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Subgroups of genus-2 quasi-Fuchsian groups and cocompact Kleinian groups

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We wish to control the geometry of some surface subgroups of a cocompact Kleinian group. More precisely, provided any genus-2 quasi-Fuchsian group Γ and cocompact Kleinian group G, then for any $\nu > 0$ we will find a surface subgroup H of G that is $(1+\nu)$-quasiconformally conjugate to a finite-index subgroup $F < \Gamma$.

22E40, 30C62, 30F40, 57K32; 37A25

1 Introduction

Main result We aim to study the geometry of compact hyperbolic 3-manifolds $M = \mathbb{H}^3/G$. We say that $H < G$ is a surface subgroup if there exist a closed surface S_g and a continuous map $f : S_g \to M$ such that the induced homomorphism f_* between the fundamental groups is injective and $f_*(\pi_1(S_g)) = H$. We say a surface subgroup $H < G$ is K-quasi-Fuchsian if there is a cocompact Fuchsian group of $\mathrm{PSL}(2,\mathbb{R})$ whose action on $\partial\mathbb{H}^3$ is K-quasiconformally conjugate to the action of H on $\partial\mathbb{H}^3$. Here is our main result:

Theorem 1.1 *Let Γ be a genus-2 quasi-Fuchsian group and G be a cocompact Kleinian group. For any $\nu > 0$, there is a surface subgroup $H < G$ that is $(1+\nu)$-quasiconformally conjugate to a finite-index subgroup $F < \Gamma$.*

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The cocompactness of G implies that G is finitely generated. Subsequently, applying the Selberg lemma [1960], we know G has a finite-index subgroup that is torsion-free. Therefore for the remainder of this paper, we reduce to the case where G is torsion-free.

Related results We say that a collection of quasi-Fuchsian surface subgroups is ubiquitous if for any pair of hyperbolic planes Π and Π' in $\mathbb{H}^3$ with distance $d(\Pi, \Pi') > 0$, there is a surface subgroup in the collection whose boundary circle lies between $\partial\Pi$ and $\partial\Pi'$.

Kahn and Marković [2012b] not only proved the original surface subgroup conjecture, that every closed hyperbolic 3-manifold $\mathbb{H}^3/G$ contains an immersed π_1-injective surface, but also implied that for all $K > 1$, the collection of K-quasi-Fuchsian surface subgroups of G is ubiquitous as well. Later, Kahn and Wright [2021] showed that for a finite-volume but not compact hyperbolic 3-manifold $\mathbb{H}^3/G$ and for all $K > 1$, the collection of K-quasi-Fuchsian surface subgroups of G is ubiquitous.

Kahn and Marković [2012a] proved a counting result stating that the number of genus-g surface subgroups grows as g^{2g}. Masters and Zhang [2008; 2009] and Baker and Cooper [2015] demonstrated that a complete finite-volume hyperbolic 3-manifold with cusps has a surface subgroup. Cooper and Futer [2019] showed that the set of closed immersed quasi-Fuchsian surfaces in a complete finite-volume hyperbolic 3-manifold is ubiquitous.

The surfaces constructed in [Kahn and Marković 2012b; Kahn and Wright 2021] are almost geodesic. On the other hand, [Kahn and Marković 2012a; Masters and Zhang 2008; Baker and Cooper 2015; Cooper and Futer 2019] produced surfaces, which are usually not nearly Fuchsian, using different techniques. Here we construct a large amount of surfaces with more geometric control. The geometry of such a surface is somewhat uniform in a sense, as it is close to a finite covering of some genus-2 quasi-Fuchsian surface.

Moreover, one may relate our result to the Hausdorff dimension of limit sets of quasi-Fuchsian groups. Brock [2003] proved that there are quasi-Fuchsian groups with the Hausdorff dimension of limit sets arbitrarily close to 2. Gehring and Väisälä [1973] discussed how the Hausdorff dimension changes under a quasiconformal mapping between 2-dimensional spaces. With these two results and Theorem 1.1, we can prove the following result:

Theorem 1.2 *Given a cocompact Kleinian group G and a real number $1 \le \alpha < 2$, for any $\epsilon > 0$ there exists a surface subgroup $H < G$ such that*

$$|\text{H-dim}(\Lambda(H)) - \alpha| < \epsilon,$$

where $\Lambda(H)$ is the limit set of H and H-dim *denotes the Hausdorff dimension.*

Bowen [2009] proved a similar result but with free groups, which says for a cocompact $G < \mathrm{PSL}(2, \mathbb{C})$ and a free group F, then G has a subgroup that is close to a finite-index subgroup of F in some sense. Then as a quick application, he showed that the set of Hausdorff dimensions of limit sets of free subgroups of any cocompact $G < \mathrm{PSL}(2, \mathbb{C})$ is dense in $[0, 2]$.

Recently, the above theorem was cited by Kahn, Marković and Smilga [Kahn et al. 2023]. They proved the remarkable result that every topological limiting measure is totally scarring (ie supported on the totally geodesic locus), while geometrical limiting measures are never totally scarring, when the closed hyperbolic 3-manifold contains at least one totally geodesic subsurface. Our main result was used in the proof of the part concerning geometrical limiting measures. More specifically, they wanted the lower bound of the genus of a surface with certain area. Then by Gauss–Bonnet, a surface with average intrinsic curvature less than $-(1+K_1\epsilon^2)$ was needed, for all ϵ small enough and some $K_1>0$. Theorem 1.1 reduced this to the simpler problem of finding a "model" genus-2 quasi-Fuchsian surface with the same property which could be constructed by hand.

Future work As mentioned in [Kahn and Wright 2021], one can regard [Kahn and Marković 2012b; 2015; Hamenstädt 2015; Kahn et al. 2018; Kahn and Wright 2021] as special cases of the general question: whether a lattice L in a Lie group $\mathcal{G}$ contains surface subgroups which are close to lying in a given subgroup of $\mathcal{G}$ isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. Hamenstädt [2015] concerns the case where $\mathcal{G}$ is a rank-1 Lie group, $\mathcal{G}\neq\mathrm{SO}(2m,1)$, and L is cocompact. In [Kahn et al. 2018], $\mathcal{G}$ is a center-free, complex simple Lie group of noncompact type and L is cocompact. Other cases, including higher-rank Lie groups like $\mathcal{G}=\mathrm{SL}(n,\mathbb{R})$, are also of special interest.

Here we care about a different version where $\mathrm{PSL}(2,\mathbb{R})$ is replaced with some discrete group Γ_0. Theorem 1.1 addresses the case where $\mathcal{G}=\mathrm{PSL}(2,\mathbb{C})$, L is cocompact and Γ_0 is a genus-2 quasi-Fuchsian group. The genus-2 condition is of great significance due to the existence of the hyperelliptic involution of genus-2 surfaces. This prompts the question of whether it is possible to generalize our result beyond genus-2. We hope that good pants homology can help us deal with this difficulty. For further insights into this, please refer to Remark 8.1. In the case where L has cofinite volume instead of being cocompact, we expect that the methods developed in [Kahn and Wright 2021] may be applied.

When $\mathcal{G}$ is a Lie group other than $\mathrm{PSL}(2,\mathbb{C})$, for instance $\mathcal{G}=\mathrm{SO}(2m+1,1)$, L is cocompact and Γ_0 is a discrete genus-2 surface subgroup of $\mathcal{G}$ which is deformed from some Fuchsian group, our method should work well. The analogous problem as the one indicated in the last paragraph will arise when dealing with the case beyond the limitation of genus 2.

Sketch of the proof Roughly speaking, we divide the proof into three parts.

(1) Our first objective is to obtain a pants decomposition[1] of the given genus-2 quasi-Fuchsian group Γ with specific desired characteristics, by using the so-called *spinning construction*.[2] These characteristics play a significant role in the proof of π_1-injectivity in step (3). It is worth noting that the pants we construct, however, cannot be as standard as those pants used in [Kahn and Wright 2021], since quasi-Fuchsian surfaces do not possess as good qualities as Riemann surfaces do.

[1] Pants decompositions of quasi-Fuchsian groups will be defined in Section 4.8.

[2] We will provide a detailed introduction to the spinning construction in Section 5.1.

To provide further details, we begin by taking any nonseparating pants decomposition σ_1 of the given genus-2 quasi-Fuchsian group Γ. We then apply the spinning construction on σ_1 to derive a subsequent pants decomposition σ_2, such that the cuff lengths of σ_2 are nearly equal. Next, by using a specific ideal triangulation of the pants within σ_2, together with the spinning construction, we construct a new pants decomposition σ_3 such that

- the cuff lengths can be as large as we want and almost the same size,
- the real part of the three shears are positive and bounded from above and below.

(2) The above pants decomposition σ_3 is actually nonseparating, which yields two isometric pants, so we have a model pants P_0. We want to find those pants that have similar geometric structures to P_0 in the hyperbolic 3-manifold M, and these pants are then assembled together in a pattern similar to σ_3.

A pair of pants P can be determined by a cuff γ and a third connection[3] α to γ. By the Margulis argument, we can estimate the number of (R_i, ϵ)-good curves.[4] Therefore, in order to count $(R_i, \epsilon)_{i=1}^3$-good pants[5] with γ a boundary curve, we count the corresponding third connections to γ and prove that these pants are distributed along γ almost evenly. Then by Hall's marriage theorem, we can find a permutation with good behavior determined by the shears in step (1), on the set of all expected pants with boundary cuff γ. By applying the doubling trick,[6] we can assemble these pants along γ together and finally get a good assembly.[7]

The idea and the method applied in this part are derived from [Kahn and Wright 2021, Sections 3 and 5.2], with only a few constants being modified.

(3) For the good assembly we get in step (2), we define its perfect model by replacing all the good pants with the perfect ones[8] and gluing along the corresponding geodesics in the pattern in step (1). Then this perfect model is actually a finite covering of the given genus-2 quasi-Fuchsian surface in step (1). We construct a map from the perfect model to the good assembly and lift it to the universal cover. By an important result for matrix multiplication, we can show some quantitative property of the lift map. Then it can be extended to a quasiconformal mapping from $\mathbb{C}$ to itself, which conjugates a finite-index subgroup of Γ to a surface subgroup of G.

The important result for matrix multiplication, proved and used in [Kahn and Wright 2021, Appendix], control the distortion of a map between semilinear sequences.[9] In particular, when considering a mapping from a linear sequence to a semilinear sequence, it is relatively straightforward to confirm the necessary

[3] Third connections will be defined in Section 4.3.

[4] Good curves will be defined in Section 3.1.

[5] Good pants will be defined in Section 4.4.

[6] The doubling trick will be introduced in Section 4.7.

[7] Assemblies and good assemblies will be defined in Sections 4.6 and 6, respectively.

[8] Perfect pants will be defined in Section 4.4.

[9] Semilinear and linear geodesic sequences will be defined in Section 7.3.

requirements. This ease of verification arises from the positional order within linear sequences. However, the major challenge in our problem lies in the fact that semilinear sequences live in $\mathbb{H}^3$, so they have a significantly higher degree of flexibility in position. In order to have a more accurate control on a semilinear sequence, we establish a relation to a linear sequence by quasi-isometries. By doing so, the result for matrix multiplication can be effectively applied to our particular scenario.

Organization Section 2 presents the left and right actions of $\mathrm{PSL}(2,\mathbb{C})$ on the frame bundle of $\mathbb{H}^3$. In Section 3, we build the inefficiency theory for 3-dimensional hyperbolic spaces, based on the work in [Kahn and Marković 2015; Liu and Marković 2015]. The theory is used to study a specific family of broken geodesics in a hyperbolic 3-manifold, where the segments are alternately long enough and relatively short. We also estimate the complex length of the geodesic representative for such a broken geodesic.

Section 4 introduces basics of pants, assemblies and ideal triangles. Some definitions and results are recalled from [Kahn and Wright 2021]. Sections 2, 3 and 4 serve as preliminary materials essential for the subsequent sections.

Section 5 covers step (1) of the sketch of the proof. We will construct a desired pants decomposition of the given genus-2 quasi-Fuchsian surface through the spinning construction, where the cuff lengths are large enough and of the same size and the complex shears have bounded and positive real parts. The cuff lengths and shears are estimated by results in Section 3.

In Section 6, we collect and generalize results from [Kahn and Wright 2021] on construction, counting and equidistribution of good pants, corresponding to step (2) of the sketch of the proof.

Section 7 is composed of step (3) of the sketch of the proof, which is the heart of this paper. In this section, we apply the idea in [Kahn and Wright 2021, Appendix] to prove that a good assembly is close to the given quasi-Fuchsian group. Finally, we build the surface and prove the main result in Section 8.

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2 Right action of $\mathrm{PSL}(2,\mathbb{C})$ on the frame bundle of hyperbolic 3-space

In this paper, we always use the upper half-plane model for $\mathbb{H}^2$ and the upper half-space model for $\mathbb{H}^3$.

In $\mathbb{H}^2$, each point can be written as $z = x + iy$ with $x, y \in \mathbb{R}$ and $y > 0$. It turns out that $\mathrm{PSL}(2,\mathbb{R})$ is the isometry group of $\mathbb{H}^2$, where there is a left action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}^2$ by

$$T(z) = \frac{az+b}{cz+d},$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}).$$

Since the left action is an isometry, we have the left action of $\mathrm{PSL}(2,\mathbb{R})$ on the unit tangent bundle $T^1(\mathbb{H}^2)$. Based on the left action on $T^1(\mathbb{H}^2)$, we want to describe another action of $\mathrm{PSL}(2,\mathbb{R})$ on $T^1(\mathbb{H}^2)$ which is called the right action.

We first fix the basepoint $z_0 = i \in \mathbb{H}^2$ and the base unit tangent vector $\boldsymbol{u}_0 = (i,i) \in T^1_{z_0}(\mathbb{H}^2)$. Then there is a unique right action such that

(1) for any $g \in \mathrm{PSL}(2,\mathbb{R})$, $g \cdot \boldsymbol{u}_0 = \boldsymbol{u}_0 \cdot g$,

(2) for any $\boldsymbol{v} \in T^1(\mathbb{H}^2)$ and $g,h \in \mathrm{PSL}(2,\mathbb{R})$, $g \cdot (\boldsymbol{v} \cdot h) = (g \cdot \boldsymbol{v}) \cdot h$.

It is easy to see that this is a transitive and faithful action.

Since the left action is transitive, for any $\boldsymbol{v} \in T^1(\mathbb{H}^2)$ there is a $g \in \mathrm{PSL}(2,\mathbb{R})$ such that $\boldsymbol{v} = g \cdot \boldsymbol{u}_0$. Therefore for $h \in \mathrm{PSL}(2,\mathbb{R})$,

$$\boldsymbol{v} \cdot h = (g \cdot \boldsymbol{u}_0) \cdot h = g \cdot (\boldsymbol{u}_0 \cdot h).$$

Hence we can use the right action of h on $\boldsymbol{u}_0$, which is the same as the left action of h on $\boldsymbol{u}_0$, to describe the right action of h on $\boldsymbol{v} = g \cdot \boldsymbol{u}_0$. Here are some useful examples:

(a) **Geodesic flow** Let

$$A(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}).$$

Then $\boldsymbol{u}_0 \cdot A(t) = A(t) \cdot \boldsymbol{u}_0 = (ie^{2t}, ie^{2t})$. Therefore $A(t)\colon T^1(\mathbb{H}^2) \to T^1(\mathbb{H}^2)$ is the geodesic flow.

(b) **Rotation** Let

$$B(\theta) = \begin{pmatrix} \cos\left(\frac{1}{2}\theta\right) & \sin\left(\frac{1}{2}\theta\right) \\ -\sin\left(\frac{1}{2}\theta\right) & \cos\left(\frac{1}{2}\theta\right) \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}).$$

Then $\boldsymbol{u}_0 \cdot B(\theta) = B(\theta) \cdot \boldsymbol{u}_0 = (i, -\sin\theta + i\cos\theta)$. Thus the action of $B(\theta)$ on any $\boldsymbol{v} \in T^1(\mathbb{H}^2)$ is the counterclockwise rotation by θ.

For an oriented n-dimensional Riemannian manifold M, a point $x \in M$ together with an orthonormal basis of T_xM with positive orientation is called an n-frame in M. We denote by $\mathcal{F}M$ the set of all n-frames in M, which forms a fiber bundle over M and is called the frame bundle of M. Now we can define the right action of $\mathrm{PSL}(2,\mathbb{C})$ on the frame bundle $\mathcal{F}\mathbb{H}^3$ of $\mathbb{H}^3$ based on the corresponding left action by isometries. Take the upper half-space model of $\mathbb{H}^3 = \{(z,t) : z \in \mathbb{C}, t > 0\}$ and let

$$\mathcal{F}\mathbb{H}^3 = \{\langle p, u, v\rangle : p \in \mathbb{H}^3, u, v \in T^1_p\mathbb{H}^3, u \perp v\}.$$

We also fix the base frame

$$\Psi_0 = \langle p_0, \boldsymbol{u}_0, \boldsymbol{v}_0\rangle = \langle (0,1), (0,1), (1,0)\rangle$$

at $(0,1) \in \mathbb{H}^3$. Then the right action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathcal{F}\mathbb{H}^3$ is such that

(1) for any $g \in \mathrm{PSL}(2,\mathbb{C})$, $g \cdot \Psi_0 = \Psi_0 \cdot g$,

(2) for any $\Psi \in \mathcal{F}\mathbb{H}^3$ and $g,h \in \mathrm{PSL}(2,\mathbb{C})$, $g \cdot (\Psi \cdot h) = (g \cdot \Psi) \cdot h$.

For $\Psi \in \mathcal{F}\mathbb{H}^3$, there exists $g \in \mathrm{PSL}(2,\mathbb{C})$ such that $\Psi = g \cdot \Psi_0$. Then for any $h \in \mathrm{PSL}(2,\mathbb{C})$, we have

$$\Psi \cdot h = (g \cdot \Psi_0) \cdot h = g \cdot (\Psi_0 \cdot h).$$

Thus the right action on Ψ_0 will tell us the right action on any other frames. Here are the examples of geodesic flow and rotations, which will be used in Section 3:

(a) **Geodesic flow** Let

$$A(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$$

with $x \in \mathbb{R}$. Then

$$\Psi_0 \cdot A(x) = A(x) \cdot \Psi_0 = \langle (0, e^{2x}), (0, e^{2x}), (e^{2x}, 0) \rangle.$$

Therefore $A(x)\colon \mathcal{F}\mathbb{H}^3 \to \mathcal{F}\mathbb{H}^3$ is the geodesic flow.

(b) **Rotation along $\boldsymbol{u_0}$** Let

$$A(iy) = \begin{pmatrix} e^{iy} & 0 \\ 0 & e^{-iy} \end{pmatrix}$$

with $y \in \mathbb{R}$. Then

$$\Psi_0 \cdot A(iy) = A(iy) \cdot \Psi_0 = \langle (0,1), (0,1), (e^{i2y}, 0) \rangle.$$

Hence $A(iy)$ is the rotation along $\boldsymbol{u}_0$.

(c) **Rotation along $\boldsymbol{u_0} \times \boldsymbol{v_0}$** Let

$$B(\theta) = \begin{pmatrix} \cos\left(\frac{1}{2}\theta\right) & \sin\left(\frac{1}{2}\theta\right) \\ -\sin\left(\frac{1}{2}\theta\right) & \cos\left(\frac{1}{2}\theta\right) \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}),$$

with $\theta \in \mathbb{R}$. Then

$$\Psi_0 \cdot B(\theta) = B(\theta) \cdot \Psi_0 = \langle (0,1), (-\sin(\theta), \cos(\theta)), (\cos(\theta), \sin(\theta)) \rangle.$$

So the action of $B(\theta)$ is the rotation along $\boldsymbol{u}_0 \times \boldsymbol{v}_0$.

3 The theory of inefficiency in dimension 3

In this section, we will build the theory of inefficiency in 3-dimensional hyperbolic manifolds, which is used to estimate the cuff lengths of the desired pants decomposition of Γ in the next section. Some definitions and results are directly derived from [Kahn and Marković 2015, Section 4; Liu and Marković 2015, Section 4].

3.1 Terminology

Suppose M is an oriented hyperbolic 3-manifold. We introduce some concepts in the geometry of framed geodesic segments.

Definition 3.1 An *oriented framed segment* in M is a triple

$$\mathfrak{s} = (s, \vec{n}_{\mathrm{ini}}, \vec{n}_{\mathrm{ter}})$$

such that s is an oriented immersed compact geodesic segment, and that $\vec{n}_{\mathrm{ini}}$ and $\vec{n}_{\mathrm{ter}}$ are two unit normal vectors at the initial endpoint and terminal endpoint of s, respectively.

- The *carrier segment* is the oriented segment s.
- The *initial endpoint* $p_{\mathrm{ini}}(\mathfrak{s})$ and the *terminal endpoint* $p_{\mathrm{ter}}(\mathfrak{s})$ are the initial endpoint and the terminal endpoint of s, respectively.
- The *initial framing* $\vec{n}_{\mathrm{ini}}(\mathfrak{s})$ and the *terminal framing* $\vec{n}_{\mathrm{ter}}(\mathfrak{s})$ are the unit normal vectors $\vec{n}_{\mathrm{ini}}$ and $\vec{n}_{\mathrm{ter}}$.
- The *initial direction* $\vec{t}_{\mathrm{ini}}(\mathfrak{s})$ and the *terminal direction* $\vec{t}_{\mathrm{ter}}(\mathfrak{s})$ are the unit tangent vectors in the direction of s at the initial point and the terminal point, respectively.
- The *orientation reversal* of $\mathfrak{s}$ is defined to be

$$\bar{\mathfrak{s}} = (\bar{s}, \vec{n}_{\mathrm{ter}}, \vec{n}_{\mathrm{ini}}),$$

where $\bar{s}$ is the orientation reversal of s. The *framing flipping* of $\mathfrak{s}$ is defined as

$$\mathfrak{s}^* = (s, -\vec{n}_{\mathrm{ini}}, -\vec{n}_{\mathrm{ter}}).$$

It follows from the definition that

$$\overline{\mathfrak{s}^*} = \bar{\mathfrak{s}}^*.$$

Definition 3.2 Suppose that $\mathfrak{s}$ is an oriented framed segment in M. We define the *length* of $\mathfrak{s}$, denoted by $l(\mathfrak{s}) \in [0, +\infty)$, to be the length of the unframed segment s carrying $\mathfrak{s}$. The *phase* of $\mathfrak{s}$, denoted by $\phi(\mathfrak{s}) \in \mathbb{R}/2\pi\mathbb{Z}$, is defined as the angle from the initial framing $\vec{n}_{\mathrm{ini}}$ to the parallel transportation of $\vec{n}_{\mathrm{ter}}$ to the initial endpoint of $\mathfrak{s}$ via s, signed with respect to the normal orientation induced from $\vec{t}_{\mathrm{ini}}$ and the orientation of M. We define the *complex length* of $\mathfrak{s}$ as

$$\boldsymbol{l}(\mathfrak{s}) = l(\mathfrak{s}) + i\phi(\mathfrak{s}),$$

and observe that the length and the phase remain unchanged under orientation reversal and framing flipping.

For an oriented closed geodesic curve c, we can also talk about its *length*, *phase*, or *complex length*, by taking an arbitrary unit normal vector $\vec{n}$ at a point $p \in c$, and regarding c as a framed segment obtained by cutting c at p and assigned with framing $\vec{n}$ at both endpoints. If the complex length of c satisfies the inequality

$$|\boldsymbol{l}(c) - 2R| < 2\epsilon,$$

for some $R \in \mathbb{C}/2\pi i\mathbb{Z}$ and $\epsilon > 0$, then we say c is an (R, ϵ)-*good curve*. Here we mean that there is a lift of $\boldsymbol{l}(c)$ in $\mathbb{C}$ that makes the inequality hold. In this paper, we always consider a small value for ϵ, making the lift unique when it exists.

Definition 3.3 Let $0 \le \delta < \frac{1}{3}\pi$, $L > d > 0$, and $0 < \theta < \pi$ be constants.

(1) An oriented framed segment $\mathfrak{s}$ is said to be *δ-consecutive* to another oriented framed segment $\mathfrak{s}'$ if

- the terminal endpoint of $\mathfrak{s}$ is the initial endpoint of $\mathfrak{s}'$,
- the terminal framing of $\mathfrak{s}$ is δ-close to the initial framing of $\mathfrak{s}'$.

We say $\mathfrak{s}$ is *consecutive* to $\mathfrak{s}'$ if δ is 0. When $\mathfrak{s}$ is δ-consecutive to $\mathfrak{s}'$, the *bending angle* between $\mathfrak{s}$ and $\mathfrak{s}'$ is the angle between the terminal direction of $\mathfrak{s}$ and the initial direction of $\mathfrak{s}'$. This angle falls within the range $[0, \pi]$ and is denoted by $\angle(\mathfrak{s}, \mathfrak{s}')$.

(2) A *δ-consecutive chain* of oriented framed segments is a finite sequence $\mathfrak{s}_1\mathfrak{s}_2\ldots\mathfrak{s}_m$ such that each $\mathfrak{s}_i$ is δ-consecutive to $\mathfrak{s}_{i+1}$. It is a *δ-consecutive cycle* if furthermore $\mathfrak{s}_m$ is δ-consecutive to $\mathfrak{s}_1$. A δ-consecutive cycle $\mathfrak{s}_1\ldots\mathfrak{s}_m$ is called (L, θ)-tame if the length of each $\mathfrak{s}_i$ is greater than $2L$ and the bending angle between $\mathfrak{s}_i$ and $\mathfrak{s}_{i+1}$ is less than θ.

(3) For a δ-consecutive chain $\mathfrak{s}_1\ldots\mathfrak{s}_m$, we define its *reduced concatenation*, denoted by $\langle\mathfrak{s}_1\mathfrak{s}_2\ldots\mathfrak{s}_m\rangle$, to be the oriented framed segments as follows. The carrier segment of $\langle\mathfrak{s}_1\mathfrak{s}_2\ldots\mathfrak{s}_m\rangle$ is the geodesic arc which is homotopic to the concatenation of the carrier segments of the $\mathfrak{s}_i$, relative to the initial endpoint of $\mathfrak{s}_1$ and the terminal endpoint of $\mathfrak{s}_m$; the initial framing of $\langle\mathfrak{s}_1\mathfrak{s}_2\ldots\mathfrak{s}_m\rangle$ is the unit normal vector closest to the initial framing of $\mathfrak{s}_1$; the terminal framing of $\langle\mathfrak{s}_1\mathfrak{s}_2\ldots\mathfrak{s}_m\rangle$ is the unit normal vector closest to the terminal framing of $\mathfrak{s}_m$.

