Geometry & Topology

Volume 29 (2025)

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Geometry & Topology **29**:2 (2025) 687–789
DOI: 10.2140/gt.2025.29.687
Published: 12 March 2025

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We prove that all 3D steady gradient Ricci solitons are $O(2)$-symmetric. The $O(2)$-symmetry is the most universal symmetry in Ricci flows with any type of symmetries. Our theorem is also the first instance of symmetry theorem for Ricci flows that are not rotationally symmetric. We also show that the Bryant soliton is the unique 3D steady gradient Ricci soliton with positive curvature that is asymptotic to a ray.

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1 Introduction

1.1 Statement of the main results

The concept of Ricci solitons was introduced by Hamilton [49]. Ricci solitons generate self-similar solutions of Hamilton's Ricci flow [48], and often arise as singularity models of Ricci flows; see also Hamilton [50; 51], Cao [22] and Chen and Zhu [31]. They can be viewed as the fixed points under the Ricci flow in the space of Riemannian metrics modulo rescalings and diffeomorphisms. Ricci solitons are also natural generalizations of the Einstein metrics and constant curvature metrics.

A complete Riemannian manifold (M, g) is called a Ricci soliton, if there exist a vector field X and a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} = \tfrac{1}{2}\mathscr{L}_X g + \lambda g.$$

The soliton is called *shrinking* if $\lambda > 0$, *expanding* if $\lambda < 0$, and *steady* if $\lambda = 0$. Moreover, if the vector field X is the gradient of some smooth function f, then we say it is a *gradient Ricci soliton*, and f is the potential function. In particular, a steady gradient Ricci soliton satisfies the equation

$$\mathrm{Ric} = \nabla^2 f.$$

The goal of this paper is to study steady gradient Ricci solitons with bounded curvature in dimension 3. We assume they are nonflat.

In dimension 2, the only steady gradient Ricci soliton is Hamilton's cigar soliton [49], which is rotationally symmetric. In dimension $n \geq 3$, Bryant [19] constructed a steady gradient Ricci soliton which is rotationally symmetric. See Cao [23], Feldman, Ilmanen and Knopf [47] and Lai [60] for more examples of Ricci solitons in dimension $n \geq 4$.

In dimension 3, we know that all steady gradient Ricci solitons are nonnegatively curved (Chen [30]), and they are asymptotic to sectors of angle $\alpha \in [0, \pi]$. In particular, the Bryant soliton is asymptotic to a ray ($\alpha = 0$), and the soliton $\mathbb{R} \times$ cigar is asymptotic to a half-plane ($\alpha = \pi$). If the soliton has positive curvature, it must be diffeomorphic to $\mathbb{R}^3$ (Petersen [67]), and asymptotic to a sector of angle in $[0, \pi)$ (Lai [60]). If the curvature is not strictly positive, then it is a metric quotient of $\mathbb{R} \times$ cigar (Morgan and Tian [63]).

Hamilton conjectured that there exists a 3D steady gradient Ricci soliton that is asymptotic to a sector with angle in $(0, \pi)$, which is called a *3D flying wing*; see Cao [23], Cao and He [26], Catino, Mastrolia and Monticelli [27], Chow, Chu, Glickenstein, Guenther, Isenberg, Ivey, Knopf, Lu, Luo and Ni [35], Deng and Zhu [42] and Chow, Lu and Ni [38]. The author [60] confirmed this conjecture by constructing a family of $\mathbb{Z}_2 \times O(2)$-symmetric 3D flying wings. More recently, the author showed that the asymptotic cone angles of these flying wings can take arbitrary values in $(0, \pi)$. It is then interesting to see whether a 3D steady gradient Ricci soliton with positive curvature must be either a flying wing or the Bryant soliton. This is equivalent to asking whether the Bryant soliton is the unique 3D steady gradient Ricci soliton with positive curvature that is asymptotic to a ray. Our first main theorem gives an affirmative answer to this.

Theorem 1.1 (uniqueness theorem) *Let (M, g) be a 3D steady gradient Ricci soliton with positive curvature. If (M, g) is asymptotic to a ray, then it must be isometric to the Bryant soliton up to a scaling.*

We mention that there are many other uniqueness results for the 3D Bryant soliton under various additional assumptions. First, Bryant [19] showed in his construction that the Bryant soliton is the unique rotationally symmetric steady gradient Ricci solitons. More recently, a well-known theorem by Brendle [12] proved that the Bryant soliton is the unique steady gradient Ricci soliton that is noncollapsed in dimension 3.

See also Deng and Zhu [42], Cao and Chen [25], Cao, Catino, Chen, Mantegazza and Mazzieri [24], Chen and Wang [32], Munteanu, Sung and Wang [65] and Catino, Mastrolia and Monticelli [27] for more uniqueness theorems for the Bryant soliton and cigar soliton.

Our Theorem 1.1 is the Ricci flow analog of X J Wang's well-known theorem [71] in mean curvature flow, which proves that the bowl soliton is the unique entire convex graphical translator in $\mathbb{R}^3$. Note that the analog of 3D steady Ricci solitons in mean curvature flow are convex translators in $\mathbb{R}^3$, where the rotationally symmetric solutions are called bowl solitons. Moreover, a 3D steady Ricci soliton asymptotic to a ray can be compared to a convex graphical translator whose definition domain is the entire $\mathbb{R}^2$.

There have been many exciting symmetry theorems in geometric flows; see for instance Huisken and Sinestrari [56], Brendle, Huisken and Sinestrari [17], Angenent, Brendle, Daskalopoulos and Šešum [2], Bamler and Kleiner [8], Brendle [13], Zhu [73; 74], Bourni, Langford and Tinaglia [10; 11], Brendle and Choi [14; 15], Brendle and Naff [18], Brendle, Daskalopoulos, Naff and Sesum [16] and Du and Haslhofer [44]. If one views the rotational symmetry as the "strongest" symmetry, then the $O(2)$-symmetry is naturally the "weakest", and the most universal symmetry in all ancient Ricci flow solutions. For example, in dimension 2, the nonflat ancient Ricci flows are the shrinking sphere, the cigar soliton, and the sausage solution (see Daskalopoulos, Hamilton and Sesum [40] Daskalopoulos and Sesum [41]), and they are all rotationally symmetric (ie $O(2)$-symmetric). In dimension 3, the author's flying wing examples and Fateev's examples [46] (see also Bakas, Kong and Ni [3]) are all $O(2)$-symmetric but not rotationally symmetric (ie $O(3)$-symmetric).

It was conjectured by Hamilton and Cao that the 3D flying wings are $O(2)$-symmetric. Our second main theorem confirms this conjecture. In particular, this is the first instance of a symmetry theorem for Ricci flows that are not rotationally symmetric.

Theorem 1.2 *Let (M,g) be a 3D flying wing. Then (M,g) is $O(2)$-symmetric.*

Here we say a complete 3D manifold is $O(2)$-symmetric if it admits an effective isometric $O(2)$-action, and the action fixes a complete geodesic Γ, such that the metric is a warped product metric on $M \setminus \Gamma$ with S^1-orbits. It is easy to see the Bryant soliton and $\mathbb{R} \times$ cigar are also $O(2)$-symmetric. Therefore, combining Theorems 1.1 and 1.2, we see that all 3D steady gradient Ricci solitons are $O(2)$-symmetric.

Theorem 1.3 *Let (M,g) be a 3D steady gradient Ricci soliton, then (M,g) is $O(2)$-symmetric.*

In mean curvature flow, the "weakest" symmetry is the $\mathbb{Z}_2$-symmetry, which are usually obtained using the standard maximum principle method. More precisely, if we compare 3D steady gradient Ricci solitons with convex translators in $\mathbb{R}^3$, then the $O(2)$-symmetry is compared with the $\mathbb{Z}_2$-symmetry (reflectional symmetry) there. The convex translators in $\mathbb{R}^3$ have been classified to be the tilted Grim Reapers, the flying wings, and the bowl soliton, all of which are $\mathbb{Z}_2$-symmetric; see Hoffman, Ilmanen, Martín and White [55]. However, as its analogy in Ricci flow, the $O(2)$-symmetry is not "discrete" at all, and no maximum principle is available.

We also obtain some geometric properties for the 3D flying wings. First, we show that the scalar curvature R always attains its maximum at some point, which is also the critical point of f. The analog of this statement in mean curvature flow is that the graph of the convex translator has a maximum point, which relies on the well-known convexity theorem by Spruck and Xiao [70].

Theorem 1.4 *Let (M, g, f) be a 3D steady gradient Ricci soliton with positive curvature. Then there exists $p \in M$ which is a critical point of the potential function f, and the scalar curvature R achieves the maximum at p.*

We study the asymptotic geometry of 3D flying wings. First, we show that the soliton is $\mathbb{Z}_2$-symmetric at infinity, in the sense that the limits of R at the two ends of Γ are equal to a same positive number. Here Γ is a complete geodesic fixed by the $O(2)$-isometry. After a rescaling we may assume this positive number is 4, then we show that there are two asymptotic limits, one is $\mathbb{R}\times$ cigar with $R(x_{\mathrm{tip}}) = 4$, and the other is $\mathbb{R}^2 \times S^1$ with the diameter of the S^1-factor equal to π. Note that in a cigar soliton where $R = 4$ at the tip, the diameter of the S^1-fibers in the warped-product metric converges to π at infinity. See Hamilton [51], Kotschwar and Wang [57], Deruelle [43], Chow, Deng and Ma [37] and Lai [60] for more discussions on the asymptotic geometry of Ricci solitons.

Theorem 1.5 ($\mathbb{Z}_2$-symmetry at infinity) *Let (M, g, f) be a 3D flying wing. Then, after a rescaling,*

$$\lim_{s\to\infty} R(\Gamma(s)) = \lim_{s\to-\infty} R(\Gamma(s)) = 4.$$

For any sequence of points $p_i \to \infty$, the pointed manifolds (M, g, p_i) smoothly converge to either $\mathbb{R}\times$ cigar with $R(x_{\mathrm{tip}}) = 4$, or $\mathbb{R}^2 \times S^1$, with the diameter of the S^1-factor equal to π. Moreover, if $p_i \in \Gamma$, then the limit is $(\mathbb{R}\times \mathrm{cigar}, x_{\mathrm{tip}})$, and if $d_g(\Gamma, p_i) \to \infty$, then the limit is $\mathbb{R}^2 \times S^1$.

We also obtain a quantitative relation between the limit of R along Γ, the asymptotic cone angle, and $R(p)$, where p is the critical point of f. This is also true in the Bryant soliton, and thus is true for all 3D steady gradient Ricci solitons with positive curvature.

Theorem 1.6 *Let (M, g, f, p) be a 3D steady gradient Ricci soliton with positive curvature. Assume (M, g) is asymptotic to a sector with angle α. Then we have*

$$\lim_{s\to\infty} R(\Gamma(s)) = \lim_{s\to-\infty} R(\Gamma(s)) = R(p)\sin^2\left(\tfrac{1}{2}\alpha\right).$$

It has been conjectured that there is a dichotomy of the curvature decay rate of steady gradient solitons, that is, that the curvature decays either exactly linearly or exponentially; see Munteanu, Sung and Wang [65], Deng and Zhu [42], Chan and Zhu [29] and Chan [28]. In dimension 3, the curvature of Bryant soliton decays linearly in the distance to the tip, and the curvature in $\mathbb{R}\times$ cigar decays exponentially in distance to the line of cigar tips. In this paper, we prove that in a 3D flying wing, the curvature decays faster than any polynomial function in r, and slower than an exponential function in r, where r is the distance function to Γ.

Theorem 1.7 *Let (M, g, f, p) be a 3D flying wing. Suppose $\lim_{s\to\infty} R(\Gamma(s)) = 4$. Then for any $\epsilon_0 > 0$ there exists $C(\epsilon_0) > 0$, and for any $k \in \mathbb{N}$ there exists $C_k > 0$, such that for all $x \in M$,*

$$C^{-1}e^{-2(1+\epsilon_0)\,d_g(x,\Gamma)} \le R(x) \le C_k\, d_g^{-k}(x,\Gamma).$$

1.2 Outline of difficulties and proofs

In Ricci flow, the "strongest" symmetry, ie the rotational symmetry, was first studied by Brendle [12; 13] in dimension 3 with many novel ideas that successfully generalize to prove rotational symmetry in higher dimensions; see also Brendle and Naff [18] and Brendle, Daskalopoulos, Naff and Sesum [16]. In contrast to the rotational symmetry, one of the major difficulties in studying any weaker symmetries is the nonuniqueness of asymptotic limits. For the $O(2)$-symmetry, this requires us to study the two different asymptotic limits $\mathbb{R}\times$ cigar and $\mathbb{R}^2 \times S^1$ separately and combine these estimates in a delicate way. Our $O(2)$-symmetry theorem is the first instance of tackling this issue in Ricci flow. Our method may be generalized to study the $O(n-k)$-symmetry for $k = 1, \dots, n-2$, which is weaker than the rotational symmetry, ie $O(n)$-symmetry. For example, for each dimension $n \ge 4$, the author [60] constructed n-dimensional steady solitons with $\mathrm{Rm} \ge 0$ that are noncollapsed, $O(n-1)$-symmetric but not $O(n)$-symmetric. It is then interesting to see whether noncollapsed 4D steady solitons with $\mathrm{Rm} \ge 0$. In mean curvature flow, Choi, Haslhofer and Hershkovits [34; 33] recently classified all noncollapsed translators in $\mathbb{R}^4$, and in particular showed that they are all $O(2)$-symmetric.

Therefore, we need to develop new tools and methods to prove the $O(2)$-symmetry. Some of our methods are independent of the soliton structure and the dimension, such as the distance distortion estimates, the curvature estimates, and the symmetry improvement theorem, which may have more applications in studying ancient Ricci flows. In particular, as a consequence of the nonuniqueness issue of limits, Brendle's construction of the Killing field is not applicable in our setting. So we introduce a new stability method to construct the Killing field; see more in Sections 7 and 8. Our stability method generalizes Brendle's method (see [12, Lemma 4.1]) in the sense that it also works on ancient flows but not only on steady solitons. For example, our method was recently applied by Zhao and Zhu [72] to study the rigidity of the noncollapsed Bryant soliton in any dimension $n \ge 4$.

We now outline the structure of the paper. In the following we assume (M, g, f) is a 3D steady gradient Ricci soliton that is not the Bryant soliton. Let $\{\phi_t\}_{t\in\mathbb{R}}$ be the diffeomorphisms generated by ∇f, $\phi_0 = \mathrm{id}$, and let $g(t) = \phi_{-t}^* g$. Then $(M, g(t))$ with $t \in (-\infty, \infty)$ is the Ricci flow of the soliton.

In Section 2 we give most of the definitions and standard Ricci flow results that will be used in the following proofs.

In Section 3 we study the asymptotic geometry of the soliton in this section. First, by the splitting theorem of 3D Ricci flow we can show that for any sequence of points $x_i \in M$ going to infinity as $i \to \infty$, the rescaled manifolds $(M, r^{-2}(x_i)g, x_i)$ converge to a smooth limit which splits off a line, where $r(x_i) > 0$

is some noncollapsing scale at x_i. We show that such an asymptotic limit is either isometric to $\mathbb{R}\times$cigar or $\mathbb{R}^2\times S^1$. Moreover, we find two integral curves Γ_1,Γ_2 of ∇f tending to infinity at one end, such that the asymptotic limits are isometric to $\mathbb{R}\times$cigar along them, and are $\mathbb{R}^2\times S^1$ away from them. The two integral curves also correspond to the two edge rays in the sector which is the blowdown limit of the soliton.

Second, we prove the existence of the maximum point of R in Theorem 1.4. This is also the unique critical point of f. We do this by a contradiction argument. Suppose there does not exist a maximum of R. Then we can find an integral curve γ of ∇f which goes to infinity at both ends. We show that R is nonincreasing along the curve and has a positive limit at one end. Using that the asymptotic limits along both ends of γ are isometric to $\mathbb{R}\times$cigar, we can compare the geometry at the two ends, and by a convexity argument, we can show that R is actually constant along γ, so that the soliton is isometric to $\mathbb{R}\times$cigar. This is a contradiction to our positive curvature assumption. So we have a closed subset Γ, which is the union of the critical point and two integral curves of ∇f, such that Γ is invariant under the diffeomorphisms generated by ∇f, and the soliton converges to $(\mathbb{R}\times\text{cigar}, x_{\text{tip}})$ under rescalings along the two ends of Γ.

Next, we prove a quadratic curvature decay away from the edge Γ. This corresponds to the case when $k=2$ in Theorem 1.7. The proof uses Perelman's curvature estimate, which gives the upper bound $R(x,0)\le C/r^2$ on scalar curvature in a nonnegatively curved Ricci flow $(M,g(t))$ with $t\in[-r^2,0]$, assuming the flow is noncollapsed at x on scale r. In our situation, we will show by methods of metric comparison geometry that the soliton is not collapsed at x on scale $d_g(x,\Gamma)$ in a local universal covering. So we can apply Perelman's curvature estimates on the local universal covering and obtain the desired quadratic decay $R(x)\le C/d_g^2(x,\Gamma)$.

Lastly, we prove Theorems 1.1 and 1.5. In proving the two theorems, we will work in the backwards Ricci flow $(M,g(-\tau))$ with $\tau\ge 0$, and reduce the change of various geometric quantities to the distortion of distances and lengths under the flow. More specifically, for a fixed point $x\in M$ at which $(M,g(-\tau))$ is close to $\mathbb{R}^2\times S^1$ on scale $h(\tau)$. Let $H(\tau)$ be the $g(-\tau)$-distance from x to Γ. Then we can show

$$H'(\tau)\ge C^{-1}\cdot h^{-1}(\tau)\quad\text{and}\quad h'(\tau)\le C\cdot H^{-2}(\tau)\cdot h(\tau).$$

We can then show that $h(\tau)$ stays bounded, and $H(\tau)$ grows at least linearly as $\tau\to\infty$. Using this we can show that R has two positive limits R_1,R_2 at the two ends of Γ, and the asymptotic cone angle is nonzero. To show $R_1=R_2$, first we can find two points x_1,x_2 at which the soliton (M,g) is $\epsilon(\tau)$-close to $\mathbb{R}^2\times S^1$, on the scales $2R_1^{-1/2}$ and $2R_2^{-1/2}$. Here $\epsilon(\tau)\to 0$ as $\tau\to\infty$. Then we can show that x_1 and x_2 stay in a bounded distance to each other as we move backwards long the Ricci flow, and hence $(M,g(-\tau))$ is $\epsilon(\tau)$-close to $\mathbb{R}^2\times S^1$ at x_1,x_2 on a uniform scale. So $R_1=R_2$ follows by controlling the scale change at x_1,x_2 in the flow. Theorem 1.5 is a key ingredient in proving the $O(2)$-symmetry theorem.

In Section 4 we prove Theorem 1.7 of the curvature estimates in this section. It is needed in the proof of the $O(2)$-symmetry. First, we derive the exponential curvature lower bound of R, which needs an improved Harnack inequality for nonnegatively curved Ricci flows. For a Ricci flow solution with nonnegative curvature operator, the following conventional integrated Harnack inequality can be obtained by integrating Hamilton's differential Harnack inequality and using the inequality $\mathrm{Ric}(v,v) \le |v|^2 R$:

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \ge \exp\left(-\frac{1}{2}\frac{d^2_{g(t_1)}(x_1,x_2)}{t_2-t_1}\right).$$

See for example Morgan and Tian [63, Theorem 4.40]. We observe that the inequality $\mathrm{Ric}(v,v) \le |v|^2 R$ can be improved to $\mathrm{Ric}(v,v) \le \frac{1}{2}|v|^2 R$, using which we can prove the improved Harnack inequality

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \ge \exp\left(-\frac{1}{4}\frac{d^2_{g(t_1)}(x_1,x_2)}{t_2-t_1}\right).$$

Using this improved Harnack inequality and some distance distortion estimates we can prove the exponential curvature lower bound. Note that this exponential lower bound $C^{-1}(\epsilon_0)e^{-(2+\epsilon_0)\,d_g(\cdot,\Gamma)}$ is sharp, because ϵ_0 can be arbitrarily small, and in $\mathbb{R}\times$cigar where $R=4$ at the cigar tip, we have $\Gamma=\mathbb{R}\times\{x_{\mathrm{tip}}\}$ and R decays like $e^{-2\,d_g(\cdot,\Gamma)}$.

Next, we derive the polynomial upper bound of R, which states that R decays faster than $d_g^{-k}(\cdot,\Gamma)$ for any $k\in\mathbb{N}$. We prove this by induction. First, the case of $k=2$ is proved in Section 3. Now assume by induction that $R \le C_k\, d_g^{-k}(\cdot,\Gamma)$ for some $k \ge 2$. Since R evolves by $\partial_t R = \Delta R + 2|\mathrm{Ric}|^2(x,t)$ under the Ricci flow, for all $s<t$ we have the reproduction formula

$$R(x,t) = \int_M G(x,t;y,s)R(y,s)\,d_s y + 2\int_s^t\int_M G(x,t;z,\tau)|\mathrm{Ric}|^2(z,\tau)\,d_\tau z\,d\tau,$$

where G is the heat kernel of the heat equation $\partial_t u = \Delta u$. Using a heat kernel estimate on G and the inductive assumption, we can show the first term goes to zero as we choose $s\to-\infty$, and the second term is bounded by $C\cdot d_{g(t)}^{-(2k-1)}(x,\Gamma)$. Note that $k\ge 2$ implies $2k-1>k$, completing the induction process.

In Section 5 we prove a local stability theorem, which is another key ingredient of the $O(2)$-symmetry theorem. It states that the degree of SO(2)-symmetry improves as we move forward in time along the Ricci flow of the soliton. Here SO(2)-symmetric means that the manifold admits an isometric SO(2)-action whose principal orbits are circles.

First, we prove the symmetry improvement theorem in the linear case. For a symmetric 2-tensor h on an SO(2)-symmetric manifold, it has a decomposition as a sum of a rotationally invariant mode and an oscillatory mode. We show that if h satisfies the linearized Ricci De Turck flow $\partial_t h = \Delta_L h$ on the cylindrical plane $\mathbb{R}^2\times S^1$, then the oscillatory mode of h decays exponentially in time. By a limiting argument, we generalize this theorem to the nonlinear case for the Ricci De Turck flow perturbation, whose background is an SO(2)-symmetric Ricci flow that is sufficiently close to $\mathbb{R}^2\times S^1$.

Moreover, the symmetry improvement theorem also describes the decay of $|h|$ in the case that it is bounded by an exponential function instead of a constant. More precisely, for $x_0 \in M$, if $|h|(\cdot,0) \le e^{\alpha\, d_g(x_0,\cdot)}$ for any $\alpha \in [0, 2.02]$, then $|h|(x_0,T) \le e^{-\delta_0 T} \cdot e^{2\alpha T}$ holds for some $\delta_0 > 0$. Applying the theorem to a 3D flying wing in which R limits to 4 along the edges, the increasing factor $e^{2\alpha T}$ will be compensated for by the cigar tip contracting along the edges under the Ricci flow. It is crucial that α can be slightly greater than 2, using which we can construct an SO(2)-symmetric approximating metric in Section 6, so that the error decays like $e^{-(2+\delta)\, d_g(\cdot,\Gamma)}$ for some small but positive δ. So the error decays faster than that of R by the exponential lower bound in Theorem 1.7. We will use this fact to construct a Killing field in Sections 7 and 8.

In Section 6 we construct an approximating SO(2)-symmetric metric $\bar{g}$ satisfying suitable error estimates. First, we construct an SO(2)-symmetric metric $\bar{g}_1$ away from Γ which satisfies

$$|\bar{g}_1 - g|_{C^{100}} \le e^{-(2+\epsilon_0)\, d_g(\cdot,\Gamma)} \tag{1-1}$$

for some $\epsilon_0 > 0$. To show this, we impose the following inductive assumption.

Inductive assumption one There are a constant $\delta \in (0, 0.01)$ and an increasing arithmetic sequence $\alpha_n > 0$, with $\delta \le \alpha_{n+1} - \alpha_n \le 0.01$, such that if $\alpha_n \le 2.02$, then there is an SO(2)-symmetric metric $\hat{g}_n$ such that

$$|\hat{g}_n - g|_{C^{100}} \le e^{-\alpha_n d_g(\cdot,\Gamma)}. \tag{1-2}$$

If this is true for all $n \in \mathbb{N}$, then $\hat{g}_N$ will satisfy (1-1) for a large enough $N \in \mathbb{N}$.

Now assume inductive assumption one holds for n. To show it also holds for $n+1$, we want to apply the symmetry improvement theorem to the Ricci flow of the soliton. After applying the symmetry improvement theorem i times, the error to a symmetric metric will decay by C^{-i} for some $C > 0$. So for points at larger distance to Γ, we need to apply the symmetry improvement theorem more times to achieve the error estimate $e^{-\delta\, d_g(\cdot,\Gamma)}$. Therefore, we need a second induction to apply the symmetry improvement theorem infinitely many times, so that eventually the error estimate $e^{-\delta\, d_g(\cdot,\Gamma)}$ holds everywhere.

Inductive assumption two There is a sequence of SO(2)-symmetric metrics $\{\hat{g}_{n,k}\}_{k=1}^{\infty}$ such that $\hat{g}_{n,k}$ satisfies (1-2), and for some $C > 0$ we have

$$|\hat{g}_{n,k} - g|_{C^{100}} \le e^{-\alpha_n d_g(\cdot,\Gamma)} \cdot C^{-i} \quad \text{on } \Gamma_{\ge iD} \text{ for } i = 0, \dots, k, \tag{1-3}$$

where $\Gamma_{\ge iD} = \{x \in M : d_g(x,\Gamma) \ge iD\}$. If inductive assumption two is true for all $k \in \mathbb{N}$, we take $\hat{g}_{n+1}$ to be a subsequential limit of $\hat{g}_{n,k}$ as $k \to \infty$, then $\hat{g}_{n+1}$ satisfies (1-2) for $n+1$.

Inductive Assumption Two clearly holds for $k = 0$ by taking $\hat{g}_{n,0} = \hat{g}_n$ and using Inductive Assumption One for n. Now assume it holds for $k \ge 0$, we verify it for $k+1$ by applying the symmetry improvement theorem. More precisely, we consider the harmonic map heat flow from $(M, g(t))$ to the Ricci flow $\hat{g}_{n,k}(t)$ starting from $\hat{g}_{n,k}$ on $[0,T]$. Then the error between $g(t)$ and $\hat{g}_{n,k}(t)$ is then described by the

Ricci De Turck flow perturbation. Let $\hat{g}_{n,k+1}$ be the final-time metric $\hat{g}_{n,k}(T)$ modulo the rotationally invariant part of the error and a diffeomorphism. Since oscillatory part of the error decays exponentially in time by the symmetry improvement theorem, we can show that $\hat{g}_{n,k+1}$ satisfies (1-3). This completes the two inductions and hence we obtain an SO(2)-symmetric metric $\bar{g}_1$ satisfying (1-1).

Lastly, we modify the metric $\bar{g}_1$ to obtain the desired approximating SO(2)-symmetric metric $\bar{g}$, which satisfies both (1-1) and

$$|\bar{g}-g|_{C^{100}}(x)\to 0 \quad \text{as } x\to\infty. \tag{1-4}$$

Note that $\bar{g}_1$ already satisfies (1-4) as we move away from Γ, we just need to extend this estimate near Γ. Since the soliton converges along the two ends of Γ to $\mathbb{R}\times$cigar which is SO(2)-symmetric, we can obtain $\bar{g}$ by gluing up $\bar{g}_1$ with $\mathbb{R}\times$cigar in suitable neighborhoods of the two ends of Γ.

The goal of Sections 7 and 8 is to construct a nontrivial Killing field of the soliton. We do this by a global stability argument using a heat kernel method, which is consistent with our curvature estimates and estimates of approximating metrics.

In this section, we study the solution to the following initial value problem of the linearized Ricci De Turck flow equation,

$$\begin{cases} \partial_t h(t)=\Delta_{L,g(t)}h(t), \\ h(0)=\mathscr{L}_X g, \end{cases} \tag{1-5}$$

where X is the Killing field of the approximating SO(2)-symmetric metric obtained in Section 6. By the conditions (1-1) and (1-4), and the exponential lower bound from Theorem 1.7, we can deduce that $|\mathscr{L}_X g/R|(x)\to 0$ as $x\to\infty$. We show that $|h(t)|\to 0$ as $t\to\infty$.

To prove this, we first observe by Anderson–Chow curvature pinching [1] that $|h|$ satisfies the inequality

$$|h(x,t)|\le \int_M G(x,t;y,0)|h|(y,0)\,d_0 y, \tag{1-6}$$

where $G(x,t;y,s)$ is the heat kernel to the heat-type equation

$$\partial_t u=\Delta u+\frac{2|\mathrm{Ric}|^2(x,t)}{R(x,t)}u. \tag{1-7}$$

Our key estimate is to show a vanishing theorem of the heat kernel $G(x,t;y,s)$ for any fixed pair (y,s) at $t=\infty$. Using this vanishing theorem we can show that the integral in (1-6) in any compact subset is arbitrarily small when $t\to\infty$. For the integral outside the compact subset, by the initial condition it is an arbitrary small multiple of R integrated against the heat kernel G, which is bounded by the maximum of R, seeing that R is also a solution to (1-7).

In Section 8 we construct a Killing field of the soliton metric. Let X be the Killing field of the approximating SO(2)-symmetric metric obtained in Section 6. Let $(M,g(t))$ be the Ricci flow of the soliton.

Let $Q(t)=\partial_t\phi_{t*}(X)-\Delta_{g(t)}\phi_{t*}(X)-\mathrm{Ric}_{g(t)}(\phi_{t*}(X))$ and $Y(t)$ be a time-dependent vector field which solves

$$\begin{cases}\partial_t Y(t)-\Delta Y(t)-\mathrm{Ric}(Y(t))=Q(t),\\ Y(0)=0.\end{cases}$$

Moreover, let $X(t):=\phi_{t*}(X)-Y$. Then $X(t)$ solves the initial value problem

$$\begin{cases}\partial_t X(t)-\Delta X(t)-\mathrm{Ric}(X(t))=0,\\ X(0)=X,\end{cases}$$

and the symmetric 2-tensor field $\mathscr{L}_{X(t)}g(t)$ satisfies the equation (1-5). Therefore, by the result from Section 7, we see that $\mathscr{L}_{X(t)}g(t)$ tends to zero as $t\to\infty$. So the limit of $X(t)$ as $t\to\infty$ is a Killing field of (M,g).

To show that the Killing field is nonzero, we first show that $Q(t)$ satisfies a polynomial decay away from Γ as a consequence of (1-1) and the polynomial curvature upper bound from Theorem 1.7. Then by some heat kernel estimates on $|Y(t)|$ we show that it also satisfies the polynomial decay away from Γ, which guarantees the nonvanishing of the limit of $X(t)$ as $t\to\infty$.

In Section 9 we prove Theorem 1.2 of the $O(2)$-symmetry. First, let X be the Killing field obtained in Section 8, and χ_θ for $\theta\in\mathbb{R}$ be the isometries generated by X. We show that the χ_θ commute with the diffeomorphisms ϕ_t generated by ∇f. Then we show that χ_θ is an SO(2)-isometry. This uses the existence of a maximum point of R, which must be fixed by the isometries χ_θ. Since the maximum point of R is also a critical point of f, it follows that f is invariant under the isometries. Using this we can show that χ_θ is an SO(2)-isometry and fixes the edge Γ.

Lastly, in order to show that the soliton is also $O(2)$-symmetric, it remains to show that the curvature form of the SO(2)-isometry vanishes everywhere. By using the soliton equation and the curvature formula under the SO(2)-isometry, we can reduce this to the vanishing of a scaling invariant quantity at a point on Γ. By a limiting argument and the scaling invariance, this can be further reduced to the Euclidean space $\mathbb{R}^3$, where the SO(2)-isometry is the rotation around the z-axis, and hence the vanishing assertion clearly holds.

Acknowledgements I would like to thank Richard Bamler for his constant help and encouragement, and also Otis Chodosh, Huai-Dong Cao, Bennett Chow, Xiaohua Zhu, Brian White, Timothy Buttsworth and Hongyi Liu for helpful discussions and comments.

The author was supported by NSF grant DMS-2203310.

2 Preliminaries

In the following we present most of the definitions and concepts that are needed in the statement and proofs of the main results of this paper.

2.1 Steady gradient Ricci solitons

Definition 2.1 (steady Ricci soliton) We say a smooth complete Riemannian manifold (M, g) is a steady Ricci soliton if it satisfies

$$\mathrm{Ric} = \tfrac{1}{2}\mathscr{L}_X g \tag{2-1}$$

for some smooth vector field X. If, moreover, the vector field is the gradient of some smooth function f, then we say it is a steady gradient Ricci soliton, and f is the potential function. In this case, the soliton satisfies the equation

$$\mathrm{Ric} = \nabla^2 f.$$

By a direct computation using (2-1), the family of metrics $g(t) = \phi_t^*(g)$, with $t \in (-\infty, \infty)$, satisfies the Ricci flow equation, where $\{\phi_t\}_{t\in(-\infty,\infty)}$ is the one-parameter group of diffeomorphisms generated by $-\nabla f$ with ϕ_0 the identity. We say $g(t)$ is the Ricci flow of the soliton.

Throughout the paper, we use the triple (M, g, f) to denote a steady gradient soliton (M, g) and a potential function f, and use the quadruple (M, g, f, p) to denote the soliton when $p \in M$ is a critical point of f.

For 3D steady gradient Ricci solitons, by the maximum principle they must have nonnegative sectional curvature [30]. Moreover, by the strong maximum principle, see eg [63, Lemma 4.13, Corollary 4.19], we see that a 3D steady gradient Ricci soliton must be isometric to quotients of $\mathbb{R} \times$ cigar if the curvature is not strictly positive everywhere. Therefore, throughout the paper we will assume our soliton has positive curvature. So by the soul theorem, the manifold is diffeomorphic to $\mathbb{R}^3$; see eg [67].

There are several important identities for the steady gradient Ricci solitons due to Hamilton; see eg [38]. In particular, we will use frequently

$$\langle \nabla R, \nabla f\rangle = -2\,\mathrm{Ric}(\nabla f, \nabla f) \quad \text{and} \quad R + |\nabla f|^2 = \text{const}. \tag{2-2}$$

By the second equation, a critical point of f must be the maximum point of R. For a 3D steady gradient Ricci soliton, since the Ricci curvature is positive, the first equation implies that a maximum point of R is also a critical point of f. We will show in Section 3 that the critical point exists in all 3D steady gradient Ricci solitons.

Hamilton's cigar soliton [49] is the first example of Ricci solitons. It is rotationally symmetric and has positive curvature. The cigar soliton is an important notion in this paper. In the following we review the definition of the cigar soliton and some properties we will use, including a precise description of the curvature decay and the tip contracting rate.

Definition 2.2 (cigar soliton; cf [49]) Hamilton's cigar soliton is a complete Riemannian surface $(\mathbb{R}^2, g_c, f)$, where

$$g_c := \frac{dx^2 + dy^2}{1 + x^2 + y^2} \quad \text{and} \quad f = \log(1 + x^2 + y^2).$$

As a solution of Ricci flow, its time-dependent version is

$$g_c(t) := \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}.$$

Let s denote the distance to the cigar tip $(0,0)$. Then we may rewrite g_c as

$$g_c = ds^2 + \tanh^2 s\, d\theta^2, \tag{2-3}$$

and the scalar curvature of g_c is

$$R_\Sigma = 4\operatorname{sech}^2 s = \frac{4}{(e^s+e^{-s})^2}. \tag{2-4}$$

In particular, $R(x_{\mathrm{tip}}) = 4$ and $K(x_{\mathrm{tip}}) = 2$. For a fixed $\theta_0 \in [0, 2\pi)$, the curve $\gamma(s) := (\theta_0, s)$ is a unit-speed ray starting from the tip, and we can also compute that

$$\int_0^r \mathrm{Ric}_\Sigma(\gamma'(s),\gamma'(s))\, ds = \int_0^r 2\operatorname{sech}^2 s\, ds = 2\tanh s|_0^r = 2\left(1-\frac{2}{e^{2r}+1}\right), \tag{2-5}$$

which converges to 2 as $r \to \infty$. Note that this integral is the speed of a point at distance r drifts away from the tip in the backward Ricci flow $g_c(t)$.

Throughout the paper, by abuse of notation, we will use g_c to denote both the metric on the cigar, and also the product metric on $\mathbb{R}\times$ cigar such that $R(x_{\mathrm{tip}}) = 4$; and use g_{stan} to denote the product metrics on $\mathbb{R}\times S^1$ and $\mathbb{R}^2\times S^1$ such that the length of the S^1-fiber is equal to 2π.

With this convention, it is easy to see from (2-3) that for any sequence of points $x_i \to \infty$, the pointed manifolds (cigar, g_c, x_i) smoothly converge to $(\mathbb{R}\times S^1, g_{\mathrm{stan}})$ in the Cheeger–Gromov sense.

Next, we introduce the concept of collapsing and noncollapsing.

Definition 2.3 (collapsing and noncollapsing) Let (M^n, g) be an n-dimensional complete Riemannian manifold. We say that it is noncollapsed (resp. collapsed) if there exists (resp. does not exist) a constant $\kappa > 0$ such that the following holds: For all $x \in M$, if $|\mathrm{Rm}|(x) < r^{-2}$ on $B_g(x,r)$ for some $r > 0$, then

$$\mathrm{vol}_g(B_g(x,r)) \geq \kappa r^n.$$

It is easy to see that an n-dimensional Riemannian manifold is collapsed if there is an asymptotic limit isometric to $\mathbb{R}^{n-2}\times$ cigar.

Lemma 2.4 *Let (M^n, g) be an n-dimensional complete Riemannian manifold. Suppose there exists a sequence of points $x_i \in M$ and constants $r_i > 0$ such that the pointed manifolds $(M, r_i^{-2}g, x_i)$ smoothly converge to $\mathbb{R}^{n-2}\times$ cigar. Then (M, g) is collapsed.*

Proof By assumption we may choose a sequence of points $y_i \in M$ such that $(M, r_i^{-2}g, y_i)$ converge to $\mathbb{R}^{n-1}\times S^1$. So there is a sequence of constants $A_i \to \infty$ such that $|\mathrm{Rm}|_{r_i^{-2}g} \leq A_i^{-2}$ on $B_{r_i^{-2}g}(y_i, A_i)$, and

$$\lim_{i\to\infty} \frac{\mathrm{vol}_{r_i^{-2}g}\, B_{r_i^{-2}g}(y_i, A_i)}{A_i^n} = 0.$$

After rescaling, this implies $|\mathrm{Rm}|_g \le (A_i r_i)^{-2}$ on $B_g(y_i, A_i r_i)$ and

$$\lim_{i\to\infty} \frac{\mathrm{vol}_g\, B_g(y_i, A_i r_i)}{(A_i r_i)^n} = 0.$$

So (M, g) is collapsed. □

In Section 3 we will see that the converse of Lemma 2.4 is also true for all 3D steady gradient Ricci solitons, ie any collapsed solitons have an asymptotic limit isometric to $\mathbb{R}\times$cigar. Note all 3D steady gradient Ricci solitons except the Bryant soliton are collapsed [12].

2.2 Local geometry models

We will show in Section 3 that both $\mathbb{R}\times$cigar and $\mathbb{R}^2\times S^1$ are asymptotic limits in any 3D steady gradient Ricci solitons that are not Bryant solitons. In this subsection we define ϵ-necks, ϵ-cylindrical planes and ϵ-tip points, which are local geometry models corresponding to these asymptotic limits. Moreover, to obtain the asymptotic limits, we need to rescale the soliton by factors that are comparable to volume scale at the points.

Definition 2.5 (volume scale) Let (M, g) be a 3D Riemannian manifold. Define the volume scale $r(\cdot)$ to be

$$r(x) = \sup\{s > 0 : \mathrm{vol}_g(B_g(x, s)) \ge \omega_0 s^3\},$$

where $\omega_0 > 0$ is chosen such that $r(x) = 1$ for all $x \in \mathbb{R}^2\times S^1$. It is clear that ω_0 is less than the volume of the radius one ball in the Euclidean space $\mathbb{R}^3$.

We measure the closeness of two pointed Riemannian manifolds by using the following notion of ϵ-isometry.

Definition 2.6 (ϵ-isometry between manifolds) Let $\epsilon > 0$ and $m \in \mathbb{N}$. Let (M_i^n, g_i) for $i = 1, 2$ be n-dimensional Riemannian manifolds, and let $x_i \in M_i$. We say a smooth map $\phi\colon B_{g_1}(x_1, \epsilon^{-1}) \to M_2$ with $\phi(x_1) = x_2$ is an ϵ-isometry in the C^m-norm if it is a diffeomorphism onto the image, and

$$|\nabla^k(\phi^* g_2 - g_1)| \le \epsilon \quad \text{on } B_{g_1}(x_1, \epsilon^{-1}) \text{ for } k = 0, 1, \dots, m, \tag{2-6}$$

where the covariant derivatives and norms are taken with respect to g_1. We also say (M_2, g_2, x_2) is ϵ-close to (M_1, g_1, x_1) in the C^m-norm. In particular, if $m = [\epsilon^{-1}]$, then we simply say (M_2, g_2, x_2) is ϵ-close to (M_1, g_1, x_1) and ϕ is an ϵ-isometry.

Definition 2.7 (ϵ-isometry between Ricci flows) Let $\epsilon > 0$. Let $(M_i^n, g_i(t))$ with $t \in [-\epsilon^{-1}, \epsilon^{-1}]$ for $i = 1, 2$ be an n-dimensional Ricci flow, and let $x_i \in M_i$. We say a smooth map $\phi\colon B_{g_1(0)}(x_1, \epsilon^{-1}) \to M_2$ such that $\phi(x_1) = x_2$ is an ϵ-isometry between the two Ricci flows if it is a diffeomorphism onto the image, and for $k = 0, 1, \dots, [\epsilon^{-1}]$,

$$|\nabla^k(\phi^* g_2(t) - g_1(t))| \le \epsilon \quad \text{on } B_{g_1(0)}(x_1, \epsilon^{-1}) \times [-\epsilon^{-1}, \epsilon^{-1}],$$

where the covariant derivatives and norms are taken with respect to $g_1(0)$. We also say $(M_2, g_2(t), x_2)$ is ϵ-close to $(M_1, g_1(t), x_1)$.

In the following, we will choose the target manifolds to be the cylinder and cylindrical plane, and call the regions that are close to them the ϵ-neck and ϵ-cylindrical plane.

Definition 2.8 (ϵ-neck) We say a 2D Riemannian manifold (N, g) is an ϵ-neck at $x_0 \in M$ for some $\epsilon > 0$ if there exists a constant $r > 0$ such that $(N, r^{-2}g, x_0)$ is ϵ-close to the cylinder $(\mathbb{R} \times S^1, (0,0))$.

We say $B_g(x_0, r\epsilon^{-1})$ is the ϵ-neck, r is the scale, x_0 is the center, and the preimage of $\{0\} \times S^1$ under the ϵ-isometry is the central circle. On $\mathbb{R} \times S^1$, let the function $x\colon \mathbb{R} \times S^1 \to \mathbb{R}$ be the projection onto the $\mathbb{R}$-factor, and let ∂_x be the vector field. Then by abuse of notation, we will denote the corresponding function and vector field on N modulo the ϵ-isometry by x and ∂_x.

Definition 2.9 (ϵ-cap) Let (M, g) be a complete 2D Riemannian manifold. We say a compact subset $\mathscr{C} \subset M$ is an ϵ-cap if $\mathscr{C}$ is diffeomorphic to a 2-ball and the boundary $\partial\mathscr{C}$ is the central circle of an ϵ-neck N in (M, g). We say that the points in $\mathscr{C} \setminus N$ are centers of the ϵ-cap.

Definition 2.10 (ϵ-cylindrical plane) Let $\epsilon > 0$. We say a 3D Riemannian manifold (M, g) is an ϵ-cylindrical plane at $x_0 \in M$ if there exists a constant $r > 0$ such that $(M, r^{-2}g, x_0)$ is ϵ-close to the cylindrical plane $\mathbb{R}^2 \times S^1$. By abuse of notation, we also refer to the image subset of the ϵ-isometry as an ϵ-cylindrical plane of g.

We say r is the scale of the ϵ-cylindrical plane, and x_0 is the center of the ϵ-cylindrical plane. Let $x, y\colon \mathbb{R}^2 \times S^1 \to \mathbb{R}$ and $\theta\colon \mathbb{R}^2 \times S^1 \to S^1$ be the projection to the three product factors. By abuse of notation, we use $\partial_x, \partial_y, \partial_\theta$ to denote the corresponding vector fields on M modulo the ϵ-isometry. In particular, we call ∂_θ the SO(2)-Killing field of the ϵ-cylindrical plane.

At the center of an ϵ-cylindrical plane, we introduce another scale in the following definition, which is comparable to the volume scale. Since this scale is measured by the length of curves, it is more useful than the volume scale in the Ricci flow of the soliton when combined with suitable distance distortion estimates.

Definition 2.11 (scale at an ϵ-cylindrical plane) Let (M, g) be a 3D Riemannian manifold. Suppose $x \in M$ is the center of an ϵ-cylindrical plane. We denote by $h(x)$ the infimum length of all closed smooth curves at x that are homotopic to the S^1-factor of the ϵ-cylindrical plane in $B(x, 1000r(x))$, where $r(x) > 0$ is the volume scale at x. It is clear that $h(x)$ is achieved by a geodesic loop at x.

Moreover, by the definition of volume scale we have $r(x) = 1$ and $h(x) = 2\pi$ for all $x \in \mathbb{R}^2 \times S^1$. So when ϵ is sufficiently small, we have

$$1.9\pi r(x) \le h(x) \le 2.1\pi r(x).$$

Next we describe points on 2D manifolds that are close to the tip of the cigar.

Definition 2.12 (ϵ-tip point) Let $\epsilon > 0$. Let (M, g) be a 2D Riemannian manifold, $x \in M$. If there is an ϵ-isometry from $(M, r^{-2}(x)g, x)$ to $(\text{cigar}, r^{-2}(x_0)g_c, x_0)$, for some $x_0 \in$ cigar such that $d_{g_c}(x_0, x_{\text{tip}}) \le \epsilon$, then we say x is an ϵ-tip point.

Similarly, if (M,g) is a 3D Riemannian manifold, $x\in M$. Suppose there is an ϵ-isometry from $(M,r^{-2}(x)g,x)$ to $(\mathbb{R}\times\text{cigar},r^{-2}(x_0)g_c,x_0)$ for some $x_0\in\mathbb{R}\times\text{cigar}$ such that $d_{g_c}(x_0,x_{\text{tip}})\le\epsilon$, where x_{tip} is the tip of the cigar with the same $\mathbb{R}$-coordinate as x_0. Then we say that x is an ϵ-tip point.

2.3 Distance distortion estimates and curvature estimates

In this subsection, we review some standard distance distortion estimates and curvature estimates, which are originally due to Hamilton [51] and Perelman [66]. A key observation in obtaining these estimates is the following. Let $(M,g(t))$ with $t\in[0,T]$ be a Ricci flow. Assume $\gamma\colon[0,1]\to M$ is a smooth curve. For any $t\in[0,T]$, write $L(t)$ for the length of γ with respect to $g(t)$. Then

(2-7) $$L'(t)=-\int_0^1\frac{\mathrm{Ric}_{g(t)}(\gamma'(s),\gamma'(s))}{|\gamma'(s)|_{g(t)}}\,ds.$$

In particular, the following lemma gives an upper bound on the speed of distance shrinking between two points, using only local curvature bounds near the two points. The proof uses the second variation formula; see eg [36, Theorem 18.7].

Lemma 2.13 *Let $(M,g(t))_{t\in[0,T]}$ be a Ricci flow of dimension n. Let $K,r_0>0$.*

(1) *Let $x_0\in M$ and $t_0\in(0,T)$. Suppose that $\mathrm{Ric}\le(n-1)K$ on $B_{t_0}(x_0,r_0)$. Then the distance function $d(x,t)=d_t(x,x_0)$ satisfies, in the outside of $B_{t_0}(x_0,r_0)$, the inequality*
$$(\partial_t-\Delta)|_{t=t_0}d\ge-(n-1)\big(\tfrac{2}{3}Kr_0+r_0^{-1}\big).$$

(2) *Let $t_0\in[0,T)$ and $x_0,x_1\in M$. Suppose*
$$\mathrm{Ric}(x,t_0)\le(n-1)K$$
for all $x\in B_{t_0}(x_0,r_0)\cup B_{t_0}(x_1,r_0)$. Then
$$\partial_t|_{t=t_0}d_t(x_0,x_1)\ge-2(n-1)(Kr_0+r_0^{-1}).$$

The next lemmas control how fast a metric ball shrinks and expands along Ricci flow, using the nearby curvature assumptions. They can be proved by using the Ricci flow equation; see eg [69, Lemmas 2.1 and 2.2].

