\geq 1}\}$ is not a finite set. Possibly passing to a subsequence, we may assume that $\{\operatorname{mld}(X_i', B_{X_i'})\}_{i=1}^{\infty}$ is strictly increasing. $\square$

Proof of Theorem 1.1 First, we prove the case of klt flips in dimension d. Let

$$\begin{tikzcd}[column sep=small]
(X, B) := & (X_0, B_0) \arrow[rr, dashed] \arrow[dr] & & (X_1, B_1) \arrow[dl] \arrow[r, dashed] & \cdots \arrow[r, dashed] & (X_i, B_i) \arrow[rr, dashed] \arrow[dr] & & (X_{i+1}, B_{i+1}) \arrow[dl] \arrow[r, dashed] & \cdots \\
& & Z_0 & & & & Z_i & &
\end{tikzcd}$$

be a sequence of klt flips of dimension d. Since $a(E, X_i, B_i) \le a(E, X_j, B_j)$ for any $i \le j$ and any prime divisor E over X, possibly truncating to a subsequence, we may assume that there exist exceptional prime divisors $E_1, \dots, E_k$ over X, such that $D(X_i, B_i)_{\le 1} = D(X, B)_{\le 1} = \{E_1, \dots, E_k\}$ for any i. Then $\{a(E_l, X_i, B_i) \mid 1 \le l \le k\}_{i=1}^{\infty}$ is a DCC set and the coefficients of B_i belong to a finite set. By Lemma 4.3 and the ACC for (global) mlds of enc pairs with finite coefficients, $\{a(E_l, X_i, B_i) \mid 1 \le l \le k\}_{i=1}^{\infty}$ satisfies the ACC. It follows that $\{a(E_l, X_i, B_i) \mid 1 \le l \le k\}_{i=1}^{\infty}$ is a finite set. Thus possibly truncating to a subsequence, we may assume that $a(E_l, X_i, B_i) = a(E_l, X_j, B_j)$ for any i, j.

Let $f_0 : Y_0 \to X_0$ be the birational morphism which extracts exactly $E_1, \dots, E_k$ for each i such that Y_0 is $\mathbb{Q}$-factorial, and let $K_{Y_0} + B_{Y_0} := f_0^*(K_{X_0} + B_0)$. We claim that we may construct a sequence of $\mathbb{Q}$-factorial terminal flips on $K_{Y_0} + B_{Y_0}$. Suppose that we have constructed (Y_i, B_{Y_i}) with $f_i \colon Y_i \to X_i$ such that $K_{Y_i} + B_{Y_i} := f_i^*(K_{X_i} + B_i)$, and $(Y_0, B_{Y_0}) \dashrightarrow (Y_1, B_{Y_1}) \cdots \dashrightarrow (Y_i, B_{Y_i})$ consists of a sequence of terminal flips on $K_{Y_0} + B_{Y_0}$. By [Birkar et al. 2010, Corollary 1.4.3], we may run a $(K_{Y_i} + B_{Y_i})$-MMP over Z_i which terminates with a minimal model $(Y_{i+1}, B_{Y_{i+1}})$. Since (X_{i+1}, B_{i+1}) is the log canonical model of (Y_i, B_{Y_i}) over Z_i, there exists an induced morphism $f_{i+1} \colon Y_{i+1} \to X_{i+1}$, such that $K_{Y_{i+1}} + B_{Y_{i+1}} := f_{i+1}^*(K_{X_{i+1}} + B_{i+1})$. Since $a(E_l, X_i, B_i) = a(E_l, X_{i+1}, B_{i+1})$ for any l, E_l is not contracted in the MMP $Y_i \dashrightarrow Y_{i+1}$, and $Y_i \dashrightarrow Y_{i+1}$ only consists of a sequence of flips. Thus we finish the proof of the claim by induction. Now the termination follows from the termination of flips for $\mathbb{Q}$-factorial terminal pairs.

Finally, we prove the general case. By Lemma 4.1, assumptions in Theorem 1.1 also hold for $\le d-1$. So we may do induction on d, and assume that any sequence of lc flips in dimension $\le d-1$ terminates. Now the termination follows from the klt case and the special termination (see [Shokurov 2004, Corollary 4; Fujino 2007; Chen and Tsakanikas 2023, Lemma 2.17(1); Han and Li 2022]). □

Proof of Theorem 1.2 This follows from Theorem 1.1 and the termination of canonical fourfold flips [Fujino 2004]. Note that [Fujino 2004] only deals with the $\mathbb{Q}$-coefficients case but the same argument works for the $\mathbb{R}$-coefficients case. □

4.2 Proof of Theorem 1.3

Lemma 4.4 *Let (X, B) be a pair. Then the set*

$$\{\mathrm{mld}(X \ni x, B) \mid x \in X\}$$

is finite. In particular, $\min_{x \in S} \mathrm{mld}(X \ni x, B)$ *is well-defined for any subset S of X.*

Proof Let $f : Y \to X$ be a log resolution of (X, B), and $K_Y + B_Y := f^*(K_X + B)$. Then

$$\{\mathrm{mld}(X \ni x, B) \mid x \in X\} \subseteq \{\mathrm{mld}(Y \ni y, B_Y) \mid y \in Y\},$$

and the latter is a finite set by [Chen and Han 2020, Lemma 3.3]. □

Definition 4.5 ([see Fulton 1984, Examples 19.1.3–19.1.6; Kollár 1996, Chapter II.(4.1.5)]) Let X be a reduced projective scheme and let k be a nonnegative integer. We denote by $Z_k(X)_{\mathbb{Q}}$ the group of k-dimensional algebraic cycles on X with rational coefficients. All cycles which are numerically equivalent to zero form a subgroup of $Z_k(X)_{\mathbb{Q}}$, and we denote by $N_k(X)_{\mathbb{Q}}$ the quotient group. Then $N_k(X)_{\mathbb{Q}}$ is a finite-dimensional $\mathbb{Q}$-vector space.

Lemma 4.6 *Let k be a positive integer, and $f: X \dashrightarrow Y$ a dominant rational map of reduced projective schemes. Suppose that f induces a birational map on each irreducible component of X and Y, and f^{-1} does not contract any k-dimensional subvariety of Y. Then*

(1) $\dim N_k(X)_{\mathbb{Q}} \geq \dim N_k(Y)_{\mathbb{Q}}$, *and*

(2) *if f contracts some k-dimensional subvariety of X, then* $\dim N_k(X)_{\mathbb{Q}} > \dim N_k(Y)_{\mathbb{Q}}$.

Proof (1) This follows from the fact that $f_*: N_k(X)_{\mathbb{Q}} \to N_k(Y)_{\mathbb{Q}}$ is surjective.

(2) Let $W \subset X$ be a subvariety of dimension k which is contracted by f. Then the cycle $[W]$ satisfies $[W] \not\equiv 0$ in $Z_k(X)_{\mathbb{Q}}$, $f_*[W] \equiv 0$ in $Z_k(Y)_{\mathbb{Q}}$, and (2) is proved. □

Shokurov proved that the ACC and the LSC conjectures for mlds imply the termination of flips [Shokurov 2004]. The following slightly stronger result actually follows from similar arguments as his proof. For the reader's convenience, we give a proof in details here.

Theorem 4.7 *Let d be a positive integer, a a nonnegative real number, and $\Gamma \subset [0,1]$ a finite set. Suppose that*

(1) *the set of mlds*

$$\{\mathrm{mld}(X \ni x, B) \mid \mathrm{mld}(X, B) > a \text{ (resp. } \geq a), \dim X = d, \mathrm{coeff}(B) \subseteq \Gamma\}$$

satisfies the ACC, and

(2) *for any pair (X, B) of dimension d such that* $\mathrm{mld}(X, B) > a$ (*resp.* $\geq a$),

$$x \to \mathrm{mld}(X \ni x, B)$$

is lower-semicontinuous for closed points x.

Then for any pair (X, B) with $\dim X = d$ *and* $\mathrm{mld}(X, B) > a$ (*resp.* $\geq a$), *any sequence of (K_X+B)-flips terminates.*

Proof **Step 1** In this step, we introduce some notation. Suppose that there exists an infinite sequence of (K_X+B)-flips,

$$(X, B) := (X_0, B_0) \overset{f_0}{\dashrightarrow} (X_1, B_1) \overset{f_1}{\dashrightarrow} \cdots (X_{i-1}, B_{i-1}) \overset{f_{i-1}}{\dashrightarrow} (X_i, B_i) \overset{f_i}{\dashrightarrow} (X_{i+1}, B_{i+1}) \cdots .$$

For each $i \geq 0$, denote by $\phi_i \colon X_i \to Z_i$ and $\phi_i^+ \colon X_{i+1} \to Z_i$ the corresponding flip contraction and flipped contraction between quasiprojective normal varieties, respectively. Let

$$a_i := \min_{x_i \in \operatorname{Exc}(\phi_i)} \operatorname{mld}(X_i \ni x_i, B_i), \quad \text{and} \quad \alpha_i := \inf\{a_j \mid j \geq i\}.$$

There exists an exceptional prime divisor E_i over X_i, such that $a_i = a(E_i, X_i, B_i)$, and $\operatorname{center}_{X_i} E_i \subseteq \operatorname{Exc}(\phi_i)$. We note that $a_i > a$ (resp. $a_i \geq a$) as $\operatorname{mld}(X_i, B_i) \geq \operatorname{mld}(X, B)$ and $\operatorname{codim} \operatorname{Exc}(\phi_i) \geq 2$.

Step 2 In this step, for each $i \geq 0$, we show that $\alpha_i = a_{n_i}$ for some $n_i \geq i$.

For each $i \geq 0$, let $\alpha_i^l := \min\{a_j \mid i \leq j \leq l\}$ for $l \geq i$. For each $l \geq i$, there exist $i \leq i_l \leq l$ and an exceptional prime divisor E_{i_l} over X_{i_l}, such that $a_{i_l} = \alpha_i^l = a(E_{i_l}, X_{i_l}, B_{i_l})$, and $\operatorname{center}_{X_{i_l}} E_{i_l} \subseteq \operatorname{Exc}(\phi_{i_l})$. Suppose that $\alpha_i \neq \alpha_i^l$ for any $l \geq i$, then there are infinitely many $l \geq i+1$, such that $i_l > i_{l-1}$. For each such l, and any $i \leq j < i_l$, we have $\operatorname{center}_{X_j} E_{i_l} \notin \operatorname{Exc}(\phi_j)$, otherwise $a(E_{i_l}, X_j, B_j) < a(E_{i_l}, X_{i_l}, B_{i_l}) = \alpha_i^l \leq a_j$, which contradicts the definition of a_j. Thus $\operatorname{mld}(X_i \ni x_{i,i_l}, B_i) = a(E_{i_l}, X_i, B_i) = a(E_{i_l}, X_{i_l}, B_{i_l}) = a_{i_l}$, where $x_{i,i_l} = \operatorname{center}_{X_i} E_{i_l}$. In particular, $\{\operatorname{mld}(X_i \ni x_{i,i_l}, B_i) \mid l \geq i\}$ is an infinite set which contradicts Lemma 4.4. Thus $\alpha_i = a_{n_i}$ for some $n_i \geq i$.

Step 3 In this step, we show that possibly passing to a subsequence of flips, there exists a nonnegative real number $a' > a$ (resp. $a' \geq a$), such that

- $a_i \geq a'$ for any $i \geq 0$, and
- $a_i = a'$ for infinitely many i.

Since $\operatorname{mld}(X_i, B_i) \geq \operatorname{mld}(X, B) > a$ (resp. $\operatorname{mld}(X_i, B_i) \geq \operatorname{mld}(X, B) \geq a$) for any i, by Step 2 and (1),

$$\{\alpha_i \mid i \in \mathbb{Z}_{\geq 0}\} = \{a_{n_i} \mid i \in \mathbb{Z}_{\geq 0}\}$$

is a finite set. In particular, there exist a nonnegative integer N, and a unique nonnegative real number $a' > a$ (resp. $a' \geq a$), such that $a_{n_i} \geq a'$ for any $i \geq N$, and $a_{n_i} = a'$ for infinitely many i.

Step 4 In this step, we construct k, S_i, W_i, and show some properties.

Possibly passing to a subsequence, we may assume there exists a nonnegative integer k satisfying the following:

- For any i, any point $x_i \in \operatorname{Exc}(\phi_i)$ with $(X_i \ni x_i, B_i) = a'$ satisfies $\dim x_i \leq k$.
- For infinitely many i, there exists a k-dimensional point $x_i \in \operatorname{Exc}(\phi_i)$ such that $(X_i \ni x_i, B_i) = a'$.

Let S_i be the set of the k-dimensional points $x_i \in X_i$ with $\operatorname{mld}(X_i \ni x_i, B_i) \leq a'$, and $W_i \subset X_i$ the Zariski closure of S_i. Then by (2) and [Ambro 1999, Proposition 2.1], any k-dimensional point $x_i \in W_i$ belongs to S_i.

