Geometry & Topology

Volume 29 (2025)

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Geometry & Topology **29**:2 (2025) 863–902
DOI: 10.2140/gt.2025.29.863
Published: 21 April 2025

Parametric inequalities and Weyl law for the volume spectrum

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We show that the Weyl law for the volume spectrum in a compact Riemannian manifold conjectured by Gromov can be derived from parametric generalizations of two famous inequalities: the isoperimetric inequality and the coarea inequality. We prove two such generalizations in low dimensions and obtain the Weyl law for 1-cycles in 3-manifolds. We also give a new proof of the Almgren isomorphism theorem.

35P20, 53A10, 53C23

1 Introduction

Consider the space of k-dimensional Lipschitz cycles in a Riemannian manifold M with coefficients in an abelian group G. One can define the flat seminorm $\mathcal{F}(x)$ on this space as the infimum of volumes of Lipschitz chains with boundary x; define flat distance as $\mathcal{F}(x,y)=\mathcal{F}(x-y)$, as in Fleming [7] and Guth [15, Appendix 1]. If we consider the quotient of this space by the equivalence relation $x\sim y$ if $\mathcal{F}(x,y)=0$ and then take its completion, then we obtain the space of flat k-cycles $\mathcal{Z}_k(M;G)$ with flat distance $\mathcal{F}$. This is a natural space to consider when trying to find submanifolds that solve a certain calculus of variations problem (like minimal surfaces) without a priori specifying the topological type of the solution.

In the 1960s Almgren initiated a program of developing Morse theory on the space $\mathcal{Z}_k(M;G)$. It follows from Almgren's work that there is a nontrivial cohomology class $\alpha\in H^{n-k}(\mathcal{Z}_k(M;\mathbb{Z}_2);\mathbb{Z}_2)$ corresponding to a sweepout of M by an $(n-k)$-parameter family of k-cycles $F\colon X\to\mathcal{Z}_k(M;\mathbb{Z}_2)$ with

$F^*(\alpha) \neq 0 \in H^{n-k}(X;\mathbb{Z}_2)$. Moreover, all cup powers α^p are also nontrivial and the corresponding families of cycles we call p-sweepouts. Corresponding to each p we can define the p-width

$$\omega_p^k(M) = \inf\Bigl\{\sup_{x\in X} \mathrm{Vol}_k(F(x)) : F \text{ is a } p\text{-sweepout}\Bigr\}.$$

The p-widths correspond to volumes of certain generalized minimal submanifolds that arise via a min-max argument. In dimension $k = n-1$, where $3 \le n \le 7$, they are smooth minimal hypersurfaces and for $k = 1$ they are stationary geodesic nets. The study of p-widths led to many remarkable breakthroughs in recent years, including the resolution of the Willmore conjecture (see Marques and Neves [23]), and Yau's conjecture on existence of infinitely many minimal surfaces (see Irie, Marques and Neves [16] and Song [30]). See also Zhou [37], Chodosh and Mantoulidis [3] and Marques and Neves [25].

Gromov [11; 12; 13] suggested viewing these widths, or the "volume spectrum", as nonlinear analogues of the eigenvalues of the Laplacian. This framework is very useful for obtaining new results about widths and some other nonlinear min-max geometric quantities; see, for example, Glynn-Adey and Liokumovich [9], Liokumovich [19] and Mazurowski [27]. In line with this analogy Gromov conjectured that the ω_p^k satisfy an asymptotic Weyl law

$$\lim_{p\to\infty} \frac{\omega_p^k(M)}{p^{(n-k)/n}} = a(n,k)\,\mathrm{Vol}(M)^{k/n}. \tag{1-1}$$

For $k = n-1$ this was proved in Liokumovich, Marques and Neves [21].

In this paper we show that for $k < n-1$, Gromov's conjecture can be reduced to proving parametric versions of two famous inequalities in geometry: the isoperimetric inequality and the coarea inequality. We prove these parametric inequalities in low dimensions and as a consequence obtain the Weyl law for 1-cycles in 3-manifolds.

Theorem 1.1 *For every compact Riemannian 3-manifold M,*

$$\lim_{p\to\infty} \frac{\omega_p^1(M)}{p^{2/3}} = a(3,1)\,\mathrm{Vol}(M)^{1/n}.$$

We do not know the value of the constant $a(3,1)$. Chodosh and Mantoulidis [4] proved that optimal sweepouts of the round 2-sphere are realized by zero sets of homogeneous polynomials and that $a(2,1) = \sqrt{\pi}$, but for all other values of $n > k \ge 1$, the constants $a(n,k)$ remain unknown.

1.1 Some applications of the Weyl law

Irie, Marques and Neves [16] used the Weyl law to prove that for a generic Riemannian metric on an n-dimensional manifold, where $3 \le n \le 7$, the union of embedded minimal hypersurfaces forms a dense set. Marques, Neves and Song [26] proved a stronger equidistribution property of minimal hypersurfaces; their proof is based on the idea that one can "differentiate" both sides of (1-1) with respect to some cleverly chosen variations of metric. Similar results were proved by Gaspar and Guaraco [8] using the Weyl law in the Allen–Cahn setting; see also Dey [5]. Li [18] used the Weyl law to prove existence of infinitely many minimal hypersurfaces for generic metrics in higher dimensions, $n > 7$. In [30], Song proved some Weyl law type asymptotic estimates for certain noncompact manifolds and used them to prove that in dimensions $3 \le n \le 7$ for all Riemannian metrics on a compact manifold (not only generic ones) there exist infinitely many minimal hypersurfaces; see also Song and Zhou [31], where similar generalizations were applied to prove a "scarring" result for minimal hypersurfaces.

Song [29] showed that the density result for generic metrics can be obtained without the full strength of the Weyl law. This observation was used in Liokumovich and Staffa [22] (together with a bumpy metrics theorem for stationary geodesic nets proved in Staffa [32]) to prove that for a generic Riemannian metric on a compact manifold M^n, with $n \ge 2$, stationary geodesic nets form a dense set, even though the Weyl law for 1-cycles in dimensions $n \ge 4$ was not known at the time. Li and Staffa [17] used Theorem 1.1 to prove an equidistribution result for stationary geodesic nets in 3-manifolds, analogous to that of Marques, Neves and Song [26] for minimal hypersurfaces.

1.2 Parametric isoperimetric inequality

The isoperimetric inequality of Federer and Fleming asserts that given a k-dimensional Lipschitz cycle z in $\mathbb{R}^n$, there exists a $(k+1)$-dimensional Lipschitz chain τ such that $\partial\tau = z$ and

$$\mathrm{Vol}_{k+1}(\tau) \le c(n)\,\mathrm{Vol}_k(z)^{(k+1)/k}. \tag{1-2}$$

This inequality is useful when z lies in a ball of large radius. If $z \subset B_R$ with $R \le \mathrm{Vol}_k(z)^{1/k}$, then the inequality simply follows by taking a cone over z.

In this paper we will be interested in the situation when a mod 2 cycle z is contained in a small cube (say, of side length 1) and has very large volume. In this case, we can subdivide the cube into smaller cubes of side length $\mathrm{Vol}(z)^{-1/(n-k)}$. Applying the Federer–Fleming deformation in this lattice, we can find a "deformation chain" that pushes z into the k-dimensional skeleton of this lattice. Since the total volume of the $(k+1)$-skeleton of the lattice is $\sim (1/\mathrm{Vol}(z)^{1/(n-k)})^{k+1}\,\mathrm{Vol}(z)^{n/(n-k)}$, we can then obtain a filling τ of z with

$$\mathrm{Vol}_{k+1}(\tau) \le c(n)\,\mathrm{Vol}_k(z)^{(n-k-1)/(n-k)}. \tag{1-3}$$

Note that when $\mathrm{Vol}(z) \gg 1$ this is a much better bound than (1-2) or the cone inequality.

We would like to prove a parametric version of inequality (1-3); namely, given a family of relative cycles $F\colon X \to \mathcal{Z}_k([0,1]^n, \partial[0,1]^n; G)$, we would like to find a continuous family of $(k+1)$-chains $H\colon X \to I_{k+1}([0,1]^n; G)$ with $\partial H(x) - F(x)$ in $\partial[0,1]^n$ and such that the mass of $H(x)$ is controlled in terms of the mass of $F(x)$ as in (1-3).

First, observe that we need to assume that map F is contractible, since otherwise a continuous family of fillings H does not exist. Secondly, existence of a family of fillings with controlled mass depends on the choice of coefficients G. In Section 3 we show that for every integer N there exists a contractible 1-parameter family of 0-cycles in $\mathcal{Z}_0(S^1, \mathbb{Z})$ of mass ≤ 2 such that every continuous family of fillings must contain a chain of mass $> N$. This example can be generalized to higher dimensions and codimensions. However, for $G = \mathbb{Z}_2$ we can optimistically conjecture:

Optimistic Conjecture 1.2 *Let $F\colon X \to \mathcal{Z}_k([0,1]^n, \partial[0,1]^n; \mathbb{Z}_2)$ be a contractible family. Then there exists a family of $(k+1)$-chains $H\colon X \to I_{k+1}([0,1]^n; \mathbb{Z}_2)$ such that $\partial H(x) - F(x)$ is supported in $\partial[0,1]^n$ and*

$$\boldsymbol{M}(H(x)) \leq c(n) \max\{1, \boldsymbol{M}(F(x))^{(n-k-1)/(n-k)}\}.$$

The actual conjecture that we prove in the case of low dimensions and use for the proof of the Weyl law is somewhat more technical. We will be interested in the situation where maximal mass $\boldsymbol{M}(F(x)) \sim p^{(n-k)/n}$. In this case we have

$$p^{(n-k-1)/n} + \boldsymbol{M}(F(x))p^{-1/n} \sim \boldsymbol{M}(F(x))^{(n-k-1)/(n-k)}.$$

We also require that the family $F(x)$ is δ-localized, a regularity condition that can always be guaranteed after a small perturbation; see Section 2.3. Finally, it will be convenient to state the conjecture for more general domains in $\mathbb{R}^n$ with piecewise smooth boundary satisfying a certain regularity condition on the boundary; see Section 2.1. In particular, it is satisfied if $\partial\Omega$ is smooth.

Conjecture 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a connected domain with piecewise smooth boundary $\partial\Omega$ with θ-corners for some $\theta \in (0, \pi)$, and let $0 \leq k < n$. There exist constants $c(\Omega) > 0$ and $\delta(\Omega, L, p) > 0$ with the following property. Let $F\colon X^p \to \mathcal{Z}_k(\Omega, \partial\Omega; \mathbb{Z}_2)$ be a continuous $\delta(\Omega, L, p)$-localized contractible p-dimensional family with $\boldsymbol{M}(F(x)) \leq L$. Then there exists a map $H\colon X \to I_{k+1}(\Omega; \mathbb{Z}_2)$ such that*

- *$\partial H(x) - F(x) \subset \partial\Omega$ for all x, and*
- *$\boldsymbol{M}(H(x)) \leq c(\Omega)\big(p^{(n-k-1)/n} + \boldsymbol{M}(F(x))p^{-1/n}\big)$.*

We also want to state a parametric isoperimetric inequality conjecture for cycles in $\mathbb{R}^n$. It doesn't have immediate applications to the Weyl law, but is interesting on its own. Unlike the conjectures above we expect it to be true for coefficients in every abelian group G. For simplicity we state it for $G = \mathbb{Z}$.

Conjecture 1.4 *Let $F\colon X^p \to \mathcal{Z}_k(\mathbb{R}^n; \mathbb{Z})$ be a continuous family of cycles, with $k \geq 1$. There exists a family of chains $H\colon X^p \to I_{k+1}(\mathbb{R}^n; \mathbb{Z})$ such that $\partial H(x) = F(x)$ and*

$$\boldsymbol{M}(H(x)) \leq c(n)\boldsymbol{M}(F(x))^{(k+1)/k}.$$

The main difficulty in this conjecture seems to be proving a bound for $\boldsymbol{M}(H(x))$ that is independent of p. An inequality of this type with constant $c(n, p)$ that depends on p subexponentially would also be of interest. In another direction, one can ask if it is possible to replace $c(n)$ with $c(k)$ and extend this result to infinite-dimensional Banach spaces as in Gromov [10] (see also Wenger [35]) or prove a similar inequality for the Hausdorff content as in Liokumovich, Lishak, Nabutovsky and Rotman [20].

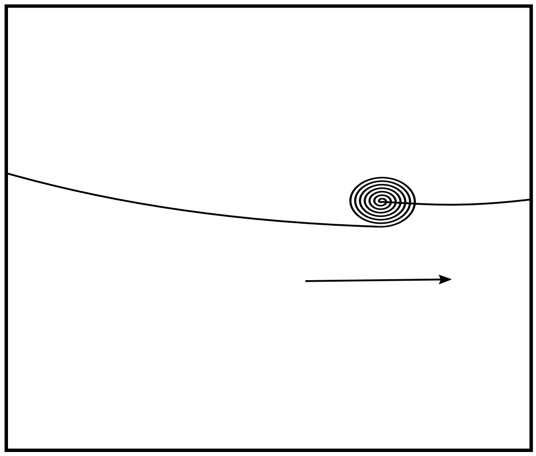

Figure 1

1.3 Parametric coarea inequality

One can similarly formulate a parametric version of the coarea inequality. Let M be a Riemannian manifold with boundary and let $M_\varepsilon = M \setminus \mathrm{cl}(N_\varepsilon(\partial M))$ be M minus a closed ε-neighborhood of its boundary. If z is a Lipschitz relative cycle in M and $r_0 > 0$, then the coarea inequality applied to the distance function implies the existence of some $r \in [0, r_0]$ such that $\mathrm{Vol}(\partial(z \llcorner M_r)) \le \mathrm{Vol}(z)/r_0$.

Very optimistically, one could conjecture that given a family of relative cycles $\{z_x\}_{x \in X}$ and $r_0 > 0$ there exists $r \in [0, r_0]$ such that $\mathrm{Vol}(\partial(z_x \llcorner M_r)) \le \mathrm{Vol}(z_x)/r_0$ for all $x \in X$.

This is not always the case. Consider a 1-parameter family $\{z_x\}_{x \in [0,1]}$ of relative cycles in $M = [0,1]^2$ with a small but tightly wound spiral moving from the center to the boundary of $[0,1]^2$; see Figure 1. No matter how you choose r, for some value of $x \in [0,1]$ the spiral will hit the boundary of M_r and the intersection may have arbitrarily many points. Letting r vary as a continuous function of x doesn't help.

However, what can help is if we allow ourselves to modify the family $\{z_x\}$ in the boundary ∂M_r, while only increasing its mass by a small amount. It seems natural to conjecture that the amount of extra mass that we may have to add should be less than or comparable to the size of an optimal p-sweepout of the boundary by k-cycles, that is, $\lesssim p^{(n-k-1)/(n-1)}$. Note that this is much smaller than the mass of an optimal p-sweepout of the n-dimensional interior by k-cycles ($\gtrsim p^{(n-k)/n}$).

Also, observe the following: if the family of relative cycles is a p-sweepout and if it can be represented by a family of chains whose boundaries are a continuous family in ∂M_r, then the family of boundaries is a p-sweepout of ∂M_r. On the other hand, we know that an optimal p-sweepout of $[0,1]^2$ by 1-cycles has maximal mass of $\mathrm{const} \cdot \sqrt{p}$, while every p-sweepout of $\partial[0,1]^2$ must have 0-cycles of mass $\ge p$. It follows that if we want the family of boundary cycles to be continuous, then an $\boldsymbol{M}(z_x)/r_0$ bound is not sufficient.

Motivated by this we state the following optimistic conjecture:

Optimistic Conjecture 1.5 *Let $n > k \geq 1$. Let $F: X^p \to \mathcal{Z}_k([0,1]^n, \partial[0,1]^n; \mathbb{Z}_2)$ be a continuous family of cycles and $r \in \left(0, \frac{1}{2}\right)$. Then there exists a family $F': X \to I_k([r, 1-r]^n; \mathbb{Z}_2)$ such that*

(1) $F'(x) \llcorner (r, 1-r)^n = F(x) \llcorner (r, 1-r)^n$,

(2) $\boldsymbol{M}(F'(x)) \leq \boldsymbol{M}(F(x)) + c(n) p^{(n-k-1)/(n-1)}$,

(3) $\partial F'(x)$ *is a continuous family of* $(k-1)$*-cycles in* $\partial[r_0, 1-r]^n$,

(4) $\boldsymbol{M}(\partial F'(x)) \leq c(n)\big(\boldsymbol{M}(F(x))/r + p^{(n-k)/(n-1)}\big)$.

In our more down-to-earth conjecture that we use to prove the Weyl law, we allow the $F'(x)$ to be a small perturbation of $F(x)$ in the interior of $(r, 1-r)^n$. In addition, we require that the family satisfies a certain technical assumption that the mass does not concentrate at a point. We also state the result for more general domains with piecewise smooth boundary with θ-corners (their definition is given in Section 2.1).

Conjecture 1.6 *Let $\Omega \subset \mathbb{R}^n$ be a connected domain with $\partial\Omega$ a piecewise smooth boundary with θ-corners for some $\theta \in (0, \pi)$. Fix $\eta > 0$. For all $p \geq p_0(\Omega)$, $r \in (0, r_0(\Omega))$ and $\delta > 0$ the following holds. Let $F: X^p \to \mathcal{Z}_k(\Omega, \partial\Omega; \mathbb{Z}_2)$ be a continuous map with no concentration of mass. There exists a map $F': X \to I_k(\mathrm{cl}(\Omega_r); \mathbb{Z}_2)$ such that*

(1) $\partial F'(x)$ *is a continuous* δ*-localized family in* $\mathcal{Z}_{k-1}(\partial\Omega_r; \mathbb{Z}_2)$,

(2) $\mathcal{F}(F(x) \llcorner \Omega_r, F'(x)) < \eta$,

(3) $\boldsymbol{M}(F'(x)) \leq \boldsymbol{M}(F(x)) + \big(\boldsymbol{M}(F(x))/r\big) p^{-1/(n-1)} + C(\Omega) \boldsymbol{M}(\partial\Omega)^{k/(n-1)} p^{(n-k-1)/(n-1)}$,

(4) $\boldsymbol{M}(\partial F'(x)) \leq c(\Omega)\big(\boldsymbol{M}(F(x))/r + p^{(n-k)/(n-1)}\big)$.

Moreover, if F is a p-sweepout of Ω, then $\partial F': X \to \partial\Omega_r$ is a p-sweepout of $\partial\Omega_r$ by $(k-1)$-cycles.

We note that the parametric coarea inequality conjecture for k-cycles in n-domains together with the parametric isoperimetric inequality for $(k-1)$-cycles in $(n-1)$-domains can be used to prove the parametric isoperimetric inequality for k-cycles in n-domain.

