Geometry & Topology

Volume 29 (2025)

Independence of singularity type for numerically effective Kähler–Ricci flows

HOSEA WONDO

ZHOU ZHANG

msp

Geometry & Topology **29**:1 (2025) 479–494
DOI: 10.2140/gt.2025.29.479
Published: 22 January 2025

Independence of singularity type for numerically effective Kähler–Ricci flows

Hosea Wondo
Zhou Zhang

We show that the singularity type of solutions to the Kähler–Ricci flow on a numerically effective manifold does not depend on the initial metric. More precisely, if there exists a type-III solution to the Kähler–Ricci flow, then any other solution starting from a different initial metric will also be type III. This generalizes a previous result by Yashan Zhang for the semiample case, and confirms a conjecture by Valentino Tosatti.

53C55; 32Q15

1 Introduction

The Kähler–Ricci flow is a geometric evolution equation used to study Kähler geometry. It has proven to be a powerful tool in constructing canonical metrics, such as Kähler–Einstein metrics [Cao 1985; Phong et al. 2007; Munteanu and Székelyhidi 2011; Phong and Sturm 2006; Tian 2012].

A primary research direction is the study of the analytic minimal model program in birational geometry [Matsuki 2002], where the flow takes a central role in producing minimal models. Initiated by Song and Tian [2007], the program proposes to emulate the procedure carried out by Hamilton and Perelman in resolving the Poincaré conjecture. The idea is to run the flow on algebraic manifolds and perform "algebraic surgeries" such as flips and contractions each time the flow encounters a singularity [Song and Weinkove 2013a; 2014; Song and Yuan 2012]. After finite iterations of this procedure, the resulting manifold would be numerically effective (nef), and hence a minimal model. At this stage, the normalized version of the flow will evolve the metric to a Kähler–Einstein metric or a generalized Kähler–Einstein metric. Significant progress has been made in this direction; see [Song and Tian 2012; 2016; 2017; Tian 2019; Guo et al. 2016; Fong 2015; Zhang 2019; Tosatti et al. 2018; Tian and Zhang 2016].

Let X be a compact Kähler manifold with numerically effective line bundle, which corresponds to the final step of the analytic minimal model program. A family on Kähler metrics $\omega(t)$ is said to evolve by the normalized Kähler–Ricci flow if

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t), \quad \omega(0) = \omega_0. \tag{1-1}$$

Since the flow is weakly parabolic, we have access to interior estimates, which smooths the metric for short times. In the long run, nonlinearities make the metric nonpositive, forming a singularity. The precise time of singularity formation is determined by the class of the evolving metric in the cohomology given by

$$H^{1,1}_{\bar\partial}(M,\mathbb{R}) = \frac{\{\bar\partial\text{-closed real }(1,1)\text{-forms}\}}{\{\bar\partial\text{-exact real }(1,1)\text{-forms}\}}. \tag{1-2}$$

The singular time, T, is characterized in [Tian and Zhang 2006]:

$$T := \sup\{t \mid [\omega(t)] \text{ is Kähler}\}. \tag{1-3}$$

By formally taking the cohomology class of (1-1) and solving the resulting ODE, we obtain the solution

$$[\omega(t)] = e^{-t}[\omega_0] - (1-e^{-t})c_1(X). \tag{1-4}$$

Since X is numerically effective, $-c_1(X)$ lies in the closure of the Kähler cone, the set of classes with a Kähler representative, and thus the solution exists for all time. Furthermore, for any initial metric, the cohomology class converges to the first Chern class $-c_1(X)$ in the limit as $t \to \infty$. This is independent of any starting metric ω_0.

The singularity formation reveals the manifold's underlying topology; therefore, the curvature blowup should be independent of the starting metric. To describe the curvature blowup, we utilize the following terminology traditionally used in parabolic geometric flows, such as the Ricci flow and mean curvature flow.

Definition 1.1 We say that a long-time solution to the normalized Kähler–Ricci flow (1-1) develops a type-III singularity if

$$\sup_{X\times[0,\infty)} |\mathrm{Rm}(\omega(t))|_{\omega(t)} < +\infty, \tag{1-5}$$

and a type-IIb singularity if

$$\sup_{X\times[0,\infty)} |\mathrm{Rm}(\omega(t))|_{\omega(t)} = +\infty. \tag{1-6}$$

The precise conjecture for metric independence is stated by Tossati [2018].

Conjecture 1.2 [Tosatti 2018] *Let X be a compact Kähler manifold with K_X nef, so every solution of the Kähler–Ricci flow exists for all positive time. Then the singularity type at infinity does not depend on the choice of the initial metric ω_0.*

Progress towards this conjecture and the analysis of the flow in general has been achieved under the assumption that K_X is semiample.

Definition 1.3 A line bundle over a compact Kähler manifold (X,ω) is semiample if there exists $k>0$ such that for each point $p \in X$, there exists a global section of $L^{\otimes k}$ that does not vanish at p.

The global sections of $L^{\otimes k}$ generate a holomorphic map,

$$f\colon X \to X_{\mathrm{can}} \subset \mathbb{CP}^N. \tag{1-7}$$

The image X_{can} is possibly a singular variety, and the dimension of X_{can} corresponds to the Kodaira dimension of X. This setup provides a fibration structure in which the flow collapses along the fibers onto the base space X_{can}, enabling a more intricate analysis; see [Tosatti 2018].

In the semiample case, Tossati and Yuguang Zhang [2015] give an almost-complete classification of singularity type based on the topology of X. More precisely, they show the following:

Theorem 1.4 (Tosatti and Yuguang Zhang [2015]) *Let X be a compact Kähler manifold with semiample K_X and consider a solution of the Kähler–Ricci flow.*

(1) *Suppose* $\mathrm{kod}(X) = 0$.
   - *If X is a finite quotient of a torus, then the solution is of type III.*
   - *If K_X is not a finite quotient of a torus, then the solution is of type IIb.*

(2) *Suppose* $\mathrm{kod}(X) = n$.
   - *If K_X is ample, then the solution is type III.*
   - *If K_X is not ample, then the solution is type IIb.*

(3) *Suppose* $0 < \mathrm{kod}(X) < n$.
   - *If X_y is not a finite quotient of a torus, then the solution is of type IIb.*
   - *If X_y is a finite quotient of a torus and $V = \varnothing$, then the solution is of type III.*

(4) *Suppose $n = 2$ and $\mathrm{kod}(X) = 1$. Then the solution is of type III if and only if the only singular fibers on f are of type mI_0 for $m > 1$.*

An alternative proof to (1), (2) and a special case of (3) was given by Yashan Zhang [2020], where he deduces these results by setting up a comparison of some well-behaved metric and an arbitrary metric. Moreover, he showed that for semiample K_X the singularity type does not depend on the initial metric, thereby confirming Conjecture 1.2 in the semiample case. Furthermore, Fong and Yashan Zhang [2020] showed that the set of singular fibers of the semiample fibration on which the Riemann curvature blows up at time infinity is independent of the choice of the initial Kähler metric. For more results pertaining to the curvature behavior of semiample Kähler–Ricci flows, see [Zhang 2009b; Song and Tian 2016; Jian 2020].