(4) For a δ-consecutive cycle $\mathfrak{s}_1\ldots\mathfrak{s}_m$, we define its *reduced cyclic concatenation*, denoted by

$$[\mathfrak{s}_1\mathfrak{s}_2\ldots\mathfrak{s}_m],$$

to be the unframed oriented closed geodesic curve freely homotopic to the concatenation of the carrier segments of each $\mathfrak{s}_i$, assuming the result is nontrivial.

(5) A *continuous chain* of oriented framed segments is a consecutive chain $\mathfrak{s}_1\ldots\mathfrak{s}_m$ with all bending angles $\frac{1}{2}\pi$ and

$$\vec{n}_{\text{ini}}(\mathfrak{s}_{i+1}) = \vec{n}_{\text{ter}}(\mathfrak{s}_i) = \vec{t}_{\text{ter}}(\mathfrak{s}_i) \times \vec{t}_{\text{ini}}(\mathfrak{s}_{i+1}),$$

for $i = 1, \ldots, m-1$. It is a *continuous cycle* if, additionally, $\mathfrak{s}_m$ is consecutive to $\mathfrak{s}_1$ with bending angle $\frac{1}{2}\pi$ and

$$\vec{n}_{\text{ini}}(\mathfrak{s}_1) = \vec{n}_{\text{ter}}(\mathfrak{s}_m) = \vec{t}_{\text{ter}}(\mathfrak{s}_m) \times \vec{t}_{\text{ini}}(\mathfrak{s}_1).$$

(6) A continuous cycle $\mathfrak{s}_1\ldots\mathfrak{s}_{2m}$ is called *(L, d, Δ)-tame* if

- the length of each $\mathfrak{s}_{2i-1}$ is greater than $2L$,
- the length of each $\mathfrak{s}_{2i}$, which is allowed to be 0, is no greater than $2d$,
- $\left|\ln\left|\sinh\left(\frac{1}{2}\boldsymbol{l}(\mathfrak{s}_{2i})\right)\right|\right|$ is bounded by Δ.

The geometric meaning of $\left|\ln\left|\sinh\left(\frac{1}{2}\boldsymbol{l}(\mathfrak{s}_{2i})\right)\right|\right|$ will be explained later.

3.2 Inefficiency of framed segments

We first recall that for any bending angle θ, we have the *inefficiency* of θ, defined as

$$I(\theta) = 2\ln\bigl(\sec\bigl(\tfrac{1}{2}\theta\bigr)\bigr).$$

The next lemma interprets the geometric meaning of the inefficiency of angles:

Lemma 3.4 [Liu and Marković 2015, Lemma 4.10] *For any $0 < \theta < \pi$ and $\epsilon > 0$, there exists $L > 0$ such that the following holds. Suppose that ΔABC is a geodesic triangle in $\mathbb{H}^3$ with $|CA|, |CB| > L$ and $\angle C = \pi - \theta$. Then*

(1) $\angle A + \angle B < \epsilon$,

(2) $I(\theta) - \epsilon < |CA| + |CB| - |AB| < I(\theta)$.

Now we want to define the inefficiency of a framed segment with complex length $d \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}/2\pi\mathbb{Z}$, which can be viewed as a subset of $\mathbb{C}/2\pi i\mathbb{Z}$, by the following lemma:

Lemma 3.5 *For any $\Delta > 0$ and $\epsilon > 0$, there exists $L > 0$ such that the following holds. Suppose that $\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3$ is a continuous chain of framed segments satisfying $\boldsymbol{l}(\mathfrak{s}_2) = 2d$ with $d \neq 0 \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}/2\pi\mathbb{Z}$, $|\ln|\sinh(d)|| < \Delta$ and $l(\mathfrak{s}_1), l(\mathfrak{s}_3) > 2L$. Then*

(1) $\angle(\mathfrak{s}_3, \overline{\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle}), \angle(\overline{\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle}, \mathfrak{s}_1) > \pi - \epsilon$,

(2) $|l(\mathfrak{s}_1) + l(\mathfrak{s}_2) + l(\mathfrak{s}_3) - l(\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle) - (\operatorname{Re}(2d) - 2\ln|\sinh(d)|)| < \epsilon$,

(3) $|\phi(\mathfrak{s}_1) + \phi(\mathfrak{s}_2) + \phi(\mathfrak{s}_3) - \phi(\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle) - (\operatorname{Im}(2d) - 2\operatorname{Arg}(\sinh(d)))| < \epsilon$,

where $|\cdot|$ on $\mathbb{R}/2\pi\mathbb{Z}$ is understood as the distance from 0 valued in $[0,\pi]$ and Arg is the principal value of the argument.

Proof Let $\mathfrak{s}_4 = \overline{\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle}$, so $\mathfrak{s}_4$ and $\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle$ have the same length and phase. For $i \in \mathbb{Z}/4\mathbb{Z}$, let A_i be the joint point of $\mathfrak{s}_{i-1}$ and $\mathfrak{s}_i$, so $A_1A_2A_3A_4$ is a hyperbolic quadrilateral with two right angles, which also can be regarded as a degenerate right-angled hexagon. Let $\vec{n}_1 = \vec{t}_{\mathrm{ter}}(\mathfrak{s}_4) \times \vec{t}_{\mathrm{ini}}(\mathfrak{s}_1)$ and $\vec{n}_4 = \vec{t}_{\mathrm{ter}}(\mathfrak{s}_3) \times \vec{t}_{\mathrm{ini}}(\mathfrak{s}_4)$ be the common normal vectors at A_1 and A_4, respectively. We then can define the complex length of A_iA_{i+1}, denoted by l_i, in this right-angled hexagon, where $l_2 = 2d$. By the hyperbolic cosine law for right-angled hexagons, we have

$$\cosh(2d) = \frac{\cosh(l_1)\cosh(l_3) + \cosh(l_4)}{\sinh(l_1)\sinh(l_2)}.$$

Therefore

$$\frac{\cosh(l_4)}{e^{(l_1+l_3)}} = \cosh(2d)\frac{\sinh(l_1)}{e^{l_1}}\frac{\sinh(l_3)}{e^{l_3}} - \frac{\cosh(l_1)}{e^{l_1}}\frac{\cosh(l_3)}{e^{l_3}}.$$

When $\operatorname{Re}(l_1), \operatorname{Re}(l_3) \to +\infty$,

$$\frac{\cosh(l_4)}{e^{l_1+l_3}} \to \tfrac{1}{4}(\cosh(2d) - 1).$$

Since $d \neq 0$, we have $\operatorname{Re}(l_4) \to +\infty$. So when $\operatorname{Re}(l_1), \operatorname{Re}(l_3) \to +\infty$,

$$e^{l_4 - l_1 - l_3} \to \sinh^2(d), \tag{3-6}$$

where $|\sinh(d)|$ is bounded above and below. By $\mathrm{Re}(l_i) = l(\mathfrak{s}_i)$ and (3-6), there exists $L_1 > 0$ such that when $l(\mathfrak{s}_1), l(\mathfrak{s}_3) > L_1$,

(3-7) $$|l(\mathfrak{s}_1) + l(\mathfrak{s}_3) - l(\mathfrak{s}_4) + 2\ln|\sinh(d)|| < \epsilon.$$

By $l(\mathfrak{s}_2) = \mathrm{Re}(d)$ and (3-7), we get (2).

By the hyperbolic cosine law again, we have

$$\cos(\angle A_1) = \frac{\cosh(l_1)\cosh(l_4) + \cosh(l_3)}{\sinh(l_1)\sinh(l_4)}.$$

When $\mathrm{Re}(l_1), \mathrm{Re}(l_3) \to +\infty$, by (3-6),

$$\cos(\angle A_1) \to \frac{e^{l_1+l_4} + e^{l_3}}{e^{l_1+l_4}} \to 1.$$

Hence there exists $L_2 > 0$ such that $\angle A_1 < \epsilon$ when $l(\mathfrak{s}_1), l(\mathfrak{s}_3) > L_2$. Then by $\angle(\vec{t}_{\mathrm{ini}}(\mathfrak{s}_1), \vec{t}_{\mathrm{ter}}(\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle)) = \pi - \angle A_1$,

$$\angle(\vec{t}_{\mathrm{ini}}(\mathfrak{s}_1), \vec{t}_{\mathrm{ter}}(\langle \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\rangle)) > \pi - \epsilon.$$

The second part of (1) is true by similar reasoning.

For (3), we notice that $\angle A_1, \angle A_4 < \epsilon$ when $\mathrm{Re}(l_1), \mathrm{Re}(l_3) > L_2$. Thus $\mathfrak{s}_3$ is ϵ-consecutive to $\mathfrak{s}_4$ and $\mathfrak{s}_4$ is ϵ-consecutive to $\mathfrak{s}_1$. Hence

(3-8) $$|(\phi(\mathfrak{s}_1) + \phi(\mathfrak{s}_3) - \phi(\mathfrak{s}_4)) - (\mathrm{Im}(l_1) + \mathrm{Im}(l_3) - \mathrm{Im}(l_4))| < 2\epsilon.$$

By (3-6), there exists $L_3 > 0$ such that when $l(\mathfrak{s}_1), l(\mathfrak{s}_3) > L_3$,

$$|\mathrm{Im}(l_1) + \mathrm{Im}(l_3) - \mathrm{Im}(l_4) + 2\,\mathrm{Arg}(\sinh(d))| < \epsilon.$$

Together with (3-8), we have

$$|(\phi(\mathfrak{s}_1) + \phi(\mathfrak{s}_3) - \phi(\mathfrak{s}_4)) + 2\,\mathrm{Arg}(\sinh(d))| < 3\epsilon.$$

Hence the proof is completed since $\phi(\mathfrak{s}_2) = \mathrm{Im}(d)$. □

For $d \in \mathbb{C}$, we define the length inefficiency of d as

$$I_l(d) = \mathrm{Re}(2d) - 2\ln|\sinh(d)|$$

and the phase inefficiency of d as

$$I_\phi(d) = \mathrm{Im}(2d) - 2\,\mathrm{Arg}(\sinh(d)).$$

Then the above lemma illustrates the geometric meaning of these definitions.

When $d = i\theta$ is purely imaginary, we have

$$I_l(i\theta) = -2\ln|\sinh(i\theta)| = 2\ln\csc(\theta),$$

which implies $I_l(i\theta)$ coincides with the inefficiency of angle $\pi - 2\theta$ since the bending angle is defined as the exterior angle.

3.3 Inefficiency of framed segment cycles

For sufficiently tame approximately consecutive framed segment cycles, Liu and Marković estimated their inefficiency in sense of length and phase. We restate the results:

Lemma 3.9 [Liu and Marković 2015, Lemma 4.8] *Given any $\delta \geq 0$, $\pi > \theta > 0$ and $\epsilon > 0$, there exists $L > 0$ such that the following holds. If $\mathfrak{s}_1 \dots \mathfrak{s}_m$ is an (L,θ)-tame δ-consecutive cycle of oriented framed segments, let $\theta_i \in [0, \pi-\theta]$ be the bending angle between $\mathfrak{s}_i$ and $\mathfrak{s}_{i+1}$ with $\mathfrak{s}_{m+1}$ equal to $\mathfrak{s}_1$. Then*

$$\left| l([\mathfrak{s}_1 \dots \mathfrak{s}_m]) - \sum_{i=1}^{m} l(\mathfrak{s}_i) + \sum_{i=1}^{m} I(\theta_i) \right| < \epsilon$$

and

$$\left| \phi([\mathfrak{s}_1 \dots \mathfrak{s}_m]) - \sum_{i=1}^{m} \phi(\mathfrak{s}_i) \right| < m\delta + \epsilon,$$

where $|\cdot|$ on $\mathbb{R}/2\pi\mathbb{Z}$ is understood as the distance from 0 valued in $[0,\pi]$.

As a matter of fact, the above lemma, which works for (L,θ)-tame δ-consecutive cycles, can be generalized to the following lemma for (L,d,θ)-tame continuous cycles:

Lemma 3.10 (sum of inefficiencies lemma) *Given any $m \in \mathbb{Z}_+$, $\Delta > 0$, $d > 0$ and $\frac{1}{4} > \epsilon > 0$, there exists $L > 0$ such that the following holds. If $\mathfrak{s}_1 \dots \mathfrak{s}_{2m}$ is an (L,d,Δ)-tame continuous cycle of oriented framed segments, let $2d_i$ be the complex length of $\mathfrak{s}_{2i}$. Then*

$$\left| l([\mathfrak{s}_1 \dots \mathfrak{s}_{2m}]) - \sum_{i=1}^{2m} l(\mathfrak{s}_i) + \sum_{i=1}^{m} I_l(d_i) \right| < \epsilon \tag{3-11}$$

and

$$\left| \phi([\mathfrak{s}_1 \dots \mathfrak{s}_{2m}]) - \sum_{i=1}^{2m} \phi(\mathfrak{s}_i) + \sum_{i=1}^{m} I_\phi(d_i) \right| < \epsilon, \tag{3-12}$$

where $|\cdot|$ on $\mathbb{R}/2\pi\mathbb{Z}$ is understood as the distance from 0 valued in $[0,\pi]$.

Proof Let $\epsilon_1 = \epsilon/(2m+1)$. We write

$$\Delta l(\mathfrak{s}_1 \dots \mathfrak{s}_k) = \begin{cases} \sum_{i=1}^{k} l(\mathfrak{s}_i) - l(\langle \mathfrak{s}_1 \dots \mathfrak{s}_k \rangle) & \text{if } \mathfrak{s}_1, \dots, \mathfrak{s}_k \text{ is a chain,} \\ \sum_{i=1}^{k} l(\mathfrak{s}_i) - l([\mathfrak{s}_1 \dots \mathfrak{s}_k]) & \text{if } \mathfrak{s}_1, \dots, \mathfrak{s}_k \text{ is a cycle,} \end{cases}$$

and

$$\Delta \phi(\mathfrak{s}_1 \dots \mathfrak{s}_k) = \begin{cases} \sum_{i=1}^{k} \phi(\mathfrak{s}_i) - \phi(\langle \mathfrak{s}_1 \dots \mathfrak{s}_k \rangle) & \text{if } \mathfrak{s}_1, \dots, \mathfrak{s}_k \text{ is a chain,} \\ \sum_{i=1}^{k} \phi(\mathfrak{s}_i) - \phi([\mathfrak{s}_1 \dots \mathfrak{s}_k]) & \text{if } \mathfrak{s}_1, \dots, \mathfrak{s}_k \text{ is a cycle,} \end{cases}$$

for convenience.

For each $1 \leq i \leq m$, write $\mathfrak{s}_{2i-1}$ as the concatenation of two consecutive oriented framed segments $\mathfrak{s}_{2i-1}^-$ and $\mathfrak{s}_{2i-1}^+$ of equal length and phase. Let $\tilde{\mathfrak{s}}_i = \langle \mathfrak{s}_{2i-1}^+ \mathfrak{s}_{2i} \mathfrak{s}_{2i+1}^- \rangle$ with $\mathfrak{s}_{2m+1} = \mathfrak{s}_1$. Since the phase of each $\mathfrak{s}_{2i}$ is at least θ away from 0, then by Lemma 3.5 there exists $L_1 > 0$ such that

$$|\Delta l(\mathfrak{s}_{2i-1}^+ \mathfrak{s}_{2i} \mathfrak{s}_{2i+1}^-) - I_l(d_i)| < \epsilon_1 \tag{3-13}$$

and

$$|\Delta\phi(\mathfrak{s}_{2i-1}^{+}\mathfrak{s}_{2i}\mathfrak{s}_{2i+1}^{-}) - I_{\phi}(d_i)| < \epsilon_1, \tag{3-14}$$

for $i = 1, \dots, m$ and where $\tilde{\mathfrak{s}}_i$ is ϵ_1-consecutive to $\tilde{\mathfrak{s}}_{i+1}$.

Now $\tilde{\mathfrak{s}}_1 \dots \tilde{\mathfrak{s}}_m$ is an ϵ_1-consecutive cycle of framed segments. Let $\theta_i = \angle(\tilde{\mathfrak{s}}_i, \tilde{\mathfrak{s}}_{i+1})$ be the bending angle between $\tilde{\mathfrak{s}}_i$ and $\tilde{\mathfrak{s}}_{i+1}$. By Lemma 3.5, the unsigned angle between $\vec{t}_{\text{ini}}(\tilde{\mathfrak{s}}_i)$ and $\vec{t}_{\text{ini}}(\mathfrak{s}_{2i-1}^{+})$ is less than ϵ_1, and the same for the unsigned angle between $\vec{t}_{\text{ter}}(\tilde{\mathfrak{s}}_{i-1})$ and $\vec{t}_{\text{ter}}(\mathfrak{s}_{2i-1}^{-})$. Thus θ_i is less than $2\epsilon_1$. By Lemma 3.9, there exists $L_2 > 0$ such that when $l(\mathfrak{s}_i) > L_2$, we have

$$\left|\Delta l(\tilde{\mathfrak{s}}_1 \dots \tilde{\mathfrak{s}}_m) - \sum_{i=1}^{m} I(\theta_i)\right| < \epsilon_1 \tag{3-15}$$

and

$$|\Delta\phi(\tilde{\mathfrak{s}}_1 \dots \tilde{\mathfrak{s}}_m)| < m\epsilon_1 + \epsilon_1. \tag{3-16}$$

Since $\epsilon_1 < \epsilon < \frac{1}{4}$, we have

$$I(\theta_i) < I(2\epsilon_1) = 2\ln(\sec(\epsilon_1)) < \epsilon_1.$$

Together with (3-15), we get

$$|\Delta l(\tilde{\mathfrak{s}}_1 \dots \tilde{\mathfrak{s}}_m)| < (m+1)\epsilon_1. \tag{3-17}$$

By (3-13) and (3-17),

$$\left|\Delta l(\mathfrak{s}_1 \dots \mathfrak{s}_{2m}) - \sum_{i=1}^{m} I_l(d_i)\right| = \left|\sum_{i=1}^{m}(\Delta l(\mathfrak{s}_{2i-1}^{+}\mathfrak{s}_{2i}\mathfrak{s}_{2i+1}^{-}) - I_l(d_i)) + \Delta l(\tilde{\mathfrak{s}}_1 \dots \tilde{\mathfrak{s}}_m)\right| \le (2m+1)\epsilon_1 = \epsilon,$$

which proves (3-11). By (3-14) and (3-16), we have

$$\left|\Delta\phi(\mathfrak{s}_1 \dots \mathfrak{s}_{2m}) - \sum_{i=1}^{m} I_{\phi}(d_i)\right| = \left|\sum_{i=1}^{m}(\Delta\phi(\mathfrak{s}_{2i-1}^{+}\mathfrak{s}_{2i}\mathfrak{s}_{2i+1}^{-}) - I_{\phi}(d_i)) + \Delta\phi(\tilde{\mathfrak{s}}_1 \dots \tilde{\mathfrak{s}}_m)\right| \le (2m+1)\epsilon_1 = \epsilon,$$

which proves (3-12). □

3.4 Zigzag geodesics

In this subsection, we aim to study the cycles of four framed segments. These cycles will appear several times in the subsequent sections.

Definition 3.18 For $L, \epsilon > 0$, a continuous cycle of four framed segments $\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4$ is called (L, ϵ)-*zigzag*, if the following conditions are satisfied:

1. $l(\mathfrak{s}_i) > 2L$, for $i = 1, 3$.
2. $|\boldsymbol{l}(\mathfrak{s}_i) - i\pi| < 2\epsilon$, for $i = 2, 4$.

In Lemma 3.10, we estimated the length of the reduced cyclic concatenation of tame-enough framed segment cycles. It turns out that we can also detect its location if furthermore it is zigzag, which is also the last lemma of this section.

Lemma 3.19 *For any $\delta > 0$, there exist constants L_0 and ϵ_0 such that the following holds. Suppose M is an oriented hyperbolic 3-manifold and $\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4$ is (L,ϵ)-zigzag in M, for some $L > L_0$ and $0 < \epsilon < \epsilon_0$. Then the Hausdorff distance between the carriers of $\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4$ and $[\mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4]$ is less than δ.*

Proof For convenience, let $\mathfrak{s} = \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4$, $l_{2i-1} = \boldsymbol{hl}(\mathfrak{s}_{2i-1})$ and $l_{2i} = \boldsymbol{hl}(\mathfrak{s}_{2i}) - \frac{1}{2}i\pi$ for $i = 1, 2$, with $|l_{2i}| < \epsilon$. Let A_i be the intersection of $\mathfrak{s}_i$ and $\mathfrak{s}_{i+1}$ for $i = 1, 2, 3, 4$, where $\mathfrak{s}_5 = \mathfrak{s}_1$. Without loss of generality, we assume that $p_0 = (0, 1) \in \mathbb{H}^3$ in the upper half-space model, also denoted by $\widetilde{A}_1$, is a lift of A_1, the initial direction $\vec{t}_{\mathrm{ini}}(\mathfrak{s}_1)$ at A_1 lifts to $\boldsymbol{u}_0 = (0, 1) \in T_{p_0}\mathbb{H}^3$ and the initial framing $\vec{n}_{\mathrm{ini}}(\mathfrak{s}_1)$ lifts to $\boldsymbol{v}_0 = (1, 0) \in T_{p_0}\mathbb{H}^3$. Let $\tilde{\mathfrak{s}}$ be the lift of $\mathfrak{s}$ which passes through $\widetilde{A}_1$. Now we want to prove that there is a lift of $[\mathfrak{s}]$ which is close to $\tilde{\mathfrak{s}}$.

Since $[\mathfrak{s}]$ is the geodesic representative of $\mathfrak{s}$, we know that the right action along $\mathfrak{s}$ is conjugate to the right action along $[\mathfrak{s}]$ in $\mathrm{PSL}(2,\mathbb{C})$. That is,

$$X = A(l_1)B\big(\tfrac{1}{2}\pi\big)A\big(l_2 + \tfrac{1}{2}i\pi\big)B\big(\tfrac{1}{2}\pi\big)A(l_3)B\big(\tfrac{1}{2}\pi\big)A\big(l_4 + \tfrac{1}{2}i\pi\big)B\big(\tfrac{1}{2}\pi\big) \sim A(l) \in \mathrm{PSL}(2,\mathbb{C}), \tag{3-20}$$

where $l = \boldsymbol{hl}([\mathfrak{s}])$. Let $Y \in \mathrm{PSL}(2,\mathbb{C})$ such that $X = YA(l)Y^{-1}$. Then $\Psi_0 \cdot Y$ is a frame on a lift of $[\mathfrak{s}]$ with $\boldsymbol{u}_0 \cdot Y = Y \cdot \boldsymbol{u}_0$ a tangent vector. The right action and the left action have the same results when acting on Ψ_0, and hence $Y \cdot \widetilde{A}_1$ is a point on a lift of $[\mathfrak{s}]$. Moreover, the collection of all $Y \cdot \widetilde{A}_1$ for Y conjugating X to $A(l)$ is a lift of $[\mathfrak{s}]$. This lift is also the fixed geodesic of X as a left action in $\mathbb{H}^3$. We denote this lift by $[\tilde{\mathfrak{s}}]$. Then the distance from A_1 to $\mathfrak{s}$ is exactly the distance from $\widetilde{A}_1$ to $[\tilde{\mathfrak{s}}]$.

