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The Dehn function of a metric space measures the area necessary in order to fill a closed curve of controlled length by a disc. As a main result, we prove that a length space has curvature bounded above by κ in the sense of Alexandrov if and only if its Dehn function is bounded above by the Dehn function of the model surface of constant curvature κ. This extends work of Lytchak and the second author (2018) from locally compact spaces to the general case. A key ingredient in the proof is the construction of minimal discs with suitable properties in certain ultralimits. Our arguments also yield quantitative local and stable versions of our main result. The latter has implications on the geometry of asymptotic cones.

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1 Introduction

1.1 Main result

The main result of this paper is an analytic characterization of CAT(κ) spaces — complete metric spaces with curvature bounded above by κ in the sense of Alexandrov. For simplicity, we first describe the result for CAT(0) spaces. We say that a metric space X satisfies the Euclidean isoperimetric inequality for curves if any closed Lipschitz curve γ in X bounds a Lipschitz disc v in X such that

$$\mathrm{Area}(v) \leq \frac{1}{4\pi} \cdot \ell^2(\gamma).$$

Here, Area(v) denotes the parametrized Hausdorff area of v and $\ell(\gamma)$ is the length of γ. The fact that CAT(0) spaces satisfy the Euclidean isoperimetric inequality for curves is a well-known consequence of Reshetnyak's majorization theorem [1968] and the isoperimetric inequality in the Euclidean plane. Vice versa, for locally compact metric spaces X, it was proved in [Lytchak and Wenger 2018b] that a Euclidean isoperimetric inequality implies the zero upper curvature bound. Moreover, in [Lytchak and Wenger 2018b], this isoperimetric characterization of upper curvature bounds was extended to nonzero bounds. In order to formulate our main result, it is convenient to introduce *Dehn functions*. In general, if X is a metric space, a Dehn function $\delta_X(r)$ controls the area needed in order to fill a closed curve of length at most r in X by a disc. A precise definition requires a choice of area and types of discs.

For instance, choosing parametrized Hausdorff area and Lipschitz discs results in the *Lipschitz Dehn function* δ_X^{Lip}. More precisely, let $\delta\colon (0,\infty) \to [0,\infty]$ be a function such that any Lipschitz circle of length at most r bounds a Lipschitz disc of area at most $\delta(r)$. Then δ_X^{Lip} is the greatest lower bound for all such functions.

Denote by M_κ^2 the complete simply connected surface of constant sectional curvature κ. Thus, up to scaling, we see the round 2-sphere $M_1^2 = \mathbb{S}^2$, the Euclidean plane $M_0^2 = \mathbb{R}^2$, and the hyperbolic plane $M_{-1}^2 = \mathbb{H}^2$. Let us denote by D_κ the diameter of M_κ^2 and by δ_κ the Lipschitz Dehn function of M_κ^2.

Theorem A *Let X be a complete length space and $r_0 \in (0, 2D_\kappa]$. Suppose that*

$$\delta_X^{\mathrm{Lip}}(r) \leq \delta_\kappa(r) \tag{1-1}$$

holds for all $r \in (0, r_0)$. Then every closed ball of radius at most $\frac{1}{4}r_0$ in X is convex and CAT(κ). Moreover, if $r_0 = 2D_\kappa$, then X is a CAT(κ) space.

This extends the main theorem in [Lytchak and Wenger 2018b] from locally compact spaces to the general case and at the same time provides a quantitative local version. For even more general results, see Section 1.3 below. Accordingly, CAT(κ) geometry is as much an analytic theory as it is a geometric theory. In particular, upper curvature bounds can be identified without the knowledge of geodesics or angles. One encounters such situations in many geometric settings [Alexander and Bishop 2004; 2016; Lytchak and Stadler 2019; 2020; 2023; Lytchak and Wagner 2024; Petrunin and Stadler 2019; Ricks 2021].

Beckenbach and Radó [1933] first discovered a relationship between isoperimetric inequalities and upper curvature bounds on smooth 2-dimensional Riemannian manifolds. Later, this was generalized to nonsmooth metric surfaces by Reshetnyak [1961]. In [Lytchak and Wenger 2018b], for locally compact spaces, the isoperimetric inequality is translated to an upper curvature bound in three steps:

(1) Solve the Plateau problem for a given Jordan curve.

(2) Show that the solution is a minimal disc satisfying the same isoperimetric inequality as the surrounding space.

(3) Prove that the minimal disc is intrinsically a metric surface with the desired upper curvature bound.

If we were able to solve the Plateau problem in the setting of Theorem A, then the strategy of [Lytchak and Wenger 2018b] would still be successful. Inspection of the proof shows that it would even be enough to solve the following problem: for a given Jordan curve $\Gamma \subset X$, construct a larger space $Y \supset X$ which satisfies (1-1) and such that the Plateau problem for Γ is solvable in Y.

Even though there are natural candidates for the space Y, such as ultracompletions X_ω of X, we are unable to do this. As a matter of fact, it is possible to construct for a given Jordan curve $\Gamma \subset X$ a minimal disc in X_ω filling Γ but we do not know whether X_ω satisfies inequality (1-1); see Section 1.5. Despite these obstacles, our proof of Theorem A still employs minimal surface theory. We sidestep the fact that we are unable to answer this question by producing minimal discs in ultralimits whose intrinsic structure appears as if the Dehn function of the surrounding space would satisfy the correct bounds; see the next section.

1.2 Minimal surfaces in ultralimits of locally noncompact spaces

As announced, we prove a general result which serves as an appropriate substitute for the solvability of the Plateau problem in locally compact spaces. This is the technical heart behind Theorem A and its generalizations discussed below. We state it here in a very simplified setting; for the version in full generality see Theorem 8.1. In the context of the Plateau problem, it is natural to enlarge the class of admissible discs filling a given circle from Lipschitz to Sobolev discs; see Section 3.1 for precise definitions. As in the classical case of Riemannian manifolds, this leads to better compactness properties relative to energy bounds. From now on we will focus on the (Sobolev) Dehn function $\delta_X(r)$ instead of the Lipschitz Dehn function $\delta_X^{\mathrm{Lip}}(r)$. Since every Lipschitz disc is Sobolev, we obtain the natural inequality $\delta_X(r) \le \delta_X^{\mathrm{Lip}}(r)$. Thus, results which only involve upper bounds on the Dehn function are a priori stronger than their Lipschitz counterparts.

Theorem B *Let X be a complete length space, $r_0 > 0$ and $\kappa \in \mathbb{R}$. Suppose $\Gamma \subset X$ is a Jordan curve with $\ell(\Gamma) < r_0$ and the Dehn function of X satisfies $\delta_X(r) \le \delta_\kappa(r)$ for all $r \in (0, r_0)$. Then there exists an ultracompletion X_ω of X and a continuous map $v\colon \overline{D} \to X_\omega$ which is a solution to the Plateau problem for Γ in X_ω and satisfies*

$$\mathrm{Area}(v|_\Omega) \le \delta_\kappa(\ell(v|_{\partial\Omega})) \tag{1-2}$$

for every Jordan domain $\Omega \subset D$ such that $\ell(v|_{\partial\Omega}) < r_0$.

Recall that an ultracompletion X_ω is a certain metric space that contains an isometric copy of X and which is constructed with the help of a nonprincipal ultrafilter ω on the natural numbers; see Section 2.3 for details. Note that in Theorem B we do not gain control on the Dehn function δ_{X_ω} of X_ω. However, the "intrinsic isoperimetric inequality" (1-2) has the effect that v behaves as if δ_{X_ω} would be bounded above by δ_κ.

Here is how we deduce Theorem A from Theorem B in the case that X is geodesic and $\kappa = 0$: For a Jordan triangle $\triangle \subset X$ we obtain from Theorem B a minimal disc v filling $\triangle$ in some ultracompletion X_ω. As follows from [Lytchak and Wenger 2018a], the map v factors as

$$\overline{D} \xrightarrow{P} Z_v \xrightarrow{\bar{v}} X_\omega,$$

where Z_v is a metric disc and $\bar{v}$ is a 1-Lipschitz map which restricts to an arclength preserving homeomorphism $\partial Z_v \to \Gamma$. Moreover, for every Jordan domain $O \subset Z_v$,

$$\mathcal{H}^2(O) \le \delta_\kappa(\ell(\partial O)).$$

We deduce from [Lytchak and Wenger 2018b] that Z_v is CAT(0) and then Reshetnyak's majorization theorem implies that Δ satisfies the CAT(0) comparison and hence X itself is CAT(0).

1.3 Generalizations

We prove a stable version of Theorem A for sequences of metric spaces. The setting involves metric spaces without any control on small scales. This requires us to adjust the way we measure area in the definition of the Dehn function. Instead of parametrized Hausdorff area we will use the *Riemannian inscribed area* Area_{μ^i}, originally defined by Ivanov [2008], and the associated *Riemannian Dehn function* $\delta_X^{\mu^i}$. Informally speaking, instead of assigning to a unit ball in a normed plane its area, one assigns the maximal area of an inscribed ellipsoid — the *John ellipsoid*. The precise definition and a discussion of basic properties can be found in Section 3.2. Here we only mention that for many geometrically interesting spaces these subtleties disappear and the equality $\text{Area}_{\mu^i} = \text{Area}$ holds. For instance, this is the case for all metric spaces with curvature locally bounded either above or below; in particular, it holds for all smooth Riemannian manifolds.

Theorem C *Let (X_n) be a sequence of complete length spaces and $\kappa \in \mathbb{R}$. Suppose $r_0 \in (0, 2D_\kappa]$ and*

$$\delta_{X_n}^{\mu^i}(r) \leq (1+\epsilon_n)\cdot\delta_\kappa(r)+\epsilon_n$$

holds for all $r \in (0, r_0)$ and some sequence $\epsilon_n \to 0$. Then every ultralimit X_ω is locally CAT(κ). More precisely, every closed ball of radius at most $\frac{1}{4}r_0$ in X_ω is convex and CAT(κ).

Note that Theorem A really is a special case. Indeed, in the setting of Theorem A it can be shown that $\delta_X^{\mu^i}(r) = \delta_X(r) \leq \delta_X^{\text{Lip}}(r)$. Then Theorem C applies to the constant sequence X and provides the curvature bound for all of its ultracompletions and therefore for X itself.

The result also has implications for asymptotic cones.

Theorem D *Let X be a complete length space such that*

$$\limsup_{r\to\infty}\frac{\delta_X^{\mu^i}(r)}{r^2} \leq \frac{1}{4\pi}.$$

Then every asymptotic cone of X is a CAT(0) space. Moreover, if the inequality is strict, then every asymptotic cone of X is a tree. In particular, in this case, X is Gromov hyperbolic.

The first statement is a version of [Wenger 2011, Theorem 1.1] for locally noncompact spaces and the second statement is a variant of [Wenger 2008, Theorem 1.1]. For a localized version of the second statement in Theorem D see Theorem 9.2.

Theorem C can also be used to turn fine infinitesimal information on the Dehn function into curvature bounds:

Theorem E *Let $\delta\colon (0, r_0) \to \mathbb{R}$ be a continuous nondecreasing function with*

$$\limsup_{r\to 0}\frac{\delta(r) - r^2/(4\pi)}{r^4} \leq 0.$$

Suppose that (X_n) is a sequence of complete length spaces such that the Riemannian Dehn functions satisfy

$$\delta_{X_n}^{\mu^i}(r) \leq \delta(r) + \epsilon_n$$

on $(0, r_0)$ *for some sequence* $\epsilon_n \to 0$*. Then any ultralimit* X_ω *is locally CAT*(0)*. More precisely, there exists* $\tilde{r} > 0$ *depending only on the function* δ *such that every closed ball in* X_ω *of radius at most* $\tilde{r}$ *is convex and CAT*(0)*.*

1.4 Motivation and strategy for Theorem B

For simplicity, we restrict this discussion to the case $\kappa = 0$.

Recall that in the locally noncompact case we are unable to solve the Plateau problem in the traditional sense. However, if X was known to be CAT(0), then the Plateau problem would be solvable, X being locally compact or not. For this one chooses a minimizing sequence of discs filling a given Jordan curve $\Gamma \subset X$ and constructs a suitable limit v in an ultracompletion X_ω. Since X is CAT(0) there exists a 1-Lipschitz retraction $X_\omega \to X$ and hence we can push v back to X without increasing energy or area [Guo and Wenger 2020; Stadler 2021].

While for a locally noncompact space X which satisfies the Euclidean isoperimetric inequality we are not able to a priori show the existence of such retractions, this still motivates the search for minimal discs in ultracompletions.

To prove Theorem B we start with a minimizing sequence (v_n) of Sobolev discs filling $\Gamma \subset X$. We then reparametrize using Morrey's ϵ-conformality lemma to make energy and area almost equal. Using Rellich–Kondrachov compactness, we select a subsequence which L^2-converges to a Sobolev disc v in some auxiliary metric space. Then comes the critical step: we use the isoperimetric information to show that the filling area of Γ cannot drop in X_ω. Using our setup, this allows us to show that the limit is in fact a minimal disc and that areas and energies converge. Once this is established, we obtain strong convergence in the Sobolev norm. Next we apply Fuglede's lemma to see length convergence $\ell(v_n|_\gamma) \to \ell(v|_\gamma)$ for most paths $\gamma \subset D$. Using area convergence and the bound on the Dehn function we conclude the intrinsic isoperimetric inequality for the limit v, thus Theorem B.

1.5 Further questions

As already mentioned above, it is not known whether an isoperimetric inequality passes to ultracompletions. More generally, we pose the following problem:

Problem 1 *Let* $\delta\colon (0, r_0) \to \mathbb{R}$ *be a continuous nondecreasing function. Let* (X_n) *be a sequence of complete length spaces such that the Dehn functions satisfy*

$$\delta_{X_n}(r) \le \delta(r)$$

on $(0, r_0)$*. Does the Dehn function of an ultralimit of* (X_n) *satisfy the same inequality?*

Note that if there exists $C > 0$ and the sequence (X_n) consists of locally compact geodesic spaces with $\delta_{X_n}(r) \le C \cdot r^2$, then the Dehn function of any ultralimit of (X_n) does satisfy the same inequality [Lytchak et al. 2020, Theorem 1.8].