We prove Conjecture 1.2 without assuming K_X is semiample. More precisely, we show that if a solution develops a type-III singularity on a numerically effective manifold, then every other solution develops a type-III singularity. Consequently, if a solution develops a type-IIb singularity, then any other solution must be type IIb.

Theorem 1.5 *Let $\tilde{\omega}(t)$ be a solution to the normalized Kähler–Ricci flow (1-1) on a manifold X with numerically effective K_X. Suppose that*

$$|\mathrm{Rm}(\tilde{\omega}(t))|^2_{\tilde{\omega}(t)} \leqslant C_0. \tag{1-8}$$

Then for any other solution to the Kähler–Ricci flow, $\omega(t)$*, there exists a constant* $C > 0$ *such that*

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C\widetilde{\omega}(t), \tag{1-9}$$

and

$$|\mathrm{Rm}(\omega(t))|^2_{\omega(t)} \leqslant C \tag{1-10}$$

for all $t \geqslant 0$*.*

Remark 1.6 Any solution to the Kähler–Ricci flow on a numerically effective manifold exists for all time. Indeed, the estimate (1-10) is impossible for finite-time singularity due to Zhou Zhang [2010].

The proof builds on the results in [Zhang 2020], but the major difference is that no assumptions on the semiampleness of X need to be made.

We end the introduction by outlining the structure of this paper. In Section 2, we show a special case of Theorem 1.5, when the two initial metrics ω_0 and $\widetilde{\omega}_0$ are related by a simple scaling factor $\lambda_0 > 0$. We first derive a Monge–Ampère equation where the potential function relates the two evolving metrics under rescaling of space and time. By deriving bounds on the potential function and because of the curvature assumption on $\widetilde{\omega}$, we derive metric equivalence between the known type-III metric $\widetilde{\omega}$ and the arbitrary metric ω. Curvature estimates are then obtained as done in [Zhang 2020]. In Section 3, we utilize Lemma 2.1 to prove Theorem 1.5. Lemma 2.1 plays a vital role in constructing "comparison" solutions to the arbitrary solution $\omega(t)$, which will be needed to obtain a favorable sign when carrying out maximum principle arguments. As a result, a potential function bound is derived and used to obtain metric equivalence. With a similar argument as carried out in Section 2, curvature bounds are obtained for the general theorem. Finally, in Section 4, we utilize Theorem 1.5 for examples from [Zhang 2020] in the numerically effective case.

Acknowledgement The authors would like to thank Valentino Tosatti, Feng Wang, Yashan Zhang and Zhenlei Zhang for their interest and feedback on this work. They would also like to thank the reviewer for their helpful comments and feedback. Finally, they would like to thank the School of Mathematics and Statistics at The University of Sydney for providing a great study and research environment.

2 Preliminary results

In the first part of this section, we restrict ourselves to the special case of Theorems 1.5, which will be utilized in the proof of the general theorem. More precisely, we show that if there exists a solution to the Kähler–Ricci flow on a numerically effective manifold X with uniformly bounded curvature, then any scaling of the initial metric produces a solution that also evolves with uniformly bounded curvature.

In the second part of this section, we recall the proof in [Zhang 2019] which allows us to derive a bound on the curvature of a flow $\omega(t)$, given that it is equivalent to a type-III flow $\widetilde{\omega}(t)$.

2.1 Flow from a scaled initial metric

Lemma 2.1 *Let X be a numerically effective Kähler manifolds such that there exists $\widetilde{\omega}(t)$ a solution to the Kähler–Ricci flow with bounded curvature tensor*:

$$|\mathrm{Rm}(\widetilde{\omega}(t))|^2_{\widetilde{\omega}(t)} \leqslant C_0. \tag{2-1}$$

Then for any $\lambda_0 > 0$, there exists $C > 0$ such that the solution $\omega(t)$ starting from $\omega(0) = \lambda_0\widetilde{\omega}(0)$ satisfies

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C\widetilde{\omega}(t), \tag{2-2}$$

$$|\mathrm{Rm}(\omega(t))|^2_{\omega(t)} \leqslant C, \tag{2-3}$$

for all $t \geqslant 0$.

We first reduce the problem to a Monge–Ampère equation. The reduction will differ slightly from what is usually done in the literature. Since the initial metrics are equal up to some scaling factor, we scale the solution in space and time by $\lambda(t)$ and $\tau(t)$, respectively, so that

$$[\omega(t)] = \lambda(t)[\widetilde{\omega}(\tau(t))] \tag{2-4}$$

for all $t \geqslant 0$ and $\omega_0 = \lambda_0\widetilde{\omega}_0$ where $\lambda_0 > 0$. For any solution to the Kähler–Ricci flow, the cohomology class evolves as

$$[\omega(t)] = e^{-t}[\omega_0] - (1 - e^{-t})c_1(X). \tag{2-5}$$

Then from (2-4), we have

$$\lambda_0 e^{-t}\widetilde{\omega}_0 - (1 - e^{-t})c_1(X) = \lambda(t)e^{-\tau(t)}\widetilde{\omega}_0 - (1 - e^{-\tau(t)})\lambda(t)c_1(X),$$

from which we deduce that

$$\lambda(t)e^{-\tau(t)} = \lambda_0 e^{-t}, \quad \lambda(t) - e^{-\tau(t)}\lambda(t) = 1 - e^{-t}. \tag{2-6}$$

Solving this system yields

$$\lambda(t) = e^{-t}(\lambda_0 - 1) + 1 \quad \text{and} \quad \tau(t) = t + \ln\left(\frac{\lambda(t)}{\lambda_0}\right). \tag{2-7}$$

Furthermore, we check that at $t = 0$, the initial class satisfies $[\omega_0] = \lambda_0[\widetilde{\omega}_0]$.

