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Hölder continuity of tangent cones in RCD(K, N) spaces and applications to nonbranching

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We study the structure theory of metric measure spaces (X, d, m) satisfying the synthetic lower Ricci curvature bound condition $\mathrm{RCD}(K, N)$. We prove that such a space is nonbranching and that tangent cones from the same sequence of rescalings are Hölder continuous along the interior of every geodesic in X. More precisely, we show that the geometry of balls of small radius centered in the interior of any geodesic changes in at most a Hölder continuous way along the geodesic in pointed Gromov–Hausdorff distance. This improves a result in the Ricci limit setting by Colding and Naber where the existence of at least one geodesic with such properties between any two points is shown. As in the Ricci limit case, this implies that the regular set of an $\mathrm{RCD}(K, N)$ space has m-a.e. constant dimension, a result recently established by Brué and Semola, and is m-a.e convex. It also implies that the top dimension regular set is weakly convex, and therefore connected. In proving the main theorems, we develop in the $\mathrm{RCD}(K, N)$ setting the expected second-order interpolation formula for the distance function along the regular Lagrangian flow of some vector field using its covariant derivative.

51K10, 53B20

1 Introduction

We prove $\mathrm{RCD}(K, N)$ spaces are nonbranching and generalize to the $\mathrm{RCD}(K, N)$ setting an improved version of the main result of Colding and Naber [30]. We begin by stating two reformulations of this result, which are our main technical results:

Theorem 1.1 (Hölder continuity of geometry of small balls with the same radius) *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$ and $N\in(1,\infty)$. Let $p,q\in X$ and $d(p,q)=\ell$. Define $K'=\big(K/(-(N-1))\vee 1\big)^{1/2}$ and $\ell'=\ell\wedge 1$. For any unit-speed geodesic $\gamma\colon[0,\ell]\to X$ between p and q, there exist constants $C(N)$, $\alpha(N)$ and $r_0(N)>0$ such that for any $\delta>0$ with $0<r<r_0(\delta\ell'/K')$ and $\delta\ell<s<t<\ell-\delta\ell$,*

$$d_{pGH}\big(\big(B_r(\gamma(s)),\gamma(s)\big),\big(B_r(\gamma(t)),\gamma(t)\big)\big)<\frac{CK'}{\delta\ell'}r|s-t|^{\alpha}. \tag{1}$$

In order to pass the result to tangents, we use the following terminology: Let $x_1,x_2\in X$, $(Y,d_Y,m_Y,y)\in\mathrm{Tan}(X,d,m,x_1)$ and $(Z,d_Z,m_Z,z)\in\mathrm{Tan}(X,d,m,x_2)$. We say Y and Z come from the *same sequence of rescalings* if there exists $s_j\downarrow 0$ such that

$$(X,s_j^{-1}d,m_{s_j}^{x_1},x_1)\xrightarrow{pmGH}(Y,d_Y,m_Y,y)\quad\text{and}\quad(X,s_j^{-1}d,m_{s_j}^{x_2},x_2)\xrightarrow{pmGH}(Z,d_Z,m_Z,z). \tag{2}$$

The following estimate on tangents from the same sequence of rescalings follows from Theorem 1.1:

Theorem 1.2 (Hölder continuity of tangent cones) *In the notation of Theorem 1.1, for any unit-speed geodesic $\gamma\colon[0,\ell]\to X$ between p and q, there exist constants $C(N),\alpha(N)>0$ such that if $(Y_s,d_{Y_s},m_{Y_s},y_s)\in\mathrm{Tan}(X,d,m,\gamma(s))$ and $(Y_t,d_{Y_t},m_{Y_t},y_t)\in\mathrm{Tan}(X,d,m,\gamma(t))$ come from the same sequence of rescalings, then*

$$d_{pGH}\big((B_r(y_s),y_s),(B_r(y_t),y_t)\big)<\frac{CK'}{\delta\ell'}r|s-t|^{\alpha} \tag{3}$$

for all $r>0$.

To prove these we first construct at least one geodesic between any two points satisfying the conclusion of Theorem 1.1, which is the main result of [30]. We then use this construction to prove $\mathrm{RCD}(K,N)$ spaces, and so, in particular, Ricci limits, are nonbranching, in Section 6.1.

Theorem 1.3 *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$ and $N\in(1,\infty)$. Then (X,d,m) is nonbranching.*

This has been a natural open problem for Ricci limits since the seminal work of Cheeger and Colding in [26; 27; 28; 29]. Potential branching coming from some simple examples was ruled out in [28, Section 5] and some partial results were obtained in [30] for noncollapsed limits of manifolds with uniform two-sided Ricci curvature bounds. We point out nonbranching would follow from the results of [30] and the proof of Theorem 1.3 from Section 6.1 if one were able to, for example, prove all geodesics in any Ricci limit are limit geodesics. Our intrinsic construction of a geodesic in Section 5.2 which satisfies the conclusion of Theorem 1.1 offers a little more freedom than the extrinsic construction of limit geodesics in [30] and this was enough to prove Theorem 1.3. Theorem 1.3 is then itself used to pass from the existence of a geodesic between any two points which satisfies the conclusion of Theorem 1.1 to Theorem 1.1 in full generality for all geodesics. We mention that, in the reverse direction, it follows easily from nonbranching that every geodesic in a Ricci limit space is a limit geodesic.

In the case of Ricci limits, the Hölder continuity of tangent cones had several key applications; see [30, Section 1.4] for a definition of the renormalized limit measure.

Theorem 1.4 [30, Theorems 1.18, 1.20 and 1.21] *Let (X, d, p) be the Ricci limit of $(M_i^n, g_i, p_i)_{i\in\mathbb{N}}$ and m be the renormalized limit measure. Then*:

(I) *There is a unique $k \in \mathbb{N}$ with $0 \le k \le n$ such that $m(X\backslash\mathcal{R}_k) = 0$, where $\mathcal{R}_k$ is the k-dimensional regular set.*

(II) *$\mathcal{R}_k$ from* (I) *is m-a.e. convex and weakly convex. In particular, $\mathcal{R}_k$ is connected.*

(III) *The isometry group of X is a Lie group.*

Statements (I) and (III) have since been proved by other means in the case of $\mathrm{RCD}(K, N)$ spaces; see Brué and Semola [15] for (I) and Guijarro and Santos-Rodríguez [49] and Sosa [69] for (III). We prove (II) in Section 6.2 following [30]. Since the proofs of (I) and (II) are intricately related, we will prove (I) as well.

Outline of paper and proof We begin this subsection by introducing the strategy of the proof by Colding and Naber [30], which we will largely follow. We then discuss the issues that arise when extending this to the metric measure setting and give an outline of their solutions.

The existence of at least one geodesic satisfying the conclusion of Theorem 1.1 was shown in [30]. The proof there was extrinsic and obtained by proving the theorem for manifolds. As such, consider a Riemannian manifold (M^n, g) with $\mathrm{Ric}_M \ge -(n-1)$ and a unit-speed-minimizing geodesic $\gamma : [0, 1] \to M$ from some p to q. Fix some $\delta > 0$ and $\delta < s_0 < s_1 < 1-\delta$. The desired Gromov–Hausdorff approximations in the proof of Theorem 1.1 for γ are constructed on a large subset of the ball $B_r(\gamma(s_1))$ from the gradient flow Ψ of $-d_p$ (more on the definition of this later). This is not altogether surprising since in the interior of γ, d is a smooth function and the Laplacian of d has a two-sided bound. A simple argument (originating from Calabi [19]) applying the Bochner formula to d gives

$$\int_{\delta}^{1-\delta} |\mathrm{Hess}\, d_p|^2(\gamma(t))\, dt \le \frac{c(n)}{\delta}. \tag{4}$$

The fact that $\mathrm{Hess}\, d_p = \frac{1}{2}\mathscr{L}_{\nabla d_p} g$ shows $D(\Psi_{s_1-s_0})\colon T_{\gamma(s_1)}M \to T_{\gamma(s_0)}M$ satisfies the estimates of Theorem 1.2. Of course, Theorem 1.2 is completely trivial in the case of Riemannian manifolds, but the point is this map comes from a construction on the manifold itself. Smoothness then allows one to use $\Psi_{s_1-s_0}$ to construct Gromov–Hausdorff approximations from $B_r(\gamma(s_1))$ to $B_r(\gamma(s_0))$, where r needs to be sufficiently small depending on γ and δ instead of just n and δ. This property does not pass through Ricci limits and so the challenge, then, is to remove the dependence on γ.

One might consider using $\operatorname{Hess} d_p$ to control the geometry of balls of uniform (ie independent of γ) radius under Ψ. However, we do not have estimates on $\operatorname{Hess} d_p$ for such balls along γ; we do not even have smoothness of d_p. We mention that due to this lack of smoothness, Ψ is not globally defined. The integral curves of Ψ starting at any $x \in M$ should be thought of simply as a (choice of) unit-speed geodesic from x to p. In this way Ψ is defined locally away from p for a definite amount of time.

Nevertheless, it turns out that one can still control the geometry under Ψ by utilizing the next formula, which follows from the standard second-order interpolation formula using the Hessian:

$$\left|\frac{d}{dt} d(\sigma_1(t), \sigma_2(t))\right| \leq |\nabla h - \sigma_1'|(\sigma_1(t)) + |\nabla h - \sigma_2'|(\sigma_2(t)) + \inf \int_{\gamma_{\sigma_1(t),\sigma_2(t)}} |\operatorname{Hess} h|. \tag{5}$$

This holds for a.e. t, where σ_1 and σ_2 are unit-speed geodesics in M, $h\colon M \to \mathbb{R}$ is a smooth function and inf is taken across all minimizing geodesics connecting $\sigma_1(t)$ and $\sigma_2(t)$. This means, as long as we can closely approximate d_p by a smooth function h where we have reasonable control on $|\nabla h - \nabla d_p|$ along two geodesics and on $\operatorname{Hess} h$, we can control the distance between those two geodesics.

We mention a similar strategy that was used to prove the almost splitting theorem of Cheeger and Colding [26]. In that setting, a single ball $B_r(\gamma(s))$ of small radius for a fixed $s \in (\delta, 1-\delta)$ is considered. It turns out that the correct approximation to take for d_p is the harmonic replacement b of d_p on $B_{2r}(\gamma(s))$. One is able to obtain the better than scale invariant estimate

$$\fint_{B_r(\gamma(s))} |\operatorname{Hess} b|^2 \, dV \leq c(n,\delta) r^{-2+\alpha(n)} \tag{6}$$

using the Bochner formula, which is enough to prove almost splitting. The almost splitting theorem has since been proved through other means for RCD spaces by Gigli [37]; see also Mondino and Naber [61].

For our purposes, (6) and the resulting almost splitting theorem is not good enough because it only allows one to compare two balls of radius r that are distance r away from each other. As discussed in detail in [30, Section 2], this estimate blows up as $r \to 0$ if one iterates along γ in r-length intervals. The crucial idea in [30] was then to use the heat-flow approximation h to (some cutoff of) d_p instead. For such an approximation, they were able to obtain the estimate

$$\int_\delta^{1-\delta} \fint_{B_r(\gamma(t))} |\operatorname{Hess} h|^2 \, dV \, dt \leq c(n,\delta), \tag{7}$$

where h is the heat flow taken to some time on the scale of r^2; see Theorem 4.12(IV). Moreover, $\int_\delta^{1-\delta} |\nabla d_p - \nabla h|$ can also be bounded to the correct order (see Lemma 4.13) for most geodesics. These estimates can then be used along with the segment inequality of Cheeger and Colding [26, Theorem 2.11] and (5) to control the total integral change in distances between elements of two sets of large measure in $B_r(\gamma(s_1))$ under the flow Ψ. This is ultimately good enough to construct a Gromov–Hausdorff approximation using $\Psi_{s_1-s_0}$. We mention that since we are using segment inequality and integral bounds, the smaller the relative measure of the sets compared to the region where we have Hessian estimates, the worse the control we have on total distance change for those sets under Ψ.

A crucial detail in this is that in order to make use of estimate (7), it is important that most of $B_r(\gamma(t))$ stays close, on the scale of r, to γ under the gradient flow for an amount of time independent of r and γ. Since the control one has over distance is for sets of large relative measure and γ is trivial in measure, one cannot guarantee using the argument outlined in the previous paragraph that most of $B_r(\gamma(t))$ does not simply drift away from γ quickly. Colding and Naber [30] overcome this by using (4). As mentioned previously, by smoothness, (4) implies balls of sufficiently small radius depending on γ stay close to γ under Ψ for some fixed amount of time depending only on n and δ. Induction with geometrically increasing radii along with the argument from the previous paragraph can then be used to guarantee large proportions of balls up to some radius independent of γ also stay close to γ under the flow Ψ.

We now outline the issues with extending this argument to the metric measure setting and the ideas we will use to resolve them:

- (5) is essential in utilizing the Hessian and gradient estimates of the approximating function to control the geometry under the flow Ψ. In the smooth setting, it stems from the first variation formula along σ_1 and σ_2 and the following interpolation formula along a unit-speed geodesic α, which should be thought of as going between $\sigma_1(t)$ and $\sigma_2(t)$ for some t in this application:

$$\langle \alpha'(\tau_1), V\rangle - \langle \alpha'(\tau_0), V\rangle = \int_{\tau_0}^{\tau_1} \langle \nabla_{\alpha'(\tau)} V, \alpha'(\tau)\rangle \, d\tau. \tag{8}$$

Here V is a vector field along α and ∇V is its covariant derivative. We will apply (5) to control the integral distance change between all elements of two sets under the flow Ψ, and so an integral version of the formula suffices. The first variation formula for "almost every" pair of σ_1 and σ_2 we are interested in follows easily from the first-order differentiation formula for Wasserstein geodesics; see Gigli [37].

In the direction of (8), the same formula (with obvious changes) was proved along Wasserstein geodesics with bounded density of Gigli and Tamanini [46, Theorem 5.13]. While this does most of the work, a suitable interpretation is required to obtain the integral interpolation formula between two sets S_1 and S_2. To see the difficulty, one might try to decompose the set of all geodesics between S_1 and S_2 by grouping together all geodesics that start at the same $x \in S_1$. In this way, one obtains a family of Wasserstein geodesics parametrized by $x \in S_1$. However, these end at a δ measure and therefore the interpolation formula of [46, Theorem 5.13] does not apply. This is, of course, expected because $\langle \nabla d_x, V\rangle$ is not well defined at x. The correct decomposition then is to break all the geodesics between S_1 and S_2 down the middle, parametrize the half that start in S_2 by the elements of S_1 they each go toward and vice versa. We point out the same decomposition is used in the proof of the segment inequality. Some work then needs to be done to check the boundary terms that arise in interpolating between each of the halves match correctly. These are the contents of Section 3.

- The Hessian estimate (7) and several other estimates on the heat flow approximation of the distance function need to be shown in the RCD(K, N) setting. This simply comes down to verifying the proofs of [30] all translate to the metric measure setting with minor adjustments. These are the contents of Section 4.

- Lastly, the argument in [30] relies on (4) to obtain estimates for small balls centered along in the interior γ under Ψ in order to start an induction process. Such an inequality is not available in the RCD setting since the Hessian is a measure-theoretic object, although progress has been made in this direction; see Bianchini and Cavalletti [13], Cavalletti [20] and Cavalletti and Mondino [23; 24]. Even if it were well defined, one does not have Jacobi fields or smoothness arguments to translate such an inequality to a statement about tangent cones or small balls along γ. This is, in many ways, the main obstruction to extending the arguments of [30]. We will not attempt to develop all this theory. The key observation is that in fact we can do without the start of induction in radius.

Recall that the need for this start of induction argument stems from the failure of (7) to control distance between a set of small measure and another set under Ψ. As such, it is possible for most of $\Psi_t(B_r(\gamma(s_1)))$ to distance from $\gamma(s_1-t)$ quickly, after which we can no longer apply (7) to control the geometry of $\Psi_t(B_r(\gamma(s_1)))$. To deal with this, consider for each $x \in B_r(\gamma(s_1))$ a piecewise geodesic which goes from p to x and then x to q. It was shown in [30] that for a significant amount of x (those with relatively low excess) and their corresponding piecewise geodesics, the estimate (7) still holds on the scale of r. The same is true for $\mathrm{RCD}(-(N-1),N)$ spaces; see Theorem 4.12. Using this, one can make an induction argument in time instead. Suppose for some small time t most of $\Psi_t(B_r(\gamma(s_1)))$ stays close to $\gamma(s_1-t)$, after which it leaves. Due to the control we had on the geometry of $\Psi_t(B_r(\gamma(s_1)))$ in that time, we can guarantee that $\Psi_t(B_r(\gamma(s_1)))$ is still very close to one of (in fact, much of) these other piecewise geodesics with a good estimate (7). Therefore we can use the estimate for that piecewise geodesic for a little longer. The start of induction is trivial since the integral curves of Ψ are 1-Lipschitz. In this way, we arrive at an $x \in B_r(\gamma(s_1))$ whose trajectory under Ψ well represents the behavior of $B_r(\gamma(s_1))$ under Ψ, in the sense that most of $B_r(\gamma(s_1))$ stays close to x on the scale of r under Ψ for a definite amount of time. Multiple limiting and gluing arguments then allow for the selection of a geodesic from p to q, perhaps different from γ, which well represents the behaviors of small balls centered in its interior under Ψ. The original argument of [30] gives the required Gromov–Hausdorff approximations in the interior of such a geodesic. Notice that, analogously to [30], we have at this point only shown the existence of a geodesic between p and q which satisfies the main theorem. These are the contents of Section 5.

The ideas outlined above overcome the difficulties of generalizing the arguments of Colding and Naber [30] to the RCD setting. To finish, we will first show RCD spaces are nonbranching before proving Theorem 1.1. In order to prove nonbranching, first notice that any two geodesics having the property above cannot branch. To see this, let γ_1 and γ_2 be two branching geodesics starting at some $p \in X$ which can be constructed by the methods of Section 5. In the interior, most of an arbitrarily small ball centered around γ_1 (resp. γ_2) must stay close to γ_1 (resp. γ_2) for some definite amount of time under the flow of Ψ, where the closeness is Hölder dependent on time. Moreover, it is possible to control how the volumes of balls change along each geodesic. Combining these observations with the essentially nonbranching property of $\mathrm{RCD}(K,N)$ spaces shows there cannot be any splitting because there is simply not enough room to flow disjoint small balls around γ_1 and γ_2 into a small ball around a branching point. While we do not initially claim all geodesics can be constructed with the methods of Section 5, our construction does give a certain amount of freedom. For any $\delta > 0$, it allows us to construct a geodesic γ^δ with nice properties on $[\delta, 1-\delta]$ which agrees with the initial geodesic γ at δ. As it turns out, combining this with the previous observation is enough to show that in fact no pair of geodesics can branch. Theorem 1.1 follows easily from the results of Section 5 and nonbranching. These are the contents of Section 6.1. In Section 6.2, we generalize to the RCD setting the applications of the main result for Ricci limits outlined in [30] using verbatim arguments.

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2 Preliminaries

2.1 Curvature-dimension condition preliminaries

A *metric measure space* is a triple (X, d, m) where (X, d) is a complete separable metric space and m is a nonnegative locally finite Borel measure. As a matter of convention, m-measurable here means measurable with respect to the completion of $(X, \mathscr{B}(X), m)$. We take the same convention for all other Borel measures as well.

Given a complete and separable metric space (X, d), we denote by $\mathcal{P}(X)$ the set of Borel probability measures and by $\mathcal{P}_2(X)$ the set of Borel probability measures with finite second moment, that is, the set of $\mu \in \mathcal{P}(X)$ where $\int_X d(x, x_0)^2 \, d\mu(x) < \infty$ for some $x_0 \in X$. Given $\mu_1, \mu_2 \in \mathcal{P}_2(X)$, the *$L^2$-Wasserstein distance* W_2 between them is defined as

$$W_2^2(\mu_1, \mu_2) := \inf_\gamma \int_{X \times X} d^2(x, y) \, d\gamma(x, y),$$

where the infimum is taken over all $\gamma \in \mathcal{P}(X \times X)$ with $(\pi_1)_*(\gamma) = \mu_1$ and $(\pi_2)_*(\gamma) = \mu_2$. Such measures γ are called *admissible plans* for the pair (μ_1, μ_2). We call $(\mathcal{P}_2(X), W_2)$ the L^2-Wasserstein space of (X, d); it has been well studied in the theory of optimal transportation. A W_2-geodesic between $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is any path $(\mu_t)_{t \in [0,1]}$ in $\mathcal{P}_2(X)$ satisfying $W_2(\mu_s, \mu_t) = |s - t| W_2(\mu_0, \mu_1)$ for any $s, t \in [0, 1]$. If (X, d) is a geodesic space then $(\mathcal{P}_2(X), W_2)$ is as well. A c-concave solution φ to the corresponding dual problem of maximizing $\int \varphi \, d\mu_0 + \int \varphi^c \, d\mu_1$ is called a *Kantorovich potential*. We refer to [3; 72] for definitions and details.

The various notions of the classical curvature-dimension condition were first proposed independently in [60] and [70; 71], and are defined as certain convexity conditions on the L^2-Wasserstein space of a metric measure space. We follow closely the formulations of [11].

Given a metric measure space (X,d,m), for any $\mu\in\mathcal{P}(X)$, the *Shannon–Boltzmann entropy* is defined as

$$\mathrm{Ent}_m(\cdot)\colon\mathcal{P}(X)\to(-\infty,\infty],\quad \mathrm{Ent}_m(\mu):=\int\log\rho\,d\mu\quad\text{if } \mu=\rho m \text{ and } (\rho\log\rho)_- \text{ is } m\text{-integrable}$$

and ∞ otherwise.

Definition 2.1 (CD(K,∞) condition) Let $K\in\mathbb{R}$. A metric measure space (X,d,m) is a CD(K,∞) space if and only if for any two absolutely continuous measures $\mu_0,\mu_1\in\mathcal{P}(X)$ with bounded support, there exists a W_2-geodesic $\{\mu_t\}_{t\in[0,1]}$ such that for any $t\in[0,1]$,

$$\mathrm{Ent}_m(\mu_t)\le(1-t)\,\mathrm{Ent}_m(\mu_0)+t\,\mathrm{Ent}_m(\mu_1)-\tfrac{1}{2}Kt(1-t)W_2^2(\mu_0,\mu_1).$$

The *N-Rényi entropy* is defined as

$$S_N(\cdot\,|m)\colon\mathcal{P}(X)\to(-\infty,0],\quad S_N(\mu|m):=-\int\rho^{1-(1/N)}\,dm,$$

if $\rho^{1-(1/N)}\in L^1(m)$, where $\mu=\rho m$, and 0 otherwise.

Letting $K\in\mathbb{R}$ and $N\in[1,\infty)$, the distortion coefficients $\sigma_{K,N}^{(t)}$ and $\tau_{K,N}^{(t)}$ are defined by

$$(t,\theta)\in[0,1]\times\mathbb{R}^+\to\sigma_{K,N}^{(t)}(\theta):=\begin{cases}\infty & \text{if } K\theta^2\ge N\pi^2,\\ \sin(t\theta\sqrt{K/N})/\sin(\theta\sqrt{K/N}) & \text{if } 0<K\theta^2<N\pi^2,\\ t & \text{if } K\theta^2=0,\\ \sinh(t\theta\sqrt{K/N})/\sinh(\theta\sqrt{K/N}) & \text{if } 0<K\theta^2<0,\end{cases}$$

and

$$\tau_{K,N}^{(t)}(\theta):=t^{1/N}\sigma_{K,N}^{(t)}(\theta)^{1-(1/N)}.$$

The standard finite-dimensional *curvature-dimension* condition was introduced in [71; 60].

Definition 2.2 (CD(K,N) condition) Let $K\in\mathbb{R}$ and $N\in[1,\infty)$. We say that a metric measure space (X,d,m) is a CD(K,N) space if for any two absolutely continuous measures $\mu_0=\rho_0 m,\mu_1=\rho_1 m\in\mathcal{P}(X)$ with bounded support there exists a W_2-geodesic $\{\mu_t\}_{t\in[0,1]}$ and an associated optimal coupling π between μ_0 and μ_1 such that for any $t\in[0,1]$ and $N'\ge N$,

$$S_N(\mu_t|m)\le-\int\big(\tau_{K,N'}^{(1-t)}(d(x,y))\rho_0(x)^{-1/N'}+\tau_{K,N'}^{(t)}(d(x,y))\rho_1(y)^{-1/N'}\big)\,d\pi(x,y).$$

The *reduced curvature-dimension* condition CD$^*(K,N)$ was introduced in [11] for its seemingly better tensorization and globalization properties. It is defined by replacing τ with σ in Definition 2.2. The CD(K,N) and CD$^*(K,N)$ conditions generalize to the metric measure setting the notion of Ricci curvature bounded below by K and dimension bounded above by N. Examples include (possibly weighted) Riemannian manifolds [71], Finsler manifolds [63] and Alexandrov spaces [64].

