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# Discrete subgroups with finite Bowen–Margulis–Sullivan measure in higher rank

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Let  $G$  be a connected semisimple real algebraic group and  $\Gamma < G$  be a Zariski dense discrete subgroup. We prove that if  $\Gamma \backslash G$  admits any finite Bowen–Margulis–Sullivan measure, then  $\Gamma$  is virtually a product of higher rank lattices and discrete subgroups of rank one factors of  $G$ . This may be viewed as a measure-theoretic analogue of the classification of convex cocompact actions by Kleiner and Leeb (*Invent. Math.* 163 (2006) 657–676) and Quint (*Geom. Dedicata* 113 (2005) 1–19), which was conjectured by Corlette in 1994. The key ingredients in our proof are the product structure of leafwise measures and the high entropy method of Einsiedler, Katok and Lindenstrauss (*Ann. of Math.* 164 (2006) 513–560). In a companion paper jointly with Edwards and Oh (*C. R. Math. Acad. Sci. Paris* 362 (2024) 1873–1880) we use this result to show that the bottom of the  $L^2$  spectrum has no atom in any infinite volume quotient of a higher rank simple algebraic group.

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## 1 Introduction

Let  $G$  be a higher rank semisimple Lie group. The lattices of  $G$  are more or less classified thanks to the Margulis arithmeticity theorem. In contrast, the structure of infinite covolume subgroups remains mysterious, with only a few completely general results available; see eg Fraczyk and Gelander [13]. Special classes of subgroups, most notably Anosov subgroups, have been investigated in more depth; see eg Bochi, Potrie and Sambarino [2], Bridgeman, Canary, Labourie and Sambarino [3], Fock and Goncharov [12], Guéritaud, Guichard, Kassel and Wienhard [14], Guichard and Wienhard [15], Hitchin [16], Kapovich, Leeb and Porti [17] and Labourie [20], as well as Kassel [18] and Wienhard [28] for survey articles. Despite these efforts, our understanding of general “large” discrete subgroups of  $G$  remains limited. Not many constructions are known, and we still lack the theory that would explain the scarcity of examples. Quint [25; 24] has shown that Patterson–Sullivan theory [22; 27] can be applied to any discrete subgroup  $\Gamma < G$ , providing a way to study discrete subgroups of  $G$  via their conformal measures and the associated Bowen–Margulis–Sullivan (BMS) measures on the quotient space  $\Gamma \backslash G$ . These measures are invariant under the action of a maximal real split torus of  $G$ , which, in our setting, has rank at least two. Einsiedler and Katok [9] and Einsiedler, Katok and Lindenstrauss [10] have shown that, under mild positive entropy and certain recurrence assumptions, the actions of higher rank tori on homogeneous spaces exhibit many rigidity phenomena that are absent in the rank one case. Here we show how one can leverage this rigidity to gain a better understanding of discrete subgroups of  $G$ .

In our main result ([Theorem 1.6](#)) we classify subgroups  $\Gamma$  that admit a finite BMS measure, which provides a measured analogue of the theorem of Kleiner–Leeb and Quint classifying higher rank convex cocompact actions. We believe that our approach using entropy rigidity of higher rank actions will yield further results in this area.

**Higher rank convex cocompact actions** Let  $X$  be a Riemannian symmetric space of noncompact type. It is known that the group  $G$  consisting of all orientation preserving isometries of  $X$  is a connected semisimple real algebraic group.<sup>1</sup> Let  $\Gamma < G$  be a Zariski dense discrete subgroup. We say  $\Gamma$  is *convex cocompact* if there exists a nonempty  $\Gamma$ -invariant closed convex subset  $C \subset X$  on which  $\Gamma$  acts cocompactly.

A conjecture of Corlette from 1994, now resolved by the work of Kleiner and Leeb [[19](#)], says that the only convex cocompact actions in higher rank symmetric spaces are those arising from the rank one constructions. More precisely, we have the following:

**Theorem 1.1** (Kleiner and Leeb) *If  $C \subset X$  is a nonempty  $\Gamma$ -invariant closed convex subset on which  $\Gamma$  acts cocompactly, then  $C = C_1 \times X_2$  for some nonempty closed subset  $C_1 \subset X_1$  where  $X_1$  (resp.  $X_2$ ) denotes the product of rank one (resp. rank  $\geq 2$ ) factors of  $X$ , so that  $X = X_1 \times X_2$ .*

**Remark 1.2** The original article deals with a more general setup from which [Theorem 1.1](#) follows as a special case. The more general version takes into account additional Euclidean factor for  $X$ , and also provides more information about the structure of  $C_1$ . We refer the readers to [[19](#), Theorem 1.1] for details.

In particular, if there exists a convex cocompact subgroup of  $G$  which is not a cocompact lattice,  $G$  necessarily has a rank one factor.

On the other hand, independently in [[26](#)], Quint established an obstruction to a higher rank convex cocompact action using a different definition. We give a precise statement of his result since it is quite close to the statement we are going to prove. Let  $G$  and  $\Gamma$  be as before,  $A$  be a maximal real split torus of  $G$ ,  $Z$  be the centralizer of  $A$ , and  $M$  be the maximal compact subgroup of  $Z$  such that  $Z = AM$ .

Let  $\Omega_\Gamma$  denote the smallest closed subset of  $\Gamma \backslash G/M$  containing all periodic orbits of one-parameter subgroups of  $A$ . When  $G$  is of rank one, the  $A$  action corresponds to the geodesic flow and  $\Omega_\Gamma$  is precisely the nonwandering set of the geodesic flow. We have:

**Theorem 1.3** (Quint) *If  $\Omega_\Gamma$  is compact, then there exist connected subgroups  $G_0, G_1, \dots, G_q$  with corresponding Zariski dense discrete subgroups  $\Gamma_0, \Gamma_1, \dots, \Gamma_q$  such that:*

- (1)  $G = G_0 G_1 \cdots G_q$  and  $\Gamma$  is commensurable<sup>2</sup> with  $\Gamma_0 \Gamma_1 \cdots \Gamma_q$ .
- (2)  $\Gamma_0$  is a cocompact lattice in  $G_0$ .
- (3) For all  $1 \leq i \leq q$ ,  $G_i$  is a simple Lie group of rank one, and  $\Gamma_i$  is a convex cocompact subgroup of  $G_i$ .

<sup>1</sup>By this we mean the connected component of the set of real points of a semisimple algebraic group defined over  $\mathbb{R}$ .

<sup>2</sup>For  $\Gamma, \Gamma' < G$ , we say  $\Gamma$  is commensurable with  $\Gamma'$  if  $\Gamma \cap \Gamma'$  is a subgroup of finite index in both  $\Gamma$  and  $\Gamma'$ .

It is known that [Theorem 1.3](#) can be used to deduce [Theorem 1.1](#) [[26](#), [Theorem 5.1](#)].

Here we obtain a measure-theoretic analogue of [Theorem 1.3](#). We are concerned with the question “which discrete subgroup  $\Gamma$  of a semisimple real algebraic group  $G$  can admit a finite *geometric measure* on  $\Gamma \backslash G$ ”.

**Bowen–Margulis–Sullivan measures** The Bowen–Margulis–Sullivan measure was first introduced by Margulis [[21](#)] in his 1970 thesis for the study of Anosov flows on compact manifolds. Since then it has become an important object in the field of ergodic theory and dynamics, and it has been studied extensively in various contexts. For a general semisimple Lie group, it can be defined as follows:

Let  $G$ ,  $A$ ,  $M$  and  $\Gamma$  be as before. Choose a closed Weyl chamber  $A^+ \subset A$ , and denote by  $\text{int } A^+$  its interior. The maximal horospherical subgroups associated to the choice of  $A^+$  are given by

$$N^\pm = \{g \in G : a^k g a^{-k} \rightarrow e \text{ as } k \rightarrow \pm\infty \text{ for some } a \in \text{int } A^+\}.$$

Let  $P^\pm = MAN^\pm$  be the corresponding minimal parabolic subgroups. For all  $x \in \Gamma \backslash G$ , let

$$W^s(x) = xN^-, \quad W^{\text{cu}}(x) = xP^+,$$

be the (global) stable and central-unstable manifold passing through  $x$  corresponding to an element  $a \in \text{int } A^+$ . For  $\varepsilon > 0$ , let

$$W_\varepsilon^s(x) = xN_\varepsilon^-, \quad W_\varepsilon^{\text{cu}}(x) = xP_\varepsilon^+,$$

where  $N_\varepsilon^-$  and  $P_\varepsilon^+$  denotes the  $\varepsilon$ -neighborhood of the identity in  $N^-$  and  $P^+$  respectively. If  $\varepsilon$  is sufficiently small, then for all  $y \in W_\varepsilon^s(x)$  and  $z \in W_\varepsilon^{\text{cu}}(x)$ ,  $W_{2\varepsilon}^{\text{cu}}(y)$  and  $W_{2\varepsilon}^s(z)$  meet in a single point. Denoting this intersection by  $\iota(y, z)$ , the map

$$(1.4) \quad \iota : W_\varepsilon^s(x) \times W_\varepsilon^{\text{cu}}(x) \rightarrow \Gamma \backslash G$$

is a homeomorphism in a neighborhood of  $x$ .

**Definition 1.5** A locally finite  $AM$ -invariant Borel measure  $m$  on  $\Gamma \backslash G$  is called a *Bowen–Margulis–Sullivan* (BMS) measure if there exists a linear form  $\psi \in \mathfrak{a}^*$  and a family of measures  $\{m_x^s\}$  and  $\{m_x^{\text{cu}}\}$  (for  $x \in \Gamma \backslash G$ ) such that for all  $x \in \Gamma \backslash G$ ,

- (1)  $m_x^s$  (resp.  $m_x^{\text{cu}}$ ) is supported on  $W^s(x)$  (resp.  $W^{\text{cu}}(x)$ ),
- (2)  $m_x^s \cdot a = e^{-\psi(\log a)} m_{x \cdot a}^s$  for all  $a \in A^+$ ,
- (3)  $m = \iota.(m_x^s \otimes m_x^{\text{cu}})$  when restricted to the image of  $\iota$ .