We reduce the flow equation to a Monge–Ampère equation where the rescaled metric takes the role of the reference metric. Using (2-4), there exists $u\colon X \to \mathbb{R}$ such that

$$\omega(t) = \lambda(t)\widetilde{\omega}(\tau(t)) + \sqrt{-1}\partial\bar{\partial}u(t). \tag{2-8}$$

For ease of notation, we drop the t dependence for τ and λ. Taking a t–time derivative,

$$\frac{\partial\omega}{\partial t}(t) = \lambda'(t)\widetilde{\omega}(\tau) + \lambda\frac{\partial\tau}{\partial t}\frac{\partial\widetilde{\omega}}{\partial\tau} + \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t}\right).$$

Using $\lambda' + \lambda = 1$, $\tau' = \lambda^{-1}$ and (1-1) in τ, the above expression becomes

$$\frac{\partial\omega}{\partial t}(t) = \lambda'(t)\widetilde{\omega}(\tau) + \sqrt{-1}\partial\bar{\partial}\log\widetilde{\omega}(\tau)^n - \widetilde{\omega}(\tau) + \sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t}\right).$$

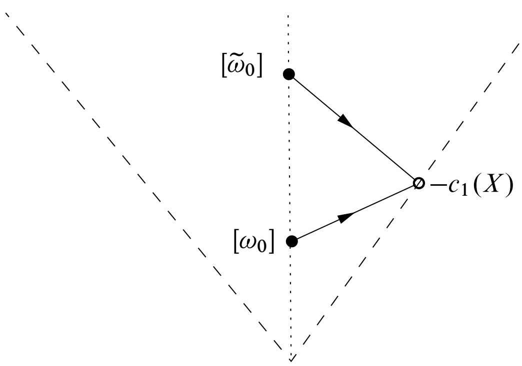

Figure 1: Scaling of initial metric.

On the other hand, (1-1) in t is

$$\frac{\partial \omega}{\partial t}(t) = \operatorname{Ric}(\omega(t)) - \omega(t) = \sqrt{-1}\partial\bar{\partial}\log\omega(t)^n - \lambda\widetilde{\omega}(\tau) - \sqrt{-1}\partial\bar{\partial}u(t).$$

Combining the previous two equations and using $\lambda' + \lambda = 1$ leads to

$$\sqrt{-1}\partial\bar{\partial}\Big(\frac{\partial u}{\partial t}\Big) = \sqrt{-1}\partial\bar{\partial}\log\omega(t)^n - \sqrt{-1}\partial\bar{\partial}\log\widetilde{\omega}(\tau)^n = \sqrt{-1}\partial\bar{\partial}u(t).$$

Integrating over the compact manifold X yields the complex Monge–Ampère equation

(2-9) $$\frac{\partial u}{\partial t} = \log\Big(\frac{\omega(t)^n}{\widetilde{\omega}(\tau)^n}\Big) - u(t), \quad u(0) = 0.$$

At $t = 0$, we have

(2-10) $$\frac{\partial u}{\partial t}\Big|_{t=0} = n\log(\lambda_0),$$

which is independent of $x \in X$. Since $u(0) \equiv 0$, the function $u(t)$ is constant over X, and thus (2-9) is an ordinary differential equation and (2-8) is

(2-11) $$\omega(t) = \lambda(t)\widetilde{\omega}(\tau).$$

Proof of Lemma 2.1 Equation (2-11) implies that there exist $C > 0$ such that

(2-12) $$C^{-1}\widetilde{\omega}(\tau) \leqslant \omega(t) \leqslant C\widetilde{\omega}(\tau).$$

Thus to show (2-2), we need to show that for some $C > 0$,

(2-13) $$C^{-1}\widetilde{\omega}(t) \leqslant \widetilde{\omega}(\tau) \leqslant C\widetilde{\omega}(t)$$

for all $t \geqslant 0$. To show the above, we use (2-1) to find a $C_0 > 0$ such that

(2-14) $$-C_0\widetilde{\omega} \leqslant \operatorname{Ric}(\widetilde{\omega}) \leqslant C_0\widetilde{\omega}.$$

This implies that (1-1) can be estimated by

(2-15) $$-C_0\widetilde{\omega} \leqslant \frac{\partial \widetilde{\omega}}{\partial t} + \widetilde{\omega} \leqslant C_0\widetilde{\omega}.$$

The lower bound in (2-15) implies

$$0 \leqslant \frac{\partial}{\partial t}(e^{(C_0+1)t}\widetilde{\omega})$$

and the upper bound in (2-15) implies

$$\frac{\partial}{\partial t}(e^{(1-C_0)t}\widetilde{\omega}) \leqslant 0$$

for all $t \geqslant 0$.

We now take cases. Suppose that $0 < \lambda_0 < 1$ and assume that $t \geqslant T$ for T large enough such that $\tau = t + \ln(\lambda(t)/\lambda_0) > t$. Using that $e^{(1-C_0)t}\widetilde{\omega}(t)$ is decreasing in t, we have

$$\widetilde{\omega}(\tau) \leqslant e^{(1-C_0)(t-\tau)}\widetilde{\omega}(t) = e^{C_0-1}\frac{\lambda(t)}{\lambda_0}\widetilde{\omega}(t) \leqslant C\widetilde{\omega}(t),$$

where $C > 0$. On the other hand, $e^{(C_0+1)t}\widetilde{\omega}(t)$ is increasing in t, which implies

$$\widetilde{\omega}(\tau) \geqslant e^{(C_0+1)(t-\tau)}\widetilde{\omega}(t) = e^{-(C_0+1)}\frac{\lambda(t)}{\lambda_0}\widetilde{\omega}(t) \geqslant C^{-1}\widetilde{\omega}(t)$$

for some $C > 0$. Combining these gives us (2-13). If $\lambda_0 > 1$ we can derive the estimate using the same method with $\tau < t$.

Combining (2-13) with (2-12) yields (2-2) for all $t \geqslant T$, for some large $T > 0$. Interior estimates imply (2-2) for all $t \geqslant 0$.

The curvature bounds (2-3) follow immediately from (2-11) and the scaling properties of the curvature tensor. □

2.2 General result for higher-order estimates

Using the metric equivalence, we derive higher-order estimates for $\omega(t)$ given that the curvature tensor is bounded for $\widetilde{\omega}$. The method is identical to that of Yashan Zhang [2020]; for convenience we sketch the argument here. When the evolving metric $\omega(t)$ is equivalent to a fixed metric ω_0, see Song and Weinkove's notes [2013b].

Since we have shown the inequality (2-2), we consider the two metrics with the same time scale t and drop the dependence of the metrics on t for notational convenience.