Lemma 2.14 *Let $(M,g(t))_{t\in[0,T]}$ be a Ricci flow of dimension n, and let $x_0\in M$. Let $K,A>0$.*

(1) *Suppose $\mathrm{Ric}_{g(t)}\ge-K$ on $B_t(x_0,A)\Subset M$ for all $t\in[0,T]$. Then for all $t\in[0,T]$,*
$$B_0(x_0,A\,e^{-KT})\subset B_t(x_0,A\,e^{-K(T-t)}).$$

(2) *Suppose $\mathrm{Ric}_{g(t)}\le K$ on $B_t(x_0,R)\Subset M$ for all $t\in[0,T]$. Then for all $t\in[0,T]$,*
$$B_t(x_0,A\,e^{-Kt})\subset B_0(x_0,A).$$

The following curvature estimate is also due to Perelman [66, Corollary 11.6]. It provides a curvature upper bound at points in a Ricci flow, if the local volume has a positive lower bound. For a more general version of this estimate, see [5, Proposition 3.2].

Lemma 2.15 (Perelman's curvature estimate) *For any $\kappa > 0$ and $n \in \mathbb{N}$, there exists $C > 0$ such that the following holds. Let $(M^n, g) \times [-T, 0]$ be an n-dimensional Ricci flow (not necessarily complete). Let $x \in M$ be a point with $B_g(x,r) \times [-r^2, 0] \Subset M \times [-T, 0]$ for some $r > 0$. Assume also $\mathrm{vol}(B_g(x,r)) \geq \kappa r^n$. Then $R(x,0) \leq C/r^2$.*

2.4 Metric comparisons

We need the following notions and facts from metric comparison geometry; see [21]. Let (M, g) be a complete n-dimensional Riemannian manifold with nonnegative sectional curvature.

Lemma 2.16 (monotonicity of angles) *For any triple of points $o, p, q \in M$, the comparison angle $\widetilde{\measuredangle} poq$ is the corresponding angle formed by minimizing geodesics with lengths equal to $d_g(o,p)$, $d_g(o,q)$ and $d_g(p,q)$ in Euclidean space.*

Let op and oq be two minimizing geodesics in M between o, p and o, q, and let $\measuredangle poq$ be the angle between them at o. Then $\measuredangle poq \geq \widetilde{\measuredangle} poq$. Moreover, for any $p' \in op$ and $q' \in oq$, we have $\widetilde{\measuredangle} p'oq' \geq \widetilde{\measuredangle} poq$.

In a nonnegatively curved complete noncompact Riemannian manifold, we can equip a length metric on the space of geodesic rays. Moreover, a blowdown sequence of this manifold converges to the metric cone over the space of rays in the Gromov–Hausdorff sense; see eg [63, Proposition 5.31].

Let γ_1 and γ_2 be two rays with unit speed starting from a point $p \in M$. The limit $\lim_{r\to\infty} \widetilde{\measuredangle}\gamma_1(r) p \gamma_2(r)$ exists by the monotonicity of angles, and we say it is the angle at infinity between γ_1 and γ_2, and denote it as $\measuredangle(\gamma_1, \gamma_2)$.

Lemma 2.17 (space of rays) *Let $p \in M$ and $S_\infty(M, p)$ be the space of equivalent classes of rays starting from p, where two rays are equivalent if and only if the angle at infinity between them is zero, and the distance between two rays is the limit of the angle at infinity between them. Then $S_\infty(M, p)$ is a compact length space.*

Lemma 2.18 (asymptotic cone) *Let $p \in M$ and $\mathcal{T}(M, p)$ be the metric cone over $S_\infty(M, p)$. Then for any $\lambda_i \to \infty$, the sequence of pointed manifolds $(M, \lambda_i^{-1} g, p)$ converges to $\mathcal{T}(M, p)$, with p converging to the cone point in the pointed Gromov–Hausdorff sense. Moreover, for $p, q \in M$, we have that $\mathcal{T}(M, p)$ is isometric to $\mathcal{T}(M, q)$. We say (M, g) is asymptotic to $\mathcal{T}(M, p)$, and $\mathcal{T}(M, p)$ is the asymptotic cone of M.*

It is clear that the asymptotic cone $\mathcal{T}(M, p)$ is in fact independent of the choice of p. It is easy to see that the asymptotic cones of the Bryant soliton and $\mathbb{R} \times$ cigar are a ray and a half-plane. In [60], the author constructed a family of 3D steady gradient Ricci solitons that are asymptotic to metric cones over an interval $[0, a]$, where $a \in (0, \pi)$. We will show in Section 3 that this is true for all 3D steady gradient Ricci soliton with positive curvature that is not a Bryant soliton.

In the rest of this subsection we introduce a very useful technical notion called strainer [21]. It is similar to the notion of an orthogonal frame in the Euclidean space $\mathbb{R}^n$, that provides a local coordinate system in the metric space.

Definition 2.19 ((m,δ)-strainer) Let $\delta > 0$ and $1 \le m \in \mathbb{N}$. A $2m$-tuple $(a_1, b_1, \dots, a_m, b_m)$ of points in a metric space (X, d) is called an (m,δ)-strainer around a point $x \in X$ if

$$\begin{cases} \tilde{\measuredangle} a_i x b_j > \frac{1}{2}\pi - \delta & \text{for all } i, j = 1, \dots, m \text{ with } i \neq j, \\ \tilde{\measuredangle} a_i x a_j > \frac{1}{2}\pi - \delta & \text{for all } i, j = 1, \dots, m \text{ with } i \neq j, \\ \tilde{\measuredangle} b_i x b_j > \frac{1}{2}\pi - \delta & \text{for all } i, j = 1, \dots, m \text{ with } i \neq j, \\ \tilde{\measuredangle} a_i x b_i > \pi - \delta & \text{for all } i = 1, \dots, m. \end{cases}$$

The strainer is said to have size r if $d(x, a_i) = d(x, b_i) = r$ for all $i = 1, \dots, m$. It is said to have size at least r if $d(x, a_i) \ge r$ and $d(x, b_i) \ge r$ for all $i = 1, \dots, m$.

We also introduce the notion of $\left(m+\frac{1}{2}, \delta\right)$-strainers. Similarly, this notion provides a local coordinate system in the metric space that looks like a half-plane $\mathbb{R}^n_+ = \mathbb{R}^n \times \mathbb{R}_+$.

Definition 2.20 ($\left(m+\frac{1}{2}, \delta\right)$-strainer) Let $\delta > 0$ and $1 \le m \in \mathbb{N}$. We call a $(2m+1)$-tuple of points $(a_1, b_1, \dots, a_m, b_m, a_{m+1})$ in a metric space (X, d) an $\left(m+\frac{1}{2}, \delta\right)$-strainer around a point $x \in X$ if

$$\begin{cases} \tilde{\measuredangle} a_i x b_j > \frac{1}{2}\pi - \delta & \text{for all } i = 1, \dots, m+1 \text{ and } j = 1, \dots, m \text{ with } i \neq j, \\ \tilde{\measuredangle} a_i x a_j > \frac{1}{2}\pi - \delta & \text{for all } i, j = 1, \dots, m+1 \text{ with } i \neq j, \\ \tilde{\measuredangle} b_i x b_j > \frac{1}{2}\pi - \delta & \text{for all } i, j = 1, \dots, m \text{ with } i \neq j, \\ \tilde{\measuredangle} a_i x b_i > \pi - \delta & \text{for all } i = 1, \dots, m. \end{cases}$$

The strainer is said to have size r if $d(x, a_i) = d(x, b_j) = r$ for all $i = 1, \dots, m+1$ and $j = 1, \dots, m$. It is said to have size at least r if $d(x, a_i) \ge r$ for all $i = 1, \dots, m+1$ and $d(x, b_j) \ge r$ for all $j = 1, \dots, m$.

2.5 Heat kernel estimates

We prove a few lemmas using the standard heat kernel estimates of the heat equations under the Ricci flows. Let $G(x, t; y, s)$, where $x, y \in M$ and $s < t$, be the heat kernel of the heat equation $\partial_t u = \Delta u$ under $g(t)$, that is,

$$\partial_t G(x, t; y, s) = \Delta_{x,t} G(x, t; y, s), \quad \lim_{t \searrow s} G(\cdot, t; y, s) = \delta_y. \tag{2-8}$$

It is easy to see that $G(x, t; y, s)$ is also the heat kernel of the conjugate heat equation, that is,

$$-\partial_s G(x, t; y, s) = \Delta_{y,s} G(x, t; y, s) - R(y, s)\, G(x, t; y, s), \quad \lim_{s \nearrow t} G(x, t; \cdot, s) = \delta_x.$$

We have the following Gaussian upper bound for G.

Lemma 2.21 (upper bound of the heat kernel for an evolving metric, cf [36, Theorem 26.25]) *Let $(M^n, g(t))$ be a complete Ricci flow on $[0,T]$ with $|\mathrm{Rm}| \le K$. There exists a constant $C_1 < \infty$ depending only on n, T and K such that the conjugate heat kernel satisfies*

$$G(x,t;y,s) \le \frac{C_1}{\mathrm{vol}^{1/2}\big(B_s\big(x, \sqrt{\tfrac{1}{2}(t-s)}\big)\big) \cdot \mathrm{vol}^{1/2}\big(B_s\big(y, \sqrt{\tfrac{1}{2}(t-s)}\big)\big)} \exp\Big(-\frac{d_s^2(x,y)}{C_1(t-s)}\Big) \tag{2-9}$$

for any $x, y \in M$ and $0 \le s < t \le T$.

Using this we can prove the following lemma, which gives a time- and distance-dependent upper bound on nonnegative subsolutions to the heat equation, depending on initial upper bounds.

Lemma 2.22 *For any $K, T, \alpha > 0$, there is a constant $C(K,T,\alpha) > 0$ such that the following holds:*

Let $(M, g(t), y_0)$ be a complete Ricci flow on $[0,T]$. Assume that

(1) *$|\mathrm{Rm}|(x,t) \le K$ for all $x \in M$ and $t \in [0,T]$, and*
(2) *$u\colon M \times [0,T] \to [0,\infty)$ is a smooth function with $\partial_t u \le \Delta u$ and*

$$u(x,0) \le e^{\alpha\, d_0(x,y_0)}.$$

Then for all $D > 0$, $u(x,t) \le C e^{(\alpha+1)D}$ for any $(x,t) \in B_0(y_0, D) \times [0,T]$.

Proof First, by the curvature assumption $|\mathrm{Rm}| \le K$, it follows immediately from the Ricci flow equation that the metrics at different times are comparable to each other, ie

$$e^{-CK} g(s) \le g(t) \le e^{CK} g(s).$$

From now on we will use C to denote all constants that depend only on T, K and α, whose values may vary.

By the reproduction formula we have

$$u(x,t) \le \int_M G(x,t;y,0) \cdot u(y,0)\, d_0 y. \tag{2-10}$$

Now let $D > 0$ and assume $x \in B_0(x_0, D)$, and split the integral into two parts

$$u(x,t) = \int_{B_0(x,1)} G(x,t;y,0) \cdot u(y,0)\, d_0 y + \int_{M \setminus B_0(x,1)} G(x,t;y,0) \cdot u(y,0)\, d_0 y.$$

For the first part, note that for any $y \in B_0(x,1)$, we have

$$d_0(y,x_0) \le d_0(y,x) + d_0(x,x_0) \le 1 + D.$$

So by the assumption on u we have $u(y,0) \le e^{\alpha(1+D)}$, and hence

$$\int_{B_0(x,1)} G(x,t;y,0) \cdot u(y,0)\, d_0 y \le e^{\alpha(1+D)} \int_{B_0(x,1)} G(x,t;y,0)\, d_0 y \le e^{\alpha(1+D)}. \tag{2-11}$$

To estimate the second part in (2-10), we first claim that for any $y \in M \setminus B_0(x,1)$,

$$G(x,t;y,0) \le C \cdot \exp\left(-\frac{d_0^2(x,y)}{Ct}\right). \tag{2-12}$$

To this end, we note that if $t \ge 1$, the volumes of the two balls $B_0(x,\sqrt{t/2})$ and $B_0(y,\sqrt{t/2})$ are bounded below by C^{-1}. So the claim follows immediately from (2-9). If $t < 1$, then by the assumption on the injectivity radius and the curvature, we see that the volumes of the two balls $B_0(x,\sqrt{t/2})$ and $B_0(y,\sqrt{t/2})$ are bounded below by $C^{-1}(t/2)^{n/2}$. Note also that for large enough C we have

$$\left(\tfrac{1}{2}t\right)^{-n/2} \le C \cdot \exp\left(\frac{1}{2C_1 t}\right) \le C \cdot \exp\left(\frac{d_0^2(x,y)}{2C_1 t}\right),$$

where we used $d_0(x,y) \ge 1$, and $C_1 > 0$ is the constant from (2-9). So by Lemma 2.21 this implies

$$G(x,t;y,0) \le C\left(\tfrac{1}{2}t\right)^{-n/2} \exp\left(-\frac{d_0^2(x,y)}{C_1 t}\right) \le C \cdot \exp\left(-\frac{d_0^2(x,y)}{2C_1 t}\right),$$

which proves the claim.

Note for any $y \in M \setminus B_0(x,1)$, we have

$$u(y,0) \le e^{\alpha d_0(x_0,y)} \le e^{\alpha(d_0(x_0,x)+d_0(x,y))} \le e^{\alpha D} \cdot e^{\alpha d_0(x,y)}.$$

Combining this with (2-12), we see that the second part in (2-10) satisfies

$$\begin{aligned} \int_{M\setminus B_0(x,1)} G(x,t;y,0)\cdot u(y,0)\,d_0 y &\le Ce^{\alpha D}\int_M \exp\left(-\frac{d_0^2(x,y)}{Ct}\right)\cdot e^{\alpha d_0(x,y)}\,d_0 y \\ &\le Ce^{\alpha D}\int_M \exp\left(-\frac{d_0^2(x,y)}{Ct}\right)d_0 y \\ &\le Ce^{\alpha D}, \end{aligned} \tag{2-13}$$

where in the last inequality we used the curvature bound $|\mathrm{Rm}| \le K$ and $t \le T$, which allows us to apply a volume comparison to estimate the last integral term; see also [58, Lemma 2.8]. Combining the two inequalities (2-11) and (2-13) we get $|u|(x,t) \le Ce^{(\alpha+1)D}$, which proves the lemma. □

3 Asymptotic geometry at infinity

In this section, we study the asymptotic geometry of a 3D steady gradient soliton that is not a Bryant soliton. We show that it dimension reduces to the cigar soliton along two integral curves of ∇f. Moreover, we show that the asymptotic cone of the soliton is isometric to a metric cone over an interval $[0,a]$, where $a \in (0,\pi)$. We also prove a few other geometric properties of the soliton, one of which is that the scalar curvature attains its maximum at some point.

3.1 Classification of asymptotic limits

The main result in this subsection is Lemma 3.3, which states that there are two asymptotic limits in the soliton, which are $\mathbb{R}^2 \times S^1$ and $\mathbb{R} \times$ cigar. In taking limits, we will often need to rescale the metrics by the volume scale $r(x_i)$ at the basepoint x_i, to guarantee the limit is smooth and also not trivial. Recall by Definition 2.5, volume scale is the maximum radius such that the volume ratio is not less than $\omega_0 > 0$, and volume scale is 1 everywhere on $\mathbb{R}^2 \times S^1$.

We will use the following lemma to show that $\mathbb{R} \times T^2$ cannot be an asymptotic limit.

Lemma 3.1 *Let (M, g) be a 3D complete Riemannian manifold with positive curvature, and suppose $\mathbb{R} \times T^2 = \mathbb{R} \times S^1 \times S^1$ is equipped with some product metric g_0 (in which the lengths of the two S^1-fibers are not necessarily equal), and let $x_0 \in \mathbb{R} \times T^2$. Then there exists an $\epsilon_0 > 0$ such that for any $y \in M$, the pointed manifold $(M, r^{-2}(y)g, y)$ is not ϵ_0-close to $(\mathbb{R} \times T^2, r^{-2}(x_0)g_0, x_0)$.*

Proof Suppose ϵ_0 does not exist. Then there exists a sequence of points $y_k \in M$ such that the sequence $(M, r^{-2}(y_k)g, y_k)$ smoothly converges to $(\mathbb{R} \times T^2, r^{-2}(x_0)g_0, x_0)$. It is clear that $y_k \to \infty$, because otherwise the manifold is isometric to $\mathbb{R} \times T^2$, a contradiction to the positive curvature. So there exists a sequence $\epsilon_k \to 0$, an open neighborhood U_k of y_k and a diffeomorphism ϕ_k such that $\phi_k : [-\epsilon_k^{-1}, \epsilon_k^{-1}] \times T^2 \to U_k \subset M$, which maps $x_0 \in \{0\} \times T^2$ to y_k, and is such that ϕ_k is an ϵ_k-isometry. We say $T_k^2 := \phi_k(\{0\} \times T^2)$ is the central torus, which is homeomorphic to T^2.

In the rest of the proof we show that a connected component of the level set of the function $d_{y_0}(\cdot) := d_g(y_0, \cdot)$ at y_k is homeomorphic to a 2-torus. Suppose this claim is true. Then for k sufficiently large, $d_g(y_0, y_k)$ is sufficiently large, which contradicts the fact that a level set of a distance function to a fixed point in a positively curved 3D Riemannian manifold must be homeomorphic to a 2-sphere at all large distances; see eg [63, Corollary 2.11]. Then this will prove the lemma.

To show the claim, without loss of generality we may assume after a rescaling that $r(y_k) = 1$. Let $s : \mathbb{R} \times T^2 \to \mathbb{R}$ be the coordinate function in the $\mathbb{R}$-direction of $\mathbb{R} \times T^2$, and let $X = \phi_k(\partial_x)$ be a vector field on $U_k \subset M$. For any small $\epsilon > 0$, it is easy to see that for all large k, the angle formed by any minimizing geodesic from y_0 to a point $p \in U_k$ and $X(p)$ is less than ϵ. Let χ_μ, with $\mu \in (-100, 100)$, be the flow generated by X on $\phi_k((-100, 100) \times T^2) \subset M$. Then the distance function $d_{y_0}(\cdot)$ increases along χ_μ at a rate bounded below by $1 - C_0\epsilon$, where $C_0 > 0$ is a universal constant. In particular, an integral curve of X intersects a level set of $d_{y_0}(\cdot)$ in a single point.

Therefore, there is a continuous function $\tilde{s} : T_k^2 \to \mathbb{R}$ such that for any $x \in T_k^2$, we have $d_{y_0}(\chi_{\tilde{s}(x)}(x)) = d_{y_0}(y_k)$. Let $F : T_k^2 \to d_{y_0}^{-1}(d_{y_0}(y_k))$ be defined by $F(x) = \chi_{\tilde{s}(x)}(x)$, then F is continuous. We show that F is an injection: suppose $F(x_1) = F(x_2) = y$. Then $x_i = \chi_{-\tilde{s}(x_i)}(y)$ for $i = 1, 2$. Since $x_1, x_2 \in T_k^2$, it follows that $(\phi_k^{-1})^* s(x_1) = (\phi_k^{-1})^* s(x_2) = 0$ and

$$0 = (\phi_k^{-1})^* s(x_1) - (\phi_k^{-1})^* s(x_2) = \tilde{s}(x_2) - \tilde{s}(x_1).$$

So $\widetilde{s}(x_2) = \widetilde{s}(x_1)$, and hence $x_1 = x_2$. Since T_k^2 is compact, F is a homeomorphism from the 2-torus onto the image which is a connected component of $d_{y_0}^{-1}(d_{y_0}(y_k))$. This proves the claim. □

The following lemma will be used to show that all asymptotic limits split off a line.

Lemma 3.2 *Let (M, g) be a complete Riemannian manifold with nonnegative sectional curvature and let $\{y_k\}_{k=0}^{\infty} \subset M$ be a sequence of points with $d_g(y_0, y_k) \to \infty$. Then after passing to a subsequence of $\{y_k\}_{k=0}^{\infty}$, there exists a ray $\sigma\colon [0,\infty) \to M$ with $\sigma(0) = y_0$ and a sequence of numbers $s_k \to \infty$ such that for $z_k = \sigma(s_k)$, we have $d_g(z_k, y_k) = d_g(y_0, y_k)$ and*

$$\widetilde{\measuredangle} y_0 y_k z_k \to \pi \quad \text{as } k \to \infty.$$

Proof A standard metric comparison argument. See for example [4, Lemma 5.1.5]. □

Now we prove the main result in this subsection.

Lemma 3.3 *Let (M, g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let $p \in M$ be a fixed point. Then for any $\epsilon > 0$, there is $D(\epsilon) > 0$ such that for all $x \in M \setminus B_g(p, D)$, the pointed manifold $(M, r^{-2}(x)g, x)$ is ϵ-close to exactly one of the following:*

(1) $(\mathbb{R} \times \text{cigar}, r^{-2}(\widetilde{x})g_c, \widetilde{x})$, *or*
(2) $(\mathbb{R}^2 \times S^1, g_{\text{stan}}, \widetilde{x})$.

We call these two limits the asymptotic limits of the soliton.

Proof First, we show that for all $x \in M \setminus B_g(p, D)$, there exists $D > 0$ such that $(M, r^{-2}(x)g, x)$ is ϵ-close to some product space with an $\mathbb{R}$-factor. This suffices to show that for any sequence of points $\{y_k\}_{k=0}^{\infty}$, the rescaled Ricci flows $(M, r^{-2}(y_k)g(r^2(y_k)t), (y_k, 0))$ converge to an ancient Ricci flow that split off a line.

First, we claim $r(y_k)/d_g(y_0, y_k) \to 0$ as $k \to \infty$. If this is not true, then (M, g) has Euclidean volume growth, and hence is flat by Perelman's curvature estimate (Lemma 2.15); a contradiction. So by passing to a subsequence we may assume $(M, r^{-2}(y_k)g(r^2(y_k)t), (y_k, 0))$ converges to an ancient 3D Ricci flow; see [60, Lemma 3.3].

So by Lemma 3.2 and the strong maximum principle [63, Lemma 4.13, Corollary 4.19], the limit flow splits off an $\mathbb{R}$-factor. Therefore, by the classification of ancient 2D Ricci flows [39; 41], the limit flow must be isometric to one of the following Ricci flows up to a rescaling:

(1) $\mathbb{R} \times \text{cigar}$,
(2) $\mathbb{R}^2 \times S^1$,
(3) $\mathbb{R} \times T^2$,
(4) $\mathbb{R} \times S^2$,
(5) $\mathbb{R} \times \text{sausage solution}$.

First, item (3) is impossible by Lemma 3.1. Second, we can argue in the same way as [60, Theorem 3.7] to exclude item (5): In [60, Theorem 3.7], we argued under the $O(2)$-symmetry assumption, and used the curve $\Gamma(s)$ fixed by the $O(2)$-isometry to define a function $F(s)$ which characterizes the diameter of the nonflat factor in the asymptotic limits at $\Gamma(s)$. We showed $\lim_{s\to\infty} F(s)=\infty$. In our current setting, we can find a curve $\widetilde{\Gamma}(s)\colon [0,\infty)\to M$ going to infinity, such that for all $s\ge 0$, $(M, R(\widetilde{\Gamma}(s))g, \widetilde{\Gamma}(s))$ is ϵ-close to either $\mathbb{R}\times S^2$, or $\mathbb{R}\times\text{cigar}$ or a time-slice of $\mathbb{R}\times\text{sausage}$ solution at the tip. Then we can define the diameter function $F(s)$ by replacing $\Gamma(s)$ by $\widetilde{\Gamma}(s)$, and show $\lim_{s\to\infty} F(s)=\infty$ in the same way. This implies that $\mathbb{R}\times\text{sausage}$ solution cannot appear in the blowup limit.

Finally, we exclude item (4) as follows: Suppose $\mathbb{R}\times S^2$ is an asymptotic limit. We claim that all asymptotic limits are $\mathbb{R}\times S^2$. If so, then it is clear that the soliton is noncollapsed, and thus has to be the Bryant soliton by Brendle's result [12], so we get a contradiction. To show the claim, suppose by contradiction that there is another limit of type (1) or (2). First, for any $\epsilon>0$, after scaling down by sufficiently large $C>0$, the metric spaces of $\mathbb{R}\times\text{cigar}$, $\mathbb{R}^2\times S^1$ and $\mathbb{R}\times S^2$ are ϵ-close in the pointed Gromov–Hausdorff sense to an ϵ^{-1}-ball in $\mathbb{R}\times\mathbb{R}_+$, $\mathbb{R}^2$ or $\mathbb{R}$, respectively. So for points z sufficiently far away from a fixed point, the metric space of $(M, C^{-1}r^{-2}(z)g, z)$ is 2ϵ-close to a $(2\epsilon)^{-1}$-ball in $\mathbb{R}\times\mathbb{R}_+$, $\mathbb{R}^2$ or $\mathbb{R}$.

Next, note that there exists $\epsilon_0>0$ such that $\mathbb{R}$ is not ϵ_0-close to either $\mathbb{R}\times\mathbb{R}_+$ or $\mathbb{R}^2$. Since $\mathbb{R}\times S^2$ is not the unique asymptotic limit, we can find points $z_k\to\infty$ as $k\to\infty$, such that $(M, C^{-1}r^{-2}(z_k)g, z_k)$ for sufficiently large $C>0$ is not $\frac{1}{3}\epsilon_0$-close to any of $\mathbb{R}\times\mathbb{R}_+$, $\mathbb{R}^2$ or $\mathbb{R}$. This contradiction proves the claim. □

The following lemma shows that in the Ricci flow of the soliton, the closeness of a time-slice to the asymptotic limit leads to the closeness in a parabolic region of a certain size.

Corollary 3.4 *Suppose that (M,g) is a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let $p\in M$ be a fixed point. Then for any $\epsilon>0$, there is a $D(\epsilon)>0$ such that for all $x\in M\setminus B_g(p,D)$, the pointed rescaled Ricci flow $(M, r^{-2}(x)g(r^2(x)t), x)$ is ϵ-close to exactly one of the following two Ricci flows*:

(1) $(\mathbb{R}\times\text{cigar}, r^{-2}(\tilde{x})g_c(r^2(\tilde{x})t), \tilde{x})$,

(2) $(\mathbb{R}^2\times S^1, g_{\text{stan}}, \tilde{x})$.

Proof Note that the Ricci flows $g_{\mathbb{R}\times\text{cigar}}(t)$ and $g_{\mathbb{R}^2\times S^1}(t)$ on $\mathbb{R}\times\text{cigar}$ and $\mathbb{R}^2\times S^1$ are both eternal Ricci flows which have bounded curvature. The assertion now follows from Lemma 3.3 by a standard limiting argument as in [59, Lemma 3.4]. □

In the rest of this section, we show that $\mathbb{R}\times\text{cigar}$ is a stable asymptotic limit when we move forward in the Ricci flow of the soliton, in the sense that a region close to the $\mathbb{R}\times\text{cigar}$ stays close to it until it is not close to any asymptotic limits.

This is based on the observation that the Ricci flow of the cigar soliton contracts all points to the tip when we move forward in time along the flow. By using this and the closeness to the Ricci flow of cigar, we show in the next lemma that an ϵ-tip point x (see Definition 2.12) stays a 2ϵ-tip point outside a compact subset, when we move along an integral curve $\phi_{-t}(x)$ of $-\nabla f$. Note that this amounts to moving forward along the Ricci flow of the soliton, since $g(t)=\phi_{-t}^{*}g$ satisfies the Ricci flow equation, where $\{\phi_t\}_{t\in\mathbb{R}}$ are the diffeomorphisms generated by ∇f.

Lemma 3.5 *Fix some $p\in M$. For any $\epsilon>0$, there exists $D(\epsilon)>0$ such that the following holds.*

For any point $x\in M\setminus B_g(p,D)$, let $\phi_{-t}(x)$ be the integral curve of $-\nabla f$ for $t\in[0,\infty)$. Suppose x is an ϵ-tip point. Then $\phi_{-t}(x)$ is a 2ϵ-tip point for all $t\in[0,t(x))$, where $t(x)\in(0,\infty]$ is the supremum of t such that $\phi_{-t}(x)\in M\setminus B_g(p,D)$.

Proof For the fixed ϵ, we choose $T(\epsilon)>0$ to be the constant such that in the Ricci flow of the cigar soliton, the metric ball of radius 2ϵ at time 0 centered at the tip contracts to a metric ball of radius ϵ at time $T(\epsilon)$. Let $0<\epsilon_1\ll\epsilon$ be a constant whose value will be chosen later, then choose $D(\epsilon_1)>0$ to be the constant from Corollary 3.4.

If $x_0\notin B_g(p,D)$ is an ϵ-tip point, in the following we will show that $\phi_{-r^2(x_0)\,t}(x_0)$ is a 2ϵ-tip point for all $t\in[0,T(\epsilon)]$ such that $\phi_{-r^2(x_0)\,t}(x_0)\notin B_g(p,D)$, and $\phi_{-r^2(x_0)\,T(\epsilon)}(x_0)$ is again an ϵ-tip point. Suppose this is true, then the lemma follows by induction immediately.

By Corollary 3.4, there is an ϵ_1-isometry ψ between the two pointed Ricci flows

$$(M,r^{-2}(x_0)g(r^2(x_0)t),x_0)\quad\text{and}\quad(\mathbb{R}\times\text{cigar},r^{-2}(\psi(x_0))g_0(r^2(\psi(x_0))t),\psi(x_0)).$$

Note that ψ is also a $100\epsilon_1$-isometry between time-slices, and hence an ϵ-isometry for $\epsilon_1<\frac{1}{100}\epsilon$.

Let x_{tip} be the tip of the cigar in $\mathbb{R}\times$cigar which has the same $\mathbb{R}$-coordinate as $\psi(x_0)$. Then by taking ϵ_1 sufficiently small depending on ϵ, and using the distance shrinking of the cigar, it is easy to see that $d_t(\psi(x_0),x_{\mathrm{tip}})<2\epsilon$ in the Ricci flow $r^{-2}(\psi(x_0))g_0(r^2(\psi(x_0))t)$, for all $t\geq 0$. This implies the first half of the claim, that $\phi_{-r^2(x_0)\,t}(x_0)$ is a 2ϵ-point. The second half of the claim follows by the choice of $T(\epsilon)$ and taking ϵ_1 sufficiently small that $\epsilon_1^{-1}>T(\epsilon)$. □

3.2 The geometry near the edges

We study the local and global geometry at points that look like $\mathbb{R}\times$cigar. First, we show in Lemma 3.14 that there are two chains of infinitely many topological 3-balls that cover all ϵ-tip points. Using this we show in Lemma 3.16 that the asymptotic cone of the soliton is a metric cone over an interval $[0,a]$, for some $a\in[0,\pi)$, and the points in these two chains correspond to the boundary points of the cone. Next, in Lemma 3.17 we construct two smooth curves going to infinity inside the two chains, such that they are two integral curves of ∇f or $-\nabla f$, and along them the soliton converges to $(\mathbb{R}\times\text{cigar},x_{\mathrm{tip}})$.

Fix a point $p \in M$, in the following technical lemma we show that the minimizing geodesic from p to an ϵ-tip point q is almost parallel to the $\mathbb{R}$-direction in $\mathbb{R}\times\text{cigar}$ at q. The idea is to study the geometry near an ϵ-tip point q in three different scales: In the largest scale $d(p,q)$, the soliton looks like its asymptotic cone; in the smallest scale by the volume scale $r(q)$, it looks like $\mathbb{R}\times\text{cigar}$; in some intermediate scale between $r(q)$ and $d(p,q)$, it looks like a two-dimensional upper half-plane. Note when there is no confusion, we will omit the subscript g and write $d_g(\cdot,\cdot)$ as $d(\cdot,\cdot)$ and $B_g(p,\cdot)$ as $B(p,\cdot)$.

Lemma 3.6 *Let (M,g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let $p \in M$ be a fixed point. For any $\delta > 0$, there exists $\epsilon > 0$ such that the following holds:*

Let $q \in M$ be an ϵ-tip point, and ϕ be an ϵ-isometry from $\mathbb{R}\times\text{cigar}$ to M. Let ∂_r be the unit vector field in the $\mathbb{R}$-direction in $\mathbb{R}\times\text{cigar}$. Let $\gamma\colon[0,1]\to M$ be a minimizing geodesic from p to q. Then $\big||\cos\measuredangle(\gamma'(1),\phi_*(\partial_r))|-1\big|\le\delta$.

Proof Suppose not, then there are $\delta>0$, $\epsilon_i\to 0$ and ϵ_i-tip points $q_i\to\infty$ such that

$$\measuredangle(\gamma_i'(1),\phi_{i*}(\partial_r))\in(\delta,\pi-\delta), \tag{3-1}$$

where $\gamma_i\colon[0,1]\to M$ is a minimizing geodesic from p to q_i, and ϕ_i is an ϵ_i-isometry at q_i. For convenience, we will use ϵ_i to denote any sequence $C\epsilon_i$ where $C>0$ is a constant independent of i.

Since $d_g(p,q_i)\to\infty$ and the curvature is positive, after passing to a subsequence we may assume that the rescaled manifold $(M,d^{-2}(q_i,p)g,p)$ converges to the asymptotic cone in the Gromov–Hausdorff sense; see Lemma 2.18. So we can find a point $z_i\in M$ such that the pair of points (p,z_i) is a $(1,\epsilon_i)$-strainer at q_i of size $d_g(q_i,p)$. Let p_i be a point on γ_i and w_i be a point on the minimizing geodesic connecting q_i and z_i such that $d(w_i,q_i)=d(p_i,q_i)=r_i/\delta_i$, where $\delta_i>r(q_i)/d(q_i,p)$ is a sequence converging to zero, which we may adjust later. Since $r(q_i)/\delta_i<d(q_i,p)$, by the monotonicity of angles, (p_i,w_i) is a $(1,\epsilon_i)$-strainer at q_i of size $r(q_i)/\delta_i$. So

$$\tilde{\measuredangle}p_iq_iw_i\ge\pi-\epsilon_i. \tag{3-2}$$

Next, consider the rescaled pointed manifold $(M,r^{-2}(q_i)g,q_i)$, which is ϵ_i-close to $(\mathbb{R}\times\text{cigar},x_{\text{tip}})$. Then there is a sequence of points $o_i\in M$ with $d(o_i,q_i)=r(q_i)/\delta_i$ such that the minimizing geodesic $\sigma_i\colon[0,1]\to M$ from q_i to o_i satisfies $\measuredangle(\sigma_i'(0),\phi_{i*}(\partial_r))\to 0$ as $i\to\infty$. Combining this fact with (3-1) we get

$$\measuredangle p_iq_io_i\in\left(\tfrac12\delta,\pi-\tfrac12\delta\right).$$

Since $\epsilon_i\to 0$, by choosing $\delta_i\to 0$ properly we have $|\tilde{\measuredangle}p_iq_io_i-\measuredangle p_iq_io_i|<\frac{1}{10}\delta$, and hence

$$\tilde{\measuredangle}p_iq_io_i\in\left(\tfrac14\delta,\pi-\tfrac14\delta\right) \tag{3-3}$$

for all sufficiently large i.

Now consider the rescaled pointed manifold $(M,\delta_i^2r^{-2}(q_i)g,q_i)$. Since $\delta_i\to 0$, after passing to a subsequence we may assume that it converges to the upper half-plane $\mathbb{R}\times\mathbb{R}_+$ in the Gromov–Hausdorff sense, with $q_i\to(0,0)$ and $o_i\to(1,0)\in\mathbb{R}\times\mathbb{R}_+$ modulo the approximation maps. Assume p_i converges to $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$. Then by (3-3) we have $y_0 > \frac{1}{100}\delta$. On the other hand, (3-2) implies that (x_0, y_0) is one point in a $(1,0)$-strainer at $(0,0)$ of size 1. So it is clear that $|x_0| = 1$ and $y_0 = 0$, a contradiction. This proves the lemma. □

In the next few definitions, we introduce the concept of ϵ-solid cylinders. These are topological 3-balls that look like a large neighborhood of the tip in $\mathbb{R} \times$ cigar. A chain of ϵ-solid cylinder is a sequence of these cylinders meeting nicely. In this subsection, we will show in Lemma 3.14 that all ϵ-tip points are covered by exactly two such chains.

Definition 3.7 (ϵ-solid cylinder) Let $x \in M$ be an ϵ-tip point, and ϕ_x be the corresponding ϵ-isometry. We say $\nu_{L,D} := \phi_x([-L, L] \times B_{g_c}(x_{\mathrm{tip}}, D))$, is an ϵ-solid cylinder centered at x, where $L, D > 0$ are constants.

In order to make sure that the union of two intersecting ϵ-solid cylinders is still a topological 3-ball, we want them to meet nicely. So we introduce the concept of good intersection between two ϵ-solid cylinders; see eg [64, Section 5.6].

Definition 3.8 (good intersection) Let y_1 and y_2 be two ϵ-tip points, and ϕ_{y_i} be the corresponding ϵ-isometries. Let $\tilde{\gamma}_i : [-L_i, L_i] \to M$ be a curve passing through y_i such that $\phi_{y_i}^{-1}(\tilde{\gamma}_i(s)) = (s, x_{\mathrm{tip}})$. We say $\nu(i) = \phi_{y_i}([-L_i, L_i] \times B_{g_c}(x_{\mathrm{tip}}, D_i))$ for $i = 1, 2$ have good intersection if, after possibly reversing the directions of either or both of the $\mathbb{R}$-factors, the following hold:

(1) The projection r_1 in the direction of $\mathbb{R}$ is an increasing function along $\tilde{\gamma}_2$ at any point of $\tilde{\gamma}_2 \cap \nu(1)$.

(2) There is a point in the negative end of $\nu(2)$ that is contained in $\nu(1)$, and the positive end of $\nu(2)$ is disjoint from $\nu(1)$.

(3) Either $1.1 D_1 r(y_1) \le D_2 r(y_2)$, or $D_2 r(y_2) \le 0.9 D_1 r(y_1)$.

With the notion above, if two ϵ-solid cylinders have good intersection, then the intersection is homeomorphic to a 3-ball; see [64, Lemma 5.19].

Definition 3.9 (chain) Suppose that we have a sequence of ϵ-solid cylinders $\{\nu(1), \dots, \nu(k)\}$, for some $k \in \mathbb{N} \cup \{\infty\}$, with the curves $\tilde{\gamma}_i$ from Definition 3.8. We say that they form a chain of ϵ-solid cylinders if the following hold:

(1) For each $1 \le i < k$ the open sets $\nu(i)$ and $\nu(i+1)$ have a good intersection with the given orientations.

(2) If $\nu(i) \cap \nu(j) \neq \varnothing$ for some $i \neq j$, then $|i - j| = 1$.

Lemma 3.10 (cf [64, Lemma 5.22]) *Suppose that $\{\nu(1), \dots, \nu(k)\}$ is a chain of ϵ-solid cylinders. Then $\nu(1) \cup \dots \cup \nu(k)$ is homeomorphic to a 3-ball and its boundary is the union of the negative end of $\nu(1)$, the positive end of $\nu(k)$, and an annulus A.*

For an ϵ-tip point x, let ν be an ϵ-solid cylinder centered at it. In Lemma 3.11, we construct a chain of ϵ-solid cylinders starting from ν, which extends to infinity on one end. Moreover, the ϵ-solid cylinder on the other end meets a 2-sphere metric sphere $\partial B(p, D_0)$ in a spanning disk, where $D_0 > 0$ is independent of x.

Lemma 3.11 (extending an ϵ-solid cylinder to a chain) *Let (M, g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let $p \in M$ be a fixed point. There exists $\overline{D} > 0$ such that the following is true.*

For any ϵ-tip point $x \in M \setminus B(p, \overline{D})$, there exist an integer $N \in \mathbb{N}$ and a sequence of ϵ-tip points $\{x_i\}_{i=-N}^{\infty}$ going to infinity such that $x_0 = x$ and $x_{-N} \in \partial B(p, \overline{D})$, and an infinite chain of ϵ-solid cylinders $\{\nu(i)\}_{i=-N}^{\infty}$ centered at x_i, where $\nu(i) = \phi_{x_i}([-900, 900] \times B_{g_c}(x_{\mathrm{tip}}, L_i))$ with $L_i \in [500, 1000]$, such that if $\nu(i)$ intersects $\partial B(p, D)$ for some $D \geq \overline{D}$, then $\nu(i)$ meets $\partial B(p, D)$ in a spanning disk.

Proof By Lemma 3.3 we can choose $\overline{D} > 0$ sufficiently large that the rescaled soliton at any point in $M \setminus B\big(p, \frac{1}{2}\overline{D}\big)$ is ϵ-close to the asymptotic limits. For an ϵ-tip point $x_0 \in M \setminus B(p, \overline{D})$, assume $d(p, x_0) = D_0$. Let ϕ_{x_0} be the ϵ-isometry. Then $\nu(0) := \phi_{x_0}([-900, 900] \times B_{g_c}(x_{\mathrm{tip}}, 1000))$ is an ϵ-solid cylinder. Let r be the projection onto the $\mathbb{R}$-direction in $\mathbb{R} \times$cigar. By Lemma 3.6, after possibly replacing r by $-r$, we may assume $\measuredangle(\nabla d(p, \cdot), \phi_{x_0 *}(\partial_r)) < \epsilon$. So the two points $y_{\pm} := \phi_{x_0}((\pm 1000, x_{\mathrm{tip}})) \in M$ satisfy

$$d(y_+, p) \geq (1-\epsilon)1000\, r(x_0) + D_0 \quad \text{and} \quad d(y_-, p) \leq -(1-\epsilon)1000\, r(x_0) + D_0.$$

By the choice of $\overline{D}$ we can find a pair of ϵ-tip points $x_{\pm 1} \in B(y_{\pm}, r(x_0))$. In particular, we have $x_{\pm 1} \notin \phi_{x_0}([-900, 900] \times B_{g_c}(x_{\mathrm{tip}}, 1000))$, and

$$d(x_1, p) > 900\, r(x_0) + D_0 \quad \text{and} \quad d(x_{-1}, p) < -900\, r(x_0) + D_0.$$

Similarly, let $\phi_{x_{\pm 1}}$ be the ϵ-isometry at $x_{\pm 1}$. Then $\nu(\pm 1) := \phi_{x_{\pm 1}}([-900, 900] \times B_{g_c}(x_{\mathrm{tip}}, 500))$ is an ϵ-solid cylinder at $x_{\pm 1}$. It is clear that $\nu(0)$ and $\nu(\pm 1)$ have good intersections.

Repeating this, we can obtain a sequence of ϵ-tip points $\{x_i\}_{i=-N}^{\infty}$, with $x_{-N} \in B(p, \overline{D})$, and a sequence of ϵ-solid cylinders $\nu(i) := \phi_{x_i}([-900, 900] \times B_{g_c}(x_{\mathrm{tip}}, D_i))$ centered at x_i, where $D_i = 500$ when i is odd, and $D_i = 1000$ when i is even, such that $x_{i+1} \in B(\phi_{x_i}((1000, x_{\mathrm{tip}})), 10 r(x_i))$. Therefore, by triangle inequalities it is easy to see that $\nu(i)$ only intersects with $\nu(i-1)$ and $\nu(i+1)$, and has good intersections with them. In particular, for all $i \geq 0$ we have

$$d(x_i, p) > 900 \sum_{k=0}^{i-1} r(x_k) + D_0 \to \infty.$$

So $\{\nu(i)\}_{i=1}^{\infty}$ is an infinite chain of ϵ-solid cylinders. Moreover, we see that $\nu(i)$ meets all metric spheres $\partial B(p, D)$ with $D \geq \overline{D}$ in a spanning disk, since by Lemma 3.6 we have $\measuredangle(\nabla d(p, \cdot), \phi_{x_i *}(\partial_r)) < \epsilon$ for all i. □

In the following, we use Lemma 3.11 to show that all ϵ-tip points outside of a compact subset are contained in the union of finitely many chains of ϵ-solid cylinders.

Lemma 3.12 (disjoint chains containing all ϵ-tip points) *Under the same assumption as in Lemma 3.11, and let $\bar{D}$ be from Lemma 3.11. There exist k chains of ϵ-solid cylinders $\mathscr{C}_1, \dots, \mathscr{C}_k$ for some $k \in \mathbb{N}$, each of which satisfies the conclusions in Lemma 3.11. Moreover, all ϵ-tip points in $M \setminus B(p, \bar{D})$ are contained in the union of $\mathscr{C}_1, \dots, \mathscr{C}_k$.*

Proof Assume x_1 is an ϵ-tip point, $x_1 \in M \setminus B(p, \bar{D})$. Then let $\mathscr{C}_1$ be a chain of ϵ-solid cylinders produced by Lemma 3.11, whose union contains x_1. If there exists an ϵ-tip point $x_2 \in M \setminus (\mathscr{C}_1 \cup B(p, \bar{D}))$, then by Lemma 3.11 we can construct a new chain $\mathscr{C}_2$ of ϵ-solid cylinders containing x_2. We claim that $\mathscr{C}_1 \cap \mathscr{C}_2 = \varnothing$, ie the union of all ϵ-solid cylinders in $\mathscr{C}_1$ is disjoint from that in $\mathscr{C}_2$.

First, for two ϵ-tip points x, y with $d(x,y) \le 1000 r(x)$, it is easy to see that $d(x,y) \le 0.1 r(x)$ when ϵ is sufficiently small. Using this fact we see that x_2 is at least $900 r(x_2)$-away from $\mathscr{C}_1$, and hence $\mathscr{C}_1 \cap \mathscr{C}_2 \cap \partial B(p, D_1) = \varnothing$, where $D_1 = d(x_2, p) \ge \bar{D}$. Let $a, b > 0$ be the infimum and supremum of $\{r : \mathscr{C}_1 \cap \mathscr{C}_2 \cap \partial B(p, r) = \varnothing\}$. Then $a < D_1 < b$. Suppose by contradiction that $a \ge \bar{D}$. Then $\mathscr{C}_1 \cap \mathscr{C}_2 \cap \partial B(p, a) \ne \varnothing$, and we can find an ϵ-tip $y \in \mathscr{C}_1 \cap \mathscr{C}_2 \cap \partial B(p, a)$. However, by Lemma 3.6 this implies that ν the ϵ-solid cylinder centered at y is contained in $\mathscr{C}_1 \cap \mathscr{C}_2$, and the positive end of ν is at distance $a + 1000 r(y)$ from p, which is greater than a. This contradicts the infimum assumption of a. By a similar argument, we can show $b = \infty$. This proves the claim.

Repeating this procedure, we obtain a sequence of chains of ϵ-solid cylinders whose unions are disjoint. This must stop in finitely many steps, because these chains intersect with $\partial B(p, \bar{D})$ in spanning disks whose areas are uniformly bounded below. So we may assume these chains are $\mathscr{C}_1, \dots, \mathscr{C}_k$ and they contain all ϵ-tip points. □

Now we show that the number of these chains is exactly equal to two. To do this, we need the following lemma, which enables us to glue ϵ-cylindrical planes that intersect, and produce a global S^1-fibration on their union.

Lemma 3.13 (global S^1-fibration [64, Proposition 4.4]) *Let M be a 3D Riemannian manifold. Given $\epsilon' > 0$, the following holds for all $\epsilon > 0$ less than a positive constant $\epsilon_1(\epsilon')$. Suppose $K \subset M$ is a compact subset and each $x \in K$ is the center of an ϵ-cylindrical plane. Then there is an open subset V containing K and a smooth S^1-fibration structure on V.*

Furthermore, if U is an ϵ-cylindrical plane that contains a fiber F of the fibration on V, then F is ϵ'-close to a vertical S^1-factor in U, and F generates the fundamental group of U. In particular, the diameter of F is at most twice the length of any circle in the ϵ-cylindrical plane centered at any point of F.

Lemma 3.14 (two chains containing all ϵ-tip points) *Let (M, g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let $p \in M$ be a fixed point. There exists a $\bar{D} > 0$ such that the following is true.*

There are exactly 2 chains of ϵ-solid cylinders $\mathscr{C}_1$ and $\mathscr{C}_2$, each of which satisfies the conclusions in Lemma 3.11. Moreover, all ϵ-tip points in $M \setminus B(p, \bar{D})$ are contained in the union of $\mathscr{C}_1$ and $\mathscr{C}_2$.

Proof First, we show $k \geq 1$, which is equivalent to the $\mathbb{R} \times$ cigar being indeed an asymptotic limit of the soliton M. Suppose this is not the case. On the one hand, since all ϵ-tip points are contained in $\mathcal{C}_1, \dots, \mathcal{C}_k$, the complement of them in $M \setminus B(p, \overline{D})$ is covered by ϵ-cylindrical planes. Then by Lemma 3.13 we can find a connected open subset V_1 containing $B(p, 2\overline{D}) - B(p, \overline{D})$ which carries a smooth S^1-fibration. Consider the homotopy exact sequence

$$\cdots \to \pi_2(V_{1,0}) \to \pi_1(S^1) \to \pi_1(V_1) \to \pi_1(V_{1,0}) \to \cdots .$$

Since the base space $V_{1,0}$ of this fibration is connected and noncompact, we have $\pi_2(V_{1,0}) = 0$, and it follows that the S^1-fiber is incompressible in V_1, ie the map $\pi_1(S^1) \to \pi_1(V_1)$ is an injection; see eg [52].

