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The Deligne–Mostow 9-ball, and the monster

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The "monstrous proposal" of the first author is that the quotient of a certain 13-dimensional complex hyperbolic braid group, by the relations that its natural generators have order 2, is the "bimonster" $(M \times M) \rtimes 2$. Here M is the monster simple group. We prove that this quotient is either the bimonster or $\mathbb{Z}/2$. In the process, we give new information about the isomorphism, found by Deligne and Mostow, between the moduli space of 12-tuples in $\mathbb{C}P^1$ and a quotient of the complex 9-ball. Namely, we identify which loops in the 9-ball quotient correspond to the standard braid generators.

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1 Introduction

This paper continues our pursuit of a conjectured relationship between complex hyperbolic geometry and the largest sporadic finite simple group, the monster M [Basak 2007; 2016; Allcock 2009b Allcock and Basak 2016; 2018]. In [Allcock 2009a], the first author conjectured a "monstrous proposal": that the quotient of a certain complex hyperbolic braid group G, by some natural relations $S = 1$ described below, coincides with the semidirect product $(M \times M) \rtimes 2$; see [Allcock 2009a] or the discussion just before Theorem 1.1. This group was named the bimonster in [Conway et al. 1988]; the $\mathbb{Z}/2$ exchanges the monster factors in the obvious way. Here we prove the conjecture up to one other possibility: G/S is either the bimonster or $\mathbb{Z}/2$.

Before describing G, we recall the source of the "complex hyperbolic braid group" terminology. The classical n-strand braid group can be described as follows [Fox and Neuwirth 1962]. Start with $\mathbb{C}^n$, remove the mirrors (fixed-point sets) of the reflections in the symmetric group S_n, quotient what remains by the action of S_n and then take the fundamental group. There are many generalizations of this construction, got by replacing S_n by some other reflection group and (optionally) replacing $\mathbb{C}^n$ by some other space. Three examples are Artin groups [van der Lek 1983; Charney and Davis 1995], the braid groups of the finite complex reflection groups [Bessis 2015] and the fundamental groups of certain discriminant complements arising in singularity theory [Brieskorn 1981; Looijenga 1984]. Complex reflections play a special role because their mirrors have real codimension two: removing them leaves a space which is connected but not simply connected.

We apply this general construction to a specific group $P\Gamma$, generated by complex reflections acting on complex hyperbolic space $\mathbb{B}^{13}$. We will give a precise description of $P\Gamma$ in Section 2.4, but the details are not important yet. We define $\mathcal{H}$ as the union of the mirrors of the complex reflections in $P\Gamma$. The braid group we will study is the orbifold fundamental group

$$G = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma).$$

A generic point of $\mathcal{H}$ has stabilizer generated by a single triflection (order-3 complex reflection), so its image in $\mathbb{B}^{13}/P\Gamma$ has a neighborhood which is smooth and whose intersection with $\mathcal{H}/P\Gamma$ is a smooth hypersurface. By a *meridian* we mean a small loop in $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ encircling this hypersurface once positively, or any loop freely homotopic to such a loop. The language comes from knot theory. The analogous definition for the classical braid group gives the conjugacy class containing the standard generators. It is well known that killing the squares of the standard generators reduces the n-strand braid group to the symmetric group S_n. We study the analogous quotient for G:

Theorem 1.1 *The complex hyperbolic braid group $G = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma)$, modulo the subgroup S generated by the squares of all meridians, is isomorphic to either the bimonster or $\mathbb{Z}/2$.*

Ruling out the case $G/S \cong \mathbb{Z}/2$ would prove the monstrous proposal.

Although we do not know a presentation for G, we do know that it is a quotient of a certain Artin group. The Artin group $\text{Art}(\Delta)$ of a graph Δ is defined as follows. It has one generator for each node of the graph. Its defining relations are that two of these generators, x and y, braid ($xyx = yxy$) or commute ($xy = yx$) according to whether their nodes are joined or not. The associated Coxeter group $\text{Cox}(\Delta)$ is defined by imposing the additional relations that the squares of the standard Artin generators be trivial.

We will write $P^2\mathbb{F}_3$ for the incidence graph of the 13 points and 13 lines of the projective plane over the field $\mathbb{F}_3$ of order 3 (see Section 2.4 for details). Theorem 1.2 of [Allcock and Basak 2018] says that G is a quotient of $\text{Art}(P^2\mathbb{F}_3)$, with the standard Artin generators mapping to meridians.

We juxtapose this with a result of Conway and Simons [2001, Theorem 2.1]: $\text{Cox}(P^2\mathbb{F}_3)$, modulo specific relations, is the bimonster $(M \times M) \rtimes 2$. The relations can be described as follows. Inside the incidence graph of $P^2\mathbb{F}_3$, one can find 12 vertices $\rho_0, \dots, \rho_{11}$ with the property that each is joined to its predecessor and successor (reading subscripts mod 12), and to no others. By a 12-gon we mean the subgraph spanned by any such set of 12 vertices. Each 12-gon is a copy of the affine Dynkin diagram $\widetilde{A}_{11}$, so it describes an embedding of the affine Weyl group $\text{Cox}(\widetilde{A}_{11}) \cong \mathbb{Z}^{11} \rtimes S_{12}$ into $\text{Cox}(P^2\mathbb{F}_3)$. The $\mathbb{Z}^{11}$ consists of the affine translations in the standard realization of $\text{Cox}(\widetilde{A}_{11})$. Conway and Simons show that $\text{Cox}(P^2\mathbb{F}_3)$, modulo the $\mathbb{Z}^{11}$'s arising this way from all 12-gons in $P^2\mathbb{F}_3$, is the bimonster. One can give explicit relations as follows. It is easy to show (see the proof of Theorem 1.1 in Section 7) that $\mathbb{Z}^{11}$ is normally generated in $\text{Cox}(\widetilde{A}_{11})$ by the word

$$\rho_{10} \cdots \rho_0 (\rho_{11} \cdots \rho_1)^{-1}. \tag{1-1}$$

So the quotient of $\mathrm{Cox}(P^2\mathbb{F}_2)$, by the subgroup normally generated by all words (1-1) arising from 12-gons in $P^2\mathbb{F}_3$, is the bimonster.

We will prove Theorem 1.1 by lifting these relations to G. For each ρ_i in the previous paragraph, we write τ_i for the corresponding generator of $\mathrm{Art}(P^2\mathbb{F}_3)$. This makes sense since the generators of $\mathrm{Cox}(P^2\mathbb{F}_3)$ and of $\mathrm{Art}(P^2\mathbb{F}_3)$ are indexed by the same set. We will prove

$$\tau_{10}\cdots\tau_0(\tau_{11}\cdots\tau_1)^{-1}=1 \tag{1-2}$$

in G. Therefore G is the quotient of $\mathrm{Art}(P^2\mathbb{F}_3)$ by these relations (one for each 12-gon) and possibly additional relations. It follows that G/S in Theorem 1.1 satisfies all the relations of the Conway–Simons quotient of $\mathrm{Cox}(P^2\mathbb{F}_3)$, ie of the bimonster. By the simplicity of M, the only quotients of the bimonster are the bimonster itself, $\mathbb{Z}/2$ and the trivial group. A short automorphic forms argument rules out the trivial case (Lemma 7.4), yielding Theorem 1.1.

Part of the appeal of the monstrous proposal is that the relations (1-2) have a simple and well-known geometric meaning, which we recall in detail in Section 3. Briefly, $\mathrm{Art}(\widetilde{A}_{11})$ is a subgroup of the 12-strand braid group $\mathrm{Br}_{12}(\mathbb{C}-\{0\})$ of the punctured plane. (It is the group of braids with total winding number 0.) Adjoining the relation (1-2) reduces $\mathrm{Art}(\widetilde{A}_{11})$ to the usual 12-strand braid group, corresponding to the natural map $\mathrm{Br}_{12}(\mathbb{C}-\{0\})\to\mathrm{Br}_{12}(\mathbb{C})$ got by filling in the puncture.

This geometric interpretation arises in our work, via the appearance of $\mathrm{Br}_{12}(\mathbb{C}P^1)$ as a subquotient of our braid group G. The key is that $\mathbb{B}^{13}$ contains a 9-ball $\mathbb{B}^9_{\mathrm{DM}}$, whose stabilizer acts on it as a group $P\Gamma_{\mathrm{DM}}$ discovered by Deligne and Mostow [1986; Mostow 1986]. They showed that $P\Gamma_{\mathrm{DM}}$ uniformizes the moduli space of unordered 12-tuples in $\mathbb{C}P^1$. To state this precisely, we write $\mathcal{H}_{\mathrm{DM}}$ for the hyperplane arrangement in $\mathbb{B}^9_{\mathrm{DM}}$ got by restricting $\mathcal{H}$. Then $(\mathbb{B}^9_{\mathrm{DM}}-\mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ is isomorphic, as a complex analytic orbifold, to the moduli space $\mathcal{M}^\circ_{12}$ of unordered 12-tuples of distinct points of $\mathbb{C}P^1$. The connection to braids is that $\pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12})\cong\mathrm{Br}_{12}(\mathbb{C}P^1)/(\mathbb{Z}/2)$. See Section 5 for background and technical details.

Our strategy will be to choose a tubular neighborhood U of $\mathbb{B}^9_{\mathrm{DM}}$, invariant under the setwise stabilizer $P\Gamma^{\mathrm{sw}}_{\mathrm{DM}}$ of $\mathbb{B}^9_{\mathrm{DM}}$. It develops that $(U-\mathcal{H})/P\Gamma^{\mathrm{sw}}_{\mathrm{DM}}$ fibers over $(\mathbb{B}^9_{\mathrm{DM}}-\mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$, with fiber that can be understood by work of Orlik and Solomon. This leads to an exact sequence

$$1\to\mathrm{Br}_5\to\pi_1^{\mathrm{orb}}((U-\mathcal{H})/P\Gamma^{\mathrm{sw}}_{\mathrm{DM}})\to\pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12})\to 1. \tag{1-3}$$

We work out the details of this (nonsplit) group extension. This yields several explicit relations in the Artin generators of G. This is done in Sections 6 and 7. After the preliminary Sections 2 and 3, it is possible to jump to Section 6, armed only with the statements of Theorems 4.1 and 5.2.

While our result gives the bimonster as an upper bound for G/S, Looijenga [2023] has found evidence that the bimonster is also a lower bound for it. If G/S were the bimonster, then the connected covering space of $(\mathbb{B}^{13}-\mathcal{H})/P\Gamma$ corresponding to S would have the bimonster as its deck group. Then every open subset of $(\mathbb{B}^{13}-\mathcal{H})/P\Gamma$ would have an orbifold cover with this deck group, namely its preimage.

Looijenga has found a fairly large open subset, over which this hoped-for cover exists unconditionally and is connected.

Heckman [2015] gave an argument that G is a quotient of the bimonster, but unfortunately it had a gap. Our Section 4 can be used to bridge that gap, bypassing some of our details; see Remark 7.3 for Heckman's argument. The largest difference from our work is that he seeks extra relations in G/S, rather than in G itself, as we do. Our fibration argument also applies to some other ball quotients with distinguished subballs, yielding analogues of the exact sequence (1-3). For example, we hope to give a new presentation of the orbifold fundamental group of the moduli space of smooth cubic surfaces, in terms of the Artin group of the Petersen graph.

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2 Background, conventions and notation

As far as possible we maintain notation and conventions used in [Allcock and Basak 2016; 2018]. For convenience we review the most important ones. We follow the convention that a hermitian form $\langle \cdot \mid \cdot \rangle$ is linear in its first variable and conjugate-linear in its second. The norm of a vector x means $\langle x \mid x \rangle$, sometimes written x^2. Often we use the same symbol for a vector and the point of projective space it represents. A superscript $^\perp$ indicates an orthogonal complement, the precise meaning depending on context. It might mean a sublattice, a linear subspace or a complex hyperbolic space inside a larger complex hyperbolic space. When describing groups, we often use names from the Atlas [Conway et al. 1985], and the numeral "2" as shorthand for $\mathbb{Z}/2$. For example, the Atlas writes $L_3(3)$ for $\mathrm{PSL}_3(\mathbb{F}_3) = \mathrm{PGL}_3(\mathbb{F}_3)$, so we write the semidirect product $\mathrm{PGL}_3(\mathbb{F}_3) \rtimes (\mathbb{Z}/2)$ as $L_3(3) \rtimes 2$.

The only material here not already in [Allcock and Basak 2018] is the construction of lattices from graphs (Section 2.2), the notation L_4 for the Eisenstein E_8 lattice, and the Deligne–Mostow lattice (Sections 2.3 and 2.7).

2.1 Complex hyperbolic space

A hermitian form is called Lorentzian if it has signature $(n, 1)$. For example, $\mathbb{C}^{n,1}$ denotes $\mathbb{C}^{n+1}$ equipped with the hermitian form

$$\langle x \mid y \rangle = -x_0 \bar{y}_0 + x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n.$$

Let V be a complex vector space equipped with a Lorentzian hermitian form. Then $\mathbb{B}(V)$ means the set of negative-definite complex lines in V. This is an open ball in $P(V)$. If M is a subspace of V such that the restriction of the hermitian form is also Lorentzian, then there is a natural inclusion $\mathbb{B}(M) \to \mathbb{B}(V)$. If M has codimension 1, then we sometimes call $\mathbb{B}(M)$ a hyperplane in $\mathbb{B}(V)$. The complex ball $\mathbb{B}(V)$ has a natural Riemannian metric with negative sectional curvature, called the Bergman metric, and is sometimes called complex hyperbolic space. The restriction of this metric to $\mathbb{B}(M)$ coincides with the Bergman metric of $\mathbb{B}(M)$. Suppose v and w are negative-norm vectors in V. Then they represent points of $\mathbb{B}(V)$ whose distance is

$$d(v,w) = \cosh^{-1}\sqrt{\frac{|\langle v \mid w\rangle|^2}{v^2w^2}}. \tag{2-1}$$

If in addition $\langle v \mid w\rangle$ is real and negative, then the real line segment joining v and w in V projects to the geodesic in $\mathbb{B}(V)$ between these points. Similarly, if $v, s \in V$ have negative and positive norm, respectively, then

$$d(v,\mathbb{B}(s^{\perp})) = \sinh^{-1}\sqrt{-\frac{|\langle v \mid s\rangle|^2}{v^2s^2}}. \tag{2-2}$$

If $x, y \in \mathbb{B}(V)$ are distinct, then there is a unique complex 1-ball containing them, which we denote by $\mathbb{B}^1(x, y)$. Some authors call this the complex geodesic through x and y.

2.2 Eisenstein lattices

Let $\omega = e^{2\pi i/3}$ and $\theta = \omega - \bar{\omega} = \sqrt{-3}$. Let $\mathcal{E}$ be the ring $\mathbb{Z}[\omega]$ of Eisenstein integers. An Eisenstein lattice K means a free $\mathcal{E}$-module equipped with a hermitian form $\langle\cdot\mid\cdot\rangle\colon K \times K \to \mathbb{Q}(\omega)$. We abbreviate $K \otimes_{\mathcal{E}} \mathbb{C}$ to $K \otimes \mathbb{C}$, and usually think of it as containing K. If K is Lorentzian, then we write $\mathbb{B}(K)$ for $\mathbb{B}(K \otimes \mathbb{C})$. If K is nondegenerate, then its dual lattice is defined as $K^* = \{x \in K \otimes \mathbb{C} : \langle x \mid k\rangle \in \mathcal{E} \text{ for all } k \in K\}$.

Let Δ be a directed graph without self-loops or multiple edges. Consider the free $\mathcal{E}$-module $\mathcal{E}\Delta$ on the vertex set Δ with basis vectors $\{e_\alpha : \alpha \in \Delta\}$. Define an $\mathcal{E}$-valued hermitian form on $\mathcal{E}\Delta$ by

$$\langle e_\alpha \mid e_\beta\rangle = \begin{cases} 3 & \text{if } \alpha = \beta, \\ \theta & \text{if } \Delta \text{ has an edge from } \beta \text{ to } \alpha, \\ 0 & \text{if } \Delta \text{ has no edge between } \beta \text{ and } \alpha. \end{cases}$$

Suppose Δ is connected, E is an edge whose removal would disconnect Δ, and Δ' is got from Δ by reversing the orientation on E. Then $\mathcal{E}\Delta$ and $\mathcal{E}\Delta'$ are isometric. (Choose one component of the graph got by removing E from Δ, negate all the e_α in that component and leave the others unchanged. The resulting vectors have the inner products specified by Δ'.) Applying this repeatedly shows that if Δ is a tree, then the isometry class of $\mathcal{E}\Delta$ depends only on the underlying undirected graph.

2.3 The Eisenstein E_8 lattice and the hyperbolic cell

Applying this construction to the Dynkin diagram A_4 (with any orientations of the edges) yields the $\mathcal{E}$-lattice whose underlying real form, under the bilinear form $\frac{2}{3}\operatorname{Re}\langle x \mid y\rangle$, is the E_8 root lattice. We called it $E_8^{\mathcal{E}}$ in [Allcock and Basak 2018], but here we will call it L_4. (Note: $\operatorname{Re}\langle x \mid y\rangle$ lies in $\frac{3}{2}\mathbb{Z}$. The natural scale for L_4 resp. E_8 is the smallest scale at which all inner products lie in $\mathcal{E}$ resp. $\mathbb{Z}$. This is the reason for the $\frac{2}{3}$ factor.) The self-duality of the E_8 lattice leads to the property $\theta L_4^* = L_4$. A useful consequence of this is that if K is an $\mathcal{E}$-lattice in which all inner products are divisible by θ, then every copy of L_4 in K is an orthogonal direct summand.

The property $\theta K^* = K$ of an Eisenstein lattice K turns out to characterize K if K is indefinite [Basak 2007, Lemma 2.6]. Two examples are the lattices L and L_{DM} below. Another is the hyperbolic cell, which means the $\mathcal{E}$-span of two null vectors with inner product θ.

2.4 The lattice L

The following lattice L will play a central role. The quickest way to define it is to refer to the uniqueness property just stated. Namely, L is the unique $\mathcal{E}$-lattice of signature $(13, 1)$ that satisfies $\theta L^* = L$; see [Basak 2007, Lemma 2.6]. The quickest concrete construction is to define L as the sum of a hyperbolic cell and three copies of L_4. In [Allcock and Basak 2016; 2018] we also used a different direct sum description of L, with the Leech lattice in place of L_4^3, but this plays no role here. The only 13-ball we will discuss is $\mathbb{B}(L)$, so we will write $\mathbb{B}^{13}$ for it. All complex hyperbolic geometry will take place in $\mathbb{B}^{13}$.

We will use the following model of L for most calculations. It was implicit in [Basak 2007, (25) in the proof of Proposition 6.1] and made explicit in [Allcock 2009b]. We write $P^2\mathbb{F}_3$ for the incidence graph of the finite projective plane over $\mathbb{F}_3$. It has 26 vertices, corresponding to the 13 points and the 13 lines of this finite projective plane. Whenever β is a line and α is a point on it, we direct the edge from the line to the point. One verifies that the associated rank-26 lattice $\mathcal{E}P^2\mathbb{F}_3$ defined in Section 2.2 has radical of rank 12; see [Basak 2016, Section 2.5]. One may define L as the quotient by the radical. Obviously all inner products are divisible by θ, ie $L \subseteq \theta L^*$, and one can check that this inclusion is equality.

