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Algebraic *K*-theory of elliptic cohomology

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We calculate the mod-(p, v_1, v_2) homotopy $V(2)_*\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ of the topological cyclic homology of the truncated Brown–Peterson spectrum $\mathrm{BP}\langle 2\rangle$, at all primes $p \geq 7$, and show that it is a finitely generated and free $\mathbb{F}_p[v_3]$-module on $12p+4$ generators in explicit degrees within the range $-1 \leq * \leq 2p^3+2p^2+2p-3$. At these primes $\mathrm{BP}\langle 2\rangle$ is a form of elliptic cohomology, and our result also determines the mod-(p, v_1, v_2) homotopy of its algebraic K-theory. Our computation is the first that exhibits chromatic redshift from pure v_2-periodicity to pure v_3-periodicity in a precise quantitative manner.

19D50, 19D55, 55P43, 55Q51; 55N20, 55N34, 55N91, 55Q10, 55T25

1 Introduction

Let p be a prime, let $V(n)$ denote a Smith–Toda complex with $\mathrm{BP}_*V(n) = \mathrm{BP}_*/(p, \dots, v_n)$, and let $\mathrm{BP}\langle n\rangle$ with $\pi_*\mathrm{BP}\langle n\rangle = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ denote a truncated Brown–Peterson spectrum equipped with the

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E_3 BP-algebra structure of Hahn and Wilson [2022, Theorem A]. Let $P(x) = \mathbb{F}_p[x]$ and $E(x)$ denote the polynomial and exterior $\mathbb{F}_p$-algebras on a generator x, and let $\mathbb{F}_p\{x\}$ denote the $\mathbb{F}_p$-module generated by x.

We confirm the quantitative form of the chromatic redshift conjecture of [Rognes 2000, page 8] in the case of $\mathrm{BP}\langle 2\rangle$ at $p \geq 7$, showing that $V(2)_* \operatorname{TC}(\mathrm{BP}\langle 2\rangle)$ is finitely generated and free as a $P(v_3)$-module. Hence the topological cyclic homology functor takes the "pure fp-type 2" ring spectrum $\mathrm{BP}\langle 2\rangle$ with $V(1)_*\mathrm{BP}\langle 2\rangle$ finitely generated and free as a $P(v_2)$-module to a "pure fp-type 3" ring spectrum $\operatorname{TC}(\mathrm{BP}\langle 2\rangle)$ with $V(2)_* \operatorname{TC}(\mathrm{BP}\langle 2\rangle)$ finitely generated and free as a $P(v_3)$-module, dilating the wavelength of periodicity[1] from $|v_2| = 2p^2 - 2$ to $|v_3| = 2p^3 - 2$.

Theorem 1.1 *Let $p \geq 7$. There is a preferred isomorphism*

$$\begin{aligned} V(2)_* \operatorname{TC}(\mathrm{BP}\langle 2\rangle) \cong P(v_3) &\otimes E(\partial, \lambda_1, \lambda_2, \lambda_3) \\ &\oplus P(v_3) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{\Xi_{1,d} \mid 0 < d < p\} \\ &\oplus P(v_3) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{\Xi_{2,d} \mid 0 < d < p\} \\ &\oplus P(v_3) \otimes E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p\{\Xi_{3,d} \mid 0 < d < p\} \end{aligned}$$

of $P(v_3) \otimes E(\lambda_1, \lambda_2, \lambda_3)$-modules. This is a finitely generated and free $P(v_3)$-module on $12p+4$ explicit generators in degrees $-1 \leq * \leq 2p^3 + 2p^2 + 2p - 3$.

The close relation between algebraic K-theory and topological cyclic homology for p-complete ring spectra leads to the following application; cf Theorem 12.20.

Theorem 1.2 *Let $p \geq 7$. There is an exact sequence of $P(v_3) \otimes E(\lambda_1, \lambda_2, \lambda_3)$-modules*

$$0 \to \Sigma^{-2}\mathbb{F}_p\{\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_1\bar{\tau}_2\} \to V(2)_* K(\mathrm{BP}\langle 2\rangle_p) \xrightarrow{\mathrm{trc}_*} V(2)_* \operatorname{TC}(\mathrm{BP}\langle 2\rangle) \to \Sigma^{-1}\mathbb{F}_p\{1\} \to 0$$

with $|\bar{\tau}_i| = 2p^i - 1$. The localization homomorphism

$$V(2)_* K(\mathrm{BP}\langle 2\rangle_p) \to v_3^{-1} V(2)_* K(\mathrm{BP}\langle 2\rangle_p)$$

is an isomorphism in degrees $* \geq 2p^2 + 2p$, and the target is a finitely generated and free $P(v_3^{\pm 1})$-module on $12p + 4$ generators.

The proven Lichtenbaum–Quillen conjecture for $K(\mathbb{Z}_{(p)})$ and $K(\mathbb{Z}_p)$ also lets us pass from the p-complete version to the p-local version of $\mathrm{BP}\langle 2\rangle$; cf Theorem 12.21.

Theorem 1.3 *Let $p \geq 7$. The p-completion map induces a $(2p^2+2p-2)$-coconnected homomorphism*

$$V(2)_* K(\mathrm{BP}\langle 2\rangle) \xrightarrow{\kappa_*} V(2)_* K(\mathrm{BP}\langle 2\rangle_p).$$

The localization homomorphism

$$V(2)_* K(\mathrm{BP}\langle 2\rangle) \to v_3^{-1} V(2)_* K(\mathrm{BP}\langle 2\rangle)$$

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[1] See also Remark 1.9 regarding the recent resolution by Burklund, Schlank and Yuan [Burklund et al. 2022] of the (weaker) qualitative form of the redshift conjecture, in the case of E_∞ ring spectra.

is an isomorphism in degrees $* \geq 2p^2 + 2p$*, and the target is a finitely generated and free* $P(v_3^{\pm 1})$*-module on* $12p + 4$ *generators.*

Remark 1.4 An alternative title for this paper could be *Topological cyclic homology modulo* p*,* v_1 *and* v_2 *of the second truncated Brown–Peterson spectrum*. In earlier work [Ausoni and Rognes 2002] we referred to the calculation of $V(1)_* \operatorname{TC}(\mathrm{BP}\langle 1\rangle)$ as (an essential step toward) a calculation of the "algebraic K-theory of topological K-theory". The relation between $\mathrm{BP}\langle 1\rangle$ and topological K-theory is analogous to that between $\mathrm{BP}\langle 2\rangle$ and elliptic cohomology, so we hope the reader grants us the poetic license presumed by our choice of title.

The v_1- and v_2-periodic families in $\pi_* V(0)$ and $\pi_* V(1)$, respectively, are related to the well-known α-family visible to topological K-theory and the fairly well understood β-family visible to elliptic cohomology. The v_3-periodic families emerging from our calculation are related to the third family of Greek letter elements, the γ-family, which is less well understood, and for which there is currently no known detecting cohomology theory with a geometric interpretation of the cohomology classes. Our result suggests that algebraic K-theory of elliptic cohomology may be such a detecting cohomology theory.

We now explain Theorem 1.1 in more detail. For each E_3 ring spectrum B we have maps of E_2 ring spectra

$$S \to K(B) \xrightarrow{\mathrm{trc}} \operatorname{TC}(B) \xrightarrow{\pi} \operatorname{THH}(B)^{h\mathbb{T}} \to \operatorname{THH}(B)$$

from the sphere spectrum to the topological Hochschild homology $\operatorname{THH}(B)$ of B, via its algebraic K-theory $K(B)$, topological cyclic homology $\operatorname{TC}(B)$ and the $\mathbb{T}$-homotopy fixed points of $\operatorname{THH}(B)$. For $p \geq 7$, the Smith–Toda spectrum $V(2)$ exists as a homotopy commutative and associative ring spectrum, with a periodic class $v_3 \in \pi_{2p^3-2} V(2)$. In Section 3 we recall that

$$V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle) = E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu),$$

with $|\lambda_i| = 2p^i - 1$ for $i \in \{1, 2, 3\}$ and $|\mu| = 2p^3$. In Sections 5 and 6 we use E_2 ring spectrum power operations to show that the THH-classes λ_i lift to K-theory classes $\lambda_i^K \in V(2)_* K(\mathrm{BP}\langle 2\rangle)$, with $\operatorname{tr}(\lambda_i^K) = \lambda_i$. We also write λ_i for their images in $V(2)_* \operatorname{TC}(\mathrm{BP}\langle 2\rangle)$ and $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$. In Sections 8–11 we determine the structure of the $\mathbb{T}$-homotopy fixed point spectral sequence

$$E^2(\mathbb{T}) = H^{-*}(\mathbb{T}, V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) = P(t) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu) \Rightarrow V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}.$$

The image of v_3 in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$ is detected by $t\mu$. The homotopy classes

$$\Xi_{i,d} \in V(2)_* \operatorname{TC}(\mathrm{BP}\langle 2\rangle)$$

for $i \in \{1, 2, 3\}$ and $0 < d < p$ are constructed in Section 12 so that

$$\pi(\Xi_{i,d}) = \sum_{n=0}^{\infty} \xi_{i+3n,d}$$

in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$. In this convergent series, each $\xi_{k,d}$ is a specific $V(2)$-homotopy element detected by a class

$$x_{k,d} = t^{\frac{d}{p} r(k)} \lambda_{[k]} \mu^{\frac{d}{p} r(k-3)} \in E^\infty(\mathbb{T}).$$

Here $[k] \in \{1,2,3\}$ satisfies $k \equiv [k] \bmod 3$, and $r(k) = p^k + p^{k-3} + \cdots + p^{[k]}$ for $k \geq 1$. In particular, both

$$\pi(\Xi_{i,d}) \text{ and } \xi_{i,d} \in \{t^{dp^{i-1}}\lambda_i\}$$

are detected by $t^{dp^{i-1}}\lambda_i$ in $E^\infty(\mathbb{T})$, for $i \in \{1,2,3\}$. Letting ∂ denote the generator of $V(2)_{-1}\operatorname{TC}(\mathrm{BP}\langle 2\rangle)$, and noting that $\lambda_i \cdot \Xi_{i,d} = 0$ for each i and d, this concludes our specification of the notation in Theorem 1.1, which appears as Theorem 12.17 in the body of the text. One way to summarize the grading of the module generators is to say that the Poincaré series of $V(3)_*\operatorname{TC}(\mathrm{BP}\langle 2\rangle)$ is

$$\begin{aligned}(1+x^{-1})&(1+x^{2p-1})(1+x^{2p^2-1})(1+x^{2p^3-1})\\ &+(1+x^{2p^2-1})(1+x^{2p^3-1})(x+x^3+\cdots+x^{2p-3})\\ &+(1+x^{2p-1})(1+x^{2p^3-1})(x^{2p-1}+x^{4p-1}+\cdots+x^{2p^2-2p-1})\\ &+(1+x^{2p-1})(1+x^{2p^2-1})(x^{2p^2-1}+x^{4p^2-1}+\cdots+x^{2p^3-2p^2-1}).\end{aligned}$$

Remark 1.5 The seminal calculation in this field was made by Bökstedt and Madsen [1994; 1995]. For the Eilenberg–MacLane spectrum $\mathrm{BP}\langle 0\rangle = H\mathbb{Z}_{(p)}$ at $p \geq 3$ they established an isomorphism

$$\begin{aligned}V(0)_*\operatorname{TC}(\mathbb{Z}_{(p)}) \cong{}& P(v_1) \otimes E(\partial, \lambda_1)\\ &\oplus P(v_1) \otimes \mathbb{F}_p\{\Xi_{1,d} \mid 0 < d < p\}\end{aligned}$$

of free $P(v_1)$-modules of rank $p+3$, where $\Xi_{1,d}$ is detected by $t^d\lambda_1$. The (then unproven) Lichtenbaum–Quillen conjecture for $K(\mathbb{Q}_p)$ could be deduced from this, showing that the natural homomorphism

$$V(0)_*K(\mathbb{Q}_p) \to V(0)_*K(\overline{\mathbb{Q}}_p)^{hG_{\mathbb{Q}_p}}$$

is 0-coconnected, where $G_{\mathbb{Q}_p} = \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the absolute Galois group. In particular, the $P(v_1)$-module generators of $V(0)_*\operatorname{TC}(\mathbb{Z}_{(p)})$ correspond in a precise manner to a basis for the Galois cohomology groups in the descent spectral sequence

$$E^2_{-s,t} = H^s_{\mathrm{Gal}}(\mathbb{Q}_p; \mathbb{F}_p(t/2)) \Rightarrow V(0)_{-s+t}K(\overline{\mathbb{Q}}_p)^{hG_{\mathbb{Q}_p}}.$$

The fact that $V(0)_*\operatorname{TC}(\mathbb{Z}_{(p)})$ is $P(v_1)$-torsion free is thus a reflection of Suslin's theorem [1984] that $V(0)_*K(\overline{\mathbb{Q}}_p) \cong V(0)_*ku = \mathbb{F}_p[u]$ is $P(v_1)$-torsion free, and the finite generation and grading of $V(0)_*\operatorname{TC}(\mathbb{Z}_{(p)})$ corresponds to precise information about the Galois (or motivic) cohomology of $\mathbb{Q}_p$.

For the Adams summand $\mathrm{BP}\langle 1\rangle = \ell$ of $ku_{(p)}$ at $p \geq 5$, Ausoni and Rognes [2002] thereafter obtained an isomorphism

$$\begin{aligned}V(1)_*\operatorname{TC}(\ell) \cong{}& P(v_2) \otimes E(\partial, \lambda_1, \lambda_2)\\ &\oplus P(v_2) \otimes E(\lambda_2) \otimes \mathbb{F}_p\{\Xi_{1,d} \mid 0 < d < p\}\\ &\oplus P(v_2) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{\Xi_{2,d} \mid 0 < d < p\}\end{aligned}$$

of free $P(v_2)$-modules of rank $4p+4$, where $\Xi_{1,d}$ is detected by $t^d\lambda_1$ and $\Xi_{2,d}$ is detected by $t^{dp}\lambda_2$. Moreover, Ausoni [2010] proceeded to calculate $V(1)_*\operatorname{TC}(ku)$, and showed [Ausoni 2005] that

$$V(1)_*K(\ell_p) \to V(1)_*K(ku_p)^{h\Delta}$$

is an isomorphism. Rognes [2014, Section 5] viewed this as computational evidence for the existence of a descent spectral sequence, converging to $V(1)_*K(\ell_p)$, from a form of motivic cohomology defined for E_∞ ring spectra such as ℓ_p. The fact that $V(1)_*\,\mathrm{TC}(\ell)$ is $P(v_2)$-torsion free would then reflect an analog of Suslin's theorem, and the finite generation and grading of $V(1)_*\,\mathrm{TC}(\ell)$ would correspond to specific information about this spectrally defined motivic cohomology.[2]

Our present conclusions about $V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ and $V(2)_*K(\mathrm{BP}\langle 2\rangle_p)$ as $P(v_3)$-modules continue this pattern, and further suggest the existence of a descent spectral sequence from a motivic cohomology defined for less commutative ring spectra, such as the E_3 ring spectrum $\mathrm{BP}\langle 2\rangle_p$. If so, Theorem 1.1 provides information about these (at the time of writing, hypothetical) motivic cohomology groups.

Remark 1.6 Our calculations in $V(2)$-homotopy involve the homotopy element $v_3 \in \pi_{2p^3-2}V(2)$ and its v_2-Bockstein image $i_2 j_2(v_3) \in \pi_{2p^3-2p^2-1}V(2)$, closely related to the first element $\gamma_1 \in \pi_{2p^3-2p^2-2p-1}S$ in the third Greek letter family. To make a similar computation of $V(3)_*\,\mathrm{TC}(\mathrm{BP}\langle 3\rangle)$ as a $P(v_4)$-module would require knowing the existence of a homotopy element $v_4 \in \pi_{2p^4-2}V(3)$, mapping to the class with the same name in $\mathrm{BP}_*V(3) = \mathrm{BP}_*/(p, \dots, v_3)$. The existence of v_4 is presently not known for any prime p; cf [Ravenel 2004, Section 5.6 and (5.6.13)]. Conceivably, a calculation could be made of $V_*\,\mathrm{TC}(\mathrm{BP}\langle 3\rangle)$ as a $P(w)$-module for another type 4 finite ring spectrum V, with v_4 self-map $w\colon \Sigma^d V \to V$. Something similar was carried out for the Eilenberg–MacLane spectrum $\mathrm{BP}\langle 0\rangle = H\mathbb{Z}_{(2)}$ at $p = 2$ in [Rognes 1999], calculating $(S/2)_*\,\mathrm{TC}(\mathbb{Z}_{(2)})$ and $(S/4)_*\,\mathrm{TC}(\mathbb{Z}_{(2)})$ in tandem.

Remark 1.7 Let $T(3) = v_3^{-1}V(2)$ be the telescopic localization of the type 3 complex $V(2)$, and let $V(3)$ be the mapping cone of $v_3\colon \Sigma^{2p^2-2}V(2) \to V(2)$. The three theorems above imply that

$$T(3)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle) \cong T(3)_*K(\mathrm{BP}\langle 2\rangle_p) \cong T(3)_*K(\mathrm{BP}\langle 2\rangle)$$

are all nontrivial $P(v_3^{\pm 1})$-modules, so that the Bousfield $T(3)$-localizations

$$L_{T(3)}\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle) \simeq L_{T(3)}K(\mathrm{BP}\langle 2\rangle_p) \simeq L_{T(3)}K(\mathrm{BP}\langle 2\rangle)$$

are all nontrivial spectra. Moreover, the graded abelian groups

$$V(3)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle) \leftarrow V(3)_*K(\mathrm{BP}\langle 2\rangle_p) \leftarrow V(3)_*K(\mathrm{BP}\langle 2\rangle)$$

are all finite, so

$$\mathrm{TC}(\mathrm{BP}\langle 2\rangle)_p \leftarrow K(\mathrm{BP}\langle 2\rangle_p)_p \leftarrow K(\mathrm{BP}\langle 2\rangle)_p$$

are all of fp-type 3 in the sense of [Mahowald and Rezk 1999]. These qualitative statements confirm a weaker form of the chromatic redshift conjecture for $\mathrm{BP}\langle 2\rangle$, roughly as formulated in [Ausoni and Rognes 2008, Conjecture 1.3], but do not contain the information that $V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ is free over $P(v_3)$, ie

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[2] See also Remark 1.9 regarding the recent discovery by Hahn, Raksit and Wilson [Hahn et al. 2022] of such a cohomology theory, in the case of E_∞ ring spectra.

that $\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ is of "pure fp-type 3" in the sense of [Rognes 2000], nor the quantitative information about its precise rank and generating basis.

In groundbreaking work, Hahn and Wilson [2022, Theorem B] confirmed the qualitative form of the chromatic redshift conjecture for all $\mathrm{BP}\langle n\rangle$, at all primes p. However, as outlined in Remark 1.5, we take the view that the precise $P(w)$-module structure of $V_* \mathrm{TC}(\mathrm{BP}\langle n\rangle)$, where V is some type $(n{+}1)$ finite complex with v_{n+1} self-map $w\colon \Sigma^d V \to V$, will be an essential ingredient of an understanding of it and $V_* K(\mathrm{BP}\langle n\rangle_p)$ as being obtained by descent from a form of motivic cohomology for ring spectra.

Remark 1.8 Ausoni and Rognes [2002] had outlined a calculation of $V(n)_* \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ as a $P(v_{n+1})$-module, under the strong hypotheses that $V(n)$ exists as a ring spectrum (with a homotopy element v_{n+1}) and that $\mathrm{BP}\langle n\rangle$ admits an E_∞ ring spectrum structure. As in the case $n=1$, the sketched argument used a homotopy Cartan formula for E_∞ power operations, and was carried out in the range of degrees where the comparison homomorphism $\widehat{\Gamma}_{1*}\colon V(n)_* \mathrm{THH}(\mathrm{BP}\langle n\rangle) \to V(n)_* \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$ is an isomorphism. When $n=2$ and $p \ge 7$, this homomorphism is $(2p^2{+}2p{-}3)$-coconnected, as we show in Theorem 8.1, so that the calculation would determine $V(2)_* \mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ for $* > 2p^2+2p-3$.

There is a $(2p^2{-}2)$-connected map $\mathrm{BP}\langle 2\rangle \to \mathrm{BP}\langle 1\rangle$ inducing a $(2p^2{-}1)$-connected map

$$V(2)_* \mathrm{TC}(\mathrm{BP}\langle 2\rangle) \to V(2)_* \mathrm{TC}(\mathrm{BP}\langle 1\rangle)$$

(cf [Bökstedt and Madsen 1994, Proposition 10.9; Dundas 1997] and Proposition 12.19). Hence the known calculation of $V(1)_* \mathrm{TC}(\mathrm{BP}\langle 1\rangle)$ does account for $V(2)_* \mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ in degrees $* < 2p^2-1$. This leaves a gap in degrees $2p^2-1 \le * \le 2p^2+2p-3$, where the traditional arguments do not determine $V(2)_* \mathrm{TC}(\mathrm{BP}\langle 2\rangle)$. (This is a new phenomenon for $n \ge 2$; there is no such gap for $n \in \{0,1\}$.)

Around the year 2000 it was only known that $\mathrm{BP}\langle n\rangle$ could be realized as an E_1 ring spectrum [Baker and Jeanneret 2002, Corollary 3.5], so the calculations were hypothetical, even for $n=2$ and $p \ge 7$. With the much more recent Hahn–Wilson construction of an E_3 ring structure on $\mathrm{BP}\langle n\rangle$, it has finally become possible to carry out most of the original program, as we show in this paper. The lower order of commutativity has, however, required us to also develop a homotopy Cartan formula for certain E_2 power operations, which we do in Section 5.

The original Bökstedt–Hsiang–Madsen presentation [Bökstedt et al. 1993] of $\mathrm{TC}(B)$ was given in terms of fixed point spectra $\mathrm{THH}(B)^C$ for finite subgroups $C \subset \mathbb{T}$, using the language of genuinely equivariant stable homotopy theory. However, almost all calculations were made using the naively equivariant homotopy fixed points $\mathrm{THH}(B)^{hC}$ and Tate constructions $\mathrm{THH}(B)^{tC}$, and were therefore only known to be valid in the range of degrees where the comparison map $\widehat{\Gamma}_1$ induces an isomorphism.

The new Nikolaus–Scholze presentation [2018] of topological cyclic homology promoted the ingredients that were previously used for calculations into definitions. Hence $\mathrm{TC}(B)$ was redefined in terms of the homotopy fixed points $\mathrm{THH}(B)^{h\mathbb{T}}$ and Tate construction $\mathrm{THH}(B)^{t\mathbb{T}}$, and the key role of the (naively

$\mathbb{T}$-equivariant) map $\widehat{\Gamma}_1$, now called the p-cyclotomic structure map φ_p, was greatly clarified. Moreover, Nikolaus and Scholze proved that the old and new definitions agree when $\mathrm{THH}(B)$ is bounded below, eg for connective B. This means that by carrying out the homotopy fixed point and Tate construction calculations in all degrees, we can now fully calculate $V(2)_* \operatorname{TC}(\mathrm{BP}\langle 2\rangle)$, eliminating the gap of degrees discussed above. We compare the old and new terminologies in Section 4.

Remark 1.9 After the present paper was first posted in preprint form, Hahn, Raksit and Wilson [Hahn et al. 2022] introduced a motivic filtration on $\operatorname{TC}(R)$, for so-called chromatically quasisyntomic E_∞ ring spectra R, whose associated graded realizes the form of motivic cohomology that was predicted to exist in Remark 1.5. This new cohomology theory for E_∞ ring spectra generalizes the syntomic cohomology for quasisyntomic commutative rings introduced by Bhatt, Morrow and Scholze [Bhatt et al. 2019, Section 7.4].

In the same year, Burklund, Schlank and Yuan [Burklund et al. 2022, Theorem E], building on [Yuan 2024, Theorem A], proved that if R is an E_∞ ring spectrum such that $K(n)_* R \neq 0$ and $K(n+1)_* R = 0$, then $K(n+1)_* K(R) \neq 0$. Combined with previous work of Land, Meier, Mathew and Tamme [Land et al. 2024, Corollary B] and Clausen, Mathew, Naumann and Noel [Clausen et al. 2024] on the vanishing of $K(m)_* K(R)$ for $m \geq n+2$, this proves that algebraic K-theory of an E_∞ ring spectrum increments chromatic complexity by precisely one, thus establishing a very general form of qualitative redshift.

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2 Smith–Toda and truncated Brown–Peterson spectra

Let $\mathcal{A}_*$ be the mod-p dual Steenrod algebra, and write $H_* X = H_*(X; \mathbb{F}_p)$ for the mod-p homology of a spectrum X, viewed as an $\mathcal{A}_*$-comodule. Likewise, let $H = H\mathbb{F}_p$ denote the mod-p Eilenberg–MacLane (E_∞ ring) spectrum.

By a Smith–Toda complex $V(n)$ we mean a finite and p-local spectrum with $H_* V(n) = E(\tau_0, \dots, \tau_n) \subset \mathcal{A}_*$. The spectra $V(0) = S \cup_p e^1$, $V(1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$ and $V(2)$ exist for $p \geq 2$, $p \geq 3$ and $p \geq 5$, respectively; see Smith [1970, Section 4] and Toda [1971, Theorem 1.1]. In the stable homotopy category there are unital multiplications $\mu_0 \colon V(0) \wedge V(0) \to V(0)$, $\mu_1 \colon V(1) \wedge V(1) \to V(1)$ and $\mu_2 \colon V(2) \wedge V(2) \to V(2)$ for $p \geq 3$, $p \geq 5$ and $p \geq 7$, respectively; cf [Yanagida and Yosimura 1977, Sections 1.4, 2.4 and 3.3]. These are unique, and therefore commutative. They are also associative, with the exception of μ_0 at $p = 3$. Toda [1971, Theorem 4.4] showed that $V(3)$ exists for $p \geq 7$ and admits a unital multiplication for $p \geq 11$. The spectra $V(n)$ for $n \geq 4$ are not known to exist at any prime p; cf [Ravenel 2004, (5.6.13)]. We use the following notation for some of the resulting homotopy cofiber sequences:

$$S \xrightarrow{p} S \xrightarrow{i_0} V(0) \xrightarrow{j_0} \Sigma S, \tag{2-1}$$

$$\Sigma^{2p-2} V(0) \xrightarrow{v_1} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1} V(0), \tag{2-2}$$

$$\Sigma^{2p^2-2} V(1) \xrightarrow{v_2} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{2p^2-1} V(1), \tag{2-3}$$

$$\Sigma^{2p^3-2} V(2) \xrightarrow{v_3} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{2p^3-1} V(2). \tag{2-4}$$

The unital multiplications on $V(0)$, $V(1)$ and $V(2)$ are also regular, in the sense that the respective Bockstein operators $i_0 j_0 \colon V(0) \to \Sigma V(0)$, $i_1 j_1 \colon V(1) \to \Sigma^{2p-1} V(1)$ and $i_2 j_2 \colon V(2) \to \Sigma^{2p^2-1} V(2)$ act as derivations; see [Araki and Toda 1965, Theorem 5.9; Yosimura 1977, Propositions 1.1 and 1.2].

The complex cobordism spectrum MU is a prototypical E_∞ ring spectrum. Basterra and Mandell [2013, Theorem 1.1] proved that the p-local Brown–Peterson spectrum BP is a retract up to homotopy of $\mathrm{MU}_{(p)}$ in the category of E_4 ring spectra, and that the E_4 ring structure on BP is unique up to equivalence. By an n^{th} truncated Brown–Peterson spectrum $\mathrm{BP}\langle n\rangle$ we mean a complex orientable p-local ring spectrum such that the composite

$$\mathbb{Z}_{(p)}[v_1, \dots, v_n] \subset \pi_*\mathrm{BP} \to \pi_* \mathrm{MU}_{(p)} \to \pi_*\mathrm{BP}\langle n\rangle$$

is an isomorphism, following [Lawson and Naumann 2014, Definition 4.1]. It follows, as in [Lawson and Naumann 2014, Theorem 4.4], that $H_*\mathrm{BP}\langle n\rangle = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k > n)$ as a subalgebra of the dual Steenrod algebra. According to recent work by Hahn and Wilson [2022], there exist towers

$$\cdots \to \mathrm{BP}\langle n+1\rangle \to \mathrm{BP}\langle n\rangle \to \cdots \to \mathrm{BP}\langle 0\rangle = H\mathbb{Z}_{(p)}$$

of E_3 BP-algebra spectra, for all p, where each $\mathrm{BP}\langle n\rangle$ is an n^{th} truncated Brown–Peterson spectrum. Hence THH(BP) is an E_3 ring spectrum with cyclotomic structure, in the sense to be recalled in Section 4, and there are towers

$$\cdots \to \mathrm{THH}(\mathrm{BP}\langle n+1\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \cdots \to \mathrm{THH}(\mathbb{Z}_{(p)})$$

of E_2 THH(BP)-algebra spectra with cyclotomic structure. The availability of these $\mathbb{T}$-equivariant ring spectrum structures is an essential prerequisite for our calculations.

Chadwick and Mandell [2015, Corollary 1.3] showed that the Quillen map $\mathrm{MU}_{(p)} \to \mathrm{BP}$ is an E_2 ring map, and it follows from [Basterra and Mandell 2013] that it exhibits BP as a retract up to homotopy of $\mathrm{MU}_{(p)}$ in the category of E_2 ring spectra. It is not known whether the Basterra–Mandell and Quillen/Chadwick–Mandell E_2 ring spectrum splittings can be chosen to agree, but the induced splittings of π_* THH(BP) off from π_* $\mathrm{THH}(\mathrm{MU}_{(p)})$, in the category of differential graded algebras, must agree, modulo addition of decomposables and multiplication by p-local units. Hence the calculation in [Rognes 2020, Theorem 5.6] of the σ-operator on π_* THH(BP), induced by the $\mathbb{T}$-action on THH(BP), is valid also for the Basterra–Mandell splitting, up to decomposables and p-local units.

3 Topological Hochschild homology

Let p be an odd prime. We use the conjugate pair of presentations

$$\mathcal{A}_* = P(\xi_k \mid k \geq 1) \otimes E(\tau_k \mid k \geq 0) = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0) = H_* H$$

of the dual Steenrod algebra [Milnor 1958], with $\bar{\xi}_k = \chi(\xi_k)$ in degree $2(p^k - 1)$ and $\bar{\tau}_k = \chi(\tau_k)$ in degree $2p^k - 1$. The Hopf algebra coproduct is given by

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i} \quad \text{and} \quad \psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}.$$

The mod-p homology Bockstein satisfies $\beta(\bar{\tau}_k) = \bar{\xi}_k$. The same formulas give the $\mathcal{A}_*$-coaction ν and Bockstein operation on the subalgebras

$$H_*\mathrm{BP} = P(\bar{\xi}_k \mid k \geq 1) \quad \text{and} \quad H_*\mathrm{BP}\langle n\rangle = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k > n)$$

of $\mathcal{A}_*$. For each E_1 ring spectrum (or S-algebra) B, the topological Hochschild homology $\mathrm{THH}(B)$ has a natural $\mathbb{T}$-action, which induces σ-operators

$$\sigma \colon H_* \,\mathrm{THH}(B) \to H_{*+1} \,\mathrm{THH}(B) \quad \text{and} \quad \sigma \colon \pi_* \,\mathrm{THH}(B) \to \pi_{*+1} \,\mathrm{THH}(B)$$

in homology and homotopy. Since BP and the $\mathrm{BP}\langle n\rangle$ are (at least) E_3 ring spectra, we can make the following homology computations:

Proposition 3.1 [McClure and Staffeldt 1993, Remark 4.3; Angeltveit and Rognes 2005, Theorem 5.12] *There are $\mathcal{A}_*$-comodule algebra isomorphisms*

$$H_* \,\mathrm{THH}(\mathrm{BP}) \cong H_*\mathrm{BP} \otimes E(\sigma\bar{\xi}_k \mid k \geq 1)$$

and

$$H_* \,\mathrm{THH}(\mathrm{BP}\langle n\rangle) \cong H_*\mathrm{BP}\langle n\rangle \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_{n+1}) \otimes P(\sigma\bar{\tau}_{n+1}).$$

Each class $\sigma\bar{\xi}_k$ is $\mathcal{A}_*$-comodule primitive, while $\nu(\sigma\bar{\tau}_{n+1}) = 1 \otimes \sigma\bar{\tau}_{n+1} + \bar{\tau}_0 \otimes \sigma\bar{\xi}_{n+1}$.

Passing to homotopy, recall that $\pi_*\mathrm{BP} = \mathbb{Z}_{(p)}[v_n \mid n \geq 1]$ with $|v_n| = 2p^n - 2$. To be definite, we take the v_n to be the Hazewinkel generators.

Proposition 3.2 [McClure and Staffeldt 1993, Remark 4.3; Rognes 2020, Proposition 4.6, Theorem 5.6] *There is an algebra isomorphism*

$$\pi_* \,\mathrm{THH}(\mathrm{BP}) \cong \pi_*\mathrm{BP} \otimes E(\lambda_n \mid n \geq 1),$$

where λ_n has degree $|\lambda_n| = 2p^n - 1$ and (mod-p) Hurewicz image $h(\lambda_n) = \sigma\bar{\xi}_n$. Here $\sigma(\lambda_n) = 0$ for each n. The first few $\sigma(v_n)$ satisfy

$$\sigma(v_1) = p\lambda_1, \quad \sigma(v_2) = p\lambda_2 - (p+1)v_1^p\lambda_1,$$
$$\sigma(v_3) = p\lambda_3 - (pv_1v_2^{p-1} + v_1^{p^2})\lambda_2 - (v_2^p - (p+1)v_1^{p+1}v_2^{p-1} + p^2v_1^{p^2-1}v_2 + pv_1^{p^2+p})\lambda_1.$$

The specific choice of $\lambda_n \in \pi_{2p^n-1}\,\mathrm{THH}(\mathrm{BP})$ made in [Rognes 2020] is the unique class detected by $t_n \in \pi_{2p^n-2}(\mathrm{BP}\wedge\mathrm{BP})$ in filtration degree 1 of the spectral sequence associated to the skeleton filtration of THH(BP). The claim that its Hurewicz image equals $\sigma\bar{\xi}_n \in H_{2p^n-1}\,\mathrm{THH}(\mathrm{BP})$ follows from the proof of [Zahler 1972, Lemma 3.7].