On a more technical level, it is also not clear whether the use of the Riemannian inscribed area in Theorem C is really necessary. The crucial point in the proof where we use it is in the form of Morrey's ϵ-conformality lemma which allows us to reparametrize a Sobolev disc such that area and energy become almost equal. Morrey's lemma fails for the Hausdorff area as can be seen in non-Euclidean normed spaces. However, constructing a counterexample to Theorem C where the Riemannian Dehn function is replaced by the Hausdorff Dehn function seems very difficult. The examples would have to involve locally noncompact spaces.

1.6 Organization

In Section 2 we set notation and collect the necessary background from metric geometry. In Section 3 we recall relevant parts of the Sobolev theory in metric spaces. In several subsections we discuss energy, area, isoperimetric inequalities, quasiconformality and regularity results. In Section 4 we introduce Dehn functions, construct universal thickenings of metric spaces and recall the Plateau problem and the intrinsic structure of its solutions. Section 5 is devoted to an infinitesimal Euclidean property for metric spaces — property (ET) — and how to ensure it holds, given some isoperimetric control. In Section 6 we begin working towards our main theorem. We prove a result (Proposition 6.1) which ensures that the filling area cannot drop in an ultralimit of a sequence of metric spaces, assuming certain bounds on the Dehn functions. In Section 7 we prove a technical result (Proposition 7.1) stating that a certain limit of Sobolev discs satisfies an intrinsic isoperimetric inequality if each individual Sobolev disc lives in a space which supports an isoperimetric inequality. In Section 8 we solve a version of the Plateau problem for locally noncompact spaces (Theorem 8.1). In the final Section 9 we provide proofs of our main results.

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2 Basics on metric spaces

2.1 Notation

Connected open subsets of $\mathbb{R}^n$ will be called *domains*. A domain $\Omega \subset \mathbb{R}^n$ is a *Lipschitz domain* if its boundary $\partial\Omega$ can locally be written as the graph of a Lipschitz function. When $n = 2$, this is equivalent to the requirement that $\partial\Omega$ is locally bilipschitz to an open interval [Tukia 1980]. The open unit disc in $\mathbb{R}^2$ is denoted by D and the standard annulus $S^1 \times [0, 1]$ by A. We call any domain $U \subset \mathbb{R}^2$ homeomorphic to A an annulus. We denote the two *boundary circles* of an annulus $U \subset \mathbb{R}^2$ by $\partial^{\pm} U$. By [Tukia 1980], U is bilipschitz to A if $\partial^{\pm} U$ are bilipschitz curves.

We will denote distances in a metric space X by d or d_X. Let $X = (X, d)$ be a metric space. The open ball in X of radius r and center $x_0 \in X$ is denoted by

$$B_r(x_0) = \{x \in X \mid d(x_0, x) < r\}.$$

More generally, for any subset $A \subset X$ we denote its open tubular neighborhood of radius r by

$$N_r(A) = \{x \in X \mid d(A, x) < r\}.$$

A *Jordan curve* in X is a subset $\Gamma \subset X$ homeomorphic to S^1. Given a Jordan curve $\Gamma \subset X$, a continuous map $\gamma\colon S^1 \to X$ with image Γ is called a *weakly monotone parametrization* of Γ if it has connected fibers.

For $m \geq 0$, the m-dimensional Hausdorff measure on X is denoted by $\mathcal{H}^m = \mathcal{H}^m_X$. The normalizing constant is chosen in such a way that on Euclidean space $\mathbb{R}^m$ the Hausdorff measure $\mathcal{H}^m$ equals the Lebesgue measure.

The length of a curve γ in a metric space X will be denoted by $\ell_X(\gamma)$ or simply by $\ell(\gamma)$. A continuous curve of finite length is called *rectifiable*. A (local) *geodesic* in a space X is a (locally) isometric map from an interval to X. A space X is called *a geodesic space* if any pair of points in X is connected by a geodesic.

For $\epsilon > 0$ we call a Lipschitz curve $c\colon [a, b] \to X$ an *ϵ-geodesic*, if it has constant speed and satisfies

$$\ell(c) \leq (1 + \epsilon) \cdot d(c(a), c(b)).$$

A space X is a *length space* if for all $x, y \in X$ the distance $d(x, y)$ equals $\inf\{\ell_X(\gamma)\}$, where γ runs over the set of all curves connecting x and y. In a length space any pair of points is connected by an ϵ-geodesic for every $\epsilon > 0$.

2.2 Intrinsic metric of a map

We refer the reader to [Burago et al. 2001; Lytchak and Wenger 2018a; Petrunin and Stadler 2019] for discussions of the following construction and related topics. Let Z be a topological space and X a metric space. Let $u\colon Z \to X$ be a continuous map. The *intrinsic distance associated with u* is the function $d_u\colon Z \times Z \to [0, \infty]$ defined by

$$d_u(z_1, z_2) = \inf\{\ell_X(u \circ \gamma) \mid \gamma \text{ is a path in } Z \text{ connecting } z_1 \text{ and } z_2\}.$$

If it only takes finite values, then it defines a pseudometric. The associated metric space Z_u, which arises from identifying pairs of points at zero d_u-distance, is a length space. We will call it the *intrinsic metric space associated with the map u*.

By construction, the space Z_u associated with the map u comes with a canonical, possibly noncontinuous, surjective projection $P\colon Z \to Z_u$ and a 1-Lipschitz map $\bar{u}\colon Z_u \to X$ such that $u = \bar{u} \circ P$.

If X is a metric space in which any pair of points is connected by a curve of finite length, then the *length space X^i associated to X* is the special case $X^i = Z_u$ of the above construction where u is the identity map $u = \mathrm{Id}\colon X \to X$. The completeness of X implies that X^i is complete as well. The 1-Lipschitz map $\bar{u}\colon X^i \to X$ from above is the identity in this case. The map $P = \bar{u}^{-1}\colon X \to X^i$ need not be continuous, but it sends curves of finite length in X to continuous curves of the same length in X^i.

2.3 Ultralimits

We refer the reader to [Alexander et al. 2024] for an extended treatment of ultralimits in the context of metric geometry. For a nonprincipal ultrafilter ω on $\mathbb{N}$ and a sequence of pointed metric spaces (X_n, x_n), we will often consider their ultralimit

$$(X_\omega, x_\omega) = \omega\text{-lim}(X_n, x_n),$$

which is a pointed metric space whose elements are equivalence classes of sequences (p_n) which are bounded relative to (x_n). The metric d_ω on (X_ω, x_ω) is induced by the metrics d_n on X_n via $d_\omega((p_n), (q_n)) = \omega\text{-lim}\, d_n(p_n, q_n)$. Often the choice of basepoints x_n is irrelevant in our considerations and then we neglect it in our notation. The ultralimit of the constant sequence X with fixed basepoint $x \in X$ will be called *ultrapower* or *ultracompletion* of X with respect to ω. Ultrapowers do not depend on the choice of basepoint. Note that any ultralimit is a complete metric space. Moreover, any metric space admits a canonical isometric embedding into its ultrapowers via constant sequences. If every X_n is a length space, then any ultralimit X_ω is geodesic.

For any sequence of subsets $A_n \subset X_n$ we denote by $A_\omega \subset X_\omega$ their ultralimit. So $A_\omega \subset X_\omega$ corresponds precisely to those points in X_ω which can be represented as a sequence (a_n) with $a_n \in X_n$ and $\sup d(a_n, x_n) < \infty$. Note that A_ω is always closed, even if the individual A_n might not be.

If (X_n) and (Y_n) are sequences of metric spaces such that $X_n \subset Y_n$ and $Y_n \subset N_{\epsilon_n}(X_n)$ for some sequence $\epsilon_n \to 0$, then $\omega\text{-lim}(X_n, x_n) = \omega\text{-lim}(Y_n, x_n)$ for any choice of basepoints $x_n \in X_n$.

The following lifting result is similar to [Lytchak et al. 2020, Corollary 2.6].

Lemma 2.1 *Let X_ω be an ultralimit of a sequence of length spaces X_n. Let $I \subset \mathbb{R}$ be a compact interval and $\gamma_\omega\colon I \to X_\omega$ a curve parametrized by arclength. Then there exists a sequence (γ_n) of uniformly Lipschitz curves $\gamma_n\colon I \to X_n$ such that $\gamma_\omega = \omega\text{-lim}\,\gamma_n$ and $\ell(\gamma_\omega) = \omega\text{-lim}\,\ell(\gamma_n)$.*

Proof Let $\{q_k \mid k \in \mathbb{N}\} \subset I$ be a countable dense subset. For every $q_k \in I$ choose a bounded sequence of points $x_{k,n} \in X_n$ such that $\gamma_\omega(q_k) = [(x_{k,n})]$. For every $j \geq 2$ set

$$\delta_j := \min\{|q_k - q_l| \mid 1 \leq k < l \leq j\}.$$

Define inductively a decreasing sequence $\mathbb{N} = N_1 \supset N_2 \supset \cdots$ of subsets N_j such that for each $j \geq 2$ we have $\omega(N_j) = 1$ and

$$|d_\omega(\gamma_\omega(q_k), \gamma_\omega(q_l)) - d_n(x_{k,n}, x_{l,n})| < \frac{\delta_j}{j}$$

for all $1 \le k < l \le j$ and all $n \in N_j$. For every $j \in \mathbb{N}$ define $M_j := N_j \setminus N_{j+1}$ and note that, together with $M_\infty := \bigcap_{k=1}^{\infty} N_k$, this defines a partition of $\mathbb{N}$. For $n \in \mathbb{N}$ let $j(n) = n$ if $n \in M_\infty$ and otherwise let $j(n) \in \mathbb{N}$ be the unique number with $n \in M_{j(n)}$. Then define $\gamma_n : \{q_1, \dots, q_{j(n)}\} \to X_n$ by $\gamma_n(q_k) := x_{k,n}$ and note that

$$d_n(\gamma_n(q_k), \gamma_n(q_l)) \le d_\omega(\gamma_\omega(q_k), \gamma_\omega(q_l)) + \frac{\delta_{j(n)}}{j(n)} < \left(1 + \frac{1}{j(n)}\right)|q_k - q_l|$$

holds for all $1 \le k < l \le j(n)$. Since X_n is a length space we can extend γ_n to a $(1+1/j(n))^2$-Lipschitz curve, again denoted by γ_n. In particular, $\ell(\gamma_n) \le (1+1/j(n))^2 \ell(\gamma_\omega)$. Since the sequence (γ_n) is bounded, it has a well-defined ultralimit $\omega\text{-lim}\,\gamma_n : I \to X_\omega$ which is a 1-Lipschitz curve. We claim that $\gamma_\omega = \omega\text{-lim}\,\gamma_n$. For all $k \in \mathbb{N}$, we have $\gamma_n(q_k) = x_{k,n}$ for all $n \in N_k$ since $j(n) \ge k$. Since $\omega(N_k) = 1$ we have $\gamma_\omega(q_k) = [(x_{k,n})] = \omega\text{-lim}\,\gamma_n(q_k)$ for every $k \in \mathbb{N}$ and therefore $\gamma_\omega = \omega\text{-lim}\,\gamma_n$. □

Lemma 2.2 *Let X_ω be an ultralimit of a sequence of length spaces X_n. Suppose that $\Gamma_\omega \subset X_\omega$ is a rectifiable Jordan curve. Then, for ω-a.e. $n \in \mathbb{N}$, there exists a rectifiable Jordan curve $\Gamma_n \subset X_n$ with $\omega\text{-lim}\,\Gamma_n = \Gamma_\omega$ and $\omega\text{-lim}\,\ell(\Gamma_n) = \ell(\Gamma_\omega)$.*

Proof Let $c_\omega : S^1 \to X_\omega$ be a constant speed parametrization of Γ_ω. By Lemma 2.1, there is a sequence of uniformly Lipschitz maps $c_n : S^1 \to X_n$ which ultraconverges to c_ω, and such that $\omega\text{-lim}\,\ell(c_n) = \ell(c_\omega)$. Now we will find the desired Jordan curves Γ_n inside the images of the c_n. Choose three equidistant points $t^j \in S^1$, with $j = 1, 2, 3$, and set $z_n^j = c_n(t^j)$. Denote by $\alpha^j \subset S^1$ the closure of the component of $S^1 \setminus \{t^1, t^2, t^3\}$ which does not contain the point t^j. Next, inside the image $c_n(\alpha^j)$, choose a minimizing geodesic γ_n^j from z_n^{j+1} to z_n^{j+2}. Note that these geodesics ultraconverge to $c_\omega(\alpha^j)$. For every $\delta > 0$ and large enough n, the geodesics γ_n^j are disjoint away from the δ-neighborhood of $\{z_n^1, z_n^2, z_n^3\}$. Now we orient the three geodesics and pass to subsegments $\hat{\gamma}_n^j \subset \gamma_n^j$ where the starting point of $\hat{\gamma}_n^j$ is the last point of γ_n^j on $\hat{\gamma}_n^{j-1}$. Finally, put $\Gamma_n = \bigcup_{j=1}^3 \hat{\gamma}_n^j$. □

2.4 CAT(κ)

Let κ be a real number. Recall that a CAT(κ) space is a complete metric space X where any pair of points at distance strictly less than D_κ is joined by a geodesic and such that distances between points on a geodesic triangle $\triangle \subset X$ of perimeter strictly less than $2D_\kappa$ are bounded above by the distances between corresponding points on the comparison triangle $\widetilde{\triangle} \subset M_\kappa^2$. In order to check if a geodesic space is CAT(κ) it suffices to prove the CAT(κ) comparison for *Jordan triangles* — geodesic triangles which are Jordan curves; see [Lytchak and Wenger 2018b, Lemma 3.1]. A more flexible characterization of CAT(κ) spaces which does not refer to geodesics can be provided using majorizations. Recall that a rectifiable Jordan curve Γ in a metric space X admits a *κ-majorization in X*, if there is a closed convex region $C \subset M_\kappa^2$ and a 1-Lipschitz map $C \to X$ which restricts to an arclength preserving homeomorphism $\partial C \to \Gamma$. If κ is clear from the context we will simply speak of majorizations.

