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Gianluca Faraco

Subhojoy Gupta

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Monodromy of Schwarzian equations with regular singularities

GIANLUCA FARACO
SUBHOJOY GUPTA

Let S be a punctured surface of finite type and negative Euler characteristic. We determine all possible representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ that arise as the monodromy of the Schwarzian equation on S with regular singularities at the punctures. Equivalently, we determine the holonomy representations of complex projective structures on S whose Schwarzian derivatives, with respect to some uniformizing structure, have poles of order at most two at the punctures. Following earlier work that dealt with the case when there are no apparent singularities, our proof reduces to the case of realizing a degenerate representation with apparent singularities. This mainly involves explicit constructions of complex affine structures on punctured surfaces, with prescribed holonomy. As a corollary, we determine the representations that arise as the holonomy of spherical metrics on S with cone points at the punctures.

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1 Introduction

Consider the Schwarzian equation

$$y'' + \tfrac{1}{2}qy = 0 \tag{1}$$

on a punctured Riemann sphere $X = \mathbb{C}\mathrm{P}^1 \setminus \{p_1, p_2, \dots, p_k\}$, where the prescribed meromorphic coefficient function q has poles of order at most two at the punctures. This is a second-order complex linear differential equation with regular singularities and thus admits meromorphic solutions that span a complex vector space of dimension two; see for example [Ince 1944, Section 15.3]. The monodromy of the solutions determines a representation $\rho\colon \pi_1(X) \to \mathrm{PSL}_2(\mathbb{C})$, and determining which monodromy groups appear has been a subject of classical work, eg [Poincaré 1884].

More generally, let S be an arbitrary punctured Riemann surface of finite-type and negative Euler characteristic, and let $\Pi = \pi_1(S)$ be its fundamental group. When S is equipped with a complex structure, the Schwarzian equation (1) makes sense by passing to the universal cover; the coefficient q is then the lift of a holomorphic quadratic differential on the surface that has poles of order at most two at the punctures. Equivalently, the solutions of (1) also determine a *complex projective* (or $\mathbb{C}\mathrm{P}^1$-) *structure* on the surface, which is a geometric structure on the surface modeled on $\mathbb{C}\mathrm{P}^1$; the monodromy of the solutions is then the holonomy or monodromy of the geometric structure; see Section 2 for a discussion. It has been a longstanding problem to determine which conjugacy classes of representations $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ arise as the monodromy of such a $\mathbb{C}\mathrm{P}^1$-structure, when one is allowed to vary the marked complex structure on S. The work in [Gupta 2021] provided an answer under the assumption that there is no *apparent singularity*; see Definition 1.2. This clarified, in particular, a remark of Poincaré [1884, page 218] concerning the case of the punctured Riemann sphere, where he wrote,

> "On peut *en général* trouver une équation du 2^d ordre, sans points à apparence singulière qui admette un groupe donné."

Indeed, we showed in [Gupta 2021] that the monodromy groups that do arise are exactly those that are *nondegenerate* as in Definition 1.1.

In this article, we drop the assumption that there are no apparent singularities, and solve the problem, providing a complete characterization of the monodromy groups of the Schwarzian equation with regular singularities. In other words, for a surface $S_{g,k}$ of genus g and $k \geq 1$ punctures that has negative Euler characteristic, we determine the image of the monodromy map

$$\Psi\colon \mathcal{P}_g(k) \to \mathrm{Hom}\big(\pi_1(S_{g,k}), \mathrm{PSL}_2(\mathbb{C})\big) /\!\!/ \mathrm{PSL}_2(\mathbb{C}), \tag{2}$$

where $\mathcal{P}_g(k)$ is the space of meromorphic projective structures on $S_{g,k}$, with respect to a choice of a marked complex structure, such that each puncture corresponds to a pole of order at most two, and the target space is the $\mathrm{PSL}_2(\mathbb{C})$-*representation variety*, the space of surface-group representations into $\mathrm{PSL}_2(\mathbb{C})$ up to the action by conjugation. To be more precise, two representations are equivalent, $\rho_1 \sim \rho_2$, if the *closures* of their $\mathrm{PSL}_2(\mathbb{C})$-orbits intersect; this coincides with geometric invariant theory quotient, for example see [Newstead 2009]. We note here that with our condition on the orders of the poles, the dimensions of the two spaces in the domain and range of the monodromy map Ψ coincide. The question of determining the image of the above monodromy map was mentioned in [Luo 1993, Question 4, page 554]. Our main result (Theorem A) below answers this completely.

To state our results, we recall the following definitions from [Gupta 2021].

Definition 1.1 A representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ is said to be *degenerate* if $\rho(\Pi)$ preserves a set F on $\mathbb{C}\mathrm{P}^1$ where either

(a) $F = \{p\}$ (ie a global fixed point) and the monodromy around each puncture is a parabolic fixing p or the identity element, or

(b) $F = \{p, q\}$ and the monodromy around each puncture fixes p and q.

Otherwise, ρ is said to be *nondegenerate*.

Definition 1.2 A representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ is said to have an *apparent singularity* at a puncture of $S_{g,k}$ if $\rho(\gamma) = \mathrm{Id}$ for the peripheral loop γ around the puncture. We also say a complex projective structure has an *apparent singularity* at a puncture, if the monodromy around it is trivial. In that case the puncture is either a branch point or a regular point of the structure; see Definition 2.2 and the discussion following it.

We shall prove:

Theorem A *Let Π be the fundamental group of a surface $S_{g,k}$ of genus g and $k \geq 1$ punctures, where $2-2g-k < 0$. A representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ arises as the monodromy of a $\mathbb{C}\mathrm{P}^1$-structure in $\mathcal{P}_g(k)$ if and only if one of the following hold:*

(i) *ρ is a nondegenerate representation, or*

(ii) *ρ is a degenerate representation, with at least one apparent singularity, with the only exceptions being*

- *the trivial representation, when $g > 0$ and $k = 1$ or 2, and*
- *a representation whose image is a group of order two, when $g > 0$ and $k = 1$.*

The case of a *closed* surface S_g where $g \geq 2$ was handled by the work of Gallo, Kapovich and Marden in [Gallo et al. 2000], who showed that nonelementary representations that lift to $\mathrm{SL}_2(\mathbb{C})$ are exactly those that arise as monodromy representations of $\mathbb{C}\mathrm{P}^1$-structures on S_g. Note that a nonelementary representation is automatically nondegenerate as defined above (for a comparison between these notions see [Gupta 2021, Section 2.4]). Gallo, Kapovich and Marden in [Gallo et al. 2000, Problem 12.2.1] also stated the problem of what happens for punctured surfaces. This paper solves this — the theorem above answers the "existence" part of their problem, and for the "nonuniqueness" part, we shall observe that the constructions in the paper imply the following:

Corollary B *The nonempty fibers of the monodromy map (2) are infinite in cardinality.*

We mention some further directions to "explore the nonuniqueness" (quoting Gallo, Kapovich and Marden) at the end of this introduction.

The case of a nondegenerate representation (case (i) in Theorem A) follows from the work in [Gupta 2021]; hence the present article, though independent of that paper, can be considered as its sequel. We provide a new and simplified discussion of the construction of that paper in Section 3, that used a geometric interpretation of certain cross-ratio coordinates introduced (in a more general context) by Fock and Goncharov [2006]. We also use one of the results of Allegretti and Bridgeland [2020] which implies that in the case of no apparent singularities, the monodromy representation is necessarily nondegenerate; see Theorem 2.5. Indeed, together with Lemma 4.3 concerning the case when the entire monodromy representation is trivial, and Lemma 5.1 concerning the case when the monodromy group, ie the image of the monodromy representation, has order two, this establishes the "only if" direction in Theorem A.

The main work in this paper is to handle the remaining case of degenerate representations, namely, to construct $\mathbb{C}\mathrm{P}^1$-structures on the punctured surface with a specified monodromy representation as in case (ii) in Theorem A. Note that this is specific to punctured surfaces, as when the surface is closed, the analogue of the degenerate case, ie elementary representations, does not arise. In case (a) of Definition 1.1 of a degenerate representation, namely when the image of a degenerate representation has a global fixed point on $\mathbb{C}\mathrm{P}^1$, it can be conjugated into the subgroup of complex affine maps

$$\mathrm{Aff}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}.$$

In this case, the strategy of our proof is to construct a (*complex*) *affine structure* on $S_{g,k}$, whose monodromy is the prescribed representation. Indeed, we prove the following result concerning the monodromy groups of affine structures on a punctured surface, which we feel is of independent interest:

Theorem C *Let Π be the fundamental group of $S_{g,k}$ as in Theorem A, such that the number of punctures $k \geq 2$. Then any nontrivial representation $\rho\colon \Pi \to \mathrm{Aff}(\mathbb{C})$ with at least one apparent singularity arises as the monodromy of a complex affine structure on $S_{g,k}$. If $k \geq 3$, then every such representation is realizable as the monodromy of a complex affine structure on $S_{g,k}$.*

Our construction of an affine structure involves considering unbounded polygons on $\mathbb{C}$ with sides paired by affine maps, and certain gluing methods which we introduce (eg Definition 4.17). The resulting affine surface acquires branch points that arise from the vertices of the polygon after the identification, that we delete to obtain an affine structure on a punctured surface. Since there is always an additional puncture at infinity, our proof of Theorem C requires at least two punctures.

In the case that the surface has exactly one puncture, we rely on a refinement of the construction alluded to above. Our strategy then is to use the action of the mapping class group to choose a generating set of the fundamental group so that the corresponding polygonal curve bounds an immersed disk in $\mathbb{C}\mathrm{P}^1$. When the representation is *coaxial*, ie when the pair of points in (b) of Definition 1.1 is globally fixed, we also introduce some further new operations (eg Definition 5.12). In these cases, even when the monodromy is affine, we obtain a *projective* structure with the desired monodromy, which is not necessarily affine.

In fact, the recent preprint of Le Fils [2023] that appeared at the time of writing this article shows that in the case of a once-punctured surface, there are some further exceptions to the existence of an affine structure; see [loc. cit., Proposition 1.5]. It turns out that this case can also be handled using the main result of [loc. cit.]; we describe this alternative approach in Section 5.4.4, which involves proving the existence of a projective structure with the desired monodromy and a single branch point, except when the monodromy group is of order two; cf Lemma 5.1. The remaining degenerate representations still fall into case (b) of Definition 1.1, but the pair of points is $\mathbb{C}\mathrm{P}^1$ is preserved, not globally fixed by such a representation; in this case our argument relies on similar techniques.

One feature of our methods is that they are constructive, and can be potentially implemented in an algorithm. The constructions we introduce also yield results on other special classes of $\mathbb{C}\mathrm{P}^1$-structures that are geometric structures in their own right. We mention some of these below, and leave the discussion of others (eg half-translation structures, see Definition 4.15) for future papers.

Translation structures A special case of a complex affine structure is a *translation structure* which has monodromy in the (smaller) subgroup of complex translations $\{z \mapsto z + a \mid a \in \mathbb{C}\} \cong \mathbb{C}$. Such a structure acquires a holomorphic (abelian) differential ω that is the pullback of the differential dz on $\mathbb{C}$ via the charts, and prescribing its monodromy is equivalent to prescribing the periods of ω. In the recent work [Chenakkod et al. 2022] and its follow-up [Chen and Faraco 2024], for a punctured surface the possible periods for meromorphic differentials has been determined in the connected components of stratum with prescribed orders of zeroes and poles. In this paper, we include a proof of the translation-structure case of Theorem C. This is in part to motivate the proof of the more general affine-structure case, which needed significant new constructions as mentioned above. Our treatment here of the translation-structures case differs from that in [Chenakkod et al. 2022], although we employ the same proof strategy.

Spherical structures Another class of projective structures that are interesting in their own right are *spherical cone-metrics* on a surface. For such a structure, the developing map is to the round sphere $\mathbb{S}^2$, and the monodromy lies in $\mathrm{PSU}(2) \cong \mathrm{SO}(3,\mathbb{R})$. Note that the monodromy around any cone point of angle α is an elliptic rotation by that angle. The space $\mathcal{MS}ph_{g,k}(\vartheta)$ of such spherical cone-metrics on $S_{g,k}$ with a set of prescribed cone-angles ϑ at the punctures admits a forgetful projection to the moduli space $\mathcal{M}_{g,k}$; this has been a subject of much study — see for instance Mondello and Panov [2016; 2019]. What has been less studied is the monodromy map in this context, namely, the forgetful projection to the space of surface-group representations into $\mathrm{SO}(3,\mathbb{R})$.

Here, we provide the following corollary to our main theorem:

Corollary D *Let Π be the fundamental group of $S_{g,k}$ as in Theorem A and let $\rho\colon \Pi \to \mathrm{SO}(3,\mathbb{R})$ be a representation. Then ρ is the monodromy of some spherical cone-metric on $S_{g,k}$ with cone points only at the punctures, if and only if it satisfies conditions (i) and (ii) in the statement of Theorem A.*

It worth noting that prescribing the monodromy around the punctures determines the cone-angles only modulo 2π; prescribing these cone-angles *in addition to* the monodromy is a more delicate problem. For a punctured sphere, the work of Eremenko [2020] solves this for the case when the prescribed monodromy is coaxial (which, in this context, is equivalent to being degenerate).

Branched $\mathbb{C}\mathrm{P}^1$-structures Finally, $\mathbb{C}\mathrm{P}^1$-structures on a surface which are allowed to have additional branch points are of independent interest; see Definition 2.2. These were first studied in [Mandelbaum 1973]; see [Calsamiglia et al. 2014] or [Biswas et al. 2019] for more recent results. As a consequence of Theorem A we obtain the following result concerning such branched projective structures on a punctured surface:

Corollary E *Suppose that* Π *is the fundamental group of* $S_{g,k}$ *as in Theorem A. Every representation* $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ *arises as the monodromy of some (possibly branched)* $\mathbb{C}\mathrm{P}^1$*-structure on* $S_{g,k}$*, with at most two branch points.*

The recent work of Nascimento [2024] also proves similar results, and develops completely different techniques for constructing branched projective structures with prescribed monodromy.

It remains to understand the fibers of the monodromy map (2) better, for example how projective structures in the same fiber are related, and we plan to address this in future work. The recent work of Ballas, Bowers, Casella and Ruffoni [Ballas et al. 2024] deals with the case when the surface is the thrice-punctured sphere $S_{0,3}$. For a closed surface, this was handled in the work of Baba [2017], and it is conceivable that one can develop the analogues of the techniques there, in the punctured case. The main result of [Luo 1993] implies that the fiber $\Psi^{-1}(\rho)$ is a discrete set when the monodromy representation ρ is irreducible, and the monodromy around each puncture is nontrivial and not parabolic. In contrast, in the case all the punctures are branch points and ρ is quasi-Fuchsian, the fibers can be connected, and the work in [Calsamiglia et al. 2014] determines when that happens.

Plan of the paper The paper is organized as follows. In Section 2 we begin by recalling the basic background of meromorphic projective structures on surfaces and their monodromy representations. In Section 3, we deal with the case of nondegenerate representations which essentially follows by [Gupta 2021], an earlier work by the second author. The proof of Theorem A for degenerate representations is the main core of the present paper and it is developed along Sections 4–6. In Section 4 we provide a proof of Theorem C which handles affine representations in the case of surfaces with at least two punctures. In Section 5, instead, we deal with the complementary case of affine representations for once-punctured surfaces. Finally, in Section 6 we deal with dihedral representations. The final Section 7 concludes with the proofs of Corollaries B, D and E.

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2 Preliminaries

Let S be a surface of finite type (open or closed) and of negative Euler characteristic.

2.1 Meromorphic projective structures

A $\mathbb{C}\mathrm{P}^1$-*structure* or a (*complex*) *projective structure* on a surface S is a maximal atlas of charts to $\mathbb{C}\mathrm{P}^1$ such that the transition functions are Möbius maps, ie elements of $\mathrm{PSL}_2(\mathbb{C})$; in other words, it is a (G, X)-structure, where $G = \mathrm{PSL}_2(\mathbb{C})$ and $X = \mathbb{C}\mathrm{P}^1$. Such a geometric structure can be equivalently described in terms of the *developing map* $f : \widetilde{S} \to \mathbb{C}\mathrm{P}^1$ and the *monodromy* (or holonomy) representation $\rho \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$; the developing map is a ρ-equivariant immersion, satisfying $f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$ for each $x \in \widetilde{S}$ and each $\gamma \in \pi_1(S)$. Note that the developing map is determined up to a postcomposition by a Möbius map A, and ρ is defined up to conjugation, so that the pair (dev, ρ) and $(A \cdot \mathrm{dev}, A \cdot \rho \cdot A^{-1})$ define the same $\mathbb{C}\mathrm{P}^1$-structure.

An example of a $\mathbb{C}\mathrm{P}^1$-structure on S is a *hyperbolic* structure, where the developing map develops into the upper hemisphere of $\mathbb{C}\mathrm{P}^1$ (that can be identified with the hyperbolic plane $\mathbb{H}^2$), and the monodromy is a discrete (Fuchsian) representation $\rho \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$.

We often consider a $\mathbb{C}\mathrm{P}^1$-structure with an additional *marking*, that is, an additional choice of a homeomorphism that identifies S with our complex projective surface, such that two marked structures that differ by a homeomorphism homotopic to the identity are considered equivalent.

A change of marking on a marked $\mathbb{C}\mathrm{P}^1$-structure is effected by the action of an element of the mapping class group of S; this changes the monodromy representation by precomposing with the corresponding automorphism of $\pi_1(S)$, which is the action of the mapping class group on the $\mathrm{PSL}_2(\mathbb{C})$-representation variety. Note that for a punctured surface, an element of its mapping class group can permute the punctures. Thus, while seeking a $\mathbb{C}\mathrm{P}^1$-structure with a prescribed monodromy representation ρ, it suffices to find one whose monodromy lies in the mapping class group orbit of ρ. We shall use this to our advantage later in the paper.

The relation with the Schwarzian equation arises through the *Schwarzian derivative* $\widetilde{q} = S(f)\, dz^2$ of the developing map f, where

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{3}$$

The quadratic differential thus defined is invariant when f is precomposed by a Möbius transformation; since f is ρ-equivariant, $\widetilde{q}$ descends to a quadratic differential q on S, that is holomorphic with respect to the complex structure induced by the $\mathbb{C}\mathrm{P}^1$-structure.

Conversely, fix a complex structure on S such that $S = \mathbb{H}^2/\Gamma$, where Γ is a Fuchsian representation. Let y_0 and y_1 be two linearly independent solutions of the Schwarzian equation

$$y'' + \tfrac{1}{2}\widetilde{q}\, y = 0 \tag{4}$$

on the universal cover $\widetilde{S} = \mathbb{H}^2$, where $\widetilde{q}$ is a Γ-invariant holomorphic quadratic differential. Then the ratio $f := y_0/y_1$ defines a developing map to $\mathbb{C}\mathrm{P}^1$, which is equivariant with respect to a monodromy representation $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{C})$, defining a $\mathbb{C}\mathrm{P}^1$-structure on S. It is an exercise to show that then the Schwarzian derivative of f equals $\widetilde{q}$, so that this is indeed an inverse construction.

For a *closed* surface S, the deformation space $\mathcal{P}(S)$ of marked $\mathbb{C}\mathrm{P}^1$-structures on S can be identified, via the correspondence sketched above, with the space of holomorphic quadratic differentials $\mathcal{Q}(S)$ that forms a bundle over Teichmüller space $\mathcal{T}(S)$, where the fiber over a marked Riemann surface X is the vector space $\mathcal{Q}(X)$ of holomorphic quadratic differentials on X. For details, see for example [Dumas 2009] or [Gunning 1967]; for more on the geometry of these spaces, see [Faraco 2020].

In the case that S is not closed, the above correspondence still holds, except that now the space of holomorphic quadratic differentials on an open Riemann surface is infinite-dimensional. However, we can restrict to *meromorphic* quadratic differentials, where the punctures of S are either removable singularities or poles of finite order:

Definition 2.1 A *meromorphic projective structure* on a surface $S_{g,k}$ of negative Euler characteristic is a $\mathbb{C}\mathrm{P}^1$-structure such that if we equip $S_{g,k}$ with a complete hyperbolic metric of finite volume so that the universal cover $\widetilde{S}_{g,k} \cong \mathbb{H}^2$, then the Schwarzian derivative of the developing map $f : \mathbb{H}^2 \to \mathbb{C}\mathrm{P}^1$ descends to a holomorphic quadratic differential on the punctured surface, with poles of finite order at the puncture. In other words, it defines a meromorphic quadratic differential on the closed surface S_g. As described in the introduction, in this paper we shall consider the space $\mathcal{P}_g(k)$ of marked meromorphic projective structures on $S_{g,k}$ whose corresponding meromorphic quadratic differentials have poles of order at most two at the punctures.

Remark The above definition, and the property that the poles have order at most two, is independent of the choice of complete hyperbolic structure, which merely serves as a "reference" projective structure. See also [Allegretti and Bridgeland 2020, Definition 3.1].

Any $\mathbb{C}\mathrm{P}^1$-structure on the closed surface S_g becomes an example of a meromorphic projective structures in $\mathcal{P}_g(k)$ after k points are deleted; another set of examples include *branched* projective structures on S_g mentioned in the introduction (see, for example, [Mandelbaum 1973]) with k branch points (as defined below) after they are deleted to form the k punctures.

Definition 2.2 (branch point) A *branch point* of a branched projective structure is a point around which the developing map is of the form $z \mapsto z^n$ in local coordinates, where $n > 1$. Note that its Schwarzian derivative, as computed by (3), has a pole of order two at such a point. Away from the branch points, a branched projective structure is a $\mathbb{C}\mathrm{P}^1$-structure in the usual sense, ie the developing map is an immersion and each point is *regular* or unbranched.

For an account of the solutions of the Schwarzian equation (1) around a pole of order two of q, see [Gupta 2021, Section 2.3] or [Ballas et al. 2024, Section 4.1] for a summary, and for more details see [de Saint-Gervais 2016, Chapter IX]. Here we give a concise summary of the discussion in [Gupta 2021, Section 2.3]: For a regular singularity at $z = 0$ of the form

$$\frac{1-\theta^2}{2z^2} + \cdots,$$

we either have

(a) $\theta \notin \mathbb{Z}$, in which case there are two linearly independent local solutions of the form $z^{\pm\theta}$, and the monodromy around the singularity is nontrivial (multiplication by $e^{2\pi i\theta}$), or

(b) $\theta \in \mathbb{Z}$, in which case the two solutions are of the form $z^{|\theta|}$ and $z^{-|\theta|} + C \ln z$. In this latter case the monodromy around 0 is trivial only if $C = 0$, and parabolic otherwise.

Thus, the definition of an *apparent singularity* as in Definition 1.2 is equivalent to saying that we are in $C = 0$ in case (b) above, and the Schwarzian equation has meromorphic solutions at the singularity. In particular, as already noted in Definition 1.2, the $\mathbb{C}\mathrm{P}^1$-structure will have a branch point (as above) or a regular point at such a puncture; we shall use this later in the paper.

We mention here that one can also define spaces of meromorphic projective structures with poles of order *greater* than two, where the corresponding Schwarzian equation has *irregular* singularities; see [Gupta and Mj 2021] for an account, including a description of the asymptotic of the developing map around such a singularity. The (framed) monodromy representations of such structures (cf the next section) were characterized in [Gupta and Mj 2020].

2.2 Framed representations

Following Fock and Goncharov [2006], a *framed representation* of $\Pi = \pi_1(S_{g,k})$ to $\mathrm{PSL}_2(\mathbb{C})$ is a representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ together with a *framing*, which, roughly speaking, is an assignment of a point in $\mathbb{C}\mathrm{P}^1$ (which is a "flag" in $\mathbb{C}^2$) to each puncture of $S_{g,k}$. More precisely, let the *Farey set* F_∞ be the set of points on the ideal boundary of the universal cover of $S_{g,k}$, that corresponds to the lifts of the punctures; a *framing* of a $\mathbb{C}\mathrm{P}^1$-structure on $S_{g,k}$ is a ρ-equivariant map $\beta\colon F_\infty \to \mathbb{C}\mathrm{P}^1$. The *moduli space of framed representations* $\hat{\chi}(\Pi)$ is then the orbit space of such pairs under the action of $\mathrm{PSL}_2(\mathbb{C})$ that identifies $(\rho, \beta) \sim (A \cdot \rho \cdot A^{-1}, A \cdot \beta)$ for each $A \in \mathrm{PSL}_2(\mathbb{C})$. The space $\hat{\chi}(\Pi)$ is in fact a moduli stack; see [Fock and Goncharov 2006, Lemma 1.1 and Definition 2.1], or [Allegretti and Bridgeland 2020, Section 4.1 and Lemma 9.1].

Definition 2.3 (Allegretti and Bridgeland [2020]; see also Gupta and Mj [2020]) A *degenerate framed representation* is a pair $(\rho, \beta) \in \hat{\chi}(\Pi)$ that satisfies one of the following conditions:

(i) There is a set of two points $F = \{p_-, p_+\} \in \mathbb{C}\mathrm{P}^1$ such that the image of β lies in F, and for any peripheral loop γ, $\rho(\gamma)$ fixes the points in F.

(ii) There is a single point $p_0 \in \mathbb{C}\mathrm{P}^1$ such that the image of β lies in F, and for any peripheral loop γ, $\rho(\gamma)$ is a parabolic element fixing p_0, or is the identity element.

A framed representation is *nondegenerate* if it is not degenerate.

Remark Since properties (i) or (ii) above are invariant under the $\mathrm{PSL}_2(\mathbb{C})$-action described above, the notion is well-defined for elements of $\hat{\chi}(\Pi)$.

The following result relates this to Definition 1.1:

Proposition 2.4 [Gupta 2021, Proposition 3.1] *A nondegenerate representation ρ can be equipped with a framing β such that the pair (ρ, β) is a nondegenerate framed representation.*

Remark It is worth mentioning that [Gupta 2021, Proposition 3.1] also shows that if, in addition, there are no apparent singularities, then forgetting the framing of a degenerate (resp. nondegenerate) framed representation yields a representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ that is degenerate (resp. nondegenerate). This allows us to go back and forth between a nondegenerate framed representation and a nondegenerate representation, in the case that there are no apparent singularities.

The following result is a direct consequence of [Allegretti and Bridgeland 2020, Theorem 6.1] and [Gupta 2021, Proposition 3.1] (see the above remark):

Theorem 2.5 *If $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ is the monodromy representation of a meromorphic $\mathbb{C}\mathrm{P}^1$-structure in $\mathcal{P}_g(k)$ with no apparent singularities, then ρ is nondegenerate.*

3 Proof of Theorem A: nondegenerate representations

In this section we shall fix an arbitrary nondegenerate representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$, and show that there is a meromorphic projective structure in the space $\mathcal{P}_g(k)$ that has monodromy ρ. This proves the "if" direction of Theorem A in case (i) of the statement of the theorem. The proof is exactly the same as that in [Gupta 2021]; the key observation is that our assumption in that paper that ρ has no apparent singularity in fact plays no role in our construction. In what follows we provide a condensed (and simplified) account of this construction, and refer at times to [Gupta 2021] for further discussion.

The first observation is that by Proposition 2.4 we can define a framing $\beta\colon F_\infty \to \mathbb{C}\mathrm{P}^1$ such that the pair (ρ, β) is a nondegenerate framed representation in the sense of Definition 2.3.

3.1 Fock–Goncharov coordinates

Fock and Goncharov [2006] described coordinates on the moduli space of framed representations $\hat{\chi}(\Pi)$ as follows.

Let $\hat{\rho} = (\rho, \beta)$ be a framed representation. Choose an ideal triangulation T of $S_{g,k}$; in the universal cover; this lifts to an ideal triangulation $\widetilde{T}$ of the universal cover, with ideal vertices in the Farey set F_∞. For each edge $e \in T$, choose a lift $\tilde{e} \in \widetilde{T}$. Let p_1, p_2, p_3, p_4 be the four ideal vertices of the two ideal triangles adjacent to $\widetilde{e}$, in counterclockwise order on the ideal boundary $\partial_\infty \widetilde{S}_{g,k} \cong \mathbb{S}^1$. Associated with e we can then define a (*complex*) *cross-ratio*

$$C(\widehat{\rho}, e) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_1 - z_4)}, \tag{5}$$

where $z_i = \beta(p_i)$ for $i = 1, 2, 3, 4$. Note that this is well-defined since by the equivariance of the framing β, a different choice of lift of e would yield a quadruple of points that differs by a Möbius transformation, which has the same cross-ratio.

The set of such cross-ratios defines an element of $(\mathbb{C}^*)^N$, where N is the number of edges of T, and this tuple is said to be the *Fock–Goncharov coordinates* of the framed representation $\widehat{\rho}$, with respect to our choice of ideal triangulation.