If $V(n)$ exists as a finite spectrum with

$$H_*V(n) = E(\tau_0,\dots,\tau_n),$$

then $H_*(V(n)\wedge \mathrm{BP}\langle n\rangle) \cong \mathcal{A}_*$, so that $V(n)\wedge\mathrm{BP}\langle n\rangle \simeq H$. We write $h_n\colon V(n)_*X \to H_*X$ for the (generalized) Hurewicz homomorphism induced by the map $V(n)\to H$ extending the unit $S\to H$.

Proposition 3.3 [Ausoni and Rognes 2012, Lemma 4.1; Angelini-Knoll et al. 2024, Proposition 2.9] *Suppose that $V(n)$ exists as a ring spectrum. Then*

$$V(n)_*\,\mathrm{THH}(\mathrm{BP}\langle n\rangle) = \pi_*(V(n)\wedge\mathrm{THH}(\mathrm{BP}\langle n\rangle)) = E(\lambda_1,\dots,\lambda_{n+1})\otimes P(\mu_{n+1})$$

maps isomorphically to the subalgebra of $\mathcal{A}_*$-comodule primitives in

$$H_*(V(n)\wedge\mathrm{THH}(\mathrm{BP}\langle n\rangle)) \cong H_*V(n)\otimes H_*\,\mathrm{THH}(\mathrm{BP}\langle n\rangle) \cong \mathcal{A}_*\otimes E(\sigma\bar{\xi}_1,\dots,\sigma\bar{\xi}_{n+1})\otimes P(\sigma\bar{\tau}_{n+1}).$$

Here each λ_k is the image of $\lambda_k \in \pi_{2p^k-1}$ THH(BP) *under the natural map induced by $S\to V(n)$ and* $\mathrm{BP}\to\mathrm{BP}\langle n\rangle$*, with Hurewicz images*

$$h(\lambda_k) = 1\wedge\sigma\bar{\xi}_k \quad\text{and}\quad h_n(\lambda_k) = \sigma\bar{\xi}_k.$$

Moreover, μ_{n+1} in degree $|\mu_{n+1}| = 2p^{n+1}$ is the class with Hurewicz images

$$h(\mu_{n+1}) = 1\wedge\sigma\bar{\tau}_{n+1} + \tau_0\wedge\sigma\bar{\xi}_{n+1} \quad\text{and}\quad h_n(\mu_{n+1}) = \sigma\bar{\tau}_{n+1}.$$

Note that the $\mathcal{A}_*$-coaction sends $h(\mu_{n+1})$ to

$$1\otimes(1\wedge\sigma\bar{\tau}_{n+1}) + \bar{\tau}_0\otimes(1\wedge\sigma\bar{\xi}_{n+1}) + 1\otimes(\tau_0\wedge\sigma\bar{\xi}_{n+1}) + \tau_0\otimes(1\wedge\sigma\bar{\xi}_{n+1}) = 1\otimes h(\mu_{n+1}),$$

so that this class is $\mathcal{A}_*$-comodule primitive. We spell out these definitions a little more explicitly in the case of main interest to us.

Definition 3.4 For $p\ge 7$ let

$$\lambda_1,\lambda_2,\lambda_3,\mu_3 \in V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)$$

denote the classes in degrees $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|\lambda_3| = 2p^3-1$ and $|\mu_3| = 2p^3$ with Hurewicz images $h(\lambda_1) = 1\wedge\sigma\bar{\xi}_1$, $h(\lambda_2) = 1\wedge\sigma\bar{\xi}_2$, $h(\lambda_3) = 1\wedge\sigma\bar{\xi}_3$ and $h(\mu_3) = 1\wedge\sigma\bar{\tau}_3 + \tau_0\wedge\sigma\bar{\xi}_3$. Then

$$V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle) = E(\lambda_1,\lambda_2,\lambda_2)\otimes P(\mu_3),$$

which has at most one monomial generator in each degree. We generally abbreviate μ_3 to μ when only discussing $\mathrm{BP}\langle 2\rangle$.

The $V(n)$-homotopy classes μ_{n+1} should not be confused with the ring spectrum multiplications $\mu_n\colon V(n)\wedge V(n)\to V(n)$, which hereafter appear explicitly only in the proof of Proposition 5.10.

4 Cyclotomic nomenclature

We review some notation in common use from 1994 to 2017, including [Hesselholt and Madsen 1997; Rognes 1999; Ausoni and Rognes 2002; Hesselholt and Madsen 2003; Ausoni and Rognes 2012]. For each $\mathbb{T}$-spectrum X there is a natural map

$$r\colon X^{C_p} \to \Phi^{C_p} X$$

of $\mathbb{T}/C_p$-spectra from the categorical C_p-fixed points to the geometric C_p-fixed points. The latter were introduced, as "spacewise C_p-fixed points", in [Lewis et al. 1986, Definition II.9.7], essentially as a left Kan extension. This definition agrees with what has later been called the monoidal geometric fixed points [Mandell and May 2002]. Recall the $\mathbb{T}$-equivariant homotopy cofiber sequence

$$E\mathbb{T}_+ \xrightarrow{c} S^0 \xrightarrow{e} \widetilde{E\mathbb{T}}.$$

In the commutative square

$$\begin{tikzcd}
X^{C_p} \arrow[r, "r"] \arrow[d, "e"'] & \Phi^{C_p}(X) \arrow[d, "\simeq"] \\
(\widetilde{E\mathbb{T}} \wedge X)^{C_p} \arrow[r, "\simeq"] & \Phi^{C_p}(\widetilde{E\mathbb{T}} \wedge X)
\end{tikzcd}$$

the right-hand and lower maps are $\mathbb{T}/C_p$-equivariant equivalences. The expression $(\widetilde{E\mathbb{T}} \wedge X)^{C_p}$ is therefore sometimes [Hesselholt and Madsen 1997] taken as a definition of the geometric fixed points, but this construction is not strictly monoidal. The commutative square

$$\begin{tikzcd}
X^{C_p} \arrow[r, "r"] \arrow[d, "c^*"'] & \Phi^{C_p}(X) \arrow[d, "c^*"] \\
F(E\mathbb{T}_+, X)^{C_p} \arrow[r, "r"] & \Phi^{C_p}(F(E\mathbb{T}_+, X))
\end{tikzcd}$$

is $\mathbb{T}/C_p$-equivariantly homotopy Cartesian. Note that $\Phi^{C_p}(F(E\mathbb{T}_+, X)) \simeq [\widetilde{E\mathbb{T}} \wedge F(E\mathbb{T}_+, X)]^{C_p} = X^{tC_p}$ defines the C_p-Tate $\mathbb{T}/C_p$-spectrum. These $\mathbb{T}/C_p$-spectra are hereafter viewed as $\mathbb{T}$-spectra via the p^{th} root isomorphism $\rho\colon \mathbb{T} \cong \mathbb{T}/C_p$, which we omit from the notation.

The $\mathbb{T}$-spectra $X = \mathrm{THH}(B)$ are cyclotomic, in the sense that there are $\mathbb{T}$-equivalences $\Phi^{C_p}(\mathrm{THH}(B)) \simeq \mathrm{THH}(B)$. Hence [Bökstedt and Madsen 1994, (6.1)] there are vertical maps of horizontal homotopy cofiber sequences

$$\begin{tikzcd}
\mathrm{THH}(B)_{hC_{p^n}} \arrow[r, "N"] \arrow[d, equal] & \mathrm{THH}(B)^{C_{p^n}} \arrow[r, "R"] \arrow[d, "\Gamma_n"'] & \mathrm{THH}(B)^{C_{p^{n-1}}} \arrow[d, "\hat{\Gamma}_n"] \\
\mathrm{THH}(B)_{hC_{p^n}} \arrow[r, "N^h"] & \mathrm{THH}(B)^{hC_{p^n}} \arrow[r, "R^h"] & \mathrm{THH}(B)^{tC_{p^n}}
\end{tikzcd}$$

known as the norm–restriction sequences, for all n. Here the (Witt vector restriction) maps R are given by

$$r^{C_{p^{n-1}}}\colon \mathrm{THH}(B)^{C_{p^n}} \to \Phi^{C_p}(\mathrm{THH}(B))^{C_{p^{n-1}}} \simeq \mathrm{THH}(B)^{C_{p^{n-1}}}.$$

The norm maps N are given by the Adams transfer equivalence $\mathrm{THH}(B)_{hC_{p^n}} \simeq [E\mathbb{T}_+ \wedge \mathrm{THH}(B)]^{C_{p^n}}$, followed by the map induced by $c\colon E\mathbb{T}_+ \to S^0$. The right-hand homotopy Cartesian squares are compatible with the (Witt vector Frobenius) maps $F\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$ that forget some invariance. The Witt vector terminology is motivated by the effects of these maps on π_0 for connective B, in view of the isomorphisms $\pi_0 \mathrm{THH}(B)^{C_{p^n}} \cong W_{n+1}(\pi_0(B))$ of [Hesselholt and Madsen 1997, Theorem 3.3].

The homotopy restriction map R^h is induced by $e\colon S^0 \to \widetilde{E\mathbb{T}}$, and induces a map of spectral sequences from the C_{p^n}-homotopy fixed point spectral sequence to the C_{p^n}-Tate spectral sequence. The map Γ_n is the comparison map from fixed points to homotopy fixed points, and $\hat{\Gamma}_n$ denotes its Tate analog. Passing to homotopy limits over the maps F, and implicitly p-completing, one obtains a map of homotopy cofiber sequences

$$\begin{tikzcd}
\Sigma \mathrm{THH}(B)_{h\mathbb{T}} \arrow[r, "N"] \arrow[d, equal] & \mathrm{TF}(B) \arrow[r, "R"] \arrow[d, "\Gamma"'] & \mathrm{TF}(B) \arrow[d, "\hat{\Gamma}"] \\
\Sigma \mathrm{THH}(B)_{h\mathbb{T}} \arrow[r, "N^h"] & \mathrm{THH}(B)^{h\mathbb{T}} \arrow[r, "R^h"] & \mathrm{THH}(B)^{t\mathbb{T}}.
\end{tikzcd}$$

Again, R^h is induced by $e\colon S^0 \to \widetilde{E\mathbb{T}}$ and induces a map of spectral sequences from the $\mathbb{T}$-homotopy fixed point spectral sequence to the $\mathbb{T}$-Tate spectral sequence. The topological cyclic homology

$$\mathrm{TC}(B) \xrightarrow{\pi} \mathrm{TF}(B) \underset{R}{\overset{1}{\rightrightarrows}} \mathrm{TF}(B)$$

was originally defined by Bökstedt, Hsiang and Madsen [Bökstedt et al. 1993] as the homotopy equalizer of the identity $1\colon \mathrm{TF}(B) \to \mathrm{TF}(B)$ and the restriction map $R\colon \mathrm{TF}(B) \to \mathrm{TF}(B)$. We refer to the preferred lifts $\mathrm{trc}\colon K(B) \to \mathrm{TC}(B)$ and $\mathrm{tr}_{\mathbb{T}} = \Gamma \circ \pi \circ \mathrm{trc}\colon K(B) \to \mathrm{THH}(B)^{h\mathbb{T}}$ of the Bökstedt trace map $\mathrm{tr}\colon K(B) \to \mathrm{THH}(B)$ as the cyclotomic trace map and the circle trace map, respectively.

Some important recent papers give new emphasis to many of these objects. Hesselholt [2018] writes

$$\mathrm{TP}(B) = \mathrm{THH}(B)^{t\mathbb{T}}$$

for the circle Tate construction on $\mathrm{THH}(B)$ and calls it the periodic topological cyclic homology of B. (One might also say topological periodic homology.) Nikolaus and Scholze [2018] write

$$\mathrm{TC}^-(B) = \mathrm{THH}(B)^{h\mathbb{T}}$$

for the circle homotopy fixed points of $\mathrm{THH}(B)$ and call it the topological negative cyclic homology, write

$$\varphi_p = \hat{\Gamma}_1\colon \mathrm{THH}(B) \to \mathrm{THH}(B)^{tC_p}$$

for the comparison map and call it the p-cyclotomic structure map, and write

$$\mathrm{can}\colon \mathrm{TC}^-(B) \to \mathrm{TP}(B)$$

for the homotopy restriction map

$$R^h\colon \mathrm{THH}(B)^{h\mathbb{T}} \to \mathrm{THH}(B)^{t\mathbb{T}}$$

and refer to it as the canonical map. The structure map

$$\epsilon\colon X \to (X^{\wedge p})^{tC_p} = R_+(X)$$

to the topological Singer construction, from [Bruner et al. 1986, Section II.5; Lunøe-Nielsen and Rognes 2012], is now called the Tate diagonal.

In the definition of $\mathrm{TC}(B)$ as a homotopy equalizer, Nikolaus and Scholze replace $\mathrm{TF}(B)$ in the source by $\mathrm{THH}(B)^{h\mathbb{T}}$ via Γ, and replace $\mathrm{TF}(B)$ in the target by $\mathrm{THH}(B)^{t\mathbb{T}}$ via $\hat{\Gamma}$. In view of the commutative square

$$\begin{tikzcd}
\mathrm{TF}(B) \arrow[r, "\hat{\Gamma}"] \arrow[d, "\Gamma"'] & \mathrm{THH}(B)^{t\mathbb{T}} \arrow[d, "G"', "\simeq"] \\
\mathrm{THH}(B)^{h\mathbb{T}} \arrow[r, "(\hat{\Gamma}_1)^{h\mathbb{T}}"] & (\mathrm{THH}(B)^{tC_p})^{h\mathbb{T}}
\end{tikzcd}$$

from [Hesselholt and Madsen 1997, page 68; Ausoni and Rognes 2002, page 27] the identity map $1\colon \mathrm{TF}(B) \to \mathrm{TF}(B)$ is then replaced with the circle homotopy fixed points $(\hat{\Gamma}_1)^{h\mathbb{T}} = \varphi_p^{h\mathbb{T}}$ of the p-cyclotomic structure map, suppressing the (still implicitly p-complete) equivalence

$$G\colon \mathrm{THH}(B)^{t\mathbb{T}} = (\mathrm{THH}(B)^{tC_p})^{\mathbb{T}} \to (\mathrm{THH}(B)^{tC_p})^{h\mathbb{T}}$$

from the notation. The fact that G is an equivalence for connective B was shown by computation in the first instances considered, and then proved in [Bökstedt et al. 2014, Proposition 3.8] under the assumption that H_*B is of finite type. It reappears in the new terminology as the Tate orbit lemma [Nikolaus and Scholze 2018, Lemma I.2.1], since $(\mathrm{THH}(B)_{hC_p})^{t\mathbb{T}} \simeq *$ is equivalent to $\Sigma\,\mathrm{THH}(B)_{h\mathbb{T}} \to (\mathrm{THH}(B)_{hC_p})^{h\mathbb{T}}$ being an equivalence, which in turn is equivalent to G being an equivalence.

Likewise, the restriction map $R\colon \mathrm{TF}(B) \to \mathrm{TF}(B)$ is replaced with the homotopy restriction map $R^h = \mathrm{can}$. Combining these replacements,

$$\mathrm{TC}(B) \xrightarrow{\pi} \mathrm{THH}(B)^{h\mathbb{T}} \underset{R^h}{\overset{G^{-1}(\hat{\Gamma}_1)^{h\mathbb{T}}}{\rightrightarrows}} \mathrm{THH}(B)^{t\mathbb{T}}$$

is redefined as the homotopy equalizer of $G^{-1} \circ (\hat{\Gamma}_1)^{h\mathbb{T}}$ and $R^h = \mathrm{can}$, much as in [Ausoni and Rognes 2012, page 1072], or (in order not to need to invert G) as the homotopy equalizer

$$\mathrm{TC}(B) \xrightarrow{\pi} \mathrm{THH}(B)^{h\mathbb{T}} \underset{GR^h}{\overset{(\hat{\Gamma}_1)^{h\mathbb{T}}}{\rightrightarrows}} (\mathrm{THH}(B)^{tC_p})^{h\mathbb{T}}$$

of $(\hat{\Gamma}_1)^{h\mathbb{T}} = \varphi_p^{h\mathbb{T}}$ and $G \circ R^h$. The old and new definitions of $\mathrm{TC}(B)$ agree for connective B, by [Nikolaus and Scholze 2018, Theorem II.3.8].

5 Homotopy power operations

Let B be an E_{n+1} ring spectrum. Using the Boardman–Vogt tensor product of operads [Dunn 1988], we may view B as an E_n algebra in the category of E_1 ring spectra (or S-algebras). There are then natural E_n

algebra structures on the algebraic K-theory spectrum $K(B)$ and on the cyclotomic spectrum $\mathrm{THH}(B)$, and these are respected by the trace map $K(B) \to \mathrm{THH}(B)$, as well as its cyclotomic refinements.

For each E_2 ring spectrum R, there is a natural "top" homology power operation

$$\xi_1 \colon H_{2k-1} R \to H_{2pk-1} R$$

introduced in [Cohen et al. 1976, Theorem III.1.3]. If R is an E_3 ring spectrum, then $\xi_1 = Q^k$ is the Araki–Kudo/Dyer–Lashof/Cohen operator, as defined in [Cohen et al. 1976, Theorem III.1.1]; we will also use this notation in the E_2 ring spectrum case, to emphasize the dependence on k (and to avoid confusion with the element ξ_1 in the dual Steenrod algebra). Let β denote the mod-p homology Bockstein operator. Ausoni and Rognes [2002, Section 1.5] discussed a homotopy power operation

$$P^k \colon \pi_{2k-1} R \to V(0)_{2pk-1} R$$

lifting Q^k (see Lemma 5.5), in the context of E_∞ ring spectra. Here we will extend its definition to E_2 ring spectra, and construct a homotopy power operation

$$P^k \colon V(0)_{2k-1} R \to V(1)_{2pk-1} R$$

also lifting Q^k (see Lemma 5.6).

To define these operations for E_2 ring spectra R, we make use of the little 2-cubes operad $\mathcal{C}_2$ encoding E_2 algebra structures. For a spectrum X let

$$\mathrm{Br}_p X = D_{2,p} X = \mathcal{C}_2(p) \ltimes_{\Sigma_p} X^{\wedge p}$$

denote the p^{th} braided-extended power of X. Note that $\mathrm{Br}_p \Sigma^2 X \cong \Sigma^{2p} \mathrm{Br}_p X$ by [Cohen et al. 1978, Theorem 1]. In the case $X = S^{2k-1}$, with $H_* X = \mathbb{F}_p\{x_{2k-1}\}$,

$$H_* \mathrm{Br}_p S^{2k-1} = \mathbb{F}_p\{\beta Q^k(x_{2k-1}), Q^k(x_{2k-1})\} \tag{5-1}$$

follows from [Cohen et al. 1976, Theorem III.5.3]; cf [Cohen 1981, Proposition II.1.2]. Hence there is an (implicitly p-complete) equivalence $\bar{\eta}_0 \colon \Sigma^{2pk-1} DV(0) \simeq \mathrm{Br}_p S^{2k-1}$, with right adjoint

$$\eta_0 \colon S^{2pk-1} \to V(0) \wedge \mathrm{Br}_p S^{2k-1}.$$

Here $DV(0) \simeq \Sigma^{-1} V(0)$ denotes the Spanier–Whitehead dual of $V(0)$, and $h_0(\eta_0) = Q^k(x_{2k-1})$.

For typographical reasons we will often simply write g for the maps $1 \wedge g \colon A \wedge B \to A \wedge C$ and $g \wedge 1 \colon B \wedge D \to C \wedge D$, for suitable A, $g \colon B \to C$ and D.

Definition 5.1 Let R be an E_2 ring spectrum. The homotopy power operation

$$P^k \colon \pi_{2k-1} R \to V(0)_{2pk-1} R$$

sends each map $f \colon S^{2k-1} \to R$ to the composite

$$P^k(f) \colon S^{2pk-1} \xrightarrow{\eta_0} V(0) \wedge \mathrm{Br}_p S^{2k-1} \xrightarrow{\mathrm{Br}_p f} V(0) \wedge \mathrm{Br}_p R \xrightarrow{\theta} V(0) \wedge R,$$

where $\theta \colon \mathrm{Br}_p R \to R$ is part of the E_2 ring structure.

In the case $X = \Sigma^{2k-1}DV(0)$, where $H_*X = \mathbb{F}_p\{x_{2k-2}, x_{2k-1}\}$ with $\beta(x_{2k-1}) = x_{2k-2}$, there is an inclusion

$$\mathbb{F}_p\{x^p_{2k-2}, \beta Q^k(x_{2k-1}), Q^k(x_{2k-1})\} \subset H_* \operatorname{Br}_p \Sigma^{2k-1}DV(0)$$

of left $\mathcal{A}_*$-comodules, or of right $\mathcal{A}$-modules. Of the dual Steenrod operations, only β and $\mathcal{P}^1_*$ act nontrivially on the left-hand side, with

$$\mathcal{P}^1_* Q^k(x_{2k-1}) = 0 \quad \text{and} \quad \mathcal{P}^1_* \beta Q^k(x_{2k-1}) = -x^p_{2k-2},$$

according to the spectrum-level Nishida relations; see [Cohen et al. 1976, Theorems III.1.1(6) and III.1.3(3); Bruner et al. 1986, Theorem III.1.1(8)]. As in [Toda 1971], let

$$V(1/2) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1},$$

so that $V(0) \subset V(1/2) \subset V(1)$ and $DV(1/2) \simeq \Sigma^{1-2p}(S \cup_{\alpha_1} e^{2p-2} \cup_p e^{2p-1})$. The following construction refines a map discussed by Toda [1968, Lemma 3]:

Lemma 5.2 *There exists a* (p*-complete*) *map*

$$\bar{\eta}_{1/2} \colon \Sigma^{2pk-1}DV(1/2) \to \operatorname{Br}_p \Sigma^{2k-1}DV(0)$$

realizing the inclusion of $\mathbb{F}_p\{x^p_{2k-2}, \beta Q^k(x_{2k-1}), Q^k(x_{2k-1})\}$ *in homology.*

Proof We can choose a minimal cell structure on $\operatorname{Br}_p \Sigma^{2k-1}DV(0)$ with a $(2pk-1)$-cell representing $Q^k(x_{2k-1})$ that is attached by a degree-p map to a $(2pk-2)$-cell representing $\beta Q^k(x_{2k-1})$. The $(2pk-1)$-cell is not attached to the $(2pk-2p+1)$-skeleton, since $\mathcal{P}^1_* Q^k(x_{2k-1}) = 0$. We can orient the $(2pk-2p)$-cell so that the $(2pk-2)$-cell is attached to it by α_1, since $\mathcal{P}^1_* \beta Q^k(x_{2k-1}) = -x^p_{2k-2}$. □

We fix a choice of $\bar{\eta}_{1/2}$ for each integer k, but see Remark 5.4. This specifies a composite map

$$\bar{\eta}_1 \colon \Sigma^{2pk-1}DV(1) \to \Sigma^{2pk-1}DV(1/2) \xrightarrow{\bar{\eta}_{1/2}} \operatorname{Br}_p \Sigma^{2k-1}DV(0), \tag{5-2}$$

with homology image $\mathbb{F}_p\{x^p_{2k-2}, \beta Q^k(x_{2k-1}), Q^k(x_{2k-1})\}$, and we write

$$\eta_1 \colon S^{2pk-1} \xrightarrow{\eta_{1/2}} V(1/2) \wedge \operatorname{Br}_p \Sigma^{2k-1}DV(0) \to V(1) \wedge \operatorname{Br}_p \Sigma^{2k-1}DV(0)$$

for its right adjoint.

Definition 5.3 Let R be an E_2 ring spectrum. The homotopy power operation

$$P^k \colon V(0)_{2k-1}R \to V(1/2)_{2pk-1}R \to V(1)_{2pk-1}R$$

sends each map $f \colon S^{2k-1} \to V(0) \wedge R$, with left adjoint $\bar{f} \colon \Sigma^{2k-1}DV(0) \to R$, to the composite

$$P^k(f) \colon S^{2pk-1} \xrightarrow{\eta_1} V(1) \wedge \operatorname{Br}_p \Sigma^{2k-1}DV(0) \xrightarrow{\operatorname{Br}_p \bar{f}} V(1) \wedge \operatorname{Br}_p R \xrightarrow{\theta} V(1) \wedge R.$$

Remark 5.4 We discuss the nonuniqueness of $\bar{\eta}_{1/2}$ and the resulting ambiguity in the operation P^k just defined. For brevity, let $U = \mathrm{Br}_p\,\Sigma^{2k-1}DV(0)$. By [Cohen et al. 1976, Theorem III.3.1] we have

$$H_*U \cong \mathbb{F}_p\{x_{2k-2}^p, x_{2k-2}^{p-1}x_{2k-1}, x_{2k-2}^{p-2}y_{4k-3}, x_{2k-2}^{p-2}y_{4k-2}, x_{2k-2}^{p-3}x_{2k-1}y_{4k-3}\}$$

in degrees $2pk-2p \le * \le 2pk-2p+2$, plus classes in higher degrees, where

$$y_{4k-3} = [x_{2k-2}, x_{2k-2}]_1 \quad \text{and} \quad y_{4k-2} = [x_{2k-2}, x_{2k-1}]_1$$

are E_2 ring spectrum Browder brackets. (We write $[x, y]_1$ in place of the traditional $\lambda_1(x, y)$ in order to avoid confusion with the homotopy class λ_1.) The (additive) indeterminacies in $\bar{\eta}_{1/2}$ and η_1 are maps $\Sigma^{2pk-1}DV(1/2) \to U$ and $m\colon S^{2pk-1} \to V(1) \wedge U$, respectively, that induce zero in homology. The Atiyah–Hirzebruch spectral sequence for $V(1)_*U$ shows that $m = \alpha_1 \cdot n$ for a class $n \in V(1)_{2pk-2p+2}U \cong H_{2pk-2p+2}U$, generated by $x_{2k-2}^{p-2}y_{4k-2}$ and $x_{2k-2}^{p-3}x_{2k-1}y_{4k-3}$. These generators map to zero in $V(1)_*R$ if the E_2 ring structure on R extends to an E_3 ring structure. Hence any two different choices of maps $\bar{\eta}_{1/2}$ will give operations P^k that differ at most by a multiple of α_1, and which strictly agree if R is an E_3 ring spectrum. This means that for all of the assertions we will make about these homotopy power operations, the choice of $\bar{\eta}_{1/2}$ makes no difference: in Lemma 5.6 the Hurewicz homomorphism h_1 annihilates α_1-multiples, and in Proposition 5.10 we assume that R is an E_∞ ring spectrum.

Lemma 5.5 *Let R be an E_2 ring spectrum. The square*

$$\begin{tikzcd}
\pi_{2k-1}R \arrow[r, "P^k"] \arrow[d, "h"'] & V(0)_{2pk-1}R \arrow[d, "h_0"] \\
H_{2k-1}R \arrow[r, "Q^k"] & H_{2pk-1}R
\end{tikzcd}$$

commutes.

Proof Let $t_0\colon H \to H \wedge DV(0)$ and $b_0\colon H \wedge V(0) \to H$ be H-module maps that split off the top and bottom cells, respectively. Then $b_0 h = h_0\colon V(0) \to H$, and the following diagram commutes.

$$\begin{tikzcd}
S^{2pk-1} \arrow[r] \arrow[d, "h"] \arrow[rr, bend left, "\eta_0"] & V(0)\wedge DV(0)\wedge S^{2pk-1} \arrow[r, "\bar{\eta}_0"] \arrow[d, "h"] & V(0)\wedge \mathrm{Br}_p\, S^{2k-1} \arrow[r, "\mathrm{Br}_p\, f"] \arrow[d, "h"] & V(0)\wedge \mathrm{Br}_p\, R \arrow[r, "\theta"] \arrow[d, "h"] & V(0)\wedge R \arrow[d, "h"] \\
H\wedge S^{2pk-1} \arrow[r] \arrow[dr, "t_0"'] & H\wedge V(0)\wedge DV(0)\wedge S^{2pk-1} \arrow[r, "\bar{\eta}_0"] \arrow[d, "b_0"] & H\wedge V(0)\wedge \mathrm{Br}_p\, S^{2k-1} \arrow[r, "\mathrm{Br}_p\, f"] \arrow[d, "b_0"] & H\wedge V(0)\wedge \mathrm{Br}_p\, R \arrow[r, "\theta"] \arrow[d, "b_0"] & H\wedge V(0)\wedge R \arrow[d, "b_0"] \\
 & H\wedge DV(0)\wedge S^{2pk-1} \arrow[r, "\bar{\eta}_0"] & H\wedge \mathrm{Br}_p\, S^{2k-1} \arrow[r, "\mathrm{Br}_p\, f"] \arrow[d, leftrightarrow, "\cong"] & H\wedge \mathrm{Br}_p\, R \arrow[r, "\theta"] \arrow[d, leftrightarrow, "\cong"] & H\wedge R \\
 & & \mathrm{Br}_p^H(H\wedge S^{2k-1}) \arrow[r, "\mathrm{Br}_p^H\, \bar{g}"] & \mathrm{Br}_p^H(H\wedge R) &
\end{tikzcd}$$

Here $\bar{g} = 1 \wedge f \colon H \wedge S^{2k-1} \to H \wedge R$ denotes the H-module map that is left adjoint to the Hurewicz image $g = hf \colon S^{2k-1} \to H \wedge R$, and Br_p^H denotes the p^{th} braided-extended power construction in the category of H-modules. The upper composite $S^{2pk-1} \to H \wedge R$ then represents $h_0 P^k(f)$, while the lower composite represents

$$Q_{p-1}(g) = \theta_*(e_{p-1} \otimes g^{\otimes p}),$$

up to a known unit in $\mathbb{F}_p$, with notation as in [May 1970, Definition 2.2; Cohen et al. 1976, Section I.1]. This equals $Q^k(hf)$. □

Lemma 5.6 *Let R be an E_2 ring spectrum. The square*

$$\begin{tikzcd}
V(0)_{2k-1} R \arrow[r, "P^k"] \arrow[d, "h_0"'] & V(1)_{2pk-1} R \arrow[d, "h_1"] \\
H_{2k-1} R \arrow[r, "Q^k"] & H_{2pk-1} R
\end{tikzcd}$$

commutes.

Proof Let $t_1 \colon H \to H \wedge DV(1)$ and $b_1 \colon H \wedge V(1) \to H$ be H-module maps that split off the top and bottom cells, respectively. Then $b_1 h = h_1 \colon V(1) \to H$ and the following diagram commutes, up to units in $\mathbb{F}_p$.

$$\begin{tikzcd}
S^{2pk-1} \arrow[r] \arrow[d, "h"] \arrow[rr, bend left, "\eta_1"] & V(1) \wedge DV(1) \wedge S^{2pk-1} \arrow[r, "\bar{\eta}_1"] \arrow[d, "h"] & V(1) \wedge \mathrm{Br}_p\, \Sigma^{2k-1} DV(0) \arrow[r, "\mathrm{Br}_p\, \bar{f}"] \arrow[d, "h"] & V(1) \wedge \mathrm{Br}_p\, R \arrow[r, "\theta"] \arrow[d, "h"] & V(1) \wedge R \arrow[d, "h"] \\
H \wedge S^{2pk-1} \arrow[r] \arrow[dr, "t_1"'] \arrow[ddr, bend right, "e_{p-1}"'] & H \wedge V(1) \wedge DV(1) \wedge S^{2pk-1} \arrow[r, "\bar{\eta}_1"] \arrow[d, "b_1"] & H \wedge V(1) \wedge \mathrm{Br}_p\, \Sigma^{2k-1} DV(0) \arrow[r, "\mathrm{Br}_p\, \bar{f}"] \arrow[d, "b_1"] & H \wedge V(1) \wedge \mathrm{Br}_p\, R \arrow[r, "\theta"] \arrow[d, "b_1"] & H \wedge V(1) \wedge R \arrow[d, "b_1"] \\
 & H \wedge DV(1) \wedge S^{2pk-1} \arrow[r, "\bar{\eta}_1"] & H \wedge \mathrm{Br}_p\, \Sigma^{2k-1} DV(0) \arrow[r, "\mathrm{Br}_p\, \bar{f}"] \arrow[d, leftrightarrow, "\cong"] & H \wedge \mathrm{Br}_p\, R \arrow[r, "\theta"] \arrow[d, leftrightarrow, "\cong"] & H \wedge R \\
 & H \wedge \mathrm{Br}_p\, S^{2k-1} \arrow[ur, "\mathrm{Br}_p\, t_0"] \arrow[d, leftrightarrow, "\cong"] & \mathrm{Br}_p^H (H \wedge \Sigma^{2k-1} DV(0)) \arrow[r, "\mathrm{Br}_p^H\, \bar{f}"] & \mathrm{Br}_p^H (H \wedge R) & \\
 & \mathrm{Br}_p^H (H \wedge S^{2k-1}) \arrow[ur, "\mathrm{Br}_p^H\, t_0"] \arrow[urr, bend right, "\mathrm{Br}_p^H\, \bar{g}"'] & & &
\end{tikzcd}$$

Here $\bar{g} \colon H \wedge S^{2k-1} \to H \wedge R$ denotes the H-module map extending the $V(0)$-Hurewicz image $g = h_0 f \colon S^{2k-1} \to H \wedge R$. The two maps from $H \wedge S^{2pk-1}$ to $H \wedge \mathrm{Br}_p\, \Sigma^{2k-1} DV(0)$ agree, up to a known unit in $\mathbb{F}_p$, because the map $\bar{\eta}_{1/2}$ in Lemma 5.2 sends the top cell to $Q^k(x_{2k-1})$. The two maps from $\mathrm{Br}_p^H(H \wedge S^{2k-1})$ to $\mathrm{Br}_p^H(H \wedge R)$ agree because $\bar{g}$ is homotopic through H-module maps to $\bar{f} t_0$. The upper composite $S^{2pk-1} \to H \wedge R$ in the diagram represents $h_1 P^k(f)$, while the lower composite represents a known unit times $Q_{p-1}(g)$, which equals $Q^k(h_0 f)$. □

The following homotopy Cartan formula generalizes the one proved for E_∞ ring spectra in [Ausoni and Rognes 2002, Lemma 1.6]:

Proposition 5.7 *Let R be an E_3 ring spectrum. For $x \in \pi_{2i} R$ and $y \in \pi_{2j-1} R$, the relation*

$$P^k(xy) = x^p P^j(y)$$

holds in $V(0)_{2pk-1} R$, where $k = i + j$.