By Reshetnyak's majorization theorem [1968], every rectifiable Jordan curve in a CAT(κ) space can be κ-majorized. On the other hand, if a Jordan triangle in a metric space admits a κ-majorization, then it clearly satisfies the CAT(κ) comparison. We will make use of the following, which is a consequence of [Alexander et al. 2024, Radial lemma 9.52].

Lemma 2.3 *Let X be a complete geodesic space. Suppose there is $r_0 \in (0, 2D_\kappa]$ such that every Jordan triangle of perimeter strictly less than r_0 satisfies the CAT(κ) comparison. Then every closed r-ball with $r \leq \frac{1}{4}r_0$ is convex and CAT(κ). Moreover, if $r_0 = 2D_\kappa$, then X is a CAT(κ) space.*

3 Sobolev theory

3.1 Basics

We recall basic definitions of Sobolev maps with values in a metric space and refer to [Ambrosio 1990; Heinonen et al. 2015; Jost 1994; Korevaar and Schoen 1993; Lytchak and Wenger 2017a; 2018a; Reshetnyak 2007] for further information. Let Ω be a bounded domain in $\mathbb{R}^n$ and (X, d) a complete metric space. For $p > 1$ let $L^p(\Omega, X)$ be the set of measurable and essentially separably valued maps $u\colon \Omega \to X$ such that for some and thus every $x \in X$ the function $u_x(z) := d(x, u(z))$ belongs to the classical space $L^p(\Omega)$ of p-integrable functions on Ω.

Definition 3.1 A map $u \in L^p(\Omega, X)$ belongs to the Sobolev space $W^{1,p}(\Omega, X)$ if there exists $h \in L^p(\Omega)$ such that u_x is in the classical Sobolev space $W^{1,p}(\Omega)$ for every $x \in X$ and its weak gradient satisfies $|\nabla u_x| \leq h$ almost everywhere. The localized spaces $W^{1,p}_{\mathrm{loc}}(\Omega)$ are defined similarly.

There are several natural notions of energy of a map $u \in W^{1,p}(\Omega, X)$. Throughout this paper we will use the *Reshetnyak p-energy* defined by

$$E_+^p(u) := \inf\{\|h\|^p_{L^p(\Omega)} \mid h \text{ as in the definition above}\}.$$

Recall that a family $\mathscr{C}$ of curves in Ω is called *p-exceptional*, if there exists a p-integrable Borel function $\sigma\colon \Omega \to [0, \infty]$ such that for every locally rectifiable curve $\gamma \in \mathscr{C}$ the path integral satisfies

$$\int_\gamma \sigma \, ds = \infty.$$

We say that a property holds for *p-a.e. curve* in Ω if the family of curves on which the property fails is p-exceptional. For instance, if a property holds for p-a.e. curve in Ω and $F\colon [0,1]^{n-1} \times [0,1] \to \Omega$ is a bilipschitz embedding, then for almost all $x \in [0,1]^{n-1}$ the property holds true for the curve $\gamma_x(t) = F(x, t)$.

A map $u \in L^p(\Omega, X)$ is contained in the Sobolev space $W^{1,p}(\Omega, X)$ if and only if there exist a Lebesgue representative $\bar{u}$ of u and a Borel function $\rho \in L^p(\Omega)$ such that for *p-a.e. curve* $\gamma\colon [0,1] \to \Omega$ the composition $\bar{u} \circ \gamma$ is continuous and

$$\ell_X(\bar{u} \circ \gamma) \leq \int_\gamma \rho \, ds. \tag{3-1}$$

In what follows, we will always choose such a representative $\bar{u}$ of u and will simply denote it by u.

There exists a minimal function $\rho = \rho_u$ satisfying the condition above, uniquely defined up to sets of measure zero. It will be called the *generalized gradient* or *minimal weak upper gradient* of u. By [Lytchak and Wenger 2017a; Reshetnyak 2007], the p^{th} power of the L^p-norm of ρ_u coincides with the Reshetnyak p-energy defined above,

$$E_+^p(u) = \|\rho_u\|_{L^p(\Omega)}^p = \int_\Omega \rho_u^p(z)\,dz.$$

If Ω is a Lipschitz domain, then every $u \in W^{1,p}(\Omega, X)$ has a canonically defined *trace* $\operatorname{tr}(u) \in L^p(\partial\Omega, X)$; see [Korevaar and Schoen 1993]. For instance, if Ω is the open unit ball in $\mathbb{R}^n$, then for almost every $z \in S^{n-1}$ the map $t \mapsto u(tz)$ is in $W^{1,p}\big((\tfrac{1}{2}, 1), X\big)$ and

$$\operatorname{tr}(u)(z) = \lim_{t \to 1} u(tz).$$

For a general Lipschitz domain Ω, if u has a continuous extension $\hat{u}$ to $\overline{\Omega}$ then $\operatorname{tr}(u)$ is just the restriction of $\hat{u}$ to $\partial\Omega$.

If a Lipschitz domain Ω is a union of two disjoint Lipschitz subdomains $\Omega^\pm$ and the Lipschitz boundary $T = \partial\Omega^- \cap \partial\Omega^+$, and $u^\pm \in W^{1,p}(\Omega^\pm, X)$ have the same trace on T, then one obtains a map $u \in W^{1,p}(\Omega, X)$ by gluing $u^\pm$ along T; see [Korevaar and Schoen 1993, Theorem 1.12.3].

3.2 Length, energy, and area

Every map $u \in W^{1,p}(\Omega, X)$ has an *approximate metric derivative* at almost every point $z \in \Omega$ in the following sense; see [Karmanova 2007] and [Lytchak and Wenger 2017a]. There exists a unique seminorm on $\mathbb{R}^n$, denoted by $\operatorname{ap}\operatorname{md} u_z$, such that

$$\operatorname{ap}\lim_{z' \to z} \frac{d(u(z'), u(z)) - \operatorname{ap}\operatorname{md} u_z(z' - z)}{|z' - z|} = 0,$$

where $\operatorname{ap}\lim$ denotes the approximate limit; see [Evans and Gariepy 2015]. If u is Lipschitz, then the approximate limit can be replaced by an honest limit. The map $z \mapsto \operatorname{ap}\operatorname{md} u_z$ into the space of seminorms has a Borel measurable representative [Lytchak and Wenger 2017a]. For p-a.e. absolutely continuous curve $\gamma\colon I \to \Omega$, we have

$$\ell_X(u \circ \gamma) = \int_I \operatorname{ap}\operatorname{md} u_{\gamma(t)}(\gamma'(t))\,dt. \tag{3-2}$$

Moreover, for almost every $z \in \Omega$ we have $\rho_u(z) = \sup_{v \in S^{n-1}} \operatorname{ap}\operatorname{md} u_z(v)$. It follows from [Lytchak and Wenger 2017a] that

$$E_+^p(u) = \int_\Omega \mathcal{I}_+^p(\operatorname{ap}\operatorname{md} u_z)\,dz,$$

where for a seminorm s on $\mathbb{R}^n$ we have set

$$\mathcal{I}_+^p(s) := \max\{s(v)^p \mid |v| = 1\}.$$

Later on, we will make use of the following lemma which is a slight variant of [Heinonen et al. 2015, Theorem 7.3.9].

Lemma 3.2 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,p}(\Omega, X)$ which converges in $L^p(\Omega, X)$ to a function u. Suppose that the corresponding sequence $(\rho_{u_n})_{n\in\mathbb{N}}$ of minimal weak upper gradients is uniformly bounded in $L^p(\Omega)$ and converges weakly in $L^p(\Omega)$ to a function ρ. Then $u \in W^{1,p}(\Omega, X)$ and ρ is a weak upper gradient of u. In particular,*

$$\|\rho_u\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|\rho_{u_n}\|_{L^p(\Omega)}.$$

Proof Let us isometrically embed X into a Banach space V. By Mazur's lemma [Heinonen et al. 2015, Section 2.3], we can form sequences $(\tilde{u}_l)_{l\in\mathbb{N}}$ and $(\tilde{\rho}_l)_{l\in\mathbb{N}}$ of convex combinations of $(u_n)_{n\in\mathbb{N}}$ and $(\rho_{u_n})_{n\in\mathbb{N}}$, respectively, such that $\tilde{u}_l \to u$ in $L^p(\Omega, V)$, $\tilde{\rho}_l \to \rho$ in $L^p(\Omega)$, and for every $l \in \mathbb{N}$ and p-a.e. rectifiable path γ in Ω,

$$\ell_V(\tilde{u}_l \circ \gamma) \leq \int_\gamma \tilde{\rho}_l \, ds.$$

By [Heinonen et al. 2015, Proposition 7.3.7], $u \in W^{1,p}(\Omega, V)$ with ρ as a weak upper gradient. Since a subsequence of $(u_n)_{n\in\mathbb{N}}$ converges pointwise almost everywhere to u, we obtain $u \in W^{1,p}(\Omega, X)$. The last statement follows from the semicontinuity of the norm with respect to weak convergence since $\|\rho_u\|_{L^p(\Omega)} \leq \|\rho\|_{L^p(\Omega)}$. □

We will mostly be interested in Sobolev discs and, more generally, Sobolev images of planar domains. So let us for the rest of this section specialize to the case where Ω is a domain in $\mathbb{R}^2$.

Definition 3.3 The (parametrized Hausdorff) area of a map $u \in W^{1,2}(\Omega, X)$ is defined by

$$\operatorname{Area}(u) := \int_\Omega \operatorname{Jac}(\operatorname{ap}\operatorname{md} u_z) \, dz,$$

where the Jacobian $\operatorname{Jac}(s)$ of a seminorm s on $\mathbb{R}^2$ is the Hausdorff 2-measure in $(\mathbb{R}^2, s)$ of the Euclidean unit square if s is a norm and $\operatorname{Jac}(s) = 0$ otherwise.

We will furthermore need a somewhat different definition of parametrized area, also known as the *inscribed Riemannian area* defined as follows. The μ^i-Jacobian $\operatorname{Jac}_{\mu^i}(s)$ of a norm s on $\mathbb{R}^2$ is given by

$$\operatorname{Jac}_{\mu^i}(s) := \frac{\pi}{|L|},$$

where $|L|$ is the Lebesgue measure of the ellipse of maximal area contained in $\{v \in \mathbb{R}^2 \mid s(v) \leq 1\}$. If s is a degenerate seminorm then we set $\operatorname{Jac}_{\mu^i}(s) = 0$. The inscribed Riemannian area of a map $u \in W^{1,2}(\Omega, X)$ is defined by

$$\operatorname{Area}_{\mu^i}(u) := \int_\Omega \operatorname{Jac}_{\mu^i}(\operatorname{ap}\operatorname{md} u_z) \, dz.$$

The two notions of area are related by

$$\frac{\pi}{4} \cdot \operatorname{Area}_{\mu^i}(u) \leq \operatorname{Area}(u) \leq \operatorname{Area}_{\mu^i}(u)$$

for every $u \in W^{1,2}(\Omega, X)$; see [Lytchak and Wenger 2017b, Section 2.4]. The following semicontinuity properties of energy and area are essential.

Proposition 3.4 [Reshetnyak 2007, Theorem 4.2; Lytchak and Wenger 2017a, Corollary 5.8] *For $p \geq 2$, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,p}(\Omega, X)$ which converges in $L^p(\Omega, X)$ to $u \in W^{1,p}(\Omega, X)$. Then*

$$E_+^p(u) \leq \liminf_{n\to\infty} E_+^p(u_n) \quad \text{and} \quad \operatorname{Area}(u) \leq \liminf_{n\to\infty} \operatorname{Area}(u_n).$$

The second inequality moreover holds with the Hausdorff area replaced by the inscribed Riemannian area.

The following definition is of central importance.

Definition 3.5 A metric space X supports a (C, l_0)*-isoperimetric inequality*, if for every Lipschitz curve $\gamma\colon S^1 \to X$ of length $l < l_0$ there exists a Sobolev disc $u \in W^{1,2}(D, X)$ with $\operatorname{tr}(u) = \gamma$ and

$$\operatorname{Area}(u) \leq C \cdot l^2.$$

Note that changing the definition above from Hausdorff to Riemannian inscribed area will only change the constant C.

Isoperimetric inequalities as above have an important effect on the regularity of minimal discs [Lytchak and Wenger 2017a]. Moreover, such inequalities allow for flexible reparametrizations. To explain this, recall from [Lytchak and Wenger 2017a] the following terminology. Let $T \subset \mathbb{R}^2$ be a subset which is bilipschitz to an open interval I. A map $w\colon T \to X$ belongs to $W^{1,2}(T, X)$ if $w \circ \varphi \in W^{1,2}(I, X)$ for some (and thus any) bilipschitz map $\varphi\colon I \to T$. Such Sobolev curves arise for instance as follows. If $u \in W^{1,2}(D, X)$ and $F\colon (0,1)^2 \to D$ is a bilipschitz embedding, then for almost every $s \in (0,1)$ the curve $u|_{T_s}$ lies in $W^{1,2}(T_s, X)$ where $T_s = F(\{s\} \times (0,1))$. The definition of $W^{1,2}(T, X)$ naturally extends to the case where T is bilipschitz to S^1. Even more generally, if $T \subset \mathbb{R}^2$ is a finite union of bilipschitz curves T_i as above then a map $w\colon T \to X$ is in $W^{1,2}(T, X)$ if w has a continuous representative and $w|_{T_i} \in W^{1,2}(T_i, X)$ for every i.