1.4 Weyl law

In the proof of the Weyl law for $(n-1)$-cycles in a compact n-manifold in Liokumovich, Marques and Neves [21], the manifold M is subdivided into small domains U_i and then a family of hypersurfaces $\{\Sigma_t\}$ is defined that, restricted to each U_i, is a p-sweepout $\{\Sigma_t^i = \Sigma_t \cap U_i\}$ of U_i. However, these hypersurfaces have boundary in ∂U_i and do not form a family of cycles in M (only a family of chains). A crucial step in the proof is to glue this family of chains into a family of cycles in M by adding a family of $(n-1)$-chains inside $\bigcup \partial U_i$. Since we are using $\mathbb{Z}_2$ coefficients their total volume is bounded by the volume of $\bigcup \partial U_i$, and is negligible compared to $p^{1/n}$ for p very large.

In the case of higher codimension we can similarly define a family of k-chains which restrict to a p-sweepout of U_i by relative cycles. However, to carry out the second step we need to find a family of chains in $\bigcup \partial U_i$ of controlled volume that we can add to Σ_t to turn it into a family of cycles. The problem is twofold:

(1) The volumes of $\partial \Sigma_t^i$ can be arbitrarily large.

(2) Even if we find a way to control the volume of $\partial \Sigma_t^i$, we also need to construct a continuous family of chains filling $\partial \Sigma_t^i$ of volume that is negligible compared to the maximal volume of Σ_t^i.

Together the parametric coarea inequality and parametric isoperimetric inequality solve these two problems.

1.5 Organization

In Section 2 we prove some important results on how families of cycles can be approximated with families that have some nice special properties, which we call δ-localized families. In Section 3 we use these results to give a new simpler proof of the Almgren isomorphism theorem. In Section 4 we prove a parametric coarea inequality for 1-cycles in 3-dimensional domains. In Section 5 we prove a parametric isoperimetric inequality for 0-cycles in 2-dimensional domains. In Section 6 we show how the Weyl law can be proved using the parametric coarea and isoperimetric inequality conjectures.

Remark 1.7 After an earlier draft of this paper appeared on arXiv, Bruno Staffa [33; 34] proved Conjectures 1.6 and 1.3 for $k = 1$ and all n. Consequently, Staffa obtained the Weyl law (Theorem 6.1) for 1-cycles in all dimensions.

Acknowledgements Liokumovich was partially supported by an NSF grant, NSERC Discovery grant and Accelerator Award and Sloan Fellowship, and would like to thank Alexander Kupers, Fernando Marques and Andre Neves for stimulating discussions. We are grateful to the referee for numerous corrections to the first draft of this article. We would like to thank Bruno Staffa for finding a gap in the proof of Proposition 2.7 in an earlier draft of this paper and suggesting a way to fix it.

2 Approximation results for families of cycles

2.1 Piecewise smooth boundaries with corners

It will be convenient to work with manifolds that have piecewise smooth boundaries with no cusps. In this subsection we give a more precise definition of the "no cusps" property.

Let M be a Riemannian manifold with piecewise smooth boundary. Given a subset $Y \subset M$, let $N_r(Y) = \{x : \mathrm{dist}(x, Y) < r\}$ denote the r-neighborhood of Y.

Given $\theta \in (0, \pi)$, we will say that ∂M is a boundary with θ-corners if for every $p \in \partial M$ and every $\varepsilon > 0$ there exists an open ball B containing p, and a $(1+\varepsilon)$-bilipschitz diffeomorphism Φ from B to a subset of $\mathbb{R}^n$, such that $\Phi(p) = 0$ and $\Phi(B)$ lies in the intersection H of at most n halfspaces with $\Phi(\partial M \cap B) \subset \partial H$. Moreover, the hyperplanes $P_1, \dots, P_j$ in ∂H are in general position (that is, the intersection of any k hyperplanes is a linear subspace of codimension $k+1$), such that the dihedral angle $\angle(P_i, P_{i'}) \geq \theta$ for all $i \neq i'$.

We will need the following useful lemma.

Lemma 2.1 *Suppose ∂M is a boundary with θ-corners, with $\theta \in (0, \pi)$. There exists $r_0 > 0$ such that for all $r \in (0, r_0]$ there exists a $c(\theta)$-bilipschitz diffeomorphism $E \colon \partial M \times [0, r_0] \to U$, with $N_{r_0}(\partial M) \subset U \subset N_{11r_0/10}(\partial M)$, such that E is the identity map on ∂M.*

Proof If ∂M is smooth, the we can choose E to be the normal exponential map $E = \exp_{\partial M}$.

Otherwise, consider a stratification of ∂M into 0-dimensional, 1-dimensional, …, $(n-1)$-dimensional smooth strata $\{S_i\}$. Inductively, we can define a continuous vectorfield V on ∂M so that

- V is smooth and $|V(x)| = 1$ on S_i for each i,
- $\angle(V(x), \partial M) \geq \frac{1}{3}\theta$.

We can extend V to a smooth vectorfield in the interior of M and define $E \colon \partial M \times [0, r] \to N_r(\partial M)$ to be a smooth map with $\partial E/\partial t|_{(x,t)=(x,0)} = V(x)$. Then for r sufficiently small this map will be the desired bilipschitz diffeomorphism. □

Let $r_0 > 0$ and E be as in the lemma above. For any $r \in (0, r_0)$, let $M_r = M \setminus E\big(\partial M \times [0, r)\big)$.

We have the following consequence of the coarea formula for flat chains.

Theorem 2.2 *Let $\tau \in I_k(M; G)$. Then there exists a constant $c(\varepsilon)$ with $\lim_{\varepsilon \to 0} c(\varepsilon) = 1$ such that for every $\varepsilon > 0$, there exists $r \in (0, \varepsilon)$ such that $\tau \llcorner M_r \in I_k(M_r; G)$ and*

$$\boldsymbol{M}(\tau \llcorner \partial M_r) \leq c(\varepsilon) \frac{\boldsymbol{M}(\tau)}{\varepsilon}.$$

Moreover, if $\tau \in \mathfrak{X}_k(M, \partial M; G)$, then $\tau \llcorner M_r \in \mathfrak{X}_k(M_r, \partial M_r; G)$ and $\partial(\tau \llcorner M_r) \in \mathfrak{X}_{k-1}(\partial M_r; G)$.

2.2 Cubical complexes

We will be considering families of flat chains parametrized by a cubical complex. We start with some definitions that will be convenient when working with these complexes, following [21].

Given $m \in \mathbb{N}$, let I^m denote the m-dimensional cube $[0, 1]^m$. For each $j \in \mathbb{N}$, $I(1, j)$ denotes the cube complex on I^1 whose 1-cells and 0-cells (vertices) are, respectively,

$$[0, 3^{-j}],\ [3^{-j}, 2 \cdot 3^{-j}],\ \dots,\ [1 - 3^{-j}, 1] \quad \text{and} \quad [0],\ [3^{-j}],\ \dots,\ [1 - 3^{-j}],\ [1].$$

We denote by $I(m, j)$ the cell complex on I^m:

$$I(m, j) = \overbrace{I(1, j) \otimes \cdots \otimes I(1, j)}^{m \text{ times}}.$$

Then $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ is a k-cell of $I(m, j)$ if and only if α_i is a cell of $I(1, j)$ for each i, and $\sum_{i=1}^{m} \dim(\alpha_i) = k$. We will abuse notation by identifying a k-cell α with its support: $\alpha_1 \times \cdots \times \alpha_m \subset I^m$.

Let X be a cubical subcomplex of $I(m, j)$. For $i \geq j$ let $X(i)$ denote a subcomplex of $I(m, i)$, whose cells lie in a cell of X. In other words, $X(i)$ is obtained from X by subdividing each k-cell of X into $3^{q(i-j)}$ smaller cells. We use the notation $X(i)_k$ to denote the set of all k-cells in $X(i)$. If E is a cell of $X(i)$, to simplify notation we will write E_k to denote the k-skeleton of the cell E (dropping (i)) and $E(i')_k$ to denote the k-skeleton of $X(i+i') \cap E$. We write X^p to denote a cubical subcomplex with cells of maximal dimension p.

2.3 Approximation theorem

Let $G = \mathbb{Z}$ or $\mathbb{Z}_p$. Assume that M is a manifold with piecewise smooth boundary ∂M with θ-corners for some $\theta \in \left(0, \frac{\pi}{2}\right)$. Let $\mathcal{Z}_k(M, \partial M; G)$ denote the space of relative cycles with coefficients in group G. Let $\mathcal{F}$ denote the flat distance in $\mathcal{Z}_k(M, \partial M; G)$.

Each individual cycle in a continuous family $\{F(x)\}$ of flat cycles can be well-approximated by a polyhedral cycle. However, close to every nice looking cycle in the family there could be a sequence of some wilder and wilder looking wiggly cycles converging to it. For example, their supports may converge in Hausdorff topology to the whole space. This makes it hard to apply some continuous deformations or surgeries to the family. It is desirable to have an approximation theorem which tells us that we can pick a sufficiently fine discrete subset of the parameter space, and the values of F on that discrete subset uniquely determine the homotopy class of the family, and, moreover, if we apply some continuous deformation to this discrete family so that it satisfies a certain property, we could interpolate to obtain a continuous family which also satisfies this property.

This gives rise to discrete and continuous settings in min-max theory; see eg [28; 23; 24].

The main result of this section is the following approximation theorem.

Theorem 2.3 (1) *There exists $c = c(M, p) > 0$ such that the following holds. For every $\eta > 0$ there exists $\varepsilon_0 > 0$ such that if $F : X^p(q)_0 \to \mathcal{Z}_k(M, \partial M; G)$ is a map with $\mathcal{F}(F(a), F(b)) < \varepsilon_0$ for every pair of adjacent vertices a and b, then there exists an extension of F to a continuous map $F : X \to \mathcal{Z}_k(M, \partial M; G)$ such that for each cell C in $X(q)$ and $x, y \in C$, we have $\mathcal{F}(F(x), F(y)) < c\eta \sup_{w \in X(q)_0} \boldsymbol{M}(F(w))$.*

(2) *There exists $\varepsilon(M, p) > 0$ such that if $F_0 : X \to \mathcal{Z}_k(M, \partial M; G)$ and $F_1 : X \to \mathcal{Z}_k(M, \partial M; G)$ are two maps with $\mathcal{F}(F_0(x), F_1(x)) < \varepsilon$ for all x and $\sup_{x \in X} \boldsymbol{M}(F_i(x)) < \infty$ for $i = 0, 1$, then F_0 and F_1 are homotopic.*

Informally, Theorem 2.3 means that we can replace a continuous family of cycles with a sufficiently dense (in the parameter space) discrete family. We can then perturb each of the cycles in the discrete family and complete them to a new continuous family. This new continuous family will be homotopic to the original family. In general, our procedure may increase the mass of cycles by a constant $c(p, M)$.

In the last subsection we show that if we assume that the family of cycles satisfies no concentration of mass condition (and without this condition in the case of 0-cycle), then the mass will only increase by an amount that goes to 0 as ε goes to 0.

Our method is to first modify the family so that for a small ball B in the parameter space and any two points $x, y \in B$, the difference $F(x) - F(y)$ lies in some fixed (and depending only on B) collection of disjoint convex sets. We will call such families "δ-localized". Then we can interpolate between $F(x)$ and $F(y)$ using radial contraction inside these convex sets.

2.4 Admissible collections of open sets in M

Pick $r_0 < \operatorname{inj}\operatorname{rad}(M)$. If M has smooth boundary, let $E_{\partial M} \colon \partial M \times [0, r_0] \to M$ denote the normal exponential map from the boundary of M.

More generally, let $E_{\partial M}$ denote the map E from Lemma 2.1.

We say that a collection of open set $U_l \subset M$ is r-admissible if they are all disjoint and:

(1) If U_l is disjoint from ∂M, then U_l is a ball of radius r_l.

(2) If U_l is not disjoint from ∂M, then $U_l = E_{\partial M}(B_{r_l} \times [0, r_l])$.

(3) $\sum r_l < r$.

We need the following two elementary lemmas about collections of r-admissible sets.

Lemma 2.4 *Let $\mathcal{B}_1 = \{\beta_i\}$ be a finite set of balls in ∂M and $\mathcal{B}_2 = \{B_j\}$ be a finite set of disjoint balls contained in the interior of M. There exists an r-admissible collection $\{U_l\}$ such that*

$$\bigcup U_l \supset \Big(\bigcup E_{\partial M}\big(\beta_i \times [0, \operatorname{rad}(\beta_i))\big)\Big) \cup \bigcup B_j$$

and $r \le \sum \operatorname{rad}(\beta_i) + 3 \sum \operatorname{rad}(B_j)$.

Proof If β_{i_1} and β_{i_2} intersect, then we can find a ball $\beta \subset \partial M$ of radius $\le \operatorname{rad}(\beta_{i_1}) + \operatorname{rad}(\beta_{i_2})$ that contains their union. Hence, we may replace $\{\beta_i\}$ with a collection of disjoint balls containing the union of original balls without increasing the sum of their radii. Let's call this replacement operation $(*)$.

If B_j intersects $E_{\partial M}(\beta_i \times [0, \operatorname{rad}(\beta_i)])$, then there exists a β' of radius $\le \operatorname{rad}(\beta_i) + 3\operatorname{rad}(B_j)$, so that $B_j \subset E(\beta' \times [0, \operatorname{rad}(\beta')])$. (Note that instead of the factor 3 we could have used $2 + \varepsilon$ for $\varepsilon \to 0$ as $\operatorname{rad}(B_j) \to 0$.) We define new sets $\mathcal{B}_1' = \mathcal{B}_1 \setminus \{\beta_i\} \cup \{\beta'\}$ and $\mathcal{B}_2' = \mathcal{B}_2 \setminus \{B_j\}$. By $(*)$ we can replace $\mathcal{B}_1'$ with a collection of disjoint balls $\mathcal{B}_1''$. Let $(**)$ denote the operation of replacing $\mathcal{B}_1$ and $\mathcal{B}_2$ with $\mathcal{B}_1''$ and $\mathcal{B}_2'$. Note that $(**)$ decreases the number of balls in $\mathcal{B}_2$ by 1, decreases the sum of radii of balls in $\mathcal{B}_2$ by $\operatorname{rad}(B_j)$ and increases the sum of radii of balls in $\mathcal{B}_1$ by $3\operatorname{rad}(B_j)$.

We perform $(**)$ repeatedly, until $E_{\partial M}(\beta_i \times [0, \operatorname{rad}(\beta_i)])$ and B_j are all disjoint. The process terminates since the number of balls in $\mathcal{B}_2$ was finite. □

Lemma 2.5 *Let $\{U_l^1\}_{l=1}^{v_1}$ be an r_1-admissible and $\{U_l^2\}_{l=1}^{v_2}$ be an r_2-admissible collection of open sets. Then there exists an r-admissible collection $\{U_l\}_{l=1}^{v}$ with $v \le v_1 + v_2$, $\bigsqcup U_l^1 \cup \bigsqcup U_l^2 \subset \bigcup U_l$ and $r \le 3(r_1 + r_2)$.*

Proof We replace the collection of all $\{U_{l_i}^1\}$ and $\{U_{l_j}^2\}$, which are balls in the interior of M, with a collection $\{B_i\}$ of disjoint balls that contain their union. We can do this with

$$\sum \operatorname{rad}(B_i) \le \sum \operatorname{rad}(U_{l_i}^1) + \sum \operatorname{rad}(U_{l_j}^2).$$

Some of the balls B_i may intersect ∂M. We replace them with $\exp(\beta_i \times [0, r']) \supset B_i$ for $\beta_i \subset \partial M$ and $r' \le 3 \operatorname{rad}(B_i)$.

Now we are in the situation where we can apply Lemma 2.4. $\square$

2.5 Localized families

Let $V \subset X(q)$. We will say that a map $F\colon V \to \mathcal{Z}_k(M, \partial M; G)$ is ε-fine if for every cell C in $X(q)$ and $a, b \in C \cap V$ we have $\mathcal{F}(F(a), F(b)) \le \varepsilon$. We will say that a map $F\colon V \to \mathcal{Z}_k(M, \partial M; G)$ is δ-localized if for every cell C of $X(q)$ there exists a δ-admissible collection of open sets $\{U_l^C\}$ such that for every pair $a, b \in C \cap V$ we have the support $\operatorname{supp}(F(a) - F(b)) \subset \bigcup U_l^C$. Given a cell C in $X(q)$ let $C_0 = C \cap X(q)_0$.

Proposition 2.6 *There exist $\delta_0(M, p) > 0$, $c(n, k, p) > 0$ and $\varepsilon_0(M, p) > 0$ with the following property. Let $F\colon X^p(q)_0 \to \mathcal{Z}_k(M, \partial M; G)$ be a δ-localized map, with $\delta < \delta_0$.*

(I) *There exists an extension map $F\colon X(q) \to \mathcal{Z}_k(M, \partial M; G)$ such that:*

  (1) *The extension is $c\delta$-localized.*

  (2) *For every cell C and $x, y \in C$, we have*

$$\mathcal{F}(F(x), F(y)) < c\delta \max_{w \in C_0} \boldsymbol{M}\Big(F(w) \llcorner \bigcup U_l^C\Big).$$

  (3) *For every cell C and $x \in C$, we have*

$$\boldsymbol{M}(F(x)) \le \max_{w \in C_0} \boldsymbol{M}(F(w)) + c \max_{w \in C_0} \boldsymbol{M}\Big(F(w) \llcorner \bigcup U_l^C\Big).$$

(II) *Suppose in addition that F is defined on some subcomplex $Y(q) \subset X(q)$ and $F\colon Y \to \mathcal{Z}_k(M, \partial M; G)$ is δ-localized and ε-fine, where $\delta < \delta_0$ and $\varepsilon < \varepsilon_0$. Then there exists an extension map*

$$F\colon X(q) \to \mathcal{Z}_k(M, \partial M; G)$$

such that:

  (1) *The extension is $c\delta$-localized.*

  (2) *For every cell C in $X(q)$ and $x, y \in C$, we have*

$$\mathcal{F}(F(x), F(y)) < c\delta \sup_{w \in C_0 \cup (C \cap Y)} \boldsymbol{M}(F(w))$$

  and $\boldsymbol{M}(F(x)) \le c \max_{w \in C_0 \cup (C \cap Y)} \boldsymbol{M}(F(w))$.

Proof We prove (I). Part (II) follows with some straightforward modifications.

We start by extending to the 1-skeleton.

Let C be an m-dimensional cell. Assume we have extended F for all $x \in \partial C$. For each $(m-1)$-dimensional cell $C' \subset \partial C$ assume, inductively, that there exists a $c(m-1)\delta$-admissible collection $\mathcal{B}(C') = \{U_1^{C'}, \dots, U_{l(C')}^{C'}\}$ with $\operatorname{supp}(F(x) - F(y)) \subset \bigcup U_i^{C'}$ for all $x, y \in C'$.

By Lemma 2.5 there exists a $c(m) > 0$ and a $c(m)\delta$-admissible collection $\mathcal{B}(C) = \{U_1^C, \dots, U_{l(C)}^C\}$ of open sets such that

$$\bigcup_{i;\, C' \subset \partial C} U_i^{C'} \subset \bigcup_i U_i^C.$$

For each i we define contraction maps R_t^i as follows. If $U_i^C = B_{r_i}(p)$ is a ball centered at p, we let $R_t^i \colon B_{r_i}(p_i) \to B_{tr_i}(p_i)$ be the radial contraction map. If $U_i^C = E(\beta_{r_i}(p_i) \times [0, r_i])$, we define $R_t^i(E(x,r)) = E(tx, tr)$.