Lemma 2.2 *Suppose that ω and $\widetilde{\omega}$ are solutions to (1-1) and $|\mathrm{Rm}(\widetilde{\omega})|_{\widetilde{\omega}} \leqslant C$. The evolution of the trace satisfies*

(2-16) $$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)\operatorname{tr}_\omega\widetilde{\omega} \leqslant C\operatorname{tr}_\omega\widetilde{\omega} + C(\operatorname{tr}_\omega\widetilde{\omega})^2 - g^{\bar{j}i}g^{\bar{q}p}\tilde{g}^{\bar{b}a}\nabla_i\tilde{g}_{p\bar{b}}\nabla_{\bar{j}}\tilde{g}_{a\bar{q}}.$$

Proof The time derivative is given by

(2-17) $$\frac{\partial}{\partial t}(\operatorname{tr}_\omega\widetilde{\omega}) = \frac{\partial g^{\bar{j}i}}{\partial t}\tilde{g}_{i\bar{j}} + g^{\bar{j}i}\frac{\partial\tilde{g}_{i\bar{j}}}{\partial t} = R^{\bar{j}i}\tilde{g}_{i\bar{j}} - \operatorname{tr}_\omega\operatorname{Ric}(\widetilde{\omega}).$$

The laplacian of the trace is given by the standard calculation carried out in [Yau 1978; Lu 1967]:

(2-18) $$\Delta_\omega\operatorname{tr}_\omega\widetilde{\omega} = R^{\bar{j}i}\tilde{g}_{i\bar{j}} - g^{\bar{j}i}g^{\bar{q}p}\widetilde{R}_{i\bar{j}p\bar{q}} + g^{\bar{j}i}g^{\bar{q}p}\tilde{g}^{\bar{b}a}\nabla_i\tilde{g}_{p\bar{b}}\nabla_{\bar{j}}\tilde{g}_{a\bar{q}}.$$

Combining the previous two equations yields

$$\left(\frac{\partial}{\partial t}-\Delta_{\omega(t)}\right)\operatorname{tr}_\omega\tilde{\omega}=-\operatorname{tr}_\omega \operatorname{Ric}(\tilde{\omega})+g^{\bar{j}i}g^{\bar{q}p}\tilde{R}_{i\bar{j}p\bar{q}}-g^{\bar{j}i}g^{\bar{q}p}\tilde{g}^{\bar{b}a}\nabla_i\tilde{g}_{p\bar{b}}\nabla_{\bar{j}}\tilde{g}_{a\bar{q}}. \tag{2-19}$$

The desired inequality follows from the assumption (2-1). □

Proposition 2.3 (Yashan Zhang [2020]) *Let $\tilde{\omega}(t)$ be a type-III solution to the normalized Kähler–Ricci flow and $\omega(t)$ be an arbitrary solution such that for some $C_0>0$,*

$$C_0^{-1}\tilde{\omega}(t)\leqslant\omega(t)\leqslant C_0\tilde{\omega}(t) \tag{2-20}$$

for all $t\geqslant 0$. Then there exists $C>0$ such that

$$|\operatorname{Rm}(\omega(t))|_{\omega(t)}\leqslant C \tag{2-21}$$

for all $t\geqslant 0$.

Proof Let $\Psi=(\Psi_{ij}^k)$ where $\Psi_{ij}^k:=\Gamma_{ij}^k-\tilde{\Gamma}_{ij}^k$ and $S=|\Psi|^2_\omega$. In local coordinates,

$$S=g^{\bar{j}i}g^{\bar{l}k}g^{\bar{q}p}\tilde{\nabla}_i g_{k\bar{q}}\overline{\tilde{\nabla}_j g_{l\bar{p}}}. \tag{2-22}$$

For the last term in (2-16), we have

$$\begin{aligned}g^{\bar{j}i}g^{\bar{q}p}\tilde{g}^{\bar{b}a}\nabla_i\tilde{g}_{p\bar{b}}\nabla_{\bar{j}}\tilde{g}_{a\bar{q}}&=g^{\bar{j}i}g^{\bar{q}p}\tilde{g}^{\bar{b}a}(\nabla_i-\tilde{\nabla}_i)\tilde{g}_{p\bar{b}}(\nabla_{\bar{j}}-\tilde{\nabla}_{\bar{j}})\tilde{g}_{a\bar{q}}\\&=g^{\bar{j}i}g^{\bar{q}p}\tilde{g}^{\bar{b}a}(-\Psi_{ip}^d)\tilde{g}_{d\bar{b}}(-\overline{\Psi}_{jq}^e)\tilde{g}_{a\bar{e}}\geqslant C^{-1}S.\end{aligned} \tag{2-23}$$

Hence, we have the estimate

$$\left(\frac{\partial}{\partial t}-\Delta_\omega\right)tr_\omega\tilde{\omega}\leqslant C-C^{-1}S. \tag{2-24}$$

The evolution of the tensor S is given by

$$(\partial_t-\Delta_\omega)S=S-|\nabla\Psi|^2_\omega-|\overline{\nabla}\Psi|^2_\omega+\langle\tilde{\nabla}_i\tilde{R}_p^k-\Psi*\operatorname{Rm}(\tilde{\omega})-g^{\bar{b}a}\tilde{\nabla}_a\tilde{R}^k_{i\bar{b}p},\Psi\rangle_\omega, \tag{2-25}$$

where $\langle\cdot,\cdot\rangle_\omega$ is the tensor inner product induced by the metric $g(t)$ associated with $\omega(t)$. Following Hamilton's argument [1982] (or see [Song and Weinkove 2013b]), the assumption (2-1) on the curvature tensor of $\tilde{\omega}$ implies

$$|\nabla_{\tilde{\omega}(t)}\operatorname{Rm}(\tilde{\omega}(t))|_{\tilde{\omega}(t)}\leqslant C$$

for some $C>0$. In light of this, we estimate (2-25) by

$$\left(\frac{\partial}{\partial t}-\Delta_\omega\right)S\leqslant CS+C-|\nabla\Psi|^2_\omega-|\overline{\nabla}\Psi|^2_\omega. \tag{2-26}$$

We now have all the pieces to set up a maximum principle argument. Let $Q:=S+A\operatorname{tr}_\omega\tilde{\omega}$ for some sufficiently large $A>0$. Using (2-24) and (2-26), there exists $C>0$ such that

$$\left(\frac{\partial}{\partial t}-\Delta_\omega\right)Q\leqslant -S+C. \tag{2-27}$$

By a standard maximum principle argument and noting that $\operatorname{tr}_{\omega(t)}\tilde{\omega}(t)\leqslant C$, we find a constant $C\geqslant 1$ such that $S\leqslant C$.