Now suppose X supports a (C, l_0)-isoperimetric inequality and $J \subset D$ is a bilipschitz Jordan curve enclosing a Jordan domain Ω. Then for every $\gamma \in W^{1,2}(J, X)$ of length $l < l_0$ there exists $u \in W^{1,2}(\Omega, X)$ with $\operatorname{Area}(u) \leq C \cdot l^2$ and $\operatorname{tr}(u) = \gamma$ [Lytchak and Wenger 2018a, Lemma 4.6].

3.3 Conformal and almost conformal Sobolev discs

Let $\Omega \subset \mathbb{R}^2$ be a domain. A map $u \in W^{1,2}(\Omega, X)$ is called *quasiconformal* if there exists a constant $Q \geq 1$ such that at almost all $z \in \Omega$ we have

$$\operatorname{ap}\operatorname{md} u_z(v) \leq Q \cdot \operatorname{ap}\operatorname{md} u_z(w)$$

for all $v, w \in S^1$. In this case we say that u is Q*-quasiconformal*, and if Q can be chosen equal to 1, we call u *conformal*. In this case, $\operatorname{ap}\operatorname{md} u_z$ is a multiple $f(z) \cdot s_0$ of the standard Euclidean norm s_0 on $\mathbb{R}^2$. The function $f \in L^2(\Omega)$ will be called the *conformal factor* of u. The conformal factor f of a conformal map $u \in W^{1,2}(\Omega, X)$ coincides with the generalized gradient ρ_u.

For every Sobolev disc $u \in W^{1,2}(D, X)$,

$$\operatorname{Area}_{\mu^i}(u) \leq E_+^2(u).$$

Moreover, if equality holds then u is $\sqrt{2}$-quasiconformal [Lytchak and Wenger 2017b, Corollary 3.3].

Recall from [Lytchak and Wenger 2017a] that a complete metric space X is said to have *property* (*ET*) if for every Sobolev disc in X the approximate metric derivative comes from a possibly degenerate inner product at almost every point. If X is a complete metric space with property (ET), then parametrized area and inscribed Riemannian area of a Sobolev disc $u \in W^{1,2}(D, X)$ coincide,

$$\operatorname{Area}_{\mu^i}(u) = \operatorname{Area}(u).$$

Moreover, the equality $\operatorname{Area}(u) = E_+^2(u)$ implies that u is conformal.

The following version of Morrey's ϵ-conformality lemma allows to find good parametrizations of Sobolev discs.

Theorem 3.6 [Fitzi and Wenger 2020, Theorem 1.4] *Let X be a complete metric space and let $u \in W^{1,2}(D, X)$. Then for every $\epsilon > 0$ there exists a diffeomorphism $\varphi\colon D \to D$ such that*

$$E_+^2(u \circ \varphi) \leq \operatorname{Area}_{\mu^i}(u) + \epsilon.$$

Moreover, there is such a map φ which extends to a diffeomorphism of $\overline{D}$ and is conformal in a neighborhood of the boundary.

Note that the use of inscribed Riemannian area is essential here.

3.4 Regularity of quasiconformal Sobolev discs

The following interior regularity result is a consequence of [Lytchak and Wenger 2017a, Propositions 8.4 and 8.7].

We say that a property holds for almost every bilipschitz Jordan curve in D, if whenever $\varphi\colon S^1 \times [0,1] \to D$ is a bilipschitz annulus, then the property holds true for almost every circle $\varphi(S^1, t)$.

Theorem 3.7 *Let Z be a complete metric space and $u \in W^{1,2}(D, Z)$ a Q-quasiconformal Sobolev disc. Suppose that there exist $C, l_0 > 0$ such that for almost every bilipschitz Jordan curve γ in D with $\ell(u|_\gamma) < l_0$ we have*

$$\operatorname{Area}(u|_{\Omega_\gamma}) \leq C \cdot \ell^2(u|_\gamma), \tag{3-3}$$

where Ω_γ is the Jordan domain enclosed by γ. Then the following statements hold:

(1) *There exists $p > 2$ such that $u \in W^{1,p}_{\mathrm{loc}}(D, Z)$. In particular, u has a continuous representative $\bar{u}$ which moreover satisfies Lusin's property* (*N*).

(2) *The representative $\bar{u}$ is locally Hölder continuous. In fact, for every $\delta \in (0,1)$ there exists $L > 0$ such that for all $z_1, z_2 \in \overline{B}_\delta(0)$ there exists a path γ in $\overline{B}_\delta(0)$ such that*

$$\ell_X(\bar{u} \circ \gamma) \leq L \cdot |z_1 - z_2|^\alpha,$$

where $\alpha = 1/(4\pi Q^2 C)$.

(3) *If $\mathrm{tr}(u)$ has a continuous representative then $\bar{u}$ continuously extends to $\overline{D}$.*

The proof can be assembled from arguments in [Lytchak and Wenger 2017a]. It basically follows from [Lytchak and Wenger 2017a, Theorems 8.2 and 9.1, Proposition 8.7]. These results assume the map u to be minimal and the space X to support a (C, l_0)-isoperimetric inequality. While this differs from our setting, these additional assumptions are only used in [Lytchak and Wenger 2017a] to prove [Lytchak and Wenger 2017a, Lemma 8.6] which ensures that inequality (3-3) holds.

For the convenience of the reader let us give a more detailed account on how to obtain Theorem 3.7 from the results in [Lytchak and Wenger 2017a]. We start by proving (1) which corresponds to [Lytchak and Wenger 2017a, Proposition 8.4], namely higher integrability and therefore Hölder continuity of u. Next we prove (2) which corresponds to [Lytchak and Wenger 2017a, Proposition 8.7], the intrinsic Hölder regularity. The proofs in [Lytchak and Wenger 2017a] apply because they only use the quasiconformality of u and inequality (3-3) for balls entirely contained in D. We are left with (3) which corresponds to [Lytchak and Wenger 2017a, Theorem 9.1]. To show that u extends continuously to a point $z \in S^1$, the proof in [Lytchak and Wenger 2017a] considers a conformal diffeomorphism $\varphi\colon D \to \Omega$ where $\Omega = D \cap B_r(z)$ for some $z \in S^1$ and a suitable $r \in (0,1)$. To make the argument from [Lytchak and Wenger 2017a] work, we need that

$$\mathrm{Area}(u \circ \varphi|_{B_s(x)}) \leq C \cdot \ell^2(u \circ \varphi|_{\partial B_s(x)})$$

holds for all $x \in D$ and almost all $s < 1 - |x|$. However, this is guaranteed since φ is bilipschitz on every compact subset of D and inequality (3-3) holds for almost every bilipschitz Jordan curve by assumption.

4 Dehn functions, thickenings, and minimal discs

4.1 Filling area and Dehn functions

Recall that for a Jordan curve Γ in a complete metric space X we denote by $\Lambda(\Gamma, X)$ the family of Sobolev discs $u \in W^{1,2}(D, X)$ whose traces have representatives which are weakly monotone parametrizations of Γ. We set

$$\mathrm{Fill}_X(\Gamma) := \inf\{\mathrm{Area}(u) \mid u \in \Lambda(\Gamma, X)\}.$$

Similarly, if $c\colon S^1 \to X$ is a curve, we set

$$\mathrm{Fill}_X(c) := \inf\{\mathrm{Area}(u) \mid u \in W^{1,2}(D, X),\ \mathrm{tr}(u) = c\}.$$

Notice that if c is a weakly monotone parametrization of a Jordan curve Γ, then, by definition,

$$\mathrm{Fill}_X(\Gamma) \le \mathrm{Fill}_X(c).$$

The *Dehn function* of X is given by

$$\delta_X(r) := \sup\{\mathrm{Fill}_X(c) \mid c\colon S^1 \to X \text{ Lipschitz}, \ \ell(c) \le r\}$$

for all $r > 0$. Similarly, we define the Riemannian versions

$$\mathrm{Fill}_X^{\mu^i}(\Gamma), \quad \mathrm{Fill}_X^{\mu^i}(c), \quad \delta_X^{\mu^i}(r)$$

by replacing the Hausdorff area of Sobolev discs by the inscribed Riemannian area. Clearly, Dehn functions are nondecreasing in r. Notice that $\delta_X(r) \le \delta_X^{\mu^i}(r)$, and equality holds for example when X has property (ET).

As mentioned in the introduction, we denote by δ_κ the Dehn function of the model surface M_κ^2. Explicitly, for $r \in (0, 2D_\kappa)$ we have

$$\delta_\kappa(r) = \begin{cases} 2\pi/\kappa - \sqrt{(2\pi/\kappa)^2 - r^2/\kappa} & \text{if } \kappa > 0, \\ r^2/4\pi & \text{if } \kappa = 0, \\ \sqrt{(2\pi/\kappa)^2 + r^2/\kappa} - 2\pi/\kappa & \text{if } \kappa < 0, \end{cases}$$

as follows for instance from the isoperimetric inequality in M_κ^2; see [Osserman 1978]. Note that for $\kappa > 0$ we have the quadratic bound $\delta_\kappa(r) \le r^2/(2\pi\kappa)$ for $r \in (0, 2D_\kappa)$.

4.2 Universal thickenings

Let X and Y be metric spaces and $\epsilon > 0$. We say that Y is an *ϵ-thickening* of X if there exists an isometric embedding $\iota\colon X \to Y$ such that the Hausdorff distance between $\iota(X)$ and Y is at most ϵ. The embedding ι is then a $(1,\epsilon)$-quasiisometry. We will make use of the following which can be proved in the same way as [Lytchak et al. 2020, Proposition 3.5]. Recall that a metric space X is said to be *L-Lipschitz 1-connected up to scale* λ_0 for some $L \ge 1$ and $\lambda_0 > 0$ if every λ-Lipschitz curve $c\colon S^1 \to X$ with $\lambda < \lambda_0$ extends to an $L\lambda$-Lipschitz map on $\overline{D}$. If the scale λ_0 is not important, we simply say X is L-Lipschitz 1-connected up to some scale. We clearly have $\delta_X(r) \le \delta_X^{\mathrm{Lip}}(r)$ for all $r > 0$, where δ_X^{Lip} denotes the Lipschitz Dehn function defined by filling Lipschitz curves by Lipschitz discs. However, equality holds for all complete length spaces which are Lipschitz 1-connected up to some scale [Lytchak et al. 2020, Proposition 3.1]. The same applies to the μ^i-versions of the Sobolev and Lipschitz Dehn functions.

Proposition 4.1 *There exists $L \ge 1$ with the following property. Let X be a complete length space and let $\epsilon, r_0 > 0$. Suppose $\delta\colon (0, r_0) \to \mathbb{R}$ is continuous and*

$$\delta_X^{\mu^i}(r) \le \delta(r) + \epsilon^2$$

for all $r \in (0, r_0)$. Then there exists a complete length space Y which is an ϵ-thickening of X and L-Lipschitz 1-connected up to scale ϵ/L and satisfies

$$\delta_Y^{\mu^i}(r) \le \delta(r) + Lr^2$$

for all $r \in (0, r_0)$ *and if* $r \in (\sqrt{\epsilon}, r_0)$ *then*

$$\delta_Y^{\mu^i}(r) \leq \delta(r) + \sqrt{\epsilon} r^2.$$

Remark 4.1 If $\delta(r) = O(r^2)$ then there exist $C \geq 1$ and $r_1 \in (0, r_0)$ depending only on L and the function δ such that Y has a (C, r_1)-isoperimetric inequality.

Definition 4.2 For a length space X we call an ϵ-thickening as in Proposition 4.1 a *universal* ϵ*-thickening*.

4.3 Plateau problem and intrinsic minimal discs

Definition 4.3 Let X be a complete metric space and $\Gamma \subset X$ a Jordan curve. We call $u \in \Lambda(\Gamma, X)$ a *solution to the Plateau problem* for the curve Γ, if $\operatorname{Area}(u) = \operatorname{Fill}(\Gamma)$ and u has minimal energy among all area minimizers in $\Lambda(\Gamma, X)$. A solution to the Plateau problem will sometimes simply be called a *minimal disc*.

In this section we consider the following setting. Let X be a complete length space which satisfies property (ET). Let $\Gamma \subset X$ be a rectifiable Jordan curve and $u \in \Lambda(\Gamma, X)$ a solution to the Plateau problem. In particular, u is conformal [Lytchak and Wenger 2017a, Theorem 11.3]. We assume that u satisfies inequality (3-3) and therefore, by Theorem 3.7, has an (intrinsically) locally Hölder continuous representative which continuously extends to $\overline{D}$. Denote this representative still by u. Then [Lytchak and Wenger 2018a, Theorem 1.1] yields the following structure for the intrinsic minimal disc Z_u; see Section 2.2. The setting in [Lytchak and Wenger 2018a] asks X to support a (C, l_0)-isoperimetric inequality. However, the proof of [Lytchak and Wenger 2018a, Theorem 1.1] only uses the quasiconformality of u and inequality (3-3).

Theorem 4.4 *The intrinsic minimal disc* Z_u *is a compact geodesic space. The canonical projection* $P: \overline{D} \to Z_u$ *is continuous. The map* $u: \overline{D} \to X$ *has a canonical factorization* $u = \bar{u} \circ P$, *where* $\bar{u}: Z_u \to X$ *is* 1*-Lipschitz. For any curve* γ *in* $\overline{D}$ *the lengths of* $P \circ \gamma$ *and* $u \circ \gamma$ *coincide*; *thus* $\bar{u}$ *preserves the length of* $P \circ \gamma$.