Let $v(C)$ denote the vertex in 0-skeleton of C that is closest to 0 (thinking of C as the subset of the unit cube).

Define a function $\sigma(x,t)$ on $\partial C \times [0,1]$ by

$$\sigma(x,t) = F(x) + \sum_i R_t^i\big((F(v(C)) - F(x)) \llcorner U_i^C\big).$$

Since $\sigma(x,1) = F(v(C))$ for all $x \in \partial C$, this gives a well-defined extension of F to C.

Now we would like to verify the bounds on $\mathcal{F}$ and the mass. By taking a cone over $F(x) - F(y) \subset \bigcup U_l^C$ we verify $\mathcal{F}(F(x), F(y)) \le \delta\big(\boldsymbol{M}\big(F(x) \llcorner \bigcup U_l^C\big) + \boldsymbol{M}\big(F(y) \llcorner \bigcup U_l^C\big)\big) \le c(p)\delta \max_{w \in C_0} \boldsymbol{M}\big(F(w) \llcorner \bigcup U_l^C\big)$. Inductively we can check that

$$\boldsymbol{M}(F(x)) \le \max_{w \in C_0} \boldsymbol{M}(F(w)) + c(p) \max_{w \in C_0} \boldsymbol{M}\Big(F(w) \llcorner \bigcup U_l^C\Big)$$

for sufficiently large $c(p)$. □

The following proposition is the key technical result necessary for the proof of Theorem 2.3. It asserts that every sufficiently fine (in the flat topology) discrete family of cycles can be approximated by a δ-localized discrete family of cycles.

Proposition 2.7 *Given $\delta > 0$, there exists $\varepsilon_0(p, M, \delta) > 0$ with the following property. Let $f : X^p(q)_0 \to \mathcal{Z}_k(M, \partial M; G)$ be a map with $\mathcal{F}(f(x), f(y)) < \varepsilon$ for any two adjacent vertices x and y and $\varepsilon \in (0, \varepsilon_0)$. Then there exists $\tilde{q}(p, M, \delta)$ and a map $F : X(q + \tilde{q})_0 \to \mathcal{Z}_k(M, \partial M; G)$ that coincides with f on $X(q)_0$, such that for every cell C in $X(q)$, we have:*

1. *$\mathcal{F}(f(x), F(y)) < c(M, p, \delta)\varepsilon$ for all $x \in C \cap X(q)_0$ and $y \in C \cap X(q + \tilde{q})_0$.*
2. *$\boldsymbol{M}(F(y)) \le \sup_{w \in C \cap X(q)_0} \boldsymbol{M}(f(w)) + c(M, p, \delta)\varepsilon$ for $y \in C \cap X(q + \tilde{q})_0$.*
3. *F is δ-localized. Moreover, if $E \subset C$ is a cell of $X(q + \tilde{q})$, then for the corresponding δ-admissible collection $\{U_i^E\}_{i=1}^{n(p)}$, we have $\boldsymbol{M}\big(F(y) \llcorner \bigcup U_i^E\big) \le \sup_{w \in C \cap X(q)_0} \boldsymbol{M}\big(f(w) \llcorner \bigcup U_i^E\big)$.*

Proof First we define F on $X(q)_0$ setting $F(x) = f(x)$.

We will define a sequence of numbers $q = q_0 < \cdots < q_p = \tilde{q}$ and maps $F_j\colon X(q+q_j)_0 \cap X(q)_j \to \mathfrak{Z}_k(M, \partial M; G)$ for $j = 0, \dots, p$. The map $F_p\colon X(q+\tilde{q})_0 \to \mathfrak{Z}_k(M, \partial M; G)$ will be the desired localized extension.

We organize our proof as follows. First, we collect all definitions that will be needed in the proof. Then we prove a key lemma constructing a localized family of chains filling a given localized family of cycles. Finally, we give a proof of the inductive step of the construction.

Definitions and notation

- Let $r_1(M)$ be sufficiently small, so that every ball of radius r_1 that lies in M is 1.01-bilipschitz diffeomorphic to a convex set in $\mathbb{R}^n$, and for every ball $\beta(r_1)$ in ∂M we have that $E_{\partial M}$ is a $c(\theta)$-bilipschitz diffeomorphism on $\beta(r_1) \times [0, r]$.

- Let $r_0(M, \delta) = \min\{\frac{1}{2} r_1, \delta\}$ and consider a collection of sets $\{B_1, \dots, B_N\}$ covering M such that each B_i is either a closed ball of radius r_0, or $B_i = E_{\partial M}(\beta_i, [0, r_0])$, where β_i is a closed ball of radius r_0 in ∂M. We will call such sets "generalized balls". Note that we can choose the covering $\{B_i\}$ so that the number N of generalized balls in the covering satisfies $N \leq c(M)/\delta^n$. Let tB_i with $t > 0$ denote the concentric generalized ball with the same center point and the radius tr_0.

Given a chain τ by the coarea inequality and Lemma 2.1 there exist generalized balls B_i' with $B_i \subset B_i' \subset 2B_i$ such that

$$\boldsymbol{M}(\tau \llcorner \partial B_i') \leq \frac{c(\theta)}{r_0} \boldsymbol{M}(\tau \llcorner 2B_i). \tag{2-1}$$

It will be convenient to use $\{B_i'\}$ to define a collection of sets with disjoint interiors as follows: we let $U_1 = B_1'$ and let U_i be the closure of $B_i' \setminus \bigcup_{j=1}^{i-1} B_j'$ for $i > 1$. We will say that $\{U_i\}$ is a coarea covering for τ. Note that $U_i \subset B_i' \subset 2B_i$ for each i, and

$$\boldsymbol{M}\Bigl(\tau \llcorner \bigcup_{i=1}^{N} \partial U_i\Bigr) \leq \boldsymbol{M}\Bigl(\tau \llcorner \bigcup_{i=1}^{N} \partial B_i'\Bigr) \leq \frac{c(\theta)N}{r_0} \boldsymbol{M}(\tau).$$

More generally, given a finite collection $\{\tau_j\}_{j=1}^{L}$, applying coarea inequality to $\sum \tau_j$, there exists a collection of generalized balls B_i' with $B_i \subset B_i' \subset 2B_i$ such that for each j we have

$$\boldsymbol{M}(\tau_j \llcorner \partial B_i') \leq \frac{c(\theta)L}{r_0} \max_{i=1,\dots,L} \boldsymbol{M}(\tau_i \llcorner 2B_i). \tag{2-2}$$

As above we set $U_1 = B_1'$ and $U_i = B_i' \setminus \bigcup_{j=1}^{i-1} B_j'$ for $i > 1$ and call $\{U_i\}$ a coarea covering for $\{\tau_j\}_{i=1}^{L}$. We then have

$$\boldsymbol{M}\Bigl(\tau \llcorner \bigcup_{i=1}^{N} \partial U_i\Bigr) \leq \frac{c(\theta)NL}{r_0} \max_{j=1,\dots,L} \boldsymbol{M}(\tau_j). \tag{2-3}$$

- It will be convenient to define a way of chopping away a chain τ_j by intersecting it with sets from the coarea covering $\{U_i\}$ of τ_j. Given a collection of chains $\{\tau_j\}_{j=1}^L$ and a coarea covering $\{U_i\}_{i=1}^N$ for $\{\tau_j\}$, we let

$$d_i(\tau_j, \{U_i\}) = \tau_j - \tau_j \llcorner \bigcup_{j=1}^{i} U_j. \tag{2-4}$$

We then have the following mass bound for the boundary of $\partial d_i(\tau_j)$:

$$\boldsymbol{M}(\partial d_i(\tau_j)) \le \boldsymbol{M}(\partial \tau_j) - \sum_{l=1}^{i} \boldsymbol{M}(\partial \tau \llcorner U_l) + \frac{c(\theta)NL}{r_0} \boldsymbol{M}(\tau_j). \tag{2-5}$$

- For each cell $C \subset X(q)$ let $x(C) \in C \cap X(q)_0$ denote the vertex of C that is closest to $(0, \dots, 0)$ (thinking of $X(q)$ as contained in the ambient unit cube of large dimension).

- $\mathrm{dist}_\infty(x, y) = \min_j \{|x_j - y_j|\}$.

- For a face $E^l \subset X(q)_l$, define $\mathrm{Center}(E)$ to be the collection of all points x in E with $\mathrm{dist}_\infty(x, \partial E) > 1/3^{q+1}$.

- It will be useful to define maps between 0-skeleta of subdivisions of cells with nice properties. These constructions are similar to [23, Appendix C].

Given a cell E of $X(q)$ and $x \in E$ we let $[x]_{q'}$ denote the closest point of $E(q')_0$ to x; if there is more than one vertex of $E(q')_0$ that minimizes $d_\infty(x, \cdot)$, then we set $[x]_{q'}$ to be the vertex closest to $(0, \dots, 0)$ in the ambient cube $I^m \supset X \supset E$.

- We will say that a map $G\colon Y(q')_0 \to Y(q'')_0$, for $q' \ge q''$, is an adjacency-preserving extension if
  1. $G(x) = x$ for all $x \in Y(q'')_0$,
  2. for any two $x, y \in Y_1(q')_0$ that lie in the same face of $Y_1(q')$, we have that $G(x)$ and $G(y)$ are in the same face of $Y_2(q'')$.

- Given a cell E of $Y(q)$ and $q' > q''$ define $H_{q',q''}\colon E(q')_0 \to E_0(q'')_0$ by $H_{q',q''}(x) = [x]_{q''}$. We can extend this definition to a map $H_{q',q''}\colon X(q+q')_0 \to X(q+q'')_0$ by inductively applying $H_{q',q''}$ on each face. We observe that $H_{q',q''}$ is an adjacency-preserving extension. It will be convenient to write $H_{q'}$ for $H_{q',0}$.

- Let $L_t(E) = \{x \in E : d_\infty(x, \partial E) = t\}$ and let $\phi_t\colon L_t(E) \to \partial E$ be a bijective homothety from the center point of E. We define maps

$$P\colon E(q'+1)_0 \setminus \mathrm{Center}(E) \to \partial E(q')_0 \times I^1(q')_0, \quad Q\colon \partial E(q')_0 \times I^1(q')_0 \to E(q')_0 \setminus \mathrm{Center}(E),$$

by setting

$$P_{q'}(x) = ([L_{d_\infty(x,\partial E)}(x)]_{q'}, 3^{q+1} d_\infty(x, \partial E)), \tag{2-6}$$

$$Q_{q'}(x, t) = [L^{-1}_{t/3^{q+1}}(x)]_{q'}. \tag{2-7}$$

Note that the composition map

$$Q \circ P(x)\colon E(q'+1)_0 \setminus \mathrm{Center}(E) \to E(q')_0 \setminus \mathrm{Center}(E)$$

is an adjacency-preserving extension. Moreover, we have $Q \circ P(x) = H_{q'+1,q'}(x)$ for $x \in \partial E(q')_0$.

Lemma 2.8 *There exists $\varepsilon_0(k,j,M,\delta) > 0$ with the following property. Suppose that $F\colon Y^j(q)_0 \to \mathcal{Z}_k(M,\partial M;G)$ is a δ-localized map with $\mathcal{F}(F(x),0) < \varepsilon$ for all $x \in Y^j(q)_0$ and some $\varepsilon < \varepsilon_0(k,j,M,\delta)$. Then there exist $\overline{q}(k,j,M,\delta)$, $c_1(k,j,M)$, $c_2(k,j,M,\delta)$ and a map $\tau\colon Y^j(q+\overline{q})_0 \to I_{k+1}(M;G)$ such that τ is $c_1\delta$-localized, $\boldsymbol{M}(\tau(x)) < c_2\varepsilon$ and $\partial\tau(x) = F \circ H_{\overline{q}}(x)$.*

Proof The proof is by induction on $n-k$. If $n = k$, then taking $\tau(x) = 0$ gives the desired result.

Assume the lemma holds for families of $(k+1)$-cycles. To prove the result for families of k-cycles, we now proceed by induction on j.

If $j = 0$ the result is immediate. Assume the lemma holds for $(j-1)$-dimensional families. Applying it to the $(j-1)$-skeleton $Y(q)_{j-1}$ of $Y(q)$ we obtain a $c_1(k,j-1,M)\delta$-localized family

$$\tau_{j-1}\colon Y(q)_j \cap Y(q+q_{j-1})_0 \to I_{k+1}(M;G)$$

with $\partial\tau(x) = F \circ H_{q_{j-1}}(x)$ and a mass bound

$$\boldsymbol{M}(\tau_{j-1}(x)) \le c_2(k,j-1,M,\delta)\varepsilon.$$

Let C^j be a j-dimensional face of $Y(q)$. Let

$$Q\colon \partial C^j(q_{j-1})_0 \times I^1(q_{j-1})_0 \to C(q_{j-1})_0 \setminus \mathrm{Center}(C)$$

be the map defined as in (2-7). We will now construct a map

$$\tau_{\partial C\times I}\colon \partial C^j(q_{j-1})_0 \times I^1(q_{j-1})_0 \to I_{k+1}(M;G).$$

We set $\tau_{\partial C\times I}(x,0) = \tau_{j-1} \circ Q(x)$.

Since F is δ-localized there exists a δ-admissible collection of sets $\{U_i^C\}$ with $F(x) - F(x(C))$ supported in $\bigcup U_i^C$ for all $x \in C_0$. Hence, there exists a $(k+1)$-chain fill(x) supported in $\bigcup U_i^C$ with $\boldsymbol{M}(\mathrm{fill}(x)) < 2\varepsilon$ and $\partial\,\mathrm{fill}(x) = F(x) - F(x(C))$ for all $x \in C_0$. We let

$$\tau_{\partial C\times I}(x,1) = \tau_{j-1}(x(C)) + \mathrm{fill}(H_{q_{j-1}}(x)).$$

We define

$$\tilde{\tau}(x) = \tau_{\partial C\times I}(x,1) - \tau_{\partial C\times I}(x,0).$$

From the definition we have that $\{\tilde{\tau}(x)\}_{x\in\partial C(q_{j-1})_0}$ is a family of relative cycles.

Since $\{\tau_{\partial C\times I}(x)\}_{x\in\partial C(q_{j-1})_0}$ is $c(k,j-1,M)\delta$-localized, applying Lemma 2.5 we have that $\{\tilde{\tau}(x)\}$ is $c'\delta$-localized for $c' < 20c_1(j-1,k,M)$. We also have the mass bounds

$$\boldsymbol{M}(\tilde{\tau}(x)) \le \boldsymbol{M}(\tau_{\partial C\times I}(x,0)) + \boldsymbol{M}(\tau_{\partial C\times I}(x,1)) \le (2c_2(j-1,k,M,\delta)+1)\varepsilon.$$

Hence, we can apply the inductive assumption for families of $(k+1)$-cycles. We obtain that there exists a $c_1(k+1, j-1, M)c'\delta$-localized family $\{\sigma(x)\}_{x\in\partial C(q_{j-1}+q')_0}$ of $(k+2)$-chains, such that $\partial\sigma(x) = \widetilde{\tau} \circ H_{q_{j-1}+q', q_{j-1}}(x)$. Let $\{U_i\}$ be a coarea covering for the family $\{\sigma(x)\}_{x\in\partial C(q_{j-1}+q')_0}$.

Increasing q' if necessary we may assume that $q_{j-1} + q' \geq \log_3(N) + 1$. Let

$$n(t) = 3^{q_{j-1}+q'+1} \operatorname{dist}_\infty(t, \partial C \times \{1\}).$$

For $t \in I(q_{j-1} + q')_0$ we have

$$\tau'(x,t) = \tau_{\partial C\times I}(x,0) + \partial d_{n(t)}(\sigma(x), \{U_i\}).$$

Observe the following properties of τ':

(1) $\partial\tau'(x,t) = F_{j-1} \circ H_{q_{j-1}+q'}(x)$ for $t < 1$.

(2) $\tau'(x,0) = \tau_{\partial C\times I}(x)$.

(3) $\tau'(x,1) = \tau_{\partial C\times I}(H_{q_{j-1}+q', q_{j-1}}(x), 1) = \tau_{j-1}(x(C)) + \operatorname{fill}(H_{q_{j-1}+q'}(x))$.

By Lemma 2.5 we also have that this family is $c_1(k, j, M)\delta$-localized for $c_1(k, j, m)$ bounded in terms of $c_1(k, j-1, M)$, $c_1(k+1, j-1, M)$ and our choice of r_0. From (2-5) we have that

$$\boldsymbol{M}(\partial\tau'(x,t)) \leq c_2(k, j-1, M, \delta)\varepsilon + c_2(k+1, j-1, M, \delta)\frac{c(\theta)N3^{q'}}{r_0}\varepsilon.$$

Since r_0 was determined by δ and M and our choice of q' depended only on k, j and M, we can bound this expression by $c_2(k, j, M, \delta)\varepsilon$.

Hence, we've obtained a $c_1\delta$-localized family

$$\{\tau'(x,t)\}_{(x,t)\in\partial C(q_{j-1}+q')_0\times I(q_{j-1}+q')_0}.$$

We let $\overline{q} = q_{j-1} + q' + 1$. We can then define a family τ on $C(\overline{q})_0 \setminus \operatorname{Center}(C)$ by setting $\tau(x) = \tau' \circ P$ for the map $P\colon C(\overline{q})_0 \to \partial C(q_{j-1} + q')_0 \times I(q_{j-1} + q')_0$. For $x \in \operatorname{Center}(C) \cap C(\overline{q})_0$ we set $\tau(x) = \tau_{j-1}(x(C)) + \operatorname{fill}(H_{\overline{q}}(x))$. It follows from the above estimates that $\{\tau(x)\}$ is a $c_1(k, j, M)\delta$-localized family with $\boldsymbol{M}(\tau(x)) \leq c_2(k, j, M, \delta)\varepsilon$. Defining τ in this way on each j-face we obtain the desired family. □

We can now prove Proposition 2.7. In the calculations below let c_1 denote a constant that only depends on M, p and k. The value of c_1 may change from line to line.

We will define a sequence of $c_1\delta$-localized maps $F_j\colon X(q)_j \cap X(q_j)_0 \to \mathcal{Z}_k(M, \partial M; G)$ with $q_0 = q$ and satisfying $F_j(x) = F_{j-1} \circ H_{q_j, q_{j-1}}(x)$ for $x \in X(q)_j \cap X(q + q_{j-1})_0$ and such that for every cell E^l of $X(q)$ for $l \leq j-1$, we have

(i) $\mathcal{F}(f(x), F(y)) < c(M, j-1, \delta)\varepsilon$ for $x \in E \cap X(q)_0$ and $y \in E \cap X(q + q_{j-1})_0$,

(ii) $\boldsymbol{M}(F(y)) \leq \sup_{w\in E\cap X(q)_0} \boldsymbol{M}(f(w)) + c(M, p, \delta)\varepsilon$ for $y \in E \cap X(q + q_{j-1})_0$,

(iii) if $D \subset E$ is a cell of $X(q + q_{j-1})$ and $\{U_i^D\}$ is the corresponding $c_1\delta$-admissible collection, then $\boldsymbol{M}\big(F(y) \llcorner \bigcup U_i^D\big) \leq \sup_{w\in E\cap X(q)_0} \boldsymbol{M}\big(f(w) \llcorner \bigcup U_i^D\big)$ for $y \in E \cap X(q + q_{j-1})_0$.