The right action along $\mathfrak{s}$ is given by

$$\begin{aligned} X &= A(l_1)B\big(\tfrac{1}{2}\pi\big)A\big(l_2 + \tfrac{1}{2}i\pi\big)B\big(\tfrac{1}{2}\pi\big)A(l_3)B\big(\tfrac{1}{2}\pi\big)A\big(l_4 + \tfrac{1}{2}i\pi\big)B\big(\tfrac{1}{2}\pi\big) \\ &= \begin{pmatrix} e^{l_1} & 0 \\ 0 & e^{-l_1} \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} ie^{l_2} & 0 \\ 0 & -ie^{-l_2} \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \\ &\qquad \cdot \begin{pmatrix} e^{l_3} & 0 \\ 0 & e^{-l_3} \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} ie^{l_4} & 0 \\ 0 & -ie^{-l_4} \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \\ &= -\begin{pmatrix} e^{l_1} & 0 \\ 0 & e^{-l_1} \end{pmatrix}\begin{pmatrix} \cosh(l_2) & \sinh(l_2) \\ -\sinh(l_2) & -\cosh(l_2) \end{pmatrix}\begin{pmatrix} e^{l_3} & 0 \\ 0 & e^{-l_3} \end{pmatrix}\begin{pmatrix} \cosh(l_4) & \sinh(l_4) \\ -\sinh(l_4) & -\cosh(l_4) \end{pmatrix} \\ &\overset{\mathrm{PSL}(2,\mathbb{C})}{=} \begin{pmatrix} e^{l_1} & 0 \\ 0 & e^{-l_1} \end{pmatrix}\begin{pmatrix} \cosh(l_2) & \sinh(l_2) \\ -\sinh(l_2) & -\cosh(l_2) \end{pmatrix}\begin{pmatrix} e^{l_3} & 0 \\ 0 & e^{-l_3} \end{pmatrix}\begin{pmatrix} \cosh(l_4) & \sinh(l_4) \\ -\sinh(l_4) & -\cosh(l_4) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{aligned} \tag{3-21}$$

where

$$\begin{aligned} a(l_1,l_2,l_3,l_4) &= e^{l_1}(e^{l_3}\cosh(l_2)\cosh(l_4) - e^{-l_3}\sinh(l_2)\sinh(l_4)), \\ b(l_1,l_2,l_3,l_4) &= e^{l_1}(e^{l_3}\cosh(l_2)\sinh(l_4) - e^{-l_3}\sinh(l_2)\cosh(l_4)), \\ c(l_1,l_2,l_3,l_4) &= e^{-l_1}(e^{-l_3}\cosh(l_2)\sinh(l_4) - e^{l_3}\sinh(l_2)\cosh(l_4)), \\ d(l_1,l_2,l_3,l_4) &= e^{-l_1}(e^{-l_3}\cosh(l_2)\cosh(l_4) - e^{l_3}\sinh(l_2)\sinh(l_4)), \end{aligned} \tag{3-22}$$

are functions of l_1, l_2, l_3 and l_4. Let $\lambda = e^l$. Then

(3-23) $$z_1 = \frac{b}{\lambda - a} = \frac{\lambda - d}{c} \quad \text{and} \quad z_2 = \frac{b\lambda}{1 - a\lambda} = \frac{1 - d\lambda}{c\lambda}$$

are the two fixed points of X, as a left action, on $\mathbb{C} = \partial\mathbb{H}^3$. Hence we want to determine the distance D from $\widetilde{A}_1 = (0, 1)$ to the geodesic connecting z_1 and z_2, which is $[\tilde{\gamma}]$. Consider Möbius transformation

$$T : z \mapsto \frac{z - z_1}{z - z_2}$$

which sends $[\mathfrak{s}]$ to the positive t-axis. We then have

$$T((0,1)) = \left(\frac{1 + z_1\bar{z}_2}{1 + |z_2|^2}, \frac{|z_1 - z_2|}{1 + |z_2|^2} \right).$$

Hence D is also the distance from $T((0,1))$ to the positive t-axis, and it is given by

$$\begin{aligned} e^D &= \frac{|(1 + z_1\bar{z}_2)/(1 + |z_2|^2)| + \sqrt{|(1 + z_1\bar{z}_2)/(1 + |z_2|^2)|^2 + |(|z_1 - z_2|)/(1 + |z_2|^2)|^2}}{(|z_1 - z_2|)/(1 + |z_2|^2)} \\ &= \frac{|1 + z_1\bar{z}_2| + \sqrt{|1 + z_1\bar{z}_2|^2 + |z_1 - z_2|^2}}{|z_1 - z_2|}. \end{aligned}$$

Therefore

(3-24) $$\sinh(D) = \left| \frac{1 + z_1\bar{z}_2}{z_1 - z_2} \right|.$$

By (3-23),

$$|z_1 - z_2| = \left| \frac{\lambda - d}{c} - \frac{1 - d\lambda}{c\lambda} \right| = \left| \frac{\lambda^2 - 1}{c\lambda} \right| = \left| \frac{1}{c}\left(\lambda - \frac{1}{\lambda}\right) \right| = \left| \frac{\sqrt{(a+d)^2 - 4}}{c} \right| = \frac{\sqrt{|(a+d)^2 - 4|}}{|c|}$$

and

$$\begin{aligned} |1 - z_1\bar{z}_2|^2 &= (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) = 1 + |z_1 z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \\ &= 1 + \left| \frac{(\lambda - d)(1 - d\lambda)}{c^2\lambda} \right|^2 - \frac{\lambda - d}{c} \cdot \frac{1 - \bar{d}\bar{\lambda}}{\bar{c}\bar{\lambda}} - \frac{\bar{\lambda} - \bar{d}}{\bar{c}} \cdot \frac{1 - d\lambda}{c\lambda} \\ &= 1 + \left| \frac{(1 - ad)\lambda}{c^2\lambda} \right| - \frac{1}{|c|^2}\left(\frac{\lambda}{\bar{\lambda}} + \frac{\bar{\lambda}}{\lambda} + 2|d|^2 - d\left(\bar{\lambda} + \frac{1}{\bar{\lambda}}\right) - \bar{d}\left(\lambda + \frac{1}{\lambda}\right) \right) \\ &= 1 + \frac{|b|^2}{|c|^2} + \frac{1}{|c|^2}\left(a\bar{d} + \bar{a}d - \frac{\lambda}{\bar{\lambda}} - \frac{\bar{\lambda}}{\lambda} \right) \\ &\le 1 + \frac{|b|^2}{|c|^2} + \frac{2|ad| + 2}{|c|^2} = 1 + \frac{|b|^2}{|c|^2} + \frac{2|bc + 1| + 2}{|c|^2} \le \frac{(|b| + |c|)^2 + 4}{|c|^2} \\ &\le \left| \frac{|b| + |c| + 2}{c} \right|^2. \end{aligned}$$

Together with (3-24) we have

(3-25) $$\sinh(D) \le \frac{|b| + |c| + 2}{\sqrt{|(a+d)^2 - 4|}}.$$

Take $L_1 = 10^{10}$ and $\epsilon_1 = 10^{-10}$. Then by (3-22), $L > L_1$ and $0 < \epsilon < \epsilon_1$, we have

$$\text{(3-26)}\quad \begin{aligned} |b|+|c|+2 &= |e^{l_1}(e^{l_3}\cosh(l_2)\sinh(l_4) - e^{-l_3}\sinh(l_2)\cosh(l_4))| \\ &\qquad + |e^{-l_1}(e^{-l_3}\cosh(l_2)\sinh(l_4) - e^{l_3}\sinh(l_2)\cosh(l_4))| + 2 \\ &\le 4(3\epsilon)e^{\mathrm{Re}(l_1+l_3)} + 2 \end{aligned}$$

and

$$\text{(3-27)}\quad \begin{aligned} |(a+d)^2-4| &= |(\cosh(l_2)\cosh(l_4)\cosh(l_1+l_3) + \sinh(l_2)\sinh(l_4)\cosh(l_1-l_3))^2 - 4| \\ &\ge (|\cosh(l_2)\cosh(l_4)\cosh(l_1+l_3)| - |\sinh(l_2)\sinh(l_4)\cosh(l_1-l_3)|)^2 - 4 \\ &\ge \tfrac{1}{4}e^{\mathrm{Re}(l_1+l_3)}. \end{aligned}$$

Thus by (3-25), (3-26) and (3-27),

$$\text{(3-28)}\quad \sinh(D) \le \frac{12\epsilon e^{\mathrm{Re}(l_1+l_3)} + 2}{(1/4)e^{\mathrm{Re}(l_1+l_3)}} = 48\epsilon + 8e^{-\mathrm{Re}(l_1+l_3)} \le 48\epsilon + 8e^{-2L}.$$

Hence for any $\delta > 0$, we can find L_2 and ϵ_2 such that when $L > L_2$ and $0 < \epsilon < \epsilon_2$, we have

$$\text{(3-29)}\quad 48\epsilon + 8e^{-2L} < \delta.$$

So let $L_0 = \max\{L_1, L_2\}$ and $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. Then by (3-28) and (3-29), we get $D < \sinh(D) < \delta$ when $L > L_0$ and $0 < \epsilon < \epsilon_0$.

Similarly, we have the same result for $\widetilde{A}_3$. For $\widetilde{A}_2$ and $\widetilde{A}_4$, we only need to reverse the orientation of each $\mathfrak{s}_i$ and use the same technique for $\bar{\mathfrak{s}} = (\bar{\mathfrak{s}}_4)(\bar{\mathfrak{s}}_3)(\bar{\mathfrak{s}}_2)(\bar{\mathfrak{s}}_1)$. Then by the property of the distance function between geodesics in $\mathbb{H}^3$, we know that for each point $\widetilde{A}$ on $\tilde{\mathfrak{s}}$, the distance from $\widetilde{A}$ to $[\tilde{\mathfrak{s}}]$ is less than δ.

On the other hand, we consider the projection map ρ from $\tilde{\mathfrak{s}}$ to $[\tilde{\mathfrak{s}}]$ sending each point on $\tilde{\mathfrak{s}}$ to its closest point on $[\tilde{\mathfrak{s}}]$. Then ρ is a continuous map, so it is onto since $\tilde{\mathfrak{s}}$ and $[\tilde{\mathfrak{s}}]$ have the same endpoint on the boundary at infinity of $\mathbb{H}^3$. For each $\widetilde{B}$ on $[\tilde{\mathfrak{s}}]$, let $\widetilde{B}'$ be a preimage of $\widetilde{B}$ under ρ. Since $\widetilde{B}$ is the closest point of $\widetilde{B}'$ on $[\tilde{\mathfrak{s}}]$, we have $d(\widetilde{B}, \widetilde{B}') < \delta$. Hence the distance from $\widetilde{B}$ to $\tilde{\mathfrak{s}}$ is less than δ.

To conclude, when $L > L_0$ and $0 < \epsilon < \epsilon_0$, the Hausdorff distance between $\tilde{\mathfrak{s}}$ and $[\tilde{\mathfrak{s}}]$ is less than δ in $\mathbb{H}^3$, so the same result holds for $\mathfrak{s}$ and $[\mathfrak{s}]$ in M. □

4 Pants, assemblies and ideal triangles

4.1 Normal bundles and complex distances

Suppose $\gamma\colon T \to M$ is a geodesic in M with unit speed and constant velocity, where T is a Riemannian 1-manifold. Then we define the unit normal bundle by

$$N^1(\gamma) = \{(u,v) : u \in U,\ v \in T_{\gamma(u)}M,\ \langle v, \gamma'(u)\rangle = 0,\ \|v\| = 1\}.$$

Then $N^1(\gamma)$ is a torsor for $\mathbb{C}/(\boldsymbol{l}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$ when γ is a closed geodesic.

Suppose γ is an oriented geodesic in M and $\boldsymbol{u}, \boldsymbol{v} \in N^1(\gamma)$. Let $p \in \gamma$ be the basepoint of $\boldsymbol{u}$ and q be the basepoint of $\boldsymbol{v}$. By $\boldsymbol{d}_\gamma(\boldsymbol{u}, \boldsymbol{v})$ we denote the complex distance from $\boldsymbol{u}$ to $\boldsymbol{v}$ along γ. Here the real part of $\boldsymbol{d}_\gamma(\boldsymbol{u}, \boldsymbol{v})$ is the signed distance between p and q along γ, and its imaginary part is the signed angle between $\boldsymbol{u}$ and the parallel transport of $\boldsymbol{v}$ from q to p. The signed angle is determined by the orientation of γ. Then the complex distance is well defined with value in $\mathbb{C}/2\pi i\mathbb{Z}$. We have

$$\boldsymbol{d}_\gamma(\boldsymbol{u}, \boldsymbol{v}) = -\boldsymbol{d}_\gamma(\boldsymbol{v}, \boldsymbol{u}) = -\boldsymbol{d}_{\bar{\gamma}}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{d}_\gamma(-\boldsymbol{u}, \boldsymbol{v}) + i\pi,$$

where $-\boldsymbol{u} \in N^1(\gamma)$ shares the same basepoint with $\boldsymbol{u}$ and points the opposite direction to $\boldsymbol{u}$. When γ is a closed geodesic, it should be specified whether the complex distance from $\boldsymbol{u}$ to $\boldsymbol{v}$ is following or against the orientation of γ.

4.2 Right-angled hexagons

An *oriented* right-angled hexagon $\mathfrak{s}$ in M is a cycle of six oriented framed segments $\mathfrak{s} = \mathfrak{s}_1\mathfrak{s}_2\mathfrak{s}_3\mathfrak{s}_4\mathfrak{s}_5\mathfrak{s}_6$ such that, for $i \in \mathbb{Z}/6\mathbb{Z}$,

- $\vec{n}_{\mathrm{ini}}(\mathfrak{s}_i) = \vec{t}_{\mathrm{ter}}(\mathfrak{s}_{i-1})$,
- $\vec{n}_{\mathrm{ter}}(\mathfrak{s}_i) = -\vec{t}_{\mathrm{ini}}(\mathfrak{s}_{i+1})$,
- $\mathfrak{s}$ is nullhomotopic as a closed curve.

Then $\mathfrak{s}_i$ is a connection between $\mathfrak{s}_{i-1}$ and $\mathfrak{s}_{i+1}$.

By hyperbolic trigonometry, we have

$$\cosh(\mathfrak{s}_i) = \frac{\cosh(\mathfrak{s}_{i-1})\cosh(\mathfrak{s}_{i+1}) + \cosh(\mathfrak{s}_{i+3})}{\sinh(\mathfrak{s}_{i-1})\sinh(\mathfrak{s}_{i+1})}.$$

Thus $\mathfrak{s}$ is determined by the complex lengths of any three mutually nonadjacent framed segments, up to isometries, and vice versa.

4.3 Orthogeodesics, connections and feet

Suppose there are two oriented closed geodesics α_i, for $i = 1, 2$, in M with unit speed, constant velocity and distinct images. Suppose that $\eta\colon [0, l] \to M$ is a geodesic segment satisfying

- η has unit speed and constant velocity,
- $\eta(0) \in \alpha_1$ and $\eta(l) \in \alpha_2$,
- $\eta(0, l)$ meets with the α_i orthogonally for $i = 1, 2$.

We then call η an *orthogeodesic* or a *connection* between α_1 and α_2. In the case where α_1 and α_2 coincide, we will call η a *third connection*. Besides, we define the complex distance between α_1 and α_2 along η to be the complex length of η by

$$\boldsymbol{d}_\eta(\alpha_1, \alpha_2) = \boldsymbol{d}_\eta\big(\alpha_1'(\eta(0)), \alpha_2'(\eta(1))\big).$$

If α_1 and α_2 intersect, η_1 is the constant function; otherwise we can reparametrize η_1 to be $\hat{\eta}\colon [0,l] \to M$ with unit speed. Then we let $\mathrm{foot}_{\alpha_1}(\eta) = \hat{\eta}'(0) \in N^1(\alpha_1)$ and $\mathrm{foot}_{\alpha_2}(\eta) = -\hat{\eta}'(l) \in N^1(\alpha_2)$ be the *feet* of $\hat{\eta}$ on α_0 and α_1, respectively. The basepoint of a foot is called a *footpoint*.

Similarly, we can define the orthogeodesic, complex distance and feet for two distinct oriented geodesics in $\mathbb{H}^3$. Here the orthogeodesic between them is always unique, so we omit the subscript for complex distance in this case.

4.4 Pants, half-lengths and third connections

Let P_0 be an oriented topological pants as a manifold with boundary. We say a *pants* in M is an injective homomorphism $\rho\colon \pi_1(P_0) \to G$, up to conjugacy. A pants in M is determined by a continuous map $f\colon P_0 \to M$ up to homotopy, and vice versa, so we can also call f or $f(P_0)$ a pants. Let the C_i be the boundary component of P_0, and orient C_i so that P_0 is on the left of C_i, for $i \in \mathbb{Z}/3\mathbb{Z}$. Then there is a unique oriented closed geodesic γ_i in M that is freely homotopic to $f(C_i)$. We can homotope f so that f maps C_i to γ_i, and we call such f a *nice* pants. From now on, when talking about pants in M, we will always assume that they are nice.

Choose any simple nonseparating arc α_i connecting $\mathbb{C}_{i-1}$ and C_{i+1}. We then can homotope f so that it maps α_i to an orthogeodesic η_i between γ_{i-1} and γ_{i+1}. Here η_i does not depend on the choice of α_i and f. We refer to the γ_i and η_i as the *cuffs* and the *short orthogeodesic* of the pants f, respectively, and the feet of η_{j-1} and η_{j+1} on γ_i are called the *feet* of f on γ_i. When we talk about a cuff γ, the information of the associated pants is also carried by γ.

We notice that the pants is divided into two oriented right-angled hexagons sharing three sides the η_i, where the orientations are determined by the orientations of the γ_i. Moreover, these two right-angled hexagons are isometric. Let $\boldsymbol{u}_i^j$ be the foot of η_j on γ_i for $i \in \mathbb{Z}/3\mathbb{Z}$ and $j \neq i$. In this case, the complex distance $\boldsymbol{d}_{\gamma_i}(\boldsymbol{u}_i^j, \boldsymbol{u}_i^k)$ from $\boldsymbol{u}_i^j$ to $\boldsymbol{u}_i^j$ is always following the orientation of γ_i for $j \neq i$, $k \neq i$ and $j \neq k$. Since the two right-angled hexagons are isometric, we have

$$\boldsymbol{d}_{\gamma_i}(\boldsymbol{u}_i^{i-1}, \boldsymbol{u}_i^{i+1}) = \boldsymbol{d}_{\gamma_i}(\boldsymbol{u}_i^{i+1}, \boldsymbol{u}_i^{i-1}).$$

Thus we let

$$\boldsymbol{hl}(\gamma_i) = \boldsymbol{d}_{\gamma_i}(\boldsymbol{u}_i^{i-1}, \boldsymbol{u}_i^{i+1})$$

be the *half-length* of γ_i. It is not hard to see $2\boldsymbol{hl}(\gamma_i) = \boldsymbol{l}(\gamma_i)$.

Suppose γ is a closed geodesic in M. Then by the discussion of [Kahn and Wright 2021, Section 3.2], a third connection η on γ uniquely determines and is determined by a pants Π of which γ is a cuff. Under this circumstance, η is called the third connection of Π on γ.

Suppose $R, R_i \in \mathbb{C}$ with positive real parts and $\epsilon > 0$. For a pants $P \in M$ with cuffs C_1, C_2 and C_3, we say P is (R,ϵ)-*good* if

$$|\boldsymbol{hl}(C_i) - R| < \epsilon,$$

for $i = 1, 2, 3$. We say P is $(R_i, \epsilon)_{i=1}^3$-*good* if

$$|\boldsymbol{hl}(C_i) - R_i| < \epsilon,$$

for $i = 1, 2, 3$. Here for each inequality above, we mean that there is a lift of the half-length to $\mathbb{C}$ to make the inequality hold. This convention will be consistently upheld throughout the remainder of this paper.

When $\epsilon = 0$, P is called an R-perfect or $(R_i)_{i=1}^3$-perfect pants. Sometime we might omit the constants for convenience and simply refer to P as a perfect pants, when there is no ambiguity.

4.5 Shears

Suppose that γ is a cuff in M. Then there is a involution σ on $N^1(\gamma)$ by $\sigma(x) = x + \boldsymbol{hl}(\gamma)$. Here "+" is well defined since $N^1(\gamma)$ is a torsor for $\mathbb{C}/(\boldsymbol{l}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$. We denote by $\mathbb{N}^1(\sqrt{\gamma})$ the quotient of $N^1(\gamma)$ by σ, and call it the quotient unit normal bundle of γ, which is a torsor for $\mathbb{C}/(\boldsymbol{hl}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$. When γ is a cuff of a pants Π, by the discussion in Section 4.4, the two feet of Π on γ descend to the same point in $N^1(\sqrt{\gamma})$. We denote by $\boldsymbol{foot}_\gamma(\Pi)$ their image in $N^1(\sqrt{\gamma})$ and call it the foot of Π on γ in $N^1(\sqrt{\gamma})$.

Suppose Π_1 and Π_2 are two pants satisfying that

- Π_1 and Π_2 share a cuff γ,
- the half-length of γ in these two pants are equal,
- Π_1 is on the left of γ and Π_2 is on the right of γ.

We define the *short shear* $s(\gamma)$ from Π_2 to Π_1 along γ by

$$s(\gamma) = \boldsymbol{foot}_\gamma(\Pi_1) - \boldsymbol{foot}_\gamma(\Pi_2) - i\pi.$$

Then $s(\gamma) \in \mathbb{C}/(\boldsymbol{hl}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$. If we reverse the orientation of γ, then the roles of Π_1 and Π_2 exchange, which will lead to the same value in $\mathbb{C}/(\boldsymbol{hl}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$.

Suppose Π_1 and Π_2 meet the assumption above. Suppose that η_i is the third connection of Π_i on γ, and $\boldsymbol{v}_i^1$ and $\boldsymbol{v}_i^2$ are the feet of η_i on γ, for $i = 1, 2$. We then let

$$(ls^1(\gamma), ls^2(\gamma)) = (\boldsymbol{v}_1^1 - \boldsymbol{v}_2^1 - i\pi, \boldsymbol{v}_1^2 - \boldsymbol{v}_2^2 - i\pi)$$

be the *long shears* from Π_2 and Π_1 along γ, which is a 2-tuple in $\mathbb{C}/(\boldsymbol{l}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$. Moreover, we call $ls(\gamma) = ls^1(\gamma) + ls^2(\gamma)$ the *sum long shear*. We observe that the value of long shears depends on the choice of the superscript of feet of third connections. On the other hand, the sum long shear is well defined, since we have the equality

$$ls(\gamma) = (\boldsymbol{v}_1^1 - \boldsymbol{v}_2^1) + (\boldsymbol{v}_1^2 - \boldsymbol{v}_2^2) = 2s(\gamma).$$

Here $s(\gamma)$ is lifted to $\mathbb{C}/(\boldsymbol{l}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$ and we use the fact that the foot of a short orthogeodesic of a pants on a cuff is the midpoint of the two feet of the third connection on this cuff. Hence the value of $2s(\gamma)$ is unique.

For $R > 0$ large enough, $\epsilon > 0$ and two (R, ϵ)-good pants Π_1 and Π_2, the lengths of the short orthogeodesics are exponentially small, but the lengths of third connections are about $\frac{1}{2}R$. Thus short shears are named "short" because they are the shears between feet of short orthogeodesics, and long shears are named "long" because they are the shears between third connections.

4.6 Assemblies

Suppose that $\mathcal{F} = \{f_i \colon P_i \to M\}$ is a multiset of pants in M satisfying that

- there is a fixed-point-free involution τ on the multiset $\bigcup_i \partial P_i$,
- for each $\alpha \in \bigcup_i \partial P_i$, α and $\tau(\alpha)$ are mapped to the same geodesic in M.

Then we call $\mathcal{A} = (\mathcal{F}, \tau)$ an *assembly*. By identifying α and $\tau(\alpha)$ with the help of the identification of their associated geodesics in M, we get a closed topological surface $S_{\mathcal{A}} = \bigl(\bigcup P_i\bigr)/\tau$, which is formed by joining the P_i. Define $f_{\mathcal{A}} \colon S_{\mathcal{A}} \to M$ by joining the f_i via τ. When $S_{\mathcal{A}}$ is connected, let $(f_{\mathcal{A}})_* \colon \pi_1(S_{\mathcal{A}}) \to G$ be the induced homomorphism between fundamental groups, and let $\rho_{\mathcal{A}} \colon \pi_1(S_{\mathcal{A}}) \to \mathrm{PSL}(2, \mathbb{C})$ be the associated representation.

4.7 The doubling trick

Suppose now that we have a family of pants $\mathcal{F} = \{f_i \colon P_i \to M\}$, where the P_i are orientable. We then take two copies of each $f_i \colon P_i \to M$, one of each orientation of P_i, denoted by $f_i^+ \colon P_i^+ \to M$ and $f_i^- \colon P_i^- \to M$, and obtain a family of oriented pants $2\mathcal{F}$.

For each geodesic $\gamma \in M$, let $\mathcal{F}(\gamma)$ be the submultiset of $\bigcup_i \partial P_i$ composed of those elements that are mapped to the unoriented γ in M. $\mathcal{F}(\gamma)$ may be empty for some γ. Now suppose that there is a self-bijection σ_γ on $\mathcal{F}(\gamma)$. Then we can make up $2\mathcal{F}(\gamma)$ by taking two copies of each $\alpha \in \mathcal{F}(\gamma)$ with opposite orientation, which is a submultiset of $\bigl(\bigcup_i \partial P_i^+\bigr) \cup \bigl(\bigcup_i \partial P_i^-\bigr)$. For $\alpha \in \mathcal{F}(\gamma)$, let $\alpha^+ \in 2\mathcal{F}(\gamma)$ be mapped to γ and $\alpha^- \in 2\mathcal{F}(\gamma)$ be mapped to γ^{-1}. Hence we can define an involution τ_γ on $2\mathcal{F}(\gamma)$ by

$$\tau_\gamma(\alpha^+) = (\sigma_\gamma(\alpha))_-$$

and

$$\tau_\gamma(\alpha^-) = (\sigma_\gamma^{-1}(\alpha))_+.$$

By joining all such involution, we get an involution τ on $\bigl(\bigcup_i \partial P_i^+\bigr) \cup \bigl(\bigcup_i \partial P_i^-\bigr)$, which will lead to an assembly.

4.8 Pants decompositions of quasi-Fuchsian surface groups

Suppose that S_g is the orientable closed topological surface of genus g, where $g \geq 2$. A *pants decomposition* of S_g is a maximal set $\mathcal{C}$ of disjoint, nontrivial and nonhomotopic simple closed curves on S_g. A pants decomposition $\mathcal{C}$ of S_g is called *nonseparating* if each element of $\mathcal{C}$ is a nonseparating curve on S_g. For a genus-2 hyperbolic surface S, we observe that a nonseparating pants decomposition of S yields us two isometric pants, due to the hyperelliptic involution of S.

Suppose Γ is a quasi-Fuchsian surface group of genus $g \geq 2$ and $\mathcal{C}$ is a pants decomposition of S_g. Then $\mathcal{C}$ has $3g-3$ elements, so we let $\mathcal{C} = \{\gamma_1, \gamma_2, \dots, \gamma_{3g-3}\}$. By the theory of pleated surfaces by Thurston, we can construct a pleated surface $f : S_g \to \mathrm{CM}(\Gamma)$ such that the $\gamma_i' = f(\gamma_i)$ are contained in the pleated locus, where $\mathrm{CM}(\Gamma)$ is the convex core of $\mathbb{H}^3/\Gamma$. Let $\mathcal{C}' = \{\gamma_1', \gamma_2', \dots, \gamma_{3g-3}'\}$ and $S_{\Gamma,\mathcal{C}} = f(S_g)$. Then each component of $S_{\Gamma,\mathcal{C}'} - \mathcal{C}'$ is a pants with geodesic boundaries in $\mathrm{CM}(\Gamma)$. By [Tan 1994, Theorem 1], $\{(\boldsymbol{hl}(\gamma_i'), s(\gamma_i'))$ for $i = 1, 2, \dots, 3g-3\}$ is the complex Fenchel–Nielsen coordinate of Γ corresponding to $\mathcal{C}$. Additionally, we refer to a *pants decomposition* of Γ as a pants decomposition $\mathcal{C}$ of S_g together with its corresponding complex Fenchel–Nielsen coordinate. For more details see [Tan 1994; Kourouniotis 1994].

For a (quasi-)Fuchsian surface S, a pants decomposition $\mathcal{C}$ of S is called (R,ϵ)-*good* if each component of this pants decomposition is (R,ϵ)-good. $\mathcal{C}$ is called $(R_i,\epsilon)_{i=1}^3$-*good* if each component of this pants decomposition is $(R_i,\epsilon)_{i=1}^3$-good.

4.9 Ideal triangulations

An *ideal triangle* in a hyperbolic manifold N is the image of an injective local isometry from an ideal triangle in $\mathbb{H}^3$ to N, where the dimension of N is 2 or 3 in our paper. We notice that all ideal triangles have hyperbolic structures, and all ideal triangles in $\mathbb{H}^3$ are isometric. Suppose S is a closed surface with a hyperbolic structure. An *ideal triangulation* of S is a lamination with finitely many leaves whose complementary components are ideal triangles.