It is possible to label the points and lines of $P^2\mathbb{F}_3$ by $p_1, \dots, p_{13}$ and $l_1, \dots, l_{13}$ so that the points on l_j are p_j, p_{j+1}, p_{j+3} and p_{j+9}. Subscripts here should be read mod 13. We use the same symbols for the vectors in L corresponding to the points and lines. Because they are roots in the sense of Section 2.5, we call them the point- and line-roots. Since points (resp. lines) are not joined to each other, the point-roots (resp. line-roots) are mutually orthogonal. One may introduce coordinates $(x_0; x_1, \dots, x_{13})$ on $\mathbb{C}^{13,1}$ such that $p_1 = (0; \theta, 0, \dots, 0)$, $l_1 = (1; 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0)$ and rightward cyclic permutation of the last 13 coordinates increases subscripts by 1.

We write Γ for the isometry group of L. Obviously it contains the group $L_3(3) := \mathrm{PGL}_3(\mathbb{F}_3)$, permuting the points and lines of $P^2\mathbb{F}_3$ via the standard action of $\mathrm{GL}_3(\mathbb{F}_3)$ on $\mathbb{F}_3^3$. There is additional symmetry. From an incidence-preserving exchange of points with lines, one can construct an isometry of L that sends the point-roots to line-roots and the line-roots to negated point-roots. Together with scalars and $L_3(3)$, this generates a subgroup of Γ whose image in $P\Gamma$ is $L_3(3) \rtimes 2$. We will use this $L_3(3) \rtimes 2$ many times.

2.5 Roots, mirrors and the hyperplane arrangement $\mathcal{H}$

A root of L means a lattice vector of norm 3. Roots are special because their triflections (complex reflections of order 3) give elements of Γ. Namely, if s is a root, then we define ω-reflection in s to be the isometry of $L \otimes \mathbb{C}$ that fixes $s^\perp$ pointwise and multiplies s by the cube root of unity ω. A formula is $x \mapsto x - (1-\omega)(\langle x \mid s\rangle / s^2)s$. Using $L \subseteq \theta L^*$, one can show that this preserves L. Although we don't need it, we remark that Γ is generated by the triflections in the 26 point- and line-roots; see [Basak 2007, Theorem 1.1] or [Allcock 2009b, Theorem 1].

The hyperplane $\mathbb{B}(s^\perp) \subseteq \mathbb{B}(L)$ is called the mirror of s, the name reflecting the fact that it is the fixed-point set of a reflection. When the meaning is clear, we sometimes abbreviate $\mathbb{B}(s^\perp)$ to $s^\perp$. We use the word mirror exclusively for hyperplanes orthogonal to roots. The union of all mirrors of L is called $\mathcal{H}$. This hyperplane arrangement is central to the paper: our goal is to understand the orbifold fundamental group of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$.

2.6 Special points in $\mathbb{B}^{13}$

We have already introduced the point-roots $p_1, \dots, p_{13}$ and line-roots $l_1, \dots, l_{13}$. We call their mirrors the point- and line-mirrors. The p_i (resp. l_i) are mutually orthogonal, and we write p_∞ (resp. l_∞) for the point of $\mathbb{B}^{13}$ orthogonal to all of them. This turns out to be represented by the norm -3 vector $p_\infty = (\bar{\theta}; 0, \dots, 0)$ (resp. $l_\infty = (4; 1, \dots, 1)$).

A convenient basepoint for the orbifold fundamental group (see Section 2.8) of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ is the midpoint τ of the geodesic segment joining p_∞ and l_∞. It is represented by the vector

$$\tau = l_\infty + i p_\infty = (4 + \sqrt{3}; 1^{13})$$

of norm $-6 - 8\sqrt{3}$. The corresponding point of $\mathbb{B}^{13}$ is the unique fixed point of $L_3(3) \rtimes 2 \subseteq P\Gamma$. The mirrors closest to τ are exactly the 26 point- and line-mirrors; see [Basak 2007, Proposition 1.2], where τ was called $\bar{\rho}$, or [Allcock and Basak 2018, Lemma A.5]. Two consequences of this are that no mirror passes through τ, and that $L_3(3) \rtimes 2$ is the full $P\Gamma$-stabilizer of τ.

2.7 The Deligne–Mostow lattice L_{DM}

The lattice L_{DM} is both a sublattice of L and a lower-dimensional analogue of L. We start with its role as an analogue. Consider the $\widetilde{A}_{11}$ Dynkin diagram (a 12-gon), with its edges' orientations alternating. The corresponding lattice (Section 2.2) has 2-dimensional radical, and L_{DM} is defined as the quotient by it; see [Allcock 2000, Section 5], which also displays an isometry between L_{DM} and the sum of a hyperbolic cell and two copies of L_4. L_{DM} has signature $(9, 1)$ and satisfies $\theta L_{\mathrm{DM}}^* = L_{\mathrm{DM}}$. We define $\mathbb{B}^9_{\mathrm{DM}}$ as $\mathbb{B}(L_{\mathrm{DM}})$, and Γ_{DM} as the isometry group of L_{DM}; see Theorem 5.2 for a refinement of the famous relationship found by Deligne and Mostow between $\mathbb{B}^9_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}}$ and the moduli space of unordered 12-tuples in $\mathbb{C}P^1$. By construction, $P\Gamma_{\mathrm{DM}}$ contains a dihedral group D_{24} of order 24 that permutes the vertices of the 12-gon. (Because of the orientations of edges, automorphisms of the diagram only provide a D_{12}. The following is also an isometry: cyclically permute the 12 roots, and negate every other one. Together with D_{12} and scalars, this generates a subgroup of Γ_{DM} whose image in $P\Gamma_{\mathrm{DM}}$ is D_{24}. This is similar to how we enlarged $L_3(3)$ to $L_3(3) \rtimes 2$ in Section 2.4.) We write ρ for the unique fixed point in $\mathbb{B}^9_{\mathrm{DM}}$ of this dihedral group. We define roots as for L, and define $\mathcal{H}_{\mathrm{DM}} \subseteq \mathbb{B}^9_{\mathrm{DM}}$ as the union of the mirrors of the roots of L_{DM}. The mirrors closest to ρ turn out to be the ones corresponding to the nodes of the 12-gon (Lemma 4.6).

L_{DM} appears as a sublattice of L because the oriented graph $P^2\mathbb{F}_3$ described in Section 2.4 contains a 12-gon with alternating orientations on edges. Because $L_{\mathrm{DM}} = \theta L^*_{\mathrm{DM}}$ and all inner products in L are divisible by θ, L_{DM} is a summand, so $\Gamma_{\mathrm{DM}} \subseteq \Gamma$. We make a particular choice of 12-gon in Section 4. The $(L_3(3)\rtimes 2)$-stabilizer of this 12-gon is the D_{24} just mentioned. The projection of $\tau \in \mathbb{B}^{13}$ to $\mathbb{B}^9_{\mathrm{DM}}$ is the point ρ. Part of our work will involve moving a basepoint along the geodesic segment $\overline{\rho\tau}$. We prove in Lemma 5.3 that $\mathcal{H}_{\mathrm{DM}}$ is the restriction of $\mathcal{H} \subseteq \mathbb{B}^{13}$ to $\mathbb{B}^9_{\mathrm{DM}}$.

2.8 Orbifold fundamental groups

Let A be a group acting properly discontinuously on a path-connected manifold X. Choose a basepoint $b \in X$. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(X/A, b)$ is defined to be equivalence classes of pairs (γ, g), where $g \in A$, γ is a path in X from b to gb, and (γ, g) is equivalent to (γ', g') if $g = g'$ and γ and γ' are homotopic in X, rel endpoints. The group operation is

$$(\gamma, g)\cdot(\gamma', g') = (\gamma \text{ followed by } g \circ \gamma', gg')$$

Inversion in $\pi_1^{\mathrm{orb}}(X/A, b)$ is given by $(\gamma, g)^{-1} = (g^{-1} \circ \mathrm{reverse}(\gamma), g^{-1})$. Projection of (γ, g) to g defines a homomorphism $\pi_1^{\mathrm{orb}}(X/A, b) \to A$. It is surjective because X is path connected. The kernel is obviously $\pi_1(X, b)$, yielding the exact sequence

$$1 \to \pi_1(X, b) \to \pi_1^{\mathrm{orb}}(X/A, b) \to A \to 1. \tag{2-3}$$

The local group at b means the set of $(\gamma, g) \in \pi_1^{\mathrm{orb}}(X/A, b)$ for which γ is homotopic to the constant path at b. This is obviously the same as the A-stabilizer of b.

2.9 Meridians

Meridians are distinguished elements of $\pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, b)$ or $\pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, b)$, where $b \in \mathbb{B}^{13} - \mathcal{H}$ or $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ is a basepoint. If s is a root and S is the ω-reflection in s, then the corresponding meridian $M_{b,s}$ is defined as $(\mu_{b,s}, S)$, where $\mu_{b,s}$ is the following path; let p be the projection of b into the mirror $s^\perp$. In this paper the real geodesic segment $\overline{bp}$ from b to p never meets any other mirror. Choose a ball around p small enough to miss all other mirrors, and let q be a point of $\overline{bp} - \{p\}$ in this ball. Then $\mu_{b,s}$ is the geodesic $\overline{bq}$, followed by the positive circular arc in $\mathbb{B}^1(b, q)$ of angle $\frac{2}{3}\pi$ centered at p, followed by $S(\overline{qb})$.

The extra generality about meridians that we developed in [Allcock and Basak 2016] is not needed here. In particular, no detours of the sort considered there are needed. The only basepoints that we will need are $\rho \in \mathbb{B}^9_{\mathrm{DM}}$ for $\pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}})$, and a variable point $\sigma \in \overline{\rho\tau} - \{\rho\} \subseteq \mathbb{B}^{13}$ when working with $\pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H}), P\Gamma)$. Once one of these basepoints is fixed, the meridians associated to the point- and line-roots are called the point- and line-meridians. No other meridians will appear.

3 Braid groups

In this section we assemble some well-known material about braid groups in a form exhibiting dihedral symmetry D_{2n}, where n is the number of strands, which will remain constant. The $n = 12$ case will be key in later sections. Although our presentations are slightly different, [Birman 1974] is an excellent general reference.

For a Riemann surface Σ, we define $X(\Sigma)$ as Σ^n, the space of ordered n-tuples in Σ. We restrict attention to the cases $\Sigma = \mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. The n-strand pure braid space of Σ means

$$X^\circ(\Sigma) := \{(x_0, \dots, x_{n-1}) \in X(\Sigma) \mid x_j \neq x_k \text{ whenever } j \neq k\}$$

The symmetric group S_n acts on $X(\Sigma)$, and this action is free on $X^\circ(\Sigma)$. Let

$$Y(\Sigma) = X(\Sigma)/S_n \quad \text{and} \quad Y^\circ(\Sigma) = X^\circ(\Sigma)/S_n.$$

The latter space $Y^\circ(\Sigma)$ is called the n-strand braid space of Σ, and its fundamental group is called the n-strand braid group of Σ, written $\mathrm{Br}_n(\Sigma)$. The ordinary braid group Br_n refers to the case $\Sigma = \mathbb{C}$. Setting $\zeta = e^{2\pi i/n}$, we will use the tuple $T = (1, \zeta, \zeta^2, \dots, \zeta^{n-1})$ as the basepoint of $X^\circ(\Sigma)$, and its image in $Y^\circ(\Sigma)$ as the basepoint there. Usually one specifies a loop in $Y^\circ(\Sigma)$ by writing down a path in $X^\circ(\Sigma)$ from T to one of its S_n-images.

We now define specific braids $\rho_j \in \mathrm{Br}_n(\mathbb{C}^*)$, whose subscripts should be read modulo n, by specifying the motion of points of T. Informally: the points beginning at ζ^{j-1} and ζ^j approach each other, move around each other in a counterclockwise direction, and then continue to ζ^j and ζ^{j-1}, respectively. The remaining points do not move. For a precise definition, choose $r : [0,1] \to [1,\infty)$ continuous with $r(0) = r(1) = 1$ and $r\left(\frac{1}{2}\right) > 1$. Then ρ_j is the element of $\mathrm{Br}_n(\mathbb{C}^*)$ represented by the path $(x_0(t), \dots, x_{n-1}(t))$ in $X^\circ(\mathbb{C}^*)$, where

$$x_{j-1}(t) = \zeta^{j-1} r(t) e^{2\pi i t/n}, \qquad x_j(t) = \zeta^j r(t)^{-1} e^{-2\pi i t/n} \qquad \text{and} \qquad x_k(t) = \zeta^k \quad \text{if } k \neq j-1, j.$$

We define the "increasing" and "decreasing" words

$$I_j = \rho_j \rho_{j+1} \cdots \rho_{j+n-2} \quad \text{and} \quad D_j = \rho_j \rho_{j-1} \cdots \rho_{j-n+2}.$$

(Following our conventions for orbifold fundamental groups in Section 2.8, I_j means ρ_j followed by ρ_{j+1}, followed by etc, and similarly for D_j. Because S_{12} acts freely on X°, the ordinary and orbifold fundamental groups of Y° coincide. So we identify elements of $\pi_1^{\mathrm{orb}}(Y^\circ)$ with their underlying paths.) One can establish the following relations by examining how the specified braids move the ζ_j around in $\mathbb{C}^*$:

$$I_j \rho_k I_j^{-1} = \rho_{k+1} \quad \text{for all } k \neq j-1, j-2, \tag{3-1}$$

$$D_j \rho_k D_j^{-1} = \rho_{k-1} \quad \text{for all } k \neq j+1, j+2. \tag{3-2}$$

The inclusions $\mathbb{C}^* \to (\mathbb{C}^* \cup \{0 \text{ or } \infty\}) \to \mathbb{C}P^1$ induce homomorphisms

$$\mathrm{Br}_n(\mathbb{C}^*) \to \mathrm{Br}_n(\mathbb{C}^* \cup \{0 \text{ or } \infty\}) \to \mathrm{Br}_n(\mathbb{C}P^1).$$

Sometimes we speak of the ρ_j as though they were elements of these other groups, meaning their images there.

Theorem 3.1 (braid groups) (a) *The subgroup of* $\mathrm{Br}_n(\mathbb{C}^*)$ *generated by* $\rho_0, \dots, \rho_{n-1}$ *has defining relations* $\rho_j \rho_k \rho_j = \rho_k \rho_j \rho_k$ *or* $\rho_j \rho_k = \rho_k \rho_j$, *according to whether* $k \in \{j \pm 1\}$ *or not, for each* j *and* k. *In particular, sending the standard generators of the Artin group* $\mathrm{Art}(\widetilde{A}_{n-1})$ *to* $\rho_0, \dots, \rho_{n-1}$ *embeds* $\mathrm{Art}(\widetilde{A}_{n-1})$ *in* $\mathrm{Br}_n(\mathbb{C}^*)$.

(b) *Adjoining to (a) the relations that all* D_j *coincide yields* $\mathrm{Br}_n(\mathbb{C}^* \cup \{0\})$. *Then* $D\rho_k D^{-1} = \rho_{k-1}$ *for all* k, *where* D *is the common image of all* D_j.

(c) *Adjoining to (a) the relations that all* I_j *coincide yields* $\mathrm{Br}_n(\mathbb{C}^* \cup \{\infty\})$. *Then* $I\rho_k I^{-1} = \rho_{k+1}$ *for all* k, *where* I *is the common image of all* I_j.

(d) *Adjoining to (a), (b) and (c) the relation* $ID = 1$ *yields* $\mathrm{Br}_n(\mathbb{C}P^1)$.

Remarks The proof of (b) shows that adjoining to (a) any single relation $D_j = D_{j'}$, with $j \neq j'$, implies the equality of all D_j; similarly for (c).

The proof of (d) shows that in the presence of the relations (a) and any one relation $I_k D_j = 1$, the relations in (b) imply those in (c) and vice versa. So $\mathrm{Br}_n(\mathbb{C}P^1)$ can be got by adjoining two relations to $\mathrm{Art}(\widetilde{A}_{n-1})$, for example $I_0 = I_1$ and $I_1 D_{n-1} = 1$.

In particular, one could omit either (b) or (c) from the presentation (d) of $\mathrm{Br}_n(\mathbb{C}P^1)$. (This omission would leave one of I and D undefined, which is why we write $I_k D_j = 1$ not $ID = 1$.) We prefer to keep both (b) and (c), because in Theorem 6.6 we will meet an extension of $\pi_1^{\mathrm{orb}}(\mathcal{M}_{12}^\circ) \cong \mathrm{Br}_{12}(\mathbb{C}P^1)/(\mathbb{Z}/2)$ to which these relations lift but the relation $ID = 1$ does not.

Proof (a) The map $(x_0, \dots, x_{n-1}) \mapsto x_0 \cdots x_{n-1}$ fibers $X^\circ(\mathbb{C}^*)$ over $\mathbb{C}^*$. This descends to a fibration $Y^\circ(\mathbb{C}^*) \to \mathbb{C}^*$. All the ρ_j are paths in a single fiber, and it is well known that the fundamental group of each fiber is the Artin group of type $\widetilde{A}_{n-1}$, with the ρ_j corresponding to the standard Artin generators; see for example [van der Lek 1983, Theorem 3.8].

The existence of this fibration shows $\mathrm{Br}_n(\mathbb{C}^*) \cong \mathrm{Art}(\widetilde{A}_{n-1}) \rtimes \mathbb{Z}$, where the $\mathbb{Z}$ is generated by any braid with total winding number 1 around 0. For example, the braid t that rotates the roots of unity one position counterclockwise; formally, $x_j(u) = \zeta^j e^{2\pi i u/n}$. Drawing a picture shows $t\rho_k t^{-1} = \rho_{k-1}$ for all k.

(b) From the inclusion $Y^\circ(\mathbb{C}^*) \to Y^\circ(\mathbb{C})$, one can work out the map $\mathrm{Br}_n(\mathbb{C}^*) \to \mathrm{Br}_n(\mathbb{C})$ by using Van Kampen's theorem. It is surjective, with kernel normally generated by the following braid: every $x_k(u)$ is the constant path at ζ^k except $x_{n-1}(u)$, which starts at ζ^{n-1}, approaches 0, encircles it once negatively and then returns to ζ^{n-1}. Formally, $\mathrm{Br}_n(\mathbb{C})$ is the quotient of $\mathrm{Br}_n(\mathbb{C}^*)$ by the relation $D_{n-1}t^{-1} = 1$.

Another way to say this is that $\mathrm{Br}_n(\mathbb{C})$ is the quotient of $\mathrm{Art}(\widetilde{A}_{n-1})$ by the relations that D_{n-1} conjugates the generators of the Artin group in the same way that t does. That is, by the relations $D_{n-1}\rho_k D_{n-1}^{-1} =$

ρ_{k-1} for all k. Repeatedly conjugating D_{n-1} by itself therefore yields $D_{n-2}, D_{n-3}, \ldots$. On the other hand, repeatedly conjugating D_{n-1} by itself also yields $D_{n-1}, D_{n-1}, \ldots$. This establishes the relations in (b).