Note, however, that the cross-ratio associated with e above is well-defined only when the quadruple of points z_1, z_2, z_3, z_4 are *distinct*, and hence for a fixed T, the Fock–Goncharov coordinates are only well-defined for a *generic* framed representation. The following result follows from Proposition 2.4 and [Allegretti and Bridgeland 2020, Theorem 9.1]:

Theorem 3.1 *For any nondegenerate representation ρ, there exists a framing β and an ideal triangulation T such that the Fock–Goncharov coordinates of the framed representation $\widehat{\rho} = (\rho, \beta)$ with respect to T are well-defined.*

Remark Given a nondegenerate framed representation, [Allegretti and Bridgeland 2020, Theorem 9.1] in fact asserts the existence of a "signed" triangulation (T, ϵ) for which the Fock–Goncharov coordinates are well-defined. Here ϵ denotes a choice of a sign (± 1) at each puncture; switching a sign at a puncture with loxodromic monodromy around it amounts to changing the framing at the puncture by assigning it the other fixed point.

Henceforth, in this section, we shall assume that we have fixed such a choice of ideal triangulation T, for the nondegenerate framed representation $\widehat{\rho} = (\rho, \beta)$ we fixed at the beginning of this section.

3.2 Pleated planes in $\mathbb{H}^3$

Recall that $\mathbb{C}\mathrm{P}^1$ is the ideal boundary of hyperbolic 3-space $\mathbb{H}^3$. The Fock–Goncharov coordinates of a framed representation $\widehat{\rho} = (\rho, \beta)$ with respect to an ideal triangulation T can be interpreted as defining a geometric object, namely a *pleated plane* in $\mathbb{H}^3$. This pleated plane is a ρ-equivariant map

$$\Psi \colon \widetilde{S} \to \mathbb{H}^3, \tag{6}$$

defined on the universal cover of our surface $S_{g,k}$, such that each ideal triangle $\Delta \in \widetilde{T}$ with, say, ideal vertices p_1, p_2, p_3 maps to the totally geodesic ideal triangle in $\mathbb{H}^3$ with ideal vertices $\beta(p_1), \beta(p_2), \beta(p_3)$. Here we assume that Ψ preserves orientation, so that the image is an oriented surface in $\mathbb{H}^3$ that is "piecewise totally geodesic".

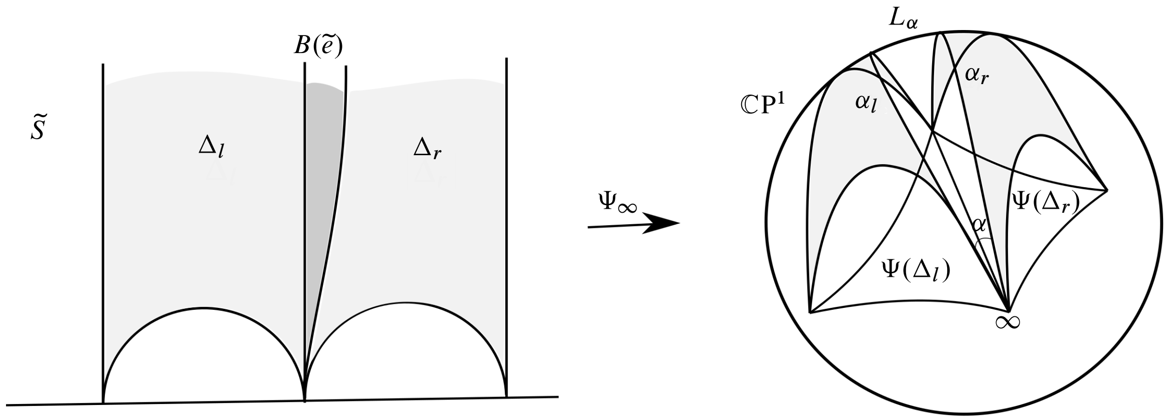

Figure 1: The map Ψ_∞ maps the grafted region $B(\widetilde{e})$ to the "lune" L_α on $\mathbb{C}\mathrm{P}^1$ bounded by the circular arcs α_l and α_r. The shaded regions on the right are the images of Δ_l and Δ_r under Ψ^0_∞; see Section 3.3.

3.3 Constructing a projective structure

Given a nondegenerate framed representation $\widehat{\rho} = (\rho, \beta)$ as in the beginning of the section, we now describe how to construct a projective structure in $\mathcal{P}_g(k)$ with monodromy ρ. This is exactly as in [Gupta 2021], however, here we provide a condensed and simplified discussion. In particular, here we shall avoid the intermediate steps of "straightening" the pleated plane and then "grafting".

The main idea is that a pleated plane Ψ constructed in the previous subsection also defines a projective structure by considering its "shadow" at the boundary at infinity of $\mathbb{H}^3$. More precisely, on each totally geodesic ideal triangle $\Psi(\Delta)$ we can consider the hyperbolic Gauss map $\mathcal{G}_\Delta$, in the normal direction consistent with the orientation of Ψ. On each ideal triangle $\Delta \in \widetilde{T}$, define the map $\Psi^0_\infty = \mathcal{G}_\Delta \circ \Psi$. Note that the image of each ideal triangle is a triangle on $\mathbb{C}\mathrm{P}^1$ with sides that are circular arcs that form "cusps" at the three vertices.

Note that this defines a map $\Psi^0_\infty \colon \widetilde{S} \to \mathbb{C}\mathrm{P}^1$ that is ρ-equivariant. However it may not be continuous: suppose Δ_l and Δ_r are two adjacent triangles such that $\Psi(\Delta_l)$ and $\Psi(\Delta_r)$ lie in totally geodesic planes that intersect at an angle $\alpha \in (0, 2\pi)$ along a common geodesic line l (which is the Ψ-image of the common edge $\widetilde{e}$ between Δ_l and Δ_r). In that case, the respective images of $\widetilde{e}$ under $\mathcal{G}_{\Delta_l}$ and $\mathcal{G}_{\Delta_r}$ are arcs α_l and α_r of great circles on $\mathbb{C}\mathrm{P}^1$ with a common pair of endpoints, that differ by an elliptic rotation of an angle α. We shall call a region in $\mathbb{C}\mathrm{P}^1$ bounded by such a pair of arcs a "lune".

We can, however, modify the map Ψ^0_∞ to obtain a *continuous* map

$$\Psi_\infty \colon \widetilde{S} \to \mathbb{C}\mathrm{P}^1 \tag{7}$$

as follows: for each pair of adjacent triangles Δ_l, Δ_r in the domain $\widetilde{S}$ as above, cut along the common edge $\widetilde{e}$, and glue in a bigon $B(\widetilde{e})$ such that the sides of the bigon are identified with the resulting two sides $\widetilde{e}_{\pm}$. Define a map from $B(\widetilde{e})$ to $\mathbb{C}\mathrm{P}^1$ such that the two sides map to the arcs α_l, α_r described above, and the image is the lune bounded by $\alpha_l \cup \alpha_r$, such that the resulting map from $\Delta_l \cup B(\widetilde{e}) \cup \Delta_r$ is a smooth orientation-preserving immersion to $\mathbb{C}\mathrm{P}^1$.

We can define the maps on the "grafted" bigons in a Π-equivariant manner, so that the final map Ψ_∞ as in (7) is ρ-equivariant, and a smooth immersion. Hence, it defines a projective structure P on the surface $S_{g,k}$ with monodromy ρ.

3.4 Schwarzian derivative at the punctures

It remains to show that the projective structure P we just constructed on $S_{g,k}$ in fact lies in the space $\mathcal{P}_g(k)$, that is, the Schwarzian derivative of the developing map with respect to a suitable reference projective structure has a pole of order at most two.

To see this, we uniformize the underlying Riemann surface structure on $S_{g,k}$ to obtain a hyperbolic metric of finite area, such that each puncture is a cusp. This will serve as a reference projective structure.

Let $\mathbb{D}^* = \{0 < |w| < 1\}$ be a conformal punctured-disk neighborhood of a puncture. Lifting to the universal cover $\widetilde{S}$, a neighborhood of an ideal point $p \in F_\infty$ would look exactly like a neighborhood of ∞ in the upper half-space model of $\mathbb{H}^2$, where the edges of the triangulation $\widetilde{T}$ are the vertical geodesics, and the deck translation corresponding to the parabolic element around the cusp is a (positive) translation. Moreover, can choose this conformal identification with the model $\mathbb{H}^2$ such that the translation is $z \mapsto z+1$, namely $\mathbb{H}^2/\langle z \mapsto z+1\rangle \cong \mathbb{D}^*$ via the map $z \mapsto w := e^{2\pi i z}$. If there are n geodesic sides of the triangulation T asymptotic to that cusp, then a fundamental domain Δ of the action will comprise n ideal triangles $\Delta_1, \Delta_2, \dots, \Delta_n$ together with n "bigons" $B(\widetilde{e}_1), B(\widetilde{e}_1), \dots, B(\widetilde{e}_n)$ that were grafted in. Here, $\widetilde{e}_i$ is the "right-hand" edge of Δ_i for each $1 \le i \le n$.

From our definition of the map Ψ_∞ above, the images of $B(\widetilde{e}_1), B(\widetilde{e}_2), \dots, B(\widetilde{e}_n)$ are each a lune in $\mathbb{C}\mathrm{P}^1$ with a common endpoint $\beta(p)$, and the images of $\Delta_1, \Delta_2, \dots, \Delta_n$ in a neighborhood of ∞ are regions bounded by circular arcs that form a "cusp" (of angle zero) at $\beta(p)$. Each successive region shares a common circular arc with the preceding one. Hence the union of their images near $\beta(p)$ looks like a region bounded by two circular arcs that intersect at some angle $\alpha \in \mathbb{R}^+$ (if $\alpha > 2\pi$ then the lune is immersed in $\mathbb{C}\mathrm{P}^1$). Here, the angle α is the sum of the angles at $\beta(p)$ of the lunes that are the images of the grafted bigons.

The rest of the argument is exactly as in the proof of Proposition 3.5 in [Gupta 2021]. Consider first the case when the total "bending angle" α around $\beta(p)$ is positive. We can assume, by postcomposing with an appropriate Möbius map, that $\beta(p) = \infty \in \mathbb{C}\mathrm{P}^1$. The conformal developing map from a neighborhood of ∞ in Δ then maps to a neighborhood of ∞ of a lune $L_\alpha \subset \mathbb{C}\mathrm{P}^1$ that has vertices at $0, \infty$. Such a conformal map has the asymptotic form $f(z) = e^{-i\alpha z}$ for $|z| \gg 1$, and hence on the punctured disk $\mathbb{D}^*$, has the expression $\widetilde{f}(w) = w^{-\alpha/2\pi}$. A computation of the Schwarzian derivative using (3) then yields that $S(\widetilde{f})$ has a pole of order 2 at the puncture, as desired.

In the case when the total bending angle $\alpha = 0$, the conformal developing map is the identity map $f(z) = z$, and the Schwarzian derivative thus yields the constant quadratic differential dz^2 on Δ that has the expression $-(1/4\pi^2)w^{-2}\,dw^2$ on $\mathbb{D}^*$, once again with a pole of order two.

The discussion in this section thus proves:

Proposition 3.2 *Let Π be the fundamental group of a surface $S_{g,k}$ of genus g and $k \geq 1$ punctures, where $2-2g-k<0$. Any nondegenerate representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ arises as the monodromy of a $\mathbb{C}\mathrm{P}^1$-structure in $\mathcal{P}_g(k)$.*

4 Proof of Theorem C: affine holonomy with at least two punctures

In this section we start to deal with the complementary case when $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ is a degenerate representation; see Definition 1.1. Here we shall consider surfaces $S_{g,k}$ with at least two punctures, that is, $k \geq 2$. Our main goal in this section is to prove our Theorem C, that is, the case (ii) of Theorem A for affine representations when $S_{g,k}$ has at least two punctures.

Theorem 4.1 *Let Π be the fundamental group of $S_{g,k}$ as in Theorem A, such that the number of punctures $k \geq 2$. Then any nontrivial representation $\rho\colon \Pi \to \mathrm{Aff}(\mathbb{C})$ arises as the monodromy of a complex affine structure on $S_{g,k}$. If $k \geq 3$, then every representation is realizable as the monodromy of a complex affine structure on $S_{g,k}$.*

In Section 4.1 we shall handle the case when ρ is the trivial representation, in Section 4.2 the case when $\rho(\Pi)$ has a single global fixed point $p \in \mathbb{C}\mathrm{P}^1$ and the monodromy of each element is a translation. In Section 4.3 we deal with the more general case when the monodromy of each element is an *affine* map. This would include also include the case when $\rho(\Pi)$ fixes a set $\{p, q\} \subset \mathbb{C}\mathrm{P}^1$, and is "coaxial". Note that the necessity of the presence of one apparent singularity follows from the work of Allegretti and Bridgeland; see Theorem 2.5.

4.1 Trivial representation

We prove the first of the exceptional cases mentioned in Theorem A. To state the first result, we introduce the following terminology:

Definition 4.2 (handles, handle-generators) On a marked surface $S_{g,k}$ of some positive genus $g > 0$, a *handle* is an embedded subsurface Σ that is homeomorphic to $S_{1,1}$. A *handle-generator* for Σ is any simple closed nonseparating curve on it. A *pair of handle-generators* for a handle will refer to a pair of simple closed curves $\{\alpha, \beta\}$ that generate $H_1(\Sigma, \mathbb{Z})$; in particular, α and β intersect once.

Lemma 4.3 *Suppose $S_{g,k}$ is a surface where $g > 0$ and $k = 1$ or 2. Then the monodromy ρ of any $\mathbb{C}\mathrm{P}^1$-structure in $\mathcal{P}_g(k)$ is nontrivial. Moreover, there is a handle Σ and a handle-generator γ_0 for Σ such that $\rho(\gamma_0)$ is nontrivial.*

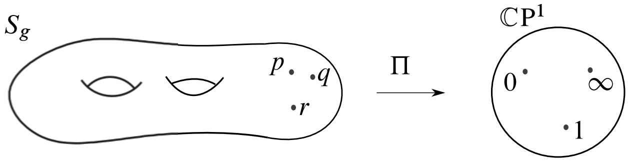

Figure 2: The map Π in Lemma 4.4 is a branched cover over the sphere with exactly three critical points and three branch points.

Proof Suppose there is a $\mathbb{C}\mathrm{P}^1$-structure on $S_{g,k}$ (where $k = 1$ or 2) such that the monodromy representation is trivial, ie $\rho(\gamma) = \mathrm{Id}$ for all $\gamma \in \Pi$. In particular, any puncture is an apparent singularity, and since this projective structure corresponds to the Schwarzian equation with regular singularities, is either a regular point or branch point.

Since the monodromy is trivial, the developing map (defined on the universal cover) in fact descends to a well-defined map $\overline{\mathrm{dev}}\colon S_{g,k} \to \mathbb{C}\mathrm{P}^1$ (where recall $k = 1$ or 2). Since the developing map is an immersion, this map can be thought of as a covering map from the closed surface S_g to $\mathbb{C}\mathrm{P}^1$ that is possibly branched at one or two points. That is, such a map has at most two critical values on $\mathbb{C}\mathrm{P}^1$. Since $g > 0$, it follows from the Riemann–Hurwitz formula that there cannot be such a branched covering.

Fix a standard decomposition of $S_{g,k}$ into g handles, and a disk with k punctures; cf Figure 3. If we assume that $\rho(\gamma) = \mathrm{Id}$ whenever γ is one of the $2g$ loops that are the generators of the handles, then we can in fact show that the representation ρ is trivial: this is immediate if $k = 1$, and if $k = 2$, note that we already know that there is one apparent singularity; the monodromy around the other singularity is then the product of commutators of the handle-generators, and hence also trivial. □

We now show that in all other cases, one can construct a projective structure with trivial monodromy:

Lemma 4.4 *Let $S_{g,k}$ be a surface where $k \geq 3$. Then there is a $\mathbb{C}\mathrm{P}^1$-structure in $\mathcal{P}_g(k)$ whose monodromy representation is trivial.*

Proof In the case that $g = 0$, it is an easy matter to define a projective structure with trivial monodromy on $S_{0,k}$, in fact it is sufficient to consider $\mathbb{C}\mathrm{P}^1$ with k punctures. Otherwise, we use the fact that the Hurwitz problem of the existence of branched coverings with prescribed branching data, that is solved for $g > 0$; see [Edmonds et al. 1984, Proposition 3.3] and also [Husemoller 1962].

In particular, that implies that there is a branched covering $\Pi\colon S_g \to \mathbb{C}\mathrm{P}^1$ of degree $2g + 1$ that is branched over $0, 1, \infty \in \mathbb{C}\mathrm{P}^1$, each of ramification order $2g + 1$, and has exactly three critical points, say p, q, r on S_g, such that $p = \Pi^{-1}(0)$, $q = \Pi^{-1}(1)$ and $r = \Pi^{-1}(\infty)$; see Figure 2.

Then, the standard projective structure on $\mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$ naturally pulls back to a projective structure on $S_g \setminus \{p, q, r\}$ that has trivial holonomy. Note that one can also equip $\mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$ with the standard translation structure (see Section 4.2) and pull it back to $S_g \setminus \{p, q, r\}$; the induced abelian differential ω extends to the closed surface S_g and has two zeroes at p and q, and a pole at r. □

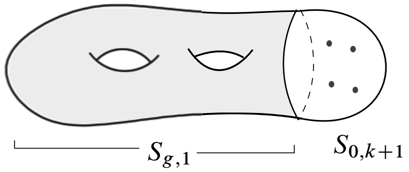

Figure 3: The surface $S_{g,k}$ can be divided into a subsurface homeomorphic to $S_{g,1}$ (shown shaded) and its complement, homeomorphic to $S_{0,k+1}$.

4.2 Translation structures

In this section we shall assume that $\rho\colon \Pi \to \mathbb{C}$ is a nontrivial representation, where

$$\mathbb{C} \cong \{z \mapsto z + c \mid c \in \mathbb{C}\}$$

is the subgroup of $\mathrm{PSL}_2(\mathbb{C})$ comprising translations. Note that since this subgroup is abelian, ρ factors through the first homology group $\Gamma_{g,k} := H_1(S_{g,k}, \mathbb{Z})$, and can be thought of as a period character (or simply character) $\chi_{g,k}\colon \Gamma_{g,k} \to \mathbb{C}$.

Note that the residue theorem implies that:

Lemma 4.5 *For any homomorphism $\chi_{g,k}\colon \Gamma_{g,k} \to \mathbb{C}$ as above, sum of the values around the peripheral curves is zero, namely*

$$\sum_{i=1}^{k} \chi_{g,k}(\gamma_i) = 0, \tag{8}$$

where γ_i is the simple closed curve around the i^{th} puncture.

Remark We shall say that a "puncture has trivial monodromy" if $\chi_{g,k}(\gamma) = 0$, where γ is the loop around that puncture.

In the case that $g > 0$ and $k > 1$ we shall also divide the character into $\chi_{g,1}$ and $\chi_{0,k+1}$ as follows: choose a separating loop γ that divides the surface $S_{g,k}$ into a subsurface $S_{g,1}$ containing all the handles and one boundary component, namely γ, and a subsurface $S_{0,k+1}$ that is topologically a punctured sphere, where one of the punctures is actually a boundary component γ; see Figure 3.

By restricting to $S_{g,1}$ and $S_{0,k+1}$, the character $\chi_{g,k}$ determines homomorphisms

$$\chi_{g,1}\colon \Gamma_{g,1} \to \mathbb{C}, \tag{9}$$

$$\chi_{0,k+1}\colon \Gamma_{0,k+1} \to \mathbb{C}, \tag{10}$$

respectively, where $\Gamma_{g,1}$ and $\Gamma_{0,n+1}$ are the homology groups of the subsurfaces $S_{g,1}$ and $S_{0,k+1}$ respectively. Note that in either case, the boundary loop γ is trivial in homology.

We shall need the following observation:

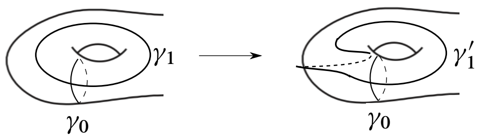

Figure 4: A Dehn twist changes the pair of handle-generators $\{\gamma_0, \gamma_1\}$ to the pair $\{\gamma_0, \gamma_1'\}$, where $\gamma_1' = \gamma_0 + \gamma_1$ in homology.

Lemma 4.6 *Let $\chi_{g,k}\colon \Gamma_{g,k} \to \mathbb{C}$ be a character and let its associated homomorphism $\chi_{g,1}$ as in (9) be a nontrivial representation. Then there is a change of homology basis $A \in \mathrm{Sp}(2g, \mathbb{Z})$ such that $\chi'_{g,k} = \chi_{g,k} \circ A$ has an associated restriction $\chi'_{g,1}$ such that $\chi'_{g,1}(\gamma) \neq 0$ for each handle-generator γ on $S_{g,1}$.*

Proof Since $\chi_{g,1}$ is nontrivial, there is some simple closed curve γ_0 that is a handle-generator on $S_{g,1}$ and is such that $\chi_{g,1}(\gamma_0) \neq 0$. Let $\{\gamma_0, \gamma_1\}$ be the handle-generators of such a handle. Now suppose $\chi_{g,1}(\gamma_1) = 0$. We can replace the handle-generator γ_1 with the curve γ_1' which is obtained by Dehn-twisting γ_1 around γ_0; see Figure 4. In homology, $\gamma_1' = \gamma_0 + \gamma_1$, and hence we now have $\chi_{g,1}(\gamma_1) \neq 0$. Thus, we have a basis of homology such that at least one of the handles has both its handle-generators with nontrivial monodromy.

Now suppose there is a handle with handle-generators $\{\alpha, \beta\}$ such that $\chi_{g,1}(\alpha) = \chi_{g,1}(\beta) = 0$, then consider the element of $\mathrm{Sp}(2g, \mathbb{Z})$ that changes the homology basis $\{\gamma_0, \gamma_1, \alpha, \beta\}$ of the two handles to the new homology basis $\{\gamma_0 - \alpha, \gamma_1 + m \cdot \alpha, \alpha, \beta + m \cdot \gamma_0 + \gamma_1\}$, for an integer m, leaving the other generators unchanged. (See [Martens 1985] for this, and related elements of $\mathrm{Sp}(2g, \mathbb{Z})$.) The handle thus acquires a new pair of generators $\{\alpha, \beta + m \cdot \gamma_0 + \gamma_1\}$, where one of the new handle-generators has nontrivial monodromy for some m, ie $\chi_{g,1}(\beta + m \cdot \gamma_0 + \gamma_1) \neq 0$ for some m, since $\chi_{g,1}(\gamma_0) \neq 0$. The holonomy of the other handle remains unchanged since $\chi_{g,1}(\alpha) = 0$. By a further change of basis as in the previous paragraph, by Dehn twisting α around the curve representing $\beta + \gamma_0$, we can ensure there is a pair of generators of the handle which are *both* nontrivial in monodromy.

Applying either of these changes of bases to each of the handles, we can ensure that we obtain a homology basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$ via a change of basis matrix $A \in \mathrm{Sp}(2g, \mathbb{Z})$ such that the homomorphism $\chi'_{g,k} = \chi_{g,k} \circ A$ satisfies $\chi'_{g,1}(\alpha_i) \neq 0$ and $\chi'_{g,1}(\beta_i) \neq 0$ for each $1 \leq i \leq g$. □

Remark Recall from the discussion in Section 2.1 that we only need to realize a monodromy representation up to the action of the mapping class group. Hence, since the change of basis above is realized by a mapping class, we can assume that the handle-generators are each nontrivial once we know that $\chi_{g,1}$ is nontrivial.

In this section, our strategy would be to define a *translation structure* on $S_{g,k}$ with the prescribed holonomy that lies in $\mathbb{C}$. Recall that it is a special case of a complex projective structure, comprising an atlas of charts to $\mathbb{C}$ such that the transition maps are translations. Note that such an atlas equips the resulting surface with a complex structure. and a translation structure is then equivalent to a nonvanishing holomorphic vector field on the Riemann surface, or equivalently, a nonvanishing holomorphic 1-form ω. Note that the punctures could be regular points, or zeroes or poles of ω; a related problem when we require some punctures to be poles, and others to be zeroes of ω, with prescribed orders, is dealt with in [Chenakkod et al. 2022]. Such a translation structure can be defined by gluing sides of a polygon, as we now describe:

Definition 4.7 (translation surface) A *translation surface* is obtained by starting with a collection of (possibly noncompact) polygons in $\mathbb{C}$ bounded by straight lines and/or straight line segments and/or rays, and identifying such sides pairwise by translations. The resulting surface Σ thus acquires a Euclidean metric, with possible cone singularities (with cone-angles an integer multiple of 2π at points arising from the by identifications of the vertices of the polygons. In the complement of such cone points, we then obtain a translation structure as defined above. The standard differential dz on the polygons descends to the holomorphic 1-form ω on the surface, and the zeroes of ω are precisely at the cone points, which are branch points of the translation surface. The periods of ω define a representation $\chi\colon H_1(\Sigma,\mathbb{Z})\to\mathbb{C}$ that is the *holonomy* of the translation surface Σ; note that if the branch points are removed from Σ, then each additional puncture has trivial monodromy around it.

Remark In the case that the polygons are noncompact, the translation structure will have at least one puncture "at infinity", where the abelian differential ω has a pole. The order of such a pole can be determined from the flat geometry of the corresponding end: if the end is cylindrical, then the pole has order one, if it is a planar end (ie like that of $\mathbb{C}$) then the pole is of order two, and a pole of order $n>2$ has an end which is isometric to an $(n-1)$-fold cover of a planar end, branched at ∞.

The following construction allows us to define a new translation surface by gluing together two translation surfaces with poles.

Definition 4.8 (gluing along a ray) Suppose Σ_1 and Σ_2 are two translation surfaces, each with at least one pole. Let $l_i\subset\Sigma_i$ for $i=1,2$ be an embedded straight line ray that starts from a cone singularity (or a regular point) and ends in a pole. Assume that l_1 and l_2 develop onto infinite rays on $\mathbb{C}$ that are parallel. Then we can define a translation surface Σ as follows: slit each ray l_i and denote the resulting sides by l_i^+ and l_i^-; then identify l_1^+ with l_2^- and l_1^- and l_2^+ by a translation. If Σ_i is homeomorphic to S_{g_i,k_i} for $i=1,2$, then the resulting surface Σ is homeomorphic to $S_{g_1+g_2,k_1+k_2-1}$. Note that the starting points of the rays are identified to a branch point on Σ with a cone-angle that is the sum of the corresponding angles on Σ_1 and Σ_2, and the other endpoints (at infinity) are identified with a higher-order pole. See Figure 5.

We finally introduce the notion of *algebraic volume* of a representation χ as follows:

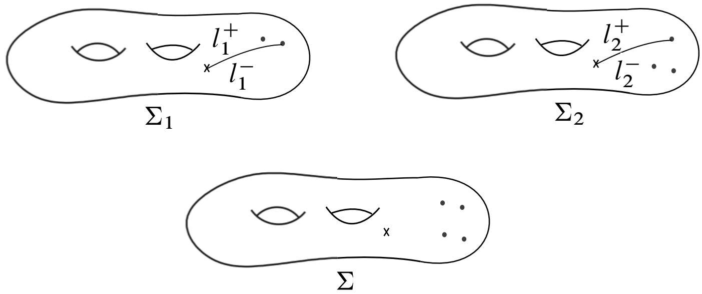

Figure 5: Gluing Σ_1 and Σ_2 along rays results in a new surface Σ; see Definition 4.8.

Definition 4.9 (algebraic volume) Let Σ be a translation structure on a once-punctured surface $S_{g,1}$ with holonomy $\chi\colon H_1(\Sigma,\mathbb{Z})\to\mathbb{C}$. The *algebraic volume* of χ is defined as the quantity

$$\mathrm{Vol}(\chi)=\sum_{i=1}^{g}\Im\big(\overline{\chi(\alpha_i)}\,\chi(\beta_i)\big), \tag{11}$$

where $\{\alpha_1,\beta_1,\dots,\alpha_g,\beta_g\}$ is any symplectic basis of $H_1(\Sigma,\mathbb{Z})$, and $\Im(c)$ denotes the imaginary part of a complex number c. If Σ is a translation structure on a generic surface $S_{g,k}$, we define the algebraic volume of χ as the algebraic volume of the subrepresentation $\chi_{g,1}$ introduced in (9). We have emphasized above the term *algebraic* in order to distinguish this notion of volume from its geometric counterpart. In what follows we do not need to consider the geometric volume and henceforth for simplicity we abridge the terminology to just "volume".

In our construction, we shall also need:

Definition 4.10 (volume of a quadrilateral) Let a and b be two complex numbers and let $\mathcal{Q}$ be the (possibly degenerate) quadrilateral spanned by the corresponding vectors. The volume of $\mathcal{Q}$ is then defined as

$$\mathrm{Vol}(\mathcal{Q})=\mathrm{Vol}(a,\,b)=\Im(\overline{a}b). \tag{12}$$

In particular, the volume is null if and only if $a=\lambda b$ for some $\lambda\in\mathbb{R}$.