Proof We use the following nearly commutative diagram, where δ_p is the operadic diagonal from [Bruner et al. 1986, Section I.2],

$$D_{n,p} X = \mathcal{C}_n(p) \ltimes_{\Sigma_p} X^{\wedge p}$$

denotes the p^{th} E_n-extended power, and $\sigma_1 \colon X^{\wedge p} \simeq D_{1,p} X \to D_{2,p} X = \mathrm{Br}_p\, X$ and $\sigma_2 \colon \mathrm{Br}_p\, X \to D_{3,p} X$ are stabilization maps.

$$\begin{tikzcd}
\Sigma^{2pk-1} DV(0) \arrow[r, "\bar\eta_0"] \arrow[d, "\cong"] & \mathrm{Br}_p\, S^{2k-1} \arrow[r, "\mathrm{Br}_p(f \cdot g)"] \arrow[d, "\cong"] & \mathrm{Br}_p\, R \arrow[r, "\theta"] & R \\
S^{2pi} \wedge \Sigma^{2pj-1} DV(0) \arrow[d, "\simeq"] & \mathrm{Br}_p(S^{2i} \wedge S^{2j-1}) \arrow[r, "\mathrm{Br}_p(f \wedge g)"] \arrow[d, "\delta_p"] & \mathrm{Br}_p(R \wedge R) \arrow[u, "\mathrm{Br}_p\, \phi"'] \arrow[d, "\delta_p"] & \\
D_{1,p} S^{2i} \wedge \mathrm{Br}_p\, S^{2j-1} \arrow[r, "\sigma_1 \wedge 1"] & \mathrm{Br}_p\, S^{2i} \wedge \mathrm{Br}_p\, S^{2j-1} \arrow[r, "\mathrm{Br}_p\, f \wedge \mathrm{Br}_p\, g"] \arrow[d, "\sigma_2 \wedge \sigma_2"] & \mathrm{Br}_p\, R \wedge \mathrm{Br}_p\, R \arrow[r, "\theta \wedge \theta"] \arrow[d, "\sigma_2 \wedge \sigma_2"'] & R \wedge R \arrow[uu, "\phi"'] \\
 & D_{3,p} S^{2i} \wedge D_{3,p} S^{2j-1} \arrow[r, "D_{3,p} f \wedge D_{3,p} g"] & D_{3,p} R \wedge D_{3,p} R \arrow[ur, "\theta \wedge \theta"'] &
\end{tikzcd}$$

We may view the E_3 ring spectrum R as an E_2 algebra in the category of E_1 ring spectra. The ring spectrum pairing $\phi \colon R \wedge R \to R$ is then an E_2 ring spectrum map, and therefore the right-hand rectangle commutes. Moreover, the right-hand triangle commutes, because the E_3 operad action extends the E_2 action.

Let $f \colon S^{2i} \to R$ and $g \colon S^{2j-1} \to R$ be maps representing x and y. The composite

$$f \cdot g \colon S^{2k-1} \cong S^{2i} \wedge S^{2j-1} \xrightarrow{f \wedge g} R \wedge R \xrightarrow{\phi} R$$

then represents xy, and the upper square commutes by functoriality of the braided-extended power. The central and lower squares commute by naturality of δ_p and σ_2.

We do not know whether the left-hand rectangle commutes. However, we do claim that the two composites $\Sigma^{2pk-1} DV(0) \to \mathrm{Br}_p\, S^{2i} \wedge \mathrm{Br}_p\, S^{2j-1}$ become homotopic after composition with $\sigma_2 \wedge \sigma_2$. This implies that the composite along the upper edge, which is adjoint to the map representing $P^k(xy)$, is homotopic to the composite along the left-hand, lower and right-hand edges, which in turn is homotopic to the central composite via $\mathrm{Br}_p\, f \wedge \mathrm{Br}_p\, g$; this is adjoint to the map representing $x^p P^j(y)$.

To justify the claim, we compute in homology. Recall the expression (5-1) for $H_* \operatorname{Br}_p S^{2k-1}$, which has an evident analog for $H_* \operatorname{Br}_p S^{2j-1}$. In the case $X = S^{2i}$, with $H_* X = \mathbb{F}_p\{x_{2i}\}$,

$$H_* \operatorname{Br}_p S^{2i} = \mathbb{F}_p\{x_{2i}^p, \alpha_{2pi+1}\} \tag{5-3}$$

follows from [Cohen et al. 1976, Theorem III.5.2]. Here $\alpha_{2pi+1} = -x_{2i}^{p-2}[x_{2i}, x_{2i}]_1$ is a class given in terms of the E_2 Browder bracket, and $\beta\alpha_{2pi+1} = 0$ according to [loc. cit., Theorem III.1.2(7)]. Note that α_{2pi+1} maps to zero under σ_2.

Along one route, the right $\mathcal{A}$-module generator x_{2pk-1} in $H_{2pk-1}\Sigma^{2pk-1}DV(0)$ maps to $x_{2pi} \otimes x_{2pj-1}$ in the homology of $S^{2pi} \wedge \Sigma^{2pj-1}DV(0)$, and thereafter to $x_{2i}^p \otimes Q^j(x_{2j-1})$ in the homologies of $D_{1,p}S^{2i} \wedge \operatorname{Br}_p S^{2j-1}$, $\operatorname{Br}_p S^{2i} \wedge \operatorname{Br}_p S^{2j-1}$ and $D_{3,p}S^{2i} \wedge D_{3,p}S^{2j-1}$.

Along the other route, x_{2pk-1} maps to $Q^k(x_{2k-1})$ in the homology of $\operatorname{Br}_p S^{2k-1}$, and to $Q^k(x_{2i} \otimes x_{2j-1})$ in the homologies of $\operatorname{Br}_p(S^{2i} \wedge S^{2j-1})$ and $D_{3,p}(S^{2i} \wedge S^{2j-1})$. By the E_3 ring spectrum Cartan formula [loc. cit., Theorem III.1.1(4)], it maps to

$$Q^i(x_{2i}) \otimes Q^j(x_{2j-1}) = x_{2i}^p \otimes Q^j(x_{2j-1})$$

in the homology of $D_{3,p}S^{2i} \wedge D_{3,p}S^{2j-1}$.

It follows that the two composites $\bar{\ell}_1, \bar{\ell}_2 \colon \Sigma^{2pk-1}DV(0) \to D_{3,p}S^{2i} \wedge D_{3,p}S^{2j-1}$ induce the same homomorphism in homology. Hence their adjoints $\ell_1, \ell_2 \colon S^{2pk-1} \to V(0) \wedge D_{3,p}S^{2i} \wedge D_{3,p}S^{2j-1}$ also agree in homology. Since $D_{3,p}S^{2i} \wedge D_{3,p}S^{2j-1}$ is $(2pk-3)$-connected and $h_0 \colon V(0) \to H$ is $(2p-3)$-connected, it follows that ℓ_1 and ℓ_2 are homotopic. Therefore $\bar{\ell}_1$ and $\bar{\ell}_2$ are also homotopic. □

Remark 5.8 This proof also shows that

$$\delta_{p*}Q^k(x_{2i} \otimes x_{2j-1}) = x_{2i}^p \otimes Q^j(x_{2j-1}) + c \cdot \alpha_{2pi+1} \otimes \beta Q^j(x_{2j-1})$$

in the homology of $\operatorname{Br}_p S^{2i} \wedge \operatorname{Br}_p S^{2j-1}$, for some unknown coefficient $c \in \mathbb{F}_p$. If $c \neq 0$ then the two maps $\Sigma^{2pk-1}DV(0) \to \operatorname{Br}_p S^{2i} \wedge \operatorname{Br}_p S^{2j-1}$ induce different homomorphisms in homology, and the left-hand rectangle does not commute.

Corollary 5.9 *Let*

$$R \xrightarrow{s} T \xrightarrow{r} R$$

be spectrum maps with rs homotopic to the identity. Assume that R is an E_2 ring spectrum, that T is an E_3 ring spectrum and that s or r is an E_2 ring map. Then $P^k(xy) = x^p P^j(y)$ in $V(0)_{2pk-1}R$ for $x \in \pi_{2i}R$, $y \in \pi_{2j-1}R$ and $k = i + j$.

Proof Replace Proposition 5.10 with Proposition 5.7 in the proof of Corollary 5.12. □

We will also need a homotopy Cartan formula for the power operations from Definition 5.3:

Proposition 5.10 *Let R be an E_∞ ring spectrum. For $x \in V(0)_{2i}R$ and $y \in V(0)_{2j-1}R$, the relation*

$$P^k(xy) = x^p P^j(y)$$

holds in $V(1)_{2pk-1}R$, where $k = i + j$.

Proof We use the following nearly commutative diagram, where

$$D_pX = \mathcal{C}_\infty(p) \ltimes_{\Sigma_p} X^{\wedge p}$$

denotes the p^{th} (unqualified) extended power, and $\sigma_2' \colon \mathrm{Br}_p X \to D_pX$ is the infinite stabilization map. We write $\mu_0^p \colon V(0)^{\wedge p} \to V(0)$ for the $(p-1)$-fold iterate of the ring spectrum multiplication, and let $m = \mu_1(i_1 \wedge 1) \colon V(0) \wedge V(1) \to V(1)$ denote the left $V(0)$-module action on $V(1)$.

$$\begin{tikzcd}
\Sigma^{2pk-1}DV(1) \arrow[r, "\bar\eta_1"] \arrow[d, "Dm"] & \mathrm{Br}_p\,\Sigma^{2k-1}DV(0) \arrow[r, "\mathrm{Br}_p\,\overline{f\cdot g}"] \arrow[d, "\mathrm{Br}_p\,D\mu_0"] & \mathrm{Br}_p\,R \arrow[r, "\theta"] & R \\
\Sigma^{2pi}DV(0)\wedge\Sigma^{2pj-1}DV(1) \arrow[d, "D\mu_0^p\wedge\bar\eta_1"] & \mathrm{Br}_p(\Sigma^{2i}DV(0)\wedge\Sigma^{2j-1}DV(0)) \arrow[r, "\mathrm{Br}_p(\bar f\wedge\bar g)"] \arrow[d, "\delta_p"] & \mathrm{Br}_p(R\wedge R) \arrow[u, "\mathrm{Br}_p\,\phi"'] \arrow[d, "\delta_p"] & \\
D_{1,p}\Sigma^{2i}DV(0)\wedge\mathrm{Br}_p\,\Sigma^{2j-1}DV(0) \arrow[r, "\sigma_1\wedge 1"] & \mathrm{Br}_p\,\Sigma^{2i}DV(0)\wedge\mathrm{Br}_p\,\Sigma^{2j-1}DV(0) \arrow[r, "\mathrm{Br}_p\,\bar f\wedge\mathrm{Br}_p\,\bar g"] \arrow[d, "\sigma_2'\wedge\sigma_2'"] & \mathrm{Br}_p\,R\wedge\mathrm{Br}_p\,R \arrow[r, "\theta\wedge\theta"] \arrow[d, "\sigma_2'\wedge\sigma_2'"'] & R\wedge R \arrow[uu, "\phi"'] \\
 & D_p\Sigma^{2i}DV(0)\wedge D_p\Sigma^{2j-1}DV(0) \arrow[r, "D_p\bar f\wedge D_p\bar g"] & D_pR\wedge D_pR \arrow[ur, "\theta\wedge\theta"'] &
\end{tikzcd}$$

The right-hand rectangle and triangle commute as before, replacing E_3 by E_∞.

Let $f \colon S^{2i} \to V(0) \wedge R$ and $g \colon S^{2j-1} \to V(0) \wedge R$ be maps representing x and y, with adjoints $\bar f \colon \Sigma^{2i}DV(0) \to R$ and $\bar g \colon \Sigma^{2j-1}DV(0) \to R$. The composite

$$\overline{f\cdot g} \colon \Sigma^{2k-1}DV(0) \xrightarrow{D\mu_0} \Sigma^{2i}DV(0) \wedge \Sigma^{2j-1}DV(0) \xrightarrow{\bar f\wedge\bar g} R \wedge R \xrightarrow{\phi} R$$

is then adjoint to the map $f \cdot g \colon S^{2k-1} \to V(0) \wedge R$ that represents xy, and the upper square commutes by functoriality of the braided-extended power. The central and lower squares commute by naturality of δ_p and σ_2'.

As before we do not know whether the left-hand rectangle commutes. However, we claim that the two composites

$$\Sigma^{2pk-1}DV(1) \to \mathrm{Br}_p\,\Sigma^{2i}DV(0) \wedge \mathrm{Br}_p\,\Sigma^{2j-1}DV(0)$$

become homotopic after composition with $\sigma_2' \wedge \sigma_2'$ to

$$W = D_p\Sigma^{2i}DV(0) \wedge D_p\Sigma^{2j-1}DV(0).$$

This implies that the composite along the upper edge, which is adjoint to the map representing $P^k(xy)$, is homotopic to the composite along the left-hand, lower and right-hand edges, which in turn is homotopic to the central composite via $\mathrm{Br}_p\,\bar f \wedge \mathrm{Br}_p\,\bar g$; this is adjoint to the map representing $x^pP^j(y)$.

To justify the claim, we first compute in homology, using [Cohen et al. 1976, Theorem I.4.1]. Writing $H_*\Sigma^{2i}DV(0) = \mathbb{F}_p\{x_{2i-1}, x_{2i}\}$ and $H_*\Sigma^{2j-1}DV(0) = \mathbb{F}_p\{x_{2j-2}, x_{2j-1}\}$, with $\beta x_{2i} = x_{2i-1}$ and $\beta x_{2j-1} = x_{2j-2}$, we have

$$H_* D_p \Sigma^{2i} DV(0) = \mathbb{F}_p\{\beta Q^i(x_{2i-1}), Q^i(x_{2i-1}), x_{2i-1}x_{2i}^{p-1}, x_{2i}^p, \beta Q^{i+1}(x_{2i-1}), Q^{i+1}(x_{2i-1}), \dots\}$$

in degrees $* \geq 2pi - 2$, and

$$H_* D_p \Sigma^{2j-1} DV(0) = \mathbb{F}_p\{x_{2j-2}^p, x_{2j-2}^{p-1}x_{2j-1}, \beta Q^j(x_{2j-2}), Q^j(x_{2j-2}), \beta Q^j(x_{2j-1}), Q^j(x_{2j-1}), \dots\}$$

in degrees $* \geq 2pj - 2p$. Their tensor product is H_*W, which is concentrated in degrees $2pk - 2p - 2 \leq * \leq 2pk - 2p + 1$ and $* \geq 2pk - 4$.

On one hand, the right $\mathcal{A}$-module generator x_{2pk-1} in $H_{2pk-1}\Sigma^{2pk-1}DV(1)$ maps to $x_{2pi} \otimes x_{2pj-1}$ in the homology of $\Sigma^{2pi}DV(0) \wedge \Sigma^{2pj-1}DV(1)$, and thereafter to $x_{2i}^p \otimes Q^j(x_{2j-1})$ in the homologies of $D_{1,p}\Sigma^{2i}DV(0) \wedge \mathrm{Br}_p\, \Sigma^{2j-1}DV(0)$, $\mathrm{Br}_p\, \Sigma^{2i}DV(0) \wedge \mathrm{Br}_p\, \Sigma^{2j-1}DV(0)$ and W. On the other hand, x_{2pk-1} maps to $Q^k(x_{2k-1})$ in the homology of $\mathrm{Br}_p\, \Sigma^{2k-1}DV(0)$, and to $Q^k(x_{2i} \otimes x_{2j-1})$ in the homologies of $\mathrm{Br}_p(\Sigma^{2i}DV(0) \wedge \Sigma^{2j-1}DV(0))$ and $D_p(\Sigma^{2i}DV(0) \wedge \Sigma^{2j-1}DV(0))$. By the E_∞ ring spectrum Cartan formula [loc. cit., Theorem I.1.1(6)] it maps to $Q^i(x_{2i}) \otimes Q^j(x_{2j-1}) = x_{2i}^p \otimes Q^j(x_{2j-1})$ in H_*W.

It follows that the two composites $\bar{m}_1, \bar{m}_2\colon \Sigma^{2pk-1}DV(1) \to W$ induce the same homomorphism in homology. Let $\bar{m} = \bar{m}_2 - \bar{m}_1$ be their difference, inducing zero in homology. The homological Atiyah–Hirzebruch spectral sequence for $V(1)_*W = [DV(1), W]_*$ shows that $\bar{m}$ is nullhomotopic, since $H_{2pk-2p+2}(W; \pi_{2p-3}V(1)) = H_{2pk-2p+2}W = 0$. Hence $\bar{m}_1$ and $\bar{m}_2$ are homotopic, as claimed. □

Remark 5.11 A similar proof goes through if R is an E_n ring spectrum with $n \geq 6$, replacing W with $W_n = D_{n,p}\Sigma^{2i}DV(0) \wedge D_{n,p}\Sigma^{2j-1}DV(0)$. For $3 \leq n \leq 5$ the group $H_{2pk-2p+2}W_n$ will be nonzero, due to the presence of E_n Browder bracket terms in this degree, so $\bar{m}$ might map the top cell of $\Sigma^{2pk-1}DV(1)$ via α_1 to a $(2pk-2p+2)$-cell of W_n, and hence be essential. For simplicity we assume $n = \infty$, since this will suffice for our application.

Corollary 5.12 *Let*

$$R \xrightarrow{s} T \xrightarrow{r} R$$

be spectrum maps with rs homotopic to the identity. Assume that R is an E_2 ring spectrum, that T is an E_∞ ring spectrum and that r or s is an E_2 ring map. Then $P^k(xy) = x^p P^j(y)$ in $V(1)_{2pk-1}R$ for $x \in V(0)_{2i}R$, $y \in V(0)_{2j-1}R$ and $k = i + j$.

Proof Apply Proposition 5.10 for T to see that

$$r_*(P^k(s_*x \cdot s_*y)) = r_*((s_*x)^p \cdot P^j(s_*y))$$

in $V(1)_{2k-1}(R)$. If r is an E_2 ring map, then naturality of the products and homotopy power operations with respect to r implies $P^k(r_*s_*x \cdot r_*s_*y) = (r_*s_*x)^p \cdot P^j(r_*s_*y)$. If s is an E_2 ring map, then naturality of the products and homotopy power operations with respect to s implies $r_*s_*(P^k(x \cdot y)) = r_*s_*(x^p \cdot P^j(y))$. In either case the conclusion follows from $r_*s_* = 1$. □

6 Some $V(0)$- and $V(1)$-homotopy classes

The homotopy power operations introduced in Definitions 5.1 and 5.3 apply for $R = S$ with its E_∞ ring structure. The E_2-term of its mod-p Adams spectral sequence

$$E_2^{s,t}(S) = \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}(S)_p^\wedge$$

contains classes traditionally denoted by

$$a_0 = [\tau_0] \quad \text{and} \quad h_i = [\xi_1^{p^i}],$$

for $i \geq 0$, in bidegrees $(s,t) = (1,1)$ and $(1, 2p^i(p-1))$, respectively. Here τ_0 is dual to β and a_0 detects $p \in \pi_0(S)_p^\wedge \cong \mathbb{Z}_p$, while $\xi_1^{p^i}$ is dual to $\mathcal{P}^{p^i}$ and h_0 detects the generator $\alpha_1 \in \pi_{2p-3}(S)_p^\wedge \cong \mathbb{Z}/p$. The classes h_i for $i \geq 1$ support nonzero d_2-differentials [Liulevicius 1962] in the Adams spectral sequence for S, but some of these map to permanent cycles in the corresponding spectral sequences for $V(0)$ and $V(1)$, detecting interesting homotopy classes.

Definition 6.1 Let

$$\beta_1^\circ = P^{p-1}(\alpha_1) \in \pi_{2p^2-2p-1} V(0) \quad \text{and} \quad \gamma_1^\circ = P^{p^2-p}(\beta_1^\circ) \in \pi_{2p^3-2p^2-1} V(1).$$

The ring/circle superscripts indicate that these classes are constructed using the E_2 ring spectrum structure.

Lemma 6.2 *The classes β_1° and γ_1° are detected by $i_0(h_1) = [\xi_1^p]$ and $i_1 i_0(h_2) = [\xi_1^{p^2}]$ in the Adams spectral sequences for $V(0)$ and $V(1)$, respectively.*

Proof The case of β_1° is due to Toda [1968, Lemma 4]. It suffices to prove that the dual Steenrod operation $\mathcal{P}_*^p$ acts nontrivially in the homology of the mapping cone $C\bar{\beta}$, where

$$\bar{\beta}\colon \Sigma^{2p^2-2p-1} DV(0) \simeq \mathrm{Br}_p\, S^{2p-3} \xrightarrow{\mathrm{Br}_p\, \alpha_1} \mathrm{Br}_p\, S \xrightarrow{\theta} S$$

is left adjoint to β_1°. There are natural maps

$$C\bar{\beta} \xleftarrow{\tilde{\theta}} C(\mathrm{Br}_p\, \alpha_1) \xrightarrow{D_{\alpha_1}} \mathrm{Br}_p(C\alpha_1)$$

that are induced by θ and the canonical nullhomotopy in a cone, respectively. By an analog of [Toda 1968, Theorem 2] for braided-extended powers we have

$$D_{\alpha_1 *}((e_{p-1} \otimes x^{\otimes p})^\wedge) = e_0 \otimes (x^\wedge)^{\otimes p},$$

up to a unit in $\mathbb{F}_p$, where $x^\wedge \in H_{2p-2}C\alpha_1$ lifts the generator $x \in H_{2p-3}S^{2p-3}$ and $(e_{p-1} \otimes x^{\otimes p})^\wedge \in H_{2p^2-2p}C(\mathrm{Br}_p\,\alpha_1)$ lifts $e_{p-1} \otimes x^{\otimes p} \in H_{2p^2-2p-1}\,\mathrm{Br}_p\,S^{2p-3}$. Since $\mathcal{P}^1_*(x^\wedge)$ generates $H_0C\alpha_1$, it follows from the homology Cartan formula that $\mathcal{P}^p_*(e_0 \otimes (x^\wedge)^{\otimes p}) = e_0 \otimes \mathcal{P}^1_*(x^\wedge)^{\otimes p}$ generates $H_0\,\mathrm{Br}_p(C\alpha_1)$. By naturality with respect to Toda's map D_{α_1}, it follows that $\mathcal{P}^p_*((e_{p-1} \otimes x^{\otimes p})^\wedge)$ generates $H_0C(\mathrm{Br}_p\,\alpha_1)$, and by naturality with respect to $\tilde{\theta}$ it follows that

$$\mathcal{P}^p_* : H_{2p^2-2p}C\bar{\beta} \to H_0C\bar{\beta}$$

is nonzero.

The proof for γ_1° is similar. It suffices to prove that the dual Steenrod operation $\mathcal{P}^{p^2}_*$ acts nontrivially in the homology of the mapping cone $C\bar{\gamma}$, where

$$\bar{\gamma} : \Sigma^{2p^3-2p^2-1}DV(1) \xrightarrow{\bar{\eta}_1} \mathrm{Br}_p(\Sigma^{2p^2-2p-1}DV(0)) \xrightarrow{\mathrm{Br}_p\,\bar{\beta}} \mathrm{Br}_p\,S \xrightarrow{\theta} S$$

is left adjoint to γ_1°. Here $\bar{\eta}_1$ was defined in (5-2). There are natural maps

$$C\bar{\gamma} \xrightarrow{\tilde{\eta}_1} C(\theta \circ \mathrm{Br}_p\,\bar{\beta}) \xleftarrow{\tilde{\theta}} C(\mathrm{Br}_p\,\bar{\beta}) \xrightarrow{D_{\bar{\beta}}} \mathrm{Br}_p(C\bar{\beta})$$

induced by $\bar{\eta}_1$, θ and the canonical nullhomotopy, respectively. By [Toda 1968, Theorem 2] again, we have

$$D_{\bar{\beta}*}((e_{p-1} \otimes y^{\otimes p})^\wedge) = e_0 \otimes (y^\wedge)^{\otimes p},$$

up to a unit in $\mathbb{F}_p$, where $y^\wedge \in H_{2p^2-2p}C\bar{\beta}$ lifts the generator

$$y \in H_{2p^2-2p-1}(\Sigma^{2p^2-2p-1}DV(0))$$

and $(e_{p-1} \otimes y^{\otimes p})^\wedge \in H_{2p^3-2p^2}C(\mathrm{Br}_p\,\bar{\beta})$ lifts

$$e_{p-1} \otimes y^{\otimes p} \in H_{2p^3-2p^2-1}\,\mathrm{Br}_p(\Sigma^{2p^2-2p-1}DV(0)).$$

Since $\mathcal{P}^p_*(y^\wedge)$ generates $H_0C\bar{\beta}$, it follows that $\mathcal{P}^{p^2}_*(e_0 \otimes (y^\wedge)^{\otimes p}) = e_0 \otimes \mathcal{P}^p_*(y^\wedge)^{\otimes p}$ generates $H_0\,\mathrm{Br}_p(C\bar{\beta})$. Naturality with respect to $D_{\bar{\beta}}$ implies that

$$\mathcal{P}^{p^2}_*((e_{p-1} \otimes y^{\otimes p})^\wedge)$$

generates $H_0C(\mathrm{Br}_p\,\bar{\beta})$, and naturality with respect to $\tilde{\theta}$ and $\tilde{\eta}_1$ implies that

$$\mathcal{P}^{p^2}_* : H_{2p^3-2p^2}C\bar{\gamma} \to H_0C\bar{\gamma}$$

is nonzero. □

The first Greek letter element $\alpha_1 \in \pi_{2p-3}S$ is the image under $j_0 \colon V(0) \to S^1$ of a class $v_1 \in \pi_{2p-2}V(0)$ detected by the class of the cobar cocycle $[\tau_1]1 + [\xi_1]\tau_0$ in bidegree $(s,t) = (1, 2p-1)$ of the Adams spectral sequence

$$E_2^{s,t}(Y) = \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p, H_*Y) \Rightarrow \pi_{t-s}(Y^\wedge_p)$$

for $Y = V(0)$. Similarly, $\beta_1 \in \pi_{2p^2-2p-2}S$ is the image under $j_0 j_1 \colon V(1) \to S^{2p}$ of a class $v_2 \in \pi_{2p^2-2}V(1)$, and $\gamma_1 \in \pi_{2p^3-2p^2-2p-1}S$ is the image under $j_0 j_1 j_2 \colon V(2) \to S^{2p^2+2p-1}$ of a class $v_3 \in \pi_{2p^3-2}V(2)$.

Lemma 6.3 *The groups $\pi_{2p-2}V(0) \cong \mathbb{Z}/p$ for $p \geq 3$, $\pi_{2p^2-2}V(1) \cong \mathbb{Z}/p$ for $p \geq 3$ and $\pi_{2p^3-2}V(2) \cong \mathbb{Z}/p$ for $p \geq 5$ are generated by classes v_1, v_2 and v_3, respectively, each in Adams filtration* 1.

Proof The claim for $V(0)$ is well known. The claim for $V(1)$ is contained in [Toda 1971, Theorem 5.2 and (5.7)]. The claim for $V(2)$ can be deduced from [Toda 1971, Section 3], as follows. Let $\mathcal{P} \subset \mathcal{A}$ be the sub-Hopf algebra of the mod-p Steenrod algebra generated by the Steenrod operations $\mathcal{P}^i$. Let $K = \mathbb{F}_p\{Q_3, \beta Q_3, \dots\}$ be the kernel of the surjection $\mathcal{A} \otimes_{\mathcal{P}} \mathbb{F}_p \to H^*V(2) = E(\beta, Q_1, Q_2)$, where Q_i denotes the Milnor primitive, and consider the long exact sequence

$$\cdots \to \operatorname{Ext}_{\mathcal{A}}^{s-1,t}(K, \mathbb{F}_p) \xrightarrow{\delta} \operatorname{Ext}_{\mathcal{A}}^{s,t}(H^*V(2), \mathbb{F}_p) \to \operatorname{Ext}_{\mathcal{P}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \to \cdots.$$

Using the May spectral sequence, Toda [1971, Section 3] calculated an upper bound for $\operatorname{Ext}_{\mathcal{P}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p)$ in the range $t < 2(p^2+2p+3)(p-1)+4$, which shows that these groups are trivial in topological degrees $t-s = 2p^3-3$ and $2p^3-2$. Hence $\delta(Q_3^*)$ in cohomological degree $s = 1$ is the only generator of $E_2(V(2)) = \operatorname{Ext}_{\mathcal{A}}(H^*V(2), \mathbb{F}_p)$ in topological degree $2p^3-2$. Moreover, there is no possible target for an Adams differential on this class, which must therefore detect v_3. □

Lemma 6.4 *For $p \geq 3$, the classes β_1° and $j_1(v_2) = \beta_1'$ in $\pi_{2p^2-2p-1}V(0)$ agree modulo (a nonzero multiple of) $\alpha_1 v_1^{p-1}$. Hence $i_1(\beta_1^\circ) = i_1(\beta_1')$ in $\pi_{2p^2-2p-1}V(1)$, and $j_0(\beta_1^\circ) = \beta_1 = j_0(\beta_1')$ in $\pi_{2p^2-2p-2}S$ is the first element in the β-family.*

For $p \geq 5$, the classes γ_1° and $j_2(v_3) = \gamma_1''$ in $\pi_{2p^3-2p^2-1}V(1)$ agree modulo $\alpha_1 v_2^{p-1}$. Hence $i_2(\gamma_1^\circ) = i_2(\gamma_1'')$ in $\pi_{2p^3-2p^2-1}V(2)$.

Proof The cobar cocycle $[\tau_2]1 + [\xi_2]\tau_0 + [\xi_1^p]\tau_1$ detects $v_2 \in \pi_{2p^2-2}V(1)$. The $\mathcal{A}_*$-comodule homomorphism $j_{1*}\colon H_*V(1) \to H_{*-2p+1}V(0)$ sends 1 and τ_0 to zero, and maps τ_1 to 1. Hence $j_1\colon E_2^{1,*}(V(1)) \to E_2^{1,*-2p+1}(V(0))$ sends $[\tau_2]1 + [\xi_2]\tau_0 + [\xi_1^p]\tau_1$ to $[\xi_1^p]1 = i_0(h_1)$. This is also the class detecting β_1°, by Lemma 6.2. Therefore $j_1(v_2) = \beta_1'$ and β_1° agree modulo Adams filtration ≥ 2, ie modulo $\alpha_1 v_1^{p-1}$. (We will see in Remark 7.5 that $v_1\beta_1^\circ \neq 0$, while $v_1\beta_1' = 0$, so $\beta_1^\circ - \beta_1'$ is a nonzero multiple of $\alpha_1 v_1^{p-1}$.) Nonetheless, $j_0(\beta_1^\circ) = j_0(\beta_1')$, since $j_0(\alpha_1 v_1^{p-1}) \doteq \alpha_1\alpha_{p-1} = 0$.

The cobar cocycle $[\tau_3]1 + [\xi_3]\tau_0 + [\xi_2^p]\tau_1 + [\xi_1^{p^2}]\tau_2$ detects $v_3 \in \pi_{2p^3-2}V(2)$. The $\mathcal{A}_*$-comodule homomorphism $j_{2*}\colon H_*V(2) \to H_{*-2p^2+1}V(1)$ sends 1, τ_0 and τ_1 to zero, and maps τ_2 to 1. Hence $j_2\colon E_2^{1,*}(V(2)) \to E_2^{1,*-2p^2+1}(V(1))$ sends $[\tau_3]1 + [\xi_3]\tau_0 + [\xi_2^p]\tau_1 + [\xi_1^{p^2}]\tau_2$ to $[\xi_1^{p^2}]1 = i_1 i_0(h_2)$. This is also the class detecting γ_1°, by Lemma 6.2. Therefore $j_2(v_3) = \gamma_1''$ and γ_1° agree modulo Adams filtration ≥ 2, ie modulo $\alpha_1 v_2^{p-1}$. □

Remark 6.5 One way to see that $\alpha_1 v_1^{p-1}$ and $\alpha_1 v_2^{p-1}$ generate Adams filtration ≥ 2 in $\pi_{2p^2-2p-1}V(0)$ and $\pi_{2p^3-2p^2-1}V(1)$, respectively, is to compare with the corresponding Adams–Novikov spectral sequences. By the beginning calculations in [Ravenel 2004, Section 4.4] the classes h_{11} and $h_{10}v_1^{p-1}$ generate the Adams–Novikov E_2-term for $V(0)$ in topological degree $2p^2-2p-1$, while the classes h_{12}

and $h_{10}v_2^{p-1}$ generate the Adams–Novikov E_2-term for $V(1)$ in topological degree $2p^3-2p^2-1$. The formula $\eta_R(v_{n+1}) = v_{n+1} + v_n t_1^{p^n} - v_n^p t_1$ in $\mathrm{BP}_*\mathrm{BP}/I_n$ from [Ravenel 2004, Corollary 4.3.21] shows that $j_n(v_{n+1})$ in $\pi_* V(n-1)$ is detected by $h_{1n} - h_{10}v_n^{p-1}$ when $v_{n+1} \in \pi_* V(n)$ exists, while $\alpha_1 v_n^{p-1}$ is detected by $h_{10}v_n^{p-1}$.