Now suppose $\delta: (0, r_0) \to \mathbb{R}$ is a continuous nondecreasing function and there exists $0 < r_1 < r_0$ such that $\ell(\Gamma) < r_1$ and for every Jordan domain $\Omega \subset D$ with $\ell(u|_{\partial\Omega}) < r_1$,

$$\operatorname{Area}(u|_{\Omega}) \leq \delta(\ell(u|_{\partial\Omega})). \tag{4-1}$$

In this setting, the arguments in [Lytchak and Wenger 2018a] provide strong topological and isoperimetric properties of the intrinsic minimal disc:

Theorem 4.5 *The map* $P: \overline{D} \to Z_u$ *is a uniform limit of homeomorphisms. For every Jordan domain* $\Omega \subset Z_u$ *with* $\ell(\partial\Omega) < r_0$,

$$\mathcal{H}^2(\Omega) \leq \delta(\ell(\partial\Omega)). \tag{4-2}$$

The proof can be assembled from [Lytchak and Wenger 2018a]. By [Lytchak and Wenger 2018a, Lemmas 6.3 and 6.4, Corollary 4.5, Theorem 8.1] the natural projection satisfies the first statement. This part relies on Moore's recognition theorem for 2-manifolds [Lytchak and Wenger 2018a, Theorem 7.11]. The second statement follows from [Lytchak and Wenger 2018a, Theorem 8.2], and this is where the continuity of δ is needed. We emphasize that unlike in inequality (4-1), Theorem 4.5 does not require that the length of $\partial\Omega$ be bounded by r_1. Indeed, [Lytchak and Wenger 2018a, Theorem 1.1] shows that $\ell(\partial Z_u) = \ell(\Gamma) < r_1$ and therefore $\mathcal{H}^2(Z_u) \leq \delta(r_1)$. In particular, even if $\Omega \subset Z_u$ is a Jordan domain with $r_1 < \ell(\partial\Omega) < r_0$, we still have $\mathcal{H}^2(\Omega) \leq \delta(r_1) \leq \delta(\ell(\partial\Omega))$ by the monotonicity of δ.

5 Euclidean tangent planes

Recall once again that a complete metric space X is said to have property (ET) if for every $u \in W^{1,2}(D, X)$ the approximate metric derivative $\operatorname{ap}\operatorname{md} u_z$ comes from a possibly degenerate inner product at almost every $z \in D$.

The proof of the following proposition is very similar to that of [Wenger 2019, Theorem 3.1], which was originally inspired by [Wenger 2008, Theorem 5.1]. The statement generalizes [Lytchak and Wenger 2018b, Theorem 5.2] and [Wenger 2019, Theorem 3.1].

Proposition 5.1 *Let $r_0 > 0$ and let $\delta\colon (0, r_0) \to \mathbb{R}$ be a continuous nondecreasing function satisfying*

$$\limsup_{r\to 0} \frac{\delta(r)}{r^2} \leq \frac{1}{4\pi}.$$

Let (X_n) be a sequence of complete length spaces satisfying

$$\delta_{X_n}(r) \leq (1+\epsilon_n)\cdot\delta(r) + \epsilon_n$$

for all $r \in (0, r_0)$, where $\epsilon_n > 0$ tends to zero as $n \to \infty$. Then every ultralimit of (X_n) has property (ET).

We provide the proof for the convenience of the reader.

Proof Let $X_\omega = (X_\omega, d_\omega)$ be an ultralimit of the sequence (X_n) and suppose, by contradiction, that X_ω does not have property (ET). By [Wenger 2019, Lemma 3.2] there exists a non-Euclidean norm $\|\cdot\|$ on $\mathbb{R}^2$ with the following property. For every finite set $\{v^1, \dots, v^m\} \subset \mathbb{R}^2$ and every $\lambda > 1$ there exist points $x^1, \dots, x^m \in X_\omega$ and $\eta > 0$ arbitrarily small such that

$$\lambda^{-1}\eta\|v^k - v^l\| \leq d_\omega(x^k, x^l) \leq \lambda\eta\|v^k - v^l\| \tag{5-1}$$

for all $k, l = 1, \dots, m$. We denote by V the normed space $(\mathbb{R}^2, \|\cdot\|)$ and let $\mathbb{I}_V \subset V$ be an isoperimetric subset for V, that is, $\mathbb{I}_V$ is convex and has largest area among all convex subsets of V with prescribed boundary length. Since V is non-Euclidean we have

$$\mathcal{H}^2_V(\mathbb{I}_V) > \frac{1}{4\pi}\cdot\ell^2_V(\partial\mathbb{I}_V); \tag{5-2}$$

see for example [Lytchak and Wenger 2018b, Lemma 5.1].

Let $\gamma\colon S^1 \to V$ be a constant speed parametrization of $\partial\mathbb{I}_V$. Choose $\lambda > 1$ sufficiently close to 1 and $m \in \mathbb{N}$ sufficiently large, both to be determined later. For $k = 1, \dots, m$ set $z_k = e^{2\pi ik/m} \in S^1$ and let

$v^k := \gamma(z_k)$. By the above, there exist $x^1, \dots, x^m \in X_\omega$ and $\eta > 0$ arbitrarily small such that (5-1) holds. If $\eta > 0$ is small enough then, after replacing the norm $\|\cdot\|$ by the rescaled norm $\eta \cdot \|\cdot\|$, we may assume that $\eta = 1$, that $r_1 := \lambda^4 \ell_V(\partial \mathbb{I}_V)$ satisfies $r_1 < r_0$ and $\delta(r) \le \frac{\lambda}{4\pi} \cdot r^2$ for all $0 < r \le r_1$.

For $k = 1, \dots, m$ write x^k as $x^k = [(x_n^k)]$ with $x_n^k \in X_n$. There exists $N \subset \mathbb{N}$ with $\omega(N) = 1$ and such that

$$\lambda^{-1} d_\omega(x^k, x^l) \le d_n(x_n^k, x_n^l) \le \lambda \cdot d_\omega(x^k, x^l)$$

for all $k, l = 1, \dots, m$ and all $n \in N$. Fix $n \in N$ large enough, to be determined later, and let $c\colon S^1 \to X_n$ be a Lipschitz curve such that $c(z_k) = x_n^k$ and such that c is a $(\lambda-1)$-geodesic on the segment of S^1 between z_k and z_{k+1} for each k. It follows from the above that

$$\ell(c) \le \lambda^3 \cdot \ell_V(\partial \mathbb{I}_V).$$

Let $u \in W^{1,2}(D, X_n)$ be such that $\operatorname{tr}(u) = c$ and

$$\operatorname{Area}(u) \le (1 + \epsilon_n) \cdot \delta(\ell(c)) + 2\epsilon_n \le \frac{1 + \epsilon_n}{4\pi} \cdot \lambda^7 \ell_V^2(\partial \mathbb{I}_V) + 2\epsilon_n.$$

View V as a linear subspace of the space ℓ^∞ of bounded sequences, equipped with the supremum norm. Since ℓ^∞ is an injective metric space there exists a λ^2-Lipschitz map $\varphi\colon X_n \to \ell^\infty$ which maps x_n^k to v^k for all k. It follows that the map $\varphi \circ u$ is Sobolev with $\operatorname{Area}(\varphi \circ u) \le \lambda^4 \operatorname{Area}(u)$. Hence, by [Lytchak et al. 2020, Proposition 3.1], there exists a Lipschitz map $v\colon \overline{D} \to \ell^\infty$ with $v|_{S^1} = \varphi \circ c$ and

$$\operatorname{Area}(v) \le \operatorname{Area}(\varphi \circ u) + \epsilon_n \le \frac{1 + \epsilon_n}{4\pi} \cdot \lambda^{11} \cdot \ell_V^2(\partial \mathbb{I}_V) + 2\lambda^4 \epsilon_n + \epsilon_n.$$

Finally, one constructs exactly as in the proof of [Wenger 2019, Theorem 3.1] a Lipschitz homotopy $\varrho\colon S^1 \times [0,1] \to \ell^\infty$ between $\varphi \circ c$ and γ with

$$\operatorname{Area}(\varrho) \le \frac{C(1 + \lambda^3)^2}{m} \cdot \ell_V^2(\partial \mathbb{I}_V),$$

where C is a universal constant. Gluing ϱ and v we obtain a Lipschitz map $w\colon \overline{D} \to \ell^\infty$ with $w|_{S^1} = \gamma$ and such that

$$\operatorname{Area}(w) \le \left(\frac{1 + \epsilon_n}{4\pi} \cdot \lambda^{11} + \frac{C(1 + \lambda^3)^2}{m} \right) \cdot \ell_V^2(\partial \mathbb{I}_V) + 2\lambda^4 \epsilon_n + \epsilon_n. \tag{5-3}$$

Since $\mathcal{H}_V^2(\mathbb{I}_V) \le \operatorname{Area}(w)$ by the quasiconvexity of the Hausdorff 2-measure — see [Burago and Ivanov 2012] — inequality (5-3) clearly contradicts inequality (5-2) for $\lambda > 1$ sufficiently close to 1, and m and n sufficiently large. □

6 Filling area in ultralimits

Let $C, r_1 > 0$ and for $n \in \mathbb{N}$ let X_n be a complete length space which is C-Lipschitz 1-connected up to some scale and admits a (C, r_1)-isoperimetric inequality. Suppose furthermore that

$$\delta_{X_n}^{\mu^i}(r) \le (1 + \epsilon_n) \cdot \delta(r) + \epsilon_n r^2$$

for all $r \in (\epsilon_n, r_0)$, where $\delta\colon (0, r_0) \to \mathbb{R}$ is continuous and nondecreasing with

$$\limsup_{r \to 0} \frac{\delta(r)}{r^2} \le \frac{1}{4\pi}$$

and $\epsilon_n \to 0$. Let X_ω be the ultralimit of (X_n) with respect to some basepoints and a nonprincipal ultrafilter ω. The following proposition plays a key role in the proofs of our main theorems.

Proposition 6.1 *Let $\Gamma_\omega \subset X_\omega$ be a rectifiable Jordan curve which is the ultralimit of a sequence of rectifiable Jordan curves $\Gamma_n \subset X_n$ such that $\limsup_{n\in\mathbb{N}} \ell(\Gamma_n) < 2\ell(\Gamma_\omega)$. Then*

$$\omega\text{-lim}\,\mathrm{Fill}^{\mu^i}_{X_n}(\Gamma_n) \le \mathrm{Fill}_{X_\omega}(\Gamma_\omega).$$

Recall that we always have $\mathrm{Fill}_{X_n}(\Gamma_n) \le \mathrm{Fill}^{\mu^i}_{X_n}(\Gamma_n)$ and thus

$$\omega\text{-lim}\,\mathrm{Fill}_{X_n}(\Gamma_n) \le \mathrm{Fill}_{X_\omega}(\Gamma_\omega)$$

holds as well. Note that this inequality may be strict, whereas

$$\mathrm{Fill}_{X_\omega}(\Gamma_\omega) = \mathrm{Fill}^{\mu^i}_{X_\omega}(\Gamma_\omega)$$

holds by Proposition 5.1. A similar remark applies to the next proposition.

Before we turn to the proof, we begin with a version for parametrized curves.

Proposition 6.2 *Let (c_n) be a bounded sequence of Lipschitz curves $c_n\colon S^1 \to X_n$ with uniformly bounded Lipschitz constants. Let $c = \omega\text{-lim}\, c_n$ be the ultralimit of this sequence. Then*

$$\omega\text{-lim}\,\mathrm{Fill}^{\mu^i}_{X_n}(c_n) \le \mathrm{Fill}_{X_\omega}(c).$$

This is a variant of [Wenger 2019, Theorem 5.1] and the proof therein applies with minor modifications.

Note that there is no obvious way how to deduce Proposition 6.1 from Proposition 6.2. In the presence of a (local) quadratic isoperimetric inequality one can relate the filling area of any parametrization of a Jordan curve to the filling area of its arclength parametrization [Lytchak and Wenger 2018a, Lemma 4.8]. However, we do not know whether X_ω admits a (local) quadratic isoperimetric inequality. In particular, there may exist a non-Lipschitz parametrization of Γ_ω whose filling area is much smaller than the filling area of any Lipschitz parametrization. To overcome this difficulty we need some preparation. In particular, the following notion of framed collar will be useful; see Figure 1 for an illustration.

Definition 6.3 A *framed collar* $U \subset \overline{D}$ is a finite union of closed balls $B_i = \overline{B}_{r_i}(p_i) \cap \overline{D}$, $i = 1, \dots, m$, centered at points $p_i \in S^1$ such that the open balls cover S^1 and nonconsecutive balls are disjoint. In particular, U is a topological annulus whose boundary circles are given by $\partial^- U = S^1$ and a bilipschitz

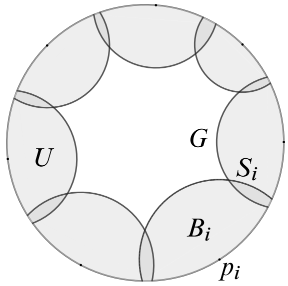

Figure 1: A framed collar U with frame G.

Jordan curve $\partial^+ U$. The *frame* G of U is the finite graph given by

$$G = S^1 \cup \bigcup_{i=1}^{m} S_i,$$

where $S_i = D \cap \partial B_i$. The set $U \setminus G$ is a disjoint union of *complementary* (*open*) *discs* Ω_j for $j = 1, \dots, 2m$.

Let us make two simple comments which will be used implicitly later on. First, if a collar U is contained in the ϵ-tubular neighborhood of S^1 for some $\epsilon > 0$, then every complementary disc Ω_j has diameter at most 2ϵ. Secondly, if U has $2m$ complementary discs, then for every small enough $t \in (0, 1)$ the intersection $\partial B_{1-t}(0) \cap G$ consists of $2m$ points.

We will use framed collars in combination with the isoperimetric inequality to produce Sobolev homotopies of small area.