We begin with the case that $k=2$, ie there are exactly two punctures; note that our assumption of negative Euler characteristic implies that $g>0$. We show:

Proposition 4.11 *Let $g>0$. Any nontrivial representation $\chi_{g,2}\colon\Gamma_{g,2}\to\mathbb{C}$ with at least one puncture with trivial monodromy appears as the holonomy of some translation structure on $S_{g,2}$, where one of the punctures corresponds to a zero of the abelian differential.*

Proof Note that when there are exactly two punctures, and one of them has trivial monodromy, then from (8) it follows that the other puncture also has trivial monodromy. Since $\chi_{g,2}$ is nontrivial, the associated representation $\chi_{g,1}$ as in (9) is nontrivial. By Lemma 4.6 we can assume that each handle-generator maps to a nonzero complex number.

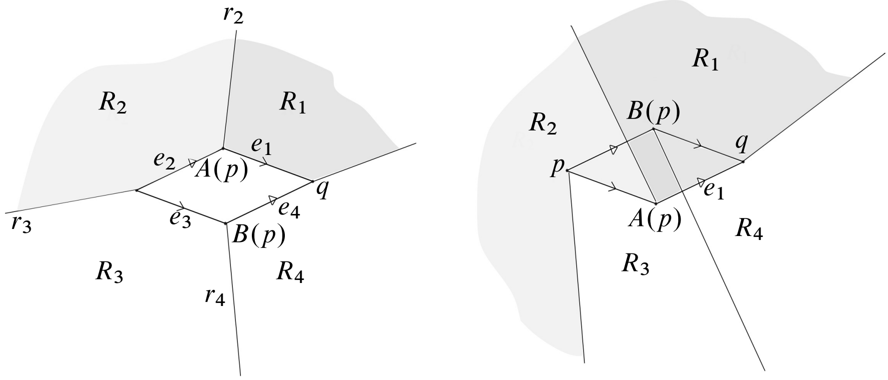

Figure 6: The identifications described in Proposition 4.11 result in a translation surface homeomorphic to $S_{1,1,}$, a torus with a puncture (at infinity). The regions R_1, R_2, R_3, R_4 are chosen to lie on the left or right side of the closed polygonal curve $L = \overline{e_1} \cup \overline{e_2} \cup e_3 \cup e_4$ depending on whether the volume of the handle is nonpositive or positive (the figures on the left and right, respectively).

Now, construct a translation surface Σ homeomorphic to $S_{g,1}$ and having holonomy $\chi := \chi_{g,1}$ as follows: First, we consider the case when $g = 1$; let the pair of handle-generators be $\{\alpha, \beta\}$, and let $A, B : \mathbb{C} \to \mathbb{C}$ be the translations $z \mapsto z + \chi(\alpha)$ and $z \mapsto z + \chi(\beta)$, respectively. Note that A and B commute, and neither is the identity map since the monodromy around each handle-generator is nontrivial.

Define four closed directed straight line segments $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{C}$ as follows: let $p \in \mathbb{C}$ be any point and let $q := AB(p) = BA(p)$. Then define

$$e_1 := \overline{A(p)q}, \quad e_2 := \overline{pA(p)}, \quad e_3 := \overline{pB(p)}, \quad e_4 := \overline{B(p)q}.$$

Note that $A(e_3) = e_1$ and $B(e_2) = e_4$. If $\overline{e_i}$ denotes the line segment e_i with its direction reversed, then we see that $L := \overline{e_1} \cup \overline{e_2} \cup e_3 \cup e_4$ is a closed directed loop based at q. Notice that L bounds a (possibly degenerate) quadrilateral.

Now, we consider four embedded polygons R_1, R_2, R_3, R_4 in $\mathbb{C}$, where

(i) each R_i is bounded by two infinite rays r_i and r_{i+1}, and e_i (where the indices $1 \le i \le 4$ are cyclically ordered, such that r_5 is actually r_1), and

(ii) R_1 and R_3 lie on opposite sides of e_1 and e_3 respectively, and R_2 and R_4 lie on opposite sides of e_2 and e_4 respectively.

Since R_1 and R_2 share the boundary ray r_2, either both lie on the left sides of the directed edges e_1 and e_2 respectively, or both lie on the right sides of those edges. Similarly, the pair of adjacent regions R_3 and R_4 either are both on the right sides of e_3 and e_4, or are both on their left sides. Thus, the choice of which side (left or right) of e_1 that R_1 should lie on determines which side of e_i the region R_i lies on for the remaining $i = 2, 3, 4$. This choice is determined by the final requirement:

(iii) R_1 lies on the left-hand side of e_1.

Note that if $\mathrm{Vol}\big(\chi(\alpha), \chi(\beta)\big) = 0$, then the four segments e_1, e_2, e_3, e_4 are colinear, and can be thought of as lying on the boundary of a slit, the exterior of which is the union of regions $R_1 \cup R_2 \cup R_3 \cup R_4$.

From requirement (i) above we know R_i and R_{i+1} already share a boundary ray r_{i+1} (for the cyclically ordered indices $1 \le i \le 4$), and it follows that $R_1 \cup R_2 \cup R_3 \cup R_4$ is topologically a punctured disk immersed in $\mathbb{C}$, with boundary L and a puncture at ∞.

The translation surface Σ is obtained by identifying the remaining boundary sides (the segments along the loop L) as follows: e_3 is identified with e_1 via the translation A, and e_2 is identified with e_4 via the translation B. Note that requirement (ii) above ensures that we obtain a surface. It is easy to see that Σ is homeomorphic to the punctured torus $S_{1,1}$ (with the puncture at ∞), and the holonomy equals χ. Note that the holonomy is with respect to a fixed pair of oriented loops that are the handle-generators on $S_{1,1}$; the choice in (iii) results in the desired orientation of these loops on Σ.

The puncture at ∞ has trivial monodromy, and the induced abelian differential ω has a pole of order two at that point. There is one branch point on Σ (or equivalently, one zero of ω) — namely, the one obtained from the endpoints of the segments e_1, e_2, e_3, e_4 after the identifications. Thus if we remove the branch point, we obtain a surface homeomorphic to $S_{1,2}$, equipped with a translation structure, with holonomy χ, where both punctures have trivial monodromy, as desired.

In the case that $g \ge 2$, we shall proceed as follows. Let $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \gamma_2\}$ be a generating set of $\pi_1(S_{g,2})$, where $\{\alpha_i, \beta_i\}$ is a pair of handle generators for the i^{th} handle for $1 \le i \le g$. Let A_i and B_i denote the images of α_i and β_i via $\chi_{g,2}$, respectively. We may assume that any handle generator is nontrivial as a consequence of Lemma 4.6. For any point $p \in \mathbb{C}$, each pair $\{A_i,\ B_i\}$ determines a (possibly degenerate) quadrilateral $\mathcal{Q}_i \subset \mathbb{C}$. We can order these quadrilaterals in such a way that the volume of $\mathcal{Q}_i$ is positive for $1 \le i \le h \le g$ and nonpositive for the remaining (possibly none) $g - h$ handles. Note that this notion does not depend on the choice of the basepoint. In what follows we shall place the quadrilaterals on the complex plane according to the following rule:

> *The rightmost vertex of $\mathcal{Q}_i$ is identified with the leftmost vertex of $\mathcal{Q}_{i+1}$. If $\mathcal{Q}_i$ has more than one rightmost vertex, ie some edges are vertical, then the topmost is chosen. If $\mathcal{Q}_{i+1}$ has more than one leftmost vertex, then the bottommost is chosen.*

Suppose there are $h \le g$ positive handles, and let q be the vertex that $\mathcal{Q}_h$ and $\mathcal{Q}_{h+1}$ have in common. Notice that such a point is unique because of the rule above. Let ℓ be a straight line passing through the point q and such that q is the only point of intersection with the chain of quadrilaterals constructed above. Note that the existence of such a straight line also follows from the above rule of placing successive quadrilaterals; indeed, the vertical line suffices unless one of the edges incident at q is already vertical, in which case one applies a slight counterclockwise tilt. We introduce an orientation on ℓ in such a way that the handles with nonpositive volume are on the right of ℓ and the handles with positive volume are on the left of ℓ. Moreover, according to this orientation, the point $q \in \ell$ divides the straight line into two rays,

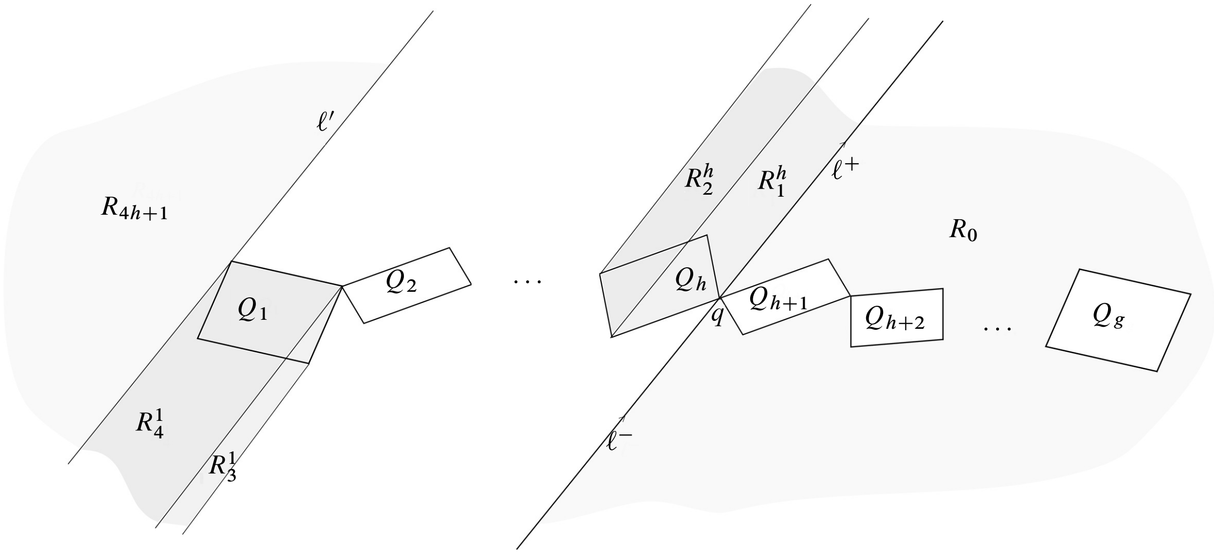

Figure 7: The chain of g quadrilaterals corresponds to the positive handles ($1 \le i \le h$) each like the right-hand figure in Figure 6, and the nonpositive handles ($h+1 \le i \le g$) each like the left-hand figure in Figure 6.

the upper one ℓ^+ and the lower one ℓ^-. The right-hand side of ℓ is a half-plane H in $\mathbb{C}$ containing $g-h$ quadrilaterals; let R_0 be the complement of $\mathcal{Q}_{h+1} \cup \cdots \cup \mathcal{Q}_g$ in H. If there are no handles with nonpositive volume then R_0 is just the half-plane H. Similarly, if there are no handles with positive volume then the complement of R_0 in $\mathbb{C}$ includes the left half-plane $\mathbb{C} \setminus H$.

Suppose $h \ge 1$, on the left-hand side of ℓ we consider a chain of h embedded quadrilaterals obeying the rule above such that the i^{th} quadrilateral Q_i has edges $\{e_1^i, e_2^i, e_3^i, e_4^i\}$ defined by

$$e_1^i := \overline{A_i(p_i)p_{i+1}}, \quad e_2^i := \overline{p_i A_i(p_i)}, \quad e_3^i := \overline{p_i B_i(p_i)}, \quad e_4^i := \overline{B_i(p_i)p_{i+1}}, \tag{13}$$

where p_i is the unique point such that $p_{i+1} = A_i B_i(p_i)$, where $p_{h+1} = q$ by definition.

For each $1 \le i \le h$, consider the four polygonal regions $\{R_1^i, R_2^i, R_3^i, R_4^i\}$ in $\mathbb{C}$ satisfying (i)–(iii) above, bounded by $e_1^i, e_2^i, e_3^i, e_4^i$ respectively, together with infinite rays that are parallel to ℓ (ie a translated copy of either ℓ^+ or ℓ^-). See Figure 7.

By construction, R_1^i and R_3^i lie on opposite sides of e_1^i and e_3^i respectively, and R_2^i and R_4^i lie on opposite sides of e_2^i and e_4^i respectively; cf Figure 6. Moreover, each R_1^i lies on the left of the corresponding side e_1^i. The two rays from the leftmost point of $\mathcal{Q}_1$ form a straight line ℓ' to ℓ. By introducing on ℓ' an orientation coherent with that of the straight line ℓ, it makes sense to say that ℓ' has the entire chain of quadrilaterals to its right. Finally, we define R_{4h+1}, the half-plane on the left of ℓ'. We can now proceed as above. The set

$$R_0 \cup \left(\bigcup_{i=1}^{h} (R_1^i \cup R_2^i \cup R_3^i \cup R_4^i) \right) \cup R_{4h+1} \tag{14}$$

is topologically a punctured disk immersed on the Riemann sphere, with a puncture at infinity, and boundary the chain of quadrilaterals $\mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \cdots \cup \mathcal{Q}_g$. The translation surface Σ is obtained by identifying the remaining boundary sides $\{e^i_j\}$ as follows: e^i_3 is identified with e^i_1 via the translation A_i, and e^i_2 is identified with e^i_4 via the translation B_i. It is easy to see that Σ is homeomorphic to a surface $S_{g,1}$ with the puncture at infinity, corresponding to a pole of order 2, and one branch point of order $2g$. We finally delete the branch point in order to get a translation surface on $S_{g,2}$ with the desired holonomy. □

Remark In the case that $g \geq 2$, we also have the following alternative construction, which is easier. Namely, we construct a translation surface Σ_j homeomorphic to $S_{1,1}$ exactly as above for each handle (so $1 \leq j \leq g$) such that the holonomy of Σ_j is precisely the holonomy of the j^{th} handle in the original character $\chi_{g,1}$. Note that each Σ_j has a pole of order two, and exactly one branch point. Choose an embedded infinite ray in each, between the cone point and the pole such that each develops into an infinite ray in $\mathbb{C}$ in the same direction. We can then glue Σ_j to Σ_{j+1} along these rays as in Definition 4.8, for each $1 \leq j < g$. The resulting translation surface Σ is homeomorphic to $S_{g,1}$, with holonomy $\chi_{g,1}$, and has one pole of order $g+1$ and one branch point. As before, deleting the branch point results in a surface homeomorphic to $S_{g,2}$, equipped with a translation structure having the desired monodromy $\chi_{g,2}$. However, notice that this construction results in a pole of order greater than 2. In Proposition 5.2 below we shall need to consider a translation surface Σ on a two-punctured genus g surface with one pole of order *exactly* two. This motivates the more complicated argument above.

In the case when $g = 0$, we have the following construction of a translation structure realizing a prescribed monodromy:

Proposition 4.12 *If $k \geq 2$, any representation $\chi_{0,k+1}\colon \Gamma_{0,k+1} \to \mathbb{C}$ with at least one puncture with trivial monodromy is the holonomy of some translation structure on $S_{0,k+1}$, where one of the punctures corresponds to a zero of the abelian differential.*

Proof If $\chi_{0,k+1}$ is trivial, let $S_{0,k+1}$ be the complex plane $\mathbb{C}$ with k punctures. This trivially defines a translation structure, since the charts to $\mathbb{C}$ are given by the inclusion map (and the transition maps are all identity maps). The points removed are clearly apparent singularities, and so is the order-two pole at infinity.

Now consider the case when $\chi_{0,k+1}$ is nontrivial. Since by removing regular points it is easy to add punctures with trivial monodromy to a translation surface, we can assume without loss of generality that each puncture, except one, has nontrivial monodromy. We shall build a translation surface Σ that is homeomorphic to $S_{0,k}$ with one branch point, and is such that after removing the branch point, the monodromy of the translation structure on the resulting surface is $\chi_{0,k+1}$; note that the monodromy around the branch point would be trivial.

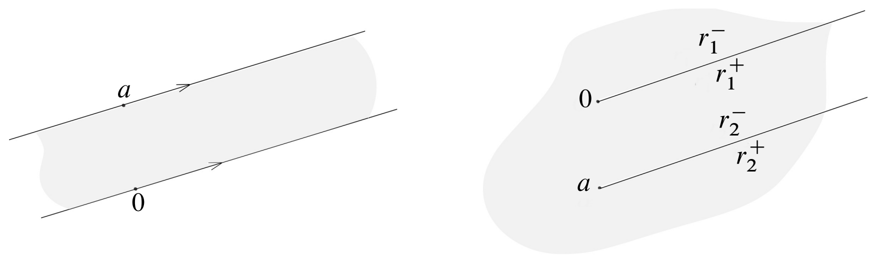

Figure 8: The construction of a translation surface homeomorphic to $S_{0,2}$, with prescribed holonomy.

We now describe two such constructions for $k = 2$; note that since one puncture has trivial monodromy, by (8) the other two punctures have monodromies a and $-a$, respectively, where a is a nonzero complex number.

(i) Choose a direction θ that is not parallel to the line passing through 0 and a, ie $\theta \neq \pm\arg(a)$. Consider the infinite strip S in $\mathbb{C}$ in the direction θ, with an orientation induced from the complex plane, such that the two boundary components differ by the translation $z \mapsto z + a$. Then let A be the translation surface obtained by identifying the two boundary components of S via the translation.

(ii) As before, choose a direction θ that is different from the direction determined by a. Consider the complex plane $\mathbb{C}$ with slits along two infinite rays r_1 and r_2, in the direction θ, that start from 0 and a, respectively. Let r_i^- and r_i^+ be the upper and lower sides of the slits, respectively, where $i = 1, 2$. We then identify r_1^- with r_2^+, and r_1^+ with r_2^-, each by the translation $z \mapsto z + a$.

In all these cases, the translation surface we obtain is homeomorphic to an annulus. Removing a regular point from this annulus, we obtain a puncture with trivial monodromy, and hence we obtain a translation structure on a surface homeomorphic to $S_{0,3}$ that has the desired monodromy. The abelian differential ω has poles at the two punctures with nontrivial monodromy: in construction (i), both poles are simple (ie order one), whereas in construction (ii), one of them has order two (and the other is simple).

The difference between constructions (i) and (ii) is the following: if you choose regular point on the resulting translation surface and consider an arc to a puncture that develops onto a ray in $\mathbb{C}$ in the direction θ, then if that puncture it is incident to is the one with monodromy a in (i), it is the one with monodromy $-a$ in (ii), and vice versa.

For $k > 2$, let the translations around the k punctures with nontrivial monodromy, as determined by $\chi_{0,k+1}$, be $A_1, A_2, \ldots, A_k$. Let $A_i(z) = z + a_i$, where $a_i \in \mathbb{C}^*$ for $1 \le i \le k$. Observe that by (8) we have

$$a_k = -\sum_{i=1}^{k-1} a_i.$$

Consider the cyclically ordered sequence of points $\{p_1, p_2, \ldots, p_k\}$ in $\mathbb{C}$, where $p_1 = 0$ and $p_i = A_{i-1}(p_{i-1})$ for each $i = 2, \ldots, k$. From our observation it follows that $p_1 = A_k(p_k)$. Choose a direction θ which is not parallel to $\overline{0\,a_i}$ for any i, and let r_i be the infinite ray in $\mathbb{C}$ that starts from p_i and has direction θ. Now for each $1 \le i \le k$, we construct a translation surface Σ_i homeomorphic to an annulus, either by (i) or (ii) above, such that an arc from a regular point to the puncture with monodromy A_i develops onto the ray r_i on $\mathbb{C}$. We then glue Σ_i to Σ_{i+1} along the rays r_i and r_{i+1}, as in Definition 4.8, where $i \in \{1, 2, 3, \dots, k\}$ is cyclically ordered so that Σ_k is glued with Σ_1. The resulting translation surface is homeomorphic to $S_{0,k}$ with k punctures having monodromy $A_1, A_2, \dots, A_k$ respectively, and a single branch point of angle $2\pi k$ (which is the point corresponding to the p_i after identifications). Removing this branch point, we obtain a translation structure on a surface homeomorphic $S_{0,k+1}$ with monodromy equal to $\chi_{0,k+1}$, as desired. □

Remark Although our main result concerns punctured surfaces with negative Euler characteristic, it is worth noting that in the case of the surface $S_{0,2}$ (ie $g = 0$, $k = 2$), *any* representation $\chi\colon \pi_1(S_{0,2}) \cong \mathbb{Z} \to \mathbb{C}$ can be realized as the monodromy of some translation structure on $S_{0,2}$. In case χ is trivial the desired structure is the complex plane punctured at any point. Otherwise, if $\chi(1) = \alpha \in \mathbb{C}^*$, we can proceed exactly as in the case (i) above; the resulting Euclidean cylinder is the desired structure on $S_{0,2}$. Note that such a nontrivial representation χ, though degenerate, is the monodromy of a complex projective structure *without* apparent singularities. Theorem 2.5 shows that this cannot happen in the case of negative Euler characteristic.

We have thus been able to deal with two cases: one with number of punctures $k = 2$ but genus $g > 0$, and the other with genus $g = 0$ but an arbitrary number of punctures, ie $k \ge 2$. Using the gluing construction along rays once again, we can now prove the following more general statement:

Proposition 4.13 *Let Γ be the first homology group of a surface $S_{g,k}$, where $k \ge 3$. Let $\chi\colon \Gamma \to \mathbb{C}$ be a nontrivial representation such that there is at least one puncture with trivial monodromy. Then there is a translation structure on $S_{g,k}$ whose monodromy is χ, and is such that the corresponding abelian differential ω on $S_{g,k}$ extends to a meromorphic abelian differential $\overline{\omega}$ on the closed surface S_g.*

Proof We have already seen the necessity of a puncture with trivial monodromy. Our construction of a translation structure with the prescribed monodromy χ splits into a few different cases. Note that the case for $g = 0$ is already done in Proposition 4.12. Henceforth we shall assume that $g > 0$.

Let $\chi_{0,n+1}\colon \Gamma_{0,n+1} \to \mathbb{C}$ be the representation as in (10), obtained by restricting χ to the subsurface of $S_{g,k}$ that contains all the punctures with nontrivial monodromy. Here we assume that there are $0 < n < k$ such punctures, and hence the subsurface is homeomorphic to $S_{0,n+1}$. Note that $\chi_{0,n+1}$ has exactly one puncture with trivial monodromy, corresponding to the boundary of the subsurface (which is trivial in homology). Let the rest of the monodromies be represented by the corresponding translation vectors $a_1, a_2, \dots, a_n \in \mathbb{C}^*$. By Proposition 4.12 there is a translation surface Σ_0 homeomorphic to $S_{0,n}$ with a single branch point p, and at least one of the other punctures is at infinity (in the induced flat metric), such that the monodromy around the n punctures are the translations by $a_1, a_2, \dots, a_n$. Then $\Sigma_0 \setminus \{p\}$ is homeomorphic to $S_{0,n+1}$, and carries a translation structure realizing the character $\chi_{0,n+1}$.

Now let $\chi_{g,1}\colon \Gamma_{g,1} \to \mathbb{C}$ be the representation as in (9), obtained by restricting χ to the subsurface of $S_{g,k}$ that contains all the handles. If $\chi_{g,1}$ is a nontrivial representation, then by Lemma 4.6 we can assume that each handle-generator maps to a nonzero complex number. From the proof of Proposition 4.11 there is a translation surface Σ_1 homeomorphic to $S_{g,1}$, with one branch point and one puncture which is a pole of order two, with holonomy $\chi_{g,1}$. We then glue Σ_0 and Σ_1 along a suitable choice of rays from the branch point on each surface to a pole, that develop to infinite rays in $\mathbb{C}$ that are parallel; see Definition 4.8. The resulting translation surface is then homeomorphic to $S_{g,n}$ and has one branch point. Removing the branch point, and $k-n-1$ additional regular points, we obtain the desired surface homeomorphic to $S_{g,k}$ equipped with a translation structure having monodromy χ, as desired.

If $\chi_{g,1}$ is the trivial representation, there are two cases:

Case 1 (χ admits at least two punctures on $S_{g,k}$ with trivial monodromy) We start with the translation structure on $S_{g,3}$ that realizes the trivial representation; see Lemma 4.4. Recall from that construction that such a translation structure is obtained from a translation surface Σ_1 with trivial holonomy that has two branch points, and a single pole of higher order at infinity in the induced flat metric. We can now glue Σ_0 and Σ_1 along suitably chosen rays from a branch point to a pole at infinity, on either surface, as in Definition 4.8. Note that the resulting surface is homeomorphic to $S_{g,n}$ and has two branch points. Removing these two branch points, and an additional $k-n-2$ regular points if necessary, we obtain a surface homeomorphic to $S_{g,k}$, equipped with a translation structure having monodromy χ.

Case 2 (there is exactly one puncture on $S_{g,k}$ with trivial monodromy) By our assumption, there are exactly $k-1$ punctures with nontrivial monodromy, which we denote by $a_1, a_2, \dots, a_{k-1}$ as before. In this case, we first construct a translation structure on $S_{g,3}$ that realizes the representation for which all handle-generators have trivial monodromy, exactly one puncture (call it p) has trivial monodromy, and the other two punctures q and r have monodromy a_1 and $-a_1$, respectively: For this, we use the same covering map $\Pi\colon S_g \to \mathbb{C}\mathrm{P}^1$ as in the proof of Lemma 4.4, namely one that has three ramification points $0, 1, \infty \in \mathbb{C}\mathrm{P}^1$ and three critical points (the preimages of $0, 1, \infty$) on the domain surface. This time, we equip $\mathbb{C}\mathrm{P}^1$ with a translation structure for which the abelian differential ω has a simple pole at 0 and ∞ (and residues $a_1/2g$ and $-a_1/2g$, respectively), and 1 is a regular point. Note that as a translation surface, the target is just a Euclidean cylinder with a distinguished regular point, and its pullback via Π is then a translation surface Σ_1 homeomorphic to $S_{g,2}$ with one cone point (which is the preimage of 1) and two punctures which are two simple poles of $\Pi^*\omega$. Removing the branch point, we obtain the desired translation structure on $S_{g,3}$. Note that in the case that $k=3$, the above construction completes the proof.

We now assume that $k>3$; note that we can then assume without loss of generality that $a_1 \neq -a_2$.

To Σ_1 we shall glue a translation surface Σ_0' homeomorphic to $S_{0,k-2}$ with a single branch point, where we construct Σ_0' such that the monodromies around the punctures are $a_1 + a_2, a_3, \dots, a_{k-1}$. Such a translation surface exists by Proposition 4.12; in that construction, we can also ensure that the puncture p' with monodromy $a_1 + a_2$ is a simple pole. The gluing is now along a choice of a ray on Σ_0' from the branch point to p', and of a ray on Σ_1 from the branch point there to the puncture r, which has monodromy $-a_1$. Once again, we choose the rays so that they develop onto parallel rays on $\mathbb{C}$. The translation surface obtained after this gluing is homeomorphic to $S_{g,k-1}$; note that the pole obtained by identifying the endpoints of the rays now has holonomy $-a_1 + (a_1 + a_2) = a_2$, and the other endpoints define a single branch point after the identification. Removing this branch point, we obtain a translation structure on $S_{g,k}$ with monodromy χ, as desired. Thus, in all cases, we are able to construct the desired translation structure, and we are done. □

4.3 Affine surfaces with at least two punctures

In this section (and the next) we shall assume that $\rho\colon \Pi \to \mathrm{Aff}(\mathbb{C})$ is a nontrivial representation, where

$$\mathrm{Aff}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{C}^* \text{ and } b \in \mathbb{C}\}$$

is the subgroup of $\mathrm{PSL}_2(\mathbb{C})$ comprising affine transformations. Here, recall that Π denotes the fundamental group of the surface $S_{g,k}$. Since $\mathrm{Aff}(\mathbb{C})$ is precisely the subgroup of $\mathrm{PSL}_2(\mathbb{C})$ that stabilizes the point $\infty \in \mathbb{C}\mathrm{P}^1$, any degenerate representation into $\mathrm{PSL}_2(\mathbb{C})$ that has a global fixed point can be conjugated to an affine representation as ρ above. Note that this includes the case of coaxial monodromy, when the representation globally fixes *two* points in $\mathbb{C}\mathrm{P}^1$.

In the language of geometric structures, an *affine structure* on $S_{g,k}$ is an atlas of charts to $\mathbb{C}$ such that the transition maps are affine maps; notice that translation structures (see Section 4.2) are a special case. Recalling our Definition 4.7, in the same spirit we describe an affine structure as follows:

Definition 4.14 An *affine surface* is a surface obtained by identifying sides of a (possibly disconnected, and possibly noncompact) polygon in $\mathbb{C}$ by affine maps. Note that the vertices after identification may result in branch points; a neighborhood of a branch point on an affine surface develops to $\mathbb{C}$ as the map $z \mapsto z^n$ for some $n > 1$. Unlike in Definition 4.7, however, the resulting surface Σ may not have an induced Euclidean metric. Removing the set of branch points B, we obtain an affine structure on the punctured surface with the punctures in B having trivial monodromy (ie apparent singularities of the affine structure).