The homotopy power operations also apply to $R = K(\mathrm{BP})$ and $R = \mathrm{THH}(\mathrm{BP})$, with their E_3 ring structures derived from the E_4 ring structure on BP, and to $R = K(\mathrm{BP}\langle n\rangle)$ and $R = \mathrm{THH}(\mathrm{BP}\langle n\rangle)$, with their E_2 ring structures derived from the E_3 ring structure on $\mathrm{BP}\langle n\rangle$. (For $n \le 1$ these are E_∞ ring structures.)

$$\begin{tikzcd}
\pi_* K(\mathrm{BP}) \arrow[r] \arrow[d, "\mathrm{tr}"'] & \pi_* K(\mathrm{BP}\langle n\rangle) \arrow[r] \arrow[d, "\mathrm{tr}"'] & \pi_* K(\mathbb{Z}_{(p)}) \arrow[d, "\mathrm{tr}"'] \\
\pi_* \mathrm{THH}(\mathrm{BP}) \arrow[r] & \pi_* \mathrm{THH}(\mathrm{BP}\langle n\rangle) \arrow[r] & \pi_* \mathrm{THH}(\mathbb{Z}_{(p)})
\end{tikzcd}$$

According to [Bökstedt and Madsen 1994, Theorem 10.14; Rognes 1998, Theorem 1.1] we can find a class $\lambda_1^K \in \pi_{2p-1} K(\mathbb{Z})$ with $\mathrm{tr}(\lambda_1^K) = \lambda_1 \in \pi_{2p-1} \mathrm{THH}(\mathbb{Z})$, having Hurewicz image $h(\lambda_1) = \sigma\bar{\xi}_1 \in H_{2p-1} \mathrm{THH}(\mathbb{Z})$. The same statements apply with $\mathbb{Z}$ replaced by $\mathrm{BP}\langle 0\rangle = H\mathbb{Z}_{(p)}$. The E_4 ring spectrum map $\mathrm{BP} \to H\mathbb{Z}_{(p)}$ is $(2p-2)$-connected, and induces a $(2p-1)$-connected map $K(\mathrm{BP}) \to K(\mathbb{Z}_{(p)})$ by [Bökstedt and Madsen 1994, Proposition 10.9]. Hence we can lift λ_1^K to $\pi_{2p-1} K(\mathrm{BP})$. Its trace image $\mathrm{tr}(\lambda_1^K) \in \pi_{2p-1} \mathrm{THH}(\mathrm{BP}) = \mathbb{Z}_{(p)}\{\lambda_1\}$ then maps to the generator $\lambda_1 \in \pi_{2p-1} \mathrm{THH}(\mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$. It follows that we can scale the choice of $\lambda_1^K \in \pi_{2p-1} K(\mathrm{BP})$ by a p-local unit so as to ensure that $\mathrm{tr}(\lambda_1^K) = \lambda_1$ in $\pi_{2p-1} \mathrm{THH}(\mathrm{BP})$.

Definition 6.6 We fix a choice of a class $\lambda_1^K \in \pi_{2p-1} K(\mathrm{BP})$ with $\mathrm{tr}(\lambda_1^K) = \lambda_1$ in $\pi_{2p-1} \mathrm{THH}(\mathrm{BP})$. These map to classes with the same names in $\pi_{2p-1} K(\mathrm{BP}\langle n\rangle)$ and $\pi_{2p-1} \mathrm{THH}(\mathrm{BP}\langle n\rangle)$, respectively, for each $n \ge 0$.

The choice of $\lambda_1^K \in \pi_{2p-1} K(\mathrm{BP})$ made here is equivalent to the selection of $\lambda_1^K \in \pi_{2p-1} K(\mathrm{BP}\langle 1\rangle)$ discussed in [Ausoni and Rognes 2002, Section 1.2], since $\mathrm{BP} \to \mathrm{BP}\langle 1\rangle = \ell$ is $(2p^2-2)$-connected, which ensures that $K(\mathrm{BP}) \to K(\mathrm{BP}\langle 1\rangle)$ is $(2p^2-1)$-connected.

Definition 6.7 Let $\lambda_2^K = P^p(\lambda_1^K) \in V(0)_{2p^2-1} K(\mathrm{BP})$, mapping to classes with the same name in $V(0)_{2p^2-1} K(\mathrm{BP}\langle n\rangle)$ for each $n \ge 1$.

By naturality of P^p for E_2 ring spectrum maps, this definition agrees with the case $n = 1$ discussed in [Ausoni and Rognes 2002, Section 1.7].

Lemma 6.8 *The classes* $\mathrm{tr}(\lambda_2^K)$ *and* $i_0(\lambda_2)$ *in* $V(0)_{2p^2-1} \mathrm{THH}(\mathrm{BP})$ *both have Hurewicz image* $\sigma\bar{\xi}_2$ *in* $H_{2p^2-1} \mathrm{THH}(\mathrm{BP})$*. Hence they agree modulo* $v_1^p \lambda_1$*, and have the same image in* $V(1)_{2p^2-1} \mathrm{THH}(\mathrm{BP})$*.*

Proof We have $\mathrm{tr}(\lambda_2^K) = \mathrm{tr}(P^p(\lambda_1^K)) = P^p(\mathrm{tr}(\lambda_1^K)) = P^p(\lambda_1)$ by naturality of P^p with respect to tr, and $h_0 P^p(\lambda_1) = Q^p h(\lambda_1) = Q^p(\sigma\bar{\xi}_1)$ by Lemma 5.5. Moreover, $Q^p(\sigma\bar{\xi}_1) = \sigma Q^p(\bar{\xi}_1) = \sigma\bar{\xi}_2$ by [Angeltveit and Rognes 2005, Proposition 5.9; Bruner et al. 1986, Theorem III.2.3]. □

Definition 6.9 Let $\lambda_3^K = P^{p^2}(\lambda_2^K) \in V(1)_{2p^3-1} K(\mathrm{BP})$, mapping to classes with the same name in $V(1)_{2p^3-1} K(\mathrm{BP}\langle n\rangle)$ for each $n \geq 2$.

Lemma 6.10 *The classes*

$$\mathrm{tr}(\lambda_3^K), \quad i_1 i_0(\lambda_3) \quad \textit{and} \quad P^{p^2}(i_0(\lambda_2))$$

in $V(1)_{2p^3-1}\,\mathrm{THH}(\mathrm{BP})$ *all have Hurewicz image* $\sigma\bar{\xi}_3$ *in* $H_{2p^3-1}\,\mathrm{THH}(\mathrm{BP})$. *Hence they agree modulo* $v_2^p\lambda_1$ *and have the same image in* $V(2)_{2p^3-1}\,\mathrm{THH}(\mathrm{BP})$.

Proof We have $\mathrm{tr}(\lambda_3^K) = \mathrm{tr}(P^{p^2}(\lambda_2^K)) = P^{p^2}(\mathrm{tr}(\lambda_2^K))$ by naturality of P^{p^2} with respect to tr, and $h_1 P^{p^2}(\mathrm{tr}(\lambda_2^K)) = Q^{p^2} h_0(\mathrm{tr}(\lambda_2^K)) = Q^{p^2}(\sigma\bar{\xi}_2)$ by Lemmas 5.6 and 6.8. Likewise, $h_1 P^{p^2}(i_0(\lambda_2)) = Q^{p^2} h_0(i_0(\lambda_2)) = Q^{p^2}(\sigma\bar{\xi}_2)$. Finally, $Q^{p^2}(\sigma\bar{\xi}_2) = \sigma Q^{p^2}(\bar{\xi}_2) = \sigma\bar{\xi}_3$ by the same two references as in the previous lemma. □

Let us summarize these results, for later reference:

Proposition 6.11 *Let* $p \geq 7$. *The trace map* $\mathrm{tr}\colon K(B) \to \mathrm{THH}(B)$ *induces ring homomorphisms*

$$V(2)_* K(\mathrm{BP}) \to V(2)_*\,\mathrm{THH}(\mathrm{BP}) \quad \textit{and} \quad V(2)_* K(\mathrm{BP}\langle 2\rangle) \to V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle),$$

each mapping $i_2 i_1 i_0(\lambda_1^K)$, $i_2 i_1(\lambda_2^K)$ *and* $i_2(\lambda_3^K)$ *to* λ_1, λ_2 *and* λ_3, *respectively.*

Proof The claims for BP follow from Definition 6.6 and Lemmas 6.8 and 6.10. The image classes in $V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)$ coincide with the classes from Definition 3.4 since their Hurewicz images in $H_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)$ agree. □

7 Approximate homotopy fixed points

For $C = C_{p^n}$ or $\mathbb{T}$ we have multiplicative homotopy fixed point spectral sequences

$$E^2(C) = H^{-*}(C; V(2)_*\,\mathrm{THH}(B)) \Rightarrow V(2)_*\,\mathrm{THH}(B)^{hC}$$

(cf [Hedenlund and Rognes 2024, Section 5]) and multiplicative Tate spectral sequences

$$\widehat{E}^2(C) = \widehat{H}^{-*}(C; V(2)_*\,\mathrm{THH}(B)) \Rightarrow V(2)_*\,\mathrm{THH}(B)^{tC}$$

(see [loc. cit., Section 6]). Here $H^*(\mathbb{T}) = P(t)$ and $\widehat{H}^*(\mathbb{T}) = P(t^{\pm 1})$ with $t \in H^2 \cong \widehat{H}^2$, while $H^*(C_{p^n}) = E(u_n) \otimes P(t)$ and $\widehat{H}^*(C_{p^n}) = E(u_n) \otimes P(t^{\pm 1})$ with $u_n \in H^1 \cong \widehat{H}^1$. Note that for $B = \mathrm{BP}\langle 2\rangle$, each bidegree of $E^2(C)$ and $\widehat{E}^2(C)$ is either 0 or $\mathbb{F}_p$. This section is devoted to the proof of the following collection of detection results:

Proposition 7.1 *The unit map $S \to K(B)$ and the circle trace map $\mathrm{tr}_{\mathbb{T}} : K(B) \to \mathrm{THH}(B)^{h\mathbb{T}}$ induce ring homomorphisms*

$$V(2)_* \to V(2)_* K(\mathrm{BP}) \to V(2)_* \mathrm{THH}(\mathrm{BP})^{h\mathbb{T}} \to V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$$

mapping $i_2 i_1 i_0(\alpha_1)$, $i_2 i_1(\beta_1^\circ)$, $i_2(\gamma_1^\circ)$ and v_3 to classes detected by $t\lambda_1$, $t^p\lambda_2$, $t^{p^2}\lambda_3$ and $t\mu$, respectively.

Proof By Proposition 7.3 the circle trace image of β_1° is detected by $t^p\lambda_2$ in the $\mathbb{T}$-homotopy fixed point spectral sequence for $V(0) \wedge \mathrm{THH}(\mathrm{BP})$, hence also for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)$.

By Proposition 7.4 the image of γ_1° is detected by $t^{p^2}\lambda_3$ in the spectral sequence for $V(1) \wedge \mathrm{THH}(\mathrm{BP})$, hence also for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)$.

By Proposition 7.6 the image of v_3 is detected by $t\mu$ in the spectral sequence for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)$.

A simpler case of this last argument shows that the image of α_1 is detected by $t\lambda_1$ in the spectral sequence for $\mathrm{THH}(\mathrm{BP})$, hence also for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)$, but this is also readily deduced from the previously known case of $\mathrm{THH}(\mathbb{Z})$. □

Notation 7.2 For any spectral sequence $E^2_{*,*} \Rightarrow G_*$ and nonzero element $x \in E^\infty_{*,*}$ we write $\{x\}$ for the coset of elements $\xi \in G_*$ that are detected by x. Sometimes we will write $[\![x]\!]$ for a specific choice of such an element ξ, so that $[\![x]\!] \in \{x\}$. Similar conventions appear in [Barratt et al. 1970, Proposition 3.1.5; Bruner and Rognes 2021, Theorem 11.61].

For each $\mathbb{T}$-spectrum X and integer $m \geq 0$ we have an m^{th}-order approximate $\mathbb{T}$-homotopy fixed point spectral sequence

$$E^2_{*,*} = \mathbb{Z}[t]/(t^{m+1}) \otimes \pi_*(X) \Rightarrow \pi_* F(S_+^{2m+1}, X)^{\mathbb{T}},$$

obtained by truncating the $\mathbb{T}$-homotopy fixed point spectral sequence to (horizontal) filtration degrees $-2m \leq * \leq 0$.

Proposition 7.3 *Consider the p^{th}-order approximate $\mathbb{T}$-homotopy fixed point spectral sequence*

$$E^2_{*,*} = \mathbb{Z}[t]/(t^{p+1}) \otimes \pi_* \mathrm{THH}(\mathrm{BP}) \Rightarrow \pi_* F(S_+^{2p+1}, \mathrm{THH}(\mathrm{BP}))^{\mathbb{T}}$$

for THH(BP), *and its analog for $V(0) \wedge \mathrm{THH}(\mathrm{BP})$. The circle trace image of $\alpha_1 \in \pi_{2p-3}(S)$ in*

$$\pi_* F(S_+^{2p+1}, \mathrm{THH}(\mathrm{BP}))^{\mathbb{T}}$$

factors as a product $[\![t]\!] \cdot [\![\lambda_1]\!]$, with $[\![t]\!] \in \{t\}$ and $[\![\lambda_1]\!] \in \{\lambda_1\}$ detected by t and λ_1, respectively. Moreover, the image of $\beta_1^\circ \in \pi_{2p^2-2p-1} V(0)$ in

$$V(0)_* F(S_+^{2p+1}, \mathrm{THH}(\mathrm{BP}))^{\mathbb{T}}$$

is the unique class detected by $t^p\lambda_2$.

Proof The p^{th}-order approximate $\mathbb{T}$-homotopy fixed point spectral sequence is multiplicative, and has E^2-term

$$\mathbb{Z}[t]/(t^{p+1}) \otimes \mathbb{Z}_{(p)}\{1, v_1, \lambda_1, v_1^2, v_1\lambda_1, \dots\},$$

with generators as listed in vertical degrees $* < 6p-6$. Here $d^2(v_1) = t \cdot \sigma(v_1) = t \cdot p\lambda_1$, as in Proposition 3.2, and $E^3 = E^\infty$ in this range of degrees. Hence t, λ_1 and $t\lambda_1$ are all infinite cycles, detecting homotopy classes with indeterminacies $\mathbb{Z}_{(p)}\{t^p v_1\}$, $\mathbb{Z}/p\{t^{p-1}v_1\lambda_1\}$ and $\mathbb{Z}/p\{t^p v_1\lambda_1\}$, respectively. The unit map $S \to F(S_+^{2p+1}, \mathrm{THH(BP)})^{\mathbb{T}}$ takes α_1 to a class detected by $t\lambda_1$; cf [Rognes 1998, Theorem 1.4]. Since each element in the indeterminacy of $\{t\lambda_1\}$ factors as an element in the indeterminacy of $\{t\}$ times λ_1 (and also factors as t times an element in the indeterminacy of $\{\lambda_1\}$), it follows that the image of α_1 can be factored as a product $[\![t]\!] \cdot [\![\lambda_1]\!]$ in $\{t\} \cdot \{\lambda_1\}$.

Let $\lambda_2^\circ = \mathrm{tr}(\lambda_2^K) = P^p(\lambda_1)$ in $V(0)_* \mathrm{THH(BP)}$. By the homotopy Cartan formula from Proposition 5.7, applied for the E_3 ring spectrum $F(S_+^{2p+1}, \mathrm{THH(BP)})^{\mathbb{T}}$, the circle trace image of $\beta_1^\circ = P^{p-1}(\alpha_1)$ is

$$P^{p-1}([\![t]\!] \cdot [\![\lambda_1]\!]) = [\![t]\!]^p \cdot P^p([\![\lambda_1]\!]).$$

Here $P^p([\![\lambda_1]\!]) \in \{\lambda_2^\circ\}$ is a class detected by λ_2°, by naturality of P^p with respect to the edge homomorphism induced by $F(S_+^{2p+1}, \mathrm{THH(BP)})^{\mathbb{T}} \to \mathrm{THH(BP)}$. It follows that $[\![t]\!]^p \cdot P^p([\![\lambda_1]\!])$ is detected by $t^p\lambda_2^\circ$, with zero indeterminacy since this class lives in the lowest filtration degree.

To complete the proof, note that $t^p\lambda_2^\circ = t^p\lambda_2$ at the $V(0)$-homotopy E^3-term, since these classes differ by a multiple of $d^2(t^{p-1}v_2) = -t^p v_1^p \lambda_1$ by Proposition 3.2 and Lemma 6.8. □

Proposition 7.4 *Consider the $(p^2)^{th}$-order approximate $\mathbb{T}$-homotopy fixed point spectral sequence*

$$E^2_{*,*} = \mathbb{Z}[t]/(t^{p^2+1}) \otimes V(0)_* \mathrm{THH(BP)} \Rightarrow V(0)_* F(S_+^{2p^2+1}, \mathrm{THH(BP)})^{\mathbb{T}}$$

for $V(0) \wedge \mathrm{THH(BP)}$ and its analog for $V(1) \wedge \mathrm{THH(BP)}$. The circle trace image of $\beta_1^\circ \in \pi_{2p^2-2p-1} V(0)$ in

$$V(0)_* F(S_+^{2p^2+1}, \mathrm{THH(BP)})^{\mathbb{T}}$$

factors as a product $[\![t^p]\!] \cdot [\![\lambda_2]\!]$, with $[\![t^p]\!] \in \{t^p\}$ and $[\![\lambda_2]\!] \in \{\lambda_2\}$ detected by t^p and λ_2, respectively. Moreover, the image of $\gamma_1^\circ \in \pi_{2p^3-2p^2-1} V(1)$ in

$$V(1)_* F(S_+^{2p^2+1}, \mathrm{THH(BP)})^{\mathbb{T}}$$

is the unique class detected by $t^{p^2}\lambda_3$.

Proof Our first goal will be to show that t^p times the indeterminacy in $\{\lambda_2\}$ and λ_2 times the indeterminacy in $\{t^p\}$, in combination, span the indeterminacy in $\{t^p\lambda_2\}$ in the $(p^2)^{\text{th}}$-order spectral sequence for $V(0) \wedge \mathrm{THH(BP)}$. To do this, we compare the m^{th}-order approximate $\mathbb{T}$-homotopy fixed point spectral sequences for the three $\mathbb{T}$-spectra

$$V(1) \wedge \mathrm{THH(BP)}, \quad V(0) \wedge \mathrm{THH(BP)} \quad \text{and} \quad \mathrm{THH(BP)},$$

via the morphisms induced by $i_0 \colon S \to V(0)$, $i_1 \colon V(0) \to V(1)$ and $j_1 \colon V(1) \to \Sigma^{2p-1} V(0)$.

We begin with the $V(1)$-homotopy spectral sequence, which is easiest to understand. The m^{th}-order spectral sequence for $V(1) \wedge \mathrm{THH}(\mathrm{BP})$ has E^2-term

$$P_{m+1}(t) \otimes P(v_2, \dots) \otimes E(\lambda_1, \lambda_2, \dots),$$

where the omitted generators have vertical degree $* \geq 2p^3 - 2$. Here v_2, λ_1 and λ_2 are infinite cycles, since multiplication by v_2 is realized by a self-map of $V(1)$ and since λ_1 and λ_2 detect the circle trace images of λ_1^K and λ_2^K, respectively. For $m = p$ it follows that this spectral sequence collapses at the E^2-term, in vertical degrees $* < 2p^3 - 2$.

For $m > p$ there are nonzero d^{2p}-differentials generated by

$$d^{2p}(t) \doteq t^{p+1}\lambda_1,$$

where $x \doteq y$ means that x is a unit (in $\mathbb{F}_p$) times y. This differential is present already in the $\mathbb{T}$-homotopy fixed point spectral sequence for THH(BP), and lifts that of [Bökstedt and Madsen 1994, Theorem 5.8(i)] for $\mathrm{THH}(\mathbb{Z}_{(p)})$ over the morphism of spectral sequences induced by $\mathrm{BP} \to H\mathbb{Z}_{(p)}$. It follows that the m^{th}-order E^{2p+1}-term equals

$$\mathbb{F}_p\{t^i \mid 0 \leq i \leq m,\ p \mid i\} \otimes P(v_2) \otimes E(\lambda_1, \lambda_2)$$

in vertical degrees $* < 2p^3 - 2$, plus some extra classes in even filtrations $-2m \leq * < -2m + 2p$ and $-2p < * \leq 0$ that survive due to being close to the truncation limits. Moreover, for $m < p^2 + p$ the spectral sequence must collapse at this stage, for these vertical degrees, since there is no room for a differential on t^p.

For later use, note that when $m = 3p - 2$ no classes survive in total degree $* = 2p^2 - 2p - 2i$ for $2 \leq i < p$, since the classes $t^{i+p-1}v_2$ support differentials and the classes $t^{i+2p-1}\lambda_1\lambda_2$ are hit by differentials. Hence $V(1)_* F(S_+^{2m+1}, \mathrm{THH}(\mathrm{BP}))^{\mathbb{T}}$ is zero in these degrees. Moreover, for $i = 1$ only the classes $t^{2p}\lambda_1\lambda_2$ and $t^p v_2$ survive in total degree $* = 2p^2 - 2p - 2$, and here $i_1 j_1(t^p v_2)$ is detected by $t^{2p}\lambda_2 \neq 0$, so only $t^{2p}\lambda_1\lambda_2$ can be (and is) in the image of i_1, since $j_1 i_1 = 0$. Hence the image of i_1 is isomorphic to $\mathbb{Z}/p$ in this degree.

We now turn to the $V(0)$-homotopy spectral sequence. The $(p^2)^{\text{th}}$-order approximate $\mathbb{T}$-homotopy fixed point spectral sequence for $V(0) \wedge \mathrm{THH}(\mathrm{BP})$ has E^2-term

$$P_{p^2+1}(t) \otimes P(v_1, v_2, \dots) \otimes E(\lambda_1, \lambda_2, \dots),$$

where the omitted generators have vertical degree $* \geq 2p^3 - 2$. Here t, v_1, λ_1 and λ_2 are d^2-cycles, while $d^2(v_2) = -tv_1^p\lambda_1$ by Proposition 3.2. Hence the E^3-term equals

$$P_{p^2+1}(t) \otimes (P(v_1)\{1\} \oplus P_p(v_1)\{\lambda_1, v_2\lambda_1, \dots, v_2^{p-1}\lambda_1\}) \otimes E(\lambda_2)$$

in vertical degrees $* < 2p^3 - 2p$, except that there are some additional classes in filtration degrees 0 and $-2p^2$; see Figure 1, which is drawn for $p = 3$, and hence is not quite to scale for the primes $p \geq 7$ under

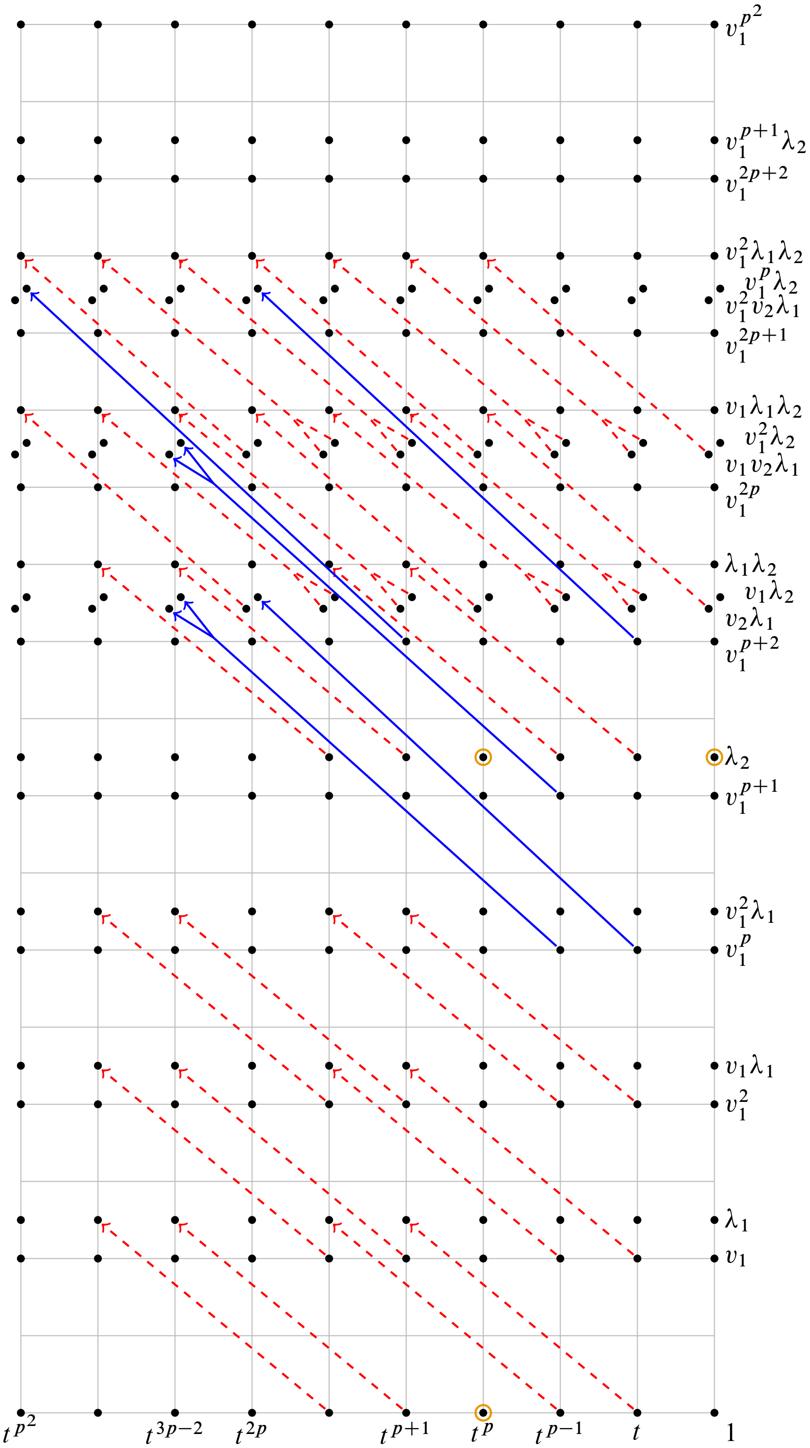

Figure 1: $E^3 \Rightarrow V(0)_*F(S^{2p^2+1},\mathrm{THH}(\mathrm{BP}))^{\mathbb{T}}$ in vertical degrees $* < 4p^2+2p-5$, with all d^{2p}-differentials (dashed) and selected d^{4p-2}-differentials (solid).

consideration. As above, we know that the classes v_1, λ_1 and λ_2 are infinite cycles. The next nonzero differentials are

$$d^{2p}(t) \doteq t^{p+1}\lambda_1 \quad \text{and} \quad d^{2p}(v_2\lambda_1) \doteq t^p v_1\lambda_1\lambda_2.$$

The d^{2p}-differential on t for $V(0) \wedge \mathrm{THH(BP)}$ follows, as above, from the one for THH(BP). The earlier differential $d^{2p-2}(v_2\lambda_1) \in \mathbb{F}_p\{t^{p-1}v_1^{p+3}\}$ must vanish by tv_1^p-linearity, since $tv_1^p \cdot v_2\lambda_1 = 0$ and $tv_1^p \cdot t^{p-1}v_1^{p+3} \neq 0$. If $d^{2p}(v_2\lambda_1)$ were zero, $v_2\lambda_1$ would detect a class in $V(0)_*F(S_+^{2p+1}, \mathrm{THH(BP)})^{\mathbb{T}}$ that maps under $i_1 : V(0) \to V(1)$ to the class in $V(1)_*F(S_+^{2p+1}, \mathrm{THH(BP)})^{\mathbb{T}}$ detected by $v_2\lambda_1$. However, the latter class maps under $i_1 j_1 : V(1) \to \Sigma^{2p-1}V(1)$ to the nonzero class $i_1 j_1(v_2\lambda_1) = i_1(\beta_1')\lambda_1 = i_1(\beta_1^\circ)\lambda_1$ detected by $t^p\lambda_2 \cdot \lambda_1 = -t^p \cdot \lambda_1\lambda_2$, as follows from Lemma 6.4 and Proposition 7.3. This contradicts $j_1 i_1 = 0$, and proves that $d^{2p}(v_2\lambda_1)$ is nonzero in $\mathbb{F}_p\{t^p v_1\lambda_1\lambda_2\}$.

It follows that the E^{2p+1}-term equals

$$P_{p+1}(t^p) \otimes (P(v_1)\{1, \lambda_2\} \oplus P_p(v_1)\{\lambda_1\} \oplus \mathbb{F}_p\{\lambda_1\lambda_2, v_1^{p-1}v_2\lambda_1\}) \\ \oplus \mathbb{F}_p\{t^i \mid 0 < i < p^2,\ p \nmid i\} \otimes (P(v_1)\{v_1^p, v_1\lambda_2 + cv_2\lambda_1\} \oplus \mathbb{F}_p\{v_1^{p-1}v_2\lambda_1\})$$

in vertical degrees $* < 4p^2 + 2p - 5$, plus some extra classes in even filtrations $-2p^2 \le * < -2p^2 + 2p$ and $-2p < * \le 0$. In the expression $v_1\lambda_2 + cv_2\lambda_1$, the coefficient c (which will vary with the t-exponent i) is some unit in $\mathbb{F}_p$.

The next differentials include

$$d^{4p-2}(tv_1^p) \doteq t^{2p}v_1\lambda_2 \quad \text{and} \quad d^{4p-2}(t^i v_1^p) \doteq t^{i+2p-1}(v_1\lambda_2 + cv_2\lambda_1)$$

for $2 \le i < p$. To see that these are nonzero, we compare the m^{th}-order spectral sequences for $V(0) \wedge \mathrm{THH(BP)}$ and $V(1) \wedge \mathrm{THH(BP)}$, in the particular case $m = 3p-2$. If $d^{4p-2}(t^i v_1^p)$ were zero in the former, then $t^i v_1^p$ would survive to detect a class in degree $2p^2 - 2p - 2i$ of $V(0)_*F(S_+^{2m+1}, \mathrm{THH(BP)})^{\mathbb{T}}$ that cannot be a v_1-multiple, for filtration reasons, and which must therefore have nonzero image in $V(1)_*F(S_+^{2m+1}, \mathrm{THH(BP)})^{\mathbb{T}}$. However, for $2 \le i < p$ we checked above that this graded abelian group is zero in these degrees. The assumption that $d^{4p-2}(t^i v_1^p)$ is zero therefore leads to a contradiction, which shows that this class is nonzero in $\mathbb{F}_p\{t^{i+2p-1}(v_1\lambda_2 + cv_2\lambda_1)\}$, as claimed.

Furthermore, for $i = 1$ it is not possible that both $t^{2p}\lambda_1\lambda_2$ and tv_1^p survive to E^∞, since then the image of i_1 in degree $2p^2 - 2p - 2$ would have order p^2, rather than the order p that we established above. Hence $d^{4p-2}(tv_1^p)$ must be nonzero in $\mathbb{F}_p\{t^{2p}v_1\lambda_2, t^{2p}v_2\lambda_1\}$. Extending to the case $m = 3p$ shows that $d^{4p-2}(tv_1^p)$ must be nonzero in $\mathbb{F}_p\{t^{2p}v_1\lambda_2\}$, as claimed.

We can now conclude that t^p is an infinite cycle in the spectral sequence converging to

$$V(0)_*F(S_+^{2p^2+1}, \mathrm{THH(BP)})^{\mathbb{T}},$$

since there are no possible targets for later differentials, and the indeterminacy in $\{t^p\}$ is generated by (classes detected by)

$$t^{p^2-p+1}v_1^{p-1} \quad \text{and} \quad t^{p^2}v_1^p.$$

The class $t^p\lambda_2$ is also an infinite cycle, detecting the circle trace image of β_1° by Proposition 7.3, and has indeterminacy generated by (a subset of)

$$t^{2p-1}(v_1\lambda_2 + cv_2\lambda_1), \quad t^{p^2-p+1}v_1^{p-1}\lambda_2 \quad \text{and} \quad t^{p^2}v_1^{p-1}v_2\lambda_1.$$

Likewise, λ_2 is an infinite cycle, detecting the circle trace image of λ_2^K plus some multiple of $v_1^p\lambda_1^K$ according to Lemma 6.8, with indeterminacy generated by (a subset of)

$$t^{p-1}(v_1\lambda_2 + cv_2\lambda_1), \quad t^{2p-2}(v_1^2\lambda_2 + cv_1v_2\lambda_1), \quad t^{p^2-p}v_1^{p-1}v_2\lambda_1 \quad \text{and} \quad t^{p^2-1}v_1^{p+1}\lambda_2.$$

Here $t^{p-1}(v_1\lambda_2 + cv_2\lambda_1)$ might support a nonzero d^r-differential and not be an infinite cycle. However, there are no possible targets in filtrations $-2p^2 \le * < -2p^2 + 2p$ of such a d^r-differential, since $d^{4p-2}(t^{p^2-1}v_1^{2p+2}) = t^{p^2+2p-2}v_1^{p+3}\lambda_2 \neq 0$ in the full $\mathbb{T}$-homotopy fixed point spectral sequence. Hence in this case $t^{2p-1}(v_1\lambda_2 + cv_2\lambda_1)$ will also support a nonzero differential, of the same length, and also not be an infinite cycle. Similarly, if $t^{p^2-p}v_1^{p-1}v_2\lambda_1$ is hit by a d^r-differential, then $t^{p^2}v_1^{p-1}v_2\lambda_1$ will be hit by a differential of the same length.

It follows that t^p times the indeterminacy in $\{\lambda_2\}$, together with the class $t^{p^2-p+1}v_1^{p-1}\lambda_2$, span the indeterminacy in $\{t^p\lambda_2\}$. That extra class lies in the indeterminacy of $\{t^p\}$ times λ_2. Hence we have achieved our first goal, as formulated at the outset of the proof.

Now choose classes x and y in $V(0)_*F(S_+^{2p^2+1}, \mathrm{THH}(\mathrm{BP}))^{\mathbb{T}}$, detected by t^p and λ_2, respectively. Then the difference between the circle trace image of β_1° and the product xy lies in the indeterminacy of $\{t^p\lambda_2\}$. By modifying the choices of x and y, within the indeterminacies of $\{t^p\}$ and $\{\lambda_2\}$, respectively, we can reduce the filtration of this difference until it becomes zero. Let $[\![t^p]\!] = x$ and $[\![\lambda_2]\!] = y$ be the final values of $x \in \{t^p\}$ and $y \in \{\lambda_2\}$, so that the circle trace image of β_1° equals the product $[\![t^p]\!] \cdot [\![\lambda_2]\!]$.