Step 5 In this step, we prove that f_i induces

- a bijective map $S_i \setminus \operatorname{Exc}(\phi_i) \to S_{i+1}$, and
- a dominant morphism $f_i' \colon W_i \setminus \operatorname{Exc}(\phi_i) \to W_{i+1}$.

It suffices to show the first assertion as the second one follows from the first one.

For any $x_i \in S_i \setminus \mathrm{Exc}(\phi_i)$, $f_i(x_i) \in S_{i+1}$ as

$$\mathrm{mld}(X_{i+1} \ni f_i(x_i), B_{i+1}) = \mathrm{mld}(X_i \ni x_i, B_i) \le a'.$$

For any $x_{i+1} \in S_{i+1}$, suppose that $x_{i+1} \in \mathrm{Exc}(\phi_i^+)$. Then

$$\min_{x_i \in \mathrm{Exc}(\phi_i)} \mathrm{mld}(X_i \ni x_i, B_i) < \mathrm{mld}(X_{i+1} \ni x_{i+1}, B_{i+1}) \le a',$$

contradicting Step 3. It follows that $x_{i+1} \notin \mathrm{Exc}(\phi_i^+)$ and f_i induces a bijective map $S_i \setminus \mathrm{Exc}(\phi_i) \to S_{i+1}$.

Step 6 In this step, we derive a contradiction, and finish the proof.

By Step 5, the number of the irreducible components of W_i is nonincreasing. Thus possibly passing to a subsequence, we may assume that f_i induces a birational map on each irreducible component of W_i. On the one hand, by Step 5 and Step 4, f_i^{-1} does not contract any k-dimensional subvariety of W_{i+1}. On the other hand, by construction of k in Step 4, there exist infinitely many i such that $\mathrm{mld}(X_i \ni x_i, B_i) = a'$ for some k-dimensional point $x_i \in \mathrm{Exc}(\phi_i)$. For such i and x_i, by Step 5, $x_i \in W_i$ is contracted by f_i, which contradicts Lemma 4.6. □

Theorem 4.8 (Theorem 1.3) *Let d be a positive integer. Assume that*

(1) *the ACC for mlds of terminal pairs with finite coefficients in dimension d, ie*

$$\{\mathrm{mld}(X \ni x, B) \mid \mathrm{mld}(X, B) > 1, \dim X = d, \mathrm{coeff}(B) \subseteq \Gamma\}$$

satisfies the ACC for any finite set Γ, and

(2) *the LSC for mlds of terminal pairs in dimension d, ie for any pair (X, B) of dimension d such that* $\mathrm{mld}(X, B) > 1$,

$$x \to \mathrm{mld}(X \ni x, B)$$

is lower-semicontinuous for closed points x.

Then any sequence of terminal flips in dimension d terminates. Moreover, if we additionally assume that Conjecture 1.9(2′) holds in dimension d, then any sequence of lc flips in dimension $\le d$ terminates.

Proof This follows from Theorem 4.7 when $a = 1$ and Theorem 1.1. □

Remark 4.9 Generalized pairs, introduced in [Birkar and Zhang 2016], have become central topics in birational geometry in recent years. By [Hacon and Liu 2023], we can run MMPs for any $\mathbb{Q}$-factorial lc generalized pair. Therefore, studying the termination of flips for generalized pairs is also intriguing. It is important to note that the proofs in this section are expected to work for generalized pairs as well. For instance, [Chen et al. 2024, Theorem 4.8] provides a proof of Theorem 4.7 for generalized pairs when $a = 0$. Consequently, we anticipate that Theorems 1.1, 1.2, and 1.3 will also apply to generalized pairs by using similar arguments to those in this section. For the sake of brevity and the reader's convenience, we omit the detailed proofs here.

5 Theorems 1.4 and 1.6 for canonical threefolds

In this section, we prove Theorem 1.4 when X is canonical.

We first prove Theorem 1.4 when (X, B) is noncanonical in codimension 2. More precisely, we have:

Lemma 5.1 *Let $\Gamma \subset [0,1]$ be a DCC set. Assume that (X, B) is an enc pair of dimension 3 and E a prime divisor over X, such that*

(1) $\mathrm{coeff}(B) \subseteq \Gamma$,

(2) $\dim \mathrm{center}_X E = 1$, *and*

(3) $a(E, X, B) \le 1$.

Then $\mathrm{mld}(X, B)$ belongs to an ACC set.

Proof Since (X, B) is enc, $a(E, X, B) = \mathrm{mld}(X, B)$. Let $C := \mathrm{center}_X E$, H a general hyperplane on X which intersects C, and $K_H + B_H := (K_X + B + H)|_H$. Then the coefficients of B_H belong to a DCC set. By [Kollár and Mori 1998, Lemma 5.17(1)],

$$\mathrm{mld}(H, B_H) \ge \mathrm{mld}(X, B+H) \ge a(E, X, B+H) = a(E, X, B) = \mathrm{mld}(X, B).$$

By [Birkar et al. 2010, Corollary 1.4.5], $a(E, X, B+H) \ge \mathrm{mld}(H, B_H)$. Thus $\mathrm{mld}(H, B_H) = \mathrm{mld}(X, B)$. By [Alexeev 1993, Theorem 3.8], $\mathrm{mld}(H, B_H)$ belongs to an ACC set, hence $\mathrm{mld}(X, B)$ belongs to an ACC set. □

5.1 Strictly canonical threefolds

In this subsection, we prove Theorem 1.4 when X is strictly canonical. More precisely, we have:

Theorem 5.2 *Let $\Gamma \subset [0,1]$ be a DCC set. Then*

$$\left\{\mathrm{mld}(X \ni x, B) \mid \dim X = 3,\ (X \ni x, B) \textit{ is enc},\ X \textit{ is strictly canonical},\ \mathrm{coeff}(B) \subseteq \Gamma\right\}$$

satisfies the ACC.

Proof Suppose that the statement does not hold. Then there exists a sequence of enc pairs $(X_i \ni x_i, B_i)$ of dimension 3, such that

- $\mathrm{mld}(X_i) = 1$, and
- $a_i := \mathrm{mld}(X_i \ni x_i, B_i)$ is strictly increasing.

Possibly taking a small $\mathbb{Q}$-factorialization, we may assume that X_i is $\mathbb{Q}$-factorial. By Lemma 5.1, we may assume that x_i is a closed point for each i.

Let E_i be the unique prime divisor that is exceptional over X_i such that $a(E_i, X_i, B_i) \le 1$. Then $a(E_i, X_i, B_i) = \mathrm{mld}(X_i \ni x_i, B_i) = a_i$ and $\mathrm{center}_{X_i} E_i = x_i$. For any prime divisor $F_i \ne E_i$ that is exceptional over X_i, $a(F_i, X_i, 0) \ge a(F_i, X_i, B_i) > 1$. Since X_i is strictly canonical, $a(E_i, X_i, 0) = 1$.

By Theorem 3.13, there exists a positive integer n and a finite set $\Gamma_0 \subset (0,1]$ depending only on Γ, such that for any i, there exists an (n, Γ_0)-decomposable $\mathbb{R}$-complement $(X_i \ni x_i, B_i^+)$ of $(X_i \ni x_i, B_i)$. Possibly passing to a subsequence, we may assume that there exists a positive real number a, such that $a(E_i, X_i, B_i^+) = a$ for any i. Since a_i is strictly increasing and $a = a(E_i, X_i, B_i^+) \le a(E_i, X_i, B_i) = a_i$, possibly passing to a subsequence, we may assume that there exists a positive real number δ, such that $a_i - a > \delta$ for any i.

Let $f_i\colon Y_i \to X_i$ be the divisorial contraction which extracts E_i, and let B_{Y_i} and $B_{Y_i}^+$ be the strict transforms of B_i and B_i^+ on Y_i respectively. Then

$$K_{Y_i} + B_{Y_i} + (1 - a_i)E_i = f_i^*(K_{X_i} + B_i).$$

By the length of extremal rays, there exists a $(K_{Y_i} + B_{Y_i} + (1 - a^+)E_i)$-negative extremal ray R_i over a neighborhood of x_i which is generated by a rational curve C_i, such that

$$0 > (K_{Y_i} + B_{Y_i} + (1 - a)E_i) \cdot C_i \ge -6$$

(see [Fujino 2017, Theorem 4.5.2(5)]). Since $(K_{Y_i} + B_{Y_i} + (1 - a_i)E_i) \cdot C_i = 0$, we have

$$0 < (a_i - a)(-E_i \cdot C_i) \le 6.$$

Thus

$$0 < (-E_i \cdot C_i) < \frac{6}{\delta}.$$

By [Kawakita 2015b, Theorem 1.1], $60K_{Y_i}$ is Cartier over a neighborhood of x_i. Since X_i is enc, Y_i is terminal. By Lemma 3.14, $60D_i$ is Cartier over a neighborhood of x_i for any Weil divisor D_i on Y_i. In particular, $-E_i \cdot C_i$ belongs to the finite set $\frac{1}{60}\mathbb{Z}_{\ge 1} \cap \left(0, \frac{6}{\delta}\right)$.

We may write $B_i = \sum_j b_{i,j} B_{i,j}$, where $B_{i,j}$ are the irreducible components of B_i, and let $B_{Y_i,j}$ be the strict transform of $B_{i,j}$ on Y_i for each i, j. Then $B_{Y_i,j} \cdot C \in \frac{1}{60}\mathbb{Z}_{\ge 0}$ for every i, j. Since $K_{Y_i} = f_i^* K_{X_i}$, $K_{Y_i} \cdot C_i = 0$. Thus

$$a(E_i, X_i, B_i) = 1 - \frac{(K_{Y_i} + B_{Y_i}) \cdot C_i}{(-E_i \cdot C_i)} = 1 - \sum_j b_{i,j} \frac{(B_{Y_i,j} \cdot C_i)}{(-E_i \cdot C_i)}.$$

Since $b_{i,j} \in \Gamma$, $a(E_i, X_i, B_i)$ belongs to an ACC set, this leads to a contradiction. □

5.2 Terminal threefolds

In this subsection, we study Theorem 1.4 when X is terminal. At the moment, we cannot prove Theorem 1.4 in full generality, but we can prove the finite coefficient case, and reduce the DCC coefficient case to the finite coefficient case (but possibly losing the condition that X is terminal).

Lemma 5.3 *Let I be a positive integer and $\Gamma \subset [0,1]$ be a DCC set. Assume that $(X \ni x, B)$ is a threefold pair, such that*

(1) *X is terminal*,

(2) $\operatorname{coeff}(B) \subseteq \Gamma$,

(3) *$(X \ni x, B)$ is enc, and*

(4) *IK_X is Cartier near x.*

Then $\operatorname{mld}(X \ni x, B)$ *belongs to an ACC set.*

Proof Possibly replacing X with a small $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial. By Lemma 5.1, we may assume that x is a closed point.

Suppose that the statement does not hold. By Theorem 3.10, there exist a nonnegative integer m, a real number $a \in (0,1]$, a strictly increasing sequence of real numbers $a_i \in (0,1]$ such that $\lim_{i\to+\infty} a_i = a$, and a sequence of $\mathbb{Q}$-factorial threefold germs $\left(X_i \ni x_i, B_i = \sum_{j=1}^{m} b_{i,j} B_{i,j}\right)$, such that for any i,

- $(X_i \ni x_i, B_i)$ is enc and X_i is terminal,
- $b_{i,j} \in \Gamma$ for any j,
- $\operatorname{mld}(X_i \ni x_i, B_i) = a_i$,
- $B_{i,1}, \dots, B_{i,m}$ are the irreducible components of B_i,
- $x_i \in \operatorname{Supp} B_{i,j}$ for any j, and
- IK_{X_i} is Cartier near x_i.

Possibly passing to a subsequence, we may assume that $b_{i,j}$ is increasing for any fixed j, and let $b_j := \lim_{i\to+\infty} b_{i,j}$. We let $\bar{B}_i := \sum_{j=1}^{m} b_j B_{i,j}$ for each i. By [Hacon et al. 2014, Theorem 1.1], possibly passing to a subsequence, we may assume that $(X_i \ni x_i, \bar{B}_i)$ is lc for each i.