In the course of this section we need to also consider another particular kind of affine structure, namely:

Definition 4.15 A *half-translation structure* is obtained by starting with a collection of (possibly noncompact) polygons in $\mathbb{C}$ bounded by straight lines and/or rays and/or segments, and identifying such sides by half-translations, ie maps of the form $z \mapsto \pm z + c$. The resulting surface Σ acquires a Euclidean metric, with possible cone points with cone-angles $k\pi$, where $k \in \mathbb{Z}^+$. On the complement of the cone points, we obtain a Euclidean structure locally modeled on $\mathbb{C}$ and such that the transition maps are affine maps of the form $z \mapsto \pm z + c$. These structures naturally come equipped with a (possibly meromorphic) quadratic differential q, induced from the quadratic differential dz^2 on $\mathbb{C}$. A zero of q of order $m \geq 1$ corresponds to a cone point of angle $(m+2)\pi$, and a simple pole is a cone point π.

In this section, our strategy is to construct affine structures with a given affine monodromy $\rho\colon \Pi \to \mathrm{Aff}(\mathbb{C})$. For this, we prove analogues of the results and constructions in Section 4.2.

Just like we did for translation surfaces in Definition 4.8, we can glue affine surfaces along rays as follows:

Definition 4.16 (gluing affine surfaces) Let Σ_1 and Σ_2 be two affine surfaces with embedded arcs l_1 and l_2, respectively, from a branch point or regular point to a puncture, that each develop onto an infinite ray in $\mathbb{C}$. Then we can define a new affine surface Σ by making a slit along l_1 and l_2, and gluing crosswise to obtain an affine surface Σ. This gluing is exactly as in Definition 4.8, except that now the identifications between sides of the slit are by an affine map and its inverse. (In particular, the two rays that are the developed images of the lifts of l_1 and l_2 need not be parallel.) As before, if Σ_i is homeomorphic to S_{g_i,k_i} for $i = 1, 2$, then Σ is homeomorphic to $S_{g_1+g_2,k_1+k_2-1}$. Moreover, the starting points of the rays determine a branch point on Σ.

However, the above gluing has a disadvantage: if the two rays in the developing image are not identical but related by an affine map A, the holonomy of the resulting affine surface Σ might be affected by A. For example, if one of the endpoints of the arc being slit is a puncture with nontrivial monodromy M_1 on one surface, and the corresponding puncture on the other surface has nontrivial monodromy M_2, then the monodromy around the puncture on Σ will be $M_1 A M_2 A^{-1}$. Note that this issue does not arise in the case of a translation surface (cf Definition 4.8) since the holonomy then is abelian.

To handle this, we introduce the following variant of the gluing procedure that ensures that after gluing, the holonomies on the two constituent subsurfaces remain unchanged.

Definition 4.17 (gluing preserving holonomy) Let Σ_0 and Σ_1 be affine surfaces, as before, with rays r_0 and r_1 respectively, each from a (possibly branched) point of the surface to the puncture at infinity. The only requirement will be that the developing map of either surface takes the starting point of the ray to a common point $p \in \mathbb{C}$. In the gluing procedure we shall use the complex plane (thought of as an affine surface) together with a choice of a ray $r_\star$ leaving from p, which we denote by $(\mathbb{C}, r_\star)$. The ray r_0 develops onto a ray $\overline{r}_0$ leaving from p. We slit Σ_0 along r_0 and $(\mathbb{C}, r_\star)$ along $\overline{r}_0$ and then we identify the resulting boundary rays crosswise, as in the Definition 4.16, to obtain an affine surface Σ_0' with holonomy ρ_0. In the same fashion, the ray r_1 develops on a ray $\overline{r}_1$ leaving from p and hence we glue the affine surfaces Σ_1 and $(\mathbb{C}, r_\star)$ along rays to obtain an affine surface Σ_1' with holonomy ρ_1. By construction, the new surfaces Σ_0' and Σ_1' each contain a ray, say r_0' and r_1' respectively, from the branch point to a puncture at infinity that develop onto the same ray $r_\star \subset \mathbb{C}$. We slit Σ_0' and Σ_1' along these two rays and glue as in Definition 4.16 to obtain the final affine surface Σ. If $\Sigma_0 \cong S_{g_0,k_0}$ and $\Sigma_1 \cong S_{g_1,k_1}$

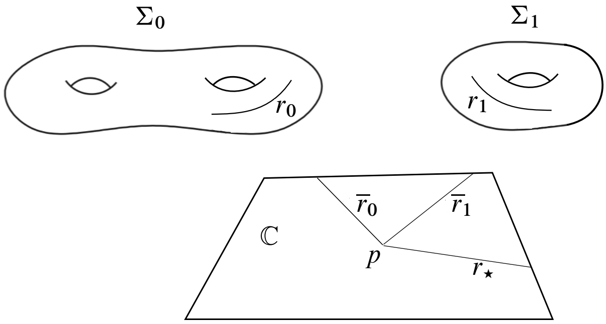

Figure 9: In the gluing preserving holonomy (Definition 4.17), we introduce an intermediate copy of the complex plane.

then the resulting surface Σ is homeomorphic to $S_{g_0+g_1,k_0+k_1-1}$. Note that the starting points of the arcs get identified to a branch point on Σ. In this construction the rays on the two surfaces being glued develop into the same ray $r_\star$, therefore the resulting affine surface has monodromy ρ that restricts to ρ_0 and ρ_1 on the subsurfaces corresponding to Σ_0 and Σ_1 respectively.

We start with the analogue of Proposition 4.12, that handles the case when $g = 0$:

Proposition 4.18 *If $k \geq 2$, any representation $\rho\colon \pi_1(S_{0,k+1}) \to \mathrm{Aff}(\mathbb{C})$ with at least one puncture with trivial monodromy is the holonomy of some affine structure on $S_{0,k+1}$.*

Proof Let us start by assuming $k = 2$ and let $\rho\colon \pi_1(S_{0,3}) \to \mathrm{Aff}(\mathbb{C})$ have at least one puncture with trivial monodromy. If all the punctures have trivial monodromy then $\mathbb{C} \setminus \{q_1, q_2\}$ provides the desired structure for any pair of points $q_1, q_2 \in \mathbb{C}$. We therefore assume the existence of at least one puncture with nontrivial monodromy, say $A \in \mathrm{Aff}(\mathbb{C})$. The remaining puncture necessarily has monodromy A^{-1}.

Let $p_0 \in \mathbb{C}$ be any point. Let r_0 be a ray leaving from p_0 and let $A(r_0)$ be the image of r_0 leaving from $A(p_0)$. We can always choose r_0 such that the rays r_0 and $A(r_0)$ are not contained in each other, ie $r_0 \not\subset A(r_0)$ and $A(r_0) \not\subset r_0$. However, the rays r_0 and $A(r_0)$ may intersect at some point $s \in r_0 \cap A(r_0)$. If this is the case, we replace p_0 with a point $p_\star \in r_0$ such that $p_\star \notin \overline{p_0 s}$. Notice that the segment $\overline{p_0 s}$ may be degenerate, that is a point, if $p_0 = \mathrm{Fix}(A)$. Let $r_\star$ be the subray of r_0, leaving from $p_\star$. By construction, $r_\star$ is disjoint from its image $A(r_\star)$.

We slit $\mathbb{C}$ along $r_\star$ and let $r_\star^+$ and $r_\star^-$ be the right and the left copy of $r_\star$ respectively. In the same fashion, we slit $\mathbb{C}$ along $A(r_\star)$ and denote by $A(r_\star)^+$ and $A(r_\star)^-$ the right and the left copy of $A(r_\star)$ respectively. We glue $r_\star^+$ with $A(r_\star)^-$ and similarly glue $r_\star^-$ with $A(r_\star)^+$ by using the affine map A; see Figure 10.

The resulting surface is a two-punctured sphere which carries an affine structure with one branch point of magnitude 4π arising from the identification of the points $p_\star$ and $A(p_\star)$. One of the punctures has monodromy A and the other puncture has monodromy A^{-1} by construction. We eventually delete the branch point in order to get an affine structure Σ on $S_{0,3}$ with the desired monodromy.

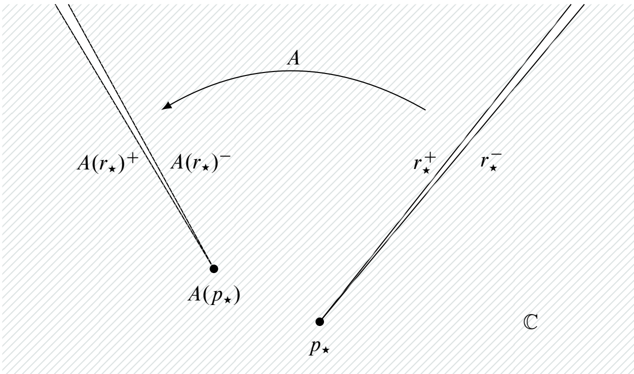

Figure 10

We now consider the general case when $k \geq 3$. Let $\rho\colon \pi_1(S_{0,k+1}) \to \mathrm{Aff}(\mathbb{C})$ be a representation such that at least one puncture has trivial monodromy. There is no loss of generality in assuming that *exactly* one puncture has trivial monodromy. In fact, as already observed in Proposition 4.12, it is easy to add further punctures with trivial monodromy to an affine structure by deleting some regular points. The idea for the general case is to define $k-1$ affine structures on $S_{0,3}$ as above and then glue them together to obtain an affine structure on $S_{0,k+1}$ with one apparent singularity.

Let $A_1, A_2, \dots, A_k, A_{k+1}$ be the monodromies of the punctures. We may assume $A_{k+1} = \mathrm{Id}$ and then observe that $A_k = (A_1 A_2 \cdots A_{k-1})^{-1}$. Let $p_0 \in \mathbb{C}$ be any point and let r_0 be a ray leaving from p_0. For any $i = 1, \dots, k-1$, we define $p_i = A_i(p_0)$ and $r_i = A_i(r_0)$. Clearly, r_i is a ray leaving from p_i. Notice that r_0 can be chosen such that $r_i \not\subset r_0$ and $r_0 \not\subset r_i$ for $i = 1, \dots, k-1$. However, the ray r_i may still intersect r_0 at some point, say s_i, as observed above. We claim the existence of some good point $\overline{p}_0 \in r_0$ and a subray $\overline{r}_0 \subseteq r_0$ leaving from $\overline{p}_0$ such that, upon setting $\overline{r}_i = A_i(\overline{r}_0)$, one has $\overline{r}_i \subseteq r_i$ and the rays $\overline{r}_0$ and $\overline{r}_i$ are disjoint.

We briefly show why the claim above is true. Let $\xi_0\colon [0,\infty) \to \mathbb{C}$ be a parametrization of r_0. We can easily see that any parameter $\tau \in [0,\infty)$ determines a subray $r_\tau \subseteq r_0$. Moreover, given two parameters τ_1, τ_2 such that $\tau_1 < \tau_2$, the corresponding rays are such that $r_{\tau_2} \subset r_{\tau_1}$. We now define $\xi_i : [0,\infty) \to \mathbb{C}$ to be the parametrization of r_i satisfying the equation $\xi_i = A_i \circ \xi_0$. Upon setting $t_0 = 0$, as we showed above, it is possible to find a time t_1 such that the subray $r_{t_1} \subset r_0$ leaving from $\xi_0(t_1)$ is disjoint from the ray $A_1(r_{t_1})$ leaving from $\xi_1(t_1)$. However, it may happen that the rays r_{t_1} and $A_2(r_{t_1})$ still intersect. We apply again the same reasoning to these rays. There is a time $t_2 \geq t_1$ such that $A_2(r_{t_2}) \subseteq A_2(r_{t_1}) \subset r_2$, leaving from $\xi_2(t_2)$, is disjoint from the subray $r_{t_2} \subset r_0$ leaving from $\xi_0(t_2)$. By proceeding in the same fashion at most k times, it is then possible to find a time $\overline{t} \geq t_0$ such that each subray $\overline{r}_i = A_i(r_{\overline{t}})$, leaving from $\xi_i(\overline{t})$, is disjoint from $r_{\overline{t}}$ for any $i = 1, \dots, k-1$.

By replacing p_0 with $\overline{p}_0 = \xi_0(\overline{t})$, we may assume without loss of generality that each ray r_i is disjoint from r_0. For each $i = 1, \dots, k-1$, we consider a copy of the $\mathbb{C}$ along with the rays r_0 and $r_i = A_i(r_0)$

that are disjoint. For the i^{th} copy of $\mathbb{C}$, we can proceed as above by slitting $\mathbb{C}$ along them and regluing to obtain a branched affine structure Σ_i on a two-punctured sphere with one branch point arising from the identification of the points p_0 and $p_i = A_i(p_0)$. One of the punctures has monodromy A_i and the other has monodromy A_i^{-1}.

We now explain how to glue these $k-1$ surfaces. Let ℓ_i be an arc on Σ_i from the branch point to the puncture with monodromy A_i^{-1} that develops into an infinite ray in $\mathbb{C}$ starting from p_0. We can now glue $\Sigma_1, \Sigma_2, \dots, \Sigma_{k-1}$ successively along these rays, as in Definition 4.17. The resulting surface is homeomorphic to $S_{0,k}$ and carries an affine structure Σ. By construction, the punctures have monodromy $A_1,\ A_2, \dots, A_{k-1}$, and $A_k = (A_1 \cdots A_{k-1})^{-1}$. The branch points on each Σ_i get identified to a single branch point $P \in \Sigma$. By deleting that point, the surface $\Sigma \setminus \{P\}$ carries an affine structure with monodromy ρ, as desired. □

For the case when $g > 0$, we begin with the following observation:

Lemma 4.19 *Assume that $g > 0$, and let $\rho\colon \Pi \to \mathrm{Aff}(\mathbb{C})$ be a representation such that $\rho(\gamma) \neq \mathrm{Id}$ for at least one handle-generator γ_0 on $S_{g,k}$. Then there exists $\phi \in \mathrm{MCG}(S_{g,k})$ with an associated outer automorphism $\phi_*\colon \Pi \to \Pi$ such that $\rho \circ \phi_*(\gamma) \neq \mathrm{Id}$ for each handle-generator γ.*

Sketch of the proof The proof is exactly the same as that of Lemma 4.6: note that the standard basis of homology can be considered as a generating set for Π, and changes of homology basis used in the proof of Lemma 4.6 can be realized by a mapping class. □

Henceforth, we shall assume that in the case $g > 0$, the representation ρ maps each handle-generator to a nontrivial affine map; cf the remark following Lemma 4.6. The following is the analogue of Proposition 4.11:

Proposition 4.20 *Let $g > 0$ and let $\rho\colon \pi_1(S_{g,2}) \to \mathrm{Aff}(\mathbb{C})$ be a nontrivial representation such that there is at least one puncture with trivial monodromy. Then there is an affine structure on $S_{g,2}$ with monodromy ρ, obtained by puncturing an affine surface Σ with a unique branch point.*

Proof We shall now describe the construction of the affine surface Σ by gluing of polygons, as in Proposition 4.11. A crucial difference from the construction there is that in the case of affine holonomy, the commutator of two elements (eg a pair of handle-generators) need not map to the trivial (identity) element via ρ. Depending on whether the image of ρ is finite of order two, we shall have to distinguish two cases.

Assume $\mathrm{Im}(\rho)$ is not finite of order two. We start with the case when $g = 1$. Let A and B be the affine maps that are the monodromies around the handle-generators α and β respectively, ie $\rho(\alpha) = A$ and $\rho(\beta) = B$. Note that by the assumption of nontriviality of ρ, and by Lemma 4.19, we can assume that neither A

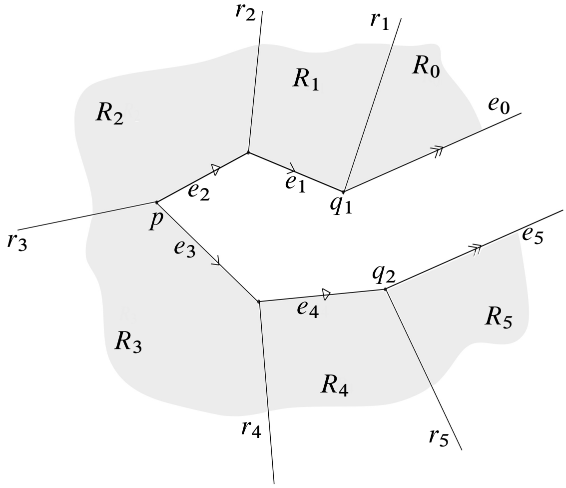

Figure 11: The construction of an affine surface Σ homeomorphic to $S_{1,1}$ where the puncture is at infinity, and Σ has a single branch point; see the proof of Proposition 4.20.

nor B is the identity map. As already observed above, the commutator $C := [A, B] = ABA^{-1}B^{-1}$ need not be the identity map; however, it is easy to verify that C is always a translation, and since $\rho\colon \pi_1(S_{1,2}) \to \mathrm{Aff}(\mathbb{C})$ is a homomorphism, the remaining puncture has monodromy C^{-1} around it.

Fix a point $p \in \mathbb{C}$ that is not a fixed point of A or B, and let $q_1 := AB(p)$ and $q_2 := BA(p)$. Note that $q_1 = q_2$ if A and B commute. Consider the four directed line segments $e_1 := \overline{A(p)q_1}$, $e_2 := \overline{pA(p)}$, $e_3 := \overline{pB(p)}$ and $e_4 := \overline{B(p)q_2}$. Note that $B(e_2) = e_4$ and $A(e_1) = e_3$. Consider two additional infinite rays e_0 and e_5 with starting points q_1 and q_2 respectively; if $q_1 = q_2$ (when A and B commute), we take $e_0 = e_5$. The directed curve $L := \overline{e_0} \cup \overline{e_1} \cup \overline{e_2} \cup e_3 \cup e_4 \cup e_5$ is then an immersed polygonal curve in $\mathbb{C}$.

As in the proof of Proposition 4.11, we then choose a collection of infinite rays $\mathcal{R} = \{r_1, r_2, \dots, r_5\}$ with starting points at the vertices of the segments defined above, and consider embedded region R_i for each $i \in \{0, 1, \dots, 5\}$, bounded by the segment e_i and one or two infinite rays from the collection $\mathcal{R}$. As before, there are two choices of such a region, since the union of e_i and the ray(s) from its endpoint(s) separates the complex plane; we choose the one that results in the correct orientation of the handle-generators α, β in the affine surface Σ that we shall define below. Note that each region R_i has one ideal vertex at ∞ on $\mathbb{C}$, and their union $R := R_0 \cup R_1 \cup \cdots \cup R_5$ is an immersed disk with a puncture at ∞ and boundary $\partial R = L$.

Define the affine surface Σ to be the quotient of R obtained by identifying the boundary segments e_1 and e_3 via the affine map A, e_2 and e_4 via the affine map B, and e_0 and e_5 via the translation $[B, A]$ (which is the identity map if A, B commute, compatible with our requirement that $e_0 = e_5$ in such a case). The resulting surface Σ is homeomorphic to a punctured torus, where the puncture is the point at ∞ on $\mathbb{C}$. From our construction, the pairs of segments $\{e_2, e_4\}$ and $\{e_1, e_3\}$ after identifications define the handle-generators α, β; here note that the regions R_i are chosen to lie on the correct "side" of these directed segments so that α and β have the desired orientation on the punctured torus. Moreover, the handle-generators α and β on Σ have holonomy A and B respectively, and the monodromy around the puncture at ∞ is the translation C. Note that the vertices of the segments $\{e_i\}_{1\le i\le 4}$ get identified to a single branch point q on Σ. The surface $\Sigma\setminus\{q\}$ is thus homeomorphic to $S_{1,2}$ and acquires an affine structure with the desired monodromy ρ, in which q is a puncture with trivial monodromy.

The higher-genus case, ie $g\ge 2$, relies on the preceding construction. Let $\{\alpha_1,\beta_1,\dots,\alpha_g,\beta_g,\gamma_1,\gamma_2\}$ be a generating set of $\pi_1(S_{g,2})$, where for $1\le i\le g$, $\{\alpha_i,\beta_i\}$ is a pair of handle-generators of the i^{th} handle. For any $i=1,\dots,g$ there is an injection $\jmath_i\colon\langle\alpha_i,\beta_i\rangle\to\pi_1(S_{g,2})$ and we define ρ_i as $\rho\circ\jmath_i$. By Lemma 4.19, we assume that each of these affine maps is nontrivial, ie not the identity map.

Let $p\in\mathbb{C}\mathrm{P}^1\setminus\{\mathrm{Fix}(A)\mid A\in\mathrm{Im}(\rho)\}$ be any point. For each i, let Σ_i be the affine surface homeomorphic to the punctured torus, with monodromy ρ_i, obtained from the construction above based at p. In order to glue these g affine surfaces together we have to find g rays, one for each Σ_i, that all develop on the same ray in $\mathbb{C}$. For this, we employ the construction in Definition 4.17, which we now spell out in more detail.

Recall that $\mathbb{C}$ in its own right can be regarded as an affine structure with trivial monodromy on a disk. We single out on such a structure a ray, say r_0, leaving from the point p above towards infinity. The pair $(\mathbb{C},r_0)$ is an affine structure with a marked r_0. On each surface Σ_i, we choose an infinite ray r_i from the unique branch point to the puncture at infinity and such that it develops on $\mathbb{C}$ along a ray, say $\overline{r}_i$, leaving from p towards infinity. Notice that a copy of the ray $\overline{r}_i$ is contained even in $(\mathbb{C},r_0)$. For any i, we glue together the affine surfaces Σ_i and $(\mathbb{C},r_0)$ along the rays r_0 and $\overline{r}_i$ according to our Definition 4.16. The resulting surface is still homeomorphic to $S_{1,1}$ but carries a new affine structure Σ_i' with monodromy ρ_i. Moreover, on each surface Σ_i' we can single out a copy of the ray r_0.

We can now glue together affine surfaces $\Sigma_1',\dots,\Sigma_g'$ along these copies of r_0 (again according to our Definition 4.16) to obtain an affine surface Σ. This surface Σ is homeomorphic to $S_{g,1}$ and has a unique branch point q which develops on $p\in\mathbb{C}$ and it is the starting-point of the rays, after the identifications. Removing q from Σ, we obtain the desired surface homeomorphic to $S_{g,2}$ and equipped with an affine structure with monodromy ρ.

Let us finally assume $\mathrm{Im}(\rho)\cong\mathbb{Z}_2$. Notice that we can always find a basis $\alpha_1,\beta_1,\dots,\alpha_g,\beta_g$ such that

$$\rho(\alpha_i)=1\quad\text{and}\quad\rho(\beta_i)=-1\quad\text{for any }1\le i\le g. \tag{15}$$

Even in this case the proof is based on an inductive process, therefore we start with the case $g=1$. There exists a half-translation structure on $\mathbb{C}\mathrm{P}^1$, see Definition 4.15, associated to a meromorphic quadratic differential ϕ having one zero of order 2 at 0, two poles of order -1 at ±1 and, finally, one pole of order -4 at $\infty\in\mathbb{C}\mathrm{P}^1$; see Definition 4.15 and the subsequent remark. Recall that, given a quadratic differential, a pole of order one corresponds to a cone point of angle π and a zero of order two corresponds to a branch point of magnitude 4π. It is possible to verify that such a structure has nontrivial holonomy given by a representation $\chi\colon\pi_1(\mathbb{C}\mathrm{P}^1\setminus\{\pm1\})\to\mathbb{Z}_2\cong\{z\to\pm z\}$. We now make use of this structure to realize a half-translation structure Σ with the desired holonomy. Take two copies of the structure $(\mathbb{C}\mathrm{P}^1, \phi)$ and slit both along the segment $e_i = [-1, 0]$ and the infinite ray $r_i = [1, \infty]$ for $i = 1, 2$. Denote the resulting sides by $e_i^{\pm}$ and $r_i^{\pm}$. We define Σ to be the half-translation structure on a torus obtained by identifying e_1^+ with e_2^+, e_1^- with e_2^- and, in the same fashion, r_1^+ with r_2^+ and r_1^- with r_2^-. Such a structure is naturally associated to a meromorphic quadratic differential q having one zero and one pole of order 6. By removing the singularities of q we obtain an affine structure on $S_{1,2}$ with the desired monodromy. The general case $g \geq 2$ now proceeds as follows. From our construction, there always exists an infinite ray $\overline{r} \subset \Sigma$ joining the two punctures. Then consider g copies of Σ slit along $\overline{r}$, and glue along rays (as in Definition 4.17) in succession. The resulting surface is homeomorphic to $S_{g,2}$ and carries a half-translation structure with holonomy ρ, as desired. □

Using the previous two propositions, together with the gluing construction as in Definition 4.16, we can now prove the analogue of Proposition 4.13:

Proposition 4.21 *Let $g > 0$ and $k > 2$, and let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{Aff}(\mathbb{C})$ be a nontrivial representation such that there is at least one puncture with trivial monodromy. Then there is an affine structure on $S_{g,k}$ with monodromy ρ, obtained by puncturing an affine surface Σ with a unique branch point.*

Proof We shall follow the strategy of the proof of Proposition 4.13. Our construction of the affine surface Σ will split into two cases.

Case 1 (the representation ρ has at least two punctures with trivial monodromy) Define $\rho_0\colon \pi_1(S_{g,2}) \to \mathrm{Aff}(\mathbb{C})$ as the restriction of ρ to a subsurface of $S_{g,k}$ homeomorphic to $S_{g,2}$ that contains all the handles and one puncture with trivial monodromy. Let $\rho_1\colon \pi_1(S_{0,k}) \to \mathrm{Aff}(\mathbb{C})$ be the restriction of ρ to the complementary subsurface, that contains all the other punctures. Note that by our assumption ρ_1 has at least one puncture with trivial monodromy. There are two subcases:

Subcase (i) (the representation ρ_0 is nontrivial) We shall build Σ by gluing together two affine surfaces Σ_0 and Σ_1, where:

- Σ_0 is homeomorphic to $S_{g,1}$ and has exactly one branch point p, and the monodromy of the affine structure on the surface $\Sigma_0 \setminus \{p\}$ is ρ_0. If ρ_0 is nontrivial, such an affine surface exists by Proposition 4.20.
- Σ_1 is homeomorphic to $S_{0,k-1}$, and has holonomy ρ_1 and exactly one branch point. Such an affine surface exists by Proposition 4.18.

Recall that both the affine surfaces Σ_0 and Σ_1 depend on the choice of an initial basepoint. We choose the same basepoint, say p for both structures. It follows from the constructions in the proofs of Propositions 4.18 and 4.20 that the branch points on both surfaces develop to p. For the gluing, we slit along rays, say r_0 and r_1, on Σ_0 and Σ_1 respectively, from the branch point to a puncture at infinity.

These rays may develop into different rays on $\mathbb{C}$ with the same starting point p. We glue along these rays as in Definition 4.17 to obtain a surface Σ homeomorphic to $S_{g,k-1}$ and having a single branch point. Removing the branch point, we obtain an affine surface homeomorphic to $S_{g,k}$ that has holonomy ρ.

Subcase (ii) (the representation ρ_0 is trivial) At least two punctures are trivial; let $A_3, A_4, \dots, A_k$ be the affine maps that are the monodromy around the remaining punctures. We can exclude the case that $k = 3$ here, since then the triviality of ρ_0 would imply that ρ is trivial, contradicting our assumption. According to our Lemma 4.4, there is a branched projective structure on S_g with three branch points, one of which develops at $\infty \in \mathbb{CP}^1$. We first construct a (branched) affine surface Σ_0 homeomorphic to $S_{g,1}$ with two branch points by removing the branch point at infinity. Let r_0 be a ray starting from one of the branch points to the puncture at infinity and let $\overline{r}_0$ be its developed image on $\mathbb{C}$. It is an infinite ray leaving from a point $p \in \mathbb{C}$. Also, construct an affine surface Σ_1 homeomorphic to $S_{0,k-2}$ with exactly one branch point, such that the monodromy around the punctures are $A_3, \dots, A_k$; such a surface exists by Proposition 4.18. Recall that the construction is subject to the choice of a basepoint. By choosing p as the basepoint, it follows by construction that the branch point of Σ_1 develops at p. Let $r_1 \subset \Sigma_1$ be any ray from the branch point to a puncture at infinity and let $\overline{r}_1$ be the developed ray leaving from p. As before, we now glue preserving holonomy, as in Definition 4.17. Namely, we slit Σ_1 along r_1 and then glue a copy of the marked affine structure $(\mathbb{C}, \overline{r}_0)$ slit along $\overline{r}_1$. Notice that the gluing is possible because r_1 develops on $\overline{r}_1 \subset \mathbb{C}$ by construction. The resulting surface is homeomorphic to $S_{0,k-2}$ but carries a new branched affine structure Σ_1' containing a whole copy of $\mathbb{C}$ with the embedded ray $\overline{r}_0$. We slit Σ_0 along r_0 and Σ_1' along $\overline{r}_0$ and then we identify the resulting boundary rays crosswise to obtain an affine surface Σ homeomorphic to $S_{g,k-2}$ and two branch points. Removing the branch points we obtain the desired affine structure on $S_{g,k}$ with monodromy ρ.