Remark 2.3 CD(K,N) implies CD(K',N') and CD$^*(K',N')$ for all $K'\le K$ and $N'\ge N$ as well as CD(K',∞). A host of results that we cite were shown in the RCD$^*(K,N)$ and RCD(K,∞) settings (see Definition 2.12) and therefore apply in the RCD(K,N) setting. Going in the other direction, it was shown in [21] that $\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ when $m(X)<\infty$ (this was extended to general σ-finite m in [57]). We mention that our proofs carry forward without modification to the $\mathrm{RCD}^*(K,N)$ setting. However, since several papers we cite use the stronger RCD assumption (though it can be checked this is not needed for the particular results we cite from them), we will do so as well to ease the burden of exposition.

It is known that if (X,d,m) is $\mathrm{CD}(K,N)$ then $\mathrm{supp}(m)$ is a geodesic space which also satisfies the $\mathrm{CD}(K,N)$ condition. Due to this, we will always assume $X=\mathrm{supp}(m)$. One can check that for any $\lambda,c>0$, if (X,d,m) is $\mathrm{CD}(K,N)$, then $(X,\lambda d,cm)$ is $\mathrm{CD}(K/\lambda^2,N)$. $\mathrm{CD}(K,N)$ spaces, like their smooth counterparts, satisfy the standard Bishop–Gromov volume comparison.

Theorem 2.4 (Bishop–Gromov volume comparison [71, Theorem 2.3]) *Let (X,d,m) be a $\mathrm{CD}(K,N)$ space for some $K\in\mathbb{R}$ and $N\in(1,\infty)$. Then for all $x_0\in X$ and all $0<r<R\le\pi\sqrt{N-1/(K\vee 0)}$,*

$$\frac{m(B_r(x_0))}{m(B_R(x_0))}\ge V_{K,N}(r,R) \tag{9}$$
$$:=\begin{cases}\int_0^r(\sin(t\sqrt{K/(N-1)}))^{N-1}\,dt\Big/\int_0^R(\sin(t\sqrt{K/(N-1)}))^{N-1}\,dt & \text{if } K>0,\\ (r/R)^N & \text{if } K=0,\\ \int_0^r(\sinh(t\sqrt{K/(N-1)}))^{N-1}\,dt\Big/\int_0^R(\sinh(t\sqrt{K/(N-1)}))^{N-1}\,dt & \text{if } K<0.\end{cases}$$

In contexts where K and N are clear, we will simply write $V(r,R)$ for $V_{K,N}(r,R)$.

For $N<\infty$, $\mathrm{CD}(K,N)$ spaces are locally doubling, by Theorem 2.4, and are therefore proper. They satisfy a 1-1 Poincaré inequality by [65, Theorem 1.1].

2.2 First-order calculus on metric measure spaces

We follow the framework for calculus on metric measure space developed by Ambrosio, Gigli and Savaré in [5; 6; 7; 38; 39]. Let (X,d,m) be a metric measure space. Let $\mathrm{lip}(X)$, $\mathrm{lip}_{\mathrm{loc}}(X)$ and $\mathrm{lip}_b(X)$ be its classes of Lipschitz, locally Lipschitz and bounded Lipschitz functions, respectively. Given $f\in\mathrm{lip}_{\mathrm{loc}}(X)$, the *local Lipschitz constant* (or *local slope*) $\mathrm{lip}(f)\colon X\to\mathbb{R}$ is defined by

$$\mathrm{lip}(f)(x):=\limsup_{y\to x}\frac{|f(y)-f(x)|}{d(y,x)}. \tag{10}$$

By convention, $\mathrm{lip}(f)(x):=0$ at any isolated point x. Given $f\in L^2(m)$, a function $g\in L^2(m)$ is called a *relaxed gradient* if there exists a sequence $f_n\in\mathrm{lip}(X)$ and $\tilde g\in L^2(m)$ such that

(I) $f_n\to f$ in $L^2(m)$ and $\mathrm{lip}(f_n)$ converges weakly to $\tilde g$ in $L^2(m)$,

(II) $g\ge\tilde g$ m-a.e.

A *minimal relaxed gradient* is a relaxed gradient that is minimal in L^2-norm in the family of relaxed gradients of f. If this family is nonempty, one can check that such a function exists and is unique m-a.e.

The minimal relaxed gradient is denoted by $|Df|$. The *domain of the Cheeger energy* $D(\mathrm{Ch}) \subseteq L^2(m)$ is the subset of L^2 functions with a minimal relaxed gradient. For $f \in L^2(m)$, the Cheeger energy is defined as

$$\mathrm{Ch}(f) = \begin{cases} \frac{1}{2}\int |Df|^2\,dm & \text{if } f \in D(\mathrm{Ch}), \\ \infty & \text{otherwise.} \end{cases}$$

Ch is a convex and lower semicontinuous functional on $L^2(m)$. The Cheeger energy, first introduced in [25], was defined using a slightly different relaxation procedure. It is also possible to define a similar functional using the idea of *minimal weak upper gradients*; see [6, Section 5.1]. It is shown in [6, Section 6] that under mild assumptions on the metric measure space (satisfied by spaces meeting the various aforementioned curvature-dimension conditions in particular), all these notions are equivalent.

Remark 2.5 Let (X,d,m) be a $\mathrm{CD}(K,N)$ space with $N<\infty$. For any Lipschitz function f on X, $\mathrm{lip}(f) = |Df|$ m-a.e. This follows from [25], where it is shown that a metric measure space satisfying a Poincaré inequality and a doubling inequality has $\mathrm{lip}(f) = |Df|$ m-a.e.

$W^{1,2}(X) := D(\mathrm{Ch})$ is a Banach space endowed with the norm $\|f\|^2_{W^{1,2}(X)} := \|f\|^2_{L^2(m)} + \||Df|\|^2_{L^2(m)}$. We define $W^{1,2}_{\mathrm{Loc}}(X)$ as the space of all functions $f \in L^2(X,m)$ such that $gf \in W^{1,2}(X)$ for every compactly supported Lipschitz g. By the strong locality property of the minimal relaxed gradient (ie $|Dg| = |Dh|$ m-a.e. in $\{g=h\}$ for any $g,h \in W^{1,2}(X)$), any $f \in W^{1,2}_{\mathrm{Loc}}(X)$ has an associated differential $|Df| \in L^2_{\mathrm{loc}}(m)$.

We call (X,d,m) *infinitesimally Hilbertian* if $W^{1,2}(X)$ is a Hilbert space. In this case, for $f,g \in W^{1,2}(X)$, one may define $\langle Df, Dg\rangle$ using polarization: $\langle Df, Dg\rangle := \frac{1}{2}(|D(f+g)|^2 - |Df|^2 - |Dg|^2) \in L^1(m)$.

From here one can define the *Laplacian*: $f \in W^{1,2}(X)$ is said to be in the *domain of the Laplacian* ($f \in D(\Delta)$) if there exists $\Delta f \in L^2(m)$ such that

$$\int g\Delta f\,dm + \int \langle Dg, Df\rangle\,dm = 0 \quad \text{for any } g \in W^{1,2}(X).$$

Given a subspace $V \in L^2(m)$, we define $D_V(\Delta) := \{f \in D(\Delta) : \Delta(f) \in V\}$. More generally, one may define the *measure-valued Laplacian*:

Definition 2.6 (measure-valued Laplacian [39, Definition 3.1.2]) The space $D(\mathbf{\Delta}) \subset W^{1,2}(X)$ is the space of $f \in W^{1,2}(X)$ such that there is a signed Radon measure μ satisfying

$$\int g\,d\mu = -\int \langle Dg, Df\rangle\,dm \quad \text{for all } g\colon X \to \mathbb{R} \text{ Lipschitz with bounded support.}$$

In this case the measure μ is unique and is denoted by $\mathbf{\Delta} f$.

2.3 Tangent, cotangent and tensor modules

A technical framework for describing first-order calculus on metric measure spaces and second-order calculus on $\mathrm{RCD}(K,N)$ spaces was developed by Gigli in [39]. While aspects of second-order calculus can be effectively developed without this framework (see for example [68; 8]), [39] crucially gives constructions which generalize the notion of tensor fields. In the next few subsections, we will quickly introduce, sometimes informally, the necessary definitions given in [39] and refer to the original article for details and insights.

Let (X, d, m) be a metric measure space. The various collections of tensor fields of interest will be objects in the category of *$L^p(m)$-normed $L^\infty(m)$-modules*.

Definition 2.7 (L^p-normed L^∞-premodules [39, Definitions 1.2.1 and 1.2.10]) Let $p \in [0, \infty]$. Let $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ be a Banach space endowed with a bilinear map $L^\infty(m) \times \mathcal{M} \ni (f, v) \mapsto f \cdot v \in \mathcal{M}$ and a function $|\cdot| \colon \mathcal{M} \to L^p_+(m)$. We say $(\mathcal{M}, \|\cdot\|_{\mathcal{M}}, \cdot, |\cdot|)$ is an L^p-normed L^∞-premodule if and only if

(I) $(fg) \cdot v = f \cdot (g \cdot v)$ for all $f, g \in L^\infty(m)$ and $v \in \mathcal{M}$,

(II) $\mathbf{1} \cdot v = v$ for all $v \in \mathcal{M}$, where $\mathbf{1}$ is the constant function equal to 1,

(III) $\||v|\|_{L^p(m)} = \|v\|_{\mathcal{M}}$ for all $v \in \mathcal{M}$,

(IV) $|f \cdot v| = |f||v|$ m-a.e. for all $f \in L^\infty(m)$ and $v \in \mathcal{M}$.

We will often simply write fv for $f \cdot v$ and call $|\cdot|$ the pointwise norm. If an *$L^p(m)$-normed $L^\infty(m)$-premodule* satisfies additional locality and gluing properties [39, Definition 1.2.1], we say it is an *$L^p(m)$-normed $L^\infty(m)$-module*. One may localize such an object to some $A \in \mathscr{B}(X)$ by defining $\mathcal{M}|_A := \{v \in \mathcal{M} : |v| = 0 \ m\text{-a.e. on } A^c\}$, which is again canonically an $L^p(m)$-normed $L^\infty(m)$-module.

An L^2-normed L^∞-module which is a Hilbert space under $\|\cdot\|_{\mathcal{M}}$ is called a *Hilbert module*. In this case one can define a pointwise inner product by polarizing the pointwise norm $|\cdot|$. The prototypical example of a Hilbert module one has in mind is the collection of L^2 vector fields on a Riemannian manifold, where $|\cdot|$ is the Riemannian pointwise norm.

Given L^∞-modules $\mathcal{M}$ and $\mathcal{N}$, we say a map $T \colon \mathcal{M} \to \mathcal{N}$ is a *module morphism* if it is a bounded linear map between $\mathcal{M}$ and $\mathcal{N}$ as Banach spaces satisfying in addition $T(fv) = fT(v)$ for all $f \in L^\infty(m)$ and $v \in \mathcal{M}$. The *dual module* $\mathcal{M}^*$ is the space of all module morphisms between $\mathcal{M}$ and $L^1(m)$ and is an $L^{p_*}(m)$-normed $L^\infty(m)$-module, where $1/p + 1/p_* = 1$. A Hilbert module $\mathcal{H}$ is canonically isomorphic to its dual.

Given two Hilbert modules $\mathcal{H}_1$ and $\mathcal{H}_2$, one can construct new Hilbert modules: the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ [39, Definition 1.5.1] and the exterior product $\mathcal{H}_1 \wedge \mathcal{H}_2$ [39, Definition 1.5.4].

Definition 2.8 [39, Definition 2.19] Let $\mathcal{M}$ be $L^2(m)$-normed and let $V \subseteq \mathcal{M}$. $\mathrm{Span}(V)$ is defined as the collection of $v \in \mathcal{M}$ for which there is a Borel decomposition $(X_n)_{n \in \mathbb{N}}$ of X and, for each $n \in \mathbb{N}$, collections $v_{1,n}, \dots, v_{k_n,n} \in V$ and $f_{1,n}, \dots, f_{k_n,n} \in L^\infty(m)$ such that

$$\mathbf{1}_{X_n} v = \sum_{i=1}^{k_n} f_{i,n} v_{i,n} \quad \text{for each } n \in \mathbb{N},$$

where $\mathbf{1}_{X_n}$ is the characteristic function of X_n. We say that V generates $\mathcal{M}$ if and only if $\overline{\mathrm{Span}\, V} = X$.

From this point we will assume (X, d, m) is an infinitesimally Hilbertian metric measure space. We now define the tangent and cotangent modules of (X, d, m).

Theorem 2.9 [39, Proposition 2.2.5] *There exists a unique, up to isomorphism, Hilbert module $\mathcal{H}$ endowed with a linear map $d : W^{1,2}(X) \to \mathcal{H}$ satisfying*

(I) $|df| = |Df|$ m-a.e. *for all* $f \in W^{1,2}(X)$,

(II) $d(W^{1,2}(X))$ *generates* $\mathcal{H}$.

Such an $\mathcal{H}$ is called the **cotangent module** *of* (X, d, m) *and denoted by* $L^2(T^*X)$.

The dual of $L^2(T^*X)$ is called the **tangent module** *of* (X, d, m) *and denoted by* $L^2(TX)$. *Elements of* $L^2(TX)$ *are called* **vector fields***.*

We denote by ∇f the dual of df (ie the unique element $\nabla f \in L^2(TX)$ such that $v(\nabla f) = \langle v, df\rangle$ for all $v \in L^2(T^*X)$).

Notice $\langle df, dg\rangle = df(\nabla g) = \langle \nabla f, \nabla g\rangle = \langle Df, Dg\rangle$ m-a.e. We will use these interchangeably in the rest of the paper depending on the convention of the theorems we are quoting. For a discussion of the philosophical differences of the objects involved, see [6, Section 2.2].

Definition 2.10 [39, Definition 2.3.11] $D(\mathrm{div}) \subseteq L^2(TX)$ is the space of all vector fields $v \in L^2(TX)$ for which there exists $f \in L^2(m)$ such that for any $g \in W^{1,2}(X)$ the equality

$$\int fg\, dm = -\int dg(v)\, dm$$

holds. In this case f is called the *divergence of* v and denoted by $\mathrm{div}(v)$. In particular, if $f \in D(\Delta)$, then $\nabla f \in D(\mathrm{div})$ and $\mathrm{div}(\nabla f) = \Delta f$.

Definition 2.11 $L^2((T^*)^{\otimes 2}(X))$ denotes the tensor product of $L^2(T^*X)$ with itself; see [39, Definition 1.5.1]. Similarly, $L^2(T^{\otimes 2}(X))$ denotes the tensor product of $L^2(TX)$ with itself. We will use $|\cdot|_{\mathrm{HS}}$ and $\cdot : \cdot$ to denote the pointwise norm (Hilbert–Schmidt norm) and the pointwise inner product of Hilbert modules which arise from tensors.

$L^2((T^*)^{\otimes 2}(X))$ and $L^2(T^{\otimes 2}(X))$ are Hilbert module duals of each other. We mention that for any element $A \in L^2((T^*)^{\otimes 2}(X))$, we will often write $A(V, W) = A(V \otimes W)$. We will sometimes write this even when V and W are such that $V \otimes W$ is not in $L^2(T^{\otimes 2}(X))$. In all these cases, $V \otimes W$ when multiplied by the characteristic function of a compact set will be in $L^2(T^{\otimes 2}(X))$, and so $A(V, W)$ is well defined as a measurable function by locality and satisfies

$$A(V, W) \le |A|_{\mathrm{HS}}|V||W|. \tag{11}$$

We will usually have additional assumptions on $|V|$ and $|W|$, so that $A(V, W) \in L^1_{\mathrm{loc}}(m)$.

2.4 RCD(K, N) and Bakry–Émery conditions

We now introduce the notion of RCD spaces, which are our main objects of interest. These were proposed and carefully analyzed in a series of papers including [7; 38; 4; 35; 9].

Definition 2.12 [7; 38] Let (X, d, m) be a metric measure space, $K \in \mathbb{R}$, and $N \in [1, \infty)$. We say (X, d, m) satisfies the *Riemannian curvature-dimension condition* $\mathrm{RCD}(K, N)$ if and only if (X, d, m) satisfies the $\mathrm{CD}(K, N)$ condition and is infinitesimally Hilbertian. Similarly one defines the Riemannian curvature-dimension conditions $\mathrm{RCD}^*(K, N)$ and $\mathrm{RCD}(K, \infty)$ using $\mathrm{CD}^*(K, N)$ and $\mathrm{CD}(K, \infty)$, respectively.

The RCD condition is stable under measured Gromov–Hausdorff convergence and tensorization. Examples of RCD spaces include Ricci limits and Alexandrov spaces, but non-Riemannian Finsler geometries are ruled out. We now state some equivalent formulations of the $\mathrm{RCD}(K, N)$ property. We will in general assume (X, d, m) is infinitesimally Hilbertian in this subsection.

As in [8], define the *Carré du champ operator* for $f \in D_{W^{1,2}(X)}(\Delta)$ and $\varphi \in D_{L^\infty(m)\cap L^2(m)}(\Delta) \cap L^\infty(m)$ by

$$\Gamma_2(f; \varphi) := \int \tfrac{1}{2}|\nabla f|^2 \Delta\varphi \, dm - \int \langle \nabla f, \nabla \Delta f \rangle \varphi \, dm.$$

This enables us to state the nonsmooth Bakry–Émery condition $\mathrm{BE}(K, N)$:

Definition 2.13 (Bakry–Émery condition [8; 35]) Let $K \in \mathbb{R}$ and $N \in [1, \infty]$. We say (X, d, m) satisfies the $\mathrm{BE}(K, N)$ condition if and only if

$$\Gamma_2(f; \varphi) \geq \frac{1}{N} \int (\Delta f)^2 \varphi \, dm + K \int |\nabla f|^2 \varphi \, dm.$$

$\mathrm{BE}(K, N)$ is closely related to $\mathrm{CD}(K, N)$. We say (X, d, m) satisfies the *Sobolev-to-Lipschitz property* [41, Definition 3.15] if any function $f \in W^{1,2}(X)$ with $|\nabla f| \in L^\infty(m)$ has a Lipschitz representative $\tilde{f} = f$ m-a.e. with Lipschitz constant equal to $\operatorname{ess\,sup}(|\nabla f|)$.

Theorem 2.14 [8; 9; 35] *Let (X, d, m) be a metric measure space satisfying an exponential growth condition* (see [8, Section 3]), $K \in \mathbb{R}$ *and* $N \in (1, \infty)$. *Then (X, d, m) is* $\mathrm{RCD}(K, N)$ *if and only if (X, d, m) is infinitesimally Hilbertian, and satisfies the Sobolev-to-Lipschitz property and the* $\mathrm{BE}(K, N)$ *condition.*

2.5 Heat flow and Bakry–Ledoux estimates

By applying the theory of the gradient flow of convex functionals on Hilbert spaces to Ch as in [6], one obtains for each $f \in L^2(m)$ a unique continuous curve $(H_t(f))_{t\in[0,\infty)}$ in $L^2(m)$ which is locally absolutely continuous in $(0, \infty)$ with $H_0(f) = f$ such that

$$\frac{d}{dt} H_t(f) = \Delta' H_t(f) \quad \text{for a.e. } t \in (0, \infty), \tag{12}$$

where $\Delta' g$ is defined as the minimizer in L^2 energy in $\partial^-(\mathrm{Ch})$ at g provided it is nonempty; see [6, Section 4.2].

If (X,d,m) is infinitesimally Hilbertian, then for any $t>0$, $H_t(f)\in D(\Delta)$, and one has the a priori estimates

$$\|H_t(f)\|_{L^2}\le\|f\|_{L^2},\quad \||DH_t(f)|\|_{L^2}^2\le\frac{\|f\|_{L^2}^2}{2t^2},\quad \|\Delta H_t(f)\|_{L^2}\le\frac{\|f\|_{L^2}}{t}.$$

$H_t(f)$ is linear and satisfies $\Delta(H_t(f))=H_t(\Delta(f))$ for any $t>0$. In particular, (12) is true for all $t\in(0,\infty)$ and

$$H_t(f)=f+\int_0^t\Delta(H_s(f))\,ds.$$

If (X,d,m) is $\mathrm{RCD}(K,\infty)$, H_t can be identified with $\mathscr{H}_t$, the gradient flow of Ent_m on $\mathcal{P}_2(X)$. Due to contraction properties coming from the RCD condition, $\mathscr{H}_t$ can be extended from $D(\mathrm{Ent}_m)$ to all of $\mathcal{P}_2(X)$.

Definition 2.15 For $t>0$ and $x\in X$, $\mathscr{H}_t(\delta_x)$ is absolutely continuous with respect to m. Then $\mathscr{H}_t(\delta_x)=H_t(x,\cdot)m$, where $H_t(\cdot,\cdot)$ is the *heat kernel*.

$H_t(x,y)$ is symmetric and continuous in both variables. For each $f\in L^2(m)$, one has the representation formula [7, Theorem 6.1]

$$H_t(f)(x)=\int f(y)H_t(x,y)\,dm. \tag{13}$$

The $\mathrm{RCD}(K,N)$ condition implies the Bakry–Ledoux estimate, which is a finite-dimensional analogue of the Bakry–Émery contraction estimate [7, Theorem 6.2]. Moreover, it was shown in [35] that one has equivalence in Theorem 2.14 with the $\mathrm{BE}(K,N)$ condition replaced by the (K,N) Bakry–Ledoux estimate.

Theorem 2.16 (dimensional Bakry–Ledoux L^2 gradient-Laplacian estimate [35, Theorem 4.3]) *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$ and $N\in[1,\infty)$. For any $f\in W^{1,2}(X)$ and $t>0$,*

$$|\nabla(H_t(f))|^2+\frac{4Kt^2}{N(e^{2Kt}-1)}|\Delta H_t(f)|^2\le e^{-2Kt}H_t(|\nabla(f)|^2)\quad m\text{-a.e.}.$$

Remark 2.17 If $|\nabla f|\in L^\infty$, one can take continuous representatives of $\Delta H_t(f)$ and $H_t(|\nabla(f)|^2)$ and identify $|\nabla(H_t(f))|$ canonically with the local Lipschitz constant of $H_t(f)$ to obtain a pointwise Bakry–Ledoux bound; see [35, Proposition 4.4].

This implies the Sobolev-to-Lipschitz property by [7, Theorem 6.2]. H_t also has an L^∞-to-Lipschitz property by [7, Theorem 6.8], where it was shown for $t>0$ and $f\in L^2(m)$,

$$2I_{2K}(t)|\nabla H_t(f)|^2\le H_t(f^2)\quad m\text{-a.e.}, \tag{14}$$

where $I_{2K}(t):=\int_0^t e^{2Ks}\,ds$.

2.6 Second-order calculus and improved Bochner inequality

The class of *test functions* was introduced in [68] as

$$\mathrm{TestF}(X) := \{f \in D(\Delta) \cap L^\infty : |Df| \in L^\infty \text{ and } \Delta f \in W^{1,2}(X)\}.$$

It is known that $\mathrm{TestF}(X)$ is an algebra and, on $\mathrm{RCD}(K,\infty)$ spaces, it was shown by the results of [7] mentioned in the previous subsection that the heat flow approximations of an $L^\infty \cap L^2$ function are test functions and so $\mathrm{TestF}(X)$ is dense in $W^{1,2}(X)$.

In [50; 68], it was shown that under the $\mathrm{BE}(K,N)$ condition, $|\nabla f|^2 \in D(\mathbf{\Delta})$ for any $f \in \mathrm{TestF}(X)$ and so one may define $\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle$. One can then define a Hessian for $f \in \mathrm{TestF}(X)$ and show the improved Bochner inequality (see Theorem 2.19). In [39, Section 3], the same calculations were carried out in the framework proposed therein. We outline the main definitions and results from there. In this subsection, we assume (X,d,m) is an $\mathrm{RCD}(K,N)$ space.