For two ideal triangles Δ_1 and Δ_2 which share a common edge γ in N, drop the altitude ξ_i of Δ_i at the remaining vertex to γ, and let $\boldsymbol{u}_i$ be the foot of ξ_i on γ, for $i = 1, 2$. We then define $\mathrm{ts}(\gamma) = \boldsymbol{u}_1 - \boldsymbol{u}_2 - i\pi$ to be the *shear* from Δ_2 and Δ_1 along γ.

We refer the readers to [Bestvina et al. 2013; Zhu 2017] for more discussion about ideal triangles in hyperbolic spaces.

4.10 Slow and constant turning normal fields

Suppose γ is a closed geodesic in M. A unit normal vector field along γ, denoted by X, is a section of the unit normal bundle $N^1(\gamma)$ over γ. Since $N^1(\gamma)$ is a torsor for $\mathbb{C}/(\boldsymbol{l}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$, X yields a curve in $\mathbb{C}/(\boldsymbol{l}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z})$ up to translation. Consequently, we can define the *slope* of X at each point along γ when X is a smooth field.

A *constant turning normal field* along γ is a smooth unit normal field X along γ with constant slope. The constant slope equals $(\theta + 2k\pi)/a$ for some $k \in \mathbb{Z}$, where $\boldsymbol{l}(\gamma) = a + i\theta$. Furthermore, we refer to X as a *slow and constant turning normal field* if

$$-\pi < \theta + 2k\pi \leq \pi.$$

We observe that a slow and constant turning normal field along γ is uniquely determined by its value at a single point. As a result, the space of slow and constant turning normal fields along γ is a circle.

5 Pants decomposition of genus-2 quasi-Fuchsian groups

In this section, we will construct a pants decomposition of Γ with long cuffs and bounded shears. From now on, assume Γ is a genus-2 K-quasi-Fuchsian group for some given $K > 1$, and $\mathrm{CM}(\Gamma)$ is the convex core of $\mathbb{H}^3/\Gamma$ which is homotopy equivalent to a topological genus-2 oriented surface S.

5.1 Good pants decomposition

In this subsection, our primary focus will be on the cuff lengths of pants decompositions of Γ, and we aim to prove the following theorem:

Theorem 5.1 *There is a constant $m > 0$ such that for any positive real number R_0, there exist $R > R_0$ and a nonseparating (R, m)-good pants decomposition of Γ.*

Proof Suppose $\{C_1', C_2', C_3'\}$, a set of three disjoint oriented simple closed curves on S, is a nonseparating pants decomposition of S. Let $f : S \to \mathrm{CM}(\Gamma)$ be a pleated map such that the $C_i = f(C_i')$ are closed geodesics. Then $\{C_1, C_2, C_3\}$ is a nonseparating pants decomposition of Γ, and there are two isometric immersed pairs of pants P_1 and P_2 in $\mathrm{CM}(\Gamma)$ whose boundary components are both $\{C_1, C_2, C_3\}$. Suppose that the C_i are oriented so that P_1 is on the right side. In each P_k, let $\gamma_{i,k}$ be the short orthogonal geodesic between C_{i+1} and C_{i+2}, oriented from C_{i+1} to C_{i+2} for $i \in \mathbb{Z}/3\mathbb{Z}$ and $k \in \mathbb{Z}/2\mathbb{Z}$. Then let $\eta_{i,i+1}$ and $\eta_{i,i+2}$ be the orthogonal geodesic between $\gamma_{i,1}$ and $\gamma_{i,2}$ along C_{i+1} and C_{i+2}, respectively. We orient $\eta_{i,i+1}$ and $\eta_{i,i+2}$ from $\gamma_{i,1}$ to $\gamma_{i,2}$. Notice that $\eta_{i+1,i}$ and $\eta_{i-1,i}$ are the short shear between P_1 and P_2 along C_i.

We want to assign frames at endpoints of these oriented geodesic arcs to make them framed segments and use them to construct new pants decompositions of Γ. For a tuple of positive integers $(n_1, n_2, n_3) \in \mathbb{Z}_+^3$, consider the sets of geodesic arcs $A_i(n_1, n_2, n_3) = \{\eta_{i,i+1} C_{i+1}^{n_{i+1}}, \gamma_{i,2}, C_{i-1}^{-n_{i-1}} \eta_{i,i-1}^{-1}, \gamma_{i,1}^{-1}\}$, for $i \in \mathbb{Z}/3\mathbb{Z}$. Then the four geodesic arcs in each $A_i(n_1, n_2, n_3)$ can form a piecewise geodesic curve, because the concatenation of $\eta_{i,i+1}$ and $C_{i+1}^{n_{i+1}}$ can be regarded as a smooth geodesic. We can find a unique framing at each joint point such that there are framed segments $\mathfrak{a}_i^{(n_{i+1})}$, $\mathfrak{b}_i$, $\mathfrak{c}_i^{(n_{i-1})}$ and $\mathfrak{d}_i$, whose carriers are $\eta_{i,i+1} C_{i+1}^{n_{i+1}}$, $\gamma_{i,2}$, $C_{i-1}^{-n_{i-1}} \eta_{i,i-1}^{-1}$ and $\gamma_{i,1}^{-1}$, forming a zigzag continuous cycle. Let $\mathfrak{s}_i^{(n_{i+1}, n_{i+2})} = \mathfrak{s}_i^{(n_{i+1}, n_{i-1})} = \mathfrak{a}_i^{(n_{i+1})} \mathfrak{b}_i \mathfrak{c}_i^{(n_{i-1})} \mathfrak{d}_i$.

To see that $\{[\mathfrak{s}_1^{(n_2,n_3)}], [\mathfrak{s}_2^{(n_3,n_1)}], [\mathfrak{s}_3^{(n_1,n_2)}]\}$ gives us a pants decomposition of Γ, it suffices to prove that the pullback homotopy classes on S form a pants decomposition of S. Thus we just need to find disjoint representatives in these three homotopy classes on S.

We notice that $\mathfrak{s}_1^{(n_2,n_3)}$ and $\mathfrak{s}_i^{(n_3,n_1)}$ both spin around C_3 by n_3 times, so we perturb these two curves in a tubular neighborhood of C_3 as indicated in Figure 1.

We can use the same technique on C_1 and C_2, so we have three disjoint simple closed curves in the three free homotopy classes. Therefore the geodesic representatives of these three homotopy classes are disjoint

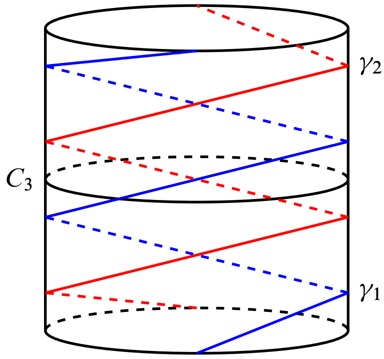

Figure 1: A neighborhood of C_3.

simple closed geodesics on S, which helps us prove that $\{[\mathfrak{s}_1^{(n_2,n_3)}], [\mathfrak{s}_2^{(n_3,n_1)}], [\mathfrak{s}_3^{(n_1,n_2)}]\}$ is indeed a pants decomposition of Γ.

Next we want to modify the real lengths of $\{[\mathfrak{s}_1^{(n_2,n_3)}], [\mathfrak{s}_2^{(n_3,n_1)}], [\mathfrak{s}_3^{(n_1,n_2)}]\}$ to be almost the same size by adjusting the value of (n_1, n_2, n_3). Since

$$\boldsymbol{l}(\mathfrak{a}_i^{(n_{i+1})}) = \boldsymbol{l}(\eta_{i,i+1}) + n_{i+1}\boldsymbol{l}(C_{i+1}) \quad \text{and} \quad \boldsymbol{l}(\mathfrak{c}_i^{(n_{i-1})}) = \boldsymbol{l}(\eta_{i,i-1}) + n_{i-1}\boldsymbol{l}(C_{i-1}),$$

by Lemma 3.10, for any $\epsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that when $n_1, n_2, n_3 > N$,

$$\textbf{(5-2)} \qquad \left| l([\mathfrak{s}_i^{(n_{i+1},n_{i+2})}]) - l(\mathfrak{a}_i^{(n_{i+1})}) - l(\mathfrak{b}_i) - l(\mathfrak{c}_i^{(n_{i-1})}) - l(\mathfrak{d}_i) + I_l\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{b}_i)\big) + I_l\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{d}_i)\big) \right| < \epsilon$$

and

$$\textbf{(5-3)} \quad \left| \phi([\mathfrak{s}_i^{(n_{i+1},n_{i+2})}]) - \phi(\mathfrak{a}_i^{(n_{i+1})}) - \phi(\mathfrak{b}_i) - \phi(\mathfrak{c}_i^{(n_{i-1})}) - \phi(\mathfrak{d}_i) + I_\phi\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{b}_i)\big) + I_\phi\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{d}_i)\big) \right| < \epsilon,$$

for $i \in \mathbb{Z}/3\mathbb{Z}$. To simplify the notation, we let

$$\sigma_i = \boldsymbol{l}(\eta_{i,i+1}) + \boldsymbol{l}(\eta_{i,i-1}) - \big(I_l\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{b}_i)\big) + I_l\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{d}_i)\big)\big) - i\big(I_\phi\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{b}_i)\big) + I_\phi\big(\tfrac{1}{2}\boldsymbol{l}(\mathfrak{d}_i)\big)\big).$$

Then by (5-2) and (5-3), when $n_1, n_2, n_3 > N$,

$$\textbf{(5-4)} \qquad |\boldsymbol{l}([\mathfrak{s}_i^{(n_{i+1},n_{i+2})}]) - n_{i+1}\boldsymbol{l}(C_{i+1}) - n_{i-1}\boldsymbol{l}(C_{i-1}) - \sigma_i| < \sqrt{2}\epsilon,$$

for $i \in \mathbb{Z}/3\mathbb{Z}$.

Consider the lattice $g\colon \mathbb{Z}^3 \to \mathbb{R}^3$ defined by

$$g(n_1, n_2, n_3) = \big(\mathrm{Re}(n_1\boldsymbol{l}(C_1) - \sigma_1), \mathrm{Re}(n_2\boldsymbol{l}(C_2) - \sigma_2), \mathrm{Re}(n_2\boldsymbol{l}(C_2) - \sigma_2)\big).$$

Then there is a constant $m_1 > 0$ such that for any point $P(x, y, z) \in \mathbb{R}^3$, we can find $n_1, n_2, n_3 \in \mathbb{Z}$ satisfying

$$d(g(n_1, n_2, n_3), P) < m_1.$$

Here d is the standard distance function in $\mathbb{R}^3$. Especially, for R' big enough, we can find n_1, n_2 and n_3 such that the distance between $g(n_1, n_2, n_3)$ and (R', R', R') is less than m_1. Since R' is big enough, we can assume the n_i are positive and $n_i > N$; here N comes from (5-4). Therefore there is $R_1 > 0$ such that for any $R' > R_1$, there exist $n_i \in \mathbb{Z}$ and $n_i > N$ satisfying

$$|\mathrm{Re}(n_i \boldsymbol{l}(C_i) - \sigma_i) - R'| < m_1, \tag{5-5}$$

for $i \in \mathbb{Z}/3\mathbb{Z}$.

Let $m_2 = \frac{1}{2}(m_1 + \sqrt{2}\epsilon)$. Then for any $R_0 > 0$, let $R' = \max\{R_1, 2R_0 + 2\sum_{i=1}^{3}|\sigma_i|, 3m_1\} + 1$. Thus we can find n_i satisfying (5-5). Let $2R = \mathrm{Re}(n_1\boldsymbol{l}(c_1) + n_2\boldsymbol{l}(c_2) + n_3\boldsymbol{l}(c_3)) - R'$. We then have

$$|\mathrm{Re}(n_{i+1}\boldsymbol{l}(C_{i+1}) + n_{i+2}\boldsymbol{l}(C_{i+2}) + \sigma_i) - 2R| = |-\mathrm{Re}(n_i\boldsymbol{l}(C_i)) + \mathrm{Re}(\sigma_i) + R'| < m_1. \tag{5-6}$$

Combining (5-4) and (5-6), we have

$$|\mathrm{Re}(\boldsymbol{l}([\mathfrak{s}_i^{(n_{i+1},n_{i+2})}])) - 2R| < \sqrt{2}\epsilon + m_1 = 2m_2,$$

so

$$|\mathrm{Re}(\boldsymbol{hl}([\mathfrak{s}_i^{(n_{i+1},n_{i+2})}])) - R| < m_2.$$

We also have

$$\begin{aligned} 2R &= \mathrm{Re}(n_1\boldsymbol{l}(c_1) + n_2\boldsymbol{l}(c_2) + n_3\boldsymbol{l}(c_3)) - R' \\ &> (\mathrm{Re}(\sigma_1) + R' - m_1) + (\mathrm{Re}(\sigma_2) + R' - m_1) + (\mathrm{Re}(\sigma_3) + R' - m_1) - R' \\ &= \sum_{i=1}^{3}\mathrm{Re}(\sigma_i) + 2R' - 3m_1 = \left(R' + \sum_{i=1}^{3}\mathrm{Re}(\sigma_i)\right) + (R' - 3m_1) > 2R_0. \end{aligned}$$

Letting $m = m_2 + \pi$ finishes the proof. □

Remark 5.7 The technique used above to build new pants decompositions out of a given one is called the *spinning construction*.

5.2 Ideal triangulation and shears

The main theorem of Section 5 is a stronger version of Theorem 5.1, with a control on shears:

Theorem 5.8 *Suppose Γ is a genus-2 quasi-Fuchsian group. Then there exist $B^+ > B^- > 0$ and $\delta > 0$ such that for any $R_0 > 0$, there exists $\bar{R} > R_0$ such that Γ admits a nonseparating $(\bar{R}, \delta)$-good pants decomposition with the real parts of twists in the interval (B^-, B^+).*

Before proving this theorem, we first introduce a lemma to estimate the shears:

Lemma 5.9 *There exist $W_0, \omega_0 > 0$ and a function $\kappa(\omega)$ satisfying*

$$\lim_{\omega \to 0} \kappa(\omega) = 0,$$

such that when $0 < \omega < \omega_0$, the following statement holds: Suppose that δ_1 and δ_2 are two nonintersecting geodesics in $\mathbb{H}^3$ with ξ the orthogeodesic between them and $Y_i = \delta_i \cap \xi$ for $i \in \mathbb{Z}/2\mathbb{Z}$. Suppose that A_1B_1 and A_2B_2 are two geodesic segments and ζ_i is a geodesic orthogonal to the geodesic segment

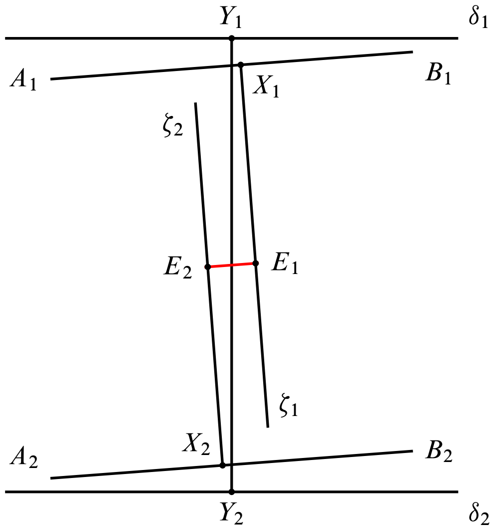

Figure 2: A picture for Lemma 5.9.

A_iB_i at X_i for $i \in \mathbb{Z}/2\mathbb{Z}$. Furthermore, suppose that ζ_1 and ζ_2 are distinct. Let τ be the common orthogonal between ζ_1 and ζ_2, and the orientation of ζ_i be from A_iB_i to τ. Let Z_1Z_2 be the common orthogonal between A_1B_1 and A_2B_2 with Z_i on the infinite geodesic A_iB_i for $i = 1, 2$. If

(1) $d(Y_1, Y_2), d(A_1, B_1), d(A_2, B_2) > W_0$,

(2) $d(A_i, \delta_i), d(B_i, \delta_i) < \omega$ *for* $i \in \mathbb{Z}/2\mathbb{Z}$,

(3) ζ_1 *and* ζ_2 *are disjoint*,

(4) $|d_{\mathbb{C}}(\zeta_1, \zeta_2) - i\pi| < \omega$,

then $d(X_i, Y_i), d(Y_i, Z_i), d(Z_i, X_i) < \kappa(\omega)$.

Proof We will prove this lemma by contradiction and limit process. We choose W_0 large enough and ω_0 small enough, for example $W_0 > 10^{10}$ and $\omega_0 < 10^{-10}$. Let $E_i = \tau \cap \zeta_i$ for $i = 1, 2$. A set of geodesics and geodesic segments $\{\delta_1, \delta_2, \xi, A_1B_1, A_2B_2, Z_1Z_2, \zeta_1, \zeta_2, \tau\}$ is called a *picture of* ω, denoted by $\mathcal{P}(\omega)$, if the above requirements (1)–(4) are satisfied.

We first study $d(X_i, Y_i)$. Suppose the inequality is not correct. Then there exist a decreasing sequence $\{\omega^{(n)}\}$ which converges to 0, a sequence of pictures $\{\mathcal{P}^{(n)}(\omega^{(n)})\}$ and $\kappa_0 > 0$ such that in each $\mathcal{P}^{(n)}(\omega^{(n)})$, $d(X_1^{(n)}, Y_1^{(n)}) < \kappa_0$ and $d(X_2^{(n)}, Y_2^{(n)}) < \kappa_0$ cannot hold together. Without loss of generality, we can assume that $d(X_1^{(n)}, Y_1^{(n)}) \geq \kappa_0 > 0$ for all n by passing to a subsequence.

Now we take the compactified hyperbolic plane $\overline{\mathbb{H}}^3$ and study $\{\mathcal{P}^{(n)}(\omega^{(n)})\}$ on $\overline{\mathbb{H}}^3$. Since $\overline{\mathbb{H}}^3$ is compact, a sequence of points in $\overline{\mathbb{H}}^3$ will have a convergent subsequence, and its limit can be a point in $(\overline{\mathbb{H}}^3)^\circ$ or a point on $\partial\overline{\mathbb{H}}^3$ at infinity. Besides, a sequence of geodesics also has a convergent subsequence, and its limit can be a geodesic in $(\overline{\mathbb{H}}^3)^\circ$ or a point on $\partial\overline{\mathbb{H}}^3$ at infinity as a degenerate geodesic.

Because each object in the picture $\mathcal{P}^{(n)}(\omega^{(n)})$ is either a point or a geodesic, we can assume $\{\mathcal{P}^{(n)}(\omega^{(n)})\}$ has a limit $\{\mathcal{P}^{(\infty)}\}$ by passing to subsequences. By applying isometries of $\mathbb{H}^3$, we can fix $E_1^{(i)}$ and $\zeta_1^{(i)}$, ie

$$E_1^{(1)} = E_1^{(2)} = \cdots = E_1^{(n)} = \cdots = E_1^{(\infty)}$$

and

$$\zeta_1^{(1)} = \zeta_1^{(2)} = \cdots = \zeta_1^{(n)} = \cdots = \zeta_1^{(\infty)}.$$

Then by

$$d(E_1^{(\infty)}, E_2^{(n)}) = d(E_1^{(n)}, E_2^{(n)}) = l(\tau^{(n)}) < \omega^{(n)},$$

we have

$$E_2^{(\infty)} = \lim_{n\to+\infty} E_2^{(n)} = E_1^{(\infty)}.$$

Moreover, since $\tau = E_1E_2$ is perpendicular to both ζ_1 and ζ_2, and $|\boldsymbol{l}(\tau^{(n)})| < \omega^{(n)}$, we know

$$\zeta_2^{(\infty)} = \lim_{n\to+\infty} \zeta_2^{(n)} = \zeta_1^{(\infty)}.$$

We let $\zeta^{(\infty)} = \zeta_1^{(\infty)} = \zeta_2^{(\infty)}$. Then since $X_1^{(n)} = \zeta_1^{(n)} \cap \delta_1^{(n)}$ and $X_2^{(n)} = \zeta_2^{(n)} \cap \delta_2^{(n)}$, $X_1^{(\infty)}$ and $X_2^{(\infty)}$ are on $\zeta^{(\infty)}$.

By $d(A_1^{(n)}, \delta_1^{(n)}) < \omega^{(n)}$ and $\{\omega^{(n)}\}$ decreasing to 0, we know that $A_1^{(\infty)} \in \delta_1^{(\infty)}$. Similarly

$$X_1^{(\infty)}, B_1^{(\infty)} \in \delta_1^{(\infty)} \quad \text{and} \quad A_2^{(\infty)}, X_2^{(\infty)}, B_2^{(\infty)} \in \delta_2^{(\infty)}.$$

Therefore $\zeta^{(\infty)}$ is orthogonal to $\delta_1^{(\infty)}$ and $\delta_2^{(\infty)}$, because $\zeta_i^{(n)}$ is perpendicular to $A_i^{(n)}B_i^{(n)}$. Hence $\zeta^{(\infty)}$ is the orthogeodesic between $\delta_1^{(\infty)}$ and $\delta_2^{(\infty)}$, ie $\zeta^{(\infty)}$ coincides with $\xi^{(\infty)}$. Next we show that the distance between $X_1^{(\infty)}$ and $Y_1^{(\infty)}$ is 0.

Case I If $X_1^{(\infty)}$ is in $(\overline{\mathbb{H}}^3)^\circ$, then

$$Y_1^{(\infty)} = \xi^{(\infty)} \cap \delta_1^{(\infty)} = \zeta^{(\infty)} \cap A_1^{(\infty)}B_1^{(\infty)} = X_1^{(\infty)}.$$

Case II If $X_1^{(\infty)}$ is on $\partial\overline{\mathbb{H}}^3$, ie the boundary at infinity of $\mathbb{H}^3$, then since $\delta_1^{(\infty)}$ is perpendicular to $\zeta^{(\infty)}$ and passes through $X_1^{(\infty)}$, we know $\delta_1^{(\infty)}$ is a point on $\partial\overline{\mathbb{H}}^3$ which coincides with $X_1^{(\infty)}$. Moreover, $X_1^{(\infty)} = Y_1^{(\infty)}$ on $\partial\overline{\mathbb{H}}^3$. Now we translate each $\{\mathcal{P}^{(n)}(\omega^{(n)})\}$ along $\zeta_1^{(n)}$ by isometry and denote the result by $\{\mathcal{P}'^{(n)}(\omega^{(n)})\}$, so that

$$X'^{(1)}_1 = X'^{(2)}_1 = \cdots = X'^{(n)}_1 = \cdots = X'^{(\infty)}_1.$$

Then $\{\mathcal{P}'^{(n)}(\omega^{(n)})\}$ has a limit picture $\mathcal{P}'^{(\infty)}$ in $\overline{\mathbb{H}}^3$ by passing to subsequences.

Since $X_1^{(\infty)} \in \mathcal{P}^{(n)}(\omega^{(n)})$ is on $\partial\overline{\mathbb{H}}^3$ and $E_1^{(\infty)} \in \mathcal{P}^{(n)}(\omega^{(n)})$ is in $(\overline{\mathbb{H}}^3)^\circ$, we know $E'^{(\infty)}_1 \in \mathcal{P}'^{(n)}(\omega^{(n)})$ is on $\partial\overline{\mathbb{H}}^3$ after translations. Therefore $E'^{(\infty)}_1 = E'^{(\infty)}_2$ on $\partial\overline{\mathbb{H}}^3$. Isometries will keep the distance between any pair of points, so if two sequences of points have the same limits in $\mathcal{P}^{(\infty)}$, then their limits are also

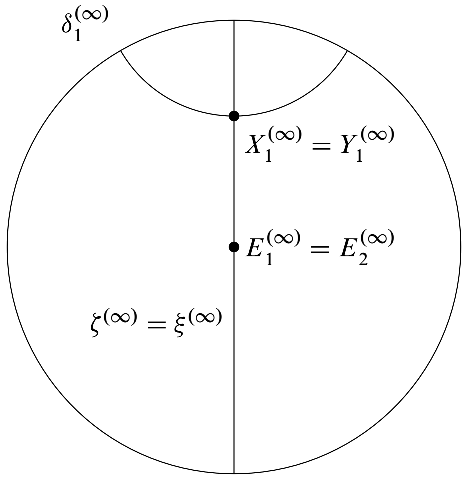

Figure 3: $\mathcal{P}^{(\infty)}$ in Case I.

the same point in $\mathcal{P}'^{(\infty)}$. Therefore we still have $\zeta'^{(\infty)} = \zeta'^{(\infty)}_1 = \zeta'^{(\infty)}_2$. Since $\delta'^{(\infty)}_2$ is perpendicular to $\zeta'^{(\infty)}_2 = \zeta'^{(\infty)}$, we know $\zeta'^{(\infty)}$ is orthogonal to both $\delta'^{(\infty)}_1$ and $\delta'^{(\infty)}_2$. Hence $\zeta'^{(\infty)} = \xi'^{(\infty)}$, and we have

$$Y'^{(\infty)}_1 = \xi'^{(\infty)} \cap \delta'^{(\infty)}_1 = \zeta'^{(\infty)} \cap \delta'^{(\infty)}_1 = X_1^{(\infty)}.$$

Thus $d(X_1^{(\infty)}, Y_1^{(\infty)}) = 0$.

Putting these two cases together, we always have $d(X_1^{(\infty)}, Y_1^{(\infty)}) = 0$, which contradicts the assumption that $d(X_1^{(n)}, Y_1^{(n)}) \geq \kappa_0 > 0$ for all n.