Note that we may view  $m$  as a measure on  $\Gamma \backslash G/M$  as well.

**Finite BMS measures** When  $G$  is of rank one, the space  $\Gamma \backslash G/M$  represents a unit tangent bundle of a rank one locally symmetric space, on which the one-dimensional group  $A = \{a_t\}$  acts as a geodesic flow. In this case, the construction of Sullivan [[27](#)], which extends the idea of Patterson [[22](#)], shows that there exists a measure  $m$  supported on the nonwandering set of  $\{a_t\}$ , and a family of measures  $m_x^s$  and  $m_x^{\text{cu}}$  verifying the conditions in [Definition 1.5](#).

Moreover, one can produce many examples of infinite covolume subgroups with finite BMS measures. Examples include a large class of subgroups such as convex cocompact subgroups, and geometrically finite subgroups of rank one Lie groups [27]. And still, the list is far from being exhaustive, as the example of Dal'bo, Otal and Peigné [5] suggests.

Turning to the case when  $G$  is of higher rank, there always exists a BMS measure  $m$  supported on  $\Omega_\Gamma$ . In fact, Quint shows that one can find a BMS measure for every linear form  $\psi \in \mathfrak{a}^*$  which is tangent to the growth indicator  $\psi_\Gamma$  (see Proposition 2.6). As a consequence, we have an abundance of examples of a measure  $m$  as in Definition 1.5. However, in contrast to the rank one case, there was no known example of discrete subgroups admitting a finite BMS measure except for lattices.

The following theorem is our main result, in which we classify all discrete subgroups admitting finite BMS measures on  $\Gamma \backslash G$ :

**Theorem 1.6** *If  $\Gamma \backslash G$  admits a finite BMS measure, then there exist connected subgroups  $G_0, G_1, \dots, G_q$  with corresponding Zariski dense discrete subgroups  $\Gamma_0, \Gamma_1, \dots, \Gamma_q$  such that:*

- (1)  $G = G_0 G_1 \cdots G_q$  and  $\Gamma$  is commensurable with  $\Gamma_0 \Gamma_1 \cdots \Gamma_q$ .
- (2)  $\Gamma_0$  is a lattice in  $G_0$ .
- (3) For all  $1 \leq i \leq q$ ,  $G_i$  is a simple Lie group of rank one, and  $\Gamma_i \backslash G_i$  admits a finite BMS measure.

The theorem leads to a new criterion for detecting lattices among the discrete subgroups of higher rank Lie groups.

**Corollary 1.7** *If  $G$  is a connected semisimple real algebraic group with no rank one factor and  $\Gamma < G$  is a Zariski dense discrete subgroup, then  $\Gamma$  is a lattice if and only if  $\Gamma \backslash G$  admits a finite BMS measure, in which case the BMS measure is necessarily the Haar measure.*

As another application, we obtain the following theorem, in a joint work with Edwards and Oh [6] using their previous work [8]:

**Theorem 1.8** (Edwards, Fraczyk, Lee and Oh [6]) *Let  $\Gamma < G$  be as in Corollary 1.7. If  $\Gamma \backslash G$  has infinite volume, then there is no positive  $L^2$ -integrable Laplace eigenfunction on the associated locally symmetric space  $\Gamma \backslash X$ .*

Note that Theorem 1.6 also recovers Quint's theorem (Theorem 1.3), since there always exists a BMS measure supported on  $\Omega_\Gamma$ .

**Idea of the proof** We describe the proof in the case when  $G$  is a simple higher rank group, or is of the form  $G = G_1 \times G_2$  and  $\Gamma$  is irreducible, ie has dense projections onto each factor.

To show that  $\Gamma$  is a lattice in  $G$ , one needs to verify that the volume of  $\Gamma \backslash G$  is finite with respect to the Haar measure. We achieve this by proving that any finite BMS measure  $m$  on  $\Gamma \backslash G$  is actually Haar. The proof is based on the high entropy method of Einsiedler and Katok [9] and Einsiedler, Katok and Lindenstrauss [10],

and the product structure of leafwise measures shown by Einsiedler and Lindenstrauss [11]. These methods allow us to establish additional invariance of a finite  $A$ -invariant measure  $\mu$  on a quotient  $\Gamma \backslash G$ , provided that sufficiently many elements in  $A$  act on  $(\Gamma \backslash G, \mu)$  with positive entropy.

If  $G$  is a simple higher rank group, the presence of the local product structure of  $m$  together with the product structure of leafwise measures  $m_x^{N^\pm}$  of  $m$  along  $N^\pm$  (Proposition 2.13) shows that either  $m_x^{N^\pm}$  is supported on a proper Zariski closed subset of  $N^\pm$  or the leafwise measures along all root subgroups of  $N^\pm$  are nontrivial. The first case can be excluded solely using the Zariski density of  $\Gamma$ . In the second case we finish the proof by applying the high entropy method.

Now assume that  $G = G_1 \times G_2$  and  $\Gamma$  is irreducible. Associated with  $m$ , there are  $\Gamma$ -conformal measures on  $G/P^\pm$  underlying the definition of  $m$  (2.2). The product structure of leafwise measures implies that the conformal measure on  $G/P^\pm$  is itself a product of measures on  $G_i/P_i^\pm$ , where  $P_i^\pm = G_i \cap P^\pm$  for  $i = 1, 2$ . These measures are “almost” conformal under the projections of  $\Gamma$  to  $G_i$  for  $i = 1, 2$ . Now, the density of the projections allows us to deduce that the measures are  $G_i$ -conformal, and hence Lebesgue (Proposition 4.13). It follows that  $m$  is Haar.

**Outline** In Section 2 we gather preliminary results on semisimple Lie groups, Bowen–Margulis–Sullivan measures and the machinery of leafwise measures developed by Einsiedler, Katok and Lindenstrauss. Section 3 is devoted to the study of leafwise measures of BMS-measures. Here, we identify the leafwise measures with respect to the unipotent radicals of minimal parabolic subgroups in terms of the Patterson–Sullivan measures of  $\Gamma$  (3.7). We also show that when  $\Gamma$  has a discrete projection onto a factor of  $G$ , the problem can be handled factor by factor (Lemma 3.16). In Section 4 we use the product structure of leafwise measures (Lemma 4.7) and Zariski density of  $\Gamma$  to show that leafwise measures with respect to all root subgroups are nontrivial (Lemma 4.8). We then deduce the additional invariance of the BMS measure (Propositions 4.12 and 4.13) and prove the main theorem.

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## 2 Preliminaries

In this section, we will fix notation and recall background that will be used throughout the paper.

Let  $G$  be a connected semisimple real algebraic group, and  $\Gamma < G$  be a Zariski dense discrete subgroup. Let  $\mathfrak{g}$  denote the Lie algebra of  $G$ ,  $\Theta: \mathfrak{g} \rightarrow \mathfrak{g}$  be a Cartan involution, and  $\mathfrak{k}$  and  $\mathfrak{p}$  respectively be  $+1$

and  $-1$  eigenspaces of  $\Theta$  so that we have the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Fix a maximal abelian subspace  $\mathfrak{a} \subset \mathfrak{p}$  and a closed positive Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$ . Let  $A = \exp(\mathfrak{a})$ ,  $A^+ = \exp(\mathfrak{a}^+)$ ,  $\text{int } A^+$  be the interior of  $A^+$ , and  $K$  be the maximal compact subgroup of  $G$  whose Lie algebra is  $\mathfrak{k}$ . Let

$$N^\pm = \{g \in G : a^k g a^{-k} \rightarrow e \text{ as } k \rightarrow \pm\infty \text{ for some } a \in \text{int } A^+\}$$

be a pair of maximal horospherical subgroups. We also let  $P^\pm = MAN^\pm$  be the corresponding minimal parabolic subgroups, where  $M$  is the centralizer of  $A$  in  $K$ .

**Conformal measures** Let  $\mathcal{F}$  be the set of all minimal parabolic subgroups of  $G$ , which can be identified with  $G/P^\pm$ . We define  $\sigma : G \times \mathcal{F} \rightarrow \mathfrak{a}$  as follows: for  $g \in G$  and  $\xi \in \mathcal{F}$ ,  $\sigma(g, \xi) \in \mathfrak{a}$  is the unique element satisfying

$$(2.1) \quad gk_\xi \in K \exp(\sigma(g, \xi))N^-,$$

where  $k_\xi \in K$  is any element such that  $\xi = k_\xi P^-$ . The definition does not depend on the choice of representative  $k_\xi \in K$  for  $\xi$ , and  $\sigma$  is called the *Iwasawa cocycle*.

Given a linear form  $\psi \in \mathfrak{a}^*$ , a finite Borel measure  $\nu$  on  $\mathcal{F}$  is called a  $(\Gamma, \psi)$ -conformal measure if, for any  $\gamma \in \Gamma$  and  $\xi \in \mathcal{F}$ ,

$$(2.2) \quad \frac{d(\gamma.\nu)}{d\nu}(\xi) = e^{-\psi(\sigma(\gamma^{-1}, \xi))},$$

where  $\gamma.\nu(E) = \nu(\gamma^{-1}E)$  for any Borel subset  $E \subset \mathcal{F}$ . We will refer to  $\nu$  as a  $\Gamma$ -conformal measure if it is a  $(\Gamma, \psi)$ -conformal measure for some  $\psi \in \mathfrak{a}^*$ .

**Bowen–Margulis–Sullivan measures** Let  $N_K(\mathfrak{a})$  and  $Z_K(\mathfrak{a})$  respectively denote the normalizer and the centralizer of  $\mathfrak{a}$  in  $K$ . The Weyl group  $\mathcal{W}$  is then defined as

$$(2.3) \quad \mathcal{W} = N_K(\mathfrak{a}) / Z_K(\mathfrak{a}).$$

Let  $i : \mathfrak{a} \rightarrow \mathfrak{a}$  be the opposition involution; it is given by  $i = -\text{Ad}_{w_0}$ , where  $w_0$  is the longest element of the Weyl group  $\mathcal{W}$ .