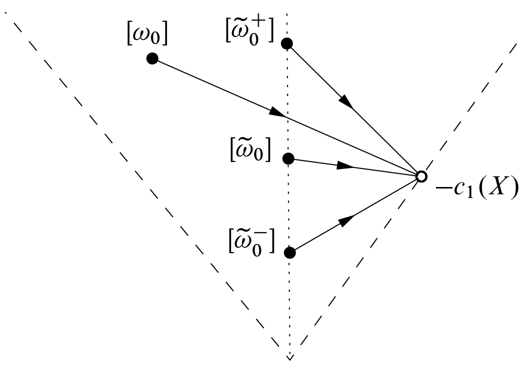

Figure 2: Scaling of type-III solutions.

The last term in (2-26) can be estimated by

$$|\bar{\nabla}\Psi|^2_\omega = |\widetilde{R}^k_{i\bar{b}p} - R^k_{i\bar{b}p}|^2_\omega \geqslant \tfrac{1}{2}|\mathrm{Rm}(\omega)|^2_\omega - C. \tag{2-28}$$

Thus

$$\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big) S \leqslant C - \tfrac{1}{2}|\mathrm{Rm}(\omega)|^2_\omega. \tag{2-29}$$

The evolution of curvature can be estimated by

$$\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big)|\mathrm{Rm}(\omega)|_\omega \leqslant C|\mathrm{Rm}(\omega)|^2_\omega - \tfrac{1}{2}|\mathrm{Rm}(\omega)|_\omega. \tag{2-30}$$

Let $Q := |\mathrm{Rm}(\omega)|_\omega + AS$ for some large constant $A > 0$, we combine the previous two estimates to obtain

$$\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big)(|\mathrm{Rm}(\omega)|_\omega + AS) \leqslant -\tfrac{1}{2}|\mathrm{Rm}(\omega)|_\omega + C. \tag{2-31}$$

Then by the maximum principle, there exists $C > 0$ such that

$$\sup_{X\times[0,\infty)} |\mathrm{Rm}(\omega)|_\omega \leqslant C. \tag{2-32}$$

□

3 Proof of Theorem 1.5

Using Lemma 2.1, we scale a type-III solution $\widetilde{\omega}(t)$ twice to $\omega^+(t)$ and $\omega^-(t)$, so that $\omega_0^- := \lambda_0^-\widetilde{\omega}_0 \leqslant \widetilde{\omega}_0 \leqslant \lambda_0^+\widetilde{\omega}_0 := \omega_0^+$. We then choose λ_0^- small enough and λ_0^+ large enough so that $\omega_0^- \leqslant \omega_0 \leqslant \omega_0^+$. Furthermore, these solutions starting from these scaled metrics are type III. We treat these new solutions as reference metrics, where either $\omega^+(t)$ or $\omega^-(t)$ is chosen to obtain a favorable sign when carrying out maximum principle arguments.

3.1 Reduction to a Monge–Ampère equation

Let $\chi \in [\omega_\infty]$ be a representative of the limiting class under the normalized Kähler–Ricci flow. Using the usual reference metrics, we have

$$\begin{gathered} \omega(t) = e^{-t}\omega_0 + (1-e^{-t})\chi + \sqrt{-1}\partial\bar{\partial}\varphi, \\ \omega^+(t) = e^{-t}\omega_0^+ + (1-e^{-t})\chi + \sqrt{-1}\partial\bar{\partial}\varphi^+ \quad \text{and} \quad \omega^-(t) = e^{-t}\omega_0^- + (1-e^{-t})\chi + \sqrt{-1}\partial\bar{\partial}\varphi^-. \end{gathered} \tag{3-1}$$

In our proof of Theorem 1.5, we require three Monge–Ampère equations relating pairs of $\omega(t)$, $\omega^+(t)$ and $\omega^-(t)$. Taking a time derivative of $\omega(t)$ and $\omega^-(t)$ in (3-1), and using the flow equation, we arrive at a Monge–Ampère equation for $u := \varphi - \varphi^-$ given by

$$\frac{\partial u}{\partial t} = \log\left(\frac{\omega^n}{(\omega^-)^n}\right) - u, \tag{3-2}$$

where

$$\omega(t) = \omega^-(t) + e^{-t}(\omega_0 - \omega_0^-) + \sqrt{-1}\partial\bar{\partial}u. \tag{3-3}$$

Equation (3-2) will be the primary Monge–Ampère equation utilized in the proof of the main theorem. Note that the construction of ω_0^- ensures $\omega_* := \omega_0 - \omega_0^- > 0$. Similarly, for $\psi := \varphi^+ - \varphi^-$ we have

$$\frac{\partial \psi}{\partial t} = \log\left(\frac{(\omega^+)^n}{(\omega^-)^n}\right) - \psi. \tag{3-4}$$

The function ψ relates the two evolving metrics by

$$\omega^+(t) = \omega^-(t) + e^{-t}(\omega_0^+ - \omega_0^-) + \sqrt{-1}\partial\bar{\partial}\psi. \tag{3-5}$$

Finally, for $v := \varphi - \varphi^+$, we have the Monge–Ampère equation

$$\frac{\partial v}{\partial t} = \log\left(\frac{\omega^n}{(\omega^+)^n}\right) - v, \tag{3-6}$$

where

$$\omega(t) = \omega^+(t) + e^{-t}(\omega_0 - \omega_0^+) + \sqrt{-1}\partial\bar{\partial}v. \tag{3-7}$$

3.2 Potential estimates

Our aim is to derive potential estimates for u and its time derivative. We begin by calculating the evolution of u and $\partial u/\partial t$. As before, we drop the t dependence on the metrics and potential functions for ease of notation. We denote geometric quantities associated with $\omega^+(t)$ and $\omega^-(t)$ by their respective symbol in superscripts.