Let $\lambda_3^\circ = P^{p^2}(\lambda_2)$ in $V(1)_*\,\mathrm{THH}(\mathrm{BP})$. We apply the Cartan formula from Corollary 5.12 in the case of the E_3 ring spectrum retract $F(S_+^{2p^2+1}, \mathrm{THH}(\mathrm{BP}))^{\mathbb{T}}$ of $F(S_+^{2p^2+1}, \mathrm{THH}(\mathrm{MU}_{(p)}))^{\mathbb{T}}$, where the latter is an E_∞ ring spectrum. It asserts that the circle trace image of $\gamma_1^\circ = P^{p^2-p}(\beta_1^\circ)$ is

$$P^{p^2-p}([\![t^p]\!] \cdot [\![\lambda_2]\!]) = [\![t^p]\!]^p \cdot P^{p^2}([\![\lambda_2]\!]).$$

Here $P^{p^2}([\![\lambda_2]\!]) \in \{\lambda_3^\circ\}$ is a class detected by λ_3°, by naturality of P^{p^2} with respect to the edge homomorphism induced by $F(S_+^{2p^2+1}, \mathrm{THH}(\mathrm{BP}))^{\mathbb{T}} \to \mathrm{THH}(\mathrm{BP})$. It follows that $[\![t^p]\!]^p \cdot P^{p^2}([\![\lambda_2]\!])$ is detected by $t^{p^2}\lambda_3^\circ$, with zero indeterminacy since this class lives in the lowest filtration degree.

To complete the proof, note that $t^{p^2}\lambda_3^\circ = t^{p^2}\lambda_3$ at the $V(1)$-homotopy E^3-term, since these classes differ by a multiple of $d^2(t^{p^2-1}v_3) = -t^{p^2}v_2^p\lambda_1$ by Proposition 3.2 and Lemma 6.10. □

Remark 7.5 In the course of the previous proof, we have seen that the circle trace image of $\beta_1^\circ \in V(0)_*$ is detected by $t^p\lambda_2$, and that $t^pv_1\lambda_2$ is not a boundary in the (approximate) $\mathbb{T}$-homotopy fixed point spectral sequence, which implies that $v_1 \cdot \beta_1^\circ \neq 0$. This confirms a claim made in the proof of Lemma 6.4.

Proposition 7.6 *Consider the first-order (approximate $\mathbb{T}$-homotopy fixed point) spectral sequence*

$$E^2_{*,*} = \mathbb{Z}[t]/(t^2) \otimes V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle) \Rightarrow V(2)_* F(S^3_+, \operatorname{THH}(\mathrm{BP}\langle 2\rangle))^{\mathbb{T}}$$

for $V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)$. The circle trace image of $v_3 \in \pi_{2p^3-2} V(2)$ in

$$V(2)_* F(S^3_+, \operatorname{THH}(\mathrm{BP}\langle 2\rangle))^{\mathbb{T}}$$

is the unique class detected by $t\mu$.

Proof The line of argument is the same as for the case of $v_2 \in \pi_{2p^2-2} V(1)$ in [Ausoni and Rognes 2002, Proposition 4.8]. For brevity, let $Y = F(S^3_+, \operatorname{THH}(\mathrm{BP}\langle 2\rangle))^{\mathbb{T}}$. We have a map of mod-$p$ Adams spectral sequences

$$E_2(V(2)) = \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, H_* V(2)) \to \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(V(2) \wedge Y)) = E_2(V(2) \wedge Y),$$

where v_3 is detected in the source in bidegree $(s,t) = (1, 2p^3 - 1)$ by the class of the cobar cocycle

$$x = [\tau_3]1 + [\xi_3]\tau_0 + [\xi_2^p]\tau_1 + [\xi_1^{p^2}]\tau_2$$

in $E_1^{1,*}(V(2)) = \bar{\mathcal{A}}_* \otimes H_* V(2)$. (As usual, $\bar{\mathcal{A}}_*$ denotes the cokernel of the unit $\mathbb{F}_p \to \mathcal{A}_*$.) We claim that this cocycle does not become a coboundary when mapped to $E_1^{1,*}(V(2) \wedge Y) = \bar{\mathcal{A}}_* \otimes H_*(V(2) \wedge Y)$. This implies that the image of v_3 is nonzero in $V(2)_*(Y)$, and in view of Proposition 3.3 the only possible detecting class in its total degree is $t\mu$.

To prove the claim we use the first-order spectral sequence for $H \wedge V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)$, which reduces to a long exact sequence, leading to an extension

$$0 \to \operatorname{cok}(\sigma) \to H_*(V(2) \wedge Y) \to \ker(\sigma) \to 0$$

of $\mathcal{A}_*$-comodules. Here

$$\sigma\colon H_*(V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \to H_{*+1}(V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle))$$

acts on $H_*(V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \cong \mathcal{A}_* \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \sigma\bar{\xi}_3) \otimes P(\sigma\bar{\tau}_3)$, as per Proposition 3.1. The cocycle x is a cobar coboundary only if there is a class $y \in E_1^{0,*}(V(2) \wedge Y) = H_*(V(2) \wedge Y)$ with $\mathcal{A}_*$-comodule coaction $\nu(y)$ containing the term $\tau_3 \otimes 1$.

There is no such class $y \in \operatorname{cok}(\sigma)$, since this $\mathcal{A}_*$-subcomodule does not contain the algebra unit 1. Moreover, since $\sigma(\bar{\tau}_3) = \sigma\bar{\tau}_3 \neq 0$, the class $\bar{\tau}_3$ is not in $\ker(\sigma)$. Hence $\ker(\sigma)$ in total degree $2p^3 - 1$ is generated by polynomials in $\bar{\tau}_0$, $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{\xi}_1$, $\bar{\xi}_2$, $\bar{\xi}_3$, $\sigma\bar{\xi}_1$, $\sigma\bar{\xi}_2$ and $\sigma\bar{\xi}_3$, none of which have $\mathcal{A}_*$-coaction that involves τ_3. This proves that no such class y exists, and x is not a coboundary. □

8 The C_p-Tate spectral sequence

We now establish an effective version of the C_p-equivariant Segal conjecture (or homotopy limit property) for $V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)$, by direct computation. The corresponding results for the groups C_{p^n} and $\mathbb{T}$ then follow from a theorem of Tsalidis. The analogous results for $\mathrm{BP}\langle 0\rangle = H\mathbb{Z}_{(p)}$ and $\mathrm{BP}\langle 1\rangle = \ell$ were proved in [Bökstedt and Madsen 1994, Theorem 5.8(i)] and [Ausoni and Rognes 2002, Theorem 5.5], respectively.

Theorem 8.1 *The C_p-Tate spectral sequence*

$$\widehat{E}^2(C_p) = \widehat{H}^{-*}(C_p; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$$

has E^2-term

$$\widehat{E}^2(C_p) = E(u_1) \otimes P(t^{\pm 1}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

There are differentials

$$d^{2p}(t^{1-p}) \doteq t\lambda_1, \quad d^{2p^2}(t^{p-p^2}) \doteq t^p\lambda_2, \quad d^{2p^3}(t^{p^2-p^3}) \doteq t^{p^2}\lambda_3 \quad \text{and} \quad d^{2p^3+1}(u_1 t^{-p^3}) \doteq t\mu,$$

and the classes $\lambda_1, \lambda_2, \lambda_3$ and $t^{\pm p^3}$ are permanent cycles. The E^∞-term

$$\widehat{E}^\infty = P(t^{\pm p^3}) \otimes E(\lambda_1, \lambda_2, \lambda_3)$$

is the associated graded of

$$V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p} \cong E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm 1}).$$

The comparison map $\widehat{\Gamma}_1\colon \operatorname{THH}(\mathrm{BP}\langle 2\rangle) \to \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$ induces the localization homomorphism

$$V(2)_*\widehat{\Gamma}_1\colon E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu) \to E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm 1}),$$

which is $(2p^2+2p-3)$-coconnected.

Proof The circle trace map $K(B) \to \operatorname{THH}(B)^{h\mathbb{T}}$ lifts the trace map, so by Proposition 6.11 the classes λ_i^K for $i \in \{1, 2, 3\}$ map to classes in $V(2)_* \operatorname{THH}(B)^{h\mathbb{T}}$ detected by the λ_i. Similarly, by Proposition 7.1 the class v_3 in $\pi_* V(2)$ maps to a class detected by $t\mu$. Hence these detecting classes are infinite cycles in all of the C-homotopy fixed point and C-Tate spectral sequences. This means that in order to determine the d^r-differentials in one of these spectral sequences, it suffices to determine $d^r(x)$ for x ranging through a $P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)$-module basis for the E^r-term.

The unit map $S \to \operatorname{THH}(B)$ factors through B, and $V(2)_*\mathrm{BP}\langle 2\rangle = \mathbb{F}_p$, so the images of α_1, β_1°, γ_1° and v_3 in $\pi_* V(2)$ map to zero in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)$ and $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$. Hence the four classes $t\lambda_1$, $t^p\lambda_2$, $t^{p^2}\lambda_3$ and $t\mu$ must all be boundaries in the C_p-Tate spectral sequence.

The first possible (nonzero) d^r-differentials on u_1 and $t^{\pm 1}$ in $\widehat{E}^2(C_p)$ have $r = 2p$. We know that $t\lambda_1$ is a boundary, so

$$d^{2p}(t^{1-p}) \doteq t\lambda_1.$$

Also $d^{2p}(u_1) \in \mathbb{F}_p\{u_1 t^p \lambda_1\}$, so $d^{2p}(u_1 t^{m_1}) = 0$ for some integer m_1 defined mod p. Hence

$$\widehat{E}^{2p+1}(C_p) = E(u_1 t^{m_1}) \otimes P(t^{\pm p}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

The next possible d^r-differentials on $u_1 t^{m_1}$ and $t^{\pm p}$ have $r = 2p^2$. We know that $t^p\lambda_2$ is a boundary, so

$$d^{2p^2}(t^{p-p^2}) \doteq t^p\lambda_2.$$

Also $d^{2p^2}(u_1 t^{m_1}) \in \mathbb{F}_p\{u_1 t^{m_1+p^2}\lambda_2\}$, so $d^{2p^2}(u_1 t^{m_2}) = 0$ for some integer m_2 defined mod p^2, with $m_2 \equiv m_1 \bmod p$. Then

$$\widehat{E}^{2p^2+1}(C_p) = E(u_1 t^{m_2}) \otimes P(t^{\pm p^2}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

If $m_2 \equiv -p \mod p^2$ then the first possible differential on $u_1 t^{m_2}$ is $d^r(u_1 t^{m_2}) \in \mathbb{F}_p\{t^{m_2+p^2+p}\lambda_1\lambda_2\}$ with $r = 2p^2+2p-1$. Otherwise, the first possible differential on $u_1 t^{m_2}$ has $r = 2p^3$.

By naturality with respect to the group cohomology transfer (Verschiebung), with $V(t^i) = 0$ and $V(u_1 t^i) = u_2 t^i$, the first possible d^r-differential on $t^{\pm p^2}$ cannot take a value of the form $u_1 x$, and hence has $r = 2p^3$; cf [Ausoni and Rognes 2002, Lemma 5.2].

We know that $t^{p^2}\lambda_3$ is a boundary, and the only possible sources in $\widehat{E}^2(C_p)$ of a d^r-differential with this target are $t^{-p^3+2p^2+p-1}\lambda_1\lambda_2$ with $r = 2p^3-2p^2-2p+2$, $u_1 t^{-p^3+2p^2-1}\lambda_2$ with $r = 2p^3-2p^2+1$, $u_1 t^{-p^3+p^2+p-1}\lambda_1$ with $r = 2p^3-2p+1$ and $t^{-p^3+p^2}$ with $r = 2p^3$. The first source is not present in $\widehat{E}^{2p^2+1}(C_p)$, and the second and third sources are present there only if $m_2 \equiv -1 \mod p^2$ or $m_2 \equiv p-1 \mod p^2$, respectively. In both of these cases $m_2 \not\equiv -p \mod p^2$, so $u_1 t^{m_2}$ survives to the E^{2p^3}-term. In the second case

$$d^{2p^3-2p^2+1}(u_1 t^{-p^3+2p^2-1}\lambda_2) = d^{2p^3-2p^2+1}(u_1 t^{-p^3+2p^2-1})\lambda_2 = 0,$$

while in the third case

$$d^{2p^3-2p+1}(u_1 t^{-p^3+p^2+p-1}\lambda_1) = d^{2p^3-2p+1}(u_1 t^{-p^3+p^2+p-1})\lambda_1 = 0.$$

Hence the fourth option,

$$d^{2p^3}(t^{-p^3+p^2}) \doteq t^{p^2}\lambda_3,$$

is the only possibility.

We also know that $t\mu$ is a boundary, and the only possible sources of a d^r-differential with this target are $u_1 t^{-p^3+p^2+p-1}\lambda_1\lambda_2$ with $r = 2p^3-2p^2-2p+3$, $t^{-p^3+p^2}\lambda_2$ with $r = 2p^3-2p^2+2$, $t^{-p^3+p}\lambda_1$ with $r = 2p^3-2p+2$ and $u_1 t^{-p^3}$ with $r = 2p^3+1$. The first source is only present in $\widehat{E}^{2p^2+1}(C_p)$ if $m_2 \equiv p-1 \mod p^2$, in which case $u_1 t^{m_2}$ survives to the E^{2p^3}-term, and

$$d^{2p^3-2p^2-2p+3}(u_1 t^{-p^3+p^2+p-1}\lambda_1\lambda_2) = d^{2p^3-2p^2-2p+3}(u_1 t^{-p^3+p^2+p-1})\lambda_1\lambda_2 = 0.$$

In the second case

$$d^{2p^3-2p^2+2}(t^{-p^3+p^2}\lambda_2) = d^{2p^3-2p^2+2}((t^{-p^2})^{p-1})\lambda_2 = 0,$$

since $t^{\pm p^2}$ survive to the E^{2p^3}-term. The third source is not present in $\widehat{E}^{2p^2+1}(C_p)$. This leaves the fourth option,

$$d^{2p^3+1}(u_1 t^{-p^3}) \doteq t\mu,$$

as the only possibility. It follows that $d^{2p^3}(u_1 t^{-p^3}) = 0$. In particular, $u_1 t^{-p^3}$ must be present in $\widehat{E}^{2p^2+1}(C_p)$, and we may take $m_1 = m_2 = 0$ in the formulas above. Then

$$\widehat{E}^{2p^3+1}(C_p) = E(u_1 t^{-p^3}) \otimes P(t^{\pm p^3}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

Here $d^{2p^3+1}(t^{-p^3})$ lies in a trivial group, so

$$\widehat{E}^{2p^3+2}(C_p) = P(t^{\pm p^3}) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

This equals $\widehat{E}^{\infty}(C_p)$, since there are no further targets for differentials on t^{-p^3}.

We claim that $\widehat{\Gamma}_1(\mu)$ in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$ is detected by a unit times t^{-p^3}. To see this, we can use naturality with respect to the map $\mathrm{BP}\langle 2\rangle \to \mathrm{BP}\langle 1\rangle$, as in the commutative diagram below.

$$\begin{tikzcd}
H_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle) \arrow[d] & V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle) \arrow[l, "h_2"'] \arrow[r, "\widehat{\Gamma}_1"] \arrow[d] & V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p} \arrow[d] \\
H_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle) \arrow[d, equal] & V(2)_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle) \arrow[l, "h_2"'] \arrow[r, "\widehat{\Gamma}_1"] & V(2)_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle)^{tC_p} \\
H_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle) & V(1)_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle) \arrow[l, "h_1"'] \arrow[r, "\widehat{\Gamma}_1"] \arrow[u, "i_2"] & V(1)_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle)^{tC_p} \arrow[u, "i_2"]
\end{tikzcd}$$

Recall from Proposition 3.3 that $V(1)_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle) = E(\lambda_1, \lambda_2) \otimes P(\mu_2)$, where $h_1(\mu_2) = \sigma\bar{\tau}_2$ in $H_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle)$, and that $\widehat{\Gamma}_1(\mu_2)$ in $V(1)_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle)^{tC_p}$ is detected by a unit times t^{-p^2} by the proof of [Ausoni and Rognes 2002, Theorem 5.5]. It follows that μ maps to $i_2(\mu_2^p)$ in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 1\rangle)$, since $h_2(\mu) = \sigma\bar{\tau}_3$ maps to $h_1(\mu_2^p) = (\sigma\bar{\tau}_2)^p = \sigma\bar{\tau}_3$. By naturality, $\widehat{\Gamma}_1(\mu)$ maps to a class detected by a unit times $(t^{-p^2})^p = t^{-p^3}$, which proves the claim.

The highest-degree class in $E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm 1})$ that is not in the image from $E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu)$ is $\lambda_1\lambda_2\lambda_3\mu^{-1}$, in degree $(2p-1)+(2p^2-1)+(2p^3-1)-(2p^3) = 2p^2+2p-3$. Hence $V(2)_*\widehat{\Gamma}_1$ is injective in this degree, and an isomorphism in all higher degrees. □

Corollary 8.2 [Tsalidis 1998, Theorem 2.4; Bökstedt et al. 2014, Theorem 2.8] *The comparison maps*

$$\Gamma_n \colon V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{C_{p^n}} \to V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{hC_{p^n}},$$
$$\widehat{\Gamma}_n \colon V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{C_{p^{n-1}}} \to V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^n}},$$

for $n \geq 1$, *and their homotopy limits*

$$\Gamma \colon V(2) \wedge \operatorname{TF}(\mathrm{BP}\langle 2\rangle) \to V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}} \quad \textit{and} \quad \widehat{\Gamma} \colon V(2) \wedge \operatorname{TF}(\mathrm{BP}\langle 2\rangle) \to V(2) \wedge \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}},$$

are all $(2p^2+2p-3)$*-coconnected.*

9 The C_{p^2}-Tate spectral sequence

Our next goal is to determine the differential structure of the C_{p^n}-Tate spectral sequence converging to $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^n}}$, for each $n \geq 2$. There are some minor differences between the cases $n = 2$ and $n \geq 3$, so we spell out the C_{p^2} case in this section, including some motivation, and leave the notationally more elaborate cases $n \geq 3$ for the next section.

We first determine the structure of the C_p-homotopy fixed point spectral sequence from that of the C_p-Tate spectral sequence, using the homotopy restriction morphism (also known as the canonical morphism)

$$R^h \colon E^r(C_p) \to \widehat{E}^r(C_p).$$

It is algebraically simpler to work with the localized spectral sequence $\mu^{-1}E^r(C_p)$, keeping in mind that

$$E^r(C_p) \to \mu^{-1}E^r(C_p)$$

is $(2p^2+2p-3)$-coconnected. In view of Theorem 8.1, the μ-localized C_p-homotopy fixed point spectral sequence for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)$ is isomorphic to the C_p-homotopy fixed point spectral sequence for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$.

Proposition 9.1 *The μ-localized C_p-homotopy fixed point spectral sequence*

$$\mu^{-1}E^2(C_p) = H^{-*}(C_p; \mu^{-1}V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow \mu^{-1}V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{hC_p}$$

has E^2-term

$$\mu^{-1}E^2(C_p) = E(u_1) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm 1}).$$

There are differentials

$$d^{2p}(\mu) \doteq (t\mu)^p \lambda_1 \mu^{1-p}, \qquad d^{2p^2}(\mu^p) \doteq (t\mu)^{p^2} \lambda_2 \mu^{p-p^2},$$
$$d^{2p^3}(\mu^{p^2}) \doteq (t\mu)^{p^3} \lambda_3 \mu^{p^2-p^3}, \quad d^{2p^3+1}(u_1 \mu^{p^3}) \doteq (t\mu)^{p^3+1},$$

and the classes $t\mu$, λ_1, λ_2, λ_3 and $\mu^{\pm p^3}$ are permanent cycles.

Proof The composite relations

$$\begin{aligned}
d^{2p}(\mu) \cdot \mu^p &= d^{2p}(t\mu \cdot t^{-1}) \cdot \mu^p \doteq t\mu \cdot t^{p-1}\lambda_1 \cdot \mu^p = (t\mu)^p \lambda_1 \mu, \\
d^{2p^2}(\mu^p) \cdot \mu^{p^2} &= d^{2p^2}((t\mu)^p \cdot t^{-p}) \cdot \mu^{p^2} \doteq (t\mu)^p \cdot t^{p^2-p}\lambda_2 \cdot \mu^{p^2} = (t\mu)^{p^2} \lambda_2 \mu^p, \\
d^{2p^3}(\mu^{p^2}) \cdot \mu^{p^3} &= d^{2p^3}((t\mu)^{p^2} \cdot t^{-p^2}) \cdot \mu^{p^3} \doteq (t\mu)^{p^2} \cdot t^{p^3-p^2}\lambda_3 \cdot \mu^{p^3} = (t\mu)^{p^3} \lambda_3 \mu^{p^2}, \\
d^{2p^3+1}(u_1 \mu^{p^3}) &= d^{2p^3+1}((t\mu)^{p^3} \cdot u_1 t^{-p^3}) \doteq (t\mu)^{p^3} \cdot t\mu = (t\mu)^{p^3+1}
\end{aligned}$$

lift to the C_p-homotopy fixed point spectral sequence and can be rewritten as claimed after inverting μ. □

The first differential leaves

$$\begin{aligned}
\mu^{-1}E^{2p+1}(C_p) = {} & E(u_1) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm p}) \\
& \oplus E(u_1) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{\lambda_1 \mu^j \mid v_p(j) = 0\}.
\end{aligned}$$

The second leaves

$$\begin{aligned}
\mu^{-1}E^{2p^2+1}(C_p) = {} & E(u_1) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm p^2}) \\
& \oplus E(u_1) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{\lambda_1 \mu^j \mid v_p(j) = 0\} \\
& \oplus E(u_1) \otimes P_{p^2}(t\mu) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{\lambda_2 \mu^j \mid v_p(j) = 1\}.
\end{aligned}$$

The third leaves

$$\begin{aligned}
\mu^{-1}E^{2p^3+1}(C_p) = {} & E(u_1) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm p^3}) \\
& \oplus E(u_1) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{\lambda_1 \mu^j \mid v_p(j) = 0\} \\
& \oplus E(u_1) \otimes P_{p^2}(t\mu) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{\lambda_2 \mu^j \mid v_p(j) = 1\} \\
& \oplus E(u_1) \otimes P_{p^3}(t\mu) \otimes E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p\{\lambda_3 \mu^j \mid v_p(j) = 2\}.
\end{aligned}$$

The final differential leaves

$$\begin{aligned}\mu^{-1} E^{2p^3+2}(C_p) = P_{p^3+1}(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm p^3}) \\ \oplus E(u_1) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{\lambda_1 \mu^j \mid v_p(j) = 0\} \\ \oplus E(u_1) \otimes P_{p^2}(t\mu) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{\lambda_2 \mu^j \mid v_p(j) = 1\} \\ \oplus E(u_1) \otimes P_{p^3}(t\mu) \otimes E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p\{\lambda_3 \mu^j \mid v_p(j) = 2\},\end{aligned}$$

which equals $\mu^{-1} E^{\infty}(C_p)$.

Next we use the commutative diagram

$$\begin{tikzcd}
\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{hC_p} \arrow[d, "F"] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{C_p} \arrow[l, "\Gamma_1"'] \arrow[r, "\widehat{\Gamma}_2"] \arrow[d, "F"] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^2}} \arrow[d, "F"] \\
\mathrm{THH}(\mathrm{BP}\langle 2\rangle) \arrow[r, equal] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle) \arrow[r, "\widehat{\Gamma}_1"] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}
\end{tikzcd}$$

and what is known about $V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{hC_p}$ above degree $2p^2 + 2p - 3$ to pin down the differential pattern of the C_{p^2}-Tate spectral sequence leading to $V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^2}}$:

Theorem 9.2 *The C_{p^2}-Tate spectral sequence*

$$\widehat{E}^2(C_{p^2}) = \widehat{H}^{-*}(C_{p^2}; V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^2}}$$

has E^2-term

$$\widehat{E}^2(C_{p^2}) = E(u_2) \otimes P(t^{\pm 1}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

There are differentials

$$\begin{gathered} d^{2p}(t^{1-p}) \doteq t\lambda_1, \quad d^{2p^2}(t^{p-p^2}) \doteq t^p \lambda_2, \quad d^{2p^3}(t^{p^2-p^3}) \doteq t^{p^2}\lambda_3, \\ d^{2p^4+2p}(t^{p^3-p^4}) \doteq t^{p^3}(t\mu)^p \lambda_1, \qquad d^{2p^5+2p^2}(t^{p^4-p^5}) \doteq t^{p^4}(t\mu)^{p^2}\lambda_2, \\ d^{2p^6+2p^3}(t^{p^5-p^6}) \doteq t^{p^5}(t\mu)^{p^3}\lambda_3, \quad d^{2p^6+2p^3+1}(u_2 t^{-p^6}) \doteq (t\mu)^{p^3+1}, \end{gathered}$$

and the classes $t\mu$, λ_1, λ_2, λ_3 and $t^{\pm p^6}$ are permanent cycles.

Proof According to [Ausoni and Rognes 2002, Lemma 5.2], naturality with respect to Frobenius and Verschiebung maps forces the first three differentials, showing that

$$\widehat{E}^{2p^3+1}(C_{p^2}) = E(u_2) \otimes P(t^{\pm p^3}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

To proceed, we shall make use of the summands

$$P_p(t\mu) \otimes \mathbb{F}_p\{\lambda_1\mu\}, \quad P_{p^2}(t\mu) \otimes \mathbb{F}_p\{\lambda_2\mu^p\}, \quad P_{p^3}(t\mu) \otimes \mathbb{F}_p\{\lambda_3\mu^{p^2}\} \quad \text{and} \quad P_{p^3+1}(t\mu) \otimes \mathbb{F}_p\{\mu^{p^3}\}$$

in $E^{\infty}(C_p)$, which is equal to $\mu^{-1} E^{\infty}(C_p)$ in these degrees. There are almost no classes in the same total degrees and of lower filtration than the vanishing products

$$(t\mu)^p \cdot \lambda_1\mu, \quad (t\mu)^{p^2} \cdot \lambda_2\mu^p, \quad (t\mu)^{p^3} \cdot \lambda_3\mu^{p^2} \quad \text{and} \quad (t\mu)^{p^3+1} \cdot \mu^{p^3}.$$

The only exception is the class $(t\mu)^{p^2+p-1}\lambda_1\lambda_2\lambda_3$ in the same total degree as $(t\mu)^{p^2} \cdot \lambda_2\mu^p$. However, this class is itself a $(t\mu)^{p^2}$-multiple, so there is no room for a hidden $v_3^{p^2}$-extension on $\lambda_2\mu^p$. Hence $\lambda_1\mu$ detects a v_3^p-torsion class x_1, $\lambda_2\mu^p$ detects a $v_3^{p^2}$-torsion class x_2, $\lambda_3\mu^{p^2}$ detects a $v_3^{p^3}$-torsion class x_3, μ^{p^3} detects a $v_3^{p^3+1}$-torsion class x_4 in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{hC_p}$, and these v_3-power torsion orders are all exact.

By Corollary 8.2 the maps Γ_1 and $\widehat{\Gamma}_2$ are $(2p^2+2p-3)$-coconnected. Hence the classes x_i lift uniquely to classes y_i in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{C_p}$ with $\Gamma_1(y_i) = x_i$, and we let $z_i = \widehat{\Gamma}_2(y_i)$ denote their images in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^2}}$. Since $\widehat{\Gamma}_1(\mu) = t^{-p^3}$, up to a unit in $\mathbb{F}_p$ that we hereafter often omit to mention, $F(z_1)$ is detected by $t^{-p^3}\lambda_1$, $F(z_2)$ is detected by $t^{-p^4}\lambda_2$, $F(z_3)$ is detected by $t^{-p^5}\lambda_3$ and $F(z_4)$ is detected by t^{-p^6} in $\widehat{E}^\infty(C_p)$.

We claim that there are no classes in $\widehat{E}^\infty(C_{p^2})$ in the same total degrees and of higher filtrations than

$$t^{-p^3}\lambda_1, \quad t^{-p^4}\lambda_2, \quad t^{-p^5}\lambda_3 \quad \text{and} \quad t^{-p^6}.$$

This will imply that the z_i are detected by precisely these classes. Already at the (known) E^{2p^3+1}-term the only exception to the claim is $u_2 t^{-p^3-p^5}$ in the same total degree as $t^{-p^5}\lambda_3$, and we shall see below that this class supports a nonzero d^{2p^4+2p}-differential, hence does not survive to the E^∞-term. It then follows that the products

$$(t\mu)^p \cdot t^{-p^3}\lambda_1, \quad (t\mu)^{p^2} \cdot t^{-p^4}\lambda_2, \quad (t\mu)^{p^3} \cdot t^{-p^5}\lambda_3 \quad \text{and} \quad (t\mu)^{p^3+1} \cdot t^{-p^6}$$

must detect zero, and therefore be boundaries in the C_{p^2}-Tate spectral sequence $\widehat{E}^r(C_{p^2})$. We shall prove that these boundaries must be

$$d^{2p^4+2p}(t^{-p^3-p^4}) \doteq (t\mu)^p \cdot t^{-p^3}\lambda_1, \qquad d^{2p^5+2p^2}(t^{-p^4-p^5}) \doteq (t\mu)^{p^2} \cdot t^{-p^4}\lambda_2,$$
$$d^{2p^6+2p^3}(t^{-p^5-p^6}) \doteq (t\mu)^{p^3} \cdot t^{-p^5}\lambda_3, \quad d^{2p^6+2p^3+1}(u_2 t^{-2p^6}) \doteq (t\mu)^{p^3+1} \cdot t^{-p^6},$$

and the asserted formulas follow readily.

We shall make use of the following lemma. Each cyclic $P(t\mu)$-module is either free or torsion, being isomorphic to a suspension of $P(t\mu)$ or of its truncation $P_h(t\mu) = P(t\mu)/((t\mu)^h)$ at some height $h \geq 1$, according to the case. Here and below exponents ϵ and ϵ_i are always assumed to lie in $\{0, 1\}$.

Lemma 9.3 *For each $r \geq 2p^3 + 1$ the C_{p^2}-Tate E^r-term $\widehat{E}^r(C_{p^2})$ is a direct sum of cyclic $P(t\mu)$-modules, generated by classes of the form $u_2^\epsilon t^i \cdot \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}$ with $p^3 \mid i$. The d^r-differential maps free summands to free summands, and is zero on the torsion summands.*

Proof We proceed by induction on $r \geq 2p^3 + 1$, assuming that the $P(t\mu)$-module structure of the E^r-term is as stated.

Suppose that there is a d^r-differential $d^r(a) = b$ hitting a nonzero $t\mu$-torsion class. Then $t\mu \cdot b = b_0 = d^{r_0}(a_0)$ must have been hit by an earlier d^{r_0}-differential, where a_0 is a generator of the form

$u_2^{\epsilon} t^i \cdot \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \lambda_3^{\epsilon_3}$ with $p^3 \mid i$. Hence a must lie in the same total degree as the formal product $(t\mu)^{-1} \cdot a_0$, but in a higher filtration. At the E^2-term, this could happen in three cases:

- if $a_0 = u_2^{\epsilon} t^i \cdot \lambda_1 \lambda_2 \lambda_3$, with a in the bidegree of $u_2^{\epsilon} t^{i-p} \cdot \lambda_2$, $u_2^{\epsilon} t^{i-p^2} \cdot \lambda_1$, $u_2 t^{i-p^2-p} \cdot 1$ or $t^{i-p^2-p+1} \cdot 1$,
- if $a_0 = u_2^{\epsilon} t^i \cdot \lambda_2 \lambda_3$, with a in the bidegree of $u_2 t^{i-p^2+p-1} \cdot \lambda_1$, $t^{i-p^2+p} \cdot \lambda_1$ or $u_2^{\epsilon} t^{i-p^2} \cdot 1$,
- if $a_0 = u_2^{\epsilon} t^i \cdot \lambda_1 \lambda_3$, with a in the bidegree of $u_2^{\epsilon} t^{i-p} \cdot 1$.

However, in none of these cases is the prescribed t-exponent ($i-p$, $i-p^2$, etc) a multiple of p^3. Hence there are no nonzero classes in these bidegrees of $\widehat{E}^{2p^3+1}(C_{p^2})$, and therefore also not in $\widehat{E}^r(C_{p^2})$ for $r \geq 2p^3+1$.

It follows that no differentials hit the torsion summands, so each nonzero differential maps a free summand to another free summand. Its kernel is then zero, while its cokernel creates a torsion summand in the E^{r+1}-term, which is still generated by a class of the required form. This proves the inductive statement for $r+1$. □

The remainder of the proof of Theorem 9.2 can be separated into five steps:

(1) We start with z_1, which we know is detected by $t^{-p^3}\lambda_1$. Checking bidegrees in $\widehat{E}^{2p^3+1}(C_{p^2})$, the next possible differentials on u_2 and t^{-p^3} are

$$\begin{aligned} d^{2p^4+2p-1}(t^{-p^3}) &\in \mathbb{F}_p\{u_2 t^{-p^3+p^4} (t\mu)^{p-1} \lambda_1 \lambda_3\}, \\ d^{2p^4+2p}(t^{-p^3}) &\in \mathbb{F}_p\{t^{-p^3+p^4} (t\mu)^p \lambda_1\}, \\ d^{2p^4+2p}(u_2) &\in \mathbb{F}_p\{u_2 t^{p^4} (t\mu)^p \lambda_1\}. \end{aligned}$$

Since $t\mu$ and the λ_i are infinite cycles, we must have $d^r = 0$ for $2p^3+1 \leq r < 2p^4+2p-1$. Moreover, $(t\mu)^p \cdot t^{-p^3}\lambda_1 \doteq d^{r_1}(a_1)$ in vertical degree $2p^4+2p-1$ must be a boundary, and the only possible source of such a d^{r_1}-differential with $r_1 \geq 2p^4+2p-1$ is $a_1 = t^{-p^3-p^4}$ with $r_1 = 2p^4+2p$. It follows that $d^{2p^4+2p-1}(t^{-p^3}) = 0$ vanishes and furthermore that $d^{2p^4+2p}(t^{-p^3}) \doteq t^{-p^3+p^4}(t\mu)^p \lambda_1$ is nonzero.