Definition 2.18 [39, Definition 3.3.1] $W^{2,2}(X) \subseteq W^{1,2}(X)$ is the space of all functions $f \in W^{1,2}(X)$ for which there exists $A \in L^2((T^*)^{\otimes 2}(X))$ such that for any $g_1, g_2, h \in \mathrm{TestF}(X)$ the equality

$$2\int hA(\nabla g_1, \nabla g_2)\,dm = \int -\langle \nabla f, \nabla g_1\rangle \operatorname{div}(h\nabla g_2) - \langle \nabla f, \nabla g_2\rangle \operatorname{div}(h\nabla g_1) - h\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle\,dm$$

holds. In this case A is called the *Hessian of* f and denoted by $\operatorname{Hess} f$. $W^{2,2}(X)$ is a Hilbert space under the norm

$$\|f\|^2_{W^{2,2}(X)} := \|f\|^2_{L^2(m)} + \|df\|^2_{L^2(T^*X)} + \|\operatorname{Hess} f\|_{L^2((T^*)^{\otimes 2}(X))}.$$

It turns out that the test functions are contained in $W^{2,2}(X)$ and one has the improved Bochner inequality as in [68; 50].

Theorem 2.19 (improved Bochner inequality [39, Theorem 3.3.8]) *Let* (X,d,m) *be an* $\mathrm{RCD}(K,N)$ *space for* $K \in \mathbb{R}$, *and* $N \in [1,\infty)$ *and* $f \in \mathrm{TestF}(X)$. *Then* $f \in W^{2,2}(X)$ *and*

$$\mathbf{\Gamma}_2(f) \geq [K|\nabla f|^2 + |\operatorname{Hess}(f)|^2_{\mathrm{HS}}]m.$$

$H^{2,2}(X)$ is then defined as the closure of $\mathrm{TestF}(X)$ in $W^{2,2}(X)$. An approximation argument gives the following:

Corollary 2.20 [39, Corollary 3.3.9] $D(\Delta) \subseteq W^{2,2}(X)$ *and for* $f \in D(\Delta)$,

$$\int |\operatorname{Hess} f|^2_{\mathrm{HS}}\,dm \leq \int [(\Delta f)^2 - K|\nabla f|^2]\,dm.$$

Finally, we introduce the analogue of vector fields which have a first-order (covariant) derivative:

Definition 2.21 [39, Definition 3.4.1] $W^{1,2}_C(TX) \subseteq L^2(TX)$ is the space of all $v \in L^2(TX)$ for which there exists $T \in L^2(T^{\otimes 2}(X))$ such that for any $g_1.g_2, h \in \mathrm{TestF}(X)$, the equality

$$\int hT : (\nabla g_1 \otimes \nabla g_2) = \int -\langle v, \nabla g_2\rangle \operatorname{div}(h\nabla g_1) - h \operatorname{Hess}(g_2)(v, \nabla g_2)\,dm$$

holds. In this case T is called the *covariant derivative* of v and denoted by ∇v. $W_C^{1,2}(TX)$ is a Hilbert space under the norm

$$\|v\|^2_{W_C^{1,2}(TX)} := \|v\|^2_{L^2(TX)} + \|\nabla v\|^2_{L^2(T^{\otimes 2}(X))}.$$

The class of *test vector fields* is defined as

$$\mathrm{TestV}(X) := \left\{ \sum_{i=1}^n g_i \nabla f_i : n \in \mathbb{N},\ f_i, g_i \in \mathrm{TestF}(X) \right\}.$$

By [39, Theorem 3.4.2], $\mathrm{TestV}(X) \subseteq W_C^{1,2}(TX)$ and so $H_C^{1,2}(TX)$ is defined to be the closure of $\mathrm{TestV}(X)$ in $W_C^{1,2}(TX)$. For any $f \in W^{2,2}(X)$, $\mathrm{Hess}\, f$ and $\nabla(\nabla f)$ are dual under the duality of $L^2((T^*)^{\otimes 2}(X))$ and $L^2(T^{\otimes 2}(X))$.

2.7 Nonbranching and essentially nonbranching spaces

Given a geodesic metric space (X,d), we define the space of constant-speed geodesics

$$\mathrm{Geo}(X) := \{\gamma \in C([0,1],X) : d(\gamma(s),\gamma(t)) = |s-t|\, d(\gamma(0),\gamma(1))\ \forall s,t \in [0,1]\}.$$

For each $t \in [0,1]$, $e_t : \mathrm{Geo}(X) \to X$ defined by $e_t(\gamma) := \gamma(t)$ denotes the evaluation map at time t. On a complete and separable metric space (X,d), any W_2-geodesic has a *lifting* to a measure on the space of geodesics in the following sense:

Theorem 2.22 [58, Theorem 3.2] *Let $(\mu_t)_{t\in[0,1]}$ be a W_2-geodesic. Then there exists $\boldsymbol{\pi} \in \mathcal{P}(\mathrm{Geo}(X))$ such that*

$$(e_t)_*(\boldsymbol{\pi}) = \mu_t \quad \text{for all } t \in [0,1], \qquad |\dot\mu_t|^2 = \int |\dot\gamma_t|^2 \, d\boldsymbol{\pi}(\gamma) \quad \text{for a.e. } t \in [0.1],$$

where $e_t(\gamma) := \gamma(t)$ is the evaluation map at time t.

These are called *optimal dynamical plans*. This motivates the following definition: for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0,\mu_1)$ the space of all optimal dynamical plans from μ_0 to μ_1.

Definition 2.23 Given two geodesics $\gamma^1 \neq \gamma^2$ on a geodesic metric space (X,d), assume γ^1 and γ^2 are constant speed and parametrized on the unit interval. We say γ^1 and γ^2 *branch* if there exists $0 < t < 1$ such that $\gamma^1_s = \gamma^2_s$ for all $s \in [0,t]$. A subset $S \subseteq \mathrm{Geo}(X)$ is called a *set of nonbranching geodesics* if there are no branching pairs in S. A geodesic metric space for which $\mathrm{Geo}(X)$ is itself a set of nonbranching geodesics is called *nonbranching*.

Many results were shown for various types of CD spaces under the additional nonbranching assumption. These include the local-to-global property, tensorization property and local Poincaré inequality; see [70; 11; 59]. A weaker assumption was introduced in [66] for which these results generalize.

Definition 2.24 A metric measure space (X, d, m) is called *essentially nonbranching* if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ absolutely continuous with respect to m, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of nonbranching geodesics.

$\mathrm{RCD}(K, \infty)$ spaces are shown to be essentially nonbranching by the results of [7; 32; 66]. We will frequently refer to the following theorem from [44], shown for finite-dimensional $\mathrm{RCD}(K, N)$ spaces; see also [65; 66] for related results and [22] for the same result in the case of essentially nonbranching $\mathrm{MCP}(K, N)$ spaces.

Theorem 2.25 [22, Theorem 1.1; 44] *Let (X, d, m) be an $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. If $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with $\mu_0 = \rho_0 m \ll m$, then there exists a unique $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$. Also $(e_t)_*(\nu) \ll m$ for any $t \in [0, 1)$ and such ν is given by a unique map $S : \mathrm{supp}(\mu_0) \to \mathrm{Geo}(X)$ in the sense that $\nu = S_*(\mu)$. Moreover, if μ_0 and μ_1 have bounded support and $\|\rho_0\|_{L^\infty(m)} < \infty$, then*

$$\|\rho_t\|_{L^\infty(m)} \le \frac{1}{(1-t)^N} e^{Dt\sqrt{(N-1)K^-}} \|\rho_0\|_{L^\infty(m)} \quad \text{for all } t \in [0, 1), \tag{15}$$

where $D := \mathrm{diam}(\mathrm{supp}(\mu_0) \cup \mathrm{supp}(\mu_1))$ and $K^- := \max\{-K, 0\}$.

Remark 2.26 In particular, this implies for any $p \in X$, there is a unique geodesic between p and x for m-a.e. $x \in X$. Using the Kuratowski and Ryll-Nardzewski measurable selection theorem, one may select constant-speed geodesics $\gamma_{x,p}$ from all $x \in X$ to p so that the map $X \times [0, 1] \ni (x, t) \mapsto \gamma_{x,p}(t)$ is Borel. This then guarantees such a choice is unique up to a set of measure 0. Similarly, one may select constant-speed geodesics $\gamma_{x,y}$ for all $x, y \in X$ so that the map $X \times X \times [0, 1] \ni (x, y, t) \mapsto \gamma_{x,y}(t)$ is Borel. Using, for example, the arguments in [20, Section 4], the set of points $(x, y) \in X \times X$ connected by nonunique geodesics is analytic. The $(m \times m)$-almost everywhere uniqueness of the Borel selection then follows using Fubini's theorem. In cases where we fix geodesics in this manner, we say we take a *Borel selection* of geodesics $\gamma_{x,p}$ from all $x \in X$ to p (or $\gamma_{x,y}$ from all $x \in X$ to all $y \in X$).

2.8 RCD(*K*, *N*) structure theory

We will assume basic familiarity with pointed Gromov–Hausdorff (pGH) and pointed measure Gromov–Hausdorff (pmGH) convergence and refer to [17; 43; 72] for details.

A notion of considerable interest for $\mathrm{RCD}(K, N)$ spaces is that of measured tangents. Similar objects have been well studied in the setting of Alexandrov spaces and Ricci limit; see for example [18; 31] for an overview. Given a metric measure space (X, d, m), $\bar{x} \in X$ and $r \in (0, 1)$, consider the *normalized rescaled pointed metric measure space* $(X, r^{-1}d, m_r^{\bar{x}}, \bar{x})$ where

$$m_r^{\bar{x}} := \left(\int_{B_r(\bar{x})} 1 - \frac{d(x, y)}{r} \, dm(y) \right)^{-1} m.$$

In what follows let (X, d, m) be an $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. We define:

Definition 2.27 (the collection of tangent spaces $\mathrm{Tan}(X,d,m,\bar{x})$) Let $\bar{x} \in X$. A normalized rescaled pointed metric measure space $(Y,d_Y,m_Y,\bar{y})$ is called a *tangent* (*cone*) of (X,d,m) at $\bar{x}$ if there exists a sequence of radii $r_i \downarrow 0$ such that $(X, r_i^{-1}d, m_{r_i}^{\bar{x}}, \bar{x}) \to (Y,d_Y,m_Y,\bar{y})$ as $i \to \infty$ in the pmGH topology. The collection of all tangents of (X,d,m) at $\bar{x}$ is denoted by $\mathrm{Tan}(X,d,m,\bar{x})$.

A standard compactness argument by Gromov shows $\mathrm{Tan}(X,d,m,\bar{x})$ is nonempty for any $\bar{x} \in X$. The rescaling and stability properties of the $\mathrm{RCD}(K,N)$ condition under pmGH convergence [7, Theorem 6.11; 43, Theorem 7.2] show that every element of $\mathrm{Tan}(X,d,m,\bar{x})$ is an $\mathrm{RCD}(0,N)$ space.

Let $c_k = \int_{B_1(0)} 1-|x|\,d\mathscr{L}(x)$ be the normalization constant of the k-dimensional Lebesgue measure and define the *k-dimensional regular set* $\mathcal{R}_k$ by

$$\mathcal{R}_k := \{x \in X : \mathrm{Tan}(X,d,m,x) = \{(\mathbb{R}^k, d_E, c_k\mathscr{H}^k, 0^k)\}\}.$$

Define $\mathcal{R}_{\mathrm{reg}} := \bigcup_{k=1}^{\lfloor N\rfloor} \mathcal{R}_k$ the *regular set* of X and $\mathcal{S} := X\backslash\mathcal{R}_{\mathrm{reg}}$ the *singular set*. In [42] it was shown for m-a.e. $x \in X$ there exists some $k \in \mathbb{N}$ for $1 \le k \le N$ such that $(\mathbb{R}^k, d_E, c_k\mathscr{H}^k, 0^k) \in \mathrm{Tan}(X,d,m,x)$. This was improved in [61], where it was shown that each $\mathcal{R}_k$ is m-measurable and $m(\mathcal{S}) = 0$. A final improvement was made in [15] with the following theorem:

Theorem 2.28 (constancy of the dimension [15, Theorem 3.8]) *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ metric measure space for some $K \in \mathbb{R}$ and $1 \le N < \infty$. Assume X is not a point. There exists a unique $n \in \mathbb{N}$ for $1 \le n \le N$ such that $m(X\backslash\mathcal{R}_n) = 0$.*

This was proved in the case of Ricci limits in [30] from the main result there, and as such also follows from Theorem 1.2; see Theorem 6.2. By [54, Theorem 1.2], see also [52, Theorem 1.9] in the case of Ricci limits, it is known that the unique n in Theorem 2.28 is also the largest integer n for which $\mathcal{R}_n$ is nonempty.

2.9 Additional RCD(K, N) theory

In this subsection we record several theorems for $\mathrm{RCD}(K,N)$ spaces which will be of use later.

Define the coefficients $\tilde{\sigma}_{K,N}(\cdot)\colon [0,\infty) \to \mathbb{R}$ by

$$\tilde{\sigma}_{K,N}(\theta) := \begin{cases} \theta\sqrt{K/N}\,\mathrm{cotan}(\theta\sqrt{K/N}) & \text{if } K > 0, \\ 1 & \text{if } K = 0, \\ \theta\sqrt{-K/N}\,\mathrm{cotanh}(\theta\sqrt{-K/N}) & \text{if } K < 0. \end{cases}$$

Then one has the following sharp bound on the measure-valued Laplacian of distance functions:

Theorem 2.29 (Laplacian comparison for the distance function [38, Corollary 5.15]) *Let (X,d,m) be a compact $\mathrm{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and $N \in (1,\infty)$. For $x_0 \in X$ denote by $d_{x_0}\colon X \to [0,\infty)$ the function $x \mapsto d(x,x_0)$. Then*

$$\tfrac{1}{2}d_{x_0}^2 \in D(\mathbf{\Delta}) \quad \textit{with} \quad \mathbf{\Delta}\tfrac{1}{2}d_{x_0}^2 \le N\tilde{\sigma}_{K,N}(d_{x_0})m \quad \textit{for all } x_0 \in X$$

and

$$d_{x_0} \in D(\mathbf{\Delta}, X\backslash\{x_0\}) \quad \textit{with} \quad \mathbf{\Delta}d_{x_0}|_{X\backslash\{x_0\}} \le \frac{N\tilde{\sigma}_{K,N}(d_{x_0}) - 1}{d_{x_0}}m \quad \textit{for all } x_0 \in X.$$

Here $\mathbf{\Delta} d_{x_0}|_{X\setminus\{x_0\}}$ is defined similarly to Definition 2.6, the difference being that the test functions g must be compactly supported in $X\setminus\{x_0\}$.

Remark 2.30 We use the fact that when X is noncompact one can make essentially the same statement. Some small adjustments are needed since the Laplacians of d_{x_0} on $X\setminus\{x_0\}$ and $d_{x_0}^2$ on X are not naturally guaranteed to be signed Radon measures. To accommodate this, a weakening of the definition of the measure-valued Laplacian was given in [24, Definitions 2.11 and 2.12]. This definition allows the Laplacian to be, more generally, a Radon functional (ie in $(C_C(X))'$). The difference is that a Radon functional has a representation as the difference of two possibly infinite positive Radon measures by the Riesz–Markov–Kakutani representation theorem, whereas in the case of a signed Radon measure, at least one of these must be finite. It was shown in [24, Corollaries 4.17 and 4.19] that the Laplacian comparison for the distance function holds as stated with this weaker definition. As such, we will, by a slight abuse of notation, treat the Laplacian of the distance function d_{x_0} on $X\setminus\{x_0\}$ as a signed Radon measure in the few instances where we use Theorem 2.29 in integration against compactly supported functions. Note that due to the comparison theorem, this Laplacian is locally a signed Radon measure, having at most an infinite negative part.

We will also need the Li–Yau Harnack inequality [56] and the Li–Yau gradient inequality [12; 56]. These were proven for the RCD setting in the finite measure case in [36], and in general in [51].

Theorem 2.31 (Li–Yau Harnack inequality [36; 51]) *Let (X, d, m) be an $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f \in L^p(m)$ for some $p \in [1, \infty)$ be nonnegative. If $K \geq 0$, then for every $x, y \in X$ and $0 < s < t$,*

$$(H_t f)(y) \geq (H_s f)(x) e^{(d^2(x,y))/(4(t-s)e^{2Ks/3})} \left(\frac{1-e^{(2K/3)s}}{1-e^{(2K/3)t}}\right)^{\frac{N}{2}}.$$

If instead $K < 0$, then

$$(H_t f)(y) \geq (H_s f)(x) e^{(d^2(x,y))/(4(t-s)e^{2Kt/3})} \left(\frac{1-e^{(2K/3)s}}{1-e^{(2K/3)t}}\right)^{\frac{N}{2}}.$$

Theorem 2.32 (Li–Yau gradient inequality [36; 51]) *Let (X, d, m) be an $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $f \in L^p(m)$ for some $p \in [1, \infty)$ be nonnegative. Then for every $t > 0$,*

$$|\nabla H_t f|^2 \leq e^{-2Kt/3} (\Delta H_t f) H_t f + \tfrac{1}{3} NK \frac{e^{-4Kt/3}}{1-e^{-2Kt/3}} (H_t f)^2 \quad m\text{-a.e.}$$

$\mathrm{RCD}(K, N)$ spaces also satisfy the parabolic maximum principle; see [48, Section 4.1; 55, Section 3] for full details.

Definition 2.33 [55, Definition 3.1] Let (X, d, m) be an $\mathrm{RCD}(K, N)$ space with $K \in \mathbb{R}$ and $N \in (1, \infty)$. Let I be an open interval in $\mathbb{R}$, Ω be an open subset of X, and $g \in L^2(\Omega)$. We say that a function $u\colon I \to W^{1,2}(\Omega)$ satisfies the parabolic equation

$$\frac{\partial}{\partial t} u - \Delta u \leq g \quad \text{weakly in } I \times \Omega,$$

if for every $t \in I$, the Fréchet derivative of u, denoted by $(\partial/\partial t)u$, exists in $L^2(\Omega)$ and for any nonnegative function $\psi \in W^{1,2}(\Omega)$,

$$\int_\Omega \frac{\partial}{\partial t} u(t,\cdot)\psi\, dm + \mathcal{E}(u(t,\cdot),\psi) \le \int_\Omega g\psi\, dm.$$

Theorem 2.34 (parabolic maximum principle [55, Lemma 3.2]) *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space with $K \in \mathbb{R}$ and $N \in (1,\infty)$. Fix $T \in (0,\infty]$ and open subset $\Omega \subseteq X$. Assume that a function $u\colon (0,T) \to W^{1,2}(\Omega)$, with $u_+(t,\cdot) = \max\{u(t,\cdot),0\} \in W^{1,2}(\Omega)$ for any $t \in (0,T)$, satisfies the following equation with initial value condition*:

$$\begin{cases} (\partial/\partial t)u - \Delta u \le 0 & \text{weakly in } (0,T)\times\Omega, \\ u_+(t,\cdot) \to 0 & \text{in } L^2(\Omega) \text{ as } t \to 0. \end{cases}$$

Then $u(t,x) \le 0$ for any t in $(0,T)$ and m-a.e. x in Ω.

2.10 Mean value and integral excess inequalities

We refer to [30] in the smooth case, and [61] in the RCD case, for the proofs of the statements in this subsection. We start with the existence of good cutoff functions.

Lemma 2.35 (existence of good cutoff functions [61, Lemma 3.1]) *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and $N \in [1,\infty)$. Then for every $x \subset X$, for every $R > 0$ and $0 < r < R$ there exists a Lipschitz function $\psi^r\colon X \to \mathbb{R}$ satisfying*

- (I) $0 \le \psi^r \le 1$ *on* X, $\psi^r \equiv 1$ *on* $B_r(x)$ *and* $\mathrm{supp}(\psi) \subset B_{2r}(x)$,
- (II) $r^2|\Delta\psi^r| + r|\nabla\psi^r| \le C(K,N,R)$ m*-a.e.*

For any subset C in a metric space, we denote by $T_r(C)$ the r-tubular neighborhood of C and for $r_1 > r_0 > 0$, we define $A_{r_0,r_1}(C) := T_{r_1}(C)\backslash\overline{T_{r_0}(C)}$ the (r_0,r_1)-annular neighborhood of C.

Lemma 2.36 (existence of good cutoff functions on annular neighborhoods [61, Lemma 3.2]) *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space for some $K \in \mathbb{R}$ and $N \in [1,\infty)$. Then for every closed subset $C \subset X$, for every $R > 0$ and $0 < 10r_0 < r_1 \le R$ there exists a Lipschitz function $\psi\colon X \to \mathbb{R}$ satisfying*

- (I) $0 \le \psi \le 1$ *on* X, $\psi \equiv 1$ *on* $A_{3r_0,r_1/3}(C)$ *and* $\mathrm{supp}(\psi) \subset A_{2r_0,r_1/2}(C)$,
- (II) $r_0^2|\Delta\psi| + r_0|\nabla\psi| \le C(K,N,R)$ m*-a.e. on* $A_{2r_0,3r_0}(C)$,
- (III) $r_1^2|\Delta\psi| + r_1|\nabla\psi| \le C(K,N,R)$ m*-a.e. on* $A_{r_1/3,r_1/2}(C)$.

Remark 2.37 The gradient bounds in Lemmas 2.35 and 2.36 are naturally m-a.e. However, since the proof involves only Theorem 2.16, by Remark 2.17, choosing continuous representatives and using the local Lipschitz constant of ψ for $|\nabla\psi|$, these statements can be made pointwise.

As demonstrated in [30], and later for the RCD setting in [61], several key estimates, including heat kernel bounds, the mean value, L^1-Harnack and integral Abresch–Gromoll inequalities can be proved starting from the existence of good cutoff functions and the Li–Yau Harnack inequality, Theorem 2.31.

Lemma 2.38 (heat kernel bounds [61, lemma 3.3]) *Let (X,d,m) be an* $\mathrm{RCD}(K,N)$ *space for some $K\in\mathbb{R}$ and $N\in(1,\infty)$, and let $H_t(x,y)$ be the heat kernel for some $x\in X$. Then for every $R>0$, for all $0<r<R$ and $t\le R^2$,*

(I) *if $y\in B_{10\sqrt{t}}(x)$, then* $C^{-1}(K,N,R)/m(B_{10\sqrt{t}}(x))\le H_t(x,y)\le C(K,N,R)/m(B_{10\sqrt{t}}(x))$,

(II) $\int_{X\setminus B_r(x)} H_t(x,y)\,dm(y)\le C(K,N,R)r^{-2}t$.

Lemma 2.39 (mean value and the L^1-Harnack inequality [61, Lemma 3.4]) *Let (X,d,m) be an* $\mathrm{RCD}(K,N)$ *space for some $K\in\mathbb{R}$, $N\in(1,\infty)$ and let $0<r<R$. If $u\colon X\times[0,r^2]\to\mathbb{R}$ given by $u(x,t)=u_t(x)$, is a nonnegative Borel function with compact support at each time t and satisfying $(\partial_t-\Delta)u\ge -c_0$ in the weak sense, then,*

$$\fint_{B_r(x)} u_0\le C(K,N,R)[u_{r^2}(x)+c_0r^2]\quad\text{for } m\text{-a.e. } x.$$

More generally the following L^1-Harnack inequality holds:

$$\fint_{B_r(x)} u_0\le C(K,N,R)\Big[\operatorname*{ess\,inf}_{y\in B_r(x)} u_{r^2}(y)+c_0r^2\Big]\quad\text{for all } x\in X.$$

Remark 2.40 Lemma 3.4 of [61], whose proof follows [30, Lemma 2.1], treats the continuous case of u. However, in what follows we will want to use this inequality for $u_t=|\nabla h_t(f)|$, which is not known to have a continuous representative. The proof of this statement for Borel u follows exactly as in the continuous case with the obvious measure-theoretic adjustments.