To finish the proof of (b) it is enough to show that adjoining the relations $D_j = D_{j'}$ for all j and j' to $\mathrm{Art}(\widetilde{A}_{n-1})$ implies $D_{n-1}\rho_k D_{n-1}^{-1} = \rho_{k-1}$ for all k. In fact it is enough to adjoin any single relation $D_j = D_{j'}$ with $j' \neq j$; write D for the common image of D_j and $D_{j'}$ in the quotient. If $j' \notin \{j \pm 1\}$, then (3-2) establishes $D\rho_k D^{-1} = \rho_{k-1}$ for all k. Reusing the argument of the previous paragraph, repeatedly conjugating D by itself yields $D_{j-1} = D$, $D_{j-2} = D$, $D_{j-3} = D$ and so on. This implies $D_{n-1}\rho_k D_{n-1}^{-1} = \rho_{k-1}$ for all k, as desired. In the remaining case $j' \in \{j \pm 1\}$, we may suppose $j' = j+1$ by exchanging j and j' if necessary. This time (3-2) shows only that $D\rho_k D^{-1} = \rho_{k-1}$ for all $k \neq j+2$. But $D_j = D_{j+1}$ implies $D_j\rho_{j+2} = D_{j+1}\rho_{j+2}$, which picture drawing shows is equal to $\rho_{j+1}D_j$. So again we have $D\rho_{j+2}D^{-1} = \rho_{j+1}$, and the same argument applies.

(c) Inversion across the unit circle exchanges 0 with ∞, preserves t, inverts every ρ_k and exchanges D_{n-1} with I_1^{-1}. Therefore we may quote the argument for (b) to obtain two descriptions of $\mathrm{Br}_n(\mathbb{C}^* \cup \{\infty\})$. First, it is the quotient of $\mathrm{Br}_n(\mathbb{C}^*)$ by the relation $I_1^{-1}t^{-1} = 1$. Second, it is the quotient of $\mathrm{Art}(\widetilde{A}_{n-1})$ by the relations that all I_j are equal.

(d) Using Van Kampen's theorem twice shows that $\mathrm{Br}_n(\mathbb{C}P^1)$ is the quotient of $\mathrm{Br}_n(\mathbb{C}^*)$ by both relations $D_{n-1}t^{-1} = 1$ and $I_1^{-1}t^{-1} = 1$. Therefore $\mathrm{Br}_n(\mathbb{C}P^1)$ can be described as the quotient of $\mathrm{Art}(\widetilde{A}_{n-1})$ by $I_1 D_{n-1} = 1$ and either the relations (b) or (c). If we assume (b), then the fact that D centralizes I_1 forces $I_0 = I_1$, $I_{n-1} = I_1$, $I_{n-2} = I_1$ etc, establishing (c), and vice versa. So the relation $I_1 D_{n-1} = 1$ can be written $ID = 1$. □

For any elements $g_1, \ldots, g_m$ of a group, we define

$$\Delta(g_1, \ldots, g_m) = (g_1 g_2 \cdots g_m)(g_1 g_2 \cdots g_{m-1}) \cdots (g_1 g_2) g_1.$$

In Br_{n+1}, $\Delta(\rho_1, \ldots, \rho_n)$ is called the "fundamental element"; it conjugates each ρ_j to ρ_{n+1-j}, and its square generates the center of Br_{n+1}.

Using the obvious $\mathrm{PGL}_2\,\mathbb{C}$ action on $X(\mathbb{C}P^1)$, we define the moduli space $\mathcal{M}_n^\circ$ of n-point subsets of $\mathbb{C}P^1$ as

$$\mathcal{M}_n^\circ = X^\circ(\mathbb{C}P^1)/(S_n \times \mathrm{PGL}_2\,\mathbb{C}) = Y^\circ(\mathbb{C}P^1)/\mathrm{PGL}_2\,\mathbb{C}.$$

We assume $n \geq 3$ to avoid degenerate cases. Then $\mathrm{PGL}_2(\mathbb{C})$ acts freely and properly on $X^\circ(\mathbb{C}P^1)$, so the quotient is a manifold. Because $\mathcal{M}_n^\circ$ is the quotient of this manifold by the finite group S_n, it is an orbifold. Recall that the tuple T is our basepoint for $X^\circ(\mathbb{C}P^1)$. We take its images in $X^\circ(\mathbb{C}P^1)/\mathrm{PGL}_2\,\mathbb{C}$ and $\mathcal{M}_n^\circ$, also denoted by T, as our basepoints when discussing their orbifold fundamental groups. It is well known that the map $\mathrm{Br}_n(\mathbb{C}P^1) \to \pi_1^{\mathrm{orb}}(\mathcal{M}_n^\circ)$ induced by $Y^\circ(\mathbb{C}P^1) \to \mathcal{M}_n^\circ$ is surjective, with kernel equal to the center of $\mathrm{Br}_n(\mathbb{C}P^1)$, which is isomorphic to $\mathbb{Z}/2$ with generator $I^n = D^{-n}$; see for example [Birman 1974, Theorem 4.5]. So Theorem 3.1 implies the following theorem (the final assertion is obvious in the presence of the fourth relation):

Theorem 3.2 *The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{M}_n^\circ, T)$ is generated by $\rho_0, \dots, \rho_{n-1}$, with defining relations*

(a) $\rho_j \rho_k \rho_j = \rho_k \rho_j \rho_k$ or $\rho_j \rho_k = \rho_k \rho_j$, *according to whether $k \in \{j \pm 1\}$ or not, for each j and k,*

(b) *all the $I_j := \rho_j \rho_{j+1} \cdots \rho_{j+n-1}$ coincide,*

(c) *all the $D_j := \rho_j \rho_{j-1} \cdots \rho_{j-n+1}$ coincide,*

(d) $ID = 1$, *where I (resp. D) is the common image of the I_j (resp. D_j).*

(e) $I^n = D^n = 1$.

When n is even, (e) *may be replaced by the relation $I^{n/2} = D^{n/2}$.* □

4 Change of basepoint in $\mathbb{B}^{13}$

In this section, we begin working in the hyperplane arrangement complement $\mathbb{B}^{13} - \mathcal{H}$, with an emphasis on how the Deligne–Mostow ball $\mathbb{B}^9_{\mathrm{DM}}$ lies inside $\mathbb{B}^{13}$. We fix an A_4 subdiagram of the incidence graph of $P^2\mathbb{F}_3$, whose 26 nodes are the point- and line-roots p_j and l_j from Section 2.4. Any two such subdiagrams are equivalent, but for concreteness we choose l_1, p_2, l_2 and p_3. These roots are mutually orthogonal, except that $\langle p_2 \,|\, l_1 \rangle = \langle p_2 \,|\, l_2 \rangle = \langle p_3 \,|\, l_2 \rangle = \theta$. Therefore their integral span is a copy of the Eisenstein lattice L_4 (see Section 2.3). We call it $L_4^{(1)}$ to distinguish it from two other copies of L_4 introduced below. As indicated in Section 2.7, we will write L_{DM} for the orthogonal complement of $L_4^{(1)}$ in L, and $\mathbb{B}^9_{\mathrm{DM}}$ for the corresponding 9-ball. See Section 5 for more information about the connection of the Deligne–Mostow ball quotient to moduli of 12-tuples in $\mathbb{C}P^1$. A mirror orthogonal to a root s of L contains $\mathbb{B}^9_{\mathrm{DM}}$ if and only if s is a root of the positive definite lattice $L_4^{(1)}$. The scaled real form of $L_4^{(1)}$ is E_8 (see Section 2.3), which has 240 roots. So exactly 40 mirrors contain $\mathbb{B}^9_{\mathrm{DM}}$, corresponding to the scalar classes of the 240 roots of $L_4^{(1)}$.

The nodes of $P^2\mathbb{F}_3$ that are not joined to the A_4 form an $\widetilde{A}_{11}$ diagram, ie a 12-gon, namely

$$p_6, l_{10}, p_{13}, l_4, p_7, l_{11}, p_{12}, l_9, p_9, l_8, p_8, l_5 \tag{4-1}$$

in cyclic order. We introduce alternative notation $s_0, \dots, s_{11}$ for them, in this order. We also write s_{A}, s_{B}, s_{C} and s_{D} for the roots l_1, p_2, l_2 and p_3 forming the A_4 diagram. As explained in Section 2.7, the $\mathcal{E}$-span of $s_0, \dots, s_{11}$ is L_{DM}. We write ρ for the projection of τ to $\mathbb{B}^9_{\mathrm{DM}}$, and let σ be any point of $\overline{\tau\rho} - \{\rho\}$. We abbreviate the meridians $M_{\tau,s_0}, \dots, M_{\tau,s_{11}}, M_{\tau,s_{\mathrm{A}}}, \dots, M_{\tau,s_{\mathrm{D}}}$ to $\tau_0, \dots, \tau_{11}, \tau_{\mathrm{A}}, \dots, \tau_{\mathrm{D}}$, and similarly with σ in place of τ. In Section 5 we will extend this notation by writing $\rho_0, \dots, \rho_{11}$ for the meridians based at ρ and associated to $s_0, \dots, s_{11}$. But these will represent elements of a different complex hyperbolic braid group $\pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$. Because ρ lies in $\mathcal{H}$, it doesn't make sense to speak of $\pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \rho)$.

The following theorem is the main result of this section. Lemma 4.6 is the only other result referenced later.

Theorem 4.1 *The segment $\overline{\tau\rho}$ meets $\mathcal{H}$ only at ρ. For any $\sigma \in \overline{\tau\rho} - \{\rho\}$, the change-of-basepoint isomorphism*

$$\pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \tau) \cong \pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \sigma),$$

induced by the segment $\overline{\tau\sigma}$, identifies each meridian $\tau_0, \dots, \tau_{11}, \tau_{\mathrm{A}}, \dots, \tau_{\mathrm{D}}$ based at τ with the corresponding meridian $\sigma_0, \dots, \sigma_{11}, \sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ based at σ.

Proof This follows immediately from the next lemma: fixing $j = 0, \dots, 11, \mathrm{A}, \dots, \mathrm{D}$, and letting σ vary over $\overline{\tau\rho} - \{\rho\}$, the surface swept out by the meridians σ_j misses $\mathcal{H}$. □

Lemma 4.2 (a) *Fix $j = 0, \dots, 11$, and let Q be the totally real quadrilateral with vertices τ, ρ and the projections τ' and ρ' of these points to $s_j^\perp$. Then Q meets $s_j^\perp$ in $\overline{\tau'\rho'}$, meets the 40 mirrors containing $\mathbb{B}^9_{\mathrm{DM}}$ in $\overline{\rho\rho'}$, and misses all other mirrors.*

(b) *Fix $j = \mathrm{A}, \dots, \mathrm{D}$, and let T be the totally real triangle with vertices τ, ρ and the projection τ' of τ to $s_j^\perp$. Then T meets $s_j^\perp$ in $\overline{\tau'\rho}$, meets the other 39 mirrors containing $\mathbb{B}^9_{\mathrm{DM}}$ at ρ only, and misses all other mirrors.*

We will start the proof of Lemma 4.2 after some preparation that includes Lemma 4.3. The stabilizer $L_3(3) \rtimes 2$ of τ contains an order-24 dihedral group D_{24} that preserves $\{s_{\mathrm{A}}^\perp, \dots, s_{\mathrm{D}}^\perp\}$ and acts faithfully and transitively on the set of mirrors $s_0^\perp, \dots, s_{11}^\perp$. Half of these transformations exchange $s_{\mathrm{A}}^\perp$ with $s_{\mathrm{D}}^\perp$ and $s_{\mathrm{B}}^\perp$ with $s_{\mathrm{C}}^\perp$. Therefore it suffices to prove the s_{A} and s_{B} cases of (b) and the s_0 case of (a). We restrict attention to these three cases.

We will need to know the corners of the polygons explicitly. All three have $\tau = (4 + \sqrt{3}; 1, \dots, 1)$ as a vertex, with norm $-6 - 8\sqrt{3}$. Now, the projection of τ to $L_4^{(1)} \otimes \mathbb{C}$ is

$$-(3 + 2\sqrt{3})(l_1 + i p_3) - (5 + 3\sqrt{3})(l_2 + i p_2).$$

One checks this by computing this vector's inner products with l_1, p_2, l_2 and p_3, and comparing with $\langle \tau \mid p_j \rangle = -\theta$ and $\langle \tau \mid l_j \rangle = -\sqrt{3}$. Subtracting it from τ gives the other vertex that is shared by all three polygons, namely

$$\rho = (6\lambda; 2\lambda, 0, 0, 2\lambda, 3\lambda, 1, 1, 1, 1, 2\lambda, 3\lambda, 1, 1) \quad \text{where } \lambda = 2 + \sqrt{3}.$$

Computation shows $\rho^2 = \langle \rho \mid \tau \rangle = -36 - 24\sqrt{3}$.

The remaining vertex or vertices are different in the three cases. First we consider the $s_{\mathrm{A}} (= l_1)$ case of Lemma 4.2(b). Then

$$\tau' = \tau - \frac{\langle \tau \mid l_1 \rangle}{\langle l_1 \mid l_1 \rangle} l_1 = \tau - \frac{-\sqrt{3}}{3} l_1 = \tau + l_1/\sqrt{3}.$$

The calculation of τ' in the s_{B} $(= p_2)$ case is the same, except that $\langle \tau \mid p_2 \rangle = -\theta$, leading to $\tau' = \tau + i p_2/\sqrt{3}$. In both cases one computes

$$(\tau')^2 = \langle \tau' \mid \tau \rangle = -7 - 8\sqrt{3} \quad \text{and} \quad \langle \tau' \mid \rho \rangle = -36 - 24\sqrt{3}.$$

Similarly, in the quadrilateral case we have $\tau' = \tau + i p_6/\sqrt{3}$,

$$(\tau')^2 = \langle \tau' \mid \tau \rangle = -7 - 8\sqrt{3} \quad \text{and} \quad \langle \tau' \mid \rho \rangle = -37 - 24\sqrt{3}.$$

The fourth vertex of Q is ρ'. Because ρ differs from τ by a vector orthogonal to $s_0 (= p_6)$, we have $\langle \rho \mid p_6 \rangle = \langle \tau \mid p_6 \rangle = -\theta$. Therefore $\rho' = \rho + i p_6/\sqrt{3}$. One computes

$$(\rho')^2 = \langle \rho' \mid \rho \rangle = \langle \rho' \mid \tau \rangle = \langle \rho' \mid \tau' \rangle = -37 - 24\sqrt{3}.$$

The reason that all these inner products turn out to be real is that $i p_1, \dots, i p_{13}, l_1, \dots, l_{13}$ have real inner products, and the real hyperbolic 13-space they span contains ρ' and all three τ'. In the three cases, we have seen that the vertices of T (resp. T, Q) are represented by negative-norm vectors whose real span is 3-dimensional and totally real (ie contains no complex subspaces). This justifies our claim in Lemma 4.2 that $Q, T \subseteq \mathbb{B}^{13}$ are totally real polygons. For use in the proof of Lemma 4.2, we also note the negativity of the pairwise inner products of the vectors representing the vertices.

Our strategy for proving Lemma 4.2 derives from [Allcock and Basak 2018, Appendix A]. There, we showed that various totally real triangles are covered by balls of various finite radii centered at the point $p_\infty = (\theta; 0, \dots, 0) \in \mathbb{B}^{13}$, where all 13 point-mirrors meet. For each of these balls, we used a computer to enumerate the finitely many mirrors that meet it. Then we worked out whether and how these mirrors meet each triangle of interest. We will use the same strategy, but we will also need balls centered around another point $c \in \mathbb{B}^{13}$. To define it, note that there is a unique way to choose two A_4 diagrams in the $\widetilde{A}_{11}$ that are not joined to each other or to s_0. Namely, we define $L_4^{(2)}$ as the span of $s_2, \dots, s_5$, and $L_4^{(3)}$ as the span of $s_7, \dots, s_{10}$. We define c as the point of $\mathbb{B}^{13}$ that is orthogonal to $\langle s_0 \rangle \oplus L_4^{(1)} \oplus L_4^{(2)} \oplus L_4^{(3)}$. Just like p_∞, c is represented by a norm -3 lattice vector, namely

$$c = \theta(4; 1, 0, 0, 2, 2, 0, 0, 0, 0, 1, 2, 1, 0).$$

As in [Allcock and Basak 2018, Appendix A], we call an open ball centered at p_∞ a critical ball if its boundary is tangent to some mirror. Even though some mirrors pass through p_∞, we do not count the radius-0 ball as critical. The distance from p_∞ to the mirror of a root s is given by

$$\sinh^{-1} \sqrt{-\frac{|\langle p_\infty \mid s \rangle|^2}{p_\infty^2 s^2}} = \sinh^{-1} \sqrt{\frac{|\langle p_\infty \mid s \rangle|^2}{9}} = \sinh^{-1} \sqrt{\tfrac{1}{3}(0, 1, 3, 4, 7, \dots)}.$$

The last equality comes from $\langle p_\infty \mid s \rangle \in \theta\mathcal{E}$. The numerical values of the first few critical radii are

$$r_1, r_2, r_3, r_4, \dots \approx 0.549,\ 0.881,\ 0.987,\ 1.210, \dots.$$

We call s a batch-n root, and $s^\perp$ a batch-n mirror (around p_∞) if $s^\perp$ is tangent to the n^{th} critical ball (around p_∞). We extend this language to "batch-0" in the case $s \perp p_\infty$. All these considerations apply verbatim with c in place of p_∞.