Case 2 (the representation ρ has exactly one puncture with trivial monodromy) Consider the subsurface of $S_{g,k}$ that contains all the handles, and the puncture with trivial monodromy; such a surface is homeomorphic to $S_{g,2}$. Let $\rho_0\colon \pi_1(S_{g,2}) \to \mathrm{Aff}(\mathbb{C})$ be the restriction of ρ to that surface. Note that ρ_0 has trivial monodromy for one of the punctures, and the other puncture has monodromy C that is the product of the commutators of the holonomies around the handle-generators, for each handle. If this product is the identity map, then we can use the same constructions as in Case 1 to finish the construction of the desired affine surface Σ. In what follows, we shall assume that C is not the identity element (and is therefore some nontrivial translation).

Either using Proposition 4.20 if ρ_0 is nontrivial, or Case 2 of the proof of Proposition 4.13 if ρ_0 is trivial, we can then build an affine surface Σ_0 such that

(a) it is homeomorphic to $S_{g,1}$ and a single branch point, say q,

(b) on removing the branch point, the affine structure on $\Sigma \setminus \{q\}$ has monodromy ρ_0, and

(c) there is a ray r_0 from q to the puncture at infinity which has monodromy C by construction. This ray develops on an infinite ray $\overline{r}_0$ leaving from a point $p \in \mathbb{C}$.

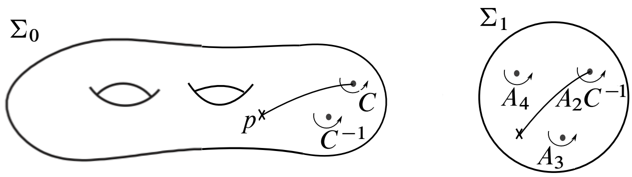

Figure 12: The construction of an affine surface Σ by gluing Σ_0 and Σ_1 along rays, in Case 2 in the proof of Proposition 4.21.

Let the monodromy around the remaining punctures be $A_2, A_3, \dots, A_k$. Given the point p as above, by Proposition 4.18 there is an affine surface Σ_1 that is homeomorphic to $S_{0,k-1}$ such that

(a) there is exactly one branch point which develops to the point p, and

(b) the monodromy around the punctures are $A_2C^{-1}, A_3, A_4, \dots, A_k$.

Let r_1 be a ray from the branch point to the puncture with holonomy A_2C^{-1} and let $\overline{r}_1$ be its developed image. We glue Σ_0 and Σ_1 along the rays r_0 and r_1, as in Definition 4.17. Recall that in that gluing-preserving holonomy, we in fact first attach copies of the affine surface $\mathbb{C}$ to r_0 and r_1 respectively, and then glue along the same ray $r_\star$ in these copies via the identity map. The resulting affine surface Σ is homeomorphic to $S_{g,k-1}$ and has one branch point p where the starting points of the rays get identified. Since the final gluing (along the ray $r_\star$) is by the identity map, the other endpoints of the rays get identified to a puncture with holonomy $A_2C^{-1} \cdot C = A_2$; cf the discussion just before Definition 4.17. Hence the monodromy of the affine structure on $\Sigma \setminus \{q\}$ is precisely ρ, as desired. □

5 Affine holonomy and a single puncture

In this section we deal with the case when the representation ρ is into the affine group $\mathrm{Aff}(\mathbb{C})$, as in the previous section, for once-punctured surfaces of positive genus, that is, $k = 1$ and $g > 0$. For this, we need to modify the construction in Proposition 4.20 such that the "puncture at infinity" for Σ is a regular point when viewed as a *projective* structure. We can then "fill in" that puncture to obtain a surface equipped with a projective structure (away from a single branch point).

5.1 Necessary conditions

We start by showing the necessity of assuming the image of ρ is not a finite group of order two in Theorem A.

Lemma 5.1 *Let $\rho\colon \pi_1(S_{g,1}) \to \mathrm{Aff}(\mathbb{C})$ be a nontrivial representation such that the puncture has trivial monodromy and the image of ρ is finite of order two. Then ρ does not appear as the monodromy of any projective structure $S_{g,1}$.*

Proof Let $G = \ker(\rho)$ and let $\widehat{S}_{g,1}$ be the covering of $S_{g,1}$ associated to G. The group G is a subgroup of $\pi_1(S_{g,1})$ of index two, and hence the covering map $f : \widehat{S}_{g,1} \to S_{g,1}$ turns out to be a Galois covering map of degree two. In particular, $\widehat{S}_{g,1}$ is homeomorphic to $S_{2g-1,2}$. Let us now assume the existence of a complex projective structure on $S_{g,1}$. Then we may lift this structure to a complex projective structure on $S_{2g-1,2}$ with monodromy determined by the composition $\rho \circ f_*$, where $f_* : \pi_1(S_{2g-1,2}) \to \pi_1(S_{g,1})$. Since the image of f_* is nothing but $\ker(\rho)$, the representation $\rho \circ f_*$ is just the trivial one. Therefore, by our Lemma 4.3, such a structure does not exist and, in turn, there is no complex projective structure on $S_{g,1}$ with monodromy ρ. □

5.2 Once-punctured translation surfaces

The case of translation structures on once-punctured surfaces is actually subsumed by the construction in the proof of Proposition 4.11, provided we only require a projective structure, and not a translation structure, on the surface.

Proposition 5.2 *Let $S_{g,1}$ be a surface of genus $g > 0$ and exactly one puncture, and let $\Gamma_{g,1}$ be its first homology group. Any nontrivial representation $\chi : \Gamma_{g,1} \to \mathbb{C}$ is the monodromy of some projective structure on $S_{g,1}$.*

Proof From (8), we know that the puncture must have trivial monodromy. We can then construct a translation surface Σ homeomorphic to $S_{g,1}$ exactly as in the proof of Proposition 4.11, that has holonomy χ, one branch point p and one pole of order two. Note that a pole of order two is the point at ∞ in a standard planar end of $\mathbb{C}$; thus in particular, ∞ is a regular point of $\mathbb{C}\mathrm{P}^1$. Thus, we can consider $\widehat{\Sigma} = \Sigma \cup \{\infty\}$ to be a surface equipped with a projective structure, with exactly one branch point (namely, p); the surface $\widehat{\Sigma} \setminus \{p\}$ is then the desired surface homeomorphic to $S_{g,1}$ equipped with a projective structure having monodromy χ. □

Remark If a nontrivial representation $\chi : \Gamma_{g,1} \to \mathbb{C}$ is the monodromy of a *translation* structure, then the corresponding abelian differential ω must extend to an abelian differential with exactly one zero on the closed surface S_g. The recent work of Le Fils [2022] and Bainbridge, Johnson, Judge and Park [Bainbridge et al. 2022] generalizing Haupt's theorem (see [Haupt 1920] and [Kapovich 2020]) provides necessary and sufficient conditions on χ for the existence of such a structure.

5.3 Once-punctured affine torus

For the once-punctured torus, the problem of finding an projective structure with prescribed affine holonomy is handled by the following result:

Proposition 5.3 *Let $\rho : \pi_1(S_{1,1}) \to \mathrm{Aff}(\mathbb{C})$ be a nontrivial representation such that the puncture has trivial monodromy. Assume $\rho\big(\pi_1(S_{1,1})\big)$ is not finite of order two. Then there is a projective structure on $S_{1,1}$ with monodromy ρ.*

Proof Let $\rho\colon \pi_1(S_{1,1}) \to \mathrm{Aff}(\mathbb{C})$ be a nontrivial affine representation, let α and β denote two handle-generators and let $\gamma = [\alpha, \beta]$ be a curve enclosing the puncture. The monodromy around the puncture is assumed to be trivial, ie $\rho(\gamma) = I$. This implies in particular that $\rho(\alpha) = A$ and $\rho(\beta) = B$ commute and hence the representation ρ is abelian. Up to conjugation, we may assume without loss of generality that

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}$$

where $a, b \notin \{0, 1\}$. In fact, a and b cannot be both equal to one as the representation is assumed to be nontrivial and, whenever A or B is the identity matrix, a suitable change of basis puts the matrices in the desired form. Given any point $p_0 \in \mathbb{C}$, we define the points p_i for $i = 1, \dots, 3$ as follows: $p_1 = A(p_0)$, $p_2 = AB(p_0) = BA(p_0)$, and finally $p_3 = B(p_0)$. The polygon

$$p_0 \mapsto p_1 \mapsto p_2 \mapsto p_3 \mapsto p_0 \tag{17}$$

bounds a possibly self-intersecting and possibly degenerate quadrilateral $\mathcal{Q}$ on the complex plane. As already done before, we shall denote the directed edges as follows:

$$e_1 = \overline{p_1\, p_2}, \quad e_2 = \overline{p_0\, p_1}, \quad e_3 = \overline{p_0\, p_3}, \quad e_4 = \overline{p_3\, p_2}.$$

The edges of this polygon are related by the maps A, B as follows: $A(e_3) = e_1$ and $B(e_2) = e_4$. Given the matrices A and B as in (16), it is convenient to choose $p_0 = 1$. As a consequence, $p_1 = a$, $p_2 = ab$ and $p_3 = b$. We observe that the points $\{1, a, b, ab\} \subset \mathbb{C}$ are all aligned if and only if they are all real and the quadrilateral $\mathcal{Q}$ degenerates to a segment. According to this property, we shall divide the discussion into two cases.

Case 1 (the points $1, a, b, ab$ are not reals) In this case, the points $1, a, b, ab$ are the vertices of some possibly self-intersecting quadrilateral $\mathcal{Q}$. As done before in Proposition 4.11, we can choose a collection of infinite rays $\mathcal{R} = \{r_0, r_1, r_2, r_3\}$ with starting points at the vertices p_i of $\mathcal{Q}$ and consider embedded region R_i, for each $i \in \{0, \dots, 3\}$, bounded by the segment e_i and two infinite rays from the collection $\mathcal{R}$. Even in this case there are two choices of each such a region, since the union of e_i and the ray from its endpoints separates the complex plane; we choose the one that results in the correct orientation of the handle-generators α, β in the affine surface Σ that we shall define below; see Figure 6. Each region R_i has one ideal vertex at $\infty \in \mathbb{C}\mathrm{P}^1$ and the union of the regions determines an immersed disc R on the Riemann sphere with boundary $\partial R = \overline{e_1} \cup \overline{e_2} \cup e_3 \cup e_4$. We define Σ to be quotient of the region R by identifying the boundary segments e_1 and e_3 via the affine map A, and the segments e_2 and e_4 via the affine map B. The resulting surface is homeomorphic to a punctured torus endowed with an affine structure on a punctured torus with one branch point of magnitude 6π and one pole of order two. We can fill up the puncture by adding a complex projective chart locally modeled at $\infty \in \mathbb{C}\mathrm{P}^1$ and eventually remove the (only) branch point. The final surface is a punctured torus endowed with a complex structure — but not affine — having monodromy ρ.

Case 2 (the complex numbers $1, a, b, ab$ are reals) In this case the four points $1, a, b, ab$ are aligned, and a similar construction works. Recall that in this case $a, b \notin \{\pm 1\}$ in the light of Lemma 5.1 above.

The main difference from Case 1 is that the quadrilateral $\mathcal{Q}$ degenerates to a segment on the real axis. Whenever either a or b is greater than zero, then we can still find a collection of rays and regions R_i with the desired properties and thence one can proceed as above. However, when both a, b are negative it turns out to be impossible to find rays and regions as desired, regardless of the choice of the basepoint p_0. In this case, we first need to change the handle-generators in order to make either a or b a positive real. For instance, we may replace $\{\alpha, \beta\}$ with $\{\alpha, \alpha\beta\}$. Then we can proceed as above. □

Remark Here is a construction, inspired by [Mondello and Panov 2019, Lemma 2.2], of a projective structure (in fact, a spherical structure) on $S_{1,1}$ such that the image of the monodromy representation is a finite cyclic group of order $k \geq 3$. Let C be a great circle in $\mathbb{C}\mathrm{P}^1$ and let α be the "orthogonal" geodesic line in $\mathbb{H}^3$ (thought of as the unit ball, with $\partial_\infty \mathbb{H}^3 = \mathbb{C}\mathrm{P}^1$) passing through the origin. On C we can single out two adjacent segments, say l_1 and l_2, each of length $2\pi/k$ in the spherical metric. Of course, l_1 and l_2 are related by the elliptic element E that is a rotation of angle $2\pi/k$ around the axis α. Slit $\mathbb{C}\mathrm{P}^1$ along l_1 and l_2. The resulting space is a bigon with two vertices each of angle 2π. Then reglue l_1^+ with l_2^- and l_1^- with l_2^+. The final surface is a torus equipped with a spherical structure and a single branch point of angle 6π. By deleting the branch point we end up with the desired structure on $S_{1,1}$ having the desired monodromy, since the monodromy of each handle-generator is $E^{\pm 1}$.

5.4 Higher-genus affine surfaces

Let us finally consider the general case of punctured surfaces with genus $g \geq 2$. Our goal is to realize the given representation to the affine group as the monodromy of some branched projective structure with a single branch point. By deleting such a point, we end up with a complex projective structure on $S_{g,1}$ as desired. Note that although the monodromy is into the affine group $\mathrm{Aff}(\mathbb{C})$, the projective structure obtained might not be an affine structure. Namely, we prove the following:

Proposition 5.4 *Let $g \geq 2$ and let $\rho\colon \pi_1(S_{g,1}) \to \mathrm{Aff}(\mathbb{C})$ be a nontrivial representation such that the puncture has trivial monodromy. Assume $\rho\big(\pi_1(S_{g,1})\big)$ is not finite of order two. Then there is a $\mathbb{C}\mathrm{P}^1$-structure on $S_{g,1}$ with monodromy ρ.*

Our proof shall deal with the coaxial case and noncoaxial case separately. In the final subsection we also provide an alternative proof using Le Fils' results [2023]. Before moving to the proof of Proposition 5.4, we shall need some technical results.

5.4.1 Some technical lemmata In order to state and prove those lemmata we shall need, we begin by introducing the following definitions.

Definition 5.5 (unitary part, linear part) Given a coaxial representation ρ, its *unitary part* $\rho_u : \Pi \to U(1)$ is defined by $\rho_u(\gamma) = \exp(i \arg(\rho(\gamma)))$ for each $\gamma \in \Pi$. Note that if ρ is unitary then $\rho = \rho_u$. This notion easily extends to any affine representation as follows. In fact, there is a natural projection $\mathrm{Li}\colon \mathrm{Aff}(\mathbb{C}) \to \mathbb{C}^*$ that associates to any mapping $A(z) = az + b$ its linear part, ie $\mathrm{Li}(A) = az$. Notice that if ρ is coaxial,

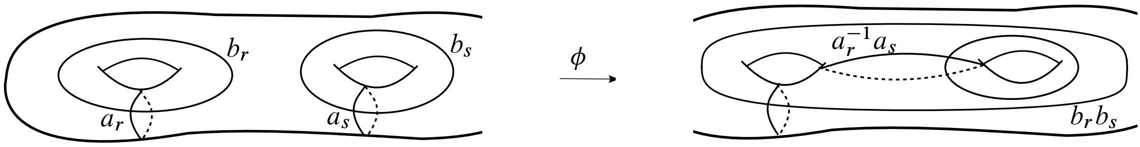

Figure 13: The mapping class ϕ changes the pairs of handle-generators $\{\alpha_r, \beta_r\}$ and $\{\alpha_s, \beta_s\}$ to $\{\alpha_r, \beta_r\beta_s\}$ and $\{\alpha_r^{-1}\alpha_s, \beta_s\}$, respectively.

then $\mathrm{Li} \circ \rho = \rho$. The *unitary part* of a generic representation ρ is a representation $\rho_u : \pi_1(S_{1,1}) \to U(1)$, defined as $\rho_u(\gamma) = \exp\bigl(i \arg(\mathrm{Li} \circ \rho(\gamma))\bigr)$.

Definition 5.6 Given a coaxial representation ρ, a handle on $S_{g,1}$ generated by a pair $\{\alpha, \beta\}$ of simple closed curves intersecting only once will be called *rational* if $\rho_u(\alpha)$ and $\rho_u(\beta)$ generate a discrete subgroup of $U(1)$. Alternatively, if the dilation factors of $\rho(\alpha)$ and $\rho(\beta)$ are a and b respectively, then the handle is rational if their arguments $\arg a$ and $\arg b$ are in $2\pi\mathbb{Q}$ (the *dilation factor* of an affine map $A(z) = az$ is $a \in \mathbb{C}^*$). We will say the handle is *irrational* if it is not rational.

The following lemma concerns affine representation with dense unitary part.

Lemma 5.7 *Let $\rho \colon \Pi \to \mathrm{Aff}(\mathbb{C})$ be an affine representation such that $\mathrm{Li} \circ \rho$ is not unitary. Then there exist handle-generators $\{\alpha_j, \beta_j\}_{1 \le j \le g}$ on $S_{g,1}$ such that $|a_j|, |b_j| > 1$, where a_j and b_j are the dilation factors of $\rho(\alpha_j)$ and $\rho(\beta_j)$ respectively. Moreover, in the case the unitary part ρ_u has a dense image in $U(1)$, we can also ensure that $\arg a_j, \arg b_j \notin 2\pi\mathbb{Q}$ for each $1 \le j \le g$ and, for any $\epsilon > 0$, we can choose a set of handle-generators that satisfy, in addition to the above properties, $|\arg a_j|, |\arg b_j| < \epsilon$ for each j.*

Proof First of all we notice that it is sufficient to prove the lemma for coaxial representations. In fact, the general case follows by replacing ρ with $\mathrm{Li} \circ \rho$. Choose an initial set of handle-generators $\{\alpha_j, \beta_j\}_{1 \le j \le g}$; in the following argument, whenever we modify this set of generators via a mapping class, we shall rename and continue denoting the resulting set by the same notation (viz. α_j, β_j). We shall also denote by a_j and b_j the dilation factors of $\rho(\alpha_j)$ and $\rho(\beta_j)$ respectively. We shall first show that we can choose handle-generators so that $|a_j| \ne 1$ and $|b_j| \ne 1$ for each j.

Since ρ is not unitary, it follows that $|a_r| \ne 1$ for some r. Note that if $|b_r| \ne 1$ instead, we can interchange the handle-generators via the mapping class that takes the pair $\{\alpha_r, \beta_r\} \mapsto \{\beta_r, \alpha_r^{-1}\}$. Now if $|b_r| = 1$ we can change this pair via the mapping class that takes $\{\alpha_r, \beta_r\} \mapsto \{\alpha_r, \alpha_r\,\beta_r\}$; such a mapping class is supported on that handle, and Dehn-twists around α_r. This makes the modulus of the dilation factor of the ρ-image of the second generator also different from 1. If $|a_s| = 1$ for some other index s, then we change the two handles (the r^{th} and s^{th}) via the mapping class ϕ that takes $\{\alpha_r, \beta_r\} \mapsto \{\alpha_r, \beta_r\,\beta_s\}$ and $\{\alpha_s, \beta_s\} \mapsto \{\alpha_r^{-1}\alpha_s, \beta_s\}$; see Figure 13. In this way we can make sure that the modulus of the dilation factor of the first generator of the second handle is $|a_r^{-1} a_s| \neq 1$, and so we can continue the process as above, until all handle-generators have their corresponding dilation factors not equal to 1.

To ensure the dilation factors are strictly greater than 1 in modulus, we perform the following modifications:

Claim 1 *For any handle, there is a change of generators by a mapping class such that their dilation factors satisfy* $|a| > 1$ *and* $|b| > 1$.

Proof Assume that $|b| < 1$. Recall that we have already ensured above that $|a| \neq 1$. We can use the change of generators $(A, B) \mapsto (A, A^n B)$ for $n \in \mathbb{Z}$, that is effected by a (power of a) Dehn twist around the handle-generator corresponding to A. Note that the dilation factor of $A^n B$ is $a^n b$. Thus for a suitable choice of sign of n, and for $|n|$ large enough, the dilation factor of $A^n B$ is strictly greater than 1 in modulus. Now the second generator $B' = A^n B$ has the desired property. If the first generator (which remains unchanged) still has $|a| < 1$, then we perform the change of generators $(A, B') \mapsto (B', A^{-1})$, which is again effected by an element of $\mathrm{SL}(2, \mathbb{Z})$ (and hence a mapping class). □

Now assume that the image of the unitary part ρ_u is dense in $U(1)$. We shall perform a change of generators exactly as in the first part of the proof, so that the arguments of the dilation factors are all irrational; we shall only observe that these modifications do not change the property that the dilation factors are greater than 1 in modulus.

Assume there is some $r \in \{1, 2, \dots, g\}$ such that $\arg a_r \notin 2\pi\mathbb{Q}$; if instead there is some r such that $\arg b_r \notin 2\pi\mathbb{Q}$, then we can switch the roles of α_r and β_r in what follows (eg instead of Dehn twists around α_r we perform Dehn twists around β_r). If $\arg b_r \in 2\pi\mathbb{Q}$ we can change this pair of handle-generators to $\{\alpha_r, \alpha_r^n \beta_r\}$ for any $n \in \mathbb{Z}$. It is easy to see that for any $n > 0$, the resulting new handle-generator will satisfy $\arg b_r \notin 2\pi\mathbb{Q}$, and since $|a_r| > 1$, we also have $|b_r| > 1$. Now let $\arg a_s \in 2\pi\mathbb{Q}$ for some s; as before, we use the mapping class ϕ to change the two pairs of generators $\{\alpha_r, \beta_r\} \mapsto \{\alpha_r, \beta_r \beta_s\}$ and $\{\alpha_s, \beta_s\} \mapsto \{\alpha_r^{-1} \alpha_s, \beta_s\}$. Again, we rename these new pairs as $\{\alpha_r, \beta_r\}$ and $\{\alpha_s, \beta_s\}$ respectively. The new generator of the latter handle now has $\arg a_s \notin 2\pi\mathbb{Q}$. Note that by the change $\{\alpha_s, \beta_s\} \mapsto \{\alpha_s \beta_s^n, \beta_s\}$ for $n \gg 0$ (achieved by Dehn twists along β_s on the s^{th} handle), we could have arranged that prior to acting by ϕ, the modulus of the dilation factor satisfied $|a_s| \gg |a_r|$, so that after acting by ϕ, the dilation factor still satisfies $|a_s| > 1$.

Finally, fix $\epsilon > 0$. We shall show that for the j^{th} handle for any $1 \leq j \leq g$, there is a change of generators by a mapping class supported on the handle, such for the resulting pair of generators we have $|\arg a_j|, |\arg b_j| < \epsilon$. Indeed, we can perform Dehn twists as usual to change the handle-generators to $\{\alpha_j, \alpha_j^n \beta_j\}$ for any $n \in \mathbb{Z}$. As before, for any $n > 0$ the dilation factors remain greater than 1 in modulus. Although the new argument could lie in $2\pi\mathbb{Q}$ for some integer, say N, it cannot be in $2\pi\mathbb{Q}$ for any $n \neq N$. (If $N \arg a_j + \arg b_j \in 2\pi\mathbb{Q}$ and $M \arg a_j + \arg b_j \in 2\pi\mathbb{Q}$ for $N \neq M$ then $(N - M) a_j \in 2\pi\mathbb{Q}$, which is a contradiction.) Since $\rho(\alpha_r)$ is an irrational rotation of the circle, we can choose $n > N$ such that $|\arg b_j| < \epsilon$. Similarly, we perform a power of a Dehn twist around b_j, to ensure that $|\arg a_j| < \epsilon$. □

Remark Let ρ be a *Euclidean* representation, namely an affine representation with unitary linear part, that is, $\rho_u = \mathrm{Li} \circ \rho$. Assume the image of ρ_u to be dense in $U(1)$. It worth noticing that, although the first claim of Lemma 5.7 never holds for Euclidean representations, it is still possible to find a basis of handle generators such that the linear parts of the ρ-images have arbitrarily small argument.

Corollary 5.8 *Let $\rho\colon \Pi \to \mathrm{Aff}(\mathbb{C})$ be an affine representation. If the unitary part ρ_u has a dense image in $U(1)$, then for any $\epsilon > 0$ there exist handle-generators $\{\alpha_j, \beta_j\}_{1\le j\le g}$ on $S_{g,1}$ such that the inequalities $-\epsilon < \arg b_j < 0$ and $0 \le \arg a_j + \arg b_j < \epsilon$ hold for each j.*

Proof The first thing we notice is that the second part of the proof above works mutatis mutandis even when $\mathrm{Li} \circ \rho$ is unitary, with the only exception being that the dilatation factors are always equal to one. What follows is nothing but a refinement of Lemma 5.7. Again, we shall assume for simplicity that ρ is coaxial, and the general case comes by replacing ρ with $\mathrm{Li} \circ \rho$. In fact, in the same notation as above, $\rho(\alpha_r)$ is an irrational rotation and we can choose $n > N$ such that $|\arg b_j| < \epsilon$ and $-\epsilon < \arg b_j < 0$. We now perform a Dehn twist around b_j to ensure that $0 < -\arg b_j < \arg a_j < \epsilon$. The result follows. □

We now consider affine representations whose unitary part is discrete in $U(1)$. We begin with the following

Lemma 5.9 *Let $\rho\colon \Pi \to \mathrm{Aff}(\mathbb{C})$ be an affine representation whose unitary part ρ_u has a discrete image in $U(1)$. Then there exist handle-generators $\{\alpha_j, \beta_j\}_{1\le j\le g}$ on $S_{g,1}$ such that*

$$\rho_u(\alpha_j) = \exp(2\pi i/m) \quad \text{and} \quad \rho_u(\beta_j) = 1 \quad \text{for each } j, \tag{18}$$

where $\rho_u(\Pi) \cong \mathbb{Z}_m$. In fact, for any surjective homomorphism $h\colon \Pi \to \mathbb{Z}_m$, we can find handle-generators such that $\rho_u(\alpha_j) = h(\alpha_j)$ and $\rho_u(\beta_j) = h(\beta_j)$ for each j.

Remark Before proving the stated lemma, we recall the following result of Edmonds [1982]. For a finite abelian group G consider the representation space $\mathrm{Hom}(\Pi, G)$. The natural action of $\mathrm{MCG}(S)$ on the representation space by precomposition yields a natural injection from the orbit space $\mathrm{Hom}(\Pi, G)/\mathrm{MCG}(S)$ to $H_2(G, \mathbb{Z})$; see [Edmonds 1982, Theorem 1.2] and [Nielsen 1937] for the case $G = \mathbb{Z}_m$. Since $H_2(\mathbb{Z}_m, \mathbb{Z}) = \{0\}$, it follows that $\mathrm{MCG}(S)$ acts transitively on the representation space. For further details the reader may also consult [Le Fils 2023, Proposition 3.2].

Proof of Lemma 5.9 Since $U(1)$ is abelian, ρ_u factors through the homology group $\Gamma = H_1(S_{g,1}, \mathbb{Z})$. Fix a set of handle-generators $\{\alpha'_j, \beta'_j\}_{1\le j\le g}$; this is also a set of generators of Γ. Since the image in $U(1)$ is a discrete group, it must be cyclic, say of order $m \ge 2$. Thus, we can think of the unitary part as a surjective homomorphism $\rho_u\colon \Gamma \to \mathbb{Z}_m$, where $\mathbb{Z}_m$ is the cyclic subgroup of $U(1)$ generated by $\exp(2\pi i/m)$. Then, by the remark above, there is $A \in \mathrm{Sp}(2g, \mathbb{Z}) \cong \mathrm{Aut}^+(\Gamma)$ such that the representation $\rho_u \circ A\colon \Gamma \to \mathbb{Z}_m$ satisfies $\rho_u(\alpha'_j) = 1$ and $\rho_u(\beta'_j) = 0$ for each $1 \le j \le g$. Indeed, for any surjective homomorphism $h_u\colon \Gamma \to \mathbb{Z}_m$ there exists an automorphism A of Γ such that the equation $\rho_u \circ A = h_u$ holds. The automorphism A is induced by a mapping class $\phi\colon S_{g,1} \to S_{g,1}$, and defining $\alpha_j := \phi(\alpha'_j)$ and $\beta_j := \phi(\beta'_j)$ then defines our desired set of handle-generators. □

From now on, the case of $m=2$ is ruled out by our assumption that the image is not of order two; cf Lemma 5.1. Therefore the condition $m \geq 3$ will be take as a blanket assumption unless stated otherwise. We finally conclude with a lemma specific to noncoaxial representations. We shall make use of the following result in Section 5.4.3.

Lemma 5.10 *Suppose that* $\rho\colon \pi_1(S_{g,1}) \to \mathrm{Aff}(\mathbb{C})$ *is a nontrivial representation as in the statement of Proposition 5.4. If* ρ *is not coaxial, then we can choose pairs of handle-generators* $\{\alpha_i, \beta_i\}_{1\leq i\leq g}$ *whose commutators are all nontrivial, ie* $\rho([\alpha_i,\beta_i]) \neq I$ *for each* $1 \leq i \leq g$.

Proof An affine map $A(z) = az+b$ has a fixed point $b/(1-a) \in \mathbb{C}$, unless $a=1$, ie A is a translation, in which case the fixed-point set satisfies $\mathrm{Fix}(A) = \varnothing$. Two affine maps A and B commute if and only if their fixed-point sets are identical. We shall also use the following elementary fact: if A and B are affine maps with different fixed-point sets, then the fixed-point set of $B \circ A$ is different from that of A. Therefore, it suffices to show that one can choose each pair $\{\alpha_i, \beta_i\}$ of handle-generators so that their fixed-point sets are not identical; for the purposes of this proof we shall call a pair having this property (and the corresponding handle) "good".