Lemma 6.4 *Let X be a complete metric space and $u \in W^{1,2}(D, X)$. Then for every $\epsilon > 0$ there exists a framed collar $U \subset N_\epsilon(S^1)$ in $\overline{D}$ with complementary discs Ω_i such that*

- $u|_{G \cap D} \in W^{1,2}(G \cap D, X)$ *and*
$$\sum_{i=1}^{2m} \ell^2(u|_{\partial \Omega_i \cap D}) < \epsilon;$$
- $u|_{\partial^+ U} = \mathrm{tr}(u|_{D \setminus U})$.

Moreover, G can be chosen to omit a given 2-exceptional family of curves in D.

The proof relies on a construction used in the proof of [Lytchak and Wenger 2018a, Lemma 4.8].

Proof We may assume that u is absolutely continuous on 2-a.e. curve in D; see Section 3.2. Fix $\epsilon > 0$ and let $\delta > 0$ be a constant whose size will be determined in terms of ϵ. We choose a small $\rho = \sin(2\pi/m) < \epsilon$ such that the restriction of u to the ρ-neighborhood of S^1 in D has energy at most δ. Next, we choose equidistant points $p_1, \dots, p_m$ on S^1 with pairwise Euclidean distance $\rho < \epsilon$.

Denote by E_i the energy of the restriction of u to $B_\rho(p_i) \cap D$. Note that

$$\sum_{i=1}^{m} E_i \le 2 \cdot E_+^2(u|_{N_\rho(S^1)}) \le 2\delta.$$

By [Lytchak and Wenger 2018a, Lemma 3.5], we find subsets $R_i \subset \left(\frac{2}{3}\rho, \rho\right)$ of positive measure, such that the following holds true for every $r_i \in R_i$. The restriction of u to the distance circle S_i of radius r_i around p_i in D is a continuous curve in $W^{1,2}(S_i, X)$ and its length ℓ_i satisfies $\ell_i^2 < 6\pi \cdot E_i$. We define B_i to be the ball $\overline{B}_{r_i}(p_i)$. By construction, $U = \bigcup_{i=1}^m \overline{B}_i$ is a framed collar with frame $G := S^1 \cup \bigcup_{i=1}^m S_i$. The domain U is subdivided by the circular arcs S_i in $2m$ Lipschitz discs Ω_j. The boundary of any Ω_j consists of two or three parts of consecutive circles S_i and a part $\partial\Omega_j \cap S^1$. By construction, $u|_{G\cap D} \in W^{1,2}(G \cap D, X)$ and

$$\sum_{j=1}^{2m} \ell^2(u|_{\partial\Omega_j \cap D}) \leq 6 \cdot \sum_{i=1}^{m} \ell_i^2 \leq 36\pi \cdot \sum_{i=1}^{m} E_i \leq 72\pi \cdot \delta.$$

Hence we can choose $\delta = \delta(\epsilon)$ small enough to guarantee the claimed length bound.

The last two statements hold since we can choose r_i freely from the positive measure set R_i. □

In the proof of Proposition 6.1 we will make use of the following auxiliary result.

Lemma 6.5 *Let $\Gamma_\omega \subset X_\omega$ be a rectifiable Jordan curve which is the ultralimit of a sequence of rectifiable Jordan curves $\Gamma_n \subset X_n$ such that $\limsup_{n\in\mathbb{N}} \ell(\Gamma_n) < 2\ell(\Gamma_\omega) - 2\epsilon$ for some $\epsilon > 0$. Let $\{t_1, \dots, t_k\} \subset S^1$ be an ordered tuple and $c_\omega\colon S^1 \to \Gamma_\omega$ a Lipschitz parametrization with $p_i := c_\omega(t_i)$ and such that $\ell(c_\omega|_{[t_i,t_{i+1}]}) < \epsilon$ for $1 \leq i \leq k$. For ω-a.e. $n \in \mathbb{N}$ let $\{p_{1,n}, \dots, p_{k,n}\} \subset \Gamma_n$ be points such that $p_i = \omega\text{-}\lim p_{i,n}$ for $1 \leq i \leq k$, and denote by $\alpha_{i,n} \subset \Gamma_n$ the shorter of the two components of $\Gamma_n \setminus \{p_{i,n}, p_{i+1,n}\}$. If $c_n\colon S^1 \to \Gamma_n$ is the map with $c_n(t_i) := p_{i,n}$ for $1 \leq i \leq k$, and such that $c_n|_{[t_i,t_{i+1}]}$ is a constant speed parametrization of $\alpha_{i,n}$, then c_n is uniformly Lipschitz and has degree 1.*

Proof By the semicontinuity of length and the bound $\limsup_{n\in\mathbb{N}} \ell(\Gamma_n) < 2\ell(\Gamma_\omega) - 2\epsilon$, for ω-a.e. $n \in \mathbb{N}$ the arc $\alpha_{i,n}$ is well defined and $\ell(\alpha_{i,n}) < \ell(\Gamma_\omega) - \epsilon$. In particular, the curves c_n are uniformly Lipschitz. Set

$$c'_\omega := \omega\text{-}\lim c_n\colon S^1 \to \Gamma_\omega.$$

Then the length bound guarantees that $c'_\omega|_{[t_i,t_{i+1}]}$ and $c_\omega|_{[t_i,t_{i+1}]}$ are homotopic relative boundary. In particular, c'_ω has degree 1. We claim that this implies that c_n has degree 1 for ω-a.e. $n \in \mathbb{N}$. Indeed, the length bound implies that the ultralimit γ_ω of a sequence of arclength parametrizations γ_n of Γ_n has degree 1. Now suppose that the degree of c_n is equal to $d \in \mathbb{N}$ for ω-a.e. $n \in \mathbb{N}$. Concatenating with an appropriate power of γ_n we obtain maps $\mu_n := c_n * \gamma_n^{-d}$ of degree 0. Thus μ_n extends to a map $u_n\colon \overline{D} \to \Gamma_n$ with uniform Lipschitz constant. In particular, $\omega\text{-}\lim \mu_n$ has degree 0 and therefore d is equal to the degree of c'_ω which is 1. □

Proof of Proposition 6.1 Fix $\epsilon_0 > 0$. We may assume that $\Lambda(\Gamma_\omega, X_\omega)$ is nonempty, otherwise there is nothing to show. Choose $u \in \Lambda(\Gamma_\omega, X_\omega)$ such that $\operatorname{Area}(u) < \operatorname{Fill}_{X_\omega}(\Gamma_\omega) + \frac{1}{2}\epsilon_0$. By Lemma 6.4, for every $\epsilon > 0$, we find a framed collar $U \subset N_\epsilon(S^1)$ with frame G and complementary discs $\Omega_i \subset U$ such that $u|_{G\cap D} \in W^{1,2}(G \cap D, X)$ with

$$\sum_{i=1}^{l} \ell^2(u|_{\partial\Omega_i\cap D}) < \epsilon.$$

Since G can be chosen to omit a given 2-exceptional family of curves in D, we may furthermore assume that the curves $u|_{\partial\Omega_i\cap D}$ and $\operatorname{tr}(u)|_{\partial\Omega_i\cap S^1}$ form a closed loop. By choosing ϵ small enough, we can arrange

$$\sum_{i=1}^{l} \ell^2(f|_{\partial\Omega_i}) < \frac{\epsilon_0}{16C},$$

where $f \in W^{1,2}(G, X_\omega)$ denotes the map given by u on $G \cap D$ and a constant speed parametrization of $\operatorname{tr}(u)|_{\partial\Omega_i\cap S^1}$ on every interval $\partial\Omega_i \cap S^1$. Let $\bar f : G \to X_\omega$ be the Lipschitz reparametrization of f which has constant speed on every edge of G. We may assume $\limsup_{n\in\mathbb{N}} \ell(\Gamma_n) < 2\ell(\Gamma_\omega) - 2\epsilon$. By [Lytchak et al. 2020, Lemma 2.6] we can lift $\bar f$ to Lipschitz graphs $\bar f_n : \bar G \cap D \to X_n$ with uniformly controlled Lipschitz constant such that $\ell(\bar f_n) \le 2 \cdot \ell(\bar f)$. Since producing such lifts only involves extensions from finite sets, we can arrange that for ω-a.e. $n \in \mathbb{N}$ the $\bar f_n$ send the boundary points of G to Γ_n. We extend $\bar f_n$ further to S^1 to a piecewise constant speed map as in Lemma 6.5. Set $\eta = \partial^+ U$. Then η is a bilipschitz Jordan curve in D and $(\bar f_n|_\eta)$ is a bounded sequence of Lipschitz curves with uniformly controlled Lipschitz constants and, by construction,

$$\bar f|_\eta = \omega\text{-lim}\, \bar f_n|_\eta.$$

By [Lytchak and Wenger 2016, Lemma 2.6] and Proposition 6.2 there exists $N \subset \mathbb{N}$ with $\omega(N) = 1$ such that

$$\operatorname{Fill}^{\mu^i}_{X_n}(\bar f_n|_\eta) \le \operatorname{Fill}_{X_\omega}(u|_\eta) + \tfrac14\epsilon_0$$

for every $n \in N$. By our choice of u, we have

$$\operatorname{Fill}_{X_\omega}(u|_\eta) \le \operatorname{Fill}_{X_\omega}(\Gamma_\omega) + \tfrac12\epsilon_0.$$

On the other hand, we can fill the curves $\bar f_n|_{\partial\Omega_i}$ using the (C, r_1)-isoperimetric inequality to produce Sobolev annuli $h_n \in W^{1,2}(A, X_n)$ which join $\bar f_n|_\eta$ and $\bar f_n|_{S^1}$ with

$$\operatorname{Area}_{\mu^i}(h_n) \le C \cdot \sum_{i=1}^{l} \ell^2(\bar f_n|_{\partial\Omega_i}) < \tfrac14\epsilon_0.$$

Thus, since $\bar f_n|_{S^1}$ is a degree 1 Lipschitz map to Γ_n by Lemma 6.5, we have

$$\operatorname{Fill}^{\mu^i}_{X_n}(\Gamma_n) \le \operatorname{Fill}^{\mu^i}_{X_n}(\bar f_n|_\eta) + \tfrac14\epsilon_0.$$

Putting things together, we obtain

$$\operatorname{Fill}^{\mu^i}_{X_n}(\Gamma_n) \le \operatorname{Fill}_{X_\omega}(\Gamma_\omega) + \epsilon_0$$

for all large enough $n \in N$. □

7 Intrinsic isoperimetric inequality

Let $r_0, r_1 > 0$, $C \geq 1$ and $\beta_n \to 0$. Let $\delta\colon (0, r_0) \to \mathbb{R}$ be a continuous nondecreasing function. Suppose that (Y_n) is a sequence of complete length spaces, each admitting a (C, r_1)-isoperimetric inequality, and such that

$$\delta^{\mu^i}_{Y_n}(r) \leq (1+\beta_n)\cdot\delta(r) + \beta_n r^2$$

if $r \in (\beta_n, r_0)$.

Proposition 7.1 *Suppose that all Y_n are isometrically embedded in a single complete metric space Z. Let $v \in W^{1,2}(D, Z)$ be a continuous map which admits a continuous extension $v\colon \overline{D} \to Z$. Further, let (v_n) be a sequence in $W^{1,2}(D, Y_n)$ such that*

- *$v_n \to v$ pointwise almost everywhere;*
- *$\mathrm{Area}_{\mu^i}(v_n|_\Omega) \to \mathrm{Area}_{\mu^i}(v|_\Omega)$ for every Jordan domain $\Omega \subset D$ with $v|_{\partial\Omega}$ rectifiable;*
- *$\ell(v_n|_\gamma) \to \ell(v|_\gamma)$ for 2-a.e. curve γ in D;*
- *$\mathrm{Area}_{\mu^i}(v_n) \leq \mathrm{Fill}^{\mu^i}_{Y_n}(\mathrm{tr}(v_n)) + \epsilon_n$ for some $\epsilon_n \to 0$.*

Let $\Omega \subset D$ be a Jordan domain such that $v|_{\partial\Omega}$ is rectifiable with $\ell(v|_{\partial\Omega}) < r_0$. Then

$$\mathrm{Area}_{\mu^i}(v|_\Omega) \leq \delta(\ell(v|_{\partial\Omega})).$$

Moreover, if $\ell(v|_{\partial\Omega}) = 0$, then $\mathrm{Area}_{\mu^i}(v|_\Omega) = 0$.

Proof Let $F\colon D \to \Omega$ be a conformal diffeomorphism and set $u_n := v_n \circ F$ and $u := v \circ F$. Recall that by Carathéodory's theorem, F extends to a homeomorphism $\overline{D} \to \overline{\Omega}$. Since v is continuous on $\overline{D}$, the map $u\colon \overline{D} \to Z$ is continuous and lies in $W^{1,2}(D, Z)$.

Since F preserves 2-exceptional families of curves [Lytchak and Wenger 2020, Proposition 3.5], we have $\ell(u_n|_\gamma) \to \ell(u|_\gamma)$ for 2-a.e. curve γ in D.

By Lemma 6.4, for every $\epsilon > 0$ there exists a framed collar $U \subset N_\epsilon(S^1)$ with frame G and complementary discs Ω_i. Moreover, $u|_{G\cap D} \in W^{1,2}(G \cap D, Z)$ and

$$\sum_{i=1}^{l} \ell^2(u|_{\partial\Omega_i \cap D}) < \epsilon$$

holds. In addition, $u|_{\partial^+ U} = \mathrm{tr}(u|_{D\setminus U})$. Since G can be chosen to omit a given 2-exceptional family of curves in D, we may additionally assume

- $u_n|_{G\cap D} \in W^{1,2}(G \cap D, Z)$ for all $n \in \mathbb{N}$;
- $u_n|_{G\cap D} \to u|_{G\cap D}$ pointwise $\mathcal{H}^1$-almost everywhere;
- $\ell(u_n|_{G\cap D}) \to \ell(u|_{G\cap D})$.

By the area convergence assumption and the semicontinuity of area (Proposition 3.4), we may also assume $\mathrm{Area}_{\mu^i}(u_n|_U) < \epsilon$ for almost all $n \in \mathbb{N}$.