Lemma 4.3 *The triangles in the s_{A} and s_{B} cases of Lemma 4.2(b), and the quadrilateral in the s_0 case of Lemma 4.2(a), are covered by the union of the fourth critical balls around p_∞ and c.*

Proof In the quadrilateral case, we define points on two edges of Q, namely $m=\frac{1}{2}(\tau+\rho)\in\overline{\tau\rho}$ and $m'=\frac{1}{2}(\tau'+\rho')\in\overline{\tau'\rho'}$. Because balls and (totally real) polygons are convex, it is enough to check that τ, τ', m and m' lie at distance $< r_4=\sinh^{-1}\sqrt{\frac{7}{3}}\approx 1.210$ from p_∞, and that m, m', ρ and ρ' lie at distance $< r_4$ from c. In the triangle cases we use the same argument, except that we define $m'=\frac{1}{2}(\tau'+\rho)$, and we replace the quadrilateral m, m', ρ, ρ' by the triangle m, m', ρ. Here are the data shared by all three cases:

$$d(p_\infty,\tau)=\cosh^{-1}\sqrt{\tfrac{1}{2}+\tfrac{2}{3}\sqrt{3}}\approx .740,\qquad d(p_\infty,m)=\cosh^{-1}\sqrt{\tfrac{1303}{1034}+\tfrac{1676}{1551}\sqrt{3}}\approx 1.172,$$
$$d(c,m)=\cosh^{-1}\sqrt{\tfrac{794}{517}+\tfrac{1376}{1551}\sqrt{3}}\approx 1.161,\qquad d(c,\rho)=\cosh^{-1}\sqrt{\tfrac{5}{4}+\tfrac{13}{18}\sqrt{3}}\approx 1.032.$$

In the triangle (resp. quadrilateral) cases we have

$$d(c,m')=\cosh^{-1}\sqrt{\tfrac{1828}{1195}+\tfrac{1056}{1195}\sqrt{3}}\approx 1.158\qquad\text{resp.}\quad \cosh^{-1}\sqrt{\tfrac{1994}{1319}+\tfrac{1152}{1319}\sqrt{3}}\approx 1.151.$$

In the s_{A} case (resp. the other two cases) we have

$$d(p_\infty,\tau')=\cosh^{-1}\sqrt{\tfrac{320}{429}+\tfrac{96}{143}\sqrt{3}}\approx .848\qquad\text{resp.}\quad \cosh^{-1}\sqrt{\tfrac{59}{143}+\tfrac{96}{143}\sqrt{3}}\approx .700.$$

In the s_{A}, s_{B} and quadrilateral cases respectively we have

$$d(p_\infty,m')=\cosh^{-1}\sqrt{\tfrac{4996}{3585}+\tfrac{256}{239}\sqrt{3}}\approx 1.195,\quad \cosh^{-1}\sqrt{\tfrac{1483}{1195}+\tfrac{1296}{1195}\sqrt{3}}\approx 1.170,$$

and

$$\cosh^{-1}\sqrt{\tfrac{3103}{2638}+\tfrac{1452}{1319}\sqrt{3}}\approx 1.163.$$

Finally, in the quadrilateral case

$$d(c,\rho')=\cosh^{-1}\sqrt{\tfrac{443}{359}+\tfrac{256}{359}\sqrt{3}}\approx 1.024.$$

One may use the identity $\cosh^2=\sinh^2+1$ to avoid approximation when checking that these distances are less than $\sinh^{-1}\sqrt{\frac{7}{3}}$. This reduces one to checking that each radicand is less than $\frac{10}{3}$. □

Proof of Lemma 4.2 As mentioned above, it suffices to prove this for the quadrilateral and two triangles treated in Lemma 4.3. By that lemma, the only mirrors that can meet these polygons are the 0^{th}, 1^{st}, 2^{nd} and 3^{rd} batch mirrors around p_∞ and c. There are only finitely many mirrors in each batch, and in [Allcock and Basak 2018, Lemma A.12] we explained how to use a computer to iterate over the mirrors in these batches around p_∞. Below we will explain the corresponding enumeration around c. Now suppose given a triangle $T\subseteq\mathbb{B}^{13}$ whose vertices are represented by vectors whose norms and inner products are negative. Following [Allcock and Basak 2018, Lemma A.1], T meets a mirror $s^\perp$ if and only if the origin lies in the triangle $\langle T\mid s\rangle\subseteq\mathbb{C}$ whose vertices are the inner products of these vectors with s. By examining the 0^{th} through 3^{rd} batch mirrors, around p_∞ and c, we found all the mirrors meeting the polygons. We carried out all calculations using exact arithmetic in the field $\mathbb{Q}(\theta,\sqrt{3})$. The results are as stated in the lemma. □

It remains to explain the enumeration of mirrors near c. We work with respect to the basis

$$c, s_0;\quad s_{\mathrm{A}}, s_{\mathrm{B}}, s_{\mathrm{C}}, s_{\mathrm{D}};\quad s_5, s_4, s_3, s_2;\quad s_7, s_8, s_9, s_{10}$$

for $\mathbb{C}^{13,1}$, whose inner product matrix is

$$\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} \oplus \begin{pmatrix} 3 & \bar\theta & 0 & 0 \\ \theta & 3 & \theta & 0 \\ 0 & \bar\theta & 3 & \bar\theta \\ 0 & 0 & \theta & 3 \end{pmatrix} \oplus \begin{pmatrix} 3 & \bar\theta & 0 & 0 \\ \theta & 3 & \theta & 0 \\ 0 & \bar\theta & 3 & \bar\theta \\ 0 & 0 & \theta & 3 \end{pmatrix} \oplus \begin{pmatrix} 3 & \bar\theta & 0 & 0 \\ \theta & 3 & \theta & 0 \\ 0 & \bar\theta & 3 & \bar\theta \\ 0 & 0 & \theta & 3 \end{pmatrix}.$$

(The backwards ordering of the 4-tuple $s_5, \dots, s_2$ makes its inner product matrix coincide with those of the other 4-tuples.) To avoid confusion, we will use square brackets when writing components of vectors with respect to this basis.

Lemma 4.4 *L consists of all vectors $[a/\theta, b/\theta; \vec v_1; \vec v_2; \vec v_3]$ where $a, b \in \mathcal{E}$ are congruent mod θ and $\vec v_1, \vec v_2, \vec v_3 \in L_4$.*

Proof This is just a computation, but we indicate why it should be true. The 4-tuple in each of the last three blocks of basis vectors generates a copy of L_4. Recall that L_4 is a summand of any lattice that contains it and that has all inner products divisible by θ. Therefore the semicolons delimit summands of L. The only possibility for the remaining summand is $\left(\begin{smallmatrix} 0 & \bar\theta \\ \theta & 0 \end{smallmatrix}\right)$. Therein, c is a primitive lattice vector in the orthogonal complement of the norm -3 vector s_0, and therefore must have norm 3 (making visible a fact we already used). There are only two proper enlargements of the lattice $\left(\begin{smallmatrix} -3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ to a lattice with all inner products divisible by θ, got by adjoining $(s_0 \pm c)/\theta$. We chose the sign of c so that $(s_0 + c)/\theta \in L$. □

Lemma 4.5 *The roots in batches $0, \dots, 3$ around c appear in Table 1.*

batch	a	b	$\lvert b\rvert^2$	norms of $\vec v_1, \vec v_2, \vec v_3$	mirrors
0	0	0	0	3, 0, 0	120
	0	$\pm\omega^j\theta$	3	0, 0, 0	1
1	1	ω^j	1	3, 0, 0	2 160
	1	$-2\omega^j$	4	0, 0, 0	3
2	θ	0	0	6, 0, 0	6 480
	θ	0	0	3, 3, 0	172 800
	θ	$\pm\omega^j\theta$	3	3, 0, 0	4 320
3	-2	ω^j	0	6, 0, 0	6 480
	-2	ω^j	0	3, 3, 0	518 400
	-2	$-2\omega^j$	4	3, 0, 0	2 160
	-2	$\omega^j(3 + (\omega \text{ or } \bar\omega))$	7	0, 0, 0	6

Table 1: The batch $0, \dots, 3$ mirrors around c; see Lemma 4.5. We list all batch-0 roots, and one root from each scalar class of roots in batches 1, 2 and 3. Listed roots have the form $[a/\theta, b/\theta; \vec v_1; \vec v_2; \vec v_3]$ where a and b appear in the table, and the norms of $\vec v_1, \vec v_3, \vec v_3 \in L_4$ are as specified, up to permutation. When present, j varies over $\{0, 1, 2\}$. The last column gives the number of mirrors arising from the listed roots.

Proof Suppose $s = [a/\theta, b/\theta; \vec{v}_1; \vec{v}_2; \vec{v}_3]$ is a root in batch 0 (resp. 1, 2, 3). Then $|\langle s \mid c\rangle|^2 = 0$ (resp. 3, 9, 12). Since $\langle s \mid c\rangle = -3(a/\theta)$, this shows $|a|^2 = 0$ (resp. 1, 3, 4). We may suppose $a = 0$ (resp. 1, θ, -2), after scaling by a unit. Since $s^2 = 3$,

$$3\left|\frac{b}{\theta}\right|^2 + \vec{v}_1^2 + \vec{v}_2^2 + \vec{v}_3^2 = 3 + 3\left|\frac{a}{\theta}\right|^2 = 3 \text{ (resp. } 4, 6, 7). \tag{4-2}$$

In particular, $|b|^2$ is at most the right side. After imposing the condition

$$b \equiv a \equiv 0 \text{ (resp. } 1, 0, 1) \mod \theta,$$

b must be one of the possibilities listed in the table. From (4-2) follows $\vec{v}_1^2 + \vec{v}_2^2 + \vec{v}_3^2 = 3 + |a|^2 - |b|^2$. Since the vectors of L_4 have norms $0, 3, 6, 9, \ldots$, the norms of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are as stated in Table 1, up to permutation. Counting the mirrors of each type uses the fact that L_4 has 240 vectors of norm 3 and 2160 of norm 6. □

To carry out the proof of Lemma 4.2, we prepared lists of the norm 3 and 6 vectors in L_4, and used these to make lists of all possibilities for $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, which in turn we used to construct the roots in Table 1. We converted these roots to our usual coordinates, and then computed their mirrors' intersections with the polygons. The enumeration of mirrors near c also allows us work out ρ's nearest mirrors:

Lemma 4.6 *The components of $\mathcal{H}$ that come nearest ρ, other than those that pass through it, are $s_0^\perp, \ldots, s_{11}^\perp$.*

Proof One can check

$$\sinh^{-1}\sqrt{-\tfrac{1}{12} + \tfrac{1}{18}\sqrt{3}} + \cosh^{-1}\sqrt{\tfrac{5}{4} + \tfrac{13}{18}\sqrt{3}} < \sinh^{-1}\sqrt{\tfrac{7}{3}}.$$

The first and second terms on the left are the distances from ρ to $s_j^\perp$ and c, respectively. They are approximately .113 and 1.032. The right side is $r_4 \approx 1.210$. It follows that the mirrors nearest ρ, subject to missing it, lie in batches $0, \ldots, 3$ around c. Examining these batches proves the lemma. To avoid approximation when checking the inequality, apply sinh to both sides and then use its "angle sum" formula. □

5 The Deligne–Mostow 9-ball quotient

In their celebrated papers, Deligne and Mostow [1986; Mostow 1986] related many ball quotients to various moduli spaces of tuples in $\mathbb{C}P^1$. Their work has been revisited from several perspectives [Thurston 1998; Couwenberg et al. 2005]. We recommend Looijenga's expository paper [2007] and use it for most of our references. We will establish notation, then extract what we need from the literature, and then explain our refinements.

We recall and extend the notation of Section 3 in the case $\Sigma = \mathbb{C}P^1$ and $n = 12$. Namely, X is the space $(\mathbb{C}P^1)^{12}$ of ordered 12-tuples in $\mathbb{C}P^1$, and X° is the subspace in which the 12 points are all distinct. We also define X^{st} as the subspace of X consisting of ordered 12-tuples in $\mathbb{C}P^1$ with no points of multiplicity > 5. To the previously defined

$$Y = X/S_{12}, \quad Y^\circ = X^\circ/S_{12} \quad \text{and} \quad \mathcal{M}_{12}^\circ = Y^\circ/\mathrm{PGL}_2\,\mathbb{C}$$

we add

$$Y^{\mathrm{st}} = X^{\mathrm{st}}/S_{12} \quad \text{and} \quad \mathcal{M}_{12}^{\mathrm{st}} = Y^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}.$$

The superscript st indicates stability in the sense of geometric invariant theory (GIT). The only part of stability that is important for us is that $\mathrm{PGL}_2\,\mathbb{C}$ acts properly on X^{st} (and hence on Y^{st}). This follows from the elementary fact that $\mathrm{PGL}_2\,\mathbb{C}$ acts simply transitively on ordered triples in $\mathbb{C}P^1$, which also shows that $\mathrm{PGL}_2\,\mathbb{C}$ acts freely on X°. Therefore $X^\circ/\mathrm{PGL}_2\,\mathbb{C}$ is a manifold. Quotienting again, this time by S_{12}, yields the orbifold $Y^\circ/\mathrm{PGL}_2\,\mathbb{C}$.

Theorem 5.1 (Deligne and Mostow) *There is an isomorphism $f\colon \mathcal{M}_{12}^{\mathrm{st}} \to \mathbb{B}_{\mathrm{DM}}^9/P\Gamma_{\mathrm{DM}}$ of complex analytic varieties with the following properties*:

(a) *The restriction of f to $\mathcal{M}_{12}^\circ = Y^\circ/\mathrm{PGL}_2\,\mathbb{C}$ is a complex analytic orbifold isomorphism onto $(\mathbb{B}_{\mathrm{DM}}^9 - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$.*

(b) *If $z \in \mathbb{B}_{\mathrm{DM}}^9$ lies in exactly one component of $\mathcal{H}_{\mathrm{DM}}$, then its image in $\mathbb{B}_{\mathrm{DM}}^9/P\Gamma_{\mathrm{DM}}$ corresponds via f to a 12-tuple with exactly one multiple point, that point having multiplicity 2.*

(c) *Write χ for the antiholomorphic involution of Y^{st} corresponding to inversion across the unit circle. We use the same notation for the induced antiholomorphic involution of $\mathcal{M}_{12}^{\mathrm{st}}$. Suppose $y \in Y^\circ$ is χ-invariant, and write $f(y)$ for its image under the composition of projection to $\mathcal{M}_{12}^\circ$ and $f\colon \mathcal{M}_{12}^\circ \to (\mathbb{B}_{\mathrm{DM}}^9 - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$. Suppose $\widetilde{f(y)} \in \mathbb{B}_{\mathrm{DM}}^9 - \mathcal{H}_{\mathrm{DM}}$ lies over $f(y)$. Then there is an antiholomorphic involution $\hat{\chi}$ of $\mathbb{B}_{\mathrm{DM}}^9$ that fixes $\widetilde{f(y)}$ and lies over $f \circ \chi \circ f^{-1}$.*

Proof We refer to the example after Looijenga's Theorem 3.9 in [2007]. In his notation we take $n = 10$, and weights $\mu = (\mu_0, \dots, \mu_{11}) = \left(\frac{1}{6}, \dots, \frac{1}{6}\right)$. The hyperbolic case of that theorem asserts

(5-1) $$\text{there exists a } \Gamma\text{-equivariant isomorphism } F\colon \widetilde{\mathcal{S}_\mu \backslash \mathcal{Q}_\mu^{\mathrm{st}}} \to \mathbb{B}_{n-1}.$$

The map F is called the Schwartz map. By an isomorphism, he means an isomorphism of complex analytic orbifolds.

Looijenga's $\mathbb{B}_{n-1} = \mathbb{B}_9$ is a complex 9-ball, and his Γ is the image of a certain monodromy representation, in this case a homomorphism from $\pi_1(Y^\circ) \cong \mathrm{Br}_{12}(\mathbb{C}P^1)$ to the group $\mathrm{PU}(9,1)$ of biholomorphisms of $\mathbb{B}_9$. This homomorphism sends each standard braid generator to a complex reflection whose nonidentity eigenvalue is $e^{2\pi i/3}$. This follows from Looijenga's Corollary 2.2. Unfortunately its statement gives the wrong eigenvalue $e^{2\pi i/6}$. However, his argument gives the correct eigenvalue $-e^{-2\pi i/6} = e^{2\pi i/3}$. The same calculation, with this same result, appears in the proof of [Mostow 1986, Lemma 3.9]. The fact that the standard generators act by complex reflections with this eigenvalue, and satisfy the braid and commutation relations, uniquely determines the monodromy representation, and identifies Looijenga's Γ

with our $P\Gamma_{\mathrm{DM}}$. This is the content of [Allcock 2000, Section 5]. Because this representation is irreducible, the Γ- and $P\Gamma_{\mathrm{DM}}$-invariant hermitian forms of signature $(9,1)$ are uniquely determined, up to scale. Therefore they are identified with each other, up to scale. This identifies Looijenga's $\mathbb{B}_9$ with our $\mathbb{B}^9_{\mathrm{DM}}$.

We phrase Looijenga's constructions in our notation. To the kernel of $\pi_1(Y^\circ) \to P\Gamma_{\mathrm{DM}}$ corresponds a covering space $\widetilde{Y}^\circ$ of Y°, with deck group $P\Gamma_{\mathrm{DM}}$. Looijenga calls this space $\widetilde{V}^\circ_{10}$ and describes it concretely in his Section 1.5. He defines a $P\Gamma_{\mathrm{DM}}$-equivariant holomorphic map $F\colon \widetilde{Y}^\circ \to \mathbb{B}^9_{\mathrm{DM}}$ in terms of the periods of certain integrals. There is a canonical extension of $\widetilde{Y}^\circ \to Y^\circ$ to a ramified covering space $\widetilde{Y}^{\mathrm{st}} \to Y^{\mathrm{st}}$, and a unique extension $F\colon \widetilde{Y}^{\mathrm{st}} \to \mathbb{B}^9_{\mathrm{DM}}$. Furthermore, $\widetilde{Y}^{\mathrm{st}}$ is smooth. This is the content of the first sentence of [Looijenga 2007, Proposition 3.8]. These spaces fit into a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{Y}^\circ & \hookrightarrow & \widetilde{Y}^{\mathrm{st}} & \longrightarrow & \widetilde{Y}^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C} & \xrightarrow{\;F\;} & \mathbb{B}^9_{\mathrm{DM}} \\
{\scriptstyle \text{quotient by } P\Gamma_{\mathrm{DM}}}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y^\circ & \hookrightarrow & Y^{\mathrm{st}} & \longrightarrow & Y^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C} & \xrightarrow{\;f\;} & \mathbb{B}^9_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}} \\
 & & & & \parallel & & \\
 & & & & \mathcal{M}^{\mathrm{st}}_{12} & &
\end{array}$$

in which vertical arrows indicate quotients by $P\Gamma_{\mathrm{DM}}$. In our notation, (5-1) asserts that the top right map F is a $P\Gamma_{\mathrm{DM}}$-equivariant isomorphism of complex analytic orbifolds. Our f is the induced map got from this by quotienting the domain and range by $P\Gamma_{\mathrm{DM}}$.

Technical note (1) Looijenga quotients by $\mathrm{SL}_2\,\mathbb{C}$ rather than $\mathrm{PGL}_2\,\mathbb{C}$. He is careful not to let the subgroup $\{\pm 1\}$ contribute to the orbifold fundamental groups, even though it acts trivially. The way he does this is by defining his space $\mathcal{Q}^\circ_\mu$ as the smooth *variety* (not orbifold) $X^\circ/\,\mathrm{SL}_2\,\mathbb{C}$. See his page 230 for this definition and explicit charts defining the variety structure. The presence of $\{\pm 1\}$, acting trivially, has no effect on this quotient. So for our purposes it is harmless to replace $\mathrm{SL}_2\,\mathbb{C}$ by $\mathrm{PGL}_2\,\mathbb{C}$. However, we may no longer use the simple connectivity of $\mathrm{SL}_2\,\mathbb{C}$ to automatically lift the $\mathrm{SL}_2\,\mathbb{C}$-action from Y° to $\widetilde{Y}^\circ$. We must check that the $\mathrm{PGL}_2\,\mathbb{C}$-action on Y° lifts. We do this as follows, where $\delta\colon S^1 \to \mathrm{PGL}_2\,\mathbb{C}$ is a 1-parameter subgroup representing the nontrivial element of $\pi_1(\mathrm{PGL}_2\,\mathbb{C})$: We must show that, regarding δ as a member of $\mathrm{Br}_{12}(\mathbb{C}P^1)$ by applying it to some 12-tuple, its image in $P\Gamma_{\mathrm{DM}}$ is trivial. By construction, δ is the braid commonly called the "full twist". Formally, δ is the nontrivial central element of $\mathrm{Br}_{12}(\mathbb{C}P^1)$. So its image in $P\Gamma_{\mathrm{DM}}$ is central there, and hence preserves every mirror. Therefore it is trivial.

(2) We have described extending the domain of F from $\widetilde{Y}^\circ$ to $\widetilde{Y}^{\mathrm{st}}$, and then quotienting by $\mathrm{PGL}_2\,\mathbb{C}$. Looijenga does this in the other order. This makes no difference, because the constructions are $\mathrm{PGL}_2\,\mathbb{C}$-invariant. Our formulation makes it easier to explain the orbifold isomorphism in (a). His $\mathcal{S}_\mu\backslash\mathcal{Q}^\circ_\mu$, also written $\mathcal{S}_\mu\backslash V^\circ_n$ to draw on earlier parts of his paper, is our $Y^\circ/\mathrm{PGL}_2\,\mathbb{C}$, and similarly with superscripts changed to $^{\mathrm{st}}$ or a tilde added.