We start with some set of handle-generators $\{\alpha_i, \beta_i\}_{1\leq i\leq g}$; note that these $2g$ elements generate the fundamental group $\pi_1(S_{g,1})$. By Lemma 4.19 we can also assume that $\rho(\alpha_i)$ and $\rho(\beta_i)$ are nontrivial affine maps for each $1 \leq i \leq g$. In what follows we shall modify this initial choice of generators by mapping class group elements until the pair of generators of each handle is good. The basic idea of this modification is the following: suppose $\{\alpha_i, \beta_i\}$ is a good pair, and $\{\alpha_j, \beta_j\}$ is not. Then we replace these two pairs of handle-generators by the pairs $\{\alpha_i, \beta_i\beta_j\}$ and $\{\alpha_i^{-1}\alpha_j, \beta_j\}$. Note that this change of handle-generators is effected by a mapping class $\phi\colon S_{g,1} \to S_{g,1}$; see Figure 13. Let F be the common fixed-point set of $\rho(\alpha_j)$ and $\rho(\beta_j)$. We divide into two cases:

Case A ($F \neq \phi$) In this case F is a single point; in what follows we assume that $F = 0 \in \mathbb{C}$ to simplify our computations, because the general case can be reduced to this via a conjugation. Let $z \mapsto a_j z$ and $z \mapsto b_j z$ be the ρ-images of the generators α_j and β_j respectively, where $a_j, b_j \in \mathbb{C} \setminus \{0,1\}$. We can assume without loss of generality that the fixed-point sets of $\rho(\alpha_i)$ and $\rho(\beta_i)$ are both distinct from F, since otherwise, if say $\mathrm{Fix}(\rho(\alpha_i)) = F$, then we can perform a Dehn twist in that handle around β_i to change its generators $\{\alpha_i, \beta_i\} \mapsto \{\alpha_i\beta_i, \beta_i\}$. By the elementary fact noted above, $\mathrm{Fix}(\rho(\alpha_i\beta_i)) \neq F$, so this new pair of generators has the required property. The same fact implies that the new second handle obtained after acting by the mapping class ϕ, ie generated by $\{\alpha_i^{-1}\alpha_j, \beta_j\}$, is good. However, it could still happen that for the new first handle, the ρ-images of the generators, namely $\rho(\alpha_i)$ and $\rho(\beta_i\beta_j)$, have the same fixed-point set. If $\rho(\alpha_i)$ is the affine map $z \mapsto a_i z + c_i$ and $\rho(\beta_i)$ is the affine map $z \mapsto b_i z + d_i$, then this happens when

$$\mathrm{Fix}(\rho(\alpha_i)) = \frac{c_i}{1-a_i} = \frac{d_i}{1-b_ib_j} = \mathrm{Fix}(\rho(\beta_i\beta_j)). \tag{19}$$

In this case, we first change the generators of the second handle at the very beginning of the construction, by a Dehn twist around α_j, namely $\{\alpha_j,\beta_j\}\mapsto\{\alpha_j,\alpha_j\beta_j\}$. This does not change the property that the fixed-point set of both generators is F; however after acting by the mapping class ϕ, the new two pairs of handle-generators are now $\{\alpha_i,\beta_i\alpha_j\beta_j\}$ and $\{\alpha_i^{-1}\alpha_j,\alpha_j\beta_j\}$. The latter is a good pair for the same reason as before; the former is not a good pair only if

$$\frac{c_i}{1-a_i}=\frac{d_i}{1-b_ia_jb_j}. \tag{20}$$

It is easy to check that (19) and (20) cannot simultaneously hold, since by our assumption $a_j\neq 1$.

Case B $(F=\phi)$ In this case both $\rho(\alpha_j)$ and $\rho(\beta_j)$ are translations, say $z\mapsto z+v$ and $z\mapsto z+w$ respectively. As above, let $z\mapsto a_iz+c_i$ and $z\mapsto b_iz+d_i$ be $\rho(\alpha_i)$ and $\rho(\beta_i)$ respectively. Then the pair of generators $\{\alpha_i^{-1}\alpha_j,\beta_j\}$ of the second handle is good since one generator maps to a translation, while the other does not. The new pair of generators $\{\alpha_i,\beta_i\beta_j\}$ of the first handle is either also good, in which case we are done, or else

$$\operatorname{Fix}(\rho(\alpha_i))=\frac{c_i}{1-a_i}=\frac{b_iw+d_i}{1-b_i}=\operatorname{Fix}(\rho(\beta_i\beta_j)). \tag{21}$$

In the latter case, we proceed as in Case A, namely, we first replace $\{\alpha_j,\beta_j\}$ with $\{\alpha_j,\alpha_j\beta_j\}$ to the second handle, at the beginning of the construction. The new second generator of the second handle is now the translation $z\mapsto z+v+w$, and after acting by the mapping class ϕ, the new two pairs of handle-generators are $\{\alpha_i,\alpha\beta_i\alpha_j\beta_j\}$ and $\{\alpha_i^{-1}\alpha_j,\alpha_j\beta_j\}$. The latter is a good pair for the same reason as before, namely because one generator maps to a translation while the other does not. The first pair must also be good, because otherwise

$$\frac{c_i}{1-a_i}=\frac{b_iv+b_iw+d_i}{1-b_i}, \tag{22}$$

which contradicts (21) since we know $v\neq 0$, as none of the handle-generators map to the identity element.

Thus, if there is one good pair of handle-generators, we can use the above modification repeatedly to make each handle good. To complete the argument, we need to show that there exists a good handle. For this, note that since ρ is not coaxial, there exist two elements from the initial set of generators that do not have the same fixed-point set. If they are generators for the same handle, then we already have one good pair. If not, suppose they belong to two handles neither of which is good; namely, suppose there are two pairs of handle-generators $\{\alpha_i,\beta_i\}$ and $\{\alpha_j,\beta_j\}$ such that the elements in each pair have the same fixed-point set, but the fixed-point sets for the pairs are not identical. Then we change the pair of handles by the mapping class ϕ exactly as above, namely where the two new pairs of handle-generators are $\{\alpha_i,\beta_i\beta_j\}$ and $\{\alpha_i^{-1}\alpha_j,\beta_j\}$. It follows from the elementary fact observed at the beginning of the proof that both of these are now good pairs. □

5.4.2 Proof of Proposition 5.4: coaxial representations We shall divide this proof into three cases:

(i) ρ is not unitary, ie its image in $\mathrm{PSL}_2(\mathbb{C})$ does not lie in the circle subgroup

$$U(1) = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta}) \mid \theta \in \mathbb{R}\}/\{\pm I\}.$$

(ii) ρ is unitary, but the image of ρ is dense in $U(1)$.

(iii) ρ is unitary, and the image of ρ is finite, but not of order two.

Recall that as in the previous section, we are considering representations from $\Pi = \pi_1(S_{g,1})$.

Case (i) (nonunitary case) First consider the quadrilaterals $\{Q_1, Q_2, \dots, Q_g\}$, where for each $1 \le i \le g$, the quadrilateral Q_i is constructed exactly as in Proposition 5.3 by taking the ρ-images A_i and B_i of the i^{th} handle, such that the basepoint p_i of Q_i is also a vertex of Q_{i-1} for each $i \ge 2$.

The key idea is that we can do this so that each these quadrilaterals are pairwise disjoint, except for adjacent quadrilaterals which intersect only at a single vertex, eg $Q_{i-1} \cap Q_i = \{p_i\}$. In order to show that we can do this, we consider two subcases, involving the unitary part ρ_u of ρ, see Definition 5.5.

Subcase 1 (ρ_u has dense image) Recall from Proposition 5.3 that the quadrilateral Q corresponding to a handle (with generators mapping to A and B) and basepoint p is defined by the oriented polygon $p \mapsto A(p) \mapsto AB(p) = BA(p) \mapsto B(p) \mapsto p$; see equation (17). It follows that if A and B have dilation factors each of modulus strictly greater than 1, and with argument sufficiently small, then

(a) Q lies entirely to the right of the vertical line passing through p, and

(b) the "rightmost" point of Q is $AB(p) = BA(p)$.

Thus, to construct the desired nonoverlapping "chain" of quadrilaterals $\{Q_1, Q_2, \dots, Q_g\}$, we choose handle-generators as in Lemma 5.7. We choose a basepoint for Q_1, corresponding to the first handle, to be a point $p \in \mathbb{R}^+ \subset \mathbb{C}$, and then define the basepoint for each successive quadrilateral, corresponding to the next handle, to be the rightmost point of the preceding quadrilateral, as in (b) above.

This oriented chain of quadrilaterals bounds an immersed punctured disk on its right, where the puncture is at the point at infinity. In other words, $Q_1 \cup Q_2 \cup \dots \cup Q_g$ bounds an immersed disk in $\mathbb{C}\mathrm{P}^1$ that contains the point ∞. For each quadrilateral, we can identify pairs of sides using the affine maps corresponding to the generators of that handle; this results in a genus g surface equipped with a branched projective structure. Moreover, there is a unique branch point, namely the point where all the vertices of the

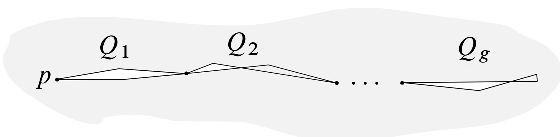

Figure 14: In Subcase 1, the chain of quadrilaterals proceeds towards the right and remains close to the positive real axis.

quadrilaterals get identified to; removing the branch point we obtain the desired surface homeomorphic to $S_{g,1}$ with a projective structure having monodromy ρ.

Subcase 2 (ρ_u has discrete image) Assume that the image is a cyclic group of order $m \geq 1$. Recall that we are in the case when ρ is not unitary. We now observe that on each handle, we can perform the change of the pair of generators such that our Claim 1 holds, and the handle-generators still satisfy the conclusion of Lemma 5.9. To see that this is true, observe that for the change of handle-generators $\{\alpha, \beta\} \mapsto \{\alpha, \alpha^n \beta\}$ for $n \in \mathbb{Z}$, if $\rho_u(\alpha) = 1$ and $\rho_u(\beta) = 0$, then $\rho_u(\alpha^n \beta) = 1$ whenever $n \equiv m (\text{mod } 1)$. Recall from the proof of Claim 1 that such Dehn twists around α can ensure that the resulting new generator satisfies $|b| > 1$. For the other generator, switch the roles of α and β; namely, consider $\{\alpha, \beta\} \mapsto \{\alpha\beta^n, \beta\}$ for some $n \in \mathbb{Z}$. Note that in this case, the ρ_u-images of the new generators remain unchanged. Thus, we can assume that the handle-generators satisfy, for each $1 \leq j \leq g$,

(1) $|a_j|, |b_j| > 1$ and

(2) $\rho_u(\alpha_j) = \exp(2\pi i/m)$ and $\rho_u(\beta_j) = 1$.

Let Q_j be the quadrilateral corresponding to the j^{th} handle, with basepoint $p_j \in \mathbb{C}$. Note that the edges of Q_j are

$$p_j \mapsto |a_j| \exp\left(\frac{2\pi i}{m}\right) p_j \mapsto |a_j||b_j| \exp\left(\frac{2\pi i}{m}\right) p_j \mapsto |b_j| p_j \mapsto p_j.$$

In other words, in polar coordinates on $\mathbb{C}$, if $|p_j| = R$ and $\arg p_j = \theta_0$, the quadrilateral Q_j bounds the rectangular region

$$\left\{(r, \theta) \;\middle|\; R \leq r \leq |a_j||b_j| R \text{ and } \theta_0 \leq \theta \leq \theta_0 + \frac{2\pi}{m}\right\}.$$

Note that the third vertex of Q_j is an extreme point of the region, furthest from the origin.

We choose the basepoint of Q_1 to be $p_1 = 1$, and for each successive quadrilateral Q_j, define the basepoint p_j to be the third (ie extreme) vertex of Q_{j-1}. Note that $|p_j| > |p_{j-1}|$ for each $2 \leq j \leq g$. The quadrilateral Q_j can only intersect Q_{j-1} at that common vertex, since all the remaining vertices of the preceding quadrilaterals $Q_1, Q_2, \ldots, Q_{j-1}$ lie in the interior of the disk of radius $|p_j|$ around the origin.

The resulting sequence of quadrilaterals $Q_1, Q_2, \ldots, Q_g$ thus forms a nonoverlapping chain, as we desired. Note that in the special case that $m = 1$ (ie ρ_u is the trivial representation), each quadrilateral is degenerate, as its sides lie along the real line, and they form a chain along the positive real axis; otherwise, if $m > 1$, this chain is "spiraling" as shown in Figure 15. As in Subcase 1, this (oriented) chain bounds an immersed (in fact an embedded) disk in $\mathbb{C}\mathrm{P}^1$ on its right containing the point ∞. Identifying pairs of sides of each quadrilateral Q_j using the affine maps $\rho(\alpha_j)$ and $\rho(\beta_j)$, we obtain a surface of genus g equipped with a branched projective structure and monodromy ρ, with a unique branch point where all the vertices of the chain get identified. Removing this branch point, we obtain the desired projective structure on $S_{g,1}$.

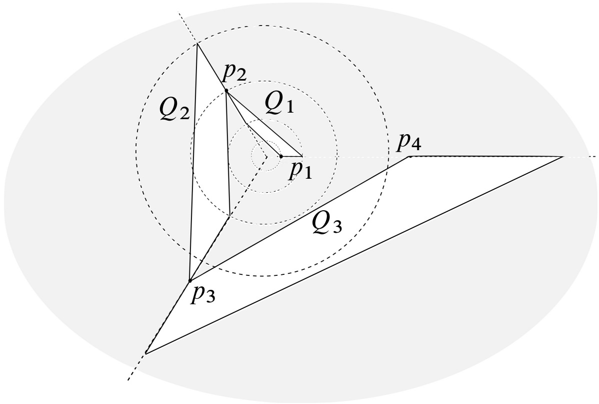

Figure 15: In Subcase 2, the chain of quadrilaterals is spiraling if the order of the discrete image is $m > 1$. (Shown above for $m = 3$.)

Case (ii) (unitary with dense image) Let us assume now that ρ is unitary (ie $\rho = \rho_u$), with an image that is dense in $U(1)$. Fix an $\epsilon > 0$, the choice of which shall be made clearer later. We can apply the proof of the second part of Lemma 5.7 to obtain a change of generators (effected by some mapping class) such that the resulting handle-generators $\{\alpha_j, \beta_j\}_{1 \le j \le g}$ satisfy, for each $1 \le j \le g$,

1. $\arg a_j, \arg b_j \notin 2\pi\mathbb{Q}$, and
2. $|\arg a_j|, |\arg b_j| < \epsilon$.

This change of generators is exactly as in [Chenakkod et al. 2022, Lemmata 11.4 and 11.5].

Thus, we can assume that for the j^{th} handle, for each $1 \le j \le g$, the pair of generators map to the elements of the form

$$z \mapsto \exp(i\theta_j^1)z \quad \text{and} \quad z \mapsto \exp(i\theta_j^2)z$$

of $U(1)$ respectively, where $0 < \theta_j^1, \theta_j^2 < \epsilon$. Choose the basepoint for the first handle to be on the positive real axis, say $p_1 = R \in \mathbb{R}^+$. Then the quadrilateral Q_1 corresponding to the first handle, given by $p_1 \mapsto \exp(i\theta_j^1)p_1 \mapsto \exp(i(\theta_j^1 + \theta_j^2))p_1 \mapsto \exp(i\theta_j^2)p_1 \mapsto p_1$, has vertices on the circle of radius R, and lies in the sector bounded by rays at angles 0 and $\theta_1^1 + \theta_1^2$. Choose the basepoint of the next handle to be $p_2 = \exp(i(\theta_j^1 + \theta_j^2))p_1$; the quadrilateral Q_2 then lies in an adjacent sector of angular width $\theta_2^1 + \theta_2^2 < 2\epsilon$. We can continue placing quadrilaterals for successive handles, choosing the basepoint of each to be the extreme point for the previous quadrilateral; each is contained in a sector of angular width less than 2ϵ. Our initial choice of $\epsilon > 0$ can be made so that g such sectors fit without overlapping, ie $2g\epsilon < 2\pi$. We thus obtain an oriented chain of quadrilaterals $Q_1, Q_2, \dots, Q_g$, as in Case (i), where successive handles intersect at a common vertex, and every other pair is disjoint. Their union then bounds an immersed disk in $\mathbb{CP}^1$ on its right, containing the point ∞. As in Case (i), we then identify pairs of edges of each quadrilateral using the maps corresponding to the generators, to obtain a surface homeomorphic to S_g, equipped with a branched projective structure with a unique branch point. Deleting the branch point, we obtain the desired projective structure on $S_{g,1}$ with monodromy ρ.

Case (iii) We now consider the remaining case when ρ is coaxial, but the image of ρ is a finite group in $U(1)$. Let the order of this finite group be $m \geq 3$; here, recall that $m \neq 2$ by Lemma 5.1.

We first apply Lemma 5.9 to obtain handle-generators $\{\alpha_j, \beta_j\}_{1 \leq j \leq g}$ such that

$$\rho(\alpha_j) = \rho(\beta_j) = \exp\left(\frac{2\pi i}{m}\right) \quad \text{for each } j.$$

For any handle generated by $\{\alpha_j, \beta_j\}$, and a choice of a basepoint $p \in \mathbb{C}$, the quadrilateral Q with edges

$$p \mapsto \rho(\alpha_j)p \mapsto \rho(\alpha_j\beta_j)p \mapsto \rho(\beta_j)p \mapsto p$$

is a degenerate "V"-shaped quadrilateral, since the second and fourth vertices coincide. Such a quadrilateral bounds an immersed (in fact embedded) disk in $\mathbb{C}\mathrm{P}^1$ in its exterior. However, all vertices lie on the circle of radius $|p|$ centered at 0, and they span an angle $4\pi/m$ at the origin. This makes it difficult to form a nonoverlapping chain of quadrilaterals, as we were able to do in Cases (i) and (ii). We resolve this difficulty by using a "grafting" construction that we shall describe next, the idea of which is similar to Definition 4.17.

First, we need to introduce the following definitions:

Definition 5.11 (projective handle) A *projective handle* will refer to a branched projective structure on a torus with a single branch point, obtained by identifying pairs of sides of a quadrilateral Q in $\mathbb{C}\mathrm{P}^1$ that bounds an immersed disk (recall that the preceding cases have involved constructing such projective handles).

Definition 5.12 (grafting in a handle) Let S be a surface equipped with a branched projective structure, and let γ be an embedded arc on S from a branch point p to itself that develops onto an embedded arc $\hat{\gamma}$ on $\mathbb{C}\mathrm{P}^1$. Suppose H is a projective handle that corresponds to a quadrilateral Q on $\mathbb{C}\mathrm{P}^1$ such that $\hat{\gamma}$ lies in the disk in $\mathbb{C}\mathrm{P}^1$ bounded by Q, and an endpoint of $\hat{\gamma}$ is a vertex of Q. Consider the one-holed torus T obtained by introducing a slit in H along the arc that develops onto $\hat{\gamma}$; let the resulting two sides of the slit be σ^+ and σ^-. Then cut along the arc γ on S and identifying the resulting sides with the boundary arc σ^+ and σ^- on T respectively, so that the genus of the resulting surface S' is one more than that of S. Here, the identification is such that the developing maps to $\mathbb{C}\mathrm{P}^1$ are precisely the same; the surface S' thus acquires a (branched) projective structure. See Figure 16.

Note that in the construction above,

- there are no new branch points that are introduced, but the order of the branch point p increases by two, and
- the monodromy representations when restricted to the subsurfaces S and H remain unchanged.

We can now construct the projective structure on $S_{g,1}$ with monodromy ρ (which is coaxial, unitary and discrete) by successively grafting in handles, as we now describe.

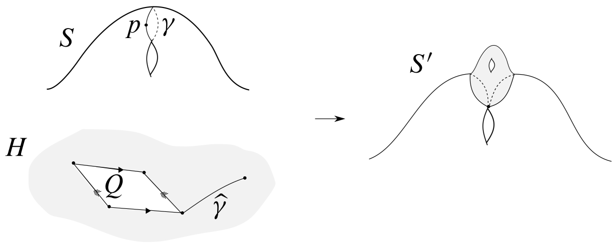

Figure 16: In Definition 5.12 the handle H is slit along $\widehat{\gamma}$ and grafted in on S along γ.

Start with the basepoint $p_1 \in \mathbb{R}^+ \subset \mathbb{C}$ and the quadrilateral Q_1 corresponding to the first handle, generated by $\{\alpha_1, \beta_1\}$. The projective handle corresponding to Q_1 is our initial surface S_1. We shall successively graft in g handles as in Definition 5.12; in what follows we describe the j^{th} step, where we assume we have a surface S_j of genus $1 \le j < g$ equipped with a branched projective structure with a unique branch point.

Let Q_j be the quadrilateral for the j^{th} handle, which by our construction will be a subsurface of S_j. Let H be the projective handle corresponding to the quadrilateral Q_{j+1} of the next handle, when the basepoint p_{j+1} for that is taken to be the third (ie extreme) point of Q_j. Note that $\gamma = \alpha_j$ or β_j is an embedded arc from the branch point on S_j to itself, and we can choose a developing image that is an embedded arc $\widehat{\gamma}$ in $\mathbb{C}\mathrm{P}^1$, namely one of the sides of Q_j that is incident to p_{j+1}. Moreover, $\widehat{\gamma}$ lies in the exterior of the quadrilateral Q_{j+1} for H; see Figure 17.

Hence the construction in Definition 5.12 can be applied, that is, we can cut along γ on S_j and graft in the handle H; the resulting surface is S_{j+1}.

At the end of g such steps, we obtain a genus-g surface S_g with a branched projective structure with a unique branch point. Removing the branch point, we obtain a projective surface homeomorphic to $S_{g,1}$ with monodromy ρ, ie with $\rho(\eta) = \exp(2\pi i/m)$ for any handle-generator η.

We have dealt with Cases (i)–(iii), and this completes the first part of the proof of Proposition 5.4.

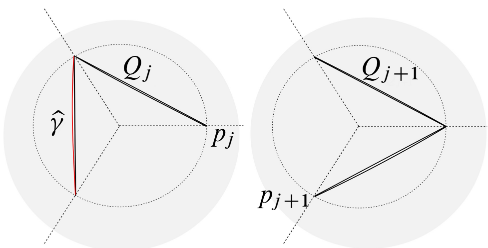

Figure 17: Two successive quadrilaterals in the case $m = 3$, shown here on two different copies of $\mathbb{C}\mathrm{P}^1$; in the inductive step the projective handle corresponding to the latter is grafted in by Definition 5.12.

5.4.3 Proof of Proposition 5.4: noncoaxial representations Let $\rho\colon \pi_1(S_{g,1}) \to \mathrm{Aff}(\mathbb{C})$ be a nontrivial representation that is not coaxial. Since the puncture has trivial monodromy, we regard ρ as a representation $\overline{\rho}\colon \pi_1(S_g) \to \mathrm{Aff}(\mathbb{C})$ and we shall realize it as the monodromy of a branch projective structure with a single branch point. We shall eventually delete the branch point to get a complex projective structure on $S_{g,1}$ with the desired monodromy.

Step 1 (commutators determine a convex polygon $\mathcal{C}$) Given a noncoaxial representation ρ, Lemma 5.10 applies and hence we can assume the existence of a basis of handle-generators $\{\alpha_i, \beta_i\}_{1\le i\le g}$ such that $\rho([\alpha_i, \beta_i]) \neq I$ for each i. It is easy to see that these commutators are all translations, which we denote by $t_1, t_2, \dots, t_n$. It is also not hard to show that there exists a permutation of the handles (realized by a mapping class) such that there is a (possibly degenerate) oriented convex polygon $\mathcal{C} \subset \mathbb{C}$ with g sides such that for each $1 \le i \le g$, the endpoints of the i^{th} side differ by the translation t_i. Indeed, one permutation that works is the one that puts the arguments of the translations in increasing order, ie if $t_i|t_i|^{-1}$ are in counterclockwise order on the unit circle; cf [Chenakkod et al. 2022, Proof of Proposition 6.1]. With respect to this choice, the piecewise linear curve $\partial\mathcal{C}$ bounds the polygon $\mathcal{C}$ on its *left*. Furthermore, the convex polygon $\mathcal{C}$ can be placed everywhere in $\mathbb{C}$. In fact, given any starting point $p_1 \in \mathbb{C}$, the i^{th} side is from p_i to $p_{i+1} = p_i + t_i$, where $i \in \{1, 2, \dots, g\}$ in the reordered set of handles and $p_{g+1} = p_1$ because $t_1 + \dots + t_g = 0$.

Step 2 (realizing a one-holed torus) Let $S_{1,1} \cong H \subset S_g$ be any handle (see Definition 4.2) such that the representation $\rho_{|H}$ induced by the inclusion $H \hookrightarrow S_g$ is not abelian. In this step we show how to realize $\rho_{|H}\colon \pi_1(S_{1,1}) \to \mathrm{Aff}(\mathbb{C})$ as the holonomy of a branched projective structure on a one-holed torus T with linear boundary, except for at most one corner point, and no interior branch point.

For this, we shall need an immersed disk in $\mathbb{CP}^1$ containing ∞ bounded by a Euclidean pentagon $\mathcal{P}$. Such a pentagon is determined by a choice of a basepoint p and the $\rho_{|H}$-images of the handle-generators, that we denote by A and B. Recall that, here, we shall assume that A and B do not commute. Thus, given a point $p \in \mathbb{C}$, the pentagon $\mathcal{P} \subset \mathbb{CP}^1$ is defined as the region containing the infinity ∞ and bounded by the chain

$$p \mapsto [A, B](p) \mapsto B^{-1}(p) \mapsto A^{-1}B^{-1}(p) \mapsto BA^{-1}B^{-1}(p) \mapsto p \tag{23}$$

on the *right*, so that the basepoint p is an extremal point of the segment σ corresponding to the commutator.

We denote the oriented sides of $\mathcal{P}$ as follows:

$$e_1 = \overline{BA^{-1}B^{-1}(p)p}, \qquad e_2 = \overline{A^{-1}B^{-1}(p)BA^{-1}B^{-1}(p)},$$
$$e_3 = \overline{A^{-1}B^{-1}(p)B^{-1}(p)}, \quad e_4 = \overline{B^{-1}(p)[A, B](p)},$$

and, finally, $\sigma = \overline{p[A, B](p)}$. Notice that a similar construction already appeared in Proposition 4.20, see Figure 6, where the basepoint has been chosen to be a different vertex. It is not clear a priori that this pentagon bounds an immersed disk for a suitable choice of p, therefore the key assertion here is the following:

Lemma 5.13 *Let $p_0 \in \mathbb{C}$ be any point with positive imaginary part sufficiently large. For $t > 0$ define*

$$p_t = \begin{cases} (1+t)\Re(p_0) + i\,\Im(p_0) & \text{if } \Re(p_0) > 0, \\ (1-t)\Re(p_0) + i\,\Im(p_0) & \text{if } \Re(p_0) < 0. \end{cases} \tag{24}$$

Then there is a basis $\{\alpha, \beta\}$ of $\pi_1(S_{1,1})$ and $t_0 > 0$ such that the pentagon (23)

$$p_t \mapsto [A, B](p_t) \mapsto B^{-1}(p_t) \mapsto A^{-1}B^{-1}(p_t) \mapsto BA^{-1}B^{-1}(p_t) \mapsto p_t$$

based at p_t bounds an immersed disk in $\mathbb{C}\mathrm{P}^1$ containing the infinity on its right for any $t \geq t_0$, where $A = \rho(\alpha)$ and $B = \rho(\beta)$.

(Here $\Re(z)$ and $\Im(z)$ are the real and imaginary parts respectively, of the complex number z.)

The polygon (23) and its shape highly depend on the choice of a basepoint p_0. A priori, there are *no* restrictions on such a choice. The lemma above says that if the basepoint p_0 is taken with positive imaginary part sufficiently large then, by moving p_0 horizontally, we eventually find a time t_0 such that the polygon (23) bounds on its right an immersed pentagon in $\mathbb{C}\mathrm{P}^1$ containing the infinity. How large the imaginary part of p_0 must be will be clear in context, case by case. Due to the technicality of the Lemma 5.13, we postpone its proof to the end of the current Section 5.4.

Suppose Lemma 5.13 holds, ie there is such an immersed disk. The desired branched projective structure is then obtained by identifying the oriented sides e_1 and e_3 via the affine map A and the sides e_2 and e_4 via the affine map B. The resulting surface is a one-holed torus T, where the side σ after the identifications forms the boundary ∂T; the vertices of $\mathcal{P}$ get identified to the unique corner-point that lies on that boundary. We shall consider handles thus constructed in the next step below.

Step 3 (gluing handles to the polygon $\mathcal{C}$) Let $\{\alpha_i, \beta_i\}_{1 \leq i \leq g}$ be a set of handle-generators as given in Step 1, namely, such that $t_i = \rho([\alpha_i, \beta_i]) \neq I$ for each i. We can order the handles cyclically so that the translations t_i form a convex polygon $\mathcal{C} \subset \mathbb{C}$ and $\partial\mathcal{C}$ bounds the polygon on its left.