(2) We turn to z_4, which we know is detected by t^{-p^6}. Thus t^{p^6} and its inverse are permanent cycles. The nonzero product $v_3^{p^3} \cdot z_4$ is detected by $b_4 = (t\mu)^{p^3} \cdot t^{-p^6}$ or, if this product is a boundary, by another class in the same total degree as b_4 but of lower filtration. Let b_4' denote the actual detecting class. Then $t\mu \cdot b_4' \doteq d^{r_4}(a_4)$ in total degree $4p^6-2$ detects $v_3^{p^3+1} \cdot z_4 = 0$, and hence is a boundary. By Lemma 9.3, the source of this differential is of the form $a_4 = u_2^{\epsilon} t^i \cdot \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \lambda_3^{\epsilon_3}$, with $p^3 \mid i$. (If a_4 were a $t\mu$-multiple at the E^2-term, then it would be a $t\mu$-multiple at the E^{r_4}-term, by the lemma. Then b_4' would be a d^{r_4}-boundary, which is impossible since it detects $v_3^{p^3} \cdot z_4 \neq 0$.) The total degree of a_4 is $4p^6-1$, so the only possibilities are $t^{p^3-2p^6}\lambda_3$ with $r_4 \geq 2p^6+2$, or $u_2 t^{-2p^6}$ with $r_4 \geq 2p^6+2p^3+1$. However, we showed in (1) that $t^{p^3-2p^6}\lambda_3$ supports a nonzero (shorter) d^{2p^4+2p}-differential. Hence $a_4 = u_2 t^{-2p^6}$

survives at least to the $E^{2p^6+2p^3+1}$-term, and $d^{r_4}(a_4) \neq 0$ for some $r_4 \geq 2p^6 + 2p^3 + 1$. Since t^{p^6} is an infinite cycle it follows that u_2 also survives to the $E^{2p^6+2p^3+1}$-term. Hence

$$\begin{aligned}\hat{E}^{2p^4+2p+1}(C_{p^2}) = {} & E(u_2) \otimes P(t^{\pm p^4}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ & \oplus E(u_2) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{t^i\lambda_1 \mid v_p(i) = 3\}.\end{aligned}$$

In particular, $u_2 t^{-p^3-p^5}$ is not an infinite cycle, and cannot detect z_3, confirming our earlier claim.

(3) We continue with z_2, which we know is detected by $t^{-p^4}\lambda_2$. Checking bidegrees in $\hat{E}^{2p^4+2p+1}(C_{p^2})$, the next possible differentials on t^{-p^4} are

$$d^{2p^5+2p^2-1}(t^{-p^4}) \in \mathbb{F}_p\{u_2 t^{-p^4+p^5}(t\mu)^{p^2-1}\lambda_2\lambda_3\} \quad \text{and} \quad d^{2p^5+2p^2}(t^{-p^4}) \in \mathbb{F}_p\{t^{-p^4+p^5}(t\mu)^{p^2}\lambda_2\},$$

while u_2 survives at least to $\hat{E}^{2p^6+2p^3+1}(C_{p^2})$ by (2). The differentials on $t\mu$, the λ_i and the torsion summand are zero. Hence $d^r = 0$ for $2p^4 + 2p + 1 \leq r < 2p^5 + 2p^2 - 1$. Moreover, $(t\mu)^{p^2} \cdot t^{-p^4}\lambda_2 = d^{r_2}(a_2)$ in vertical degree $2p^5 + 2p^2 - 1$ must be a boundary, and the only possible source of such a d^{r_2}-differential with $r_2 \geq 2p^5 + 2p^2 - 1$ is $a_2 \doteq t^{-p^4-p^5}$ with $r_2 = 2p^5 + 2p^2$. It follows that $d^{2p^5+2p^2-1}(t^{-p^4}) = 0$ vanishes and that $d^{2p^5+2p^2}(t^{-p^4}) \doteq t^{-p^4+p^5}(t\mu)^{p^2}\lambda_2$ is nonzero. Hence

$$\begin{aligned}\hat{E}^{2p^5+2p^2+1}(C_{p^2}) = {} & E(u_2) \otimes P(t^{\pm p^5}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ & \oplus E(u_2) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{t^i\lambda_1 \mid v_p(i) = 3\} \\ & \oplus E(u_2) \otimes P_{p^2}(t\mu) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{t^i\lambda_2 \mid v_p(i) = 4\}.\end{aligned}$$

(4) We know from (2) that z_3 is detected by $t^{-p^5}\lambda_3$. Checking bidegrees in $\hat{E}^{2p^5+2p^2+1}(C_{p^2})$, the next possible differential on t^{-p^5} is

$$d^{2p^6+2p^3}(t^{-p^5}) \in \mathbb{F}_p\{t^{-p^5+p^6}(t\mu)^{p^3}\lambda_3\},$$

while u_2 survives to the $E^{2p^6+2p^3+1}$-term by (2). The differentials on $t\mu$, the λ_i and the torsion summands are zero. Hence $d^r = 0$ for $2p^5 + 2p^2 + 1 \leq r < 2p^6 + 2p^3$. Moreover, $(t\mu)^{p^3} \cdot t^{-p^5}\lambda_3 = d^{r_3}(a_3)$ in vertical degree $2p^6+2p^3-1$ must be a boundary, and the only possible source of such a differential is $a_3 \doteq t^{-p^5-p^6}$ with $r_3 = 2p^6 + 2p^3$. It follows that $d^{2p^6+2p^3}(t^{-p^5}) \doteq t^{-p^5+p^6}(t\mu)^{p^3}\lambda_3$ is nonzero. Hence

$$\begin{aligned}\hat{E}^{2p^6+2p^3+1}(C_{p^2}) = {} & E(u_2) \otimes P(t^{\pm p^6}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ & \oplus E(u_2) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{t^i\lambda_1 \mid v_p(i) = 3\} \\ & \oplus E(u_2) \otimes P_{p^2}(t\mu) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{t^i\lambda_2 \mid v_p(i) = 4\} \\ & \oplus E(u_2) \otimes P_{p^3}(t\mu) \otimes E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p\{t^i\lambda_3 \mid v_p(i) = 5\}.\end{aligned}$$

(5) Finally, we return to z_4. Since $b_4 = (t\mu)^{p^3} \cdot t^{-p^6}$ is nonzero, in vertical degree $2p^6$ of the $E^{2p^6+2p^3+1}$-term, it can no longer become a boundary. We can therefore strengthen the conclusions in (2) to conclude that $v_3^{p^3} \cdot z_4$ is detected by $b'_4 = b_4$, and that $t\mu \cdot b_4 = (t\mu)^{p^3+1} \cdot t^{-p^6}$ is a unit times

$d^{r_4}(a_4)$, with $r_4 = 2p^6 + 2p^3 + 1$ and $a_4 = u_2 t^{-2p^6}$. It follows that $d^{2p^6+2p^3+1}(u_2 t^{-p^6}) \doteq (t\mu)^{p^3+1}$, since t^{p^6} is an infinite cycle. Hence

$$\begin{aligned}\hat{E}^{2p^6+2p^3+2}(C_{p^2}) = P(t^{\pm p^6}) \otimes P_{p^3+1}(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ \oplus E(u_2) \otimes P_p(t\mu) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{t^i \lambda_1 \mid v_p(i) = 3\} \\ \oplus E(u_2) \otimes P_{p^2}(t\mu) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{t^i \lambda_2 \mid v_p(i) = 4\} \\ \oplus E(u_2) \otimes P_{p^3}(t\mu) \otimes E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p\{t^i \lambda_3 \mid v_p(i) = 5\}.\end{aligned}$$

No free summands remain, so by Lemma 9.3 there are no further differentials, and this E^r-term equals $\hat{E}^\infty(C_{p^2})$. □

10 The C_{p^n}-Tate spectral sequences

The following notations will be convenient when we now determine the differential structure of the C_{p^n}-Tate spectral sequence.

Definition 10.1 [Ausoni and Rognes 2002, Definition 2.5; Angelini-Knoll et al. 2024, (5.8)] Let $r(k) = 0$ for $k \in \{0, -1, -2\}$ and set $r(k) = p^k + r(k-3)$ for $k \geq 1$. Thus $r(3n-2) = p^{3n-2} + \cdots + p$, $r(3n-1) = p^{3n-1} + \cdots + p^2$ and $r(3n) = p^{3n} + \cdots + p^3$, with n terms in each sum.

Let $[k] \in \{1, 2, 3\}$ be defined by $k \equiv [k] \bmod 3$, so that $\{\lambda_{[k]}, \lambda_{[k+1]}, \lambda_{[k+2]}\} = \{\lambda_1, \lambda_2, \lambda_3\}$.

Theorem 10.2 *The C_{p^n}-Tate spectral sequence*

$$\hat{E}^2(C_{p^n}) = \hat{H}^{-*}(C_{p^n}; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^n}}$$

has E^2-term

$$\hat{E}^2(C_{p^n}) = E(u_n) \otimes P(t^{\pm 1}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3).$$

There are differentials

$$d^{2r(k)}(t^{p^{k-1}-p^k}) \doteq t^{p^{k-1}} (t\mu)^{r(k-3)} \lambda_{[k]}$$

for each $1 \leq k \leq 3n$, and

$$d^{2r(3n)+1}(u_n t^{-p^{3n}}) \doteq (t\mu)^{r(3n-3)+1}.$$

The classes $t\mu$, λ_1, λ_2, λ_3 and $t^{\pm p^{3n}}$ are permanent cycles.

For $n = 1$, this is Theorem 8.1. We prove the statement for general n by induction, assuming the statement for a value $n \geq 2$, and deducing that it also holds for $n + 1$. The case $n = 2$ is provided by Theorem 9.2.

The distinct terms of the C_{p^n}-Tate spectral sequence are

$$\begin{aligned}\hat{E}^{2r(m)+1}(C_{p^n}) = E(u_n) \otimes P(t^{\pm p^m}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ \oplus \bigoplus_{k=4}^{m} E(u_n) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}\end{aligned}$$

for $1 \le m \le 3n$. To see this, note that the differential $d^{2r(k)}$ only affects the summand

$$E(u_n) \otimes \mathbb{F}_p\{t^i \mid v_p(i) = k-1\} \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3),$$

and here its homology is $E(u_n) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}$. Thereafter,

$$\begin{aligned}\hat{E}^{2r(3n)+2}(C_{p^n}) = P(t^{\pm p^{3n}}) \otimes P_{r(3n-3)+1}(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)\\ \oplus \bigoplus_{k=4}^{3n} E(u_n) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}.\end{aligned}$$

To see this, note that $d^{2r(3n)+1}$ only affects the summand $E(u_n) \otimes P(t^{\pm p^{3n}}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)$, and that its homology is $P(t^{\pm p^{3n}}) \otimes P_{r(3n-3)+1}(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)$. For bidegree reasons, the remaining differentials are zero, so $\hat{E}^{2r(3n)+2}(C_{p^n}) = \hat{E}^{\infty}(C_{p^n})$ and the classes $t^{\pm p^{3n}}$ are permanent cycles.

We obtain the differential structure of the C_{p^n}-homotopy fixed point spectral sequence $E^r(C_{p^n})$ for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)$ from that of the C_{p^n}-Tate spectral sequence $\hat{E}^r(C_{p^n})$ by restricting to the second quadrant, and write $\mu^{-1}E^r(C_{p^n})$ for its localization given by inverting (a power of) μ. It follows from Theorem 8.1 that $\mu^{-1}E^r(C_{p^n})$ is isomorphic to the C_{p^n}-homotopy fixed point spectral sequence for $V(2) \wedge \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$.

Proposition 10.3 *The μ-localized C_{p^n}-homotopy fixed point spectral sequence*

$$\mu^{-1}E^2(C_{p^n}) = H^{-*}(C_{p^n}; \mu^{-1}V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow \mu^{-1}V(2)_* \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{hC_{p^n}}$$

has E^2-term

$$\mu^{-1}E^2(C_{p^n}) = E(u_n) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm 1}).$$

There are differentials

$$d^{2r(k)}(\mu^{p^{k-1}}) \doteq (t\mu)^{r(k)} \lambda_{[k]} \mu^{p^{k-1}-p^k}$$

for each $1 \le k \le 3n$, and

$$d^{2r(3n)+1}(u_n \mu^{p^{3n}}) \doteq (t\mu)^{r(3n)+1}.$$

The classes $t\mu$, λ_1, λ_2, λ_3 and $\mu^{\pm p^{3n}}$ are permanent cycles.

Proof This follows from Theorem 10.2 by comparison along the morphism

$$R^h \colon E^r(C_{p^n}) \to \hat{E}^r(C_{p^n})$$

of spectral sequences induced by the homotopy restriction map (also known as the canonical map), and the $(2p^2+2p-3)$-coconnected localization morphism

$$E^r(C_{p^n}) \to \mu^{-1}E^r(C_{p^n}).$$

Algebraically, the translation is achieved through multiplication with appropriate powers of $t\mu$. □

The distinct terms of the μ-localized C_{p^n}-homotopy fixed point spectral sequence are

$$\mu^{-1}E^{2r(m)+1}(C_{p^n}) = E(u_n)\otimes P(t\mu)\otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu^{\pm p^m}) \\ \oplus\bigoplus_{k=1}^{m} E(u_n)\otimes P_{r(k)}(t\mu)\otimes E(\lambda_{[k+1]},\lambda_{[k+2]})\otimes \mathbb{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j)=k-1\}$$

for $1\le m\le 3n$. To see this, note that the differential $d^{2r(k)}$ only affects the summand

$$E(u_n)\otimes P(t\mu)\otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes \mathbb{F}_p\{\mu^j \mid v_p(j)=k-1\},$$

and here its homology is $E(u_n)\otimes P_{r(k)}(t\mu)\otimes E(\lambda_{[k+1]},\lambda_{[k+2]})\otimes\mathbb{F}_p\{\lambda_{[k]}\mu^j\mid v_p(j)=k-1\}$. Thereafter

$$\mu^{-1}E^{2r(3n)+2}(C_{p^n}) = P_{r(3n)+1}(t\mu)\otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu^{\pm p^{3n}}) \\ \oplus\bigoplus_{k=1}^{3n} E(u_n)\otimes P_{r(k)}(t\mu)\otimes E(\lambda_{[k+1]},\lambda_{[k+2]})\otimes \mathbb{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j)=k-1\}.$$

As before, $d^{2r(3n)+1}$ only affects the summand $E(u_n)\otimes P(t\mu)\otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu^{\pm p^{3n}})$, and its homology is $P_{r(3n)+1}(t\mu)\otimes E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu^{\pm p^{3n}})$. For bidegree reasons the remaining differentials are zero, so $\mu^{-1}E^{2r(3n)+2}(C_{p^n}) = \mu^{-1}E^{\infty}(C_{p^n})$ and the classes $\mu^{\pm p^{3n}}$ are permanent cycles.

To achieve the inductive step we use the commutative diagram

$$\begin{tikzcd}
\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{hC_{p^n}} \arrow[d, "F^n"] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{C_{p^n}} \arrow[l, "\Gamma_n"'] \arrow[r, "\widehat{\Gamma}_{n+1}"] \arrow[d, "F^n"] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^{n+1}}} \arrow[d, "F^n"] \\
\mathrm{THH}(\mathrm{BP}\langle 2\rangle) \arrow[r, equal] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle) \arrow[r, "\widehat{\Gamma}_1"] & \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}
\end{tikzcd} \tag{10-1}$$

and what is known about $V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{hC_{p^n}}$ above degree $2p^2+2p-3$ to determine the differential pattern of the $C_{p^{n+1}}$-Tate spectral sequence converging to $V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^{n+1}}}$.

Proof of Theorem 10.2 We must show that the $C_{p^{n+1}}$-Tate spectral sequence

$$\widehat{E}^2(C_{p^{n+1}}) = \widehat{H}^{-*}(C_{p^{n+1}}; V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^{n+1}}}$$

has the asserted differential pattern. By naturality with respect to (Tate spectrum) Frobenius and Verschiebung morphisms,

$$F\colon \widehat{E}^r(C_{p^{n+1}}) \rightleftarrows \widehat{E}^r(C_{p^n}) : V,$$

it follows as in [Ausoni and Rognes 2002, Lemma 5.2] that the left-hand spectral sequence has differentials

$$d^{2r(k)}(t^{p^{k-1}-p^k}) \doteq t^{p^{k-1}}(t\mu)^{r(k-3)}\lambda_{[k]}$$

for all $1\le k\le 3n$, leading via the $E^{2r(3)+1} = E^{2p^3+1}$-term

$$\widehat{E}^{2p^3+1}(C_{p^{n+1}}) = E(u_{n+1})\otimes P(t^{\pm p^3})\otimes P(t\mu)\otimes E(\lambda_1,\lambda_2,\lambda_3)$$

to the $E^{2r(3n)+1}$-term

$$\hat{E}^{2r(3n)+1}(C_{p^{n+1}}) = E(u_{n+1}) \otimes P(t^{\pm p^{3n}}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ \oplus \bigoplus_{k=4}^{3n} E(u_{n+1}) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}.$$

We shall prove that this spectral sequence contains three more families of even-length differentials, followed by one family of odd-length differentials, after which it collapses.

Note that the E^{2p^3+1}-term is free as a $P(t\mu)$-module. Replacing C_{p^2} with $C_{p^{n+1}}$ and u_2 with u_{n+1} in the proof of Lemma 9.3, with no other changes, establishes the more general statement:

Lemma 10.4 *For each $r \geq 2p^3 + 1$ the $C_{p^{n+1}}$-Tate E^r-term $\hat{E}^r(C_{p^{n+1}})$ is a direct sum of cyclic $P(t\mu)$-modules, generated by classes of the form $u_{n+1}^{\epsilon} t^i \cdot \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \lambda_3^{\epsilon_3}$ with $p^3 \mid i$. The d^r-differential maps free summands to free summands, and is zero on the torsion summands.* □

By our inductive hypothesis, the abutment $E^{\infty}(C_{p^n})$, which is isomorphic to $\mu^{-1} E^{\infty}(C_{p^n})$ above degree $2p^2 + 2p - 3$, contains summands

$$P_{r(3n-2)}(t\mu) \otimes \mathbb{F}_p\{\lambda_1 \mu^{p^{3n-3}}\}, \quad P_{r(3n-1)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 \mu^{p^{3n-2}}\},$$
$$P_{r(3n)}(t\mu) \otimes \mathbb{F}_p\{\lambda_3 \mu^{p^{3n-1}}\}, \quad P_{r(3n)+1}(t\mu) \otimes \mathbb{F}_p\{\mu^{p^{3n}}\}.$$

Moreover, $\mu^{-1} E^{\infty}(C_{p^n})$ is generated as a $P(t\mu)$-module by classes in filtrations -1 and 0. Hence any class in $E^{\infty}(C_{p^n})$ in the same total degree as, but of lower filtration than, one of the vanishing products

$$(t\mu)^{r(3n-2)} \cdot \lambda_1 \mu^{p^{3n-3}}, \quad (t\mu)^{r(3n-1)} \cdot \lambda_2 \mu^{p^{3n-2}},$$
$$(t\mu)^{r(3n)} \cdot \lambda_3 \mu^{p^{3n-1}}, \quad (t\mu)^{r(3n)+1} \cdot \mu^{p^{3n}},$$

must itself be divisible by (at least) the indicated power of $t\mu$. It follows that there are no hidden v_3-power extensions present, so that $\lambda_1 \mu^{p^{3n-3}}$ detects a $v_3^{r(3n-2)}$-torsion class $x_1 \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{hC_{p^n}}$, $\lambda_2 \mu^{p^{3n-2}}$ detects a $v_3^{r(3n-1)}$-torsion class x_2, $\lambda_3 \mu^{p^{3n-1}}$ detects a $v_3^{r(3n)}$-torsion class x_3 and $\mu^{p^{3n}}$ detects a $v_3^{r(3n)+1}$-torsion class x_4, and these v_3-power torsion orders are exact.

By Corollary 8.2 there are unique classes $y_i \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{C_{p^n}}$ and $z_i \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_{p^{n+1}}}$ with $\Gamma_n(y_i) = x_i$ and $\hat{\Gamma}_{n+1}(y_i) = z_i$ for each i. Moreover, $z_1, \ldots, z_4$ are v_3-power torsion classes of orders precisely $r(3n-2)$, $r(3n-1)$, $r(3n)$ and $r(3n)+1$, respectively.

Applying Frobenius maps F^n as in (10-1), and the fact from Theorem 8.1 that $\hat{\Gamma}_1$ maps μ to t^{-p^3} (up to the usual implicit unit) and preserves the λ_i, we deduce that $F^n(z_1), \ldots, F^n(z_4)$ are detected by the classes $t^{-p^{3n}}\lambda_1$, $t^{-p^{3n+1}}\lambda_2$, $t^{-p^{3n+2}}\lambda_3$ and $t^{-p^{3n+3}}$ in $\hat{E}^{\infty}(C_p)$. Hence $z_1, \ldots, z_4$ are detected in $\hat{E}^{\infty}(C_{p^{n+1}})$ in the same total degrees as these classes, in equal or higher filtration. However, since $n \geq 2$, there are no possible detecting classes of strictly higher filtration present in $\hat{E}^{2r(3n)+1}(C_{p^{n+1}})$. We can therefore conclude that $z_1, \ldots, z_4$ are detected by

$$t^{-p^{3n}}\lambda_1, \quad t^{-p^{3n+1}}\lambda_2, \quad t^{-p^{3n+2}}\lambda_3 \quad \text{and} \quad t^{-p^{3n+3}},$$

respectively, in $\hat{E}^\infty(C_{p^{n+1}})$. (The only problematic class at the $E^{2r(3)+1}$-term, $u_{n+1}t^{-p^3-p^{3n+2}}$ in the same total degree as $t^{-p^{3n+2}}\lambda_3$, is now known to support a $d^{2r(4)}$-differential, as in the C_{p^2} case.)

It follows that the products

$$(t\mu)^{r(3n-2)}\cdot t^{-p^{3n}}\lambda_1,\quad (t\mu)^{r(3n-1)}\cdot t^{-p^{3n+1}}\lambda_2,\quad (t\mu)^{r(3n)}\cdot t^{-p^{3n+2}}\lambda_3 \quad\text{and}\quad (t\mu)^{r(3n)+1}\cdot t^{-p^{3n+3}}$$

must detect zero, and therefore be boundaries, in the $C_{p^{n+1}}$-Tate spectral sequence. We shall prove that these boundaries must be

$$\begin{aligned} d^{2r(3n+1)}(t^{-p^{3n}-p^{3n+1}}) &\doteq (t\mu)^{r(3n-2)}\cdot t^{-p^{3n}}\lambda_1,\\ d^{2r(3n+2)}(t^{-p^{3n+1}-p^{3n+2}}) &\doteq (t\mu)^{r(3n-1)}\cdot t^{-p^{3n+1}}\lambda_2,\\ d^{2r(3n+3)}(t^{-p^{3n+2}-p^{3n+3}}) &\doteq (t\mu)^{r(3n)}\cdot t^{-p^{3n+2}}\lambda_3,\\ d^{2r(3n+3)+1}(u_{n+1}t^{-2p^{3n+3}}) &\doteq (t\mu)^{r(3n)+1}\cdot t^{-p^{3n+3}}. \end{aligned}$$

In view of the Leibniz rule, the first three can be rewritten as

$$d^{2r(k)}(t^{p^{k-1}-p^k}) \doteq t^{p^{k-1}}(t\mu)^{r(k-3)}\lambda_{[k]}$$

for $3n+1\le k\le 3n+3$, while the fourth is equivalent to

$$d^{2r(3n+3)+1}(u_{n+1}t^{-p^{3n+3}}) \doteq (t\mu)^{r(3n)+1}.$$

As for Theorem 9.2, the remainder of the proof of Theorem 10.2 will consist of five steps, but for $n\ge 2$ we can start with z_4 in place of z_1, and this simplifies the discussion of the class u_{n+1}.

(1) We know that z_4 is detected by $t^{-p^{3n+3}}$. Thus $t^{p^{3n+3}}$ and its inverse are permanent cycles. The nonzero product $v_3^{r(3n)}\cdot z_4$ is detected by $b_4=(t\mu)^{r(3n)}\cdot t^{-p^{3n+3}}$ or, if this product is a boundary, by another class in the same total degree as b_4 but of lower filtration. Let b_4' denote the actual detecting class. Then $t\mu\cdot b_4'$ in total degree $4p^{3n+3}-2$ and vertical degree $\ge 2r(3n+3)$ detects $v_3^{r(3n)+1}\cdot z_4=0$, and hence is a boundary. We write $t\mu\cdot b_4'\doteq d^{r_4}(a_4)$. By Lemma 10.4, the source of this differential is of the form $a_4=u_{n+1}^{\epsilon}t^i\cdot\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}$, with $p^3\mid i$. The total degree of a_4 is $4p^{3n+3}-1$, so the only possible sources are $t^{p^3-2p^{3n+3}}\lambda_3$, with $r_4\ge 2r(3n+3)-2p^3+2$, or $u_{n+1}t^{-2p^{3n+3}}$. However, since $n\ge 2$, the first of these possibilities is no longer present in $\hat{E}^{2r(3n)+1}(C_{p^{n+1}})$. Hence $a_4=u_{n+1}t^{-2p^{3n+3}}$ survives at least to the $E^{2r(3n+3)+1}$-term, and $d^{r_4}(a_4)\ne 0$ for some $r_4\ge 2r(3n+3)+1$. Since $t^{p^{3n+3}}$ is an infinite cycle it also follows that $d^r(u_{n+1})=0$ for all $r\le 2r(3n+3)$.

(2) We continue with z_1, which is detected by $t^{-p^{3n}}\lambda_1$. The nonzero product $v_3^{r(3n-2)-1}\cdot z_1$ is detected by $b_1=(t\mu)^{r(3n-2)-1}\cdot t^{-p^{3n}}\lambda_1$ or, if this product is a boundary, by another class in the same total degree as b_1 but of lower filtration. Let b_1' denote the detecting class. Then $t\mu\cdot b_1'$ in total degree $2p^{3n+1}+2p^{3n}-1$ and vertical degree $\ge 2r(3n+1)-1$ detects $v_3^{r(3n-2)}\cdot z_1=0$, and hence is a boundary. We write $t\mu\cdot b_1'\doteq d^{r_1}(a_1)$. By Lemma 10.4 the source of this differential is of the form $a_1=u_{n+1}^{\epsilon}t^i\cdot\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}$ with $p^3\mid i$. The only such class in the correct total degree is $a_1=t^{-p^{3n}-p^{3n+1}}$. Considering vertical degrees,

it follows that $r_1 \ge 2r(3n+1)$. Since the torsion summands in $\widehat{E}^{2r(3n)+1}(C_{p^{n+1}})$ are not affected by later differentials, the λ_i and $t\mu$ are infinite cycles and u_{n+1} survives to the $E^{2r(3n+3)+1}$-term by (1), it follows that $d^r = 0$ for $2r(3n) < r < 2r(3n+1)$. After this, b_1 is in too low a vertical degree to be a boundary. Hence $b_1' = b_1$ and $r_1 = 2r(3n+1)$. It follows that $d^{2r(3n+1)}(t^{-p^{3n}}) \doteq t^{-p^{3n}+p^{3n+1}}(t\mu)^{r(3n-2)}\lambda_1$. This establishes the first new even-length differential, and leads to the $E^{2r(3n+1)+1}$-term

$$\begin{aligned}\widehat{E}^{2r(3n+1)+1}(C_{p^{n+1}}) &= E(u_{n+1}) \otimes P(t^{\pm p^{3n+1}}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)\\ &\quad \oplus \bigoplus_{k=4}^{3n+1} E(u_{n+1}) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}.\end{aligned}$$

(3) Next[3] we turn to z_2, which is detected by $t^{-p^{3n+1}}\lambda_2$. The nonzero product $v_3^{r(3n-1)-1} \cdot z_2$ is detected by $b_2 = (t\mu)^{r(3n-1)-1} \cdot t^{-p^{3n+1}}\lambda_2$ or, if this product is a boundary, by another class in the same total degree as b_2 but of lower filtration. Let b_2' denote the detecting class. Then $t\mu \cdot b_2' \doteq d^{r_2}(a_2)$ detects $v_3^{r(3n-1)} \cdot z_2 = 0$, and hence is a boundary. The source a_2 of this differential is of total degree $2p^{3n+2} + 2p^{3n+1}$, and by Lemma 10.4 it has the usual form $u_{n+1}^{\epsilon} t^i \cdot \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}$, so $a_2 = t^{-p^{3n+1}-p^{3n+2}}$ is the only possibility, with $r_2 \ge 2r(3n+2)$. It follows as in (2) that $d^r = 0$ for $2r(3n+1) < r < 2r(3n+2)$. After this, b_2 lies too close to the horizontal axis to be a boundary, so $b_2' = b_2$ and $r_2 = 2r(3n+2)$. It then follows that $d^{2r(3n+2)}(t^{-p^{3n+1}}) \doteq t^{-p^{3n+1}+p^{3n+2}}(t\mu)^{r(3n-1)}\lambda_2$. This establishes the second new even-length differential, and gives the $E^{2r(3n+2)+1}$-term

$$\begin{aligned}\widehat{E}^{2r(3n+2)+1}(C_{p^{n+1}}) &= E(u_{n+1}) \otimes P(t^{\pm p^{3n+2}}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)\\ &\quad \oplus \bigoplus_{k=4}^{3n+2} E(u_{n+1}) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}.\end{aligned}$$

(4) Carrying on, we consider z_3, which is detected by $t^{-p^{3n+2}}\lambda_3$. The nonzero product $v_3^{r(3n)-1} \cdot z_3$ is detected by $b_3 = (t\mu)^{r(3n)-1} \cdot t^{-p^{3n+2}}\lambda_3$, unless this class is a boundary, in which case the product is detected by another class in the same total degree as b_3, but of lower filtration. Let b_3' denote the detecting class. Then $t\mu \cdot b_3' \doteq d^{r_3}(a_3)$ detects $v_3^{r(3n)} \cdot z_3 = 0$, and must be a boundary. The source a_3 of this differential is of total degree $2p^{3n+3} + 2p^{3n+2}$, and by Lemma 10.4 it has the usual form $u_{n+1}^{\epsilon} t^i \cdot \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}$ not involving μ. The only possibility is $a_3 = t^{-p^{3n+2}-p^{3n+3}}$, with $r_3 \ge 2r(3n+3)$. It follows as above that $d^r = 0$ for $2r(3n+2) < r < 2r(3n+3)$, after which b_3 is in too low a vertical degree to become a boundary, so $b_3' = b_3$ and $r_3 = 2r(3n+3)$. Hence $d^{2r(3n+3)}(t^{-p^{3n+2}}) \doteq t^{-p^{3n+2}+p^{3n+3}}(t\mu)^{r(3n)}\lambda_3$. This establishes the third new even-length differential, and leaves the $E^{2r(3n+3)+1}$-term

$$\begin{aligned}\widehat{E}^{2r(3n+3)+1}(C_{p^{n+1}}) &= E(u_{n+1}) \otimes P(t^{\pm p^{3n+3}}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)\\ &\quad \oplus \bigoplus_{k=4}^{3n+3} E(u_{n+1}) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}.\end{aligned}$$

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[3] Steps (3) and (4) are very similar to (2), but we believe the arguments are easier to follow when written out separately.

(5) Finally, we return to z_4. Since $b_4 = (t\mu)^{r(3n)} \cdot t^{-p^{3n+3}}$ is nonzero, in vertical degree $2r(3n+3)-2p^3$ of the $E^{2r(3n+3)+1}$-term, it cannot be a boundary, and hence is equal to the class b_4' from step (1). Thus $t\mu \cdot b_4' = (t\mu)^{r(3n)+1} \cdot t^{-p^{3n+3}} \doteq d^{r_4}(a_4)$ with $a_4 = u_{n+1}t^{-2p^{3n+3}}$ and $r_4 = 2r(3n+3)+1$. It follows that $d^{2r(3n+3)+1}(u_{n+1}t^{-p^{3n+3}}) \doteq (t\mu)^{r(3n)+1}$, since $t^{p^{3n+3}}$ is an infinite cycle. This establishes the claimed new odd-length differential, and leaves

$$\begin{aligned}\hat{E}^{2r(3n+3)+2}(C_{p^{n+1}}) = {} & P(t^{\pm p^{3n+3}}) \otimes P_{r(3n)+1}(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ & \oplus \bigoplus_{k=4}^{3n+3} E(u_{n+1}) \otimes P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}.\end{aligned}$$

No free summands remain, so by Lemma 10.4 the remaining differentials are all zero, and this E^r-term equals $\hat{E}^\infty(C_{p^{n+1}})$. This completes the nth inductive step. □

11 The $\mathbb{T}$-Tate spectral sequence

We can now make the differential structure of the spectral sequences

$$\begin{aligned} E^2(\mathbb{T}) &= H^{-*}(\mathbb{T}; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}, \\ \mu^{-1} E^2(\mathbb{T}) &= H^{-*}(\mathbb{T}; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}) \Rightarrow V(2)_* (\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}, \\ \hat{E}^2(\mathbb{T}) &= \hat{H}^{-*}(\mathbb{T}; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}} \end{aligned}$$

fully explicit.

Theorem 11.1 *The $\mathbb{T}$-Tate spectral sequence*

$$\hat{E}^2(\mathbb{T}) = \hat{H}^{-*}(\mathbb{T}; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}}$$

has E^2-term

$$\hat{E}^2(\mathbb{T}) = P(t^{\pm 1}) \otimes P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3),$$

there are differentials

$$d^{2r(k)}(t^{p^{k-1}-p^k}) \doteq t^{p^{k-1}} (t\mu)^{r(k-3)} \lambda_{[k]}$$

for each $k \geq 1$, the classes $t\mu$, λ_1, λ_2 and λ_3 are permanent cycles, and the E^∞-term is

$$\begin{aligned}\hat{E}^\infty(\mathbb{T}) = {} & P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ & \oplus \bigoplus_{k \geq 4} P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mid v_p(i) = k-1\}.\end{aligned}$$

Proof This follows by passage to the limit over n from Theorem 10.2. □

Remark 11.2 We saw in Propositions 9.1 and 10.3 that for each $n \geq 1$, some positive power of $\mu \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)$ lifts to $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{hC_{p^n}}$, so that the μ-localized C_{p^n}-homotopy fixed point spectral sequence converges to a localization $\mu^{-1} V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{hC_{p^n}}$. However, no such power of μ lifts to $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$, and we therefore instead express the abutment of the μ-localized $\mathbb{T}$-homotopy fixed point spectral sequence in terms of $\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$, with $\mu^{-1} V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle) \cong V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$, as per Theorem 8.1.