First we set $F_0(x) = F(x)$ for $x \in X(q)_0$. Assume we have defined F_{j-1} on $X(q)_{j-1} \cap X(q_{j-1})_0$. For each j-dimensional face C of X, by Lemma 2.8 there exists a $c_1\delta$-localized family $\{\tau(x)\}_{x \in \partial C(q_{j-1}+\overline{q})_0}$ such that $\partial\tau(x) = F(x(C)) - F_{j-1} \circ H_{q_{j-1}+\overline{q},q_{j-1}}(x)$. We let $q_j = q_{j-1} + \overline{q} + 1$ and define

$$F_j(x) = F_{j-1} \circ H_{q_j,q_{j-1}}(x) = F \circ H_{q_j}(x) \quad \text{on } X(q)_{j-1} \cap X(q_j)_0. \tag{2-8}$$

We now fix j-dimensional cell C and describe the extension of F_j from $\partial C \cap X(q_j)_0$ to $C \cap X(q_j)_0$.

Let $\{U_i\}$ be a coarea covering for the family $\{\tau(x)\}_{x \in \partial C(q_{j-1}+\overline{q})_0}$. To obtain the desired mass bound for F_j we will need the following construction. We defined F_j by composing F_{j-1} with an adjacency-preserving extension on ∂C and now we'd like to extend to the interior of C by "chopping away" $\tau(x)$ using (2-4), taking the boundary of the resulting $(k+1)$-chain and adding it to the value of F_j on the boundary, until we deform the family of $\{F_j(x)\}_{x \in \partial C(q_j)_0}$ to the constant family equal to $F(x(C))$ on Center(C). The problem with this approach is that the restriction $F_j(x) \llcorner U_i$ for $x \in \partial C(q_j)_0$ can be large for some values of i, while $F(x(C)) \llcorner U_i$ can be large for other values of i, so the interpolation may increase the total mass to more than $\boldsymbol{M}\big(F(x(C))\big) + \boldsymbol{M}(F(y))$ (for some $y \in C_0$), violating mass bounds (ii) and (iii) above. Instead, we will make a mass-minimizing choice in each U_i that will guarantee that the mass bounds are satisfied.

For each $i = 1, \dots, N$ we let $m_i = \tau(x(i)) \llcorner U_i$, where $x(i) \in \partial C(q_{j-1} + \overline{q})_0$ is such that

$$\boldsymbol{M}\big(\partial\tau(x(i)) \llcorner U_i - F(x(C)) \llcorner U_i\big) \le \boldsymbol{M}\big(\partial\tau(x) \llcorner U_i - F(x(C)) \llcorner U_i\big)$$

for all $x \in \partial C(q_{j-1} + \overline{q})_0$. Define

$$b\big(\{\tau(x)\}, F(x(C)), \{U_i\}\big) = \sum_{i=1}^{N} m_i.$$

From estimate (2-5) we have the following mass bounds for $b\big(\{\tau(x)\}, F(x(C)), \{U_i\}\big)$ and its boundary:

$$\boldsymbol{M}\big(b\big(\{\tau(x)\}, F(x(C)), \{U_i\}\big)\big) \le \sum_{x \in C(q_{j-1}+\overline{q})_0} \boldsymbol{M}(\tau(x)), \tag{2-9}$$

$$\begin{aligned} \boldsymbol{M}\big(\partial b\big(\{\tau_j\}, F(x(C)), \{U_i\}\big) - F(x(C))\big) \le {} & \min_{x \in C(q_{j-1}+\overline{q})_0} \boldsymbol{M}\big(\partial\tau(x) - F(x(C))\big) \\ & + \frac{2^j N 3^{q_{j-1}+\overline{q}}}{r_0} \max_{x \in C(q_{j-1}+\overline{q})_0} \boldsymbol{M}(\tau(x)). \end{aligned} \tag{2-10}$$

We define a map

$$\tau' \colon \partial C(q_{j-1} + \overline{q})_0 \times I(q_{j-1} + \overline{q})_0 \to I_{k+1}(M; G)$$

as follows. Let $n(t) = 3^{q_{j-1}+\overline{q}}(1-t)$; without any loss of generality we may assume $N < 3^{q_{j-1}+\overline{q}}$. Set

$$\tau'(x,t) = \tau \circ H_{q_j,q_j-1}(x) - d_{n(t)}\big(\tau \circ H_{q_j,q_j-1}(x) - b\big(\{\tau(x)\}, F(x(C)), \{U_i\}\big)\big). \tag{2-11}$$

Finally, we let

$$F_j(x) = F(x(C)) - \partial\tau' \circ P_{q_j-1}(x) \quad \text{for } x \in C(q_j)_0 \setminus \text{Center}(C).$$

Observe that this definition coincides with (2-8) on $\partial C(q_j)_0$. For $x \in C(q_j)_0 \cap \mathrm{Center}(C)$ we define

$$F_j(x) = F(x(C)) - \partial b\bigl(\{\tau(x)\}, F(x(C)), \{U_i\}\bigr).$$

The flat distance bound (i) follows from (2-9); the mass bounds (ii) and (iii) follow from (2-10) and our construction of b. □

Remark 2.9 If M is a PL manifold, then we can perform the above construction starting with a polyhedral approximation $F(x)$ for each cycle $f(x)$, with $x \in X(q)_0$. The construction in the proof of Proposition 2.7 then gives a discrete family $F\colon X(q')_0 \to \mathcal{Z}_k^{\mathrm{PL}}(M, \partial M; G)$ of polyhedral cycles, and Proposition 2.6 gives a continuous family $F\colon X \to \mathcal{Z}_k^{\mathrm{PL}}(M, \partial M; G)$ of polyhedral cycles. Hence, whenever it is convenient, we can replace a Riemannian metric with a PL approximation and a family of flat cycles with an approximating family of polyhedral cycles. All relative k-dimensional cycles in our construction may be chosen so that $F(x) \cap \partial M = \partial F(x)$ and is a $(k-1)$-dimensional polyhedral cycle in ∂M.

Combining Proposition 2.6 and 2.7 we obtain the following proposition.

Proposition 2.10 *For all $\delta > 0$ sufficiently small there exists $\varepsilon_1 > 0$ such that the following holds. Given a map $F\colon X(q)_0 \to \mathcal{Z}_k(M, \partial M; G)$, which is ε_1-fine, there exists q' and a continuous extension $F\colon X(q+q') \to \mathcal{Z}_k(M, \partial M; G)$ such that:*

(1) *$F\colon X(q+q') \to \mathcal{Z}_k(M, \partial M; G)$ is δ-localized.*

(2) *For every cell C of $X(q)$, $x \in C_0$ and $y \in C \cap X(q+q')_0$, we have*

$$\mathcal{F}(F(x), F(y)) < c(M, p)\, \delta \max_{w \in X(q)_0} \boldsymbol{M}(F(w)).$$

(3) *$\boldsymbol{M}(F(x)) \le c(M, p)(1+\delta) \max_{w \in C_0} \boldsymbol{M}(F(w))$.*

Proof Follows by applying Propositions 2.7 and 2.6. □

Theorem 2.3(1) directly follows from Proposition 2.10. To prove Theorem 2.3(2) we need the following lemma.

Lemma 2.11 *For every sufficiently small $\delta > 0$, there exists $\varepsilon_2(M, p, \delta) > 0$ such that the following holds. Let $F_0\colon X(q) \to \mathcal{Z}_k(M, \partial M; G)$ and $F_1\colon X(q) \to \mathcal{Z}_k(M, \partial M; G)$ be two δ-localized maps. Suppose $\mathcal{F}(F_0(x), F_1(x)) \le \varepsilon \le \varepsilon_2$. Suppose*

$$m = \max\Bigl\{\sup_{w \in X} \boldsymbol{M}(F_0(w)), \sup_{w \in X} \boldsymbol{M}(F_1(w))\Bigr\} < \infty.$$

Then there exists a homotopy $F\colon X(q) \times I(1, q) \to \mathcal{Z}_k(M, \partial M; G)$ between F_0 and F_1 such that

(1) *$\mathcal{F}(F(x,t), F_0(x)) < c(p, M) m\delta + \varepsilon_2$ for all $x \in X$,*

(2) *$\boldsymbol{M}(F(x,t)) \le c(p, M) m$ for all $(x,t) \in X \times I$.*

Proof First we will define F restricted to $X(\tilde{q}) \times \{0\} \cup X(\tilde{q})_0 \times I(1,\tilde{q})_0 \cup X(\tilde{q}) \times \{1\}$ for some $\tilde{q} \geq q$ using the construction from Proposition 2.7. Then we will apply Proposition 2.6(II) to extend F to a continuous map defined everywhere.

Let $F|_{X(q)\times 0} = F_0$ and $F|_{X(q)\times 1} = F_1$. Applying the inductive step extension argument from the proof of Proposition 2.7, we define a $c\delta$-localized map F^1 on $v \times I(q_1)_0$ for each $v \in X(q)_0$, such that $F^1(v,t) = F(v,t) = F_t(v)$ for $t = 0, 1$. Note that $\mathcal{F}(F(v,t), F_0(v)) \leq \varepsilon_2$. Using the same extension argument we can then extend F^1 to a δ-localized map on $(X(q+q_1)_0 \times I(q_1)_0) \cap (X(q)_1 \times I(q_1)_0) \setminus X(q) \times \{0,1\}$. If we define $F^1(x) = F(x)$ on $X(q) \times \{0,1\}$, then by Lemma 2.5, F^1 will still be $c\delta$-localized. This defines F^1 on $(X(q) \times I(q))_1 \cap (X(q_1) \times I(q_1))_0$. We apply the inductive step from the proof of Proposition 2.7 to define F^i for $i = 2, \dots, p$, until we obtain a $c\delta$-localized map on $X(q_p)_0 \times I(q_p)$. We can then apply Proposition 2.6(II) to extend F to a continuous map defined everywhere. The mass and flat norm estimates follow from the corresponding estimates in Proposition 2.6 and the inductive step in the proof of Proposition 2.7. □

Let F_0 and F_1 be two maps as in Theorem 2.3(2). Let $m = \sup_{w\in X} \boldsymbol{M}(F_0(w)) + \sup_{w\in X} \boldsymbol{M}(F_1(w))$.

By Proposition 2.10, for every sufficiently small $\delta > 0$, we can define δ-localized maps $\bar{F}_0$ and $\bar{F}_1$ which agree with F_0 and F_1 on the 0-skeleton of some subdivision $X(q')_0$. By Proposition 2.10 we then have

$$\mathcal{F}(\bar{F}_0(x), \bar{F}_1(x)) \leq c(M,p)m\delta + \varepsilon.$$

For $\delta > 0$ and $\varepsilon > 0$ sufficiently small we have that $\bar{F}_0$ and $\bar{F}_1$ are homotopic by Lemma 2.11.

Now we would like to construct a homotopy between $\bar{F}_i$ and F_i for $i = 0, 1$.

Let δ_i be a decreasing sequence of positive numbers converging to 0 and $\eta_i = \min\{\delta_i, \varepsilon_2(\delta_i)\}$, where $\varepsilon_2(\delta_i) > 0$ is from Proposition 2.10. Let $q_i \geq q$ be an increasing sequence of integers, so that $F_0\colon X(q_i) \to \mathcal{Z}_k(M,\partial M;G)$ is ε_i-fine. Applying Proposition 2.10 we obtain a sequence of δ_i-localized maps $G_i\colon X(q_i') \to \mathcal{Z}_k(M,\partial M;G)$. We claim that $G_i \to F_0$. Given $x \in X$, let x_i denote the closest point of $X(q_i)_0$ to x. Then

$$\begin{aligned}\mathcal{F}(G_i(x), F_0(x)) &\leq \mathcal{F}(G_i(x), G_i(x_i)) + \mathcal{F}(G_i(x_i), F_0(x_i)) + \mathcal{F}(F_0(x_i), F_0(x)) \\ &\leq c(M,p)m\delta_i + \varepsilon_i.\end{aligned} \tag{2-12}$$

The convergence follows by standard compactness results for flat cycles [6, 4.2.17].

To finish the construction we need to define a homotopy between G_i and G_{i+1}. Observe that if G_i is δ_i-localized with domain $X(q_i')$, then it is also δ_i-localized with domain $X(q_{i+1}')$ for $q_{i+1}' > q_i'$. It follows from inequality (2-12) that $\mathcal{F}(G_{i+1}(x), G_i(x)) \to 0$. Assuming that δ_i is small enough we can apply Lemma 2.11 to define a homotopy $H_i(x,t)$ between maps $G_i\colon X(q_{i+1}') \to \mathcal{Z}_k(M,\partial M;G)$ and $G_{i+1}\colon X(q_{i+1}') \to \mathcal{Z}_k(M,\partial M;G)$ for all $i > 0$. Moreover, the flat norm estimates from Proposition 2.10 and Lemma 2.11 imply that for each x and t the sequence $H_i(x,t)$ is Cauchy in $\mathcal{F}$. This implies continuity of the homotopy from $\bar{F}_0$ to F_0. The same construction works for the homotopy from $\bar{F}_1$ to F_1.

2.6 Homotopies with better estimates for the mass

We need to prove an analogue of Theorem 2.3 with a better estimate for the mass of cycles when the family satisfies some additional assumptions.

In the case of 0-cycles we prove approximation theorem with an optimal bound.

Proposition 2.12 *There exist $\varepsilon_0(M,p)>0$ and $c(M,p)>0$ such that the following holds.*

Let $F\colon X(q)_0\to\mathcal{Z}_0(M,\partial M;\mathbb{Z}_2)$ be a map satisfying $\mathcal{F}(F(a),F(b))<\varepsilon\le\varepsilon_0$ for every pair of adjacent vertices a and b. Then there exists an extension of F to a continuous map $F\colon X\to\mathcal{Z}_0(M,\partial M;\mathbb{Z}_2)$ with the following properties. For each cell C in $X(q)$, and $x,y\in C$, we have

(a) $\mathcal{F}(F(x),F(y))<c\sup_{w\in X_0}\boldsymbol{M}(F(w))\varepsilon$,

(b) $\boldsymbol{M}(F(x))\le\sup_{w\in X_0}\boldsymbol{M}(F(w))$.

Proof We observe that every ε-fine family of 0-cycles $F\colon X(q)_0\to\mathcal{Z}_0(M,\partial M;\mathbb{Z}_2)$ is δ-localized for some $\delta\le c(p)\varepsilon$. Hence, there is no need for Proposition 2.7.

We can prove an analogue of Proposition 2.6 with a better bound for the mass. We do it as follows.

First we extend to a 1-skeleton as follows. Let $E\subset X(q)_1$ and let $\{U_j^E\}$ be a δ-admissible collection of open sets containing $F(x)-F(y)$, where $\partial E=x-y$. Therefore, $\boldsymbol{M}(F(x)\llcorner U_j^E)=\boldsymbol{M}(F(y)\llcorner U_j^E) \bmod 2$. Let $z_j=0$ if $\boldsymbol{M}(F(x)\llcorner U_j^E)=0 \bmod 2$ and $z_j=\{c_l\}$ if $\boldsymbol{M}(F(x)\llcorner U_j^E)=1 \bmod 2$, where c_j is a center point of U_j^E.

Let e denote the midpoint of E. We define

$$F(e)=F(x)-\sum F(x)\llcorner U_j^E+\sum z_j=F(y)-\sum F(y)\llcorner U_j^E+\sum z_j.$$

We homotop $F(x)$ to the midpoint of E by radially contracting $F(x)\llcorner U_j^E$ in each U_j^E. We do the same for $F(y)$.

Similarly, given a cell C and a δ-admissible collection $\{U_j^C\}$ such that $F(x)-F(y)\subset\bigcup U_j^C$ for any pair $x,y\in\partial C$, we define cycles z_j and radially contract the family $\{F(x)\}_{x\in\partial C}$ to

$$F(e)=F(x)-\sum F(x)\llcorner U_j^C+\sum z_j,$$

where e is the center point of C. □

For higher-dimensional cycles we obtain a better bound for families that have no concentration of mass.

The map $F\colon X\to\mathcal{Z}_k(M,\partial M;G)$ is said to have no concentration of mass if

$$\lim_{r\to 0}\sup\{\boldsymbol{M}(F(x)\llcorner B_r(p)):p\in M,x\in X\}=0$$

for every $x\in X$; see [24, Theorem 3.7].

Proposition 2.13 *Let $F\colon X(q)\to \mathcal{Z}_k(M,\partial M;G)$ be a map with no concentration of mass and such that $\sup_{x\in X}\boldsymbol{M}(F(x))<\infty$. For every $\varepsilon>0$ and $\delta>0$, there exists a δ-localized map $F'\colon X^p(q+\overline{q})\to \mathcal{Z}_k(M,\partial M;G)$ such that*

(a) $\mathcal{F}(F(x),F'(x))<\varepsilon$,

(b) $\boldsymbol{M}(F'(x))\le \sup_{w\in X_0}\boldsymbol{M}(F(x))+\varepsilon$.

Proof Let $m=\sup_{x\in X}\boldsymbol{M}(F(x))$. Let $c_0>0$ be larger than the constant $c(n,p)$ appearing in Proposition 2.6. We will choose $\delta'>0$ and $\varepsilon'>0$ satisfying the following conditions.

Conditions for δ':

(1) $\delta'<\delta/c_0$.

(2) If $\{U_l\}_{l=1}^{n(p)}$ is a δ'-admissible collection of open sets constructed in Proposition 2.7, then (using no concentration of mass condition) we have

$$\sup_{x\in X}\sum_{l=1}^{n(p)}\boldsymbol{M}\Big(F(x)\llcorner\bigcup U_l\Big)<\frac{\varepsilon}{10c_0}.$$

(3) $\delta'<\varepsilon/(c_0 m)$.

(4) $\delta'<\delta/c_0$.

Conditions for ε':

(1) $\varepsilon'<\varepsilon_0(M,p,\delta')$, where ε_0 is from the statement of Proposition 2.7.

(2) $\varepsilon'<\varepsilon/(10c(M,p,\delta'))$, where $c(M,p,\delta')$ is the constant from Proposition 2.7.