On the other hand, let U be an ϵ-cylindrical plane contained in $B(p, 2\overline{D}) - B(p, \overline{D})$, let $x \in U$ be the center of the ϵ-cylindrical plane, and let F be the S^1-fiber of the fibration on V that passes through x. Then by Lemma 3.13, the fiber F is ϵ-close to the vertical S^1-factor in U, and hence F is also contained in U, and hence contained in $B(p, 2\overline{D}) - B(p, \overline{D})$. Note that the metric spheres $\partial B(p, \overline{D})$ and $\partial B(p, 2\overline{D})$ are diffeomorphic to 2-spheres, and $B(p, 2\overline{D}) - B(p, \overline{D})$ is diffeomorphic to $\mathbb{R} \times S^2$. So if $k = 0$, we see that $B(p, 2\overline{D}) - B(p, \overline{D}) - \mathcal{C}_1 - \cdots - \mathcal{C}_k$ is diffeomorphic to $\mathbb{R} \times S^2$, and if $k = 1$, it is diffeomorphic to $\mathbb{R}^3$, both of which have trivial fundamental groups. Therefore, F bounds a disk in $B(p, 2\overline{D}) - B(p, \overline{D})$. However, this contradicts with the above fact that the fiber F is incompressible. So $k \geq 2$.

Next, we show $k = 2$. Suppose not; then $k \geq 3$. The subset $K := B(p, 2\overline{D}) - B(p, \overline{D}) - \mathcal{C}_1 - \mathcal{C}_2 - \cdots - \mathcal{C}_k$ is covered by ϵ-cylindrical planes, so by Lemma 3.13 we can obtain a smooth S^1-fibration on a subset V containing K. Let C_1, C_2 be two circles of the fibration on V that are contained in $\mathcal{C}_1, \mathcal{C}_2$, respectively. Then by Lemma 3.13, C_i bounds a spanning disk D_i in $\mathcal{C}_i$ for $i = 1, 2$. Let $A \subset B(p, 2\overline{D}) - B(p, \overline{D})$ be an annulus bounded by C_1 and C_2 which is saturated by S^1-fibers. Then the union $S := D_1 \cup D_2 \cup A$ is a 2-sphere which is isotopic to the two metric spheres $\partial B(p, \overline{D})$ and $\partial B(p, 2\overline{D})$. In particular, S separates $\partial B(p, \overline{D})$ from $\partial B(p, 2\overline{D})$.

Since $k \geq 3$, it follows that $\mathcal{C}_3$ intersects with S at some ϵ-tip point y. First, y cannot be in the annulus A because A is covered by ϵ_0-cylindrical planes for a very small $\epsilon_0 > 0$. Second, y cannot be in either of the two spanning disks D_1 and D_2, because $\mathcal{C}_3$ is disjoint from $\mathcal{C}_i$ for $i = 1, 2$. So this contradiction proves $k = 2$. □

It is clear that at the volume scale, an ϵ-tip point is different from a point at which the manifold is an ϵ-cylindrical plane. We show in the following technical lemma that they can also be distinguished from each another at an even larger scale. Roughly speaking, the former point looks like a boundary point in the half-plane, while the latter looks like a point in the plane.

Lemma 3.15 *Let (M, g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let $p \in M$ be a fixed point. For any $\delta > 0$, there exists an $\epsilon > 0$ such that for any ϵ-tip point x, there exists a constant $0 < r_x < \delta d(p, x)$ such that there is no 2-strainer at x of size larger than r_x.*

Proof For x sufficiently far away from p, we have by volume comparison that $r(x)/d(p,x) \le \delta^2$. Take $r_x = r(x)/\delta$. Then $r_x < \delta d(p,x)$, and the rescaling $(M, r_x^{-2} g, x)$ is the scaling down of $(M, r^{-2}(x) g, x)$ by δ^{-1}. The scaling down of $(\mathbb{R}\times\text{cigar}, x_{\text{tip}})$ by δ^{-1} is $C\delta$-close to the two-dimensional upper half-plane $(\mathbb{R}\times\mathbb{R}_+, (0,0))$ in the pointed Gromov–Hausdorff sense. Take $\epsilon < \delta$. By the definition of ϵ-tip point, we see that $(M, r^{-2}(x) g, x)$ is ϵ-close to $(\mathbb{R}\times\text{cigar}, x_{\text{tip}})$. Therefore, $(M, r_x^{-2} g, x)$ is $C\epsilon\delta$-close in the Gromov–Hausdorff sense to the metric ball $B((0,0),1)$ in the two-dimensional upper half-plane $(\mathbb{R}\times\mathbb{R}_+, (0,0))$.

It is easy to see that there is no $(2,\epsilon_0)$-strainer at $(0,0)$ in $\mathbb{R}\times\mathbb{R}_+$ of size larger than 1, which is the same in $(M, r_x^{-2} g, x)$. Scaling back, it follows that there is no $(2,\epsilon_0)$-strainer at x in (M,g) of size larger than r_x. □

Recall that $S_\infty(M,p)$ is the metric space of equivalent classes of rays starting from p. We show in the following lemma that $S_\infty(M,p)$ is a closed interval $[0,\theta]$ for some $\theta \in [0,\pi)$. Moreover, the two chains of ϵ-solid cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ are two "edges" of the soliton (M,g), in the sense that the two rays γ_1 and γ_2 correspond to the two end points in $[0,\theta]$. We will show $\theta > 0$ in Corollary 3.43, so that the soliton is indeed a flying wing.

Lemma 3.16 *Let (M,g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let $p \in M$ be a fixed point. Then $S_\infty(M,p)$ is isometric to an interval $[0,\theta]$ for some $\theta \in [0,\pi)$.*

Moreover, for $i = 1,2$, and for any sequence of points $p_k^{(i)} \in \mathcal{C}_i$ such that $p_k^{(i)} \to \infty$ as $k \to \infty$, the minimizing geodesics $pp_k^{(i)}$ subsequentially converge to two rays γ_1 and γ_2 such that $[\gamma_1] = 0$ and $[\gamma_2] = \theta$, after possibly switching γ_1 and γ_2.

Proof Fix some $p \in M$. We shall say that two quantities $d_1, d_2 > 0$ are comparable if $C^{-1} d_1 < d_2 < C d_1$ for some universal constant $C > 0$. By the noneuclidean volume growth of (M,g), the asymptotic cone is a two-dimensional metric cone over $S_\infty(M,p)$, where $S_\infty(M,p)$ is a one-dimensional Alexandrov space, and hence is an interval or a circle. In the latter case, we can find a $(2,\epsilon)$-strainer of size comparable to $d(p,x)$ at any point $x \in M$. However, this is impossible at an ϵ-tip point by Lemma 3.15. So $S_\infty(M,p)$ is isometric to an interval $[0,\theta]$, for some $\theta \in [0,\pi]$. Moreover, we have $\theta < \pi$, because otherwise the manifold splits off a line and is isometric to $\mathbb{R}\times$cigar, which does not have strictly positive curvature.

First, we show that for any points going to infinity in $\mathcal{C}_1$, the minimizing geodesics from p to them subsequentially converge to rays that are in the same equivalence class in $S_\infty(M,p)$. Suppose by contradiction that this is not true. Then we can find two sequences of points $p_{1k}, p_{2k} \in \mathcal{C}_1$ going to infinity, such that the minimizing geodesics pp_{1k}, pp_{2k} converge to two rays σ_1, σ_2 with $\tilde{\measuredangle}(\sigma_1,\sigma_2) > 0$.

Let $\beta_k : [0,1] \to \mathcal{C}_1$ be a smooth curve joining p_{1k}, p_{2k}, which consists of ϵ-tip points. By Lemma 3.6 we may assume that $d(p,\beta_k(s))$ is monotone in s. Let $\theta_{ik}(s) = \tilde{\measuredangle}\sigma_i(d(p,\beta_k(s)))p\beta_k(s)$ for $i = 1,2$.

Then it is clear that as $k\to\infty$, $\theta_{1k}(0)$ and $\theta_{2k}(1)$ converge to 0, and $\theta_{1k}(1)$ and $\theta_{2k}(0)$ converge to $\widetilde{\measuredangle}(\sigma_1,\sigma_2)>0$. So for sufficiently large k, we have

$$\frac{\theta_{1k}(0)}{\theta_{2k}(0)}<\epsilon \quad \text{and} \quad \frac{\theta_{1k}(1)}{\theta_{2k}(1)}>\epsilon^{-1}.$$

By continuity, there exists a point $q_k=\beta_k(s_k)$ with $s_k\in(0,1)$ such that $\theta_{1k}(s_k)=\theta_{2k}(s_k)$, and hence

$$\widetilde{\measuredangle}\sigma_1(d(p,q_k))\,p\,q_k=\widetilde{\measuredangle}\sigma_2(d(p,q_k))\,p\,q_k.$$

Since $d(p,p_{1k}),d(p,p_{2k})\to\infty$, we have $d(p,q_k)\to\infty$. Therefore, after passing to a subsequence, pq_k converges to a ray σ_3, which satisfies $\widetilde{\measuredangle}(\sigma_3,\sigma_1)=\widetilde{\measuredangle}(\sigma_3,\sigma_2)$. Since $S_\infty(M,p)$ is an interval, this implies

$$\widetilde{\measuredangle}(\sigma_3,\sigma_1)=\widetilde{\measuredangle}(\sigma_3,\sigma_2)=\tfrac{1}{2}\widetilde{\measuredangle}(\sigma_1,\sigma_2).$$

So for sufficiently large k, we can find a $(2,\epsilon)$-strainer at q_k of size comparable to $d(q_k,p)$, which is impossible by Lemma 3.15 because q_k is an ϵ-tip point.

Now it remains to show that the ray σ_1 corresponds to one of the two end points in $S_\infty(M,p)=[0,\theta]$. This can be shown by a similar argument: Suppose $\theta>0$ and $[\sigma_1]\in(0,\theta)$. Then we can find $(2,\epsilon_0)$-strainers at a sequence of ϵ-tip points $x_k\to\infty$ at scales comparable to $d(p,x_k)$, contradicting Lemma 3.15. □

Now we prove our main result in this subsection. For $i=1,2$ we will construct two smooth curves $\Gamma_i:[0,\infty)\to\mathscr{C}_i$ tending to infinity, which are integral curves of either ∇f or $-\nabla f$, such that the rescaled manifold converges to $\mathbb{R}\times\text{cigar}$ pointed at the tip along Γ_i.

Lemma 3.17 *Let (M,g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. There exists a smooth curve $\Gamma_i:[0,\infty)\to\mathscr{C}_i$ which is an integral curve of ∇f or $-\nabla f$, such that $\lim_{s\to\infty}\Gamma_i(s)=\infty$, and for any sequence of points $x_i\to\infty$ along Γ_i, the pointed manifolds $(M,r^{-2}(x_i)g,x_i)$ smoothly converge to $(\mathbb{R}\times\text{cigar},r^{-2}(x_{\text{tip}})g_c,x_{\text{tip}})$.*

Proof Fix some point $p\in M$, and let $D_0>0$ be a large constant such that if f has a critical point (which must be unique), then $B(p,D_0)$ contains the critical point. We will use $\epsilon(D)>0$ denote all positive constants depending on D such that $\epsilon\to 0$ as $D\to\infty$.

Take a sequence of ϵ_k-tip points $p_k\in\mathscr{C}_1$ going to infinity, where $\epsilon_k\to 0$. Assume $p_k\notin B(p,D_0)$ for all k. Denote by $\gamma_{p_k}:[0,s_k]\to M$ the integral curve of $-\nabla f$ starting from p_k, where $s_k\in(0,\infty]$ is the smallest value of s such that $\gamma_{p_k}(s)\in\partial B(p,D_0)$. It is easy to see that $s_k\to\infty$ as $k\to\infty$. By Lemma 3.5 we see that $\gamma_{p_k}([0,s_k])\subset\mathscr{C}_1$ and if $\gamma_{p_k}(s)\notin B(p,D)$ for some $D>0$, then $\gamma_{p_k}(s)$ is an $\epsilon(D)$-tip point, where $\epsilon(D)\to 0$ as $D\to\infty$ is independent of k.

First, assume s_k is finite for all k. Let $q_k=\gamma_{p_k}\in\mathscr{C}_1\cap\partial B(p,D_0)$. Then, after passing to a subsequence, q_k converges to a point $q_\infty\in\mathscr{C}_1\cap\partial B(p,D_0)$. Denote by $\widetilde{\gamma}_{q_k}$ the integral curve of ∇f starting from q_k.

Then $\widetilde{\gamma}_{q_k}$ converges to the integral curve $\Gamma_1\colon[0,\infty)\to\mathcal{C}_1$ of ∇f starting from q_∞, which satisfies all the assertions.

Now assume $s_k=\infty$ for some $k=k_0$. Since the critical point of f (if it exists) is contained in $B(p,D_0)$, this implies that $\gamma_{p_{k_0}}(s)\to\infty$ as $s\to\infty$. Therefore, we may take $\gamma_{p_{k_0}}$ to be Γ_1, which is an integral curve of $-\nabla f$ and satisfies all assertions as a consequence of Lemma 3.5. By the same argument, we can find $\Gamma_2\subset\mathcal{C}_2$ satisfying the assertions. □

Remark 3.18 When (M,g) is isometric to $\mathbb{R}\times$cigar, and the potential function f could be a nonconstant linear function in the $\mathbb{R}$-direction, then one of Γ_1 and Γ_2 is the integral curve of ∇f and the other is of $-\nabla f$. However, if (M,g) has strictly positive curvature, we will show in Theorem 3.31 that f has a unique critical point, and Γ_1 and Γ_2 are both integral curves of ∇f.

Let $\Gamma=\Gamma_1([0,\infty))\cup\Gamma_2([0,\infty))$. The following lemma shows that the distance from sufficiently far away points to the subset Γ must be achieved at the interior, so that the minimizing geodesics connecting them to Γ are orthogonal to Γ by the first variation formula.

Lemma 3.19 *Under the same assumptions as in Lemma 3.17, suppose there are a sequence of points $p_i\to\infty$ and a constant $s_0>0$ such that one of the following conditions holds*:

(1) $d(p_i,\Gamma)=d(p_i,\Gamma_1([0,s_0])\cup\Gamma_2([0,s_0]))$,

(2) $d(p_i,\Gamma_1)=d(p_i,\Gamma_1([0,s_0])$, *or*

(3) $d(p_i,\Gamma_2)=d(p_i,\Gamma_2([0,s_0]))$.

*Then M is isometric to $\mathbb{R}\times$*cigar.

Proof We prove the lemma under condition (1), and the two other cases follow in the same way. We will show that after passing to a subsequence, the minimizing geodesics from p to p_i converge to a geodesic ray γ such that $\measuredangle(\gamma,\gamma_i)\ge\frac{1}{2}\pi$, where γ_i are the two rays corresponding to the two end points in $S_\infty(M,p)$; see Lemma 3.16. Then by Lemma 3.16 this implies $S_\infty(M,p)=[0,\pi]$ and the assertion of the lemma follows immediately. In the proof we use ϵ to denote a general positive constant that goes to zero as $i\to\infty$.

Let $q_i\in\Gamma$ be such that $d(p_i,\Gamma([-s_0,s_0]))=d(p_i,q_i)$. Choose a sequence of points $o_i\in\Gamma_1$ with $d(q_i,o_i)=\alpha_i d(p_i,q_i)$ and such that $\alpha_i\to0$ and $d(q_i,o_i)\to\infty$. Since $\alpha_i\to0$, we have $\widetilde{\measuredangle}o_ip_iq_i\to0$, and hence

$$\widetilde{\measuredangle}o_iq_ip_i+\widetilde{\measuredangle}q_io_ip_i\ge\pi-\epsilon.$$

Since $d(p_i,o_i)\ge d(p_i,\Gamma)=d(p_i,q_i)$, the segment o_iq_i is the longest in the comparison triangle $o_ip_iq_i$, so it must be opposite to the largest comparison angle, ie

$$\widetilde{\measuredangle}o_iq_ip_i\ge\widetilde{\measuredangle}q_io_ip_i.$$

So the last two inequalities imply

$$\widetilde{\measuredangle} o_i q_i p_i \geq \tfrac{1}{2}\pi - \epsilon. \tag{3-4}$$

By Lemma 3.16 the minimizing geodesics po_i converge to the ray γ_1. After passing to a subsequence we may assume the pp_i converge to a ray γ. Then by the boundedness of $\{q_i\}$ and (3-4), it is easy to see that $\measuredangle(\gamma_1, \gamma) \geq \frac{1}{2}\pi$. By a symmetric argument, we also have $\measuredangle(\gamma_2, \gamma) \geq \frac{1}{2}\pi$. So $\measuredangle(\gamma_1, \gamma_2) \geq \pi$ by Lemma 3.17, which implies a splitting of (M, g), and hence it is isometric to $\mathbb{R} \times$ cigar. □

3.3 Quadratic curvature decay

The main result in this subsection is the following theorem of the quadratic curvature decay, which corresponds to the upper bound in Theorem 1.7. We show that there is a uniform $C > 0$ such that the scalar curvature has the upper bound $R \leq C/d^2(\cdot, \Gamma)$, where $\Gamma = \Gamma_1([0,\infty)) \cup \Gamma_2([0,\infty))$, and Γ_1 and Γ_2 are from Lemma 3.17.

Theorem 3.20 *Let (M, g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. There exists $C > 0$ such that for any $x \in M \setminus \Gamma$,*

$$R(x) \leq \frac{C}{d^2(x, \Gamma)}.$$

Proof First, we introduce some constants that will be used throughout this subsection, and we may further adjust their values. The dependence of these constants is subject to the order

$$\delta, \quad \alpha, \quad D_2, \quad \epsilon, \quad D_1, \quad \omega, \quad C,$$

such that each constant is chosen depending only on the preceding ones. More precisely, we will choose D_1 and D_2 sufficiently large that δ can be arbitrarily close to zero.

Let $\epsilon > 0$ be some very small number, and fix a point $p_0 \in M$. Then by Lemmas 3.3, 3.14 and 3.17 we see that there are $D_1 > D_2 > 0$ such that the following holds: First, the soliton is an ϵ-cylindrical plane at all points x which satisfies $d(x, p_0) \geq \frac{1}{2} D_1$ and $d(x, \Gamma) \geq \frac{1}{2} D_2\, r(x)$. Second, the soliton is ϵ-close to $(\mathbb{R} \times \text{cigar}, r^{-2}(x_0) g_c, x_0)$ for some x_0 for which $d_{g_c}(x_{\text{tip}}, x_0) < 10 D_2$.

By compactness we may choose $C > 0$ large enough, so that Theorem 3.20 holds in $B(p_0, D_1)$. Moreover, for points $x \in M \setminus B(p_0, D_1)$ which satisfy $d(x, p_0) \geq D_1$ and $d(x, \Gamma) \leq D_2\, r(x)$, by the definition of the volume scale and a volume comparison argument, it is clear that there exists $\omega > 0$ such that

$$\operatorname{vol}(B(\tilde{x}, d(x, \Gamma))) \geq \omega\, d(x, \Gamma)^3.$$

So Theorem 3.20 holds at x by Perelman's curvature estimate; see Lemma 2.15.

Therefore, from now on we assume that

$$d(x, p_0) \geq D_1 \quad \text{and} \quad d(x, \Gamma) \geq D_2\, r(x). \tag{3-5}$$

So the metric ball $B\big(x, \frac{1}{2}d(x,\Gamma)\big)$ is covered by ϵ-cylindrical planes. By the S^1-fibration Lemma 3.13, we see that there is an open subset $U \supset B\big(x, \frac{1}{2}d(x,\Gamma)\big)$ which has a global S^1-fibration, and the S^1-fibers are incompressible.

The following lemma gives a uniform lower bound for the volume ratio on the universal coverings of U. So by applying Perelman's curvature estimate (Lemma 2.15) in the universal covering we obtain the inequality in Theorem 3.20 at a lift of x, which implies the same inequality at x. So Theorem 3.20 reduces to the following lemma by Perelman's curvature estimates (Lemma 2.15). □

Lemma 3.21 *There exists $\omega > 0$ such that in the universal cover $\widetilde{U}$ of U, we have $B\big(\tilde{x}, \frac{1}{2}d(x,\Gamma)\big) \Subset \widetilde{U}$, and*

$$\mathrm{vol}\big(B\big(\tilde{x}, \tfrac{1}{2}d(x,\Gamma)\big)\big) \geq \omega\, d^3(x,\Gamma),$$

where $\tilde{x} \in \widetilde{U}$ is a lift of x.

Proof To prove the lemma, we will follow the idea in [6, Lemma 2.2] to construct a $(3,\delta)$-strainer near $\tilde{x}$ of size comparable to $d(x,\Gamma)$ for some small $\delta > 0$, then use this to obtain a lower bound on the volume ratio as in [6, Lemma 2.2] and [20, Theorem 10.8.18]. We first construct a $(2,\delta)$-strainer at $\tilde{x}$ using Claims 3.22 and 3.23.

Claim 3.22 *There exist $p, q \in M$ such that the triple of points $\{p, q, x\}$ forms a $\frac{1}{6}\pi$-triangle and $d(x,p) = d(x,\Gamma)$. Here by a μ-triangle we mean that $\mu > 0$ and*

$$\tilde{\measuredangle} xpq, \tilde{\measuredangle} xqp, \tilde{\measuredangle} pxq > \mu.$$

Proof Let $\epsilon \in \big(0, \frac{1}{100}\big)$ be some fixed small number. Let $p \in \Gamma$ to be the closest point to x on Γ. Then by Lemma 3.19, p is an interior point in Γ, and p is an ϵ-tip point by Lemma 3.17. Therefore, it is easy to see that the minimizing geodesic px and $\psi_{p*}(\partial_r)$ is almost orthogonal at p, where ψ_p is the ϵ-isometry to $(\mathbb{R} \times \text{cigar}, r^{-2}(x_{\text{tip}})g_c, x_{\text{tip}})$ at p, ie

$$\big|\measuredangle(px, \psi_{p*}(\partial_r)) - \tfrac{1}{2}\pi\big| \leq \epsilon. \tag{3-6}$$

Note that $d(x,\Gamma) > D_2\, r(x)$ and $d(x,p) = d(x,\Gamma)$. We claim that

$$d(x,\Gamma) > \tfrac{1}{10} D_2 r(p), \tag{3-7}$$

because otherwise we may choose D_1 sufficiently large (depending on D_2) so that $d(x,p) \leq \frac{1}{10}D_2 r(p)$ must imply $r(p) \leq 10 r(x)$, which implies $d(x,p) \leq D_2\, r(x)$, a contradiction. So (3-7) holds.

By Lemma 3.17 we can take D_1 to be large so that ϵ is sufficiently small, where ϵ is determined by D_2 such that the following holds: By (3-6) and the ϵ-closeness to $\mathbb{R} \times$ cigar in the region containing the $B(p, D_2\, r(p))$, we have for all points $y \in B(p, D_2\, r(p))$ that

$$\tilde{\measuredangle} ypx' \leq \tfrac{1}{2}\pi + \tfrac{1}{100} \leq \tfrac{2}{3}\pi, \tag{3-8}$$

where x' is a point on the minimizing geodesic px such that

$$d(p,x') = \tfrac{1}{10} D_2 r(p) < d(x,\Gamma).$$

Now let $q \in \Gamma$ be a point such that $d(q,p) = d(x,\Gamma)$. By (3-7) we can choose a point q' on the minimizing geodesic pq such that

$$d(p,q') = \tfrac{1}{10} D_2\, r(p) < d(x,\Gamma).$$

Then using (3-8) we obtain

$$\widetilde{\measuredangle} q' p x' \le \tfrac{1}{2}\pi + \tfrac{1}{100} \le \tfrac{2}{3}\pi.$$

By the monotonicity of angles, the last three inequalities imply

$$\widetilde{\measuredangle} q p x \le \tfrac{1}{2}\pi + \tfrac{1}{100} \le \tfrac{2}{3}\pi.$$

Note $d(x,q) \ge d(x,\Gamma) = d(x,p) = d(p,q)$. The segment $|\tilde{x}\tilde{q}|$ is the longest in the comparison triangle $\widetilde{\triangle}\tilde{x}\tilde{q}\tilde{p}$, and thus it must be opposite to the largest comparison angle, ie

$$\widetilde{\measuredangle} q p x \ge \widetilde{\measuredangle} q x p = \widetilde{\measuredangle} p q x.$$

This combined with $\widetilde{\measuredangle} q p x + \widetilde{\measuredangle} q x p + \widetilde{\measuredangle} p q x = \pi$ implies

$$\widetilde{\measuredangle} q p x \ge \widetilde{\measuredangle} q x p = \widetilde{\measuredangle} p q x \ge \tfrac{1}{6}\pi.$$

So $\{p,q,x\}$ forms a $\frac{1}{6}\pi$-triangle. □

Claim 3.23 *There exists $\alpha > 0$ such that when D_1 and D_2 are sufficiently large, the following holds for all $x \in M$ satisfying (3-5): There is a point $x' \in B\big(x, \frac{1}{10} d(x,\Gamma)\big)$ such that there is a $(2,\delta)$-strainer at x' of size $\alpha d(x,\Gamma)$.*

Proof Suppose this is not true. Then there is a sequence of points $x_i \in M$ going to infinity and $d(x_i,\Gamma) \to \infty$ such that the claim fails. Let p_i, q_i be points from Claim 3.22 which form a $\frac{1}{6}\pi$-triangle together with x_i, and such that $d(p_i,x_i) = d(x_i,\Gamma)$.

After passing to a subsequence, the pointed manifolds $(M, d^{-2}(x_i,\Gamma)g, x_i)$ converge to a complete noncompact length space (X, d_X), which is an Alexandrov space with nonnegative curvature [21]. The triples (p_i,q_i,x_i) converge to a triple $(p_\infty,q_\infty,x_\infty)$ in the limit space, which forms a $\frac{1}{6}\pi$-triangle. So the Alexandrov dimension of the limit space is at least two, because otherwise X must be isometric to a ray or a line, which does not contain any $\frac{1}{6}\pi$-triangles.

Since the set of $(2,\delta)$-strainers is open and dense in an Alexandrov space of dimension 2, we can find a point $x' \in B_{d_X}\big(x, \frac{1}{10}\big)$ such that there is a $(2,\delta)$-strainer at x' of size α for some $\alpha > 0$. This induces a contradiction for sufficiently large i. □

Assume we can show that the volume of $B(x',1)$ is bounded below. Then by a volume comparison using that $x' \in B\big(x, \frac{1}{10} d(x,\Gamma)\big)$, this would imply a lower bound on the volume of $B(x,1)$. Therefore, we may assume without loss of generality that there is a $(2,\delta)$-strainer (a_1,b_1,a_2,b_2) at x of size $\alpha d(x,\Gamma)$.

From now on we will work with the rescaled metric $h = d^{-2}(x,\Gamma)g$ and bound the 1-ball around a lift of x in the universal cover from below by a universal constant.

Let $\pi : \widetilde{U} \to U$ be a universal covering, and $\tilde{x}, \tilde{a}_i, \tilde{b}_i$ be lifts of x, a_i, b_i in the universal cover $\widetilde{U}$ such that

$$d(\tilde{x}, \tilde{a}_i) = d(x, a_i) = d(\tilde{x}, \tilde{b}_i) = d(x, b_i) = \alpha. \tag{3-9}$$

Then since the covering map π is 1-Lipschitz, we have

$$d(\tilde{a}_i, \tilde{b}_j) \geq d(a_i, b_j), \quad d(\tilde{a}_i, \tilde{a}_j) \geq d(a_i, a_j), \quad d(\tilde{b}_i, \tilde{b}_j) \geq d(b_i, b_j). \tag{3-10}$$

So the comparison angles between $\tilde{x}, \tilde{a}_i, \tilde{b}_j$ at $\tilde{x}$ are at least as large as those between x, a_i, b_j at x, ie

$$\tilde{\measuredangle} \tilde{a}_i \tilde{x} \tilde{b}_j \geq \tilde{\measuredangle} a_i x b_j, \quad \tilde{\measuredangle} \tilde{a}_i \tilde{x} \tilde{b}_j \geq \tilde{\measuredangle} a_1 x a_2, \quad \tilde{\measuredangle} \tilde{a}_i \tilde{x} \tilde{b}_j \geq \tilde{\measuredangle} b_1 x b_2.$$

So $(\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2)$ is a $(2, \delta)$-strainer at $\tilde{x}$.

Next, we will extend the $(2, \delta)$-strainer to a $\left(2+\frac{1}{2}, \delta\right)$-strainer at $\tilde{x}$. Since the S^1-fiber in U is incompressible, we can find a sequence $\tilde{x}_i$ of lifts of x that is unbounded. We may assume that the consecutive distances of $\{\tilde{x}_i\}$ are at most $D_2^{-1/2}$. Because otherwise, there would be two points $\tilde{x}_i, \tilde{x}_j$ such that $d(\tilde{x}_i, \tilde{x}_j) \geq D_2^{-1/2}$. Rescaling back to the metric g and by Lemma 3.13, we see that the length of the circle in the ϵ-circle plane at x is at least $D_2^{-1/2} d(x, \Gamma)$. This implies $r(x) \geq D_2^{-1/2} d(x, \Gamma)$, which contradicts our assumption (3-5). So we can find an $i \in \mathbb{N}$ such that with $\tilde{y} = \tilde{x}_i$, we have

$$|d(\tilde{y}, \tilde{x}) - D_2^{-1/4}| < D_2^{-1/2}.$$

Claim 3.24 *The tuple $(\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2, \tilde{y})$ is a $\left(2+\frac{1}{2}, \delta\right)$-tuple at $\tilde{x}$ of size at least $D_2^{-1/4} - D_2^{-1/2}$.*

Proof Note that in the triangle $\triangle \tilde{y} \tilde{x} \tilde{a}_i$, the segment $|\tilde{y} \tilde{a}_i|$ has the longest length, and thus must be opposite to the largest comparison angle, ie

$$\tilde{\measuredangle} \tilde{a}_i \tilde{x} \tilde{y} \geq \tilde{\measuredangle} \tilde{x} \tilde{y} \tilde{a}_i.$$

Since $d(\tilde{x}, \tilde{y}) \to 0$ as $D_2 \to \infty$, we find

$$\tilde{\measuredangle} \tilde{y} \tilde{a}_i \tilde{x} < \delta. \tag{3-11}$$

We also have

$$\tilde{\measuredangle} \tilde{y} \tilde{a}_i \tilde{x} + \tilde{\measuredangle} \tilde{a}_i \tilde{x} \tilde{y} + \tilde{\measuredangle} \tilde{x} \tilde{y} \tilde{a}_i = \pi. \tag{3-12}$$

So the last three inequalities imply $\tilde{\measuredangle} \tilde{a}_i \tilde{x} \tilde{y} \geq \frac{1}{2}\pi - \delta$. The same is true with $\tilde{a}_i$ replaced by $\tilde{b}_i$. So

$$\tilde{\measuredangle} \tilde{a}_i \tilde{x} \tilde{y} \geq \tfrac{1}{2}\pi - \delta \quad \text{and} \quad \tilde{\measuredangle} \tilde{b}_i \tilde{x} \tilde{y} \geq \tfrac{1}{2}\pi - \delta, \tag{3-13}$$

and hence the claim holds. □

Claim 3.25 *The tuple $(\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2, \tilde{x})$ is a $\left(2+\frac{1}{2}, \delta\right)$-tuple at $\tilde{y}$ of size at least $D_2^{-1/4} - D_2^{-1/2}$.*

Proof Since $|d(\tilde{y}, \tilde{a}_i) - d(\tilde{x}, \tilde{a}_i)| < D_2^{-1/4}$ and $|d(\tilde{y}, \tilde{b}_i) - d(\tilde{x}, \tilde{b}_i)| < D_2^{-1/4}$, we see that $(\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2)$ is a $(2, \delta)$-strainer at $\tilde{y}$ of size at least $\alpha - D_2^{-1/4} - 2D_2^{-1/2}$. We may assume that D_2 is sufficiently large that this is at least $\frac{1}{2}\alpha > D_2^{-1/4} - D_2^{-1/2}$.

By metric comparison we have

$$\widetilde{\measuredangle}\tilde{a}_i\tilde{x}\tilde{y}+\widetilde{\measuredangle}\tilde{b}_i\tilde{x}\tilde{y}+\widetilde{\measuredangle}\tilde{a}_i\tilde{x}\tilde{b}_i\le 2\pi,$$

which by (3-13) and $\widetilde{\measuredangle}\tilde{a}_i\tilde{x}\tilde{b}_i\ge\pi-\delta$ implies

$$\widetilde{\measuredangle}\tilde{a}_i\tilde{x}\tilde{y}\le\tfrac{1}{2}\pi+\delta\quad\text{and}\quad\widetilde{\measuredangle}\tilde{b}_i\tilde{x}\tilde{y}\le\tfrac{1}{2}\pi+\delta.$$

Combining this with (3-12) and (3-11) we obtain

$$\widetilde{\measuredangle}\tilde{x}\tilde{y}\tilde{a}_i\ge\tfrac{1}{2}\pi-\delta\quad\text{and}\quad\widetilde{\measuredangle}\tilde{x}\tilde{y}\tilde{b}_i\ge\tfrac{1}{2}\pi-\delta.$$

So the claim holds. □

Now take $\tilde{m}$ be the midpoint of a minimizing geodesic between $\tilde{y}$ and $\tilde{x}$.

Claim 3.26 *The tuple $(\tilde{a}_1,\tilde{a}_2,\tilde{b}_1,\tilde{b}_2,\tilde{y},\tilde{x})$ is a $(3,\delta)$-strainer at $\tilde{m}$ of size at least $\frac{1}{2}D_2^{-1/4}-D_2^{-1/2}$.*

Proof First, by the monotonicity of comparison angles we have

$$\widetilde{\measuredangle}\tilde{m}\tilde{x}\tilde{a}_i\ge\widetilde{\measuredangle}\tilde{y}\tilde{x}\tilde{a}_i\ge\tfrac{1}{2}\pi-\delta\quad\text{and}\quad\widetilde{\measuredangle}\tilde{m}\tilde{x}\tilde{b}_i\ge\widetilde{\measuredangle}\tilde{y}\tilde{x}\tilde{b}_i\ge\tfrac{1}{2}\pi-\delta.$$

Then repeating the same argument as in Claim 3.25 replacing $\tilde{y}$ by $\tilde{m}$, we see that

$$\widetilde{\measuredangle}\tilde{a}_i\tilde{m}\tilde{x},\widetilde{\measuredangle}\tilde{b}_i\tilde{m}\tilde{x}>\tfrac{1}{2}\pi-\delta.$$

Replacing $\tilde{x}$ by $\tilde{y}$, similarly we can obtain

$$\widetilde{\measuredangle}\tilde{a}_i\tilde{m}\tilde{y},\widetilde{\measuredangle}\tilde{b}_i\tilde{m}\tilde{y}>\tfrac{1}{2}\pi-\delta.$$

Moreover, as before we can see that $(\tilde{a}_1,\tilde{a}_2,\tilde{b}_1,\tilde{b}_2)$ is a $(2,\delta)$-strainer at $\tilde{m}$. Finally, $\widetilde{\measuredangle}\tilde{y}\tilde{m}\tilde{x}=\pi$ is trivially true. So the claim holds. □

Using the $(3,\delta)$-strainer in Claim 3.26, one can construct a 100-bilipschitz map $f:B(\tilde{m},\lambda D_2^{-1/4})\to\mathbb{R}^3$ for some sufficiently small λ as in [6, Lemma 2.2(i)] and [20, Theorem 10.8.18], which implies

$$\mathrm{vol}(B(\tilde{m},\lambda D_2^{-1/4}))>c(\lambda D_2^{-1/4})^3$$

for some universal $c>0$. This implies Lemma 3.21 by volume comparison. □

3.4 Existence of a critical point

The main result in this subsection is Theorem 3.31, which proves the existence of the maximum point of the scalar curvature, which is also the critical point of the potential function.

In Lemmas 3.27 and 3.28, we study the geometry of level sets of the potential function f. To start, we first note that the second fundamental form of a level set of f satisfies

$$\mathrm{II}=-\frac{\nabla^2 f}{|\nabla f|}\bigg|_{f^{-1}(a)}=-\frac{\mathrm{Ric}}{|\nabla f|}\bigg|_{f^{-1}(a)}\le 0. \tag{3-14}$$

Recall the Gauss equation, that for a manifold N embedded in a Riemannian manifold (M, g), the curvature tensor Rm_N of N with induced metric can be expressed using the second fundamental form and Rm_M, the curvature tensor of M:

$$\langle \mathrm{Rm}_N(u,v)w,z\rangle = \langle \mathrm{Rm}_M(u,v)w,z\rangle + \langle \mathrm{II}(u,z), \mathrm{II}(v,w)\rangle - \langle \mathrm{II}(u,w), \mathrm{II}(v,z)\rangle.$$

So (3-14) implies that the level sets of f with induced metric have positive curvature.

More precisely, for a level set of f at an ϵ-tip point, Lemma 3.27 shows that at a point that is sufficiently far away from this ϵ-tip point in the level set, the soliton is close to $\mathbb{R}^2 \times S^1$ under a suitable rescaling. We show this by a limiting argument.

Lemma 3.27 *Let (M, g, f) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. For any $\delta > 0$ and $C_0 > 0$, there exists $\overline{D} > 0$ such that the following holds:*

For all $D \geq \overline{D}$, there exists $\epsilon > 0$ such that the following holds: Suppose $q \in M$ is an ϵ-tip point with $|\nabla f|(q) \geq C_0^{-1}$, and let $\Sigma = f^{-1}(f(q))$ be the level set passing through q. Suppose also that $z \in \Sigma$ is a point with $d_\Sigma(q, z) = DR^{-1/2}(q)$, where d_Σ is the length metric of the induced metric on Σ. Then the soliton (M, g) is a δ-cylindrical plane at z.

Proof Suppose this is false. Then there is some $\delta > 0$ such that for any large $\overline{D}$, there exist a constant $D > \overline{D}$, a sequence of constants $\epsilon_k \to 0$, and a sequence of ϵ_k-tip points $q_k \to \infty$, and a sequence of points $z_k \in \Sigma_k := f^{-1}(f(q_k))$ with $d_{\Sigma_k}(q_k, z_k) = DR^{-1/2}(q_k)$ such that the soliton (M, g) is not a δ-cylindrical plane at z_k.

In the following we will show that when $\overline{D}$ is large enough, the level sets Σ_k under suitable rescalings converge to a level set of a smooth function on $\mathbb{R} \times \text{cigar}$, and z_k converge to a point at which $\mathbb{R} \times \text{cigar}$ is a $\frac{1}{2}\delta$-cylindrical plane, which will imply a contradiction for sufficiently large k. We now divide the discussion into three cases depending on the limit of $R(q_k)$. To start, note that $R + |\nabla f|^2 = C_1^2$ for some $C_1 > 0$. So $R \leq C_1^2$ and $|\nabla f| \leq C_1$.

Case 1 ($\limsup_{k\to\infty} R(q_k) > C^{-2} > 0$ for some $C > 0$ which may depend on the sequence) In this case we can deduce from the soliton equation $\nabla^2 f = \mathrm{Ric}$ that derivatives of order at least two of the functions $\tilde{f}_k := f - f(q_k)$ are uniformly bounded. Moreover, at q_k we have $\tilde{f}_k(q_k) = 0$ and

$$C_0^{-1} \leq |\nabla \tilde{f}_k|(q_k) = |\nabla f|(q_k) \leq C_1.$$

Let $\tilde{\nabla}$ denote the covariant derivatives of $\tilde{g}_k = \frac{1}{16} R(q_k) g$, then we have that $|\tilde{\nabla} \tilde{f}_k|_{\tilde{g}_k} = 4R^{-1/2}(q_k)|\nabla f|$ is uniformly bounded above, and in particular at q_k we have

$$|\tilde{\nabla} \tilde{f}_k|(q_k) \geq 4C_0^{-1} R^{-1/2}(q_k) \geq 4C_0^{-1} C_1^{-1}.$$

So after passing to a subsequence we may assume that the functions $\tilde{f}_k$ converge to a smooth function f_∞ on $\mathbb{R} \times \text{cigar}$, which satisfies $f_\infty(x_{\text{tip}}) = 0$, $\nabla^2 f_\infty = \mathrm{Ric}$ and

$$|\nabla f_\infty|(x_{\text{tip}}) \geq 4C_0^{-1} C_1^{-1}.$$

Note that C_0 and C_1 are constants that only depend only on the soliton but not the sequence.

Since $\nabla^2 f_\infty = \mathrm{Ric}$, by the uniqueness of the potential function on the cigar soliton we see that f_∞ is the sum of the potential function on cigar which vanishes at the tip and a linear function along the $\mathbb{R}$-factor whose derivative has absolute value at least $4C_0^{-1}C_1^{-1}$ and vanishes at x_{tip}. In particular, 0 is a regular value of f_∞ and the level set $\Sigma_\infty := f_\infty^{-1}(0)$ is a noncompact complete rotationally symmetric 2D manifold.

Therefore, after passing to a subsequence, as the manifolds $\bigl(M, \frac{1}{16}R(q_k)g, q_k\bigr)$ smoothly converge to $(\mathbb{R}\times\mathrm{cigar}, g_c, x_{\mathrm{tip}})$, the level sets $\bigl(\Sigma_k, \frac{1}{16}R(q_k)g_{\Sigma_k}, q_k\bigr)$ of $\tilde{f}_k$ with the induced metrics smoothly converge to the level set $(\Sigma_\infty, g_{\Sigma_\infty}, x_{\mathrm{tip}})$ of f_∞, and $z_k \in \Sigma_k$ converge to a point $z_\infty \in \Sigma_\infty$ with $d_{\Sigma_\infty}(x_{\mathrm{tip}}, z_\infty) = \frac{1}{4}D$. Since $\mathbb{R}\times\mathrm{cigar}$ is a $\frac{1}{2}\delta$-cylindrical plane at z_∞ when $\bar{D}$ is sufficiently large depending on δ and $4C_0^{-1}C_1^{-1}$, we obtain a contradiction for all sufficiently large k.

Case 2 $(\lim_{k\to\infty} R(q_k) = 0)$ Consider the rescaled metrics $\tilde{g}_k = \frac{1}{16}R(q_k)g$ and the rescaled functions

$$\tilde{f}_k := \frac{f - f(q_k)}{4R^{-1/2}(q_k)},$$

then $\tilde{f}_k$ satisfies $\tilde{f}_k(q_k) = 0$ and

$$\tilde{\nabla}^2 \tilde{f}_k = \nabla^2 \tilde{f}_k = \frac{\nabla^2 f}{4R^{-1/2}(q_k)} = \frac{\mathrm{Ric}}{4R^{-1/2}(q_k)} = \frac{\widetilde{\mathrm{Ric}}}{4R^{-1/2}(q_k)}, \tag{3-15}$$

and also

$$|\tilde{\nabla}\tilde{f}_k|_{\tilde{g}_k} = 4R^{-1/2}(q_k)|\nabla \tilde{f}_k| = |\nabla f| \le C_1.$$

In particular, at q_k we have

$$|\tilde{\nabla}\tilde{f}_k|_{\tilde{g}_k}(q_k) = |\nabla f|(q_k) \in [C_0^{-1}, C_1]. \tag{3-16}$$

By (3-15) and (3-16), the derivatives of $\tilde{f}_k$ are uniformly bounded. Thus there is a subsequence of $\tilde{f}_k$ converging to a smooth function f_∞ on $\mathbb{R}\times\mathrm{cigar}$ with $f_\infty(x_{\mathrm{tip}}) = 0$.

By (3-15) and (3-16) we have $\nabla^2 f_\infty = 0$ and $|\nabla f_\infty|(x_{\mathrm{tip}}) > 0$. So f_∞ is a nonconstant linear function in the $\mathbb{R}$-direction. In particular, 0 is a regular value of f_∞, and the level set $\Sigma_\infty := f_\infty^{-1}(0)$ is equal to $\{a\}\times\mathrm{cigar} \subset \mathbb{R}\times\mathrm{cigar}$ for some $a \in \mathbb{R}$.

Therefore, after passing to a subsequence, as the manifolds $\bigl(M, \frac{1}{16}R(q_k)g, q_k\bigr)$ smoothly converge to $(\mathbb{R}\times\mathrm{cigar}, g_c, x_{\mathrm{tip}})$, the level sets $\bigl(\Sigma_k, \frac{1}{16}R(q_k)g_{\Sigma_k}, q_k\bigr)$ of $\tilde{f}_k$ with induced metrics smoothly converge to the level set $(\Sigma_\infty, g_{\Sigma_\infty}, x_{\mathrm{tip}})$ of f_∞, and the points $z_k \in \Sigma_k$ converge to a point $z_\infty \in \Sigma_\infty$ with $d_{\Sigma_\infty}(x_{\mathrm{tip}}, z_\infty) = \frac{1}{4}D$. So it follows when $\bar{D}$ is sufficiently large that $\mathbb{R}\times\mathrm{cigar}$ is a $\frac{1}{2}\delta$-cylindrical plane at z_∞. This is a contradiction for large k. ◻

The next lemma shows that for a point at which the soliton looks sufficiently like the cylindrical plane $\mathbb{R}^2\times S^1$, the level set of f passing through it looks like the cylinder $\mathbb{R}\times S^1$. We prove this lemma by a limiting argument.

Lemma 3.28 *Let (M, g, f) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. For any $\delta > 0$ and $C_0 > 0$, there exists $\epsilon > 0$ such that if M is an ϵ-cylindrical plane at $x \in M$ and $r(x) \ge C_0^{-1}$, then the level set $f^{-1}(f(x))$ of f passing through x is a δ-neck at x at scale $r(x)$.*

Proof Suppose the conclusion is not true. Then for some $\delta > 0$ and $C_0 > 0$, there is a sequence of points $x_i \in M$ at which (M, g) is an ϵ_i-cylindrical plane, with $\epsilon_i \to 0$, such that $\Sigma_i := f^{-1}(f(x_i))$ is not a δ-neck at x_i at scale $r(x_i)$.

Consider the rescalings of the metrics $\tilde{g}_i := r^{-2}(x_i)g$, and the rescalings of the functions

$$\tilde{f}_i := \frac{f_i - f_i(x_i)}{r(x_i)}.$$

We have $\tilde{f}_i(x_i) = 0$ and

$$\tilde{\nabla}^{k+2}\tilde{f}_i = r^{-1}(x_i)\tilde{\nabla}^{k+2} f_i = r^{-1}(x_i)\tilde{\nabla}^k(\tilde{\nabla}^2 f_i) = r^{-1}(x_i)\tilde{\nabla}^k(\widetilde{\mathrm{Ric}}) \quad \text{for } k \ge 0,$$

and also

$$\tilde{\nabla}\tilde{f}_i = r^2(x_i)\nabla\tilde{f}_i = r(x_i)\nabla f_i.$$

Therefore, using $r(x_i) \ge C_0^{-1}$ we obtain

$$|\tilde{\nabla}^{k+2}\tilde{f}_i|_{\tilde{g}_i} = r^{-1}(x_i)|\tilde{\nabla}^k(\widetilde{\mathrm{Ric}})|_{\tilde{g}_i} \le C_0|\tilde{\nabla}^k(\widetilde{\mathrm{Ric}})|_{\tilde{g}_i},$$

which goes to zero for each $k \ge 0$ since $\epsilon_i \to 0$. Note that $R + |\nabla f|^2 = C_1^2$ for some $C_1 > 0$, and we also have $|\nabla f| \le C_1$ and

$$|\tilde{\nabla}\tilde{f}_i|_{\tilde{g}_i} = |\nabla f| \le C_1.$$

In particular, since $r(x_i) \ge C_0^{-1}$ and $\epsilon_i \to 0$, it follows that $R(x_i) \to 0$ and $|\nabla f|(x_i) \ge \frac{1}{2}C_1$ for all large i. So at x_i we have

$$|\tilde{\nabla}\tilde{f}_i|_{\tilde{g}_i}(x_i) = |\nabla f|(x_i) \in \left[\tfrac{1}{2}C_1, C_1\right].$$

So after passing to a subsequence we may assume that the manifolds $(M, \tilde{g}_i, x_i)$ smoothly converge to $(\mathbb{R}^2 \times S^1, g_{\mathrm{stan}}, x_\infty)$, and the functions $\tilde{f}_i$ converge to a smooth function f_∞ on $\mathbb{R}^2 \times S^1$, which satisfies $f_\infty(x_\infty) = 0$ and

$$|\nabla^{k+2} f_\infty| = 0 \quad \text{for } k \ge 0, \qquad \text{and} \qquad |\nabla f_\infty|(x_\infty) \in \left[\tfrac{1}{2}C_1, C_1\right]. \tag{3-17}$$

By (3-17) it is easy to see that f_∞ is a constant in each S^1-factor in $\mathbb{R}^2 \times S^1$, and a nonconstant linear function on the $\mathbb{R}^2$-factor. After a possible rotation on $\mathbb{R}^2$, we may assume the level set $\Sigma_\infty := f_\infty^{-1}(0) = \{(x, 0, \theta) : x \in \mathbb{R}, \theta \in [0, 2\pi)\}$. In particular, 0 is a regular value of f_∞, and Σ_∞ is isometric to $(\mathbb{R} \times S^1, g_{\mathrm{stan}})$. Therefore, the level sets $(\Sigma_i, r^{-2}(x_i)g_{\Sigma_i}, x_i)$ of $\tilde{f}_i$ smoothly converge to the level set $(\Sigma_\infty, g_{\mathrm{stan}}, x_\infty)$ of f_∞. This implies that Σ_i is a δ-neck when i is sufficiently large, a contradiction. □

The following lemma compares the value of f at two points x and y, when the minimizing geodesic from x to y is orthogonal to ∇f at y.