Since $(X_i \ni x_i, B_i)$ is enc, we may let E_i be the unique prime divisor over $X_i \ni x_i$ which computes $\operatorname{mld}(X_i \ni x_i, B_i)$. By Lemma 3.14 and [Nakamura 2016, Theorem 1.2], possibly passing to a subsequence, we may assume that $a' := a(E_i, X_i, \bar{B}_i) \geq 0$ is a constant, and we may pick a strictly decreasing sequence of real numbers ϵ_i, such that $(1+\epsilon_i)B_i \geq \bar{B}_i \neq B_i$ for each i and $\lim_{i\to+\infty} \epsilon_i = 0$. Therefore,

$$a' = a(E_i, X_i, \bar{B}_i) < a(E_i, X_i, B_i) = a_i < a,$$

and

$$\lim_{i\to+\infty} \epsilon_i \operatorname{mult}_{E_i} B_i \geq \lim_{i\to+\infty} (a(E_i, X_i, B_i) - a(E_i, X_i, \bar{B}_i)) = a - a' > 0.$$

Hence $\lim_{i\to+\infty} \operatorname{mult}_{E_i} B_i = +\infty$. Let $t_i := \operatorname{ct}(X_i \ni x_i, 0; B_i)$. Since $(X_i \ni x_i, B_i)$ is enc and $a(E_i, X_i, B_i) < 1$, we have that $t_i < 1$. By [Han et al. 2022, Lemma 2.12(1)], $a(E_i, X_i, t_i B_i) = 1$ for each i. Since

$$1 = a(E_i, X_i, t_i B_i) = a(E_i, X_i, B_i) - (1-t_i)\operatorname{mult}_{E_i} B_i < a - (1-t_i)\operatorname{mult}_{E_i} B_i,$$

we have

$$1 > t_i > 1 - \frac{a-1}{\mathrm{mult}_{E_i} B_i}.$$

Thus $\lim_{i\to+\infty} t_i = 1$ and $t_i < 1$ for each i, which contradicts Theorem 3.6. □

Definition 5.4 Let $(X \ni x, B)$ be an lc germ. A *terminal blow-up* of $(X \ni x, B)$ is a birational morphism $f\colon Y \to X$ which extracts a prime divisor E over $X \ni x$, such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$, $-E$ is f-ample, and Y is terminal.

Lemma 5.5 [Han et al. 2022, Lemma 2.35] *Let $(X \ni x, B)$ be a germ such that X is terminal and* $\mathrm{mld}(X \ni x, B) = 1$. *Then there exists a terminal blow-up $f\colon Y \to X$ of $(X \ni x, B)$. Moreover, if X is $\mathbb{Q}$-factorial, then Y is $\mathbb{Q}$-factorial.*

Theorem 5.6 *Let $\Gamma \subset [0,1]$ be a finite set. Then*

$$\bigl\{\mathrm{mld}(X \ni x, B) \mid \dim X = 3,\ (X \ni x, B) \text{ is enc},\ X \text{ is terminal},\ \mathrm{coeff}(B) \subseteq \Gamma\bigr\}$$

satisfies the ACC.

Proof **Step 1** We construct some functional pairs in this step.

Possibly replacing X with a small $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial. By Lemma 5.1, we may assume that x is a closed point. By Lemmas 5.1 and 5.3, we may assume that $X \ni x$ is a cA/n type singularity for some positive integer $n \geq 3$. Let E be the unique prime divisor over $X \ni x$ such that $\mathrm{mld}(X \ni x, B) = a(E, X, B)$.

Let $t := \mathrm{ct}(X \ni x, 0; B)$. By Theorem 3.6, there exists a positive real number ϵ_0 depending only on Γ, such that $t \leq 1 - \epsilon_0$.

By our assumptions and [Han et al. 2024, Theorem 5.6], there exist two positive integers l, m, real numbers $1, v_0^1, \dots, v_0^l$ that are linearly independent over $\mathbb{Q}$, $\boldsymbol{v}_0 := (v_0^1, \dots, v_0^l) \in \mathbb{R}^l$, an open set $U \ni \boldsymbol{v}_0$ of $\mathbb{R}^l$, and $\mathbb{Q}$-linear functions $s_1, \dots, s_m\colon \mathbb{R}^{l+1} \to \mathbb{R}$ depending only on Γ, and Weil divisors $B^1, \dots, B^m \geq 0$ on X, such that:

(1) $s_i(1, \boldsymbol{v}_0) > 0$ and $x \in \mathrm{Supp}\, B^i$ for each $1 \leq i \leq m$.

(2) Let $B(\boldsymbol{v}) := \sum_{i=1}^m s_i(1, \boldsymbol{v}) B^i$ for any $\boldsymbol{v} \in \mathbb{R}^l$. Then $B(\boldsymbol{v}_0) = B$ and $(X \ni x, B(\boldsymbol{v}))$ is lc for any $\boldsymbol{v} \in U$.

(3) Possibly shrinking U, we may assume that $\bigl(1 + \frac{\epsilon_0}{2}\bigr) B \geq B(\boldsymbol{v}) \geq \bigl(1 - \frac{\epsilon_0}{2}\bigr) B$ for any $\boldsymbol{v} \in U$.

We may pick vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{l+1} \in U \cap \mathbb{Q}^l$ and real numbers $b_1, \dots, b_{l+1} \in (0,1]$ depending only on Γ, such that $\sum_{i=1}^{l+1} b_i = 1$ and $\sum_{i=1}^{l+1} b_i \boldsymbol{v}_i = \boldsymbol{v}_0$. We let $B_i := B(\boldsymbol{v}_i)$ for each i. Then there exists a positive integer M depending only on Γ, such that MB_i is integral for any i.

Let $t_i := \mathrm{ct}(X \ni x, 0; B_i)$ for each i. By [Han et al. 2022, Lemma 2.12(2)], $\mathrm{mld}(X \ni x, t_i B_i) = 1$. Since $B_i \geq \left(1 - \frac{\epsilon_0}{2}\right) B$ and $t \leq 1 - \epsilon_0$, we have

$$t_i \leq \frac{1-\epsilon_0}{1-\frac{\epsilon_0}{2}} < 1.$$

Thus

$$B \geq \frac{1-\epsilon_0}{1-\frac{\epsilon_0}{2}}\left(1+\frac{\epsilon_0}{2}\right) B \geq \frac{1-\epsilon_0}{1-\frac{\epsilon_0}{2}} B_i \geq t_i B_i.$$

Since $(X \ni x, B)$ is enc, $a(E, X, t_i B_i) = 1$ and $a(F, X, t_i B_i) > 1$ for any prime divisor $F \neq E$ over $X \ni x$.

Step 2 Construct divisorial contractions.

By Lemma 5.5, there exists a terminal blow-up $f : Y \to X$ of $(X \ni x, t_i B_i)$ which extracts a prime divisor E over $X \ni x$. By [Kawakita 2005, Theorem 1.3], f is of ordinary type as $n \geq 3$. By [Han et al. 2022, Theorem 2.31(1)], we may take suitable local coordinates x_1, x_2, x_3, x_4 of $\mathbb{C}^4$, an analytic function $\phi \in \mathbb{C}\{x_1, x_2, x_3, x_4\}$, and positive integers r_1, r_2, a, b and d satisfying

- $\gcd(b, n) = 1$,
- $(X \ni x) \cong (\phi(x_1, x_2, x_3, x_4) = 0) \subset (\mathbb{C}^4 \ni o)/\frac{1}{n}(1, -1, b, 0)$,
- ϕ is semi-invariant under the group action
$$\boldsymbol{\mu} := (x_1, x_2, x_3, x_4) \to (\xi x, \xi^{-1} x_2, \xi^b x_3, x_4),$$
where $\xi = e^{2\pi i/n}$,
- f is a weighted blow-up at $x \in X$ with the weight vector $w := \frac{1}{n}(r_1, r_2, a, n)$,
- $a \equiv b r_1 \mod n$, $\gcd((a - b r_1)/n, r_1) = 1$, $r_1 + r_2 = adn$,
- $\phi(x_1, x_2, x_3, x_4) = x_1 x_2 + g(x_3^n, x_4)$, and
- $z^{dn} \in g(x_3^n, x_4)$ and $w(\phi) = adn$.

There are two cases:

Case 1 $d \geq 4$ or $a \geq 4$.

Case 2 $d \leq 3$ and $a \leq 3$.

Step 3 In this step we deal with Case 1, that is, the case when $d \geq 4$ or $a \geq 4$. In this case, we can pick three positive integers r_1', r_2' and a', such that

- $r_1' + r_2' = a' d n$,
- $a' \equiv b r_1' \mod n$,
- $r_1', r_2' > n$, and
- $\frac{1}{n}(r_1', r_2', a', n) \neq \frac{1}{n}(r_1, r_2, a, n)$.

In fact, when $a \geq 4$, we may take $a' = 3$. When $d \geq 4$, we may take $a' = 1$ and $(r_1', r_2') \neq (r_1, r_2)$.

Let $w' := \frac{1}{n}(r_1', r_2', a', n)$. Since $a \geq a'$, by [Han et al. 2022, Lemma C.7], the weighted blow-up with the weight vector w' at $x \in X$ under analytic local coordinates x_1, x_2, x_3, x_4 extracts an exceptional prime divisor $E' \neq E$, such that $w'(X \ni x) = a'/n$. By our assumptions,

$$a(E', X, B) = 1 + \frac{a'}{n} - \sum_{i=1}^{l+1} b_i w'(B_i) > 1,$$

hence $\sum_{i=1}^{l+1} b_i w'(B_i) < a'/n$. Since MB_i is integral for each i and $x \in \operatorname{Supp} B_i$, $w'(B_i) \in \frac{1}{Mn}\mathbb{Z}_{\geq 1}$ for each i.

Let $b_0 := \min\{b_i \mid i = 1, 2, \dots, l+1\}$. By Lemma 5.3, we may assume that $n > 3M/\gamma_0$. Since MB_i a Weil divisor, $MB_i = (h_i = 0) \subset \mathbb{C}^4/\frac{1}{n}(1, -1, b, 0)$ for some analytic function h_i. Since $w'(x_1) > 1$, $w'(x_2) > 1$, $w'(x_4) = 1$, $a' \leq 3$, $b_i \geq b_0$, and $n > 3M/\gamma_0$, for each i, there exists a positive integer $1 \leq p_i < Ma'/\gamma_0$, such that $x_3^{p_i} \in h_i$ and $w'(x_3^{p_i}) = w'(h_i) = p_i a'/n$, and

$$\sum_{i=1}^{l+1} b_i \frac{p_i a'}{Mn} = \sum_{i=1}^{l+1} b_i w'(B_i) < \frac{a'}{n}.$$

In particular, $\sum_{i=1}^{l+1} b_i p_i/M < 1$. We have

$$w(B) = \sum_{i=1}^{l+1} b_i w(B_i) = \sum_{i=1}^{l+1} \frac{b_i}{M} w(h_i) \leq \sum_{i=1}^{l+1} \frac{b_i}{M} w(x_3^{p_i}) = \sum_{i=1}^{l+1} \frac{b_i p_i}{M} \cdot \frac{a}{n} < \frac{a}{n},$$

hence $a(E, X, B) = a(E, X, 0) - \operatorname{mult}_E B = 1 + \frac{a}{n} - w(B) > 1$, a contradiction.

Step 4 In this step, we deal with Case 2, that is, the case when $d \leq 3$ and $a \leq 3$, hence we conclude the proof. In this case, since $a \equiv br_1 \bmod n$ and $\gcd(b, n) = 1$, $\gcd(r_1, n) = \gcd(a, n) \leq a \leq 3$, so $\gcd(r_1, n) \mid 6$. Since $r_1 + r_2 = adn$,

$$\gcd(r_1, r_2) = \gcd(r_1, adn) \mid ad \gcd(r_1, n) \mid 216.$$

Since MB_i is a Weil divisor, nMB_i is Cartier near x. Thus

$$\frac{a}{n} = a(E, X, 0) - a(E, X, t_i B_i) = t_i \operatorname{mult}_E B_i = \frac{t_i}{nM} \operatorname{mult}_E nMB_i \in \frac{t_i}{nM}\mathbb{Z}_{\geq 1},$$

which implies that $aM/t_i \in \mathbb{Z}_{\geq 1}$. Then $(aM/t_i)(K_X + t_i B_i)$ is a Weil divisor for any i, by [Han et al. 2022, Lemma 5.3], $(216aM/t_i)(K_X + t_i B_i)$ is Cartier near x.

By [Shokurov 1993, 4.8 Corollary], there exists a prime divisor $E' \neq E$ over $X \ni x$ such that $a(E', X, 0) = 1 + \frac{a'}{n}$ for some integer $a' \in \{1, 2\}$. Since $(216aM/t_i)(K_X + t_i B_i)$ is Cartier near x and $a(E', X, t_i B_i) > 1$ for any i, there exists a positive integer k_i such that

$$a(E', X, t_i B_i) = 1 + \frac{k_i t_i}{216aM}.$$

Thus

$$\operatorname{mult}_{E'} B_i = \frac{1}{t_i}(a(E', X, 0) - a(E', X, t_i B_i)) = \frac{a'}{nt_i} - \frac{k_i}{216aM}.$$

Since MnB_i is Cartier near x and $x \in \operatorname{Supp} B_i$, there exist positive integers a'_i, such that $\operatorname{mult}_{E'} B_i = a'_i/(Mn)$. Since $\big(X \ni x, B = \sum_{i=1}^{l+1} b_i B_i\big)$ is enc and $a(E, X, B) < 1,\ a(E', X, B) > 1$, hence

$$\frac{\sum_{i=1}^{l+1} b_i a'_i}{Mn} = \operatorname{mult}_F B < \frac{a'}{n}. \tag{5-1}$$

Thus the a'_i belong to a finite set depending only on Γ.