Let  $\psi \in \mathfrak{a}^*$  be a linear form and  $\nu_\psi$  and  $\nu_{\psi \circ i}$  be a pair of  $\Gamma$ -conformal measures on  $\mathcal{F}$  associated to  $\psi$  and  $\psi \circ i$ . By means of the Hopf parametrization

$$(2.4) \quad G/M \rightarrow G/P^- \times G/P^+ \times \mathfrak{a}, \quad gM \mapsto (gP^-, gP^+, \sigma(g, P^-)),$$

the *Bowen–Margulis–Sullivan* (BMS) measure  $\tilde{m}_{\nu_\psi, \nu_{\psi \circ i}}$  is defined to be the locally finite Borel measure on  $G/M$  given by

$$(2.5) \quad \tilde{m}_{\nu_\psi, \nu_{\psi \circ i}} = e^{\psi(\mathcal{G}(\cdot, \cdot))} \nu_\psi \otimes \nu_{\psi \circ i} \otimes \text{Leb}_\mathfrak{a},$$

where  $\text{Leb}_\mathfrak{a}$  denotes the Lebesgue density on  $\mathfrak{a}$ , and  $\mathcal{G}$  is the *vector valued Gromov product* defined by

$$\mathcal{G}(gP^-, gP^+) = \sigma(g, P^-) + i\sigma(g, P^+).$$

By the relation (2.2),  $\tilde{m}_{v_\psi, v_{\psi \circ i}}$  is left  $\Gamma$ -invariant and descends to a measure on  $\Gamma \backslash G/M$ , which we denote by  $m_{v_\psi, v_{\psi \circ i}}$ .

The fact that this definition of BMS measure is equivalent to that given in the introduction can be derived from the properties shown in [7, Section 4.1].

**The limit cone and the growth indicator** By Cartan decomposition  $G = KA^+K$ , for every  $g \in G$  there exists a unique element  $\mu(g) \in \mathfrak{a}^+$  such that

$$g \in K \exp(\mu(g))K.$$

Benoist introduced the *limit cone* of  $\Gamma$  [1], denoted by  $\mathcal{L}_\Gamma$ , which is the asymptotic cone of  $\{\mu(\gamma) : \gamma \in \Gamma\}$ . He showed that if  $\Gamma$  is Zariski dense, then  $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$  is a closed convex cone with nonempty interior.

The *growth indicator* of  $\Gamma$ , denoted by  $\psi_\Gamma$ , is a function  $\psi_\Gamma : \mathfrak{a}^+ \rightarrow \mathbb{R}_{\geq 0} \cup \{-\infty\}$  defined by Quint [24] as

$$\psi_\Gamma(v) = \|v\| \inf_{v \in \mathcal{C}} h_{\mathcal{C}};$$

here  $\|\cdot\|$  denotes the norm on  $\mathfrak{a}$  induced from the Killing form on  $\mathfrak{g}$ , the infimum is taken over all open cones  $\mathcal{C} \subset \mathfrak{a}^+$  containing  $v$  and  $h_{\mathcal{C}}$  denotes the abscissa of convergence for the series  $s \mapsto \sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-s\|\mu(\gamma)\|}$ .

We say a linear form  $\psi \in \mathfrak{a}^*$  is *tangent to  $\psi_\Gamma$*  at  $v \in \mathfrak{a}^+$  if

$$\psi \geq \psi_\Gamma \quad \text{on } \mathfrak{a}^+ \quad \text{and} \quad \psi(v) = \psi_\Gamma(v)$$

The following general properties of  $\psi_\Gamma$  are known [25; 24]:

**Proposition 2.6** (Quint) (1) For any  $v \in \mathfrak{a}^+$ ,  $\psi_\Gamma(v) \geq 0$  if and only if  $v \in \mathcal{L}_\Gamma$ . Moreover,  $\psi_\Gamma$  is strictly positive in the interior of  $\mathcal{L}_\Gamma$ .

(2) If there exists a  $(\Gamma, \psi)$ -conformal measure on  $\mathcal{F}$ , then  $\psi \geq \psi_\Gamma$ .

(3) For any  $\psi \in \mathfrak{a}^*$  tangent to  $\psi_\Gamma$  at  $v \in \text{int } \mathfrak{a}^+$ , there exists a  $(\Gamma, \psi)$ -conformal measure on  $\mathcal{F}$ .

**Conditional measures** Let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra on  $\Gamma \backslash G$  and  $\mathcal{A} \subset \mathcal{B}$  be a countably generated sub- $\sigma$ -algebra. For  $x \in \Gamma \backslash G$ , let  $[x]_{\mathcal{A}}$  denote the  $\mathcal{A}$ -atom of  $x$ , which is the intersection of all elements of  $\mathcal{A}$  containing  $x$ .

Given a finite Borel measure  $\mu$  on  $\Gamma \backslash G$ , there exists a  $\mu$ -conull set  $X \subset \Gamma \backslash G$  and a family of finite measures  $\{\mu_x^{\mathcal{A}} : x \in X\}$  of total mass one with the following properties [11, Theorem 5.9]:

(1)  $\mu_x^{\mathcal{A}}$  is supported on  $[x]_{\mathcal{A}}$  and  $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$  if  $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$ ,

(2) the map  $x \mapsto \mu_x^{\mathcal{A}}(B)$  is  $\mathcal{A}$ -measurable for all  $B \in \mathcal{B}$ , and

(3) we have

$$(2.7) \quad \mu = \int_{\Gamma \backslash G} \mu_x^{\mathcal{A}} d\mu(x).$$

They are unique in the following sense: if  $\lambda_x^{\mathcal{A}}$  is another family of measures satisfying the above, then there exists a  $\mu$ -conull set  $X' \subset \Gamma \backslash G$  such that  $\mu_x^{\mathcal{A}} = \lambda_x^{\mathcal{A}}$  for all  $x \in X'$ .

**Leafwise measures** For a closed subgroup  $S < G$ , there exists a  $\mu$ -conull set  $X' \subset \Gamma \backslash G$  and a family  $\{\mu_x^S : x \in X'\}$  of locally finite Borel measures on  $S$ , well-defined up to a proportionality, called the *leafwise measure* of  $\mu$  along  $S$  [11, Theorem 6.3]. Locally, they can be described as follows:

For  $R > 0$ , let  $B_S(R)$  denote the ball of radius  $R$  in  $S$  centered at  $e$ . For any  $x \in \Gamma \backslash G$  such that the map  $s \mapsto x.s$  for  $s \in S$  is injective, there exists a local cross section  $C \subset G$  containing  $e$  such that the map

$$\phi : C \times B_S(R) \rightarrow \Gamma \backslash G, \quad (g, s) \mapsto x.(gs),$$

is injective, where  $C$  (and hence  $\phi$ ) depends on  $R$  and  $x$ . Let  $\mathcal{B}_C$  denote the Borel  $\sigma$ -algebra on  $\phi(C \times \{e\})$  induced from  $\mathcal{B}$ , and  $\{\emptyset, S\}$  the trivial  $\sigma$ -algebra on  $S$ . Up to proportionality,  $\mu_x^S$  is then characterized<sup>3</sup> as follows:

$$(2.8) \quad \phi(e, \cdot)_* \mu_x^S|_{B_S(R)} \propto (\mu|_{\text{Im } \phi})_x^{\mathcal{B}_C \otimes \{\emptyset, S\}}.$$

Now let  $\mu$  be a finite  $A$ -invariant measure. We record here several results concerning the leafwise measures of  $\mu$  that will be needed. The main reference is [11]. For a closed subgroup  $S < G$ , we have:

**Lemma 2.9** [11, Lemma 7.16] *If  $A$  normalizes  $S$ , then for all  $a \in A$ ,*

$$\theta_* \mu_x^S \propto \mu_{x.a}^S$$

for  $\mu$ -a.e.  $x$ , where  $\theta : S \rightarrow S$  is the map given by  $\theta(s) = a^{-1}sa$ .

A leafwise measure will be called *trivial* if it is proportional to a Dirac measure supported at  $e$ . By Lemma 2.9, if  $\mu$  is a finite  $A$ -invariant ergodic measure then either  $\mu_x^S$  is trivial  $\mu$ -a.e. or nontrivial  $\mu$ -a.e.

**Lemma 2.10** [11, Lemma 9.18] *Let  $L \leq S$  be a closed subgroup. Suppose that for every  $\ell \in L$ ,  $\ell.\mu_x^S = \mu_x^S$  for  $\mu$ -a.e.  $x$ . Then  $\mu$  is  $L$ -invariant.*

In particular,  $\mu$  is  $S$ -invariant if and only if  $\mu_x^S$  is Haar  $\mu$ -a.e. For a fixed unit ball  $B_1$  in  $N^+$  and  $a \in A^+$ , define

$$D_\mu(a, N^+)(x) := \lim_{k \rightarrow \infty} \frac{1}{k} \log \mu_x^{N^+}(a^{-k} B_1 a^k).$$

**Lemma 2.11** [11, Theorem 7.6] *For any  $a \in A^+$ ,  $D_\mu(a, N^+)(x)$  is defined  $\mu$ -a.e., and the measure-theoretic entropy of  $a$  with respect to  $\mu$  is given by*

$$(2.12) \quad h_\mu(a) = \int_{\Gamma \backslash G} D_\mu(a, N^+)(x) d\mu(x).$$

For  $a \in A$ , let  $Z_G(a)$  denote the centralizer of  $\langle a \rangle$ . The proposition below describes the product structure of the leafwise measures. Together with the high entropy method of [9; 10], this is the most important ingredient of our proof.

<sup>3</sup>This follows from the defining equation [11, (6.20b)] of  $\mu_x^S$ . For local cross sections, see [11, Definition 6.6].