Applying Lemma 2.39 to a function which is constant in time gives the following classical mean value inequality:

Corollary 2.41 (classical mean value inequality [61, Corollary 3.5]) *Let (X,d,m) be an* $\mathrm{RCD}(K,N)$ *space for some $K\in\mathbb{R}$, $N\in(1,\infty)$ and let $0<r<R$. If $u\colon X\to\mathbb{R}$ is a nonnegative Borel function with compact support with $u\in D(\boldsymbol{\Delta})$ and satisfies $\boldsymbol{\Delta}u\le c_0m$ in the sense of measures, then for* $0<r\le R$,

$$\fint_{B_r(x)} u\le C(K,N,R)[u(x)+c_0r^2]\quad\text{for } m\text{-a.e. } x.$$

This, in combination with the existence of good cutoff functions and Laplacian estimates on distance functions, allows one to prove an integral Abresch–Gromoll inequality. For points p and q in a metric space, we define the excess function $e_{p,q}(x):=d(p,x)+d(x,q)-d(p,q)$.

Theorem 2.42 (integral Abresch–Gromoll inequality [61, Theorem 3.6]) *Suppose (X,d,m) is an* $\mathrm{RCD}(K,N)$ *space for some $K\in\mathbb{R}$ and $N\in(1,\infty)$; let $p,q\in X$ with $d_{p,q}:=d(p,q)\le 1$ and fix* $0<\epsilon<1$.

If $x\in A_{\epsilon d_{p,q},2d_{p,q}}(\{p,q\})$ satisfies $e_{p,q}(x)\le r^2d_{p,q}\le\bar r(K,N,\epsilon)^2d_{p,q}$, then

$$\fint_{B_{rd_{p,q}}(x)} e_{p,q}(y)\,dm(y)\le C(K,N,\epsilon)r^2d_{p,q}.$$

Combined with the Bishop–Gromov volume comparison, this immediately implies the classical Abresch–Gromoll inequality [1]:

Corollary 2.43 (classical Abresch–Gromoll inequality [61, Corollary 3.7]) *Let* (X,d,m) *be an* $\mathrm{RCD}(K,N)$ *space for some* $K\in\mathbb{R}$ *and* $N\in(1,\infty)$; *let* $p,q\in X$ *with* $d_{p,q}:=d(p,q)\le 1$ *and fix* $0<\epsilon<1$.

If $x\in A_{\epsilon d_{p,q},2d_{p,q}}(\{p,q\})$ *satisfies* $e_{p,q}(x)\le r^2 d_{p,q}\le \bar r(K,N,\epsilon)^2 d_{p,q}$, *then there exists* $\alpha(N)\in(0,1)$ *such that*

$$e_{p,q}(y)\le C(K,N,\epsilon)r^{1+\alpha(N)}d_{p,q}\quad \text{for all } y\in B_{rd_{p,q}}(x).$$

3 Differentiation formulas for regular Lagrangian flows

3.1 Regular Lagrangian flow

In what follows, we will always be on some $\mathrm{RCD}(K,N)$ space (X,d,m) for $K\in\mathbb{R}$ and $N\in[1,\infty)$. In [30], the crucial idea is to understand the geometric properties of the gradient flow with respect to heat flow approximations of the distance function. We will do the same with the regular Lagrangian flow, which was introduced for the metric measure setting of Ambrosio and Trevisan [10]; see also [2; 34] for the origins of the theory in $\mathbb{R}^d$. The setup is quite general and we refer to [10] for full details. We also refer to [14; 15; 16; 33] for regularity results related to or extending the results of this section.

We will be specifically interested in vector fields contained the L^2 tangent module of an RCD space.

Definition 3.1 (time-dependent L^2 vector fields) Let $T>0$. $V:[0,T]\to L^2(TX)$ is a time-dependent L^2 vector field if and only if, for any $f\in W^{1,2}(X)$, the map $(t,x)\mapsto df(V_t)(x)$ is measurable with respect to $\mathscr{L}^1\otimes\mathscr{B}(X)$, where $\mathscr{B}(X)$ is the Borel σ-algebra of X and $\mathscr{L}^1$ is the standard Lebesgue σ-algebra of $\mathbb{R}$.

V is bounded if and only if

$$\|V\|_{L^\infty}:=\||V|\|_{L^\infty([0,T]\times X)}<\infty.$$

$V\in L^1([0,T],L^2(TX))$ if and only if

$$\int_0^T\|V_t\|_{L^2(TX)}\,dt<\infty.$$

Definition 3.2 (regular Lagrangian flow) Given a time-dependent L^2 vector field (V_t), a Borel map $F:[0,T]\times X\to X$ is a *regular Lagrangian flow* (*RLF*) to V_t if and only if the following hold:

(I) $F_0(x)=x$ and $[0,T]\ni t\mapsto F_t(x)$ is continuous for every $x\in X$.

(II) For every $f\in\mathrm{TestF}(X)$ and m-a.e. $x\in X$, $t\mapsto f(F_t(x))$ is in $W^{1,1}([0,T])$ and

$$\frac{d}{dt}f(F_t(x))=df(V_t)(F_t(x))\text{ for a.e. } t\in[0,T]. \tag{16}$$

(III) There exists a constant $C:=C(V)$ such that $(F_t)_*m\le Cm$ for all t in $[0,T]$.

Remark 3.3 In the case where $V \in L^1([0,T], L^2(TX))$ and F is an RLF of V, using a standard Fubini's theorem argument and that TestF(X) is dense in $W^{1,2}(X)$, we have for every $f \in W^{1,2}_{\mathrm{loc}}(X)$, $t \mapsto f(F_t(x))$ is in $W^{1,1}([0,T])$ and

$$\frac{d}{dt} f(F_t(x)) = df(V_t)(F_t(x)) \quad \text{for a.e. } t \in [0,T]. \tag{17}$$

The existence and uniqueness of RLFs to (V_t) in a certain class of vector fields follows from [10]. We use the following weaker formulation of their result and note that only a bound on the symmetric part of ∇V_t is needed.

Theorem 3.4 (existence and uniqueness of regular Lagrangian flow [10]) *Let $(V_t) \in L^1([0,T], L^2(TX))$ satisfy $V_t \in D(\mathrm{div})$ for a.e. $t \in [0,T]$ with*

$$\mathrm{div}(V_t) \in L^1([0,T], L^2(m)), \quad (\mathrm{div}(V_t))^- \in L^1([0,T], L^\infty(m)), \quad \nabla V_t \in L^1([0,T], L^2(T^{\otimes 2}X)).$$

There exists a unique, up to m-a.e. equality, RLF $(F_t)_{t\in[0,T]}$ for (V_t). The bound

$$(F_t)_*(m) \le \exp\biggl(\int_0^t \|\mathrm{div}(V_s)^-\|_{L^\infty(m)}\, ds\biggr) m \tag{18}$$

holds for every $t \in [0,T]$.

Remark 3.5 It was pointed out to the author by Nicola Gigli that the estimate (18) can be localized for any $S \in \mathscr{B}(X)$:

$$(F_t)_*(m|_S) \le \exp\biggl(\int_0^t \|\mathrm{div}(V_s)^-\|_{L^\infty((F_s)_*(m|_S))}\, ds\biggr) m.$$

This follows from [10, (4-22)] choosing $\beta(z) := z^p$ for $p \to \infty$. See [47, Proposition 5.3] for a precise argument.

For (F_t) an RLF to some (V_t), we will be interested in expressing $(d/dt)\, d(F_t(x), F_t(y))$ in two ways: using V_t in a first-order variation formula and ∇V_t in a second-order formula (see (8)) which we show in Section 3.3.

Proposition 3.6 (first-order differentiation formula along RLFs) *Suppose that $T > 0$ and $U, V \in L^1([0,T], L^2(TX))$. If (F_t) and (G_t) are the regular Lagrangian flows of (U_t) and (V_t), respectively, then for $(m\times m)$-a.e. $(x,y) \in X \times X$, we have $d(F_t(x), G_t(y)) \in W^{1,1}([0,T])$ and*

$$\frac{d}{dt} d(F_t(x), G_t(y)) = \langle \nabla d_{G_t(y)}, U_t\rangle(F_t(x)) + \langle \nabla d_{F_t(x)}, V_t\rangle(G_t(y)) \quad \text{for a.e. } t \in [0,T].$$

Proof It is known that $\mathrm{RCD}(K,\infty)$ spaces have the tensorization of Cheeger energy property from [7, Theorem 6.17] and the density of the product algebra property from [15, Proposition A.1]; see also [45, Definition 3.8 and 3.9] for definitions. Consider the vector field (W_t) defined by requiring, for all $f \in W^{1,2}(X \times X)$,

$$\langle W_t, \nabla f\rangle(x,y) = \langle U_t, \nabla f_y\rangle(x) + \langle V_t, \nabla f_x\rangle(y),$$

for $(m\times m)$-a.e. $(x,y)\in X\times X$. The tensorization of Cheeger energy is used implicitly in this definition and the vector field is naturally in $L^1([0,T],L^2_{\mathrm{loc}}(T(X\times X)))$. We refer to [45] for a rigorous treatment of locally L^2 vector fields and the corresponding theory of RLFs. We mention a slightly more careful, alternative definition of (W_t) was also given in [45, Proposition 3.7, Theorem 3.13], where the expected decomposition of the module $L^0(T^*(X\times X))$ was shown for spaces with tensorization of Cheeger energy and density of the product algebra properties. By [15, Proposition A.2], (F_t,G_t) is an RLF of (W_t), from which the proposition follows by definition of (W_t). □

3.2 Continuity equation

We give a brief summary of the theory of continuity equations in this section. These are intimately related to regular Lagrangian flows but provide a more convenient language for the discussion of local flows in cases where RLFs, which as defined are of a global nature, may not exist.

Definition 3.7 (curves of bounded compression [39, Definition 2.3.21]) We say a curve $(\mu_t)_{t\in[0,T]}\subseteq\mathcal{P}_2(X)$ is a curve of bounded compression if and only if

(I) it is W_2-continuous,

(II) for some $C>0$, $\mu_t\le Cm$ for every $t\in[0,T]$.

Definition 3.8 (solutions of continuity equation [39, Definition 2.3.22; 40]) Let $(\mu_t)_{t\in[0,T]}\subseteq\mathcal{P}_2(X)$ be a curve of bounded compression and $V\in L^1([0,T],L^2(TX))$. We say that (μ_t,V_t) solves the continuity equation

$$\frac{d}{dt}\mu_t+\operatorname{div}(V_t\mu_t)=0$$

if and only if, for every $f\in W^{1,2}(X)$, the map $t\mapsto\int f\,d\mu_t$ is absolutely continuous and satisfies

$$\frac{d}{dt}\int f\,d\mu_t=\int df(V_t)\,d\mu_t\quad\text{for a.e. } t\in[0,T]. \tag{19}$$

Remark 3.9 By abuse of notation we will sometimes say (μ_t) solves the continuity equation

$$(d/dt)\mu_t+\operatorname{div}(V_t\mu_t)=0$$

for some vector field (V_t) which is only locally L^2, for example, $V_t=-\nabla d_p$ for some $p\in X$. In this case, (μ_t) is always compactly supported for every t and it is understood that we cut off the vector field V_t outside of this support.

As shown in [10], RLFs are very closely related to the solutions of continuity equations (see [46, Theorem 1.4] for the sense in which we mean "a solution of continuity equation"); they can be thought of as realizations of these solutions as maps on the space itself.

Theorem 3.10 [10] *Let (V_t) satisfy the conditions of Theorem 3.4 and (F_t) be the corresponding unique regular Lagrangian flow. If $\mu_0\in\mathcal{P}_2(X)$ with bounded density, then $\mu_t:=(F_t)_*(\mu_0)$ is a solution of the continuity equation $(d/dt)\mu_t+\operatorname{div}(V_t\mu_t)=0$.*

Remark 3.11 The existence and uniqueness of solutions to $(d/dt)\mu_t + \operatorname{div}(V_t\mu_t) = 0$ starting at some μ_0 of bounded density is proved in [10] for (V_t) satisfying the conditions of Theorem 3.4. In fact, the existence and uniqueness of RLFs in Theorem 3.4 is shown in part by using the existence and uniqueness on the level of continuity equations combined with a superposition principle.

RLFs from vector fields with a two-sided divergence bound are m-a.e. invertible. To be precise:

Proposition 3.12 *Let $(V_t) \in L^1([0,T], L^2(TX))$ satisfy $V_t \in D(\operatorname{div})$ for a.e. $t \in [0,T]$ with*

$$\operatorname{div}(V_t) \in L^1([0,T], L^2(m)), \quad \operatorname{div}(V_t) \in L^1([0,T], L^\infty(m)), \quad \nabla V_t \in L^1([0,T], L^2(T^{\otimes 2}X)).$$

Let (F_t) be the unique RLF of $(V_t)_{t\in[0,T]}$ and (G_t) be the unique RLF of $(-V_{T-t})_{t\in[0,T]}$. For m-a.e. $x \in X$ and any $0 \le t \le T$,

$$G_t(F_T(x)) = F_{T-t}(x).$$

Proof We first show $G_T(F_T(x)) = x$ for m-a.e. $x \in X$. Define the time-dependent L^2 vector field $(W_t)_{t\in[0,2T]}$ by

$$W_t := \begin{cases} V_t & \text{if } 0 \le t \le T, \\ -V_{2T-t} & \text{if } T < t \le 2T. \end{cases}$$

For any μ with compact support and bounded density, $(\mu_t)_{t\in[0,2T]}$ defined by

$$\mu_t := \begin{cases} (F_t)_*(\mu) & \text{if } 0 \le t \le T, \\ (G_{t-T})_*((F_T)_*(\mu)) & \text{if } T < t \le 2T, \end{cases}$$

solves the continuity equation $(d/dt)\mu_t + \operatorname{div}(W_t\mu_t) = 0$ by Theorem 3.10. This in particular means $(\mu_t)_{t\in[0,T]}$ solves the continuity equation $(d/dt)\mu_t + \operatorname{div}(V_t\mu_t) = 0$ on $[0,T]$. It is then easy to check by definition that

$$\mu'_t := \begin{cases} \mu_t & \text{if } 0 \le t \le T, \\ \mu_{2T-t} & \text{if } T < t \le 2T, \end{cases}$$

solves the continuity equation $(d/dt)\mu'_t + \operatorname{div}(W_t\mu'_t) = 0$ as well. By uniqueness (see Remark 3.11) $(G_T)_*((F_T)_*(\mu)) = \mu$. Since this is true for any μ with compact support and bounded density, we conclude $G_T(F_T(x)) = x$ for m-a.e. $x \in X$.

By the same argument for each t in the countable set $\mathbb{Q} \cap [0,T]$, for m-a.e. $x \in X$ and any $t \in \mathbb{Q} \cap [0,T]$,

$$G_t(F_T(x)) = F_{T-t}(x).$$

The proposition follows by continuity of $G_t(x)$ and $F_t(x)$ in t for all $x \in X$. □

We recall the following result from [37] (we state an equivalent version below from [46]) which in particular implies W_2-geodesics with uniformly bounded densities are solutions of continuity equations:

Theorem 3.13 [46, Theorem 1.1] *Let μ_t be a W_2-geodesic with compact support and $\mu_t \le Cm$ for every $t \in [0,1]$ and some $C > 0$. If $f \in W^{1,2}(X)$ then the map $[0,1] \ni t \mapsto \int f\,d\mu_t$ is $C^1([0,1])$ and*

$$\frac{d}{dt}\int f\,d\mu_t = \int \langle \nabla f, \nabla \phi_t\rangle\,d\mu_t,$$

where ϕ_t is any function such that for some $s \ne t$ with $s \in [0,1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from μ_t to μ_s.

The corollary below then follows by making the same type of arguments as in [10, Section 7]:

Corollary 3.14 *Let $p \in X$ and $f \in W^{1,2}(X)$ fixing a representative. For m-a.e. $x \in X$, the map $t \mapsto f(\gamma_{x,p}(t))$ is in $W^{1,1}_{\mathrm{loc}}([0,d_{x,p}))$ and*

$$\frac{d}{dt} f(\gamma_{x,p}(t)) = -df(\nabla d_p)(\gamma_{x,p}(t)) \quad \text{for a.e. } t \in [0, d_{x,p}),$$

where $\gamma_{x,p}$ is a unit-speed geodesic from x to p.

Proof By Remark 2.26, we take a Borel selection of $\gamma_{x,p}$ which is unique for m-a.e. x. For each $x \in X$, let $\tilde\gamma_{x,p}\colon [0,1] \to X$ be the constant-speed reparametrization of γ.

First consider a Lipschitz representative of $f \in \mathrm{TestF}(X)$. Clearly $f(\tilde\gamma_{x,p}(t))$ is continuous on $t \in [0,1]$ for each x. We show

(I) for m-a.e. x, $(-d(\tilde\gamma_{x,p}(t),p)/(1-t))\langle \nabla f, \nabla d_p\rangle(\tilde\gamma_{x,p}(t)) \in L^1_{\mathrm{loc}}([0,1))$,

(II) for m-a.e. x, $f(\tilde\gamma_{x,p}(b)) - f(\tilde\gamma_{x,p}(a)) = \int_a^b (-d(\tilde\gamma_{x,p}(t),p)/(1-t))\langle \nabla f, \nabla d_p\rangle(\tilde\gamma_{x,p}(t))\,dt$ for any $0 \le a < b < 1$.

For any μ with compact support and bounded density with respect to m, define $\mu_t := (\tilde\gamma_{\cdot,p}(t))_*(\mu)$ for $t \in [0,1]$. Then $(\mu_t)_{t\in[0,1]}$ is a W_2-geodesic. By Theorem 2.25, for any $\delta > 0$, $(\mu_t)_{t\in[0,1-\delta]}$ is of uniformly bounded density. By Theorem 3.13, the map $[0,1-\delta] \ni t \mapsto \int f\,d\mu_t$ is in $C^1([0,1-\delta])$ and

$$\frac{d}{dt}\int f\,d\mu_t = \int \frac{-d(x,p)}{1-t}\langle \nabla f, \nabla d_p\rangle(x)\,d\mu_t(x). \tag{20}$$

Fix a representative of $\langle \nabla f, \nabla d_p\rangle \in L^2(m)$.

(I) The map $t \mapsto \int (-d(x,p)/(1-t))\langle \nabla f, \nabla d_p\rangle(x)\,d\mu_t(x)$ is in $L^1_{\mathrm{loc}}([0,1))$ since $\langle \nabla f, \nabla d_p\rangle(x) \in L^1_{\mathrm{loc}}(m)$ and μ_t has uniformly bounded support and locally uniformly bounded density on $[0,1)$. By Fubini's theorem, this implies for μ-a.e. x, $(-d(\tilde\gamma_{x,p}(t),p)/(1-t))\langle \nabla f, \nabla d_p\rangle(\tilde\gamma_{x,p}(t)) \in L^1_{\mathrm{loc}}([0,1))$. Since this is true starting at any measure μ with compact support and bounded density with respect to m, (I) follows.

(II) By continuity of $f(\tilde\gamma_{x,p}(t))$ in t, it is enough to show for m-a.e. x,

$$f\Big(\tilde\gamma_{x,p}\Big(\frac{k}{n}\Big)\Big) - f\Big(\tilde\gamma_{x,p}\Big(\frac{k-1}{n}\Big)\Big) = \int_{(k-1)/n}^{k/n} \frac{-d(\tilde\gamma_{x,p}(t),p)}{1-t}\langle \nabla f, \nabla d_p\rangle(\tilde\gamma_{x,p}(t))\,dt,$$

for any $n \in \mathbb{N}$ and $1 \le k \le n-1$. Assume this is not the case. Then there exist some n, k and a bounded set S with $0 < m(S) < \infty$ such that for each $x \in S$, without loss of generality,

$$f\Big(\tilde{\gamma}_{x,p}\Big(\frac{k}{n}\Big)\Big) - f\Big(\tilde{\gamma}_{x,p}\Big(\frac{k-1}{n}\Big)\Big) > \int_{(k-1)/n}^{k/n} \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} \langle \nabla f, \nabla d_p\rangle(\tilde{\gamma}_{x,p}(t))\,dt.$$

Applying (20) to the Wasserstein geodesic from the normalization of $m|_S$ to δ_p gives a contradiction.

The general case of $f \in W^{1,2}(X)$ then follows by an approximation argument. Choose a sequence $f_i \in \mathrm{TestF}(X)$ converging to f in $W^{1,2}(X)$. Using a standard diagonalization argument with the Borel–Cantelli lemma and Fubini's theorem, there exists some subsequence f_i such that for m-a.e. $x \in X$, $f_i(\tilde{\gamma}_{x,p}(t)) \to f(\tilde{\gamma}_{x,p}(t))$ in $L^1_{\mathrm{loc}}([0,1))$ as functions of t. Slightly more precisely, from the $W^{1,2}(X)$ convergence and Fubini's theorem, for each $n \in \mathbb{N}$, we can find some i_n such that if we define

$$S_n := \Big\{x \in X : \|f \circ \tilde{\gamma}_{x,p} - f_{i_n} \circ \tilde{\gamma}_{x,p}\|_{L^1([0,1-1/n])} \ge \frac{1}{n}\Big\},$$

we have that

$$m(S_n) < \frac{1}{n^2}.$$

The Borel–Cantelli lemma then allows us to conclude that f_{i_n} is a subsequence satisfying the above assertion.

For any μ with compact support and bounded density, and μ_t defined as before, we also have

$$\int \frac{-d(x,p)}{1-t} \langle \nabla f_i, \nabla d_p\rangle(x)\,d\mu_t(x) \to \int \frac{-d(x,p)}{1-t} \langle \nabla f, \nabla d_p\rangle(x)\,d\mu_t(x) \quad \text{in } L^1_{\mathrm{loc}}([0,1)).$$

Another diagonalization argument with the Borel–Cantelli lemma and Fubini's theorem gives a further subsequence f_i such that for m-a.e. $x \in X$,

$$\frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} \langle \nabla f_i, \nabla d_p\rangle(\tilde{\gamma}_{x,p}(t)) \to \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} \langle \nabla f, \nabla d_p\rangle(\tilde{\gamma}_{x,p}(t)) \quad \text{in } L^1_{\mathrm{loc}}([0,1)).$$

Combining these with (I) and (II), we have for any $f \in W^{1,2}(X)$, for m-a.e. $x \in X$, the map $t \mapsto f(\tilde{\gamma}_{x,p}(t))$ is in $W^{1,1}_{\mathrm{loc}}([0,1))$ and

$$\frac{d}{dt} f(\tilde{\gamma}_{x,p}(t)) = \frac{-d(\tilde{\gamma}_{x,p}(t),p)}{1-t} df(\nabla d_p)(\tilde{\gamma}_{x,p}(t)) \quad \text{for a.e. } t \in [0,1).$$

The corollary then follows by a reparametrization of $\tilde{\gamma}_{x,p}$. □

We will be particularly interested in the following type of object: Let $p \in X$ and $\mu \in \mathcal{P}_2(X)$ be of bounded density with respect to m. Take a Borel selection (Remark 2.26) of unit-speed geodesics $\gamma_{x,p}$ from all $x \in X$ to p and define $T := d(\mathrm{supp}(\mu), p)$. For $0 \le t \le T$, define $\mu_t := (\gamma_{\cdot,p}(t))_*(\mu)$. The (μ_t) defined this way are more naturally considered L^1-Wasserstein geodesics and are well studied in the theory of needle decomposition of RCD spaces; see [13; 20; 23]. We record some properties of these objects which will be needed later.