Since $a(E, X, 0) = 1 + \frac{a}{n}$ and $a(E, X, t_i B_i) = 1,\ \operatorname{mult}_E B_i = a/(nt_i)$. Thus

$$a(E, X, B_i) = a(E, X, 0) - \operatorname{mult}_E B_i = 1 + \frac{a}{n} - \frac{a}{nt_i}.$$

Since $(X \ni x, B_i)$ is lc, $a(E, X, B_i) > 0$, so

$$nt_i > \frac{a}{1+\frac{a}{n}} = \frac{1}{\frac{1}{a}+\frac{1}{n}} > \frac{1}{2},$$

and we have $a'/(nt_i) < 2a'$. Since

$$\frac{a'}{nt_i} - \frac{k_i}{216aM} = \operatorname{mult}_{E'} B_i = \frac{a'_i}{nM} \tag{5-2}$$

and each k_i is a positive integer, the k_i belong to a finite set depending only on Γ. By (5-1) and (5-2), we have

$$a'\bigg(\sum_{i=1}^{l+1} \frac{b_i}{nt_i} - \frac{1}{n}\bigg) - \sum_{i=1}^{l+1} \frac{k_i b_i}{216Ma} = \sum_{i=1}^{l+1}\bigg(\frac{b_i a'}{nt_i} - \frac{b_i k_i}{216Ma}\bigg) - \frac{a'}{n} = \bigg(\sum_{i=1}^{l+1} \frac{b_i a'_i}{Mn}\bigg) - \frac{a'}{n} < 0.$$

Since a', k_i, b_i, M, a, a'_i belong to a finite set depending only on Γ,

$$\sum_{i=1}^{l+1} \frac{b_i a'_i}{Mn} - \frac{a'}{n} = -\frac{1}{n}\bigg(a' - \sum_{i=1}^{l+1} \frac{b_i a'_i}{M}\bigg)$$

belongs to a DCC set depending only on Γ, and $\sum_{i=1}^{l+1} b_i/(nt_i) - \frac{1}{n}$ also belongs to a DCC set depending only on Γ. Since

$$\operatorname{mult}_E B_i = \frac{1}{t_i}(a(E, X, 0) - a(E, X, t_i B_i)) = \frac{1}{t_i}\Big(1 + \frac{a}{n} - 1\Big) = \frac{a}{nt_i},$$

we have

$$\begin{aligned} \operatorname{mld}(X \ni x, B) = a(E, X, B) = a(E, X, 0) - \operatorname{mult}_E B = a(E, X, 0) - \sum_{i=1}^{l+1} b_i \operatorname{mult}_E B_i \\ = 1 + \frac{a}{n} - \sum_{i=1}^{l+1} \frac{ab_i}{nt_i} = 1 - a\bigg(\sum_{i=1}^{l+1} \frac{b_i}{nt_i} - \frac{1}{n}\bigg). \end{aligned}$$

Thus $\operatorname{mld}(X \ni x, B)$ belongs to an ACC set depending only on Γ. $\square$

Theorem 5.7 *Let $\epsilon \in (0,1)$ be a positive real number. Suppose that*

$$\{\mathrm{mld}(X \ni x, B) \mid \dim X = 3, \mathrm{coeff}(B) \subseteq \Gamma_0, (X \ni x, B) \text{ is } \mathbb{Q}\text{-factorial enc}\} \cap [\epsilon, 1]$$

satisfies the ACC for any finite set $\Gamma_0 \subset [0,1]$. Then for any DCC set $\Gamma \subset [0,1]$,

$$\{\mathrm{mld}(X \ni x, B) \mid \dim X = 3, X \text{ is terminal}, \mathrm{coeff}(B) \subseteq \Gamma, (X \ni x, B) \text{ is enc}\} \cap [\epsilon, 1]$$

satisfies the ACC.

Proof Possibly replacing X with a small $\mathbb{Q}$-factorialization, we may assume that X is $\mathbb{Q}$-factorial. Suppose that the statement does not hold. By Theorem 3.10, there exist a positive integer m, a real number a, a strictly increasing sequence of real numbers a_i, and a sequence of $\mathbb{Q}$-factorial enc threefold pairs $\left(X_i \ni x_i, B_i = \sum_{j=1}^{m} b_{i,j} B_{i,j}\right)$, such that for any i,

- $b_{i,j} \in \Gamma$, and $B_{i,j} \geq 0$ are Weil divisors for any j,
- for any fixed j, $b_{i,j}$ is increasing,
- $\mathrm{mld}(X_i \ni x_i, B_i) = a_i$, and
- $\lim_{i\to+\infty} a_i = a \in (\epsilon, 1]$.

By [Han et al. 2022, Theorem 1.1], $a < 1$. Let $b_j := \lim_{i\to+\infty} b_{i,j}$ and $\bar{B}_i := \sum_{j=1}^{m} b_j B_{i,j}$. By [Hacon et al. 2014, Theorem 1.1], possibly passing to a subsequence, we may assume that $(X_i \ni x_i, \bar{B}_i)$ is lc for each i. Let $t_i := \mathrm{ct}(X_i \ni x_i; B_i)$ and E_i the unique prime divisor over $X_i \ni x_i$ such that $a(E_i, X_i, B_i) < 1$. Then $a(E_i, X_i, B_i) = a_i$. By [Han et al. 2022, Lemma 2.12(1)], $a(E_i, X_i, t_i B_i) = 1$, hence $\mathrm{mult}_{E_i} B_i = (1-a_i)/(1-t_i) < 1/(1-t_i)$. By construction, $t_i < 1$ for each i. By Theorem 3.6, we may assume that t_i is decreasing, hence there exists a positive real number M such that $\mathrm{mult}_{E_i} B_i < M$.

By construction, there exists a sequence of positive real numbers ϵ_i such that $(1+\epsilon_i)B_i \geq \bar{B}_i$ and $\lim_{i\to+\infty} \epsilon_i = 0$. We have

$$a_i = a(E_i, X_i, B_i) \geq a(E_i, X_i, \bar{B}_i) \geq a(E_i, X_i, (1+\epsilon_i)B_i) = a(E_i, X_i, B_i) - \epsilon_i \,\mathrm{mult}_{E_i} B_i > a_i - \epsilon_i M,$$

Since $\lim_{i\to+\infty} a_i = \lim_{i\to+\infty}(a_i - \epsilon_i M) = a$, possibly passing to a subsequence, we may assume that $\bar{a}_i := a(E_i, X_i, \bar{B}_i)$ is strictly increasing and $\lim_{i\to+\infty} \bar{a}_i = a$.

Let $f_i : Y_i \to X_i$ be the divisorial contraction which extracts E_i, and let $B_{Y_i}, \bar{B}_{Y_i}$ be the strict transforms of B_i and $\bar{B}_i$ on Y_i respectively. Then $(Y_i/X_i \ni x_i, B_{Y_i} + (1-a)E_i)$ is canonical and

$$\mathrm{coeff}(B_{Y_i} + (1-a)E_i) \subseteq \Gamma \cup \{1-a\}.$$

By Theorem 3.6, possibly passing to a subsequence, we may assume that $(Y_i/X_i \ni x_i, \bar{B}_{Y_i} + (1-a)E_i)$ is canonical.

By Theorem 3.13, there exists a positive integer N and a finite set $\Gamma_0 \subset (0,1]$, such that $(X_i \ni x_i, \bar{B}_i)$ has an (N, Γ_0)-decomposable $\mathbb{R}$-complement $(X_i \ni x_i, \bar{B}_i^+)$ for each i. In particular, $a(E_i, X_i, \bar{B}_i^+)$

belongs to a discrete, hence finite set for any i. Thus there exists a positive real number t, such that $(Y_i/X_i \ni x_i, \bar{B}_{Y_i} + (1-a+t)E_i)$ is lc. Possibly passing to a subsequence, we may assume that $a - \bar{a}_i < \frac{t}{2}$ for any i.

For any i, we let

$$\mathcal{D}_i := \{F_i \mid F_i \text{ is over } X_i \ni x_i,\ F_i \neq E_i,\ a(F_i, X_i, \bar{B}_i) < 1\}.$$

For any $F_i \in \mathcal{D}_i$, we have

$$a(F_i, Y_i, \bar{B}_{Y_i} + (1-a+t)E_i) \geq 0 \quad \text{and} \quad a(F_i, Y_i, \bar{B}_{Y_i} + (1-\bar{a}_i)E_i) < 1.$$

Since $a - \bar{a}_i < \frac{t}{2}$, $\operatorname{mult}_{F_i} E_i < \frac{2}{t}$, and

$$\begin{aligned} a(F_i, X_i, \bar{B}_i) &= a(F_i, Y_i, \bar{B}_{Y_i} + (1-\bar{a}_i)E_i) \\ &= a(F_i, Y_i, \bar{B}_{Y_i} + (1-a)E_i) + (a - a_i)\operatorname{mult}_{F_i} E_i > 1 - \frac{2(a-\bar{a}_i)}{t}. \end{aligned}$$

Possibly passing to a subsequence, we may assume that $a - \bar{a}_i < \frac{t}{2}(1-a)$ for every i. Then $a(F_i, X_i, \bar{B}_i) > a > \bar{a}_i = a(E_i, X_i, \bar{B}_i)$ for any $F_i \in \mathcal{D}_i$.

If $(X_i, \bar{B}_i)$ is not klt near x_i for infinitely many i, then we let $\phi_i \colon W_i \to X_i$ be a dlt modification of $(X_i, \bar{B}_i)$, and let $K_{W_i} + \bar{B}_{W_i} := \phi_i^*(K_{X_i} + B_i)$. Then there exists a prime divisor $H_i \subset \operatorname{Exc}(\phi_i)$ such that $\operatorname{center}_{W_i} E_i \subset H_i$. We immediately get a contradiction by applying adjunction to H_i and using the precise inversion of adjunction formula (see [Liu 2018, Lemma 3.3]) and the ACC for mlds of surfaces [Alexeev 1993, Theorem 3.8]. Therefore, possibly passing to a subsequence, we may assume that $(X_i, \bar{B}_i)$ is klt near x_i for each i.

Since $(X_i, \bar{B}_i)$ is klt near x_i, each $\mathcal{D}_i$ is a finite set, and we may assume that $\mathcal{D}_i = \{F_{i,1}, \dots, F_{i,r_i}\}$ for some nonnegative integer r_i. Note that $\lim_{i\to+\infty} a(F_{i,j}, X_i, \bar{B}_i) = 1$ for any $1 \leq j \leq r_i$, and $\lim_{i\to+\infty} a(E_i, X_i, \bar{B}_i) = a$. By [Liu 2018, Lemma 5.3], possibly reordering $F_{i,1}, \dots, F_{i,r_i}$, one of the following hold:

(1) There exists a birational morphism $g_i \colon Z_i \to X_i$ which extracts exactly $F_{i,1}, \dots, F_{i,r_i}$, such that

$$\bar{a}_i = a(E_i, Z_i, \bar{B}_{Z_i}) + \sum_{j=1}^{r_i}(1 - a(F_{i,j}, X_i, \bar{B}_i)F_{i,j}) \leq a(E_i, Z_i, \bar{B}_{Z_i}) < a$$

for each i, where $\bar{B}_{Z_i}$ is the strict transform of $\bar{B}_i$ on Z_i. In this case, we let $E_i' := E_i$ and $B_i' := \bar{B}_{Z_i}$

(2) There exists a birational morphism $g_i \colon Z_i \to X_i$ which extracts exactly $E_i, F_{i,1}, \dots, F_{i,r_i-1}$, such that

$$1 - \frac{2(a-\bar{a}_i)}{t} \leq a(F_{i,r_i}, X_i, \bar{B}_i) \leq a(F_{i,r_i}, Z_i, \bar{B}_{Z_i} + (1-a)E_{Z_i}) < 1$$

for each i, where $\bar{B}_{Z_i}, E_{Z_i}$ are the strict transforms of $\bar{B}_i$ and E_i on Z_i respectively. In this case, we let $E_i' = F_{i,r_i}$ and $B_i' := \bar{B}_{Z_i} + (1-a)E_{Z_i}$.

In either case, possibly passing to a subsequence, we may assume that $\bar{a}'_i := a(E'_i, Z_i, B'_i) < 1$ is strictly increasing. For any i, we let

$$\mathcal{D}'_i := \{F'_i \mid F'_i \text{ is over } X_i \ni x_i,\ F'_i \text{ is exceptional over } Z_i,\ a(F'_i, Z_i, B'_i) = 1\}.$$

By construction, $(Z_i/X_i \ni x_i, B'_i)$ is klt, so $\mathcal{D}'_i$ is a finite set. Let $h_i\colon V_i \to W_i$ be a birational morphism which extracts all divisors in $\mathcal{D}'_i$, and let B'_{V_i} be the strict transform of B'_i on V_i. Then (V_i, B'_{V_i}) is enc and $a'_i := a(E'_i, V_i, B_{V'_i})$ is strictly increasing. Moreover, $\lim_{i\to+\infty} a'_i = 1$ or a. Since $a > \epsilon$, possibly passing to a subsequence, we may assume that $a'_i > \epsilon$ for any i. However, the coefficients of B'_{V_i} belong to a finite set, which contradicts our assumptions. □

6 Index one cover

6.1 Enc cyclic quotient singularities

In this subsection, we prove Theorem 1.4 when X is noncanonical, with isolated singularities, and the index 1 cover of X is smooth (see Theorem 6.3).