Pick $q'>0$ large enough so that $F\colon X(q+q')\to\mathcal{Z}_k(M,\partial M;G)$ is ε'-fine. Apply Proposition 2.7 to F restricted to $X(q+q')_0$ to obtain (by our choice of $\varepsilon'<\varepsilon_0(M,p,\delta')$) a δ'-localized map

$$F'\colon X(q+q'+\tilde{q})_0\to\mathcal{Z}_k(M,\partial M;G).$$

Then we apply Proposition 2.6 to obtain a $c_0\delta'$-localized family. By our choice of δ' we have that the family is δ-localized. It follows from Propositions 2.7 and 2.6 and our choices of ε' and δ' that for every $x\in C\subset X(q+q'+\tilde{q})$, $w\in C\cap X(q+q')_0$ and $y\in C\cap X(q+q'+\tilde{q})_0$, we have

$$\begin{aligned}\mathcal{F}(F'(x),F(x))&\le\mathcal{F}(F'(x),F'(y))+\mathcal{F}(F'(y),F'(w))+\mathcal{F}(F'(w),F(w))+\mathcal{F}(F(x),F(w))\\&<c_0m\delta'+c(M,p,\delta')\varepsilon'+\varepsilon'<\varepsilon,\end{aligned}$$

$$\boldsymbol{M}(F'(x))\le\sup_x\boldsymbol{M}(F(x))+c(M,p,\delta')\varepsilon'+c_0\max_{w\in C_0}\boldsymbol{M}\Big(F(w)\llcorner\bigcup U_l^C\Big)\le\sup_x\boldsymbol{M}(F(x))+\varepsilon.\quad\square$$

3 Homotopy classes of the space of cycles

Here we use results from Section 2 to give a new proof of the Almgren isomorphism theorem [1; 2].

Let X^p be a cubical subcomplex that is also a p-dimensional manifold and $G=\mathbb{Z}_2$ or $\mathbb{Z}$. If $G=\mathbb{Z}$ we demand that X is oriented. Let M be a Riemannian manifold and let ∂M be piecewise smooth with θ-corners. Fix a PL structure on M that is a bilipschitz approximation of M. To a δ-localized map $F\colon X(q)_0 \to \mathcal{Z}_k^{\mathrm{PL}}(M,\partial M;G)$ we associate a polyhedral $(p+k)$-cycle $A_F \in \mathcal{Z}_{p+k}^{\mathrm{PL}}(X \times M;G)$ constructed as follows.

For each l-cell $C^l \subset X(q)_l$ we will define a polyhedral $(l+k)$-chain $A(C) \in I_{l+k}^{\mathrm{PL}}(X \times M;G)$. We will then define $A_F = \sum_C A(C)$, where the sum is over all p-dimensional cells of $X(q)$. The definition is by induction on l. For each $x \in X$, let $i_x\colon M \to X \times M$ be the inclusion $i_x(a) = (x,a)$. For $l=0$ we define $A(x) = i_x(F(x))$ for each $x \in X(q)_0$. For every $(l-1)$-cell $C^{l-1} \subset X(q)_{l-1}$ assume that we have already defined $A(C)$. Given a set $U \subset M$ and a bilipshitz diffeomorphism $f: U \to \mathbb{R}^n$ with $f(U)$ convex, a point $x \in U$ and a relative cycle $B \in \mathcal{Z}_i(C \times U, C \times (U \cap \partial M);G)$, set

$$\mathrm{cone}_x(B) = g^{-1}\big(\mathrm{Cone}_{g(p_C,x)}(g(B))\big),$$

where p_C is the center point of cell C, $g = (\mathrm{id}_C, f)\colon C \times U \to C \times \mathbb{R}^n$ and $\mathrm{Cone}_{g(p_C,x)}(g(B))$ denotes the cone over $g(B)$ in $C \times f(U)$ with vertex $(p_C, f(x))$. We will choose f to be the exponential map from point p if p lies in the interior of M, or the map E from Section 2.1 if $p \in \partial M$.

Define $A(\partial C) = \sum_{E \subset \partial C} (-1)^i A(E)$, where the sign $(-1)^i$ is defined in the standard way for the boundary operator on a cubical complex, $\partial C = \sum_{E \subset \partial C} (-1)^i E$. Let $\{U_i^C\}$ be an admissible collection of sets corresponding to δ-localized family $\{F(x)\}_{x \in C_0}$. We fix a point $x_i \in U_i^C$ for each i; if U_i^C intersects ∂M, then we pick $x_i \in \partial U_i^C$. We fix a vertex $v \in C_0$ and set

$$A(C) = A(v) \times C + \sum_i \mathrm{cone}_{x_i}([A(\partial C) - A(v) \times \partial C] \cap C \times U_i^C).$$

By induction and our definition of δ-localized families, $A(\partial C) - A(v) \times \partial C$ is an $(l-1)k$-dimensional polyhedral relative cycle in $\bigsqcup_i C \times U_i^C$, so $A(C)$ is well-defined.

It follows from the construction that A has the following properties:

(1) For an open and dense subset of points $x \in C$ and for an open and dense subset of points $x \in \partial C$ we have that
$$\mathrm{proj}_M(\mathrm{proj}_X^{-1}(x) \cap A(C))$$
is a polyhedral k-cycle.

(2) There exists a sequence $q = q_0 < q_1 < q_2 < \cdots$ and $\varepsilon_i \to 0$ such that discrete families of cycles $F^i: C(q_i)_0 \to \mathcal{Z}_k(M,\partial M;G)$ given by $F^i(x) = \mathrm{proj}_M(\mathrm{proj}_X^{-1}(x) \cap A(C))$ are $c(l-1)\delta$-localized (with respect to a fixed admissible collection $\{U_i^C\}$ that depends only on C) and ε_i-fine.

It follows from the definition that if F_1 and F_2 are two δ-localized families with $\mathcal{F}(F_1,F_2) < \varepsilon$ for all $x \in X(q)_0$, then for all sufficiently small $\varepsilon > 0$ the corresponding $(p+k)$-cycles A_{F_1} and A_{F_2} are homologous; cf [14, Section 1]. In particular, given a map $F\colon X \to \mathcal{Z}_k(M,\partial M;G)$, we can define $\alpha_F \in H_{p+k}(X \times M, X \times \partial M;G)$ corresponding to cycle $A_{F'}$, where F' is a δ-localized approximation of F obtained by Proposition 2.7, and α_F is independent of the choice of F'.

Theorem 3.1 *Two maps $F_0\colon X \to \mathcal{Z}_k(M, \partial M; G)$ and $F_1\colon X \to \mathcal{Z}_k(M, \partial M; G)$ are homotopic if and only if $\alpha_{F_0} = \alpha_{F_1} \in H_{p+k}(X \times M, X \times \partial M; G)$.*

Proof One direction follows immediately from the construction: if F_0 and F_1 are homotopic, then $\alpha_{F_0} = \alpha_{F_1}$. Indeed, let $F\colon [0,1] \times X \to \mathcal{Z}_k(M, \partial M; G)$ be the homotopy. By Proposition 2.7 there exists a δ-localized approximation F' of F and the restrictions $F'|_{i \times X}$ are δ-localized approximations of F_i for $i = 0, 1$. Applying our construction to F' we obtain a $(p+1+k)$-chain B with $\partial B = A_{F'|_{1\times X}} - A_{F'|_{0\times X}}$.

Now we prove the other direction. Let $F_0'\colon X(q)_0 \to \mathcal{Z}_k(M, \partial M; G)$ and $F_1'\colon X(q)_0 \to \mathcal{Z}_k(M, \partial M; G)$ be two ε-fine δ-localized maps that ε-approximate F_0 and F_1, respectively.

Define $A_{F_1'}$ and $A_{F_2'}$ as described above. Assume that they represent the same homology class. We would like to construct a homotopy from F_1' to F_2'. It is enough to construct a sequence of maps $F^i\colon X(q')_0 \to \mathcal{Z}_k(M, \partial M; G)$ for $q' \geq q$ which are

(1) ε-fine,
(2) satisfy $\mathcal{F}(F^i(x), F^{i+1}(x)) \leq \varepsilon$ for all $x \in X(q')_0$,
(3) $\mathcal{F}(F^0(x), F_1'(y) - F_0'(y)) \leq c\varepsilon$ for x, y in the same cell of $X(q)$,
(4) $F^N(x)$ is the empty cycle for all x.

By Theorem 2.3 such a sequence of maps would guarantee the existence of a continuous homotopy contracting $F_1 - F_0$.

We will define sequence F_i by defining a sequence of $(p+k)$-cycles A_i in $X \times M$ and intersecting them with fibers of proj_X.

The cycles $A_{F_1'}$ and $A_{F_2'}$ are constructed in such a way that for every $\varepsilon > 0$, picking a sufficiently fine set of discrete points $X(q'')_0$ (ie q'' sufficiently large) we can guarantee that the family $\{\mathrm{proj}^{-1}(x) \cap A_{F_j'}\}$ is ε-fine. We observe that every $(p+k)$-cycle in general position will have this property. This follows from a PL version of transversality, which we explain in more detail below.

Fix a fine PL structure on of M that is $(1+\varepsilon)$-bilipschitz to the original metric. Fix a cell $E_X^p \times E_M^n$, where $E_X \subset X$ and $E_M \subset M$. Let C be a linear m-cell contained in $E_X^p \times E_M^n$, for some $m > 0$. Let $\theta(C)$ denote the angle that C makes with fibers of $\mathrm{proj}_X\colon E_X \times E_M \to E_X$, defined as follows. Fix a point a in the interior of C. Let P_M denote the n-plane that passes through a and contains a fiber of proj_X. Let P_C denote the m-plane that passes through a and contains C. If $m > p$ let $\beta = \{v_1, \dots, v_{m-p}\}$ be a set of linearly independent vectors contained in $P_C \cap P_M$, otherwise set $\beta = \{0\}$. Define $\theta(C)$ to be the minimum over all angles between nonzero vectors $v_1 \in P_C$ perpendicular to $\mathrm{span}(\beta)$ and $v_2 \in P_M$ perpendicular to $\mathrm{span}(\beta)$. Note that this definition is independent of the choice of a.

We will say that a polyhedral chain (resp. relative cycle) A^m in $X^p \times M^n$, where $m \geq p$, is in general position if the following conditions are satisfied:

(1) $A \cap X \times \partial M$ is an $(m-1)$-dimensional polyhedral chain (resp. cycle).

(2) Each cell C^i of A is contained in the interior of a cell E^j of $X \times M$ with $j \geq i$.

(3) For each cell C^i, $i > 0$, of A we have $\theta(C) > 0$.

Lemma 3.2 *If A^m is a polyhedral relative cycle in general position, then there exists an open and dense subset $G \subset X$ such that the following hold:*

(a) *For every $x \in G$ we have that $\mathrm{proj}_X^{-1}(x) \cap A$ is a polyhedral $(m-p)$-dimensional relative cycle and $\mathrm{proj}_X^{-1}(x) \cap C \cap X \times \partial M$ is a polyhedral $(m-p-1)$-dimensional cycle.*

(b) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in G$ and $\mathrm{dist}(x, y) < \delta$, then*

$$\mathcal{F}\big(\mathrm{proj}_M(\mathrm{proj}_X^{-1}(x)), \mathrm{proj}_M(\mathrm{proj}_X^{-1}(y))\big) < \varepsilon.$$

Proof Let $(A)_0$ denote the 0-skeleton of A and let $G \subset X$ be the union of interiors of all n-dimensional faces of X minus $\mathrm{proj}_X((A)_0)$. By [36, Theorem 1.3.1] and our angle condition, $\mathrm{proj}_X^{-1}(x)$ is an $(m-p)$-dimensional polyhedral cycle for each point $x \in G$, and given $y_i \in G$ and a piecewise linear segment L_i connecting y_i to x, $\mathrm{proj}_X^{-1}(L_i)$ is a polyhedral $(m-p+1)$-chain with $\mathrm{Vol}(L_i) \to 0$ as $y_i \to x$. □

Let B denote a polyhedral $(p+k+1)$-chain with $\partial B = A_{F_1} - A_{F_2}$. There exists a fine subdivision of B such that for any $(p+k)$-cell C of B and almost every $x \in X$, we have that $z_C(x) = \mathrm{proj}_X^{-1}(x) \cap C$ is a polyhedral k-chain of volume less than ε.

Let N denote the number of $(p+k+1)$-cells in B. We can pick a large enough $q' \geq q$ so that for every two adjacent vertices $x_1, x_2 \in X(q')_0$ and every $(p+k)$-cell C of B, we have

$$\mathcal{F}\big(\mathrm{proj}_M(\mathrm{proj}_X^{-1}(x_1) \cap C), \mathrm{proj}_M(\mathrm{proj}_X^{-1}(x_2) \cap C)\big) < \frac{\varepsilon}{N}.$$

Let $\{C_i\}_{i=1}^N$ be $(p+k+1)$-cells of B. Define a sequence of maps $F_i : X(q')_0 \to \mathcal{Z}_k(M, \partial M; G)$ by

$$F_i(x) = \mathrm{proj}_M\left(\mathrm{proj}_X^{-1}(x) \cap \partial\left(B \setminus \bigcup_{j=1}^{i} C_j\right)\right).$$

By construction this defines a sequence of ε-close ε-fine discrete maps, and by Theorem 2.3 we can construct a homotopy from $F : X \to \mathcal{Z}_k(M, \partial M; G)$ given by $F(x) = F_1(x) - F_0(x)$ to the 0-map. □

Let $F_A : \pi_m(\mathcal{Z}_k(M^n; G)) \to H_{m+k}(M^n \times S^m; G)$ be a map defined as follows. Given a map $f : S^m \to \mathcal{Z}_k(M^n; G)$ representing class $[f] \in \pi_m(\mathcal{Z}_k(M^n; G))$, we set $F_A([f]) = \alpha_f \in H_{m+k}(M^n \times S^m; G)$.

Theorem 3.3 *F_A is a bijective groupoid homomorphism.*

Proof By Theorem 3.1, if $f : S^m \to \mathcal{Z}_k(M; G)$ and $g : S^m \to \mathcal{Z}_k(M; G)$ are two homotopic maps, then cycles A_f and A_g are homologous, so the map is well-defined. It also follows from the construction of A_f that F_A is a homomorphism and injectivity of F_A follows by Theorem 3.1.

Every homology class in $H_{m+k}(M^n \times S^m; G)$ can be represented by a polyhedral chain A in general position, so that for a sufficiently large q discrete family $\{\mathrm{proj}_M(\mathrm{proj}_X^{-1}(x) \cap A)\}_{x \in X(q)_0}$ represents a unique homotopy class in $\pi_m(\mathcal{Z}_k(M^n; G))$. Hence, F_A is also surjective. $\square$

By the Künneth formula we have $H_{m+k}(M^n \times S^m; G) \cong H_{m+k}(M^n; G) \oplus H_k(M^n; G)$. If we restrict the domain of F_A to maps $f: S^m \to \mathcal{Z}_k(M^n; G)$ with a fixed a basepoint cycle $z \in \mathcal{Z}_k(M; G)$ we obtain the following isomorphism.

Corollary 3.4 (Almgren isomorphism theorem) *The homotopy group $\pi_m(\mathcal{Z}_k(M^n; G); \{[z]\})$ is isomorphic to the homology group $H_{m+k}(M^n; G)$.*

For future applications we will also need the following Proposition, which allows us to replace a contractible family of relative cycles with a contractible family of absolute cycles.

Proposition 3.5 *Let $F: X(q) \to \mathcal{Z}_k(M, \partial M; G)$ be a contractible map. For every $\varepsilon > 0$ there exists a map $G: X \to \mathcal{Z}_k(M, G)$ such that*

$$\mathcal{F}(F(x) \llcorner \mathrm{int}(M), G(x) \llcorner \mathrm{int}(M)) < \varepsilon$$

Proof Let A_F denote a $(p+k)$-dimensional relative cycle in $X \times M$ corresponding to F. Since F is contractible, by Theorem 3.1 there exists a cycle B with $\partial B - A_F \subset X \times \partial M$. For ∂B in general position and q sufficiently large we can define an $\varepsilon/2$-fine map $G: X(q)_0 \to \mathcal{Z}_k(M, G)$ defined by $G(x) = \mathrm{proj}_M(\mathrm{proj}_X^{-1}(x)) \cap \partial B$.

By Theorem 2.3 we can complete G to a continuous family of absolute cycles approximating F. $\square$

We finish this section with a counterexample to parametric isoperimetric inequality for contractible families of cycles with integer coefficients.

Proposition 3.6 *Let $N > 0$. There exists a contractible family of 0-cycles $F_N: S^1 \to \mathcal{Z}_0(S^1; \mathbb{Z})$ such that*

- *$\mathbf{M}(F_N(x)) \le 2$ for all $x \in S^1$, and*
- *any family of fillings $H: S^1 \to I_1(S^1; \mathbb{Z})$ with $\partial H(x) = F_N(x)$ satisfies $\mathbf{M}(H(x_0)) > N$ for some $x_0 \in S^1$.*

Proof Consider $[0,1] \times S^1$ and let $\{\gamma_i = \{t_i\} \times S^1\}$ be a collection of $2N$ disjoint closed curves in $[0,1] \times S^1$ all oriented the same way and with $t_i \in \left(0, \frac{1}{2}\right)$; let $\{\beta_i = \{s_i\} \times S^1\}$ be a collection of $2N$ disjoint closed curves all oriented in the opposite way to γ_i and with $s_i \in \left(\frac{1}{2}, 1\right)$. Observe that we can make a PL perturbation of curves γ_i and β_i, so that fibers of the projection map $p: [0,1] \times S^1 \to [0,1]$ intersect $\bigcup \gamma_i \cup \beta_i$ in at most two points (and when the intersection has exactly two points, they have opposite orientation).

Identifying the endpoints of $[0,1]$ we can think of curves γ_i and β_i as lying in $S^1 \times S^1$. Let $X = S^1$. Then for sufficiently large q we have that $\left\{F(x) = p^{-1}(x) \cap \left(\bigcup \gamma_i \cup \beta_i\right) : x \in X(q)_0\right\}$ is an ε-fine discrete family of cycles. Since $Z = \bigcup \gamma_i \cup \beta_i$ is null-homologous as a 1-cycle in $S^1 \times S^1$, the family $F(x)$ is contractible. Every family of fillings H of F then corresponds to a filling τ of Z. It is easy to see, using a winding number argument, that either $p^{-1}(0) \cap \tau$ or $p^{-1}\left(\frac{1}{2}\right) \cap \tau$ is a circle of multiplicity $\geq N$. It follows that $\boldsymbol{M}(H(x_0)) \geq N$ for some $x_0 \in S^1$. □

4 Parametric coarea inequality

Let $\Omega \subset \mathbb{R}^3$ be a domain with $\partial\Omega$ piecewise-smooth boundary with θ-corners. Let $\Omega_\varepsilon = \Omega \setminus E(\partial\Omega \times [0, \varepsilon])$, where E is the "exponential map" from Lemma 2.1, and $\Sigma = \partial\Omega_\varepsilon$. In this section we will prove a somewhat stronger version of Conjecture 1.6 for $n = 3$ and $k = 1$.