Theorem 3.15 *Let $0 < \delta < T$ and $\mu \le Am$ for some $A > 0$. Let $(\mu_t)_{t\in[0,T-\delta]}$ be as defined in the previous paragraph and $D := \sup_{x\in\operatorname{supp}(\mu)} d(x,p) \le \overline{D}$. Then:*

(I) *$(\mu_t)_{t\in[0,T-\delta]}$ is a W_2-geodesic.*

(II) *There exists $C(K,N,\overline{D},\delta)$ such that $\mu_t \le A(1+Ct)^N m$ for all $t \in [0,T-\delta]$. In particular, the densities of $(\mu_t)_{t\in[0,T-\delta]}$ are uniformly bounded with respect to m.*

(III) *$(\mu_t)_{t\in[0,T-\delta]}$ solves the continuity equation*

$$\frac{d}{dt}\mu_t + \operatorname{div}(-\nabla d_p \mu_t) = 0.$$

Proof Statement (I) is proved in [20, Lemma 4.4], where d-monotonicity is used to show d^2-monotonicity of $\{(x,\gamma_{x,p}(T-\delta)) : x \in \operatorname{supp}\mu\}$. Statement (II) is proved in [13, Section 9]; see also [24, Section 3.2] for a discussion. Statements (I) and (II) give that (μ_t) is of bounded compression. Statement (III) then follows from Corollary 3.14. □

In order to have terminology which includes the globally defined RLFs as well as the type of locally defined flows such as the example above, we will use the following definition:

Definition 3.16 Let $S \in \mathscr{B}(X)$ with $m(S) > 0$ and $(V_t)_{t\in[0,T]} \in L^1([0,T], L^2(TX))$. A Borel map $F : [0,T]\times S \to X$ is a *local flow of V_t from S* if

(I) $F_0(x) = x$ and $[0,T] \ni t \mapsto F_t(x)$ is continuous for every $x \in S$,

(II) for every $f \in \operatorname{TestF}(X)$ and m-a.e. $x \in S$, $t \mapsto f(F_t(x))$ is in $W^{1,1}([0,T])$ and

$$\frac{d}{dt} f(F_t(x)) = df(V_t)(F_t(x)) \quad \text{for a.e. } t \in [0,T], \tag{21}$$

(III) there exists a constant $C := C(V,S)$ such that $(F_t)_*(m|_S) \le Cm$ for all t in $[0,T]$.

As before, by abuse of notation we will often say (F_t) is the local flow of (V_t) from S for some vector field V which is only locally L^2. In this case, it is understood that $F_t(S)$ is essentially bounded for each t and we cut off V_t outside of this region.

Remark 3.17 We will be primarily interested in the following examples:

(I) For any $p \in X$ and bounded $S \in \mathscr{B}(X)$, $F_t(x) := \gamma_{x,p}(t)$ defined on $(t,x) \in [0,T-\delta]\times S$, where $T := \operatorname{ess\,inf}_{x\in S} d(x,p)$, $\delta > 0$, and $\gamma_{x,p}$, a unit-speed geodesic from x to p, is Borel selected (see Remark 2.26), is a local flow of $-\nabla d_p$ from S by Corollary 3.14 and Theorem 3.15.

(II) The restriction of any RLF onto some $S \in \mathscr{B}(X)$ is a local flow of the corresponding (V_t) from S by definition.

The same argument as for RLFs in Proposition 3.6 gives the following differentiation formula:

Proposition 3.18 (first-order differentiation formula for distance along local flows) *Let $T > 0$. If $(F_t)_{t\in[0,T]}$ and $(G_t)_{t\in[0,T]}$ are local flows of (U_t) and (V_t) from S_1 and S_2, respectively, then for $(m\times m)$-a.e. $(x,y)\in S_1\times S_2$, we have $d(F_t(x),G_t(y))\in W^{1,1}([0,T])$ and*

$$\frac{d}{dt}d(F_t(x),G_t(y)) = \langle \nabla d_{G_t(y)}, U_t\rangle(F_t(x)) + \langle \nabla d_{F_t(x)}, V_t\rangle(G_t(y)) \quad \text{for a.e. } t\in[0,T].$$

We mention that if (W_t) is defined as in Proposition 3.6 from (U_t) and (V_t), then it is straightforward to check using the arguments of [15] that (F_t, G_t) is a local flow of (W_t) from $S_1\times S_2$. Again, (W_t) here naturally belongs in $L^1([0,T], L^2_{\text{loc}}(T(X\times X)))$ so Definition 3.16 needs to be altered to allow for this. We refer to [45] for relevant definitions.

The next proposition gives control on the metric speeds of the curves $t\mapsto F_t(x)$ of a local flow F. As pointed out in [46, (A.22)], it follows from a similar argument as in [39, Theorem 2.3.18] after a small adjustment since we do not a priori assume the absolute continuity of the curves $F_\cdot(x)$.

Proposition 3.19 *Let $V\in L^1([0,T], L^2(TX))$ and let (F_t) be a local flow of (V_t) from S. For m-a.e. $x\in S$ the curve $t\mapsto F_t(x)$ is absolutely continuous and its metric speed $ms_t(F_\cdot(x))$ at time t satisfies*

$$ms_t(F_\cdot(x)) = |V_t|(F_t(x)) \quad \text{for a.e. } t\in[0,T].$$

3.3 Second-order interpolation formula

The proof of a second-order interpolation formula for the distance function (see (8) and (5)) along flows requires the results of [46]. The hard work is done there and their result immediately implies an analogous second-order interpolation formula (Theorem 3.20) for the Wasserstein distance between two solutions of the continuity equation. It is our goal to pass this formula from Wasserstein distance to distance on the space itself.

For the rest of the subsection we will always be in the setting of some RCD(K,N) space (X,d,m) with $K\in\mathbb{R}$ and $N\in[1,\infty)$. We fix a Borel selection (see Remark 2.26) of constant-speed geodesics $\tilde\gamma_{x,y}$ from all $x\in X$ to all $y\in X$ parametrized on the unit interval. We denote by $\gamma_{x,y}$ the unit-speed reparametrization of $\tilde\gamma_{x,y}$ to $[0,d(x,y)]$. We start with the following formulation of the main result from [46]:

Theorem 3.20 [46, Theorem 5.13] *Let $\mu_0,\mu_1\in\mathcal{P}_2(X)$ be compactly supported and satisfy $\mu_0,\mu_1\le Cm$ for some $C>0$. Let (μ_t) be the unique W_2-geodesic connecting μ_0 to μ_1. For every $t\in[0,1]$, let ϕ_t be any function such that for some $s\neq t$ with $s\in[0,1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from μ_t to μ_s. For any $V\in H^{1,2}_C(TX)$, the map $[0,1]\ni t\mapsto \int\langle V,\nabla\phi_t\rangle\,d\mu_t$ is in $C^1([0,1])$ and*

$$\frac{d}{dt}\left(\int \langle V,\nabla\phi_t\rangle\,d\mu_t\right) = \int (\nabla V : (\nabla\phi_t\otimes\nabla\phi_t))\,d\mu_t \quad \text{for all } t\in[0,1]. \tag{22}$$

The next lemma follows from the previous theorem:

Lemma 3.21 *Let $p \in X$ and $\nu \le Cm$ be a nonnegative compactly supported measure. For any $V \in H^{1,2}_C(TX)$,*

$$\int \langle V, \nabla d_p\rangle(x)\, d\nu(x) - \int \langle V, \nabla d_p\rangle\big(\tilde{\gamma}_{p,x}\big(\tfrac{1}{2}\big)\big)\, d\nu(x) = \int_{1/2}^{1} \Big(\int d(p,x)(\nabla V : (\nabla d_p \otimes \nabla d_p))(\tilde{\gamma}_{p,x}(t))\, d\nu(x) \Big)\, dt, \tag{23}$$

where $\tilde{\gamma}_{x,y}$ is as defined in the beginning of this subsection.

Note that although $\nabla d_p \otimes \nabla d_p$ is not in $L^2(T^{\otimes 2}(X))$, it is locally (ie it is after multiplication by the characteristic function of any compact set). Therefore, by the locality properties of the objects involved, $\nabla V : (\nabla d_p \otimes \nabla d_p)$ is well defined and is in $L^2(m)$.

Proof Fix representatives for $\langle V, \nabla d_p\rangle$ and $\nabla V : (\nabla d_p \otimes \nabla d_p) \in L^2(m)$.

We claim for m-a.e. $x \in X$,

$$\langle V, \nabla d_p\rangle(x) - \langle V, \nabla d_p\rangle\big(\tilde{\gamma}_{p,x}\big(\tfrac{1}{2}\big)\big) = \int_{1/2}^{1} d(p,x)(\nabla V : (\nabla d_p \otimes \nabla d_p))(\tilde{\gamma}_{p,x}(t))\, dt. \tag{24}$$

The right side integral is finite for m-a.e. x by using a Fubini's theorem argument along with (15) in Theorem 2.25.

Suppose (24) does not hold for m-a.e. $x \in X$. Then without loss of generality we may assume there exists a bounded set S with $0 < m(S) < \infty$ such that

$$\langle V, \nabla d_p\rangle(x) - \langle V, \nabla d_p\rangle\big(\tilde{\gamma}_{p,x}\big(\tfrac{1}{2}\big)\big) > \int_{1/2}^{1} d(p,x)(\nabla V : (\nabla d_p \otimes \nabla d_p))(\tilde{\gamma}_{p,x}(t))\, dt \tag{25}$$

for each $x \in S$. Let $\mu := (1/m(S))m|_S$. Multiplying both sides of (25) by $d(p,x)$ and integrating with respect to μ, we immediately contradict Theorem 3.20. Therefore (24) holds and so the lemma follows for any ν with compact support and bounded density. □

To proceed we state the segment inequality for $L^1(m)$ functions, first introduced by Cheeger and Colding [26, Theorem 2.11]. This has been established for the metric measure setting in [67] but we will give a self-contained proof since the decomposition procedure for the family of geodesics used in the proof will be used again in the proof of Proposition 3.23.

Theorem 3.22 (segment inequality for L^1 functions on RCD spaces) *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space with $K \in \mathbb{R}$ and $N \in [1,\infty)$. Let $\bar{R} > 0$. Let $f \in L^1_{\mathrm{loc}}(X)$ be nonnegative and $\mu \le A(m \times m)$ be a nonnegative measure on $X \times X$ supported on $B_R(p) \times B_R(p)$ for some $0 < R \le \bar{R}$ and $p \in X$. Then*

$$\int_0^1 \Big(\int f(\tilde{\gamma}_{x,y}(t)) d(x,y)\, d\mu(x,y) \Big)\, dt \le \mathrm{AC}(K,N,\bar{R}) R\big[m\big(\pi_1(\mathrm{supp}(\mu))\big) + m\big(\pi_2(\mathrm{supp}(\mu))\big)\big] \int_{B_{2R}(p)} f(z)\, dm(z), \tag{26}$$

where π_1 and π_2 are projections onto the first and the second coordinates, respectively, and $\tilde{\gamma}_{x,y}$ is as defined in the beginning of this subsection.

Proof The Radon–Nikodym theorem implies that $\mu = g(m\times m)$ for some compactly supported $g \in L^\infty(X\times X, m\times m)$. We define $g^1_x(\cdot) := g(x,\cdot)$ and $g^2_y(\cdot) := g(\cdot\,, y)$. By Fubini's theorem, $\|g^1_x\|_{L^\infty} \le A$ for m-a.e. x and similarly, $\|g^2_y\|_{L^\infty} \le A$ for m-a.e. y.

For each $t \in \left[\frac12, 1\right]$ and m-a.e. $x \in \pi_1(\operatorname{supp}(\mu))$, by (15) in Theorem 2.25,

$$(\tilde\gamma_{x,\cdot}(t))_*(g^1_x m) \le \mathrm{AC}(K,N,\bar R) m|_{B_{2R}(p)}. \tag{27}$$

Similarly for $t \in \left[0, \frac12\right]$ and m-a.e. $y \in \pi_2(\operatorname{supp}(\mu))$,

$$(\tilde\gamma_{\cdot,y}(t))_*(g^2_y m) \le \mathrm{AC}(K,N,\bar R) m|_{B_{2R}(p)}. \tag{28}$$

We conclude

$$\begin{aligned}
&\int_0^1\Big(\int f(\tilde\gamma_{x,y}(t))\, d(x,y)\, d\mu(x,y)\Big)\, dt \\
&\qquad\le 2R\int_0^1\Big(\int f(\tilde\gamma_{x,y}(t))\, d\mu(x,y)\Big)\, dt \\
&\qquad= 2R\Big(\int_{1/2}^1\Big(\int f(\tilde\gamma_{x,y}(t))\, d\mu(x,y)\Big)\, dt + \int_0^{1/2}\Big(\int f(\tilde\gamma_{x,y}(t))\, d\mu(x,y)\Big)\, dt\Big) \\
&\qquad\le \mathrm{AC}(K,N,\bar R) R\big[m\big(\pi_1(\operatorname{supp}(\mu))\big) + m\big(\pi_2(\operatorname{supp}(\mu))\big)\big]\int_{B_{2R}(p)} f(z)\, dm(z),
\end{aligned}$$

where (27), (28) and Fubini's theorem were used in the last line. □

We now prove the main interpolation formula of this subsection. The main idea is to use Lemma 3.21 along with the decomposition procedure from the proof of the segment inequality. Notice the right side of (29) makes sense due to the segment inequality.

Proposition 3.23 (second-order interpolation formula) *Let $\mu \le C(m\times m)$ be a nonnegative and compactly supported measure on $X\times X$. Let $V \in H^{1,2}_C(TX)$. Then*

$$\begin{aligned}
\int (\langle V, \nabla d_x\rangle(y) + \langle V, \nabla d_y\rangle(x))\, d\mu(x,y) \\
= \int_0^1\int d(x,y)(\nabla V : \nabla d_x\otimes\nabla d_x)(\tilde\gamma_{x,y}(t))\, d\mu(x,y)\, dt,
\end{aligned}\tag{29}$$

where $\tilde\gamma_{x,y}$ is as defined in the beginning of this subsection.

Proof Let $\mu = g(m\times m)$ for compactly supported $g \in L^\infty(X\times X, m\times m)$ with $g^1_x(\cdot) := g(x,\cdot)$ and $g^2_y(\cdot) := g(\cdot\,, y)$. Then $\|g^1_x\|_{L^\infty} \le C$ for m-a.e. x and $\|g^2_y\|_{L^\infty} \le C$ for m-a.e. y. We have

$$\begin{aligned}
&\int \langle V, \nabla d_x\rangle(y)\, d\mu(x,y) - \int \langle V, \nabla d_x\rangle\big(\tilde\gamma_{x,y}\big(\tfrac12\big)\big)\, d\mu(x,y) \\
&\qquad= \int\int \Big(\langle V, \nabla d_x\rangle(y) - \langle V, \nabla d_x\rangle\big(\tilde\gamma_{x,y}\big(\tfrac12\big)\big)\Big)\, d(g^1_x m)(y)\, dm(x) \\
&\qquad= \int\int_{1/2}^1\Big(\int d(x,y)(\nabla V : (\nabla d_x\otimes\nabla d_x))(\tilde\gamma_{x,y}(t))\, d(g^1_x m)(y)\Big)\, dt\, dm(x) \\
&\qquad= \int_{1/2}^1\Big(\int \big(d(x,y)(\nabla V : (\nabla d_x\otimes\nabla d_x))(\tilde\gamma_{x,y}(t))\big)\, d\mu(x,y)\Big)\, dt,
\end{aligned}\tag{30}$$

where the first and third equalities follow by Fubini's theorem, and the second by Lemma 3.21. Making the analogous argument on $t \in \left[0, \frac{1}{2}\right]$, we obtain

$$\int \langle V, \nabla d_y \rangle(x)\, d\mu(x,y) - \int \langle V, \nabla d_y \rangle\big(\tilde{\gamma}_{x,y}\big(\tfrac{1}{2}\big)\big)\, d\mu(x,y) = \int_0^{1/2} \Big(\int \big(d(x,y)(\nabla V : (\nabla d_y \otimes \nabla d_y))(\tilde{\gamma}_{x,y}(t)) \big)\, d\mu(x,y) \Big)\, dt. \tag{31}$$

The claim then follows if

$$\int \langle V, \nabla d_x \rangle\big(\tilde{\gamma}_{x,y}\big(\tfrac{1}{2}\big)\big)\, d\mu(x,y) + \int \langle V, \nabla d_y \rangle\big(\tilde{\gamma}_{x,y}\big(\tfrac{1}{2}\big)\big)\, d\mu(x,y) = 0 \tag{32}$$

and

$$\int_0^{1/2} \Big(\int \big(d(x,y)(\nabla V : (\nabla d_y \otimes \nabla d_y))(\tilde{\gamma}_{x,y}(t)) \big)\, d\mu(x,y) \Big)\, dt = \int_0^{1/2} \Big(\int \big(d(x,y)(\nabla V : (\nabla d_x \otimes \nabla d_x))(\tilde{\gamma}_{x,y}(t)) \big)\, d\mu(x,y) \Big)\, dt, \tag{33}$$

which we show in Lemma 3.25 and Remark 3.26. □

The following two lemmas show that, in a measure-theoretic sense, ∇d_p and ∇d_q in the interior of a geodesic between p and q "point in opposite directions".

Lemma 3.24 *Let $p, q \in X$. Let $\gamma_{p,q} \colon [0,1] \to X$ be a constant-speed geodesic from p to q and let $z = \gamma(t_0)$ for some $t_0 \in (0,1)$. Let f be a locally Lipschitz function. Then $\operatorname{lip}(f + d_p)(z) = \operatorname{lip}(f - d_q)(z)$.*

Proof By the classical Abresch–Gromoll inequality (Corollary 2.43), for x in a sufficiently small neighborhood of z, $e_{p,q}(x) \le C d(x,z)^{1+\alpha}$. Therefore, for any such x,

$$\frac{|(f(x) + d_p(x)) - (f(z) + d_p(z))|}{d(x,z)} - \frac{|(f(x) - d_q(x)) - (f(z) - d_q(z))|}{d(x,z)} \le \frac{C d(x,z)^{1+\alpha}}{d(x,z)} = C d(x,z)^{\alpha}.$$

This shows that $\operatorname{lip}(f + d_p)(z) = \operatorname{lip}(f - d_q)(z)$, since $C d(x,z)^{\alpha} \to 0$ as $d(x,z) \to 0$. □

Lemma 3.25 *In the notation of Proposition 3.23, for any $t \in (0,1)$,*

$$\int \langle V, \nabla d_x \rangle(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y) + \int \langle V, \nabla d_y \rangle(\tilde{\gamma}_{x,y}(t))\, d\mu(x,y) = 0.$$

Proof Fix $t \in (0,1)$. Let $\mu = g(m \times m)$ for compactly supported $g \in L^{\infty}(X \times X, m \times m)$ with $g_x^1(\cdot) := g(x, \cdot)$ and $g_y^2(\cdot) := g(\cdot, y)$. Then $\|g_x^1\|_{L^\infty} \le C$ for m-a.e. x and $\|g_y^2\|_{L^\infty} \le C$ for m-a.e. y.

We first prove the claim for $V = \nabla f$ where f is locally Lipschitz. The general claim then follows by approximation. We observe the following:

(I) By Remark 2.5, for each $x \in X$,

$$\langle \nabla f, \nabla d_x \rangle = \tfrac{1}{2}(|\nabla(f + d_x)|^2 - |\nabla f|^2 - |\nabla d_x|^2) = \tfrac{1}{2}\big((\operatorname{lip}(f + d_x))^2 - (\operatorname{lip}(f))^2 - (\operatorname{lip}(d_x))^2\big) \quad m\text{-a.e.}$$

(II) Similarly, for each $y \in X$,

$$\langle \nabla f, -\nabla d_y \rangle = \tfrac{1}{2}(|\nabla(f-d_y)|^2 - |\nabla f|^2 - |-\nabla d_y|^2) = \tfrac{1}{2}\big((\mathrm{lip}(f-d_y))^2 - (\mathrm{lip}(f))^2 - (\mathrm{lip}(-d_y))^2\big) \quad m\text{-a.e.}$$

(III) $(\tilde{\gamma}_{x,\cdot}(t))_*(g_x^1 m)$ is of bounded density with respect to m for m-a.e. x by (15) from Theorem 2.25.

Lemma 3.24 and Fubini's theorem then give the claim for ∇f,

$$\begin{aligned}
\int \langle \nabla f, \nabla d_x \rangle (\tilde{\gamma}_{x,y}(t))\, d\mu(x,y) \\
&= \int \Big(\int \langle \nabla f, \nabla d_x \rangle (\tilde{\gamma}_{x,y}(t))\, d(g_x^1 m)(y) \Big)\, dm(x) \\
&= \int \Big(\int \tfrac{1}{2}\big((\mathrm{lip}(f+d_x))^2 - (\mathrm{lip}(f))^2 - (\mathrm{lip}(d_x))^2\big)(\tilde{\gamma}_{x,y}(t))\, d(g_x^1 m)(y) \Big)\, dm(x) \\
&= \int \Big(\int \tfrac{1}{2}\big((\mathrm{lip}(f-d_y))^2 - (\mathrm{lip}(f))^2 - (\mathrm{lip}(-d_y))^2\big)(\tilde{\gamma}_{x,y}(t))\, d(g_x^1 m)(y) \Big)\, dm(x) \\
&= \int \langle \nabla f, -\nabla d_y \rangle (\tilde{\gamma}_{x,y}(t))\, d\mu(x,y),
\end{aligned}$$

where the second equality follows by (I) and (III), the third by Lemma 3.24 and (III), and the fourth by (II) and (III). □

Remark 3.26 Lemma 3.25 implies (33) as well. This is because Lemma 3.21 is true for any interval $[a,b] \subseteq (0,1]$ and so one can use the same argument as in Proposition 3.23 to say that, for $0 < s < \frac{1}{2}$,

$$\begin{aligned}
\int_s^{1/2} \Big(\int \big(d(x,y)(\nabla V : (\nabla d_y \otimes \nabla d_y))(\tilde{\gamma}_{x,y}(t))\big)\, d\mu(x,y) \Big)\, dt \\
&= \int \langle V, \nabla d_y \rangle \big(\tilde{\gamma}_{x,y}\big(\tfrac{1}{2}\big)\big)\, d\mu(x,y) - \int \langle V, \nabla d_y \rangle (\tilde{\gamma}_{x,y}(s))\, d\mu(x,y) \\
&= -\int \langle V, \nabla d_x \rangle \big(\tilde{\gamma}_{x,y}\big(\tfrac{1}{2}\big)\big)\, d\mu(x,y) + \int \langle V, \nabla d_y \rangle (\tilde{\gamma}_{x,y}(s))\, d\mu(x,y) \\
&= \int_s^{1/2} \Big(\int \big(d(x,y)(\nabla V : (\nabla d_x \otimes \nabla d_x))(\tilde{\gamma}_{x,y}(t))\big)\, d\mu(x,y) \Big)\, dt.
\end{aligned}$$

Taking a limit as $s \to 0$ gives (33).

The second-order interpolation formula (Proposition 3.23) and first-order differentiation formula for distance along local flows (Proposition 3.18) immediately give an integral version of (5). They also give the following related estimate which will be used heavily in Section 5. Let $U, V \in L^1([0,T], L^2(TX))$ be bounded (see Definition 3.1) and S_1 and S_2 be bounded sets of positive measure. Let $(F_t)_{t\in[0,T]}$ and $(G_t)_{t\in[0,T]}$ be local flows of U and V from S_1 and S_2, respectively. Let $r > 0$, and for each $t \in [0,T]$, define $dt_r^{F,G}(t) \colon S_1 \times S_2 \to [0,r]$ the *distance distortion on scale r at t* by

$$dt_r^{F,G}(t)(x,y) := \min\big\{r, \max_{0\le\tau\le t} |d(x,y) - d(F_\tau(x), G_\tau(y))|\big\}. \tag{34}$$

Define $\Gamma_r^{F,G}(t) := \{(x,y) \in S_1 \times S_2 : dt_r^{F,G}(t)(x,y) < r\}$. The terminology and definition of the distance distortion function comes from [53], where it was used in a similar way as in this paper to analyze the geometry of gradient flows. We mention that in [53] both the underlying space and vector field considered are smooth and so the main technical challenge of this section was to extend such a result to the setting where both the underlying space and the vector field may be nonsmooth.

Proposition 3.27 *Let $W \in H^{1,2}_C(TX)$. The map $t \mapsto \int_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\,d(m\times m)(x,y)$ is Lipschitz on $[0,T]$ and satisfies*

$$\begin{aligned}\frac{d}{dt}\int_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\,d(m\times m)(x,y) &\le \int_{\Gamma_r^{F,G}(t)}\big(|U_t-W|(F_t(x))+|V_t-W|(G_t(y))\big)\,d(m\times m)(x,y)\\ &\quad+\int_0^1\int_{\Gamma_r^{F,G}(t)} d(F_t(x),G_t(y))|\nabla W|_{\mathrm{HS}}(\tilde\gamma_{F_t(x),G_t(y)}(s))\,d(m\times m)(x,y)\,ds\end{aligned}$$

for a.e. $t\in[0,T]$, where $\tilde\gamma_{\cdot,\cdot}$ is as defined in the beginning of this subsection.