Lemma 3.29 *Let x and y be two points in M. Suppose that a minimizing geodesic σ from y to x is orthogonal to ∇f at y. Then $f(x) \geq f(y)$.*

Proof Since $\langle \nabla f(\sigma(0)), \sigma'(0) \rangle = 0$, computing by calculus variation we have

$$\begin{aligned} f(x) - f(y) = f(\sigma(1)) - f(\sigma(0)) &= \int_0^1 \langle \nabla f(\sigma(r)), \sigma'(r) \rangle \, dr \\ &= \int_0^1 \int_0^r \nabla^2 f(\sigma'(s), \sigma(s)) \, ds \, dr \\ &= \int_0^1 \int_0^r \mathrm{Ric}(\sigma'(s), \sigma(s)) \, ds \, dr \geq 0. \end{aligned}$$ □

The following lemma compares the scales of two ϵ-necks in a positively curved noncompact complete 2D manifold. It says that the scale of the inner ϵ-neck is almost not larger than that of the outer ϵ-neck.

Lemma 3.30 *For any $\delta > 0$, there exists an $\epsilon > 0$ such that the following holds:*

Let (M, g) be a 2D complete noncompact Riemannian manifold with positive curvature and let p be a soul for it. Then for any ϵ-neck N disjoint from p, the central circle of N separates the soul from the end of the manifold. In particular, if two ϵ-necks N_1 and N_2 in M are disjoint from each other and from p, then the central circles of N_1 and N_2 are the boundary components of a region in M diffeomorphic to $S^1 \times I$.

Moreover, assume N_1 is contained in the 2-ball bounded by the central circle of N_2. Then the scales r_1 and r_2 of N_1 and N_2 satisfy

$$r_1 < (1 + \delta) r_2.$$

Proof The proof is a slight modification of the proof of [63, Lemma 2.20] using Busemann functions. □

Now we prove the critical point theorem.

Theorem 3.31 *Let (M, g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Then there exists $p \in M$ such that R attains its maximum at p and $\nabla f(p) = 0$.*

Proof Let $\epsilon > 0$ be some constant we can take arbitrarily small, and we will use $\delta > 0$ to denote all constants that converge to zero as $\epsilon \to 0$. We suppose by contradiction that $R_{\max}$ does not exists.

First, the level sets of f are noncompact: Suppose not. Then for some $a \in \mathbb{R}$, the level set $f^{-1}(a)$ is compact and hence is diffeomorphic to S^2. So $f^{-1}(a)$ separates the manifold into a compact and a noncompact connected component. Since f is convex and nonconstant, by the maximum principle it attains its minimum in the compact region. This contradicts our assumption.

Let $\Gamma_1, \Gamma_2 \colon [0, \infty) \in M$ be the two integral curves of ∇f or $-\nabla f$ from Lemma 3.17, which extend to infinity on the open ends. First, we claim that Γ_2 and Γ_1 cannot be integral curves of $-\nabla f$ at the same time. Otherwise, on the one hand, we have $d(\Gamma_1(s), \Gamma_2(s)) \to \infty$ as $s \to \infty$ by Lemma 3.16. On the other hand, since Γ_1 and Γ_2 are integral curves of $-\nabla f$, it follows by the positive curvature and distance expanding along the backwards Ricci flow of the soliton that $d(\Gamma_1(s),\Gamma_2(s)) \le d(\Gamma_1(s_0),\Gamma_2(s_0))$ for any $s \ge s_0$. This contradiction shows the claim.

So we may assume Γ_1 is an integral curve of ∇f. In Claims 3.32, 3.33 and 3.34, we will construct a complete integral curve $\Gamma\colon (-\infty,+\infty) \in M$ of ∇f such that $\Gamma([s_0,\infty)) \subset \mathscr{C}_1, \Gamma((-\infty,-s_0]) \subset \mathscr{C}_2$ for some $s_0>0$, and moreover the manifolds $(M, r^{-2}(\Gamma(s))g, \Gamma(s))$ converge to $(\mathbb{R}\times\text{cigar}, r^{-2}(x_{\text{tip}})g_c, x_{\text{tip}})$ as $s \to \pm\infty$. Note that $\Gamma((-\infty,\infty))$ is invariant under the diffeomorphisms ϕ_t generated by ∇f.

Claim 3.32 *Take $\gamma_1\colon [0,\infty) \to M$ to be the integral curve of $-\nabla f$ starting from $\Gamma_1(0)$. Then $\gamma_1(s)$ goes to infinity as $s \to \infty$.*

Proof Suppose otherwise: assume for some $s_i \to \infty$ and compact subset V that we have $\gamma_1(s_i) \in V$. By the compactness of V, there is $c>0$ such that $\text{Ric} \ge cg$ and $c \le |\nabla f| \le c^{-1}$ in V. So by the first identity in (2-2) we have

$$\frac{d}{ds}\bigg|_{s=s_i} R(\gamma_1(s)) \ge c.$$

Moreover, by the increasing of $R(\gamma_1(s))$ we get $(d/ds)|_{s=s_i} R(\gamma_1(s)) \ge 0$ for all s. It is clear that there is a uniform $C_0>0$ such that

$$\left|\frac{d^2}{ds^2} R(\gamma_1(s))\right| \le C_0 \quad \text{for all } s.$$

We may choose the sequence s_i such that $s_{i+1} > s_i + 1$. Then

$$R(\gamma_1(s_{i+1})) \ge R(\gamma(s_1)) + \sum_{k=1}^{i} (R(\gamma_1(s_{k+1})) - R(\gamma_1(s_k))) \ge R(\gamma(s_1)) + \tfrac{1}{2} i c^2 C_0^{-1} \to \infty,$$

which is impossible. □

Claim 3.33 *As $s \to \infty$, the manifolds $(M, r^{-2}(\gamma_1(s))g, \gamma_1(s))$ converge smoothly to the manifold $(\mathbb{R}\times\text{cigar}, r^{-2}(x_{\text{tip}})g_\Sigma, x_{\text{tip}})$. In particular, $\gamma_1(s)$ is an ϵ-tip point for all sufficiently large s.*

Proof It follows from Lemma 3.3 that the manifolds $(M, r^{-2}(\gamma_1(s))g, \gamma_1(s))$ converge smoothly to either $(\mathbb{R}\times\text{cigar}, r^{-2}(x_0)g_\Sigma, x_0)$ or $(\mathbb{R}^2\times S^1, g_{\text{stan}}, x_0)$. We show that it must be the first case. Since $\liminf_{s\to\infty} R(\gamma_1(s)) > 0$, by the quadratic curvature decay of Theorem 3.20, it follows that $\gamma_1(s)$ is within uniformly bounded distance to the ϵ-tip points on $\Gamma_1 \cup \Gamma_2$. So the limit must be $(\mathbb{R}\times\text{cigar}, r^{-2}(x_0)g_\Sigma, x_0)$.

Moreover, since $\gamma_1(s)$ is an integral curve of $-\nabla f$, by the distance shrinking in the cigar soliton it is easy to see that x_0 must be a tip point in $\mathbb{R}\times\text{cigar}$. □

Claim 3.34 *For all s sufficiently large, $\gamma_1(s) \subset \mathscr{C}_2$.*

Proof Since the two chains $\mathscr{C}_1$ and $\mathscr{C}_2$ contain all ϵ-tip points by Lemma 3.14, we have either $\gamma_1(s) \in \mathscr{C}_2$ or $\gamma_1(s) \in \mathscr{C}_1$ for all sufficiently large s.

Suppose $\gamma_1(s) \in \mathscr{C}_1$ for all large s. On the one hand, by Claim 3.33, $d(\gamma_1(s), \Gamma_1) \to 0$ as $s \to \infty$. Since $\Gamma_1(s)$ is the integral curve of ∇f, we see that $|\nabla f|(\Gamma_1(s))$ increases in s, and hence

$$\liminf_{s\to\infty} |\nabla f|(\gamma_1(s)) = \liminf_{s\to\infty} |\nabla f|(\Gamma_1(s)) > 0.$$

So by Claim 3.33 we have that $\measuredangle(\nabla f, \phi_*(\partial_r)) < \epsilon$ at all ϵ-tip points in $\mathscr{C}_1$, after possibly replacing r by $-r$.

On the other hand, for a fixed point p_0, by Lemma 3.6, $\measuredangle(\nabla d(p_0,\cdot), \phi_*(\partial_r)) < \epsilon$ holds at all ϵ-tip points in $\mathscr{C}_1$ after possibly replacing r by $-r$. So either $\measuredangle(\nabla f, \nabla d(p_0,\cdot)) < \epsilon$ or $\measuredangle(-\nabla f, \nabla d(p_0,\cdot)) < \epsilon$ has to hold at all ϵ-tip points in $\mathscr{C}_1$. Note that Claim 3.32 implies that $d(p_0, \gamma_1(s)) \to \infty$ as $s \to \infty$, and hence (1) must hold. But the fact that $d(p_0, \Gamma_1(s)) \to \infty$ as $s \to \infty$ implies that (2) must hold, a contradiction. $\square$

Therefore, letting $\Gamma(s) = \Gamma_1(s)$ for $s \geq s_0$, and $\Gamma(s) = \gamma_1(s_0 - s)$ for $s \leq s_0$, we get the desired complete integral curve $\Gamma\colon (-\infty, +\infty)$ of ∇f. So we may assume $\Gamma_2(s) = \gamma_1(s - s_0)$ for $s \geq s_0$. Then Γ_1 and Γ_2 still satisfy the conclusions in Lemma 3.17, and moreover satisfy the additional properties that $\lim_{s\to\infty} R(\Gamma_2(s)) > 0$, and that Γ_1 and Γ_2 are both parts of a complete integral curve Γ.

After a rescaling we may assume $\lim_{s\to\infty} R(\Gamma_2(s)) = 4$. Then for some $s_1 > 0$, whose value will be determined later, we can find a point p, which is the center of an ϵ-cylindrical plane, such that $|h(p) - 2\pi| \leq \epsilon$ and $d(p, \Gamma) = d(p, \Gamma_2) = s_1$; see Definition 2.11 for $h(\cdot)$. Let $\gamma_p(t)$ be the integral curve of ∇f starting from p. Then $d(\gamma_p(t), \Gamma)$ increases in t. In particular, $d(\gamma_p(t), \Gamma(0)) \geq d(\gamma_p(t), \Gamma) \geq s_1$. So by Lemma 3.19 we see that when s_1 is sufficiently large, the distance $d(\gamma_p(t), \Gamma)$ for any fixed t is always attained in $\Gamma((-\infty, -s_0) \cup (s_0, \infty))$, where $\Gamma(s)$ are ϵ-tip points.

In particular, the minimizing geodesic connecting $\gamma_p(t)$ to some point $y_t \in \Gamma((-\infty, -s_0) \cup (s_0, \infty))$ such that $d(\gamma_p(t), y_t) = d(\gamma_p(t), \Gamma)$ is orthogonal to Γ at the ϵ-tip point y_t, and $y_t \to \infty$ as $t \to \infty$. On the one hand, by the distance distortion estimate we have

$$\frac{d}{dt} d(\gamma_p(t), \Gamma) \geq \sup_{\gamma \in \mathscr{Z}(t)} \int_\gamma \mathrm{Ric}(\gamma'(s), \gamma'(s))\, ds \tag{3-18}$$

in the backward difference quotient sense (see [36, Lemma 18.1]), where $\mathscr{Z}(t)$ is the space of minimizing geodesics γ that realize the distance of $d(\gamma_p(t), \Gamma)$. In particular, if $y_t \in \Gamma_2$ and γ is a minimizing geodesic connecting $\gamma_p(t)$ to y_t, we have

$$\frac{d}{dt} d(\gamma_p(t), \Gamma) \geq \int_\gamma \mathrm{Ric}(\gamma'(s), \gamma'(s))\, ds \geq 2 - \epsilon \quad \text{for all } t \in [0, T], \tag{3-19}$$

where in the second inequality we used (2-5) that in a cigar soliton with $R(x_{\mathrm{tip}}) = 4$, the integral of Ricci curvature along a geodesic ray starting from the tip is equal to 2. On the other hand, we have

$$\frac{d}{dt} d(\gamma_p(t), \Gamma_1(s_0 + t)) \leq \inf_{\gamma \in \mathscr{W}(t)} \int_\gamma \mathrm{Ric}(\gamma'(s), \gamma'(s))\, ds \tag{3-20}$$

in the forward difference quotient sense, where $\mathcal{W}(t)$ is the space of all minimizing geodesics between $\gamma_p(t)$ and $\Gamma_1(s_0+t)$. Since $R(\Gamma(s))$ strictly decreases in s, because otherwise (M,g) is isometric to $\mathbb{R}\times$ cigar, we may assume that for some $c_1>0$ we have $R(\Gamma_1(s_0,\infty))\le 4-c_1$. So by (3-20) there exists some $c_2>0$ such that

$$\frac{d}{dt}d(\gamma_p(t),\Gamma_1(s_0+t))\le 2-c_2. \tag{3-21}$$

Therefore, it follows from (3-19) and (3-21) that $d(\gamma_p(t),\Gamma)=d(\gamma_p(t),\Gamma_1)$ for sufficiently large t.

Therefore, we may let

$$T=\sup\{t: d(\gamma_p(t),\Gamma)=d(\gamma_p(t),\Gamma_2)\}<\infty,$$

then $d(\gamma_p(T),\Gamma_1)=d(\gamma_p(T),\Gamma_2)$ and $d(\gamma_p(t),\Gamma)=d(\gamma_p(t),\Gamma_2)$ for all $t\le T$. Integrating (3-19) from 0 to T we obtain

$$d(\gamma_p(t),\Gamma)\ge d(p,\Gamma)+(2-\epsilon)t\ge s_1+(2-\epsilon)t.$$

Since $\gamma_p(t)$ is the integral curve of ∇f starting from p, it follows by the definition of h that $h(\gamma_p(t))$ is equal to the length of a minimizing geodesic loop at p with respect to $g(t)$. So by the Ricci flow equation, $\mathrm{Rm}\ge 0$, and $|\mathrm{Ric}|\le CR$, we see that $h(\gamma_p(t))$ is nondecreasing in t and the evolution inequality

$$\frac{d}{dt}h(\gamma_p(t))\le C\cdot R(\gamma_p(t))\cdot h(\gamma_p(t))$$

holds. Combining this with the following curvature upper bound from Theorem 3.20,

$$R(\gamma_p(t))\le\frac{C}{d^2(\gamma_p(t),\Gamma)}\le\frac{C}{(s_1+(2-\epsilon)t)^2},$$

we obtain

$$\frac{d}{dt}h(\gamma_p(t))\le\frac{C}{(s_1+(2-\epsilon)t)^2}\cdot h(\gamma_p(t)).$$

Assuming s_1 is sufficiently large and integrating this we obtain

$$h(\gamma_p(t))\le h(p)(1+\epsilon)\le 2\pi(1+\epsilon). \tag{3-22}$$

Let $q\in\Gamma_1$ be a point such that

$$d(\gamma_p(T),q)=d(\gamma_p(T),\Gamma).$$

So by Lemma 3.29 we have

$$f(q)<f(\gamma_p(T)).$$

Since $R(\Gamma_1(s))$ decreases and $f(\Gamma_1(s))$ increases in s, there is $q_2\in\Gamma_1$ such that

$$f(q_2)=f(\gamma_p(T))>f(q)\quad\text{and}\quad R(q)>R(q_2).$$

In the rest of proof, we will show $R(q_2)\ge 4-\delta$. First, if $d(\gamma_p(T),q_2)<\delta^{-1}R^{-1/2}(q_2)$, then since q_2 is an ϵ-tip point, we obtain

$$|h(\gamma_p(T))-4\pi\cdot R^{-1/2}(q_2)|\le\delta.$$

This fact combined with (3-22) gives

$$R^{-1/2}(q_2) \le \frac{1}{4\pi} h(\gamma_p(T)) + \delta \le \tfrac{1}{2} + \delta, \tag{3-23}$$

and hence $R(q_2) \ge 4 - \delta$.

So we may assume from now on that $d(\gamma_p(T), q_2) \ge \delta^{-1} R^{-1/2}(q_2)$.

Claim 3.35 *There exists an δ-cylindrical plane at some $z \in f^{-1}(f(q_2))$ at scale $r = R^{-1/2}(q_2)$.*

Proof Let $\phi\colon (\mathbb{R} \times \text{cigar}, x_{\text{tip}}) \colon\to (M, q_2)$ be an ϵ-isometry. For the interval $[-1/\delta, 1/\delta] \subset \mathbb{R}$ and the ball $B_{g_c}(x_{\text{tip}}, 1/\sqrt{\delta})$ in (cigar, g_c), we consider image of their product $U := \phi\big([-1/\delta, 1/\delta] \times B_{g_c}(x_{\text{tip}}, 1/\sqrt{\delta})\big)$. Let σ be a smooth curve connecting q_2 to $\gamma_p(T)$ in the level set $f^{-1}(f(q_2))$. Since $\gamma_p(T) \notin U$, by continuity σ must exit U at some $z \in \partial U$.

Write $\partial U_{\pm} := \phi(\{\pm 1/\delta\} \times B_{g_c}(x_{\text{tip}}, 1/\sqrt{\delta}))$. We will show that $z \in \phi([-1/\delta, 1/\delta] \times \partial B_{g_c}(x_{\text{tip}}, 1/\sqrt{\delta})) = \partial U - \partial U_- - \partial U_+$. Replacing $+$ and $-$ if necessary, we may assume $f(\phi(1/\delta)) > f(q_2) > f(\phi(-1/\delta))$.

On the one hand, for any $y \in \partial U_-$ let $\sigma\colon [0, \ell] \to M$ be a unit-speed geodesic from y to some point $q_- \in \phi([-1/\delta, 1/\delta] \times x_{\text{tip}})$ which achieves the distance from y to it. Then we have

$$\begin{aligned} f(y) - f(q_-) &= \int_0^\ell \langle \nabla f, \sigma'(r) \rangle \, dr \\ &= \int_0^\ell \int_0^r \nabla^2 f(\sigma'(s), \sigma'(s)) \, ds \, dr \\ &\le \int_0^\ell \int_0^\ell \operatorname{Ric}(\sigma'(s), \sigma'(s)) \, ds \, dr \le \frac{C}{\sqrt{\delta}}, \end{aligned}$$

where we used that the length of σ satisfies $\ell \in [\delta^{-1/2} R^{-1/2}(q_2), 2\delta^{-1/2} R^{-1/2}(q_2)]$. This then implies

$$f(y) \le f(q_-) + \frac{C}{\sqrt{\delta}} \le f(q_2) - \frac{1}{C\delta} + \frac{C}{\sqrt{\delta}} < f(q_2),$$

and hence ∂U_- is disjoint from $f^{-1}(f(q_2))$.

On the other hand, for any $y \in \partial U_+$ let q_+ be a point in $\phi([-1/\delta, 1/\delta] \times x_{\text{tip}})$ that is closest to y. Then

$$f(y) \ge f(q_+) \ge f(q_2) + \frac{1}{C\delta} > f(q_2),$$

and hence ∂U_+ is also disjoint from $f^{-1}(f(q_2))$. So the claim holds. □

By Lemma 3.28, the two δ-cylindrical planes at $\gamma_p(T)$ and z produce the two δ-necks in the level set surface $f^{-1}(f(q_2))$: One δ-neck, denoted by N_1, is centered at $\gamma_p(T)$, which has scale $2(1+\delta)$ because $h(\gamma_p(T)) \le 2\pi(1+\delta)$ by (3-23), and $h(\gamma_p(T)) \ge h(p) \ge 2\pi - \epsilon$ by the choice of p and the monotonicity of h along integral curves; and the other δ-neck, denoted by N_2, is centered at z with scale $R^{-1/2}(q_2)$.

By the choice of N_2 and $d(\gamma_p(T), q_2) \geq \delta^{-1} R^{-1/2}(q_2)$, it is clear that N_2 is in the 2-ball bounded by the central circle of N_1. Since $f^{-1}(f(q_2))$ is positively curved, we can apply Lemma 3.30 and deduce that

$$R^{-1/2}(q_2) \leq 2(1+\delta) \quad \text{and hence} \quad R(q_2) \geq 4(1-\delta).$$

Now letting ϵ go to zero, by the monotonicity of R along Γ this implies that R is a constant along Γ. So $R \equiv 2$ on Γ. By the soliton identity,

$$\mathrm{Ric}(\nabla f, \nabla f) = -\langle \nabla R, \nabla f \rangle = 0.$$

The Ricci curvature vanishes along Γ in the direction of ∇f. So the soliton splits off a line and it is isometric to $\mathbb{R} \times$ cigar, contradiction. This proves the existence of a critical point of f. □

The following corollary follows immediately from Theorem 3.31 and Lemmas 3.17 and 3.16.

Corollary 3.36 *There are two integral curves $\Gamma_i : (-\infty, \infty) \to M$ of ∇f, for $i = 1, 2$, such that the following hold*:

1. *Let p be the critical point of f. Then $\lim_{s \to -\infty} \Gamma_i(s) = p$;*
2. *As $s \to \infty$, the pointed manifolds $(M, r^{-2}(\Gamma_i(s))g, \Gamma_i(s))$ smoothly converge to the manifold $(\mathbb{R} \times \mathrm{cigar}, r^{-2}(x_{\mathrm{tip}})g_c, x_{\mathrm{tip}})$.*
3. *For any $p_k^{(i)} \in \Gamma_i$ such that $p_k^{(i)} \to \infty$ as $k \to \infty$, the minimizing geodesics $pp_k^{(i)}$ subsequentially converge to two rays γ_1 and γ_2 such that $[\gamma_1] = 0$ and $[\gamma_2] = \theta \in S_\infty(M, p) = [0, \theta]$ for some $\theta \in [0, \pi)$.*

3.5 An ODE lemma for distance distortion estimates

We will use the following ODE lemma of two time-dependent scalar functions to estimate certain distance distortions in Theorem 3.41. This method generalizes the bootstrap argument in [60, Theorem 1.3], which relies on the $O(2)$-symmetric structure of the soliton.

Lemma 3.37 (an ODE lemma) *Let $H, h : [0, T] \to (0, \infty)$ be two differentiable functions satisfying*

$$\begin{cases} H'(t) \geq C_1 \cdot h^{-1}(t), \\ h'(t) \leq C_2 \cdot H^{-2}(t) \cdot h(t), \end{cases} \tag{3-24}$$

for some constants $C_1, C_2 > 0$. Suppose

$$\frac{H(0)}{h(0)} > \frac{C_2}{C_1}. \tag{3-25}$$

Let $C_3 := C_1 h^{-1}(0) - C_2 H^{-1}(0) > 0$. Then for all $t \in [0, T]$ we have

$$\begin{cases} H(t) \geq C_3 t + H(0), \\ h(t) \leq h(0) e^{C_2/(C_3 H(0))}. \end{cases}$$

Proof Dividing both sides by $h(t)$ in the second inequality in (3-24), we get

$$\partial_t(\ln h(t)) \leq C_2 H^{-2}(t).$$

Integrating this from 0 to t we get

$$\ln h(t) \le C_2 \int_0^t H^{-2}(s)\,ds + \ln h(0),$$

and hence

$$h(t) \le h(0) e^{C_2 \int_0^t H^{-2}(s)\,ds},$$

plugging which into the first inequality in (3-24) we get

$$H'(t) \ge C_1 h(0) e^{-C_2 \int_0^t H^{-2}(s)\,ds}.$$

Let $H_0(t)$ be a solution to the following problem:

$$\begin{cases} H_0(0) = H(0), \\ H_0'(t) = C_1 h^{-1}(0) e^{-C_2 \int_0^t H_0^{-2}(s)\,ds}. \end{cases} \tag{3-26}$$

Then it is easy to see that for all $t \ge 0$,

$$H(t) \ge H_0(t) > 0. \tag{3-27}$$

The second equation in (3-26) implies

$$\ln(H_0'(t)) = \ln(C_1\, h^{-1}(0)) - C_2 \int_0^t H_0^{-2}(s)\,ds,$$

differentiating which at both sides we obtain

$$\partial_t (H_0'(t) - C_2 H_0^{-1}(t)) = 0.$$

Integrating this and using (3-26) and (3-25) we obtain

$$H_0'(t) - C_2 H_0^{-1}(t) = H_0'(0) - C_2 H_0^{-1}(0) = C_1 h^{-1}(0) - C_2 H^{-1}(0) = C_3 > 0.$$

So by (3-27) we obtain

$$H(t) \ge H_0(t) \ge C_3 t + H(0).$$

Substituting this into the second inequality in (3-24) we get

$$\partial_t (\ln h(t)) \le \frac{C_2}{(C_3 t + H(0))^2},$$

integrating which we obtain

$$h(t) \le h(0) e^{C_2/(C_3 H(0)) - C_2/(C_3(C_3 t + H(0)))} \le h(0) e^{C_2/(C_3 H(0))}. \qquad \square$$

3.6 Asymptotic cone is not a ray

Theorem 3.41 is the main result in this section. It states that the soliton is $\mathbb{Z}_2$-symmetric at infinity, in the sense that R has equal positive limits along the two ends of Γ. Moreover, assuming this positive limit is equal to 4 after a proper rescaling, then any sequence of points going to infinity converges to either $\mathbb{R}^2 \times S^1$ or $\mathbb{R} \times \text{cigar}$, without any rescalings. As a consequence of Theorem 3.41, we prove in Corollary 3.43 the uniqueness of the Bryant soliton among all 3D steady solitons on $\mathbb{R}^3$ asymptotic to a ray.

We remark that this $\mathbb{Z}_2$-symmetry at infinity is also true in mean curvature flow: a mean curvature flow flying wing in $\mathbb{R}^3$ is a graph over a finite slab. Moreover, the slab width is equal to that of its asymptotic translators, which are two tilted Grim Reaper hypersurfaces [70; 55].

We first prove a technical lemma using metric comparison.

Lemma 3.38 *There exists $\epsilon > 0$ such that the following holds: Let Σ be a 2D complete Riemannian manifold with nonnegative curvature. Then there cannot be more than two disjoint ϵ-caps.*

Moreover, suppose there are two disjoint ϵ-caps centered at p_1 and p_2, and Σ is an ϵ-neck at a point $p \in \Sigma$ such that p is not in the two ϵ-caps. Then the central circle at p separates p_1 and p_2.

Proof For the first claim, suppose by contradiction that there are three disjoint ϵ-caps $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ centered at p_1, p_2 and p_3. We shall use $\delta(\epsilon)$ to denote all constants that go to zero as ϵ goes to zero.

Assume the minimizing geodesics p_1p_2 and p_1p_3 intersect the boundary of $\mathcal{C}_1$ at x_2 and x_3 respectively, which are centers of two ϵ-necks. So we have

$$d(x_2, x_3) < \delta(\epsilon)d(x_2, p_1),$$

which by the monotonicity of angles implies

$$\widetilde{\measuredangle} p_2p_1p_3 \le \widetilde{\measuredangle} x_2p_1x_3 \le \delta(\epsilon).$$

In the same way we obtain that $\widetilde{\measuredangle} p_1p_2p_3, \widetilde{\measuredangle} p_1p_3p_2 \le \delta(\epsilon)$. But then we have

$$\widetilde{\measuredangle} p_1p_2p_3 + \widetilde{\measuredangle} p_2p_1p_3 + \widetilde{\measuredangle} p_1p_3p_2 \le 3\delta(\epsilon) < \pi,$$

which is impossible.

For the second claim, suppose the central circle at p does not separate p_1 and p_2. Denote the ϵ-isometry of the ϵ-neck at p by $\psi\colon (-\epsilon^{-1}, \epsilon^{-1}) \times S^1 \to \Sigma$. Let $\gamma_\pm = \psi(\{\pm\epsilon^{-1}\} \times S^1)$. Then, after possibly replacing $+$ with $-$, we claim that the minimizing geodesics pp_1, pp_2 and p_1p_2 are all contained in the component of Σ separated by γ_- which contains γ_+: First, since p_1 and p_2 are in the same component of Σ separated by $\psi(\{0\} \times S^1)$, suppose pp_1 intersects γ_+. Then it follows that pp_2 also intersects Σ_+, and the claim follows by the minimality of these geodesics.

By the claim, we can use a similar argument as before to deduce

$$\widetilde{\measuredangle} p_1p_2p + \widetilde{\measuredangle} p_2p_1p + \widetilde{\measuredangle} p_1pp_2 < \pi,$$

which is a contradiction. □

The following lemma is a key ingredient in the proof of Theorem 3.41.

Lemma 3.39 *There exists $C_0 > 0$ such that the following holds: Let $\epsilon > 0$ be a sufficiently small number, and $p \in M$ be an ϵ-cylindrical plane point. Let $q \in \Gamma$ be the point such that $d(p,q) = d(p,\Gamma)$. Suppose $R(q) < C_0^{-1}$. Then $r(q) \le 1200\, h(p)$, where $h(\cdot)$ is defined in Definition 2.11.*

Proof Let C denote all positive universal constants, and δ denote all positive constants that converge to zero as $\epsilon \to 0$. Let $\Sigma := f^{-1}(f(q))$. Then Σ is diffeomorphic to S^2, and it separates M into a bounded component $f^{-1}([0, f(q)))$ diffeomorphic to a 3-ball, and an unbounded component $f^{-1}((f(q),\infty))$ diffeomorphic to $\mathbb{R}\times S^2$. So we may assume $\Gamma\cap\Sigma=\{q,\overline{q}\}$, and q and $\overline{q}$ are δ-tip points. Let $\gamma\colon[0,1]\to M$ be a minimizing geodesic from q to p. Then by Lemma 3.29 we have $\min_{\gamma([0,1])} f \ge f(q)$, so $p\in f^{-1}([f(q),\infty))$. Therefore, there is a smooth nonnegative function $T\colon[0,1]\to\mathbb{R}$ such that $\overline{\gamma}(r) := \phi_{-T(r)}(\gamma(r))\in\Sigma$ for $r\in[0,1]$. Let $\overline{p}=\overline{\gamma}(1)=\phi_{-T(1)}(p)$.

Claim 3.40 *We have that $d_\Sigma(\overline{p},\overline{q}) \ge \frac{1}{2} d_\Sigma(q,\overline{p})$, where d_Σ denotes the intrinsic metric on Σ.*

Proof First, we may assume (M,g) is an ϵ-cylindrical plane at $\overline{p}$. First, by the positive curvature and distance shrinking in the Ricci flow $g(t)=\phi_{-t}^{*}g$ of the soliton, we have

$$h(\overline{p}) \le h(p). \tag{3-28}$$

Consider the smooth map $\chi\colon[0,1]\times\mathbb{R}\to M$ defined by $\chi(r,t)=\phi_t(\overline{\gamma}(r))$. Since ϕ_t is the flow of ∇f, we have $f\circ\chi(r,t)=t+f(q)$ and hence $\langle\chi_*(\partial_t),\chi_*(\partial_r)\rangle=\langle\nabla f,\chi_*(\partial_r)\rangle=0$. So we can compute

$$\begin{aligned} d(p,q)=\mathrm{Length}(\gamma) &= \int_0^1 |\chi_{*(r,T(r))}(\partial_r)+T'(r)\cdot\chi_{*(r,T(r))}(\partial_t)|\,dr \\ &\ge \int_0^1 |\chi_{*(r,T(r))}(\partial_r)|\,dr = \int_0^1 |\phi_{T(r)*}(\overline{\gamma}'(r))|\,dr \\ &\ge \int_0^1 |\overline{\gamma}'(r)|\,dr = \mathrm{Length}(\overline{\gamma}) \ge d_\Sigma(q,\overline{p}). \end{aligned} \tag{3-29}$$

Since $|\nabla f|\ge C^{-1}>0$ on $M\setminus B(x_0,1)$, we have

$$\begin{aligned} d(p,\overline{p}) \le C(f(p)-f(\overline{p})) &= C(f(p)-f(q)) \\ &= C\int_0^1\int_0^r \mathrm{Ric}(\gamma'(s),\gamma'(s))\,ds\,dr \\ &\le C\int_0^1 \mathrm{Ric}(\gamma'(r),\gamma'(r))\,dr \le \tfrac{1}{2}d(p,q), \end{aligned} \tag{3-30}$$

where in the second equality we used $\langle\nabla f,\gamma'(0)\rangle=0$, and in the last inequality we used

$$\int_0^1 \mathrm{Ric}(\gamma'(r),\gamma'(r))\,dr \le \frac{1}{2C}d(p,q),$$

which follows from Lemma 2.13(2), Theorem 3.20, and the assumption $R(q)<C_0^{-1}$ by taking C_0 sufficiently large.

Since $d(p,\overline{q})\ge d(p,\Gamma)=d(p,q)$, together with (3-29) and (3-30) this implies

$$d_\Sigma(\overline{p},\overline{q}) \ge d(\overline{p},\overline{q}) \ge d(p,\overline{q})-d(p,\overline{p}) \ge d(p,q)-d(p,\overline{p}) \ge \tfrac{1}{2}d_\Sigma(q,\overline{p}).$$

This proves the claim. □

Since q is a δ-tip point, we may assume without loss of generality that $d_\Sigma(q,\overline{p}) \geq \overline{D} R^{-1/2}(q)$, where $\overline{D}$ is from Lemma 3.27, because otherwise for sufficiently small ϵ we have $r(q) \leq 10r(\overline{p}) \leq 10r(p) \leq 100h(p)$, and thus the lemma holds. So we can find a point q_1 on the Σ-minimizing geodesic between q and $\overline{p}$ such that $d_\Sigma(q,q_1) = \overline{D} R^{-1/2}(q)$ and hence

$$r(q) \leq 10\, r(q_1). \tag{3-31}$$

By Lemma 3.27 it follows that (M,g) is a δ-cylindrical plane at q_1. So by Lemma 3.28 this implies that the level set Σ is a δ-neck at $\overline{p}$ and q_1 at scale $r(\overline{p})$ and $r(q_1)$, respectively.

We may also assume $d_\Sigma(\overline{p},q_1) > 100r(\overline{p}) > 10h(\overline{p})$, because otherwise we have $r(q) \leq 10r(q_1) \leq 20r(\overline{p}) \leq 20r(p)$, and thus the lemma holds. We will show the inequalities

$$d_\Sigma(\overline{q},\overline{p}) \leq d_\Sigma(\overline{q},q_1) \leq 3\, d_\Sigma(\overline{q},\overline{p}). \tag{3-32}$$

For the first inequality in (3-32), since $\overline{q}$ and q_1 are in two disjoint δ-caps, it follows by Lemma 3.38 that the central circle at $\overline{p}$ separates q_1 and $\overline{q}$ in Σ. So a minimizing geodesic between q_1 and $\overline{q}$ intersects the central circle at $\overline{p}$, and hence

$$d_\Sigma(\overline{q},q_1) \geq d_\Sigma(\overline{p},\overline{q}) - 10h(\overline{p}) + d_\Sigma(\overline{p},q_1) \geq d_\Sigma(\overline{p},\overline{q}),$$

where in the last inequality we used $d_\Sigma(\overline{p},q_1) > 100\, r(\overline{p}) > 10\, h(\overline{p})$. For the second inequality in (3-32), by Claim 3.40 we have

$$d_\Sigma(\overline{q},q_1) \leq d_\Sigma(\overline{q},\overline{p}) + d_\Sigma(\overline{p},q_1) \leq d_\Sigma(\overline{q},\overline{p}) + d_\Sigma(\overline{p},q) \leq 3\, d_\Sigma(\overline{q},\overline{p}).$$

By the first inequality in (3-32) and the positive curvature on Σ, we can deduce by volume comparison that

$$\frac{|\partial B_\Sigma(\overline{q}, d_\Sigma(\overline{q},q_1))|}{d_\Sigma(\overline{q},q_1)} \leq \frac{|\partial B_\Sigma(\overline{q}, d_\Sigma(\overline{q},\overline{p}))|}{d_\Sigma(\overline{q},\overline{p})}. \tag{3-33}$$

Since Σ is a δ-neck at both points $\overline{p}$ and q_1, it is easy to see that

$$\tfrac{1}{2} \leq \frac{|\partial B_\Sigma(\overline{q}, d_\Sigma(\overline{q},q_1))|}{r(q_1)} \leq 2 \quad \text{and} \quad \tfrac{1}{2} \leq \frac{|\partial B_\Sigma(\overline{q}, d_\Sigma(\overline{q},\overline{p}))|}{r(\overline{p})} \leq 2.$$

Therefore, by the second inequality in (3-32) and (3-33) we get $r(q_1) \leq 12r(\overline{p})$. Then by (3-31), $r(\overline{p}) \leq 10h(\overline{p})$ and (3-28) we get

$$r(q) \leq 120r(\overline{p}) \leq 1200h(\overline{p}) \leq 1200h(p),$$

which proves the lemma. □

Now we prove the main theorem in this section. A key step in the proof is to choose two suitable functions that evolve by the conditions in Lemma 3.37.

Theorem 3.41 *Let (M,g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Let Γ_1 and Γ_2 be the two integral curves of ∇f from Corollary 3.36. Then after a rescaling of (M,g), we have*

$$\lim_{s\to\infty} R(\Gamma_1(s)) = \lim_{s\to\infty} R(\Gamma_2(s)) = 4.$$

Moreover, for any sequence of points $q_k\to\infty$, the sequence of pointed manifolds (M,g,q_k) converge to either $(\mathbb{R}^2\times S^1, g_{\text{stan}})$ or $(\mathbb{R}\times\text{cigar}, g_c)$. In particular, if $\{q_k\}\subset\Gamma_1\cup\Gamma_2$, then (M,g,q_k) converges to $\mathbb{R}\times\text{cigar}$.

Proof We will first prove $\lim_{s\to\infty} R(\Gamma_i(s))>0$ for $i=1,2$. By Theorem 3.31 we know that f has a unique critical point x_0. Assume $f(x_0)=0$. Then it is easy to see that all level sets $f^{-1}(s)$ for all $s>0$ are diffeomorphic to 2-spheres, and the induced metrics have positive curvature by (3-14). Let Γ_1 and Γ_2 be from Corollary 3.36, and $\Gamma=\Gamma_1(-\infty,\infty)\cup\Gamma_2(-\infty,\infty)\cup\{x_0\}$. Then the subset Γ is invariant under the diffeomorphism ϕ_t. Let C denote all positive universal constants, ϵ denote all positive constants that we may take arbitrarily small, and δ denote all positive constants that converge to zero as $\epsilon\to 0$. Suppose by contradiction that the assertion $\lim_{s\to\infty} R(\Gamma_i(s))>0$ for both $i=1,2$ does not hold; we will derive a contradiction. Without loss of generality, we may assume $\lim_{s\to\infty} R(\Gamma_1(s))=0$.

Choose a point $p\in M$ such that (M,g) is an ϵ-cylindrical plane at p, and $d(p,\Gamma)=d(p,\Gamma_1)$. By Lemma 3.5, (M,g) is always an ϵ-cylindrical plane along the integral curve $\phi_t(p)$ of ∇f starting from p for $t\ge 0$. We abbreviate $\phi_t(p)$ as p_t. Let $q_t\in\Gamma$ be a point such that $d(p_t,\Gamma)=d(p_t,q_t)$. We claim that $R(q_t)\to 0$ as $t\to\infty$.

The claim is clear when $\lim_{s\to\infty} R(\Gamma_2(s))=0$, using that $q_t\to\infty$ as $t\to\infty$ by Lemma 3.19. If $\lim_{s\to\infty} R(\Gamma_2(s))>0$, we argue as follows: First, for any large $s_1>0$, by Lemma 3.19 we can choose p to be sufficiently far away from x_0, so that $d(p_t,\Gamma_i)$ for $i=1,2$ are achieved at points on $\Gamma_i((s_1,\infty))$. On the one hand, by $\lim_{s\to\infty} R(\Gamma_2(s))>0$ and (2-7), we may assume $\frac{d}{dt}d(p_t,\Gamma_2)\ge c_0$ for some $c_0>0$. On the other hand, by $\lim_{s\to\infty} R(\Gamma_1(s))=0$, Theorem 3.20, and the distance distortion estimate in Lemma 2.13(2), we may also assume $\frac{d}{dt}d(p_t,\Gamma_1)\le\frac{1}{100}c_0$ for all $t\ge 0$. So we have

$$\frac{d}{dt}d(p_t,\Gamma_2)>\frac{d}{dt}d(p_t,\Gamma_1).$$

Since $d(p,\Gamma)=d(p,\Gamma_1)$, it follows by integrating that $d(p_t,\Gamma_2)>d(p_t,\Gamma_1)=d(p_t,\Gamma)$ for all $t\ge 0$. So q_t always falls on $\Gamma_1([s_1,\infty))$, and thus $R(q_t)\to 0$. So the claim holds.

Since $(M,r^{-2}(q_t)g,q_t)$ converges to $\mathbb{R}\times\text{cigar}$ as $t\to\infty$, it follows for large t that $R\ge C^{-1}r^{-2}(q_t)$ in $B(q_t,C^{-1}r(q_t))$. Let $H(\phi_t(p))=d(\phi_t(p),\Gamma)$. Then by (2-7) this implies:

$$\partial_t H(p_t)\ge C^{-1}\cdot r^{-1}(q_t). \tag{3-34}$$

Further, by Theorem 3.20 we have $R\le CH^{-2}(p_t)$ in $B(p_t,100\,h(p_t))$, where $h(\cdot)$ is from Definition 2.11. So by (2-7) this implies

$$\partial_t h(p_t)\le C\cdot H^{-2}(p_t)\cdot h(p_t). \tag{3-35}$$

By the above claim we know $R(q_t)\to 0$ as $t\to\infty$, so we can assume $R(q_t)\le C_0^{-1}$ for $C_0>0$ from Lemma 3.39, by applying which we have $r(q_t)<1200\,h(p_t)$, and thus

$$\begin{cases}\partial_t H(p_t)\ge (1200C)^{-1}\cdot h^{-1}(q_t),\\ \partial_t h(p_t)\le C\cdot H^{-2}(p_t)\cdot h(p_t).\end{cases}$$

By taking ϵ sufficiently small, we may assume $H(p)/h(p)>1200\,C^2$. So $H(p_t)$ and $h(p_t)$ satisfy all assumptions in the ODE Lemma 3.37, applying which we can deduce

$$H(p_t)\ge C^{-1}t \quad\text{and}\quad h(p_t)\le C \quad\text{for all sufficiently large } t.$$

So we obtain $r(q_t)\le C$. Since $(M,r^{-2}(q_t)g,q_t)$ converge to $(\mathbb{R}\times\text{cigar},x_{\text{tip}})$ as $t\to\infty$, this implies $\lim_{t\to\infty}R(q_t)>0$, a contradiction to the above claim. This proves $\lim_{s\to\infty}R(\Gamma_i(s))>0$ for $i=1,2$.

Now we prove $\lim_{s\to\infty}R(\Gamma_1(s))=\lim_{s\to\infty}R(\Gamma_2(s))$ and the remaining assertions of the theorem. Note that it suffices to show that for any two ϵ-cylindrical plane points $x_1,x_2\in M$, we have $|h(x_1)-h(x_2)|\le\delta$. To show this, by replacing p_t by $\phi_t(x_i)$ for $i=1,2$ in (3-34), and noting that $r(q_t)\le C^{-1}$ as a consequence of $\lim_{s\to\infty}R(\Gamma_i(s))>0$ for $i=1,2$, it follows that $H(\phi_t(x_i))$ increases at least linearly, and thus by (3-35) we obtain

$$|h(x_i)-h(\phi_t(x_i))|\le\delta \quad\text{for } i=1,2. \tag{3-36}$$

Claim 3.42
$$\lim_{t\to\infty} d(\phi_t(x_1),\phi_t(x_2))<\infty.$$

Proof First, we have that $d(\phi_t(x_i),\Gamma)\ge d(x_i,\Gamma)+C^{-1}t$ for $i=1,2$, so by the distance distortion of Lemma 2.13 and Theorem 3.20 we have

$$\frac{d}{dt}d(\phi_t(x_1),\phi_t(x_2))\le\max\left\{\frac{C}{d(x_1,\Gamma)+C^{-1}t},\frac{C}{d(x_2,\Gamma)+C^{-1}t}\right\},$$

integrating which we obtain

$$d(\phi_t(x_1),\phi_t(x_2))\le d(x_1,x_2)+C\ln t. \tag{3-37}$$

Therefore, for any sufficiently large t, letting $\gamma\colon[0,1]\to M$ be a minimizing geodesic between $\phi_t(x_1)$ and $\phi_t(x_2)$, by triangle inequalities we have $d(\gamma([0,1],\Gamma)>C^{-1}t$. So by Theorem 3.20 we have $\sup_{s\in[0,1]}R(\gamma(s))\le C/t^2$, and hence (3-37) implies

$$\frac{d}{dt}d(\phi_t(x_1),\phi_t(x_2))\le\int_\gamma \mathrm{Ric}(\gamma'(s),\gamma'(s))\,ds\le\frac{C}{t^{3/2}},$$

integrating which we proved the claim. □

Note that $\phi_t(x_2)$ converges to a rescaling of $\mathbb{R}^2\times S^1$ as $t\to\infty$, so by Claim 3.42 we see that

$$|h(\phi_t(x_1))-h(\phi_t(x_2))|\le\delta$$

for all sufficiently large t. Combining this with (3-36), this implies

$$|h(x_1)-h(x_2)|\le\delta,$$

which proves the theorem. □

In the following we show that the soliton is asymptotic to a sector. Therefore, 3D steady gradient solitons are all flying wings except the Bryant soliton.

Corollary 3.43 (Theorem 1.1, asymptotic to a sector) *Let (M,g) be a 3D steady gradient soliton with positive curvature. If the asymptotic cone of (M,g) is a ray, then (M,g) is isometric to a Bryant soliton.*

Proof Suppose that (M,g) is not a Bryant soliton. Let $C>0$ denote all constants depending on the soliton (M,g), and let $\epsilon>0$ be some sufficiently small number. Let Γ_1 and Γ_2 be the integral curves from Corollary 3.36. By Theorem 3.41 we may assume $\lim_{s\to\infty} R(\Gamma_i(s))=4$. We write $\Gamma=\Gamma_1([0,\infty))\cup\Gamma_2([0,\infty))$. Let $p\in M$ be the center of an ϵ-cylindrical plane, then we have

$$d(\phi_t(p),\Gamma)\geq 1.9t+d(p,\Gamma)\quad \text{for } t\geq 0. \tag{3-38}$$

Suppose $\max R=R(x_0)\leq C$. Then we have

$$d(x_0,\phi_t(p))\leq 20Ct,$$

which combined with (3-38) implies

$$d(\phi_t(p),\Gamma)\geq C^{-1}d(x_0,\phi_t(p))\quad \text{for all large } t. \tag{3-39}$$

Let $q_k\in\Gamma_1$ and $\overline{q}_k\in\Gamma_2$ be sequences of points such that $d(x_0,q_k)=d(x_0,\overline{q}_k)$, and let $\sigma_k:[0,1]\to M$ be a minimizing geodesic connecting q_k and $\overline{q}_k$. Then it is easy to see that $d(x_0,\sigma_k([0,1]))\to\infty$ as $k\to\infty$. Since $d(x_0,\phi_t(p))\to\infty$ as $t\to\infty$, the integral curve $\phi_t(p)$ must pass through the S^1-factor of an ϵ-cylindrical plane centered at some point on $\sigma_k(0,1)$. In particular, we can find $t_k>0$ and $s_k\in(0,1)$ such that $t_k\to\infty$ as $k\to\infty$, and

$$d(\phi_{t_k}(p),\sigma_k(s_k))<2\pi.$$

Since $d(q_k,\overline{q}_k)\geq d(\sigma_k(s_k),q_k)\geq d(\sigma_k(s_k),\Gamma)$, this implies by the triangle inequality that

$$d(q_k,\overline{q}_k)\geq d(\phi_{t_k}(p),\Gamma)-d(\phi_{t_k}(p),\sigma_k(s_k))\geq d(\phi_{t_k}(p),\Gamma)-2\pi,$$

which together with (3-39) implies

$$d(q_k,\overline{q}_k)\geq C^{-1}d(x_0,\phi_{t_k}(p))\quad \text{for all large } k. \tag{3-40}$$

Since (M,g) is not isometric to $\mathbb{R}\times$ cigar, we have

$$d(q_k,\overline{q}_k)<(2-C^{-1})d(x_0,q_k).$$

Combining it with the triangle inequality

$$d(x_0,\phi_{t_k}(p))+d(q_k,\overline{q}_k)\geq d(x_0,q_k)+d(x_0,\overline{q}_k)-2\pi=2d(x_0,q_k)-2\pi,$$

we obtain

$$d(x_0,\phi_{t_k}(p))\geq C^{-1}d(x_0,q_k).$$

So by (3-40) this implies $d(q_k,\overline{q}_k)\geq C^{-1}d(x_0,q_k)$ and thus $\widetilde{\measuredangle} q_k x_0 \overline{q}_k\geq C^{-1}$. Lastly, by Lemma 3.16, the minimizing geodesics x_0q_k and $x_0\overline{q}_k$ converge to two rays σ_1 and σ_2 with $\widetilde{\measuredangle}(\sigma_1,\sigma_2)\geq C^{-1}>0$. So the soliton is asymptotic to a sector. □

4 Upper and lower curvature estimates

In this section, we prove Theorem 1.7 of the two-sided curvature estimates. For the lower bound, Theorem 4.8 shows that R decays at most exponentially fast away from Γ by using the improved Harnack inequality in Corollary 4.6. For the upper bound, Theorem 4.11 shows that R decays at least polynomially fast away from Γ. Theorem 4.11 is proved using the quadratic curvature decay from Theorem 3.20 and a heat kernel method.