Theorem 4.1 *Fix $\eta > 0$. For all $\varepsilon \in (0, \varepsilon_0)$, $\delta_1 > 0$ and $p \geq p_0(\Omega, \varepsilon)$ the following holds. Suppose that $F\colon X^p \to \mathcal{Z}_1(\Omega, \partial\Omega; \mathbb{Z}_2)$ is a continuous map with no concentration of mass. There exists a map $F'\colon X \to I_1(\mathrm{cl}(\Omega_\varepsilon); \mathbb{Z}_2)$ such that*

1. *$\partial F'$ is a continuous family of 0-cycles in $\partial\Omega_\varepsilon$ that is δ_1-localized on $X(q')$ for sufficiently large q',*
2. *$\mathcal{F}(F(x) \llcorner \Omega_\varepsilon, F'(x) \llcorner \Omega_\varepsilon) < \eta$,*
3. *$\boldsymbol{M}(F'(x)) \leq \boldsymbol{M}(F(x)) + c(n)\boldsymbol{M}(\partial\Omega)\left(p^{1/2} + \dfrac{\boldsymbol{M}(F(x))}{\varepsilon\sqrt{p}}\right)$,*
4. *$\boldsymbol{M}(\partial F'(x)) \leq c(n)\boldsymbol{M}(\partial\Omega)p$.*

Moreover, if F is a p-sweepout of Ω, then $\partial F'\colon X \to \mathcal{Z}_0(\partial\Omega_\varepsilon; \mathbb{Z}_2)$ is a p-sweepout of $\partial\Omega_\varepsilon$.

Proof By Proposition 2.13 we can replace family F with a δ-localized family, which is arbitrarily close to F in flat norm, while increasing the mass by an arbitrarily small amount. Hence, without any loss of generality we may assume that F is δ-localized for some very small (compared to η and ε) $\delta > 0$.

4.1 Proof strategy

We give a brief informal overview of the proof strategy. For each value of $x \in X(q)_0$ we can use the coarea inequality to find $s(x)$ such that the restriction $F(x) \llcorner \Omega_{\varepsilon - s(x)}$ satisfies a good bound on boundary mass. We then project the part sticking outside of Ω_ε onto $\Sigma = \partial\Omega_\varepsilon$, and denote the resulting chain (which will be a relative cycle in Ω_ε) by $C_{s(x)}(F(x))$; see the precise definitions in Section 4.2. This gives us a way to define the family on the 0-skeleton $X(q)_0$. The challenge is to extend this family to the higher-dimensional skeleta of $X(q)$.

For two adjacent vertices x, y we have that $I = C_{s(x)}(F(x)) - C_{s(y)}(F(y))$ is a chain supported in Σ. Recall that F is δ-localized and so we can assume that for any two adjacent vertices the values of $s(x)$ and $s(y)$ are chosen so that $F(x) - F(y)$ is supported away from $\partial\Omega_{\varepsilon - s(x)}$ and $\partial\Omega_{\varepsilon - s(y)}$. It follows that we can deform I into a chain $I' = C_{s(x)}(F(x)) - C_{s(y)}(F(x))$, and that we have a good bound on the mass of $\partial I'$. To interpolate between $C_{s(x)}(F(x))$ and $C_{s(y)}(F(y))$ we would like to contract I, while

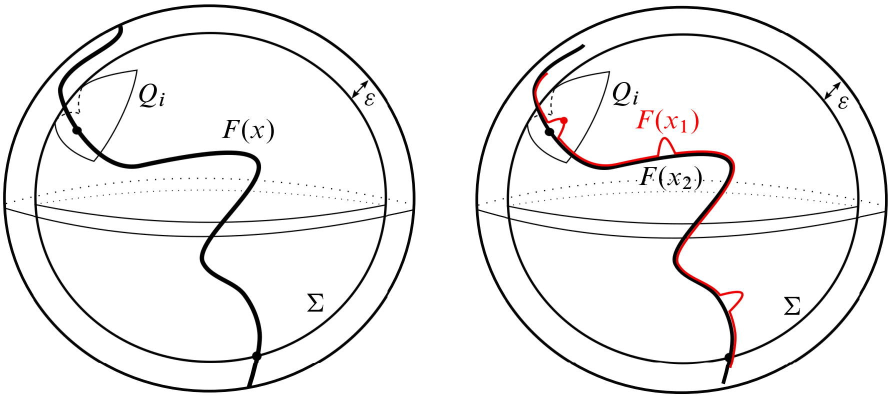

Figure 2: $F(x_1)$ and $F(x_2)$ are two cycles with $x_1, x_2 \in C \cap X(q)_0$. We use the coarea inequality to cut them in $\Omega \setminus \Omega_\varepsilon$. Then we project the part sticking out outside of Ω_ε onto $\Sigma = \partial\Omega_\varepsilon$.

controlling its mass and boundary mass. Note that the mass of I may be as large as that of $F(x)$ or $F(y)$. So to do this we subdivide Σ into small triangles $\{Q_i\}$, with the goal of contracting I radially, triangle by triangle. However, we have no control over how large the mass of the intersection of I with ∂Q_i is (but we can assume that F was slightly perturbed so that the intersection has finite mass), so contracting $(I)_i = I \llcorner Q_i$ may increase the boundary mass by an arbitrary amount. To solve this problem we add to each $C_{s(x)}(F(x))$ a chain $\sum_i f_i(x)$, where each $f_i(x)$ is supported in ∂Q_i and is a kind of "filling" of the 0-chain $C_{s(x)}(F(x)) \llcorner \partial Q_i$, reducing its mass to ≤ 1.

With this in mind we will define $F'(x) = \sum_i C_{s(x)}(F(x)) \llcorner Q_i + f_i(x) + \mathrm{Cone}(\partial F(x) \llcorner Q_i)$ on the 0-skeleton $X(q)_0$. The last term is the cone over the boundary of $C_{s(x)}(F(x))$ in Q_i with vertex $q_i \in Q_i$. Adding the cone makes some of the interpolation formulas simpler as we extend F' to higher-dimensional faces of $X(q)$. The interpolation process is illustrated in Figure 3.

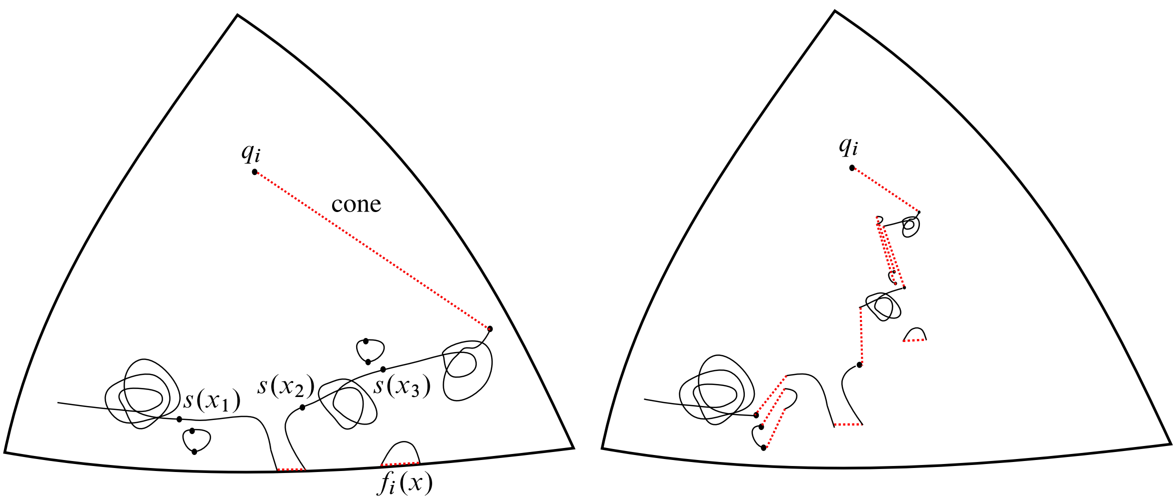

Figure 3

Let's ignore, for simplicity, the difference between values of F on adjacent vertices, since they are contained in a collection of tiny balls away from where we "cut" $F(x)$. The interpolation procedure then can be thought of as radial scaling of terms like $[C_{s(x_l)}(F(x)) - C_{s(x_j)}(F(x))] \cap Q_i + f_i(x_l) - f_i(x_j)$ for different vertices x_l and x_j in the 0-skeleton of a cell in $X(q)$. On Figure 3 on the left, we see $(F'(x_4))_i = (C_{s(x_4)}(F(x)))_i + f_i(x_4) + \mathrm{Cone}$, where $s(x_4)$ corresponds to the cut that is closest to $\partial\Omega$. Then we have cuts corresponding to $s(x_1)$, $s(x_2)$ and $s(x_3)$ indicated on the same picture. The cut corresponding to $s(x_2)$ has mass 1 and the other two cuts have mass 3. The interpolation will involve scaling the difference between these cuts as on the figure on the right, and adding a portion of the cone over q_i to connect the boundaries. Inductively we will assume that on the boundary of a cell C the family of chains is of this form, and then linearly change the scaling factors to homotop the family in Q_i to a cycle $F'(x_j) \llcorner Q_i$ that has the least boundary mass (in the case of the figure this is $x_j = x_2$ or $x_j = x_4$).

4.2 Notation

We need to make the following definitions:

- Let $\pi \colon \Omega \setminus \Omega_\varepsilon \to \Sigma$ be the projection map.
- Given a k-chain $\tau \in I_1(\Omega; \mathbb{Z}_2)$ and $s \in [0, \varepsilon]$, let

$$L_s(\tau) = \pi(\tau \llcorner (\Omega_{\varepsilon - s} \setminus \Omega_\varepsilon)) \subset \Sigma \quad \text{and} \quad C_s(\tau) = (\tau \llcorner \Omega_\varepsilon) + L_s(\tau) \subset \mathrm{cl}(\Omega_\varepsilon).$$

- Let $[\tau]_{[s_1, s_2]} = C_{s_2}(\tau) - C_{s_1}(\tau)$.
- For $x \in X(q)$ let $B(x)$ denote the cell of $X(q)$ of smallest dimension such that x lies in its interior; if $x \in X(q)_0$, then $B(x) = x$.
- For each vertex $x \in X(q)_0$ we apply the coarea inequality Theorem 2.2 (here we assume that $\varepsilon \le \varepsilon(\Omega)$ is sufficiently small) to find $s(x) \in \left[\frac{1}{2}\varepsilon, \varepsilon\right]$ such that

$$\boldsymbol{M}(\partial C_{s(x)}(F(x))) \le \frac{1.5}{\varepsilon - 4\delta} \boldsymbol{M}(F(x) \llcorner \mathrm{cl}(\Omega \setminus \Omega_\varepsilon)) \le \frac{2}{\varepsilon} \boldsymbol{M}(F(x) \llcorner \mathrm{cl}(\Omega \setminus \Omega_\varepsilon))$$

and, moreover, we have that $\partial\Omega_{\varepsilon - s(x)}$ is disjoint from a δ-admissible family $\{U_i\}$.

In particular, for any $x, y \in X(q)_0$ that lie in a common face of $X(q)$, we have that $C_{s(x)}(F(x) - F(y))$ is an absolute cycle.

- For $p > p_0(\Omega, \varepsilon)$ we can triangulate Σ by subdividing it into N' subsets Q_i, with $N' \le \mathrm{const} \cdot p \boldsymbol{M}(\partial\Omega)$, with disjoint interiors and piecewise smooth boundaries such that there exists a $(1+\varepsilon^2)$-bilipschitz diffeomorphism P_i from Q_i to a convex simplex $Q_i' \subset \mathbb{R}^2$, with $\mathrm{diam}(Q_i) \le p^{-1/2}$ and $\boldsymbol{M}(\partial Q_i) \le p^{-1/2}$.
- For each triangle Q_i, fix a point q_i in the interior of Q_i. Given a cycle z, we denote by $\mathrm{Cone}_i(z) = P_i^{-1}(\mathrm{Cone}_{P_i(q_i)}(P_i(z \llcorner Q_i)))$ the cone over $z \llcorner Q_i$ with vertex q_i.
- Let $\Psi^i_\rho \colon Q_i \to Q_i$ be the radial scaling (towards q_i) map $\Psi^i_\rho(x) = P_i^{-1}\big(P_i(q_i) + \rho(P_i(x) - P_i(q_i))\big)$, where $\rho \in [0, 1]$.

- Given a k-chain τ, $(k-1)$-cycle x and $\rho \in [0,1]$ define the "conical collar" map

$$\Phi^i_\rho(\tau,x) = \mathrm{Cone}_i(x) - \mathrm{Cone}_i(\Psi^i_\rho(x)) + \Psi^i_\rho(\tau).$$

We will drop the superscript i whenever it is clear to which region the map is being applied.

In the special case when $x = \partial\tau$, we have that $\Phi_\rho(\tau,x)$ fixes the boundary of τ, and replaces the chain with a sum of a shrunk copy of itself and a piece of the cone over $\partial\tau$. More generally, we have $\partial\Phi_\rho(\tau,x) = x - \Psi_\rho(x) + \partial\Psi_\rho(\tau)$.

- We will use the following notation: given a chain I we let $(I)_i = I \llcorner Q_i$.

- Let $k \geq 2$. Given chains $\tau_1,\dots,\tau_k$ and $0 \leq \rho_1 \leq \dots \leq \rho_{k-1} \leq 1$ define a chain in Q_i by

$$\Delta_i\big((\tau_j)_{j=1}^k, (\rho_j)_{j=1}^{k-1}\big) = (\tau_1)_i + \sum_{j=1}^{k-1} \Phi_{\rho_1\cdots\rho_j}\big((\tau_{j+1}-\tau_j)_i, (\partial\tau_j)_i\big) + \mathrm{Cone}_i\big(\Psi_{\rho_1\cdots\rho_{k-1}}((\partial\tau_k)_i)\big).$$

For $k = 1$ we can write

$$\Delta_i(\tau) = (\tau)_i + \mathrm{Cone}_i((\partial\tau)_i) = \Delta_i((\tau,\tau'),(0)),$$

where τ' can be any chain.

- For each $x \in X(q)_0$, we define $f_i(x)$ to be a 1-chain in $I_1(\partial Q_i;\mathbb{Z}_2)$ with the following property. Let V_i denote the set of vertices of Q_i, then $\partial f_i(x) - \big(C_{t(x)}(F(x))\big)_i \llcorner \partial Q_i \subset V_i$.

4.3 Definition of F' on $X(q)_0$

After an arbitrarily small perturbation of the family we may assume that $C_{s(x)}(F(x))$ intersects $\bigcup \partial Q_i$ transversely and $\partial C_{s(x)}(F(x)) \llcorner \bigcup \partial Q_i$ is empty for each $x \in X(q)_0$.

We define

$$F'(x) = F(x)\llcorner\Omega_\varepsilon + \sum_i \Delta_i(C_{s(x)}(F(x))) + f_i(x). \tag{4-1}$$

Observe that with this definition, if x and y are vertices in a cell D of $X(q)$, then there exists a cycle $e(x,y)$ of length $\leq \delta$, and contained in a δ-admissible collection of open sets in $\mathrm{cl}(\Omega_\varepsilon)$, such that

$$F'(x) = F(y)\llcorner\Omega_\varepsilon + \sum_i \Delta_i(C_{s(x)}(F(y))) + f_i(x) + e(x,y).$$

4.4 Inductive property

Assume that we defined F' on the k-skeleton of $X(q)$ so that it satisfies the following property:

Inductive property for k-skeleton $X(q)_k$ For every p'-dimensional cell $C^{p'}$ of $X(q)$ with $k \leq p' \leq p$, and point $y \in C$, the following holds. Let $(x_j)_{j=1}^{2^{p'}}$ denote the set of points in $C \cap X(q)_0$ and assume that they are numbered so that $s(x_{j_1}) \leq s(x_{j_2})$ if $j_1 < j_2$. Then there exist:

- Functions $\rho^i_j : C \cap X(q)_k \to [0,1]$ for $j \in \{1,\dots,2^{p'}-1\}$ and $i \in \{1,\dots,N'\}$. Let
$$n_i(x) = \#\left\{\prod_{j=1}^{l} \rho^i_j(x) : l \in \{1,\dots,2^{p'}-1\}\right\} - 1.$$
Here $n_i(x)$ will correspond to the number of distinct collars in Q_i. Then the following properties hold:
  (i) $\sum_{i=1}^{N'} n_i(x) \le \dim(B(x)) \le k$. Moreover, for all $x_j \notin B(x)_0$ we have
$$\rho^i_j(x) = \begin{cases} 1 & \text{if } s(x_j) < \max_{x\in B(x)_0}\{s(x)\}, \\ 0 & \text{if } s(x_j) = \max_{x\in B(x)_0}\{s(x)\}. \end{cases}$$
  (ii)
$$\begin{aligned}\sum_i (1-\rho^i_1(x))\boldsymbol{M}((\partial(C_{s(x_1)}(F(y)))_i)) &\\ +\sum_i\sum_j(\rho^i_1(x)\dots\rho^i_j(x) - \rho^i_1(x)\dots\rho^i_{j+1}(x))\boldsymbol{M}((\partial(C_{s(x_j)}(F(y)))_i)) &\\ +\sum_i \rho^i_1(x)\dots\rho^i_{2^{p'}-1}(x)\boldsymbol{M}((\partial(C_{s(x_{2^{p'}})}(F(y))))_i) &\\ \le \frac{2\boldsymbol{M}(F(x))}{\varepsilon}.&\end{aligned}$$
- A function $e : C \cap X(q)_k \to \mathcal{Z}_1(\mathrm{cl}(\Omega_\varepsilon), \Sigma; \mathbb{Z}_2)$ of relative cycles of length $\le c(k)\delta$ and contained in a $c(k)\delta$-admissible collection of sets such that for all $x \in C \cap X(q)_k$,

$$\begin{aligned}F'(x) &= F(y)\llcorner\Omega_\varepsilon + e(x)\\ &+\sum_i^{N'}\left(\Delta_i\big((C_{s(x_j)}(F(y)))_{j=1}^{2^{p'}},(\rho_j(x))_{j=1}^{2^{p'}-1}\big)+f_i(x_1)+\sum_{j=1}^{2^{p'}-1}\Psi^i_{\rho^i_1(x)\dots\rho^i_j(x)}(f_i(x_{j+1})-f_i(x_j))\right).\end{aligned}\tag{4-2}$$

We claim that our definition (4-1) for $k=0$ satisfies the inductive assumption. Indeed, let $x \in X(q)_0$ and y lie in a p'-dimensional cell D of $X(q)$ that contains x. Setting $\rho^i_j(x) = 1$ for all j such that $s(x_j) < s(x)$, and $\rho^i_j(x) = 0$ for all j such that $s(x_j) \ge s(x)$, we can write (4-1) as

$$\begin{aligned}F'(x) &= F(x)\llcorner\Omega_\varepsilon\\ &+\sum_i\left(\Delta_i\big((C_{s(x_j)}(F(x)))_{j=1}^{2^{p'}},(\rho_j(x))_{j=1}^{2^{p'}-1}\big)+f_i(x_1)+\sum_{j=1}^{2^{p'}-1}\Psi^i_{\rho^i_1(x)\dots\rho^i_j(x)}(f_i(x_{j+1})-f_i(x_j))\right).\end{aligned}$$

Let $G(x,y)$ denote the cycle obtained by replacing $F(x)$ in the expression above with $F(y)$:

$$\begin{aligned}G(x,y) &= F(y)\llcorner\Omega_\varepsilon\\ &+\sum_i\left(\Delta_i\big((C_{s(x_j)}(F(y)))_{j=1}^{2^{p}},(\rho_j(x))_{j=1}^{2^{p'}-1}\big)+f_i(x_1)+\sum_{j=1}^{2^{p'}-1}\Psi^i_{\rho^i_1(x)\dots\rho^i_j(x)}(f_i(x_{j+1})-f_i(x_j))\right).\end{aligned}$$

Then using the fact that $F(x)$ is δ-localized and our choices of $s(x)$ (that guarantee $\partial C_{s(x)}(F(x)) = \partial C_{s(x)}(F(y))$ if x and y lie in the same cell of $X(q)$) we obtain that $e(x) = F'(x) - G(x,y)$ is a cycle of length less than δ. Property (i) of the functions ρ^i_j is immediate. and property (ii) follows since $\boldsymbol{M}(\partial C_{s(x)}(F(x))) \le 2\boldsymbol{M}(F(x))/\varepsilon$.