Proposition 11.3 *The μ-localized $\mathbb{T}$-homotopy fixed point spectral sequence*

$$\mu^{-1} E^2(\mathbb{T}) = H^{-*}(\mathbb{T}; \mu^{-1} V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}$$

has E^2-term

$$\mu^{-1} E^2(\mathbb{T}) = P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm 1}),$$

there are differentials

$$d^{2r(k)}(\mu^{p^{k-1}}) \doteq (t\mu)^{r(k)} \lambda_{[k]} \mu^{p^{k-1} - p^k}$$

for each $k \geq 1$, the classes $t\mu$, λ_1, λ_2 and λ_3 are permanent cycles, and the E^∞-term is

$$\begin{aligned} \mu^{-1} E^\infty(\mathbb{T}) = {} & P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ & \oplus \bigoplus_{k \geq 1} P_{r(k)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\lambda_{[k]} \mu^j \mid v_p(j) = k-1\}. \end{aligned}$$

Proof This follows by passage to the limit over n from Proposition 10.3. □

Proposition 11.4 *The $\mathbb{T}$-homotopy fixed point spectral sequence*

$$E^2(\mathbb{T}) = H^{-*}(\mathbb{T}; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) \Rightarrow V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$$

has E^2-term

$$E^2(\mathbb{T}) = P(t) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu),$$

for each $k \geq 1$ there are differentials

$$\begin{aligned} d^{2r(k)}(\mu^{dp^{k-1}}) &\doteq (t\mu)^{r(k)} \lambda_{[k]} \mu^{(d-p)p^{k-1}} && \text{for } d > p \text{ with } p \nmid d, \\ d^{2r(k)}(\mu^{(p-d)p^{k-1}}) &\doteq t^{dp^{k-1}} (t\mu)^{r(k) - dp^{k-1}} \lambda_{[k]} && \text{for } 0 < d < p, \\ d^{2r(k)}(t^{dp^{k-1}}) &\doteq t^{dp^{k-1} + p^k} (t\mu)^{r(k-3)} \lambda_{[k]} && \text{for } d > 0 \text{ with } p \nmid d, \end{aligned}$$

the classes $t\mu$, λ_1, λ_2 and λ_3 are permanent cycles, and the E^∞-term is

$$\begin{aligned} E^\infty(\mathbb{T}) = {} & P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3) \\ & \oplus \bigoplus_{k \geq 1} P_{r(k)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\lambda_{[k]} \mu^{dp^{k-1}} \mid p \nmid d > 0\} \\ & \oplus \bigoplus_{k \geq 1} P_{r(k) - dp^{k-1}}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^{dp^{k-1}} \lambda_{[k]} \mid 0 < d < p\} \\ & \oplus \bigoplus_{k \geq 4} P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^{dp^{k-1}} \lambda_{[k]} \mid p \nmid d > p\}. \end{aligned}$$

Proof The differentials on $t^{dp^{k-1}}$ for $p \nmid d > 0$ follow as in Theorem 11.1, while those on $\mu^{dp^{k-1}}$ for $p \nmid d > p$ are as in Proposition 11.3. For $0 < d < p$,

$$d^{2r(k)}(\mu^{dp^{k-1}}) \doteq (t\mu)^{r(k)} \lambda_{[k]} \mu^{dp^{k-1}-p^k} = t^{p^k - dp^{k-1}} (t\mu)^{r(k)+dp^{k-1}-p^k} \lambda_{[k]}.$$

Replacing d by $p-d$ we obtain the claimed formula.

For each $k \geq 1$ and $p \nmid d$, the $d^{2r(k)}$-differential maps the summand

$$E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \mu^j \mid i-j = dp^{k-1} - p^k\}$$

of $E^{2r(k)}(\mathbb{T})$ injectively to the summand

$$E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^i \lambda_{[k]} \mu^j \mid i-j = dp^{k-1}\},$$

with cokernel one of the displayed summands in $E^\infty(\mathbb{T})$. Here $i \geq 0$ and $j \geq 0$ in each case. □

Following the referee's good advice, we decompose these E^∞-terms as in the next three definitions.

Definition 11.5 Let $A = P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)$, viewed as a subalgebra of $E^\infty(\mathbb{T})$. For $k \geq 1$ and $0 < d < p$ let

$$C(k,d) = P_{r(k)-dp^{k-1}}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^{dp^{k-1}} \lambda_{[k]}\}$$

be the finite A-submodule of $E^\infty(\mathbb{T})$ generated by

$$c_{k,d} = t^{dp^{k-1}} \lambda_{[k]}.$$

The class

$$x_{k,d} = (t\mu)^{\frac{d}{p} r(k-3)} \cdot c_{k,d} = t^{\frac{d}{p} r(k)} \lambda_{[k]} \mu^{\frac{d}{p} r(k-3)}$$

is an element of $C(k,d)$, is nonzero since $\frac{d}{p} r(k-3) < r(k) - dp^{k-1}$, and has total degree $|x_{k,d}| = 2p^{[k]} - 2dp^{[k]-1} - 1$. In particular,

$$x_{1,d} = c_{1,d} = t^d \lambda_1, \quad x_{2,d} = c_{2,d} = t^{dp} \lambda_2 \quad \text{and} \quad x_{3,d} = c_{3,d} = t^{dp^2} \lambda_3$$

for all $0 < d < p$. Let $C = \bigoplus_{k \geq 1, 0 < d < p} C(k,d)$, and let

$$B = \bigoplus_{k \geq 1} P_{r(k)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\lambda_{[k]} \mu^{dp^{k-1}} \mid p \nmid d > 0\},$$

$$D = \bigoplus_{k \geq 4} P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^{dp^{k-1}} \lambda_{[k]} \mid p \nmid d > p\}$$

be the indicated A-submodules of $E^\infty(\mathbb{T})$, concentrated in positive and negative total degrees, respectively. Then $E^\infty(\mathbb{T}) = A \oplus B \oplus C \oplus D$.

It should be clear from the context whether B refers to this summand in $E^\infty(\mathbb{T})$ or a generic S-algebra. The classes $x_{k,d}$ are the ones mentioned in Section 1. Their role, together with the classes $z_{k,d}$ defined just below, will only become apparent starting with Corollaries 12.7 and 12.8.

Definition 11.6 Let $A' = P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)$ as a subalgebra of $\mu^{-1}E^\infty(\mathbb{T})$. For $k \ge 1$ and $0 < d < p$, let

$$C'(k,d) = P_{r(k)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\lambda_{[k]}\mu^{-dp^{k-1}}\}$$

be the finite A'-submodule of $\mu^{-1}E^\infty(\mathbb{T})$ generated by

$$c'_{k,d} = \lambda_{[k]}\mu^{-dp^{k-1}}.$$

The class

$$z_{k,d} = (t\mu)^{\frac{d}{p}r(k)} \cdot c'_{k,d} = t^{\frac{d}{p}r(k)}\lambda_{[k]}\mu^{\frac{d}{p}r(k-3)}$$

is an element of $C'(k,d)$, is nonzero since $\frac{d}{p}r(k) < r(k)$, and has total degree $|z_{k,d}| = 2p^{[k]} - 2dp^{[k]-1} - 1$. Let $C' = \bigoplus_{k\ge1, 0<d<p} C'(k,d)$, and let

$$B' = \bigoplus_{k\ge1} P_{r(k)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\lambda_{[k]}\mu^{dp^{k-1}} \mid p \nmid d > 0\},$$

$$D' = \bigoplus_{k\ge1} P_{r(k)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\lambda_{[k]}\mu^{-dp^{k-1}} \mid p \nmid d > p\}$$

be the indicated A'-submodules of $\mu^{-1}E^\infty(\mathbb{T})$, concentrated in positive and negative total degrees, respectively. Then $\mu^{-1}E^\infty(\mathbb{T}) = A' \oplus B' \oplus C' \oplus D'$.

Definition 11.7 Let $\widehat{A} = P(t\mu) \otimes E(\lambda_1, \lambda_2, \lambda_3)$,

$$\widehat{C}(k,d) = P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^{dp^{k-1}}\lambda_{[k]}\}$$

for $k \ge 1$ and $0 < d < p$, and

$$\widehat{B} = \bigoplus_{k\ge4} P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^{-dp^{k-1}}\lambda_{[k]} \mid p \nmid d > 0\},$$

$$\widehat{C} = \bigoplus_{k\ge4, 0<d<p} \widehat{C}(k,d),$$

$$\widehat{D} = \bigoplus_{k\ge4} P_{r(k-3)}(t\mu) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{t^{dp^{k-1}}\lambda_{[k]} \mid p \nmid d > p\}.$$

Then $\widehat{E}^\infty(\mathbb{T}) = \widehat{A} \oplus \widehat{B} \oplus \widehat{C} \oplus \widehat{D}$. Note that $\widehat{C}(k,d) = 0$ for $k \in \{1,2,3\}$.

The $\mathbb{T}$-equivariant comparison map

$$\widehat{\Gamma}_1 \colon \mathrm{THH}(\mathrm{BP}\langle 2\rangle) \to \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_2}$$

(renamed the p-cyclotomic structure map φ_p in [Nikolaus and Scholze 2018]) induces a morphism of $\mathbb{T}$-homotopy fixed point spectral sequences, given at the E^2-term by the homomorphism

$$E^2(\widehat{\Gamma}_1^{h\mathbb{T}}) \colon E^2(\mathbb{T}) = P(t) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu) \to P(t) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu^{\pm1}) = \mu^{-1}E^2(\mathbb{T})$$

that inverts μ. At the E^∞-terms we have the following formulas:

Lemma 11.8 *The homomorphism*

$$E^\infty(\hat{\Gamma}_1^{h\mathbb{T}})\colon E^\infty(\mathbb{T}) \to \mu^{-1} E^\infty(\mathbb{T})$$

maps

1. *A isomorphically to A',*
2. *B isomorphically to B',*
3. *C injectively to C', and*
4. *D to zero.*

Specifically, $E^\infty(\hat{\Gamma}_1^{h\mathbb{T}})$ is injective in total degrees $* \geq -2p^3 + 2p^2$ and bijective in total degrees $* \geq 2p^2 + 2p - 2$.

Proof Cases (1) and (2) are clear. In (3), the injection $C(k,d) \to C'(k,d)$ takes $c_{k,d} = t^{dp^{k-1}} \lambda_{[k]}$ to $(t\mu)^{dp^{k-1}} \cdot c'_{k,d}$, which is annihilated by the same $t\mu$-power as $c_{k,d}$, namely $(t\mu)^{r(k)-dp^{k-1}}$. In (4), the image of D in D' is zero since $t^{dp^{k-1}} \lambda_{[k]}$ maps to $(t\mu)^{dp^{k-1}} \cdot \lambda_{[k]} \mu^{-dp^{k-1}}$, which is zero because $dp^{k-1} \geq r(k)$ for $d > p$.

The highest-degree element in the kernel of $E^\infty(\hat{\Gamma}_1^{h\mathbb{T}})$ is $t^{(p+1)p^3}(t\mu)^{p-1}\lambda_1\lambda_2\lambda_3$ in D in total degree $-2p^3 + 2p^2 - 1$, mapping to $d^{2r(4)}(t^{p^3-1}\lambda_2\lambda_3\mu^{-1})$. The highest-degree element not in the image of $E^\infty(\hat{\Gamma}_1^{h\mathbb{T}})$ is $\lambda_1\lambda_2\lambda_3\mu^{-1}$ in $C'(1,1)$, in total degree $2p^2 + 2p - 3$. □

Similarly, the homotopy restriction map

$$R^h\colon \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}} \to \mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}}$$

(renamed the canonical map in [Nikolaus and Scholze 2018]) induces a morphism of spectral sequences, given at the E^2-term by the homomorphism

$$E^2(R^h)\colon E^2(\mathbb{T}) = P(t) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu) \to P(t^{\pm 1}) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu) = \hat{E}^2(\mathbb{T})$$

that inverts t. The following lemma is similar to [Ausoni and Rognes 2002, Proposition 7.2]:

Lemma 11.9 *The homomorphism*

$$E^\infty(R^h)\colon E^\infty(\mathbb{T}) \to \hat{E}^\infty(\mathbb{T})$$

maps

1. *A isomorphically to $\hat{A}$,*
2. *B to zero,*
3. *C surjectively to $\hat{C}$, and*
4. *D isomorphically to $\hat{D}$.*

Specifically, $E^\infty(R^h)$ is surjective in total degrees $* \leq 2p^3 + 2p - 2$ and bijective in total degrees $* \leq 0$.

Proof Cases (1) and (4) are clear. In (2), the image of B in $\widehat{B}$ is zero since $\lambda_{[k]}\mu^{dp^{k-1}}$ maps to $(t\mu)^{dp^{k-1}} \cdot t^{-dp^{k-1}}\lambda_{[k]}$, which is zero because $dp^{k-1} \geq r(k-3)$ for $d > 0$. In (3), the surjection $C(k,d) \to \widehat{C}(k,d)$ takes $c_{k,d} = t^{dp^{k-1}}\lambda_{[k]}$ to $t^{dp^{k-1}}\lambda_{[k]}$, which is annihilated by a lower $t\mu$-power than $c_{k,d}$ since $r(k-3) < r(k) - dp^{k-1}$ for $0 < d < p$.

The lowest-degree element not in the image of $E^\infty(R^h)$ is $t^{-p^3}\lambda_1$ in $\widehat{B}$, in total degree $2p^3+2p-1$. The lowest-degree element in the kernel of $E^\infty(R^h)$ is $t^{p-1}\lambda_1$ in $C(1,p-1)$ in total degree 1, mapping to $d^{2p}(t^{-1})$. □

12 Topological cyclic homology and algebraic K-theory

We now pursue the calculational strategy employed in [Bökstedt and Madsen 1994; 1995; Hesselholt and Madsen 1997; Rognes 1999; Ausoni and Rognes 2002, 2012; Ausoni 2010] to identify $\mathrm{TC}(B)$ with the homotopy equalizer of the two maps GR^h and $\widehat{\Gamma}_1^{h\mathbb{T}}$ displayed below.

$$\begin{tikzcd}
\mathrm{TC}(B) \arrow[r, "\pi"] & \mathrm{THH}(B)^{h\mathbb{T}} \arrow[r, "R^h"] \arrow[dr, "\widehat{\Gamma}_1^{h\mathbb{T}}"'] & \mathrm{THH}(B)^{t\mathbb{T}} \arrow[d, "G", "\simeq"'] \\
 & & (\mathrm{THH}(B)^{tC_p})^{h\mathbb{T}}
\end{tikzcd}$$

In these papers, this identification was only known to be valid in V-homotopy in a range of sufficiently high degrees, for suitable finite spectra V. However, with the work of Nikolaus and Scholze [2018, Remark 1.6], we now know that $\mathrm{TC}(B)$ is given by the homotopy equalizer above in all degrees, whenever $\mathrm{THH}(B)$ is bounded below. (This certainly holds for all connective S-algebras B.) Let $GR^h_* = V_*(GR^h)$ and $\widehat{\Gamma}_{1*}^{h\mathbb{T}} = V_*(\widehat{\Gamma}_1^{h\mathbb{T}})$. The associated long exact sequence

$$\cdots \xrightarrow{\partial} V_*\,\mathrm{TC}(B) \xrightarrow{\pi_*} V_*\,\mathrm{THH}(B)^{h\mathbb{T}} \xrightarrow{GR^h_* - \widehat{\Gamma}_{1*}^{h\mathbb{T}}} V_*(\mathrm{THH}(B)^{tC_p})^{h\mathbb{T}} \xrightarrow{\partial} \cdots$$

leads to the short exact sequence

$$0 \to \Sigma^{-1}\operatorname{cok}(GR^h_* - \widehat{\Gamma}_{1*}^{h\mathbb{T}}) \xrightarrow{\partial} V_*\,\mathrm{TC}(B) \xrightarrow{\pi_*} \ker(GR^h_* - \widehat{\Gamma}_{1*}^{h\mathbb{T}}) \to 0.$$

In our case, the task is to calculate the kernel and cokernel of $GR^h_* - \widehat{\Gamma}_{1*}^{h\mathbb{T}}$ for $B = \mathrm{BP}\langle 2\rangle$ and $V = V(2)$, and thereby to determine $V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$. We studied the effect of $\widehat{\Gamma}_1^{h\mathbb{T}}$ and R^h at the level of spectral sequence E^∞-terms in Lemmas 11.8 and 11.9. In Proposition 12.1 we do something similar for G. Thereafter we find lifts $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ in $V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$ of the summands A, B, C and D of $E^\infty(\mathbb{T})$ from Definition 11.5, and compute the effect of $GR^h_* - \widehat{\Gamma}_{1*}^{h\mathbb{T}}$ acting upon these lifts.

Proposition 12.1 *The isomorphism*

$$G_* = V(2)_*(G)\colon V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}} \xrightarrow{\cong} V(2)_*(\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}$$

takes each class

$$\eta \in \{t^{p^3 i}(t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\}$$

detected by $y = t^{p^3 i}(t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3} \in \hat{E}^\infty(\mathbb{T})$ *to a class*

$$G_*(\eta) \in \{(t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^{-i}\}$$

detected by $z = (t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^{-i} \in \mu^{-1}E^\infty(\mathbb{T})$ *(up to a unit in* $\mathbb{F}_p$*, which we suppress). Conversely, its inverse* G_*^{-1} *takes each class*

$$\zeta \in \{(t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^{j}\}$$

to a class

$$G_*^{-1}(\zeta) \in \{t^{-p^3 j}(t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\}$$

(again, up to a unit in $\mathbb{F}_p$*, which we suppress).*

Proof We first handle the case $m = 0$, using the commutative diagram

$$\begin{tikzcd}
\mathrm{THH}(B)^{t\mathbb{T}} \arrow[r, "F^t"] \arrow[d, "G"', "\simeq"] & \mathrm{THH}(B)^{tC_{p^{n+1}}} \arrow[d, "\simeq"', "G_n"] \\
(\mathrm{THH}(B)^{tC_p})^{h\mathbb{T}} \arrow[r, "F^h"] & (\mathrm{THH}(B)^{tC_p})^{hC_{p^n}}
\end{tikzcd}$$

in the special case of $B = \mathrm{BP}\langle 2\rangle$ and $n = 0$. It is constructed by viewing the $\mathbb{T}/C_p$-equivariant C_p-fixed point spectrum

$$X = [\widetilde{E\mathbb{T}} \wedge F(E\mathbb{T}_+, \mathrm{THH}(B))]^{C_p} \simeq \mathrm{THH}(B)^{tC_p}$$

as a $\mathbb{T}$-spectrum via the p^{th} root isomorphism $\rho\colon \mathbb{T} \cong \mathbb{T}/C_p$. The comparison map $G\colon X^{\mathbb{T}} \to X^{h\mathbb{T}}$ is then compatible with the comparison map $G_n\colon X^{C_{p^n}} \to X^{hC_{p^n}}$, via the group restriction maps along $C_{p^n} \subset \mathbb{T}$.

In the case $n = 0$, the group restriction map F^t induces a morphism of spectral sequences given at the E^2-terms by the inclusion

$$E^2(F^t)\colon \hat{E}^2(\mathbb{T}) = P(t^{\pm 1}) \otimes E(\lambda_1,\lambda_2,\lambda_3) \otimes P(\mu) \to E(u_1) \otimes P(t^{\pm 1}) \otimes E(\lambda_1,\lambda_2,\lambda_3) \otimes P(\mu) = \hat{E}^2(C_p).$$

Hence each class $\eta \in V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}}$ detected by $t^{p^3 i}\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3} \neq 0$ in $\hat{E}^\infty(\mathbb{T})$ maps to a class $F_*^t(\eta)$ detected by $t^{p^3 i}\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}$ in $\hat{E}^\infty(C_p) = P(t^{\pm p^3}) \otimes E(\lambda_1,\lambda_2,\lambda_3)$, which remains nonzero there. It follows from Theorem 8.1 that $(G_0 F^t)_*(\eta) = \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^{-i}$ in $V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p}$ up to a unit in $\mathbb{F}_p$, which we suppress. This equals $(F^h G)_*(\eta)$, where the group restriction map F^h for $n = 0$ induces the edge homomorphism

$$E^\infty(F^h)\colon \mu^{-1}E^\infty(\mathbb{T}) \to E(\lambda_1,\lambda_2,\lambda_3) \otimes P(\mu^{\pm 1}).$$

Hence $G_*(\eta)$ must be detected in $\mu^{-1}E^\infty(\mathbb{T})$ by a class z mapping to $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^{-i}$ under the edge homomorphism, and the only possibility is that $z = \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^{-i}$, in filtration degree zero.

For $m \geq 1$, each class η detected by $y = t^{p^3 i}(t\mu)^m\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3} \neq 0$ in $\hat{E}^\infty(\mathbb{T})$ is of the form $\eta = v_3^m \cdot \eta_0$, with η_0 detected by $y_0 = t^{p^3 i}\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3} \in \hat{E}^\infty(\mathbb{T})$. To see this, note that each element of $\hat{E}^\infty(\mathbb{T})$ in the same total degree as y, but of lower filtration, is a $(t\mu)^m$-multiple. This follows from the case enumeration in the proof of Lemma 9.3. By the first part of the proof, $G_*(\eta_0)$ is detected by $z_0 = \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^{-i}$. Hence $G_*(\eta) = v_3^m \cdot G_*(\eta_0)$ is detected by $(t\mu)^m \cdot z_0 = z$, since this product is nonzero.

For the converse, consider any class ζ detected by $z = (t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\mu^j \neq 0$ in $\mu^{-1}E^\infty(\mathbb{T})$. Then $\eta = G_*^{-1}(\zeta)$ must be detected by some monomial y in $E^\infty(\mathbb{T})$, and $G_*(\eta) = \zeta$ is detected by z. By the first part of the proposition, this monomial must be $y = t^{-p^3 j}(t\mu)^m \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}$. □

Recall λ_1^K, λ_2^K and λ_3^K from Definitions 6.6, 6.7 and 6.9.

Definition 12.2 Let

$$\widetilde{A} = P(v_3) \otimes E(\lambda_1, \lambda_2, \lambda_3) \subset V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$$

be the subalgebra generated by the images of $i_2 i_1 i_0(\lambda_1^K), i_2 i_1(\lambda_2^K), i_2(\lambda_3^K) \in V(2)_* K(\mathrm{BP}\langle 2\rangle)$ and $v_3 \in \pi_* V(2)$ under the composites

$$S \to K(\mathrm{BP}\langle 2\rangle) \xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathrm{BP}\langle 2\rangle) \xrightarrow{\pi} \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}},$$

where trc denotes the cyclotomic trace map [Bökstedt et al. 1993]. The homomorphisms GR_*^h and $\widehat{\Gamma}_{1*}^{h\mathbb{T}}$ agree on these classes, and we let

$$\widetilde{A}' = P(v_3) \otimes E(\lambda_1, \lambda_2, \lambda_3) \subset V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}$$

be the subalgebra generated by the images of v_3, λ_1, λ_2 and λ_3, under either one of these homomorphisms.

The subalgebras $\widetilde{A}$ and $\widetilde{A}'$ are lifts to $V(2)$-homotopy of the subalgebras $A \subset E^\infty(\mathbb{T})$ and $A' \subset \mu^{-1}E^\infty(\mathbb{T})$, respectively. To choose good lifts $\widetilde{C}(k,d)$ and $\widetilde{C}'(k,d)$ in $V(2)$-homotopy of the summands $C(k,d)$ and $C'(k,d)$ we make use of the norm–restriction homotopy cofiber sequence

$$\Sigma \operatorname{THH}(\mathrm{BP}\langle 2\rangle)_{h\mathbb{T}} \xrightarrow{N^h} \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}} \xrightarrow{R^h} \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}} \xrightarrow{\partial^h} \Sigma^2 \operatorname{THH}(\mathrm{BP}\langle 2\rangle)_{h\mathbb{T}}$$

and the associated long exact sequence. The $\mathbb{T}$-Tate spectral sequence maps to a horizontally shifted $\mathbb{T}$-homotopy orbit spectral sequence

$$E^2_{*,*} = H_{*-2}(\mathbb{T}; V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)) = \mathbb{F}_p\{t^i \mid i < 0\} \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu) \\ \Rightarrow V(2)_* \Sigma^2 \operatorname{THH}(\mathrm{BP}\langle 2\rangle)_{h\mathbb{T}}, \tag{12-1}$$

concentrated in filtrations $s \geq 2$ of the first quadrant.

The $\mathbb{T}$-Tate differentials crossing the vertical line $s = 1$ are closely related to the homotopy norm map $N_*^h = V(2)_*(N^h)$; cf [Bökstedt and Madsen 1994, Theorem 2.15]. Let $R_*^h = V(2)_*(R^h)$, so that $\operatorname{im}(N_*^h) = \ker(R_*^h)$ by exactness. The following two lemmas spell out some upper bounds for $\ker E^\infty(R^h)$:

Lemma 12.3 *In the $\mathbb{T}$-Tate spectral sequence $(\widehat{E}^r(\mathbb{T}), d^r)$, the nonzero differentials from total degrees $* < 2p^3$ that cross the line $s = 1$ are of the form*

$$d^{2p}(t^{d-p}\lambda_2^{\epsilon_2}) \doteq t^d \lambda_1 \lambda_2^{\epsilon_2}, \quad d^{2p^2}(t^{dp-p^2}\lambda_1^{\epsilon_1}) \doteq t^{dp}\lambda_1^{\epsilon_1}\lambda_2, \quad d^{2p^3}(t^{dp^2-p^3}\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}) \doteq t^{dp^2}\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3,$$

for suitable $0 < d < p$ *and* $\epsilon_1, \epsilon_2 \in \{0,1\}$. *Hence in total degrees* $* \le 2p^3 - 2$ *the classes on the right-hand side generate* $\ker E^\infty(R^h)$. *These lie in filtrations* $-2(p^3 - p^2) \le s \le -2$, *and there is at most one class in each total degree* $* \le 2p^3 - 2$.

Proof The restriction to total degrees $* < 2p^3$ means we only have to consider differentials on the classes t^i for $-p^3 < i < 0$, and their λ_1- and λ_2-multiples. The d^{2p}-differentials only cross $s = 1$ for $-p < i < 0$. The d^{2p^2}-differentials are defined for $p \mid i$ and only cross $s = 1$ when $-p^2 < i < 0$. The d^{2p^3}-differentials are defined for $p^2 \mid i$ and cross $s = 1$ whenever $-p^3 < i < 0$. An explicit enumeration shows that each total degree in the range $1 \le * \le 2p^3 - 2$ occurs at most once. □

Lemma 12.4 *In the* $\mathbb{T}$*-Tate spectral sequence, the nonzero* d^r*-differentials from total degrees* $* < 4p^3 - 1$ *that cross the line* $s = 1$ *are of the form*

$$\begin{aligned} d^{2p}(t^{d-p}(t\mu)^m \lambda_2^{\epsilon_2} \lambda_3^{\epsilon_3}) &\doteq t^d (t\mu)^m \lambda_1 \lambda_2^{\epsilon_2} \lambda_3^{\epsilon_3}, \\ d^{2p^2}(t^{dp-p^2}(t\mu)^m \lambda_1^{\epsilon_1} \lambda_3^{\epsilon_3}) &\doteq t^{dp} (t\mu)^m \lambda_1^{\epsilon_1} \lambda_2 \lambda_3^{\epsilon_3}, \\ d^{2p^3}(t^{dp^2-p^3}(t\mu)^m \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2}) &\doteq t^{dp^2} (t\mu)^m \lambda_1^{\epsilon_1} \lambda_2^{\epsilon_2} \lambda_3, \\ d^{2p^4+2p}(t^{-p^3} \lambda_2^{\epsilon_2}) &\doteq t^{p^4-p^3} (t\mu)^p \lambda_1 \lambda_2^{\epsilon_2} \end{aligned}$$

for suitable $m, \epsilon_1, \epsilon_2, \epsilon_3 \in \{0,1\}$, *with* $m + \epsilon_3 \le 1$. *In the* d^{2p} *case with* $m = 1$ *we have* $d = -1$ *or* $0 < d < p - 1$, *while in the remaining* d^{2p}, d^{2p^2} *and* d^{2p^3} *cases we have* $0 < d < p$. *Hence in total degrees* $* \le 4p^3 - 3$ *the classes on the right-hand side generate* $\ker E^\infty(R^h)$. *These lie in filtrations* $-2(p^3 - p^2 + 1) \le s \le 0$, *except for the last two classes* $t^{p^4-p^3}(t\mu)^p \lambda_1 \lambda_2^{\epsilon_2}$, *which lie in filtration* $-2(p^4 - p^3 + p)$ *and total degrees* $2p^3 - 1 + \epsilon_2(2p^2 - 1)$.

Proof The restriction to total degrees $* < 4p^3 - 1$ means that we only have to consider differentials on the classes t^i for $-p^3 \le i < 0$, and some of their $t\mu$-, λ_1-, λ_2- and λ_3-multiples (without repeated factors). The resulting right-hand classes have the form $t^i y$ in Tate filtration $s = -2i$, where $0 \le i \le p^3 - p^2 + 1$, except in the last two cases. □

Recall $c_{k,d}$ and $c'_{k,d}$ from Definitions 11.5 and 11.6.

Proposition 12.5 *For each* $k \in \{1,2,3\}$ *and* $0 < d < p$ *there is a unique element*

$$\gamma_{k,d} \in \{c_{k,d}\} \subset V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$$

that satisfies

$$R^h_*(\gamma_{k,d}) = 0.$$

Moreover, $\lambda_k \cdot \gamma_{k,d} = 0$ *and* $v_3^{p^k - dp^{k-1}} \cdot \gamma_{k,d} = 0$.

Proof The tower of spectra inducing the $\mathbb{T}$-homotopy fixed point spectral sequence is obtained by restricting the tower inducing the $\mathbb{T}$-Tate spectral sequence to filtrations $s \le 0$. Hence each nonzero class $x \in \ker E^\infty(R^h) \subset E^\infty(\mathbb{T})$ can be represented by an element $\xi \in \ker(R^h_*) \subset V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$, in the sense that $\xi \in \{x\}$; see [Bökstedt and Madsen 1994, page 75; Ausoni and Rognes 2002, Lemma 7.3]. Furthermore, for x in total degree $* \le 2p^3 - 2$, the element ξ is unique. To see this, suppose that $\xi' \in \{x\}$ is also in $\ker(R^h_*)$. Then $\xi' - \xi$ in $\ker(R^h_*)$ must be detected by a class x' in $\ker E^\infty(R^h)$, in the same total degree as x, but in lower filtration. As noted in Lemma 12.3, there are no nonzero such x', so $\xi' = \xi$.

In particular, for $k \in \{1, 2, 3\}$ and $0 < d < p$ this applies to the classes $c_{k,d} = t^{dp^{k-1}} \lambda_k$ in total degrees $1 \le 2p^k - 2dp^{k-1} - 1 \le 2p^3 - 2p^2 - 1$, and uniquely defines the homotopy elements $\gamma_{k,d}$.

By exactness, we can write $\gamma_{k,d} = N^h_*(\theta_{k,d})$ with

$$\theta_{k,d} \in V(2)_* \Sigma^2 \operatorname{THH}(\mathrm{BP}\langle 2\rangle)_{h\mathbb{T}}$$

in degree $2p^k - 2dp^{k-1}$. In fact, $\theta_{k,d} \in \{t^{dp^{k-1}-p^k}\}$, up to a unit multiple, but we only need to know that $\theta_{k,d}$ must be detected in filtration $s \le 2p^k - 2dp^{k-1}$ in the shifted $\mathbb{T}$-homotopy orbit spectral sequence (12-1). Hence $v_3^{p^k - dp^{k-1}} \cdot \theta_{k,d} = 0$, for filtration reasons, which implies that $v_3^{p^k - dp^{k-1}} \cdot \gamma_{k,d} = 0$ since N^h_* is $P(v_3)$-linear.

Finally, $\lambda_k \cdot \theta_{k,d} = 0$, because $t^{dp^{k-1}-p^k} \cdot \lambda_k \doteq d^{2p^k}(t^{dp^{k-1}-2p^k})$ is a boundary and by inspection of bidegrees there are no other classes in the E^∞-term of (12-1) in the same total degree and of lower filtration. Applying N^h_* we can conclude that $\lambda_k \cdot \gamma_{k,d} = 0$. □

Proposition 12.6 *For each $k \ge 1$ and $0 < d < p$ there are elements*

$$\gamma'_{k,d} \in \{c'_{3,d}\} \subset V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}} \quad \textit{and} \quad \gamma_{k+3,d} \in \{c_{k+3,d}\} \subset V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$$

that satisfy

$$v_3^{dp^{k-1}} \cdot \gamma'_{k,d} = \hat\Gamma^{h\mathbb{T}}_{1*}(\gamma_{k,d}) \quad \textit{and} \quad GR^h_*(\gamma_{k+3,d}) = \gamma'_{k,d}.$$

Moreover, $v_3^{r(k)} \cdot \gamma'_{k,d} = 0$ and $v_3^{r(k+3)-dp^{k+2}} \cdot \gamma_{k+3,d} = 0$.

Proof We proceed by induction on $k \ge 1$, starting from Proposition 12.5. By Lemma 11.8 the image $\hat\Gamma^{h\mathbb{T}}_{1*}(\gamma_{k,d}) \in V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}$ of the previously constructed class $\gamma_{k,d} \in \{c_{d,k}\}$ is detected by $t^{dp^{k-1}}\lambda_k = (t\mu)^{dp^{k-1}} \cdot c'_{k,d}$ in $\mu^{-1}E^\infty(\mathbb{T})$, so any initial choice of $\gamma'_{k,d} \in \{c'_{k,d}\}$ will satisfy $v_3^{dp^{k-1}} \cdot \gamma'_{k,d} \equiv \hat\Gamma^{h\mathbb{T}}_{1*}(\gamma_{k,d})$, modulo classes of lower filtration. Since $\mu^{-1}E^\infty(\mathbb{T})$ is generated as a $P(t\mu)$-module by classes in filtration $s = 0$, each nonzero class in lower filtration than $c_{k,d}$, but of the same total degree, is $(t\mu)^{dp^{k-1}}$ times a class in the same total degree as $c'_{k,d}$ and of lower filtration. Hence the choice of $\gamma'_{k,d}$ can be iteratively adjusted so as to make $v_3^{dp^{k-1}} \cdot \gamma'_{k,d} = \hat\Gamma^{h\mathbb{T}}_{1*}(\gamma_{k,d})$.