Proof First fix representatives for all involved measure-theoretic objects. For any $(x,y)\in S_1\times S_2$, $dt_r^{F,G}(t)(x,y)$ is continuous, monotone nondecreasing and bounded between 0 and r as a function of $t\in[0,T]$. Therefore $dt_r^{F,G}(t)(x,y)$ is differentiable for a.e. $t\in[0,T]$ and $(d^+/dt)dt_r^{F,G}(t)(x,y)=0$ for all t where $(x,y)\notin\Gamma_r^{F,G}(t)$.

Furthermore, by boundedness of U and V, and Proposition 3.19, $F_\cdot(x)$ and $G_\cdot(y)$ are uniformly Lipschitz curves for m-a.e. $x\in S_1$ and m-a.e. $y\in S_2$, respectively. In particular, the functions $[0,T]\ni t\mapsto d(F_t(x),G_t(y))$ are uniformly Lipschitz for $(m\times m)$-a.e. $(x,y)\in S_1\times S_2$. The same is true for $t\mapsto dt_r^{F,G}(t)(x,y)$. Therefore $t\mapsto\int_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\,d(m\times m)(x,y)$ is Lipschitz on $[0,T]$ as well.

By Proposition 3.18, for $(m\times m)$-a.e. $(x,y)\in S_1\times S_2$, $d(F_t(x),G_t(y))\in W^{1,1}([0,T])$ and

$$\frac{d}{dt}d(F_t(x),G_t(y))=\langle\nabla d_{G_t(y)},U_t\rangle(F_t(x))+\langle\nabla d_{F_t(x)},V_t\rangle(G_t(y)). \tag{35}$$

At any point of differentiability for both $d(F_t(x),G_t(y))$ and $dt_r^{F,G}(t)(x,y)$, we have that

$$\frac{d}{dt}dt_r^{F,G}(t)(x,y)\le\Big(\frac{d}{dt}d(F_t(x),G_t(y))\Big)_+$$

since it follows from the definition that

$$|dt_r^{F,G}(t)(x,y)-dt_r^{F,G}(s)(x,y)|\le(d(F_t(x),G_t(y))-d(F_s(x),G_s(y)))_+$$

for any $T\ge t>s\ge 0$.

For any $t\in[0,T]$, let $\tilde\Gamma_r^{F,G}(t):=\{(x,y):\langle\nabla d_{G_t(y)},U_t\rangle(F_t(x))+\langle\nabla d_{F_t(x)},V_t\rangle(G_t(y))\ge 0\}\cap\Gamma_r^{F,G}(t)$. Then $\mu_t:=(F_t,G_t)_*((m\times m)|_{\tilde\Gamma_r^{F,G}(t)})$ is compactly supported by Proposition 3.19 and has bounded density with respect to $(m\times m)$ by Definition 3.16, the definition of a local flow. By the second-order interpolation formula (Proposition 3.23),

$$\int(\langle W,\nabla d_y\rangle(x)+\langle W,\nabla d_x\rangle(y))\,d\mu_t(x,y)=\int_0^1\int d(x,y)(\nabla W:\nabla d_x\otimes\nabla d_x)(\tilde\gamma_{x,y}(s))\,d\mu_t(x,y)\,ds. \tag{36}$$

Therefore

$$
\begin{aligned}
(37)\quad & \int (\langle U_t, \nabla d_y\rangle(x) + \langle V_t, \nabla d_x\rangle(y))\, d\mu_t(x,y) \\
&= \int (\langle U_t - W, \nabla d_y\rangle(x) + \langle V_t - W, \nabla d_x\rangle(y))\, d\mu_t(x,y) \\
&\qquad + \int_0^1 \int d(x,y)(\nabla W : \nabla d_x \otimes \nabla d_x)(\tilde{\gamma}_{x,y}(s))\, d\mu_t(x,y)\, ds && \text{by (36)} \\
&\le \int (|U_t - W|(x) + |V_t - W|(y))\, d\mu_t(x,y) \\
&\qquad + \int_0^1 \int d(x,y) |\nabla W|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(s))\, d\mu_t(x,y)\, ds && \text{since } |\nabla d_x|, |\nabla d_y| = 1 \ m\text{-a.e.} \\
&\le \int_{\Gamma_r^{F,G}(t)} \big(|U_t - W|(F_t(x)) + |V_t - W|(G_t(y))\big)\, d(m\times m)(x,y) \\
&\qquad + \int_0^1 \int_{\Gamma_r^{F,G}(t)} d(F_t(x), G_t(y)) |\nabla W|_{\mathrm{HS}}(\tilde{\gamma}_{F_t(x), G_t(y)}(s))\, d(m\times m)(x,y)\, ds,
\end{aligned}
$$

by definition of $\mu_t = (F_t, G_t)_*((m\times m)|_{\widetilde{\Gamma}_r^{F,G}(t)})$, $\widetilde{\Gamma}_r^{F,G}(t) \subseteq \Gamma_r^{F,G}(t)$ and the fact that all integrands are positive.

To conclude, for a.e. $t \in [0,T]$, for $(m\times m)$-a.e. $(x,y) \in S_1 \times S_2$, we have that $d(F_t(x), G_t(y))$ and $dt_r^{F,G}(t)(x,y)$ are both differentiable in t. For any such t,

$$
\begin{aligned}
&\frac{d}{dt} \int_{S_1\times S_2} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y) \\
&\qquad = \int_{S_1\times S_2} \frac{d}{dt} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y) \\
&\qquad = \int_{\Gamma_r^{F,G}(t)} \frac{d}{dt} dt_r^{F,G}(t)(x,y)\, d(m\times m)(x,y) \\
&\qquad \le \int_{\Gamma_r^{F,G}(t)} \Big(\frac{d}{dt} d(F_t(x), G_t(y))\Big)_+ d(m\times m)(x,y) \\
&\qquad = \int_{\widetilde{\Gamma}_r^{F,G}(t)} \langle \nabla d_{G_t(y)}, U_t\rangle(F_t(x)) + \langle \nabla d_{F_t(x)}, V_t\rangle(G_t(y))\, d(m\times m)(x,y) \\
&\qquad \le \int_{\Gamma_r^{F,G}(t)} \big(|U_t - W|(F_t(x)) + |V_t - W|(G_t(y))\big)\, d(m\times m)(x,y) \\
&\qquad\qquad + \int_0^1 \int_{\Gamma_r^{F,G}(t)} d(F_t(x), G_t(y)) |\nabla W|_{\mathrm{HS}}(\tilde{\gamma}_{F_t(x), G_t(y)}(s))\, d(m\times m)(x,y)\, ds,
\end{aligned}
$$

where the first equality follows by the dominated convergence theorem, the last equality by (35) and the definition of $\widetilde{\Gamma}$, and the last inequality by (37). □

4 Estimates on the heat flow approximations of distance and excess functions

Here we collect estimates on the heat flow approximations of distance and excess functions, all of which were established in [30]. All their arguments translate directly to the RCD setting due to the availability of the improved Bochner inequality, the Li–Yau Harnack and gradient inequalities, and the various estimates of Section 2.10. We record their proofs for the sake of completeness, making minor regularity and measure-theoretic adjustments as needed.

In this section we fix (X,d,m) an $\mathrm{RCD}(-(N-1),N)$ space for $N\in(1,\infty)$, $0<\delta<\frac{1}{2}$ and two points $p,q\in X$ with $d(p,q)\le 1$. Any time we use c it is a constant depending only on N and δ unless specified otherwise. We fix the following notation:

(I) $d_{p,q}=d(p,q)\le 1$ and for any $\epsilon>0$, $d_\epsilon:=\epsilon d_{p,q}$.

(II) $d^-(x):=d(p,x)$.

(III) $d^+(x):=d(p,q)-d(x,q)$.

(IV) $e(x):=d(p,x)+d(x,q)-d(p,q)=d^-(x)-d^+(x)$.

We will consider these functions multiplied by some appropriate cutoff functions. Let $\psi^\pm\colon X\to\mathbb{R}$ be the good cutoff functions as in Lemma 2.36 satisfying

$$\psi^-=\begin{cases}1 & \text{on } A_{(\delta/8)d_{p,q},8d_{p,q}}(p),\\ 0 & \text{on } X\setminus A_{(\delta/16)d_{p,q},16d_{p,q}}(p),\end{cases}\qquad \psi^+=\begin{cases}1 & \text{on } A_{(\delta/8)d_{p,q},8d_{p,q}}(q),\\ 0 & \text{on } X\setminus A_{(\delta/16)d_{p,q},16d_{p,q}}(q).\end{cases}$$

Let $\psi:=\psi^+\psi^-$, $e_0:=\psi e$ and $h_0^\pm:=\psi d^\pm$. We define

(V) $h_t^\pm:=H_t(h_0^\pm)$ and $e_t:=H_t(e_0)$,

(VI) $X_{r,s}:=A_{rd_{p,q},sd_{p,q}}(p)\cap A_{rd_{p,q},sd_{p,q}}(q)$.

By definition $e_0=e$ and $h_0^\pm=h^\pm$ on $X_{\delta/8,8}$ and $e_t=h_t^--h_t^+$ by uniqueness of heat flow.

We will always take the continuous representative whenever possible. This in particular applies to e_t, $h_t^\pm$, Δe_t and $\Delta h_t^\pm$ for $t>0$. We remark that since $h_t^\pm$ and e_t are Lipschitz, one can also take the local Lipschitz constant as the representative of $|\nabla h_t^\pm|$ and $|\nabla e_t|$ by Remark 2.5. These have a sufficiently nice continuity property (see Lemma 4.1), which makes most of our m-a.e. statements about $|\nabla h_t^\pm|$ and $|\nabla e_t|$ pointwise and ease certain measure-theoretic difficulties in the arguments for this section.

Lemma 4.1 *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space for $K\in\mathbb{R}$ and $N\in[1,\infty)$. Let $f\colon X\to\mathbb{R}$ be a Lipschitz function. Fix $U\subseteq X$ open and $x\in U$. Then*

$$\operatorname{lip}(f)(x)\le \operatorname*{ess\,sup}_U \operatorname{lip}(f).$$

Proof For any $\epsilon>0$, there exists $y\in U$ such that $|f(y)-f(x)|/d(y,x)\ge\operatorname{lip}(f)(x)-\frac{1}{2}\epsilon$. By continuity of f, there exists $r>0$ such that $B_r(y)\subseteq U$ and for any $z\in B_r(y)$, $|f(z)-f(x)|/d(z,x)\ge\operatorname{lip}(f)(x)-\epsilon$. Let $\gamma_{z,x}\colon[0,1]\to X$ be a constant-speed geodesic from z to x. The local Lipschitz constant is an upper gradient of f, see [6, Remark 2.7], and therefore,

$$\fint_{B_r(y)}|f(z)-f(x)|\,dm(z)\le\fint_{B_r(y)}\int_0^1 d(z,x)\operatorname{lip}(f)(\gamma_{z,x}(s))\,ds\,dm(z).$$

Since $⨍_{B_r(y)}|f(z)-f(x)| \geq \operatorname{lip}(f)(x)-\epsilon$ and for each $s<1$, $(\gamma_{\cdot,x}(s))_*(m)$ is absolutely continuous with respect to m by Theorem 2.25, we conclude $\operatorname{ess\,sup}_U \operatorname{lip}(f) \geq \operatorname{lip}(f)(x)-\epsilon$. □

We proceed with our estimates for e_t and $h_t^{\pm}$.

Lemma 4.2 *There exists a constant $c(N,\delta)$ such that for all $t>0$,*

$$\Delta h_t^-, -\Delta h_t^+, \Delta e_t \leq \frac{c(N,\delta)}{d_{p,q}}. \tag{38}$$

Proof We show the claim for e_t; the proof is analogous for the others. By the Laplacian comparison theorem for the distance function (Theorem 2.29; see also Remark 2.30), and the definition of ψ, $e_0 \in D(\mathbf{\Delta})$ with

$$\mathbf{\Delta} e_0 = \Delta\psi e m + \langle \nabla\psi, \nabla e\rangle m + \psi\mathbf{\Delta} e \leq \frac{c(N,\delta)}{d_{p,q}} m. \tag{39}$$

We know $e_t(x) = \int H_t(x,y)e_0(y)\,dm(y)$. For $t>0$, $e_t \in D(\Delta)$ and

$$\begin{aligned} \Delta e_t &= \int \Delta_x H_t(x,y)e_0(y)\,dm(y) \\ &= \int \Delta_y H_t(x,y)e_0(y)\,dm(y) && \text{by symmetry of } H_t \\ &= \int H_t(x,y)\,d\mathbf{\Delta} e_0(y) && \text{since } e_0 \text{ is compactly supported} \\ &\leq \frac{c(N,\delta)}{d_{p,q}}. \end{aligned} \tag{40}$$

□

Lemma 4.3 *There exists a constant $c(N,\delta)$ such that for all $0<\epsilon\leq\bar\epsilon(N,\delta)$ and $x\in X_{\delta/4,5}$ the following hold*:

(I) $|e_{d_\epsilon^2}(y)| \leq c(\epsilon^2 d_{p,q} + e(x))$ *for every* $y\in B_{10d_\epsilon}(x)$.

(II) $|\nabla e_{d_\epsilon^2}|(y) \leq c(\epsilon+\epsilon^{-1}e(x)/d_{p,q})$ *for* m*-a.e.* $y\in B_{10d_\epsilon}(x)$.

(III) $|\Delta e_{d_\epsilon^2}(y)| \leq c(1/d_{p,q}+\epsilon^{-2}e(x)/d_{p,q}^2)$ *for every* $y\in B_{10d_\epsilon}(x)$.

(IV) $⨍_{B_{d_\epsilon}(y)}|\operatorname{Hess} e_{d_\epsilon^2}|_{\mathrm{HS}}^2 \leq c(1/d_{p,q}+\epsilon^{-2}e(x)/d_{p,q}^2)^2$ *for every* $y\in B_{10d_\epsilon}(x)$.

Proof By definition of the heat flow and the continuity of e_s, $e_t(x) = e_0(x) + \int_0^t \Delta e_s(x)ds$ pointwise. By Lemma 4.2,

$$e_t(x) \leq e_0(x) + \frac{c}{d_{p,q}}t = e(x) + \frac{c}{d_{p,q}}t. \tag{41}$$

Setting $s=d_\epsilon^2$, $t=2d_\epsilon^2$ and $y\in B_{10d_\epsilon}(x)$ in the statement of the Li–Yau Harnack inequality (Theorem 2.31), we conclude

$$\begin{aligned} e_{d_\epsilon^2}(y) &\leq c(N)e_{2d_\epsilon^2}(x) && \text{by Theorem 2.31 and } d_{p,q}\leq 1 \\ &\leq c(e(x)+\epsilon^2 d_{p,q}) && \text{by (41).} \end{aligned} \tag{42}$$

This proves (I).

To prove (III), first notice we need only establish a lower bound on $\Delta e_{d_\epsilon^2}(y)$ since Lemma 4.2 already gives us the desired upper bound. This is an application of the Li–Yau gradient inequality (Theorem 2.32) and (I). The bound holds pointwise even though Theorem 2.32 holds only a.e. due to the existence of a continuous representative of Δe_t.

Statement (II) follows from another application of Li–Yau gradient inequality (Theorem 2.32) along with the bounds from (I) and (III).

For the last statement take good cutoff function ϕ supported on $B_{2d_\epsilon}(y)$ with $\phi \equiv 1$ on $B_{d_\epsilon}(y)$. We have

$$\begin{aligned}\int_X (\Delta e_{d_\epsilon^2})^2 \phi \, dm &= -\int_X \langle \nabla e_{d_\epsilon^2}, \nabla(\Delta e_{d_\epsilon^2}\phi)\rangle \, dm \\ &= -\int_X \langle \nabla e_{d_\epsilon^2}, \nabla \Delta e_{d_\epsilon^2}\rangle \phi \, dm - \int_X \langle \nabla e_{d_\epsilon^2}, \nabla \phi\rangle \Delta e_{d_\epsilon^2} \, dm.\end{aligned} \tag{43}$$

Integrating ϕ with $|\mathrm{Hess}\, e_{d_\epsilon^2}|^2_{\mathrm{HS}}$ and applying the improved Bochner inequality (Theorem 2.19), we get

$$\begin{aligned}\int_{B_{d_\epsilon}} |\mathrm{Hess}\, e_{d_\epsilon^2}|^2_{\mathrm{HS}} \, dm &\le \int_X |\mathrm{Hess}\, e_{d_\epsilon^2}|^2_{\mathrm{HS}} \phi \, dm \\ &\le \int_X \tfrac{1}{2}\phi \, d\mathbf{\Delta} |\nabla e_{d_\epsilon^2}|^2 - \int_X \langle \nabla e_{d_\epsilon^2}, \nabla \Delta e_{d_\epsilon^2}\rangle \phi \, dm + \int_X (N-1)|\nabla e_{d_\epsilon^2}|^2 \phi \, dm \\ &\le \int_X \tfrac{1}{2}\Delta\phi |\nabla e_{d_\epsilon^2}|^2 \, dm + \int_X (\Delta e_{d_\epsilon^2})^2 \phi \, dm + \int_X |\nabla e_{d_\epsilon^2}||\nabla \phi| \Delta e_{d_\epsilon^2} \, dm + \int_X (N-1)|\nabla e_{d_\epsilon^2}|^2 \phi \, dm,\end{aligned} \tag{44}$$

where the second inequality follows by Theorem 2.19 and the third by (43). Applying to this computation properties (I)–(III) of the lemma, property (II) of good cutoff functions (Lemma 2.35) and the Bishop–Gromov volume comparison (Theorem 2.4), we obtain (IV). □

Next we prove estimates on the heat flow approximation of the distance functions.

Lemma 4.4 *There exists $c(N,\delta)$ such that for every $\epsilon \le \bar\epsilon(N,\delta)$ and $x \in X_{\delta/4,5}$,*

$$|h^\pm_{d_\epsilon^2} - d^\pm|(x) \le c(\epsilon^2 d_{p,q} + e(x)).$$

Proof From the Laplacian bounds in Lemma 4.2, for $x \in X_{\delta/4,5}$,

$$h^-_{d_\epsilon^2}(x) - d^-(x) = \int_0^{d_\epsilon^2} \Delta h^-_t(x) \, dt \le c\epsilon^2 d_{p,q} \tag{45}$$

and

$$d^+(x) - h^+_{d_\epsilon^2}(x) = \int_0^{d_\epsilon^2} -\Delta h^+_t(x) \, dt \le c\epsilon^2 d_{p,q}. \tag{46}$$

To obtain bounds in the other direction, we note

$$h^-_{d_\epsilon^2} - d^-(x) = h^+_{d_\epsilon^2} - d^+(x) + e_{d_\epsilon^2}(x) - e(x).$$

We conclude using this with the bound $|e_{d_\epsilon^2}(x)| \le c(\epsilon^2 d_{p,q} + e(x))$ from Lemma 4.3(I) and bounds (45) or (46). □

We will end up wanting to establish appropriate gradient and Hessian bounds along curves that are close to being a geodesic between p and q. This requires the following definition:

Definition 4.5 A unit-speed piecewise-geodesic curve σ between p and q is called an ϵ-geodesic between p and q if $||\sigma| - d_{p,q}| \le \epsilon^2 d_{p,q}$, where $|\sigma|$ is the length of σ.

Remark 4.6 Notice x lies on an ϵ-geodesic if and only if $e(x) \le \epsilon^2 d_{p,q}$.

Lemma 4.4 can now be restated in terms of ϵ-geodesics:

Corollary 4.7 *There exists $c(N,\delta)$ such that for every ϵ-geodesic between p and q with $\epsilon \le \bar\epsilon(N,\delta)$ and $\frac{1}{3}\delta \le t \le 1-\frac{1}{3}\delta$,*

$$|h^{\pm}_{d_\epsilon^2} - d^{\pm}|(\sigma(t)) \le c(\epsilon^2 d_{p,q}).$$

Proof This follows from Lemma 4.4 since

(I) $e(\sigma(t)) \le \epsilon^2 d_{p,q}$,

(II) σ is unit speed and an ϵ-geodesic, as long as $\epsilon < \sqrt{\delta/12}$, and so $\sigma(t) \in X_{\delta/4,5}$. □

We establish an upper bound on the norm of the gradient of $h_t^{\pm}$ for $x \in X_{\delta/5,6}$:

Lemma 4.8 *There exists $c(N,\delta)$ such that for $\epsilon \le \bar\epsilon(N,\delta)$ and m-a.e. $x \in X_{\delta/5,6}$,*

$$|\nabla h^{\pm}_{d_\epsilon^2}| \le 1 + c d_\epsilon^2.$$

Proof By the Bakry–Ledoux estimate (Theorem 2.16), for any $t > 0$,

$$|\nabla h_t^{\pm}| \le e^{2(N-1)t} H_t(|\nabla h_0^{\pm}|) \quad m\text{-a.e.} \tag{47}$$

By definition of $h_0^{\pm} = \psi d^{\pm}$, we have the following a.e. bounds on $|\nabla h_0^{\pm}|$:

(I) $|\nabla h_0^{\pm}| = 1$ in $X_{\delta/8,8}$,

(II) $|\nabla h_0^{\pm}| = 0$ in $X \backslash X_{\delta/16,16}$,

(III) In $X_{\delta/16,16} \backslash X_{\delta/8,8}$,

$$|\nabla h_0^{\pm}| = |\nabla\psi||d^{\pm}| + |\psi||\nabla d^{\pm}| \le \frac{c(N)}{\delta d_{p,q}}|d^{\pm}| + 1 \le c(N,\delta). \tag{48}$$

Finally, for any $x \in X_{\delta/2,4}$,

$$\begin{aligned} H_t(|\nabla h_0^{\pm}|)(x) &= \int_X H_t(x,y)|\nabla h_0^{\pm}|(y)\,dm(y) \\ &= \int_{X_{\delta/16,16}\backslash X_{\delta/8,8}} H_t(x,y)|\nabla h_0^{\pm}|(y)\,dm(y) + \int_{X_{\delta/4,8}} H_t(x,y)|\nabla h_0^{\pm}|(y)\,dm(y) \\ &\le c(N,\delta)\int_{X_{\delta/16,16}\backslash X_{\delta/8,8}} \nabla H_t(x,y)\,dm(y) + 1 \\ &\le c(N,\delta)\big(\tfrac{1}{8}\delta\big)^{-2} t + 1, \end{aligned} \tag{49}$$

where the first inequality follows by (48) and the last lines use statement (II) of the heat kernel bounds (Lemma 2.38).

The lemma follows by combining (47) and (49), with $t = d_\epsilon^2$ for small ϵ. □

We will now establish some integral bounds on $|\nabla h_t^{\pm}|$. Roughly, we want to apply the L^1-Harnack inequality (Lemma 2.39) to $|\nabla h^{\pm}|$. To this effect, we give some regularity of the heat flow in the time parameter.

Lemma 4.9 *Let (X,d,m) be an $\mathrm{RCD}(K,N)$ space for some $K\in\mathbb{R}$, $N\in[1,\infty)$. Let $f\in L^2(m)$. Then:*

(I) $H_t(f)\in C^1((0,\infty),L^2(m))$ *and*

$$\frac{d}{dt}H_t(f)=\Delta H_t(f)\quad\text{for all } t>0. \tag{50}$$

(II) $H_t(f)\in C^0((0,\infty),W^{1,2}(X))$.

(III) $H_t(f)\in C^1((0,\infty),W^{1,2}(X))$ *and in particular* $|\nabla H_t(f)|^2\in C^1((0,\infty),L^1(m))$ *with*

$$\frac{d}{dt}|\nabla H_t(f)|^2=2\langle\nabla H_t(f),\nabla\Delta H_t(f)\rangle\quad\text{for all } t>0. \tag{51}$$

If f or $|\nabla f|$ is in $L^\infty(X,m)$, then $|\nabla H_t(f)|^2\in C^1((0,\infty),L^2(m))$ and the same formula holds.

Proof Since $H_t(f)=H_{t'}(f)+\int_{t'}^t\Delta H_s(f)\,ds$ for $t>t'>0$ and $\Delta H_s(f)\in C^0((0,\infty),L^2(m))$, (I) follows by the fundamental theorem of calculus.

For $t>0$,

$$\begin{aligned}\lim_{s\to0}\int_X&|\nabla H_{t+s}(f)-\nabla H_t(f)|^2\\&=\lim_{s\to0}\int_X(H_{t+s}(f)\Delta H_{t+s}(f))+2(H_{t+s}(f)\Delta H_t(f))-(H_t(f)\Delta H_t(f))=0,\end{aligned}$$

since all terms involved are continuous maps from $s\in[0,\infty)\to L^2(m)$. We already know $H_t(f)\in C^0((0,\infty),L^2(X))$, so (II) follows.