There are commuting actions of $\mathrm{PGL}_2\,\mathbb{C}$ and $P\Gamma_{\mathrm{DM}}$ on $\widetilde{Y}^{\mathrm{st}}$. We quotient by $P\Gamma_{\mathrm{DM}}$ and then $\mathrm{PGL}_2\,\mathbb{C}$, and compare to what happens when we quotient in the other order. The quotient $\widetilde{Y}^{\mathrm{st}}/P\Gamma_{\mathrm{DM}}$ is an orbifold and hence also an analytic space. *As an analytic space*, it is the complex manifold Y^{st}, essentially by the definition of the normalization of analytic spaces (which is how $\widetilde{Y}^{\mathrm{st}}$ is defined). The $\mathrm{PGL}_2\,\mathbb{C}$-action on Y^{st} is proper, making the quotient into an orbifold and hence an analytic space. These are the standard orbifold and analytic space structures on $\mathcal{M}_{12}^{\mathrm{st}}$. Now we take the quotients in the other order. Looijenga's theorem identifies $\widetilde{Y}^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$ (as an orbifold) with the manifold $\mathbb{B}^9_{\mathrm{DM}}$. Taking the quotient of this by $P\Gamma_{\mathrm{DM}}$ gives the orbifold $\mathbb{B}^9_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}}$, whose underlying analytic space has now been identified with $\mathcal{M}_{12}^{\mathrm{st}}$. This proves the claim on the first line of Theorem 5.1.

We phrased this as an isomorphism of analytic spaces, even though all the spaces involved are orbifolds. This is because it is not an orbifold isomorphism: the elements of $\widetilde{Y}^{\mathrm{st}} - \widetilde{Y}^{\circ}$ have nontrivial $P\Gamma_{\mathrm{DM}}$-stabilizers, so the *orbifold* quotient $\widetilde{Y}^{\mathrm{st}}/P\Gamma_{\mathrm{DM}}$ is not the manifold Y^{st}, even though the analytic space quotient is. However, this issue disappears when $\widetilde{Y}^{\mathrm{st}}$ is replaced by $\widetilde{Y}^{\circ}$, where $P\Gamma_{\mathrm{DM}}$ acts freely. Then the same argument establishes an orbifold isomorphism between $\mathcal{M}_{12}^{\circ}$ and its image in $\mathbb{B}^9_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}}$.

In particular, to prove (a) it is enough to show $F(\widetilde{Y}^{\circ}) = \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$. It is difficult to give a precise citation, so we will unpack some of Looijenga's constructions. The key point is that if some of the 12 points collide, then the Schwartz map F degenerates to the Schwartz map of a smaller tuple of numbers, got by fusing the μ_k's in that collision pattern; see his Section 1.4. His Theorem 3.9 also applies to that Schwartz map, establishing an isomorphism from the moduli space of stable 12-tuples with those collisions (or more) to a subball of $\mathbb{B}^9_{\mathrm{DM}}$. That subball may contain subsubballs corresponding to more-singular collision patterns, and so on. In this way, the natural stratification of Y^{st} is identified with the stratification of $\mathbb{B}^9_{\mathrm{DM}}$ by intersections of mirrors. In particular, $\widetilde{Y}^{\circ}/\mathrm{PGL}_2\,\mathbb{C}$ is identified with $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, and the smooth part of $\widetilde{Y}^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$ is identified with the smooth part of $\mathcal{H}_{\mathrm{DM}}$. Modulo details, this gives parts (a) and (b) of the current theorem.

We unpack the details that we will need. To evaluate limits of F as some of the 12 points collide, consider some $z(0) = (z_0(0), \dots, z_{11}(0)) \in X^{\mathrm{st}}$. For now the argument "(0)" is just a formal symbol, present because we will soon choose a path $z(t)$ approaching $z(0)$. By reordering, we may suppose

(i) the $z_j(0)$ that are involved in any given collision have consecutive subscripts, all less than 11.

By applying an element of $\mathrm{PGL}_2\,\mathbb{C}$, we may suppose

(ii) $z_{11}(0) = \infty$.

Now we choose a path $z\colon [0,1] \to X^{\mathrm{st}}$, with $z(0)$ equal to the point just discussed, and $z(t) \in X^{\circ}$ for all $t > 0$. It is easy to choose this path satisfying:

(iii) if $z_j(0)$ is not a multiple point of $z(0)$, then $z_j(t)$ is constant, and

(iv) if $z_{j-1}(0) = z_j(0)$, then $z_{j-1}(t)$ and $z_j(t)$ have constant imaginary part, and for all $t > 0$ the following hold: $z_{j-1}(t)$ lies to the left of $z_j(t)$, and the segment $\overline{z_{j-1}(t)z_j(t)}$ contains none of the $z_k(t)$ except its own endpoints.

Each such curve $z(t)$ can be pushed forward into Y^{st} and then lifted to a curve $\tilde{z}(t)$ in $\widetilde{Y}^{\mathrm{st}}$. The lifting uses that $\widetilde{Y}^\circ \to Y^\circ$ is a covering space (allowing one to define $\tilde{z}$ on $(0,1]$), and that $\widetilde{Y}^{\mathrm{st}} \to Y^{\mathrm{st}}$ is proper (allowing an extension to $[0,1]$). Up to the action of $\mathrm{PGL}_2\,\mathbb{C}$, every point of $\widetilde{Y}^{\mathrm{st}} - \widetilde{Y}^\circ$ has the form $\tilde{z}(0)$ for some lift $\tilde{z}(t)$ of some such curve $z(t)$. We will show that F sends every such $\tilde{z}(0)$ into $\mathcal{H}_{\mathrm{DM}}$. The same argument will show that if $z(0)$ is more degenerate than a single collision of two of the z_j, then F sends every such $\tilde{z}(0)$ into an intersection of several components of $\mathcal{H}_{\mathrm{DM}}$.

So fix a path $z(t)$ as above. To show that F sends every $\tilde{z}(0)$ into $\mathcal{H}_{\mathrm{DM}}$, it is enough to prove this for any single lift $\tilde{z}$ of z. This uses $P\Gamma_{\mathrm{DM}}$-equivariance. To specify a lift $\tilde{z}$, it is enough to choose a lift $\tilde{z}(1)$ of the other endpoint $z(1)$. One may specify $\tilde{z}(1)$ by choosing what Looijenga calls an L-slit (see his Section 1.2). This means a sequence of curves $\delta_{j,1}$ from $z_{j-1}(1)$ to $z_j(1)$ which together form an arc that is eventually horizontal and rightward-moving as it approaches $z_{11} = \infty$. We choose any L-slit with the property that if z_{j-1} and z_j collide at $t=0$, then $\delta_{j,1}$ is the horizontal segment from $z_{j-1}(1)$ to $z_j(1)$. The fact that such an L-slit exists uses (i)–(iv). There is a unique-up-to-isotopy way to deform the $\delta_{j,1}$, through paths $\delta_{j,t}$ that form an L-slit for $z(t)$, as t varies over $(0,1]$. Again using (i)–(iv), we may suppose that if z_{j-1} and z_j collide at $t=0$, then $\delta_{j,t}$ is the horizontal segment from $z_{j-1}(t)$ to $z_j(t)$, for all $t \in (0,1]$.

Our choice of L-slit allows one to write down $F(\tilde{z}(t))$ explicitly. Namely, it allows one to specify a branch of a certain multivalued 1-form $\eta_{z(t)}$ on $\mathbb{C}P^1$, with certain singularities at the $z_j(t)$; see Looijenga's Section 1.1. Then, in suitable local projective coordinates,

$$F(\tilde{z}(t)) = (F_1(\tilde{z}(t)), \dots, F_{10}(\tilde{z}(t))), \qquad \text{where} \qquad F_j(\tilde{z}(t)) = \int_{\delta_{j,t}} \eta_{z(t)} \quad \text{for } t \in (0,1].$$

(See Looijenga's Section 1.3. His F_j are certain multiples of these integrals, rather than the integrals themselves, but the constant factors do not affect our arguments.) Suppose z_{j-1} and z_j collide at $t=0$. Then Looijenga's (1.6) shows that $F_j(\tilde{z}(t)) \to 0$ as $t \to 0$. Namely, the left side of his equation is $(z_j(t) - z_{j-1}(t))^{-2/3}$, times a number of terms bounded away from 0, times $F_j(\tilde{z}(t))$. The right side is bounded. So $F_j(\tilde{z}(t)) \to 0$ as $t \to 0$.

Just before his (2.2), Looijenga writes $\sqrt{T}$ for the braid group element, supported in a disk small enough to miss all other z_k, that moves z_{j-1} and z_j around each other positively by a half-twist. His (2.2) describes how each $F_k(\tilde{z}(t))$ changes under the monodromy action of $\sqrt{T}$, namely by the addition of a multiple of $F_j(\tilde{z}(t))$. Since $F_j(\tilde{z}(t))$ approaches 0 as $t \to 0$, it follows that $F(\tilde{z}(0))$ is invariant under the monodromy action of $\sqrt{T}$. Since this action is by a triflection whose mirror is a component of $\mathcal{H}_{\mathrm{DM}}$, we have proven that $F(\tilde{z}(0))$ lies in a component of $\mathcal{H}_{\mathrm{DM}}$. It follows that $F(\widetilde{Y}^{\mathrm{st}} - \widetilde{Y}^\circ) \subseteq \mathcal{H}_{\mathrm{DM}}$. Also, if more than one collision takes place, for example if three points collide or four points degenerate in two pairs, then the argument applies to each pair that degenerates, yielding multiple mirrors that contain $F(\tilde{z}(0))$.

Now we can prove (a) and (b). Because $\mathrm{PGL}_2\,\mathbb{C}$ acts freely and properly on $\widetilde{Y}^{\mathrm{st}}$, $(\widetilde{Y}^{\mathrm{st}} - \widetilde{Y}^\circ)/\mathrm{PGL}_2\,\mathbb{C}$ is a closed analytic subspace of $\widetilde{Y}^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$, and each of its (analytic space) components has dimension 8.

By (5-1), the same holds for its F-image inside $\mathbb{B}^9_{\mathrm{DM}}$. Since this image lies in $\mathcal{H}_{\mathrm{DM}}$, whose components $P\Gamma_{\mathrm{DM}}$ permutes transitively, it follows that $F((\widetilde{Y}^{\mathrm{st}} - \widetilde{Y}^\circ)/\mathrm{PGL}_2\,\mathbb{C}) = \mathcal{H}_{\mathrm{DM}}$. Because F is an isomorphism, this implies $F(\widetilde{Y}^\circ) = \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, finishing the proof of (a). At this point we know that 12-tuples without degenerations correspond to points of $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, and 12-tuples with more than one degeneration correspond to points of $\mathbb{B}^9_{\mathrm{DM}}$ that lie in multiple mirrors. So every remaining point of $\mathcal{H}_{\mathrm{DM}}$ must correspond to a 12-tuple with exactly one degeneration. This proves (b).

We now prove (c). Because $\chi : Y^\circ \to Y^\circ$ sends meridians to inverses of meridians, it sends the monodromy representation to its complex conjugate. In particular, it preserves the kernel of this representation, and hence lifts to an antiholomorphic involution $\hat{\chi}$ of $\widetilde{Y}^\circ$. We may suppose that $\hat{\chi}$ fixes any chosen preimage $\tilde{y} \in \widetilde{Y}^\circ$ of y, say one with $F(\tilde{y}) = \widetilde{f(y)}$. Because χ normalizes $\mathrm{PGL}_2\,\mathbb{C}$, $\hat{\chi}$ does too, so $\hat{\chi}$ descends to an antiholomorphic involution of $\widetilde{Y}^\circ/\mathrm{PGL}_2\,\mathbb{C} = \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, which we will also call $\hat{\chi}$. By construction, $\hat{\chi}$ fixes $\widetilde{f(y)}$ and lies over the antiholomorphic involution $f \circ \chi \circ f^{-1}$ of $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$.

The ball $\mathbb{B}^9_{\mathrm{DM}}$ certainly admits antiholomorphic involutions. Following $\hat{\chi}$ by one of them gives a holomorphic map $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}} \to \mathbb{B}^9_{\mathrm{DM}}$, which extends to all of $\mathbb{B}^9_{\mathrm{DM}}$ by the Riemann extension theorem. It follows that $\hat{\chi}$ itself extends (as an antiholomorphic involution) to all of $\mathbb{B}^9_{\mathrm{DM}}$. □

Our goal is to make the isomorphism in Theorem 5.1 more explicit. We will say which point of $\mathbb{B}^9_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}}$ corresponds to the standard basepoint for $\mathcal{M}^\circ_{12}$, and which loops in $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ correspond to the standard generators of $\mathrm{Br}_{12}(\mathbb{C}P^1)$. This result is interesting without any monstrous connection at all.

Inside L, the sublattice L_{DM} is the orthogonal complement of the span L_4 of specific roots $s_{\mathrm{A}}, \dots, s_{\mathrm{D}}$ of L. Its isometry group Γ_{DM} is a subgroup of Γ because L_{DM} is a summand of L. Although we defined the hyperplane arrangement $\mathcal{H}_{\mathrm{DM}}$ in terms of the roots of L_{DM}, it can also be defined as the set of hyperplanes arising from intersections of $\mathbb{B}^9_{\mathrm{DM}}$ with components of $\mathcal{H}$; see Lemma 5.3. In Section 4 we defined ρ as the projection of $\tau \in \mathbb{B}^{13}$ to $\mathbb{B}^9_{\mathrm{DM}}$, and we defined specific roots $s_0, \dots, s_{11}$ of L_{DM}. By Lemma 4.6, their mirrors are the mirrors of $\mathcal{H}_{\mathrm{DM}}$ that come closest to ρ. We define

$$\rho_j := \text{the meridian } M_{\rho, s_j} \in \pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho).$$

Now we can state the main theorem of this section; as in Section 3, ζ means $e^{\pi i/6}$.

Theorem 5.2 *The orbifold isomorphism $f : \mathcal{M}^\circ_{12} \to (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ of Theorem 5.1(a) identifies the basepoint $T = \{1, \zeta, \dots, \zeta^{11}\} \in \mathcal{M}^\circ_{12}$ with the basepoint $\rho \in \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ (or rather with its image mod $P\Gamma_{\mathrm{DM}}$). Furthermore, it identifies the just-defined $\rho_j \in \pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$ with the elements of $\pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12}, T)$ denoted by the same symbols in Section 3.*

In particular, $\rho_0, \dots, \rho_{11}$ generate $\pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$, with defining relations stated in the $n = 12$ case of Theorem 3.2.

We prove our earlier claim about the two descriptions of $\mathcal{H}_{\mathrm{DM}}$, then proceed to the proof of Theorem 5.2.

Lemma 5.3 *Every mirror of L that meets $\mathbb{B}^9_{\mathrm{DM}}$ is the mirror of a root of L_{DM}, except for the* 40 *mirrors which contain $\mathbb{B}^9_{\mathrm{DM}}$.*

Proof Suppose $\mathbb{B}^9_{\mathrm{DM}}$ meets the mirror of a root r. Then the $\mathcal{E}$-span of r and L_4 is positive definite. Also, all its inner products are divisible by θ. Therefore L_4 is a summand of it. Since r^2 and the minimal norm of L_4 are both 3, $r \in L_4$ or $r \perp L_4$. In the first case $r^\perp$ contains $\mathbb{B}^9_{\mathrm{DM}}$, and in the second $r \in L_{\mathrm{DM}}$. □

Lemma 5.4 *The images of T in $\mathcal{M}^\circ_{12}$ and ρ in $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ are identified by the isomorphism f.*

Proof The main point is that ρ's $P\Gamma_{\mathrm{DM}}$-stabilizer is the dihedral group D_{24} of order 24. This uses Lemma 4.6, ie that the mirrors of $s_0, \dots, s_{11}$ are the components of $\mathcal{H}_{\mathrm{DM}}$ closest to ρ. So any element of the stabilizer permutes them, and indeed is determined by how it permutes them. (Consider its action on their points closest to ρ.) Since these roots form an $\widetilde{A}_{11}$ diagram, the stabilizer is no larger than D_{24}, and a D_{24} in the stabilizer is visible (see Section 2.7).

The orbifold isomorphism in Theorem 5.1(a) shows that ρ corresponds to a point of $\mathcal{M}^\circ_{12}$ with local group D_{24}. That is, to an unordered 12-tuple of distinct points in $\mathbb{C}P^1$, with $\mathrm{PGL}_2\,\mathbb{C}$-stabilizer isomorphic to D_{24}. Considering the finite subgroups of $\mathrm{PGL}_2\,\mathbb{C}$ shows that there is only one such tuple, up to projective equivalence, namely T. □

Let ρ' be the projection of ρ to $s_0^\perp$. The segment $\overline{\rho\rho'}$ corresponds via f to some motion in $\mathbb{C}P^1$ of the 12 points of T. The rest of the proof of Theorem 5.2 amounts to formulating this precisely and then identifying the motion. To start, observe that the orbifold covering spaces

$$X^\circ/\mathrm{PGL}_2\,\mathbb{C} \to \mathcal{M}^\circ_{12} \quad \text{and} \quad \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}} \to (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}} \cong \mathcal{M}^\circ_{12}$$

are manifold covers of the same orbifold. Therefore Theorem 5.1(a) implies there is a diffeomorphism $\tilde{f}$, from a neighborhood E_T of (the image of) T in $X^\circ/\mathrm{PGL}_2\,\mathbb{C}$ to a neighborhood E_ρ of ρ in $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, that lies over f and is D_{24}-equivariant.

Together with analytic continuation, this allows us to transfer any path in $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, beginning at ρ, to a path in $X^\circ/\mathrm{PGL}_2\,\mathbb{C}$ beginning at T. In particular, there exists a unique path $\gamma\colon [0,1) \to X^\circ/\mathrm{PGL}_2\,\mathbb{C}$, whose initial segment corresponds to $\overline{\rho\rho'}$ under $\tilde{f}$, and which has the same projection as $\overline{\rho\rho'} - \{\rho'\}$ to $\mathcal{M}^\circ_{12} \cong (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$. Because $X^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$ is finite over $Y^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$, γ extends continuously to $[0,1]$. Finally, because X^{st} is a $\mathrm{PGL}_2\,\mathbb{C}$-bundle over $X^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$, we may lift γ to a path $\beta(t) = (\beta_0, \dots, \beta_{11}) \in X^{\mathrm{st}}$ of ordered 12-tuples in $\mathbb{C}P^1$.

Our analysis of β will rely on a study of certain antiholomorphic involutions of $\mathcal{M}^\circ_{12}$ and $\mathbb{B}^9_{\mathrm{DM}}$. Following Theorem 5.1(c), write $\chi\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ for the inversion map across the unit circle, ie $z \mapsto 1/\bar{z}$. We also write χ for the induced maps on X^{st} and Y^{st} and their quotients by $\mathrm{PGL}_2\,\mathbb{C}$. As a self-map of $\mathbb{C}P^1$, the antiholomorphic involution χ preserves T pointwise and commutes with T's setwise stabilizer $D_{24} \subseteq \mathrm{PGL}_2\,\mathbb{C}$. Another way to say this is that χ's action on $X^\circ/\mathrm{PGL}_2\,\mathbb{C}$ preserves the image therein of T, and commutes with the $D_{24} \subseteq S_{12}$ stabilizing this image.