**Proposition 2.13** [11, Corollary 8.8] *Let  $U < G$  be a subgroup normalized and contracted by an element  $a \in A$ . Assume that  $T$  is a subgroup of  $Z_G(a)$  normalizing  $U$  and  $H = T \times U$ . Then for any finite  $\langle a \rangle$ -invariant measure  $\mu$  on  $\Gamma \backslash G$ ,*

$$\mu_x^H \propto \iota(\mu_x^T \times \mu_x^U)$$

for  $\mu$ -a.e.  $x$ , where  $\iota: T \times U \rightarrow H$  is the product map  $(t, u) \mapsto tu$ .

### 3 Ergodic properties of BMS measures

Let  $G$  be a connected semisimple real algebraic group and  $\Gamma < G$  be a Zariski dense discrete subgroup. We fix  $\psi \in \mathfrak{a}^*$  and a pair of  $\Gamma$ -conformal measures  $\nu_\psi$  and  $\nu_{\psi \circ i}$  on  $\mathcal{F}$ , and let

$$(3.1) \quad m = m_{\nu_\psi, \nu_{\psi \circ i}}$$

denote the associated BMS measure on  $\Gamma \backslash G$ .

**Conservative BMS measures** Let  $\{a_t\} \subset A$  be a one-parameter subgroup. We recall the following definitions:

- (1) A Borel subset  $B \subset \Gamma \backslash G$  is a *wandering set*, if for  $m$ -a.e.  $x \in B$ ,

$$\int_{\mathbb{R}} \mathbf{1}_B(xa_t) dt < \infty.$$

- (2) We say  $(\Gamma \backslash G, m, \{a_t\})$  is *conservative* if there exists no wandering set  $B$  such that  $m(B) > 0$ .

Note that if  $m(\Gamma \backslash G) < \infty$ , then  $(\Gamma \backslash G, m, \{a_t\})$  is conservative by the Poincaré recurrence theorem.

For any BMS measure on  $\Gamma \backslash G$ , the following dichotomy is known:

**Lemma 3.2** [4] *Let  $\{a_t\} \subset A$  be a one-parameter subgroup containing an element of  $\text{int } A^+$ . Then  $(\Gamma \backslash G, m, \{a_t\})$  is conservative if and only if  $(\Gamma \backslash G, m, \{a_t\} \times M)$  is ergodic. In particular, if  $m(\Gamma \backslash G) < \infty$ , then it is AM-ergodic.*

**The limit set** Let  $C(\mathcal{F})$  be the set of all continuous functions on  $\mathcal{F}$ . The *limit set*  $\Lambda_\Gamma \subset \mathcal{F}$  is defined by

$$\Lambda_\Gamma = \{\xi \in \mathcal{F} : \gamma_i \cdot \nu \rightarrow \delta_\xi \text{ for some } \gamma_i \rightarrow \infty \text{ in } \Gamma\},$$

where  $\nu$  denotes the unique finite  $K$ -invariant measure on  $\mathcal{F}$  of total mass one,  $\delta_\xi$  denotes the Dirac mass at  $\xi$  and the convergence takes place in the weak-star topology on the dual of  $C(\mathcal{F})$ . Since  $\Gamma$  is Zariski dense,  $\Lambda_\Gamma$  is the unique  $\Gamma$ -minimal subset of  $\mathcal{F}$ , and is Zariski dense in  $\mathcal{F}$  [1, Lemma 3.6]. If  $\{a_t\}$  is a one-parameter subsemigroup containing an element of  $\text{int } A^+$ , then for  $g \in G$ , the following can be checked from the definition above:

$$(3.3) \quad \text{if } [g]a_{t_i} \in \Gamma \backslash G \text{ is bounded for some } t_i \rightarrow +\infty, \text{ then } gP^- \in \Lambda_\Gamma.$$

**The set  $\Omega_\Gamma$**  Let  $\mathcal{F}^{(2)}$  denote the  $G$ -orbit of  $(P^-, P^+) \in \mathcal{F} \times \mathcal{F}$ ; it is the unique open  $G$ -orbit in  $\mathcal{F} \times \mathcal{F}$ . We define

$$(3.4) \quad \Lambda_\Gamma^{(2)} = (\Lambda_\Gamma \times \Lambda_\Gamma) \cap \mathcal{F}^{(2)} \quad \text{and} \quad \tilde{\Omega}_\Gamma = \Lambda_\Gamma^{(2)} \times \mathfrak{a}.$$

The image of the embedding (2.4) is given by  $\mathcal{F}^{(2)} \times \mathfrak{a}$ , and hence we may consider  $\tilde{\Omega}_\Gamma$  as a subset of  $G/M$ . The set  $\Omega_\Gamma \subset \Gamma \backslash G/M$  in the introduction (see Theorem 1.3) can then be identified with  $\Gamma \backslash \tilde{\Omega}_\Gamma$ .

In the proof below, for  $w \in \mathcal{W}$  (2.3), we will use the expressions

$$(3.5) \quad waw^{-1} (a \in A), \quad wP^\pm,$$

where  $w$  should be understood as a group element in  $N_K(\mathfrak{a})$  representing it; note that they do not depend on the choice of the representatives. We have:

**Lemma 3.6** *If  $m(\Gamma \backslash G) < \infty$ , then  $m$  is supported on  $\Omega_\Gamma$ .*

**Proof** Let  $\{a_t\}$  be a fixed one-parameter subsemigroup containing an element of  $\text{int } A^+$  and  $\{wa_t w^{-1}\}$  be subsemigroups for each  $w \in \mathcal{W}$ . Since  $m$  is finite, it is conservative with respect to  $\{wa_t w^{-1}\}$ , and hence

$$X_w := \{[g] \in \Gamma \backslash G : [g](wa_{t_i} w^{-1}) \text{ is bounded for some } t_i \rightarrow +\infty\}$$

is  $m$ -conull. Setting  $X_0 = \cap \{X_w : w \in \mathcal{W}\}$ , (3.3) implies that  $g w P^- \in \Lambda_\Gamma$  for all  $[g] \in X_0$  and  $w \in \mathcal{W}$ . In particular,  $g P^\pm \in \Lambda_\Gamma$  for all  $[g] \in X_0$  and the lemma follows in view of the Hopf parametrization (2.4) and (3.4). □

**Leafwise measures of  $m$  along  $N^+$**  The leafwise measures  $m_x^{N^+}$  of the BMS measure  $m$  along  $N^+$  can be explicitly described from (2.5); their densities are given by

$$(3.7) \quad dm_x^{N^+}(n) = e^{\psi(\sigma(gn, P^-))} d\nu_\psi(gn P^-),$$

where  $g \in G$  is such that  $x = [g]$ . The expression (3.7) does not depend on the choice of the representative  $g$ , by the relation (2.2).

Let  $a \in A^+$ . Now a direct computation as in [7, Lemma 4.2] shows

$$m_x^{N^+}(a^{-k} B_1 a^k) = e^{k\psi(\log a)} m_{xa^k}^{N^+}(B_1).$$

Since  $\{xa^k : k \in \mathbb{N}\}$  is recurrent for  $m$ -a.e.  $x$  by Poincaré recurrence,

$$D_m(a, N^+)(x) = \lim_{k \rightarrow \infty} \frac{1}{k} \log m_x^{N^+}(a^{-k} B_1 a^k) = \psi(\log a),$$

for  $m$ -a.e.  $x$ . By Lemma 2.11, we have

$$(3.8) \quad h_m(a) = \psi(\log a).$$

**Remark 3.9** If  $m(\Gamma \backslash G) < \infty$ , then for all  $a \in \text{int } A^+$ ,

$$h_m(a) > 0.$$

**Proof** If  $m(\Gamma \backslash G) < \infty$ , then  $(\Gamma \backslash G, m)$  is conservative for the action of any one-parameter subgroup of  $A$ . This together with [1, Lemma 4.6] implies that the limit cone  $\mathcal{L}_\Gamma$  coincides with  $\mathfrak{a}^+$ . On the other hand, since  $\psi \in \mathfrak{a}^*$  is a linear form for which there exists a  $(\Gamma, \psi)$ -conformal measure on  $\mathcal{F}$ , we have  $\psi \geq \psi_\Gamma$  by Proposition 2.6(2). Since  $\psi_\Gamma$  is strictly positive in the interior of  $\mathcal{L}_\Gamma$  by Proposition 2.6(1), so is  $\psi$ , and hence the lemma follows by (3.8).  $\square$

**Remark 3.10** Remark 3.9 provides many elements of  $a \in A^+$  such that  $h_m(a) > 0$ . By a Pesin [23] type entropy formula obtained in [11], this information can be used to prove the existence of a root subgroup  $U < N^+$  for which  $m_x^U$  is nontrivial  $m$ -a.e. However, the proof of Theorem 1.6 does not rely on the calculation of entropy. Our proof requires the knowledge of nontriviality of  $m_x^U$  for all root subgroups  $U < N^+$  (Lemma 4.8) and entropy formula as in [23] will not be sufficient to draw this conclusion.

**BMS measures on the product  $G_1 \times G_2$**  Consider the case

$$G = G_1 \times G_2,$$

where  $G_1$  and  $G_2$  are semisimple real algebraic groups without compact factors. We will also make the identifications  $G_1 \simeq G_1 \times \{e\}$  and  $G_2 \simeq \{e\} \times G_2$ . Let  $\pi_i : G \rightarrow G_i$  denote the projection map, and set  $\Gamma_i := \pi_i(\Gamma)$  for  $i = 1, 2$  to ease the notation. Denote by  $\Lambda_{\Gamma_i}$  the limit set of  $\Gamma_i$  in  $\mathcal{F}_i := G_i/P_i^-$ , where  $P_i^\pm = G_i \cap P^\pm$  for  $i = 1, 2$ . Note that  $\Gamma_i$  is Zariski dense in  $G_i$ , and hence  $\Lambda_{\Gamma_i}$  is the unique  $\Gamma_i$ -minimal set in  $\mathcal{F}_i$  by [1, Lemma 3.6].