4.1 Improved integrated Harnack inequality

In this subsection, we prove an improved integrated Harnack inequality for Ricci flows with nonnegative curvature operators. This improved Harnack inequality will be used to deduce the exponential curvature lower bound in Theorem 4.8.

First, we state Hamilton's traced differential Harnack inequality and its integrated version.

Theorem 4.1 *Let $(M, g(t))$ for $t \in (0, T]$ be an n-dimensional Ricci flow with complete time slices and nonnegative curvature operator. Assume furthermore that the curvature is bounded on compact time intervals. Then for any $(x, t) \in M \times (0, T]$ and $v \in T_x M$,*

$$\partial_t R(x,t) + \frac{R}{t} + 2\langle v, \nabla R\rangle + 2\,\mathrm{Ric}(v,v) \geq 0. \tag{4-1}$$

Moreover, integrating this inequality appropriately yields: For any $(x_1, t_1), (x_2, t_2) \in M \times (0, T]$ with $t_1 < t_2$, we have

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \geq \frac{t_1}{t_2} \exp\biggl(-\frac{1}{2}\frac{d^2_{g(t_1)}(x_1,x_2)}{t_2-t_1}\biggr). \tag{4-2}$$

Remark 4.2 By the soliton identities it is not hard to see that the equality in the differential Harnack inequality (4-1) is achieved if $(M, g(t))$ is the Ricci flow of an expanding gradient Ricci soliton with nonnegative curvature operator and $v = \nabla f_t$. We adopt the convention that the flow satisfies

$$\mathrm{Ric}(g(t)) + \frac{1}{2t} g(t) = \nabla^2 f_t \quad \text{for } t > 0.$$

See eg [38, Chapter 10.4].

Remark 4.3 In dimension 2, using $\mathrm{Ric} = \frac{1}{2} R\, g$ one can prove the slightly better integrated Harnack inequality

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \geq \frac{t_1}{t_2} \exp\biggl(-\frac{1}{4}\frac{d^2_{g(t_1)}(x_1,x_2)}{t_2-t_1}\biggr). \tag{4-3}$$

The main result of this subsection shows that (4-3) actually holds in all dimensions. Our key observation is the following curvature inequality.

Lemma 4.4 *Let (M,g) be an n-dimensional Riemannian manifold with nonnegative curvature operator. Then*

$$\mathrm{Ric} \le \tfrac{1}{2} R g. \tag{4-4}$$

Proof To show this, let $p \in M$ and choose an orthonormal basis $\{e_i\}_{i=1}^n$ of $T_p M$ under which the Ricci curvature is diagonal, so

$$\mathrm{Ric} = (\lambda_1, \dots, \lambda_n),$$

where $\lambda_1 \ge \dots \ge \lambda_n$ are the n eigenvalues. Let $k_{ij} = \mathrm{Rm}(e_i, e_j, e_j, e_i)$. Then since $\mathrm{Rm} \ge 0$, we have

$$k_{1i} \le \sum_{j \ne i} k_{ji} = \lambda_i$$

for all $i = 2, \dots, n$. So

$$\lambda_1 = k_{12} + k_{13} + \dots + k_{1n} \le \lambda_2 + \lambda_3 + \dots + \lambda_n,$$

and hence we have

$$\mathrm{Ric}(v,v) \le \lambda_1 |v|^2 \le \left(\frac{\lambda_1 + \dots + \lambda_n}{2} \right) |v|^2 = \tfrac{1}{2} |v|^2 R. \qquad \square$$

Now we prove the improved integrated Harnack inequality.

Theorem 4.5 (improved integrated Harnack inequality) *Let $(M, g(t))$ for $t \in [0,T]$ be a complete Ricci flow with nonnegative curvature operator. Suppose also that the $(M, g(t))$ have bounded curvature. Then for any $x_1, x_2 \in M$ and $0 < t_1 < t_2 \le T$, we have*

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \ge \frac{t_1}{t_2} \exp\left(-\frac{1}{4} \frac{d^2_{g(t_1)}(x_1,x_2)}{t_2 - t_1} \right).$$

Proof In the Harnack inequality (4-1), letting $v = -\nabla(\log R)(x,t)$ and $T_0 = 0$, we get

$$R^{-1} \partial_t R + \frac{1}{t} - 2|\nabla \log R|^2 + \frac{2\,\mathrm{Ric}(\nabla \log R, \nabla \log R)}{R} \ge 0. \tag{4-5}$$

Then by Lemma 4.4 we obtain

$$\frac{\partial}{\partial t} \log(tR) \ge |\nabla \log R|^2. \tag{4-6}$$

Let $\mu \colon [t_1, t_2] \to M$ be a $g(t_1)$-minimizing geodesic from x_1 to x_2. Then

$$\begin{aligned}
\log\left(\frac{t_2 R(x_2,t_2)}{t_1 R(x_1,t_1)} \right) &= \int_{t_1}^{t_2} \frac{d}{dt} \log(tR(\mu(t),t))\,dt = \int_{t_1}^{t_2} \frac{\partial}{\partial t} \log(tR) + \left\langle \nabla \log(tR), \frac{d\mu}{dt} \right\rangle dt \\
&\ge \int_{t_1}^{t_2} |\nabla \log R|^2 + \left\langle \nabla \log R, \frac{d\mu}{dt} \right\rangle dt \ge \int_{t_1}^{t_2} |\nabla \log R|^2 - |\nabla \log R| \left| \frac{d\mu}{dt} \right| dt \\
&\ge -\frac{1}{4} \int_{t_1}^{t_2} \left| \frac{d\mu}{dt} \right|^2 d\mu \ge -\frac{1}{4} \frac{d^2_{g(t_1)}(x_1,x_2)}{t_2 - t_1}.
\end{aligned} \qquad \square$$

Note that if moreover the Ricci flow $(M, g(t))$ is ancient, then (4-6) becomes

$$\frac{\partial}{\partial t}\log R \geq |\nabla \log R|^2. \tag{4-7}$$

In particular, the following integrated Harnack inequality is a direct consequence of Theorem 4.5.

Corollary 4.6 (improved Harnack inequality, ancient flow) *Let $(M, g(t))$ for $t \in (-\infty, 0]$ be a complete Ricci flow with bounded curvature. Suppose $\mathrm{Rm}_{g(t)} \geq 0$ for all $t \in (-\infty, 0]$. Then for any $x_1, x_2 \in M$ and $t_1 < t_2$, we have*

$$\frac{R(x_2,t_2)}{R(x_1,t_1)} \geq \exp\left(-\frac{1}{4}\frac{d^2_{g(t_1)}(x_1,x_2)}{t_2-t_1}\right).$$

Remark 4.7 The equality in Lemma 4.4 holds on any two-dimensional solutions. In the cigar soliton, it is easy to see that the equality in (4-7) is achieved, but the equality is lost in the integrated version. Nevertheless, the factor $\frac{1}{4}$ in Corollary 4.6 is still sharp in the sense that using it we can obtain a curvature lower bound on cigar soliton which is arbitrarily close to the actual curvature decay in the cigar at infinity:

Let (Σ, g_c) be a cigar soliton and $R(x_{\mathrm{tip}}) = 4$. Let $(\Sigma, g_c(t))$ be the Ricci flow of the soliton. For any $x \in \Sigma$, let $t = -d_0(x, x_{\mathrm{tip}})/2$. Then by the distance distortion estimate (2-5) in $g_c(t)$, we have

$$d_t(x, x_{\mathrm{tip}}) \leq d_0(x, x_{\mathrm{tip}}) + (2-\epsilon)(-t),$$

where $\epsilon > 0$ denotes all constants depending on $d_0(x, x_{\mathrm{tip}})$, such that $\epsilon \to 0$ as $d_0(x, x_{\mathrm{tip}}) \to \infty$. So applying the improved Harnack inequality we get

$$R(x,0) \geq R(x_{\mathrm{tip}}, t)\, e^{-(2-\epsilon)\, d_0(x,x_{\mathrm{tip}})} = 4\, e^{-(2-\epsilon)\, d_0(x,x_{\mathrm{tip}})}.$$

This can be compared with the curvature formula of the cigar soliton (2-4),

$$R(x,0) = \frac{16}{(e^{d_0(x,x_{\mathrm{tip}})} + e^{-d_0(x,x_{\mathrm{tip}})})^2} \leq 16\, e^{-2\, d_0(x,x_{\mathrm{tip}})}.$$

4.2 Exponential lower bound of the curvature

In this subsection we use the improved Harnack inequality to deduce the exponential curvature lower bound.

Theorem 4.8 (scalar curvature exponential lower bound) *Let (M, g, f, p) be a 3D steady gradient soliton that is not a Bryant soliton. Assume $\lim_{s\to\infty} R(\Gamma_1(s)) = \lim_{s\to\infty} R(\Gamma_2(s)) = 4$. Then for any $\epsilon_0 > 0$, there exists $C > 0$ such that*

$$R(x) \geq C^{-1} e^{-2(1+\epsilon_0) d_g(x,\Gamma)}. \tag{4-8}$$

Proof For any $\epsilon_0 > 0$, let $\epsilon > 0$ denote all small constants depending on ϵ_0, whose values may change from line to line. Let $(M, g(t))$, $t \in (-\infty, \infty)$, be the Ricci flow associated to the soliton (M, g), $g(0) = g$.

Consider the subset U consisting of all points x such that the distance $d_t(x,\Gamma)$ for all $t\le 0$ must be achieved at ϵ-tip points on Γ. Then it is clear by Lemma 3.19 that the complement of U is compact. So we can find a constant $C>0$ such that $R\ge C^{-1}$ on $M\setminus U$. Therefore, it suffices to prove the curvature lower bound (4-8) for points $(x,0)\in U\times\{0\}\subset M\times(-\infty,\infty)$.

Let $x\in U$ and $t\le 0$. By a distance distortion estimate we have

$$-\frac{d}{dt}d_t(x,\Gamma)\le \sup_{\gamma\in\mathfrak{X}(t)}\int_\gamma \mathrm{Ric}(\gamma'(s),\gamma'(s))\,ds, \tag{4-9}$$

where the derivative is the backward difference quotient, and $\mathfrak{X}(t)$ is the space of all minimizing geodesics γ which realize the distance $d_t(x,\Gamma)$. For any such γ connecting x to a point $y_t\in\Gamma$ such that $d_t(x,y_t)=d_t(x,\Gamma)$, we have that y_t is an ϵ-tip point since $x\in U$, and γ is orthogonal at y_t to Γ. Moreover, by the assumption $\lim_{s\to\infty}R(\Gamma_1(s))=\lim_{s\to\infty}R(\Gamma_2(s))=4$ we have

$$R(y_t,t)\ge 4-\epsilon. \tag{4-10}$$

So by taking ϵ small, (4-9) implies

$$-\frac{d}{dt}d_t(x,\Gamma)\le 2(1+\epsilon_0),$$

integrating which we get

$$d_t(x,y_t)=d_t(x,\Gamma)\le d_0(x,\Gamma)+2(1+\epsilon_0)(-t). \tag{4-11}$$

Now applying Corollary 4.6

(the improved Harnack inequality) and using (4-10) and (4-11) we obtain

$$R(x,0)\ge R(y_t,t)e^{-d_t^2(x,y_t)/(-4t)}\ge (4-\epsilon)e^{-\left(d_0(x,\Gamma)+2(1+\epsilon_0)(-t)\right)^2/(-4t)}.$$

Letting $t=-\frac{1}{2}d_0(x,\Gamma)$, this implies

$$R(x,0)\ge (4-\epsilon)e^{-2(1+\epsilon_0)d_0(x,\Gamma)}. \qquad \square$$

Remark 4.9 This curvature estimate is sharp: in the manifold $\mathbb{R}\times$cigar, the curvature decays like $O(e^{-2\,d_g(\cdot,\Gamma)})$, so our lower bound estimate $O(e^{-(2+\epsilon_0)\,d_g(\cdot,\Gamma)})$ gets arbitrarily close to it as the distance $d_g(\cdot,\Gamma)$ goes to infinity.

4.3 Polynomial upper bound of the curvature

In Theorem 4.11 we show that the quadratic curvature decay from Theorem 3.20 can be improved to polynomial decay at any rate. The proof relies on the following heat kernel estimate. This estimate shows that the heat kernel starting from (x,t) behaves like a Gaussian, and it is centered at the (x,s) for all $s<t-2$. For $s\in[t-2,t)$, the Gaussian bound also holds by Lemma 2.21.

Lemma 4.10 *Let (M, g, f, p) be a 3D steady gradient soliton that is not a Bryant soliton and $(M, g(t))$ be the Ricci flow of the soliton. Let $G(x, t; y, s)$ for $x, y \in M$ and $s < t-2$ be the heat kernel of the heat equation $\partial_t u = \Delta u$ under $g(t)$. Then there exists $C > 0$ such that*

$$G(x,t;y,s) \le C(t-s)^{3/2} \exp\left(-\frac{d_s^2(x,y)}{4C(t-s)}\right).$$

Proof After a rescaling we assume $\lim_{s\to\infty} R(\Gamma_1(s)) = \lim_{s\to\infty} R(\Gamma_2(s)) = 4$. We shall use C to denote all constants depending only on the soliton (M, g). Without loss of generality, we may assume $s = 0$. For any $s \in [0,1]$ and $z \in B_s(x,1)$, let $\gamma\colon [s,t] \to M$ be a curve such that $\gamma|_{[s,2]}$ is a minimizing geodesic connecting x and z with respect to $g(0)$, and $\gamma|_{[2,t]} \equiv y$.

For any $\tau \in [0, t-s]$, by Theorem 3.20 we have $R(y, t-\tau) \le C/r^2(y, t-\tau)$. Moreover, denoting $d_t(x,\Gamma)$ by $r(x,t)$, by Theorem 3.41 and distance distortion estimates we have

$$r(x,t) + 1.9(t-s) \le r(x,s) \le r(x,t) + 2.1(t-s). \tag{4-12}$$

So $R(y, t-\tau) \le C/(r(x,t) + C^{-1}\tau)^2$. For $\tau \in [t-2, t-s]$, we have $R(\gamma(t-\tau), t-\tau) \le C$ and $|\gamma'|(t-\tau) \le C$. Putting these together we can estimate the $\mathcal{L}$-length of γ,

$$\begin{aligned}\mathcal{L}(\gamma) &= \int_0^{t-2} \sqrt{\tau} R(y,\tau)\, d\tau + \int_{t-2}^{t-s} \sqrt{\tau}(R(\gamma(t-\tau), t-\tau) + |\gamma'|^2)\, d\tau \\ &\le \int_0^{t-2} \sqrt{\tau} \frac{C}{(r(x,t) + C^{-1}\tau)^2}\, d\tau + \int_{t-2}^{t-s} C\sqrt{\tau}\, d\tau \le C\sqrt{t}.\end{aligned}$$

Let $\ell(z,s) := \ell_{(x,t)}(z,s)$ be the reduced length from (x,t) to (z,s). Then

$$\ell(z,s) = \frac{\mathcal{L}_{(x,t)}(z,s)}{2\sqrt{t-s}} \le \frac{\mathcal{L}(\gamma)}{2\sqrt{t-s}} \le C.$$

Recalling the heat kernel lower bound by Perelman in [66, Corollary 9.5], we get

$$G(x,t;z,s) \ge \frac{1}{4\pi(t-s)^{3/2}} e^{-\ell(z,s)} \ge \frac{C}{t^{3/2}} \tag{4-13}$$

for all $s \in [0,1]$ and $z \in B_s(x,1)$, integrating which in $B_s(x,1)$ we get

$$\int_{B_s(x,1)} G(x,t;z,s)\, d_s z \ge \frac{C}{t^{3/2}} \quad \text{for all } s \in [0,1]. \tag{4-14}$$

Let $y \in M$, then by the multiplication inequality for the heat kernel in [54, Theorem 1.30] we have

$$\left(\int_{B_s(x,1)} G(x,t;z,s)\, d_s z\right)\left(\int_{B_s(y,1)} G(x,t;z,s)\, d_s z\right) \le C \exp\left(-\frac{(d_s(x,y)-2)^2}{4C(t-s)}\right). \tag{4-15}$$

Substituting (4-14) into (4-15) and using the distance distortion estimate $d_s(x,y) - 2 \ge C^{-1} d_0(x,y) - 2 \ge (2C)^{-1} d_0(x,y)$, we obtain

$$\left(\int_{B_s(y,1)} G(x,t;z,s)\, d_s z\right) \le C t^{3/2} \exp\left(-\frac{d_0(x,y)^2}{4C(t-s)}\right).$$

Integrating this for all $s\in[0,1]$, and then applying the parabolic mean-value inequality (see eg [36]) to $G(x,t;\cdot,\cdot)$ at $(y,0)$, we obtain

$$G(x,t;y,0)\le Ct^{\frac{3}{2}}\exp\left(-\frac{d_0^2(x,y)}{4Ct}\right).$$ □

Theorem 4.11 (scalar curvature polynomial upper bound) *Let (M,g,f) be a 3D steady gradient soliton that is not a Bryant soliton. Then for any integer $k\ge 2$, there exists $C_k>0$ such that*

$$R\le\frac{C_k}{d_g^k(\cdot,\Gamma)}.$$

Proof By Theorem 3.20 this is true for $k=2$. Let $(M,g(t))$ be the Ricci flow of the soliton. We denote $d_t(x,\Gamma)$ by $r(x,t)$. After a rescaling we assume $\lim_{s\to\infty}R(\Gamma_1(s))=\lim_{s\to\infty}R(\Gamma_2(s))=4$, so (4-12) holds.

Suppose by induction that this is true for $k\ge 2$, we will show that this is also true for $k+1$. In the following C denotes all positive constants that depend on k, the maximum of R and the limits of R at the two ends of Γ. Since R satisfies the evolution equation

$$\partial_t R=\Delta R+2|\mathrm{Ric}|^2,$$

for a fixed pair $(x,t)\in M\times(-\infty,\infty)$ we have

$$R(x,t)=\int_M G(x,t;y,s)R(y,s)\,d_sy+2\int_s^t\int_M G(x,t;z,s)|\mathrm{Ric}|^2(z,\tau)\,d_\tau z\,d\tau:=I(s)+II(s).$$

First, we claim that $\lim_{s\to-\infty}\mathrm{I}(s)=0$. To show this, we split $I(s)$ into two integrals on $B_s\big(x,\frac{1}{1000}r(x,s)\big)$ and $M\setminus B_s\big(x,\frac{1}{1000}r(x,s)\big)$, and denote them respectively by $I_1(s)$ and $I_2(s)$. Then for $I(s)$, using $\int_M G(x,t;y,s)\,d_sy=1$ we can estimate that

$$I_1(s)\le\left(\int_M G(x,t;y,s)\,d_sy\right)\cdot\left(\sup_{B_s\left(x,\frac{1}{1000}r(x,s)\right)}R(\cdot,s)\right)=\sup_{B_s\left(x,\frac{1}{1000}r(x,s)\right)}R(\cdot,s).$$

For any $y\in B_s\big(x,\frac{1}{1000}r(x,s)\big)$, we have $r(y,s)\ge r(x,s)-\frac{1}{1000}r(x,s)\ge\frac{1}{2}r(x,s)$. So by the inductive assumption, $R(y,s)\le C/r^k(x,s)$. So it follows by (4-12) that $I_1(s)\le C/r^2(x,s)$, which goes to zero as $s\to-\infty$.

For $I_2(s)$, since by (4-12) we have

$$\frac{d_s^2(y,x)}{t-s}\ge C^{-1}r(x,s)\ge C^{-1}(t-s)\quad\text{for all } y\in M\setminus B_s\big(x,\tfrac{1}{1000}r(x,s)\big),$$

it follows by the heat kernel estimates Lemma 4.10 and Lemma 2.21 that

$$G(x,t;y,s)\le Ce^{-d_s^2(y,x)/C(t-s)},$$

which implies

$$I_2(s)\le C\int_{M\setminus B_s\left(x,\frac{1}{1000}r(x,s)\right)}e^{-d_s^2(y,x)/C(t-s)}\,d_sy\le Ce^{-r(x,s)/C}\to 0\quad\text{as } s\to-\infty.$$

Next, we estimate $II(s)$. Let

$$J(\tau) := \int_M G(x,t;z,\tau)|\mathrm{Ric}|^2(z,\tau)\,d_\tau z,$$

and split it into two integrals on $B_\tau\big(x, \frac{1}{1000}r(x,\tau)\big)$ and $M \setminus B_\tau\big(x, \frac{1}{1000}r(x,\tau)\big)$, and denote them respectively by $J_1(\tau)$ and $J_2(\tau)$. Then by a similar argument as above and using the inductive assumption, we see that $J_1(\tau) \le C/r^{2k}(x,\tau)$ and $J_2(\tau) \le Ce^{-r(x,\tau)/C} \le C/r^{2k}(x,\tau)$.

Therefore, integrating $J(s)$ we obtain

$$\begin{aligned} II(s) = 2\int_s^t J(\tau)\,d\tau = 2\int_s^t (J_1(\tau)+J_2(\tau))\,d\tau &\le \int_s^t \frac{C}{r^{2k}(x,\tau)}\,d\tau \le \int_s^t \frac{C}{(r(x,t)+1.9(t-\tau))^{2k}}\,d\tau \\ &= C\left(\frac{1}{r^{2k-1}(x,t)} - \frac{1}{(r(x,t)+1.9(t-s))^{2k-1}}\right) \\ &\le \frac{C}{r^{2k-1}(x,t)}. \end{aligned}$$

Combining this with the estimate on $I(s)$, it follows that

$$R(x,t) \le \limsup_{s\to-\infty}(I(s)+II(s)) \le \frac{C}{r^{2k-1}(x,t)}.$$

Since $k \ge 2$, we have $2k-1 \ge k+1$, which proves the theorem by induction. □

5 Symmetry improvement theorems

In this section we will study the Ricci De Turck perturbations h whose background metric is an SO(2)-symmetric complete Ricci flow which is sufficiently close to the cylindrical plane $\mathbb{R}^2\times S^1$. Such symmetric 2-tensor can be decomposed as $h = h_+ + h_-$, where h_+ is the rotationally invariant mode and h_- is the oscillatory mode. We show that the oscillatory mode h_- decays in time exponentially in a certain sense. We will first prove the linear version of this symmetry improvement theorem, that is, the oscillatory mode of a linearized Ricci De Turck flow on $\mathbb{R}^2 \times S^1$ decays exponentially in time. Then we can obtain the theorem from its linear version by using a limiting argument.

More explicitly, $|h_-|$ decays exponentially in time in the following sense: First, if $|h_-|$ is initially bounded uniformly by a constant, then the theorem shows that it decays as $e^{-\delta_0 t}$ for some $\delta_0 > 0$. Moreover, if $|h_-|(\cdot,0)$ has an exponential growth in the space direction, then the theorem shows that $|h_-|(\cdot,t)$ still decays as $e^{-\delta_0 t}$ modulo the same exponential growth rate in the space direction.

5.1 SO(2)-decomposition of a symmetric 2-tensor

For a 3D Riemannian manifold (M,g), we say it is SO(2)-symmetric if it admits an effective isometric SO(2)-action. Equivalently, this means that there is a one-parameter group of isometries ψ_θ, with $\theta\in\mathbb{R}$, such that $\psi_\theta = \mathrm{id}$ if and only if $\theta = 2k\pi$ for $k\in\mathbb{Z}$. Throughout this section, we will moreover assume that

(M,g) is a 3D SO(2)-symmetric Riemannian manifold such that there exist a 2D Riemannian manifold (N,g_0) and a Riemannian submersion $\pi\colon (M,g)\to(N,g_0)$ which maps an orbit of the SO(2)-action to a point in N. This can be ensured when the SO(2)-action is free.

Let U be a local coordinate chart on N with coordinates $\rho\colon (x,y)\in U_0\subset\mathbb{R}^2\to\rho(x,y)\in U$. Take a section $s\colon U\to\pi^{-1}(U)\subset M$. Parametrize SO(2) by $\theta\colon[0,2\pi)\to\mathrm{SO}(2)$. Then we obtain a local coordinate on $\pi^{-1}(U)$ by $(x,y,\theta)\to\theta\cdot s(\rho(x,y))$.

Let h be a symmetric 2-tensor on M, and

$$h_+(y):=\frac{1}{2\pi}\int_0^{2\pi}(\theta^*h)(y)\,d\theta, \tag{5-1}$$

and $h_-:=h-h_+$. Then h_+ and h_- are two symmetric 2-tensors. For any $\theta_0\in\mathrm{SO}(2)$, we have

$$\theta_0^*h_+=\frac{1}{2\pi}\int_0^{2\pi}\theta_0^*(\theta^*h)\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}(\theta_0+\theta)^*h\,d\theta=\frac{1}{2\pi}\int_{\theta_0}^{2\pi+\theta_0}\theta^*h\,d\theta=h_+.$$

So we say h_+ is the rotationally invariant part and h_- is the oscillatory part of h, and $h=h_++h_-$ is the SO(2)-decomposition of h. Similarly, we say h is rotationally invariant if $h=h_+$, and oscillatory if $h_+=0$.

We now analyze the structure of the oscillatory mode more carefully. Since the one-forms dx, dy and $d\theta$ are invariant under the SO(2)-action, it follows that the basis $\{dx^2,dy^2,dx\,dy,dx\,d\theta,dy\,d\theta,d\theta^2\}$ of the space of all symmetric 2-tensors are rotationally invariant. So the SO(2)-decomposition of h reduces to the decomposition of components of h under this basis: h can be written as below under the local coordinates,

$$h=F_1\,dx^2+F_2\,dy^2+F_3\,dx\,dy+F_4\,dx\,d\theta+F_5\,dy\,d\theta+F_6\,d\theta^2,$$

where $F_i(x,y,\theta)\colon U_0\times S^1\to\mathbb{R}$ are functions. Let $F_{i,\pm}$ be the i^{th} component in $h_\pm$. Then

$$F_{i,+}(x,y,\theta)=\frac{1}{2\pi}\int_0^{2\pi}F_i(x,y,\theta')\,d\theta',$$

which is independent of θ, and

$$F_{i,-}(x,y,\theta)=\sum_{j=1}^{\infty}A_{i,j}(x,y)\cos(j\theta)+B_{i,j}(x,y)\sin(j\theta),$$

where

$$A_{i,j}(x,y)=\frac{1}{\pi}\int_0^{2\pi}F_i(x,y,\theta')\cos(j\theta')\,d\theta'\quad\text{and}\quad B_{i,j}(x,y)=\frac{1}{\pi}\int_0^{2\pi}F_i(x,y,\theta')\sin(j\theta')\,d\theta'.$$

We have the following observations. Suppose $\{M_i,g_i,x_i\}$ is a sequence of SO(2)-symmetric Riemannian manifolds, which smoothly converges to an SO(2)-symmetric Riemannian manifold $(M_\infty,g_\infty,x_\infty)$, and the convergence is SO(2)-equivariant. Suppose also that h_i is a sequence of symmetric 2-tensors on M_i that smoothly converges to a symmetric 2-tensor on M_∞. Write $h_i=h_{i,+}+h_{i,-}$ and $h_\infty=h_{\infty,+}+h_{\infty,-}$ for the SO(2)-decomposition. Then $h_{i,+}$ smoothly converges to $h_{\infty,+}$, and $h_{i,-}$ smoothly converges to $h_{\infty,-}$.

5.2 A symmetry improvement theorem in the linear case

Note that straightforward computation shows that the decomposition $h = h_+ + h_-$ is compatible with the linearized Ricci De Turck flow $\partial_t h = \Delta_L h$ on the cylindrical plane $\mathbb{R}^2 \times S^1$. In the following, we consider an oscillatory symmetric 2-tensor h on $\mathbb{R}^2 \times S^1$ which solves the linearized Ricci De Turck equation. Assume $|h|(\cdot, 0)$ satisfies an exponential growth bound, then the following proposition shows that $|h|$ decays exponentially in time in a certain sense.

Proposition 5.1 (on $\mathbb{R}^2 \times S^1$, linear) *Let $\delta_0 = 0.01$. There exists a $T_0 > 0$ such that for all*

$$\alpha \in [0, 2.02] \quad \textit{and} \quad T \geq T_0,$$

the following holds. Let $h(\cdot, t)$ for $t \in [0, T]$ be a continuous family of oscillatory tensors on $\mathbb{R}^2 \times S^1$, which is smooth on $t \in (0, T]$ and satisfies the linearized Ricci De Turck flow $\partial_t h = \Delta_L h$. Suppose that

$$|h(x, y, \theta, 0)| \leq A\, e^{\alpha\sqrt{x^2+y^2}} \tag{5-2}$$

for any $(x, y, \theta) \in \mathbb{R}^2 \times S^1$. Then

$$|h|(0, 0, \theta, T) \leq A\, e^{2\alpha T} \cdot e^{-\delta_0 T}.$$

Note that at $(0, 0, \theta)$, the upper bound $|h| \leq A$ at time 0 becomes $|h| \leq A\, e^{2\alpha T} \cdot e^{-\delta_0 T}$ at time T. For $\alpha = 0$, the bound at time T is $|h| \leq A \cdot e^{-\delta_0 T}$, in which case the exponential decay is clear. For $\alpha \neq 0$, there is an extra increasing factor $e^{2\alpha T}$, which seems to cancel out the effect of the decreasing factor $e^{-\delta_0 T}$. In this case, the exponential decay rate is measured by the time-dependent distance to the "basepoint" in a suitable Ricci flow. So the increasing factor $e^{2\alpha T}$ will be compensated for by the distance shrinking as going forward along the flow.

In Section 6, we will apply the nonlinear version of this proposition on the 3D flying wing with $\lim_{s\to\infty} R(\Gamma_i(s)) = 4$ for $i = 1, 2$. We will consider h satisfying the initial bound $|h| \leq e^{\alpha\, d_g(\cdot,\Gamma)}$. Since the soliton converges to $\mathbb{R} \times$ cigar along Γ, it follows that the distance to Γ shrinks at a speed arbitrarily close to 2. This will outweigh the increasing caused by $e^{2\alpha T}$. It is crucial that α can be slightly greater than 2 since we will rely on this to find an SO(2)-symmetric metric sufficiently close to the soliton metric so that the error decays like $e^{-(2+\delta)\, d_g(\cdot,\Gamma)}$ for some small but positive δ. So the error can decay faster than the scalar curvature as a consequence of Theorem 1.7.

Proof Since h is oscillatory, we can write it as

$$h(x, y, \theta, t) = F_1(x, y, \theta, t)\, dx^2 + F_2(x, y, \theta, t)\, dx\, dy + F_3(x, y, \theta, t)\, dy^2 + F_4(x, y, \theta, t)\, dx\, d\theta + F_5(x, y, \theta, t)\, dy\, d\theta + F_6(x, y, \theta, t)\, d\theta^2,$$

where F_i are in the form

$$F_i(x, y, \theta, t) = \sum_{j=1}^{\infty} A_{i,j}(x, y, t)\cos(j\theta) + B_{i,j}(x, y, t)\sin(j\theta),$$

where

$$A_{i,j}(x,y,t)=\frac{1}{\pi}\int_0^{2\pi} F_i(x,y,\theta',t)\cos(j\theta')\,d\theta',$$
$$B_{i,j}(x,y,t)=\frac{1}{\pi}\int_0^{2\pi} F_i(x,y,\theta',t)\sin(j\theta')\,d\theta'.$$

So by the assumption (5-2) we have $|F_i(x,y,\theta,0)|\le A\,e^{\alpha\sqrt{x^2+y^2}}$, and hence

$$|A_{i,j}|(x,y,0),\ |B_{i,j}|(x,y,0)\le 2Ae^{\alpha\sqrt{x^2+y^2}}. \tag{5-3}$$

Now, the tensor h satisfies $\partial_t h=\Delta_L h$, which in the coordinate (x,y,θ) is equivalent to

$$\partial_t F_i(x,y,\theta,t)=(\partial_{xx}+\partial_{yy}+\partial_{\theta\theta})F_i(x,y,\theta,t)=(\Delta_{\mathbb{R}^2}+\partial_{\theta\theta})F_i(x,y,\theta,t). \tag{5-4}$$

Solving (5-4) term by term we see

$$\partial_t A_{i,j}=\Delta_{\mathbb{R}^2}A_{i,j}-j^2A_{i,j}\quad\text{and}\quad \partial_t B_{i,j}=\Delta_{\mathbb{R}^2}B_{i,j}-j^2B_{i,j}.$$

So $A_{i,j}(x,y,t)\cdot e^{j^2(t-T)}$ and $B_{i,j}(x,y,t)\cdot e^{j^2(t-T)}$ satisfy the heat equation on $\mathbb{R}^2$. In the following, we will estimate these terms from above at $(0,0,T)$.

For convenience, we will omit the indices for a moment and let

$$u(x,y,t)=A_{i,j}(x,y,t)\cdot e^{j^2(t-T)}. \tag{5-5}$$

Then u satisfies the heat equation

$$\partial_t u=\Delta_{\mathbb{R}^2}u,$$

and by (5-3) we have

$$|u|(x,y,0)\le 2Ae^{\alpha\sqrt{x^2+y^2}}\cdot e^{-j^2T}. \tag{5-6}$$

Since $\cos 0.4\ge\frac{1}{1.1}$, it follows that for any $(x,y)\in\mathbb{R}^2$, there is

$$\sqrt{x^2+y^2}\le 1.1(x\cos\theta+y\sin\theta)\quad\text{for all }\ \theta\in[\theta_0-0.4,\theta_0+0.4],$$

where θ_0 satisfies $\cos\theta_0=x/\sqrt{x^2+y^2}$ and $\sin\theta_0=y/\sqrt{x^2+y^2}$. So for any α we have that

$$e^{\alpha\sqrt{x^2+y^2}}\le\frac{1}{0.8}\int_0^{2\pi} e^{1.1\alpha(x\cos\theta+y\sin\theta)}\,d\theta. \tag{5-7}$$

Let

$$v(x,y,t)=2Ae^{-j^2T}\cdot e^{(1.1\alpha)^2t}\cdot\frac{1}{0.8}\int_0^{2\pi} e^{(1.1\alpha)(x\cos\theta+y\sin\theta)}\,d\theta.$$

For any fixed θ, by a straightforward computation we see that the function $e^{(1.1\alpha)^2t}\cdot e^{(1.1\alpha)(x\cos\theta+y\sin\theta)}$ is a solution to the heat equation on $\mathbb{R}^2$. So it follows that v also satisfies the heat equation, ie

$$\partial_t v=\Delta_{\mathbb{R}^2}v.$$

Note by (5-6) and (5-7) we have $|u|(x,y,0) \le v(x,y,0)$. Moreover, by Lemma 2.22 we have a linear exponential growth bound on $|u|(x,y,t)$ for all later times $t \in [0,T]$, which may depend on T. This allows us to use the maximum principle (see eg [45]) and deduce that

$$|u|(0,0,T) \le v(0,0,T) = 2.5Ae^{-j^2T} \cdot e^{(1.1\alpha)^2 T}. \tag{5-8}$$

Since $\alpha \in [0, 2.02]$, it is easy to check that

$$2\alpha - (1.1\alpha)^2 + 1 \ge 0.1.$$

Take $T_0 = (\ln 2.5)/0.05$, then $e^{0.05T} > 2.5$ for all $T \ge T_0$, and hence

$$2.5\, e^{-T} \cdot e^{(1.1\alpha)^2 T} \le e^{2\alpha T} \cdot e^{-0.05\, T}.$$

Substituting this into (5-8), we obtain

$$|u|(0,0,T) \le A\, e^{-(j^2-1)T} \cdot e^{2\alpha T} \cdot e^{-0.05\, T}.$$

Restoring the indices in (5-5), we obtain

$$|A_{i,j}|(0,0,T) \le A\, e^{-(j^2-1)T} \cdot e^{2\alpha T} \cdot e^{-0.05\, T}.$$

Similarly, $|B_{i,j}|(0,0,T)$ satisfies the same inequality. Therefore, assuming $T_0 \ge (\ln 400)/0.04$, we obtain

$$\begin{aligned} |F_i|(0,0,T) \le \sum_{j=1}^{\infty} |A_{i,j}|(0,0,T) + \sum_{j=1}^{\infty} |B_{i,j}|(0,0,T) &\le 2A\, e^{2\alpha T} \cdot e^{-0.05\, T} \cdot \sum_{j=1}^{\infty} e^{-(j^2-1)T} \\ &\le 4A\, e^{2\alpha T} \cdot e^{-0.05\, T} \le \tfrac{1}{100} A\, e^{2\alpha T} \cdot e^{-0.01\, T}, \end{aligned}$$

which implies $|h|(0,0,T) \le A\, e^{2\alpha T} \cdot e^{-0.01\, T}$, and hence proves the lemma. □

5.3 A symmetry improvement theorem in the nonlinear case

In Theorem 5.3, we prove the nonlinear version of Proposition 5.1. In the theorem, h is a symmetric 2-tensor satisfying the Ricci De Turck flow perturbation with background metric $g(t)$ being an SO(2)-symmetric complete Ricci flow which is sufficiently close to $\mathbb{R}^2 \times S^1$ at a basepoint. We will show that the oscillatory part of h has a similar exponential decay in time as in Proposition 5.1.

We briefly recall some facts of Ricci De Turck flow perturbations from [9, Appendix A]. Let $(M, g(t))$ be a complete Ricci flow and h be a solution to the Ricci De Turck flow perturbation equation with background metric $g(t)$, given by

$$\nabla_{\partial_t} h = \Delta_{g(t)} h + 2\,\mathrm{Rm}_{g(t)}(h) + \mathcal{Q}_{g(t)}[h],$$

where $\mathcal{Q}_{g(t)}[h]$ is quadratic in h and its spatial derivatives, and the left-hand side contains the conventional Uhlenbeck trick,

$$(\nabla_{\partial_t} h)_{ij} = (\partial_t h)_{ij} + g^{pq}(h_{pj}\,\mathrm{Ric}_{qi} + h_{ip}\,\mathrm{Ric}_{qj}).$$

Then $\widetilde{h} := \alpha^{-1} h$ satisfies the rescaled Ricci De Turck flow perturbation equation

$$\nabla_{\partial_t} \widetilde{h} = \Delta_{g(t)} \widetilde{h} + 2\,\mathrm{Rm}_{g(t)}(\widetilde{h}) + \mathcal{Q}^{(\alpha)}_{g(t)}[\widetilde{h}],$$

which as $\alpha \to 0$ converges to the linearized Ricci De Turck equation,

$$\nabla_{\partial_t} \widetilde{h} = \Delta_{g(t)} \widetilde{h} + 2\,\mathrm{Rm}_{g(t)}(\widetilde{h}),$$

which can also be written as $\partial_t \widetilde{h} = \Delta_L \widetilde{h}$, where Δ_L is the Lichnerowicz laplacian

$$\Delta_L h_{ij} = \Delta h_{ij} + 2\, g^{kp} g^{\ell q} R_{kij\ell} h_{pq} - g^{pq}(h_{pj}\,\mathrm{Ric}_{qi} + h_{ip}\,\mathrm{Ric}_{qj}).$$

Theorem 5.3 is proved using a limiting argument: We consider a sequence of blowups of solutions h_i to the Ricci De Turck flow perturbation, and show that they converge to a solution to the linearized Ricci De Turck flow to which we can apply Proposition 5.1. To take the limit, we need to derive uniform bounds for h_i and the derivatives.

To this end, we first observe that for a solution h to the Ricci De Turck flow perturbation, $|h|^2$ satisfies the evolution inequality [9, Appendix A.1]

$$\partial_t |h|^2 \le (g+h)^{ij} \nabla^2_{ij} |h|^2 - 2(g+h)^{ij} g^{pq} g^{uv} \nabla_i h_{pu} \nabla_j h_{qv} + C(n) |\mathrm{Rm}_g| \cdot |h|^2 + C(n)|h| \cdot |\nabla h|^2,$$

where $C(n) > 0$ is some dimensional constant. Note that the elliptic operator $(g+h)^{ij} \nabla^2_{ij}$ is not exactly a laplacian of metrics. So in order to use the standard heat kernel estimates, we compare this operator with the exact laplacian $\Delta_{g(t)+h(t)}$ in the following lemma, and show that $\partial_t |h|^2 \le \Delta_{g(t)+h(t)} |h|^2 + |h|^2$.

Lemma 5.2 *For any $n \in \mathbb{N}$, there are constants $C_0(n)$ and $C_1(n)$ such that the following holds. Let $(M^n, g(t))$ with $t \in [0,T]$ be a Ricci flow (not necessarily complete) with $|\mathrm{Rm}| \le 1$ and $\mathrm{inj} \ge 1$, and let $h(t)$ be a Ricci flow perturbation with background $g(t)$. Suppose $|\nabla^k h| \le 1/C_0(n) < \frac{1}{100}$ for $k = 0, 1$. Then $|h|^2$ satisfies the evolution inequality*

$$\partial_t |h|^2 \le \Delta_{g(t)+h(t)} |h|^2 + C_1(n) |h|^2. \tag{5-9}$$

Proof In the following, the covariant derivatives and curvature quantities are taken with respect to $g(t)$, and the time-index t in $g(t)$ and $h(t)$ is suppressed. Let C denote all dimensional constants whose values may change from line to line. Let $(x^1, x^2, \dots, x^n)$ be local coordinates on an open subset $U \subset M$, such that $|\nabla^k g| \le C$ for $k = 0, 1$; for example we may choose the distance coordinates [67, Theorem 74]. By the formula of the Hessian $\nabla^2_{ij} f = \partial^2_{ij} f - \Gamma^k_{ij} \partial_k f$ for any smooth function f on U, it is easy to see

$$(g+h)^{ij} \nabla^2_{ij} |h|^2 = (g+h)^{ij} \partial^2_{ij} |h|^2 - (g+h)^{ij} \Gamma^k_{ij} \partial_k |h|^2,$$
$$\Delta_{g+h} |h|^2 = (g+h)^{ij} \partial^2_{ij} |h|^2 - (g+h)^{ij} \widetilde{\Gamma}^k_{ij} \partial_k |h|^2,$$

where $\widetilde{\Gamma}^k_{ij}$ is the Christoffel symbol of $g+h$. So we have

$$(g+h)^{ij} \nabla^2_{ij} |h|^2 - \Delta_{g+h} |h|^2 = (g+h)^{ij} (\widetilde{\Gamma}^k_{ij} - \Gamma^k_{ij}) \partial_k |h|^2.$$

Seeing that $|\widetilde{\Gamma}_{ij}^k - \Gamma_{ij}^k| \le C(|h| + |\nabla h|)$ and using the assumption $|\nabla^k h| \le 1/C_0(n)$ for $k = 0, 1$, for sufficiently large $C_0(n)$ we obtain

$$|(g+h)^{ij}\nabla_{ij}^2|h|^2 - \Delta_{g+h}|h|^2| \le C|h|^2|\nabla h| + C|h||\nabla h|^2 \le \tfrac{1}{2}|h|^2 + \tfrac{1}{2}|\nabla h|^2.$$

Combining this with Section 5.3, and noting that

$$2(g+h)^{ij} g^{pq} g^{uv} \nabla_i h_{pu} \nabla_j h_{qv} \ge 1.8|\nabla h|^2,$$

we obtain (5-9). □

Now we prove the main result of this section.

Theorem 5.3 (on almost cylindrical part, nonlinear) *There exist $\delta_0, T_0 > 0$ such that for any $T \ge T_0$, there exist $\overline{\epsilon}(T), \overline{\delta}(T), \underline{D}(T) > 0$ such that for any*

$$\alpha \in [0, 2.02], \quad \epsilon < \overline{\epsilon}, \quad \delta < \overline{\delta} \quad \text{and} \quad D_\# > \underline{D},$$

the following holds.

Let $(M, g(t), x_0)$ with $t \in [0, T]$ be a three-dimensional $\mathrm{SO}(2)$-symmetric complete Ricci flow with $|\mathrm{Rm}|_{g(t)} \le 1$, and suppose (M, g, x_0) is δ-close to $(\mathbb{R}^2 \times S^1, g_{\mathrm{stan}})$ in the C^{98}-norm. Let $h(t)$ be a Ricci De Turck flow perturbation with background metric $g(t)$ on $B_0(x_0, D_\#) \times [0, T]$ and suppose $|\nabla^k h(t)| \le \frac{1}{1000}$ for $k = 0, 1$. Note that the norms and derivatives are with respect to $g(0)$. Suppose also

$$\begin{cases} |\nabla^k h|(x,0) \le \epsilon \cdot e^{100\, d_0(x,x_0)} & \text{for } x \in B_0(x_0, D_\#) \text{ and } k = 0,1,2, \\ |h|(x,t) \le \epsilon \cdot e^{10\, D_\#} & \text{for } (x,t) \in \partial B_0(x_0, D_\#) \times [0,T]. \end{cases} \tag{5-10}$$

Suppose also

$$|h|(x,0) \le \epsilon \cdot e^{\alpha\, d_{g(0)+h(0)}(x,x_0)} \quad \text{for } x \in B_0(x_0, \underline{D}). \tag{5-11}$$

Then we have

$$|\nabla^k h_-|(x_0, T) \le \epsilon \cdot e^{-\delta_0 T} \cdot e^{2\alpha T} \quad \text{for } k = 0, 1, \dots, 100,$$

where h_- is the oscillatory part of h.