Theorem 5.1(c) provides an antiholomorphic involution $\hat{\chi}$ of $\mathbb{B}^9_{\mathrm{DM}}$ that fixes ρ and lies over the antiholomorphic involution $f \circ \chi \circ f^{-1}$ of $\mathbb{B}^9_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}}$. In fact, there are several candidates for $\hat{\chi}$, differing by composition with the involutions in D_{24}. Since $\tilde{f}\colon E_T \to E_\rho$ is an isomorphism of orbifold covering spaces, it identifies the lifts of χ to E_T with the lifts of χ to E_ρ. We define $\hat{\chi}$ to be the lift to $\mathbb{B}^9_{\mathrm{DM}}$ that corresponds to χ under $\tilde{f}$. (We can now discard E_T and E_ρ.) We remark that antiholomorphic automorphisms of the complex ball are complex hyperbolic isometries.

Lemma 5.5 *There are only two possibilities for $\hat{\chi}$: it is either the antiholomorphic involution $\hat{\chi}_0$ of $\mathbb{B}^9_{\mathrm{DM}}$ got from complex conjugation in the $P^2\mathbb{F}_3$ model of L, or the composition of $\hat{\chi}_0$ with the central involution Z of the $P\Gamma_{\mathrm{DM}}$-stabilizer D_{24} of ρ.*

Proof Since $\hat{\chi}$ preserves ρ and $\mathcal{H}_{\mathrm{DM}}$, it permutes the mirrors nearest ρ, which by Lemma 4.6 are the $s_i^\perp$. Since it commutes with D_{24}, it either preserves each of them, or else sends each $s_i^\perp$ to $s_{i+6}^\perp$. In either case, this determines the action of $\hat{\chi}$ on the points of the $s_i^\perp$ that are nearest ρ, and hence the action on all of $\mathbb{B}^9_{\mathrm{DM}}$. The first of these possibilities is realized by $\hat{\chi}_0$, and the second by $\hat{\chi}_0 \circ Z$. □

We now return to our path $\beta(t) = (\beta_0(t), \dots, \beta_{11}(t)) \in X^{\mathrm{st}}$. The key ingredient in the analysis is that $\overline{\rho\rho'}$ is $\hat{\chi}_0$-invariant. Also important is that the $\beta_i(t)$ are all distinct, for each t until $t = 1$. At $t = 1$, two of the $\beta_i(t)$ become equal, while the rest remain distinct from them and from each other. This follows from Theorem 5.1(b) and the fact that ρ' lies in exactly one component of $\mathcal{H}_{\mathrm{DM}}$. One might skip the next proof for now, since the lemma after it has a similar but simpler proof.

Lemma 5.6 $$\hat{\chi} = \hat{\chi}_0.$$

Proof Suppose to the contrary, so $\hat{\chi} = \hat{\chi}_0 \circ Z$. Because $\overline{\rho\rho'}$ is $\hat{\chi}_0$-invariant, γ is $(\chi\circ Z)$-invariant, where Z is still the central involution of D_{24}, but now acting on $X^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$. That is, there exists $g_t\colon [0,1] \to \mathrm{PGL}_2\,\mathbb{C}$ such that $(\chi \circ Z)(\beta(t)) = g_t(\beta(t))$. Since Z acts on $X = (\mathbb{C}P^1)^{12}$ by cyclically permuting the factors by six positions, we have

$$\big(\chi(\beta_6(t)), \dots, \chi(\beta_{11}(t)), \chi(\beta_0(t)), \dots, \chi(\beta_5(t))\big) = \big(g_t(\beta_0(t)), \dots, g_t(\beta_{11}(t))\big). \tag{5-2}$$

Choose i and j so that β_i, β_j, β_{i+6} and β_{j+6} are not involved in the collision as $t \to 1$. From (5-2) this 4-tuple is projectively equivalent to $\chi(\beta_{i+6}), \chi(\beta_{j+6}), \chi(\beta_i), \chi(\beta_j)$. This is projectively equivalent to $\bar{\beta}_{i+6}, \bar{\beta}_{j+6}, \bar{\beta}_i, \bar{\beta}_j$, because χ and the usual complex conjugation on $\mathbb{C}P^1$ differ by a projective transformation. This gives an equality of cross ratios:

$$\frac{(\beta_i - \beta_j)(\beta_{i+6} - \beta_{j+6})}{(\beta_i - \beta_{i+6})(\beta_j - \beta_{j+6})} = \frac{(\bar{\beta}_{i+6} - \bar{\beta}_{j+6})(\bar{\beta}_i - \bar{\beta}_j)}{(\bar{\beta}_{i+6} - \bar{\beta}_i)(\bar{\beta}_{j+6} - \bar{\beta}_j)}.$$

In particular, β_i, β_j, β_{i+6} and β_{j+6} have real cross ratio.

A 4-tuple has real cross ratio if and only if it can be carried into the unit circle by a projective transformation. So, by applying a time-dependent projective transformation, we may suppose that $\beta_i(t)$, $\beta_j(t)$, $\beta_{i+6}(t)$ and $\beta_{j+6}(t)$ lie on the unit circle, for all t. At $t = 0$, $\{\beta_i, \beta_{i+6}\}$ separates β_j and β_{j+6} from each other on the circle. These four points remain distinct during the motion, so the same holds for all t. Therefore the geodesics $\overline{\beta_i(t)\beta_{i+6}(t)}$ and $\overline{\beta_j(t)\beta_{j+6}(t)}$, in the Poincaré disk bounded by the unit circle, meet each other. By applying a time-dependent projective transformation, preserving the unit circle, we may suppose without loss of generality that the intersection point is the origin. In particular, $\beta_i(t)$ and $\beta_{i+6}(t)$ are antipodal for all t, and similarly with j in place of i.

We revisit (5-2) in light of this information. It says that g_t acts by negation on four points of the unit circle. This uniquely determines g_t as $z \mapsto -z$, for all t. It follows that $\beta_{k+6}(t) = -1/\bar{\beta}_k(t)$ for all k. In particular, the unordered 12-tuple underlying $\beta(t)$ is preserved by the fixed-point-free self-map $z \mapsto -1/\bar{z}$ of $\mathbb{C}P^1$. This is a contradiction: $\beta(1)$ has a single collision point, and hence admits no such symmetry. □

Lemma 5.7 *The path γ in $\mathcal{M}_{12}^{\mathrm{st}}$, corresponding to $\overline{\rho\rho'}$, is represented by a path $(\beta_0(t), \dots, \beta_{11}(t)) \in (\mathbb{C}P^1)^{12}$, satisfying*

(a) *$\beta_i(t)$ lies in the unit circle for all i and t,*

(b) *for each $t \in [0, 1)$, the $\beta_i(t)$ are distinct,*

(c) *two of the $\beta_i(1)$ coincide and the rest are distinct from them and each other.*

Proof By the previous two lemmas, $\hat{\chi} = \hat{\chi}_0$. So the $\hat{\chi}_0$-invariance of $\overline{\rho\rho'}$ implies the χ-invariance of γ. In other words, $\chi(\beta(t))$ is projectively equivalent to $\beta(t)$ for all t, which is to say that there exists $g_t \colon [0, 1] \to \mathrm{PGL}_2\,\mathbb{C}$ such that

$$\big(\chi(\beta_0(t)), \dots, \chi(\beta_{11}(t))\big) = \big(g_t(\beta_0(t)), \dots, g_t(\beta_{11}(t))\big) \tag{5-3}$$

for all t. We choose i, j and k not involved in the collision; in particular, $\beta_i(t)$, $\beta_j(t)$ and $\beta_k(t)$ remain distinct as $t \to 1$. Because $\mathrm{PGL}_2\,\mathbb{C}$ acts 3-transitively on $\mathbb{C}P^1$, we may apply a time-dependent automorphism of $\mathbb{C}P^1$ to suppose that $\beta_i(t)$, $\beta_j(t)$ and $\beta_k(t)$ lie on the unit circle for all t. That is, they are χ-invariant, and hence by (5-3) also g_t-invariant. Only the trivial element of $\mathrm{PGL}_2\,\mathbb{C}$ fixes three points, so g_t is the identity for all t. Now (5-3) says that $\beta_i(t)$ is fixed by χ, for all i and t. We have exhibited a representative $\beta \colon [0, 1] \to X^{\mathrm{st}}$ for $\overline{\rho\rho'}$, consisting of 12 points moving on the unit circle that remain distinct until $t = 1$, when two neighbors collide but no other collisions take place. This last part uses Theorem 5.1(b). □

Proof of Theorem 5.2 Our goal is to understand the motions of 12-tuples in $\mathbb{C}P^1$, corresponding to the meridians $\rho_0, \dots, \rho_{11} \in \pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$. By definition, ρ_0 follows $\overline{\rho\rho'}$ until very near ρ', then travels one third of the way around $s_0^{\perp}$, onto $S_0(\overline{\rho'\rho})$, and then continues along $S_0(\overline{\rho'\rho})$ until arriving at $S_0(\rho)$.

We already transferred $\overline{\rho\rho'}$ to a path γ in $X^{\mathrm{st}}/\mathrm{PGL}_2\,\mathbb{C}$. By Lemma 5.7, γ is represented by a path β of ordered 12-tuples in $S^1 \subseteq \mathbb{C}P^1$ that remain distinct until $t = 1$, when two neighbors collide, say $\beta_0(t)$ and $\beta_1(t)$. By this choice of labeling, $\gamma(1)$ lies in the component C_{01} of $(X^{\mathrm{st}} - X^{\circ})/\mathrm{PGL}_2\,\mathbb{C}$ fixed by the involution $A = (01) \in S_{12}$. Because there is only one collision among $\beta_0(1), \dots, \beta_{11}(1)$, this is the only component of $(X^{\mathrm{st}} - X^{\circ})/\mathrm{PGL}_2\,\mathbb{C}$ containing $\gamma(1)$, just as $s_0^{\perp}$ is the only component of $\mathcal{H}$ containing ρ'. Because $X^{\mathrm{st}} \to Y^{\mathrm{st}}$ has order-2 branching along C_{01}, rather than order 3, we can now identify the path in $X^\circ/\mathrm{PGL}_2\,\mathbb{C}$ corresponding to ρ_0. It begins at (the $\mathrm{PGL}_2\,\mathbb{C}$-orbit of) T, travels along $\gamma(t)$ until t is very near 1 (say $t = t_0$), so that $\gamma(t_0)$ is very near C_{01}. Then it travels halfway around C_{01}, ending at $A(\gamma(t_0))$, and then continues along $A(\mathrm{reverse}(\gamma))$ until arriving at $A(T)$.

This path is easy to lift to X°, because we have already lifted γ to β. It begins at T and follows $\beta(t)$ until $t = t_0$, so that β_0 and β_1 have almost collided. Then β_0 and β_1 move counterclockwise around each other, so that they swap places, ending at the $A(\beta(t_0))$. Then it travels along $A(\mathrm{reverse}(\beta))$ all the way to $A(T)$. This is one of the standard generators given in Section 3 for the braid group $\mathrm{Br}_{12}(\mathbb{C}P^1)$.

We have proven that the isomorphism $f : \mathcal{M}_{12}^\circ \to (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ sends the standard generators for $\mathrm{Br}_{12}(\mathbb{C}P^1)$, called ρ_i in Section 3, to the meridians in $\pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$ called ρ_i in this section, up to labeling. Considering which pairs of generators braid and which pairs commute shows that the identification may be taken to match subscripts in the natural way. □

6 A neighborhood of $\mathbb{B}^9_{\mathrm{DM}}$, modulo its stabilizer

Recall from Section 4 that $\mathbb{B}^9_{\mathrm{DM}}$ means the 9-ball orthogonal to the roots $s_{\mathrm{A}}, \dots, s_{\mathrm{D}}$ that form our chosen A_4 subdiagram of $P^2\mathbb{F}_3$. Our goal is to define a suitable neighborhood U of $\mathbb{B}^9_{\mathrm{DM}}$, invariant under

$$P\Gamma^{\mathrm{sw}}_{\mathrm{DM}} := \text{the setwise } P\Gamma\text{-stabilizer of } \mathbb{B}^9_{\mathrm{DM}},$$

and write down a presentation of $\pi_1^{\mathrm{orb}}((U - \mathcal{H})/P\Gamma^{\mathrm{sw}}_{\mathrm{DM}})$. Theorem 5.2 leads to a surjection from this group to the orbifold fundamental group of the moduli space $\mathcal{M}_{12}^\circ$ of 12-point subsets of $\mathbb{C}P^1$. We will identify the kernel as the ordinary 5-strand braid group, and then work out the details of the group extension.

We defined $\mathcal{H}_{\mathrm{DM}} \subseteq \mathbb{B}^9_{\mathrm{DM}}$ as the union of the mirrors of the roots of L_{DM}. In Lemma 5.3 we showed that these are exactly the roots of L whose mirrors meet $\mathbb{B}^9_{\mathrm{DM}}$, except for the 240 roots in the positive definite lattice $\langle s_{\mathrm{A}}, \dots, s_{\mathrm{D}}\rangle \cong L_4$, which are orthogonal to $\mathbb{B}^9_{\mathrm{DM}}$. This immediately implies:

Lemma 6.1 *There is a $P\Gamma^{\mathrm{sw}}_{\mathrm{DM}}$-invariant neighborhood U of $\mathbb{B}^9_{\mathrm{DM}}$ such that orthogonal projection $\pi : \mathbb{B}^{13} \to \mathbb{B}^9_{\mathrm{DM}}$ realizes $U - \mathcal{H}$ as a fibration over $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, with fibers as follows. The fiber over each $x \in \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ is an open ball centered at x, in the $\mathbb{B}^4$ orthogonal to $\mathbb{B}^9_{\mathrm{DM}}$ at x, minus the 40 mirrors of L_4.* □

Now we choose a basepoint $\sigma \in \overline{\tau\rho} - \{\rho\}$, close enough to ρ so that the meridians σ_j, defined in Section 4 and based at σ, lie in U for $j = 0, \dots, 11, \mathrm{A}, \dots, \mathrm{D}$. By moving σ closer to ρ we may also suppose that σ's $P\Gamma$-stabilizer is a subgroup of ρ's. We set

$$J := \pi_1^{\mathrm{orb}}((U - \mathcal{H})/P\Gamma^{\mathrm{sw}}_{\mathrm{DM}}, \sigma).$$

Most of this section concerns relations between $\sigma_0, \dots, \sigma_{11}, \sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$. The roots $s_0, \dots, s_{11}, s_{\mathrm{A}}, \dots, s_{\mathrm{D}}$ form an $\widetilde{A}_{11}A_4$ subdiagram of $P^2\mathbb{F}_3$, and so the corresponding meridians $\tau_0, \dots, \tau_{11}, \tau_{\mathrm{A}}, \dots, \tau_{\mathrm{D}}$ (all based at τ) satisfy the $\widetilde{A}_{11}A_4$ Artin relations. This is [Basak 2016, Theorem 4.4]. By Theorem 4.1, $\sigma_0, \dots, \sigma_{11}, \sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ do too.

Orthogonal projection $\mathbb{B}^{13} \to \mathbb{B}^9_{\mathrm{DM}}$ is $P\Gamma^{\mathrm{sw}}_{\mathrm{DM}}$-equivariant, inducing a map

$$(U - \mathcal{H})/P\Gamma^{\mathrm{sw}}_{\mathrm{DM}} \to (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}. \tag{6-1}$$

Here $P\Gamma_{\mathrm{DM}}$ is the group of automorphisms of $\mathbb{B}^9_{\mathrm{DM}}$ induced by $P\Gamma^{\mathrm{sw}}_{\mathrm{DM}}$. That is, $P\Gamma_{\mathrm{DM}}$ is the quotient of $P\Gamma^{\mathrm{sw}}_{\mathrm{DM}}$ by the pointwise stabilizer of $\mathbb{B}^9_{\mathrm{DM}}$. This agrees with our previous use of $P\Gamma_{\mathrm{DM}}$ to mean the projective isometry group of L_{DM}, because L_{DM} is a summand of L. The projection (6-1) induces a homomorphism

$$J \to \pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho) \cong \pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12}, T). \tag{6-2}$$

The isomorphism here uses our refinement (Theorem 5.2) of the theorem of Deligne and Mostow. It is well known that $\mathcal{M}^\circ_n$ has contractible orbifold universal cover [Birman 1974]. Therefore, applying the long exact homotopy sequence to the fibration in Lemma 6.1 yields an exact sequence

$$1 \to \pi_1^{\mathrm{orb}}(\text{fiber over } \rho, \sigma) \to J \to \pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12}, T) \to 1. \tag{6-3}$$

Next we work out the left term:

Lemma 6.2 *The kernel π_1^{orb}(fiber over ρ, σ) in* (6-3) *is isomorphic to the* 5*-strand braid group* Br_5*. The meridians $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ form a standard set of generators. That is, their A_4 Artin relations are defining relations.*

Proof We recall that $\mathrm{Aut}(L_4)$ is the finite complex reflection group numbered 32 in the Shephard–Todd list [1954, Table VII]. The fibration in Lemma 6.1 shows that the kernel in (6-3) is

$$\pi_1^{\mathrm{orb}}((B - \mathcal{H})/\mathrm{Aut}(L_4), \sigma), \tag{6-4}$$

where B is a small ball centered at ρ, in the orthogonal complement to $\mathbb{B}^9_{\mathrm{DM}}$ at ρ. This is essentially the definition of the braid group associated to $\mathrm{Aut}(L_4)$. (The standard definition [Bessis 2015] uses $\mathbb{C}^4$ in place of B, and π_1 in place of π_1^{orb}, which is no change at all because finite complex reflection groups act freely on their mirror complements.) Orlik and Solomon [1988, Theorem 2.25] showed that this is isomorphic to the standard 5-strand braid group.

It remains to show that $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ are comprise a standard set of generators. We use [Allcock and Basak 2016, Theorem 1.2] to prove generation. This requires us to check several things. First, the reflections in $s_{\mathrm{A}}, \dots, s_{\mathrm{D}}$ generate $\mathrm{Aut}(L_4)$, which is well known. Second, of the 40 mirrors of $\mathrm{Aut}(L_4)$, the ones closest to σ are the mirrors orthogonal to these four roots. This is an easy computer check. To state the third condition, we write E for the set of eight images of σ under the ω- and $\bar{\omega}$-reflections in these four mirrors. One must check, for each of the remaining 36 mirrors, that the projection of σ to that mirror is closer to some element of E than it is to σ. This is another easy computer check.

The A_4 Artin relations satisfied by $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ are the relations defining Br_5 in terms of its standard generators. So we obtain a self-surjection $\mathrm{Br}_5 \to \mathrm{Br}_5$ by sending some standard set of generators to $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$. The braid group is Hopfian [Bell and Margalit 2006, page 276], which means that any self-surjection must be an automorphism. Therefore $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ are themselves a standard set of generators. □

Remark We sketch a geometric argument that avoids the Hopfian trickery and the reliance on [Orlik and Solomon 1988]. First one proves generation, as above, and it remains to show that the braid relations on $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ define the subgroup of J that they generate. Suppose $\mathcal{R}(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}) = 1$ is a relation they satisfy. By Theorem 4.1 $\mathcal{R}(\tau_{\mathrm{A}}, \dots, \tau_{\mathrm{D}}) = 1$. Then by $L_3(3) \rtimes 2$ symmetry $\mathcal{R}(\tau_0, \dots, \tau_3) = 1$. Another application of Theorem 4.1 shows $\mathcal{R}(\sigma_0, \dots, \sigma_3) = 1$, and then projecting to $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ gives $\mathcal{R}(\rho_0, \dots, \rho_3) = 1$. Using normal forms for words in braid groups, one can show that the homomorphism $\mathrm{Br}_5(\mathbb{C}) \to \mathrm{Br}_{12}(\mathbb{C}P^1) \to \pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12})$ sending the standard generators of Br_5 to four consecutive standard generators of $\mathrm{Br}_{12}(\mathbb{C}P^1)$ is injective. It follows that $\mathcal{R}(\rho_0, \dots, \rho_3) = 1$ is a consequence of the braid relations on $\rho_0, \dots, \rho_3$. Reversing the first part of the argument shows that the same holds with $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ in place of $\rho_0, \dots, \rho_3$.