Lemma 3.1 *For $\omega_* := \omega_0 - \omega_0^- > 0$, any solution u to (3-2) satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right)u = \frac{\partial u}{\partial t} - n + \operatorname{tr}_\omega \omega^- + e^{-t}\operatorname{tr}_\omega \omega_* \tag{3-8}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right)\frac{\partial u}{\partial t} = R^- - \operatorname{tr}_\omega \operatorname{Ric}(\omega^-) - \frac{\partial u}{\partial t} - \operatorname{tr}_\omega \omega^- - e^{-t}\operatorname{tr}_\omega \omega_* + n. \tag{3-9}$$

Proof The first equation (3-8) follows from (3-1):

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right)u = \frac{\partial u}{\partial t} - \operatorname{tr}_\omega(\omega - \omega^- - e^{-t}\omega_*) = \frac{\partial u}{\partial t} - n + \operatorname{tr}_\omega \omega^- + e^{-t}\operatorname{tr}_\omega \omega_*.$$

Equation (3-9) is derived using (3-2):

$$\Big(\frac{\partial}{\partial t}-\Delta_{\omega(t)}\Big)\frac{\partial u}{\partial t}=\Big(\frac{\partial}{\partial t}-\Delta_{\omega(t)}\Big)\log\Big(\frac{\omega^n}{\omega^{-n}}\Big)-\Big(\frac{\partial}{\partial t}-\Delta_{\omega(t)}\Big)u$$

$$=R^- -\operatorname{tr}_\omega \operatorname{Ric}(\omega^-)-\frac{\partial u}{\partial t}+n-\operatorname{tr}_\omega \omega^- - e^{-t}\operatorname{tr}_\omega \omega_*. \qquad \square$$

Lemma 3.2 *Let u be a solution to (3-2). Then there exists a uniform $C>0$ such that*

$$0\leqslant u(t)\leqslant C \tag{3-10}$$

for all $t\geqslant 0$.

Proof To obtain the lower bound, we apply the maximum principle to $e^t u+At$ for any constant $A>0$. Over the region $[0,T]\times X$ for any $T>0$, if the spacetime minimum occurs at some $t>0$, at that point we have

$$0\geqslant\frac{\partial}{\partial t}(e^tu+At)=e^t\log\bigg(\frac{(\omega^-(t)+e^{-t}(\omega_0-\omega_0^-)+\sqrt{-1}\partial\bar\partial u)^n}{(\omega^-)^n}\bigg)+A\geqslant e^t\log(1)+A>0,$$

which is a contradiction. Thus the minimum must occur at $t=0$, where $e^tu+At=0$. We conclude that, for all spacetime,

$$e^tu+At\geqslant 0$$

holds for any $A>0$. Finally, we take the limit $A\to 0$ to deduce that

$$u\geqslant 0.$$

To derive an upper bound for u, we first observe that at a spatial maximum $v_{\max}$, from (3-6) we have

$$\frac{\partial v_{\max}}{\partial t}\leqslant\log\bigg(\frac{(\omega_++e^{-t}(\omega_0-\omega_0^+))^n}{(\omega^+)^n}\bigg)-v_{\max}\leqslant -v_{\max}.$$

Then by the maximum principle, we deduce that $v\leqslant 0$. On the other hand, we have $C>0$ such that

$$|\psi|\leqslant C \tag{3-11}$$

for all $t>0$. Indeed, the initial metrics ω_0^+, ω_0^- and $\tilde\omega_0$ are scalar multiples of each other, so Lemma 2.1 and (3-4) imply

$$\frac{\partial}{\partial t}(e^t\psi)=e^t\log\bigg(\frac{(\omega^+)^n}{(\omega^-)^n}\bigg)\leqslant Ce^t$$

for some $C>0$. Integrating the above gives us (3-11). It now follows that

$$u(t)=\varphi(t)-\varphi^-(t)=v(t)+\psi(t)\leqslant C,$$

where $C>0$ is independent of time. $\square$

Lemma 3.3 *Let u be a solution to (3-2). Then there exists $C>0$ such that*

$$\Big|\frac{\partial u}{\partial t}\Big|\leqslant C \tag{3-12}$$

for all $t\geqslant 0$.

Proof We first note that Lemma 2.1 implies that for some $C > 0$, we have

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega^-(t) \leqslant C\widetilde{\omega}(t) \tag{3-13}$$

and

$$|\mathrm{Rm}(\omega^-(t))|^2_{\omega^-(t)} \leqslant C, \tag{3-14}$$

for all $t \geqslant 0$. Let

$$Q := \frac{\partial u}{\partial t} - Au, \tag{3-15}$$

where $A > 0$ a constant that will be set later. By combining (3-8) and (3-9), and using (3-14), we obtain

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big)Q &= R^- - \mathrm{tr}_\omega \mathrm{Ric}(\omega^-) - (A+1)\,\mathrm{tr}_\omega\,\omega^- - (A+1)e^{-t}\,\mathrm{tr}_\omega\,\omega_* - (A+1)\frac{\partial u}{\partial t} + (A+1)n \\ &\leqslant -(A+1-C_0)\,\mathrm{tr}_\omega\,\omega^- - (A+1)e^{-t}\,\mathrm{tr}_\omega\,\omega_* - (A+1)\frac{\partial u}{\partial t} + (A+1+C_0)n.\end{aligned}$$

We choose $A = 2 + C_0$ to obtain the estimate

$$\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big)Q \leqslant -C\frac{\partial u}{\partial t} + C.$$

Applying the parabolic maximum principle yields the upper bound

$$\frac{\partial u}{\partial t} \leqslant C$$

for some $C > 0$.

For the lower bound, we instead consider the quantity

$$Q := \frac{\partial u}{\partial t} + Au,$$

for some $A > 0$ to be determined later. Once again, we use (3-8), (3-9) and (3-14) to set up a maximum principle argument:

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big)Q &= R^-(t) - \mathrm{tr}_\omega \mathrm{Ric}(\omega^-) + (A-1)\,\mathrm{tr}_\omega\,\widetilde{\omega} + (A-1)e^{-t}\,\mathrm{tr}_\omega\,\omega_* + (A-1)\frac{\partial u}{\partial t} + n(1-A) \\ &\geqslant (-C_0 + 1 - A)n + (A-1)\frac{\partial u}{\partial t} + (A - C_0 - 1)\,\mathrm{tr}_\omega\,\omega^- + (A-1)e^{-t}\,\mathrm{tr}_\omega\,\omega_*.\end{aligned}$$

Applying a similar argument to before, we choose $A = 2 + C_0$ to obtain an estimate

$$\Big(\frac{\partial}{\partial t} - \Delta_\omega\Big)Q \geqslant -C + C\frac{\partial u}{\partial t} + \mathrm{tr}_\omega\,\omega^- \geqslant -C + C\frac{\partial u}{\partial t} + n\Big(\frac{(\omega^-)^n}{\omega^n}\Big)^{1/n} = -C + C\frac{\partial u}{\partial t} + ne^{-(1/n)(\partial u/\partial t + u)}.$$

Then applying the maximum principle, using that u is uniformly bounded, shows that there exists a $C > 0$ independent of time such that

$$\frac{\partial u}{\partial t} \geqslant -C. \tag{3-16}$$

□

3.3 Proof of Theorem 1.5

We now derive a metric equivalence between the type-III solution $\widetilde{\omega}$ and the arbitrary solution ω.