Proof Let $T_0 > 0$ be from Proposition 5.1, and the value of δ_0 will be determined later. Suppose the assertion does not hold for some $T \ge T_0$ and $C_0 > 0$. Then there are sequences of numbers $\epsilon_i \to 0$, $\delta_i \to 0$ and $D_{\#,i} > \underline{D}_i \to \infty$, a sequence of $\mathrm{SO}(2)$-symmetric complete Ricci flows $(M_i, g_i(t))$ with $t \in [0, T]$, which is δ_i-close to $\mathbb{R}^2 \times S^1$ at $(x_i, 0) \in M_i \times [0, T]$, and a sequence of Ricci De Turck flow perturbation $h_i(t)$ with background metric $g_i(t)$, defined on $B_0(x_i, D_{\#,i}) \times [0, T]$, such that (5-10) holds in $B_0(x_0, D_{\#,i})$, and

$$|h_i|(x,0) \le \epsilon_i \cdot e^{\alpha\, d_{g_i(0)+h_i(0)}(x,x_i)} \quad \text{on } B_0(x_i, \underline{D}_i), \tag{5-12}$$

but there is some $k_i \in \{0, 1, \dots, 100\}$ such that

$$|\nabla^{k_i} h_{i,-}|(x_i, T) \geq \epsilon_i \cdot e^{-\delta_0 T} \cdot e^{2\alpha T}. \tag{5-13}$$

After passing to a subsequence we may assume that the pointed Ricci flows $(M_i, g_i(t), x_i)$ on $[0, T]$ converge to $(\mathbb{R}^2 \times S^1, g_{\text{stan}}, x_0)$ in the C^{96}-sense. We will show that h_i/ϵ_i converge to a solution to the linearized Ricci De Turck flow on $\mathbb{R}^2 \times S^1$. To this end, since $|\nabla^k h| \leq \frac{1}{1000}$, $k = 0, 1$, by Lemma 5.2, there is $C_0 > 0$ such that

$$\partial_t |h_i|^2 \leq \Delta_{g_i+h_i} |h_i|^2 + C_0 \cdot |h_i|^2 \quad \text{on } B_0(x_i, D_{\#,i}) \times [0, T].$$

Let $u_i := e^{-C_0 t}|h_i|^2$, then this implies $\partial_t u_i \leq \Delta_{g_i+h_i} u_i$. Moreover, (5-10) implies

$$\begin{cases} u_i(x, 0) \leq \epsilon_i^2 \cdot e^{20\, d_{g_i(0)+h_i(0)}(x, x_i)} & \text{for } x \in B_0(x_i, D_{\#,i}), \\ u_i(x, t) \leq \epsilon_i^2 \cdot e^{20\, D_{\#,i}} & \text{for } x \in \partial B_0(x_i, D_{\#,i}) \text{ and } t \in [0, T]. \end{cases}$$

Applying the heat kernel estimate Lemma 2.22 and the weak maximum principle on $B_0(x_i, D_{\#,i}) \times [0, T]$, we obtain bounds on $|u_i|$ which are independent of all i: for any $A > 0$, there exists $C(A, T) > 0$, which is uniform for all i, such that

$$|u_i|(x, t) \leq C(A, T) \cdot \epsilon_i^2 \quad \text{on } B_0(x_i, A) \times [0, T],$$

and hence

$$|h_i|(x, t) \leq C(A, T) \cdot \epsilon_i \quad \text{on } B_0(x_i, A) \times [0, T], \tag{5-14}$$

with a possibly larger $C(A, T)$. By the local derivative estimates for the Ricci De Turck flow perturbations [9, Lemma A.14], this implies bounds for higher derivatives,

$$|\nabla^m h_i| \leq C_m(A, T) \cdot \epsilon_i \cdot t^{-m/2} \quad \text{on } B_0\big(x_i, \tfrac{1}{2}A\big) \times (0, T],$$

where $m \in \mathbb{N}$ and $C_m(A, T) > 0$ are constants depending on A, T and m. Moreover, the first inequality in (5-10) implies

$$|\nabla^k h_i| \leq C(A, T) \cdot \epsilon_i \quad \text{on } B_0\big(x_i, \tfrac{1}{2}A\big) \times [0, T] \text{ for } k = 0, 1, 2.$$

Therefore, letting $H_i = h_i/\epsilon_i$,

$$|\nabla^m H_i| \leq C_m(A, T) \cdot t^{-m/2} \quad \text{on } B_0\big(x_i, \tfrac{1}{2}A\big) \times (0, T],$$

and also

$$|\nabla^k H_i| \leq C(A, T) \quad \text{on } B_0\big(x_i, \tfrac{1}{2}A\big) \times [0, T] \text{ for } k = 0, 1, 2.$$

So after passing to a subsequence, H_i converges to a symmetric 2-tensor H_∞ on $(\mathbb{R}^2 \times S^1) \times [0, T]$ in the C^0-sense, and the convergence is smooth on $(0, T]$.

On the one hand, by the contradiction assumption (5-13) there is some $k_0 \in \{0, \dots, 100\}$ such that

$$|\nabla^{k_0} H_{\infty,-}|(x_0, T) \geq e^{-\delta_0 T} \cdot e^{2\alpha T}. \tag{5-15}$$

On the other hand, since $\epsilon_i \to 0$, it follows that H_∞ satisfies the linearized Ricci De Turck equation $\partial_t H_\infty = \Delta_L H_\infty$. The initial bound (5-12) passes to the limit and implies

$$|H_\infty|(x,0) \le e^{\alpha\, d_{g_{\mathrm{stan}}}(x,x_0)} \quad \text{for all } x \in \mathbb{R}^2 \times S^1.$$

So we can apply Proposition 5.1 to the oscillatory part $H_{\infty,-}$ at every point in the backward parabolic neighborhood $U := B_T(x_0,1) \times [T-1,T] \subset (\mathbb{R}^2 \times S^1) \times [0,T]$ centered at (x_0,T), and obtain

$$|H_{\infty,-}| < e^{\alpha+0.01} \cdot e^{-0.01\,T} \cdot e^{2\alpha T} \quad \text{on } U.$$

Since the components of $H_{\infty,-}$ satisfy the heat equation (5-4), it follows by the standard derivative estimates of heat equations that for all $k = 0, 1, \dots, 100$,

$$|\nabla^k H_{\infty,-}|(x_0,T) < C_k \cdot e^{-0.01T} \cdot e^{2\alpha T} \le e^{-\delta_0 T} \cdot e^{2\alpha T}$$

for some $\delta_0 < 0.01$, which contradicts with (5-15). □

6 Construction of an approximating SO(2)-symmetric metric

The main goal in this section is to construct an approximating SO(2)-symmetric metric whose error to the soliton metric is bounded by $e^{-2(1+\epsilon_0)\, d_g(\cdot,\Gamma)}$ for some positive constant $\epsilon_0 > 0$, and moreover the error goes to zero as we move towards the infinity of the soliton. Here $\Gamma = \Gamma_1(-\infty,\infty) \cup \Gamma_2(-\infty,\infty) \cup \{p\}$, where p is the critical point of M, and Γ_1 and Γ_2 are two integral curves from Corollary 3.36.

The construction consists of two parts. First, in Section 6.4, we do an inductive construction to obtain an SO(2)-symmetric metric $\overline{g}$ that approximates the soliton metric within the error $e^{-2(1+\epsilon_0) d_g(\cdot,\Gamma)}$. Next, in Section 6.5, we extend $\overline{g}$ to a neighborhood of Γ to obtain the desired approximating metric.

In the first step, we repeat the following process in an induction scheme: We consider the harmonic map heat flows from the Ricci flow of the soliton to the Ricci flow of some approximating SO(2)-symmetric metric. The error between the two flows is characterized by the Ricci De Turck flow perturbation, whose oscillatory mode decays in time by our symmetric improvement theorem. Therefore, the accuracy of the approximation will improve by the flow, after adding the rotationally symmetric mode in the Ricci De Turck flow perturbation to the approximating metric.

Note that the norm of the perturbation could grow very fast in the compact regions since we do not have a symmetry improvement theorem there. In order to deal with this, we will do surgeries to the soliton metric g and the approximating SO(2)-symmetric metrics, by cutting off their compact regions and gluing regions that are sufficiently close to $\mathbb{R}^2 \times S^1$. The resulting manifolds are diffeomorphic to $\mathbb{R}^2 \times S^1$, and close to $\mathbb{R}^2 \times S^1$ everywhere. So the harmonic map heat flows between the flows of these manifolds exist up to a long enough time for us to apply Theorem 5.3. In the surgeries, we need to glue ϵ-cylindrical planes, and for the SO(2)-symmetric metrics we also need to preserve the SO(2)-symmetry in the resulting metrics. This needs some gluing lemmas in Section 6.1. We conduct the surgeries in Sections 6.2 and 6.3.

6.1 Glue up SO(2)-symmetric metrics

In order to do the surgeries, we need to know how to glue SO(2)-symmetric metrics. This is done in this subsection. Recall that for a 3D Riemannian manifold (M,g), we say it is SO(2)-symmetric if there is a one-parameter group of isometries ψ_θ, with $\theta\in\mathbb{R}$, such that $\psi_\theta=\mathrm{id}$ if and only if $\theta=2k\pi$, $k\in\mathbb{Z}$. In this subsection, we show how to glue several SO(2)-symmetric metrics which are close to $(\mathbb{R}^2\times S^1,g_{\mathrm{stan}})$ and also close to each other on their intersections.

Since the metrics throughout this subsection are ϵ-close to $\mathbb{R}^2\times S^1$ for some very small ϵ, we will take derivatives and measure norms using the metric g_{stan} on $\mathbb{R}^2\times S^1$, since different choices of the ϵ-cylindrical planes only cause an error of $C_0\epsilon$ for some universal constant $C_0>0$. Moreover, we say two maps are ϵ-close in C^k-norm if their images are ϵ-close, and they are ϵ-close in C^k-norm under the standard coordinates of two ϵ-cylindrical planes ϵ-close to the preimage and image spaces.

First, we show in the following lemma that if a 3D Riemannian manifold (M,g) is ϵ-close to $\mathbb{R}^2\times S^1$ at $x_0\in M$ under two ϵ-isometries ϕ_1 and ϕ_2, then the two vector fields $\phi_{1*}(\partial_\theta)$ and $\phi_{2*}(\partial_\theta)$ are $C_0\epsilon$-close. Therefore, the vector field ∂_θ is well-defined on an ϵ-cylindrical plane up to sign and an error ϵ.

Lemma 6.1 *Let $k\in\mathbb{N}$. There exists $C_0,\bar\epsilon>0$ such that the following holds for all $\epsilon<\bar\epsilon$. Let (M,g) be a 3-dimensional Riemannian manifold. Suppose (M,g,x_0) is ϵ-close to $(\mathbb{R}^2\times S^1,g_{\mathrm{stan}})$ in the C^k-norm under two ϵ-isometries $\phi_i\colon S^1\times(-\epsilon^{-1},\epsilon^{-1})\times(-\epsilon^{-1},\epsilon^{-1})\to U_i\Subset M$ for $i=1,2$, where $V:=\phi_1(S^1\times(-100,100)\times(-100,100))\subset U_2$. Then after possibly replacing ϕ_2 by $\phi_2\circ p$, where $p(\theta,x,y)=(-\theta,x,y)$ for $\theta\in[0,2\pi)$ and $x,y\in(-\epsilon^{-1},\epsilon^{-1})$, we have*

$$|\phi_{1*}(\partial_\theta)-\phi_{2*}(\partial_\theta)|_{C^{k-1}(V)}\le C_0\epsilon.$$

Proof We shall use ϵ to denote all constants $C_0\epsilon$, where $C_0>0$ is a constant depending only on k. Let $g_i=(\phi_i^{-1})^*g_{\mathrm{stan}}$ and $X_i=\phi_{i*}(\partial_\theta)$ for $i=1,2$. Let (x,y,θ) be the coordinates on U_1 induced by ϕ_1 such that g_1 can be written as $d\theta^2+dx^2+dy^2$, where $x,y\in(-\epsilon^{-1},\epsilon^{-1})$ and $\theta\in[0,2\pi)$. So $X_1=\partial_\theta$. The coordinate function θ can be lifted to a function z on the universal covering $\widetilde U_1\to U_1$, so that the metric on $\widetilde U_1$ can be written as $dz^2+dx^2+dy^2$ under the coordinates (x,y,z).

Assume ϵ is sufficiently small. Then

$$|g_1-g_2|_{C^k(V)}\le|g_1-g|_{C^k(V)}+|g_2-g|_{C^k(V)}\le\epsilon. \tag{6-1}$$

Since $\mathscr{L}_{X_2}g_2=0$, this implies

$$|\mathscr{L}_{X_2}g_1|_{C^{k-1}(V)}\le|\mathscr{L}_{X_2}(g_2+(g_1-g_2))|_{C^{k-1}(V)}=|\mathscr{L}_{X_2}(g_1-g_2)|_{C^{k-1}(V)}\le\epsilon. \tag{6-2}$$

Note that all Killing fields on $\mathbb{R}^3$ have the form

$$a_1\partial_x+a_2\partial_y+a_3\partial_z+b_1(x\partial_z-z\partial_x)+b_2(x\partial_y-y\partial_x)+b_3(z\partial_y-y\partial_z),$$

where $a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{R}$. By a direct computation using (6-2), we see that X_2 is ϵ-close in the C^{k-1}-norm to a vector field on V of the form

$$a_1\partial_x+a_2\partial_y+a_3\partial_\theta+b_1(x\partial_\theta-\theta\partial_x)+b_2(x\partial_y-y\partial_x)+b_3(\theta\partial_y-y\partial_\theta). \tag{6-3}$$

In the following, we will estimate these coefficients and show that $|a_3 \pm 1| \le \epsilon$ and other coefficients are bounded in absolute values by ϵ. First, since X_2 is smooth, it follows that the vector field in (6-3) restricted at $\theta = 0$ and $\theta = 2\pi$ must be ϵ-close. So we must have

$$|b_1| + |b_3| \le \epsilon.$$

Next, since $\nabla^{g_2} X_2 = 0$, it follows by (6-1) that $|\nabla^{g_1} X_2|_{C^{k-1}(V)} \le \epsilon$, which implies $|b_2| \le \epsilon$. So X_2 is ϵ-close in the C^{k-1}-norm to the vector

$$Y := a_1 \partial_x + a_2 \partial_y + a_3 \partial_\theta,$$

and hence the flow generated by Y, which is

$$\begin{cases} x(t; x_0, y_0, \theta_0) = x_0 + a_1 t, \\ y(t; x_0, y_0, \theta_0) = y_0 + a_2 t, \\ \theta(t; x_0, y_0, \theta_0) = \theta_0 + a_3 t, \end{cases}$$

is ϵ-close in the C^{k-1}-norm to the flow generated by X_2 on V. Since the flow of X_2 is 2π-periodic, it follows that

(6-4) $$|a_1| + |a_2| + |2\pi a_3 - 2m\pi| \le \epsilon \quad \text{for some } m \in \mathbb{N}.$$

Note that for $i = 1, 2$, X_i is a unit-speed velocity vector of the g_i-minimal geodesic loop, so it is easy to see that for sufficiently small ϵ we must have either $|X_2 - X_1| \le \frac{1}{1000}$ or $|X_2 + X_1| \le \frac{1}{1000}$. This implies $|a_3 \pm 1| \le \frac{1}{100}$, which combined with (6-4) implies

$$|a_1| + |a_2| + |a_3 \pm 1| \le \epsilon,$$

which proves the lemma. □

The next lemma is a step further than Lemma 6.1, which shows that if an SO(2)-symmetric metric is ϵ-close to an ϵ-cylindrical plane, then their Killing fields are also $C_0\epsilon$-close.

Lemma 6.2 *Let $k \in \mathbb{N}_+$. There exists $C_0, \overline{\epsilon} > 0$ such that the following holds for all $\epsilon < \overline{\epsilon}$. Let (U, g) be a 3D Riemannian manifold (which is not necessarily complete), and let $x_0 \in U$. Suppose that g is* SO(2)*-symmetric metric and X is the Killing field of the* SO(2)*-isometry. Suppose also that (U, g, x_0) is ϵ-close to $(\mathbb{R}^2 \times S^1, g_{\text{stan}})$ in the C^k-norm, and*

$$|X - \partial_\theta| \le \tfrac{1}{1000} \quad \text{on } B_g(x_0, 1000) \Subset U,$$

where ∂_θ denotes the Killing field along the S^1-direction in an ϵ-cylindrical plane. Then

$$|X - \partial_\theta|_{C^{k-1}(B_g(x_0,1000))} \le C_0 \epsilon.$$

Note that the assumption $|X - \partial_\theta| \le \frac{1}{1000}$ in this lemma is necessary even if we derive a better bound using it. For example, on $\mathbb{R}^2 \times S^1$, an SO(2)-isometry could be either a rotation in the xy-plane around the origin, or a rotation in the S^1-factor, but their Killing fields are not close to each other.

Proof We shall use ϵ to denote all constants $C_0\epsilon$, where $C_0>0$ is a constant independent of ϵ. First, let ϕ_1 be the ϵ-isometry from $(\mathbb{R}^2\times S^1, g_{\mathrm{stan}})$ to (U,g,x_0), and let (x,y,θ) with $x,y\in(-\epsilon^{-1},\epsilon^{-1})$ and $\theta\in[0,2\pi)$ be local coordinates on an open subset V containing $B_g(x_0,1000)$, such that $g_1:=(\phi_1^{-1})^*g_{\mathrm{stan}}$ can be written as $d\theta^2+dx^2+dy^2$. Then θ can be lifted to a function z on the universal covering $\widetilde{V}\to V$, and the induced metric on $\widetilde{V}$ is $dz^2+dx^2+dy^2$.

Seeing that $\mathscr{L}_X g=0$, this implies

$$|\mathscr{L}_X g_1|_{C^{k-1}(U)}=|\mathscr{L}_X(g_1-g)|_{C^{k-1}(U)}\le\epsilon.$$

As in Lemma 6.1, this implies that X is ϵ-close in the C^{k-1}-norm to the vector field on V given by

$$Y:=a_1\partial_x+a_2\partial_y+a_3\partial_\theta+b_2(x\partial_y-y\partial_x), \tag{6-5}$$

where $a_1,a_2,a_3,b_2\in\mathbb{R}$.

In the following we will show that $|a_3-1|\le\epsilon$ and $|a_1|,|a_2|,|b_2|\le\epsilon$. First, assume $b_2\neq 0$. Then the flow generated by Y is

$$\begin{cases} y(t;x_0,y_0,\theta_0)=y_0\cos b_2t+\dfrac{a_1}{b_2}(1-\cos b_2t)+\dfrac{a_2}{b_2}\sin b_2t,\\ x(t;x_0,y_0,\theta_0)=x_0\cos b_2t+\dfrac{a_2}{b_2}(\cos b_2t-1)+\dfrac{a_1}{b_2}\sin b_2t,\\ \theta(t;x_0,y_0,\theta_0)=\theta_0+a_3t.\end{cases} \tag{6-6}$$

Since the flow generated by X is 2π-periodic and Y is ϵ-close to X, it follows that

$$|x(2\pi;x_0,y_0,\theta_0)-x_0|+|y(2\pi;x_0,y_0,\theta_0)-y_0|+|2\pi a_3-2m\pi|\le\epsilon \tag{6-7}$$

for some $m\in\mathbb{N}$. First, by $|X-\partial_\theta|\le\frac{1}{1000}$ we have

$$|Y-\partial_\theta|\le|X-Y|+|X-\partial_\theta|\le\epsilon+|X-\partial_\theta|\le\tfrac{1}{500},$$

which implies $|b_2|\le\frac{1}{100}$. Moreover, by taking $y_0=100,-100$ in (6-7) and using (6-6), we get

$$200|\cos(2\pi b_2)-1|=|y(2\pi;0,100,0)-100-y(2\pi;0,-100,0)+100|\le\epsilon,$$

which combined with $|b_2|\le\frac{1}{100}$ implies $|b_2|\le\epsilon$.

So we may assume $b_2=0$ in (6-5), so X is ϵ-close to the vector field in the C^{k-1}-norm on V given by

$$Z:=a_1\partial_x+a_2\partial_y+a_3\partial_\theta, \tag{6-8}$$

which generates the flow

$$\begin{cases} y(t;x_0,y_0,\theta_0)=y_0+a_2t,\\ x(t;x_0,y_0,\theta_0)=x_0+a_1t,\\ \theta(t;x_0,y_0,\theta_0)=\theta_0+a_3t.\end{cases} \tag{6-9}$$

The 2π-periodicity of the flow of X implies immediately

$$|a_1|+|a_2|+|2\pi a_3-2m\pi|\le\epsilon.$$

Using $|X-\partial_\theta|\le\frac{1}{1000}$ again this implies $|a_3-1|\le\epsilon$, which proves the lemma. □

In the following lemma, we show that one can glue one-parameter groups of diffeomorphisms that are ϵ-close to each other on their intersections, to obtain a global one-parameter group of diffeomorphisms that are $C_0\epsilon$-close to them, where the constant C_0 does not depend on ϵ.

Lemma 6.3 *Let $m,k \in \mathbb{N}_+$. There exists $C_0,\bar{\epsilon} > 0$ such that the following holds for all $\epsilon < \bar{\epsilon}$. Let (M,g) be a 3D Riemannian manifold. Suppose (M,g) is ϵ-close to $(\mathbb{R}^2\times S^1, g_{\text{stan}})$ at all $x \in M$. Suppose $\{U_i\}_{i=1}^{\infty}$ is an open covering of M such that at most m of them intersect at one point. Additionally, suppose that there is a one-parameter group of diffeomorphisms $\{\phi_{i,\theta}\}_{\theta\in\mathbb{R}}$ on each U_i which satisfies*

(1) $\phi_{i,0} = \phi_{i,2\pi} = \mathrm{id}$,

(2) $|\phi_{i,t*}(\partial_t)-\partial_\theta| \le \frac{1}{1000}$, *where ∂_θ denotes the Killing field along the S^1-direction in an ϵ-cylindrical plane up to sign*,

(3) $|\phi_i - \phi_j|_{C^k((U_i\cap U_j)\times S^1)} \le \epsilon$, *where $\phi_i(x,\theta) = \phi_{i,\theta}(x)$ for any $(x,\theta) \in U_i \times S^1$.*

Then there exists a one-parameter group of diffeomorphisms $\{\psi_t\}_{t\in\mathbb{R}}$ on M satisfying

(1) $\psi_0 = \psi_{2\pi} = \mathrm{id}$,

(2) $|\psi - \phi_i|_{C^k(U_i\times S^1)} \le C_0\epsilon$ *for all i*,

(3) $\psi_t = \phi_{i,t}$ *on $\{x \in M : B_g(x, 1000\, r(x)) \subset U_i\}$.*

Proof In the following, C_0 denotes all positive constants that depend on k. Since (M,g) is covered by ϵ-cylindrical planes, by a standard gluing argument we can find a smooth complete surface N embedded in M such that the tangent space of N is $\frac{1}{100}$-almost orthogonal to ∂_θ in each ϵ-cylindrical plane. Equip the manifold $N \times S^1$ with a warped product metric $\bar{g} = g_N + d\theta^2$, $\theta \times [0,2\pi)$, where g_N is the induced metric of (M,g) on N.

First, we will use the local one-parameter groups $\phi_{i,\theta}$ to construct local diffeomorphisms $F_i : N\times S^1 \to M$. Let $V_i = U_i \cap N$. Then $\{V_i\}_{i=1}^{\infty}$ is an open covering of N, and at most m of them intersect at one point. Let $F_i : V_i \times S^1 \to U_i$ be defined by

$$F_i(x,\theta) = \phi_{i,\theta}(\varphi(x)).$$

Then F_i is a diffeomorphism, and

$$|F_i - F_j|_{C^k((V_i\cap V_j)\times S^1)} \le C_0\epsilon.$$

Next, we will first construct a global diffeomorphism $F : N\times S^1 \to M$ by gluing up the diffeomorphisms F_i so that F is $C_0\epsilon$-close to each F_i. Suppose $\bar{\epsilon}$ is sufficiently small such that for any $x \in M$, $B_g(x,\bar{\epsilon})$ is a convex neighborhood of x, ie the minimizing geodesics connecting any two points in $B_g(x,\bar{\epsilon})$ are unique and contained in $B_g(x,\bar{\epsilon})$. Let Δ_2 be the open neighborhood of $\{(x,x) \in M^2\}$ the diagonal of M^2,

$$\Delta_2 = \{(x,y) \in M^2 : d_g(x,y) \le \bar{\epsilon}\} \subset M^2.$$

Define the smooth map

$$\Sigma_2 : \{(s_1,s_2) \in [0,1]^2 : s_1 + s_2 = 1\} \times \Delta_2 \to M \tag{6-10}$$

as $\Sigma_2(s_1,s_2,x_1,x_2)=\gamma_{x_2,x_1}(s_1)$, where $\gamma_{x_2,x_1}\colon[0,1]\to M$ is a minimizing geodesic from x_2 to x_1. Then for all $s_1,s_2\in[0,1]$ with $s_1+s_2=1$, and $x,x_1,x_2\in M$, Σ_2 satisfies the properties

(1) $\Sigma_2(1,0,x_1,x_2)=x_1$, $\Sigma_2(0,1,x_1,x_2)=x_2$,

(2) $\Sigma_2(s_1,s_2,x,x)=x$.

Then for each $K\in\mathbb{N}$, we can inductively construct the open neighborhood

$$\Delta_K=\{(x_1,\dots,x_K)\in M^K\colon d_g(x_i,x_j)\le\overline{\epsilon}\}\subset M^K$$

of $\{(x,\dots,x)\in M^K\}$ the diagonal in M^K (see also [7]), and the smooth map

$$\Sigma_K\colon\{(s_1,\dots,s_K)\in[0,1]^K : s_1+\dots+s_K=1\}\times\Delta_K\to M,$$

by defining

$$\Sigma_K(s_1,\dots,s_K,x_1,\dots,x_K):=\Sigma_2(1-s_K,s_K,\Sigma_{K-1}(s_1,\dots,s_{K-1},x_1,\dots,x_{K-1}),x_K),$$

with the following properties for all $s_1,\dots,s_K\in[0,1]$, $s_1+\dots+s_K=1$ and $x,x_1,x_2,\dots,x_K\in M$:

(1) If for some $j\in\{1,\dots,K\}$ we have $s_j=1$ and $s_i=0$ for all $i\neq j$, then
$$\Sigma_K(s_1,\dots,s_K,x_1,\dots,x_K)=x_j.$$

(2) $\Sigma_K(s_1,\dots,s_K,x,\dots,x)=x$.

(3) If $s_{K-i+1}=\dots=s_K=0$ for some $i\ge 1$, then
$$\Sigma_K(s_1,\dots,s_K,x_1,\dots,x_K)=\Sigma_{K-1}(s_1,\dots,s_{K-i},x_1,\dots,x_{K-i}).$$

Let $\{h_i\}_{i=1}^\infty$ be a partition of unity of $N\times S^1$ subordinated to $\{V_i\times S^1\}_{i=1}^\infty$, such that h_i is constant on each S^1-factor, and $h_i\equiv 1$ on $\widetilde{V}_i\times S^1$, where $\widetilde{V}_i=\{x\in N : B_{g_N}(x,500\,r(x))\subset V_i\}\times S^1$. Then let $F\colon N\times S^1\to M$ be such that for any $x\in N\times S^1$,

$$F(x):=\Sigma_K(h_1(x),\dots,h_K(x),F_1(x),\dots,F_K(x)),$$

where $K\in\mathbb{N}$ is some integer such that $h_{K'}(x)=0$ for any $K'>K$. By the properties of Δ_K and Σ_K, we see that F is a well-defined smooth map, and it satisfies

$$|F-F_i|_{C^k(V_i\times S^1)}\le C_0\epsilon. \tag{6-11}$$

Next, we will show that F is a diffeomorphism. First, by (6-11) and the definition of $\overline{g}$ we see that for any $p\in N\times S^1$ and $v\in T_p(N\times S^1)$, we have

$$|F_{*p}(v,v)|_g\ge 0.9|v|_{\overline{g}}. \tag{6-12}$$

So F_* is nondegenerate. Next, we argue that F is injective. To see this, note that (6-11) and assumption (2) imply that F is injective on $B_{g_N}(x,2)\times S^1$ for any $x\in N$, and $F(x_1\times S^1)\cap F(x_2\times S^1)=\varnothing$ for any $x_1,x_2\in N$ such that $d_{g_N}(x_1,x_2)\ge 1$. Now suppose $F(x_1,\theta_1)=F(x_2,\theta_2)$. Then we must have $d_{g_N}(x_1,x_2)<1$, and hence $x_1=x_2,\theta_1=\theta_2$ as desired. Therefore, F is a diffeomorphism.

Next, let $\theta \in [0, 2\pi)$ be the parametrization of S^1, and let $X := F_*(\partial_\theta)$. Then X generates a 2π-periodic one-parameter group of diffeomorphisms ψ_θ with $\theta \in [0, 2\pi)$ on M. In the following we will show that ψ_θ satisfies all the properties. Write $\psi(x, \theta) = \psi_\theta(x)$ for all $(x, \theta) \in M \times S^1$. First, let $\widetilde{U}_i = \{x \in M : B_g(x, 1000\, r(x)) \subset U_i\}$. Then it is easy to see that $\widetilde{U}_i \subset F(\widetilde{V}_i \times S^1)$. Since $h_i \equiv 1$ on $\widetilde{V}_i \times S^1$, it follows that $F = F_i$, and hence $\psi_\theta = \phi_{i,\theta}$ on $\widetilde{U}_i$, which verifies property (3).

Lastly, to verify property (2), we note that for any $x \in U_i = F_i(V_i \times S^1)$, supposing $x = F_i(z, \theta_1)$, then for any $\theta \in [0, 2\pi)$, we have

$$\phi_{i,\theta}(x) = \phi_{i,\theta+\theta_1}(x) = F_i(z, \theta_1 + \theta)$$

and also

$$\psi(F \circ F_i^{-1}(x), \theta) = \psi(F(z, \theta_1), \theta) = F(z, \theta_1 + \theta).$$

Therefore, using (6-11) the closeness of F and F_i, as well as the closeness of $F \circ F_i^{-1}\colon U_i \to M$ and $\mathrm{id}\colon U_i \to M$, we can deduce that

$$|\psi - \phi_i|_{C^k(U_i \times S^1)} \le C_0 \epsilon,$$

which verifies (2). □

Now we prove the main result of this subsection, which shows that if there are some SO(2)-symmetric metrics which are ϵ-close to each other, then we can glue them together to obtain a global SO(2)-symmetric metric which is $C_0\epsilon$-close to the original metrics.

Lemma 6.4 *Let $k, m \in \mathbb{N}_+$. There exists $C_0, \overline{\epsilon} > 0$ such that the following holds for all $\epsilon < \overline{\epsilon}$. Let (M, g) be a 3D Riemannian manifold diffeomorphic to $\mathbb{R}^2 \times S^1$, which is ϵ-close to $(\mathbb{R}^2 \times S^1, g_{\mathrm{stan}})$ at all $x \in M$. Suppose $\{U_i\}_{i=1}^\infty$ is a locally finite covering of M such that at most m of them intersect at one point, and there is an SO(2)-symmetric metric g_i on $W_i := \bigcup_{x \in U_i} B_g(x, 1000 r(x))$, with the SO(2)-isometry $\{\phi_{i,\theta}\}_{\theta \in [0,2\pi)}$ and Killing field X_i, which satisfies*

$$|g_i - g|_{C^k(W_i)} \le \epsilon \quad \text{and} \quad |X_i - \partial_\theta|_{C^0(W_i)} \le \tfrac{1}{1000}, \tag{6-13}$$

where ∂_θ denotes the Killing field along the S^1-direction in an ϵ-cylindrical plane, up to sign. Let $\{h_i\}_{i=1}^\infty$ be a partition of unity subordinate to $\{U_i\}_{i=1}^\infty$. Then we can find an SO(2)-symmetric metric $\overline{g}$ on M with SO(2)-isometry $\{\psi_\theta\}_{\theta \in [0,2\pi)}$ such that

$$|\overline{g} - g|_{C^{k-1}(M)} \le C_0\epsilon \quad \text{and} \quad |\psi_\theta - \phi_{i,\theta}|_{C^{k-1}(U_i)} \le C_0\epsilon. \tag{6-14}$$

Moreover, $\overline{g} = g_i$ on the subset $\{x \in M : h_i(\phi_{i,\theta}(x)) = 1 \text{ for } \theta \in [0, 2\pi)\}$.

Proof In the following C_0 denotes all positive constants that depend on k and m. Let

$$V_i = \{x \in M : h_i(\phi_{i,\theta}(x)) = 1 \text{ for } \theta \in [0, 2\pi)\} \subset U_i \subset W_i.$$

First, applying Lemma 6.2 to g_i we have

$$|X_i - \partial_\theta|_{C^{k-1}(U_i)} \le C_0\epsilon, \tag{6-15}$$

where ∂_θ is the Killing field along the S^1-direction in an ϵ-cylindrical plane up to sign. As in Lemma 6.3, let N be a 2D complete surface smoothly embedded in M, whose tangent space is $\frac{1}{100}$-almost orthogonal to ∂_θ.

Next, we will construct a diffeomorphism $\sigma\colon N\times S^1\to M$ such that $\sigma|_{N\times\{0\}}=\mathrm{id}_N$ and $\sigma_*(\partial_\theta)$ is $\frac{1}{100}$-close to the S^1-factor of any ϵ-cylindrical planes. To do this, first we can find a covering of M by ϵ-cylindrical planes such that the number of them intersecting at any point is bounded by a universal constant. Then we claim that after reversing the θ-coordinate in certain ϵ-cylindrical planes we can arrange that the vector fields ∂_θ are $C_0\epsilon$-close in the intersections. Suppose the claim does not hold. Then by Lemma 6.1 it is easy to find an embedded Klein bottle in M. Since M is diffeomorphic to $\mathbb{R}^2\times S^1$, which can be embedded into $\mathbb{R}^3$ as a tubular neighborhood of a circle, it follows that the Klein bottle can be embedded in $\mathbb{R}^3$, which is impossible by [53, Corollary 3.25]. Now the diffeomorphism σ follows immediately from applying Lemma 6.3, since the one-parameter groups of these ϵ-cylindrical planes are $C_0\epsilon$-close.

Therefore, we can replace X_i by $-X_i$ for some i so that they are all $\frac{1}{100}$-close to $\sigma_*(\partial_\theta)$. So (6-15) implies

$$|X_i-X_j|_{C^{k-1}(U_i\cap U_j)}\le C_0\epsilon. \tag{6-16}$$

Replacing $\phi_{i,\theta}$ by $\phi_{i,-\theta}$ for such i, this then implies

$$|\phi_{i,\theta}-\phi_{j,\theta}|_{C^{k-1}(U_i\cap U_j)}\le C_0\epsilon. \tag{6-17}$$

Then by Lemma 6.3 we can construct a one-parameter group of diffeomorphisms $\{\psi_\theta\}$ on M such that $\psi_0=\psi_{2\pi}=\mathrm{id}$ and

$$|\psi_\theta-\phi_{i,\theta}|_{C^{k-1}(U_i)}\le C_0\epsilon,$$

and $\psi_\theta=\phi_{i,\theta}$ on V_i.

Let $\hat g=\sum_{i=1}^\infty h_i\cdot g_i$. Then $\hat g=g_i$ on V_i and by (6-13) we have

$$|\hat g-g|_{C^{k-1}}\le C_0\epsilon\quad\text{on } M.$$

Let

$$\bar g=\frac{1}{2\pi}\int_0^{2\pi}\psi_\theta^*\hat g\,d\theta.$$

Then $(M,\bar g)$ is SO(2)-symmetric under the isometries ψ_θ, and

$$\begin{aligned}|\bar g-g_i|_{C^{k-2}(U_i)}&\le\frac{1}{2\pi}\int_0^{2\pi}|\psi_\theta^*g-\phi_{i,\theta}^*g_i|_{C^{k-2}(U_i)}\,d\theta\\&\le\frac{1}{2\pi}\int_0^{2\pi}|\psi_\theta^*(g-g_i)|_{C^{k-2}(U_i)}+|\psi_\theta^*g_i-\phi_{i,\theta}^*g_i|_{C^{k-2}(U_i)}\,d\theta\le C_0\epsilon,\end{aligned}$$

which combined with (6-13) implies (6-14).

Moreover, if $x\in V_i$, then $\psi_\theta(x)=\phi_{i,\theta}(x)$ and $\hat g(x)=g_i(x)$, so we have

$$\bar g(x)=\frac{1}{2\pi}\int_0^{2\pi}\psi_\theta^*(\hat g(\psi_\theta(x)))\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\phi_{i,\theta}^*(g_i(\phi_{i,\theta}(x)))\,d\theta=g_i(x),$$

which finishes the proof. □

6.2 Surgery on the soliton metric

In this subsection we will conduct a surgery on the soliton by first removing a neighborhood of the edges Γ and then grafting a region covered by ϵ-cylindrical planes onto the soliton. After the surgery, we obtain a complete metric on $\mathbb{R}^2 \times S^1$, which is covered everywhere by ϵ-cylindrical planes.

We fix some conventions and notations. First, in the rest of the entire section we assume (M, g) is a 3D steady gradient soliton with positive curvature that is not a Bryant soliton. Then by Theorem 3.41 we may assume after a rescaling that

$$\lim_{s\to\infty} R(\Gamma_1(s)) = \lim_{s\to\infty} R(\Gamma_2(s)) = 4. \tag{6-18}$$

Next, let $\rho = d_g(\cdot, \Gamma)$. For any $0 < A < B$ we write

$$\Gamma_{[A,B]} = \rho^{-1}([A, B]), \quad \Gamma_{\le A} = \rho^{-1}([0, A]), \quad \Gamma_{\ge A} = \rho^{-1}([A, \infty)), \quad \Gamma_A = \rho^{-1}(A).$$

In the following, we construct a complete metric on $\mathbb{R}^2 \times S^1$ and a vector field so that they are equal to the soliton metric and ∇f on the infinite triangular region $\Gamma_{>A}$, and it is covered by ϵ-cylindrical planes. Using the flow of V we construct a flow $g'(t)$ on M' which then satisfies the Ricci flow equation in a region of $M' \times [0, \infty)$, that looks like the space beneath a staircase. In future proofs in this section, the flow $(M', g'(t))$ will be used as domains of harmonic map heat flows.

Lemma 6.5 (ϵ-grafted soliton, ϵ-grafted soliton flow) *Let (M, g) be a 3D steady gradient soliton with positive curvature that is not a Bryant soliton, satisfying (6-18). For any $\epsilon > 0$ and $m \in \mathbb{N}$, there exists a quadruple (M', g', V, A) which consists of a complete Riemannian manifold (M', g'), a smooth vector field V on M', and a constant $A > 200$, such that the following hold:*

1. *M' is diffeomorphic to $\mathbb{R}^2 \times S^1$.*
2. *There is an isometric embedding $\tau\colon (\Gamma_{>A-200}, g) \to (M', g')$. So we will identify $\Gamma_{>B}$ as a subset in M' for any $B \ge A - 200$.*
3. *(M', g') is an ϵ-cylindrical plane at any point $x \in M'$ in the C^m-norm.*
4. *$V = \nabla f$ on $\Gamma_{>A}$, and $V = 0$ on $M' \setminus \Gamma_{>A-200}$, and $|\nabla^k V| \le 1000$ for $k = 0, \dots, m$.*
5. *Let $\{\psi_t\}_{t\in\mathbb{R}}$ be the flow of diffeomorphisms generated by V with $\psi_0 = \mathrm{id}$. Let $(M', g'(t))$ be a smooth flow for $t \ge 0$, where $g'(t) = \psi_{-t}^* g'$. Then $\psi_t = \phi_t$ for $t \ge 0$ on $\Gamma_{>A}$, and*

$$\Gamma^t_{>B} := \psi_t(\Gamma_{>B}) = \phi_t(\Gamma_{>B}).$$

Moreover, $g'(t)$ satisfies the Ricci flow equation on the open subset

$$\bigcup_{t\ge 0} (\Gamma^t_{>A} \times \{t\}) \subset M' \times [0, \infty).$$

We call (M', g', V, A) the ***ϵ-grafted soliton***, *and $(M', g'(t), A)$ the* ***ϵ-grafted soliton flow***.

Proof Let $A>0$ be sufficiently large that $\Gamma_{\geq A}$ is covered by ϵ-cylindrical planes. We may furthermore increase A depending on m, ϵ and the soliton (M,g).

By using the Ricci flow equation $\partial_t g(t)=-2\operatorname{Ric}(g(t))$, where $g(t)=\phi_{-t}^* g$, the quadratic curvature upper bound in Theorem 4.11, and Shi's derivative estimates [68], we may assume when A is sufficiently large that

$$|\nabla^{\ell}(g-\phi_{-(k+1)}^* g)|\leq \epsilon \quad \text{on } \phi_k(\Gamma_{\geq A}) \text{ for } \ell=0,\dots,m,$$

where the covariant derivatives are taken with respect to g, and $k\in\mathbb{N}$. Therefore, for each $k\in\mathbb{N}_+$, by a standard gluing argument we can construct a metric g_k on $\Gamma_{\geq A}$ which satisfies

$$g_k=\phi_{-k}^* g \quad \text{on } \phi_k(\Gamma_{\geq A}),$$

and for all $i=0,1,2,\dots,k-1$,

$$|\nabla^{\ell}(g_k-g)|\leq C_0\epsilon \quad \text{on } M \text{ for } \ell=0,\dots,m, \tag{6-19}$$

where here and below C_0 denotes all positive constants that only depend on the soliton and m.

Now fixed a point $p\in\Gamma_{\geq A}\subset M$ and let $p_k:=\phi_k(p)$. Then after passing to a subsequence we may assume that the pointed manifolds $(\Gamma_{\geq A},g_k,p_k)$ converge to a smooth manifold (M',g',p'). At the same time, the isometric embeddings $\phi_k\colon(\Gamma_{\geq A},g,p)\to(\phi_k(\Gamma_{\geq A}),g_k,p_k)\subset(M,g_k,p_k)$ smoothly converge to an isometric embedding $\pi\colon(\Gamma_{\geq A},g,p)\to(M',g',p')$ in the C^m-sense.

Furthermore, by (6-19), we see that (M',g',p') is $C_0\epsilon$-close to the smooth limit of (M,g,p_k), which by Lemma 3.3 must be isometric to $\mathbb{R}^2\times S^1$ after a suitable rescaling. In particular, this implies that (M',g') is complete, diffeomorphic to $\mathbb{R}^2\times S^1$, and covered by $C_0\epsilon$-cylindrical planes.

Now assume A sufficiently large, and replace A by $A-200$ and also ϵ by ϵ/C_0. Then assertions (1), (2) and (3) are satisfied. We can find a vector field V on M' which satisfies the conditions in (4). Then (5) follows by using the fact that $g(t)=\phi_{-t}^* g$ and $V=\nabla f$ on $\Gamma_{>A}$. □

6.3 Surgery on SO(2)-symmetric metrics

The next lemma allows us to do an SO(2)-invariant extension to an SO(2)-symmetric metric $\widehat{g}$ defined on an open subset containing $\Gamma_{\geq B}$ for some large $B>0$. It extends the incomplete SO(2)-symmetric metric $\widehat{g}$ to a complete SO(2)-symmetric metric. Moreover, if $\widehat{g}$ is close to the soliton metric g, then the resulting complete metric is close to the grafted soliton metric g'.

In future proofs in this section, we will run harmonic map heat flows from the grafted soliton flow $(M',g'(t))$ to Ricci flows starting from some suitable SO(2)-symmetric metrics we obtained by the SO(2)-invariant surgery.

Lemma 6.6 (SO(2)-invariant extension) *There are constants $C_0 > 0$ such that the following holds. For any $\epsilon > 0$, let (M', g') be the ϵ-grafted soliton from Lemma 6.5 with $m = 1000$ and $A = A(\epsilon, m)$. Then for any $B > A$, suppose $\hat{g}$ is an $\mathrm{SO}(2)$-symmetric metric on an open subset $U \supset \Gamma_{\geq B}$ with the $\mathrm{SO}(2)$-isometry $\{\psi_\theta\}_{\theta\in[0,2\pi)}$ and the Killing field X, such that*

(6-20) $$|\hat{g} - g'|_{C^{100}(U)} \leq \epsilon \quad \text{and} \quad |X - \partial_\theta|_{C^0(U)} < \tfrac{1}{1000},$$

where ∂_θ is the Killing field of an ϵ-cylindrical plane (by Lemma 6.5, (M', g') is covered by ϵ-cylindrical planes). Then there is an $\mathrm{SO}(2)$-symmetric metric $\tilde{g}$ on M' with the $\mathrm{SO}(2)$-isometry $\tilde{\psi}_{\theta\in[0,2\pi)}$ and the corresponding Killing field $\tilde{X}$, such that

$$|\tilde{g} - g'|_{C^{98}(M')} \leq C_0\,\epsilon \quad \text{and} \quad |\tilde{X} - \partial_\theta|_{C^{98}(M')} \leq C_0\,\epsilon.$$

Moreover, we have $\tilde{g} = \hat{g}$ and $\tilde{\psi}_\theta = \psi_\theta$ for $\theta \in [0, 2\pi)$ on $\Gamma_{\geq B+100}$.

Proof It is easy to find a sequence of points $\{x_i\}_{i=1}^\infty \subset M' \setminus U$ such that the number of metric balls $B_{g'}(x_i, 1000)$ that intersect at any point is bounded by a universal constant, and $\{B_{g'}(x_i, 1000)\}_{i=1}^\infty$ together with U form an open covering of M'. Let $\{h_i\}_{i=0}^\infty$ be a partition of unity subordinate to $\{U_i\}_{i=0}^\infty$ such that the function h_0 satisfies $h_0(\psi_\theta(x)) = 1$ for all $x \in \Gamma_{\geq B+100} \subset U$. By the property of (M', g'), there is an ϵ-isometry $(\mathbb{R}^2 \times S^1, g_{\mathrm{stan}}) \to (M', g', x_i)$ for each i. The assertions now follow by applying Lemma 6.4 to glue all the $\mathrm{SO}(2)$-isometries from the ϵ-cylindrical planes on $B_{g'}(x_i, 1000)$ and the $\mathrm{SO}(2)$-isometry on U. □

6.4 An approximating metric away from the edge

In this subsection, we construct an SO(2)-symmetric approximating metric away from the edge Γ such that the error decays at the rate $O(e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)})$.

In the proof of Theorem 6.7, we will need to choose some constants that are sufficiently large or small such that certain requirements are satisfied. In order to show that the dependence between these constants is not circular, we introduce the parameter order

$$\delta_0, \quad C_0, \quad T, \quad \underline{D}, \quad \epsilon, \quad A, \quad D,$$

such that each parameter is chosen depending only on the preceding parameters.

Theorem 6.7 (approximation with good exponential decay) *Let (M, g, f, p) be a 3D steady gradient soliton that is not a Bryant soliton satisfying (6-18). Then there exist constants $\epsilon_1, A_1 > 0$ such that $\Gamma_{\geq A_1} \subset M$ is covered by ϵ-cylindrical planes of g, and an $\mathrm{SO}(2)$-symmetric metric $\hat{g}$ on an open subset containing $\Gamma_{\geq A_1}$, and*

$$|\nabla^m (g - \hat{g})| \leq e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)} \quad \text{on } \Gamma_{\geq A_1} \text{ for } m = 0, \dots, 100,$$

where the covariant derivatives and norms are taken with respect to g. Moreover, let X be the Killing field of the $\mathrm{SO}(2)$-isometry of $\hat{g}$, and ∂_θ be the $\mathrm{SO}(2)$-Killing field of an ϵ-cylindrical plane of g, then we have $|X - \partial_\theta| \leq \frac{1}{1000}$.

Proof We choose the following constants which satisfy the above parameter order, and whose values may be further adjusted later:

(1) Let $\delta_0 > 0$ be from Theorem 5.3.

(2) Let $C_0 > 0$ be the maximum of 1 and the constants $C_0 > 0$ from Lemma 6.1, 6.4 and 6.6.

(3) Let $T > T_0$, where T_0 is from Theorem 5.3. Assume also that $2C_0^2 e^{400} < e^{\frac{1}{2}\delta_0 T}$ and $T^{1/2} > 160/\delta_0$.

(4) (a) Let $0 < \epsilon < \min\{\frac{1}{1000}\overline{\delta}(T)/C_0^2, \overline{\epsilon}(T)/(e^{400}C_0^2)\}$, where $\overline{\delta}(T), \overline{\epsilon}(T) > 0$ are constants determined by Theorem 5.3.

(b) For any complete metric $\tilde{g}$ on $\mathbb{R}\times S^1$ which is covered by $2C_0\epsilon$-cylindrical planes in the C^m-sense, the Ricci flow $(M', \tilde{g}(t))$ starting from $\tilde{g}$ stays smooth and $|\mathrm{Rm}|_{\tilde{g}(t)} \le 1/T$ for all $t\in[0,T]$.

(c) Let $\tilde{g}_i(t)$ for $i = 1, 2$ and $t\in[0,T]$ be two smooth families of metrics on $\mathbb{R}^2\times S^1$ that are covered by $2C_0\epsilon$-cylindrical planes. Then the harmonic map heat flow

$$\{\chi_t\}\colon (\mathbb{R}^2\times S^1, \tilde{g}_1(t)) \to (\mathbb{R}^2\times S^1, \tilde{g}_2(t)), \quad \text{with } \chi_0 = \mathrm{id},$$

is smooth for $t\in[0,T]$. The existence of such ϵ is guaranteed by [9, Lemma A.24], the existence of harmonic map heat flows, and estimates of perturbations.

(5) Let $A > 0$ be sufficiently large that:

(a) (M', g', V, A) is an ϵ/C_0-grafted soliton by Lemma 6.5. In particular, the ϵ/C_0-grafted soliton flow $(M', g'(t))$ satisfies the Ricci flow equation on

$$\bigcup_{t\ge 0}\Gamma^t_{>A} \subset M'\times[0,\infty).$$

(b) By Theorem 3.41 and (6-18), we may assume that for any point $x\in\Gamma_{\ge A}\subset M$ and $t\ge 0$,

$$2 - \tfrac{1}{16}\delta_0 \le \frac{d}{dt} d_g(\phi_t(x), \Gamma) \le 2 + \tfrac{1}{16}\delta_0. \tag{6-21}$$

(6) Let $D = \max\{A, 100\ln C_0, \ln\epsilon^{-1}, 10\underline{D}, 100T^{1/2}, e^{400}\}$, where $\underline{D}(T) > 0$ is determined by Theorem 5.3.