We can obtain similar results for $d(Y_i, Z_i)$ and $d(Y_i, X_i)$, so the lemma is proved. ◻

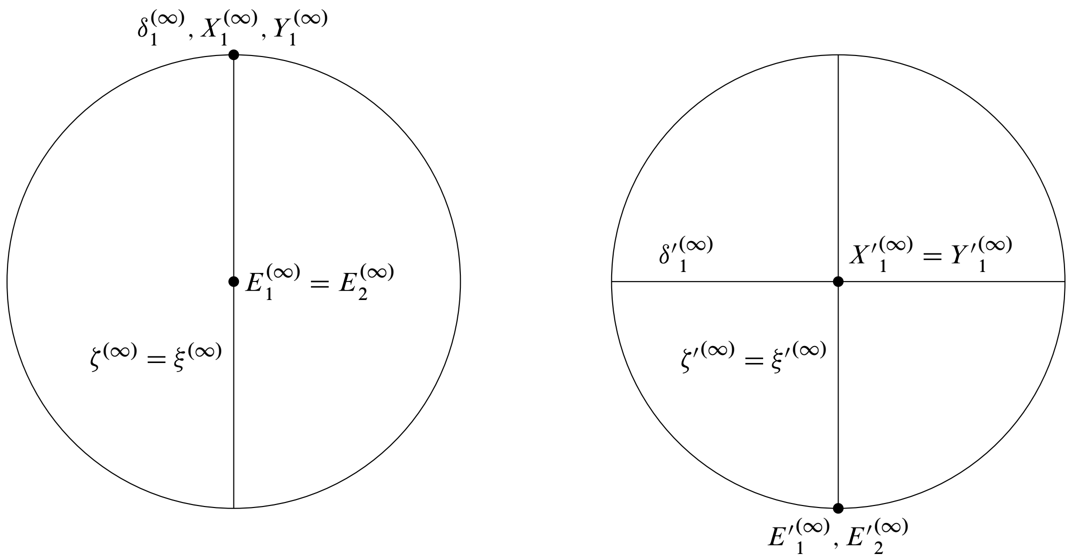

Figure 4: $\mathcal{P}^{(\infty)}$ (left) and $\mathcal{P}'^{(\infty)}$ (right) in Case II.

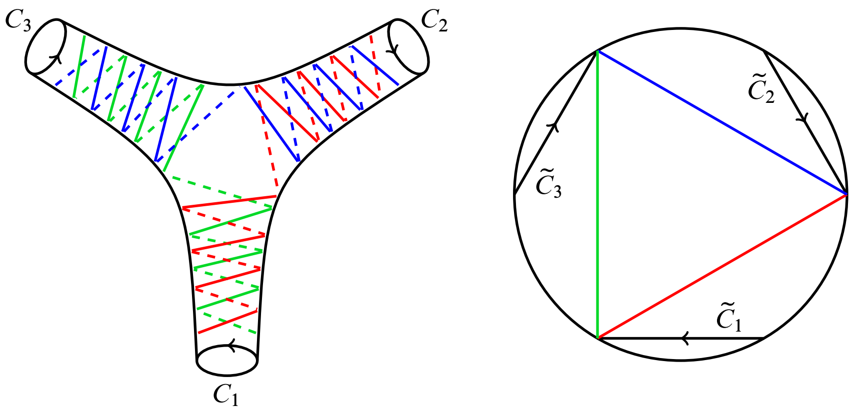

Figure 5: An ideal triangulation of P_1 and its corresponding picture in the universal cover $\mathbb{H}^3$, where $\widetilde{C}_i$ is a lift of C_i for $i = 1, 2, 3$ and all these lifts may not be on the same hyperbolic plane.

Proof of Theorem 5.8 By Theorem 5.1, we can suppose $\{P_1, P_2\}$ is an (R, m)-good nonseparating pants decomposition of Γ, where m is a constant and R is big enough and will be determined later. We apply the spinning construction to this (R, m)-good pants decomposition and inherit the notation from the proof of Theorem 5.1. Hence there is a constant δ, such that for any $R_0 > 0$, there exist $\bar{R} > R_0$ and n_1, n_2, n_3 such that $\{[\mathfrak{s}_1^{(n_2,n_3)}], [\mathfrak{s}_2^{(n_3,n_1)}], [\mathfrak{s}_3^{(n_1,n_2)}]\}$ is an $(\bar{R}, \delta)$-good nonseparating pants decomposition of Γ. Now we want to use ideal triangulations of $\{P_1, P_2\}$ to estimate the short shears of the new $(\bar{R}, \delta)$-good pants decomposition.

We take the orientations of the C_i so that P_1 is on the right side of each C_i. Then there is a unique ideal triangulation of P_1 such that the sides of ideal triangles are asymptotic to each C_j in the same direction as the orientation of this C_j. This ideal triangulation can be also determined in the hyperbolic 3-space $\mathbb{H}^3$, where one can connect the attracting points of the lifts of the C_i as shown in Figure 5. Similarly, there is a unique ideal triangulation of P_2 such that the sides of the ideal triangles are asymptotic to each C_j in the reverse direction of the orientation of this C_j. In each P_k, two ideal triangles share three sides, and we let p_k^i be the common side approaching C_{i+1} and C_{i+2} in P_k with the orientation from C_{i+1} to C_{i+2}.

For each pair $\{p_1^i, p_2^i\}$, consider all orthogeodesics between them. We can label them in a unique way as $\mu_{i+1,k}^i$ and $\mu_{i+2,k}^i$ for all $k \in \mathbb{Z}$ so that $\delta_{m,n}^i$ is freely homotopic to $[\mathfrak{s}_i^{(m,n)}]$ for any $m, n \in \mathbb{Z}$, where the $\delta_{m,n}^i$ are defined as follows. Since $\mu_{i+1,m}^i$ and $\mu_{i+2,n}^i$ are two orthogeodesics between p_1^i and p_2^i, correspondingly, p_1^i and p_2^i are two orthogeodesics between $\mu_{i+1,m}^i$ and $\mu_{i+2,n}^i$. Thus we can denote the orthogonal segment between $\mu_{i+1,m}^i$ and $\mu_{i+2,n}^i$ along p_1^i by $p_{1,m,n}^i$, and the one along p_2^i by $p_{2,m,n}^i$, with the same orientation by p_1^i and p_2^i, respectively. For $i \in \mathbb{Z}/3\mathbb{Z}$ and $m, n \in \mathbb{Z}$, we define

$$\delta_{m,n}^i = \mu_{i+1,m}^i p_{2,m,n}^i (\mu_{i+2,n}^i)^{-1} (p_{1,m,n}^i)^{-1},$$

which is a 4-piece broken geodesic. Let the orientation of $\mu_{m,n}^i$ be from p_1^i to p_2^i. Then each piece of $\delta_{n_{i+1},n_{i+2}}^i$ is oriented. Furthermore, each piece of $\delta_{m,n}^i$ is orthogonal to the two pieces connecting to it.

So there exists a unique framing at each joint point, such that there are framed segments $\mathfrak{w}_i^{(m)}$, $\mathfrak{x}_i^{(m,n)}$, $\mathfrak{y}_i^{(n)}$ and $\mathfrak{z}_i^{(m,n)}$, whose carriers are $\mu^i_{i+1,m}$, $p^i_{2,m,m}$, $(\mu^i_{i+2,n})^{-1}$ and $(p^i_{1,m,n})^{-1}$, forming a zigzag continuous cycle. Let

$$\mathfrak{t}_i^{(m,n)} = \mathfrak{w}_i^{(m)}\mathfrak{x}_i^{(m,n)}\mathfrak{y}_i^{(n)}\mathfrak{z}_i^{(m,n)}.$$

For $(n_1,n_2,n_3)\in\mathbb{Z}^3$, consider $\{[\mathfrak{t}_1^{(n_2,n_3)}],[\mathfrak{t}_2^{(n_3,n_1)}],[\mathfrak{t}_3^{(n_1,n_2)}]\}$. Because $\delta^i_{n_{i+1},n_{i+2}}$ is freely homotopic to $[\mathfrak{s}_i^{(n_{i+1},n_{i+2})}]$, we know $[\mathfrak{t}_i^{(n_{i+1},n_{i+2})}]=[\mathfrak{s}_i^{(n_{i+1},n_{i+2})}]$. Hence $\{[\mathfrak{t}_1^{(n_2,n_3)}],[\mathfrak{t}_2^{(n_3,n_1)}],[\mathfrak{t}_3^{(n_1,n_2)}]\}$ is an $(\overline{R},\delta)$-good nonseparating pants decomposition of Γ.

Next we want to estimate the length of each segment of the $\mathfrak{t}_i^{(n_{i+1},n_{i+2})}$. Without loss of generality, we assume $i=1$. Let $\boldsymbol{l}(C_1)=\lambda\in\mathbb{C}$ and $\boldsymbol{l}(\mathfrak{w}_1^{(k)})=d_k\in\mathbb{C}$ for $k\in\mathbb{Z}$. By composition with Möbius transformations, we can assume the geodesic ρ connecting -1 and 0 in $\mathbb{H}^3$ is a lift of p^1_1. Let $z_k\in\mathbb{C}$ satisfy that the geodesic ψ_k connecting ∞ and z_k is a lift of p^1_2 whose distance to ρ is d_k. Then

$$z_k = e^{-k\lambda}z_0,$$

for all $k\in\mathbb{Z}$. By elementary hyperbolic geometry, we know

$$e^{d_k} = -\bigl(1+2z_k+2\sqrt{z_k^2+z_k}\bigr).$$

Then when $k\to+\infty$, we have $|z_k|\to 0$. Therefore

$$\lim_{k\to+\infty} e^{d_k} = -1,$$

and

$$|d_k - i\pi| \sim |e^{d_k-i\pi}-1| = |2z_k+2\sqrt{z_k^2+z_k}| \sim 2e^{-k\,\mathrm{Re}(\lambda)/2}.$$

We can make similar estimates for all other $\mathfrak{w}_i^{(k)}$ and $\mathfrak{y}_i^{(k)}$. Hence for any $\epsilon>0$, there exists N_ϵ such that when $k>N_\epsilon$, we have

$$\textbf{(5-10)}\qquad |\boldsymbol{l}(\mathfrak{w}_i^{(k)})-i\pi|<\epsilon \quad\text{and}\quad |\boldsymbol{l}(\mathfrak{y}_i^{(k)})-i\pi|<\epsilon,$$

for any $i\in\mathbb{Z}/3\mathbb{Z}$. On the other hand, by the symmetry of P_1 and P_2, we have

$$l(\mathfrak{x}_1^{(n_2,n_3)}) = l(\mathfrak{z}_1^{(n_2,n_3)}).$$

Therefore

$$\begin{aligned}\textbf{(5-11)}\qquad l(\mathfrak{x}_1^{(n_2,n_3)}) = l(\mathfrak{z}_1^{(n_2,n_3)}) &= \tfrac12\bigl(l(\mathfrak{t}_1^{(n_2,n_3)})-l(\mathfrak{w}_1^{(n_2)})-l(\mathfrak{y}_1^{(n_3)})\bigr)\\ &\ge \tfrac12\bigl(l([\mathfrak{t}_1^{(n_2,n_3)}])-l(\mathfrak{w}_1^{(n_2)})-l(\mathfrak{y}_1^{(n_3)})\bigr) = \tfrac12\bigl(l([\mathfrak{s}_1^{(n_2,n_3)}])-l(\mathfrak{w}_1^{(n_2)})-l(\mathfrak{y}_1^{(n_3)})\bigr).\end{aligned}$$

Thus for any $\epsilon>0$, we can choose R_0 large enough that when $\{[\mathfrak{s}_1^{(n_2,n_3)}],[\mathfrak{s}_2^{(n_3,n_1)}],[\mathfrak{s}_3^{(n_1,n_2)}]\}$ is an $(\overline{R},\delta)$-good nonseparating pants decomposition of Γ for some $\overline{R}>R_0$, we also have $n_i>N_\epsilon$ for $i\in\mathbb{Z}/3\mathbb{Z}$. So $|\boldsymbol{l}(\mathfrak{w}_i^{(k)})-i\pi|<\epsilon$ and $|\boldsymbol{l}(\mathfrak{y}_i^{(k)})-i\pi|<\epsilon$. Then by (5-11), the $\mathfrak{t}_i^{(n_{i+1},n_{i+2})}$ are $\bigl(\tfrac12(\overline{R}-\delta-\epsilon),\tfrac12\epsilon\bigr)$-zigzag continuous cycles, for $i\in\mathbb{Z}/3\mathbb{Z}$.

Now we want to use the ideal triangle shears along p_i^j in P_i to estimate the long shears of pants decomposition $\{[\mathrm{t}_1^{(n_2,n_3)}], [\mathrm{t}_2^{(n_3,n_1)}], [\mathrm{t}_3^{(n_1,n_2)}]\}$ along $[\mathrm{t}_j^{n_{j+1},n_{j+2}}]$. Let T_i^1 and T_i^2 be the two ideal triangles in P_i for $i \in \mathbb{Z}/2\mathbb{Z}$. In T_i^k, let $q_i^{k,j}$ be the altitude to the side p_i^j with orientation as $[0,\infty)$, for $j \in \mathbb{Z}/3\mathbb{Z}$, $i \in \mathbb{Z}/2\mathbb{Z}$ and $k \in \mathbb{Z}/2\mathbb{Z}$. For pants decomposition $\{[\mathrm{t}_1^{(n_2,n_3)}], [\mathrm{t}_2^{(n_3,n_1)}], [\mathrm{t}_3^{(n_1,n_2)}]\}$, let P_1' and P_2' be two pairs of pants and ξ_{k,n_j}^j be the third connection of P_k' on $[\mathrm{t}_j^{n_{j+1},n_{j+2}}]$, for $k \in \mathbb{Z}/2\mathbb{Z}$ and $j \in \mathbb{Z}/3\mathbb{Z}$.

We label the orthogeodesics between $q_1^{k,j}$ and $q_2^{k,j}$ by $\tau_{n_j}^{k,j}$ for any k, j and n_j such that the following holds: Suppose h is any free homotopy from $\mathrm{t}_j^{n_{j+1},n_{j+2}}$ to $[\mathrm{t}_j^{n_{j+1},n_{j+2}}]$, and the restriction of h to $\partial(q_1^{k,j,n_j}\,\tau_{n_j}^{k,j}\,(q_2^{k,j,n_j})^{-1})$ extends to a homotopy of $q_1^{k,j,n_j}\,\tau_{n_j}^{k,j}\,(q_2^{k,j,n_j})^{-1}$ to get a curve L'. Then L' is homotopic to ξ_{k,n_j}^j through paths with endpoints on $[\mathrm{t}_j^{n_{j+1},n_{j+2}}]$. Here q_i^{k,j,n_j} is the segment on $q_i^{k,j}$ between $\tau_{n_j}^{k,j}$ and p_i^j.

Similar with (5-10), we have

$$\boldsymbol{l}(\tau_{n_j}^{k,j}) - i\pi \sim e^{-n_j \boldsymbol{hl}(C_j)} \to 0 \quad \text{as } n_j \to \infty \text{ for } k = 1,2. \tag{5-12}$$

Now we require $R > 100m$. Then for $\delta > 0$ with $\kappa(\delta) < \frac{1}{2}m$, where $\kappa(\delta)$ is from Lemma 5.9, let ϵ_0 and L_0 be as in Lemma 3.19 for δ. By (5-10) and (5-12) we can find n_j big enough that

$$|\boldsymbol{l}(\mu_{j,n_j}^i) - i\pi| < \epsilon_0, \quad \tfrac{1}{2}(\overline{R} - \delta - \epsilon_0) > L_0 \quad \text{and} \quad |\boldsymbol{l}(\tau_{n_j}^{k,j}) - i\pi| < \delta.$$

Let $X_i^{k,j}$ be the intersection between $q_i^{k,j}$ and p_i^j, and $Y_i^{k,j}$ be the intersection between ξ_{k,n_j}^j and $[\mathrm{t}_j^{n_{j+1},n_{j+2}}]$ in P_i. Thus by Lemmas 3.19 and 5.9, we have

$$d(X_i^{k,j}, Y_i^{k,j}) < \kappa(\delta) < \tfrac{1}{2}m, \tag{5-13}$$

for $i \in \mathbb{Z}/2\mathbb{Z}$, $k \in \mathbb{Z}/2\mathbb{Z}$ and $j \in \mathbb{Z}/3\mathbb{Z}$.

On p_i^j, $X_i^{1,j} X_i^{2,j}$ is the ideal triangle shear between T_i^1 and T_i^2 along p_i^j. Therefore

$$\boldsymbol{l}(X_i^{1,j} X_i^{2,j}) = \boldsymbol{hl}(C_{j+1}) + \boldsymbol{hl}(C_{j+2}) - \boldsymbol{hl}(C_j),$$

for $j \in \mathbb{Z}/3\mathbb{Z}$ and $i \in \mathbb{Z}/2\mathbb{Z}$. By $\{P_1, P_2\}$ being an (R,m)-good pants decomposition, we have $|\boldsymbol{hl}(C_j) - R| < m$, for $j = 1,2,3$. Hence $|\mathrm{Re}(\boldsymbol{hl}(C_j)) - R| < m$, so we have

$$R - 3m < l(X_i^{1,j} X_i^{2,j}) < R + 3m. \tag{5-14}$$

Thus by (5-13) and (5-14),

$$R - 4m < l(Y_i^{1,j} Y_i^{2,j}) < R + 4m. \tag{5-15}$$

By definition, $l(Y_i^{1,j} Y_i^{2,j})$ for $i = 1,2$ are the real parts of the long shears between P_1' and P_2' along $[\mathrm{t}_j^{n_{j+1},n_{j+2}}]$. Since the short shear s_j between P_1' and P_2' along $[\mathrm{t}_j^{n_{j+1},n_{j+2}}]$ satisfies that

$$2\,\mathrm{Re}(s_j) = l(Y_1^{1,j} Y_1^{2,j}) + l(Y_2^{1,j} Y_2^{2,j}),$$

we always have

(5-16) $$R - 4m < \mathrm{Re}(s_j) < R + 4m,$$

which gives us upper and lower bounds of real parts of short shears of the pants decomposition $\{P_1', P_2'\}$. Letting $B^- = R - 4m$ and $B^+ = R + 4m$ completes the proof. □

At the end of this section, we use Lemma 5.9 to refine Lemma 5.9 itself, in order to get a more quantitative version. The more accurate estimate will be used in the proof of our main theorem in Section 8, for the purpose of bounding the error term of the shears.

Lemma 5.17 *For any $B, m > 0$, there exist $R_0 > 0$ and $B' > 0$ such that the following statement holds: Suppose that δ_1 and δ_2 are two nonintersecting geodesics in $\mathbb{H}^3$ with ξ the orthogeodesic between them and $Y_i = \delta_i \cap \xi$ for $i \in \mathbb{Z}/2\mathbb{Z}$. Suppose that A_1B_1 and A_2B_2 are two geodesic segments and ζ_i is a geodesic orthogonal to A_iB_i at X_i for $i \in \mathbb{Z}/2\mathbb{Z}$. Furthermore, suppose that ζ_1 and ζ_2 are distinct. Let τ be the common orthogonal between ζ_1 and ζ_2, and the orientation of ζ_i be from A_iB_i to τ. If*

1. $R - m < d(Y_1, Y_2), d(A_1, B_1), d(A_2, B_2) < R + m$,
2. $d(A_i, \delta_i), d(B_i, \delta_i) < Be^{-R/2}$ *for* $i \in \mathbb{Z}/2\mathbb{Z}$,
3. ζ_1 *and* ζ_2 *are disjoint*,
4. $|d_{\mathbb{C}}(\zeta_1, \zeta_2) - i\pi| < Be^{-R/2}$,

then $d(X_i, Y_i) < B'e^{-R/2}$.

Proof Let Z_1Z_2 be the common orthogonal between A_1B_1 and A_2B_2 with Z_i on geodesic A_iB_i. Then we will estimate $d(X_i, Z_i)$ and $d(Z_i, Y_i)$ separately. By Lemma 5.9, we can take $R_1 > 0$ such that when $R > R_1$, we have $d(X_i, Y_i), d(Y_i, Z_i), d(Z_i, X_i) < 1$.

(a) Let $a = d_{\mathbb{C}}(\zeta_1, Z_1Z_2)$, $b = d_{\mathbb{C}}(\zeta_2, Z_1Z_2)$, $c = d_{\mathbb{C}}(\zeta_1, \zeta_2)$, $x = d_{\mathbb{C}}(A_1B_1, \tau)$, $y = d_{\mathbb{C}}(A_2B_2, \tau)$ and $z = d_{\mathbb{C}}(A_1B_1, A_2B_2)$. Then

(5-18) $$|c - i\pi| < Be^{-R/2}.$$

By the hyperbolic cosine rule for right-angled hexagons, we have

$$\cosh(c) = \frac{\cosh(x)\cosh(y) + \cosh(z)}{\sinh(x)\sinh(y)}.$$

Therefore

(5-19) $$\begin{aligned}\cosh(z - i\pi) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y)\cosh(c - i\pi)\\ &= \cosh(x + y) + \sinh(x)\sinh(y)(\cosh(c - i\pi) - 1).\end{aligned}$$

By the hyperbolic sine rule,

(5-20) $$\frac{\sinh(a)}{\sinh(y)} = \frac{\sinh(b)}{\sinh(x)} = \frac{\sinh(c)}{\sinh(z)} = \frac{\sinh(c - i\pi)}{\sinh(z - i\pi)}.$$

Since $d(Y_i, Z_i) < 1$ and $d(Y_1, Y_2) > R-m$, we have $d(Z_1, Z_2) > R-m-2$. So there exists $R_2 > 0$ such that when $R > R_2$,

$$|\cosh(z-i\pi)| < 2|\sinh(z-i\pi)|, \tag{5-21}$$

$$\mathrm{Re}(x+y) \geq \mathrm{Re}(z-c) \geq R-m-2-Be^{-R/2} \geq R-m-3, \tag{5-22}$$

$$|\cosh(x+y)| > \tfrac{1}{4}|e^{x+y}|, \tag{5-23}$$

$$|\sinh(c-i\pi)| < 2|c-i\pi| < 2Be^{-R/2} < \tfrac{1}{8}. \tag{5-24}$$

Because $|\sinh(t)| \leq |e^t|$ for any $\mathrm{Re}(t) \geq 0$, by (5-18)–(5-24) we get

$$\begin{aligned}
|\sinh(a)| = |\sinh(c-i\pi)| \cdot \frac{|\sinh(y)|}{|\sinh(z-i\pi)|} &< 2Be^{-R/2}\frac{|\sinh(y)|}{|\sinh(z-i\pi)|/2} \\
&= 4Be^{-R/2}\frac{|\sinh(y)|}{|\cosh(x+y)+\sinh(x)\sinh(y)(\cosh(c-i\pi)-1)|} \\
&\leq 4Be^{-R/2}\frac{|\sinh(y)|}{|\cosh(x+y)|-|\sinh(x)\sinh(y)(\cosh(c-i\pi)-1)|} \\
&\leq 4Be^{-R/2}\frac{|e^y|}{|e^{x+y}|/4-|e^x e^y|\cdot|\cosh(c-i\pi)-1|} = 4Be^{-R/2}\frac{1}{|e^x|/4-|\cosh(c-i\pi)-1|} \\
&\leq 4Be^{-R/2}\frac{1}{1/4-1/8} = 32Be^{-R/2}.
\end{aligned}$$

Hence $d(X_i, Z_i) = \mathrm{Re}(a) < B_1 e^{-R/2}$ for some $B_1 > 0$.

(b) Let $r = d(Y_1, Y_2)$ and $s = d(Z_1, Z_2)$. Since $d(X_i, Y_i), d(Z_i, X_i) < 1$, there exists $B_2 > 0$ such that

$$d(Z_i, \delta_i), d(Y_i, A_i B_i) < B_2 e^{-R/2}. \tag{5-25}$$

Then $|r-s| < 2B_2 e^{-R/2}$. Let M_i be the projection from Z_i on δ_i and N_1 be the projection from M_2 to δ_1. Then

$$d(M_2, N_1) \leq d(M_2, M_1) \leq d(M_2, Z_2) + d(Z_1, Z_2) + d(Z_1, M_1) < r + 4B_2 e^{-R/2}. \tag{5-26}$$

In the hyperbolic quadrilateral $Y_1 Y_2 M_2 N_1$, let $u = d_{\mathbb{C}}(Y_1, Y_2)$, $v = d_{\mathbb{C}}(N_1, M_2)$ and $w = d_{\mathbb{C}}(Y_2, M_2)$. Hence

$$\mathrm{Re}(u) = r \leq \mathrm{Re}(v) < r + 4B_2 e^{-R/2}, \tag{5-27}$$

and we have

$$\cosh\bigl(u + i\tfrac{1}{2}\pi\bigr) = \frac{\cosh\bigl(i\tfrac{1}{2}\pi\bigr)\cosh\bigl(w - i\tfrac{1}{2}\pi\bigr) + \cosh\bigl(v + i\tfrac{1}{2}\pi\bigr)}{\sinh\bigl(i\tfrac{1}{2}\pi\bigr)\sinh\bigl(w - i\tfrac{1}{2}\pi\bigr)},$$

which is simplified as

$$\cosh(w) = \frac{\sinh(v)}{\sinh(u)}.$$

So by (5-27) and $r > R-m$, there exist $R_3 > 0$ and $B_3 > 0$ such that when $R > R_2$, we have

$$|w| < B_3 e^{-R/4}. \tag{5-28}$$

Thus by (5-25) and (5-28),

$$d(Z_2, Y_2) < B_2 e^{-R/2} + B_3 e^{-R/4}. \tag{5-29}$$

We have a similar result for $d(Z_1, Y_1)$.

Now we take $R_0 > \max\{R_1, R_2, R_3\}$. Combining the results from (a) and (b), we can find $B' > 0$ such that when $R > R_0$, we have

$$d(X_i, Y_i) < B' e^{-R/2}.$$

□

6 Counting and matching pants in the 3-manifold

In this section, we will first count good curves and good pants in compact hyperbolic 3-manifolds, and then match good pants along each good curve by Hall's marriage theorem to get a good assembly.

By Theorem 5.8, we can take a nonseparating $(\bar{R}, \delta)$-good pants decomposition of Γ with real parts of short shears bounded by (B^-, B^+), where $\bar{R} > 0$ is big enough and will be determined later and δ is a constant. Let the C_i be the cuffs of this pants decomposition with $\boldsymbol{hl}(C_i) = R_i \in \mathbb{C}/2\pi i\mathbb{Z}$ and s_i the short shear along C_i for $i \in \mathbb{Z}/3\mathbb{Z}$. Then we have

$$|R_i - \bar{R}| < \delta \tag{6-1}$$

and

$$\mathrm{Re}(s_i) \in (B^-, B^+). \tag{6-2}$$

When we mention the R_i later, the information of $\bar{R}$ and δ is carried.

We say that an assembly $\mathcal{A}$ is $(R_i, s_i, \epsilon)_{i=1}^3$-good if:

(1) Each pair of pants in $\mathcal{A}$ is $(R_i, \epsilon)_{i=1}^3$-good.

(2) When two pants of $\mathcal{A}$ are glued along a curve γ which is (R_i, ϵ)-good for some i, let $\alpha_1, \alpha_2 \in N^1(\sqrt{\gamma})$ be two feet of these two pants. Then the following holds:

$$|\alpha_1 - \alpha_2 - (s_i + i\pi)| < \epsilon/\bar{R}.$$

Here these two pants are required to be oriented and induce opposite orientation on γ.