We have established the exact sequence

$$1 \to \mathrm{Br}_5 \to J \to \pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12}) \to 1. \tag{6-5}$$

We know that J is generated by $\sigma_0, \dots, \sigma_{11}, \sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$. Three kinds of relations suffice to define J. First are the relations defining the normal subgroup Br_5, which are words in $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$. Then there are relations saying how $\sigma_0, \dots, \sigma_{11}$ act on this normal subgroup: they centralize it. Finally there are relations of the form

$$\mathrm{word}(\sigma_0, \dots, \sigma_{11}) = \mathrm{word}(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}), \tag{6-6}$$

where the left side is one of the relators defining $\pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12})$, except written with σ_j's in place of ρ_j's. And the right side is a word in $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ that is equal to the left-side word. (Such a word exists, because the left side lies in the kernel of the projection to $\pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12})$.) It is easy to see that any word in $\sigma_0, \dots, \sigma_{11}, \sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ that is trivial in J can be reduced to the trivial word by use of these relators. So a set of defining relations for J consists of the Artin relators among $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$, their commutativity with $\sigma_0, \dots, \sigma_{11}$ and one relation of the form (6-6) for each defining relator of $\pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12})$.

It remains to work out the words (6-6) explicitly. As preparation, we record some facts about important elements of $P\Gamma$. Write $S_0, \dots, S_{11}, S_{\mathrm{A}}, \dots, S_{\mathrm{D}}$ for the ω-reflections in the roots $s_0, \dots, s_{11}, s_{\mathrm{A}}, \dots, s_{\mathrm{D}}$.

Lemma 6.3 (a) *$S_j S_{j+1} \cdots S_{j+10}$ (resp. $S_j S_{j-1} \cdots S_{j-10}$) is independent of $j \in \{0, \dots, 11\}$. It permutes the mirrors $s_k^\perp$ by incrementing (resp. decrementing) subscripts. It, and also $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})$, act on $\mathbb{B}^1(\rho, \tau)$ by the positive $\frac{1}{6}\pi$ rotation around ρ.*

(b) *The actions of $S_1 S_2 \cdots S_{10} S_{11}^2 S_{10} \cdots S_2 S_1$ and $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})^2$ on $\mathbb{B}^{13}$ coincide. Namely, both $e^{\pi i/3} S_1 \cdots S_{10} S_{11}^2 S_{10} \cdots S_1$ and $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})^2$ act trivially on L_{DM} and by the scalar $e^{\pi i/3}$ on its orthogonal complement.*

(c) *$\Delta(S_1, \dots, S_{11})$ permutes the mirrors $s_j^\perp$ by $s_j^\perp \mapsto s_{12-j}^\perp$, and acts on $\mathbb{B}^1(\rho, \tau)$ by the π rotation around ρ.*

Proof The equalities were suggested by experience with the ordinary braid groups and the braid groups of finite complex reflection groups. The roots s_j are defined, via (4-1), in Section 2.4. The formula for a triflection appears in Section 2.5. We used a computer to write down matrices for the triflections S_j and then check our assertions. □

Lemma 6.4 *The $P\Gamma$-stabilizer of σ fixes $\mathbb{B}^1(\rho,\tau)$ pointwise, has generators*

$$S_1 S_2 \cdots S_{11} \cdot \Delta(S_{\mathrm{A}},\dots,S_{\mathrm{D}})^{-1} \text{ of order } 12 \quad \text{and} \quad (S_1 S_2 \cdots S_{11})^6 \cdot \Delta(S_1,\dots,S_{11})^{-1} \text{ of order } 2,$$

and is dihedral of order 24*. Conjugation by the second word inverts the first.*

Proof We chose σ close enough to ρ so that σ's stabilizer lies in ρ's. Therefore σ's stabilizer acts trivially on $\mathbb{B}^1(\rho,\sigma)=\mathbb{B}^1(\rho,\tau)$. The stabilizer of τ is $L_3(3)\rtimes 2$, and its subgroup fixing ρ (or equivalently σ) is dihedral of order 24. That the listed elements stabilize σ follows from Lemma 6.3. Computation establishes their orders and that the second inverts the first. We remark that the exponent -1 could be removed in the second word, because $\Delta(S_1,\dots,S_{11})$ has order 2. However, the current form matches the lift of this word with σ_j's in place of S_j's, in Lemma 7.1. □

Lemma 6.5 *Let γ (resp. δ) be the positively directed circular arc in $\mathbb{B}^1(\rho,\tau)$, beginning at σ, and subtending angle $\frac{1}{6}\pi$ (resp. $\frac{1}{2}\pi$) around its center ρ. Then*

(a) *the $\sigma_j\sigma_{j+1}\cdots\sigma_{j+10}$ are equal to each other and to $(\gamma, S_j S_{j+1}\cdots S_{j+10})$,*

(b) *the $\sigma_j\sigma_{j-1}\cdots\sigma_{j-10}$ are equal to each other and to $(\gamma, S_j S_{j-1}\cdots S_{j-10})$,*

(c) *$\Delta(\sigma_{\mathrm{A}},\dots,\sigma_{\mathrm{D}})$ is equal to $(\gamma,\Delta(S_{\mathrm{A}},\dots,S_{\mathrm{D}}))$, and*

(d) *$\Delta(\sigma_1,\dots,\sigma_{11})$ is equal to $(\delta,\Delta(S_1,\dots,S_{11}))$.*

Proof of Lemma 6.5(a)–(b) Assuming for a moment one case of (a), namely

$$\sigma_1\cdots\sigma_{11}=(\gamma, S_1\cdots S_{11}), \tag{6-7}$$

we will prove the rest of (a) and all of (b) by symmetry. The $P\Gamma$-stabilizer of σ acts trivially on $\mathbb{B}^1(\rho,\tau)$, and therefore fixes γ. Lemma 6.4 shows that it contains $S_1\cdots S_{11}\cdot\Delta(S_{\mathrm{A}},\dots,S_{\mathrm{D}})^{-1}$, which by Lemma 6.3(a) permutes the mirrors $s_0^\perp,\dots,s_{11}^\perp$ by incrementing subscripts. Therefore the conjugates of (6-7) by the powers of $S_1\cdots S_{11}\cdot\Delta(S_{\mathrm{A}},\dots,S_{\mathrm{D}})^{-1}$ are the relations $\sigma_j\cdots\sigma_{j+10}=(\gamma,S_j\cdots S_{j+10})$. These relations are part of our claim in (a). The rest of (a) follows, because Lemma 6.3(a) shows that $S_j\cdots S_{j+10}$ is independent of j.

Next, Lemma 6.4 shows that the $P\Gamma$-stabilizer of σ contains $(S_1\cdots S_{11})^6\cdot\Delta(S_1,\dots,S_{11})^{-1}$. It fixes γ, for the same reason as in the previous paragraph. It reverses the cyclic ordering of the mirrors $s_0^\perp,\dots,s_{11}^\perp$ by Lemma 6.3(c). Therefore (b) follows from (a).

It remains to prove (6-7). We write β for the path underlying $\sigma_1\cdots\sigma_{11}$, namely

$$\beta := \mu_{\sigma,s_1} \text{ followed by } S_1(\mu_{\sigma,s_2}) \text{ followed by } S_1S_2(\mu_{\sigma,s_3})\dots \text{ followed by } S_1S_2\cdots S_{10}(\mu_{\sigma,s_{11}}).$$

It is obvious that $\sigma_1 \cdots \sigma_{11}$ has the same underlying element of $P\Gamma$ as $(\gamma, S_1 \cdots S_{11})$. In particular, the endpoints of β and γ coincide. So it remains to show that the loop $\beta\gamma^{-1}$ (ie β followed by reverse(γ)) is nullhomotopic in $U - \mathcal{H}$. In fact we will show that it is nullhomotopic in $V - \mathcal{H}$, where V is the intersection of U with the $\mathbb{B}^{10}$ spanned by $\mathbb{B}^9_{\mathrm{DM}}$ and τ. It follows from Lemma 6.1 that $V - \mathcal{H}$ is a punctured-disk bundle over $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$.

Our first step is to show that the projection of $\beta\gamma^{-1}$ to $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ is nullhomotopic. To do this we find braids representing the projections of $\sigma_1 \cdots \sigma_{11}$ and $(\gamma, S_1 \cdots S_{11})$ to $\pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho) = \pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12}, T)$. By Theorem 5.2, the former is represented by the braid $\rho_1 \cdots \rho_{11}$. The latter projects to

$$((\text{the constant path at } \rho), S_1 \cdots S_{11}).$$

This is represented by a braid which keeps all 12 points evenly spaced on the unit circle (because its underlying path in the moduli space is constant), and permutes them in the same way as $\rho_1 \cdots \rho_{11}$ (because the element of $P\Gamma_{\mathrm{DM}}$ underlying $\rho_1 \cdots \rho_{11}$ is $S_1 \cdots S_{11}$). That is, $(\gamma, S_1 \cdots S_{11})$ is represented by the braid in which all 12 points of T move $\frac{1}{6}\pi$ clockwise around the unit circle with uniform speed. The first braid followed by the inverse of the second is trivial in $\mathrm{Br}_{12}(\mathbb{C}P^1)$. It follows that the projection of $\beta\gamma^{-1}$ to $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ represents the trivial element of $\pi_1^{\mathrm{orb}}(\mathcal{M}^\circ_{12})$ and is therefore nullhomotopic.

Now the exact sequence on π_1, for the punctured-disk fibration $V - \mathcal{H} \to \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$, shows that $\beta\gamma^{-1}$ is homotopic into the fiber over ρ. So it will be enough to show that $\beta\gamma^{-1}$ has trivial winding number around $\mathbb{B}^9_{\mathrm{DM}}$ in $\mathbb{B}^{10}$. For this, we may replace β by its projection α to $\mathbb{B}^1(\rho, \tau)$. In summary, it will be enough to prove that the specific loop $\alpha\gamma^{-1}$ is nullhomotopic in the specific punctured disk $V \cap \mathbb{B}^1(\rho, \tau)$. This becomes obvious upon drawing it, which we now prepare to do.

First, each μ_{σ,s_i} was defined as a perturbation of the concatenation of two geodesic segments. The first segment joins σ to the projection of σ into $s_i^\perp$, which the second segment joins to $S_i(\sigma)$. The perturbation is to avoid hitting $s_i^\perp$ at the concatenation point, and can be made arbitrarily small. The concatenation point does not lie in $\mathbb{B}^9_{\mathrm{DM}}$, so the perturbation does not affect the winding number of $\beta\gamma^{-1}$ around $\mathbb{B}^9_{\mathrm{DM}}$. So, for the rest of the proof, we will ignore the perturbation and regard μ_{σ,s_i} as the concatenation of the two segments. (Or you can imagine that the perturbation is so small that it becomes invisible in Figure 1.) This makes β into a concatenation of 22 specific geodesic segments in $\mathbb{B}^{10}$.

Second, if two vectors of negative norm in $\mathbb{C}^{14}$ have negative inner product, then the segment joining them in $\mathbb{C}^{14}$ represents the geodesic joining the corresponding points in $\mathbb{B}^{13}$. Therefore μ_{σ,s_i} is represented by the line segment in $\mathbb{C}^{14}$ from σ to its linear projection onto $s_i^\perp$, followed by the line segment from there to $S_i(\sigma)$. In this way, we may regard β as a path in $\mathbb{C}^{14}$.

We orthogonally project β into $\mathbb{C}\langle \rho, \tau \rangle$ and decompose the result as a linear combination of ρ and $\tau - \rho$. These are orthogonal to each other, with norms $-36 - 24\sqrt{3}$ and $30 + 16\sqrt{3}$, respectively. Therefore

$$\text{the projection of } \beta(t) \text{ to } \mathbb{C}\langle \rho, \tau \rangle \text{ is } \frac{\langle \beta(t) \mid \rho \rangle}{-36 - 24\sqrt{3}}\rho + \frac{\langle \beta(t) \mid \tau - \rho \rangle}{30 + 16\sqrt{3}}(\tau - \rho).$$

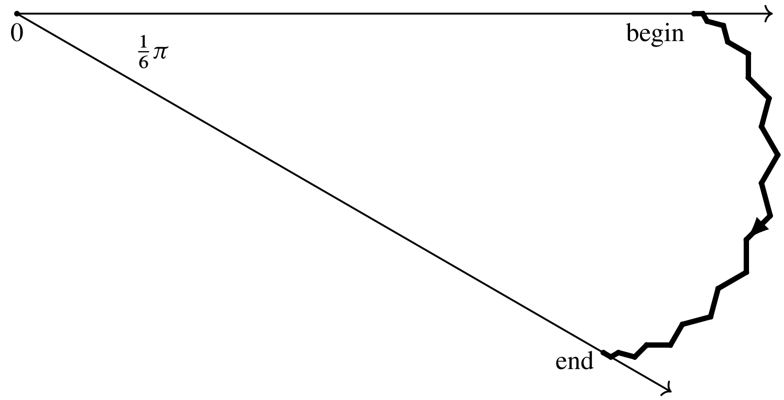

Figure 1: The piecewise-linear path $t \mapsto \langle \beta(t) \mid \rho \rangle$ in $\mathbb{C}$, up to scaling. Its reciprocal, scaled, is the path α at the heart of the proof of Lemma 6.5(a).

The coefficient of the second component is constant in time, because $\beta(t)$ differs from $\beta(0) = \sigma$ by a linear combination of $s_0, \dots, s_{11}$. These are orthogonal to $\tau - \rho$ because ρ was defined as the projection of τ to their span. The constant *does* depend on our choice of basepoint $\sigma = \beta(0)$, and is nonzero because $\sigma \in \overline{\rho\tau} - \{\rho\}$. On the other hand, the coefficient of the first component depends on time, but does not depend on the choice of $\sigma \in \overline{\rho\tau} - \{\rho\}$. This is because any two candidates for σ differ by a multiple of $\tau - \rho$.

We identify $P\mathbb{C}\langle \rho, \tau \rangle$ with $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ by sending $u\rho + v(\tau - \rho)$ to v/u. This identifies ρ with 0, and $\mathbb{B}^1(\rho, \tau)$ with an open disk centered there. (If one cares, the radius is $\sqrt{-\rho^2/(\tau-\rho)^2}$.) Most importantly, we get the simple formula

$$\alpha(t) = \frac{\text{constant depending on } \sigma}{\langle \beta(t) \mid \rho \rangle, \text{ which is independent of } \sigma},$$

where $\beta(t)$ is a concatenation of 22 segments in $\mathbb{C}^{14}$, starting at $\beta(0) = \sigma$.

The path appearing in Figure 1 is *not* α. Rather, the figure shows the piecewise linear path $t \mapsto \langle \beta(t) \mid \rho \rangle$, up to scale. It is clearly homotopic, rel endpoints, to a clockwise circular arc of angle $\frac{1}{6}\pi$ around 0. To check that the picture does not deceive, it is enough to verify that the endpoints of the segments lie in the right half-plane in $\mathbb{C}$. (This is why we work with the piecewise linear path, rather than α itself.) Therefore α is homotopic, rel endpoints, to the *counter*clockwise circular arc of angle $\frac{1}{6}\pi$ in $\mathbb{B}^1(\rho, \tau)$, starting at σ and centered at ρ. This is the same path as γ, proving that $\alpha\gamma^{-1}$ is nullhomotopic. □

If one's only goal is to prove Theorem 1.1, then one can skip the rest of this section; see the proof of that theorem in Section 7.

Proof of Lemma 6.5(c) First we claim that $(\gamma, \Delta(S_A, \dots, S_D))$ has the same image in the abelianization $\mathbb{Z}$ of Br_5 as $\Delta(\sigma_A, \dots, \sigma_D)$. To see this, observe that its twelfth power is the circle in $\mathbb{B}^1(\rho, \tau)$ centered at ρ. This loop encircles the 40 mirrors, once each. Since a loop around one of these mirrors is (conjugate to) the 3rd power of one of the standard braid generators, it follows that the image of $(\gamma, \Delta(S_A, \dots, S_D))$

in $\mathbb{Z}$ is $\frac{1}{12}(40\cdot 3) = 10$ times that of a standard generator. On the other hand, $\Delta(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}})$ is defined as a product of 10 standard generators. This proves our claim.

Next we claim that conjugation by $(\gamma, \Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}}))$ exchanges $\sigma_{\mathrm{A}} \leftrightarrow \sigma_{\mathrm{D}}$ and $\sigma_{\mathrm{B}} \leftrightarrow \sigma_{\mathrm{C}}$. That is, it permutes the generators of Br_5 in the same way that $\Delta(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}})$ does. It follows that they differ by a central element of Br_5. It is well known that $Z(\mathrm{Br}_5)$ is cyclic, generated by $\Delta(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}})^2$. Since the center maps faithfully to the abelianization, the previous paragraph shows that $\Delta(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}})$ and $(\gamma, \Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}}))$ are actually equal, not just equal up to center.

It remains to prove the claim. We will show

$$(\gamma, \Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}}))\sigma_{\mathrm{D}}(\gamma, \Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}}))^{-1} = \sigma_{\mathrm{A}}. \tag{6-8}$$

The same argument proves that $(\gamma, \Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}}))$ also sends σ_{A} to σ_{D}, and exchanges $\sigma_{\mathrm{B}} \leftrightarrow \sigma_{\mathrm{C}}$. The element of $P\Gamma$ underlying the left side of (6-8) is

$$\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})S_{\mathrm{D}}\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})^{-1}.$$

Since $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})$ exchanges $S_{\mathrm{A}} \leftrightarrow S_{\mathrm{D}}$, this simplifies to S_{A}. This agrees with the element of $P\Gamma$ underlying the right side of (6-8).

To simplify analysis of the underlying path, we write F_t for the 1-parameter subgroup of $\mathrm{U}(13,1)$ that acts trivially on $L_{\mathrm{DM}} \otimes \mathbb{C}$ and by the scalar $e^{\pi i t/6}$ on its orthogonal complement. On every complex line through ρ, which is orthogonal to $\mathbb{B}^9_{\mathrm{DM}}$, it acts by $\frac{1}{6}\pi t$ rotation. In particular, $\gamma(t) = F_t(\sigma)$. Also, although the F_t do not all preserve $\mathcal{H}$, they do preserve $U \cap \mathcal{H}$, which is the only part of $\mathcal{H}$ that will be important in this proof.