Lemma 6.1 (see [Liu and Luo 2022, Lemma 2.11; Ambro 2006, Theorem 1]) *Let d be a positive integer and $(X \ni x) = \frac{1}{r}(a_1, a_2, \dots, a_d)$ a d-dimensional cyclic quotient singularity. Let*

$$\boldsymbol{e} := \left(\left\{\frac{a_1}{r}\right\}, \left\{\frac{a_2}{r}\right\}, \dots, \left\{\frac{a_d}{r}\right\}\right),$$

$$\boldsymbol{e}_i \text{ the } i^{th} \text{ unit vector in } \mathbb{Z}^d \text{ for any } 1 \le i \le d,$$

$$N := \mathbb{Z}_{\ge 0}\boldsymbol{e} \oplus \mathbb{Z}_{\ge 0}\boldsymbol{e}_1 \oplus \mathbb{Z}_{\ge 0}\boldsymbol{e}_2 \oplus \cdots \oplus \mathbb{Z}_{\ge 0}\boldsymbol{e}_d,$$

$$\sigma := N \cap \mathbb{Q}^d_{\ge 0}, \quad \text{and} \quad \operatorname{relin}(\sigma) := N \cap \mathbb{Q}^d_{>0}.$$

The following holds.

(1) *For any prime divisor E over $X \ni x$ that is invariant under the cyclic quotient action, there exists a primitive vector $\alpha \in \operatorname{relin}(\sigma)$ such that $a(E, X, 0) = \alpha(x_1 x_2 \cdots x_d)$. In particular, there exists a unique positive integer $k \le r$, such that*

$$\alpha \in \left(1 + \frac{a_1 k}{r} - \left\lceil \frac{a_1 k}{r} \right\rceil, 1 + \frac{a_2 k}{r} - \left\lceil \frac{a_2 k}{r} \right\rceil, \dots, 1 + \frac{a_d k}{r} - \left\lceil \frac{a_d k}{r} \right\rceil\right) + \mathbb{Z}^d_{\ge 0}.$$

(2) $$\operatorname{mld}(X \ni x) = \min_{1 \le k \le r-1} \left\{\sum_{i=1}^{d} \left(1 + \frac{k a_i}{r} - \left\lceil \frac{k a_i}{r} \right\rceil\right)\right\} \le d.$$

Proof Point (1) is elementary toric geometry, and (2) follows immediately from (1). □

Lemma 6.2 *Let d be a positive integer and ϵ a positive real number. Then there exists a positive integer I, depending only on d and ϵ, satisfying the following. Let r be a positive integer and $v_1, \dots, v_d \in [0, 1]$ real numbers, such that $\sum_{i=1}^{d}(1 + (m-1)v_i - \lceil m v_i \rceil) \ge \epsilon$ for any $m \in [2, r] \cap \mathbb{Z}$. Then $r \le I$.*

Proof Suppose that the statement does not hold. Then for each $j \in \mathbb{Z}_{\geq 1}$, there exist $v_{1,j}, \dots, v_{d,j} \in [0,1]$ and positive integers r_j, such that

- $\sum_{i=1}^{d}(1+(m-1)v_{i,j}-\lceil m v_{i,j}\rceil) \geq \epsilon$ for any $m \in [2, r_j] \cap \mathbb{Z}$,
- r_j is strictly increasing, and
- $\bar{v}_i := \lim_{j\to+\infty} v_{i,j}$ exists.

Let $\boldsymbol{v} := (\bar{v}_1, \dots, \bar{v}_d)$. By Kronecker's theorem, there exist a positive integer n and a vector $\boldsymbol{u} \in \mathbb{Z}^d$ such that $\|n\boldsymbol{v}-\boldsymbol{u}\|_\infty < \min\{\frac{\epsilon}{d}, \bar{v}_i \mid \bar{v}_i > 0\}$ and $n\bar{v}_i \in \mathbb{Z}$ for any i such that $\bar{v}_i \in \mathbb{Q}$. In particular, $\lceil (n+1)\bar{v}_i\rceil = \lfloor (n+1)\bar{v}_i\rfloor + 1$ for any i such that $\bar{v}_i \in (0,1)$. Now $\lim_{j\to+\infty}(1+nv_{i,j}-\lceil (n+1)v_{i,j}\rceil) = 0$ when $\bar{v}_i = 0$ and $\lim_{j\to+\infty}(1+nv_{i,j}-\lceil (n+1)v_{i,j}\rceil) = 1+n\bar{v}_i-\lceil (n+1)\bar{v}_i\rceil$ when $\bar{v}_i > 0$. Thus

$$\begin{aligned}\lim_{j\to+\infty}\sum_{i=1}^{d}(1+nv_{i,j}-\lceil (n+1)v_{i,j}\rceil) &= \sum_{0<\bar{v}_i<1}(1+n\bar{v}_i-\lceil (n+1)\bar{v}_i\rceil)\\ &= \sum_{0<\bar{v}_i<1}(1+(n+1)\bar{v}_i-\lceil (n+1)\bar{v}_i\rceil-\bar{v}_i)\\ &= \sum_{\bar{0}<\bar{v}_i<1}(\{(n+1)\bar{v}_i\}-\bar{v}_i) < \sum_{0<\bar{v}_i<1}\frac{\epsilon}{d} \leq \epsilon.\end{aligned}$$

Thus possibly passing to a subsequence, $\sum_{i=1}^{d}(1+nv_{i,j}-\lceil (n+1)v_{i,j}\rceil) < \epsilon$ for any j, hence $n > r_j$, which contradicts $\lim_{j\to+\infty} r_j = +\infty$. □

Theorem 6.3 *Let $\Gamma \subset [0,1]$ be a DCC (resp. finite) set. Assume that $(X \ni x, B)$ is a $\mathbb{Q}$-factorial enc pair of dimension 3, such that*

(1) *$X \ni x$ is a noncanonical isolated singularity,*

(2) *$\mathrm{coeff}(B) \subseteq \Gamma$, and*

(3) *$\tilde{X} \ni \tilde{x}$ is smooth, where $\pi : (\tilde{X} \ni \tilde{x}) \to (X \ni x)$ is the index 1 cover of $X \ni x$.*

Then $\mathrm{mld}(X \ni x, B)$ belongs to an ACC set (resp. is discrete away from 0).

Proof We may assume that $\mathrm{mld}(X \ni x, B) > \epsilon$ for some fixed positive real number $\epsilon < \frac{12}{13}$. Then $\mathrm{mld}(X \ni x) > \epsilon$. Since $X \ni x$ is noncanonical and $(X \ni x, B)$ is enc, $X \ni x$ is enc. Since $X \ni x$ is an isolated singularity, by Theorem 3.15, $\mathrm{mld}(X \ni x) = \mathrm{mld}(X) \leq \frac{12}{13}$. Since $\tilde{X}$ is smooth, $(X \ni x)$ is analytically isomorphic to a cyclic quotient singularity $(Y \ni y) = \frac{1}{r}(a_1, a_2, a_3)$, where a_1, a_2, a_3, r are positive integers such that $a_i < r$ and $\gcd(a_i, r) = 1$ for each i. By Lemma 6.1, there exists a positive integer $k_0 \in [1, r-1]$ such that $\mathrm{mld}(Y \ni y) = \mathrm{mld}(X \ni x) = \sum_{i=1}^{3}\{a_i k_0/r\} \in \left[\epsilon, \frac{12}{13}\right]$, and for any positive integer $k \neq k_0$ such that $k \in [1, r-1]$, $\sum_{i=1}^{3}\{a_i k/r\} > \min\{1, 2\,\mathrm{mld}(X \ni x)\}$. Let $v_i := \{a_i k_0/r\}$ for each i. Then

$$\sum_{i=1}^{3}(1+(m-1)v_i-\lceil m v_i\rceil) = \sum_{i=1}^{3}(1+m v_i-\lceil m v_i\rceil) - \sum_{i=1}^{3} v_i > \min\{\tfrac{1}{13}, \epsilon\}$$

for any $m \in [2, r/\gcd(k_0, r) - 1] \cap \mathbb{Z}_{\geq 1}$. By Lemma 6.2, $r/\gcd(k_0, r)$ belongs to a finite set. In particular, $w := (v_1, v_2, v_3)$ belongs to a finite set, hence $\mathrm{mld}(Y \ni y)$ belongs to a finite set. Suppose that $B = \sum_i b_i B_i$, where B_i are the irreducible components of B and $b_i \in \Gamma_{>0}$, and $B_i \cong (f_i = 0)|_Y$ for some semi-invariant analytic function $f_i \in \mathbb{C}\{x_1, x_2, x_3\}$. Let E be the unique divisor over $X \ni x$ such that $a(E, X, 0) \leq 1$. Then by Lemma 3.18,

$$\begin{aligned}\mathrm{mld}(X \ni x, B) = a(E, X, B) = a(E, X, 0) - \mathrm{mult}_E B = a(E, X, 0) - \sum_i b_i \,\mathrm{mult}_E B_i \\ = \mathrm{mld}(Y \ni y) - \sum_i b_i w(f_i)\end{aligned}$$

belongs to an ACC set (resp. finite set). □

6.2 Enc cDV quotient singularities

In this subsection, we prove Theorem 1.4 when X is noncanonical, with isolated singularities, and the index 1 cover of X is cDV (see Theorem 6.8).

Theorem 6.4 *Let r be a positive integer, a, b, c, d integers, and $f \in \mathbb{C}\{x_1, x_2, x_3, x_4\}$ an analytic function, such that*

$$(X \ni x) \cong \big((f = 0) \subset (\mathbb{C}^4 \ni 0)\big)\big/\tfrac{1}{r}(a, b, c, d)$$

is an enc threefold isolated singularity. Let

$$N := \Big\{w \in \mathbb{Q}^4_{\geq 0} \;\Big|\; w \equiv \frac{1}{r}(ja, jb, jc, jd) \bmod \mathbb{Z}^4 \text{ for some } j \in \mathbb{Z}\Big\} \setminus \{\mathbf{0}\}.$$

Then there exists at most one primitive vector $\beta \in N$, such that $t := \beta(x_1x_2x_3x_4) - \beta(f) \leq 1$.

In particular, for any $\alpha \in N \backslash \{\beta, 2\beta, \dots, (k-1)\beta\}$, $\alpha(x_1x_2x_3x_4) - \alpha(f) > 1$, where $k := \lfloor \frac{1}{t} \rfloor + 1$.

Proof Assume that there exists a primitive vector $\beta \in N$ such that $\beta(x_1x_2x_3x_4) - \beta(f) \leq 1$. It suffices to show that such β is unique.

Let $Z := \mathbb{C}^4/\frac{1}{r}(a, b, c, d)$, and $\phi_\beta\colon Z_\beta \to Z$ the toric morphism induced by β which extracts an exceptional divisor E_β. Let X_β be the strict transform of X on Z_β. By [Jiang 2021, Proposition 2.1], we have

$$K_{Z_\beta} + X_\beta + (1 - t)E_\beta = \phi_\beta^*(K_Z + X).$$

Since $X \ni x$ is an isolated klt singularity and $\dim X = 3$, (Z, X) is plt by inversion of adjunction. Thus $(Z_\beta, X_\beta + (1 - t)E_\beta)$ is plt. By the adjunction formula,

$$K_{X_\beta} + B_\beta := (K_{Z_\beta} + X_\beta + (1 - t)E_\beta)|_{X_\beta} = \phi_\beta^* K_X,$$

for some $B_\beta \geq 0$, and the coefficients of B_β are of the form $1 - (1 - s(1 - t))/l$ for some positive integers l, s as E_β intersects X_β. Since X is enc, $\mathrm{Supp}\, B_\beta$ is a prime divisor, say F_β.

Let v_{F_β} be the divisorial valuation of F_β. Thus $v_{F_\beta}(\boldsymbol{x}^{\boldsymbol{m}}) = (1-(1-s(1-t))/l)\beta(\boldsymbol{x}^{\boldsymbol{m}})$ for any monomial $\boldsymbol{x}^{\boldsymbol{m}}$, where $\boldsymbol{m} \in M$, and M is the dual sublattice of $\mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{r}(a,b,c,d)$. Hence such β is unique by the primitivity. $\square$

We introduce the following setting. Roughly speaking, Theorem 6.6 below will show that if

$$(X \ni x) \cong (f=0) \subset (\mathbb{C}^4 \ni 0) \Big/ \frac{1}{r}(a_1,a_2,a_3,a_4)$$

is enc and a cyclic quotient of an isolated cDV singularity, then f, a_i, r, e, k, and β should satisfy Setting 6.5. Therefore, we can transform the ACC conjecture in this case to computations on variables that satisfy Setting 6.5.