6.1 Counting good curves, geodesic connections and pants

Now we will follow results in [Kahn and Wright 2021, Sections 3 and 5] to help us count $(R_i, \epsilon)_{i=1}^3$-good pants in $\mathbb{H}^3/G$. We do not need to consider cusps since G is cocompact.

We first want to count (R,ϵ)-good curves in $\mathbb{H}^3/G$ for any $R\in\mathbb{C}/2\pi i\mathbb{Z}$ and $\epsilon>0$, which follows from the Margulis argument. Let $\Gamma_{\epsilon,R}$ be the set of (R,ϵ)-good curves in $\mathbb{H}^3/G$. Then by [loc. cit., (3.1.1)], we have

(6-3) $$\lim_{R\to\infty}\frac{\#(\Gamma_{\epsilon,R})}{\epsilon^2e^{4\operatorname{Re}(R)}/\operatorname{Re}(R)}=c_\epsilon,$$

where c_ϵ is a nonzero constant depending on ϵ.

Our next step is to count geodesic connections between two geodesics. Let γ_0 and γ_1 be two oriented closed geodesics. Then for a connection α between γ_0 and γ_1, we let $n_i(\alpha)$ be the unit vector that points in toward α at the point where α meets γ_i and $\theta(\alpha)$ be the angle between the tangent vector to γ_1 where it meets α and the parallel transport along α of the tangent vector to γ_0 where it meets α. We define $w(\alpha)=l(\alpha)+i\theta(\alpha)$. Let

$$\mathbb{I}(\gamma_0,\gamma_1)=N^1(\gamma_0)\times N^1(\gamma_1)\times\mathbb{C}/2\pi i\mathbb{Z}.$$

Then for each α, we have a triple

$$\boldsymbol{I}(\alpha)=(n_0(\alpha),n_1(\alpha),w(\alpha))\in\mathbb{I}(\gamma_0,\gamma_1).$$

We fix the measure on $\mathbb{I}(\gamma_0,\gamma_1)$ by regarding $\mathbb{C}/2\pi i\mathbb{Z}$ as $S^1\times\mathbb{R}$ and then take the product of Lebesgue measures on the first three coordinates times $e^{2t}\,dt$ on $\mathbb{R}$. We also have a metric on $\mathbb{I}(\gamma_0,\gamma_1)$ which is the L^2 norm of the distances in each coordinate. Let $\mathcal{N}_\eta(A)$ be the set of points with distance less than η to the set A and $\mathcal{N}_{-\eta}(A)$ be the set of points with distance greater than η to the complement of A.

Theorem 6.4 [Kahn and Wright 2021, Theorem 3.2] *There exists $q>0$ depending on G such that the following holds when R^- is sufficiently large. Suppose $A\subset\mathbb{I}(\gamma_0,\gamma_1)$, and let R^- be the infimum of the fourth coordinate of values in A. Let $\eta=e^{-qR^-}$. Then the number of connections $\boldsymbol{n}(A)$ for α between γ_0 and γ_1 that have $\boldsymbol{I}(\alpha)\in A$ satisfies*

$$(1-\eta)|\mathcal{N}_{-\eta}(A)|\le 32\pi^2\boldsymbol{n}(A)|\mathbb{H}^3/G|\le(1+\eta)|\mathcal{N}_\eta(A)|.$$

By letting $\gamma_0=\gamma_1=\gamma$ be an (R_1,ϵ)-good curve in Theorem 6.4, we can have an estimate on the number of $(R_i,\epsilon)_{i=1}^3$-good pants which have γ as a boundary. Let Π_{ϵ,R_i} and Π^*_{ϵ,R_i} be the set of all unoriented and oriented $(R_i,\epsilon)_{i=1}^3$-good pants in $\mathbb{H}^3/G$, respectively, and $\Pi^*_{\epsilon,R_i}(\gamma)$ be the set of pants in Π^*_{ϵ,R_i} for which γ is a cuff. We recall that the unit normal bundle $N^1(\gamma)$ is a torsor for $\mathbb{C}/(\boldsymbol{l}(\gamma)\mathbb{Z}+2\pi i\mathbb{Z})$ and the quotient unit normal bundle $N^1(\sqrt{\gamma})$ is a torsor for $\mathbb{C}/(\boldsymbol{hl}(\gamma)\mathbb{Z}+2\pi i\mathbb{Z})$. For $P\in\Pi^*_{\epsilon,R_i}(\gamma)$, let n_0 and n_1 be the first two coordinates of $\boldsymbol{I}(\alpha)$, where α is the third connection for P. Then it turns out that we have a well-defined map $u\colon\Pi^*_{\epsilon,R_i}(\gamma)\to N^1(\sqrt{\gamma})$ by

$$u(P)=\tfrac{1}{2}(n_0+n_1).$$

The proof of the next theorem is the same as that of [loc. cit., Theorem 3.3], with slight modification on the lengths of cuffs.

Theorem 6.5 *There exists positive constant q depending on G such that for any $\epsilon > 0$ the following holds when the R_i are sufficiently large. Let γ be an (R_1, ϵ)-good curve. If $B \subset N^1(\sqrt{\gamma})$, then*

$$(1-\xi)\operatorname{Vol}(\mathcal{N}_{-\xi}(B)) \le \frac{\#\{P \in \Pi^*_{\epsilon,R_i}(\gamma) : u(P) \in B\}}{C_{\text{count}}(\epsilon)\epsilon^4 e^{2\operatorname{Re}(R_2)+2\operatorname{Re}(R_3)-l(\gamma)}/|\mathbb{H}^3/G|} \le (1+\xi)\operatorname{Vol}(\mathcal{N}_{\xi}(B)),$$

where $\xi = e^{-q\bar{R}/2}$ and $C_{\text{count}}(\epsilon) \to 1$ as $\epsilon \to 0$.

6.2 Matching pants

At each (R_i, ϵ)-good curve γ, we want to match each oriented $(R_i, \epsilon)_{i=1}^3$-good pants with γ as a cuff with another such good pants that has the opposite orientation on γ. For $j = 1, 2, 3$ and $\gamma \in \Gamma^*_{\epsilon,R_j}$, we define $\tau_j : N^1(\sqrt{\gamma}) \to N^1(\sqrt{\gamma})$ by $\tau(v) = v + i\pi + s_j$. Now we will follow the idea in [loc. cit., Section 5.2] to prove the following theorem by using Hall's marriage theorem:

Theorem 6.6 *For all $\epsilon > 0$, there exists $R_0 > 0$ such that for all $\bar{R} > R_0$ the following holds: Let γ be an oriented (R_i, ϵ)-good curve for some $i \in \{1, 2, 3\}$, where the R_i are determined at the beginning of Section 6. Then there exists a permutation $\sigma_\gamma : \Pi_{\epsilon,R_i}(\gamma) \to \Pi_{\epsilon,R_i}(\gamma)$ such that*

$$|\boldsymbol{foot}_\gamma(\sigma_\gamma(\pi)) - \tau_i(\boldsymbol{foot}_\gamma(\pi))| < \epsilon/\bar{R},$$

for all $\pi \in \Pi_{\epsilon,R_i}(\gamma)$.

Before we prove this theorem, we introduce some notation. For $A \subset N^1(\sqrt{\gamma})$, let

$$\#A := |\{\pi \in \Pi_{\epsilon,R_i}(\gamma) : \boldsymbol{foot}_\gamma(\pi) \in A\}|.$$

Proposition 6.7 [Kahn and Wright 2021, Corollary 5.6] *If $A \subset N^1(\sqrt{\gamma})$ and $|\mathcal{N}_\eta(A)| \le \frac{1}{2}|N^1(\sqrt{\gamma})|$, then*

$$\frac{|\mathcal{N}_\eta(A)|}{|A|} > 1 + \frac{\eta}{\operatorname{Re}(R_i)}.$$

Moreover,

$$\frac{|\mathcal{N}_\eta(A)|}{|A|} > 1 + \frac{\eta}{2\bar{R}},$$

when $\bar{R}$ is sufficiently large.

Proof of Theorem 6.6 Let $\eta = \epsilon/\bar{R}$, $C = C_{\text{count}}(\epsilon)\epsilon^4 e^{2\operatorname{Re}(R_2)+2\operatorname{Re}(R_3)-l(\gamma)}/|\mathbb{H}^3/G|$ and ξ be as it appears in Theorem 6.5. By Hall's marriage theorem, Theorem 6.6 follows from the statement that $\#\mathcal{N}_\eta(A) \ge \#\tau(A)$ for every finite set A. Actually when $\bar{R}$ is large, we have

$$\tfrac{1}{2}\eta = \tfrac{1}{2}\epsilon\bar{R} > 5\bar{R}\xi > \big(4\bar{R} + 1 + \tfrac{1}{2}\epsilon\bar{R}\big)\xi = \big(4\bar{R} + 1 + \tfrac{1}{2}\eta\big)\xi.$$

Therefore by Proposition 6.7,

$$\frac{|\mathcal{N}_{\eta/2}(A)|}{|\mathcal{N}_\xi(A)|} > 1 + \frac{\eta/2 - \xi}{2\bar{R}} > \frac{1+\xi}{1-\xi}. \tag{6-8}$$

The remainder of the proof is the same as that of [loc. cit., Theorem 5.7]. □

7 Existence of the quasiconformal map

The goal of this section is to show that the gluing of good pants by Theorem 6.6 forms an assembly that is close to its perfect model. The idea follows from the appendix of [Kahn and Wright 2021], where the perfect model is considered as an assembly consisting of all identical pants.

7.1 An estimate for matrix multiplication

We first quote an important theorem in [loc. cit.], which helps us estimate the error of matrix multiplication.

For any element U of a Lie algebra (for a given Lie group), we let $U(t)$ be a shorthand for $\exp(tU)$. Let

$$X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}), \quad \theta = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad Y = e^{(\pi/2)ad_\theta} X,$$

so that $Y(t) = \theta\left(\frac{1}{2}\pi\right)X(t)\theta\left(-\frac{1}{2}\pi\right)$. We consider $\mathrm{SL}_2(\mathbb{R})$ as a subset of $M_2(\mathbb{R})$. Then we can add or subtract elements of $\mathrm{SL}_2(\mathbb{R})$ from each other and take the matrix operator norm.

Theorem 7.1 *Suppose $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$, $(a_i')_{i=1}^n$ and $(b_i')_{i=1}^n$ are sequences of complex numbers, and A, B and ϵ are positive real numbers such that $\epsilon < \min(1/A, 1/e)$,*

$$\sum_{i=1}^n |a_i| e^{|b_i|} \le B,$$

and for all i,

1. $2|a_i|e^{|\mathrm{Re}(b_i)|+1} \le A$,
2. $|b_i - b_i'| < \epsilon$, *and*
3. $|a_i - a_i'| < \epsilon |a_i|$.

Then

$$\left\| \sum_{i=1}^n Y(b_i)X(a_i)Y(-b_i) - \sum_{i=1}^n Y(b_i')X(a_i')Y(-b_i') \right\| \le 12 e^{A+2B} B\epsilon.$$

7.2 Frames and distortion

We first want to introduce the distance and distortion in $\mathcal{F}\mathbb{H}^3$. Fix a left-invariant metric d on $\mathrm{Isom}(\mathbb{H}^3)$, and for any $g \in \mathrm{Isom}(\mathbb{H}^3)$, let $d(g) = d(1, g)$.

For $u, v \in \mathcal{F}\mathbb{H}^3$, there is a unique $u \to v \in \mathrm{Isom}(\mathbb{H}^3)$ such that $u \cdot (u \to v) = v$. Provided $X \subset \mathcal{F}\mathbb{H}^3$ and a map $\tilde{e}\colon X \to \mathcal{F}\mathbb{H}^3$, we say $\tilde{e}$ has *ϵ-bounded distortion to distance D* if

$$d(u \to v, \tilde{e}(u) \to \tilde{e}(v)) < \epsilon,$$

for all $u, v \in X$ with $d(u, v) < D$ (where $d(u, v) = d(u \to v)$).

Given $\tilde{e}_1, \tilde{e}_2 \colon X \to \mathcal{F}\mathbb{H}^3$, we say that $\tilde{e}_1$ and $\tilde{e}_2$ are *ϵ-related* if

$$d(\tilde{e}_1(u), \tilde{e}_2(u)) < \epsilon$$

for all $u \in X$. Here are some observations:

Lemma 7.2 *For all D, there exists D' such that if $U \in \mathrm{SL}(2,\mathbb{C})$ and $\|U\| < D$, then $d(U) < D'$. This is also true if $\|\cdot\|$ and $(\cdot)$ are interchanged.*

Lemma 7.3 *For all D and ϵ there exists δ such that if $U, V \in \mathrm{SL}(2,\mathbb{C})$, $\|U\|, \|V\| < D$ and $\|U-V\| < \delta$, then $d(U-V) < \epsilon$.*

Lemma 7.4 *For all ϵ, D and k there exists δ such that if $u_0, \dots, u_k, v_0, \dots, v_k \in \mathcal{F}\mathbb{H}^3$,*

$$d(u_i \to u_{i+1}, v_i \to v_{i+1}) < \delta,$$

and

$$d(u_i \to u_{i+1}) < D,$$

then

$$d(u_0 \to u_k, v_0 \to v_k) < \epsilon.$$

Here is some notation about frames in a 3-manifold M. Suppose γ is an oriented geodesic in M. Then any unit normal vector v to γ determines a unique 3-frame q, w, v in M, where w is the unit tangent vector to γ and the frame is positively oriented. We call this frame the associated 3-frame for v with respect to γ. We denote by $\mathcal{F}(\gamma)$ the set of all such frames. If γ is unoriented, then we let $\mathcal{F}(\gamma) = \mathcal{F}(\gamma^+) \cup \mathcal{F}(\gamma^-)$, where γ^+ and γ^- are two possible oriented versions of γ. For a pair of pants Q in M, let $\partial^{\mathcal{F}} Q$ be the union of $\mathcal{F}(\gamma)$ for every oriented $\gamma \in \partial Q$. We also let $\hat{\partial}^{\mathcal{F}}(Q)$ denote the union of the associated 3-frames for the unique slow and constant turning normal field on each boundary of Q, which is determined by the feet of the short orthogeodesics.

To briefly introduce the logic of this section, suppose that we have $(R_i, \epsilon_i)_{i=1}^3$-good pants Q_i glued along boundaries, such that each boundary component of one C_i is geometrically identified with some boundary component of another C_j, so that we can form an assembly $\mathcal{A}$. We will construct a perfect model $\hat{\mathcal{A}}$ for $\mathcal{A}$ later which provides a perfect pants $\widehat{Q}_i$ for each Q_i in $\mathcal{A}$ along with a map $h_i \colon \widehat{Q}_i \to M$ sending $\partial \widehat{Q}_i$ to ∂Q_i up to homotopy through such maps. We can also glue the $\widehat{Q}_i$ together through the geometric and isometric identifications of the boundary components of the $\widehat{Q}_i$. Therefore we obtain a surface $S_{\hat{\mathcal{A}}}$ with a quasi-Fuchsian structure and a homotopy class of maps $h \colon S_{\hat{\mathcal{A}}} \to M$ which send each $\widehat{Q}_i$ to Q_i in M. Then we lift this map to $\tilde{h} \colon \mathbb{H}^3 \to \mathbb{H}^3$, where the homotopy class of h determines the relationship of each lift of $\widehat{Q}_i$ and the corresponding lift of Q_i, and the same for boundary geodesics. Now suppose we have maps $e \colon \partial^{\mathcal{F}}(\widehat{Q}_i) \to \partial^{\mathcal{F}}(Q_i)$ that send frames over each boundary geodesic of $\widehat{Q}_i$ to frames over the corresponding boundary geodesic of Q_i. Then we can use $\hat{h}$ to get a canonical lift $\tilde{e}$ of e to $\partial^{\mathcal{F}}(\tilde{\hat{\mathcal{A}}})$. We say $e \colon \partial^{\mathcal{F}}(\hat{\mathcal{A}}) \to \partial^{\mathcal{F}}(\mathcal{A})$ has ϵ-bounded distortion to distance D if and only if $\tilde{e}$ does.

7.3 Sequences of geodesics

In this subsection we will prove a theorem which is based on Theorem 7.1. We first recall some definitions from [Kahn and Wright 2021].

A *linear sequence of geodesics* in $\mathbb{H}^2$ is a sequence $(\gamma_i)_{i=0}^n$ of disjoint geodesics in $\mathbb{H}^2$ such that each one separates those before it and those after it. The geodesics are oriented so that those following a given geodesic are to the left of that geodesic. For a linear sequence (γ_i) of geodesics and given $x_0 \in \gamma_0$, we inductively define $x_i \in \gamma_i$ such that x_{i+1} and x_i are related by the unique orientation-preserving isometry sending γ_i to γ_{i+1}. We say that the x_i form a *homologous sequence* of points on the γ_i. Similarly we can define a homologous sequence of associated frames, since the associated 2-frame to γ_0 at each point $x_0 \in \gamma_0$ is uniquely determined. A *semilinear sequence of geodesics* in $\mathbb{H}^3$ is a sequence $(\gamma_i)_{i=0}^n$ satisfying that each pair of geodesics are disjoint and have a common orthogonal. We can also define a homologous sequence of points and associated frames over a semilinear sequence.

For two (semi)linear sequences $(\gamma_i)_{i=0}^n$ and $(\gamma_i')_{i=0}^n$, let η_i be the common orthogonal to γ_i and γ_{i+1} with orientation from γ_i to γ_{i+1}, u_i be the signed complex distance from γ_i to γ_{i+1} and v_i be the signed complex distance along γ_i from η_{i-1} to η_i. We likewise define η_i', u_i' and v_i' for (γ_i'). Furthermore we can define a map $e\colon \mathcal{F}(\gamma_0) \cup \mathcal{F}(\gamma_n) \to \mathcal{F}(\gamma_0') \cup \mathcal{F}(\gamma_n')$ such that $e\colon \mathcal{F}(\gamma_0) \to \mathcal{F}(\gamma_0')$ and $e\colon \mathcal{F}(\gamma_n) \to \mathcal{F}(\gamma_n')$ are isometric embeddings and e maps the foot of η_0 on γ_0 to the foot of η_0' on γ_0'. The same holds for the foot of η_{n-1} on γ_n.

We say that two sequences $(\gamma_i)_{i=0}^n$ and $(\gamma_i')_{i=0}^n$ are $(R, B, \epsilon, B^-, B^+)$*-well-matched* if the following properties hold for each i:

(1) $B^- < \mathrm{Re}(v_i) < B^+$,

(2) $|v_i' - v_i| < (B^-\epsilon)/(2R)$,

(3) $B^{-1} < |u_i| e^{R/2} < B$,

(4) $|u_i' - u_i| < \epsilon |u_i|$.

We say that a sequence $(\gamma_i)_{i=0}^n$ is (R, B, B^-, B^+, K)*-related* to another sequence $(\gamma_i')_{i=0}^n$ if there exists a K-quasiconformal map from $\mathbb{C}$ to itself sending the attracting and repelling endpoints of γ_i to the corresponding points of γ_i' for each i, and

(1) $B^{-1} < |u_i| e^{R/2}, |u_i'| e^{R/2} < B$,

(2) $B^- < \mathrm{Re}(v_i), \mathrm{Re}(v_i') < B^+$,

(3) $|\mathrm{Re}(v_i) - \mathrm{Re}(v_i')| < 1/R$.

Theorem 7.5 *For any $B > 0$, $D > 0$, $B^+ > B^- > 0$ and $K > 1$, there exist $R_0 > 0$, ϵ_0 and C such that when $R > R_0$ and $0 < \epsilon < \epsilon_0$, the following holds. Suppose $(\gamma_i)_{i=0}^n$ and $(\gamma_i')_{i=0}^n$ are $(R, B, \epsilon, B^-, B^+)$-well-matched, and $(\gamma_i)_{i=0}^n$ is (R, B, B^-, B^+, K)-related to a linear sequence $(\gamma_i'')_{i=0}^n$. Then the map $e\colon \mathcal{F}(\gamma_0) \cup \mathcal{F}(\gamma_n) \to \mathcal{F}(\gamma_0') \cup \mathcal{F}(\gamma_n')$ has $C\epsilon$-bounded distortion to distance D.*

Before proving this theorem, we prove some lemmas.

Lemma 7.6 *For any $K > 1$, there exists $t = t(K)$ such that the following holds. Suppose f is a K-quasiconformal mapping from $\mathbb{C}$ to $\mathbb{C}$ fixing 0. Then for any $z_1, z_2 \in \mathbb{C}$ with $|z_1| < |z_2|$, we have $|f(z_1)| < t|f(z_2)|$.*

Proof Without loss of generality, we can assume $z_2 = f(z_2) = 1$. Thus we simply need to prove that if f is K-quasiconformal, then for any $|z| < 1$ we have $|f(z)| < t(K)$. First we fix $|z| < 1$ with $z \neq 0$. Let μ be the Beltrami coefficient of f. Then $\|\mu\|_\infty = (K-1)/(K+1)$. Now for any s in the unit disk, let

$$\mu_s = s \cdot \frac{\mu}{\|\mu\|_\infty},$$

and f_s be the quasiconformal automorphism of $\mathbb{C}$ fixing 0 and 1 with Beltrami coefficient μ_s. Hence we know $f_0 = \mathrm{id}$ and $f_{\|\mu\|_\infty} = f$. Let $g(s) = f_s(z)$. Then g is holomorphic on the unit disk and $g(s) \neq 0, 1$. Therefore by [Tsuji 1959, Theorem VI.19], there is an absolute constant c such that

$$|g(s)| \le \exp\left(\frac{c \ln(|g(0)| + 2)}{1 - |s|}\right).$$

By $|g(0)| = |f_0(z)| = |z| < 1$, we have

$$|f(z)| = |f_{\|\mu\|_\infty}(z)| = |g(\|\mu\|_\infty)| < \exp\left(\frac{c \ln 3}{1 - \|\mu\|_\infty}\right).$$

So $|f(z)|$ is bounded by a constant only depending on K. □

For a oriented geodesic $\gamma \in \mathbb{H}^3$ and a map f from $\partial\mathbb{H}^3$ to itself, we denote by $[f(\gamma)]$ the geodesic in $\mathbb{H}^3$ determined by the image of the endpoints of γ under f. The orientation of $[f(\gamma)]$ is determined by f and the orientation of γ.

Lemma 7.7 *For any $K > 1$ and $D > 0$, there exists $C(K, D) > 0$ such that the following holds: Suppose γ_1 and γ_2 are two disjoint geodesics in $\mathbb{H}^2$ and f is a K-quasiconformal mapping from $\mathbb{C}$ to itself. Let $\gamma_i' = [f(\gamma_i)]$ for $i = 1, 2$. Let η be the common orthogonal of γ_1' and γ_2' and $A_i \in \gamma_i'$ be a point on each geodesic. If $d(A_1, A_2) < D$, then*

$$|\boldsymbol{d}(A_1, \eta) - \boldsymbol{d}(A_2, \eta)| < C(K, D),$$

where $\boldsymbol{d}(A_i, \eta)$ is the signed distance along γ_i'.

Proof We first normalize f so that f fixes 0, 1 and ∞, and we assume that the endpoints of γ_1 are 0 and ∞ and the endpoints of γ_2 are 1 and x for some $1 \neq x > 0$. Without loss of generality, we assume $0 < x < 1$. Let $d = d(\gamma_1, \gamma_2)$, which is actually real, and we have

$$d = \ln\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right). \tag{7-8}$$

Similarly, we have

$$d_{\mathbb{C}}(\gamma_1', \gamma_2') = \pm \ln\left(\frac{1+\sqrt{f(x)}}{1-\sqrt{f(x)}}\right) \tag{7-9}$$

So there exists ϵ_1 such that when $|z| < \epsilon_1$, we have

$$|\sqrt{z}| < \left|\ln\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right| < 4|\sqrt{z}|. \tag{7-10}$$

It's also well known that given $K > 1$, for a normalized K-quasiconformal mapping f there exist $\epsilon_2 > 0$ and $M > 0$ such that if $|x| < \epsilon_2$, then

$$|f(x)| < M|x|^{1/K}. \tag{7-11}$$

Therefore by (7-8)–(7-11), there exists $\epsilon_0 > 0$ such that when $0 < d \le \epsilon_0$, we have

$$\begin{aligned}|d_{\mathbb{C}}(\gamma_1', \gamma_2')| &= \left|\ln\left(\frac{1+\sqrt{|f(x)|}}{1-\sqrt{|f(x)|}}\right)\right| < 4\sqrt{|f(x)|} < 4\sqrt{M x^{1/K}} < 4\sqrt{M}\left(\ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\right)^{1/K} \\ &= 4\sqrt{M}\, d^{1/K} \le 4\sqrt{M}\,\epsilon_0^{1/K},\end{aligned} \tag{7-12}$$

which is a constant depending only on K. If $\boldsymbol{d}(A_1, \eta)\boldsymbol{d}(A_2, \eta) > 0$, then by the triangle inequality we have

$$|\boldsymbol{d}(A_1, \eta) - \boldsymbol{d}(A_2, \eta)| \le d(A_1, A_2) < D.$$

If $\boldsymbol{d}(A_1, \eta)\boldsymbol{d}(A_2, \eta) < 0$, then by Lemma 3.5 and (7-12) there is a constant $C_1(K, D)$ depending on K and D such that

$$|\boldsymbol{d}(A_1, \eta) - \boldsymbol{d}(A_2, \eta)| = |\boldsymbol{d}(A_1, \eta)| + |\boldsymbol{d}(A_2, \eta)| < d(A_1, A_2) + C_1(K, D) < D + C_1(K, D).$$

Thus we always have

$$|\boldsymbol{d}(A_1, \eta) - \boldsymbol{d}(A_2, \eta)| < D + C_1(K, D). \tag{7-13}$$

When $d > \epsilon_0$, we first prove a statement that there exists δ depending only on K, such that

$$|d_{\mathbb{C}}(\gamma_1', \gamma_2')| > \delta$$

and

$$|d_{\mathbb{C}}(\gamma_1', \gamma_2') - i\pi| > \delta.$$

By $d > \epsilon_0$ and (7-8), there exists δ_1 such that $x > \delta_1$. Then applying (7-11) to f^{-1}, we have

$$\delta_1 < |x| < M|f(x)|^{1/K} \quad \text{or} \quad |f(x)| > \epsilon_1.$$

Thus $|f(x)| > \min\{(\delta_1/M)^K, \epsilon_1\}$. Since $0 < x < 1$, $|f(x)|$ is also bounded above by Lemma 7.6. By

$$d_{\mathbb{C}}(\gamma_1', \gamma_2') = \ln\left(1 + \frac{2\sqrt{f(x)}}{1-\sqrt{f(x)}}\right),$$

there exists δ such that $|d_{\mathbb{C}}(\gamma_1', \gamma_2')| > \delta$. To prove the second inequality, we reverse the orientation of γ_2, and then it follows from the first inequality.