We also recall that the path underlying σ_{D} is $\mu_{\sigma, s_{\mathrm{D}}}$, moving from σ to very near $s_{\mathrm{D}}^{\perp}$, then $\frac{2}{3}\pi$ of the way around $s_{\mathrm{D}}^{\perp}$, and then onward to $S_{\mathrm{D}}(\sigma)$. What remains is to consider the path underlying the left side of (6-8). This is γ, followed by the $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})$-image of $\mu_{\sigma, s_{\mathrm{D}}}$, followed by the $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})S_{\mathrm{D}}$-image of reverse$(\gamma)$. The first part is the path $t \mapsto F_t(\sigma)$ as t varies over $[0,1]$. Because $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})$ exchanges $s_{\mathrm{A}}^{\perp}$ with $s_{\mathrm{D}}^{\perp}$, the second part is $\mu_{F_1(\sigma), s_{\mathrm{A}}}$, from $F_1(\sigma)$ to $S_{\mathrm{A}}(F_1(\sigma)) = F_1(S_{\mathrm{A}}(\sigma))$. The third part is $t \mapsto F_{1-t}(S_{\mathrm{A}}(\sigma))$. We must show that this 3-part path, followed by reverse$(\mu_{\sigma, s_{\mathrm{A}}})$, bounds a disk in $U - \mathcal{H}$. This is almost obvious: consider the surface swept out by the F_t-images of $\mu_{\sigma, s_{\mathrm{A}}}$ as t varies from 0 to 1 (or equivalently, the surface swept out by the paths $\mu_{F_t(\sigma), s_{\mathrm{A}}}$). □

Proof of Lemma 6.5(d) In the ordinary 12-strand braid group, $\Delta(\sigma_1, \dots, \sigma_{11})$ can be described by saying that the 12 strands are embedded in a strip that rotates about its midline through an angle π. This represents a constant path in $\mathcal{M}^{\circ}_{12}$, so $\Delta(\sigma_1, \dots, \sigma_{11})$ is homotopic, rel endpoints, into the punctured disk which is the fiber of $V - \mathcal{H}$ over ρ. By Lemma 6.3(c), its final endpoint is halfway around this fiber from σ. To finish the proof, it is enough to show that $\Delta(\sigma_1, \dots, \sigma_{11})^2$ encircles the puncture ρ once positively. This follows from its equality with $(\sigma_1 \cdots \sigma_{11})^{12}$ in Br_{12}, and our description of $\sigma_1 \cdots \sigma_{11}$ in part (a) of this lemma. □

Theorem 6.6 *The orbifold fundamental group*

$$J = \pi_1^{\mathrm{orb}}((U - \mathcal{H})/P\Gamma_{\mathrm{DM}}^{\mathrm{sw}}, \sigma)$$

has generators $\sigma_0, \dots, \sigma_{11}, \sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$ *and defining relations*

(a) *the Artin relations of the* $\widetilde{A}_{11}A_4$ *diagram*,

(b) *all the* $I_j := \sigma_j \sigma_{j+1} \cdots \sigma_{j+10}$ *coincide*; *write* I *for their common value*,

(c) *all the* $D_j := \sigma_j \sigma_{j-1} \cdots \sigma_{j-10}$ *coincide*; *write* D *for their common value*,

(d) $ID = \Delta(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}})^2$, *and*

(e) $D^6 = I^6$.

Furthermore, $I\sigma_k I^{-1} = \sigma_{k+1}$, $D\sigma_k D^{-1} = \sigma_{k-1}$ *and* $\Delta(\sigma_1, \dots, \sigma_{11})\sigma_k \Delta(\sigma_1, \dots, \sigma_{11})^{-1} = \sigma_{12-k}$ *for all* $k = 0, \dots, 11$.

Here and henceforth we use I_j, D_j, I and D for these words in the σ_k rather than the ρ_k. To avoid confusion, we will not use these symbols in the proof until after establishing (a)–(e).

Proof We will establish the stated relations; that they are defining relations follows. This is because they include defining relations for the normal subgroup Br_5 generated by $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$, relations saying how $\sigma_0, \dots, \sigma_{11}$ conjugate $\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}}$, and also one relation of the form (6-6), for each defining relator of $\pi_1^{\mathrm{orb}}(\mathcal{M}_{12}^{\circ})$ from Theorem 3.2. (See the discussion before Lemma 6.3.) Since 12 is even, we may use the alternative form $(\rho_1 \cdots \rho_{11})^6 = (\rho_{11} \cdots \rho_1)^6$ of Theorem 3.2(e). The $\widetilde{A}_{11}A_4$ Artin relations hold by [Basak 2016, Theorem 4.4]. Lemma 6.5 establishes (b) and (c).

(d) Lemma 6.3(b) shows that the elements of $P\Gamma$ underlying

$$\sigma_1 \cdots \sigma_{10} \sigma_{11}^2 \sigma_{10} \cdots \sigma_1 \quad \text{and} \quad \Delta(\sigma_{\mathrm{A}}, \dots, \sigma_{\mathrm{D}})^2,$$

namely $S_1 \cdots S_{10} S_{11}^2 S_{10} \cdots S_1$ and $\Delta(S_{\mathrm{A}}, \dots, S_{\mathrm{D}})^2$, coincide. Lemma 6.5(a)–(c) show that both underlying paths are homotopic, rel endpoints, to the positive circular arc in $\mathbb{B}^1(\rho, \tau)$ that starts at σ and subtends angle $\frac{1}{3}\pi$ around its center ρ.

(e) Lemma 6.5(a)–(b) shows that $(\sigma_1 \cdots \sigma_{11})^6$ and $(\sigma_{11} \cdots \sigma_1)^6$ have the same underlying path, namely the positive semicircular arc in $\mathbb{B}^1(\rho, \tau)$, starting at σ and centered at ρ. And one can check that their underlying elements $(S_1 \cdots S_{11})^6$ and $(S_{11} \cdots S_1)^6$ of $P\Gamma$ are equal. This proves (e).

Because $\sigma_0, \dots, \sigma_{11}$ satisfy (b) and the $\widetilde{A}_{11}$ Artin relations, Theorem 3.1(c) shows that $I\sigma_k I^{-1} = \sigma_{k+1}$ and $D\sigma_k D^{-1} = \sigma_{k-1}$ for all k. For $k \neq 0$ in the final relation, the A_{11} Artin relations satisfied by $\sigma_1, \dots, \sigma_{11}$ are enough: this is one of the standard properties of the fundamental element of Br_{12}. To check that $\Delta(\sigma_1, \dots, \sigma_{11})$ commutes with σ_0, observe that $\sigma_k \mapsto \sigma_{12-k}$, and conjugation by $\Delta(\sigma_1, \dots, \sigma_{11})$, are automorphisms of J that agree on the generating set $\{\sigma_1, \dots, \sigma_{11}\}$. (This is a generating set because one case of (b) is $\sigma_0 \cdots \sigma_{10} = \sigma_1 \cdots \sigma_{11}$, which expresses σ_0 in terms of $\sigma_1, \dots, \sigma_{11}$.) □

7 Proof of the main theorem

We will find new relations in G by using the previous section to write down generators for the local group at σ, defined in Section 2.8, as words in the meridians $\sigma_0, \dots, \sigma_{11}, \sigma_A, \dots, \sigma_D$. Because the $P\Gamma$-stabilizer of σ lies inside the $L_3(3) \rtimes 2$ stabilizing τ, conjugation by these words must permute the point- and line-meridians, ie the 26 Artin generators of G. The new relations express this conjugation action. We remark that we may speak of "the" local group at σ, even though we have considered σ as a point of two different orbifolds, $(U - \mathcal{H})/P\Gamma_{\mathrm{DM}}^{\mathrm{sw}}$ and $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$. Their local groups at σ coincide because the full $P\Gamma$-stabilizer of σ preserves $\mathbb{B}^9_{\mathrm{DM}}$, and hence U. In the next lemma, I means the "increasing" product $\sigma_j \sigma_{j+1} \cdots \sigma_{j+10}$, which is independent of j by Lemma 6.5(a).

Lemma 7.1 *The local group at σ is dihedral of order 24, generated by*

$$I \cdot \Delta(\sigma_A, \dots, \sigma_D)^{-1} \text{ of order } 12 \quad \text{and} \quad I^6 \cdot \Delta(\sigma_1, \dots, \sigma_{11})^{-1} \text{ of order } 2.$$

Conjugation by the second inverts the first.

Proof Lemma 6.5(a) and (c) show that the path γ (defined there) underlies $I = \sigma_1 \cdots \sigma_{11}$ and also $\Delta(\sigma_A, \dots, \sigma_D)$. Furthermore, γ lies in $\mathbb{B}^1(\rho, \tau)$, on which $S_1 \cdots S_{11}$ acts by rotating $\gamma(0)$ to $\gamma(1)$. Therefore the path underlying $I \cdot \Delta(\sigma_A, \dots, \sigma_D)^{-1}$ is homotopic to γ followed by the reverse of γ. So $I \cdot \Delta(\sigma_A, \dots, \sigma_D)^{-1}$ represents the same element of J as

$$(\text{the constant path at } \sigma, S_1 S_2 \cdots S_{11} \cdot \Delta(S_A, \dots, S_D)^{-1}).$$

This gets identified with the first element of $P\Gamma$ listed in Lemma 6.4, under the correspondence between the $P\Gamma$-stabilizer and the local group of σ. A similar argument using Lemma 6.5(a) and (d) identifies $I^6 \cdot \Delta(\sigma_1, \dots, \sigma_{11})^{-1}$ with the second element listed there. □

We know from [Allcock and Basak 2018] that the point- and line-meridians generate G, and from [Basak 2016, Theorem 4.4] that they satisfy the Artin relations of $P^2\mathbb{F}_3$. Therefore G is a quotient of $\mathrm{Art}(P^2\mathbb{F}_3)$. But for current purposes it is cleaner to express it as a quotient of $\mathrm{Art}(P^2\mathbb{F}_3) \rtimes \mathrm{Aut}(P^2\mathbb{F}_3)$, where the Artin generators map to the point- and line-meridians, and $\mathrm{Aut}(P^2\mathbb{F}_3) = L_3(3) \rtimes 2$ is identified with the local group at τ.

Theorem 7.2 (new relations in G) *The orbifold fundamental group*

$$G = \pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \tau)$$

is the quotient of $\mathrm{Art}(P^2\mathbb{F}_3) \rtimes \mathrm{Aut}(P^2\mathbb{F}_3)$ by the following relations and possibly some additional (presently unknown) relations.

Consider any $\widetilde{A}_{11} A_4$ subdiagram of $P^2\mathbb{F}_3$, any labeling of the nodes of $\widetilde{A}_{11}$ by $\tau_0, \dots, \tau_{11}$ cyclically around it in either direction, and either labeling of the nodes of A_4 by $\tau_A, \dots, \tau_D$ along it. Then

(a) $\tau_1 \tau_2 \cdots \tau_{11} \cdot \Delta(\tau_A, \dots, \tau_D)^{-1}$ *equals the element of $\mathrm{Aut}(P^2\mathbb{F}_3)$ that permutes the nodes of the $\widetilde{A}_{11}$ by $\tau_j \mapsto \tau_{j+1}$, and those of the A_4 by $\tau_A \leftrightarrow \tau_D$ and $\tau_B \leftrightarrow \tau_C$,*

(b) $(\tau_1\tau_2\cdots\tau_{11})^6\cdot\Delta(\tau_1,\dots,\tau_{11})^{-1}$ *equals the element of* $\mathrm{Aut}(P^2\mathbb{F}_3)$ *that permutes the nodes of the* $\widetilde{A}_{11}$ *by* $\tau_j\mapsto\tau_{6-j}$, *and fixes each of* $\tau_{\mathrm{A}},\dots,\tau_{\mathrm{D}}$.

Proof Recall from Section 4 the 16 particular point- and line-meridians $\sigma_0,\dots,\sigma_{11},\sigma_{\mathrm{A}},\dots,\sigma_{\mathrm{D}}$ based at σ, and the corresponding meridians $\tau_0,\dots,\tau_{11},\tau_{\mathrm{A}},\dots,\tau_{\mathrm{D}}$ based at τ. These are 16 of the Artin generators, forming a labeled $\widetilde{A}_{11}A_4$ diagram as specified in the statement.

The segment $\overline{\sigma\tau}$ identifies the orbifold fundamental groups of $(\mathbb{B}^{13}-\mathcal{H})/P\Gamma$ based at σ and τ. It is fixed pointwise by the $P\Gamma$-stabilizer D_{24} of σ, and therefore identifies the local group at σ with a subgroup of $L_3(3)\rtimes 2$. By Theorem 4.1, this segment also identifies each of $\sigma_0,\dots,\sigma_{11},\sigma_{\mathrm{A}},\dots,\sigma_{\mathrm{D}}$ with the corresponding $\tau_0,\dots,\tau_{11},\tau_{\mathrm{A}},\dots,\tau_{\mathrm{D}}$. Therefore the words in Lemma 7.1, with σ_j's replaced by τ_j's, permute the 26 Artin generators. The pointwise $(L_3(3)\rtimes 2)$-stabilizer of an $\widetilde{A}_{11}A_4$ diagram is trivial. Therefore these words' actions on the 26 generators are completely determined by their actions on $\tau_0,\dots,\tau_{11},\tau_{\mathrm{A}},\dots,\tau_{\mathrm{D}}$.

We will work out this action in the case of the second word; the argument for the first is similar. Every word in $\tau_1,\dots,\tau_{11}$ centralizes $\tau_A,\dots,\tau_D$, so it is enough to show that $(\tau_1\cdots\tau_{11})^6\Delta(\tau_1,\dots,\tau_{11})^{-1}$ conjugates τ_j to τ_{6-j} for each $j=0,\dots,11$. By Theorem 4.1, it is enough to prove this with all τ_k's replaced by σ_k's, which is immediate from the identities at the end of Theorem 6.6.

We have established the lemma for one particular choice of $\tau_0,\dots,\tau_{11},\tau_{\mathrm{A}},\dots,\tau_{\mathrm{D}}$. This does not quite prove the theorem, because $L_3(3)\rtimes 2$ acts with two orbits on the set of such choices. The other orbit is represented by the labeling $\tau_0,\dots,\tau_{11},\tau_{\mathrm{D}},\dots,\tau_{\mathrm{A}}$. But this case follows from the first, by the relation $\Delta(\tau_{\mathrm{A}},\dots,\tau_{\mathrm{D}})=\Delta(\tau_{\mathrm{D}},\dots,\tau_{\mathrm{A}})$ in Br_5. □

Proof of Theorem 1.1 Consider $\delta:=\tau_0\tau_1\cdots\tau_{10}(\tau_1\tau_2\cdots\tau_{11})^{-1}$, regarded as an element of $\mathrm{Art}(\widetilde{A}_{11})\subseteq\mathrm{Art}(P^2\mathbb{F}_3)$. We call it the "deflation word" for reasons that will become clear. An immediate consequence of Theorem 7.2(a) is that δ dies in G. (In fact, the relation $\delta=1$ in G follows from Lemma 6.5(a), so the later parts of Section 6 are not needed for this theorem.) We write $\bar{\delta}$ for the image of δ in $\mathrm{Cox}(\widetilde{A}_{11})\subseteq\mathrm{Cox}(P^2\mathbb{F}_3)$.

We claim that the subgroup of $\mathrm{Cox}(\widetilde{A}_{11})\cong\mathbb{Z}^{11}\rtimes S_{12}$ normally generated by $\bar{\delta}$ is the translation subgroup $\mathbb{Z}^{11}$. This is standard: annihilating $\bar{\delta}$ expresses τ_0 as a word in $\tau_1,\dots,\tau_{11}$, so the quotient can be no larger than S_{12}. The quotient is no smaller, because the transpositions $(1\ 2),(2\ 3),\dots,(11\ 12),(12\ 1)\in S_{12}$ satisfy the relations of $\mathrm{Cox}(\widetilde{A}_{11})$ and also the deflation relation.

Now we can quote a result of Conway and Simons [2001], which depends essentially on work of Ivanov [1999] and Norton [1992]. They consider the "deflation" of $\mathrm{Cox}(P^2\mathbb{F}_3)$, meaning the quotient of this group by the subgroup normally generated by the translation subgroups of the $\mathrm{Cox}(\widetilde{A}_{11})$'s coming from all the $\widetilde{A}_{11}$ subdiagrams of $P^2\mathbb{F}_3$. They prove that this quotient is the bimonster. Because G is a quotient of $\mathrm{Art}(P^2\mathbb{F}_3)$ in which δ dies, G/S is a quotient of $\mathrm{Cox}(P^2\mathbb{F}_3)$ in which $\bar{\delta}$ dies. By the previous paragraph,

G/S is a quotient of the deflation of $\mathrm{Cox}(P^2\mathbb{F}_3)$, and hence of the bimonster $(M \times M) \rtimes 2$. Since M is simple, the only possibilities for G/S are the bimonster, $\mathbb{Z}/2$ and the trivial group. The last case is excluded by Lemma 7.4. □

Remark 7.3 We sketch an argument of Heckman [2015]. It lets one skip most of Sections 5 and 6, and still prove Theorem 1.1, although it does not establish any new relations in G itself. Killing the squares of the meridians collapses the exact sequence (6-5), namely

$$1 \to \mathrm{Br}_5 \to J \to \pi_1^{\mathrm{orb}}(\mathcal{M}_{12}^\circ) \to 1,$$

so that the left term becomes S_5 and the right term S_{12}. Because S_5 has trivial center, the extension is determined by the homomorphism $S_{12} \to \mathrm{Out}(S_5) = 1$. Therefore J becomes $S_5 \times S_{12}$. The images of $\sigma_0, \dots, \sigma_{11}$ commute with S_5, and hence lie in the S_{12} factor. Indeed, they project to the standard transpositions generating S_{12}. It follows that the deflation relation holds for the images of $\sigma_0, \dots, \sigma_{11}$ in G/S. Then Theorem 4.1 gives the same result for $\tau_0, \dots, \tau_{11}$, and one can quote Conway and Simons [2001] as in the proof above. (Unfortunately, Heckman's approach to Theorem 4.1 had a gap.)

Lemma 7.4 *The quotient of $G = \pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma)$ by the subgroup S generated by the squares of the meridians is nontrivial.*

Proof It is enough to exhibit a surjection $G \to \mathbb{Z}/2$, or equivalently a connected orbifold double cover of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$. There is a degree-4 holomorphic automorphic form Ψ_0 for $P\Gamma$, whose zero locus is $\mathcal{H}$ with multiplicity 1 along each component [Allcock 2000, Theorem 7.1]. This means: Ψ_0 is a holomorphic function on the preimage Ω of $\mathbb{B}^{13}$ in $\mathbb{C}^{14} - \{0\}$, homogeneous of degree -4, with a simple zero along each component of the preimage $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ in Ω, and no other zeros. The same reference shows that Ψ_0 transforms by a character $\Gamma \to \mathbb{Z}/3$.

Let Z be the space of pairs (ℓ, ϕ), where $\ell \subseteq \Omega$ is the preimage of a point of $\mathbb{B}^{13} - \mathcal{H}$, and ϕ is a holomorphic function on ℓ whose square coincides with $\Psi_0^3|_\ell$. Forgetting ϕ gives a projection $Z \to \mathbb{B}^{13} - \mathcal{H}$, which is obviously a degree-2 covering space. It is connected because Ψ_0^3 has zeros of odd order along $\widetilde{\mathcal{H}}$: taking a square root introduces ramification there. The Γ-invariance of Ψ_0^3 yields a Γ-action on Z, with each $g \in \Gamma$ sending (ℓ, ϕ) to $(g(\ell), \phi \circ g^{-1})$. Because ϕ is homogeneous of degree -6, the scalars in Γ act trivially, yielding a $P\Gamma$-action on Z and a $P\Gamma$-equivariant map $Z \to \mathbb{B}^{13} - \mathcal{H}$. Our promised double cover of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ is $Z/P\Gamma$. □

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