To avoid issues when gluing Sobolev maps we will approximate Ω from within by Lipschitz domains as follows. For $\rho \in (0,1)$ consider the sphere $\sigma_\rho := \partial B_{1-\rho}(0) \subset D$. For almost all small $\rho \in (0,1)$,

- the set $\sigma_\rho \cap G$ consists of l points $\{\theta_1, \dots, \theta_l\}$;
- we have pointwise convergence $u_n \to u$ on $\sigma_\rho \cap G$.

Note that $F(\sigma_\rho) \subset \Omega$ is a bilipschitz Jordan curve for every $\rho > 0$. Moreover, after possibly adjusting ϵ and U and choosing ρ small enough, we can arrange

$$\left|\sum_{i=1}^{l-1} d_Z(u(\theta_i), u(\theta_{i+1})) - \ell(v|_{\partial\Omega})\right| < \epsilon.$$

Set $\widetilde{\Omega}_i := \Omega_i \cap B_{1-\rho}(0)$. Now let $f_n \colon G \to Y_n$ be any continuous map which is an ϵ-geodesic on every topological interval $\Omega_i \cap \sigma_\rho$ and which is given by u_n on $G \cap B_{1-\rho}(0)$. Then $f_n \in W^{1,2}(G, Y_n)$ and since the diameter of Ω_i can be assumed to be arbitrary small, (after possibly adjusting ϵ and U again,) we may assume

$$\sum_{i=1}^{l} \ell^2(f_n|_{\partial\widetilde{\Omega}_i}) < \epsilon$$

for almost all $n \in \mathbb{N}$.

Using the (C, r_1)-isoperimetric inequality to fill the curves $f_n|_{\partial\widetilde{\Omega}_i}$, we obtain homotopies $h_n \in W^{1,2}(A, Y_n)$ between $u_n|_{\partial^+ U}$ and $f_n|_{\sigma_\rho}$ with

$$\mathrm{Area}_{\mu^i}(h_n) \le C \cdot \epsilon,$$

where $A = U \cap B_{1-\rho}(0)$. Note that $h_n|_{\partial^- A}$ is a piecewise ϵ-geodesic in Y_n whose vertices are the images under u_n of the points $\{\theta_1, \dots, \theta_l\} = \sigma_\rho \cap G$. By the pointwise convergence $u_n \to u$ on $\sigma_\rho \cap G$, we have $d_{Y_n}(u_n(\theta_i), u_n(\theta_{i+1})) \to d_Z(u(\theta_i), u(\theta_{i+1}))$. Thus for $n \in \mathbb{N}$ large enough, we have

$$\begin{aligned}\ell(h_n|_{\partial^- A}) &\le (1+\epsilon)\sum_{i=1}^{l} d_{Y_n}(u_n(\theta_i), u_n(\theta_{i+1}))\\ &\le (1+\epsilon)\sum_{i=1}^{l} d_Z(u(\theta_i), u(\theta_{i+1})) + \epsilon \le (1+\epsilon)\ell(v|_{\partial\Omega}) + 2\epsilon.\end{aligned}$$

Now let us first assume $\ell(v|_{\partial\Omega}) > 0$. Then for small enough $\epsilon > 0$ and large enough $n \in \mathbb{N}$ we have

$$\ell(h_n|_{\partial^- A}) \ge (1-\epsilon)\sum_{i=1}^{l} d_Z(u(\theta_i), u(\theta_{i+1})) > \beta_n.$$

Thus, by assumption, we find a filling $w_n \in W^{1,2}(D, Y_n)$ of $h_n|_{\partial^- A} \in W^{1,2}(S^1, Y_n)$ with

$$\begin{aligned}\mathrm{Area}_{\mu^i}(w_n) &\le (1+\beta_n)\cdot\delta(\ell(h_n|_{\partial^- A})) + \beta_n \ell^2(h_n|_{\partial^- A})\\ &\le (1+\beta_n)\cdot\delta((1+\epsilon)\ell(v|_{\partial\Omega}) + 2\epsilon) + \beta_n((1+\epsilon)\ell(v|_{\partial\Omega}) + 2\epsilon)^2.\end{aligned}$$

We glue the maps $v_n|_{(D\setminus\Omega)\cup F(U)}$, $h_n \circ (F|_A)^{-1}$ and w_n to obtain a new map $\tilde{v}_n \in W^{1,2}(D, Y_n)$ with $\tilde{v}_n|_{(D\setminus\Omega)\cup F(U)} = v_n|_{(D\setminus\Omega)\cup F(U)}$. Here we use that $\partial^+ U$ and $F(\sigma_\rho)$ are bilipschitz Jordan curves; see [Lytchak and Wenger 2018a, Lemma 4.6]. By the last item of our assumptions and our construction so far, we obtain for large enough $n \in \mathbb{N}$ the estimates

$$\begin{aligned}\mathrm{Area}_{\mu^i}(v_n) &\le \mathrm{Area}_{\mu^i}(\tilde{v}_n) + \epsilon_n \\ &\le \mathrm{Area}_{\mu^i}(v_n|_{D\setminus\Omega}) + \mathrm{Area}_{\mu^i}(v_n|_{F(U)}) + \mathrm{Area}_{\mu^i}(h_n) + \mathrm{Area}_{\mu^i}(w_n) + \epsilon_n \\ &\le \mathrm{Area}_{\mu^i}(v_n|_{D\setminus\Omega}) + (1+\epsilon)\cdot\delta((1+\epsilon)\ell(v|_{\partial\Omega}) + 2\epsilon) + 2C\epsilon.\end{aligned}$$

Thus,

$$\mathrm{Area}_{\mu^i}(v_n|_\Omega) \le (1+\epsilon)\cdot\delta((1+\epsilon)\ell(v|_{\partial\Omega}) + 2\epsilon) + 2C\epsilon,$$

and since $\mathrm{Area}_{\mu^i}(v_n|_\Omega) \to \mathrm{Area}_{\mu^i}(v|_\Omega)$ we obtain the claim in the case $\ell(v|_{\partial\Omega}) > 0$.

For the supplement we argue similarly. If $\ell(v|_{\partial\Omega}) = 0$ holds, then the estimate above becomes

$$\ell(h_n|_{\partial^- A}) \le 2\epsilon.$$

We then use the (C, r_1)-isoperimetric inequality to fill the curves $h_n|_{\partial^- A}$. The same construction as above provides Sobolev discs $\tilde{v}_n \in W^{1,2}(D, Y_n)$ with $\mathrm{Area}(\tilde{v}_n) \le \mathrm{Area}_{\mu^i}(v_n|_{D\setminus\Omega}) + (1 + C + 4C\epsilon)\epsilon$. Arguing as before, we conclude $\mathrm{Area}(v|_\Omega) = 0$ as claimed. □

8 Solutions to Plateau's problem in ultralimits

Let $C, r_1 > 0$ and for $n \in \mathbb{N}$ let Y_n be a complete length space which is C-Lipschitz 1-connected up to some scale and admits a (C, r_1)-isoperimetric inequality. Suppose furthermore that

$$\delta^{\mu^i}_{Y_n}(r) \le (1+\beta_n)\cdot\delta(r) + \beta_n r^2$$

for all $r \in (\beta_n, r_0)$, where $\delta\colon (0, r_0) \to \mathbb{R}$ is continuous and nondecreasing with

$$\limsup_{r\to 0} \frac{\delta(r)}{r^2} \le \frac{1}{4\pi}$$

and $\beta_n \to 0$.

Let Y_ω be the ultralimit of (Y_n) with respect to some basepoints and a nonprincipal ultrafilter ω, and let $\Gamma_\omega \subset Y_\omega$ be a rectifiable Jordan curve of length strictly less than r_0. We then have the following generalization of Theorem B.

Theorem 8.1 *There is a subsequence of (Y_n) whose ultralimit $\widehat{Y}_\omega$ (with respect to the same basepoints and ultrafilter) contains an isometric copy of Γ_ω and has the following property. There exists a continuous map $v\colon \overline{D} \to \widehat{Y}_\omega$ which is a solution of the Plateau problem for Γ_ω in $\widehat{Y}_\omega$ and satisfies*

$$\mathrm{Area}(v|_\Omega) \le \delta(\ell(v|_{\partial\Omega}))$$

for every Jordan domain $\Omega \subset D$ such that $\ell(v|_{\partial\Omega}) < r_0$.

The proof will occupy the rest of the section. By Lemma 2.2 there exist rectifiable Jordan curves $\Gamma_n \subset Y_n$ such that ω-$\lim \Gamma_n = \Gamma_\omega$ and ω-$\lim \ell(\Gamma_n) = \ell(\Gamma_\omega)$. Let $u_n \colon D \to Y_n$ be Sobolev discs $u_n \in \Lambda(\Gamma_n, Y_n)$ with $\mathrm{Area}_{\mu^i}(u_n) \le \mathrm{Fill}^{\mu^i}_{Y_n}(\Gamma_n) + 1/(2n)$. By Morrey's ϵ-conformality lemma [Fitzi and Wenger 2020, Theorem 1.4], we may assume that

$$E_+^2(u_n) \le \mathrm{Area}_{\mu^i}(u_n) + \frac{1}{2n}$$

for every $n \in \mathbb{N}$.

After passing to a subsequence, we may assume that the Γ_n converge in the Gromov–Hausdorff sense to Γ_ω. Let γ_n be an arclength parametrization of Γ_n and fix distinct points $q_1, q_2, q_3 \in S^1$. After precomposing with conformal diffeomorphisms of the disc, we may assume that $\mathrm{tr}(v_n)(q_i) = \gamma_n(q_i)$, with $i = 1, 2, 3$, for every n.

By the compactness theorem [Guo and Wenger 2020, Theorem 3.1], after possibly passing to a subsequence, there exist a complete metric space Z, isometric embeddings $\varphi_n \colon X_n \hookrightarrow Z$, a compact subset $K \subset Z$ and $v \in W^{1,2}(D, Z)$ such that $\varphi_n(\Gamma_n) \subset K$ for all $n \in \mathbb{N}$, and $v_n := \varphi_n \circ u_n$ converges to v in $L^2(D, Z)$. After passing to a further subsequence, we can ensure that $\varphi_n(\Gamma_n)$ Hausdorff converges to a set $\Gamma_Z \subset K$. Since $\varphi_n \circ \gamma_n$ converges uniformly to an arclength parametrization of Γ_Z it follows from [Wenger 2019, Lemma 6.7] that the sequence $(\mathrm{tr}(v_n))$ uniformly converges to a weakly monotone parametrization of Γ_Z.

After passing to a further subsequence, we may assume $v_n \to v$ pointwise on a full measure subset $M \subset D$ as well as $\rho_{v_n} \to \rho$ weakly in $L^2(D)$ where $\rho_{v_n} \in L^2(D)$ is the minimal weak upper gradient of v_n and $\rho \in L^2(D)$ is a weak upper gradient of v (Lemma 3.2).

We can now embed $v(M)$ isometrically into the ultralimit $\widehat{Y}_\omega := \omega$-$\lim Y_n$ and therefore $v \in W^{1,2}(D, \widehat{Y}_\omega)$. Note that possibly $\widehat{Y}_\omega \ne Y_\omega$ since we passed to subsequences several times. Recall that since $\widehat{Y}_\omega$ has property (ET) by Proposition 5.1, the Hausdorff area and the Riemannian inscribed area coincide, $\mathrm{Area}(v) = \mathrm{Area}_{\mu^i}(v)$. Our next goal is:

Proposition 8.2 *In the setting above, there exists a subsequence (v_{n_l}) such that*

$$\lim_{l\to\infty} \mathrm{Area}_{\mu^i}(v_{n_l}) = \mathrm{Area}(v), \quad \lim_{l\to\infty} E_+^2(v_{n_l}) = E_+^2(v),$$

and

$$E_+^2(v) = \mathrm{Area}(v).$$

Moreover,

$$\mathrm{Area}(v) = \mathrm{Fill}_{Y_\omega}(\Gamma_\omega) = \lim_{l\to\infty} \mathrm{Fill}^{\mu^i}_{Y_{n_l}}(\Gamma_{n_l}).$$

In particular, v is conformal and a solution to the Plateau problem for Γ_ω in $\widehat{Y}_\omega$. In addition, $\rho_{v_{n_l}} \to \rho$ in $L^2(D)$ and ρ is the minimal weak upper gradient for v, that is, $\rho = \rho_v$.