Now if $d_{\mathbb{R}}(\gamma_1', \gamma_2') > \frac{1}{2}\delta$, then by Lemma 3.5 there exists a constant $C_2(K, \delta)$ such that

(7-14) $$|\boldsymbol{d}(A_1, \eta)| + |\boldsymbol{d}(A_2, \eta)| < d(A_1, A_2) + C_2(K, \delta).$$

If $d_{\mathbb{R}}(\gamma_1', \gamma_2') \le \frac{1}{2}\delta$, then $\frac{1}{2}\delta < \mathrm{Im}(d_{\mathbb{C}}(\gamma_1', \gamma_2')) < \pi - \frac{1}{2}\delta$. Then by Lemma 3.5 again, there exists $C_3(K, \delta)$ such that

(7-15) $$|\boldsymbol{d}(A_1, \eta)| + |\boldsymbol{d}(A_2, \eta)| < d(A_1, A_2) + C_3(K, \delta).$$

Since δ only depends on K, by (7-13)–(7-15) there exists a constant $C(K, D)$ such that

$$|\boldsymbol{d}(A_1, \eta) - \boldsymbol{d}(A_2, \eta)| < C(K, D). \qquad \square$$

Lemma 7.16 [Kahn and Wright 2021, Lemma A.10] *For all ϵ, D and k, there exists δ such that when $u_0, \dots, u_k, v_0, \dots, v_k \in \mathcal{F}\mathbb{H}^3$,*

$$d(u_i \to u_{i+1}, v_i \to v_{i+1}) < \delta,$$

and

$$d(u_i \to u_{i+1}) < D,$$

then

$$d(u_0 \to u_k, v_0 \to v_k) < \epsilon.$$

Lemma 7.17 [Kahn and Wright 2021, Lemma A.13] *Suppose $(\gamma_i)_{i=0}^n$ is a linear sequence, with (u_i) and (v_i) defined as they were at the beginning of this subsection. Let $D = d(\gamma_0, \gamma_n)$, and suppose that $u_0, u_{n-1} < 1$. Then*

$$\left|\sum_{i=1}^{n-1} v_i\right| \le D + 2\ln D - \ln u_0 - \ln u_{n-1} + 3.$$

Proof of Theorem 7.5 Suppose $x \in \mathcal{F}(\gamma_0)$ and $y \in \mathcal{F}(\gamma_n)$ are such that $d(x, y) < D$. Then in particular, $d(\gamma_0, \gamma_n) < D$. By [Shiga 2005, Theorem 1.2], there is an absolute constant A and a constant C_K only depending on K, such that $d(\gamma_0'', \gamma_n'') < AKD + C_K := D'$. Then by Lemma 7.17, we have

(7-18) $$(n-1)B^- < \sum_{i=1}^{n-1} v_i'' < D' + 2\ln D' + R + 2\ln B + 3 = R + C_1(B, D, K) < 2R,$$

when $R > C_1(B, D, K)$. Let (x_i) be the homologous sequence of frames for (γ_i) with $x_0 = x$, and let (x_i') be the same for (γ_i') with $x_0' = x' = e(x)$.

We want to use Theorem 7.1 to control $d(x_0 \to x_n, x_0' \to x_n')$. Let $a_i = u_i$ and b_i be complex numbers such that $\mathrm{foot}_{\gamma_{i+1}}(\gamma_i) = x_i Y(b_i)$. Then we notice that $x_{i+1} = x_i Y(b_i) X(a_i) Y(-b_i)$ and $b_{i+1} = b_i + v_{i+1}$, and we have similar results for x_i' and b_i'. Thus $b_0' = b_0$ by the definition of the map e, and

$$b_i = b_0 + \sum_{j=1}^{i} v_j,$$

and the same for b_i' and v_i'. Hence by (7-18) we have

$$|b_i' - b_i| = \left|\sum_{j=1}^{i}(v_i' - v_i)\right| < \frac{(n-1)B^-\epsilon}{2R} < \epsilon,$$

by condition (2) of being well-matched. So we verify Theorem 7.1(2). On the other hand, Theorem 7.1(3) directly follows from condition (4) of being well-matched.

Now we want to control $|a_i|e^{|\mathrm{Re}(b_i)|}$ to satisfy the remaining conditions in Theorem 7.1. The a_i are about $e^{-R/2}$ in size and $B^- < \mathrm{Re}(v_i) < B^+$, so all we have to do is to estimate the largest $|a_i|e^{|\mathrm{Re}(b_i)|}$, which is actually either the first or the last, in order to control the sum of all $|a_i|e^{|\mathrm{Re}(b_i)|}$.

To control $|a_0|e^{|\mathrm{Re}(b_0)|}$, it suffices to give an upper bound of $d(x,\gamma_1)$. Let f be the K-quasiconformal mapping sending the endpoints of $(\gamma_i')_{i=1}^n$ to the endpoints of $(\gamma_i'')_{i=1}^n$, and $\tilde f:\mathbb{H}^3\to\mathbb{H}^3$ be the (K',C')-quasi-isometric extension of f, where K' and C' are constants depending on K. Let $\tilde g:\mathbb{H}^3\to\mathbb{H}^3$ be the approximate inverse of $\tilde f$. Then $\tilde g$ is a quasi-isometric extension of f^{-1}. Moreover $\tilde f(\gamma_i)$ is within distance C_1' of γ_i'' and $\tilde g(\gamma_i'')$ is within distance C_1' of γ_i for some constant C_1' depending only on K. Now let x'' and y'' be the projections of $\tilde f(x)$ and $\tilde f(y)$ on γ_0'' and γ_n'', respectively. We have

$$d(x'',y'') < d(x'',\tilde f(x)) + d(\tilde f(x),\tilde f(y)) + d(\tilde f(y),y'') < 2C_1' + K'D + C'.$$

Therefore

$$d(x'',\gamma_2'') \le d(x'',\gamma_n'') \le d(x'',y'') < 2C_1' + K'D + C'.$$

Hence

$$d(\tilde f(x),\gamma_2'') \le d(\tilde f(x),x'') + d(x'',\gamma_2'') < 3C_1' + K'D + C'.$$

Since we also have $d(x,\tilde g\circ\tilde f(x)) < C'$,

$$d(x,\gamma_2) < d(x,\tilde g(\gamma_2'')) + C_1' < d(\tilde g\circ\tilde f(x),\tilde g(\gamma_2'')) + C' + C_1' < K'(3C_1' + K'D + C') + C' + C_1',$$

which is a constant depending on K and D.

Now we want to control $|a_{n-1}|e^{|\mathrm{Re}(b_{n-1})|}$. By the same method as above, we obtain an upper bound of $d(y,\gamma_{n_1})$, so estimating $d(x_n,y)$ will be enough. Let $s = \frac{1}{2}\sum_{j=1}^{n-1} v_j$. Then by (7-18) and condition (3) for being (R,B,B^-,B^+,K)-related, we know

$$\mathrm{Re}(s) = \frac{1}{2}\sum_{j=1}^{n-1}\mathrm{Re}(v_j) < \frac{1}{2}\sum_{j=1}^{n-1}\Big(\mathrm{Re}(v_j'') + \frac{1}{R}\Big) < \frac{R}{2} + \frac{1}{2}C_1(B,D,K) + \frac{n-1}{R} < \frac{R}{2} + \frac{1}{2}C_1(B,D,K) + \frac{2}{B^-} = \frac{R}{2} + C_2(B,D,K,B^-). \tag{7-19}$$

Let a_0 be the basepoint of the frame $\mathrm{foot}_{\gamma_1}\gamma_0 Y(s)$, and a_i be a homologous sequence determined by a_0. Then $d(a_0,\mathrm{foot}_{\gamma_1}(\gamma_0)) = d(a_n,\mathrm{foot}_{\gamma_{n-1}}(\gamma_n)) = \mathrm{Re}(s)$. Then by (7-19) and condition (3) of being well-matched, there exists a constant $C_3(B,D,K,B^-)$ such that

$$d(a_0,a_n) \le \sum_{i=0}^{n-1} d(a_i,a_{i+1}) < C_3(B,D,K,B^-). \tag{7-20}$$

Let η be the common orthogonal between γ_0 and γ_n. Then by (7-20) and Lemma 7.7,

(7-21) $$|\boldsymbol{d}(a_0,\eta)-\boldsymbol{d}(a_n,\eta)| < C_4(B,D,K,B^-),$$

for some constant $C_4(B,D,K,B^-)$. Similarly, there is a constant $C_5(K,D)$ such that

(7-22) $$|\boldsymbol{d}(x,\eta)-\boldsymbol{d}(y,\eta)| < C_5(K,D).$$

By the definition of a homologous sequence, we know $\boldsymbol{d}(x,a_0)=\boldsymbol{d}(x_n,a_n)$ as signed distance. Thus

$$\begin{aligned}
d(x_n,y) &= |\boldsymbol{d}(x_n,\eta)-\boldsymbol{d}(y,\eta)| = |(\boldsymbol{d}(x_n,\eta)-\boldsymbol{d}(x,\eta))+(\boldsymbol{d}(x,\eta)-\boldsymbol{d}(y,\eta))| \\
&= |(\boldsymbol{d}(x_n,\eta)+d(a_n,x_n))-(d(a_0,x)+\boldsymbol{d}(x,\eta))+(\boldsymbol{d}(x,\eta)-\boldsymbol{d}(y,\eta))| \\
&= |(\boldsymbol{d}(a_n,\eta)-\boldsymbol{d}(a_0,\eta))+(\boldsymbol{d}(x,\eta)-\boldsymbol{d}(y,\eta))| < |\boldsymbol{d}(a_n,\eta)-\boldsymbol{d}(a_0,\eta)|+|\boldsymbol{d}(x,\eta)-\boldsymbol{d}(y,\eta)| \\
&< C_4(B,D,K,B^-)+C_5(K,D).
\end{aligned}$$

Hence, all the conditions in Theorem 7.1 are satisfied.

Next we consider $d(x_n\to y, x_n'\to y')$. Since x_n and y are frames on γ_n, $|b_{n-1}-b_{n-1}'|<\epsilon$ and $d(y,\eta_{n-1})=d(y',\eta_{n-1}')$ as signed distance, so $d(x_n\to y, x_n'\to y')<\epsilon$.

Now we consider x, x_n, y and x', x_n', y', and by Lemma 7.16, the theorem is proved. □

7.4 Good assemblies and the perfect model

In this subsection, we construct the perfect model $\hat{\mathcal{A}}$ for a good assembly $\mathcal{A}$, then bound the distortion of the map $e\colon \partial^{\mathcal{F}}\hat{\mathcal{A}}\to\partial^{\mathcal{F}}\mathcal{A}$ by some geometric control.

Suppose Q is an $(R_i,\epsilon)_{i=1}^3$-good pants, and $\widehat{Q}$ is its perfect model. That means the lengths of cuffs of $\widehat{Q}$ are R_1, R_2 and R_3, and we say $\widehat{Q}$ is $(R_i)_{i=1}^3$-perfect. We say Q is *ϵ-compliant* if for every short orthogeodesic η of Q and corresponding orthogeodesic $\hat{\eta}$ in $\widehat{Q}$, we have

$$|l(\eta)-l(\hat{\eta})|<\epsilon l(\hat{\eta}).$$

Then we quote and revise [Kahn and Wright 2021, Lemma A.15] for our purpose.

Lemma 7.23 *For each $\delta>0$, there is a universal constant C such that every $(R_i,\epsilon)_{i=1}^3$-good pants is $C\epsilon$-compliant with $|R_i-\overline{R}|<\delta$ for any $\overline{R}>\delta$.*

For an $(R_i,\epsilon)_{i=1}^3$-good pants Q, there is a unique map $e\colon \hat{\partial}^{\mathcal{F}}\widehat{Q}\mapsto\partial^{\mathcal{F}}Q$ satisfying the following three properties:

(1) The map from $\partial\widehat{Q}$ to ∂Q induced by e is the restriction (to $\partial\widehat{Q}$) of an orientation-preserving homeomorphism from $\widehat{Q}$ to Q.

(2) The induced map is affine (linear) on each component of $\partial\widehat{Q}$, and maps each component of $\hat{\partial}^{\mathcal{F}}\widehat{Q}$ to the frames, which are determined by a slow and constant turning normal field on ∂Q.

(3) e maps each foot of $\widehat{Q}$ to the corresponding foot of Q.

We say e is (N,ϵ)-compliant if e has ϵ-bounded distortion to distance N.

Suppose that $\mathcal{A}$ is a good assembly and $\hat{\mathcal{A}}$ is a perfect one. Then we say that $e\colon \partial^{\mathcal{F}}\hat{\mathcal{A}} \to \partial^{\mathcal{F}}\mathcal{A}$ is (N,ϵ)-compliant if the following are satisfied:

(1) For each component $\widehat{Q}$ of $\hat{\mathcal{A}}$, there is a corresponding Q of $\mathcal{A}$ such that the restriction of e on $\hat{\partial}^{\mathcal{F}}\widehat{Q}$ is an (N,ϵ)-compliant map to $\partial^{\mathcal{F}}Q$.

(2) If $\hat{\gamma}$ is a gluing boundary of $\widehat{Q}_1$ and $\widehat{Q}_2$ of $\hat{\mathcal{A}}$, and n_i are the frames in $\hat{\partial}^{\mathcal{F}}\widehat{Q}_i$ for $i=1,2$ with n_1 and n_2 sharing the same basepoint on $\hat{\gamma}$, then
$$d(n_1 \to n_2, e(n_1) \to e(n_2)) < \epsilon.$$

Before constructing the map from a good assembly to its perfect model, we want to introduce one more definition. We say two complex tuples $(R_i,s_i)_{i=1}^3$ and $(R_i',s_i')_{i=1}^3$ are K-related for some $K>1$ if there are two genus-2 quasi-Fuchsian groups Γ_1 and Γ_2 and a K-quasiconformal mapping $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that:

(1) Γ has a nonseparating pants decomposition with cuff half-lengths R_i and shears s_i for $i=1,2,3$, and the same for Γ' and $(R_i',s_i')_{i=1}^3$.

(2) f conjugates Γ_1 to Γ_2 and preserves the corresponding homotopy classes of each corresponding pants decomposition.

Theorem 7.24 *For all $M>0$, $\delta>0$, $B^+>B^->0$ and $K>1$, we can find $C, R_0>0$ such that for all $\overline{R}>R_0$ and $\epsilon>0$ the following holds: Suppose that $R_i, s_i \in \mathbb{C}$ and $R_i', s_i' \in \mathbb{R}$ for $i \in \mathbb{Z}/3\mathbb{Z}$ satisfy*

(1) $|R_i - \overline{R}| < \delta$ and $B^- < \operatorname{Re}(s_i) < B^+$,

(2) $\overline{R} - \delta < R_i' < \overline{R} + \delta$ and $B^- < s_i' < B^+$,

(3) $(R_i,s_i)_{i=1}^3$ is K-related to $(R_i',s_i')_{i=1}^3$,

(4) $|\operatorname{Re}(s_i) - s_i'| < 1/\overline{R}$.

Then for an $(R_i,s_i,\epsilon)_{i=1}^3$-good assembly $\mathcal{A}$, there is an $(R_i,s_i)_{i=1}^3$-perfect assembly $\hat{\mathcal{A}}$ and an $(M, C\epsilon)$-compliant map $e\colon \partial^{\mathcal{F}}\hat{\mathcal{A}} \to \partial^{\mathcal{F}}\mathcal{A}$.

Proof By Theorem 7.5, and $(R_i,s_i)_{i=1}^3$ being K-related to $(R_i',s_i')_{i=1}^3$, we can prove that there exists $C_1>0$ such that for each good pants Q of $\mathcal{A}$, we can construct an $(M, C_1\epsilon)$-compliant map $e\colon \hat{\partial}^{\mathcal{F}}\widehat{Q} \mapsto \partial^{\mathcal{F}}Q$ where C_1 does not depend on ϵ, as in the proof of [Kahn and Wright 2021, Theorem A.16]. Now given the whole assembly $\mathcal{A}$, we construct the perfect one as follows: if two good pants are glued along an (R_i,ϵ)-good curve, then the corresponding two perfect pants are glued along the boundary curve with length R_i and the two feet are joined with shear by s_i. Thus condition (1) of the global map e automatically holds. Condition (2) then follows from the $(R_i,s_i,\epsilon)_{i=1}^3$-goodness of $\mathcal{A}$, since the basepoints of $e(n_1)$ and $e(n_2)$ are always within $(B^+ + 1)\epsilon/R$ of each other and the difference of bending is at most ϵ. □

Now we can estimate the distortion of the map in the above theorem:

Theorem 7.25 *For all $D > 0$, $B^+ > B^- > 0$, $\delta > 0$ and $K > 1$ there exist $C, R_0, \epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and $\bar{R} > R_0$ the following holds: Suppose $R_i, s_i \in \mathbb{C}$ and $R'_i, s'_i \in \mathbb{R}$ for $i \in \mathbb{Z}/3\mathbb{Z}$ satisfy*

(1) $|R_i - \bar{R}| < \delta$ *and* $B^- < \mathrm{Re}(s_i) < B^+$,

(2) $\bar{R} - \delta < R'_i < \bar{R} + \delta$ *and* $B^- < s'_i < B^+$,

(3) $(R_i, s_i)_{i=1}^3$ *is K-related to* $(R'_i, s'_i)_{i=1}^3$,

(4) $|\mathrm{Re}(s_i) - s'_i| < 1/\bar{R}$.

Then for any $(R_i, s_i, \epsilon)_{i=1}^3$-good assembly $\mathcal{A}$, we can find an $(R_i, s_i)_{i=1}^3$-perfect assembly $\hat{\mathcal{A}}$ and a map $e \colon \partial^{\mathcal{F}} \hat{\mathcal{A}} \to \partial^{\mathcal{F}} \mathcal{A}$ which has $C\epsilon$-bounded distortion to distance D.

Proof Let $\mathcal{A}$ be as given in the statement. Then by Theorem 7.24, we have an $(R_i, s_i)_{i=1}^3$-perfect model $\hat{\mathcal{A}}$ and a $(D, C\epsilon)$-compliant map $e \colon \partial^{\mathcal{F}} \hat{\mathcal{A}} \to \partial^{\mathcal{F}} \mathcal{A}$, where C does not depend on ϵ. We will work in the universal cover of $\hat{\mathcal{A}}$, and apply Theorem 7.5 to prove that the lift of e (still denoted by e) has bounded distortion. Thus we suppose that p and q are two frames based on the boundary curves of $\hat{\mathcal{A}}$ in the universal cover with $d(p, q) < D$, and let p lie on γ and q lie on $\bar{\gamma}$.

Let f be the K-quasiconformal mapping which relates $(R_i, s_i)_{i=1}^3$ and $(R'_i, s'_i)_{i=1}^3$. Then f extends to a (K', C')-quasi-isometry $\tilde{f} \colon \mathbb{H}^3 \to \mathbb{H}^3$ and we can also assume that for each geodesic $\gamma \in \mathbb{H}^3$, $\tilde{f}(\gamma)$ is always within distance C' of $[\tilde{f}(\gamma)]$. Since $\hat{\mathcal{A}}$ corresponds to a finite-index subgroup of a genus-2 quasi-Fuchsian group Γ_1, we apply the conjugacy by f and get a finite-index subgroup of a genus-2 Fuchsian group Γ_2 which gives us an $(R'_i, s'_i)_{i=1}^3$-perfect assembly $\hat{\mathcal{A}}'$. Let $\gamma'_0 = [\tilde{f}(\gamma)]$ and $\gamma'_n = [\tilde{f}(\bar{\gamma})]$ with lifts of boundary curves $\gamma'_1, \gamma'_2, \dots, \gamma'_{n-1}$ separating them in sequence. Let $\gamma_i = [\tilde{f}^{-1}(\gamma'_i)]$ for $i = 0, 1, 2, \dots, n$. Then $\gamma = \gamma_0$ and $\bar{\gamma} = \gamma_n$. we define (η_i), (u_i), (v_i) and (η'_i), (u'_i), (v'_i) as in Section 7.3.

Let $p' \in \gamma'_0$ and $q' \in \gamma'_n$ such that $d(\tilde{f}(p), p') = d(\tilde{f}(p), \gamma'_0)$ and $d(\tilde{f}(q), q') = d(\tilde{f}(q), \gamma'_n)$. Also let $\tilde{g}$ be the approximate inverse of $\tilde{f}$, which is a quasi-isometric extension of f^{-1}. Suppose the geodesic segment $p'q'$ intersects with γ'_i at p'_i for $i = 1, 2, \dots, n-1$ and let $p' = p'_0$ and $q' = p'_n$. Now we know η_i, η'_i is a short orthogeodesic between two cuffs of a pair of pants, so we know there is a constant C_1 such that $C_1^{-1} r^{-\bar{R}/2} < |u_i|, u'_i < C_1 e^{-\bar{R}/2}$. We also know that for each i, there exists $j_i \in \{1, 2, 3\}$ such that $v_i \equiv s_{j_i} (\mathrm{mod}\ R_{j_i})$ (as complex numbers) and $v'_i \equiv s'_{j_i} (\mathrm{mod}\ R'_{j_i})$. By Lemma 7.17, there exists a constant C_2 such that if $d(\gamma'_{i-1}, \gamma'_{i+1}) < C_2$, then $v'_i = s'_{j_i}$. Moreover since $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}'$ are related by f,

$$\frac{v_i - s_{j_i}}{R_{j_i}} = \frac{v'_i - s'_{j_i}}{R'_{j_i}}.$$

So $v_i = s_{j_i}$ if and only if $v'_i = s'_{j_i}$.

Now we define a *run* to be an interval $\mathbb{Z} \cap [x, y]$ such that $v'_i = s'_{j_i}$ for all $x < i < y$, where $x, y \in \mathbb{Z}$. A run is called *maximal* if it is not a proper subinterval of another run. We allow the case that $y = x + 1$, which is a trivial run. Then we can find integers $0 = x_1 < x_2 < \cdots < x_k = n$ such that each $[x_i, x_{i+1}]$ is a maximal run, and the union of these intervals cover $\mathbb{Z} \cap [0, n]$. We know k is bounded in terms of D and K, since $d(\gamma'_{x_i-1}, \gamma'_{x_i+1})$ is bounded below and $d(p_0, p_n) < K'D + 2C'$ by $\tilde{f}$ quasi-isometric. Similarly

$d(p_i, p_{i+1}) < K'D + 2C'$. Thus there is a constant C_3 such that $d(p_i, \eta_{i-1}), d(p_i, \eta_i) < \frac{1}{2}\bar{R} + C_3$, which is the case $n = 1$ of Theorem 7.5.

For each i, $(\gamma_i)_{i=x_i}^{x_{i+1}}$ and $(\gamma_i'')_{i=x_i}^{x_{i+1}}$ are $(R, C_1, \epsilon, B^-, B^+)$-well-matched; conditions (1) and (3) are satisfied by the definition of a run, and conditions (2) and (4) hold from (R_i, s_i, ϵ)-goodness of $\mathcal{A}$. Then $(\gamma_i)_{i=x_i}^{x_{i+1}}$ is (R, C_1, B^-, B^+, K)-related to $(\gamma_i')_{i=x_i}^{x_{i+1}}$ since $(R_i, s_i)_{i=1}^3$ is K-related to $(R_i', s_i')_{i=1}^3$ and $|\mathrm{Re}(s_i) - s_i'| < 1/\bar{R}$.

Let $p_{x_i} \in \gamma_{x_i}$ be the point closest to $\tilde{g}(p_{x_i}')$ for $i = 1, 2, \dots, n-1$, and let $p_0 = p$ and $p_n = q$. Let α_i be the frame lifted from $\hat{\partial}^{\mathcal{F}} \widehat{Q}$ where two boundary curves of $\widehat{Q}$ lift to γ_{x_i} and γ_{x_i+1}, and β_i be the frame lifted from $\hat{\partial}^{\mathcal{F}} \widehat{Q}$ where two boundary curves of $\widehat{Q}$ lift to γ_{x_i} and γ_{x_i-1}.

Now we make the following claims:

(1) $d(\alpha_i \to \beta_{i+1}, e(\alpha_i) \to e(\beta_{i+1})) < \epsilon$.

(2) $d(\beta_i \to \alpha_i, e(\beta_i) \to e(\alpha_i)) < \epsilon$.

The second one directly follows from the $(D, C\epsilon)$-compliance of e. For the first one, we know the map in Theorem 7.5 is ϵ-related to e on the relevant parts of $\mathcal{F}(\gamma_{x_i})$ and $\mathcal{F}(\gamma_{x_{i+1}})$ since $d(p_j, \eta_{j-1}), d(p_j, \eta_j) < \frac{1}{2}\bar{R} + C_3$ for all j. Then the first claim follows from Theorem 7.5.

Therefore the theorem follows by Lemma 7.16. □

7.5 Extensions

We recall that $X' \subset X$ is called *A-dense* in a metric space X if $\mathcal{N}_A(X') = X$.

Theorem 7.26 *For all A, there exist B and K such that for all δ and Ω there exists ϵ such that the following holds: Suppose Λ is a K-quasicircle in $\widehat{\mathbb{C}}$ and $U \subset \mathcal{F}(C(\Lambda))$ is A-dense; here $C(\Lambda)$ is the convex hull of Λ. If $e\colon U \to \mathcal{F}(\mathbb{H}^3)$ is a map having ϵ-bounded distortion to distance B, then e is a K-quasi-isometric embedding, and e extends to $\hat{e}\colon \Lambda \to \partial(\mathbb{H}^3)$ to be a $(\Omega, 1+\delta)$-quasisymmetric embedding.*

Remark 7.27 Here, by $\hat{e}$ being $(\Omega, 1+\delta)$-quasisymmetric, we mean that for any quadruple (z_1, z_2, z_3, z_4) of four distinct points with its cross ratio in a compact set $\Omega \subset \mathbb{C}$, we have

$$|[\hat{e}(z_1), \hat{e}(z_2); \hat{e}(z_3), \hat{e}(z_4)] - [z_1, z_2; z_3, z_4]| < \delta.$$

We want to quote two theorems in [Kahn and Wright 2021] to prove the above theorem:

Theorem 7.28 *For all K and δ, there exist K' and D such that the following holds: Suppose X is a path metric space, Y is δ-hyperbolic and $f\colon X \to Y$ is such that*

$$K^{-1} d(x, x') - K < d(f(x), f(x')) < K d(x, x') + K$$

whenever $d(x, x') < D$. Then f is a K'-quasi-isometric embedding.