Step 1 In this step, we impose two inductive assumptions. In the first induction, we construct a sequence of metrics $\hat{g}_n$ such that $\hat{g}_n$ has an exponential decay rate of α_n, and the sequence α_n increases by a positive amount one by one. So the last metric $\hat{g}_N$ will satisfy all assertions of the theorem. Suppose the first inductive assumption holds for a fixed $n < N$. Then in the second induction we construct an infinite sequence of metrics $\{\hat{g}_{n,k}\}_{k=1}^{\infty}$, where $\hat{g}_{n,0} = \hat{g}_n$, and $\hat{g}_{n,k}$ has the better decay rate of α_{n+1} on larger and larger domains depending on k. The domains will eventually cover the part sufficiently far away from Γ. So letting $i\to\infty$, we can take a limit of $\hat{g}_{n,k}$ and obtain $\hat{g}_{n+1}$ that satisfies the decay rate of α_{n+1} everywhere, and thus verifies the first inductive assumption for $n+1$. We can repeat this process until $\alpha_n > 2$, which completes the proof.

Inductive Assumption One

(1) For any $n \in \mathbb{N}$, if $\alpha_n := (n \ln C_0)/D \le 2.02$, there are a sequence of increasing constants $D_n > 0$ and an SO(2)-symmetric metric $\widehat{g}_n$ defined on an open subset in $M \setminus \Gamma$ containing $\Gamma_{\ge D_n}$ such that

$$|\nabla^m(\widehat{g}_n - g)| \le \epsilon \cdot e^{-\alpha_n(d_g(\cdot,\Gamma)-D_n)} \quad \text{on } \Gamma_{\ge D_n} \text{ for } m = 0, \dots, 100.$$

(2) Moreover, let X_n be the Killing field of the SO(2)-isometry of $\widehat{g}_n$. Then on any ϵ-cylindrical plane of g, the SO(2)-Killing field ∂_θ satisfies

$$|X_n - \partial_\theta| \le \tfrac{1}{1000} \quad \text{on } \Gamma_{\ge D_n}.$$

Suppose Inductive Assumption One is true for a moment. Since $(\ln C_0)/D < 0.01$, let N be an integer such that $2 < (N \ln C_0)/D \le 2.02$. Then the metric $\widehat{g}_N$ on $\Gamma_{\ge D_N}$ satisfies all assertions of the theorem, with $\epsilon_1 = \frac{1}{2}\big((N \ln C_0)/D - 2\big)$ and $A_1 = D_N$. So the theorem follows immediately after establishing Inductive Assumption One.

First, for $n = 0$, since (M', g') is covered everywhere by ϵ/C_0-cylindrical planes and $g' = g$ on $\Gamma_{\ge A}$, by applying Lemma 6.4 we obtain an SO(2)-symmetric metric on M' which is covered by $C_0\epsilon$-cylindrical planes, and its restriction on $\Gamma_{\ge D_0}$ satisfies the inductive assumptions for some $D_0 \ge A + D$. Now suppose the inductive assumption holds for some $n \ge 0$. In the rest of the proof we show that it also holds for $n+1$. Now we impose the second inductive assumption.

Inductive Assumption Two

(1) Let $n \ge 0$ be fixed. Then for any $k \in \mathbb{N}$ there exists an SO(2)-symmetric metric $\widehat{g}_{n,k}$, defined on an open subset in $M \setminus \Gamma$ containing $\Gamma_{\ge D_n}$ which, on $\Gamma_{\ge D_n + iD}$, satisfies

$$|\nabla^m(\widehat{g}_{n,k} - g)| \le \epsilon \cdot C_0^{-i} \cdot e^{-\alpha_n(d_g(\cdot,\Gamma)-D_n)} \quad \text{for } i = 0, \dots, k \text{ and } m = 0, \dots, 100.$$

(2) Moreover, let $X_{n,k}$ be the Killing field of the SO(2)-isometry of $\widehat{g}_{n,k}$. Then the SO(2)-Killing field ∂_θ of any ϵ-cylindrical plane of g satisfies

$$|X_{n,k} - \partial_\theta| \le \tfrac{1}{1000} \quad \text{on } \Gamma_{\ge D_n}.$$

Step 2 For $k = 0$, Inductive Assumption Two clearly holds by choosing $\widehat{g}_{n,0} = \widehat{g}_n$. Now assume it is true for some integer $k \ge 0$. In this step, we will extend the SO(2)-symmetric metric $\widehat{g}_{n,k}$ to a complete SO(2)-symmetric metric $\widetilde{g}_{n,k}$. Then we will study the harmonic map heat flow between the grafted soliton flow $g'(t)$ and the Ricci flow starting from $\widetilde{g}_{n,k}$ and obtain some distance distortion estimates under these flows.

First, part (1) of Inductive Assumption Two implies $|\nabla^m(\widehat{g}_{n,k} - g)| \le \epsilon$ on $\Gamma_{\ge D_n}$ for $m = 0, \dots, 100$. So by applying Lemma 6.6 (SO(2)-invariant extension), we obtain a complete SO(2)-symmetric metric $\widetilde{g}_{n,k}$ on M' such that $\widetilde{g}_{n,k} = \widehat{g}_{n,k}$ on $\Gamma_{\ge D_n + 100}$, and

$$|\nabla^m(\widetilde{g}_{n,k} - g')| \le C_0\epsilon \quad \text{and} \quad |\widetilde{X}_{n,k} - \partial_\theta| \le C_0\epsilon \quad \text{on } M' \text{ for } m = 0, \dots, 98, \tag{6-22}$$

where the norms and derivatives are taken with respect to g', and $\widetilde{X}_{n,k}$ and ∂_θ are the SO(2)-killing fields of $\widetilde{g}_{n,k}$, and any ϵ-cylindrical plane of g'.

Next, we claim the complete SO(2)-symmetric metric $\widetilde{g}_{n,k}$ satisfies, on $\Gamma_{\geq D_n+iD}$,

(6-23) $$|\nabla^m(\widetilde{g}_{n,k}-g)|\leq C_0\cdot e^{400}\cdot\epsilon\cdot C_0^{-i}\cdot e^{-\alpha_n(d_g(\cdot,\Gamma)-D_n)} \quad \text{for } i=-1,0,\dots,k \text{ and } m=0,\dots,98.$$

We prove the claim in the following three cases:

(a) For all $i\geq 1$, since $\widetilde{g}_{n,k}=\widehat{g}_{n,k}$ on $\Gamma_{\geq D_n+100}\supset\Gamma_{\geq D_n+D}$, the claim holds by part (1) of Inductive Assumption Two for $\widehat{g}_{n,k}$.

(b) For $i=-1$, the claim follows from (6-22) by noting that $g'=g$ on $\Gamma_{\geq A}\supset\Gamma_{\geq D_n-D}$.

(c) For $i=0$, first, we have $\widetilde{g}_{n,k}=\widehat{g}_{n,k}$ on $\Gamma_{\geq D_n+100}$, where the claim holds by part (1) of Inductive Assumption Two; then, on $\Gamma_{[D_n,D_n+100]}$, the claim follows from (6-22) by noting that $g'=g$ on $\Gamma_{>A}$, and using $\alpha_n<4$ and $d_g(\cdot,\Gamma)-D_n\leq 100$.

Let $(M',\widetilde{g}_{n,k}(t))$ with $t\in[0,T]$ be the Ricci flow that starts from the complete SO(2)-symmetric $\widetilde{g}_{n,k}$. Then $|\mathrm{Rm}|_{\widetilde{g}_{n,k}(t)}\leq 1/T$ for all $t\in[0,T]$. Let $\{\chi_{n,k,t}\}\colon(M',g'(t))\to(M',\widetilde{g}_{n,k}(t))$ for $t\in[0,T]$ be a smooth harmonic map heat flow with $\chi_{n,k,0}=\mathrm{id}$, and let $h_{n,k}(t):=(\chi_{n,k,t}^{-1})^*g'(t)-\widetilde{g}_{n,k}(t)$. These are guaranteed by the choice of ϵ.

For fixed n and k, we will omit the subscripts n and k in $\chi_{n,k,t}$, $\widetilde{g}_{n,k}(t)$ and $h_{n,k}(t)$ for a moment. For any fixed $i=0,1,\dots,k+1$, let

$$x\in\Gamma^T_{\geq D_n+iD} \quad\text{and}\quad x'=\chi_T(x).$$

We first estimate distance distortions and drifts of points under the harmonic map heat flow.

Claim 6.8 *For any $L>10T^{1/2}$, we have $B_{\widetilde{g}(0)}(x',L)\subset\chi_t(B_{g'(t)}(x,10L))$ for all $t\in[0,T]$.*

Proof First, we observe that by $|h|\leq\frac{1}{1000}$ we have

(6-24) $$B_{\widetilde{g}(t)}(\chi_t(x),5L)\subset\chi_t(B_{g'(t)}(x,10L)).$$

Now let $y\in B_0(x',L)$. By the triangle inequality we have

$$d_{\widetilde{g}(t)}(y,\chi_t(x))\leq d_{\widetilde{g}(t)}(x',\chi_t(x))+d_{\widetilde{g}(t)}(x',y).$$

On the one hand, by the local drift estimate of harmonic map heat flows [9, Lemma A.18], and the curvature bound $|\mathrm{Rm}|_{\widetilde{g}(t)}<1/T$, we have for all $t\in[0,T]$ that

(6-25) $$d_{\widetilde{g}(t)}(x',\chi_t(x))=d_{\widetilde{g}(t)}(\chi_t(x),\chi_T(x))\leq 10(T-t)^{1/2}\leq 10T^{1/2}<L.$$

On the other hand, by the distance distortion estimate on the Ricci flow $(M',\widetilde{g}_k(t))$ under the curvature bound $|\mathrm{Rm}|_{\widetilde{g}(t)}<1/T$, we have

$$d_{\widetilde{g}(t)}(x',y)\leq 2e^{T\sup|\mathrm{Rm}|_{\widetilde{g}(t)}}d_{\widetilde{g}(0)}(x',y)\leq 4d_{\widetilde{g}(0)}(x',y)\leq 4L.$$

Combining the last three inequalities we obtain $d_{\widetilde{g}(t)}(y,\chi_t(x))\leq 5L$, which together with (6-24) implies the claim. □

Since $x \in \Gamma^T_{\ge D_n+iD}$, by the definition of the flow $g'(t)$ we see that $x \in \Gamma^t_{\ge D_n+iD}$ for all $t \in [0,T]$. In particular, we have $x \in \Gamma^0_{\ge D_n+iD} = \Gamma_{\ge D_n+iD}$. So taking $L = \frac{1}{10}D$ in Claim 6.8 we see that

$$B_{\tilde g(0)}(x', \underline{D}) \subset B_{\tilde g(0)}\big(x', \tfrac{1}{10}D\big) \subset B_g(x, D) \subset \Gamma_{\ge D_n+(i-1)D}, \tag{6-26}$$

which in particular implies $x' \in \Gamma_{\ge D_n+(i-1)D}$.

Step 3 In this step, we will apply Theorem 5.3 (symmetry improvement) at $(x', 0)$, with constants chosen as follows.

(a) We choose α in Theorem 5.3 to be α_n.

(b) We choose ϵ in Theorem 5.3 to be

$$\mathcal{H}(x') = C_0 \cdot e^{400} \cdot \epsilon \cdot C_0^{-(i-1)} \cdot e^{-\alpha_n (d_g(x',\Gamma)-D_n)}.$$

Note $d_g(x', \Gamma)$ is well-defined, since $x' \in \Gamma_{\ge D_n+(i-1)D} \subset \Gamma_{>A}$.

(c) We choose $D_\#$ in Theorem 5.3 to be

$$D_\# := \tfrac{1}{10}(d_{g(T)}(x,\Gamma) - D_n + D) \ge \tfrac{1}{10}D > \underline{D}.$$

First, we verify assumption (5-11). Let $y \in \Gamma_{\ge D_n+(i-1)D}$, then by comparing (6-23) with $\mathcal{H}(x')$ (note that since we choose $i = 0, \dots, k+1$, it follows that $i-1 = -1, \dots, k$, which meets the range of (6-23)), and using the triangle inequality $d_g(y,\Gamma) \ge d_g(x',\Gamma) - d_g(y,x')$, we obtain

$$|\nabla^m h|(y, 0) \le \mathcal{H}(x') \cdot e^{\alpha_n d_g(y,x')} \quad \text{for } m = 0, \dots, 98, \tag{6-27}$$

which also holds for all $y \in B_{\tilde g(0)}(x', \underline{D})$ by (6-26). This verifies the assumption (5-11).

Next, we verify the assumption of Theorem 5.3 that h restricted on $B_{\tilde g(0)}(x', D_\#) \times [0,T]$ is a Ricci De Turck flow perturbation. This follows from Claim 6.8: by taking $L = D_\#$ we have, for all $t \in [0,T]$,

$$\begin{aligned} B_{\tilde g(0)}(x', D_\#) &\subset \chi_t(B_{g'(t)}(x, d_{g(T)}(x,\Gamma) - D_n + D)) \\ &\subset \chi_t(B_{g'(t)}(x, d_{g(t)}(x,\Gamma) - D_n + D)) \subset \chi_t(\Gamma^t_{\ge D_n - D}) \subset \chi_t(\Gamma^t_{\ge A}). \end{aligned}$$

Moreover, by the local derivative estimates for Ricci De Turck flow perturbations [9, Lemma A.14], we may taking ϵ small so that $|\nabla^\ell h| \le \frac{1}{1000}$ for $\ell = 0, 1$.

Lastly, we will verify assumption (5-10) of Theorem 5.3. Recall that assumption (5-10) consists of two estimates of $|h|$ on the parabolic boundary of

$$B_{\tilde g(0)}(x', D_\#) \times [0,T] = (\partial B_{\tilde g(0)}(x', D_\#) \times [0,T]) \cup (B_{\tilde g(0)}(x', D_\#) \times \{0\}).$$

We first verify assumption (5-10) on $\partial B_{\tilde g(0)}(x', D_\#) \times [0,T]$. Note by (6-25) that $d_g(x,x') \le 10T^{1/2}$, and thus by (6-21) and the triangle inequality $d_g(x',\Gamma) \le d_g(x,\Gamma) + d_g(x,x')$ we have

$$d_g(x',\Gamma) \le d_{g(T)}(x,\Gamma) + 2.1T + 10T^{1/2} = 10D_\# + 2.1T + 10T^{1/2} + D_n.$$

Then using $\alpha_n < 4$ we obtain

$$e^{-\alpha_n(d_g(x',\Gamma)-D_n)} \geq e^{-4(10D_\# + 2.1T + 10T^{1/2})} \geq e^{-40D_\# - 10T} \geq e^{-40D_\# - D} \geq e^{-50D_\#}.$$

Using $D \geq 100 \ln C_0$ and $D_\# \geq (i+1)D$, we also have

$$C_0 \cdot e^{400} \cdot C_0^{-(i-1)} \geq e^{-(i+1)D} \geq e^{-10D_\#}.$$

Substituting these into $\mathcal{H}(x')$, we obtain

$$|h| \leq \tfrac{1}{1000} \leq e^{100D_\#} \cdot \mathcal{H}(x'). \tag{6-28}$$

Next, we verify assumption (5-10) on $B_{\tilde{g}(0)}(x', D_\#) \times \{0\}$. Let $y \in \Gamma_{[D_n+(j-1)D, D_n+jD)}$ for some $0 \leq j \leq i-1$. Then we have $d_g(x', y) \geq (i-j)D$, which implies

$$C_0^{i-j} \leq e^{((\ln C_0)/D) \cdot d_g(x',y)} \leq e^{0.01 d_g(x',y)},$$

which combined with (6-23), $\mathcal{H}(x')$ and the triangle inequality implies

$$|\nabla^m h|(y,0) \leq \mathcal{H}(x') \cdot C_0^{i-j} \cdot e^{\alpha_n(d_g(x',\Gamma) - d_g(y,\Gamma))} \leq \mathcal{H}(x') \cdot e^{4 d_g(x',y)} \tag{6-29}$$

for all $y \in \Gamma_{\geq D_n - D}$ and all $m = 0, \dots, 98$. If $y \in \Gamma_{\geq D_n+(i-1)D}$, then (6-29) also holds by (6-27). Since $B_{\tilde{g}(0)}(x', D_\#) \subset \Gamma_{\geq D_n - D}$, this together with (6-28) verifies assumption (5-10).

Step 4 In this step we will apply Theorem 5.3, get an exponential decay on the oscillatory part of h at time T, use this decay to extend Inductive Assumption Two from k to $k+1$, and deduce that Inductive Assumption One holds for $n+1$. To be concrete, by applying Theorem 5.3 (symmetric improvement) at $(x', 0)$, we obtain

$$|\nabla^m h_-|(x', T) \leq \mathcal{H}(x') \cdot e^{2\alpha_n T} \cdot e^{-\delta_0 T} \leq C_0 \cdot e^{400} \cdot \epsilon \cdot C_0^{-(i-1)} \cdot e^{-\alpha_n(d_g(x',\Gamma)-D_n)} \cdot e^{2\alpha_n T} \cdot e^{-\delta_0 T}$$

for $m = 0, \dots, 100$. We will estimate below that the last term $e^{-\delta_0 T}$ is sufficiently small that it outweighs all other factors bigger than 1. Specifically, we have

$$\begin{aligned} |\nabla^m h_-|(x', T) &\leq C_0 \cdot e^{400} \cdot \epsilon \cdot C_0^{-(i-1)} \cdot e^{-\alpha_n(d_g(x,\Gamma)-D_n)} \cdot e^{2\alpha_n T} \cdot e^{-\frac{3}{4}\delta_0 T} \\ &\leq C_0 \cdot e^{400} \cdot \epsilon \cdot C_0^{-(i-1)} \cdot e^{-\alpha_n(d_{g(T)}(x,\Gamma)-D_n)} \cdot e^{-\frac{1}{2}\delta_0 T} \\ &\leq \tfrac{1}{2}\epsilon \cdot C_0^{-i} \cdot e^{-\alpha_n(d_{g(T)}(x,\Gamma)-D_n)}, \end{aligned}$$

where in the first inequality we used the triangle inequality $d_g(x', \Gamma) - d_g(x, \Gamma) \geq -d_g(x, x')$ together with $d_g(x', x) \leq 10T^{1/2}$ and $T^{1/2} > 160/\delta_0$; in the second inequality we used the distance distortion estimate (6-21) and $\alpha_n < 4$; in the last inequality we used $2C_0^2 \cdot e^{400} < e^{\frac{1}{2}\delta_0 T}$. The above inequality also holds when the norms and derivatives are with respect to $(\chi_T^{-1})^* g'(T)$ after removing the factor $\frac{1}{2}$. So we have

$$|\nabla^m \chi_T^* h_-|(x, T) \leq \epsilon \cdot C_0^{-i} \cdot e^{-\alpha_n(d_{g(T)}(x,\Gamma)-D_n)} \quad \text{for } m = 0, \dots, 100, \tag{6-30}$$

where the norms and derivatives are with respect to $g(T)$.

Now we restore the subscripts n,k. Since

$$g'(T)=\chi^*_{n,k,T}(h_{n,k}(T)+\tilde{g}_{n,k}(T)) \quad \text{and} \quad h_{n,k}(T)=h_{n,k,+}(T)+h_{n,k,-}(T),$$

by letting

$$\hat{g}_{n,k+1}=\chi^*_{n,k,T}(h_{n,k,+}(T)+\tilde{g}_{n,k}(T)),$$

we see that $\hat{g}_{n,k+1}$ is an SO(2)-symmetric metric on M' and (6-30) implies

$$|\nabla^m(g'(T)-\hat{g}_{n,k+1})|\le \epsilon\cdot C_0^{-i}\cdot e^{-\alpha_n(d_{g(T)}(\cdot,\Gamma)-D_n)} \quad \text{on } \Gamma^T_{\ge D_n+iD}$$

for $m=0,\dots,100$, and the norms and derivatives are with respect to $g(T)$. Note that this is true for all $i=0,1,\dots,k+1$. Since $g'(T)=\phi^*_{-T}g$ and $\phi_{-T}\colon(\Gamma^T_{\ge A},g'(T))\to(\Gamma_{\ge A},g)$ is an isometry, replacing $\hat{g}_{n,k+1}$ by $\phi^*_T(\hat{g}_{n,k+1})$ we then have

$$|\nabla^m(g-\hat{g}_{n,k+1})|\le \epsilon\cdot C_0^{-i}\cdot e^{-\alpha_n(d_g(\cdot,\Gamma)-D_n)} \quad \text{on } \Gamma_{\ge D_n+iD}$$

for $m=0,\dots,98$. This verifies part (1) of Inductive Assumption Two for $k+1$. Moreover, it is easy to see that part (2) of Inductive Assumption Two for $k+1$ also holds, as a consequence of the smallness of ϵ and $|h|$.

Therefore, Inductive Assumption Two holds for all $k\ge 0$. Now let $p\in\Gamma_{>D_n}$ be some fixed point. Then we may assume, after passing to a subsequence and letting $k\to\infty$, that the pointed SO(2)-symmetric manifolds $(M',\hat{g}_{n,k},p)$ converge in the C^{98}-norm to an SO(2)-symmetric manifold $(M',\hat{g}_{n+1},p)$ which satisfies

$$|\nabla^m(g-\hat{g}_{n+1})|\le \epsilon\cdot C_0^{-i}\cdot e^{-\alpha_n(d_g(\cdot,\Gamma)-D_n)} \quad \text{on } \Gamma_{\ge D_n+iD} \tag{6-31}$$

for all $i\in\mathbb{N}$ and $m=0,\dots,98$. Moreover, part (2) of Inductive Assumption One holds for $n+1$ with $\hat{g}_{n+1}$, as a consequence of part (2) Inductive Assumption Two and the smooth convergence. It remains to verify part (1) of Inductive Assumption One. For any $i\ge 0$ and $y\in\Gamma_{[D_n+iD,D_n+(i+1)D)}$, we have $d_g(y,\Gamma)\le D_n+(i+1)D$, which together with (6-31) implies

$$|\nabla^m(g-\hat{g}_{n+1})|(y)\le \epsilon\cdot e^{-\alpha_{n+1}(d_g(y,\Gamma)-D_{n+1})},$$

where

$$D_{n+1}=\frac{(D+D_n)\ln C_0+\alpha_n D_n D}{\ln C_0+\alpha_n D}>D_n \quad \text{and} \quad \alpha_{n+1}=\frac{(n+1)\ln C_0}{D}.$$

Therefore, we have

$$|\nabla^m(g-\hat{g}_{n+1})|\le \epsilon\cdot e^{-\alpha_{n+1}(d_g(\cdot,\Gamma)-D_{n+1})} \quad \text{on } \Gamma_{\ge D_{n+1}}$$

for $m=0,\dots,100$, which verifies part (1) of Inductive Assumption One for $n+1$, and thus completes the proof. □

6.5 Extend the approximating metric near the edges

The approximating SO(2)-symmetric metric $\widehat{g}$ in Theorem 6.7 is defined on an open subset away from Γ. Next, we want to extend it to an SO(2)-symmetric metric which is also defined on a neighborhood of Γ. Seeing that the soliton dimension reduces to $\mathbb{R}\times$cigar along Γ, we can find a sequence of SO(2)-symmetric metrics close to g in balls centered at Γ whose radius goes to infinity. Then by gluing these metrics with $\widehat{g}$, we obtain an SO(2)-symmetric metric $\overline{g}$ defined everywhere outside of a compact subset of M. We will show that $\overline{g}-g$ has the same exponential decay rate as $\widehat{g}-g$, and it decays to zero at infinity in addition. In Section 6.1 we studied the gluing of SO(2)-symmetric metrics that are close to $\mathbb{R}^2\times S^1$. Here we need to glue SO(2)-symmetric metrics that are close to $\mathbb{R}\times$cigar, which needs the following gluing lemma. The lemma shows that if a vector field on $\mathbb{R}\times$cigar is almost a Killing field, and it is close to ∂_θ at a larger distance to $\mathbb{R}$, then it is also close to ∂_θ at a smaller distance.

Lemma 6.9 *For any $C_1,\epsilon_1>0$, there exists $C(C_1,\epsilon_1)>0$ such that the following holds for any $\epsilon>0$. Let $(r,s,\theta)\colon(\mathbb{R}\times\text{cigar},g_\Sigma)\to\mathbb{R}_+\times\mathbb{R}\times[0,2\pi)$ be coordinates such that $g_\Sigma=dr^2+ds^2+\varphi^2(r)\,d^2\theta$, and let r be the distance to the line $\mathbb{R}\times\{x_{\text{tip}}\}$. Suppose $Y(r,s,\theta)$ is a smooth vector field, defined for all $(r,s,\theta)\in[A,B]\times[-s_0,s_0]\times[0,2\pi)$ for some $1<A<B$ and $s_0\in[0,\infty]$, such that*

$$|\nabla^k(\mathscr{L}_Y g_\Sigma)|\le C_1\,e^{-2(1+\epsilon_1)r}\quad\text{for } k=0,\dots,98. \tag{6-32}$$

Suppose also that

$$|\nabla^k(Y-\partial_\theta)|(\cdot,B,\cdot)\le\epsilon\quad\text{for } k=0,\dots,98. \tag{6-33}$$

Then we have

$$|\nabla^k(Y-\partial_\theta)|(\cdot,A,\cdot)\le Ce^{-2(1+\epsilon_1)A}+C(B-A)\epsilon+C\epsilon\quad\text{for } k=0,\dots,96.$$

Proof Write $Y=Y^s\partial_s+Y^r\partial_r+Y^\theta\partial_\theta$ under the coordinates (s,r,θ). Then, by the formula of Lie derivatives for a symmetric 2-tensor, we have

$$\mathscr{L}_Y g_\Sigma(\partial_r,\partial_\theta)=\partial_\theta Y^r+\partial_r Y^\theta\varphi^2,\quad \mathscr{L}_Y g_\Sigma(\partial_r,\partial_s)=\partial_r Y^s+\partial_s Y^r,\quad \mathscr{L}_Y g_\Sigma(\partial_r,\partial_r)=2\,\partial_r Y^r. \tag{6-34}$$

Moreover, by assumption (6-33) we have

$$(|\nabla^k Y^s|+|\nabla^k Y^r|+|\nabla^k(Y^\theta-1)|)(\cdot,B,\cdot)\le Ce^{-2(1+\epsilon_1)A}.$$

By the third equation in (6-34) and (6-32), integrating from $r=A$ to $r=B$ and using (6-33) we see that

$$|\nabla^{k-1}Y^r|_{C^{k-1}}(\cdot,r,\cdot)\le C(e^{-2(1+\epsilon_1)r}+\epsilon)$$

for any $r\in[A,B]$. Substituting this into the first two equations in (6-34), integrating from $r=A$ to $r=B$ and using (6-33), we obtain

$$(|\nabla^{k-2}(Y^\theta-1)|+|\nabla^{k-2}Y^s|)(\cdot,A,\cdot)\le Ce^{-2(1+\epsilon_1)A}+C(B-A)\epsilon+C\epsilon,$$

which proves the lemma. □

We now prove the main result of this section.

Theorem 6.10 *Let (M, g, f, p) be a 3D steady gradient soliton that is not a Bryant soliton. Assume that $\lim_{s\to\infty} R(\Gamma_1(s)) = \lim_{s\to\infty} R(\Gamma_2(s)) = 4$. Then there exist constants $C, \epsilon_1 > 0$ and an SO(2)-symmetric metric $\overline{g}$ defined outside of a compact subset of M such that for any $k = 0, \dots, 98$,*

$$|\nabla^k (g - \overline{g})| \to 0 \quad \text{as } x \to \infty \qquad \text{and} \qquad |\nabla^k (g - \overline{g})| \le C e^{-2(1+\epsilon_1) d_g(\cdot, \Gamma)}, \tag{6-35}$$

where the covariant derivatives and norms are taken with respect to g.

Proof We will use $\epsilon(D)$ to denote a general function that goes to zero as $D \to \infty$. Let $C > 0$ denote all constants that depend only on the soliton. On the one hand, since for each $i = 1, 2$ the manifold dimension reduces along Γ_i to $\mathbb{R} \times \text{cigar}$, we can find $D_1(s) > 0$ for all large s with $D_1(s) \to \infty$ slowly enough in s so that $(M, g, \Gamma_i(s))$ is $e^{-2(1+\epsilon_1) D_1(s)}$-close to $\mathbb{R} \times \text{cigar}$ in $B_g(\Gamma_i(s), D_1(s))$, where $\epsilon_1 > 0$ is from Theorem 6.7. Then by a standard gluing argument, we can find an SO(2)-symmetric metric $\widetilde{g}_i$ on an open subset U_i containing the balls $B_g(\Gamma_i(s), D_1(s))$ for all large s, such that:

(1) For each $i = 1, 2$ we have
$$|\nabla^k (\widetilde{g}_i - g)| \le \min\{\epsilon(d_g(\cdot, p)), C e^{-2(1+\epsilon_1) d_g(\cdot, \Gamma)}\} \quad \text{on } U_i, \tag{6-36}$$
where $\epsilon(d_g(\cdot, p))$ denotes a general function that tends to 0 as $d_g(\cdot, p) \to \infty$.

(2) $U_1 \cap U_2 = \varnothing$.

(3) Let $\chi_{i,\theta}$ with $\theta \in [0, 2\pi)$ be the SO(2)-isometries for $(U_i, \widetilde{g}_i)$ for $i = 1, 2$. We can also assume that there is an embedded surface N_i in $U_i \cap \Gamma_{\ge 1000}$ which is diffeomorphic to $\mathbb{R}^2$ and intersects each S^1-orbit of $\chi_{i,\theta}$ exactly once, and its tangent space $T_x N_i$ at $x \in N_i$ is $\frac{1}{100}$-close to the orthogonal space of the S^1-orbit passing through x.

(4) For each large s, there is a smooth map $\psi_{i,s}$ from $B_g(\Gamma_i(s), D(s))$ into $(\mathbb{R} \times \text{cigar}, g_\Sigma)$ which is a diffeomorphism onto the image, such that
$$|\nabla^k (\widetilde{g}_i - \psi_{i,s}^* g_\Sigma)| \le C e^{-2(1+\epsilon_1) d_g(\cdot, \Gamma)}. \tag{6-37}$$
Let X_i be the Killing field of the SO(2)-isometry of $\widetilde{g}_i$. Then
$$|\nabla^k (X_i - (\psi_{i,s})_*^{-1}(\partial_\theta))| \le C e^{-2(1+\epsilon_1) d_g(\cdot, \Gamma)}. \tag{6-38}$$

On the other hand, by Theorem 6.7, we have an SO(2)-symmetric metric $\widehat{g}$ defined on an open subset $U \supset \Gamma_{\ge A}$ for some $A > 0$ such that
$$|\nabla^k (g - \widehat{g})| \le C e^{-2(1+\epsilon_1) d_g(\cdot, \Gamma)} \quad \text{for } k = 0, \dots, 100. \tag{6-39}$$

Let χ_θ with $\theta \in [0, 2\pi)$ be the SO(2)-isometries for $(U, \widehat{g})$, and Y be the Killing field of χ_θ. Next, we will compare the two vector fields X_i and Y on $\Gamma_{>A} \cap U_i$. First, by (6-36), (6-37), (6-39) and triangle inequalities we have
$$|\nabla^k (\widehat{g} - \psi_{i,s}^* g_\Sigma)| \le C e^{-2(1+\epsilon_1) d_g(\cdot, \Gamma)} \quad \text{for } k = 0, \dots, 100.$$

Then by $\mathscr{L}_Y \widehat{g} = 0$, this implies

$$|\nabla^k(\mathscr{L}_Y \psi_{i,s}^* g_\Sigma)| \le C e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)} \quad \text{for } k = 0, \dots, 98. \tag{6-40}$$

Moreover, by Theorem 4.11 we can see that (M, g) is a $C_k/d_g^2(\cdot,\Gamma)$-cylindrical plane in the C^k-sense and $C_k > 0$ at points where the metrics $\widehat{g}$ and $\psi_{i,s}^* g_\Sigma$ are both defined. So we can apply Lemma 6.2 and deduce

$$|\nabla^k(Y - (\psi_{i,s})_*^{-1}(\partial_\theta))| \le \frac{C}{d_g^2(\cdot,\Gamma)} \quad \text{for } k = 0, \dots, 98. \tag{6-41}$$

We can find $D_2\colon [s_0,\infty) \to \mathbb{R}_+$ for a sufficiently large s_0, such that $D_2(s) \le D_1(s)$, $B_g(\Gamma_i(s), D_2(s)) \subset U_i$ and $D_2(s) \to \infty$ sufficiently slowly as $s \to \infty$ so that

$$\frac{C}{D_1^2(s)}(D_1(s) - D_2(s)) < \frac{C}{D_1(s)} < e^{-2(1+\epsilon_1)D_2(s)}. \tag{6-42}$$

Therefore, for each $i = 1, 2$, by (6-40), (6-41) and (6-42) we can apply Lemma 6.9 with $\epsilon = C/D_1^2(s)$ and $B = D_1(s) \ge A = D_2(s)$, and deduce

$$|\nabla^k(Y - (\psi_{i,s})_*^{-1}(\partial_\theta))| \le C e^{-2(1+\epsilon_1)D_2(s)} \le C e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)}$$

on $B_g(\Gamma_i(s), D_2(s)) \cap U$ for $k = 0, \dots, 98$. Combining this with inequality (6-38) we see that for $V_i := \bigcup_{s>s_0} B_g(\Gamma_i(s), D_2(s))$, we have

$$|\nabla^k(Y - X_i)| \le C e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)} \tag{6-43}$$

on $U \cap V_i$ for $k = 0, \dots, 98$, and thus

$$|\nabla^k(\chi_\theta - \chi_{i,\theta})| \le C e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)}.$$

Therefore, by the same argument as in Lemma 6.3, we can glue the metrics $\widehat{g}$ on U, $\widetilde{g}_i$ on V_i for $i = 1, 2$, and also glue the SO(2)-isometries χ_θ and $\chi_{i,\theta}$ for $i = 1, 2$ to obtain an SO(2)-symmetric metric $\overline{g}$ defined outside of a compact subset of M, which satisfies the following properties:

(1) For each $i = 1, 2$ we have that $\overline{g} = \widetilde{g}_i$ on W_i, where $B_g(\Gamma_i(s), D_3(s)) \subset W_i \subset V_i$ is a $\chi_{i,\theta}$-invariant open subset, $D_3(s) < D_2(s)$, and $D_3(s) \to \infty$ as $s \to \infty$.

(2) We have that $\overline{g} = \widehat{g}$ on $U \setminus (V_1 \cup V_2)$.

(3) For some $D_0 > 0$, we have

$$\begin{cases} |\nabla^k(\overline{g} - g)| \le C e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)} & \text{on } \Gamma_{\ge D_0} \text{ for } k = 0, \dots, 98, \\ |\overline{g} - g| \le \epsilon(d_g(\cdot, p)) & \text{on } W_1 \cup W_2. \end{cases} \tag{6-44}$$

The first inequality implies that $\overline{g}$ satisfies the assertion of exponential decay away from Γ. Moreover, since $d_g(x, p) \to \infty$ as $d_g(x,\Gamma) \to \infty$ for any $x \in M \setminus (W_1 \cup W_2)$, the first inequality implies that $|\overline{g} - g| \le \epsilon(d_g(\cdot, p))$ also holds on $M \setminus (W_1 \cup W_2)$. So $\overline{g}$ satisfies the assertion of decaying to zero at infinity. □

7 The evolution of the Lie derivative

In this section, (M, g) is a 3D steady gradient soliton that is not a Bryant soliton, and $(M, g(t))$ is the Ricci flow of the soliton. Let $h(t)$ be a linearized Ricci De Turck flow with background metric $g(t)$. The main result is Proposition 7.3, which shows that $h(t)$ tends to zero as t goes to infinity, if the initial value $h(0)$ satisfies the condition $h(x,0)/R(x) \to 0$ as $x \to \infty$. In particular, let $\overline{g}$ be the approximating SO(2)-symmetric metric obtained from Theorem 6.10, and let ∂_θ with $\theta \in [0, 2\pi)$ be the Killing field of the SO(2)-symmetry. We show that the Lie derivative $\mathscr{L}_{\partial_\theta} g$ satisfies this initial condition, hence decays to zero as $t \to \infty$ under the linearized Ricci De Turck equation.

7.1 The vanishing of a heat kernel at infinity

In this subsection we prove a vanishing theorem of the heat kernel to a certain heat-type equation at time infinity. We will see that for a linearized Ricci De Turck flow $h(t)$, the norm $|h|(\cdot, t)$ is controlled by the convoluted integral of this heat kernel and $|h|(\cdot, 0)$.

Let G be the heat kernel of the heat-type equation

$$\partial_t H = \Delta H + \frac{2|\mathrm{Ric}|^2}{R} H. \tag{7-1}$$

That is, for any $t > s$ and $x, y \in M$,

$$\partial_t G(x,t;y,s) = \Delta_{x,t} G(x,t;y,s) + \frac{2|\mathrm{Ric}|^2(x,t)}{R(x,t)} G(x,t;y,s) \quad \text{and} \quad \lim_{t \searrow s} G(\cdot, t; y, s) = \delta_y.$$

Lemma 7.1 (vanishing of heat kernel at time infinity) *Let p be the point where R attains its maximum. For any fixed $D > 0$, let*

$$u_D(x,t) = \int_{B_0(p,D)} G(x,t;y,0)\, d_0 y.$$

Then $\sup_{B_t(p,D)} u_D(\cdot, t) \to 0$ as $t \to \infty$.

Proof Note that u_D satisfies the equation (7-1). First, we show that there exists $C_1 > 0$ such that $u_D(x,t) \le C_1$ for all $(x,t) \in M \times [0, \infty)$. Since the scalar curvature satisfies the equation

$$\partial_t R = \Delta R + \frac{2|\mathrm{Ric}|^2}{R} \cdot R,$$

by using the reproduction formula we have

$$R(x,t) = \int_M G(x,t;y,0) R(y,0)\, d_0 y.$$

By compactness we have for some $c > 0$ that $R(y,0) \ge c$ for all $y \in B_0(p,D)$, so it follows that $u_D(x,t) \le c^{-1} R(x,t) \le c^{-1} R(p)$. So we may take $C_1 = c^{-1} R(p)$.

Now suppose the lemma does not hold. Then there exist $\epsilon > 0$ and a sequence of $t_i \to \infty$ and $x_i \in B_{t_i}(p, D)$ such that $u_D(x_i, t_i) \geq \epsilon > 0$. Without loss of generality we may assume that $t_{i+1} \geq t_i + 1$. Since $u \leq C_1$, by a standard parabolic estimate we see that $|\partial_t u_D|(x,t) + |\nabla u_D|(x,t) \leq C_2$ for some $C_2 > 0$. Therefore, there exists $\delta_1 \in (0, 1)$ such that

$$\int_{B_t(p,D)} \int_{B_0(p,D)} G(x,t;y,0)\, d_0 y\, d_t x = \int_{B_t(p,D)} u_D(x,t), d_t x \geq \delta_1 \tag{7-2}$$

for all $t \in [t_i, t_i + \delta_1]$.

Recall that the (symmetric) curvature operator $\mathrm{Rm}\colon \Lambda^2 TM \to \Lambda^2 TM$ is defined so that

$$\langle \mathrm{Rm}(v_i \wedge v_j), v_k \wedge v_\ell \rangle = R(v_i, v_j, v_k, v_\ell) \quad \text{for any } v_i, v_j, v_k, v_\ell \in T_x M,$$

where here $R(\cdot,\cdot,\cdot,\cdot)$ is the curvature tensor of type $(4,0)$. We use the convention

$$R(v_i, v_j, v_i, v_j) = -\|v_i \wedge v_j\|^2 \cdot K(v_i \wedge v_j).$$

Since in dimension 3, elements of $\Lambda^2 TM$ are all simple, we can choose an orthonormal basis $\{e_1, e_2, e_3\}$ for $T_x M$, such that Rm is diagonalized by $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$. That is, there are constants $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that $\mathrm{Rm}(e_1 \wedge e_2) = -\lambda_3 e_1 \wedge e_2$, $\mathrm{Rm}(e_1 \wedge e_3) = -\lambda_2\, e_1 \wedge e_3$ and $\mathrm{Rm}(e_2 \wedge e_3) = -\lambda_1\, e_2 \wedge e_3$. Since $\mathrm{Rm} > 0$ (equivalent to positive sectional curvature in 3D), we have $\lambda_1, \lambda_2, \lambda_3 > 0$. So it is easy to see that $2|\mathrm{Ric}|^2 = 2(\lambda_2 + \lambda_3)^2 + 2(\lambda_1 + \lambda_3)^2 + 2(\lambda_1 + \lambda_2)^2$ and $R^2 = 4(\lambda_1 + \lambda_2 + \lambda_3)^2$, which implies

$$2|\mathrm{Ric}|^2 - R^2 < 0.$$

So by compactness there is $\delta_2 > 0$ such that

$$\sup_{B_t(p,D)} \frac{2|\mathrm{Ric}|^2 - R^2}{R} \leq -\delta_2. \tag{7-3}$$

Let $F_D(t) = \int_{B_0(p,D)} \int_M G(x,t;y,0)\, d_t x\, d_0 y$. Then since

$$\begin{aligned} \partial_t \int_M G(x,t;y,0)\, d_t x & \\ &= \int_M \partial_t G(x,t;y,0) - R(x,t) G(x,t;y,0)\, d_t x \\ &= \int_M \Delta_{x,t} G(x,t;y,0) + \frac{2|\mathrm{Ric}|^2(x,t)}{R(x,t)} G(x,t;y,0) - R(x,t) G(x,t;y,0)\, d_t x \\ &= \int_M \frac{G(x,t;y,0)}{R(x,t)} (2|\mathrm{Ric}|^2(x,t) - R^2(x,t))\, d_t x, \end{aligned} \tag{7-4}$$

where we used that the heat kernel G satisfies a Gaussian upper bound so that $\int_M \Delta_{x,t} G(x,t;y,0)\, d_t x$ vanishes by the divergence theorem. Hence we obtain

$$\partial_t F_D(t) = \int_{B_0(p,D)} \int_M \frac{G(x,t;y,0)}{R(x,t)} (2|\mathrm{Ric}|^2 - R^2)(x,t)\, d_t x\, d_0 y < 0, \tag{7-5}$$

and by (7-2) and (7-3) we see that $\partial_t F_D(t) \le -\delta_1\delta_2$ for all $t \in [t_i, t_i+\delta_1]$. Note that G is everywhere positive so that $F_D(t) > 0$ and $t_{i+1} \ge t_i + 1 > t_i + \delta_1$, implying

$$-F_D(0) \le \lim_{t\to\infty} F_D(t) - F_D(0) \le -\sum_{i=1}^{\infty} \int_{t_i}^{t_i+\delta_1} \delta_1 \cdot \delta_2 \, dt = -\sum_{i=1}^{\infty} \delta_1^2 \cdot \delta_2 = -\infty,$$

which is a contradiction. This proves the lemma. □

7.2 The vanishing of the Lie derivative at infinity

We prove the main result in this subsection by applying the heat kernel estimates. First, we prove a lemma using the Anderson–Chow pinching estimate.

Lemma 7.2 (cf [1]) *Let $(M, g(t))$ with $t \in [0, T]$ be a 3D complete Ricci flow with bounded curvature and positive sectional curvature. Consider a solution h to the linearized Ricci De Turck flow on $(M, g(t))$, and a positive solution H to the equation*

$$\partial_t H = \Delta H + \frac{2|\mathrm{Ric}|^2(x,t)}{R(x,t)} H. \tag{7-6}$$

Then

$$\partial_t \left(\frac{|h|^2}{H^2}\right) \le \Delta\left(\frac{|h|^2}{H^2}\right) + 2\nabla H \cdot \nabla\left(\frac{|h|^2}{H^2}\right).$$

Proof By a direct computation using $\partial_t h = \Delta_{L,g(t)} h$ and (7-6) we have (see [1])

$$\partial_t\left(\frac{|h|^2}{H^2}\right) = \Delta\left(\frac{|h|^2}{H^2}\right) + 2\nabla H \cdot \nabla\left(\frac{|h|^2}{H^2}\right) + \frac{4}{H^2}\left(R_{ijkl}h_{il}h_{jk} - \frac{|\mathrm{Ric}|^2}{R}|h|^2\right) - 2\frac{|H\nabla_i h_{jk} - (\nabla_i H)h_{jk}|^2}{H^4}.$$

Then the lemma follows immediately form the pinching estimate from [1] that for any nonzero symmetric 2-tensor h,

$$\frac{\mathrm{Rm}(h,h)}{|h|^2} \le \frac{|\mathrm{Ric}|^2}{R}.$$ □

We now prove the main result of this section. Noting that $|h|(\cdot\,, t)$ is controlled by the convoluted integral of the heat kernel of (7-1), we split the integral into two parts, where in the compact region, the integral tends to zero as a consequence of our vanishing theorem. In the noncompact subset, we use the assumption $h(x,0)/R(x) \to 0$ as $x \to \infty$ and the reproduction formula of the scalar curvature to deduce that the integral is bounded above by arbitrarily small multiples of the scalar curvature.

Proposition 7.3 *Let $(M, g(t))$ be the Ricci flow of a 3D steady gradient Ricci soliton that is not a Bryant soliton. Consider a solution $h(t)$, with $t \in [0, \infty)$, to the linearized Ricci De Turck flow on M, ie*

$$\partial_t h(t) = \Delta_{L,g(t)} h(t).$$

Suppose h satisfies the initial condition

$$\frac{|h|(x,0)}{R(x)} \to 0 \quad \text{as } x \to \infty.$$

Then for each $D > 0$,

$$\sup_{x \in B_t(p,D)} |h|(x,t) \to 0 \quad \text{as } t \to \infty.$$

Proof Let $H(x,t) = \int_M G(x,t;y,0)|h|(y,0)\, d_0 y$. Then H solves the equation

$$\partial_t H = \Delta H + \frac{2|\mathrm{Ric}|^2(x,t)}{R(x,t)} H.$$

Therefore, by Lemma 7.2 and applying the weak maximum principle to $|h|^2/H^2$, we see that $|h| \le H$, that is,

$$|h(x,t)| \le \int_M G(x,t;y,0)|h|(y,0)\, d_0 y. \tag{7-7}$$

For any $\epsilon > 0$, by the assumption on $|h|(\,\cdot\,,0)$, we can find some $D > 0$ such that $|h(y,0)| \le \epsilon R(y,0)$ for all $y \in M \setminus B_0(p,D)$. We may assume $D > 1/\epsilon$. So by (7-7) and using Lemma 7.1 (vanishing of heat kernel) we have for all sufficiently large t and $x \in B_t(p,D)$ that

$$\begin{aligned} |h(x,t)| &\le \int_{B_0(p,D)} G(x,t;y,0)|h|(y,0)\, d_0 y + \int_{M \setminus B_0(p,D)} G(x,t;y,0)|h|(y,0)\, d_0 y \\ &\le \epsilon + \epsilon \int_M G(x,t;y,0) R(y,0)\, d_0 y = \epsilon + \epsilon R(x,t) \le \epsilon(1 + R(p)). \end{aligned}$$

This implies $\sup_{B_t(p,D)} |h|(\,\cdot\,,t) \le \epsilon(1 + R(p))$ for all large t, and the assertion follows by letting $\epsilon \to 0$. □

As a direct application, we prove the following:

Corollary 7.4 *Let $\bar{g}$ be the $\mathrm{SO}(2)$-symmetric metric from Theorem 6.10, and X be a vector field on M which is a smooth extension of the Killing field of the $\mathrm{SO}(2)$-isometry of $\bar{g}$, and X has a bounded C^2-norm. Let $h(t)$ be the solution to the initial value problem of the linearized Ricci De Turck flow*

$$\begin{cases} \partial_t h(t) = \Delta_{L,g(t)} h(t), \\ h(0) = \mathcal{L}_X g. \end{cases}$$

Then for each $D > 0$,

$$\sup_{x \in B_t(p,D)} |h|(x,t) \to 0 \quad \text{as } t \to \infty.$$

Proof By Proposition 7.3 it suffices to show that $|h|(x,0)/R(x) \to 0$ as $x \to \infty$. On the one hand, by Theorem 6.10, there exist $\epsilon_1, C_1 > 0$ such that

$$\begin{cases} |h(0)| \le C_1 \cdot e^{-2(1+\epsilon_1) d_g(\cdot,\Gamma)} & \text{on } M, \\ |h(x,0)| \to 0 & \text{as } x \to \infty. \end{cases} \tag{7-8}$$

On the other hand, by Theorem 4.8 (scalar curvature exponential lower bound) we can find a constant $C_2 > 0$ depending only on ϵ_1 such that

$$R \geq C_2^{-1} e^{-2(1+\frac{1}{2}\epsilon_1)d_g(\cdot,\Gamma)}. \tag{7-9}$$

For any $\epsilon > 0$, by the second condition in (7-8) we can find $D(\epsilon) > 0$ such that $|h|(x,0) \leq \epsilon$ for all $x \in M \setminus B_g(p, D(\epsilon))$. First, if $\Gamma_{\leq L(\epsilon)} \setminus B_g(p, D(\epsilon))$, where

$$L(\epsilon) = \frac{\ln(C/\sqrt{\epsilon})}{2(1+\epsilon_1)},$$

we have

$$|h|(x,0) \leq \epsilon = \sqrt{\epsilon} \cdot C \cdot e^{-2(1+\epsilon_1)L(\epsilon)} \leq \sqrt{\epsilon} \cdot R(x).$$

Second, if $x \in \Gamma_{\geq L(\epsilon)}$, then by the first condition in (7-8) and (7-9) we obtain

$$|h|(x,0) \leq C_1 \cdot e^{-\epsilon_1\, d_g(x,\Gamma)} \cdot e^{-2(1+\frac{1}{2}\epsilon_1)\, d_g(x,\Gamma)} \leq C_1 C_2 \cdot e^{-\epsilon_1\, L(\epsilon)} \cdot R(x).$$

Note $L(\epsilon) \to \infty$ as $\epsilon \to 0$, it follows immediately that $|h|(x,0)/R(x) \to 0$ as $x \to \infty$. □

8 Construction of a Killing field

Let (M, g) be a 3D steady gradient soliton that is not a Bryant soliton, and let $(M, g(t))$ with $t \in (-\infty, \infty)$ be the Ricci flow of the soliton. In this section, we study the evolution of a vector field $X(t)$ under the equation

$$\partial_t X(t) = \Delta_{g(t)} X(t) + \mathrm{Ric}_{g(t)}(X). \tag{8-1}$$

In particular, we will choose $X(0)$ to be the Killing field of the SO(2)-isometry of the approximating metric obtained from Theorem 6.10, and show that $X(t_i)$ converges to a nonzero Killing field of the soliton (M, g) for a sequence $t_i \to \infty$.