Applying (II) to $\Delta H_\epsilon(f)$ for arbitrarily small positive ϵ, we see that $\Delta H_t(f)\in C^0((0,\infty),W^{1,2}(X))$. The first part of (III) then follows by applying the fundamental theorem of calculus to $H_t(f)=H_{t'}(f)+\int_{t'}^t\Delta H_s(f)\,ds$ viewed as a $W^{1,2}(X)$-valued Bochner integral. The second part follows by a direct computation. Notice that if f or $|\nabla f|$ is in $L^\infty(X,m)$, then $|\nabla h_t(f)|\in L^\infty$ by L^∞-to-Lipschitz regularization (14) or by the Bakry–Ledoux estimate (Theorem 2.16). □

Lemma 4.10 *Suppose $\phi\in D(\Delta)$ is nonnegative, compactly supported and time independent with $|\phi|,|\nabla\phi|,|\Delta\phi|\le K_1$. If h is the heat flow of some $h_0\in L^2(m)\cap L^\infty(m)$ and $|\nabla h|\le K_2$ on $\{\phi>0\}$, then $(\partial/\partial t-\Delta)[\phi^2|\nabla h|^2]\le c(N,K_1,K_2)$ weakly in $(0,\infty)\times X$ as in Definition 2.33.*

Proof Let $t>0$. Then $h_t\in\mathrm{TestF}(X)$ by the L^∞-to-Lipschitz regularization property of H_t (14). Let $f\in\mathrm{TestF}(X)$. By [39, Proposition 3.3.22], $\langle\nabla h_t,\nabla h_t\rangle\in W^{1,2}(X)$ and

$$\langle\nabla\langle\nabla h_t,\nabla h_t\rangle,\nabla f\rangle=2\,\mathrm{Hess}(h_t)(\nabla h_t,\nabla f)\quad m\text{-a.e.}$$

Therefore $|\nabla\langle\nabla h_t, \nabla h_t\rangle\nabla f| \le 2|\mathrm{Hess}(h_t)|_{\mathrm{HS}}|\nabla h_t||\nabla f|$ m-a.e.

Then for any $\epsilon > 0$ and m-a.e.,

$$\begin{aligned}4\phi|\langle\nabla|\nabla h_t|^2, \nabla\phi\rangle| &\le 8\phi|\mathrm{Hess}(h_t)|_{\mathrm{HS}}|\nabla h_t||\nabla\phi| \\ &\le 4\epsilon\phi^2|\mathrm{Hess}(h_t)|^2_{\mathrm{HS}}|\nabla h_t|^2 + \frac{4}{\epsilon}|\nabla\phi|^2,\end{aligned}$$

where the first inequality follows by the above and the density of $\mathrm{TestF}(X)$ in $W^{1,2}(X)$. Choosing $\epsilon > 0$ small so that $4\epsilon K_2^2 < 2$,

$$\begin{aligned}\mathbf{\Delta}(\phi^2|\nabla h_t|^2) &= \phi^2\mathbf{\Delta}|\nabla h_t|^2 + (2\langle\nabla|\nabla h_t|^2, \nabla\phi^2\rangle + |\nabla h_t|^2\Delta\phi^2)m \\ &\ge \big(2\phi^2|\mathrm{Hess}(h_t)|^2_{\mathrm{HS}} + 2\phi^2\langle\nabla\Delta h_t, \nabla h_t\rangle - 2(N-1)\phi^2|\nabla h_t|^2 + 4\phi\langle\nabla|\nabla h_t|^2, \nabla\phi\rangle \\ &\qquad + |\nabla h_t|^2\Delta\phi^2\big)m \\ &\ge (2\phi^2\langle\nabla\Delta h_t, \nabla h_t\rangle - c)m,\end{aligned}$$

where the improved Bochner inequality (Theorem 2.19) is used for line two and the previous estimate with ϵ was used for line three.

Finally, by Lemma 4.9, $(d/dt)\phi^2|\nabla h_t|^2 = 2\phi^2\langle\nabla\Delta h_t, \nabla h\rangle$. □

Theorem 4.11 *There exists a constant $c(N,\delta)$ such that for all $\epsilon \le \bar{\epsilon}(N,\delta)$,*

(I) *if $x \in X_{\delta/2,3}$ with $e(x) \le \epsilon^2 d_{p,q}$ then $⨍_{B_{10d_\epsilon}(x)} ||\nabla h^{\pm}_{d^2_\epsilon}| - 1| \le c\epsilon$,*

(II) *if σ is an ϵ-geodesic connecting p and q, then $\int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} ⨍_{B_{10d_\epsilon}(\sigma(s))} ||\nabla h^{\pm}_{d^2_\epsilon}| - 1| \le c\epsilon^2 d_{p,q}$.*

Proof We prove the theorem in the case of h^-_t. The h^+_t case is similar. We will take the local Lipschitz constant representatives for $|\nabla h^{\pm}_t|$. All statements made will be for ϵ sufficiently small depending on N and δ so we will forgo repeating this.

By Lemma 4.8, we choose $c'(N,\delta)$ so that $|\nabla h^-_t| \le 1 + c't^2$ for all $x \in X_{\delta/5,6}$ and $t \le \epsilon'(N,\delta)^2 d^2_{p,q}$. This means there exists $c''(N,\delta)$ such that

$$w_t := 1 + c''t - |\nabla h^-_t|^2 \ge 0 \quad \text{on } X_{\delta/5,6}.$$

Let $\phi = \phi^+\phi^-$, where $\phi^{\pm}$ are annular good cutoff functions (Lemma 2.36) around p and q, respectively, such that $\phi = 1$ on $X_{\delta/4,5}$ and $\phi = 0$ on $X \backslash X_{\delta/5,6}$. By Lemma 4.10,

$$(\partial_t - \Delta)|\phi^2 w_t| \ge -c \quad \text{weakly in } (0, d^2_{\epsilon'}) \times X.$$

Applying the L^1-Harnack inequality (Lemma 2.39) we have, for $x \in X_{\delta/3,4}$,

$$⨍_{B_{10d_\epsilon}(x)} w_{d^2_\epsilon} \le c\Big[\operatorname*{ess\,inf}_{B_{10d_\epsilon}(x)} w_{2d^2_\epsilon} + d^2_\epsilon\Big]. \tag{52}$$

We will show the right side is sufficiently small. Let $e(x) \le \epsilon^2 d_{p,q}$ and let $\gamma(t)$ be a unit-speed geodesic from x to p. By Corollary 4.7,

$$
\begin{aligned}
(53)\quad & |h^-_{2d_\epsilon^2}(x) - h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon)) - 10d_\epsilon| \\
& \le |h^-_{2d_\epsilon^2}(x) - d^-(x)| + |h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon)) - d^-(\gamma(10d_\epsilon))| + |d^-(x) - d^-(\gamma(10d_\epsilon)) - 10d_\epsilon| \\
& \le c\epsilon^2 d_{p,q}.
\end{aligned}
$$

The local Lipschitz constant $|\nabla h^-_{2d_\epsilon^2}|$ is an upper gradient [6, Remark 2.7]. Therefore

$$
(54)\qquad |h^-_{2d_\epsilon^2}(\sigma) - h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon))| \le \int_0^{10d_\epsilon} |\nabla h^-_{2d_\epsilon^2}|(\gamma(s))\,ds.
$$

We have

$$
\begin{aligned}
(55)\quad \int_0^{10d_\epsilon} w_{2d_\epsilon^2}(\gamma(s))\,ds &= \int_0^{10d_\epsilon} \big(1 + cd_\epsilon^2 - |\nabla h^-_{2d_\epsilon^2}|^2(\gamma(s))\big)\,ds \\
&\le 10d_\epsilon + cd_\epsilon^3 - \frac{1}{10d_\epsilon}\Big(\int_0^{10d_\epsilon} |\nabla h^-_{2d_\epsilon^2}|(\gamma(s))\,ds\Big)^2 && \text{by Cauchy–Schwarz} \\
&\le 10d_\epsilon + cd_\epsilon^3 - \frac{1}{10d_\epsilon}\big(h^-_{2d_\epsilon^2}(x) - h^-_{2d_\epsilon^2}(\gamma(10d_\epsilon))\big)^2 && \text{by (54)} \\
&\le 10d_\epsilon + cd_\epsilon^3 - \frac{1}{10d_\epsilon}(10d_\epsilon - c\epsilon^2 d_{p,q})^2 && \text{by (53)} \\
&\le c\epsilon && \text{if } \epsilon < 1.
\end{aligned}
$$

In particular, there exists $s \in [0, 10d_\epsilon]$ such that $w_{2d_\epsilon^2}(\gamma(s)) \le c\epsilon$. Applying Lemma 4.1 to $|\nabla h_t^\pm|$ and $\gamma(s) \in \overline{B_{10d_\epsilon}(x)}$, we conclude $\operatorname{ess\,inf}_{B_{10d_\epsilon}(x)} w_{2d_\epsilon^2} \le c\epsilon$ and so (I) is proved by (52).

By Corollary 4.7, arguing as in (53),

$$
(56)\qquad \Big| h^-_{2d_\epsilon^2}\Big(\sigma\big((1-\tfrac{1}{2}\delta)d_{p,q}\big)\Big) - h^-_{2d_\epsilon^2}\big(\sigma\big(\tfrac{1}{2}\delta d_{p,q}\big)\big) - (1-\delta)d_{p,q} \Big| \le c\epsilon^2 d_{p,q}.
$$

Arguing as in (54) and (55),

$$
(57)\qquad \int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} w_{2d_\epsilon^2}(\sigma(s))\,ds \le c\epsilon^2 d_{p,q}.
$$

Finally,

$$
\begin{aligned}
(58)\quad & \int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} \Big(⨍_{B_{10d_\epsilon}(\sigma(s))} \big||\nabla h^-_{d_\epsilon^2}|^2 - 1\big| \Big)\,ds \\
&\le \int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} \Big(⨍_{B_{10d_\epsilon}(\sigma(s))} w_{d_\epsilon^2} + c\epsilon^2 d_{p,q} \Big)\,ds \\
&\le c \int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} \Big(\operatorname*{ess\,inf}_{B_{10d_\epsilon^2}(\sigma(s))} w_{2d_\epsilon^2} + c\epsilon^2 d_{p,q} \Big)\,ds && \text{by (52)} \\
&\le c \int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} \big(w_{2d_\epsilon^2}(\sigma(s)) + c\epsilon^2 d_{p,q} \big)\,ds && \text{by Lemma 4.1} \\
&\le c\epsilon^2 d_{p,q}.
\end{aligned}
$$

□

We now prove the main Hessian estimate for $h_t^{\pm}$:

Theorem 4.12 *There exists a constant $c(N,\delta)$ such that for any $0<\epsilon\le\bar\epsilon(N,\delta)$, any $x\in X_{\delta/2,3}$ with $e(x)\le\epsilon^2 d_{p,q}$, or any ϵ-geodesic σ connecting p and q, there exists $r\in\left[\frac12,2\right]$ with*

(I) $|h^{\pm}_{rd_\epsilon^2}-d^{\pm}|\le c\epsilon^2 d_{p,q}$,

(II) $⨍_{B_{d_\epsilon}(x)}||\nabla h^{\pm}_{rd_\epsilon^2}|^2-1|\le c\epsilon$,

(III) $\int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}}\left(⨍_{B_{d_\epsilon}(\sigma(s))}||\nabla h^{\pm}_{rd_\epsilon^2}|^2-1|\right)ds\le c\epsilon^2 d_{p,q}$,

(IV) $\int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}}\left(⨍_{B_{d_\epsilon}(\sigma(s))}|\mathrm{Hess}\, h^{\pm}_{rd_\epsilon^2}|^2\right)ds\le c/d_{p,q}^2$.

Proof Statement (I) follows from Lemma 4.4 and (II) and (III) follow from Theorem 4.11 with Bishop–Gromov. Note any $r\in\left[\frac12,2\right]$ works in the first three statements.

Using Lemma 2.35, we fix, for each $s\in\left(\frac12\delta d_{p,q},\left(1-\frac12\delta\right)d_{p,q}\right)$, a good cutoff function ϕ with $\phi\equiv 1$ on $B_{d_\epsilon}(\sigma(s))$, vanishing outside of $B_{3d_\epsilon}(\sigma(s))$, and $d_\epsilon|\nabla\phi|$, $d_\epsilon^2|\Delta\phi|\le c(N)$. Similarly, fix $\alpha(t)$ a smooth function in time such that $0\le\alpha(t)\le 1$, $\alpha(t)\equiv 1$ for $t\in\left[\frac12 d_\epsilon^2,2d_\epsilon^2\right]$, vanishing for t outside of $\left[\frac14 d_\epsilon^2,4d_\epsilon^2\right]$, and satisfying $|\alpha'|\le 10d_\epsilon^{-2}$.

Applying the improved Bochner inequality (Theorem 2.19) to $h_t^{\pm}$, we obtain, for each s and t,

$$
\begin{aligned}
&\int\alpha(t)\phi|\mathrm{Hess}\, h_t^{\pm}|^2\,dm \\
&\quad\le\int\alpha(t)\phi\, d(\mathbf{\Delta}|\nabla h_t^{\pm}|^2)+2\int\alpha(t)\phi((N-1)|\nabla h_t^{\pm}|^2-\langle\nabla h_t^{\pm},\nabla\Delta h_t^{\pm}\rangle)\,dm \\
&\quad=\int\alpha(t)(|\nabla h_t^{\pm}|^2-1)\Delta(\phi)\,dm+2(N-1)\int\alpha(t)\phi|\nabla h_t^{\pm}|^2\,dm-\int\alpha(t)\phi\partial_t(|\nabla h_t^{\pm}|^2)\,dm.
\end{aligned}
\tag{59}
$$

In the last line, we used the definition of the Laplacians along with the fact that $\int\Delta\phi\, dm=0$ for the first term and Lemma 4.9 for the third term. Integrating in time using integration by parts and $\int_0^\infty\alpha'(t)\,dt=0$ on the third term of the previous line,

$$
\begin{aligned}
&\int_0^\infty\int\alpha(t)\phi|\mathrm{Hess}\, h_t^{\pm}|^2\,dm\,dt \\
&\quad\le\int_0^\infty\biggl(\int\alpha(t)(|\nabla h_t^{\pm}|^2-1)\Delta(\phi)\,dm+2(N-1)\int\alpha(t)\phi|\nabla h_t^{\pm}|^2\,dm \\
&\qquad\qquad+\int\alpha'(t)\phi(|\nabla h_t^{\pm}|^2-1)\,dm\biggr)\,dt.
\end{aligned}
\tag{60}
$$

Using what we know about ϕ and α and using Bishop–Gromov in line two of the following, we obtain

$$
\begin{aligned}
&\int_{d_\epsilon^2/2}^{2d_\epsilon^2}⨍_{B_{d_\epsilon}(\sigma(s))}|\mathrm{Hess}\, h_t^{\pm}|^2\,dm\,dt \\
&\quad\le\int_{d_\epsilon^2/4}^{4d_\epsilon^2}\biggl(⨍_{B_{3d_\epsilon}(\sigma(s))}\bigl((|\nabla h_t^{\pm}|^2-1)\Delta(\phi)+2(N-1)|\nabla h_t^{\pm}|^2+\alpha'(t)(|\nabla h_t^{\pm}|^2-1)\bigr)\,dm\biggr)\,dt \\
&\quad\le\int_{d_\epsilon^2/4}^{4d_\epsilon^2}\biggl(⨍_{B_{3d_\epsilon}(\sigma(s))}2(N-1)+cd_\epsilon^{-2}||\nabla h_t^{\pm}|^2-1|\,dm\biggr)\,dt.
\end{aligned}
\tag{61}
$$

Integrating across σ for $s \in \left[\frac{1}{2}\delta d_{p,q}, \left(1-\frac{1}{2}\delta\right)d_{p,q}\right]$,

$$\text{(62)}\quad \int_{d_\epsilon^2/2}^{2d_\epsilon^2}\Big(\int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} \fint_{B_{d_\epsilon}(\sigma(s))} |\mathrm{Hess}\, h_t^{\pm}|^2\, dm\, ds\Big)\, dt$$
$$\le c d_\epsilon^{-2} \int_{d_\epsilon^2/4}^{4d_\epsilon^2}\Big(\int_{(\delta/2)d_{p,q}}^{(1-\delta/2)d_{p,q}} \fint_{B_{3d_\epsilon}(\sigma(s))} c d_\epsilon^2 + ||\nabla h_t^{\pm}|^2 - 1|\, dm\, ds\Big)\, dt$$
$$\le c\epsilon^2 d_{p,q} \qquad \text{by Theorem 4.11(II).}$$

Therefore (IV) holds for some $r \in \left[\frac{1}{2}, 2\right]$ and $t = r d_\epsilon^2$. □

Lemma 4.13 *Let $\epsilon \le \bar{\epsilon}(N,\delta)$. Let $\gamma_{x,p}$ be any unit-speed geodesic from $x \in X$ to p. Then for m-a.e. $x \in X_{\delta/2,3}$ and any $0 \le t_1 < t_2 \le d_{x,p} - \frac{1}{2}\delta$, the following estimates hold:*

(I) $\int_0^{d_{x,p}-\delta/2} ||\nabla h^-_{d_\epsilon^2}|^2 - 1|(\gamma_{x,p}(s))\, ds \le c(N,\delta)/d_{p,q}(e(x) + d_\epsilon^2)$,

(II) $\int_0^{d_{x,p}-\delta/2} |\langle \nabla h^-_{d_\epsilon^2}, \nabla d^-\rangle - 1|(\gamma_{x,p}(s))\, ds \le c(N,\delta)/d_{p,q}(e(x) + d_\epsilon^2)$,

(III) $\int_{t_1}^{t_2} |\nabla h^-_{d_\epsilon^2} - \nabla d^-|(\gamma_{x,p}(s))\, ds \le c(N,\delta)\sqrt{t_2 - t_1}/\sqrt{d_{p,q}}(\sqrt{e(x)} + d_\epsilon)$.

Proof The bounds on $(|\nabla h^-_{d_\epsilon^2}|^2 - 1)_+$ and $(\langle \nabla h^-_{d_\epsilon^2}, \nabla d^-\rangle - 1)_+$ for (I) and (II) come from Lemma 4.8, Fubini's theorem and Theorem 3.15(II). The bound on the negative part comes from an estimate like (56) combined with Corollary 3.14 for (II), and then an additional application of Cauchy–Schwarz for (I). We note if one traces the proof of (56) back to Lemma 4.4, it is clear that one can obtain bounds where the excess is not related to the heat flow time as they have been for the past several claims.

For (III),

$$|\nabla h^-_{d_\epsilon^2} - \nabla d^-|^2 = |\nabla h^-_{d_\epsilon^2}|^2 + 1 - 2\langle \nabla h^-_{d_\epsilon^2}, \nabla d^-\rangle \le ||\nabla h^-_{d_\epsilon^2}|^2 - 1| + 2|\langle \nabla h^-_{d_\epsilon^2}, \nabla d^-\rangle - 1| \quad m\text{-a.e.}$$

So statements (I) and (II), Cauchy–Schwarz and an argument by Fubini's theorem using Theorem 3.15(II) give (III). □

5 Gromov–Hausdorff approximation

This section will be divided into three subsections. The main lemma, proved in the first subsection, gives a way of overcoming the lack of start of induction in the arguments of [30] generalized to the RCD setting. In the second subsection we use the main lemma to construct geodesics with nice properties in their interiors. Finally, we prove the main theorem in the third subsection. To be precise, we prove a slightly weaker version of the main theorem analogous to the main result of [30], which will be used to be prove nonbranching and, subsequently, the main theorem. We refer to the introduction for some of the main ideas behind this section.

Fix (X, d, m), an $\mathrm{RCD}(-(N-1), N)$ metric measure space for $N \in (1, \infty)$ and $p, q \in X$ with $d(p, q) = 1$. Fix $0 < \delta < 0.1$. For any $x_1, x_2 \in X$, we fix a constant-speed geodesic from x_1 to x_2 parametrized on $[0, 1]$ and denote it by $\tilde{\gamma}_{x_1,x_2}$. By Remark 2.26, we may assume the map $X \times X \times [0, 1] \ni (x_1, x_2, t) \mapsto \tilde{\gamma}_{x_1,x_2}(t)$ is Borel. The unit-speed reparametrization of $\tilde{\gamma}_{x_1,x_2}$ to the interval $[0, d(x_1, x_2)]$ will be denoted by γ_{x_1,x_2} and γ will denote $\gamma_{p,q}$. For each $x \in X$, define $\Psi\colon X \times [0, \infty) \to X$ by

$$(x, s) \mapsto \Psi_s(x) = \begin{cases} \gamma_{x,p}(s) & \text{if } d(x, p) \geq s, \\ p & \text{if } d(x, p) < s. \end{cases} \tag{63}$$

Similarly, define $\Phi\colon X \times [0, \infty) \to X$ by

$$(x, s) \mapsto \Phi_s(x) = \begin{cases} \gamma_{x,q}(s) & \text{if } d(x, q) \geq s, \\ q & \text{if } d(x, q) < s. \end{cases} \tag{64}$$

By the integral Abresch–Gromoll inequality (Theorem 2.42), for any sufficiently small $r \leq \bar{r}(N, \delta)$ and any $\delta \leq t_0 \leq 1 - \delta$,

$$⨍_{B_r(\gamma(t_0))} e \leq c_0(N, \delta) r^2.$$

Therefore there exists a subset $S \subseteq B_r(\gamma(t_0))$ such that

(I) $m(S)/m(B_r(\gamma(t_0))) \geq 1 - \frac{1}{3} V(1, 10)$ (see Theorem 2.4 for the definition of $V := V_{-(N-1),N}$),

(II) for all $z \in S$, $e(z) \leq c_1(N, \delta)^2 r^2$.

We fix such a c_1 for the rest of this section and assume in addition $c_1 > 100$.

For brevity and to give intuition, we also make use of the expression "nontrivial (in measure) compared to". In the context of what follows, we say some set A is nontrivial in measure compared to some set B if $m(A) \geq c\,m(B)$, where $c > 0$ is a constant depending only on structural constants and some parameters to be fixed, and importantly not depending on the point, geodesic or space under consideration.

In all subsections the letter c will be used to represent different constants which only depend on N and δ. Any constant which will be used repeatedly will be given a subscript. We will continue using the notation of Section 4.

5.1 Proof of main lemma

In this section we prove our main lemma, which circumvents the necessity for a start of induction step as in [30]. Roughly, we show that if we fix a point $\gamma(t_0)$ in the interior of a geodesic from p to q, then for any sufficiently small radius r we can find a point z within r of $\gamma(t_0)$ such that for a large collection of points x within r of z, the geodesic from x to p stays relatively close (say, within $2r$) to the geodesic from z to p for an amount of time (here time means the unit-speed parameter of the geodesic) that is independent of r. The main difficulty is to obtain this for an amount of time that is independent of r. Indeed, the triangle inequality alone would give closeness for a time up to r. To achieve the desired result, we use an induction in time. As mentioned, we can easily find a point satisfying the desired properties for a time of r. We then show that if we can find a point satisfying the desired properties for an amount of time t less than some constant ϵ_1 (depending only on structural constants), then we can find a point satisfying the desired properties for an amount of time up to $t+r$. This is enough to find some $z \in B_r(\gamma(t_0))$ such that the desired properties hold for any time up to ϵ_1. An intuitive way to think about this z is that it is a point within $B_r(\gamma(t_0))$ such that the geodesic between it and p captures the behavior (on the scale of r) of most geodesics from points in $B_r(\gamma(t_0))$ to p (for a short amount of time that is independent of r).

Lemma 5.1 (main lemma) *There exist $\epsilon_1(N,\delta) > 0$ and $\bar{r}_1(N,\delta) > 0$ such that for all $r \le \bar{r}_1$ and $\delta \le t_0 \le 1-\delta$, there exists $z \in B_r(\gamma(t_0))$ such that*:

(I) $V(1,100) \le m(B_r(\Psi_s(z)))/m(B_r(z)) \le 1/V(1,100)$ *for any* $s \le \epsilon_1$.