Setting 6.5 *We set up the following notation and conditions.*

(1) *Let r be a positive integer, $0 \le a_1,a_2,a_3,a_4,e < r$ integers, such that*
  - (a) $\gcd(a_i,r) \mid \gcd(e,r)$ *for any* $1 \le i \le 4$,
  - (b) $\gcd(a_i,a_j,r) = 1$ *for any* $1 \le i < j \le 4$,
  - (c) $\sum_{i=1}^4 a_i - e \equiv 1 \mod r$.

(2) $f \in \mathbb{C}\{x_1,x_2,x_3,x_4\}$ *is $\boldsymbol{\mu}$-semi-invariant, that is, $\boldsymbol{\mu}(f) = \xi^e f$, and is one of the following 3 types,*
  - (a) *(cA type)* $f = x_1x_2 + g(x_3,x_4)$ *with* $g \in \mathfrak{m}^2$,
  - (b) *(Odd type)* $f = x_1^2 + x_2^2 + g(x_3,x_4)$ *with* $g \in \mathfrak{m}^3$ *and* $a_1 \not\equiv a_2 \mod r$,
  - (c) *(cD-E type)* $f = x_1^2 + g(x_2,x_3,x_4)$ *with* $g \in \mathfrak{m}^3$,

where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{x_1,x_2,x_3,x_4\}$, and $\boldsymbol{\mu}\colon \mathbb{C}^4 \to \mathbb{C}^4$ is the action $(x_1,x_2,x_3,x_4) \to (\xi^{a_1}x_1, \xi^{a_2}x_2, \xi^{a_3}x_3, \xi^{a_4}x_4)$.

(3) *One of the two cases hold:*
  - (a) $\alpha(x_1x_2x_3x_4) - \alpha(f) > 1$ *for any $\alpha \in N$. In this case, we let $k := 1$ and $\beta := \boldsymbol{0}$.*
  - (b) *There exists an integer $k \ge 2$, and a primitive vector $\beta \in N$, such that*
    - (i) • *either* $\frac{1}{k} < \beta(x_1x_2x_3x_4) - \beta(f) \le \min\{\frac{12}{13}, \frac{1}{k-1}\}$, *or*
      • $\beta(x_1x_2x_3x_4) - \beta(f) = 1$ *and* $k = 2$,

      and
    - (ii) *for any* $\alpha \in N \backslash \{\beta, 2\beta, \dots, (k-1)\beta\}$, $\alpha(x_1x_2x_3x_4) - \alpha(f) > 1$,

where

$$N := \Big\{ w \in \mathbb{Q}^4_{\ge 0} \;\Big|\; w \equiv \frac{1}{r}(ja_1, ja_2, ja_3, ja_4) \bmod \mathbb{Z}^4 \text{ for some } j \in \mathbb{Z} \Big\} \setminus \{\boldsymbol{0}\}.$$

Moreover, if f is of cA type, then for any integer a such that $\gcd(a,r)=1$, $\frac{1}{r}(a_1,a_2,a_3,a_4,e) \not\equiv \frac{1}{r}(a,-a,1,0,0) \mod \mathbb{Z}^5$.

Theorem 6.6 *Let r be a positive integer, $0 \le a_1, a_2, a_3, a_4, e < r$ integers, $\xi := e^{2\pi i/r}$,*

$$N := \left\{ w \in \mathbb{Q}^4_{\ge 0} \mid w \equiv \frac{1}{r}(ja_1, ja_2, ja_3, ja_4) \bmod \mathbb{Z}^4 \text{ for some } j \in \mathbb{Z} \right\} \setminus \{\mathbf{0}\},$$

$\boldsymbol{\mu}: \mathbb{C}^4 \to \mathbb{C}^4$ the action $(x_1, x_2, x_3, x_4) \to (\xi^{a_1} x_1, \xi^{a_2} x_2, \xi^{a_3} x_3, \xi^{a_4} x_4)$, and $f \in \mathbb{C}\{x_1, x_2, x_3, x_4\}$ a $\boldsymbol{\mu}$-semi-invariant analytic function such that $\boldsymbol{\mu}(f) = \xi^e f$. Suppose that

$$(X \ni x) \cong (f = 0) \subset (\mathbb{C}^4 \ni 0) \Big/ \frac{1}{r}(a_1, a_2, a_3, a_4)$$

be a hyperquotient singularity such that

- *$(Y \ni y) := (f = 0) \cong (\mathbb{C}^4 \ni 0)$ is an isolated cDV singularity,*
- *$\pi: (Y \ni y) \to (X \ni x)$ is the index one cover, and*
- *$(X \ni x)$ is enc,*

then possibly replacing $\frac{1}{r}(a_1, a_2, a_3, a_4)$ with $(\{ja_1/r\}, \{ja_2/r\}, \{ja_3/r\}, \{ja_4/r\})$ for some j such that $\gcd(j, r) = 1$, and taking a $\boldsymbol{\mu}$-equivariant analytic change of coordinates and possibly permuting the coordinates x_i, we have that a_i, e, r, f satisfy Setting 6.5.

Proof By [Reid 1987, Page 394], since $\boldsymbol{\mu}$ acts freely outside y, a_i, e, r satisfy Setting 6.5(1)(a) and (1)(b). Let $s \in \omega_Y$ be a generator, then $\boldsymbol{\mu}$ acts on s by $s \to \xi^{\sum_{i=1}^4 a_i - e} s$. Since the Cartier index of K_X near x is r, $\gcd\left(\sum_{i=1}^4 a_i - e, r\right) = 1$, and a_i, e, r satisfy Setting 6.5(1)(c). By [Reid 1987, pages 394–395 and Proposition (6.7)] (see also [Jiang 2021, Proposition 4.2]), f satisfies Setting 6.5(2).

By Theorem 6.4, in order to show that f satisfies Setting 6.5(3), we only need to prove that

$$\beta(x_1 x_2 x_3 x_4) - \beta(f) \notin \left(\tfrac{12}{13}, 1\right).$$

We may assume that $r > 13$. By [Liu and Xiao 2021, Theorem 1.6, Lemmas 6.3 and 6.4] and [Jiang 2021, Lemma 2.12, Remark 2.13], if $\beta(x_1 x_2 x_3 x_4) - \beta(f) \in \left(\tfrac{12}{13}, 1\right)$, then $(X \ni x)$ and β satisfy [Jiang 2021, Section 4, Rules I–III], which is absurd according to [Jiang 2021, Section 4].

It suffices to show that a_i, e, r satisfy the "moreover" part of Setting 6.5. By [Kollár and Shepherd-Barron 1988, Theorem 6.5], if $\frac{1}{r}(a_1, a_2, a_3, a_4, e) \equiv \frac{1}{r}(a, -a, 1, 0, 0) \bmod \mathbb{Z}^5$, then $X \ni x$ is a terminal singularity, which leads to a contradiction. □

Theorem 6.7 *With notation and conditions as in Setting 6.5, either r or $\beta \neq \mathbf{0}$ belongs to a finite set depending only on k. In particular, $\beta(x_1 x_2 x_3 x_4) - \beta(f)$ belongs to a finite set depending only on k, and $\beta(g)$ belongs to a discrete set for any analytic function g.*

Proof Proving that either r or $\beta \neq \mathbf{0}$ belongs to a finite set depending only on k is elementary but requires complicated computations, so we omit the proof and refer the reader to [Han and Liu 2025, Theorem 1.2] (which was Theorem A.1 of the first arXiv version[2] of the present paper).

[2] See arXiv:2209.13122v1.

We are left to prove the "in particular"-part. There exists a positive integer n depending only on k such that $n\beta \in \mathbb{Z}^4_{\geq 0}$. Thus $\beta(g)$ belongs to the discrete set $\frac{1}{n}\mathbb{Z}_{\geq 0}$ for any analytic function g. Since $\beta(x_1x_2x_3x_4)-\beta(f) \in (0,1]$, $\beta(x_1x_2x_3x_4)-\beta(f)$ belongs to the finite set $\frac{1}{n}\mathbb{Z}_{\geq 0} \cap (0,1]$. □

Theorem 6.8 *Let* $\Gamma \subset [0,1]$ *be a DCC (resp. finite) set. Assume that* $(X \ni x, B)$ *is a* $\mathbb{Q}$*-factorial enc pair of dimension* 3, *such that*

(1) $X \ni x$ *is an isolated noncanonical singularity*,

(2) $\operatorname{coeff}(B) \subseteq \Gamma$, *and*

(3) $\widetilde{X} \ni \tilde{x}$ *is terminal but not smooth*, *where* $\pi\colon (\widetilde{X} \ni \tilde{x}) \to (X \ni x)$ *is the index* 1 *cover of* $X \ni x$.

Then $\operatorname{mld}(X \ni x, B)$ *belongs to an ACC set (resp. is discrete away from* 0).

Proof We only need to show that for any positive integer $l \geq 2$, if $\operatorname{mld}(X \ni x) \in (\frac{1}{l}, \frac{1}{l-1}]$, then $\operatorname{mld}(X \ni x, B)$ belongs to an ACC set (resp. is discrete away from 0).

There exists a positive integer r, integers $0 \leq a_1, a_2, a_3, a_4, e$, and $\xi := e^{2\pi i/r}$, such that

$$(X \ni x) \cong \big((f=0) \subset (\mathbb{C}^4 \ni 0)\big)/\boldsymbol{\mu},$$

where $\boldsymbol{\mu}\colon \mathbb{C}^4 \to \mathbb{C}^4$ is the action $(x_1,x_2,x_3,x_4) \to (\xi^{a_1}x_1, \xi^{a_2}x_2, \xi^{a_3}x_3, \xi^{a_4}x_4)$ and f is $\boldsymbol{\mu}$-semi-invariant, such that $\boldsymbol{\mu}(f) = \xi^e f$. By Setting 6.5(1)(c), possibly replacing (a_1,a_2,a_3,a_4) and e, we may assume that $a_1+a_2+a_3+a_4-e \equiv 1 \bmod r$. Moreover, possibly shrinking X to a neighborhood of x, we may write $B = \sum_{i=1}^m b_i B_i$ where B_i are the irreducible components of B and $x \in \operatorname{Supp} B_i$ for each i. Then $b_i \in \Gamma$, and we may identify B_i with $((f_i = 0) \subset (\mathbb{C}^4 \ni 0))/\boldsymbol{\mu}|_X$ for some $\boldsymbol{\mu}$-semi-invariant function f_i for each i.

Let

$$N := \Big\{w \in \mathbb{Q}^4_{\geq 0} \;\Big|\; w = \frac{1}{r}(ja_1, ja_2, ja_3, ja_4) \bmod \mathbb{Z}^4 \text{ for some } j \in \mathbb{Z}\Big\} \setminus \{\mathbf{0}\}.$$

By Setting 6.5(3), there are two cases:

Case 1 $\alpha(x_1x_2x_3x_4) - \alpha(f) > 1$ for any $\alpha \in N$. In this case, by Theorems 6.6, 6.7, r belongs to a finite set. Since $(X \ni x, B)$ is enc and $X \ni x$ is noncanonical, there exists a unique prime divisor E over $X \ni x$, such that $a(E,X,B) = \operatorname{mld}(X \ni x, B)$ and $a(E,X,0) < 1$. Since rK_X is Cartier, $r\,a(E,X,0)$ belongs to a finite set and $r \operatorname{mult}_E B_i \in \mathbb{Z}_{\geq 1}$ for each i. Thus

$$a(E,X,B) = a(E,X,0) - \sum_{i=1}^m b_i \operatorname{mult}_E B_i$$

belongs to an ACC set (resp. finite set).

Case 2 There exists a unique primitive vector $\beta \in N$ and an integer $k \geq 2$, such that

$$\frac{1}{k} < \beta(x_1x_2x_3x_4) - \beta(f) \leq \frac{1}{k-1}.$$

We consider the pair

$$(Z \ni z, X + B_Z) := \left(\mathbb{C}^4 \ni 0, (f = 0) + \sum_{i=1}^{m} b_i (f_i = 0)\right) \Big/ \boldsymbol{\mu}.$$

By [Jiang 2021, Proposition 2.1], the primitive vector $\beta \in N$ corresponds to a divisor E over $Z \ni z$, and

$$a := a(E, Z, X + B_Z) = \beta(x_1x_2x_3x_4) - \beta(f) - \sum_{i=1}^{m} b_i \beta(f_i = 0). \tag{6-1}$$

In particular, $0 < a \leq 1/(k-1) \leq 1$. Let $h\colon W \to Z$ be the birational morphism which extracts E. Then

$$K_W + X_W + B_W + (1-a)E = h^*(K_Z + X + B_Z),$$

where X_W and B_W are the strict transforms of X and B on W, respectively.