Throughout this section, we assume

$$\lim_{s\to\infty} R(\Gamma_1(s)) = \lim_{s\to\infty} R(\Gamma_2(s)) = 4,$$

and $\overline{g}$ is the SO(2)-symmetric metric defined outside of a compact subset of M from Theorem 6.10.

First we fix some notation. Let $A > 0$ be sufficiently large that $\Gamma_{\geq A}$ is covered by ϵ-cylindrical planes for some sufficiently small ϵ. In particular, the SO(2)-isometry ψ_θ of $\overline{g}$ acts freely on an open subset $U \supset \Gamma_{\geq A}$, and the length of each S^1-orbit is 100ϵ-close to 2π. So we can find a 2D manifold (N, g_N) and a Riemannian submersion $\pi: (U, \overline{g}) \to (N, g_N)$ which maps a S^1-orbit to a point in N.

Let ρ: $(-1,1)^2 \to B \subset N$ be a local coordinate at $p \in N$ such that $\rho(0,0) = p$, and s: $B \to U$ be a section of the Riemannian submersion π. Then the map Φ: $(-1,1)^2 \times [0, 2\pi) \to U$ defined by

$\Phi(x,y,\theta)=\psi_\theta(s(\rho(x,y)))$ gives a local coordinate at $s(p)\in U$ under which $\bar g$ can be written as

$$\bar g=\sum_{\alpha,\beta=x,y} g_{\alpha\beta}\,d\alpha\,d\beta+G(d\theta+A_x\,dx+A_y\,dy)^2,$$

where G, A_x, A_y and $g_{\alpha\beta}$ are functions that are independent of θ, and G is the length of S^1-orbit. Note that a change of section changes the connection form $A=A_x\,dx+A_y\,dy$ by an exact form, and leaves invariant the curvature form

$$dA=(\partial_x A_y-\partial_y A_x)\,dx\wedge dy=F_{xy}\,dx\wedge dy.$$

Lemma 8.1 *For any $k\in\mathbb{N}$, there are $C_k>0$ such that for all $q\in U$ we have*

$$|\tilde\nabla^\ell(dA)|(\pi(q))\le\frac{C_k}{d_g^k(q,\Gamma)}\quad\text{for } \ell=0,1, \tag{8-2}$$

$$|\tilde\nabla^\ell G^{1/2}|(\pi(q))\le\frac{C_k}{d_g^k(q,\Gamma)}\quad\text{for } \ell=1,2, \tag{8-3}$$

where $\tilde\nabla$ denotes the covariant derivative on the 2D manifold (N,g_N).

Proof We adopt the notation that for a tensor $\tau=\tau^{i_1\dots i_r}_{j_1\dots j_s}\,dx^{j_1}\otimes\cdots\otimes dx^{j_s}\otimes\partial_{x_{i_1}}\otimes\cdots\otimes\partial_{x_{i_r}}$ on B, we have

$$\tau^{i_1\dots i_r}_{j_1\dots j_s,k}=\partial_{x_k}(\tau^{i_1\dots i_r}_{j_1\dots j_s}),\qquad \tau^{i_1\dots i_r}_{j_1\dots j_s,k\ell}=\partial^2_{x_k,x_\ell}(\tau^{i_1\dots i_r}_{j_1\dots j_s}),$$
$$\tilde\nabla_{\partial_{x_k}}\tau=\tau^{i_1\dots i_r}_{j_1\dots j_s;k}dx^k\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}\otimes\partial_{x_{i_1}}\otimes\cdots\otimes\partial_{x_{i_r}},$$
$$\tilde\nabla^2_{\partial_{x_k,x_\ell}}\tau=\tau^{i_1\dots i_r}_{j_1\dots j_s;k\ell}dx^k\otimes dx^\ell\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_s}\otimes\partial_{x_{i_1}}\otimes\cdots\otimes\partial_{x_{i_r}}.$$

For a point p in the base manifold parametrized by x and y, it is convenient to choose the section so that $A(p)=0$. Then the nonzero components of the curvature tensor $\bar R_{IJKL}$ of $\bar g$ are given in terms of the components of the curvature tensor $R_{\alpha\beta\gamma\delta}$ of (B,g_N), the components F_{xy}, and the function G, by

$$\begin{aligned}\bar R_{\theta\alpha\theta\beta}&=-\tfrac12 G_{;\alpha\beta}+\tfrac14 G^{-1}G_{,\beta}G_{,\alpha}+\tfrac14 g^{\gamma\delta}G^2F_{\alpha\gamma}F_{\beta\delta},\\ \bar R_{\theta xyx}&=-\tfrac12 GF_{xy;x}-\tfrac34 G_{,x}F_{xy},\\ \bar R_{xyxy}&=R_{xyxy}-\tfrac34 GF_{xy}^2.\end{aligned}\tag{8-4}$$

(See [62, Section 4.2].) Let R_{IJKL} be the components of the curvature tensor of the soliton metric g, then by Theorem 4.11 we have $|R_{IJKL}|\le C_k/d_g^k(\cdot,\Gamma)$ for any $k\in\mathbb{N}$ and $C_k>0$. So by (6-35) we obtain $|\bar R_{IJKL}|\le C_k/d_g^k(\cdot,\Gamma)$ after replacing C_k by a possibly larger number. In particular, by the second equation in (8-4) we obtain

$$|\tilde\nabla(G^{3/2}\,dA)|(\pi(q))\le\frac{C_k}{d_g^k(q,\Gamma)}\quad\text{for all } q\in U. \tag{8-5}$$

By Kato's inequality this implies

$$|\tilde\nabla|G^{3/2}\,dA||(\pi(q))\le|\tilde\nabla(G^{3/2}\,dA)|(\pi(q))\le\frac{C_k}{d_g^k(q,\Gamma)}. \tag{8-6}$$

By Theorem 3.41 there exists $C>0$ such that for $t\ge 0$,

$$d_g(\phi_t(q),\Gamma)\ge d_g(q,\Gamma)+C^{-1}t$$

for any point $q\in\Gamma_{\ge A}$, where $A>0$ is sufficiently large. Note that since $(M,g,\phi_t(q))$ converges to $\mathbb{R}^2\times S^1$ as $t\to\infty$, we have $\lim_{t\to\infty}|G^{3/2}\,dA|(\pi(\phi_t(q)))=0$. Since by (8-6) we have

$$\left|\frac{d}{dt}|G^{3/2}\,dA|(\pi(\phi_t(q)))\right|=|\langle\widetilde{\nabla}|G^{3/2}\,dA|,\pi_*(\nabla f(\phi_t(q)))\rangle|\le\frac{C_k}{d_g^k(\phi_t(q),\Gamma)},\tag{8-7}$$

integrating which from 0 to ∞, we see that there is some $C_{k-1}>0$ such that

$$|G^{3/2}\,dA|(\pi(q))\le\frac{C_{k-1}}{d_g^{k-1}(q,\Gamma)}.$$

This together with (8-5) implies (8-2). This also implies $|\frac{1}{4}g^{\alpha\beta}G^2F_{xy}^2|(\pi(q))\le C_k/d_g^k(q,\Gamma)$ in the first equation in (8-4), and hence implies $|\widetilde{\nabla}^2G^{1/2}|(\pi(q))\le C_k/d_g^k(q,\Gamma)$. Similarly, we obtain $|\widetilde{\nabla}G^{1/2}|(\pi(q))\le C_k/d_g^k(q,\Gamma)$ by integrating along $\phi_t(q)$ from 0 to ∞. □

Let ∂_θ be the Killing field of the SO(2)-isometry of $\overline{g}$ outside of a compact subset of M. We can extend it to a smooth vector field Y on M such that $|Y|\le 10$. Let $Y(t)=\phi_{t*}Y$ for all $t\ge 0$, and let

$$Q(t)=-\partial_t(Y(t))+\Delta_{g(t)}Y(t)+\mathrm{Ric}_{g(t)}(Y(t)).$$

We will often abbreviate it as $Q(t)=-\partial_tY+\Delta Y+\mathrm{Ric}(Y)$ when there is no confusion. Next, we show that $Q(t)$ has a polynomial decay away from Γ.

Lemma 8.2 *For any $k\in\mathbb{N}$, there are $C_k>0$ such that*

$$|Q(x,t)|\le\frac{C_k}{d_t^k(x,\Gamma)}\quad\textit{for all } t\ge 0.$$

Proof Since $g(t)=\phi_{-t}^*g$, $\phi_t:(M,g)\to(M,g(t))$ is an isometry, it follows that

$$\partial_tY(t)=\partial_t(\phi_{t*}Y)=\phi_{t*}(\mathscr{L}_{\nabla f}Y),\quad \Delta_{g(t)}Y(t)=\phi_{t*}(\Delta Y),\quad \mathrm{Ric}_{g(t)}(Y(t))=\phi_{t*}(\mathrm{Ric}(Y)).$$

So the lemma reduces to showing

$$|-\mathscr{L}_{\nabla f}Y+\Delta Y+\mathrm{Ric}(Y)|\le\frac{C_k}{d_g^k(\cdot,\Gamma)}.$$

Let $h=\mathscr{L}_Yg$. Then by a direct computation we have the identity (see eg [12])

$$\mathrm{div}(h)-\tfrac{1}{2}\,\mathrm{tr}\,\nabla h=\Delta Y+\mathrm{Ric}(Y).\tag{8-8}$$

Since by (6-35) we have

$$|\nabla^m h|=|\nabla^m(\mathscr{L}_Yg)|=|\nabla^m(\mathscr{L}_Y(g-\overline{g}))|\le C\cdot e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)}\quad\text{for } m=0,1,$$

it then follows that

$$|\mathrm{div}(h)|+|\mathrm{tr}\,\nabla h|\le C\cdot e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)}.$$

Therefore, by (8-8) and Theorem 4.11 (scalar curvature polynomial upper bounds),

$$|\Delta Y+\mathrm{Ric}(Y)|\le C\cdot e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)}\le \frac{C_k}{d_g^k(\cdot,\Gamma)}. \tag{8-9}$$

So the lemma reduces to estimating $|\mathscr{L}_{\nabla f}Y|$.

To do this, we assume $\nabla f=F^\theta\partial_\theta+F^\alpha\partial_\alpha$ for $\alpha=x,y$, where $F^\theta=\partial_\theta f\cdot G^{-1}$. Then we can compute

$$\mathscr{L}_{\nabla f}Y=\mathscr{L}_{\nabla f}\partial_\theta=-(\partial_\theta F^\theta\cdot\partial_\theta+\partial_\theta F^\alpha\cdot\partial_\alpha). \tag{8-10}$$

We will see in the next equations that the components $\partial_\theta F^\theta$ and $\partial_\theta F^\alpha$ also appear in the components of $\mathscr{L}_{\nabla f}\bar g$ which we will compare them to. Replacing the section $\theta=0$ by $\theta+\int_0^y\int_0^x\partial_y A_x(x',y')\,dx'\,dy'=0$ and define the local coordinates using this new section, then by (8-2) we have

$$\bar g=\sum_{\alpha,\beta=x,y} g_{\alpha\beta}\,d\alpha\,d\beta+G\,d\theta^2+\bar h,$$

where $\bar h$ is a 2-tensor satisfying $|\nabla^m\bar h|\le C_k/d_g^k(\cdot,\Gamma)$ for $m=0,1$. In particular, this implies

$$|\bar g_{\alpha\theta}|+|\partial_\beta\bar g_{\alpha\theta}|\le\frac{C_k}{d_g^k(\cdot,\Gamma)}\quad\text{for }\alpha,\beta=x,y.$$

So by a direct computation we obtain

$$\begin{aligned}(\mathscr{L}_{\nabla f}\bar g)_{\theta\beta}&=2\,\partial_\theta F^\alpha\cdot\bar g_{\alpha\beta}-F^\theta\partial_\beta G+O(d_g^{-k}(\cdot,\Gamma)),\\(\mathscr{L}_{\nabla f}\bar g)_{\theta\theta}&=F^\alpha\partial_\alpha G+2(\partial_\theta F^\theta)\cdot G+O(d_g^{-k}(\cdot,\Gamma)),\end{aligned} \tag{8-11}$$

where $O(d_g^{-k}(\cdot,\Gamma))$ denotes functions that are bounded by $C_k/d_g^k(\cdot,\Gamma)$ in absolute values.

Since $\frac12\mathscr{L}_{\nabla f}g=\nabla^2 f=\mathrm{Ric}$, we have

$$|\mathscr{L}_{\nabla f}\bar g|\le|\mathscr{L}_{\nabla f}(\bar g-g)|+|\mathscr{L}_{\nabla f}g|\le C\cdot e^{-2(1+\epsilon_1)d_g(\cdot,\Gamma)}+2|\mathrm{Ric}_g|\le\frac{C_k}{d_g^k(\cdot,\Gamma)}.$$

Therefore, by comparing (8-10) and (8-11) we can deduce

$$|\mathscr{L}_{\nabla f}Y|\le\frac{C_k}{d_g^k(\cdot,\Gamma)}+C\cdot|\tilde\nabla G|,$$

which combined with (8-3) implies

$$|\mathscr{L}_{\nabla f}Y|\le\frac{C_k}{d_g^k(\cdot,\Gamma)},$$

proving the lemma. □

Let $Z(t)$ be a vector field which solves

$$\begin{cases}-\partial_t Z+\Delta Z+\mathrm{Ric}(Z)=Q(t),\\ Z(0)=0.\end{cases} \tag{8-12}$$

In the next lemma, we show that $Z(t)$ has a polynomial decay away from Γ.

Lemma 8.3 *For any $k \in \mathbb{N}$, there are $C_k > 0$ such that $|Z(t)| \le C_k / d_t^k(\cdot, \Gamma)$.*

Proof We can compute that

$$\Delta |Z|^2 = 2\langle \Delta Z, Z\rangle + 2\langle \nabla Z, \nabla Z\rangle \quad \text{and} \quad \partial_t |Z|^2 = 2\langle \partial_t Z, Z\rangle - 2\operatorname{Ric}(Z, Z),$$

combining which with (8-12) we obtain

$$\partial_t |Z(t)|^2 = \Delta_{g(t)} |Z(t)|^2 - 2|\nabla_{g(t)} Z(t)|^2_{g(t)} - 2\langle Q(t), Z(t)\rangle_{g(t)},$$

which we will often abbreviate as $\partial_t |Z|^2 = \Delta |Z|^2 - 2|\nabla Z|^2 - 2\langle Q, Z\rangle$. Similarly, we can show $\partial_t |X|^2 = \Delta |X|^2 - 2|\nabla X|^2 \le \Delta |X|^2$. By the maximum principle we get $|X(t)| \le |X(0)| \le C$, and hence

$$|Z(t)| \le |Y(t)| + |X(t)| \le C.$$

So by Lemma 8.2,

$$\partial_t |Z|^2 \le \Delta |Z|^2 + C \cdot |Q| \le \Delta |Z|^2 + \frac{C_m}{d_t^m(\cdot, \Gamma)} \quad \text{for any } m = 0, \dots, 94.$$

The lemma now follows immediately from the following lemma. □

Lemma 8.4 *Let $(M, g(t))$ be a 3D steady gradient soliton that is not a Bryant soliton, and suppose that $u\colon M \times [0, T] \to \infty$ is a smooth nonnegative function which satisfies $u(\cdot, 0) = 0$ and*

$$\partial_t u \le \Delta u + \frac{C_0}{d_t^k(\cdot, \Gamma)}$$

for some integer $k \ge 2$ and $C_0 > 0$. Then there exists $C = C(C_0, k) > 0$ such that $u(\cdot, t) \le C / d_t^{k-1}(\cdot, \Gamma)$.

Proof Let $C > 0$ denote all constants depending on C_0, k. Denote $d_t(x, \Gamma)$ by $r(x, t)$, which satisfies the distance distortion estimates (4-12). By $u(\cdot, 0) = 0$, the maximum principle, and the integral formula for solutions of heat-type equations, we obtain

$$u(x, t) \le \int_0^t \int_M G(x, t; y, s) \frac{C_0}{r^k(y, s)} \, d_s y \, ds,$$

where $G(x, t; y, s)$ is the heat kernel of the heat equation under $g(t)$; see (2-8).

For a fixed $s \in [0, t]$, we split the integral $\int_M G(x, t; y, s)(C_0 / r^k(y, s)) \, d_s y$ into two integrals on $B_s\big(x, \frac{1}{1000} r(x, s)\big)$ and $M \setminus B_s\big(x, \frac{1}{1000} r(x, s)\big)$, and denote them respectively by $I(s)$ and $II(s)$. We will estimate them similarly as in the proof of Theorem 4.11. For $\mathrm{II}(s)$, note that

$$\frac{d_s^2(y, x)}{t - s} \ge \frac{r(x, s)}{C}$$

for all $y \in M \setminus B_s\big(x, \frac{1}{1000} r(x, s)\big)$, by the heat kernel estimates of Lemmas 4.10 and 2.21 we obtain

$$II(s) \le C \int_{M \setminus B_s(x, r(x, s)/1000)} e^{-d_s^2(y, x)/(C(t-s))} \, d_s y \le C \cdot e^{-r(x, s)/C} \le \frac{C}{r^k(x, s)}.$$

For $I(s)$, we have $r(y,s) \ge \frac{1}{2} r(x,s)$ for all $y \in B_s\bigl(x, \frac{1}{1000} r(x,s)\bigr)$, and thus

$$I(s) \le C \sup_{y \in B_s(x, r(x,s)/1000)} r^{-k}(y,s) \le \frac{C}{r^k(x,s)}.$$

Therefore, by (4-12) and the estimates of $I(s)$ and $II(s)$, we obtain

$$u(x,t) \le \int_0^t \frac{C}{r^k(x,s)}\, ds \le C\left(\frac{1}{r^{k-1}(x,t)} - \frac{1}{(r(x,t)+1.9t)^{k-1}}\right) \le \frac{C}{r^{k-1}(x,t)}. \qquad \square$$

Now we prove the main result of this section, which finds a nontrivial Killing field of (M,g) as time goes to infinity.

Proposition 8.5 *There exists a vector field X_∞ such that $\mathscr{L}_{X_\infty} g = 0$ which does not vanish everywhere and has bounded norm.*

Proof Let $X(t) = Y(t) - Z(t)$ and $\widetilde{X}(t) = \phi_{-t*} X(t)$. We will show that there exists a sequence $t_i \to \infty$ such that the vector fields $\widetilde{X}(t_i)$ on M smoothly converge to a nonzero Killing field X_∞.

By the definitions of $Z(t)$ and $X(t)$, it is easy to see that

$$\begin{cases} \partial_t X(t) = \Delta X(t) + \mathrm{Ric}(X(t)), \\ X(0) = Y(0). \end{cases}$$

Let $h(t) = \mathscr{L}_{X(t)} g(t)$. Then a direct computation shows that $h(t)$ satisfies the linearized Ricci De Turck equation

$$\begin{cases} \partial_t h(t) = \Delta_L h(t), \\ h(0) = \mathscr{L}_X g = \mathscr{L}_Y g. \end{cases}$$

Note that we have the isometry

$$(M, g(t), p, X(t), h(t)) \xrightarrow{\phi_{-t}} (M, g, p, \widetilde{X}(t), \widetilde{h}(t)),$$

where $\widetilde{h}(t) = \phi_t^* h(t) = \mathscr{L}_{\widetilde{X}(t)} g$.

By Theorem 3.41, for any $\epsilon > 0$, $\Gamma_{\ge A}$ is covered by ϵ-cylindrical planes on scale 1 for sufficiently large A. So we may pick a point $q \in M$ such that $\bigl||Y|(q,0) - 1\bigr| \le \epsilon$ and $r(q,0) > 2C_1 + 1$, where $C_1 > 0$ is the constant from Lemma 8.3. Then by the definition of Y we have $|Y|(\phi_t(q), t) = |Y|(q,0) \ge 1 - \epsilon$, and by Lemma 8.3 we have $|Z|(\phi_t(q), t) = |Z|(q,0) \le \frac{1}{2}$. Therefore,

$$|\widetilde{X}|_g(q,t) = |X|_{g(t)}(\phi_t(q), t) \ge |Y|_{g(t)}(\phi_t(q), t) - |Z|_{g(t)}(\phi_t(q), t) \ge \tfrac{1}{2} - \epsilon.$$

Next, by $|X|(t) \le C$, and the standard interior estimates for linear parabolic equations [61, Theorem 7.22], we have that $|\nabla^k X|(t) \le C_k$ uniformly for all t on $(M, g(t))$, and thus $|\nabla^k \widetilde{X}|(t) \le C_k$ on (M,g) for any $k \in \mathbb{N}$. Therefore, by the Arzelà–Ascoli theorem, there exists $t_i \to \infty$ such that $\widetilde{X}(t_i)$ smoothly uniformly converges to a vector field X_∞ and correspondingly $\widetilde{h}(t_i)$ converges to a smooth symmetric 2-tensor $\mathscr{L}_{X_\infty} g$.

First, we have $X_\infty \ne 0$, because $|X_\infty|(q) = |\widetilde{X}|(q, t_i) \ge \frac{1}{2} - \epsilon$. Moreover, by Corollary 7.4 we see that $\widetilde{h}(t_i)$ converges to 0 smoothly and uniformly on any compact subsets of M, implying $\mathscr{L}_{X_\infty} g = 0$. $\square$

9 Proof of the $O(2)$-symmetry

In this section we prove the $O(2)$-symmetry for all 3D steady gradient solitons that are not the Bryant soliton. In Proposition 8.5 we find a nonzero smooth vector field X such that $\mathscr{L}_X g = 0$. We will show that X induces an isometric $O(2)$-action $\{\chi_\theta\}_{\theta\in[0,2\pi)}$. Throughout this section we assume (M, g, f, p) is a 3D steady gradient solitons with positive curvature that is not the Bryant soliton, where p is the critical point of f, and $f(p) = 0$.

First, let $\{\chi_\theta\}_{\theta\in\mathbb{R}}$, be the one-parameter group of isometries generated by X, we show that X and ∇f commute, and hence the diffeomorphisms they generate commute.

Lemma 9.1 *We have that $[X, \nabla f] = 0$, and $\chi_\theta \circ \phi_t = \phi_t \circ \chi_\theta$ for all $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$.*

Proof We first show that the potential function is invariant under χ_θ. Let p be the critical point of f. Since p is the unique maximum point of R, we have $\chi_\theta(p) = p$ for all t, and hence

$$f \circ \chi_\theta(p) = f(p) = 0, \quad \nabla(f \circ \chi_\theta)(p) = \nabla f(p) = 0, \quad \nabla^2(f \circ \chi_\theta) = \mathrm{Ric} = \nabla^2 f.$$

For any $x \in M$, let $\sigma\colon [0,1] \to M$ be a minimizing geodesic from p to x, then

$$\begin{aligned} f(\chi_\theta(x)) &= f(\chi_\theta(p)) + \int_0^1\int_0^r \nabla^2(f\circ\chi_\theta)(\sigma'(s),\sigma'(s))\,ds\,dr \\ &= f(p) + \int_0^1\int_0^r \nabla^2 f(\sigma'(s),\sigma'(s))\,ds\,dr = f(x). \end{aligned}$$

So $f \circ \chi_\theta \equiv f$. Now since $\chi_\theta^*(f) = f$ and $\chi_\theta^* g = g$, it is easy to see that $\chi_\theta^*(\nabla f) = \nabla f$. So $[X, \nabla f] = 0$ and hence $\chi_\theta \circ \phi_t = \phi_t \circ \chi_\theta$. □

Second, we show that χ_θ is an SO(2)-isometry.

Lemma 9.2 *There exists $\lambda > 0$ such that after replacing $\{\chi_\theta\}$ by $\{\chi_{\lambda\theta}\}$, we have that $\{\chi_\theta\}$ is a* SO(2)*-isometry on M.*

Proof Since f is invariant under χ_θ, it follows that the level sets of f are invariant under χ_θ. So χ_θ induces an isometry on each level set of f. Since the level sets $f^{-1}(a)$ for $a > 0$ are compact and diffeomorphic to S^2, it is easy to see that $X|_{f^{-1}(a)}$ vanishes at exactly two points, and $\chi_\theta|_{f^{-1}(a)}$ acts by rotations with two fixed points.

Therefore, for some $a > 0$, after replacing X by λX for some $\lambda > 0$ we may assume that $\chi_\theta|_{f^{-1}(a)} = \mathrm{id}$ if and only if $\theta = 2k\pi$ for $k \in \mathbb{Z}$. In particular, for a point $y \in f^{-1}(a)$, we have $\chi_{2\pi}(y) = y$, and $(\chi_{2\pi}|_{f^{-1}(a)})_{*y}$ is the identity transformation of the tangent space $T_y f^{-1}(a)$. Since χ_θ is a smooth family of diffeomorphisms, and $\chi_0 = \mathrm{id}$, it follows that $\chi_{2\pi}$ preserves the orientation. So $(\chi_{2\pi})_{*y}$ is the identity on $T_y M$, and hence $\chi_{2\pi} = \mathrm{id}$. Therefore, χ_θ for $\theta \in [0, 2\pi)$ is an SO(2)-isometry. □

Next, we show that the fixed-point set $\Gamma' = \{x \in M : X(x) = 0\} = \{x \in M : \chi_\theta(x) = x, \theta \in \mathbb{R}\}$ of the SO(2)-isometry χ_θ coincides with $\Gamma = \Gamma_1(-\infty,\infty) \cup \Gamma_2(-\infty,\infty) \cup \{p\}$, where Γ_1 and Γ_2 are two integral curves of ∇f from Corollary 3.36.

Lemma 9.3 $$\Gamma = \Gamma'.$$

Proof Note that $\Gamma \setminus \{p\}, \Gamma' \setminus \{p\}$ are both unions of two integral curves of ∇f. Let Γ_1' and Γ_2' be the two connected components of $\Gamma' \setminus \{p\}$. It suffices to show that for each $j = 1, 2$, the integral curves Γ_j and Γ_j' intersect at some point, after possibly switching the order of Γ_1 and Γ_2. To see this, note that on the one hand, by Corollary 3.36 we have that the manifolds $(M, r^{-2}(x)g, x)$ converge to $(\mathbb{R} \times \text{cigar}, r^{-2}(x_{\text{tip}})g_c, x_{\text{tip}})$ for any sequence $x \to \infty$ along Γ. On the other hand, since the points on Γ' are fixed points of the SO(2)-isometry, it is easy to see that the manifolds $(M, r^{-2}(x)g, x)$ converge to $(\mathbb{R} \times \text{cigar}, r^{-2}(x_{\text{tip}})g_c, x_{\text{tip}})$ for any sequence $x \to \infty$ along Γ'.

Therefore, for any $i \in \mathbb{N}$, after switching the order of Γ_1 and Γ_2 we may assume that there are two points $x_i \in \Gamma_1 \cap (M \setminus B_g(p, 2))$ and $y_i \in \Gamma_1' \cap (M \setminus B_g(p, 2))$ such that $d_g(x_i, y_i) < i^{-1}$. Let $t_i > 0$ be a constant such that $\phi_{-t_i}(x_i) \in B_g(p, 2) \setminus B_g(p, 1)$. Then

$$d_g(\phi_{-t_i}(x_i), \phi_{-t_i}(y_i)) = d_{g(t_i)}(x_i, y_i) \le d_g(x_i, y_i) \le i^{-1} \to 0.$$

So after passing to a subsequence we may assume $\phi_{-t_i}(x_i), \phi_{-t_i}(y_i) \to q \neq p$, and hence $q \in \Gamma_1 \cap \Gamma_1'$ and $\Gamma_1 = \Gamma_2$. Similarly we can show $\Gamma_2 = \Gamma_2'$. □

Lastly, we prove the $O(2)$-symmetry, that is, that there exist a totally geodesic surface $N \subset M$ and a diffeomorphism $\Phi\colon N \times S^1 \to M \setminus \Gamma$ such that the pullback metric $\Phi^* g$ is a warped-product metric $\Phi^* g = g_N + \varphi^2\, d\theta^2$, with $\theta \in [0, 2\pi)$, where g_N is the induced metric on N and φ is a smooth positive function on N. Note that not every SO(2)-isometry is an $O(2)$-isometry. For example, the S^1-action on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by complex scalar multiplication is an isometry, which gives $\mathbb{CP}^n = S^{2n+1}/S^1$. But it is not an $O(2)$-isometry since the curvature form is nonzero.

Lemma 9.4 *The* SO(2)*-isometry χ_θ is an $O(2)$-isometry.*

Proof Let $\Sigma = f^{-1}(a)$ for some fixed $a > 0$, and $\sigma : [0, 1] \to \Sigma$ be a minimizing geodesic in Σ connecting the two fixed points $\{x_a, \bar{x}_a\}$. Let $\Phi\colon (0, 1) \times (-\infty, \infty) \times [0, 2\pi) \to M \setminus \Gamma$ be a diffeomorphism defined as

$$\Phi(r, t, \theta) = \phi_t(\chi_\theta(\sigma(r))) = \chi_\theta(\phi_t(\sigma(r))).$$

Note that ϕ_t and χ_θ commute by Lemma 9.1. Then we can write the metric under this coordinate as

$$g = \sum_{\alpha,\beta=r,t} g_{\alpha\beta}\, dx_\alpha\, dx_\beta + G(d\theta + A)^2.$$

Since the vectors ∂_r, $\partial_t = \nabla f$ and $\partial_\theta = X$ are orthogonal at all points in

$$\Sigma \setminus \{x_a, \bar{x}_a\} = \Phi(\{(r, 0, \theta) : r \in (0, 1), \theta \in [0, 2\pi)\}),$$

the connection form $A = A_r\, dr + A_t\, dt$ vanishes at these points. Moreover, we have $A_t = 0$ everywhere because $\langle \partial_t, \partial_\theta \rangle = \langle \nabla f, X \rangle = 0$.

On the one hand, since A vanishes on $\Sigma\setminus\{x_a,\bar{x}_a\}$, by the curvature formula (8-4) for an SO(2)-symmetric metric we have

$$\mathrm{Ric}_{t\theta}=R_{\theta rtr}=-\tfrac{1}{2}GF_{rt;r}-\tfrac{3}{4}G_{,r}F_{rt}. \tag{9-1}$$

On the other hand, by the soliton equation $\mathrm{Ric}=\nabla^2 f$ we have

$$\mathrm{Ric}_{t\theta}=\nabla^2_{t\theta}f=-\langle\nabla_{\nabla f}\nabla f,\partial_\theta\rangle=-\langle\nabla_{\partial_t}\partial_t,\partial_\theta\rangle=\tfrac{1}{2}\partial_\theta\langle\partial_t,\partial_t\rangle=0, \tag{9-2}$$

where we used $\langle\partial_t,\partial_\theta\rangle=\langle X,\nabla f\rangle=0$ and $\partial_\theta\langle\partial_t,\partial_t\rangle=X(|\nabla f|^2)=0$. So by (9-1) and (9-2) we have

$$\widetilde{\nabla}_{\partial_r}(G^{3/2}\,dA)=G^{1/2}\big(GF_{rt;r}+\tfrac{3}{2}G_{,r}F_{rt}\big)\,dr\wedge dt=0\quad\text{on } M\setminus\Gamma, \tag{9-3}$$

where $\widetilde{\nabla}$ denotes the covariant derivative on the 2D submanifold $\{\phi_t(\sigma(r))\mid r\in(0,1),t\in(-\infty,\infty)\}$.

Claim 9.5 *The equality $dA=0$ holds on $M\setminus\Gamma$.*

Proof Consider the rescaled manifolds $(M,r_i^{-2}g,x_a)$, where $r_i>0$ is an arbitrary sequence going to zero. Then it is easy to see that $(M,r_i^{-2}g,x_a)$ smoothly converges to the Euclidean space $\mathbb{R}^3$, with Γ converging to a straight line, which we may assume to be the z-axis after a change of coordinates. So the SO(2)-isometry on $(M,r_i^{-2}g,x_a)$ converges to the rotation around the z-axis. Note that $|G\,dA|$ is scale invariant, so this convergence implies $|G\,dA|(r_i,0)\to 0$ as $i\to\infty$, which proves $\lim_{r\to 0}|G\,dA|(r,0)=0$. So $\lim_{r\to 0}|G^{3/2}\,dA|(r,0)=0$. Then by (9-3), we get $G^{3/2}\,dA(r,0)=0$ and $dA(r,0)=0$. So $dA=0$ on $\Sigma\setminus\{x_a,\bar{x}_a\}$.

Note that we may choose Σ to be $f^{-1}(a)$ for any $a>0$, and the same argument implies $dA=0$ everywhere on $M\setminus\Gamma$, which proves the claim. □

Now since $dA=0$ and $A_t=0$, we have $\partial A_r/\partial t=\partial A_t/\partial r=0$. Note that $A_r(r,0,\theta)=0$, and this implies $A_r(r,t,\theta)=0$ for all $t\in\mathbb{R}$. So $A=0$, and hence the metric can be written as the following warped-product form under the coordinates (r,t,θ),

$$g=\sum_{\alpha,\beta=r,t}g_{\alpha\beta}\,dx_\alpha\,dx_\beta+G\,d\theta^2.$$ □

Using the $O(2)$-symmetry and the $\mathbb{Z}_2$-symmetry at infinity, to prove Theorem 1.6 we can follow the same line as in [60, Theorem 1.5].

References

[1] **G Anderson**, **B Chow**, *A pinching estimate for solutions of the linearized Ricci flow system on 3-manifolds*, Calc. Var. Partial Differential Equations 23 (2005) 1–12 MR Zbl

[2] **S Angenent**, **S Brendle**, **P Daskalopoulos**, **N Šešum**, *Unique asymptotics of compact ancient solutions to three-dimensional Ricci flow*, Comm. Pure Appl. Math. 75 (2022) 1032–1073 MR Zbl

[3] **I Bakas**, **S Kong**, **L Ni**, *Ancient solutions of Ricci flow on spheres and generalized Hopf fibrations*, J. Reine Angew. Math. 663 (2012) 209–248 MR Zbl

[4] **R H Bamler**, *Ricci flow with surgery*, Diplomarbeit, Ludwig-Maximilians-Universität München (2007) Available at `https://math.berkeley.edu/~rbamler/Ricci-flow-with-surgery.pdf`

[5] **R H Bamler**, *Long-time behavior of* 3*-dimensional Ricci flow, A: Generalizations of Perelman's long-time estimates*, Geom. Topol. 22 (2018) 775–844 MR Zbl

[6] **R H Bamler**, *Long-time behavior of* 3*-dimensional Ricci flow, D: Proof of the main results*, Geom. Topol. 22 (2018) 949–1068 MR Zbl

[7] **R H Bamler**, *Compactness theory of the space of super Ricci flows*, Invent. Math. 233 (2023) 1121–1277 MR Zbl

[8] **R H Bamler**, **B Kleiner**, *On the rotational symmetry of* 3*-dimensional* κ*-solutions*, J. Reine Angew. Math. 779 (2021) 37–55 MR Zbl

[9] **R H Bamler**, **B Kleiner**, *Uniqueness and stability of Ricci flow through singularities*, Acta Math. 228 (2022) 1–215 MR Zbl

[10] **T Bourni**, **M Langford**, **G Tinaglia**, *Collapsing ancient solutions of mean curvature flow*, J. Differential Geom. 119 (2021) 187–219 MR Zbl

[11] **T Bourni**, **M Langford**, **G Tinaglia**, *Convex ancient solutions to mean curvature flow*, from "Differential geometry in the large" (O Dearricott, W Tuschmann, Y Nikolayevsky, T Leistner, D Crowley, editors), Lond. Math. Soc. Lect. Note Ser. 463, Cambridge Univ. Press (2021) 50–74 MR Zbl

[12] **S Brendle**, *Rotational symmetry of self-similar solutions to the Ricci flow*, Invent. Math. 194 (2013) 731–764 MR Zbl

[13] **S Brendle**, *Rotational symmetry of Ricci solitons in higher dimensions*, J. Differential Geom. 97 (2014) 191–214 MR Zbl

[14] **S Brendle**, **K Choi**, *Uniqueness of convex ancient solutions to mean curvature flow in* $\mathbb{R}^3$, Invent. Math. 217 (2019) 35–76 MR Zbl

[15] **S Brendle**, **K Choi**, *Uniqueness of convex ancient solutions to mean curvature flow in higher dimensions*, Geom. Topol. 25 (2021) 2195–2234 MR Zbl

[16] **S Brendle**, **P Daskalopoulos**, **K Naff**, **N Sesum**, *Uniqueness of compact ancient solutions to the higher-dimensional Ricci flow*, J. Reine Angew. Math. 795 (2023) 85–138 MR Zbl

[17] **S Brendle**, **G Huisken**, **C Sinestrari**, *Ancient solutions to the Ricci flow with pinched curvature*, Duke Math. J. 158 (2011) 537–551 MR Zbl

[18] **S Brendle**, **K Naff**, *Rotational symmetry of ancient solutions to the Ricci flow in higher dimensions*, Geom. Topol. 27 (2023) 153–226 MR Zbl

[19] **R Bryant**, *Ricci flow solitons in dimension three with* SO(3)*-symmetries*, preprint, Duke Univ. (2005) Available at `http://www.math.duke.edu/~bryant/3DRotSymRicciSolitons.pdf`

[20] **D Burago**, **Y Burago**, **S Ivanov**, *A course in metric geometry*, Graduate Studies in Math. 33, Amer. Math. Soc., Providence, RI (2001) MR Zbl

[21] **Y Burago**, **M Gromov**, **G Perelman**, *A D Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk 47 (1992) 3–51 MR Zbl In Russian; translated in Russian Math. Surveys 47 (1992) 1–58

[22] **H-D Cao**, *Existence of gradient Kähler–Ricci solitons*, from "Elliptic and parabolic methods in geometry" (B Chow, R Gulliver, S Levy, J Sullivan, editors), Peters, Wellesley, MA (1996) 1–16 MR Zbl

[23] **H-D Cao**, *Recent progress on Ricci solitons*, from "Recent advances in geometric analysis" (Y-I Lee, C-S Lin, M-P Tsui, editors), Adv. Lect. Math. 11, International, Somerville, MA (2010) 1–38 MR Zbl

[24] **H-D Cao**, **G Catino**, **Q Chen**, **C Mantegazza**, **L Mazzieri**, *Bach-flat gradient steady Ricci solitons*, Calc. Var. Partial Differential Equations 49 (2014) 125–138 MR Zbl

[25] **H-D Cao**, **Q Chen**, *On locally conformally flat gradient steady Ricci solitons*, Trans. Amer. Math. Soc. 364 (2012) 2377–2391 MR Zbl

[26] **H-D Cao**, **C He**, *Infinitesimal rigidity of collapsed gradient steady Ricci solitons in dimension three*, Comm. Anal. Geom. 26 (2018) 505–529 MR Zbl

[27] **G Catino**, **P Mastrolia**, **D D Monticelli**, *Classification of expanding and steady Ricci solitons with integral curvature decay*, Geom. Topol. 20 (2016) 2665–2685 MR Zbl

[28] **P-Y Chan**, *Curvature estimates for steady Ricci solitons*, Trans. Amer. Math. Soc. 372 (2019) 8985–9008 MR Zbl

[29] **P-Y Chan**, **B Zhu**, *On a dichotomy of the curvature decay of steady Ricci solitons*, Adv. Math. 404 (2022) art. id. 108458 MR Zbl

[30] **B-L Chen**, *Strong uniqueness of the Ricci flow*, J. Differential Geom. 82 (2009) 363–382 MR Zbl

[31] **B-L Chen**, **X-P Zhu**, *Complete Riemannian manifolds with pointwise pinched curvature*, Invent. Math. 140 (2000) 423–452 MR Zbl

[32] **X Chen**, **Y Wang**, *On four-dimensional anti-self-dual gradient Ricci solitons*, J. Geom. Anal. 25 (2015) 1335–1343 MR Zbl

[33] **K Choi**, **R Haslhofer**, **O Hershkovits**, *Classification of noncollapsed translators in $\mathbb{R}^4$*, Camb. J. Math. 11 (2023) 563–698 MR Zbl

[34] **K Choi**, **R Haslhofer**, **O Hershkovits**, *A nonexistence result for wing-like mean curvature flows in $\mathbb{R}4$*, Geom. Topol. 28 (2024) 3095–3134 MR Zbl

[35] **B Chow**, **S-C Chu**, **D Glickenstein**, **C Guenther**, **J Isenberg**, **T Ivey**, **D Knopf**, **P Lu**, **F Luo**, **L Ni**, *The Ricci flow: techniques and applications, I: Geometric aspects*, Math. Surv. Monogr. 135, Amer. Math. Soc., Providence, RI (2007) MR Zbl

[36] **B Chow**, **S-C Chu**, **D Glickenstein**, **C Guenther**, **J Isenberg**, **T Ivey**, **D Knopf**, **P Lu**, **F Luo**, **L Ni**, *The Ricci flow: techniques and applications, III: Geometric-analytic aspects*, Math. Surv. Monogr. 163, Amer. Math. Soc., Providence, RI (2010) MR Zbl

[37] **B Chow**, **Y Deng**, **Z Ma**, *On four-dimensional steady gradient Ricci solitons that dimension reduce*, Adv. Math. 403 (2022) art. id. 108367 MR Zbl

[38] **B Chow**, **P Lu**, **L Ni**, *Hamilton's Ricci flow*, Graduate Studies in Math. 77, Amer. Math. Soc., Providence, RI (2006) MR Zbl

[39] **S-C Chu**, *Type II ancient solutions to the Ricci flow on surfaces*, Comm. Anal. Geom. 15 (2007) 195–215 MR Zbl

[40] **P Daskalopoulos**, **R Hamilton**, **N Sesum**, *Classification of ancient compact solutions to the Ricci flow on surfaces*, J. Differential Geom. 91 (2012) 171–214 MR Zbl

[41] **P Daskalopoulos**, **N Sesum**, *Eternal solutions to the Ricci flow on $\mathbb{R}^2$*, Int. Math. Res. Not. 2006 (2006) art. id. 83610 MR Zbl

[42] **Y Deng**, **X Zhu**, *Three-dimensional steady gradient Ricci solitons with linear curvature decay*, Int. Math. Res. Not. 2019 (2019) 1108–1124 MR Zbl

[43] **A Deruelle**, *Smoothing out positively curved metric cones by Ricci expanders*, Geom. Funct. Anal. 26 (2016) 188–249 MR Zbl

[44] **W Du**, **R Haslhofer**, *Hearing the shape of ancient noncollapsed flows in $\mathbb{R}^4$*, Comm. Pure Appl. Math. 77 (2024) 543–582 MR Zbl

[45] **L C Evans**, *Partial differential equations*, Graduate Studies in Math. 19, Amer. Math. Soc., Providence, RI (1998) MR Zbl

[46] **V A Fateev**, *The sigma model (dual) representation for a two-parameter family of integrable quantum field theories*, Nuclear Phys. B 473 (1996) 509–538 MR Zbl

[47] **M Feldman**, **T Ilmanen**, **D Knopf**, *Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons*, J. Differential Geom. 65 (2003) 169–209 MR Zbl

[48] **R S Hamilton**, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982) 255–306 MR Zbl

[49] **R S Hamilton**, *The Ricci flow on surfaces*, from "Mathematics and general relativity" (J A Isenberg, editor), Contemp. Math. 71, Amer. Math. Soc., Providence, RI (1988) 237–262 MR Zbl

[50] **R S Hamilton**, *Eternal solutions to the Ricci flow*, J. Differential Geom. 38 (1993) 1–11 MR Zbl

[51] **R S Hamilton**, *The formation of singularities in the Ricci flow*, from "Surveys in differential geometry, II" (S-T Yau, editor), International, Cambridge, MA (1995) 7–136 MR Zbl

[52] **A Hatcher**, *Notes on basic 3-manifold topology*, unpublished book draft (2001) Available at `https://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html`

[53] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[54] **H-J Hein**, **A Naber**, *New logarithmic Sobolev inequalities and an ϵ-regularity theorem for the Ricci flow*, Comm. Pure Appl. Math. 67 (2014) 1543–1561 MR Zbl

[55] **D Hoffman**, **T Ilmanen**, **F Martín**, **B White**, *Graphical translators for mean curvature flow*, Calc. Var. Partial Differential Equations 58 (2019) art. id. 117 MR Zbl

[56] **G Huisken**, **C Sinestrari**, *Convex ancient solutions of the mean curvature flow*, J. Differential Geom. 101 (2015) 267–287 MR Zbl

[57] **B Kotschwar**, **L Wang**, *Rigidity of asymptotically conical shrinking gradient Ricci solitons*, J. Differential Geom. 100 (2015) 55–108 MR Zbl

[58] **Y Lai**, *Ricci flow under local almost non-negative curvature conditions*, Adv. Math. 343 (2019) 353–392 MR Zbl

[59] **Y Lai**, *Producing 3D Ricci flows with nonnegative Ricci curvature via singular Ricci flows*, Geom. Topol. 25 (2021) 3629–3690 MR Zbl

[60] **Y Lai**, *A family of 3D steady gradient solitons that are flying wings*, J. Differential Geom. 126 (2024) 297–328 MR Zbl

[61] **G M Lieberman**, *Second order parabolic differential equations*, World Sci., River Edge, NJ (1996) MR Zbl

[62] **J Lott**, *Dimensional reduction and the long-time behavior of Ricci flow*, Comment. Math. Helv. 85 (2010) 485–534 MR Zbl

[63] **J Morgan**, **G Tian**, *Ricci flow and the Poincaré conjecture*, Clay Math. Monogr. 3, Amer. Math. Soc., Providence, RI (2007) MR Zbl

[64] **J Morgan**, **G Tian**, *Completion of the proof of the geometrization conjecture*, preprint (2008) arXiv 0809.4040

[65] **O Munteanu**, **C-J A Sung**, **J Wang**, *Poisson equation on complete manifolds*, Adv. Math. 348 (2019) 81–145 MR Zbl

[66] **G Perelman**, *The entropy formula for the Ricci flow and its geometric applications*, preprint (2002) arXiv math/0211159

[67] **P Petersen**, *Riemannian geometry*, Graduate Texts in Math. 171, Springer (1998) MR Zbl

[68] **W-X Shi**, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. 30 (1989) 223–301 MR Zbl

[69] **M Simon**, **P M Topping**, *Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces*, Geom. Topol. 25 (2021) 913–948 MR Zbl

[70] **J Spruck**, **L Xiao**, *Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature*, Amer. J. Math. 142 (2020) 993–1015 MR Zbl

[71] **X-J Wang**, *Convex solutions to the mean curvature flow*, Ann. of Math. 173 (2011) 1185–1239 MR Zbl

[72] **Z Zhao**, **X Zhu**, *Rigidity of the Bryant Ricci soliton*, preprint (2022) arXiv 2212.02889

[73] **J Zhu**, SO(2) *symmetry of the translating solitons of the mean curvature flow in $\mathbb{R}^4$*, Ann. PDE 8 (2022) art. id. 6 MR

[74] **J Zhu**, *Rotational symmetry of uniformly 3-convex translating solitons of mean curvature flow in higher dimensions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (online publication October 2023)

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Received: 26 December 2022
Revised: 28 October 2023

GEOMETRY & TOPOLOGY

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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GEOMETRY & TOPOLOGY

Volume 29 Issue 2 (pages 549–1114) 2025