Therefore $v_3^{r(k)} \cdot \gamma'_{k,d} = v_3^{r(k)-dp^{k-1}} \cdot \hat\Gamma^{h\mathbb{T}}_{1*}(\gamma_{k,d}) = 0$, since $\hat\Gamma^{h\mathbb{T}}_{1*}$ is $P(v_3)$-linear and $v_3^{r(k)-dp^{k-1}} \cdot \gamma_{k,d} = 0$ by the inductive hypothesis.

The final choice of class $\gamma'_{k,d}$ is still detected by $c'_{k,d} = \lambda_{[k]}\mu^{-dp^{k-1}}$ in $C'(k,d) \subset \mu^{-1}E^\infty(\mathbb{T})$, so by Proposition 12.1, $G^{-1}_*(\gamma'_{k,d}) \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{t\mathbb{T}}$ is detected by $t^{dp^{k+2}}\lambda_{[k]}$ in $\hat C(k+3,d) \subset \hat E^\infty(\mathbb{T})$.

This class lies in negative total degree, where $E^\infty(R^h)$ is bijective by Lemma 11.9. It follows that $R^h_*(\gamma_{k+3,d}) = G_*^{-1}(\gamma'_{k,d})$ for a uniquely determined class $\gamma_{k+3,d} \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$, which is detected by $c_{k+3,d} = t^{dp^{k+2}}\lambda_{[k]}$ in $C(k+3,d) \subset E^\infty(\mathbb{T})$.

From the relation $v_3^{r(k)} \cdot \gamma'_{k,d} = 0$ and $P(v_3)$-linearity of G_* and R^h_* we deduce that $v_3^{r(k)} \cdot G_*^{-1}(\gamma'_{k,d}) = 0$ and $R^h_*(v_3^{r(k)} \cdot \gamma_{k+3,d}) = 0$. Since $\ker(R^h_*) = \operatorname{im}(N^h_*)$, we can write $v_3^{r(k)} \cdot \gamma_{k+3,d} = N^h_*(\theta_{k+3,d})$ for some $\theta_{k+3,d}$ in degree $2p^{k+3} - 2dp^{k+2}$. From the $\mathbb{T}$-Tate differential

$$d^{2r(k+3)}(t^{dp^{k+2}-p^{k+3}}) \doteq t^{dp^{k+2}}(t\mu)^{r(k)}\lambda_{[k]} = (t\mu)^{r(k)} \cdot c_{k+3,d}$$

we could prove that $\theta_{k+3,d} \in \{t^{dp^{k+2}-p^{k+3}}\}$ (up to a unit), but again we only need to know that $\theta_{k+3,d}$ must be detected in filtration $s \le 2p^{k+3} - 2dp^{k+2}$ in (12-1). Hence $v_3^{p^{k+3}-dp^{k+2}} \cdot \theta_{k+3,d} = 0$ in $V(2)_* \Sigma^2 \operatorname{THH}(\mathrm{BP}\langle 2\rangle)_{h\mathbb{T}}$, which implies that

$$v_3^{r(k+3)-dp^{k+2}} \cdot \gamma_{k+3,d} = v_3^{p^{k+3}-dp^{k+2}} \cdot N^h_*(\theta_{k+3,d}) = 0$$

in $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$, as asserted. □

Recall the classes $x_{k,d}$ and $z_{k,d}$ from Definitions 11.5 and 11.6.

Corollary 12.7 *For each $k \in \{1,2,3\}$ and $0 < d < p$ there is a unique element*

$$\xi_{k,d} \in \{x_{k,d}\} \subset V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$$

that satisfies $R^h_*(\xi_{k,d}) = 0$. Moreover, $\lambda_k \cdot \xi_{k,d} = 0$ and $v_3^{p^k-dp^{k-1}} \cdot \xi_{k,d} = 0$.

Proof Let $\xi_{k,d} = \gamma_{k,d}$ as in Proposition 12.5, noting that $x_{k,d} = c_{k,d}$. □

Corollary 12.8 *For each $k \ge 1$ and $0 < d < p$ there are unique elements*

$$\xi_{k+3,d} \in \{x_{k+3,d}\} \subset V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}} \quad \text{and} \quad \zeta_{k,d} \in \{z_{k,d}\} \subset V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}$$

that satisfy

$$GR^h_*(\xi_{k+3,d}) = \widehat{\Gamma}^{h\mathbb{T}}_{1*}(\xi_{k,d}) = \zeta_{k,d}.$$

Moreover, $\lambda_{[k]} \cdot \xi_{k+3,d} = 0$ and $v_3^{(1-\frac{d}{p})r(k+3)} \cdot \xi_{k+3,d} = 0$.

Proof For $k \ge 1$, choose elements $\gamma_{k+3,d}$ and $\gamma'_{k,d}$ as in Proposition 12.6. Recalling that $x_{k+3,d} = (t\mu)^{\frac{d}{p}r(k)} \cdot c_{k+3,d}$, we let

$$\xi_{k+3,d} = v_3^{\frac{d}{p}r(k)} \cdot \gamma_{k+3,d}.$$

Then

$$GR^h_*(\xi_{k+3,d}) = v_3^{\frac{d}{p}r(k)} \cdot GR^h_*(\gamma_{k+3,d}) = v_3^{\frac{d}{p}r(k)-dp^{k-1}} \cdot v_3^{dp^{k-1}} \cdot \gamma'_{k,d} = v_3^{\frac{d}{p}r(k-3)} \cdot \widehat{\Gamma}^{h\mathbb{T}}_{1*}(\gamma_{k,d})$$
$$= \widehat{\Gamma}^{h\mathbb{T}}_{1*}(\xi_{k,d}).$$

To see that this uniquely determines $\xi_{k+3,d} \in \{x_{k+3,d}\}$, note that any other choice of class $\xi \in \{x_{k+3,d}\}$ with $GR^h_*(\xi) = GR^h_*(\xi_{k+3,d})$ would differ from $\xi_{k+3,d}$ by an element ξ' in $\ker(R^h_*)$ that is detected by an element x' in $\ker E^\infty(R^h)$ of lower filtration than $x_{k+3,d}$, and hence of filtration $s < -2(p^3+1)$. By Lemma 12.3, no such element x' exists in total degree $|\xi_{k+3,d}| = 2p^{[k]} - 2dp^{[k]-1} - 1 \le 2p^3 - 2$.

By induction, $GR^h_*(\lambda_{[k]} \cdot \xi_{k+3,d}) = \widehat{\Gamma}^{h\mathbb{T}}_{1*}(\lambda_{[k]} \cdot \xi_{k,d}) = 0$. Hence, if $\xi'' = \lambda_{[k]} \cdot \xi_{k+3,d}$ were nonzero, it would be a class in $\ker(R^h_*)$, in total degree $4p^{[k]} - 2dp^{[k]-1} - 2 \le 4p^3 - 3$, that is detected by an element x'' in $\ker E^\infty(R^h)$ of lower filtration than that of $x_{k+3,d}$. By Lemma 12.4, treating the cases $k+3=4$ and $k+3 \ge 5$ separately, no such element x'' exists. This contradiction proves that $\lambda_{[k]} \cdot \xi_{k+3,d} = 0$. By Proposition 12.6,

$$v_3^{(1-\frac{d}{p})r(k+3)} \cdot \xi_{k+3,d} = v_3^{r(k+3)-dp^{k+2}} \cdot \gamma_{k+3,d} = 0.$$

Finally, let $\zeta_{k,d} = \widehat{\Gamma}^{h\mathbb{T}}_{1*}(\xi_{k,d})$, which is then detected by $E^\infty(\widehat{\Gamma}^{h\mathbb{T}}_1)(x_{k,d}) = z_{k,d}$. □

We now fix compatible choices of classes $\gamma_{k,d}$ and $\gamma'_{k,d}$, as in Propositions 12.5 and 12.6:

Definition 12.9 For $k \ge 1$ and $0 < d < p$ let

$$\widetilde{C}(k,d) \cong P_{r(k)-dp^{k-1}}(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\gamma_{k,d}\}$$

be the $P(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]})$-submodule of $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$ generated by $\gamma_{k,d}$, and let

$$\widetilde{C}'(k,d) \cong P_{r(k)}(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\gamma'_{k,d}\}$$

be the $P(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]})$-submodule of $V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}$ generated by $\gamma'_{k,d}$. Let

$$\widetilde{C} = \prod_{\substack{k \ge 1 \\ 0<d<p}} \widetilde{C}(k,d) \quad \text{and} \quad \widetilde{C}' = \prod_{\substack{k \ge 1 \\ 0<d<p}} \widetilde{C}'(k,d).$$

These are detected by the summands $C \subset E^\infty(\mathbb{T})$ and $C' \subset \mu^{-1} E^\infty(\mathbb{T})$, respectively.

Lemma 12.10 *The $P(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]})$-submodules*

$$\langle \xi_{k,d} \rangle \subset \widetilde{C}(k,d) \quad \text{and} \quad \langle \zeta_{k,d} \rangle \subset \widetilde{C}'(k,d)$$

generated by $\xi_{k,d} = v_3^{\frac{d}{p}r(k-3)} \cdot \gamma_{k,d}$ and $\zeta_{k,d} = v_3^{\frac{d}{p}r(k)} \cdot \gamma'_{k,d}$, respectively, are equal to the (uniquely defined) $\widetilde{A}$-submodules generated by $\xi_{k,d}$ and $\zeta_{k,d}$, with

$$\langle \xi_{k,d} \rangle \cong P_{(1-\frac{d}{p})r(k)}(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\xi_{k,d}\},$$
$$\langle \zeta_{k,d} \rangle \cong P_{(1-\frac{d}{p})r(k)}(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\zeta_{k,d}\}.$$

Proof These $P(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]})$-submodules are $\widetilde{A}$-submodules, since we proved $\lambda_{[k]} \cdot \xi_{k,d} = 0$ in Corollaries 12.7 and 12.8, which readily implies that $\lambda_{[k]} \cdot \zeta_{k,d} = 0$. □

Remark 12.11 With this notation, Proposition 12.6 shows that $\hat{\Gamma}_{1*}^{h\mathbb{T}}$ induces isomorphisms $\langle \xi_{k,d} \rangle \to \langle \zeta_{k,d} \rangle$, and injections $\widetilde{C}(k,d) \to \widetilde{C}'(k,d)$ and $\widetilde{C}(k,d)/\langle \xi_{k,d} \rangle \to \widetilde{C}'(k,d)/\langle \zeta_{k,d} \rangle$. It also shows that GR_*^h induces isomorphisms $\widetilde{C}(k+3,d)/\langle \xi_{k+3,d} \rangle \to \widetilde{C}'(k,d)/\langle \zeta_{k,d} \rangle$, and surjections $\langle \xi_{k+3,d} \rangle \to \langle \zeta_{k,d} \rangle$ and $\widetilde{C}(k+3,d) \to \widetilde{C}'(k,d)$, for all $k \ge 1$ and $0 < d < p$.

Choosing lifts of the B- and D-summands requires less precision:

Definition 12.12 For each $k \ge 1$ and $p \nmid d > 0$ choose a class

$$\beta_{k,d} \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2 \rangle)^{h\mathbb{T}}$$

detected by $\lambda_{[k]}\mu^{dp^{k-1}} \in B$, and let

$$\widetilde{B}(k,d) \cong P_{r(k)}(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\beta_{k,d}\}$$

be the $E(\lambda_{[k+1]}, \lambda_{[k+2]})$-submodule of $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2 \rangle)^{h\mathbb{T}}$ generated by $v_3^m \cdot \beta_{k,d}$ for $0 \le m < r(k)$. For each $k \ge 4$ and $p \nmid d > p$ choose a class

$$\delta_{k,d} \in V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2 \rangle)^{h\mathbb{T}}$$

detected by $t^{dp^{k-1}}\lambda_{[k]} \in D$, and let

$$\widetilde{D}(k,d) \cong P_{r(k-3)}(v_3) \otimes E(\lambda_{[k+1]}, \lambda_{[k+2]}) \otimes \mathbb{F}_p\{\delta_{k,d}\}$$

be the $E(\lambda_{[k+1]}, \lambda_{[k+2]})$-submodule of $V(2)_* \operatorname{THH}(\mathrm{BP}\langle 2 \rangle)^{h\mathbb{T}}$ generated by $v_3^m \cdot \delta_{k,d}$ for $0 \le m < r(k-3)$. Let

$$\widetilde{B} = \prod_{\substack{k \ge 1 \\ p \nmid d > 0}} \widetilde{B}(k,d) \quad \text{and} \quad \widetilde{D} = \prod_{\substack{k \ge 4 \\ p \nmid d > p}} \widetilde{D}(k,d).$$

These are detected by the summands B and D of $E^\infty(\mathbb{T})$, respectively.

Lemma 12.13 *For each $k \ge 1$ and $p \nmid d > 0$ the difference*

$$(GR_*^h - \hat{\Gamma}_{1*}^{h\mathbb{T}})(\beta_{k,d}) \in V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2 \rangle)^{tC_p})^{h\mathbb{T}}$$

is detected by $-\lambda_{[k]}\mu^{dp^{k-1}} \in B'$. For each $k \ge 4$ and $p \nmid d > p$ the difference

$$(GR_*^h - \hat{\Gamma}_{1*}^{h\mathbb{T}})(\delta_{k,d}) \in V(2)_*(\operatorname{THH}(\mathrm{BP}\langle 2 \rangle)^{tC_p})^{h\mathbb{T}}$$

is detected by $\lambda_{[k]}\mu^{-dp^{k-4}} \in D'$.

Proof On one hand, by Lemma 11.8 the image $-\hat{\Gamma}_{1*}^{h\mathbb{T}}(\beta_{k,d})$ is detected by $-\lambda_{[k]}\mu^{dp^{k-1}}$ in homotopy fixed point filtration 0, while by Lemma 11.9 and Proposition 12.1 the image $GR_*^h(\beta_{k,d})$ lies in negative filtration (or is zero). Hence $(GR_*^h - \hat{\Gamma}_{1*}^{h\mathbb{T}})(\beta_{k,d})$ is detected by the filtration 0 class.

On the other hand, by Lemma 11.9 and Proposition 12.1 the image $GR_*^h(\delta_{k,d})$ is detected by $\lambda_{[k]}\mu^{-dp^{k-4}}$ in filtration 0, while by Lemma 11.8 the image $-\hat{\Gamma}_{1*}^{h\mathbb{T}}(\delta_{k,d})$ lies in negative filtration (or is zero). Hence $(GR_*^h - \hat{\Gamma}_{1*}^{h\mathbb{T}})(\delta_{k,d})$ is detected by the filtration 0 class. □

Definition 12.14 As subgroups of $V(2)_*(\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}}$, let

$$\widetilde{B}' = (GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*})(\widetilde{B}) \quad\text{and}\quad \widetilde{D}' = (GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*})(\widetilde{D}).$$

These are detected by the summands B' and D' of $\mu^{-1}E^\infty(\mathbb{T})$, respectively.

Proposition 12.15 *The inclusions induce isomorphisms*

$$V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}} \cong \widetilde{A}\oplus\widetilde{B}\oplus\widetilde{C}\oplus\widetilde{D} \quad\textit{and}\quad V(2)_*(\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{tC_p})^{h\mathbb{T}} \cong \widetilde{A}'\oplus\widetilde{B}'\oplus\widetilde{C}'\oplus\widetilde{D}'.$$

In these terms, $GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*}$ is the direct sum of the zero homomorphism $\widetilde{A}\xrightarrow{0}\widetilde{A}'$, two isomorphisms $\widetilde{B}\xrightarrow{\cong}\widetilde{B}'$ and $\widetilde{D}\xrightarrow{\cong}\widetilde{D}'$, and the difference $\Delta\colon\widetilde{C}\to\widetilde{C}'$ between the restricted homomorphisms

$$GR^h_*\colon \prod_{\substack{k\ge 1\\ 0<d<p}} \widetilde{C}(k,d) \to \prod_{\substack{k\ge 1\\ 0<d<p}} \widetilde{C}'(k,d), \quad (\dots,\gamma_{k,d},\dots)\mapsto(\dots,\gamma'_{k-3,d},\dots)$$

and

$$\hat{\Gamma}^{h\mathbb{T}}_{1*}\colon \prod_{\substack{k\ge 1\\ 0<d<p}} \widetilde{C}(k,d) \to \prod_{\substack{k\ge 1\\ 0<d<p}} \widetilde{C}'(k,d), \quad (\dots,\gamma_{k,d},\dots)\mapsto(\dots,v_3^{dp^{k-1}}\cdot\gamma'_{k,d},\dots).$$

Here $\gamma'_{k-3,d}$ is to be interpreted as 0 for $k\in\{1,2,3\}$.

Proof The submodules $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ are detected by the direct summands A, B, C and D spanning $E^\infty(\mathbb{T})$, so $\widetilde{A}\oplus\widetilde{B}\oplus\widetilde{C}\oplus\widetilde{D}\to V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)^{h\mathbb{T}}$ is an isomorphism by strong convergence of the $\mathbb{T}$-homotopy fixed point spectral sequence. Likewise, $\widetilde{A}'$, $\widetilde{B}'$, $\widetilde{C}'$ and $\widetilde{D}'$ are detected by the direct summands A', B', C' and D' spanning $\mu^{-1}E^\infty(\mathbb{T})$.

The homomorphisms GR^h_* and $\hat{\Gamma}^{h\mathbb{T}}_{1*}$ agree on $\widetilde{A}$, since the classes v_3, λ_1, λ_2 and λ_3 come from algebraic K-theory, and hence also from topological cyclic homology. Their difference is therefore the zero homomorphism. The restricted homomorphisms $GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*}\colon\widetilde{B}\to\widetilde{B}'$ and $GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*}\colon\widetilde{D}\to\widetilde{D}'$ are isomorphisms, by the construction of the target modules, which relies on Lemma 12.13. The restricted homomorphism $GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*} = \Delta\colon\widetilde{C}\to\widetilde{C}'$ factors as asserted by Propositions 12.5 and 12.6. □

Proposition 12.16 *There are $P(v_3)\otimes E(\lambda_1,\lambda_2,\lambda_3)$-module isomorphisms*

$$\begin{aligned}\ker(GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*}) \cong P(v_3)\otimes E(\lambda_1,\lambda_2,\lambda_3) &\oplus P(v_3)\otimes E(\lambda_2,\lambda_3)\otimes\mathbb{F}_p\{\Xi_{1,d}\mid 0<d<p\}\\ &\oplus P(v_3)\otimes E(\lambda_1,\lambda_3)\otimes\mathbb{F}_p\{\Xi_{2,d}\mid 0<d<p\}\\ &\oplus P(v_3)\otimes E(\lambda_1,\lambda_2)\otimes\mathbb{F}_p\{\Xi_{3,d}\mid 0<d<p\},\\ \mathrm{cok}(GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*}) \cong P(v_3)\otimes E(\lambda_1,\lambda_2,\lambda_3).&\end{aligned}$$

Here $\Xi_{i,d}$ in degree $2p^i-2dp^{i-1}-1$ is detected by $x_{i,d}=t^{dp^{i-1}}\lambda_i\in E^\infty(\mathbb{T})$, for each $i\in\{1,2,3\}$ and $0<d<p$.

Proof Let $\Delta\colon \widetilde{C} \to \widetilde{C}'$ be as in Proposition 12.15. Then

$$\ker(GR^h_* - \widehat{\Gamma}^{h\mathbb{T}}_{1*}) = \widetilde{A} \oplus \ker(\Delta) \quad\text{and}\quad \operatorname{cok}(GR^h_* - \widehat{\Gamma}^{h\mathbb{T}}_{1*}) = \widetilde{A}' \oplus \operatorname{cok}(\Delta).$$

Consider the associated map of vertical short exact sequences

$$\begin{tikzcd}
\prod_{k\ge 1,0<d<p}\langle\xi_{k,d}\rangle \arrow[r, "\Delta'"] \arrow[d, tail] & \prod_{k\ge 1,0<d<p}\langle\zeta_{k,d}\rangle \arrow[d, tail] \\
\prod_{k\ge 1,0<d<p}\widetilde{C}(k,d) \arrow[r, "\Delta"] \arrow[d, two heads] & \prod_{k\ge 1,0<d<p}\widetilde{C}'(k,d) \arrow[d, two heads] \\
\prod_{k\ge 1,0<d<p}\widetilde{C}(k,d)/\langle\xi_{k,d}\rangle \arrow[r, "\Delta''", "\cong"'] & \prod_{k\ge 1,0<d<p}\widetilde{C}'(k,d)/\langle\zeta_{k,d}\rangle.
\end{tikzcd}$$

In the upper row, the $\widehat{\Gamma}^{h\mathbb{T}}_{1*}\colon \langle\xi_{k,d}\rangle \to \langle\zeta_{k,d}\rangle$ for $k \ge 1$ and $0 < d < p$ are isomorphisms, so we can identify $\ker(\Delta')$ with the product over $i \in \{1,2,3\}$ and $0 < d < p$ of the limit of the sequence

$$\cdots \to \langle\xi_{k+3,d}\rangle \xrightarrow{(\widehat{\Gamma}^{h\mathbb{T}}_{1*})^{-1}GR^h_*} \langle\xi_{k,d}\rangle \to \cdots \to \langle\xi_{i+3,d}\rangle \xrightarrow{(\widehat{\Gamma}^{h\mathbb{T}}_{1*})^{-1}GR^h_*} \langle\xi_{i,d}\rangle,$$

where $k \equiv i \mod 3$. Since

$$(\widehat{\Gamma}^{h\mathbb{T}}_{1*})^{-1}GR^h_*\colon \xi_{k+3,d} \mapsto \xi_{k,d},$$

this limit is isomorphic, as an $\widetilde{A}$-module, to $P(v_3) \otimes E(\lambda_{[i+1]}, \lambda_{[i+2]}) \otimes \mathbb{F}_p\{\Xi_{i,d}\}$, with

$$\Xi_{i,d} = (\dots, 0, \xi_{k+3,d}, 0, 0, \xi_{k,d}, 0, \dots)$$

detected by $x_{i,d}$ in $E^\infty(\mathbb{T})$. Similarly, we can identify $\operatorname{cok}(\Delta')$ with the (right) derived limit of this sequence, which vanishes because each $GR^h_*\colon \langle\xi_{k+3,d}\rangle \to \langle\zeta_{k,d}\rangle$ is surjective.

In the lower row, $\ker(\Delta'') = 0$ and $\operatorname{cok}(\Delta'') = 0$ because $\widetilde{C}(i,d)/\langle\xi_{i,d}\rangle = 0$ for $i \in \{1,2,3\}$ and the $GR^h_*\colon \widetilde{C}(k+3,d)/\langle\xi_{k+3,d}\rangle \to \widetilde{C}'(k,d)/\langle\zeta_{k,d}\rangle$ are isomorphisms. Taken together, this proves that

$$\ker(\Delta) = \ker(\Delta') \cong \prod_{\substack{i\in\{1,2,3\}\\ 0<d<p}} P(v_3) \otimes E(\lambda_{[i+1]}, \lambda_{[i+2]}) \otimes \mathbb{F}_p\{\Xi_{i,d}\}$$

and $\operatorname{cok}(\Delta) = 0$. □

Theorem 12.17 *Let $p \ge 7$. There is a preferred $P(v_3)\otimes E(\lambda_1,\lambda_2,\lambda_3)$-module isomorphism*

$$\begin{aligned} V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle) \cong P(v_3) \otimes E(\partial, \lambda_1, \lambda_2, \lambda_3) &\oplus P(v_3) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{\Xi_{1,d} \mid 0 < d < p\} \\ &\oplus P(v_3) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{\Xi_{2,d} \mid 0 < d < p\} \\ &\oplus P(v_3) \otimes E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p\{\Xi_{3,d} \mid 0 < d < p\}, \end{aligned}$$

with $\Xi_{i,d}$ detected by $x_{i,d} = t^{dp^{i-1}}\lambda_i$ for $i \in \{1,2,3\}$ and $0 < d < p$. Here $|v_3| = 2p^3-2$, $|\lambda_i| = 2p^i-1$, $|\partial| = -1$ and $|t| = -2$. This is a free $P(v_3)$-module on the $16 + 12(p-1) = 12p + 4$ generators

$$\partial^\epsilon\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}, \quad \lambda_2^{\epsilon_2}\lambda_3^{\epsilon_3}\Xi_{1,d}, \quad \lambda_1^{\epsilon_1}\lambda_3^{\epsilon_3}\Xi_{2,d} \quad\text{and}\quad \lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\Xi_{3,d}$$

in degrees $-1 \le * \le 2p^3 + 2p^2 + 2p - 3$, where $\epsilon, \epsilon_i \in \{0,1\}$ and $0 < d < p$.

Proof The definition of $\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ as the homotopy equalizer of $\hat{\Gamma}_1^{h\mathbb{T}}$ and GR^h leads to the short exact sequence

$$0 \to \Sigma^{-1}\operatorname{cok}(GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*}) \xrightarrow{\partial} V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle) \xrightarrow{\pi} \ker(GR^h_* - \hat{\Gamma}^{h\mathbb{T}}_{1*}) \to 0.$$

It splits as an extension of $P(v_3)\otimes E(\lambda_1,\lambda_2,\lambda_3)$-modules, since the image of ∂ is trivial in the (even) degrees of the products $\lambda_i \cdot \Xi_{i,d}$ that vanish on the right-hand side. The splitting is unique, since the left-hand side is trivial in the (zero or odd) degrees of the module generators 1 and $\Xi_{i,d}$. □

Corollary 12.18 *The classes α_1, β_1' and $\gamma_1'' \in \pi_* V(2)$ map under the unit map $S \to \mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ to the classes $\Xi_{1,1}$, $\Xi_{2,1}$ and $\Xi_{3,1}$, respectively.*

Proof These elements are detected, in pairs, by $t\lambda_1$, $t^p\lambda_2$ and $t^{p^2}\lambda_3$ in $E^\infty(\mathbb{T})$, and in these (total) degrees there are no other classes of lower filtration, nor in the image of ∂. □

Thanks to the Nikolaus–Scholze model for $\mathrm{TC}(B)$ we no longer need to recover $V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ from $V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 1\rangle)$ in low degrees, but we nonetheless have the following consistency result:

Proposition 12.19 *The E_3 BP-algebra map $\mathrm{BP}\langle 2\rangle \to \mathrm{BP}\langle 1\rangle$ induces a $(2p^2-1)$-connected surjective ring homomorphism*

$$\begin{aligned} V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle) \to V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 1\rangle) \cong E(\partial,\lambda_1,\lambda_2) \\ \oplus E(\lambda_2)\otimes \mathbb{F}_p\{\Xi_{1,d} \mid 0<d<p\} \\ \oplus E(\lambda_1)\otimes \mathbb{F}_p\{\Xi_{2,d} \mid 0<d<p\} \end{aligned}$$

mapping ∂, λ_1, λ_2, $\Xi_{1,d}$ and $\Xi_{2,d}$ to the classes with the same names, and mapping v_3, λ_3 and $\Xi_{3,d}$ to zero.

Proof This is clear for ∂, λ_1 and λ_2. Moreover, $\Xi_{1,d}$ and $\Xi_{2,d}$ in $V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ map to classes in $V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 1\rangle)^{h\mathbb{T}}$ that are detected by $t^d\lambda_1$ and $t^{dp}\lambda_2$, respectively, which characterizes their images in $V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 1\rangle)$. The classes v_3, λ_3 and $\Xi_{3,d}$ for $1 \le d \le p-2$ are mapped to trivial groups. Finally, $\Xi_{3,p-1}$ in degree $2p^2-1$ maps to zero in $V(2)_*\,\mathrm{THH}(\mathrm{BP}\langle 2\rangle)$, and hence cannot be detected by λ_2. □

We write $\mathrm{BP}\langle 2\rangle_p$ for the p-completion of the p-local E_3 ring spectrum $\mathrm{BP}\langle 2\rangle$.

Theorem 12.20 *Let $p \ge 7$. There is an exact sequence*

$$0 \to \Sigma^{-2}\mathbb{F}_p\{\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_1\bar{\tau}_2\} \to V(2)_*K(\mathrm{BP}\langle 2\rangle_p) \xrightarrow{\mathrm{trc}_*} V(2)_*\,\mathrm{TC}(\mathrm{BP}\langle 2\rangle) \to \Sigma^{-1}\mathbb{F}_p\{1\} \to 0.$$

Hence $V(2)_*K(\mathrm{BP}\langle 2\rangle_p)$ is the direct sum of a free $P(v_3)$-module on $12p+4$ generators in degrees $0 \le * \le 2p^3+2p^2+2p-3$, plus an $\mathbb{F}_p$-module with trivial v_3-action spanned by three classes in degrees $2p-3$, $2p^2-3$ and $2p^2+2p-4$. In particular, the localization homomorphism

$$V(2)_*K(\mathrm{BP}\langle 2\rangle_p) \to v_3^{-1}V(2)_*K(\mathrm{BP}\langle 2\rangle_p)$$

is an isomorphism in degrees $* \ge 2p^2+2p$.

Proof By [Dundas 1997; Hesselholt and Madsen 1997, Theorem D] there is a homotopy cofiber sequence

$$K(\mathrm{BP}\langle 2\rangle_p)_p \xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathrm{BP}\langle 2\rangle)_p \xrightarrow{\varpi} \Sigma^{-1} H\mathbb{Z}_p,$$

and hence also a long exact sequence

$$\cdots \to V(2)_* K(\mathrm{BP}\langle 2\rangle_p) \xrightarrow{\mathrm{trc}_*} V(2)_* \mathrm{TC}(\mathrm{BP}\langle 2\rangle) \xrightarrow{\varpi_*} V(2)_*(\Sigma^{-1} H\mathbb{Z}_p) \to \cdots.$$

Here $V(2)_*(H\mathbb{Z}_p) \cong E(\bar{\tau}_1, \bar{\tau}_2)$ with $|\bar{\tau}_1| = 2p-1$ and $|\bar{\tau}_2| = 2p^2-1$. The only $P(v_3)$-module generator of $V(2)_* \mathrm{TC}(\mathrm{BP}\langle 2\rangle)$ that is mapped nontrivially by ϖ_* is ∂, with $\varpi_*(\partial) \doteq \Sigma^{-1}1$. The generators $\partial\lambda_1$, $\partial\lambda_2$ and $\partial\lambda_1\lambda_2$ come from $V(0)$-homotopy, hence factor through $V(0)_*(\Sigma^{-1} H\mathbb{Z}_p)$, and therefore map to zero. The generator $\lambda_1 \Xi_{2,1}$ is the product of two classes in the image of trc_*, hence also maps to zero under ϖ. It follows that $\ker(\varpi_*)$ is freely generated as a $P(v_3)$-module by the same generators as for $V(2)_* \mathrm{TC}(\mathrm{BP}\langle 2\rangle)$, except that ∂ in degree -1 is replaced by $v_3\partial$ in degree $2p^3-3$. □

Theorem 12.21 *The p-completion map $\kappa\colon \mathrm{BP}\langle 2\rangle \to \mathrm{BP}\langle 2\rangle_p$ induces a $(2p^2+2p-2)$-coconnected homomorphism*

$$V(2)_* K(\mathrm{BP}\langle 2\rangle) \xrightarrow{\kappa_*} V(2)_* K(\mathrm{BP}\langle 2\rangle_p).$$

Hence $V(2)_* K(\mathrm{BP}\langle 2\rangle)$ is the direct sum of a free $P(v_3)$-module on $12p+4$ generators in degrees $0 \le * \le 2p^3+2p^2+2p-3$, plus an $\mathbb{F}_p$-module with trivial v_3-action concentrated in degrees $1 \le * \le 2p^2+2p-3$. In particular,

$$V(2)_* K(\mathrm{BP}\langle 2\rangle) \to v_3^{-1} V(2)_* K(\mathrm{BP}\langle 2\rangle)$$

is an isomorphism in degrees $* \ge 2p^2+2p$.

Proof By the proven Lichtenbaum–Quillen/Bloch–Kato conjectures [Voevodsky 2011] and the earlier calculation of $V(0)_* \mathrm{TC}(\mathbb{Z})$ from [Bökstedt and Madsen 1994; 1995], $V(1)_* K(\mathbb{Q})$ and $V(1)_* K(\mathbb{Q}_p)$ are concentrated in degrees $0 \le * \le 2p-2$. Hence $V(1) \wedge K(\mathbb{Q}) \to V(1) \wedge K(\mathbb{Q}_p)$ is $(2p-1)$-coconnected. It follows from the localization sequence in algebraic K-theory that $V(1) \wedge K(\mathbb{Z}_{(p)}) \to V(1) \wedge K(\mathbb{Z}_p)$ is also $(2p-1)$-coconnected, so that $V(2) \wedge K(\mathbb{Z}_{(p)}) \to V(2) \wedge K(\mathbb{Z}_p)$ is $(2p^2+2p-2)$-coconnected. By the commutative cube

$$\begin{tikzcd}
K(\mathrm{BP}\langle 2\rangle)_p \arrow[rr, "\kappa"] \arrow[dr] \arrow[dd, "\mathrm{trc}"'] & & K(\mathrm{BP}\langle 2\rangle_p)_p \arrow[dr] \arrow[dd, "\mathrm{trc}"' near start] & \\
& K(\mathbb{Z}_{(p)})_p \arrow[rr, "\kappa" near start] \arrow[dd, "\mathrm{trc}" near start] & & K(\mathbb{Z}_p)_p \arrow[dd, "\mathrm{trc}"] \\
\mathrm{TC}(\mathrm{BP}\langle 2\rangle)_p \arrow[rr, "\simeq" near start] \arrow[dr] & & \mathrm{TC}(\mathrm{BP}\langle 2\rangle_p)_p \arrow[dr] & \\
& \mathrm{TC}(\mathbb{Z}_{(p)})_p \arrow[rr, "\simeq"] & & \mathrm{TC}(\mathbb{Z}_p)_p
\end{tikzcd}$$

and [Dundas 1997] applied to the left-hand and right-hand faces, $V(2) \wedge K(\mathrm{BP}\langle 2\rangle) \to V(2) \wedge K(\mathrm{BP}\langle 2\rangle_p)$ is also $(2p^2+2p-2)$-coconnected. □

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