Let  $p_1 : G/P^- \rightarrow G_1/P_1^-$  be the projection map. For a  $\Gamma$ -conformal measure  $\nu$  supported on  $\Lambda_\Gamma$ , let

$$(3.11) \quad \nu = \int_{\Lambda_{\Gamma_1}} \nu_{\xi_1} d(p_{1*}\nu)(\xi_1),$$

where  $\nu_{\xi_1}$  denotes the conditional measure of  $\nu$  along the fiber  $p_1^{-1}(\xi_1)$ ; this is obtained from (2.7) by taking  $\mu = \nu$ , and  $\mathcal{A} = p_1^{-1}(\mathcal{B}_1)$  where  $\mathcal{B}_1$  denotes the Borel  $\sigma$ -algebra on  $G_1/P_1^-$ . A direct computation shows:

**Lemma 3.12** (1)  $p_{1*}\nu$  is a  $(\Gamma \cap G_1)$ -conformal measure whose support is contained in  $\Lambda_{\Gamma_1}$ .

(2) For  $p_{1*}\nu$ -a.e.  $\xi_1 \in \Lambda_{\Gamma_1}$ ,  $\nu_{\xi_1}$  is a  $(\Gamma \cap G_2)$ -conformal measure.

**Proof** Let  $\psi \in \mathfrak{a}^*$  be the linear form associated to  $\nu$  and  $\mathfrak{a}_i \subset \mathfrak{a}$  be the Lie algebra of  $A \cap G_i$  for  $i = 1, 2$ . We can write  $\sigma$  uniquely as a sum  $\sigma_1 + \sigma_2$  of  $\mathfrak{a}_i$ -valued cocycles  $\sigma_i : G_i \times G_i/P_i^- \rightarrow \mathfrak{a}_i$ . Note that  $\sigma(g_i, \cdot) = \sigma_i(g_i, p_i(\cdot))$  for all  $g_i \in G_i$ . Let  $\mathbf{1}_{G_i/P_i^-}$  denote the constant function on  $G_i/P_i^-$  which is identically 1.

For (1), let  $\gamma \in \Gamma \cap G_1$  be arbitrary and  $f \in C(G_1/P_1^-)$  be a continuous function. By (2.2),

$$\begin{aligned} \gamma.(p_{1*}v)(f) &= (p_{1*}v)(f(\gamma \cdot)) = v((f \otimes \mathbf{1}_{G_2/P_2^-})(\gamma \cdot)) = \int_{G/P^-} e^{-\psi(\sigma(\gamma^{-1}, \xi))} f(p_1(\xi)) dv(\xi) \\ &= \int_{G/P^-} e^{-\psi|_{\alpha_1}(\sigma_1(\gamma^{-1}, p_1(\xi)))} f(p_1(\xi)) dv(\xi) \\ &= \int_{G_1/P_1^-} e^{-\psi|_{\alpha_1}(\sigma_1(\gamma^{-1}, \xi_1))} f(\xi_1) d(p_{1*}v)(\xi_1). \end{aligned}$$

This implies that  $p_{1*}v$  is a  $(\Gamma \cap G_1, \psi|_{\alpha_1})$ -conformal measure. Recall that  $\Lambda_\Gamma$  (resp.  $\Lambda_{\Gamma_1}$ ) is the unique closed  $\Gamma$  (resp.  $\Gamma_1$ )-invariant subset of  $G/P^-$  (resp.  $G_1/P_1^-$ ). Since  $p_1$  is equivariant, it follows that  $p_1(\Lambda_\Gamma) = \Lambda_{\Gamma_1}$  and  $p_{1*}v$  is supported on  $\Lambda_{\Gamma_1}$ .

For (2), let  $\gamma \in \Gamma \cap G_2$  be arbitrary and  $f \in C(G_2/P_2^-)$  be a continuous function. Similarly as in (1), we have

$$\begin{aligned} \gamma.v(\mathbf{1}_{G_1/P_1^-} \otimes f) &= \int_{G/P^-} e^{-\psi|_{\alpha_2}(\sigma_2(\gamma^{-1}, p_2(\xi)))} f(p_2(\xi)) dv(\xi) \\ &= \int_{\Lambda_{\Gamma_1}} v_{\xi_1}(e^{-\psi|_{\alpha_2}(\sigma_2(\gamma^{-1}, \cdot))} f(\cdot)) d(p_{1*}v)(\xi_1), \end{aligned}$$

where the first equality is due to (2.2) and the second equality is due to (3.11). On the other hand, applying (3.11) in a different way gives

$$\gamma.v(\mathbf{1}_{G_1/P_1^-} \otimes f) = v(\mathbf{1}_{G_1/P_1^-} \otimes f(\gamma \cdot)) = \int_{\Lambda_{\Gamma_1}} \gamma.v_{\xi_1}(f) d(p_{1*}v)(\xi_1).$$

Since  $f$  was arbitrary, (2) follows from the uniqueness of the conditional measures (2.7) by comparing the above identities. □

Now suppose that  $m(\Gamma \backslash G) < \infty$ , and one of the  $\Gamma_i$ , say  $\Gamma_1$ , is discrete. Consider the projection  $\Gamma \backslash \tilde{\Omega}_\Gamma \rightarrow \Gamma_1 \backslash \tilde{\Omega}_{\Gamma_1}$  induced from the map  $\Gamma \backslash G \rightarrow \Gamma_1 \backslash G_1$ , which we will call  $\pi_1$  by abuse of notation. Similarly as in (3.11), let

$$m = \int_{\Gamma \backslash \tilde{\Omega}_{\Gamma_1}} m_{x_1} d(\pi_{1*}m)(x_1),$$

where  $m_{x_1}$  denotes the conditional measure of  $m$  along the fiber  $\pi_1^{-1}(x_1)$ . More precisely, denoting by  $\mathcal{B}$  the Borel sigma-algebra of  $\Gamma_1 \backslash G_1$  and  $\mathcal{A}$  the smallest countably generated sigma-algebra equivalent to  $\pi_1^{-1}(\mathcal{B})$  (cf [11, Definition 5.7, and Proposition 5.8]),  $m_{x_1}$  is obtained by taking  $\mu = m$  from (2.7). We have:

**Lemma 3.13** *Assume that  $m(\Gamma \backslash G) < \infty$  and  $\Gamma_1$  is discrete. Then  $\Gamma \cap G_2$  is Zariski dense in  $G_2$ , and  $m_{x_1}$  is isomorphic to a finite BMS measure on  $\Gamma \cap G_2 \backslash \tilde{\Omega}_{\Gamma_2}$  for  $\pi_{1*}m$ -a.e.  $x_1 \in \Gamma_1 \backslash \tilde{\Omega}_{\Gamma_1}$ .*

**Proof** Since  $\Gamma_1$  is discrete,  $\Gamma \cdot G_2 = \Gamma_1 \times G_2$  is a closed subset of  $G = G_1 \times G_2$ . Hence the map  $\Gamma \cap G_2 \backslash G_2 \rightarrow \Gamma \backslash G$  is a proper closed embedding. Recall that for  $\pi_{1*}m$ -a.e.  $x_1$ ,  $m_{x_1}$  is supported on an atom of  $\mathcal{A}$ . Because fibers of  $\pi_1$  are closed  $G_2$ -orbits,  $\mathcal{A} = \{\pi_1^{-1}(B) : B \in \mathcal{B}\}$  and atoms of  $\mathcal{A}$  are  $G_2$ -orbits. Hence each  $m_{x_1}$  can be identified with a measure on  $\Gamma \cap G_2 \backslash G_2$ .

Next, observe that  $\Gamma \cap G_2$  is a normal subgroup of  $\Gamma_2$ . Since  $\Gamma_2$  is Zariski dense in  $G_2$ , the Zariski closure of  $\Gamma \cap G_2$  is also a normal subgroup of  $G_2$ . Hence we may write  $G_2$  as an almost direct product  $G_2 = G' \cdot G''$  where both  $G'$  and  $G''$  are products of  $\mathbb{R}$ -simple factors of  $G_2$ ,  $\Gamma \cap G'$  is a Zariski dense subgroup of  $G'$  and  $\Gamma \cap G''$  is a finite group.

We claim that  $G'' = \{e\}$  and hence  $G' = G_2$ . Assume to the contrary that  $G''$  is nontrivial, in particular  $A \cap G''$  is an unbounded subgroup of  $A$ . Since  $m$  is a finite  $A$ -invariant measure on  $\Gamma \backslash G$ , by Poincaré recurrence, we can find  $\gamma_i \in \Gamma$ ,  $g \in G$  and  $a_i \rightarrow \infty$  in  $A \cap G''$  such that  $\gamma_i g a_i \in G$  is bounded. Let us write  $\gamma_i = \gamma_{i,1} \gamma_{i,2}$ , where  $\gamma_{i,1} \in G_1 \cdot G'$  and  $\gamma_{i,2} \in G''$ . Because  $a_i \in G''$ , passing to a subsequence,  $\gamma_{i,1} \in G_1 \cdot G'$  is constant. Hence we can assume that  $\gamma_{i,1} = \gamma_{0,1}$  for all  $i$ . Now note that  $\gamma_0^{-1} \gamma_i \in \Gamma \cap G''$  and  $(\gamma_0^{-1} \gamma_i) g a_i \in G$  is bounded. This is a contradiction, since  $\gamma_0^{-1} \gamma_i$  provides an unbounded (and hence infinite) sequence of elements in  $\Gamma \cap G''$ .

By the claim,  $\Gamma \cap G_2$  is a normal subgroup of  $\Gamma_2$  which is Zariski dense in  $G_2$ . It follows that  $\Lambda_{\Gamma \cap G_2} = \Lambda_{\Gamma_2}$ . Therefore  $\tilde{\Omega}_{\Gamma \cap G_2} = \tilde{\Omega}_{\Gamma_2}$ . Hence  $m_{x_1}$  is a BMS measure in this case by Lemma 3.12(2).  $\square$

In the proof of the lemma below, we will use the notation (3.5).