Next we show that the inductive property implies the desired mass and boundary mass bounds.

Lemma 4.2 *Suppose the inductive property is satisfied for $k = p' = p$ and $y \in C^p \subset X(q)$. Then*:

(1) $\mathscr{F}(F(y)\llcorner\Omega_\varepsilon, F'(y)) < \eta$.

(2) $\boldsymbol{M}(F'(y)) \leq \boldsymbol{M}(F(y)) + c(n)\left(\dfrac{\boldsymbol{M}(F(y))}{\varepsilon}\dfrac{1}{\sqrt{p}} + \boldsymbol{M}(\partial\Omega)\sqrt{p}\right)$.

(3) $\boldsymbol{M}(\partial F'(y)) \leq c(n)\boldsymbol{M}(\partial\Omega)p$.

Proof The first inequality follows immediately from (4-2).

Now we prove the second inequality. Observe that

$$\begin{aligned}\boldsymbol{M}\big(\Phi^i_\rho((C_{s(x_{j+1})}(F(y)) - C_{s(x_j)}(F(y)))_i)\big) \leq \rho\big(\boldsymbol{M}\big(([F(y)]_{[s(x_j),s(x_{j+1})]})_i\big)\big)&\\ + (1-\rho)\boldsymbol{M}\big((\partial C_{s(x_j)}(F(y)))_i\big),&\\ \boldsymbol{M}\big(\Psi^i_\rho(f_i(x_{j+1}) - f_i(x_j))\big) \leq \rho\boldsymbol{M}(\partial Q_i).&\end{aligned}$$

Unraveling the definition of Δ_i and using property (ii) of functions $\rho^i_j(y)$ we obtain

$$\begin{aligned}\boldsymbol{M}(F'(y)) &\leq \boldsymbol{M}(F(y)) + \sum_i \sum_{j=1}^{2^p-1}\Big(\boldsymbol{M}\big(\mathrm{Cone}_i\big(\Psi^i_{\rho^i_1(y)\dots\rho^i_j(y)}((\partial C_{s(x_j)})_i)\big)\big)\\ &\qquad - \boldsymbol{M}\big(\mathrm{Cone}_i\big(\Psi^i_{\rho^i_1(y)\dots\rho^i_{j+1}(y)}((\partial C_{s(x_j)})_i)\big)\big)\\ &\qquad + \boldsymbol{M}\big(\Psi^i_{\rho^i_1(y)\dots\rho^i_j(y)}(f_i(x_{j+1}) - f_i(x_j))\big)\Big)\\ &\leq \boldsymbol{M}(F(y)) + \frac{2\boldsymbol{M}(F(y))}{\varepsilon}\frac{1}{\sqrt{p}} + N'\frac{p}{\sqrt{p}}\\ &\leq \boldsymbol{M}(F(y)) + \frac{2\boldsymbol{M}(F(y))}{\varepsilon}\frac{1}{\sqrt{p}} + c(n)\boldsymbol{M}(\partial\Omega)\sqrt{p}.\end{aligned}$$

To prove inequality (3), observe that

$$\boldsymbol{M}\Big(\big(\partial C_{s(x_j)}(F(y)) + \Delta_i\big(C_{s(x_j)}(F(y)), C_{s(x_{j+1})}(F(y)), \rho\big) + \partial\Psi^i_\rho(f_i(x_{j+1}) - f_i(x_j))\big)_i\Big) \leq 4,$$

where we may have one boundary point at the tip of the cone $q_i \in Q_i$ due to an odd number of boundary points in $(\partial C_{s(x_{j+1})}(F(y)))_i$ and at most 3 boundary points $[F(y)]_{[s(x_j),s(x_{j+1})]}\llcorner\partial Q_i + f_i F(y)$ at the vertices of triangle Q_i. The total number of vertices in triangulation of Σ is bounded by $c'(n)p\boldsymbol{M}(\partial\Omega)$. Hence, using the first property of functions $\rho^i_j F(y)$, we obtain

$$\begin{aligned}\boldsymbol{M}(\partial F'(y)) \leq \sum_i^{N'}\boldsymbol{M}\big((\partial F'(x))_i\big) &\leq 4N' + \sum_{i,j}\boldsymbol{M}\big(\partial\Psi^i_{\rho^i_1(x)\dots\rho^i_j(x)}(f_i(x_{j+1}) - f_i(x_j))\big)\\ &\leq 4c'(n)p\boldsymbol{M}(\partial\Omega) + \sum_i n_i(y) \leq c(n)p\boldsymbol{M}(\partial\Omega).\end{aligned}$$ □

Now we assume that F' is defined and satisfies the inductive property on the k-skeleton $X(q)_k$. We will define F' on the $(k+1)$-skeleton and prove that it satisfies the inductive property on $X(q)_{k+1}$. By Lemma 4.2 this will finish the proof of the theorem.

4.5 Inductive step

Let D be a $(k+1)$-cell in $X(q)$. We will define map $F_D\colon \partial D \times [0, N']$ such that $F_D = F'$ on $\partial D \times \{0\}$ and $F_D(x, N')$ is a constant map for all $x \in D$. This immediately gives the desired extension of F' to D by identifying D and $\partial D \times [0, N']/\partial D \times \{N'\}$. We will show that it satisfies the inductive property and, hence, the mass and boundary mass bounds.

Let $D_0 = \{x_1, \dots, x_{2^{k+1}}\}$ denote the collection of vertices in $D \cap X(q)_0$. Assume that the vertices are numbered so that $s(x_{j_1}) \le s(x_{j_2})$ if $j_1 < j_2$. Let $y_i(D) \in D_0$ be a vertex with

$$\boldsymbol{M}\big(\big(\partial C_{s(y_i(D))}(F(x_1))\big)_i\big) = \min\big\{\boldsymbol{M}\big(\big(\partial C_{s(x)}(F(x_1))\big)_i\big) : x \in D_0\big\}.$$

(Recall that $\partial C_{s(y_i)}(F(x_1)) = \partial C_{s(y_i)}(F(x))$ for all $x \in D_0$.)

By the inductive property for all $x \in \partial D$, we can write

$$F'(x) = F(x_1)\llcorner\Omega_\varepsilon + \sum_i^{N'} \Delta_i\big(\big((C_{s(x_j)}(F(x_1)))_i\big)_{j=1}^{2^{k+1}}, (\rho_j^i(x))_{j=1}^{2^{k+1}-1}\big) + e(x) \\ + \sum_i^{N'} \sum_{j=1}^{2^{k+1}-1} \Psi^i_{\rho_1^i(x)\dots\rho_j^i(x)}(f_i(x_{j+1}) - f_i(x_j)).$$

Now we will define functions $\rho_j^i(x,t)$, $t \in [0, N']$ and $x \in \partial D$, as follows:

(1) For $t < i-1$ we set $\rho_j^i(x,t) = \rho_j^i(x)$.

(2) For $t \in [i-1, i]$, we set

$$\rho_j^i(x,t) = \begin{cases} (t-i+1) + (1-(t-i+1))\rho_j^i(x) & \text{if } s(x_j) < s(y_i(D)), \\ (1-(t-i+1))\rho_j^i(x) & \text{if } s(x_j) \ge s(y_i(D)). \end{cases}$$

(3) For $t > i$, we set

$$\rho_j^i(x,t) = \begin{cases} 1 & \text{if } s(x_j) < s(y_i(D)), \\ 0 & \text{if } s(x_j) \ge s(y_i(D)). \end{cases}$$

We define

$$F_D(x,t) = C_{s(x_1)}(F(x_1)) + \sum_i^{N'} \Delta_i\big((C_{s(x_j)}(F(y)))_{j=1}^{2^{k+1}}, (\rho_j^i(x,t))_{j=1}^{2^{k+1}-1}\big) \\ + e(x) + f_i(x_1) + \sum_i \sum_{j=1}^{2^{k+1}-1} \Psi^i_{\rho_1^i(x,t)\dots\rho_j^i(x,t)}(x_{j+1} - x_j). \tag{4-3}$$

From F_D we obtain the definitions of ρ_j^i and F' on D by identifying D and $\partial D \times [0, N']/\partial D \times \{N'\}$. We claim that with this definition the inductive property will be satisfied for F' on D. Fix a point $y \in X(q)_0$ in a cell $C^{p'}$ of $X(q)$ that intersects D, for some $k+1 \le p' \le p$. Let $\{z_l\}$ denote the vertices of C arranged, as usual, so that $(s(z_l))$ is an increasing sequence. We define $\tilde{\rho}_l^i(x) = \rho_j^i(x)$ if $z_l = x_j$. For the values of l that correspond to vertices z_l that do not lie in D_0 and every $x \in D \cap C$ we set $\widetilde{\rho}_l^i(x) = 1$ if $s(z_l) < s(x_{2^k+1})$ and $\widetilde{\rho}_l^i(x) = 0$ if $s(z_l) \geq s(x_{2^k+1})$. For $x \in D \cap C$, consider

$$G(x,y) = F(y) \llcorner \Omega_\varepsilon + \sum_i \Delta_i\big((C_{s(z_l)}(F(y)))_{l=1}^{2^{p'}}, (\widetilde{\rho}_l(x))_{l=1}^{2^{p'}-1}\big) + f_i(y_1) \\ + \sum_i \sum_{l=1}^{2^{p'}-1} \Psi^i_{\widetilde{\rho}_1^i(x)\dots\widetilde{\rho}_l^i(x)}(f_i(z_{l+1}) - f_i(z_l)).$$

It follows from the definition (4-3) and the inductive assumption of $e(x)$ that $e'(x) = F'(x) - G(x,y)$ is a family of cycles of length $\leq c(k+1)\delta$ contained in a $c(k+1)\delta$-admissible collection of sets.

We would like to show that properties (i)–(ii) of functions $\widetilde{\rho}_l^i$ hold. The first property follows from the inductive assumption and our definitions of $\rho_j^i(x,t)$ and $\widetilde{\rho}_l^i(x)$. To see that (ii) holds, observe that the choice of $y_i(D)$ as the vertex with minimal $\boldsymbol{M}((\partial C_{s(y_i(D))}(F(x_1)))_i)$ implies that for $x' = (x,t)$ as defined in (4-3), the functions $\widetilde{\rho}_l^i(x')$ satisfy

$$(1 - \widetilde{\rho}_1^i(x,t))\boldsymbol{M}((\partial(C_{s(z_1)}(F(y)))_i)) \\ + \sum_l (\widetilde{\rho}_1^i(x,t)\dots\widetilde{\rho}_l^i(x,t) - \widetilde{\rho}_1^i(x,t)\dots\widetilde{\rho}_{l+1}^i(x,t))\boldsymbol{M}((\partial(C_{s(z_l)}(F(y)))_i)) \\ + \widetilde{\rho}_1^i(x,t)\dots\widetilde{\rho}_{2^{p'}-1}^i(x,t)\boldsymbol{M}((\partial(C_{s(z_{2^{p'}})}(F(y))))_i) \\ \leq (1 - \widetilde{\rho}_1^i(x,0))\boldsymbol{M}((\partial(C_{s(z_1)}(F(y)))_i)) \\ + \sum_l (\widetilde{\rho}_1^i(x,0)\dots\widetilde{\rho}_l^i(x,0) - \widetilde{\rho}_1^i(x,0)\dots\widetilde{\rho}_{l+1}^i(x,0))\boldsymbol{M}((\partial(C_{s(z_l)}(F(y)))_i)) \\ + \widetilde{\rho}_1^i(x,0)\dots\widetilde{\rho}_{2^{p'}-1}^i(x,0)\boldsymbol{M}((\partial(C_{s(z_{2^{p'}})}(F(y))))_i).$$

Finally, we want to prove that for a given $\delta_1 > 0$ and some sufficiently large q', the map

$$\partial F' \colon X(q') \to \mathcal{Z}_0(\Sigma, \mathbb{Z}_2)$$

is δ_1-localized on $X(q')$. Observe that in the construction above, given any two x, y in a cell C of $X(q)$ and $Q_i \subset \Sigma$ we have that $(\partial F'(x) - \partial F'(y)) \llcorner Q_i$ is supported in at most $k = \sup_{v \in C_0} \boldsymbol{M}(\partial F'(x) \llcorner Q_i)$ balls of radius bounded by $c(p)\,\mathrm{dist}_\infty(x,y)$. Hence, choosing $q'\big(p, \delta_1, \sup_{v \in X(q)_0} \boldsymbol{M}(\partial F'(x))\big) \geq q$ sufficiently large, we have that the map F' is δ_1-localized on $X(q')$.

This finishes the proof of Theorem 4.1. $\square$

5 Parametric isoperimetric inequality

In this section we prove a parametric isoperimetric inequality Conjecture 1.3 for 0-cycles in a disc. One can think about this result as a quantitative version of the Dold–Thom theorem.

Let Q denote a 2-dimensional disc of radius 1.

Theorem 5.1 *There exists constant $c(n) > 0$ with the following property. Suppose that $\varepsilon > 0$ and $\delta > 0$, and let $\Phi\colon X^p \to \mathcal{Z}_0(Q, \partial Q; \mathbb{Z}_2)$ be a continuous contractible map. There exists a δ-localized map $\widetilde{\Phi}\colon X^p \to \mathcal{Z}_0(Q; \mathbb{Z}_2)$ and $\Psi\colon X \to I_1(Q; \mathbb{Z}_2)$ such that*

- $\mathcal{F}\big(\Phi(x) \llcorner \mathrm{int}(Q), \widetilde{\Phi}(x) \llcorner \mathrm{int}(Q)\big) < \varepsilon$ *for all x,*
- $\partial \Psi(x) = \widetilde{\Phi}(x)$ *for all x,*
- $\boldsymbol{M}(\Psi(x)) \leq c\big(\boldsymbol{M}(\Phi(x)) p^{-1/2} + p^{1/2}\big)$,
- $\boldsymbol{M}(\widetilde{\Phi}(x)) \leq 2\boldsymbol{M}(\Phi(x)) + cp$.

We will need the following lemma:

Lemma 5.2 *Let $F\colon X^p(q)_0 \to \mathcal{Z}_0(Q; \mathbb{Z}_2)$ be an ε-fine family and let $I \subset \partial Q$ be a small interval of length L. There exists a continuous family $F'\colon X(q) \to \mathcal{Z}_0(Q; \mathbb{Z}_2)$ with the following properties:*

(1) *F' is ε'-fine for $\varepsilon' = c(p)\big(\varepsilon + \sup_{x \in X(q)_0} \boldsymbol{M}(F(x)) L\big)$.*

(2) *$F'(x) \llcorner \mathrm{int}(Q) = F(x) \llcorner \mathrm{int}(D)$ for all $x \in X(q)_0$.*

(3) *$\boldsymbol{M}(F'(x) \llcorner I) \leq 2p$.*

Proof Let z denote the cycle of mass 1 supported on the midpoint e of I and let $B = B_L(e)$.

For every $x \in X(q)_0$ define

$$F'(x) = \begin{cases} F(x) - F(x) \llcorner B + z & \text{if } \boldsymbol{M}(F(x) \llcorner I) \text{ is odd,} \\ F(x) - F(x) \llcorner B & \text{if } \boldsymbol{M}(F(x) \llcorner I) \text{ is even.} \end{cases}$$

Inductively we extend F' to the k-skeleton of $X(q)$.

Fix edge $E \subset X(q)_1$ with $\partial E = x - y$ and let T be a 1-chain, $\boldsymbol{M}(T) \leq \varepsilon$, with $\partial T = F(x) - F(y)$. We may assume that T is a finite collection of disjoint linear arcs. We remove all arcs of T whose endpoints are contained in I and replace each arc of T connecting a point $a \in Q$ to $b \in I$ with an arc connecting a to the midpoint e of I. We call the resulting 1-chain T'. Note that by triangle inequality T' satisfies

$$\boldsymbol{M}(T') \leq \boldsymbol{M}(T) + \max\{\boldsymbol{M}(F(x)), \boldsymbol{M}(F(y))\} L.$$

We have that $\partial T' = F'(x) - F'(y)$. We contract each arc of T' one by one to obtain a family T'_t of 1-chains and define $F'(t) = F'(x) + \partial T'_t$ for $t \in E$. Observe that $\boldsymbol{M}(F'(t)) \leq \max\{\boldsymbol{M}(F'(x)), \boldsymbol{M}(F'(y))\} + 2$.

Now we assume that we have extended to the k-skeleton with $\boldsymbol{M}(F'(x) \llcorner I) \leq 2k$ and so that the family is $c(k)\varepsilon$-localized. Let C be a $(k+1)$-face. By the assumption that the family is localized, there exists a ball $B_r(e)$, with $r \in (L, L + c(k)\varepsilon)$, such that $F'(x) \llcorner \partial B_r(e) = \varnothing$ for all $x \in \partial C$. Let $R_t\colon B_r(e) \to B_{r-t}(e)$ denote a 1–Lipschitz map defined as (in polar coordinates) $R_t(\rho, \theta) = \big(((r - t - L)\rho + tL)/(r - L), \theta\big)$ for $\rho > L$ and $R_t(\rho, \theta) = (\rho, \theta)$ otherwise. Note that R_t shrinks annulus $A(L, r, e)$ onto $A(L, r - t, e)$. For $t \in [0, r - L]$ define $F'(x, t) = F'(x) - F'(x) \llcorner B_r(e) + R_t(F'(x) \llcorner B_r(e))$. For $t = r - L$ we are guaranteed that the family $F'(x, t) \llcorner B_r(e)$ satisfies $\boldsymbol{M}(F'(x, t) \llcorner B_r(e) \llcorner \partial Q^2) \leq 2k + 2$. Then we can apply radial contraction to the center point e.

This contracts the family $F'(x)\llcorner B_r(e)$, where $x \in \partial C$, to a single cycle. We contract $F'(x)$ outside of $B_r(e)$ in the usual way; see Proposition 2.12. $\square$

Lemma 5.3 *Suppose $F\colon X^p \to \mathcal{Z}_0(Q^2;\mathbb{Z}_2)$, and there exists $I \subset \partial Q^2$ such that $\boldsymbol{M}(F(t)\llcorner I) \le K$. There exists a continuous family of 1-chains $\{\tau_t\}$ such that*

(1) $\partial\tau_t - F(t) \subset \partial Q$,

(2) $\boldsymbol{M}(\tau_t) \le 2\big(\boldsymbol{M}(F(t)\llcorner \mathrm{int}(Q)) + K\big)$,

(3) $\boldsymbol{M}(\partial\tau_t) \le 2\boldsymbol{M}(F(t)\llcorner \mathrm{int}(Q)) + 2K$.