Proof Passing to a subsequence, we may assume that $(\mathrm{Fill}^{\mu^i}_{Y_n}(\Gamma_n))$ converges. From Proposition 6.1, we obtain

$$\lim_{n\to\infty} \mathrm{Fill}^{\mu^i}_{Y_n}(\Gamma_n) \le \mathrm{Fill}_{\widehat{Y}_\omega}(\Gamma_\omega).$$

By construction, we have $\mathrm{Area}_{\mu^i}(v_n) = \mathrm{Area}_{\mu^i}(u_n)$ and

$$\mathrm{Fill}^{\mu^i}_{Y_n}(\Gamma_n) \le \mathrm{Area}_{\mu^i}(v_n) \le E^2_+(v_n) \le \mathrm{Area}_{\mu^i}(v_n) + \frac{1}{2n} \le \mathrm{Fill}^{\mu^i}_{Y_n}(\Gamma_n) + \frac{1}{n}.$$

In particular, we have

$$\lim_{n\to\infty} \mathrm{Fill}^{\mu^i}_{Y_n}(\Gamma_n) = \lim_{n\to\infty} \mathrm{Area}_{\mu^i}(v_n) = \lim_{n\to\infty} E^2_+(v_n).$$

Thus, by semicontinuity of area we obtain

$$\mathrm{Fill}_{\widehat{Y}_\omega}(\Gamma_\omega) \le \mathrm{Area}(v) \le \lim_{n\to\infty} \mathrm{Area}_{\mu^i}(v_n) = \lim_{n\to\infty} \mathrm{Fill}^{\mu^i}_{Y_n}(\Gamma_n) \le \mathrm{Fill}_{\widehat{Y}_\omega}(\Gamma_\omega).$$

Therefore, equality holds throughout. Semicontinuity of energy then yields

$$E^2_+(v) \le \lim_{n\to\infty} E^2_+(v_n) = \lim_{n\to\infty} \mathrm{Area}_{\mu^i}(v_n) = \mathrm{Area}(v) \le E^2_+(v).$$

Thus,

$$\lim_{n\to\infty} E^2_+(v_n) = E^2_+(v) = \mathrm{Area}(v).$$

Since $\rho_{v_n} \to \rho$ weakly in $L^2(D)$ and $E^2_+(v_n) = \int_D \rho^2_{v_n}(z)\,dz$, we obtain $\rho_{v_n} \to \rho$ strongly in $L^2(D)$ and therefore ρ is the minimal weak upper gradient of v. Finally, the conformality of v follows from the equality $E^2_+(v) = \mathrm{Area}(v)$ and the fact that $\widehat{Y}_\omega$ has property (ET). □

We will now continue to investigate the properties of v. To make use of the convergence $v_n \to v$ we will view v as a map with values in Z. Let us assume that we have already passed to a subsequence as provided by Proposition 8.2. In particular, we have the L^2-convergence $\rho_{v_n} \to \rho_v$. This allows us to apply Fuglede's lemma [Heinonen et al. 2015]. Thus, after passing to a further subsequence, we have

$$\lim_{n\to\infty} \int_\gamma |\rho_{v_n} - \rho_v|\,ds = 0$$

for 2-a.e. curve γ in D. Since v is conformal by Proposition 8.2, we have

$$\ell(v\circ\gamma) = \int_\gamma \rho_v\,ds$$

for 2-a.e. curve γ in D. It follows that the lengths of almost all curves converge:

$$\limsup_{n\to\infty} \ell(v_n\circ\gamma) \le \lim_{n\to\infty} \int_\gamma \rho_{v_n}\,ds = \int_\gamma \rho_v\,ds = \ell(v\circ\gamma) \le \liminf_{n\to\infty} \ell(v_n\circ\gamma).$$

The last inequality holds since the well known lower semicontinuity of length with respect to pointwise convergence of curves extends to the setting of almost everywhere pointwise convergence.

Recall that every space Y_n admits a (C, r_1)-isoperimetric inequality.

Lemma 8.3 *In the above setting the map v satisfies inequality* (3-3) *with constant C: for almost every bilipschitz Jordan curve γ in D with $\ell(v|_\gamma) < r_1$,*

$$\mathrm{Area}(v|_{\Omega_\gamma}) \le C\cdot\ell^2(v|_\gamma),$$

where Ω_γ is the Jordan domain enclosed by γ.

Proof Suppose that $\gamma \subset D$ is a bilipschitz Jordan curve with Jordan domain Ω_γ and $\ell(v|_\gamma) < r_1$ and such that $\ell(v_n|_\gamma) \to \ell(v|_\gamma)$. Our choice of v_n ensures

$$\operatorname{Area}_{\mu^i}(v_n|_{\Omega_\gamma}) \leq C \cdot \ell^2(v_n|_\gamma) + \frac{1}{2n}$$

for all n large enough. Semicontinuity of area, Proposition 3.4, yields

$$\operatorname{Area}(v|_{\Omega_\gamma}) = \operatorname{Area}_{\mu^i}(v|_{\Omega_\gamma}) \leq C \cdot \ell^2(v|_\gamma). \qquad \square$$

Now Theorem 3.7 implies that the Sobolev disc v has a locally Hölder continuous representative which continuously extends to $\overline{D}$ and which we will still denote by v.

We next claim that for every Jordan domain $\Omega \subset D$ such that $v|_{\partial\Omega}$ is rectifiable we have

$$\operatorname{Area}_{\mu^i}(v_n|_\Omega) \to \operatorname{Area}_{\mu^i}(v|_\Omega). \tag{8-1}$$

Indeed, since $v|_{\partial\Omega}$ is rectifiable and hence $\mathcal{H}^2(v(\partial\Omega)) = 0$ it follows from [Lytchak and Wenger 2017a, Proposition 4.3] that $\operatorname{Area}_{\mu^i}(v|_{\partial\Omega}) = 0$. Hence, by semicontinuity of area, Proposition 3.4, any subsequence v_{n_l} satisfies

$$\begin{aligned}
\operatorname{Area}_{\mu^i}(v) &= \operatorname{Area}_{\mu^i}(v|_\Omega) + \operatorname{Area}_{\mu^i}(v|_{D\setminus\overline{\Omega}}) \\
&\leq \liminf \operatorname{Area}_{\mu^i}(v_{n_l}|_\Omega) + \liminf \operatorname{Area}_{\mu^i}(v_{n_l}|_{D\setminus\overline{\Omega}}) \\
&\leq \liminf\bigl(\operatorname{Area}_{\mu^i}(v_{n_l}|_\Omega) + \operatorname{Area}_{\mu^i}(v_{n_l}|_{\partial\Omega}) + \operatorname{Area}_{\mu^i}(v_{n_l}|_{D\setminus\overline{\Omega}})\bigr) \\
&= \operatorname{Area}_{\mu^i}(v),
\end{aligned}$$

which proves (8-1).

Now we can apply Proposition 7.1 to conclude that for every Jordan domain $\Omega \subset D$ such that $v|_{\partial\Omega}$ has length strictly less then r_0 we have

$$\operatorname{Area}(v|_\Omega) = \operatorname{Area}_{\mu^i}(v|_\Omega) \leq \delta(\ell(v|_{\partial\Omega})).$$

This complete the proof of Theorem 8.1.

9 Main applications

In this final section we provide the proofs of our main results. Here is a restatement of Theorem C:

Theorem 9.1 *Let (X_n) be a sequence of complete length spaces and $\kappa \in \mathbb{R}$. Suppose $r_0 \in (0, 2D_\kappa]$ and*

$$\delta^{\mu^i}_{X_n}(r) \leq (1+\epsilon_n)\cdot\delta_\kappa(r) + \epsilon_n$$

holds for all $r \in (0, r_0)$ and some sequence $\epsilon_n \to 0$. Then every ultralimit X_ω is locally CAT(κ). More precisely, every closed ball of radius at most $\frac{1}{4}r_0$ in X_ω is convex and CAT(κ).

Proof By Proposition 4.1, there exists a sequence (Y_n) of complete length spaces and a sequence (β_n) tending to zero such that Y_n is a universal β_n-thickening of X_n with the following properties. First, Y_n is

L-Lipschitz 1-connected up to some scale for some universal $L \geq 1$. Moreover, there exist $C \geq 1$ and $r_1 > 0$ such that each Y_n admits a (C, r_1)-isoperimetric inequality and

$$\delta_{Y_n}(r) \leq (1+\beta_n) \cdot \delta_\kappa(r) + \beta_n r^2$$

holds for all $r \in (\beta_n, r_0)$. Note that since $\beta_n \to 0$ we have ω-$\lim Y_n = X_\omega$. Let $\triangle \subset X_\omega$ be a Jordan triangle of perimeter strictly less than r_0. By Theorem 8.1, we find an ultralimit $\widehat{Y}_\omega$ of a subsequence of (Y_n) which contains an isometric copy of $\triangle$ and has the following property. There exists a continuous map $v\colon \overline{D} \to \widehat{Y}_\omega$ which is a solution of the Plateau problem for $\triangle$ in $\widehat{Y}_\omega$ and satisfies

$$\operatorname{Area}(v|_\Omega) \leq \delta_\kappa(\ell(v|_{\partial\Omega}))$$

for every Jordan domain $\Omega \subset D$ such that $\ell(v|_{\partial\Omega}) < r_0$. By Theorem 4.5, the associated intrinsic minimal disc Z_v is homeomorphic to $\overline{D}$ and for every Jordan domain $\Omega \subset Z_v$ with $\ell(\partial\Omega) < 2D_\kappa$ we have

$$\mathscr{H}^2(\Omega) \leq \delta_\kappa(\ell(\partial\Omega)).$$

It follows from the proof of [Lytchak and Wenger 2020, Theorem 1.4] that the Dehn function of Z_v is bounded above by δ_κ on $(0, 2D_\kappa)$. Thus, by [Lytchak and Wenger 2018b, Theorem 1.4], Z_v is a CAT(κ) space. Let $u = \bar{u} \circ P$ be the induced factorization; see Section 4.3. By Reshetnyak's majorization theorem, ∂Z_v admits a κ-majorization $\varphi\colon C \to Z_v$ and then $\bar{u} \circ \varphi$ provides a κ-majorization for $\triangle$. In particular, $\triangle$ satisfies the CAT(κ) triangle comparison and Lemma 2.3 implies the claim. □

Theorem 9.2 *Let $\delta\colon (0, r_0) \to \mathbb{R}$ be a continuous nondecreasing function with*

$$\limsup_{r \to 0} \frac{\delta(r)}{r^2} < \frac{1}{4\pi}.$$

Suppose that (X_n) is a sequence of complete length spaces such that the Riemannian Dehn functions satisfy

$$\delta^{\mu^i}_{X_n}(r) \leq \delta(r) + \epsilon_n$$

on $(0, r_0)$ for some sequence $\epsilon_n \to 0$. Then any ultralimit X_ω is 1-dimensional. More precisely, there exists $\tilde{r} > 0$ depending only on the function δ such that every closed ball in X_ω of radius at most $\tilde{r}$ is convex and a tree.

Proof Let X_ω be an ultralimit of the sequence (X_n). By assumption, for every $\kappa \leq 0$ there exists $\tilde{r}_\kappa > 0$ such that $\delta(r) \leq \delta_\kappa(r)$ holds for all $r \in (0, \tilde{r}_\kappa)$. From Theorem 9.1 we conclude that every closed ball of radius at most $\frac{1}{4}\tilde{r}_\kappa$ in X_ω is convex and CAT(κ). We set $\tilde{r} := \frac{1}{4}\tilde{r}_0$. Let $B \subset X_\omega$ be a closed ball of radius at most $\tilde{r}$. Then B is CAT(0) and therefore contractible. Moreover, B is locally CAT(κ) for every $\kappa < 0$. We conclude from the Cartan–Hadamard theorem [Alexander et al. 2024, Theorem 9.65] that B itself is globally CAT(κ) for every $\kappa < 0$. □

Now we obtain Theorem D:

Corollary 9.3 *Let X be a complete length space such that*

$$\limsup_{r\to\infty} \frac{\delta_X^{\mu^i}(r)}{r^2} \leq \frac{1}{4\pi}.$$

Then every asymptotic cone of X is a CAT(0) space. Moreover, if the inequality is strict, then every asymptotic cone of X is a tree. In particular, in this case, X is Gromov hyperbolic.

Proof Set $C = \limsup_{r\to\infty} \delta_X^{\mu^i}(r)/r^2$. For every positive sequence (λ_n) with $\lambda_n \to 0$ we find another positive sequence (ϵ_n) tending to zero such that

$$\delta_{X_n}^{\mu^i}(r) \leq (1+\epsilon_n)C \cdot r^2 + \epsilon_n$$

holds for all $r \geq 0$ and all $n \in \mathbb{N}$, where X_n denotes the metric space $(X, \lambda_n \cdot d)$. Thus Theorem 9.1 implies that any ultralimit of the sequence (X_n) is CAT(0). The additional statement in case $C < 1/(4\pi)$ follows from Theorem 9.2. □

It only remains to prove Theorem E.

Theorem 9.4 *Let $\delta\colon (0, r_0) \to \mathbb{R}$ be a continuous nondecreasing function with*

$$\limsup_{r\to 0} \frac{\delta(r) - r^2/(4\pi)}{r^4} \leq 0.$$

Suppose that (X_n) is a sequence of complete length spaces such that the Riemannian Dehn functions satisfy

$$\delta_{X_n}^{\mu^i}(r) \leq \delta(r) + \epsilon_n$$

on $(0, r_0)$ for some sequence $\epsilon_n \to 0$. Then any ultralimit X_ω is locally CAT(0). More precisely, there exists $\tilde{r} > 0$ depending only on the function δ such that every closed ball in X_ω of radius at most $\tilde{r}$ is convex and CAT(0).

Note that the assumption on $\delta(r)$ cannot be relaxed to

$$\limsup_{r\to 0} \frac{\delta(r)}{r^2} \leq \frac{1}{4\pi}.$$

The latter condition merely ensures property (ET) and is satisfied by every Riemannian manifold.

Proof Let X_ω be an ultralimit of the sequence (X_n). Note that for $\kappa > 0$ we have

$$\delta_\kappa(r) = \frac{1}{4\pi} \cdot r^2 + \frac{\kappa}{64\pi^3} \cdot r^4 + o(r^5).$$

By assumption, for every $\kappa > 0$ there exists $\tilde{r}_\kappa > 0$ such that $\delta(r) \leq \delta_\kappa(r)$ holds for all $r \in (0, \tilde{r}_\kappa)$. From Theorem 9.1 we conclude that every closed ball of radius at most $\frac{1}{4}\tilde{r}_\kappa$ in X_ω is convex and CAT(κ). We set $\tilde{r} := \min\{\frac{1}{4}\tilde{r}_1, \frac{\pi}{2}\}$. Let $B \subset X_\omega$ be a closed ball of radius at most $\tilde{r}$. Then B is CAT(1) and uniquely geodesic. We claim that B is CAT(0). For any $\kappa > 0$, the space B is locally CAT(κ) and geodesics depend continuously on their endpoints. Hence B is CAT(κ) for every $\kappa > 0$ [Alexander et al. 2024, Theorem 9.30] and therefore is CAT(0) [Alexander et al. 2024, Proposition 9.7]. □

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