**Lemma 3.14** *If  $m(\Gamma \backslash G) < \infty$ , then  $\Lambda_\Gamma = \Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2}$ .*

**Proof** It suffices to show  $\Lambda_\Gamma = \Lambda_1 \times \Lambda_2$  for some closed nonempty  $\Gamma_i$ -invariant subset  $\Lambda_i \subset \mathcal{F}_i$  ( $i = 1, 2$ ), since then

$$\Lambda_\Gamma \subset \Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2} \subset \Lambda_1 \times \Lambda_2 = \Lambda_\Gamma,$$

where the second inclusion is due to the fact that  $\Lambda_{\Gamma_i}$  is the unique  $\Gamma_i$ -minimal subset of  $\mathcal{F}_i$  [1, Lemma 3.6] and  $\Lambda_i$  would clearly be a nonempty closed  $\Gamma_i$ -invariant set.

As in the proof of Lemma 3.6, for each  $w \in \mathcal{W}$ , set

$$X_w := \{[g] \in \Gamma \backslash G : [g](w a_t w^{-1}) \text{ is recurrent for some one-parameter subsemigroup } \{a_t\} \text{ containing an element of } \text{int } A^+\},$$

and  $X_0 = \cap \{X_w : w \in \mathcal{W}\}$ . Then  $X_w$  is an  $m$ -conull set by the conservativity of  $m$ , and

$$(3.15) \quad g w P^- \in \Lambda_\Gamma \quad \text{for all } [g] \in X_0 \text{ and } w \in \mathcal{W}.$$

Let  $\xi = (\xi_1, \xi_2)$  and  $\eta = (\eta_1, \eta_2)$  be the coordinates in  $\mathcal{F}_1 \times \mathcal{F}_2$  of arbitrary elements  $\xi, \eta \in \Lambda_\Gamma$ . We claim that  $(\xi_1, \eta_2)$  and  $(\eta_1, \xi_2)$  belong to  $\Lambda_\Gamma$ ; note that this implies  $\Lambda_\Gamma$  is a product set. Recalling the definition (3.4), since  $\Lambda_\Gamma^{(2)}$  is dense in  $\Lambda_\Gamma \times \Lambda_\Gamma$ , it suffices to check the claim for  $(\xi, \eta) \in \Lambda_\Gamma^{(2)}$ . Now [1, Lemma 3.6(iv)] shows that there exists  $g_j \in G$  such that  $[g_j] \in X_e \cap X_{w_0}$  and  $(g_j P^-, g_j P^+) \rightarrow (\xi, \eta)$  as  $j \rightarrow \infty$ , where  $w_0 \in \mathcal{W}$  denotes the longest element. Since  $X_0$  is conull, its closure contains  $\Omega_\Gamma$  and therefore contains  $X_e \cap X_{w_0}$  (Lemma 3.6). Hence after modifying  $g_j$ , we may assume in addition that  $[g_j] \in X_0$ . Let us write  $g_j = (g_{j,1}, g_{j,2}) \in G_1 \times G_2$ . To summarize:

$$(g_{j,i} P_i^-, g_{j,i} P_i^+) \rightarrow (\xi_i, \eta_i) \quad \text{as } j \rightarrow \infty \text{ for } i = 1, 2.$$

Now the claim follows by considering  $w \in \mathcal{W}$  such that

$$g_j w P^+ = (g_{j,1} P_1^-, g_{j,2} P_2^+) \quad \text{and} \quad g_j w P^- = (g_{j,1} P_1^+, g_{j,2} P_2^-),$$

in view of the property (3.15). □

**Lemma 3.16** *Suppose  $m(\Gamma \backslash G) < \infty$  and  $\Gamma_1$  is discrete. Then  $\Gamma_2$  is also discrete,  $\Gamma$  is commensurable with  $\Gamma_1 \times \Gamma_2$  and  $m$  is a finite extension of a product of BMS measures on  $\Gamma_1 \backslash G_1$  and  $\Gamma_2 \backslash G_2$ .*

**Proof** As  $m$  is finite and  $\Gamma_1$  is discrete, by Lemma 3.13,  $\Gamma \cap G_2$  is Zariski dense in  $G_2$  and  $\Gamma \cap G_2 \backslash \tilde{\Omega}_{\Gamma_2}$  admits a finite BMS measure. We claim that  $\Gamma_2$  is discrete. Observe that  $\Gamma_2$  normalizes  $\Gamma \cap G_2$ , and so does its closure  $\bar{\Gamma}_2$ , which is a closed Lie subgroup by Cartan’s theorem. Let  $N_0 \subset N_{G_2}(\Gamma \cap G_2)$  be the connected component of  $\bar{\Gamma}_2$ . Since  $\Gamma \cap G_2$  is discrete,  $N_0$  centralizes  $\Gamma \cap G_2$ . Hence  $N_0 = \{e\}$  and the claim follows. We now apply Lemma 3.13 with  $\Gamma_2$  in place of  $\Gamma_1$ . This in turn implies  $\Gamma \cap G_1 \backslash \tilde{\Omega}_{\Gamma_1}$  admits a finite BMS measure. Since  $\tilde{\Omega}_{\Gamma} = \tilde{\Omega}_{\Gamma_1} \times \tilde{\Omega}_{\Gamma_2}$  by Lemma 3.14, we have

$$\Gamma \backslash \tilde{\Omega}_{\Gamma} = (\Gamma \backslash \tilde{\Omega}_{\Gamma_1} \times \tilde{\Omega}_{\Gamma_2}) \simeq (\Gamma_1 \backslash \tilde{\Omega}_{\Gamma_1} \times \Gamma_2 \backslash \tilde{\Omega}_{\Gamma_2}) \times (\Gamma \backslash \Gamma_1 \times \Gamma_2),$$

a measurable isomorphism which accounts for the disintegration of  $m$  over the covering map

$$\Gamma \backslash \tilde{\Omega}_{\Gamma} \rightarrow (\Gamma_1 \backslash \tilde{\Omega}_{\Gamma_1} \times \Gamma_2 \backslash \tilde{\Omega}_{\Gamma_2}).$$

The finiteness assumption on  $m$  implies that  $\#(\Gamma \backslash \Gamma_1 \times \Gamma_2) < \infty$ , and hence the lemma. □

We will need the following well-known fact:

**Lemma 3.17** *Let  $\Delta < G$  be a Zariski dense subgroup. If  $G$  is simple, then  $\Delta$  is either discrete or dense in the analytic topology.*

**Proof** Let  $H := \bar{\Delta}$  be the closure of  $\Delta$  in the analytic topology; it is a closed Lie subgroup of  $G$  by Cartan’s theorem. Let  $\mathfrak{h}$  be the Lie algebra of  $H$ . Its normalizer  $N_G(\mathfrak{h})$  is a Zariski closed subgroup containing  $\Delta$ . Since  $\Delta$  is Zariski dense, it follows that  $N_G(\mathfrak{h}) = G$ . Hence either  $\mathfrak{h} = \{0\}$  or  $\mathfrak{h} = \mathfrak{g}$ , and the lemma follows. □

### 4 Proof of Theorem 1.6

Let  $G$  be a connected semisimple real algebraic group of the form  $G = G_1 \times \dots \times G_q$ , where each  $G_i$  for  $1 \leq i \leq q$  is simple and noncompact. Let  $\Gamma < G$  be a Zariski dense subgroup, and  $m$  be as in (3.1). Throughout the section, we will always assume that

$$m(\Gamma \backslash G) < \infty.$$

Let  $\mathfrak{g}$  denote the Lie algebra of  $G$  and  $\mathfrak{g}_i$  denote the Lie algebra of  $G_i$ . We recall the notation from Section 2. While doing so, we may assume that  $\Theta: \mathfrak{g} \rightarrow \mathfrak{g}$  is chosen to stabilize  $\mathfrak{g}_i$ , so that we have a Cartan decomposition  $\mathfrak{g}_i = (\mathfrak{k} \cap \mathfrak{g}_i) \oplus (\mathfrak{p} \cap \mathfrak{g}_i)$  that is compatible with  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Set

$$N_i^\pm = N^\pm \cap G_i, \quad P_i^\pm = P^\pm \cap G_i \quad \text{and} \quad \mathcal{F}_i = G_i / P_i^-.$$

Let  $\pi_i : G \rightarrow G_i$  be the projection,  $\Gamma_i := \pi_i(\Gamma)$  and  $\Lambda_{\Gamma_i}$  denote the limit set of  $\Gamma_i$  in  $\mathcal{F}_i$ .

**Lemma 4.1** We have  $\Lambda_{\Gamma} = \prod \Lambda_{\Gamma_i}$ .

**Proof** The lemma follows from a repeated application of Lemma 3.14. □

**Remark 4.2** A major step in the proof of Theorem 1.3 in [26] was to show that  $\Lambda_{\Gamma_i} = \mathcal{F}_i$  for  $\Gamma_i < G_i$  of rank  $\geq 2$  [26, Proposition 3.1].

**Coarse Lyapunov weights** For  $\alpha \in \mathfrak{a}^*$ , let

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} : \text{ad}_Y(X) = \alpha(Y)X \text{ for all } Y \in \mathfrak{a}\}$$

and  $\Sigma = \{\alpha \in \mathfrak{a}^* : \mathfrak{g}_{\alpha} \neq \{0\}\}$ . A *coarse Lyapunov weight* is an equivalence class in  $\Sigma$  where two elements in  $\Sigma$  are equivalent when one of them is a positive multiple of the other. Denote by  $[\alpha]$  the coarse Lyapunov weight containing  $\alpha \in \Sigma$ , and let

$$\mathfrak{g}_{[\alpha]} = \bigoplus_{\beta \in [\alpha]} \mathfrak{g}_{\beta}$$

be the corresponding Lie subalgebra. For  $\alpha \in \Sigma - \{0\}$ , set

$$U_{[\alpha]} = \exp(\mathfrak{g}_{[\alpha]}).$$

Given a finite  $A$ -invariant ergodic measure  $\mu$  on  $\Gamma \backslash G$ , let  $\tilde{U}_{[\alpha]}$  denote the smallest Zariski closed  $A$ -normalized subgroups containing the support of  $\mu_x^{U_{[\alpha]}}$  for  $\mu$ -a.e.  $x$ .