Since (X, B) is klt near x, $X \ni x$ is an isolated singularity, and $\dim X = 3$, $(Z, X + B_Z)$ is plt by the inversion of adjunction. Thus $(W, X_W + B_W + (1-a)E)$ is plt. Since h is a divisorial contraction of E and $\operatorname{center}_Z E = x$, E is $\mathbb{Q}$-Cartier, and $\operatorname{Supp}(E \cap X_W)$ contains a prime divisor F which does not belong to $\operatorname{Supp}(B_W \cap E)$. By the adjunction formula,

$$a(F, X, B) = \frac{1}{n}(1 - s(1-a)) \leq a \leq \frac{1}{k-1}$$

for some positive integers n, s. Since $(X \ni x, B)$ is enc, $a(F, X, B) = \operatorname{mld}(X \ni x, B) > \frac{1}{l}$, hence $k \leq l$. Thus k belongs to a finite set. By Theorems 6.6 and 6.7, $a \leq 1$ belongs to an ACC set (resp. finite set). Thus

$$\operatorname{mld}(X \ni x, B) = a(F, X, B) = \frac{1}{n}(1 - s(1-a))$$

belongs to an ACC set (resp. is discrete away from 0), and we are done. □

7 Proofs of other main results

Lemma 7.1 *For any positive integer l, Theorem E_l implies Theorem N_l.*

Proof This follows from Lemma 4.3. □

Lemma 7.2 *For any positive integer l, Theorem C_l implies Theorem E_l.*

Proof Let $(X, B) \in \mathcal{E}(l, \Gamma)$, and E the unique exceptional prime divisor over (X, B) such that $a(E, X, B) = \operatorname{mld}(X, B)$.

By Lemma 5.1, we may assume that $x := \mathrm{center}_X\, E$ is a closed point. By Theorem 5.7, we may assume that either X is terminal and Γ is a finite set, or X is not terminal. By Theorem 5.6, we may assume that X is not terminal. By Theorem 5.2, we may assume that X is not canonical. Let $(\widetilde{X} \ni \tilde{x}) \to (X \ni x)$ be the index 1 cover of $X \ni x$. Then $\widetilde{X} \ni \tilde{x}$ is smooth, or an isolated cDV singularity, or a strictly canonical singularity. By Theorems 6.3 and 6.8, we may assume that $\widetilde{X} \ni \tilde{x}$ is strictly canonical. By Theorem C_l, $\mathrm{mld}(X, B)$ belongs to an ACC set. $\square$

Lemma 7.3 *For any positive integer $l \geq 2$, Theorem N_{l-1} implies Theorem C_l.*

Proof Let $(X \ni x, B) \in \mathcal{C}(l, \Gamma)$, $\pi\colon (\widetilde{X} \ni \tilde{x}) \to (X \ni x)$ the index 1 cover of $X \ni x$, and $\widetilde{B} := \pi^* B$. Then $\mathrm{coeff}(\widetilde{B}) \subseteq \Gamma$.

Since $\mathrm{mld}(X) < 1$, there exists an exceptional prime divisor E over X, such that $a(E, X, B) \leq a(E, X, 0) = \mathrm{mld}(X) < 1$. Thus E is the unique exceptional prime divisor over X such that $a(E, X, B) \leq 1$ as $(X \ni x, B)$ is enc. In particular, $a(E, X, B) = \mathrm{mld}(X \ni x, B)$. Hence for any exceptional prime divisor $\widetilde{E}$ over $\widetilde{X}$ such that $a(\widetilde{E}, \widetilde{X}, \widetilde{B}) \leq 1$, we have $a(\widetilde{E}, \widetilde{X}, \widetilde{B}) = r_{\widetilde{E}}\, a(E, X, B)$, where $r_{\widetilde{E}}$ is the ramification index of π along $\widetilde{E}$. Since $(\widetilde{X} \ni \tilde{x})$ is canonical,

$$1 \leq a(\widetilde{E}, \widetilde{X}, 0) = r_{\widetilde{E}}\, a(E, X, 0) < r_{\widetilde{E}},$$

so $r_{\widetilde{E}} \geq 2$ for any $\widetilde{E}$. It follows that

$$\mathrm{mld}(\widetilde{X}, \widetilde{B}) \in \bigl\{a(\widetilde{E}, \widetilde{X}, \widetilde{B}) \leq 1 \mid \widetilde{E} \text{ is exceptional over } \widetilde{X}\bigr\} \subseteq \bigl\{2\,\mathrm{mld}(X, B), \dots, (l-1)\,\mathrm{mld}(X, B)\bigr\}$$

as $\mathrm{mld}(X, B) > \frac{1}{l}$ and $(\widetilde{X} \ni \tilde{x})$ is strictly canonical. In particular,

$$1 \leq \#\bigl(\{a(\widetilde{E}, \widetilde{X}, \widetilde{B}) \mid \widetilde{E} \text{ is exceptional over } \widetilde{X}\} \cap [0, 1]\bigr) \leq l - 2.$$

Moreover, since $a(\widetilde{E}, \widetilde{X}, \widetilde{B}) = r_{\widetilde{E}}\, a(E, X, B) > \frac{2}{l} \geq 1/(l-1)$, we have $\mathrm{mld}(\widetilde{X}, \widetilde{B}) > 1/(l-1)$. Thus by Theorem N_{l-1}, $\mathrm{mld}(\widetilde{X}, \widetilde{B})$ belongs to an ACC set, which implies that $\mathrm{mld}(X, B)$ also belongs to an ACC set. $\square$

Proof of Theorems E, N, and C These follow from Lemmas 7.1, 7.2, 7.3. $\square$

Proof of Theorem 1.6 By [Han et al. 2022, Theorem 1.1], we may assume that $\mathrm{mld}(X, B) < 1$. Now the theorem follows from Theorem N. $\square$

Proof of Theorem 1.4 This follows from Theorem 1.6. $\square$

Corollary 7.4 (Corollary 1.5) *Any sequence of lc flips*

$$(X, B) := (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow \cdots (X_i, B_i) \dashrightarrow \cdots$$

terminates in dimension 3.

Proof We only need to check the conditions of Theorem 4.8 when $d = 3$. Theorem 4.8(1) follows from [Nakamura 2016, Corollary 1.5] (see also [Han et al. 2022, Theorem 1.1]), Theorem 4.8(2) follows from [Ambro 1999, Main Theorem 1] (see also [Nakamura and Shibata 2021, Theorem 1.2]), and Theorem 4.8(3) follows from Theorem 1.4. $\square$

We conjecture that Corollary 1.7 generalizes to high dimensions:

Conjecture 7.5 *Let N be a nonnegative integer, d a positive integer, and $\Gamma \subset [0,1]$ a DCC set. Then there exists an ACC set Γ' depending only on d, N and Γ satisfying the following. Assume that (X, B) is a klt pair of dimension d, such that*

(1) $\operatorname{coeff}(B) \subseteq \Gamma$, *and*

(2) *there are at most N different (exceptional) log discrepancies of (X, B) that are ≤ 1, ie*

$$\#\big(\{a(E, X, B) \mid E \text{ is exceptional over } X\} \cap [0,1]\big) \leq N,$$

then $\{a(E, X, B) \mid E$ is exceptional over $X\} \cap [0,1] \subset \Gamma'$.

Theorem 7.6 *Assume that Conjecture 1.9(1) holds in dimension d. Then Conjecture 7.5 holds in dimension d.*

Proof This follows from Lemma 4.3. $\square$

Proof of Corollary 1.7 This follows from Theorems 1.4 and 7.6. $\square$

Finally, we show the following theorem for independent interest. Theorem 7.7 implies that in order to show the 1-gap conjecture for mlds (see [Chen et al. 2021, Conjecture 5.4]), it suffices to show the 1-gap conjecture for mlds of enc pairs. We note that the 1-gap conjecture has a close relation with the birational boundedness of rationally connected klt Calabi–Yau varieties; see [Chen et al. 2021, Corollary 5.5; Han and Jiang 2024].

Theorem 7.7 *Let d be a positive integer, and $\Gamma \subset [0,1]$ a set. Then*

$$\begin{aligned}\sup\big\{\operatorname{mld}(X, B) < 1 \mid (X, B) \text{ is } \mathbb{Q}\text{-factorial enc}, \dim X = d, \operatorname{coeff}(B) \subseteq \Gamma\big\} \\ = \sup\big\{\operatorname{mld}(X, B) < 1 \mid (X, B) \text{ is klt}, \dim X = d, \operatorname{coeff}(B) \subseteq \Gamma\big\}.\end{aligned}$$

Proof Let (X, B) be a klt pair such that $\dim X = d$, $\operatorname{coeff}(B) \subseteq \Gamma$, and $\operatorname{mld}(X, B) < 1$. By [Kollár and Mori 1998, Proposition 2.36], we may assume that $E_1, E_2, \dots, E_k$ are all exceptional prime divisors over X, such that $a(E_i, X, B) < 1$. By [Liu 2018, Lemma 5.3], there exist $1 \leq i \leq k$ and a birational morphism $f : Y \to X$ which extracts exactly all $E_1, E_2, \dots, E_k$ but E_i, such that $1 > a(E_i, Y, f_*^{-1}B) \geq a(E_i, X, B)$. Possibly replacing Y with a small $\mathbb{Q}$-factorialization, we may assume that Y is $\mathbb{Q}$-factorial. Then $(Y, f_*^{-1}B)$ is a $\mathbb{Q}$-factorial enc pair with $1 > \operatorname{mld}(Y, f_*^{-1}B) \geq \operatorname{mld}(X, B)$, and we are done. $\square$

8 Further remarks

Remark 8.1 (history of enc pairs) We briefly introduce some history on the study of enc pairs. In [Liu 2018, Lemma 5.4], a class of pairs similar to enc pairs, that is, pairs (X, B) such that $\mathrm{mld}(X, B) < a$ and there exists only 1 exceptional divisor with log discrepancy $\leq a$ with respect to (X, B), was constructed. When $a = 1$, these are exactly enc pairs. However, [Liu 2018] only deals with the case when $a < 1$ and does not deal with the case when $a = 1$. [Jiang 2021, Definition 2.1] first formally introduced enc varieties X, naming them "extremely noncanonical". In dimension 3 and when $\mathrm{mld}(X) \to 1$, [Jiang 2021] systematically studied the singularities of these varieties, which played a crucial role in his proof of the 1-gap conjecture for threefolds. [Han et al. 2022] introduced enc pairs (X, B) to prove the 1-gap conjecture for threefold pairs.

Remark 8.2 (enc pairs and exceptional Fano pairs) We explain why we use the notation "exceptionally noncanonical" instead of "extremely noncanonical" as in [Jiang 2021]. The key reason is that, as suggested by Shokurov, we expect exceptionally noncanonical singularities in dimension d to have connections with the global lc thresholds in dimension $d-1$, while the latter is known to have connections with the mlds of exceptional pairs in dimension $d-1$ [Liu 2023, Theorem 1.2] (see [Han et al. 2024; Shokurov 2020]). Shokurov suggested us that the role of enc singularities in the study of klt singularities may be as important as the role of exceptional pairs in the study of Fano varieties (see [Birkar 2019]).

Theorem 7.7 could provide some evidence of this for us: when $d = 3$ and $\Gamma = \{0\}$, the 1-gap of mld is equal to $\frac{1}{13}$ (Theorem 3.15) and is reached at an enc cyclic quotient singularity $\frac{1}{13}(3, 4, 5)$. If we let $f : Y \to X$ be the divisorial contraction which extracts the unique prime divisor E over $X \ni x$ such that $a(E, X, 0) = \mathrm{mld}(X \ni x)$, then E is normal and

$$\left(\mathbb{P}(3,4,5), \tfrac{12}{13}(x_1^3x_2 + x_2^2x_3 + x_3^2x_1 = 0)\right) \cong (E, B_E),$$

where $K_E + B_E \sim_{\mathbb{Q}} f^*K_X|_E = \left(K_Y + \frac{1}{13}E\right)|_E$. On the other hand, $\frac{12}{13}$ is also expected to be[3] the largest surface global lc threshold [Liu 2023, Remark 2.5; Alexeev and Liu 2019, Notation 4.1] and can be reached by the same pair $\left(\mathbb{P}(3,4,5), \frac{12}{13}(x_1^3x_2 + x_2^2x_3 + x_3^2x_1 = 0)\right)$ [Kollár 2013, 40].

Remark 8.3 (enc pairs, Calabi–Yau varieties, and mirror symmetry) Enc pairs also have a deep relationship with Calabi–Yau varieties in different ways.

First, by Theorem 7.7, the 1-gap conjecture for mlds of enc pairs implies the 1-gap conjecture of mlds, while the latter will imply the birational boundedness of rationally connected Calabi–Yau varieties by applying similar arguments as in [Han and Jiang 2024, proof of Theorem 1.2].