Proof Pick a point $a \in \mathbb{R}^2$ outside of disc Q such that two tangent lines from a to ∂Q touch ∂Q at the endpoints of interval I. Given a point $q \in Q$ there is a unique line L_q passing through a and q. Let $H(q) = L_q \cap (\partial Q \setminus I)$. We define $\tau_t = \mathrm{Cone}_a(H(F(t))) - \mathrm{Cone}_a(F(t))$. $\square$

Now we can prove Theorem 5.1. By Proposition 3.5 we can replace Φ with an ε-close family of absolute cycles $\Phi'\colon X \to \mathcal{Z}_0(Q;\mathbb{Z}_2)$ and by Lemmas 5.3 and 5.2 we may assume that there exists a family of chains $\tau\colon X \to I_1(Q;\mathbb{Z}_2)$ such that $\partial\tau(x) = \Phi'(x)$. Note that the family will have the desired bound for the mass in the boundary.

Observe that for each x the chain τ consists of finitely many interval segments each lying on a ray from a fixed point a (point a lies outside of Q). We subdivide Q into p regions $\{Q_i\}$ of boundary length and diameter $\sim 1/\sqrt{p}$.

We perform a "bend-and-cancel" construction [14; 15] to reduce the length of chains τ_t by pushing them into the 1-skeleton of the subdivision $\bigcup \partial Q_i$.

For each $Q_i \subset Q$ let $B_{2\eta}(p_i) \subset Q_i$ be a ball chosen so that every line passing through the point a (from the proof of Lemma 5.3) intersects at most one ball $B_{2\eta}(p_i)$. Let $\mathrm{proj}_{p_i}\colon Q_i \setminus p_i \to \partial Q_i$ denote the projection map from the point p_i and define a piecewise linear map $P_i\colon Q_i \to Q_i$ such that $P_i = \mathrm{proj}_{p_i}$ on $Q_i \setminus B_{2\eta}(p_i)$ (in particular, $P_i(Q_i \setminus B_{2\eta}(p_i)) = \partial Q_i$) and $P_i(y) = y$ for all $y \in B_\eta(p_i)$. Note that P_i can be chosen so that for any line l passing through Q_i we have $\boldsymbol{M}(P_i(l \cap Q_i)) \le 2\,\mathrm{diam}(Q_i) \le 2/\sqrt{p}$. Since P_i is the identity map on ∂Q_i we can define a map $P\colon Q \to Q$ by $P(x) = P_i(x)$ for $x \in Q_i$.

Given a 0-cycle $z \subset Q$ define a continuous map $l_i\colon \mathcal{Z}_0(Q;\mathbb{Z}_2) \to I_1(Q_i;\mathbb{Z}_2)$ by setting $l_i(z)$ to be the union of linear arcs connecting each point of $z\llcorner Q_i$ to the corresponding point of $P_i(z\llcorner Q_i)$. Let $l(z) = \sum_i l_i(z)$. Define $\Psi(x) = P(\tau(x)) + l(\Phi'(x))$.

We observe that $\partial\Psi(x)\llcorner \mathrm{int}(Q) = \Phi'(x)$ and

$$\begin{aligned}\boldsymbol{M}(\Psi(x)) \le \boldsymbol{M}(P(\tau(x))) + \boldsymbol{M}(l(\Phi'(x))) &\le \boldsymbol{M}\Big(\bigcup \partial Q_i\Big) + \frac{2\boldsymbol{M}(\Phi'(x))}{\sqrt{p}} + \frac{\boldsymbol{M}(\Phi'(x))}{\sqrt{p}} \\ &\le c\bigg(\sqrt{p} + \frac{\boldsymbol{M}(\Phi'(x))}{\sqrt{p}}\bigg).\end{aligned}$$

This finishes the proof of Theorem 5.1.

We now prove Conjecture 1.3 for $k=0$ and $n=2$.

Theorem 5.4 *Let Ω be a 2-dimensional connected simply connected manifold with piecewise smooth boundary $\partial\Omega$ with θ-corners for some $\theta\in(0,\pi)$, and let $L>0$ and $p\in\mathbb{N}$. There exist constants $c(\Omega)>0$ and $\delta(L,p,\Omega)>0$ with the following property. Let $F\colon X^p\to\mathcal{Z}_0(\Omega,\partial\Omega;\mathbb{Z}_2)$ be a continuous contractible δ-localized p-dimensional family with $\sup_{x\in X}\boldsymbol{M}(F(x))\le L$. Then there exists a map $H\colon X\to I_1(\Omega;\mathbb{Z}_2)$ such that*

- $\partial H(x)-F(x)\subset\partial\Omega$ *for all x,*
- $\boldsymbol{M}(H(x))\le c(\Omega)\bigl(p^{1/2}+\boldsymbol{M}(F(x))p^{-1/2}\bigr)$.

Proof Let $F\colon X\to\mathcal{Z}_0(\Omega;\mathbb{Z}_2)$. Composing with a bilipschitz homeomorphism Π between Ω and disc Q we obtain a family F' of 0-cycles in Q. We apply Theorem 5.1 with $\varepsilon_i\to 0$ to obtain a sequence of δ-localized maps $\widetilde{F}_i$ converging uniformly to F' and a corresponding family of fillings Ψ_i of $\widetilde{F}_i$.

Pick i large enough so that $\mathcal{F}(\widetilde{F}_i(x),F'(x))<\frac{1}{2}\delta$. We will define a family $\widetilde{H}(x)$ with $\partial\widetilde{H}(x)=F'(x)-\widetilde{F}_i(x)$ and $\boldsymbol{M}(\widetilde{H}(x))\le c_0(p)L\delta$. We define $\widetilde{H}(x)$ inductively over the skeleton of X. On each vertex $x\in X_0$ define $\widetilde{H}(x)$ to be the area-minimizing filling of $F'(x)-\widetilde{F}_i(x)$. By the flat distance bound there exists a δ-admissible collection of open sets $\{U_i^x\}$ such that $\widetilde{H}(x)$ is supported in $\bigcup U_i$.

Now we prove the inductive step. Suppose we defined $\widetilde{H}$ on the k-skeleton of X and for each k-cell C the family of chains $\{\widetilde{H}(x)\}_{x\in C}$ is supported in a $c'(k)\delta$-admissible collection of sets $\{U_i^C\}$ and $\boldsymbol{M}(H(x))\le c_0(k)\delta\sup_{y\in X}\boldsymbol{M}(\widetilde{H}(y))$ for some constants c_0 and c' that depend only on k. Let D be a $(k+1)$-cell in X. By Lemma 2.5 there exists a constant $c'(k+1)>0$ and a $c'(k+1)\delta$-admissible collection of sets $\{U_i^D\}$ such that family of cycles $\{F'(x)-\widetilde{F}_i(x)\}_{x\in D}$ and family of chains $\{\widetilde{H}(x)\}_{x\in\partial D}$ are supported in $\bigcup U_i^D$. Observe that for each i we can define a continuous family of conical fillings $\tau_i(x)$, with $x\in D$, of $(F'(x)-\widetilde{F}_i(x))\llcorner U_i^D$ in U_i^D. ("Conical fillings" means a collection of linear segments from the support of $(F'(x)-\widetilde{F}_i(x))\llcorner U_i^D$ to the center point of U_i^D.) It will be convenient to fix polar coordinates (y,t), with $y\in\partial D$ and $t\in[0,1]$, on D, with $(y,1)=y$ and $(y,0)$ the center point of cell D. Let $h_i(y,t)$ denote the radial contraction towards the center point of U_i^D of relative cycle $\tau_i(y)-\widetilde{H}(y)\llcorner U_i^D$ so that $h_i(y,1)=\tau_i(y)-\widetilde{H}(y)\llcorner U_i^D$ and $h_i(y,0)=0$. Define $\widetilde{H}(y,t)=\sum_i\tau_i(y)-h_i(y,t)$. It is straightforward to check, using the inductive assumption, that it follows from this definition that $\widetilde{H}(x)$ is continuous on D, $\partial\widetilde{H}(x)=F'(x)-\widetilde{F}_i(x)$ and

$$\boldsymbol{M}(\widetilde{H}(y,t))\le 2\boldsymbol{M}(\widetilde{H}(y))+c'(k+1)\boldsymbol{M}(F'(x))\delta\le c_0(k+1)L\delta$$

for sufficiently large $c_0(k+1)$. This finishes the construction of $\widetilde{H}$.

Composing $H'=\Psi_i+\widetilde{H}$ with Π^{-1} gives the desired family H of fillings in Ω. The upper bound for $\boldsymbol{M}(H(x))$ follows from the upper bound for $\boldsymbol{M}(\Psi_i(x))$ and choosing $\delta=\delta(p,L,\Omega)$ sufficiently small. $\square$

Observe, that from Theorems 5.4 and 4.1 one can deduce the proof of Conjecture 1.3 for $k=1$ and $n=3$. We omit the details as this result is not used in the proof of the Weyl law.

6 Proof of the Weyl law

Theorem 6.1 *Let $n = 3$ and $k = 1$. Assume that Conjecture 1.6 holds for all families of k-cycles in domains in $\mathbb{R}^n$, and Conjecture 1.3 holds for all families of $(k-1)$-cycles in domains in $\mathbb{R}^{n-1}$. Then the Weyl law for k-cycles in n-manifolds holds*:

For every compact Riemannian manifold M (possibly with boundary),

$$\lim_{p\to\infty} \frac{\omega_p^k(M)}{p^{(n-k)/n}} = a(n,k)\,\mathrm{Vol}(M)^{k/n}.$$

Combined with Theorems 4.1 and 5.1 we obtain:

Corollary 6.2 *The Weyl law holds for 1-cycles in 3-manifolds*:

For every compact Riemannian 3-manifold M (possibly with boundary),

$$\lim_{p\to\infty} \frac{\omega_p^1(M)}{p^{2/3}} = a(3,1)\,\mathrm{Vol}(M)^{1/n}.$$

6.1 Proof of Theorem 6.1

In [21] it was proved that for any compact contractible domain $U \subset \mathbb{R}^n$ we have

$$\lim_{p\to\infty} \frac{\omega_p^k(U)}{p^{(n-k)/n}} = a(n,k)\,\mathrm{Vol}(U)^{k/n},$$

where $a(n,k)$ is a constant that depends only on n and k.

Let M be a compact n-manifold and consider sequence $\{\omega_p^k\}$ of k-dimensional p-widths of M. It is known that for some constants $a_1(n) \le a_2(n)$ we have

$$0 < a_1\,\mathrm{Vol}(M)^{k/n} = \liminf_{p\to\infty} \frac{\omega_p^k}{p^{(n-k)/n}} \le \limsup_{p\to\infty} \frac{\omega_p^k}{p^{(n-k)/n}} = a_2\,\mathrm{Vol}(M)^{k/n} < \infty.$$

We will show that $a_2 \le a(n,k) \le a_1$, which implies Theorem 6.1.

The inequality $a_1 \ge a(n,k)$ was proved in [21, Theorem 4.1]. It remains to prove $a_2 \le a(n,k)$.

Fix $\varepsilon > 0$. Fix a triangulation T of M so that each n-dimensional simplex U_i for $i = 1,\dots,N$ in the triangulation is $(1+\varepsilon)$-bilipschitz homeomorphic to a domain with θ-corners in $\mathbb{R}^n$, for some fixed $\theta \in \left(0, \frac{\pi}{2}\right)$.

Let $\widetilde{U}_i \subset \mathbb{R}^n$ denote the images of U_i under $(1+\varepsilon)$-bilipschitz homeomorphism and assume that they lie at a distance greater than 1 from each other. We connect sets $\widetilde{U}_i$ by tubes of very small volume to obtain a connected set $V \subset \mathbb{R}^n$. By Weyl law for domains in $\mathbb{R}^n$ for all sufficiently large p there exists a p-sweepout $\Phi\colon X^p \to \mathcal{Z}_k(V, \partial V; \mathbb{Z}_2)$ of V by k-cycles of mass bounded by $a(n,k)((1+2\varepsilon)\,\mathrm{Vol}(M))^{k/n} p^{(n-k)/n}$. Let $\widetilde{\Phi}_i\colon X \to \mathcal{Z}_k(\widetilde{U}_i, \partial\widetilde{U}_i; \mathbb{Z}_2)$ denote the restriction of Φ to $\widetilde{U}_i$ and let $\Phi_i\colon X \to \mathcal{Z}_k(U_i, \partial U_i; \mathbb{Z}_2)$ denote the map obtained by composing with bilipschitz homeomorphism from $\widetilde{U}_i$ to U_i.

Since $U_i \subset M$ has boundary with θ-corners, we can find a sufficiently small $\eta > 0$ so that there exists a $(1+c(U_i)\eta)$-bilipschitz homeomorphism $\Delta_\eta^i\colon (U_i)_\eta \to U$. (Recall that $(U_i)_\eta$ is the region obtained by removing a small neighborhood of ∂U_i from U_i, $(U_i)_\eta = U_i \setminus E([0,\eta]\times\partial U_i)$, where E is defined as in Lemma 2.1.) Let $c = \sup_i c(U_i)$. Let $V_m = \bigcup_{j=1}^m U_j$. We will define a sequence of maps $F_m\colon X \to \mathscr{Z}_k(V_m;\mathbb{Z}_2)$, satisfying

$$\text{(6-1)}\quad \boldsymbol{M}(F_m(x)) \le (1+c\eta)^k \sum_{i=1}^m \boldsymbol{M}(\Phi_i(x)) + c_m\biggl(p^{(n-k-1)/(n-1)} + \frac{1}{\eta}\sum_{i=1}^m \boldsymbol{M}(\Phi_i(x))\,p^{-1/(n-1)}\biggr),$$

$$\text{(6-2)}\quad \boldsymbol{M}(\partial F_m(x)) \le c_m\biggl(\frac{1}{\eta}\sum_{i=1}^m \boldsymbol{M}(\Phi_i(x)) + p^{(n-k)/(n-1)}\biggr).$$

Here c_m is a constant depending on m and $\{U_i\}$. Assume by induction that we defined F_{m-1} satisfying the mass bounds (6-1) and (6-2). We apply the coarea inequality Conjecture 1.6 to Φ_m to obtain a δ-localized map from X to $\mathscr{Z}_k((U_m)_\eta;\mathbb{Z}_2)$ and then compose with Δ^i_η to obtain a map $\Phi'\colon X \to \mathscr{Z}_k(U_m;\mathbb{Z}_2)$ with the following properties:

$$\boldsymbol{M}(\Phi'_m(x)) \le (1+c\eta)^k\biggl(\boldsymbol{M}(\Phi_m(x)) + cp^{(n-k-1)/(n-1)} + c\frac{\boldsymbol{M}(\Phi_m(x))}{\eta}p^{-1/(n-1)}\biggr),$$
$$\boldsymbol{M}(\partial\Phi'_m(x)) \le c(1+c\eta)^k\biggl(\frac{\boldsymbol{M}(\Phi_m(x))}{\eta} + p^{(n-k)/(n-1)}\biggr).$$

Let $S = \partial V_{m-1} \cap \partial U_m$. If S is empty, then we set $F_m = F_{m-1} + \Phi'_m$. Observe that since both $\{\partial F_{m-1}(x) \llcorner S\}$ and $\{\Phi'_m(x) \llcorner S\}$ are p-sweepouts of S by relative $(k-1)$-cycles, the family

$$\{f_S(x) = (\partial\Phi'_m(x) + \partial F_{m-1}(x)) \llcorner S\}$$

is a contractible family of relative $(k-1)$-cycles in S. (We refer to [34] for a more detailed proof of this fact.) By the parametric isoperimetric inequality Conjecture 1.3 applied to f_S there exists a family $\Psi_S\colon X \to I_k(S;\mathbb{Z}_2)$ with $\partial\Psi_S(x) - f_S(x)$ supported in ∂S and satisfying

$$\begin{aligned}\boldsymbol{M}(\Psi_S(x)) &\le c(S)p^{(n-k-1)/(n-1)} + \bigl(\boldsymbol{M}(\partial F_{m-1}(x)) + \boldsymbol{M}(\partial\Phi'_m(x))\bigr)p^{-1/(n-1)}\\ &\le (c_{m-1}+c(S))c(1+c\eta)^k\biggl(p^{(n-k-1)/(n-1)} + \sum_{i=1}^m \frac{\boldsymbol{M}(\Phi'_i(x))}{\eta}p^{-1/(n-1)}\biggr).\end{aligned}$$

When $k=1$ and $n=3$ the boundary ∂S is one-dimensional and we can find a continuous family of mod 2 one-chains $\{\overline{\Psi}_S(x)\}$ supported in ∂S with $\boldsymbol{M}(\partial\Psi(x) + f_S(x) + \partial\overline{\Psi}_S(x)) \le c(S)p$. Since $\overline{\Psi}_S(x)$ is supported in ∂S its mass is bounded by the length of ∂S. For the cases of higher dimensions and codimensions one needs to inductively apply the parametric coarea and isoperimetric inequalities to obtain a family of partial fillings in ∂S analogous to $\overline{\Psi}(x)$ (with the desired mass bounds). We define

$$F_m(x) = F_{m-1}(x) + \Phi_m(x) + \Psi_S(x) + \overline{\Psi}_S(x).$$

Observe that F_m is a continuous family of relative cycles and a p-sweepout of V_m. It is straightforward to check that the inductive assumption for the mass bounds (6-1) and (6-2) will be satisfied for some sufficiently large $c_m > c_{m-1}$ that only depends on the choice of domains U_i. For $m = N$ we obtain a p-sweepout F_N of M with

$$\begin{aligned}\frac{\boldsymbol{M}(F_N(x))}{p^{(n-k)/n}} &\le (1 + C(\varepsilon,\eta))\left(\frac{\boldsymbol{M}(\Phi(x))}{p^{(n-k)/n}} + c_N \frac{\boldsymbol{M}(\Phi(x))}{\eta p^{(n-k)/n+1/(n-1)}} + c_N p^{-k/(n(n-1))}\right)\\ &\le (1 + C(\eta,\varepsilon))a(n,k)\operatorname{Vol}(M)^{k/n} + c(\varepsilon,\eta,M)(p^{-1/(n-1)} + p^{-k/(n(n-1))})\end{aligned}$$

for a constant $C(\varepsilon,\eta)$ with $C(\varepsilon,\eta) \to 0$ as η and ε tend to 0. As $p \to \infty$ the second term goes to 0. Since η and ε can be chosen to be arbitrarily small, we conclude that $a_2 \le a(n,k)$. This finishes the proof of Theorem 6.1.

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Proposed: John Lott Received: 3 March 2022
Seconded: Dmitri Burago, David Fisher Revised: 16 September 2022

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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Geometry & Topology

Volume 29 Issue 2 (pages 549–1114) 2025