**Lemma 4.3** [11, Theorem 9.14] *Let  $\mu$  be a finite  $A$ -invariant ergodic measure on  $\Gamma \backslash G$ , and  $[\alpha]$  and  $[\beta]$  be coarse Lyapunov weights such that  $[\alpha] \neq [\beta] \neq [-\alpha]$ . Then  $\mu$  is invariant under the group generated by the commutator  $[\tilde{U}_{[\alpha]}, \tilde{U}_{[\beta]}]$ .*

Let  $\{[\alpha_1], \dots, [\alpha_l]\}$  be the set of all coarse Lyapunov weights such that  $U_{[\alpha_i]} \subset N^+$ . We may rearrange them so that for each  $i$ ,  $[\alpha_i]$  does not meet the convex cones generated by  $\{[\alpha_j] : j = i + 1, \dots, l\}$ . Let

$$\iota : U_{[\alpha_1]} \times \dots \times U_{[\alpha_l]} \rightarrow N^+$$

be the product map. Let  $\mathfrak{n}$  be the Lie algebra of  $N^+$ .

**Lemma 4.4** *The following polynomial function has a polynomial inverse:*

$$\mathfrak{g}_{[\alpha_1]} \times \dots \times \mathfrak{g}_{[\alpha_l]} \rightarrow \mathfrak{n}, \quad (X_1, \dots, X_l) \mapsto \log(e^{X_1} \dots e^{X_l}).$$

**Proof** By the Baker–Campbell–Hausdorff formula there exist constants  $c_{i_1, \dots, i_N}$  such that  $\log(e^{X_1} \dots e^{X_l})$  can be formally written as

$$(4.5) \quad X_1 + \dots + X_l + \sum_{N=1}^{\infty} \sum_{1 \leq i_1, \dots, i_N \leq l} c_{i_1, \dots, i_N} [X_{i_1}, [X_{i_2}, \dots, [X_{i_{N-1}}, X_{i_N}]]],$$

where for each multi-index  $(i_1, \dots, i_N)$ , at least two of the indices are required to be distinct. Since  $\mathfrak{n}$  is nilpotent, (4.5) has only finitely many terms and  $\Phi(x_1 + \dots + x_l) := \log(e^{x_1} \dots e^{x_l})$  is a polynomial map from  $\mathfrak{n}$  to  $\mathfrak{n}$ .

Let us extend  $\Phi$  to the complexification  $\mathfrak{n}^{\mathbb{C}} := \mathfrak{n} \otimes \mathbb{C}$ . The lemma will follow if we manage to show that the complexified map has a polynomial inverse. The complexified map is given by the same formula as  $\Phi$ , so it still satisfies the following equivariance property:

$$(4.6) \quad \text{Ad}(a)\Phi(x) = \Phi(\text{Ad}(a)x) \quad \text{for } a \in A \text{ and } x \in \mathfrak{n}^{\mathbb{C}}.$$

Since the derivative of  $\Phi$  at zero is the identity, we can find an open neighborhood  $U \subset \mathfrak{n}^{\mathbb{C}}$  of zero such that  $\Phi$  restricted to  $U$  is biholomorphic onto the image. On the other hand, for any bounded open set  $V \subset \mathfrak{n}^{\mathbb{C}}$ , there exists an  $a \in A$  such that  $\text{Ad}(a)V \subset W$ , so (4.6) implies that  $\Phi$  is biholomorphic on  $V$ . Taking exhaustive sequence of  $V$ 's we deduce that  $\Phi$  is biholomorphic on  $\mathfrak{n}^{\mathbb{C}}$  and, in particular,  $\Phi^{-1}$  is an entire function on  $\mathfrak{n}^{\mathbb{C}}$ . We will now prove it is of polynomial growth.

Let  $x = x_1 + \dots + x_l$ , with  $x_i \in \mathfrak{g}_{[\alpha_i]}^{\mathbb{C}}$ . We fix an auxiliary norm  $\|\cdot\|$  on  $\mathfrak{n}^{\mathbb{C}}$  defined by  $\|x\| = \max_{i=1, \dots, l} \|x_i\|_i$ , where  $\|\cdot\|_i$  is any chosen norm on  $\mathfrak{g}_{[\alpha_i]}^{\mathbb{C}}$ . Choose  $H_0$  in the interior of  $\mathfrak{a}^+$  and put  $a(t) = e^{tH_0} \in A$ . Choosing a representative such that  $[\alpha_i] \subset \{\alpha_i, 2\alpha_i\}$ , we can choose  $0 < t \leq c_1 \log \|x\|$ , where  $c_1 := \max_{i=1, \dots, l} (2\alpha_i(H_0))^{-1}$ , such that  $\|\text{Ad}(a(-t))(x)\| \leq 1$ . Note that

$$\Phi^{-1}(x) = \text{Ad}(a(t))(\Phi^{-1}(\text{Ad}(a(-t))(x))).$$

Putting  $C = \sup_{\|\Phi(z)\| \leq 1} \|z\|$  and  $c_2 = \max_{i=1, \dots, l} 2\alpha_i(H_0)$ , we get

$$\|\Phi^{-1}(x)\| \leq C e^{c_1 t} \leq C \|x\|^{c_1 c_2}.$$

This establishes the polynomial growth bound for  $\Phi^{-1}$ . Any entire holomorphic function of polynomial growth is a polynomial. Indeed, using the multivariate Cauchy formula we can now prove that sufficiently high derivatives of  $\Phi^{-1}$  vanish, so the Taylor series can have only finitely many nonzero terms.  $\square$

The following is then obtained by inductive applications of Proposition 2.13:

**Lemma 4.7** [11, Theorem 9.8] *If  $m(\Gamma \backslash G) < \infty$ , then for  $m$ -a.e.,*

$$m_x^{N^+} \propto \iota(m_x^{U_{[\alpha_1]}} \times \dots \times m_x^{U_{[\alpha_l]}}).$$

Using Lemma 4.7, one can show that all the leafwise measures are nontrivial. This is one of the most important ingredients of the proof.

**Lemma 4.8** *For any coarse Lyapunov weight  $[\alpha]$ , we have  $\tilde{U}_{\pm[\alpha]} = U_{\pm[\alpha]}$ . In particular,  $m_x^{U_{\pm[\alpha]}}$  are nontrivial for  $m$ -a.e.  $x$ .*

**Proof** By contradiction, suppose  $\tilde{U}_{[\alpha]}$  is a proper subgroup of  $U_{[\alpha]}$ . Without loss of generality,  $U_{[\alpha]} \subset N^+$ . Labeling the coarse Lyapunov weights as in Lemma 4.7, we have  $[\alpha] = [\alpha_i]$  for some  $i$  and

$$(4.9) \quad m_x^{N^+} \propto \iota(m_x^{U_{[\alpha_1]}} \times \dots \times m_x^{U_{[\alpha_l]}}),$$

for  $m$ -a.e.  $x$ . Fix such  $x \in \Gamma \backslash G$  and let  $g \in G$  be such that  $x = [g]$ . Since  $m_x^{U_{[\alpha]}}$  is supported on  $\tilde{U}_{[\alpha]}$ , by (4.9), the topological support of  $m_x^{N^+}$  is contained in

$$X := \iota(U_{[\alpha_1]} \times \cdots \times \tilde{U}_{[\alpha_i]} \times \cdots \times U_{[\alpha_l]}).$$

Since  $N^+$  and the  $U_{[\alpha]}$  are nilpotent groups, they are isomorphic to their Lie algebras as algebraic varieties, via the exponential map. By Lemma 4.4, it follows that  $X$  is a proper Zariski closed subset of  $N^+$ . It follows from (3.7) that  $g^{-1}\Lambda_\Gamma$  is contained in the image of  $X \subset N^+$ , under the open embedding  $N^+ \rightarrow G/P^-$ . This contradicts the Zariski density of  $\Lambda_\Gamma$  [1, Lemma 3.6].  $\square$

We also record the following two standard lemmas:

**Lemma 4.10** *Let  $\Phi$  be a simple root system (ie the root system of a complex simple Lie group) in a Euclidean space  $V$ . Suppose  $W \subset V$  is a subspace of codimension at least two. Then*

$$\Phi - W \subset \{\alpha + \beta \in \Phi : \alpha, \beta \in \Phi - W, (\mathbb{R}\alpha + \mathbb{R}\beta) \cap W = \{0\}\}.$$

**Proof** Let  $\kappa \in \Phi - W$ . By [13, Lemma 3.7], the set  $\{\eta \in \Phi : \langle \eta, \kappa \rangle \neq 0\}$  spans  $V$ , so we can find an  $\eta$  such that  $\langle \eta, \kappa \rangle \neq 0$ , and  $\kappa + W$  and  $\eta + W$  are linearly independent in  $V/W$ . The subspace  $E := \mathbb{R}\kappa + \mathbb{R}\eta$  intersects  $W$  transversally:  $E \cap W = \{0\}$ . Consider the intersection  $\Phi \cap E$ . It is a rank two root system containing a pair of nonorthogonal linearly independent roots  $\kappa$  and  $\eta$ . It follows that  $\Phi \cap E$  is of type  $A_2, C_2, D_2$  or  $G_2$ . In each of these root systems one can check by hand that  $\kappa = \pm\alpha + \pm\beta$  for some  $\alpha, \beta \in \Phi \cap E$ . Condition  $E \cap W = \{0\}$  implies that  $(\mathbb{R}\alpha + \mathbb{R}\beta) \cap W = \{0\}$ .  $\square$

**Lemma 4.11** *Assume that  $G$  is a simple Lie group of rank  $\geq 2$ . Then  $G$  is generated by the subgroups  $\{[U_{[\alpha]}, U_{[\beta]}] : \alpha, \beta \in \Sigma, [\alpha] \neq [\beta] \neq -[\alpha]\}$  and  $AM$ .*

**Proof** Let  $\mathfrak{g}'$  be the subalgebra generated by  $\{[\mathfrak{g}_{[\alpha]}, \mathfrak{g}_{[\beta]}] : \alpha, \beta \in \Sigma, \beta \neq -\alpha\}$ ,  $\mathfrak{a}$  and  $\mathfrak{m}$ , where  $\mathfrak{m}$  denotes the Lie algebra of  $M$ . It suffices to verify that  $\mathfrak{g}' = \mathfrak{g}$ , or equivalently that their complexifications satisfy  $(\mathfrak{g}')^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$ .