Second, as mentioned in Remark 8.2, the mlds of enc pairs have connections with the global lc thresholds, while the latter is related to the minimal possible mld of klt Calabi–Yau varieties. Indeed, the 1-gap of the

[3] It is proven in [Liu and Shokurov 2023, Theorem 1.1] after the first version of this paper appeared.

mlds of enc pairs is always smaller than or equal to the minimal possible mld of klt Calabi–Yau varieties of smaller dimensions, and they are expected to be the same (see [Esser et al. 2022, Proposition 6.1]). Finally, the second author was informed by Chengxi Wang that the klt Calabi–Yau variety with minimal possible mld should be associated with a klt Calabi–Yau variety with maximal possible index by mirror symmetry (see [Esser et al. 2022, Proposition 6.1]).

References

[Alexeev 1993] **V Alexeev**, *Two two-dimensional terminations*, Duke Math. J. 69 (1993) 527–545 MR Zbl

[Alexeev and Liu 2019] **V Alexeev**, **W Liu**, *Open surfaces of small volume*, Algebr. Geom. 6 (2019) 312–327 MR Zbl

[Alexeev et al. 2007] **V Alexeev**, **C Hacon**, **Y Kawamata**, *Termination of (many) 4-dimensional log flips*, Invent. Math. 168 (2007) 433–448 MR Zbl

[Ambro 1999] **F Ambro**, *On minimal log discrepancies*, Math. Res. Lett. 6 (1999) 573–580 MR Zbl

[Ambro 2006] **F Ambro**, *The set of toric minimal log discrepancies*, Cent. Eur. J. Math. 4 (2006) 358–370 MR Zbl

[Birkar 2019] **C Birkar**, *Anti-pluricanonical systems on Fano varieties*, Ann. of Math. 190 (2019) 345–463 MR Zbl

[Birkar and Zhang 2016] **C Birkar**, **D-Q Zhang**, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Publ. Math. Inst. Hautes Études Sci. 123 (2016) 283–331 MR Zbl

[Birkar et al. 2010] **C Birkar**, **P Cascini**, **C D Hacon**, **J McKernan**, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23 (2010) 405–468 MR Zbl

[Chen 2022] **J-J Chen**, *Accumulation points on 3-fold canonical thresholds*, preprint (2022) arXiv 2202.06230 To appear in J. Math. Soc. Japan

[Chen and Han 2020] **G Chen**, **J Han**, *Boundedness of (ϵ, n)-complements for surfaces*, preprint (2020) arXiv 2002.02246 Short version published in Adv. Math. 383 (2021) art. id. 107703

[Chen and Tsakanikas 2023] **G D Chen**, **N Tsakanikas**, *On the termination of flips for log canonical generalized pairs*, Acta Math. Sin. (Engl. Ser.) 39 (2023) 967–994 MR Zbl

[Chen et al. 2021] **W Chen**, **G Di Cerbo**, **J Han**, **C Jiang**, **R Svaldi**, *Birational boundedness of rationally connected Calabi–Yau 3-folds*, Adv. Math. 378 (2021) art. id. 107541 MR Zbl

[Chen et al. 2024] **W Chen**, **Y Gongyo**, **Y Nakamura**, *On generalized minimal log discrepancy*, J. Math. Soc. Japan 76 (2024) 393–449 MR

[Ein et al. 2003] **L Ein**, **M Mustaţă**, **T Yasuda**, *Jet schemes, log discrepancies and inversion of adjunction*, Invent. Math. 153 (2003) 519–535 MR Zbl

[Esser et al. 2022] **L Esser**, **B Totaro**, **C Wang**, *Calabi–Yau varieties of large index*, preprint (2022) arXiv 2209.04597

[Fujino 2004] **O Fujino**, *Termination of 4-fold canonical flips*, Publ. Res. Inst. Math. Sci. 40 (2004) 231–237 MR Zbl Addendum in Publ. Res. Inst. Math. Sci. 41 (2005) 252–257

[Fujino 2007] **O Fujino**, *Special termination and reduction to pl flips*, from "Flips for 3-folds and 4-folds" (A Corti, editor), Oxford Lect. Ser. Math. Appl. 35, Oxford Univ. Press (2007) 63–75 MR Zbl

[Fujino 2017] **O Fujino**, *Foundations of the minimal model program*, MSJ Mem. 35, Math. Soc. Japan, Tokyo (2017) MR Zbl

[Fulton 1984] **W Fulton**, *Intersection theory*, Ergebnisse der Math. 2, Springer (1984) MR Zbl

[Hacon and Liu 2023] **C D Hacon**, **J Liu**, *Existence of flips for generalized lc pairs*, Camb. J. Math. 11 (2023) 795–828 MR Zbl

[Hacon et al. 2014] **C D Hacon**, **J McKernan**, **C Xu**, *ACC for log canonical thresholds*, Ann. of Math. 180 (2014) 523–571 MR Zbl

[Han and Jiang 2024] **J Han**, **C Jiang**, *Birational boundedness of rationally connected log Calabi–Yau pairs with fixed index*, Algebraic Geom. Phys. 1 (2024) 59–79

[Han and Li 2022] **J Han**, **Z Li**, *Weak Zariski decompositions and log terminal models for generalized pairs*, Math. Z. 302 (2022) 707–741 MR Zbl

[Han and Liu 2025] **J Han**, **J Liu**, *Classification of threefold enc cDV quotient singularities*, preprint (2025) arXiv 2501.01024

[Han and Luo 2023] **J Han**, **Y Luo**, *On boundedness of divisors computing minimal log discrepancies for surfaces*, J. Inst. Math. Jussieu 22 (2023) 2907–2930 MR Zbl

[Han and Luo 2024] **J J Han**, **Y J Luo**, *A simple proof of ACC for minimal log discrepancies for surfaces*, Acta Math. Sin. (Engl. Ser.) 40 (2024) 425–434 MR Zbl

[Han et al. 2021] **J Han**, **Z Li**, **L Qi**, *ACC for log canonical threshold polytopes*, Amer. J. Math. 143 (2021) 681–714 MR Zbl

[Han et al. 2022] **J Han**, **J Liu**, **Y Luo**, *ACC for minimal log discrepancies of terminal threefolds*, preprint (2022) arXiv 2202.05287

[Han et al. 2023] **J Han**, **Y Liu**, **L Qi**, *ACC for local volumes and boundedness of singularities*, J. Algebraic Geom. 32 (2023) 519–583 MR Zbl

[Han et al. 2024] **J Han**, **J Liu**, **V V Shokurov**, *ACC for minimal log discrepancies of exceptional singularities*, Peking Math. J. (online publication November 2024)

[Hayakawa 1999] **T Hayakawa**, *Blowing ups of 3-dimensional terminal singularities*, Publ. Res. Inst. Math. Sci. 35 (1999) 515–570 MR Zbl

[Jiang 2021] **C Jiang**, *A gap theorem for minimal log discrepancies of noncanonical singularities in dimension three*, J. Algebraic Geom. 30 (2021) 759–800 MR Zbl

[Kawakita 2005] **M Kawakita**, *Three-fold divisorial contractions to singularities of higher indices*, Duke Math. J. 130 (2005) 57–126 MR Zbl

[Kawakita 2015a] **M Kawakita**, *A connectedness theorem over the spectrum of a formal power series ring*, Int. J. Math. 26 (2015) art. id. 1550088 MR Zbl

[Kawakita 2015b] **M Kawakita**, *The index of a threefold canonical singularity*, Amer. J. Math. 137 (2015) 271–280 MR Zbl

[Kawamata 1988] **Y Kawamata**, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. 127 (1988) 93–163 MR Zbl

[Kawamata 1992a] **Y Kawamata**, *The minimal discrepancy of a 3-fold terminal singularity* (1992) Zbl Appendix to V V Shokurov, “3-Fold log flips”, Izv. Ross. Akad. Nauk. Ser. Mat. 56 (1992) 105–203 (In Russian; translated in Izv. Math. 40 (1992) 93–202)

[Kawamata 1992b] **Y Kawamata**, *Termination of log flips for algebraic 3-folds*, Int. J. Math. 3 (1992) 653–659 MR Zbl

[Kawamata et al. 1987] **Y Kawamata**, **K Matsuda**, **K Matsuki**, *Introduction to the minimal model problem*, from "Algebraic geometry" (T Oda, editor), Adv. Stud. Pure Math. 10, North-Holland, Amsterdam (1987) 283–360 MR Zbl

[Kollár 1992] **J Kollár** (editor), *Flips and abundance for algebraic threefolds*, Astérisque 211, Soc. Math. France, Paris (1992) MR

[Kollár 1996] **J Kollár**, *Rational curves on algebraic varieties*, Ergebnisse der Math. 32, Springer (1996) MR Zbl

[Kollár 2013] **J Kollár**, *Moduli of varieties of general type*, from "Handbook of moduli, II" (G Farkas, I Morrison, editors), Adv. Lect. Math. 25, International, Somerville, MA (2013) 131–157 MR Zbl

[Kollár and Mori 1998] **J Kollár**, **S Mori**, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math. 134, Cambridge Univ. Press (1998) MR Zbl

[Kollár and Shepherd-Barron 1988] **J Kollár**, **N I Shepherd-Barron**, *Threefolds and deformations of surface singularities*, Invent. Math. 91 (1988) 299–338 MR Zbl

[Li et al. 2020] **C Li**, **Y Liu**, **C Xu**, *A guided tour to normalized volume*, from "Geometric analysis" (J Chen, P Lu, Z Lu, Z Zhang, editors), Progr. Math. 333, Birkhäuser, Cham (2020) 167–219 MR Zbl

[Liu 2018] **J Liu**, *Toward the equivalence of the ACC for a-log canonical thresholds and the ACC for minimal log discrepancies*, preprint (2018) arXiv 1809.04839

[Liu 2023] **J Liu**, *Remark on complements on surfaces*, Forum Math. Sigma 11 (2023) art. id. e42 MR Zbl

[Liu and Luo 2022] **J Liu**, **Y Luo**, *Second largest accumulation point of minimal log discrepancies of threefolds*, preprint (2022) arXiv 2207.04610

[Liu and Shokurov 2023] **J Liu**, **V V Shokurov**, *Optimal bounds on surfaces*, preprint (2023) arXiv 2305.19248 To appear in Algebraic Geom. and Phys.

[Liu and Xiao 2021] **J Liu**, **L Xiao**, *An optimal gap of minimal log discrepancies of threefold non-canonical singularities*, J. Pure Appl. Algebra 225 (2021) art. id. 106674 MR Zbl

[Markushevich 1996] **D Markushevich**, *Minimal discrepancy for a terminal cDV singularity is* 1, J. Math. Sci. Univ. Tokyo 3 (1996) 445–456 MR Zbl

[Mori 1985] **S Mori**, *On* 3*-dimensional terminal singularities*, Nagoya Math. J. 98 (1985) 43–66 MR Zbl

[Nakamura 2016] **Y Nakamura**, *On minimal log discrepancies on varieties with fixed Gorenstein index*, Michigan Math. J. 65 (2016) 165–187 MR Zbl

[Nakamura and Shibata 2021] **Y Nakamura**, **K Shibata**, *Inversion of adjunction for quotient singularities*, *II*: *Non-linear actions* (2021) arXiv 2112.09502 To appear in Algebr. Geom.

[Nakamura and Shibata 2022] **Y Nakamura**, **K Shibata**, *Inversion of adjunction for quotient singularities*, Algebr. Geom. 9 (2022) 214–251 MR Zbl

[Reid 1987] **M Reid**, *Young person's guide to canonical singularities*, from "Algebraic geometry" (S J Bloch, editor), Proc. Sympos. Pure Math. 46, Amer. Math. Soc., Providence, RI (1987) 345–414 MR Zbl

[Shokurov 1985] **V V Shokurov**, *A nonvanishing theorem*, Izv. Akad. Nauk SSSR Ser. Mat. 49 (1985) 635–651 MR In Russian; translated in Math. USSR-Izv. 26 (1985) 591–604

[Shokurov 1988] **V V Shokurov**, *Problems about Fano varieties*, from "Birational geometry of algebraic varieties", Open Problems 23, Taniguchi Found., Kyoto (1988) 30–32

[Shokurov 1993] **V V Shokurov**, *Semi-stable* 3*-fold flips*, Izv. Ross. Akad. Nauk Ser. Mat. 57 (1993) 165–222 MR

[Shokurov 1994] **V V Shokurov**, *ACC in codimension* 2, personal note (1994)

[Shokurov 1996] **V V Shokurov**, 3*-Fold log models*, J. Math. Sci. 81 (1996) 2667–2699 MR

[Shokurov 2004] **V V Shokurov**, *Letters of a bi-rationalist, V: Minimal log discrepancies and termination of log flips*, Tr. Mat. Inst. Steklova 246 (2004) 328–351 MR In Russian; translated in Proc. Steklov Inst. Math. 2004 (2004) 315–336

[Shokurov 2020] **V V Shokurov**, *Existence and boundedness of n-complements*, preprint (2020) arXiv 2012.06495

[Zhuang 2024] **Z Zhuang**, *On boundedness of singularities and minimal log discrepancies of Kollár components*, J. Algebraic Geom. 33 (2024) 521–565

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