Theorem 7.29 *Let X and Y be Gromov hyperbolic, and let $f\colon X \to Y$ be a quasi-isometric embedding. Then f extends continuously to an embedding $\hat{f}\colon \partial X \to \partial Y$. Moreover, $\hat{f}$ depends continuously on f with the uniform topology on $\hat{f}$ and the local uniform topology on f.*

Proof of Theorem 7.26 The proof of the first part follows from the proof of [Kahn and Wright 2021, Theorem A.19], by applying Theorems 7.28 and 7.29. Thus the only remaining task is to demonstrate that $\hat{e}$ is $(\Omega, 1+\delta)$-quasisymmetric.

Since we can change $\hat{e}$ by Möbius transformations in domain and range, we can assume that the quadruple on Λ is $(z, -1; 1, \infty)$ for some $z \in \mathbb{C}$ with $z \neq 1, -1$ and $\hat{e}(-1) = -1$, $\hat{e}(1) = 1$ and $\hat{e}(\infty) = \infty$. To show the inequality for cross ratios, it suffices to prove that there exists ϵ such that

$$|\hat{e}(z) - z| < 2\delta.$$

Suppose to the contrary such ϵ does not exist. Then we can take a sequence of maps e_n and $z_n \in \mathbb{C} - \{1, -1\}$ defined on a sequence of A-dense sets U_n, with $1/n$-bounded distortion to distance B and $|\hat{e}_n(z_n) - z_n| > 2\delta$. Since Ω is compact, $\{z_n\}$ is uniformly bounded. By passing to a subsequence, we have limits e_∞ and $z_\infty \in \mathbb{C}$, and $\hat{e}_\infty(z) = z$ for any z, because $\hat{e}_\infty$ preserves -1, 1 and ∞ and is a Möbius transformation. In particular, $\hat{e}_\infty(e_\infty) = z_\infty$, which is a contradiction to $|\hat{e}_\infty(z_\infty) - z_\infty| > 2\delta$. □

We conclude this subsection by extending the map defined on a quasicircle to a map defined on the whole Riemann sphere:

Theorem 7.30 *Given any $K \geq 1$ and $\nu > 0$, there exist a compact set $\Omega \subset \mathbb{C}$ and $\delta > 0$ such that: Suppose γ is a K-quasicircle in $\widehat{\mathbb{C}}$ and $g\colon \gamma \to \widehat{\mathbb{C}}$ is $(\Omega, 1+\delta)$-quasisymmetric. Then g extends to a $(1+\nu)$-quasiconformal map from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. Moreover, if g conjugates a group of Möbius transformations to another such group, then the extension does as well.*

Lemma 7.31 *For any $K > 1$ and $\epsilon > 0$, there exist $\delta > 0$ and Ω compact such that: Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal mapping and $g\colon f(S^1) \to \mathbb{C}$ is $(\Omega, 1+\delta)$-quasisymmetric. Suppose f and g are normalized by $f(-1) = -1 = g(-1)$ and $f(1) = 1 = g(1)$. Then for any $z \in S^1$, we have*

$$|g(f(z)) - f(z)| < \epsilon.$$

Proof This is equivalent to proving that for any $K > 1$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $f\colon \mathbb{R} \to \mathbb{C}$ is a K-quasiline and $g\colon f(\mathbb{R}) \to \mathbb{C}$ is $(\Omega, 1+\delta)$-quasisymmetric with $f(i) = i = g(i)$ for $i = 0, 1$, then for any $z \in [0, 1]$, we have

$$|g(f(z)) - f(z)| < \epsilon.$$

Given $K > 1$, we know that there exists $m \in \mathbb{Z}_+$ such that for any K-quasiconformal mapping $f\colon \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$ and $f(1) = 1$, and $x, y \in \mathbb{D}$ (the unit disc) with $|x - y| \leq 1/2^m$, we have $|f(x) - f(y)| \leq \frac{1}{2}$.

We also choose Ω to be the close disk centered at 0 with radius R, where R will be determined later. Now we define

$$T_\delta(n)=\max\left\{\left|g\left(f\left(\frac{a}{2^{nm}}\right)\right)-f\left(\frac{a}{2^{nm}}\right)\right|:0\le a\le 2^{nm},\ 2^m\nmid a,\ f \text{ is } K\text{-quasiconformal}, \right.$$
$$\left. g \text{ is } (\Omega,1+\delta)\text{-quasisymmetric},\ f(0)=0=g(0),\ f(1)=1=g(1)\right\},$$

for $n\ge 1$. We want to use recursion to prove that there exists δ such that the $T_\delta(n)$ are universally bounded by ϵ. We first consider $T_\delta(1)$. Since f is K-quasiconformal and normalized, $f(z)$ is bounded by K for $0\le z\le 1$. Then by m fixed and g being $(\Omega,1+\delta)$-quasisymmetric, we can find δ and R depending on ϵ, m and K such that

$$\left|g\left(f\left(\frac{a}{2^m}\right)\right)-f\left(\frac{a}{2^m}\right)\right|<\tfrac{1}{2}\epsilon,$$

for $a=1,2,\dots,2^m-1$, and $\delta\to 0$ as $\epsilon\to 0$. Then $T_\delta(1)\le\frac{1}{2}\epsilon$. Now for n, divide $[0,1]$ into 2^m subintervals $[i/2^m,(i+1)/2^m]$ with $i=0,1,\dots,2^m-1$. For each subinterval $[i/2^m,(i+1)/2^m]$, we divide it into 2^{nm} pieces, renormalize f and g and use the result for n. Therefore

$$T_\delta(n+1)\le T_\delta(1)+\tfrac{1}{2}T_\delta(n)\le\tfrac{1}{2}\epsilon+\tfrac{1}{2}T_\delta(n).$$

Together with $T_\delta(1)<\frac{1}{2}\epsilon$, we know $T_\delta(n)<\epsilon$ for any positive integer n. Then by continuity of f and g, we know for any $x\in[0,1]$,

$$|g(f(x))-f(x)|<\epsilon. \qquad\square$$

Proof of Theorem 7.30 Let f be a K-quasiconformal mapping from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ that sends the unit circle S^1 to γ. We normalize f and g so that $f(-1)=-1=g(-1)$ and $f(1)=1=g(1)$. Then we merely have to prove that g can be extended to the interior of $f(S^1)$ where the extension is compatible with Möbius transformations as g, since the other side can be proved by applying the inversion along S^1. We take four points A, B, C and D on $f(S^1)$ such that the modulus of the quadrilateral $ABCD$ is 1, which means the extremal distance from arc $\widehat{AB}$ to arc $\widehat{CD}$ within $f(\overline{\mathbb{D}})$ is 1. We first want to prove that the extremal distance b from arc $\widehat{g(A)g(B)}$ to arc $\widehat{g(C)g(D)}$ within $g\circ f(\overline{\mathbb{D}})$ is close to 1 when δ is small.

For $\alpha>-1$, consider the circle $S_{1+\alpha}$ centered at the origin with radius $1+\alpha$ and the disc $\mathbb{D}_{1+\alpha}$ that is bounded by $S_{1+\alpha}$. For any $z\in S^1$, let $z'=(1+\alpha)z\in S_{1+\alpha}$. For any K, we know there exists $C(K,\alpha)>0$ such that for any $z_1,z_2\in\overline{\mathbb{D}}_{1+\alpha}$, we have

$$|f(z_1)-f(z_2)|<C(K,\alpha), \tag{7-32}$$

whenever $|z_1-z_2|<\alpha$. Moreover $C(K,\alpha)\to 0$ as $\alpha\to 0$ and K is fixed. Applying (7-32) for f^{-1} and by Lemma 7.31, for any $\alpha>0$ there exists $\delta>0$ and compact Ω such that

$$|f^{-1}\circ g\circ f(z)-z|=|f^{-1}\circ g\circ f(z)-f^{-1}(f(z))|<\alpha, \tag{7-33}$$

for g being (Ω,δ)-quasisymmetric and $z\in S^1$. Then $g(f(S^1))\subset f(\mathbb{D}_{1+\alpha}-\mathbb{D}_{1-\alpha})$.

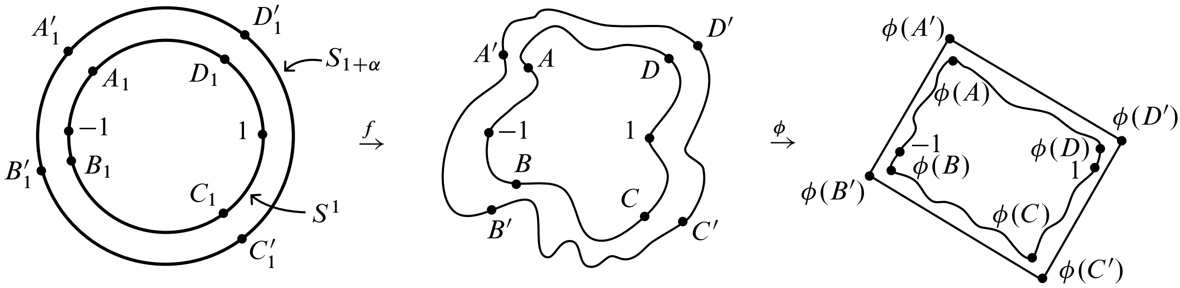

Figure 6: S^1 and $S_{1+\alpha}$, and their images under f and $\phi\circ f$.

Let $A_1, B_1, C_1, D_1 \in S^1$ be the preimages of A, B, C and D under f, A_1', B_1', C_1' and D_1' be the corresponding points on $S_{1+\alpha}$, and A', B', C' and D' be the images of A_1', B_1', C_1' and D_1' under f. We first want to study the extremal distance a from arc $\widehat{A'B'}$ to arc $\widehat{C'D'}$ within $f(\mathbb{D}_{1+\alpha})$. Let ϕ be the conformal mapping from $f(\mathbb{D}_{1+\alpha})$ to a rectangle R in $\mathbb{C}$ such that $\phi(A')$, $\phi(B')$, $\phi(C')$ and $\phi(D')$ are four vertices, $|\phi(A')\phi(D')| = a|\phi(A')\phi(B')|$ and ϕ preserves -1 and 1. Since ϕ is conformal, $\phi\circ f$ is K-quasiconformal on the disk $\mathbb{D}_{1+\alpha}$. On the other hand, the modulus of quadrilateral $ABCD$ is 1. Therefore the modulus of quadrilateral $A_1B_1C_1D_1$ is between $1/K$ and K, and so is the modulus of quadrilateral $A_1'B_1'C_1'D_1'$. Thus the modulus of quadrilateral $A'B'C'D'$ is between $1/K^2$ and K^2, and so is the modulus of R, which means $1/K^2 < a < K^2$. Thus the rectangle R admits a πa-quasiconformal reflection, which is proven in [Werner 1997]. Then $\phi\circ f$ extends to a πaK-quasiconformal mapping $\overline{\phi\circ f}$ on $\widehat{\mathbb{C}}$. Since $\pi aK < \pi K^3$, we have that $\overline{\phi\circ f}$ is πK^3-quasiconformal. Thus by (7-32) and the fact that $\overline{\phi\circ f}$ preserves -1 and 1, for any $z\in S^1$,

(7-34) $$|\phi\circ f(z) - \phi\circ f(z')| < C(\pi K^3, \alpha),$$

Then for any arc β inside $\phi(f(\mathbb{D}))$ connecting arc $\widehat{\phi(A)\phi(B)}$ and arc $\widehat{\phi(C)\phi(D)}$, we can extend β to β' which connects $\overline{\phi(A')\phi(B')}$ and $\overline{\phi(C')\phi(D')}$, and

$$l(\beta') < l(\beta) + 2C(\pi K^3, \alpha).$$

Let $|\phi(A')\phi(B')| = t$. Then $|\phi(A')\phi(D')| = at$ and $l(\beta') \ge at$. Hence $l(\beta) > at - 2C(\pi K^3, \alpha)$. Since the extremal distance from arc $\widehat{AB}$ to arc $\widehat{CD}$ within $f(\mathbb{D})$ is 1 and ϕ is conformal,

(7-35) $$\begin{aligned} 1 = \sup_\rho \frac{\inf_\beta L_\rho(\beta)}{\mathrm{Area}(\rho)} &\ge \frac{\inf_\beta l(\beta)}{\mathrm{Area}(\phi\circ f(\mathbb{D}))} \ge \frac{l(\beta)^2}{at^2} \\ &> \frac{(at - 2C(\pi K^3,\alpha))^2}{at^2} = a - \frac{4C(\pi K^3,\alpha)}{t} + \frac{4C(\pi K^3,\alpha)^2}{at^2} \ge a - \frac{4C(\pi K^3,\alpha)}{t}, \end{aligned}$$

where ρ goes through all metrics and β is among all arcs connecting $\widehat{\phi(A)\phi(B)}$ and $\widehat{\phi(C)\phi(D)}$ within $\phi\circ f(\mathbb{D})$. The rectangle R contains -1 and 1, so its diagonal has length at least 2. Then by $1/K^2 < \alpha < K^2$, we have

$$4 \le t^2 + a^2t^2 = (1+a^2)t^2 \le (1+K^4)t^2.$$

Thus $t \geq 2/\sqrt{1+K^4}$. Together with (7-35), we know

$$1 > a - \frac{4C(\pi K^3, \alpha)}{t} \geq a - 2\sqrt{1+K^4}\,C(\pi K^3, \alpha) =: a - u(K, \alpha).$$

Hence

(7-36) $$a < 1 + u(K, \alpha),$$

and

$$\lim_{\alpha \to 0} u(K, \alpha) = 0,$$

for any fixed K. Similarly, considering the opposite pair of sides of the quadrilateral,

(7-37) $$1/a < 1 + u(K, \alpha).$$

By (7-33), we know

$$|f^{-1} \circ g \circ f(z) - z'| \leq |f^{-1} \circ g \circ f(z) - z| + |z' - z| < 2\alpha.$$

Thus by (7-34),

$$|\phi(g(f(z))) - \phi(f(z'))| = \bigl|\phi\bigl(f(f^{-1}(g(f(z))))\bigr) - \phi(f(z'))\bigr| < C(\pi K^3, 2\alpha).$$

Then since $g(f(S^1)) \subset f(\mathbb{D}_{1+\alpha} - \mathbb{D}_{1-\alpha})$, we can apply the above method to the curve $g \circ f(S^1)$ to obtain

(7-38) $$a < b + u(K, 2\alpha),$$

(7-39) $$\frac{1}{a} < \frac{1}{b} + u(K, 2\alpha),$$

where b is defined at the end of the first paragraph.

Now by (7-36)–(7-39), we have

(7-40) $$\frac{1}{1+u(K,\alpha)} - u(K, 2\alpha) < b < \left(\frac{1}{1+u(K,\alpha)} - u(K, 2\alpha)\right)^{-1}.$$

Since $\lim_{\alpha \to 0} u(K, \alpha) = 0$,

(7-41) $$\lim_{\alpha \to 0^+} \left(\frac{1}{1+u(K,\alpha)} - u(K, 2\alpha)\right) = 1.$$

Next we let R_0 and R_1 be the interiors of γ and $g(\gamma)$, respectively, with $\overline{R}_0$ and $\overline{R}_1$ as their corresponding closures. Then for $i = 0, 1$, take a homeomorphism $h_i \colon \overline{R}_i \to \overline{\mathbb{D}}$ such that h_i is conformal on R_i. Here each h_i is unique up to Möbius transformations. We then have a map $\eta := h_2 \circ g \circ h_1^{-1} \colon S^1 \to S^1$. Since h_1 is conformal, we know the quadrilateral $h_1^{-1}(A)h_1^{-1}(B)h_1^{-1}(C)h_1^{-1}(D)$ has modulus 1. By (7-40) and (7-41), and since h_2 is conformal, the quadrilateral $h_2(g(A))h_2(g(B))h_2(g(C))h_2(g(D))$ has modulus approaching 1 uniformly as $\alpha \to 1$. Thus for any $\nu_1 > 0$, there exists $\alpha > 0$ such that η is $(1+\nu_1)$-quasisymmetric. On the other hand, for any $\nu > 0$, there exists $\nu_1 > 1$ such that when η is $(1+\nu_1)$-quasisymmetric, its Douady–Earle extension $E(\eta) \colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is $(1+\nu)$-quasiconformal on $\mathbb{D}$. Moreover $\hat{g} := h_2^{-1} \circ E(\eta) \circ h_1 \colon \overline{R}_0 \to \overline{R}_1$ is $(1+\nu)$-quasiconformal on R_1, which is the desired extension of g.

Finally, we only need to verify that the above extension has the natural property with Möbius transformations when g does. Suppose $\tau: G \to G'$ is an isomorphism between groups of Möbius transformations, where G preserves γ and G' preserves $g(\gamma)$, such that for any $x \in G$,

$$g \circ x = \tau(x) \circ g, \tag{7-42}$$

as maps from γ to $g(\gamma)$. Since $h_1 \circ x \circ h_1^{-1}: \mathbb{D} \to \mathbb{D}$ is conformal, there exists a Möbius transformation x_1 such that $h_1 \circ x \circ h_1^{-1} = x_1$. Thus

$$h_1 \circ x = x_1 \circ h_1 \quad \text{and} \quad x \circ h_1^{-1} = h_1^{-1} \circ x_1. \tag{7-43}$$

Similarly, there is another Möbius transformation x_2 such that

$$h_2 \circ \tau(x) = x_2 \circ h_2 \quad \text{and} \quad \tau(x) \circ h_2^{-1} = h_2^{-1} \circ x_2. \tag{7-44}$$

Hence by the natural property of Douady–Earle extension with Möbius transformations, we have

$$\begin{aligned}
\hat{g} \circ x &= h_2^{-1} \circ E(\eta) \circ h_1 \circ x \overset{\text{(7-43)}}{=} h_2^{-1} \circ E(\eta) \circ x_1 \circ h_1 = h_2^{-1} \circ E(\eta \circ x_1) \circ h_1 \\
&= h_2^{-1} \circ E(h_2 \circ g \circ h_1^{-1} \circ x_1) \circ h_1 \overset{\text{(7-43)}}{=} h_2^{-1} \circ E(h_2 \circ g \circ x \circ h_1^{-1}) \circ h_1 \\
&\overset{\text{(7-42)}}{=} h_2^{-1} \circ E(h_2 \circ \tau(x) \circ g \circ h_1^{-1}) \circ h_1 \overset{\text{(7-44)}}{=} h_2^{-1} \circ E(x_2 \circ h_2 \circ g \circ h_1^{-1}) \circ h_1 = h_2^{-1} \circ E(x_2 \circ \eta) \circ h_1 \\
&= h_2^{-1} \circ x_2 \circ E(\eta) \circ h_1 \overset{\text{(7-44)}}{=} \tau(x) \circ h_2^{-1} \circ E(\eta) \circ h_1 = \tau(x) \circ \hat{g}.
\end{aligned}$$

□

7.6 Good is close to perfect

We conclude this section with the following theorem:

Theorem 7.45 *For all $B^+ > B^- > 0$, $\delta > 0$ and $K > 1$, there exists R_0 such that for all $\nu > 0$, there exists $\epsilon > 0$ such that for all $\bar{R} > R_0$ the following holds: Suppose that $R_i, s_i \in \mathbb{C}$ and $R_i', s_i' \in \mathbb{R}$ for $i \in \mathbb{Z}/3\mathbb{Z}$, satisfy*

1. *$|R_i - \bar{R}| < \delta$ and $B^- < \mathrm{Re}(s_i) < B^+$,*
2. *$\bar{R} - \delta < R_i' < \bar{R} + \delta$ and $B^- < s_i' < B^+$,*
3. *$(R_i, s_i)_{i=1}^3$ is K-related to $(R_i', s_i')_{i=1}^3$,*
4. *$|\mathrm{Re}(s_i) - s_i'| < 1/\bar{R}$.*

Suppose that Γ is the genus-2 quasi-Fuchsian group corresponding to the $(R_i, s_i)_{i=1}^3$-perfect assembly of two components. For any $(R_i, s_i, \epsilon)_{i=1}^3$-good assembly $\mathcal{A}$ in the hyperbolic 3-manifold M such that $S_{\mathcal{A}}$ is connected, let $\rho_{\mathcal{A}}$ be the corresponding surface subgroup representation. Then $\rho_{\mathcal{A}}$ is $(1+\nu)$-quasiconformally conjugate to a finite-index subgroup of Γ.

Proof Since $(R_i, s_i)_{i=1}^3$ is K-related to $(R_i', s_i')_{i=1}^3$, we know Γ is K-quasi-Fuchsian. Hence the radius of the convex core $\mathrm{CM}(\Gamma)$ of $\mathbb{H}^3/\Gamma$ is bounded by a constant A only depending on K. Therefore the universal cover of any (R_i, s_i)-perfect assembly in $\mathbb{H}^3$ is A-dense in the convex hull $C(\Lambda(\Gamma))$. The result follows from Theorems 7.25, 7.26 and 7.30. □

8 Proof of the main result

Proof of Theorem 1.1 By Theorem 5.1, there exists an (R,m)-good nonseparating pants decomposition of Γ for some $R,m>0$. We denote the half-lengths of cuffs by r_i for $i=1,2,3$. Then there exists $K_1>1$ such that Γ is K_1-quasiconformally conjugate to a Fuchsian group Γ' with a nonseparating pants decomposition of cuff lengths $2\,\mathrm{Re}(r_i)$ for $i=1,2,3$. Then by Theorem 5.8, there exist $B^+>B^->0$ and $\delta>0$, such that for any $R_0>0$, there exists $\bar{R}>R_0$ such that Γ admits an $(\bar{R},\delta)$-good pants decomposition with cuff half-lengths R_i and twists $\mathrm{Re}(s_i)\in(B^-,B^+)$, so we can choose $\bar{R}$ sufficiently large such that all previous results related to $\bar{R}$ hold. Similarly Γ' admits a corresponding pants decomposition (R_i',s_i') by the K_1-quasiconformal mapping, and $s_i'\in(B^-,B^+)$. Moreover, by Lemma 5.17, we know

$$|\mathrm{Re}(s_i)-s_i'|\le\left|\mathrm{Re}(s_i)-\tfrac{1}{2}(\mathrm{Re}(r_{i+1})+\mathrm{Re}(r_{i+2})-\mathrm{Re}(r_i))\right|+\left|s_i'-\tfrac{1}{2}(\mathrm{Re}(r_{i+1})+\mathrm{Re}(r_{i+2})-\mathrm{Re}(r_i))\right|<1/\bar{R},$$

when $\bar{R}$ is large enough. Then for B^-, B^+, δ, $\max\{K_1,K\}$, R_0 and given $\nu>0$, let ϵ be as in Theorem 7.45.

Let

$$A=\sum_{P\in\Pi_{\epsilon,R_i}}P$$

be the formal sum of all unoriented $(R_i,\epsilon)_{i=1}^3$-good pants in the hyperbolic 3-manifold M. Then we want to use the doubling trick in Section 4.7 to match all the oriented pants together to construct a closed $(R_i,s_i,\epsilon)_{i=1}^3$-good assembly $\mathcal{A}$. To be more specific, let γ be an (R_i,ϵ)-good curve for some i, then let A_γ be the formal sum of one of each pants in $\Pi_{\epsilon,R_i}(\gamma)$ and σ_γ be the permutation in Theorem 6.6. Then we take $2A_\gamma$, where there are two copies with opposite orientation for each pants, and divide it into those π with $\partial\pi$ the same orientation as γ and those π with $\partial\pi$ the opposite orientation. Thus each $\pi\in A_\gamma$ has a π_+ and a π_- as oriented pants. We define the involution τ on $2A_\gamma$ by $\tau(\pi_+)=(\sigma_\gamma(\pi_+))_-$ and $\tau(\pi_-)=(\sigma_\gamma^{-1}(\pi_-))_+$. So this involution gives us the way of gluing pants together along each cuff, and results in a closed oriented assembly.

If $S_{\mathcal{A}}$ is not connected, we can take one connected component, which is still denoted by $S_{\mathcal{A}}$, by passing to a subassembly. Then by Theorem 7.45, we know $\rho_{\mathcal{A}}$ is K-quasiconformally conjugate to a Fuchsian group $\rho_{\hat{\mathcal{A}}}$, where $\hat{\mathcal{A}}$ is the perfect model of $\mathcal{A}$.

We know all pants in $\hat{\mathcal{A}}$ are identical to the $(R_i)_{i=1}^3$-perfect pants. Moreover, by Theorem 6.6, these pants are glued by shear s_i along the cuff with half-length R_i. Therefore $S_{\hat{\mathcal{A}}}$ is a finite covering of our original quasi-Fuchsian surface $\mathbb{H}^3/\Gamma$, which means $\rho_{\hat{\mathcal{A}}}$ is a finite-index subgroup of Γ. □

Remark 8.1 Γ is required to be a genus-2 Fuchsian group, so a nonseparating pants decomposition of Γ will have two identical pants. That is the reason why we can match those pants together by a permutation and the doubling trick, without considering the imbalance of the number of pants which are close to different model pants. In general, it is possible to use good pants homology to solve this difficulty.

We will end this paper by proving Theorem 1.2, which will follow from Theorem 1.1 and some general theorems about Hausdorff dimension.

Proof of Theorem 1.2 As stated in [Ruelle 1982, Section 7], we know that the Hausdorff dimension of the limit set is a real analytic function on the deformation space of any quasi-Fuchsian group. In particular, the function is continuous, so by [Brock 2003, Theorem 1.3], for any $1 \le \alpha < 2$ and $\epsilon > 0$, we can find a genus-2 quasi-Fuchsian group Γ such that

$$\text{H-dim}(\Lambda(\Gamma)) = \alpha.$$

By [Gehring and Väisälä 1973, Theorems 8 and 12; Astala 1994, Corollary 1.2] we can find $K > 1$ such that for any K-quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{C}$, we have

$$|\text{H-dim}(f(\lambda(\Gamma))) - \alpha| < \epsilon.$$

Hence the result follows from Theorem 1.1. □

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