Let  $\mathfrak{h} = \mathfrak{a} + i\mathfrak{b} \subset \mathfrak{g}^{\mathbb{C}}$  be a Cartan subalgebra containing  $\mathfrak{a}$ . Let  $\Phi$  be the root system of  $\mathfrak{g}^{\mathbb{C}}$  relative to  $\mathfrak{h}$ . To distinguish them from the roots in  $\Sigma$ , we will use  $\alpha', \beta'$ , etc for roots in  $\Phi$  and  $\alpha, \beta$ , etc for roots in  $\Sigma$ . For  $\alpha' \in \Phi$ , write

$$\mathfrak{g}_{\alpha'}^{\mathbb{C}} = \{X \in \mathfrak{g}^{\mathbb{C}} : \text{ad}_Y(X) = \alpha'(Y)X \text{ for all } Y \in \mathfrak{h}\}.$$

Let  $V := \text{Hom}(\mathfrak{h}, \mathbb{R})$  and let  $W := \{\xi \in V : \xi(\mathfrak{a}) = 0\}$ . Note that the higher rank assumption means that  $W$  is of codimension at least two. By Lemma 4.10, any root  $\kappa' \in \Phi$  which restricts to a nontrivial root of  $\mathfrak{a}$  (ie  $\kappa' \notin W$ ) can be written as  $\kappa' = \alpha' + \beta'$ , with  $(\mathbb{R}\alpha' + \mathbb{R}\beta') \cap W = \{0\}$ . Writing  $\alpha$  and  $\beta$  for the restrictions of  $\alpha'$  and  $\beta'$  to  $\mathfrak{a}$ , this in particular implies that  $[\alpha] \neq [\beta] \neq -[\alpha]$ . We have

$$\mathfrak{g}_{\kappa'}^{\mathbb{C}} = [\mathfrak{g}_{\alpha'}^{\mathbb{C}}, \mathfrak{g}_{\beta'}^{\mathbb{C}}] \subset [\mathfrak{g}_{[\alpha]}, \mathfrak{g}_{[\beta]}]^{\mathbb{C}} \subset (\mathfrak{g}')^{\mathbb{C}}.$$

By [13, Lemma 3.7],  $\mathfrak{g}^{\mathbb{C}}$  is generated by the root subspaces  $\mathfrak{g}_{\kappa'}^{\mathbb{C}}$  for  $\kappa' \in \Phi - W$ , so we can deduce that  $(\mathfrak{g}')^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$ .  $\square$

**Proposition 4.12** Assume that  $m(\Gamma \backslash G) < \infty$ . Then for each simple factor  $G_i$  of rank  $\geq 2$ ,  $m$  is  $G_i$ -invariant.

**Proof** By Lemmas 4.3 and 4.8,  $m$  is invariant under the subgroups  $\{[U_{[\alpha]}, U_{[\beta]}] : \alpha, \beta \in \Sigma, \beta \neq -\alpha\}$ . Since  $G_i$  is simple of rank  $\geq 2$ , Lemma 4.11 gives the conclusion.  $\square$

At this point we already completed the proof for the simple higher rank case. To deal with the products of rank one groups, we will once again take advantage of the product structure of the leafwise measures (Proposition 2.13) in the proof of the following:

**Proposition 4.13** Assume that  $m(\Gamma \backslash G) < \infty$  and  $\pi_i(\Gamma)$  is dense for each  $i$ . Then  $m$  is Haar.

**Proof** Let  $\psi$  and  $\psi \circ i$  be the linear forms on  $\mathfrak{a}$  implicitly given in (3.1). Identifying  $\mathfrak{a}_i := \mathfrak{a} \cap \mathfrak{g}_i$  with its embedded image in  $\mathfrak{g}$ ,  $\psi$  can be written uniquely as a sum of linear forms  $\psi_i$  on  $\mathfrak{a}_i$ . For any  $g \in G$ , let  $g_i \in G_i$  be its projection to  $G_i$ . Denoting by  $\sigma_i : G_i \times \mathcal{F}_i \rightarrow \mathfrak{a}_i$  the Iwasawa cocycles for  $G_i$  as in (2.1), we have

$$\sigma(g, \xi) = \sum_{i=1}^q \sigma_i(g_i, \xi_i)$$

for all  $\xi = (\xi_1, \dots, \xi_q) \in \mathcal{F}$ . Let  $\iota : N_1^+ \times \dots \times N_q^+ \rightarrow N^+$  denote the product map. Since  $m$  is  $A$ -invariant, by a repeated application of Proposition 2.13,

$$(4.14) \quad m_x^{N^+} \propto \iota(m_x^{N_1^+} \times \dots \times m_x^{N_q^+})$$

for  $m$ -a.e.  $x$ . Fix such  $x \in \Gamma \backslash G$  and let  $g = (g_1, \dots, g_q) \in G$  be such that  $x = [g]$ . We then define measures  $\nu_i$  on  $\mathcal{F}_i$  for  $1 \leq i \leq q$  by

$$(4.15) \quad d\nu_i(g_i n_i P_i^-) := e^{\psi_i(\sigma_i(g_i n_i, P_i^-))} d m_x^{N_i^+}(n_i).$$

By choosing the normalization of  $m_x^{N_i^+}$  as in (3.7), it follows from (4.14) and (4.15) that

$$\nu_\psi \propto \nu_1 \times \dots \times \nu_q.$$

In particular,  $\nu_i$  are finite measures on  $\mathcal{F}_i$ , and for all  $\gamma \in \Gamma$ ,

$$\gamma_1 \cdot \nu_1 \times \dots \times \gamma_q \cdot \nu_q \propto \gamma \cdot \nu_\psi = e^{-\psi_1(\sigma_1(\gamma_1^{-1}, \cdot))} \nu_1 \times \dots \times e^{-\psi_q(\sigma_q(\gamma_q^{-1}, \cdot))} \nu_q$$

by (2.2). It follows that there exists a multiplicative character  $\chi_i : \Gamma \rightarrow \mathbb{R}^\times$  for  $1 \leq i \leq q$  such that for all  $\gamma \in \Gamma$  and  $\xi_i \in \mathcal{F}_i$ ,

$$\frac{d(\gamma_i \cdot \nu_i)}{d\nu_i}(\xi_i) = \chi_i(\gamma) e^{-\psi_i(\sigma_i(\gamma_i^{-1}, \xi_i))}.$$

Set  $\Gamma_0 := [\Gamma, \Gamma]$  and note that each  $\chi_i$  is trivial on  $\pi_i(\Gamma_0)$ . In particular,  $\nu_i$  is  $(\pi_i(\Gamma_0), \psi_i)$ -conformal. Since  $\pi_i(\Gamma_0)$  is dense in  $G_i$ , it follows that  $\nu_i$  is  $(G_i, \psi_i)$ -conformal. This implies that  $\nu_i$  is the unique  $G_i \cap K$ -invariant measure on  $\mathcal{F}_i$  up to normalization, and  $m_x^{N_i^+}$  is Haar  $m$ -a.e. The same argument shows that  $m_x^{N_i^-}$  is Haar  $m$ -a.e. Since  $N_i^\pm$  and  $AM$  generate  $G$ , by Lemma 2.10, it follows that  $m$  is Haar.  $\square$

**Proof of Theorem 1.6** The validity of the theorem depends only on the commensurability class of  $\Gamma$ . Hence, by passing to a covering and a quotient of  $G$ , we may assume without loss of generality that  $G$  is a direct product of its simple factors, none of which are compact. This allows us to freely use the lemmas established in this section.

Let  $\pi_i : G \rightarrow G_i$  for  $i = 1, \dots, q$  denote the projection of  $G$  to each factor. Since  $\pi_i(\Gamma)$  is Zariski dense in  $G_i$ , it is either discrete or dense by Lemma 3.17. Let  $I$  be the set of indices for which  $\pi_i(\Gamma)$  is dense, and  $J$  be the complement of  $I$ . Let  $G_I$  (resp.  $G_J$ ) be the product of the  $G_i$  with  $i \in I$  (resp.  $i \in J$ ). Set

$$\Gamma_J := \prod_{j \in J} \pi_j(\Gamma) < G_J,$$

and  $\Gamma_I := \pi_I(\Gamma)$  where  $\pi_I : G \rightarrow G_I$  is the projection. Applying Lemma 3.16 to the product  $G = G_I \cdot G_J$ , it follows that  $\Gamma_I$  is discrete,  $\Gamma$  is commensurable with  $\Gamma_I \cdot \Gamma_J$  and  $m$  is a finite extension of a product of finite BMS measures on  $\Gamma_I \backslash G_I$  and  $\Gamma_J \backslash G_J$ , say  $m_I$  and  $m_J$ . Applying Lemma 3.16 repeatedly to each  $\pi_j(\Gamma)$  for  $j \in J$  shows that  $m_J$  is a product of finite BMS measures on  $\pi_j(\Gamma) \backslash \tilde{\Omega}_{\pi_j(\Gamma)}$ .

By Proposition 4.13,  $m_I$  is Haar and  $\Gamma_I$  is a lattice in  $G_I$ . By Proposition 4.12,  $\Gamma_j$  is a lattice in  $G_j$  for all  $j \in J$  such that  $G_j$  is of rank  $\geq 2$ , and the remaining indices of  $J$  correspond to discrete subgroups of rank one groups with finite BMS measures.  $\square$

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