For $i \in \{1, 2, \dots, g\}$, let H_i be the handle generated by $\{\alpha_i, \beta_i\}$. For any i, we want to apply the second step above and then obtain a one-holed torus T_i which carries a branched projective structure having holonomy $\rho_i = \rho|_{H_i}$. In fact, given a suitable starting point p_i, Lemma 5.13 above states that by perturbing p_i horizontally, ie by preserving the imaginary coordinate, the chain (23) eventually bounds an immersed pentagon $\mathcal{P}_i \subset \mathbb{C}\mathrm{P}^1$ on its right containing the point at infinity.

Recall there are no restrictions on where to place $\mathcal{C}$, that is, its shape does not depend on the basepoint; indeed, changing the basepoint changes $\mathcal{C}$ by a translation. Therefore we place it sufficiently far from the origin so that the real and imaginary parts of each vertex are sufficiently large and hence Lemma 5.13 applies for each handle. More precisely, the i^{th} vertex of $\mathcal{C}$ will serve as the basepoint for the i^{th} handle. A fundamental membrane for the developing image of T_i is R_i, the immersed pentagonal region in $\mathbb{C}\mathrm{P}^1$ on the right of the chain (23) based at the i^{th} vertex of $\mathcal{C}$. The image of the boundary ∂T_i is exactly the i^{th} side of the convex polygon $\mathcal{C}$ constructed in Step 1.

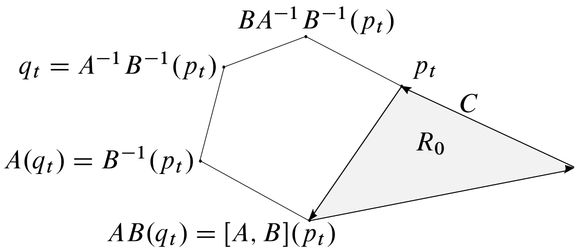

Figure 18: The vertices of the pentagon in the proof of Lemma 5.13 are shown labeled. The commutator edge $\overline{p_t[A, B](p_t)}$ coincides with one of the edges of the convex polygon $\mathcal{C}$, and the region R_0 bounded by $\mathcal{C}$ lies on its left.

Consider the region $R_0 \subset \mathbb{C}$ bounded by the polygon $\mathcal{C}$; note that R is empty if $\mathcal{C}$ is degenerate (ie all sides are colinear). Define the space $\overline{R}_0 = R_0/\sim$, where $\sim$ identifies all the vertices of $\mathcal{C}$ to a point; topologically, $\overline{R}_0$ is homotopy equivalent to a g-holed sphere, and the i^{th} side of $\mathcal{C}$ defines an i^{th} "boundary circle" c_i on $\overline{R}_0$. We now glue $\overline{R}_0$ with the one-holed tori obtained above by identifying the boundary of T_i with c_i for each $1 \le i \le g$, so that the resulting surface is homeomorphic to S_g. This surface acquires a branched projective structure with a unique branch point, and a fundamental membrane in the image of its developing map is $R_0 \cup \bigcup_{i=1}^{g} R_i$. Removing the branch point, we obtain our desired projective structure on $S_{g,1}$ with holonomy ρ.

It only remains to prove Lemma 5.13.

Proof of Lemma 5.13 Given any set of handle generators, say $\{\alpha, \beta\}$, for $\pi_1(S_{1,1})$, we shall denote the ρ-images of α and β by $A(z) = az + c$ and $B(z) = bz + d$, respectively. We observe that the proof follows as soon as we show that three consecutive edges of the pentagon are embedded in $\mathbb{C}$. This is because it is easy to verify that a closed oriented curve in $\mathbb{C} \cup \{\infty\}$ which is obtained by concatenating two embedded arcs always bounds an immersed disk on either of its sides; cf Figure 6. Let us start with some generalities.

Let $p_0 \in \mathbb{C}$ be any point on the upper half-plane such that $[A, B](p_0)$ is contained in the same half-plane. Notice that this can be made sure by choosing p_0 with positive imaginary part and greater than $2|p_0 - [A, B](p_0)|$. Let p_t be defined as in (24). As a consequence of our definition, for t running to infinity, the imaginary part of p_t remains constant and hence the points p_t and $[A, B](p_t)$ both lie on the upper half-plane for any time $t \ge 0$.

We now define $q_t = A^{-1}B^{-1}(p_t)$ for any $t \ge 0$. We observe that, for t tending to infinity, we have the limits

$$|q_t| \to \infty \quad \text{and} \quad \arg q_t \to \arg \overline{ab} = \delta. \tag{25}$$

The second limit can be easily explained as follows. We first notice that q_t can be written as

$$q_t = \frac{(p_t - (bc + d))\overline{ab}}{|ab|^2}. \tag{26}$$

By setting $w = bc + d$, we have

$$\Re(q_t) = \frac{1}{|ab|^2}\big((\Re(p_t) - \Re(w))\Re(\overline{ab}) - (\Im(p_t) - \Im(w))\Im(\overline{ab})\big), \tag{27}$$

$$\Im(q_t) = \frac{1}{|ab|^2}\big((\Re(p_t) - \Re(w))\Im(\overline{ab}) + (\Im(p_t) - \Im(w))\Re(\overline{ab})\big). \tag{28}$$

Now it is a routine exercise to check that $\arg q_t \to \delta$ for t running to infinity — recall that the imaginary part of p_t is constant as a function of t and equal to $\Im(p_0)$.

We shall now distinguish two cases, according to the image of the unitary part of ρ.

Case 1 (the unitary part ρ_u has dense image in $U(1)$) In this case our Corollary 5.8 applies and hence, for any arbitrarily small $\epsilon > 0$, there is a set of handle generators $\{\alpha, \beta\}$ such that

$$0 < \arg a < \epsilon, \quad 0 < -\arg b < \epsilon \quad \text{and} \quad 0 \le \arg a + \arg b < \epsilon.$$

The inequalities $0 \le \arg a + \arg b < \epsilon$ readily imply $-\epsilon < \delta \le 0$ since $\delta = -\arg a - \arg b$, and note that $\delta = 0$ if and only if $ab \in \mathbb{R}$. Let us start by assuming $\delta < 0$. Then, for any t large enough, the point q_t always lies on the lower half-plane and its norm can be taken to be arbitrarily large.

We now consider the point $A(q_t) = aq_t + c$. For any t large enough, $\arg A(q_t)$ is barely affected by the translational part of A, in other words $\arg A(q_t) \approx \arg aq_t$. More precisely, since $\arg q_t$ tends to δ (see formula (25)), we have that $\arg aq_t \to -\arg b > 0$ and the open ball $B(aq_t, 2|c|)$ is entirely contained in the upper half-plane for any t sufficiently big. Clearly, $A(q_t) \in B(aq_t, 2|c|)$.

In the same fashion we can observe that $\arg bq_t \to -\arg a < 0$ and the open ball $B(bq_t, 2|d|)$ is entirely contained in the lower half-plane for any t sufficiently big. Similarly to the above, it is clear that $B(q_t) \in B(bq_t, 2|d|)$.

Recall that there exists a $t_1 > 0$ such that q_t always lies in the lower half-plane for any time $t > t_1$. This necessarily forces the segment $\overline{q_t B(q_t)}$ to be contained in the lower half-plane. On the other hand, there exists a $t_2 > 0$ such that $A(q_t)$ always lies in the upper half-plane for any $t > t_2$, as already observed, and this forces the edge $\overline{A(q_t), AB(q_t)}$ to be entirely contained in the upper half-plane, where $AB(q_t) = [A, B](p_t)$. As a consequence, the chain of segments

$$AB(q_t) \to A(q_t) \to q_t \to B(q_t) \tag{29}$$

is embedded in $\mathbb{C}$. The edges $\overline{q_t B(q_t)}$ and $\overline{AB(q_t) A(q_t)}$ cannot intersect for any time $t > \max\{t_1, t_2\}$ because they lie on different half-planes. Therefore the polygon (23) bounds an immersed disk on its right containing the infinity on the Riemann sphere.

Let us now assume $\delta = 0$. This case occurs if and only if $ab \in \mathbb{R}$ and this implies $\Im(\overline{ab}) = 0$. The imaginary part $\Im(q_t)$ of q_t seen as a function of t is constant, see the formula (28), and it may be positive. Therefore the argument above might simply not apply in this case. We bypass this issue as follows. Recall that $ab \in \mathbb{R}$ if and only if $\arg a = -\arg b$. Then it is sufficient to replace the given pair of handle-generators $\{\alpha, \beta\}$ with $\{\alpha', \beta'\} = \{\alpha\beta^{-1}, \beta\}$ in order to fall into the case of $\delta < 0$. In fact, let a' and b' be the linear parts of $\rho(\alpha')$ and $\rho(\beta')$ respectively; then

$$\arg a' = \arg a - \arg b = 2 \arg a \quad \text{and} \quad \arg b' = \arg b. \tag{30}$$

Now it is an easy matter to check that both $0 < \arg a', -\arg b' < 2\epsilon$ and $0 < \arg a' + \arg b' < 2\epsilon$ hold, which imply $\arg \overline{a'b'} < 0$, as desired.

Case 2 (the unitary part ρ_u has discrete image in $U(1)$) We now suppose the unitary part ρ_u of ρ is discrete, that means $\mathrm{Im}(\rho_u) \cong \mathbb{Z}_m$ for some $m \geq 2$, or $\mathrm{Im}(\rho_u)$ is trivial. We shall consider these cases separately.

Subcase (i) (ρ_u nontrivial) Suppose the unitary part ρ_u of ρ is a nontrivial representation with discrete image isomorphic to $\mathbb{Z}_m$ for some $m \geq 2$. Our Lemma 5.9 applies and hence we can find a set of handle generators $\{\alpha, \beta\}$ such that

$$\rho_u(\alpha) = \exp\left(\frac{2\pi i}{m}\right) \quad \text{and} \quad \rho_u(\beta) = 1.$$

In this case the proof does not differ much from the previous one, but it can be simplified a little. Since $\mathrm{Li} \circ \rho(\beta) = b \in \mathbb{R}^+$, we can immediately observe that the imaginary part $\Im(A(q_t))$ is constant as a function of t. In fact, by setting $q_t = A^{-1}B^{-1}(p_t)$ as above, it is sufficient to observe that

$$A(q_t) = B^{-1}(p_t) = \frac{p_t - d}{b}.$$

In particular, since p_0 can be chosen arbitrarily, we may suppose $\Im(p_0)$ big enough to make $\Im(A(q_t)) > 0$. As a direct consequence, we can deduce that the segment $\overline{A(q_t), AB(q_t)}$ is entirely contained in the upper half-plane.

The rest of the proof now proceeds as in Case 1 above. As $t \to +\infty$ the point q_t tends to infinity and $\arg q_t \to -\arg a = -2\pi i/m < 0$. For t big enough, the open ball $B(bq_t, 2|d|)$, which contains the segment $\overline{q_t, B(q_t)}$ is entirely contained in the lower half-plane. Therefore, the chain of segments

$$AB(q_t) \to A(q_t) \to q_t \to B(q_t) \tag{31}$$

is embedded in $\mathbb{C}$ because the edges $\overline{q_t B(q_t)}$ and $\overline{AB(q_t)A(q_t)}$ cannot intersect. Therefore the polygon (23) bounds an immersed disk on its right containing the infinity on the Riemann sphere.

Subcase (ii) (ρ_u trivial) We begin by applying some reduction in order to put a and b in a more convenient form. A first important fact to note is that, whenever ρ_u is trivial, then $\mathrm{Li} \circ \rho(\alpha)$ and $\mathrm{Li} \circ \rho(\beta)$ are both real and different from zero. Moreover, a and b cannot be both equal to 1. In fact, if this was the case, then $[A, B] = I$. In the case one between a or b is equal to 1 we may apply a suitable Dehn twist to make both different from 1. Therefore, we can suppose $a, b \neq 1$. We then apply our Claim 1, if necessary, to make them both greater than one in modulus; thus we may assume $|a| > |b| > 1$. Finally, there is no loss of generality in assuming b positive, whereas a could be positive or negative.

Let $q_t = A^{-1}B^{-1}(p_t)$ as above. Notice that $\Im(\overline{ab}) = 0$ because both $a, b \in \mathbb{R}^*$. In this special case, formulæ (27) and (28) simplify as

(32) $$\Re(q_t) = \frac{1}{ab}\big(\Re(p_t) - \Re(w)\big) \quad \text{and} \quad \Im(q_t) = \frac{1}{ab}\big(\Im(p_t) - \Im(w)\big),$$

where w is defined as above. Thus the imaginary part of q_t, seen as a function of t, remains constant because it no longer depends on $\Re(p_t)$.

Let us consider the point $A(q_t)$. It is an easy matter to check that, for t running to infinity, the value $\Im(A(q_t))$ seen as a function of t remains constant. In fact, by recalling that $\Im(p_t)$ is assumed to be constant in t, it is sufficient to notice that

$$A(q_t) = aq_t + c = \left(\frac{1}{b}\big(\Re(p_t) - \Re(w)\big) + \Re(c)\right) + i\left(\frac{1}{b}\big(\Im(p_t) - \Im(w)\big) + \Im(c)\right).$$

In the same fashion, it is possible to check that

$$B(q_t) = bq_t + d = \left(\frac{1}{a}\big(\Re(p_t) - \Re(w)\big) + \Re(d)\right) + i\left(\frac{1}{a}\big(\Im(p_t) - \Im(w)\big) + \Im(d)\right),$$

and so even the imaginary part of $B(q_t)$ is constant as a function of t. Let us finally consider the points $BA(q_t) = p_t$ and $AB(q_t) = [A, B](p_t)$. As a consequence of our definitions even their imaginary parts are constant as functions of t. More precisely, we have already seen in (32) that

$$\Im(q_t) = \frac{1}{ab}\big(\Im(p_t) - \Im(w)\big),$$

and an easy computation shows that

$$\Im\big([A, B](p_t)\big) = \Im(AB(q_t)) = \big(\Im(p_t) - \Im(w)\big) + a\Im(d) + \Im(c).$$

Let us now recall that the starting point p_0 can be chose arbitrarily on the upper half-plane. This allows us to choose the initial point p_0 such that its imaginary part $\Im(p_0)$ is positive and sufficiently large to guarantee that the one of the following chain of inequalities holds for any $t \geq 0$:

$$\begin{aligned} \frac{1}{ab}\big(\Im(p_t) - \Im(w)\big) < \frac{1}{a}\big(\Im(p_t) - \Im(w)\big) + \Im(d) &< \frac{1}{b}\big(\Im(p_t) - \Im(w)\big) + \Im(c) \\ &< \big(\Im(p_t) - \Im(w)\big) + a\Im(d) + \Im(c) \quad \text{if } a > b > 1, \end{aligned}$$

or

$$\begin{aligned} \frac{1}{a}\big(\Im(p_t) - \Im(w)\big) + \Im(d) < \frac{1}{ab}\big(\Im(p_t) - \Im(w)\big) &< \frac{1}{b}\big(\Im(p_t) - \Im(w)\big) + \Im(c) \\ &< \big(\Im(p_t) - \Im(w)\big) + a\Im(d) + \Im(c) \quad \text{if } a < 0 < b, \end{aligned}$$

recalling that $1 < |b| < |a|$. Since $\overline{A(q_t)q_t}$ and $\overline{q_t B(q_t)}$ do not overlap and intersect only at q_t for t large enough, we have that the chain of segments

(33) $$AB(q_t) \to A(q_t) \to q_t \to B(q_t)$$

is embedded in $\mathbb{C}$. Therefore, even in this case, the polygon (23) bounds an immersed disk on its right containing the infinity on the Riemann sphere. □

This completes the proof of the noncoaxial case and indeed the proof of Proposition 5.4. □

5.4.4 An alternative proof Proposition 5.4 is also a consequence of the recent work of Le Fils [2023], as we shall now describe. Indeed, that paper provides necessary and sufficient conditions for a representation $\rho\colon \pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{C})$ to appear as the monodromy of some branched projective structure with prescribed singularities; here we are interested in the case of a single branch point. We shall see that assuming these conditions, the proof of Proposition 5.4 is only a few lines long, however, turning it into a constructive proof like ours would make it considerably longer.

We begin by describing the obstructions for a representation to appear as the holonomy of some branched projective structure with a *single* branch point, following the discussion in [Le Fils 2023, Section 1].

The obstructions Here, we shall assume that $g \geq 2$. It is shown in [Gallo et al. 2000, Corollary 11.2.3] that a representation $\rho\colon \pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{C})$ that arises as the monodromy of a branched projective structure on S_g with n branch points of orders $m_1, m_2, \dots, m_n$ lifts to a representation to $\mathrm{SL}_2(\mathbb{C})$ if and only if $\sum m_i$ is even. Here we are interested in affine representations which are well-known to be liftable to a representation to $\mathrm{SL}_2(\mathbb{C})$. Therefore, any representation $\rho\colon \pi_1(S_g) \to \mathrm{Aff}(\mathbb{C})$ arises as the monodromy of some branched projective structure with one single branch point of magnitude $2(m+1)\pi$ only if m is even. This yields a first obstruction.

A second obstruction arises from the fact that whenever an affine representation ρ arises as the monodromy of a branched projective structure with a single branch points of order, say m, then we need $m \geq 2g-2$. In particular the equality holds if and only if the structure is a branched affine structure. This is stated in [Le Fils 2023, Proposition 6.18].

When the image of ρ is finite of order N, a third obstruction comes from the Riemann–Hurwitz formula. Let $\widehat{S}_g$ be the cover of S_g associated to $\ker(\rho)$. The developing map yields a branched covering $\widehat{S}_g \to \mathbb{C}\mathrm{P}^1$ of degree d and the Riemann–Hurwitz formula implies that

$$N\chi(S_g) = \chi(\widehat{S}_g) = 2d - Nm. \tag{34}$$

As d cannot be smaller than $m+1$, we obtain

$$N\bigl(\chi(S_g) + m\bigr) \geq 2(m+1). \tag{35}$$

Compare with [Le Fils 2023, Section 6.2].

An affine representation ρ is said to be *Euclidean* if $\mathrm{Im}(\rho)$ is a subgroup of $\mathbb{S}^1 \ltimes \mathbb{C} < \mathrm{Aff}(\mathbb{C})$. For Euclidean representations we may define the notion of *volume* as a real number naturally attached to the representation. The volume appears as a further obstruction for realizing a representation ρ as the monodromy of a branched affine structure (and hence projective) on S_g with a branch point of order $2g-2$. However, this obstruction completely vanishes for realizing ρ as the monodromy of some branched projective structure (no longer affine) on S_g with a single branch point of order $m \geq 2g$. See [Le Fils 2023, Obstructions 4 and 5 in Section 1] for further details.

The sixth and last obstruction listed in [Le Fils 2023] concerns $g = 2$ and dihedral (but not affine) representations, and rules out the possibility of a single branch point of order two. We shall consider dihedral representations in the next section Section 6.

Remark The necessity of assuming that $\rho\colon \pi_1(S_{g,1}) \to \mathrm{Aff}(\mathbb{C})$ does not have a finite image of order two is a consequence of our Lemma 5.1. Alternatively, such a necessity can be also deduced by the obstruction (35). In fact, when $N = 2$, inequality (35) is never satisfied for any $g \geq 2$.

Alternative proof of Proposition 5.4 The proof is nothing but a direct consequence of the previous discussion. In [Le Fils 2023, Theorem 1.1] it is showed that an affine representation $\rho\colon \pi_1(S_g) \to \mathrm{Aff}(\mathbb{C})$ arises as the holonomy of some branched projective structure, not necessarily affine, with one single branch point if and only if it satisfies all the obstructions described above. By choosing m even and bigger than

$$\frac{2 + N(2g-2)}{N-2} > 2g - 2, \tag{36}$$

we can see that the conditions of all the obstructions above are met. Hence, there exists a $\mathbb{C}\mathrm{P}^1$-structure on S_g with holonomy ρ and single branch point of order m; deleting this branch point we obtain our desired projective structure on $S_{g,1}$. □

6 Dihedral representations

A representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ is called *dihedral* if there exists a pair of points $F = \{p, q\}$ in $\mathbb{C}\mathrm{P}^1$ which is globally preserved by the representation. Up to conjugation, we may assume $F = \{0, \infty\}$. Notice that a coaxial representation is, in particular, dihedral because the set F is fixed pointwise by the representation. In this section we shall assume $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ is a nontrivial, degenerate and dihedral (but not affine) representation. We recall for the reader's convenience that, according to our Definition 1.1, a dihedral representation $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ is degenerate if the monodromy around each puncture fixes the set $\{0, \infty\}$ pointwise and $\rho(\gamma)$ preserves $\{0, \infty\}$ for any $\gamma \in \Pi$.

The aim of this section is to prove the following:

Proposition 6.1 *Let Π be the fundamental group of a surface $S_{g,k}$ of negative Euler type and let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ be a nontrivial, dihedral (but not affine) and degenerate representation such that at least one puncture has trivial monodromy. Then ρ arises as the monodromy representation of a $\mathbb{C}\mathrm{P}^1$-structure in $\mathcal{P}_g(k)$.*

The following observation is an immediate consequence of our definitions:

Lemma 6.2 *If a representation $\rho\colon \pi_1(S_{0,k}) \to \mathrm{PSL}_2(\mathbb{C})$ is either*

(a) *dihedral and degenerate, or*

(b) *has image which is a cyclic group of finite order,*

then ρ is coaxial and hence affine.

In light of Lemma 6.2(a) above we need to consider only surfaces of positive genus, since the genus $g=0$ case is covered by Proposition 4.18. We shall distinguish two cases according to the number of punctures. In both cases, we shall make use of the following technical result.

Lemma 6.3 *Let Π be the fundamental group of a surface $S_{g,k}$ of genus g and $k \geq 0$ punctures. Let $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ be a dihedral (not affine) and degenerate representation. Then there is a set of handle-generators $\{\alpha_i, \beta_i\}_{1\leq i\leq g}$ such that the restriction of ρ to the handle generated by $\{\alpha_1, \beta_1\}$ is a dihedral representation and the restriction of ρ to the complementary subsurface is coaxial.*

Proof The argument of this proof shares a few similarities with the proof of Lemma 5.10. Let us begin with some observations. Being dihedral, the representation ρ globally preserves a pair of points in $\mathbb{C}\mathrm{P}^1$, which we may assume to be $\{0,\infty\}$. Since ρ is not affine, there is a simple closed curve $\delta \in \Pi$ such that $\rho(\delta)(z) = 1/az$ for some $a \in \mathbb{C}^*$. Since ρ is degenerate (Definition 1.1), $\rho(\gamma)$ is a coaxial transformation of $\mathbb{C}\mathrm{P}^1$ fixing $\{0,\infty\}$ for any simple closed curve γ enclosing a puncture. Thus δ must be a handle-generator.

We start with some set of handle-generators $\{\alpha_i, \beta_i\}_{1\leq i\leq g}$ and, in what follows, we modify this initial choice of generators by mapping class group elements until we get a new set of handle-generators with the desired property. Notice that, if $\rho(\eta)$ globally fixed $\{0,\infty\}$ for any $\eta \in \{\alpha_i, \beta_i\}$ then the representation ρ would be coaxial, hence affine, and this leads to a contradiction. Therefore there exists $i \in \{1,\dots,g\}$ such that the restriction of ρ to the handle $\{\alpha_i, \beta_i\}$ is a dihedral representation. We may even suppose that the points $\{0,\infty\}$ are pointwise fixed by $\rho(\alpha_i)$ and swapped by $\rho(\beta_i)$, ie

$$\rho(\alpha_i) = \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(\beta_i) = \begin{pmatrix} 0 & 1 \\ b_i & 0 \end{pmatrix}.$$

Let $\{\alpha_j, \beta_j\}$ be another pair and suppose that the restriction of ρ to such handle is not coaxial. Up to replacing this pair with $\{\alpha_j\beta_j, \beta_j\}$ or $\{\alpha_j, \alpha_j\beta_j\}$ if needed, we may assume that both $\rho(\alpha_j)$ and $\rho(\beta_j)$ preserve the couple $\{0,\infty\}$ by swapping the points.

The basic modification here is as follows. We replace the handle-generators $\{\alpha_i, \beta_i\}$ with $\{\alpha_i, \beta_i\beta_j\}$ and we replace the handle-generators $\{\alpha_j, \beta_j\}$ with $\{\alpha_i^{-1}\alpha_j\beta_j, \beta_j\}$. This modification is effected by a mapping class element and the handle generated by $\{\alpha_i, \beta_i\beta_j\}$ remains disjoint from the handle generated by $\{\alpha_i^{-1}\alpha_j\beta_j, \beta_j\}$. It remains to show that the restriction of ρ to one of these handles is dihedral and the restriction to the other handle is coaxial. By writing

$$\rho(\alpha_j) = \begin{pmatrix} 0 & 1 \\ a_j & 0 \end{pmatrix} \quad \text{and} \quad \rho(\beta_j) = \begin{pmatrix} 0 & 1 \\ b_j & 0 \end{pmatrix},$$

it is an easy matter now to check that ρ, once restricted to the handle $\{\alpha_i, \beta_i\beta_j\}$, is coaxial, because $\rho(\alpha_i)$ and $\rho(\beta_i)\rho(\beta_j)$ both fix the pair $\{0,\infty\}$ pointwise. In fact, $\rho(\beta_i)$ and $\rho(\beta_j)$ both swap $\{0,\infty\}$ and hence their product fixes them pointwise. It is also easy to check that ρ, once restricted to the handle $\{\alpha_i^{-1}\alpha_j\beta_j, \beta_j\}$, is dihedral, because $\rho(\beta_j)$ swaps the pair $\{0,\infty\}$. We can even observe that the transformation $\rho(\alpha_i)^{-1}\rho(\beta_i)\rho(\beta_j)$ keeps the pair $\{0,\infty\}$ pointwise fixed. An iterative argument will provide a set of handle-generators with the desired property. Notice that this argument does not involve the number of punctures, and hence the claim holds for any $k \geq 0$. □

6.1 Once-punctured surfaces

We begin by considering the case of once-punctured surfaces, ie we assume $g > 0$ and $k = 1$.

6.1.1 The once-punctured torus case In this subsection we prove the following:

Lemma 6.4 *Let $\rho\colon \pi_1(S_{1,1}) \to \mathrm{PSL}_2(\mathbb{C})$ be a nontrivial, dihedral (but not affine) and degenerate representation such that the puncture has trivial monodromy. Then there is a projective structure on $S_{1,1}$ with monodromy ρ.*

Some generalities. Let $\rho\colon \pi_1(S_{1,1}) \to \mathrm{PSL}_2(\mathbb{C})$ be a nontrivial dihedral (but not affine) and degenerate representation. We assume ρ preserves $\{0,\infty\}$. Denote by α and β two handle-generators and let $A=\rho(\alpha)$ and $B=\rho(\beta)$. We now distinguish two possible cases according on how many handle-generators act nontrivially on $\{0,\infty\}$. Without loss of generality, we may assume

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \tag{37}$$

if one handle-generator fixes $\{0,\infty\}$ pointwise, or

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \tag{38}$$

if both handle-generators act nontrivially on $\{0,\infty\}$. Notice that $a, b \in \mathbb{C}^*$. Let $\gamma = [\alpha,\beta]$ be a curve enclosing the puncture and let $\rho(\gamma) = [A, B]$.

Proof Assuming the monodromy of the puncture to be trivial, ie $\rho(\gamma) = \mathrm{Id}$, it is possible to deduce some constraints about the possible values of a and b. A simple computation shows that $a = \pm 1$ if A and B are in the form (37) or $a = \pm b$ if A and B are in the form (38). Once again, given any basepoint $p_0 \in \mathbb{C}^*$, we can define in every case a polygon

$$p_0 \mapsto p_1 \mapsto p_2 \mapsto p_3 \mapsto p_0, \tag{39}$$

where the points p_i are defined as: $p_1 = A(p_0)$, $p_2 = AB(p_0) = BA(p_0)$ and, finally, $p_3 = B(p_0)$. The polygon bounds a possibly self-intersecting and possibly degenerate quadrilateral $\mathcal{Q}$ on the complex plane. As already done before, we shall denote the directed edges as follows: $e_1 = \overline{p_1 p_2}$, $e_2 = \overline{p_0 p_1}$, $e_3 = \overline{p_0 p_3}$ and, finally, $e_4 = \overline{p_3 p_2}$. The edges of this polygon are related by the maps A and B as follows: $A(e_3) = e_1$ and $B(e_2) = e_4$. Let us now discuss case by case.