Proof From a standard calculation, using (3-14), we have for some $C > 0$ and $C_0 > 0$

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \log \operatorname{tr}_\omega \omega^- \leqslant C_0 \operatorname{tr}_\omega \omega^- + C \tag{3-17}$$

for all times $t \geqslant 0$. The uniform bound on $\dfrac{\partial u}{\partial t}$ from Lemma 3.3 implies that

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) u = \frac{\partial u}{\partial t} - n + \operatorname{tr}_\omega \omega^- + e^{-t} \operatorname{tr}_\omega(\omega_0 - \omega_0^-) \geqslant -C + \operatorname{tr}_\omega \omega^-. \tag{3-18}$$

Following a similar argument as before, we define $Q := \log \operatorname{tr}_\omega \omega^- - Au$. Then using the two inequalities, we choose A large enough so that

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q \leqslant C - \operatorname{tr}_\omega \omega^-. \tag{3-19}$$

Applying the maximum principle, and again using Lemmas 3.2 and 3.3, we obtain some $C > 0$ such that

$$\operatorname{tr}_\omega \omega^- \leqslant C \tag{3-20}$$

for all $t \geqslant 0$. Combining the potential estimates from Lemmas 3.2 and 3.3, we derive from the Monge–Ampère equation (3-2) the volume bounds

$$C^{-1} \leqslant \frac{\omega^n}{(\omega^-)^n} \leqslant C \tag{3-21}$$

for some $C > 0$ for all $t \geqslant 0$. Similar to before, we employ standard eigenvalue estimates to obtain

$$\operatorname{tr}_{\omega^-} \omega \leqslant n\left(\frac{\omega^n}{(\omega^-)^n}\right) \operatorname{tr}_\omega \omega^- \leqslant C.$$

Then the two trace bounds imply the following metric equivalence:

$$C^{-1}\omega^-(t) \leqslant \omega(t) \leqslant C\omega^-(t). \tag{3-22}$$

The above inequality and (3-13) yield (1-9). To obtain bounds on the curvature tensor, we apply Proposition 2.3 with $\omega^-(t)$ in place of $\widetilde{\omega}$. □

We can now apply Theorem 1.5 to answer Conjecture 1.2:

Theorem 3.4 *Let X be a Kähler manifold with numerically effective canonical line bundle. Then the singularity type of solutions to the Kähler–Ricci flow is independent of the initial metric.*

Proof Suppose a solution $\widetilde{\omega}$ is type III. Then by Theorem 1.5, any other solution starting from a different initial metric must also be type III. On the other hand, if a type-IIb solution exists, then any other solution must be type IIb, as otherwise Theorem 1.5 implies that all solutions must be type III, a contradiction. □

Remark 3.5 Finally, Theorem 1.5 can be interpreted as evidence for the Kähler extension of the abundance conjecture. Indeed, if a counterexample to Theorem 1.5 could be constructed, then Yashan Zhang's result [2020] implies that the manifold cannot be semiample.

4 Further remarks

Our results and techniques have applications and generalizations in many settings.

We start with some examples where an explicit solution to the normalized Kähler–Ricci flow is known. This allows us to deduce the singularity type for an arbitrary initial metric for these manifolds. These examples are obtained from [Zhang 2019].

(1) **Calabi–Yau metrics** Suppose that X admits a Calabi–Yau metric. Then under the Kähler–Ricci flow, we check that $\omega(t) = e^{-t}\omega_{CY}$ is a solution with $\omega(0) = \omega_{CY}$. For such solutions, the curvature evolves as

$$|\mathrm{Rm}(\omega(t))|_{\omega(t)} = e^{t}|\mathrm{Rm}(\omega_{CY})|_{\omega_{CY}}, \tag{4-1}$$

and therefore develops a type-III singularity if and only if ω_{CY} is a flat metric, which is possible if and only if X is a finite quotient of a torus. Therefore we conclude from Theorem 1.5 that the Kähler–Ricci flow develops a type-III solution on a numerically effective manifold X if and only if it is a finite quotient of a torus. Furthermore, this demonstrates Theorem 1.4(1) without assuming K_X is semiample.

Remark 4.1 Theorem 1.4(2) assumes K_X is big. Since nef and big imply semiample, the generalization to nef is immediate.

(2) **Product manifold** Consider the product manifold $X := B \times Y$ where Y is a Calabi–Yau manifold. This is a special case of Theorem 1.4(3). Suppose that ω is a Kähler–Einstein metric on B, then check that

$$\widetilde{\omega}(t) = e^{-t}\omega_{CY} + (1-e^{-t})\omega_B \tag{4-2}$$

is a solution to the Kähler–Ricci flow. The curvature is given by

$$|\mathrm{Rm}\,\omega(t)|^2_{\omega(t)} = e^{2t}|\mathrm{Rm}(\omega_{CY})|^2_{\omega_{CY}} + \frac{1}{(1-e^{-t})^2}|\mathrm{Rm}(\omega_B)|^2_{\omega_B}. \tag{4-3}$$

Thus we have a type-III singularity if and only if ω_{CY} is flat, which occurs if Y is a finite quotient of a torus. So a type-III singularity develops if and only if Y is a finite quotient of a torus. Hence, using Theorem 1.5, for any initial metric ω_0, there exists $C > 0$ such that

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C\widetilde{\omega}(t) \quad \text{and} \quad |\mathrm{Rm}(\omega(t))|_{\omega(t)} \leqslant C, \tag{4-4}$$

for all $t \geqslant 0$.

Furthermore, our method can be applied to other flows, for example the modified Kähler–Ricci flow as studied in [Zhang 2009a; Yuan 2011; Wu and Zhang 2022]. More precisely, the independence of infinite-time singularity type should follow from metric equivalence and higher-order estimates.