Case 1 (A and B are in the form (37)) We begin by observing that $a = 1$ implies $A = I$ and therefore the image of ρ is cyclic of order two. In particular, ρ is coaxial (and hence affine) as observed in Case (b) of Lemma 6.2. As ρ is assumed to be dihedral but not affine, it follows that $a = -1$. Given the matrices

A and B as in (37), we notice that $p_0 = -p_1$ and $p_2 = -p_3$ because $A(z) = -z$. The polygon (39) is self-intersecting and bounds an immersed disk in $\mathbb{C}\mathrm{P}^1$ containing the point ∞. Note that there always exists a good choice of p_0 such that the polygon is not degenerate (but still self-intersecting). This is because the polygon is degenerate whenever $p_0, -p_0 = p_1$ and $p_3 = B(p_0)$ are colinear, and this happens if and only if there is a real $\lambda \neq 0$ such that the equality $p_0^2(1-2\lambda) = b$ holds. It is easy to observe that then p_2 is necessarily colinear to the other three points. Then we proceed as in Propositions 4.11 and 5.3.

Case 2 (A and B are in the form (38)) In this second case we observe that $a = b$ implies $A = B$ and therefore the image of ρ is cyclic of order two. In particular, ρ is coaxial (and hence affine), as observed in Lemma 6.2. As ρ is supposed to be dihedral but not affine, it follows that $a = -b$. Given A and B as in (38), we note that $p_0 = -p_2$ and $p_1 = -p_3$. The basepoint p_0 can be chosen in such a way that the polygon (39) bounds an *embedded* disk in $\mathbb{C}\mathrm{P}^1$ containing the point ∞. More precisely, the polygon (39) is degenerate if and only if the three points $p_0, -p_0 = p_2$ and p_3 are colinear, and this happens whenever they satisfy the same relation as in Case 1 above. It is now easy to see that there is always a $p_0 \in \mathbb{C}$ such that the polygon (39) is nondegenerate. Even in this case we proceed as in Propositions 4.11 and 5.3. □

6.1.2 Higher-genus once-punctured surfaces We then consider the case of once-punctured surfaces, ie we assume $k = 1$. Since we have already handled the case $g = 1$ above, we can assume that $g \geq 2$. Here we shall prove the following:

Lemma 6.5 *Let $g \geq 2$ and let $\rho\colon \pi_1(S_{g,1}) \to \mathrm{PSL}_2(\mathbb{C})$ be a nontrivial, dihedral (but not affine) and degenerate representation such that the puncture has trivial monodromy. Then there is a projective structure on $S_{g,1}$ with monodromy ρ.*

Let $\rho\colon \pi_1(S_{g,1}) \to \mathrm{PSL}_2(\mathbb{C})$ be a nontrivial, dihedral (but not affine) representation such that the puncture has trivial monodromy. We can regard ρ as a representation $\bar{\rho}\colon \pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{C})$ and therefore the basic idea, again, is to realize this latter as the monodromy of a branched projective structure with a single branch point. By deleting such a point, we will get the desired result.

Proof By Lemma 6.3 we can assume that there is a handle (see Definition 4.2) $H \subset S_g$ generated by the pair $\{\alpha_1, \beta_1\}$ such that $\rho|_H$ is dihedral (but not affine), and the restriction of ρ to the complementary subsurface is a coaxial representation. Recall that, up to conjugation, we may assume that ρ globally preserves the pair of points $\{0, \infty\} \subset \mathbb{C}\mathrm{P}^1$. Notice that, since $\rho([\alpha_i, \beta_i]) = I$ for each index $i = 2, \dots, g$, it follows that $\rho([\alpha_1, \beta_1]) = I$. Moreover, we may even assume that $\rho_{|H}(\alpha_1)$ and $\rho_{|H}(\beta_1)$ are in the form (38). In fact, if they were in the form (37), a suitable Dehn twist puts them into the desired form.

Let $\rho_0\colon \pi_1(S_{g-1,1}) \to \mathrm{Aff}(\mathbb{C})$ be the restriction of ρ to the complement of $H \subset S_g$. It is coaxial and the puncture has trivial monodromy by construction. We can regard ρ_0 as a coaxial representation $\bar{\rho}_0\colon \pi_1(S_{g-1}) \to \mathrm{Aff}(\mathbb{C})$ and, by Proposition 5.4 (in fact from the coaxial case of the proof in Section 5.4.2), $\bar{\rho}_0$ is realized as the holonomy of a branched projective structure on S_{g-1} with a unique branch point.

Denote this projective surface by S. From the proof (see Section 5.4.2), this projective surface is in fact obtained by constructing a chain of quadrilaterals $\mathcal{C}$ in $\mathbb{C}\mathrm{P}^1$ that bounds an immersed disk (see, for example, Figures 14 or 15), and then identifying pairs of edges of these quadrilaterals. In particular, this construction defines a set of handle-generators that develop onto the edges of the quadrilaterals, which are embedded arcs in $\mathbb{C}\mathrm{P}^1$. Let γ be such a generator, which in our construction is a simple closed curve on the surface from the branch point to itself, and let $\hat{\gamma} \subset \mathbb{C}\mathrm{P}^1$ be an embedded arc that it develops onto.

Let us now consider the dihedral representation $\rho_{|H}\colon \pi_1(S_{1,1}) \to \mathrm{PSL}_2(\mathbb{C})$. The puncture has trivial holonomy by construction, Lemma 6.4 applies and therefore $\rho_{|H}$ appears as the holonomy of a complex projective structure on a punctured torus. Let Σ denote the projective handle (see Definition 5.11) obtained by filling the puncture with a branched projective chart. From our construction, this projective handle is obtained by identifying sides of a quadrilateral $\mathcal{Q}$ on $\mathbb{C}\mathrm{P}^1$ that bounds an embedded disk because the handle-generators α_1 and β_1 are chosen in such a way that $\rho_{|H}(\alpha_1)$ and $\rho_{|H}(\beta_1)$ are in the form (38). In fact, from the proof of Lemma 6.4, there was plenty of freedom in choosing the quadrilateral $\mathcal{Q}$, namely we could choose any basepoint $p_0 \in \mathbb{C}$ so that $\mathcal{Q}$, as defined by (39), is nondegenerate and indeed embedded. In particular, we can choose a basepoint p_0 such that

- $\hat{\gamma}$ lies in the embedded disk bounded by Q, and
- a vertex of Q is an endpoint of $\hat{\gamma}$.

The desired structure with holonomy ρ is then obtained by grafting the projective handle Σ on S along γ as in Definition 5.12. The resulting surface is homeomorphic to S_g and has a branched projective structure with a unique branch point; recall that grafting in a handle does not change the monodromy of H or its complement. Deleting the branch point we obtain our desired projective structure on $S_{g,1}$ with monodromy ρ. □

Remark Even in this case there is an alternative proof can be derived from the results of [Le Fils 2023]. As in the preceding discussion, regard the representation $\rho\colon \pi_1(S_{g,1}) \to \mathrm{PSL}_2(\mathbb{C})$ with trivial monodromy around the puncture as a representation $\bar{\rho}\colon \pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{C})$. As before, it suffices to realize this latter representation as the monodromy of a branched projective structure with one branch point and thus obtain the desired projective structure on $S_{g,1}$ with monodromy ρ by deleting the branch point. However, according to the main theorem in [Le Fils 2023], every dihedral representation $\pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{C})$ can be realized as the monodromy of such a branched projective structure with a single branch point of order at least three (cf the last obstruction, as mentioned in Section 5.4.4).

6.2 Surfaces with at least two punctures.

We finally consider punctured surfaces $S_{g,k}$ of genus at least one and with at least two punctures, ie $g \geq 1$ and $k \geq 2$. This subsection is devoted to proving Proposition 6.1 for these remaining cases. We start by considering the case of surfaces with exactly two punctures, and the general case with more than two punctures will follow by extending our constructions.

Lemma 6.6 *Let $g \geq 1$ and let $\rho\colon \pi_1(S_{g,2}) \to \mathrm{PSL}_2(\mathbb{C})$ be a nontrivial dihedral (but not affine) and degenerate representation such that at least one puncture has trivial monodromy. Then there is a projective structure on $S_{g,2}$ with monodromy ρ.*

Proof The case of genus $g = 1$, namely of representations $\rho\colon \pi_1(S_{1,2}) \to \mathrm{PSL}_2(\mathbb{C})$, uses the same argument as previously used for Proposition 4.20 for the genus-one case. The main difference here is that the sides are glued by elliptic transformations of order two preserving the pair $\{0, \infty\} \subset \mathbb{C}\mathrm{P}^1$. Note that we can always find infinitely many rays joining the two punctures. The case of $g \geq 2$ is handled using the $g = 1$ case, as we shall now describe.

Let $\rho\colon \pi_1(S_{g,2}) \to \mathrm{PSL}_2(\mathbb{C})$ be a dihedral and degenerate representation. By Lemma 6.3, there is a set of handle-generators $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ such that the restriction of ρ to the handle generated by $\{\alpha_1, \beta_1\}$ is dihedral of the form (37) or form (38) and the restrictions of ρ to each handle generated by $\{\alpha_i, \beta_i\}$ for $2 \leq i \leq g$ is coaxial, fixing $\{0, \infty\}$ pointwise. Assume $\rho(\gamma_1) = \mathrm{Id}$; as a consequence we have that $\rho([\alpha_i, \beta_i]) = \mathrm{Id}$ for all $i \geq 2$, and $\rho([\alpha_1, \beta_1]) = \rho(\gamma_2)$ and is a dilation fixing $\{0, \infty\}$ pointwise.

Let $\rho_0 : \pi_1(S_{g-1,2}) \to \mathrm{PSL}_2(\mathbb{C})$ be the restriction of ρ to the subsurface of $S_{g,2}$ homeomorphic to $S_{g-1,2}$ that contains all the handles with coaxial monodromy and one puncture with trivial monodromy. We notice that ρ_0 is a coaxial representation. Finally, let $\rho_1 : \pi_1(S_{1,2}) \to \mathrm{PSL}_2(\mathbb{C})$ be the restriction of ρ to the complementary subsurface that contains, in particular, the remaining puncture. The representation ρ_1 is a dihedral (but nonaffine) degenerate representation.

We may also assume the representation ρ_0 to be nontrivial. If ρ_0 was trivial then we can apply a proper change of basis $\{\alpha'_2, \beta'_2, \dots, \alpha'_g, \beta'_g\}$ so that the restriction of ρ_0 to any handle $\langle \alpha'_i, \beta'_i \rangle$ is not trivial, and fix the set $\{0, \infty\} \subset \mathbb{C}\mathrm{P}^1$ pointwise. In fact, since ρ is dihedral but not affine, we may assume $\rho(\alpha_1)$ or $\rho(\beta_1)$ to be a dilation (that is, of the form $z \mapsto cz$ for some $c \in \mathbb{C}^*$). Then we can apply the change of basis, handle by handle, as described in Lemma 4.6 in order to get the desired basis.

We start by considering the representation ρ_0. Our Proposition 4.20 applies and therefore ρ_0 can be realized as the monodromy of some complex projective structure (in fact an affine structure) on $S_{g-1,2}$ with two punctures with trivial monodromy. We briefly recall the construction. Let $p \in \mathbb{C}^*$ be a point and suppose p is not a fixed point of $A_i = \rho(\alpha_i)$ or $B_i = \rho(\beta_i)$ for any $i = 1, \dots, g$. For any $i = 2, \dots, g$, define $\mathcal{Q}_i$ to be the quadrilateral based at p whose sides are defined by

$$p \mapsto A_i(p) \mapsto A_i B_i(p) = B_i A_i(p) \mapsto B_i(p) \mapsto p, \tag{40}$$

and define Σ_i the 2-punctured torus obtained by the crosswise identification given by the mappings A_i and B_i, where we subsequently delete the branch point arising from the vertices of the polygon. Choose rays from p to the puncture at infinity in $\mathbb{C} \setminus \mathcal{Q}_i$ for each i, and glue these surfaces along the rays; see Definition 4.16. We thus obtain a surface Σ homeomorphic to $S_{g-1,2}$ equipped with a complex projective structure (in fact, an affine structure) with monodromy ρ_0.

Let us now consider ρ_1. It is dihedral, degenerate and, by construction, at least one puncture has trivial monodromy. By the $g = 1$ case handled at the beginning, ρ_1 can be realized as the monodromy of some complex projective structure on $S_{1,2}$. In fact, given $p \in \mathbb{C}^*$ as above, we proceed as in the genus-one case of Proposition 4.20: We can define an immersed polygonal curve based at p (ie the directed curve L in Proposition 4.20) and then glue the sides of such a polygon by using the mappings A_1, B_1 and $[A_1, B_1]$. The resulting surface Σ' is homeomorphic to $S_{1,2}$ and carries a complex projective structure with holonomy ρ_1. Note that there are arcs between the punctures that develop onto rays in $\mathbb{C}$ going towards the puncture at infinity.

We now glue together these two structures along the rays by using Definition 4.17, as we now describe. Let r any ray in Σ joining the punctures and let r' be any ray joining the two punctures of Σ'. By construction, the ray r develops onto a ray $\overline{r} \subset \mathbb{C}\mathrm{P}^1$ leaving from p. In the same fashion, the ray r' develops onto a ray $\overline{r}' \subset \mathbb{C}\mathrm{P}^1$ leaving from p. Note that these rays may or may not coincide, but they have the same starting point, so Definition 4.17 applies. We glue the surfaces Σ and Σ' along the rays r and r' as in that definition. The resulting surfaces is homeomorphic to $S_{g,2}$ and carries a complex projective structure with monodromy ρ, as desired. □

Corollary 6.7 *Let $g \geq 1$ and $k \geq 3$, and let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ be a nontrivial dihedral (but not affine) and degenerate representation such that at least one puncture has trivial monodromy. Then there is a projective structure on $S_{g,k}$ with monodromy ρ.*

Proof Suppose there are more than two punctures, ie $k > 2$. Let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ be a dihedral (but not affine) degenerate representation. Let $A_1, A_2, \dots, A_k$ be the monodromies of the punctures. Since ρ is a degenerate representation, we can assume without loss of generality that $A_1 = \mathrm{Id}$. Let $\rho_0\colon \pi_1(S_{g,2}) \to \mathrm{PSL}_2(\mathbb{C})$ be the restriction of ρ to the subsurface of $S_{g,k}$ homeomorphic to $S_{g,2}$ that contains one puncture with trivial monodromy. Let $C = \rho_0(\gamma_2)$ be the monodromy of the other puncture of $S_{g,2}$. We observe that $A_2 A_3 \cdots A_k C^{-1} = \mathrm{Id}$. Let $S_{0,k-1}$ be the $(k-1)$-punctured sphere and let δ_i denote a curve enclosing the i^{th} puncture. Similarly, we define $\rho_1\colon \pi_1(S_{0,k-1}) \to \mathrm{PSL}_2(\mathbb{C})$ to be the representation such that $\rho_1(\delta_i) = A_i$ for any $i = 1, \dots, k-2$ and $\rho_1(\delta_{k-1}) = A_k C^{-1}$. Note that the representation ρ_1 is by itself an affine representation.

Let us consider first the representation ρ_0. Our previous Lemma 6.6 applies, and ρ_0 can be realized as the monodromy of some complex projective structure on $S_{g,2}$. Let us now denote by Σ the surface $S_{g,2}$ equipped with such a structure. It follows by construction that there exists arcs between the punctures (in fact infinitely many) that develop onto rays in $\mathbb{C}$. Recall that one of these punctures is an apparent singularity and any neighborhood of it is locally modeled on a punctured disk centered at some point $p \in \mathbb{C}$. Let us fix any such arc $r \subset \Sigma$ joining the punctures and denote by $\overline{r}$ its developed image, which is an infinite ray on $\mathbb{C}$ leaving p towards the infinity.

Let us now consider the affine representation ρ_1. According to the proof of our Proposition 4.18, after an appropriate choice pf a basepoint, ρ_1 appears as the holonomy of some branched affine structure (and hence a branched projective structure) on $S_{0,k-1}$. We denote by Σ' the surface $S_{0,k-1}$ equipped the branched affine structure we obtain by choosing p as the basepoint, where p is the point we saw above. Let r' be an arc from the unique branch point, say q, to the puncture with holonomy $A_k C^{-1}$. This ray develops on a ray $\overline{r}'$ leaving from p. Note that $\overline{r}$ and $\overline{r}'$ are two rays based at p; in particular, they intersect only at p if they do not coincide.

It finally remains to glue together structures Σ and $\Sigma' \setminus \{q\}$, along the rays r and r' defined above, by the gluing construction described in Definition 4.17. The resulting surface, after the gluing, is homeomorphic to $S_{g,k}$ and carries a complex projective structure with holonomy ρ as desired. □

This concludes the construction of the general case and indeed the proof of Proposition 6.1. □

7 Corollaries

7.1 Infinite fibers

Here we provide a proof of Corollary B. Let $\rho\colon \Pi \to \mathrm{PSL}_2(\mathbb{C})$ be a representation that satisfies the requirements of Theorem A, so that there exists a projective structure in $\mathcal{P}_g(k)$ with monodromy ρ. Here, we shall describe how the proof of Theorem A shows that in fact, the set of such projective structures with monodromy ρ is infinite. That is, for any such ρ, the fiber of the monodromy map $\Psi^{-1}(\rho)$ is infinite in cardinality.

For this, we recall the following surgery, well-known in the context of branched projective structures; see for instance [Gallo et al. 2000, Section 12.1] or [Calsamiglia et al. 2014, Definition 2.5].

Definition 7.1 (bubbling) Let S be a surface equipped with a projective structure, and let γ be an embedded arc on S with from one puncture to another, such that the developing image is an embedded arc $\hat{\gamma}$ in $\mathbb{C}\mathrm{P}^1$. We shall call such an arc γ an *admissible arc* for the $\mathbb{C}\mathrm{P}^1$-structure on S. Take a copy of $\mathbb{C}\mathrm{P}^1$ slit along $\hat{\gamma}$, and let $\hat{\gamma}_+$ and $\hat{\gamma}_-$ be the resulting sides of the slit. Cut S along γ, and identify the resulting sides with $\hat{\gamma}_\pm$ so that the resulting surface S' is homeomorphic to S, and acquires a projective structure. The developing map of this new projective structure, when restricted to a fundamental domain, now wraps an additional time around $\mathbb{C}\mathrm{P}^1$; however, the monodromy remains unchanged. Notice that we have already implicitly used this fact in Definition 5.12. Moreover, a computation exactly as in Section 3.4 shows that the resulting projective structure is also in $\mathcal{P}_g(k)$; that is, the Schwarzian derivative of developing map has a pole of order at most two at the punctures.

Indeed, once we have an admissible arc as in the definition above, then we can perform the bubbling operation m times for any m, each time adding a new copy of $\mathbb{C}\mathrm{P}^1$ along γ, thus obtaining infinitely many projective structures with the same monodromy.

It only remains to show that there exist admissible arcs in any of the $\mathbb{C}\mathrm{P}^1$-structures we construct in the course of the proof of Theorem A. If the representation ρ is nondegenerate, then recall from Section 3.3 that the projective structure on a surface S with monodromy ρ is obtained by considering a ρ-equivariant pleated plane $\Psi\colon \widetilde{S} \to \mathbb{H}^3$, and then taking its "shadow" at the conformal boundary at infinity $\partial_\infty \mathbb{H}^3 = \mathbb{C}\mathrm{P}^1$. It follows from that construction that any of the pleating lines of Ψ is the lift of an admissible arc on S; see also [Gupta 2021, Theorem 1.3] and its proof.

For a degenerate representation ρ, note that:

- In the case of the trivial representation handled in Lemma 4.4, either $g = 0$, in which case any arc between punctures is admissible, or else $g > 0$, in which case the projective structure is obtained by taking a branched cover of $\mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$. Since we can obtain infinitely many $\mathbb{C}\mathrm{P}^1$-structures on the latter by bubbling along any arc between the three punctures, their pullbacks under the same topological branched cover defines an infinite set of points in the fiber, as desired.
- In all remaining constructions in Sections 4–6, there is a handle-generator that develops onto an edge of a polygonal curve in $\mathbb{C}\mathrm{P}^1$, and is hence admissible.

This completes the proof of Corollary B. We note that the above argument proves that each nonempty fiber is at least *countably* infinite; however, as noted at the end of Section 1, there are representations with connected (and hence uncountably infinite) fibers.

7.2 Spherical cone-metrics

Here we provide a proof of Corollary D. Since a spherical cone-metric is also a $\mathbb{C}\mathrm{P}^1$-structure on the punctured surface obtained by deleting the cone points, the "only if" direction is an immediate consequence of Theorem 2.5, and Lemmata 4.3 and 5.1. Namely, it follows from these results that the holonomy of such a structure satisfies conditions (i) and (ii) of Theorem A. In what follows, we shall prove the "if" direction by handling the cases of nondegenerate and degenerate holonomy separately.

Let $\rho\colon \Pi \to \mathrm{SO}(3,\mathbb{R})$ be a nondegenerate representation. Note that this can be thought of as a representation into $\mathrm{PSL}_2(\mathbb{C})$ that is unitary. In particular, note that the monodromy around any puncture is either elliptic or the identity element. By Proposition 3.2 one can construct a $\mathbb{C}\mathrm{P}^1$-structure P on $S_{g,k}$ with monodromy ρ. By virtue of the holonomy lying in the isometry group of the round metric on $\mathbb{C}\mathrm{P}^1$, the punctured surface acquires a spherical metric. It only remains to verify that the punctures are cone points (or regular points if the cone-angle is 2π). This is a consequence of our construction in Section 3.3; see also [Gupta 2021, Section 3]. In what follows we describe briefly how the developing map for P extends to each puncture as a branch point.

Consider the ρ-equivariant pleated plane Ψ in $\mathbb{H}^3$ (see Section 3.2); recall that the image of Ψ comprises totally geodesic ideal triangles with vertices in the image of the framing map $\beta\colon F_\infty \to \mathbb{C}\mathrm{P}^1$. The edges of these totally geodesic ideal triangles form an equivariant collection of *pleating lines*, which are geodesic lines, each with a weight in $(0, 2\pi)$ which equals the dihedral angle between the two adjacent ideal triangles adjacent at the pleating line.

Let $\overline{p}$ be a puncture on $S_{g,k}$. Since the monodromy around $\overline{p}$ is elliptic, from [Gupta 2021, Lemma 3.2] it follows that up to the equivariance, there will be finitely many pleating lines incident on any lift $p \in F_\infty$, and the sum of their weights will be positive. Interpreting this in terms of our construction of P, this implies that, in the language of Section 3.4, the "total bending angle" α around $\beta(p)$ is positive. The developing map f of the projective structure P then takes a neighborhood of the puncture into the portion of a lune L_α in $\mathbb{C}\mathrm{P}^1$ that lies in a neighborhood of one of its endpoints $\beta(p)$. The developing map can thus be extended to p by mapping it to $\beta(p)$; as explained in Section 3.4, in a conformal coordinate w on the surface in a neighborhood of the puncture $\overline{p}$, if we take $\beta(p) = 0 \in \mathbb{C}\mathrm{P}^1$ the developing map has the form $w \mapsto w^{\alpha/2\pi}$. This differs slightly from the map $\widetilde{f}$ in Section 3.4, since there we took $\beta(p) = \infty \in \mathbb{C}\mathrm{P}^1$. The puncture $\overline{p}$ is thus a cone point of angle α (and a regular point if $\alpha = 2\pi$), as desired.

Now let $\rho\colon \Pi \to \mathrm{SO}(3, \mathbb{R})$ be a degenerate representation satisfying condition (ii) of Theorem A. The constructions of Section 4 apply to produce a $\mathbb{C}\mathrm{P}^1$-structure on $S_{g,k}$ with monodromy ρ. Away from the punctures, the charts to $\mathbb{C}\mathrm{P}^1$ for this projective structure can be considered as charts to the round sphere $\mathbb{S}^2$. So as observed above, since the monodromy of any curve is an element of $\mathrm{SO}(3, \mathbb{R})$, ie an isometry of $\mathbb{S}^2$, the pullback of the spherical metric defines a spherical metric on the punctured surface. The key observation is that our constructions in Section 4 always produce projective structures where the punctures are cone points. Indeed, a puncture that is an apparent singularity is necessarily a branch point (ie with cone-angle an integer-multiple of 2π) or a regular point (when the cone-angle is exactly 2π). A puncture with nontrivial monodromy around it, say an elliptic rotation of angle α, has a cone-angle $\alpha + 2\pi n$ for some integer $n \geq 0$. In particular, the developing map always extends to the puncture and has the form $z \mapsto z^{\alpha/2\pi}$ in a coordinate disk centered at the puncture. This defines a spherical cone-metric on $S_{g,k}$ with monodromy ρ and cone points at the punctures, as desired.

7.3 Branched projective structures

We finally provide a proof of Corollary E. Our main Theorem A already covers all representations except those that are "exceptional" in the following sense.

Definition 7.2 An exceptional representation $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ is necessarily degenerate and satisfies one of the following additional conditions:

- ρ does not have any apparent singularity (in the sense of Definition 1.2), or
- ρ is trivial when $g > 0$ and $k = 1$ or 2, or
- ρ has an apparent singularity, but the image of ρ is a group of order two, when $g > 0$ and $k = 1$.

Our proof of Corollary E is then an immediate consequence of the following lemmata, that deal with each of these possibilities.

Lemma 7.3 *Let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ be a degenerate representation without apparent singularity. Then ρ arises as the monodromy of a branched $\mathbb{C}\mathrm{P}^1$-structure on $S_{g,k}$ with a single branch point.*

Proof Let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ be a degenerate representation without any apparent singularity. Note that ρ cannot be trivial. Then we may regard ρ as a representation $\overline{\rho}\colon \pi_1(S_{g,k+1}) \to \mathrm{PSL}_2(\mathbb{C})$ such that the monodromy of the extra puncture is trivial. Notice that $k+1 \geq 2$. Theorem A applies and the representation $\overline{\rho}$ arises as the monodromy of a complex projective structure with one apparent singularity. We eventually fill the apparent singularity with a (necessarily branched) complex projective chart. The resulting structure is therefore a branched projective structure with a single branch point and monodromy ρ. □

Lemma 7.4 *Let $\rho\colon \pi_1(S_{g,1}) \to \mathrm{PSL}_2(\mathbb{C})$ be a degenerate representation with the puncture having trivial monodromy such that $\mathrm{Im}(\rho) \cong \mathbb{Z}_2$. Then ρ arises as the monodromy of a branched $\mathbb{C}\mathrm{P}^1$-structure on $S_{g,1}$ with a single branch point.*

Proof Let $\rho\colon \pi_1(S_{g,1}) \to \mathrm{PSL}_2(\mathbb{C})$ be a degenerate representation such that $\mathrm{Im}(\rho) \cong \mathbb{Z}_2$. Assume the puncture has trivial monodromy. We regard ρ as a representation $\overline{\rho}\colon \pi_1(S_{g,2}) \to \mathrm{PSL}_2(\mathbb{C})$ such that the monodromy of the extra puncture is trivial. Note that both punctures have trivial monodromy. Let us consider first the case $g=1$. Let $e = \overline{pq} \subset \mathbb{C}$ be any segment such that $p, q \notin \{0, \infty\}$. Slit $\mathbb{C}\mathrm{P}^1$ along $e \cup -e$ and denote the resulting sides as $e^{\pm}$ and $-e^{\pm}$. Then glue e^+ with $-e^+$ and e^- with $-e^-$ to obtain a half-translation structure Σ on a torus and two branch points of magnitude 4π. By removing one of them we obtain a branched projective structure on $S_{1,1}$ with monodromy ρ. Assume now that $g \geq 2$. By construction, we can always find a geodesic segment r joining the two branch points on Σ. Suppose $\Sigma_1, \dots, \Sigma_g$ are g copies of Σ. For any $i = 1, \dots, g$, we slit Σ_i along r_i and denote the resulting segments r_i^+ and r_i^-. We then glue the Σ_i together by identifying r_i^- with r_{i+1}^+. The resulting surface is homeomorphic to S_g and carries a branched projective structure with two branch points, each one of magnitude $4g\pi$. By removing one of them we obtain a branched projective structure on $S_{g,1}$ with a single branch point and the desired monodromy. □

Lemma 7.5 *Let $k = 1, 2$ and let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ be the trivial representation. Then ρ arises as the monodromy of a branched $\mathbb{C}\mathrm{P}^1$-structure on $S_{g,k}$ with a single branch point if $k = 2$ or two branch points if $k = 1$.*

Proof Let $k = 1, 2$ and let $\rho\colon \pi_1(S_{g,k}) \to \mathrm{PSL}_2(\mathbb{C})$ be the trivial representation. We can regard ρ as the trivial representation $\overline{\rho}\colon \pi_1(S_{g,3}) \to \mathrm{PSL}_2(\mathbb{C})$. Lemma 4.4 applies and hence $\overline{\rho}$ appears as the monodromy of a complex projective structure on $S_{g,3}$. We eventually fill one or two punctures with a (necessarily) branched projective chart depending on whether $k = 2$ or $k = 1$, respectively. In both cases, we obtain a branched projective structure on $S_{g,k}$ for $k = 1, 2$, with trivial monodromy. □

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Dipartimento di Matematica e Applicazioni U5, Università degli Studi di Milano-Bicocca
Milan, Italy

Department of Mathematics, Indian Institute of Science
Bangalore, India

gianluca.faraco@unimib.it, subhojoy@iisc.ac.in

Proposed: Anna Wienhard
Seconded: Benson Farb, David Fisher

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