References

[Cao 1985] **H D Cao**, *Deformation of Kähler metrics to Kähler–Einstein metrics on compact Kähler manifolds*, Invent. Math. 81 (1985) 359–372 MR Zbl

[Fong 2015] **F T-H Fong**, *On the collapsing rate of the Kähler–Ricci flow with finite-time singularity*, J. Geom. Anal. 25 (2015) 1098–1107 MR Zbl

[Fong and Zhang 2020] **F T-H Fong**, **Y Zhang**, *Local curvature estimates of long-time solutions to the Kähler–Ricci flow*, Adv. Math. 375 (2020) art. id. 107416 MR

[Guo et al. 2016] **B Guo**, **J Song**, **B Weinkove**, *Geometric convergence of the Kähler–Ricci flow on complex surfaces of general type*, Int. Math. Res. Not. 2016 (2016) 5652–5669 MR Zbl

[Hamilton 1982] **R S Hamilton**, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982) 255–306 MR Zbl

[Jian 2020] **W Jian**, *Convergence of scalar curvature of Kähler–Ricci flow on manifolds of positive Kodaira dimension*, Adv. Math. 371 (2020) art. id. 107253 MR Zbl

[Lu 1967] **Y-C Lu**, *On holomorphic mappings of complex manifolds*, PhD thesis, University of California, Berkeley (1967) Available at `https://www.proquest.com/docview/302237334`

[Matsuki 2002] **K Matsuki**, *Introduction to the Mori program*, Springer (2002) MR Zbl

[Munteanu and Székelyhidi 2011] **O Munteanu**, **G Székelyhidi**, *On convergence of the Kähler–Ricci flow*, Comm. Anal. Geom. 19 (2011) 887–903 MR Zbl

[Phong and Sturm 2006] **D H Phong**, **J Sturm**, *On stability and the convergence of the Kähler–Ricci flow*, J. Differential Geom. 72 (2006) 149–168 MR Zbl

[Phong et al. 2007] **D H Phong**, **N Sesum**, **J Sturm**, *Multiplier ideal sheaves and the Kähler–Ricci flow*, Comm. Anal. Geom. 15 (2007) 613–632 MR Zbl

[Song and Tian 2007] **J Song**, **G Tian**, *The Kähler–Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. 170 (2007) 609–653 MR Zbl

[Song and Tian 2012] **J Song**, **G Tian**, *Canonical measures and Kähler–Ricci flow*, J. Amer. Math. Soc. 25 (2012) 303–353 MR Zbl

[Song and Tian 2016] **J Song**, **G Tian**, *Bounding scalar curvature for global solutions of the Kähler–Ricci flow*, Amer. J. Math. 138 (2016) 683–695 MR Zbl

[Song and Tian 2017] **J Song**, **G Tian**, *The Kähler–Ricci flow through singularities*, Invent. Math. 207 (2017) 519–595 MR Zbl

[Song and Weinkove 2013a] **J Song**, **B Weinkove**, *Contracting exceptional divisors by the Kähler–Ricci flow*, Duke Math. J. 162 (2013) 367–415 MR Zbl

[Song and Weinkove 2013b] **J Song**, **B Weinkove**, *An introduction to the Kähler–Ricci flow*, from "An introduction to the Kähler–Ricci flow" (S Boucksom, P Eyssidieux, V Guedj, editors), Lecture Notes in Math. 2086, Springer (2013) 89–188 MR Zbl

[Song and Weinkove 2014] **J Song**, **B Weinkove**, *Contracting exceptional divisors by the Kähler–Ricci flow, II*, Proc. Lond. Math. Soc. 108 (2014) 1529–1561 MR Zbl

[Song and Yuan 2012] **J Song**, **Y Yuan**, *Metric flips with Calabi ansatz*, Geom. Funct. Anal. 22 (2012) 240–265 MR Zbl

[Tian 2012] **G Tian**, *Existence of Einstein metrics on Fano manifolds*, from "Metric and differential geometry" (X Dai, X Rong, editors), Progr. Math. 297, Birkhäuser, Basel (2012) 119–159 MR Zbl

[Tian 2019] **G Tian**, *Some progresses on Kähler–Ricci flow*, Boll. Unione Mat. Ital. 12 (2019) 251–263 MR

[Tian and Zhang 2006] **G Tian**, **Z Zhang**, *On the Kähler–Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B 27 (2006) 179–192 MR Zbl

[Tian and Zhang 2016] **G Tian**, **Z Zhang**, *Convergence of Kähler–Ricci flow on lower-dimensional algebraic manifolds of general type*, Int. Math. Res. Not. 2016 (2016) 6493–6511 MR Zbl

[Tosatti 2018] **V Tosatti**, *KAWA lecture notes on the Kähler–Ricci flow*, Ann. Fac. Sci. Toulouse Math. 27 (2018) 285–376 MR Zbl

[Tosatti and Zhang 2015] **V Tosatti**, **Y Zhang**, *Infinite-time singularities of the Kähler–Ricci flow*, Geom. Topol. 19 (2015) 2925–2948 MR Zbl

[Tosatti et al. 2018] **V Tosatti**, **B Weinkove**, **X Yang**, *The Kähler–Ricci flow, Ricci-flat metrics and collapsing limits*, Amer. J. Math. 140 (2018) 653–698 MR Zbl

[Wu and Zhang 2022] **H Wu**, **Z Zhang**, *The modified Kähler–Ricci flow, II*, preprint (2022) arXiv 2210.10296

[Yau 1978] **S T Yau**, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. 100 (1978) 197–203 MR Zbl

[Yuan 2011] **Y Yuan**, *On the convergence of a modified Kähler–Ricci flow*, Math. Z. 268 (2011) 281–289 MR Zbl

[Zhang 2009a] **Z Zhang**, *A modified Kähler–Ricci flow*, Math. Ann. 345 (2009) 559–579 MR Zbl

[Zhang 2009b] **Z Zhang**, *Scalar curvature bound for Kähler–Ricci flows over minimal manifolds of general type*, Int. Math. Res. Not. 2009 (2009) 3901–3912 MR Zbl

[Zhang 2010] **Z Zhang**, *Scalar curvature behavior for finite-time singularity of Kähler–Ricci flow*, Michigan Math. J. 59 (2010) 419–433 MR Zbl

[Zhang 2019] **Y Zhang**, *Collapsing limits of the Kähler–Ricci flow and the continuity method*, Math. Ann. 374 (2019) 331–360 MR Zbl

[Zhang 2020] **Y Zhang**, *Infinite-time singularity type of the Kähler–Ricci flow*, J. Geom. Anal. 30 (2020) 2092–2104 MR Zbl

School of Mathematics and Statistics, The University of Sydney
Sydney, NSW, Australia

School of Mathematics and Statistics, The University of Sydney
Sydney, NSW, Australia

hdw24@cornell.edu, zhou.zhang@sydney.edu.au

Proposed: Gang Tian
Seconded: John Lott, Tobias H Colding

Received: 9 November 2023
Revised: 7 February 2024

Geometry & Topology

msp.org/gt

See inside back cover or msp.org/gt for submission instructions.

The subscription price for 2025 is US $865/year for the electronic version, and $1210/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

GT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

GEOMETRY & TOPOLOGY

Volume 29 Issue 1 (pages 1–548) 2025