(II) *There exists $A \subseteq B_r(z)$ with $m(A) \ge (1-V(1,10))m(B_r(z))$ and $\Psi_s(A) \subseteq B_{2r}(\Psi_s(z))$ for any $s \le \epsilon_1$.*

(III) $e(z) \le c_1^2 r^2$.

Proof Fix $\delta \le t_0 \le 1-\delta$ and a scale $r \le \bar{r}_1(N,\delta)$. We need only choose $\bar{r}_1$ smaller than the radius bounds required for the application of various theorems in the proof; most notably Theorem 2.42 and the estimates of Section 4. It will be clear that all the required radius bounds only depend on N and δ so we will not address this each time for the sake of brevity. In addition, we assume $\bar{r}_1 \le \frac{1}{10}\delta$.

By the Bishop–Gromov volume comparison (Theorem 2.4), the integral Abresch–Gromoll inequality (Theorem 2.42) and the fact that Ψ is defined using unit-speed geodesics, it is clear there exist ϵ depending on N, δ and r, and $z \in B_r(\gamma(t_0))$ satisfying

(I) $V(1,100) \le m(B_r(\Psi_s(z)))/m(B_r(z)) \le 1/V(1,100)$ for any $s \le \epsilon$,

(II) there exists $A \subseteq B_r(z)$ with $m(A) \ge (1-V(1,10))m(B_r(z))$ and $\Psi_s(A) \subseteq B_{2r}(\Psi_s(z))$ for any $s \le \epsilon$,

(III) $e(z) \le c_1^2 r^2$.

We will remove the dependence of ϵ on r.

To this effect, we will show that if (I)–(III) hold for some $z \in B_r(\gamma(t_0))$ and all $s \le \epsilon$ less than or equal to some $\epsilon_1(N,\delta)$ to be fixed later, then in fact we can choose $z' \in B_r(\gamma(t_0))$ satisfying (III) which significantly improves the estimates in (I) and (II) for $s \le \epsilon$. To be precise, we find $z' \in B_r(\gamma(t_0))$ satisfying

(I) $2V(1,100) \le m(B_r(\Psi_s(z')))/m(B_r(z')) \le 1/(2V(1,100))$ for any $s \le \epsilon$,

(II) there exists $A' \subseteq B_r(z')$ with $m(A') \ge (1-V(1,10))m(B_r(z'))$ and $\Psi_s(A') \subseteq B_{3r/2}(\Psi_s(z'))$ for any $s \le \epsilon$,

(III) $e(z') \le c_1^2 r^2$.

We a priori assume $\epsilon_1 \le \frac{1}{10}\delta$ and impose more bounds on ϵ_1 as the proof continues. Let z satisfy (I)–(III) for some $\epsilon \le \epsilon_1$.

Let w be the midpoint of z and $\gamma(t_0)$. We know $B_{r/2}(w) \subseteq B_r(\gamma(t_0)) \cap B_r(z)$. By Bishop–Gromov and since $r \le \bar{r}$, which was assumed to be less than 0.01,

$$\frac{m(B_r(\gamma(t_0)) \cap B_r(z))}{m(B_r(z))} \ge \frac{m(B_{r/2}(w))}{m(B_r(z))} \ge \frac{m(B_{r/2}(w))}{m(B_{3r/2}(w))} \ge V\big(\tfrac{1}{2}r, \tfrac{1}{2}3r\big) > V\big(\tfrac{1}{2}, \tfrac{3}{2}\big).$$

Using $m(A) \ge (1 - V(1,10))m(B_r(z)) > \big(1 - \frac{1}{3}V\big(\frac{1}{2}, \frac{3}{2}\big)\big)m(B_r(z))$ and the previous estimate,

$$\frac{m(A \cap B_r(\gamma(t_0)))}{m(B_r(z))} > \tfrac{2}{3}V\big(\tfrac{1}{2}, \tfrac{3}{2}\big).$$

Therefore

$$\begin{aligned}\frac{m(A \cap B_r(\gamma(t_0)))}{m(B_r(\gamma(t_0)))} &= \frac{m(A \cap B_r(\gamma(t_0)))}{m(B_r(z))}\frac{m(B_r(z))}{m(B_r(\gamma(t_0)))} \\ &> \tfrac{2}{3}V\big(\tfrac{1}{2}, \tfrac{3}{2}\big)V(r, 2r) > \tfrac{2}{3}V\big(\tfrac{1}{2}, \tfrac{3}{2}\big)V\big(\tfrac{3}{2}, 3\big) > \tfrac{2}{3}V(1, 10).\end{aligned} \tag{65}$$

Define the set

$$D_1 := A \cap B_r(\gamma(t_0)) \cap \{e(x) \le c_1^2 r^2\}, \tag{66}$$

where c_1 is as fixed earlier. We will choose a z' satisfying properties (I)–(III) from D_1. From (65) and the definition of c_1,

$$\frac{m(D_1)}{m(B_r(\gamma(t_0)))} > \tfrac{1}{3}V(1, 10).$$

Therefore, by Bishop–Gromov,

$$\frac{m(D_1)}{m(B_r(z))} \ge c(N). \tag{67}$$

Since $e(z) \le c_1^2 r^2$ by property (III) of z, the curve traversing $\gamma_{z,p}$ in reverse and then $\gamma_{z,q}$ is a $c_1 r$-geodesic from p to q. Fix $h^- \equiv h^-_{\rho(c_1 r)^2}$ satisfying Theorem 4.12(IV) for the balls of radius $c_1 r$ along this curve, where $\rho \in \big[\frac{1}{2}, 2\big]$.

From the above excess bound for z, by integral Abresch–Gromoll, there exists $B_{2r}(z)' \subseteq B_{2r}(z)$ such that

$$e(x) \le c(N,\delta)r^2 \quad \text{for all } x \in B_{2r}(z)' \qquad \text{and} \qquad \frac{m(B_{2r}(z)')}{m(B_{2r}(z))} \ge 1 - \tfrac{1}{2}V(1,10)^2. \tag{68}$$

For all $s \in [0, \epsilon]$ and $(x, y) \in X \times X$, define

$$dt_1(s)(x, y) := \min\Big\{r, \max_{0 \le \tau \le s} |d(x, y) - d(\Psi_\tau(x), \Psi_\tau(y))|\Big\} \tag{69}$$

and

$$U_1^s := \{(x, y) \in D_1 \times B_{2r}(z)' : dt_1(s)(x, y) < r\}. \tag{70}$$

Consider $\int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)\,d(m\times m)(x,y)$ for $0\le s\le\epsilon$. Since $r\le\bar r_1\le\frac{1}{10}\delta$ we have $\epsilon\le\epsilon_1\le\frac{1}{10}\delta$, and $t_0\ge\delta$, $(\Psi_s)_{s\in[0,\epsilon]}$ is a local flow of $-\nabla d_p$ from both D_1 and $B_{2r}(z)'$. Therefore, $s\mapsto\int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)\,d(m\times m)(x,y)$ is Lipschitz and

$$\begin{aligned}(71)\quad \frac{d}{ds}\int_{D_1\times B_{2r}(z)'} & dt_1(s)(x,y)\,d(m\times m)(x,y)\\ &\le\int_{U_1^s}\big(|\nabla h^- -\nabla d_p|(\Psi_s(x))+|\nabla h^- -\nabla d_p|(\Psi_s(y))\big)\,d(m\times m)(x,y)\\ &\quad+\int_0^1\int_{U_1^s} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau,\end{aligned}$$

for a.e. $s\in[0,\epsilon]$ by Proposition 3.27.

For any $s\in[0,\epsilon]$ and $(x,y)\in U_1^s$,

(I) $d(x,y)<3r$ since $D_1\subseteq B_r(z)$ and $B_{2r}(z)'\subseteq B_{2r}(z)$,

(II) $d(\Psi_s(x),\Psi_s(z))<2r$ since $D_1\subseteq A$ by definition (66),

(III) $dt_1(s)(x,y)<r$ by definition of U_1^s (70).

Therefore $\Psi_s(y)\in B_{6r}(\Psi_s(z))$ by the triangle inequality, and so

$$(\Psi_s,\Psi_s)(U_1^s)\subseteq B_{(c_1/2)r}(\Psi_s(z))\times B_{(c_1/2)r}(\Psi_s(z)),$$

since we assumed $c_1>100$. Therefore

$$\begin{aligned}(72)\quad \int_0^1\int_{U_1^s} & d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\\ &\le c(N,\delta)\int_0^1\int_{(\Psi_s,\Psi_s)(U_1^s)} d(x,y)|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde\gamma_{x,y}(\tau))\,d(m\times m)(x,y)\,d\tau\\ &\le c(N,\delta)rm(B_{(c_1/2)r}(\Psi_s(z)))\int_{B_{c_1r}(\Psi_s(z))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm\\ &\le c(N,\delta)rm(B_r(z))^2\fint_{B_{c_1r}(\Psi_s(z))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm,\end{aligned}$$

where the first inequality follows by Theorem 3.15(II), the second by the segment inequality (Theorem 3.22), and the last by Bishop–Gromov and property (I) of z. Integrating in $s\in[0,\epsilon]$,

$$\begin{aligned}(73)\quad \int_0^\epsilon\Big(\int_0^1\int_{U_1^s} & d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\Big)\,ds\\ &\le crm(B_r(z))^2\int_0^\epsilon\fint_{B_{c_1r}(\Psi_s(z))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm\,ds\\ &\le c(N,\delta)rm(B_r(z))^2\sqrt{\epsilon},\end{aligned}$$

where the last line follows from the definition of h^-, Theorem 4.12(IV) and Cauchy–Schwarz.

By Lemma 4.13(III), the excess bound on the elements of D_1 (66) (and therefore also on the elements $\Psi_s(D_1)$) and Bishop–Gromov,

$$(74)\qquad \int_0^\epsilon\int_{U_1^s}|\nabla h^- -\nabla d_p|(\Psi_s(x))\,d(m\times m)(x,y)\,ds\le c(N,\delta)rm(B_r(z))^2\sqrt{\epsilon}.$$

Similarly, by the excess bounds on the elements of $B_{2r}(z)'$ (68),

$$\int_0^{\epsilon}\int_{U_1^s}|\nabla h^- - \nabla d_p|(\Psi_s(y))\,d(m\times m)(x,y)\,ds \le c(N,\delta)rm(B_r(z))^2\sqrt{\epsilon}. \tag{75}$$

Combining (73)–(75) with the bound (71) on $(d/ds)\int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)$, we obtain

$$\begin{aligned}\int_{D_1\times B_{2r}(z)'} dt_1(\epsilon)(x,y)\,d(m\times m)(x,y) &= \int_0^{\epsilon}\Big[\frac{d}{ds}\int_{D_1\times B_{2r}(z)'} dt_1(s)(x,y)\,d(m\times m)(x,y)\Big]\,ds\\ &\le c(N,\delta)rm(B_r(z))^2\sqrt{\epsilon}.\end{aligned} \tag{76}$$

Since D_1 takes a nontrivial portion of the measure of $B_r(z)$ by (67),

$$⨍_{D_1}\int_{B_{2r}(z)'} dt_1(\epsilon)(x,y)\,dm(y) \le c(N,\delta)rm(B_r(z))\sqrt{\epsilon}.$$

In particular, there exists $z'\in D_1$ such that

$$\int_{B_{2r}(z)'} dt_1(\epsilon)(z',y)\,dm(y) \le crm(B_r(z))\sqrt{\epsilon}.$$

By definition of D_1, property (III) of z' is satisfied.

We next check property (II) is satisfied for the chosen z' as well if ϵ is sufficiently small. Define $B_r(z')' := B_r(z')\cap B_{2r}(z)'$. By the previous estimate and Bishop–Gromov,

$$\int_{B_r(z')'} dt_1(\epsilon)(z',y)\,dm(y) \le crm(B_r(z))\sqrt{\epsilon} \le c(N,\delta)rm(B_r(z'))\sqrt{\epsilon}. \tag{77}$$

Using this, we bound ϵ_1 sufficiently small depending on N and δ so that for $\epsilon\le\epsilon_1$,

$$\int_{B_r(z')'} dt_1(\epsilon)(z',y)\,dm(y) \le \tfrac{1}{4}rm(B_r(z'))V(1,10). \tag{78}$$

For example, $\epsilon_1 \le (V(1,10)/(4c))^2$ suffices, where c is the last one from (77). Moreover, $B_{2r}(z)'$ takes significant mass in $B_{2r}(z)$ from (68) and so by Bishop–Gromov,

$$\frac{m(B_r(z')')}{m(B_r(z'))} \ge 1-\bigg(\frac{m(B_{2r}(z)')}{m(B_{2r}(z))}\frac{m(B_{2r}(z))}{m(B_r(z'))}\bigg) \ge 1-\bigg(\frac{1}{2}\frac{(V(1,10)^2)}{V(1,3)}\bigg) \ge 1-\frac{1}{2}V(1,10). \tag{79}$$

Combining (78) and (79), we conclude there exists $A'\subseteq B_r(z')'$ such that

$$\frac{m(A')}{m(B_r(z'))} \ge 1-V(1,10) \qquad\text{and}\qquad dt_1(\epsilon)(z',y)\le \tfrac{1}{2}r \quad \text{for all } y\in A'. \tag{80}$$

The latter implies $\Psi_s(A')\subseteq B_{3r/2}(\Psi_s(z'))$ for any $s\le\epsilon$ and so property (II) of z' is satisfied.

This also gives one direction of the bound in property (I) for z'. For each $s\le\epsilon$,

$$\begin{aligned}\frac{m(B_r(\Psi_s(z')))}{m(B_r(z'))} &\ge V\big(1,\tfrac{3}{2}\big)\frac{m(B_{3r/2}(\Psi_s(z')))}{m(B_r(z'))} && \text{by Bishop–Gromov}\\ &\ge V\big(1,\tfrac{3}{2}\big)\frac{m(\Psi_s(A'))}{m(B_r(z'))}\\ &\ge V\big(1,\tfrac{3}{2}\big)(1+c(N,\delta)s)^{-N}\frac{m(A')}{m(B_r(z'))} && \text{by Theorem 3.15(II)}\\ &\ge V\big(1,\tfrac{3}{2}\big)(1+c(N,\delta)s)^{-N}(1-V(1,10)).\end{aligned}$$

We bound ϵ_1 sufficiently small depending on N and δ so that for $s \le \epsilon \le \epsilon_1$, the last line is greater than $2V(1,100)$.

The other direction of the bound in property (I) of z' will be proved similarly by sending a sufficiently large portion of $B_r(\Psi_s(z'))$ close to z' (in fact z) using a flow which does not decrease measure significantly and then using Bishop–Gromov. To do this, we first use the RLF associated to $-\nabla h^-$ to send a portion of $B_r(z')$ close to $\Psi_s(z')$. We then use the inverse flow (ie the RLF associated to ∇h^-) on the image of that portion to make sure a large enough portion of $B_r(\Psi_s(z'))$ indeed ends up close to z' under the inverse flow.

Then $|\nabla h_0^-| \in L^\infty(m)$ by (48) and so $|\nabla h^-|, \Delta h^- \in L^\infty(m)$ by the Bakry–Ledoux estimate (Theorem 2.16) and $h^- \in W^{2,2}(X)$ by Corollary 2.20 of the improved Bochner inequality. Therefore the time-independent vector fields $-\nabla h^-$ and ∇h^- are bounded and satisfy the conditions of the existence and uniqueness of RLFs (Theorem 3.4). Let $(\widetilde{\Psi}_t)_{t\in[0,1]}$ and $(\widetilde{\Psi}_{-t})_{t\in[0,1]}$ be the associated RLFs of $-\nabla h^-$ and ∇h^- for $t \in [0,1]$, respectively. The choice of notation is due to Proposition 3.12, which says $\widetilde{\Psi}_{-t}$ and $\widetilde{\Psi}_t$ are m-a.e. inverses of each other.

Since $e(z') \le c_1^2 r^2$ and $m(A') \ge (1 - V(1,10)) m(B_r(z'))$, integral Abresch–Gromoll gives $A'' \subseteq A'$ such that

$$e(x) \le c(N,\delta) r^2 \quad \text{for all } x \in A'' \qquad \text{and} \qquad \frac{m(A'')}{m(B_r(z'))} \ge 1 - 2V(1,10). \tag{81}$$

For all $s \in [0,\epsilon]$ and $(x,y) \in X \times X$, define

$$dt_2(s)(x,y) := \min\bigl\{r, \max_{0\le\tau\le s} |d(x,y) - d(\Psi_\tau(x), \widetilde{\Psi}_\tau(y))|\bigr\} \tag{82}$$

and

$$U_2^s := \{(x,y) \in A'' \times B_r(z') : dt_2(s)(x,y) < r\}. \tag{83}$$

Consider $\int_{A''\times B_r(z')} dt_2(s)(x,y)\, d(m\times m)(x,y)$ for $0 \le s \le \epsilon$. By Proposition 3.27, for a.e. $s \in [0,\epsilon]$,

$$\begin{aligned} \frac{d}{ds} \int_{A''\times B_r(z')} & dt_2(s)(x,y)\, d(m\times m)(x,y) \\ &\le \int_{U_2^s} |\nabla h^- - \nabla d_p|(\Psi_s(x))\, d(m\times m)(x,y) \\ &\quad + \int_0^1 \int_{U_2^s} d(\Psi_s(x), \widetilde{\Psi}_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\widetilde{\Psi}_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau. \end{aligned} \tag{84}$$

For any $s \in [0,\epsilon]$ and $(x,y) \in U_2^s$,

(I) $d(x,y) < 2r$ since $A'' \subseteq B_r(z')$,

(II) $d(\Psi_s(x), \Psi_s(z)) \le d(\Psi_s(x), \Psi_s(z')) + d(\Psi_s(z'), \Psi_s(z)) < \frac{7}{2} r$ by definition of A' (80) and $z' \in D_1 \subseteq A$ (66),

(III) $dt_2(s)(x,y) < r$ by definition of U_2^s (84).

Therefore $\widetilde{\Psi}_s(y) \in B_{13r/2}(\Psi_s(z))$ by the triangle inequality, and so

$$(\Psi_s, \widetilde{\Psi}_s)(U_2^s) \subseteq B_{(c_1/2)r}(\Psi_s(z)) \times B_{(c_1/2)r}(\Psi_s(z))$$

since $c_1 > 100$. Therefore

$$\begin{aligned}(85)\quad &\int_0^1 \int_{U_2^s} d(\Psi_s(x), \widetilde{\Psi}_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\widetilde{\Psi}_s(y)}(\tau))\, d(m \times m)(x,y)\, d\tau \\ &\le c(N,\delta) \int_0^1 \int_{(\Psi_s,\widetilde{\Psi}_s)(U_2^s)} d(x,y) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\, d(m \times m)(x,y) \\ &\le c(N,\delta) r m(B_{(c_1/2)r}(\Psi_s(z))) \int_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \\ &\le c(N,\delta) r m(B_r(z))^2 \fint_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm,\end{aligned}$$

where the first inequality follows from Theorem 3.15(II), Lemma 4.2 and (18) of Theorem 3.4, the second from the segment inequality (Theorem 3.22), and the last from Bishop–Gromov and property (I) of z. Integrating in $s \in [0, \epsilon]$,

$$\begin{aligned}(86)\quad &\int_0^\epsilon \Big(\int_0^1 \int_{U_2^s} d(\Psi_s(x), \widetilde{\Psi}_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\widetilde{\Psi}_s(y)}(\tau))\, d(m \times m)(x,y)\, d\tau \Big)\, ds \\ &\le c r m(B_r(z))^2 \int_0^\epsilon \fint_{B_{c_1 r}(\Psi_s(z))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm\, ds \\ &\le c(N,\delta) r m(B_r(z))^2 \sqrt{\epsilon},\end{aligned}$$

where the last line follows from the definition of h^-, Theorem 4.12(IV) and Cauchy–Schwarz.

By Lemma 4.13(III), the excess bound on the elements of A'' (81), and Bishop–Gromov,

$$(87)\quad \int_0^\epsilon \int_{U_2^s} |\nabla h^- - \nabla d_p|(\Psi_s(x))\, d(m \times m)(x,y)\, ds \le c(N,\delta) r m(B_r(z))^2 \sqrt{\epsilon}.$$

Combining (86) and (87) with the bound (84) we obtain

$$\begin{aligned}(88)\quad \int_{A'' \times B_r(z')} dt_2(\epsilon)(x,y)\, d(m \times m)(x,y) &= \int_0^\epsilon \Big[\frac{d}{ds} \int_{A'' \times B_r(z')} dt_2(s)(x,y)\, d(m \times m)(x,y) \Big]\, ds \\ &\le c(N,\delta) r m(B_r(z))^2 \sqrt{\epsilon}.\end{aligned}$$

A'' is comparable in measure to $B_r(z')$ by (81) and hence also to $B_r(z)$ by Bishop–Gromov. Therefore there exists $z_1 \in A''$ such that

$$\int_{B_r(z')} dt_2(\epsilon)(z_1, y)\, dm(y) \le c(N,\delta) r m(B_r(z)) \sqrt{\epsilon}.$$

By Bishop–Gromov, $m(B_r(z))/m(B_r(z')) \le c(N)$ and so

$$\fint_{B_r(z')} dt_2(\epsilon)(z_1, y)\, dm(y) \le c(N,\delta) r \sqrt{\epsilon}.$$

Using this, we bound ϵ_1 sufficiently small depending on N and δ so that there exists $D_2 \subseteq B_r(z')$ with

$$(89)\quad m(D_2)/m(B_r(z')) \ge 1 - V(1,10) \qquad \text{and} \qquad dt_2(\epsilon)(z_1, y) \le \tfrac{1}{2} r \quad \text{for all } y \in D_2.$$

For each $y \in D_2$ and $s \in [0, \epsilon]$,

(I) $d(z_1, y) < 2r$,

(II) $d(\Psi_s(z_1),\Psi_s(z)) \le d(\Psi_s(z_1),\Psi_s(z')) + d(\Psi_s(z'),\Psi_s(z)) < \frac{7}{2}r$,

(III) $dt_2(\epsilon)(z_1,y) \le \frac{1}{2}r$,

and so

$$\widetilde{\Psi}_s(D_2) \subseteq B_{4r}(\Psi_s(z')) \subseteq B_{6r}(\Psi_s(z)). \tag{90}$$

Moreover, $\widetilde{\Psi}_s(D_2)$ is nontrivial in measure compared to $B_r(z)$:

$$\begin{aligned}\frac{m(\widetilde{\Psi}_s(D_2))}{m(B_r(z))} &\ge e^{-c(N,\delta)(\delta/10)}\frac{m(D_2)}{m(B_r(z))} && \text{by Lemma 4.2, (18) of Theorem 3.4 and } \epsilon_1 \le \tfrac{1}{10}\delta \\ &\ge c(N,\delta) && \text{by definition of } D_2 \text{ and Bishop–Gromov.}\end{aligned} \tag{91}$$

We will now flow $\widetilde{\Psi}_s(D_2)$ back by $\widetilde{\Psi}_{-t}$ and use that to control the flow of $B_r(\Psi_s(z'))$ under $\widetilde{\Psi}_{-t}$. Fix $s\in[0,\epsilon]$. By Proposition 3.12, we may assume, up to choosing a full-measure subset, that D_2 satisfies

$$\widetilde{\Psi}_{-t}(\widetilde{\Psi}_s(x)) = \widetilde{\Psi}_{s-t}(x) \quad \text{for all } t\in[0,s] \text{ and for all } x\in D_2. \tag{92}$$

For all $t\in[0,s]$ and $(x,y)\in X\times X$, define

$$dt_3(t)(x,y) := \min\bigl\{r, \max_{0\le\tau\le t}|d(x,y) - d(\widetilde{\Psi}_{-\tau}(x),\widetilde{\Psi}_{-\tau}(y))|\bigr\} \tag{93}$$

and

$$U_3^t := \{(x,y)\in\widetilde{\Psi}_s(D_2)\times B_r(\Psi_s(z')) : dt_3(t)(x,y) < r\}. \tag{94}$$

We note U_3^t implicitly depends on s. Consider $\int_{\widetilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(t)(x,y)\,d(m\times m)(x,y)$ for $0\le t\le s$. By Proposition 3.27, for a.e. $t\in[0,s]$,

$$\begin{aligned}&\frac{d}{dt}\int_{\widetilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(t)(x,y)\,d(m\times m)(x,y)\\ &\qquad\le \int_0^1\int_{U_3^t} d(\widetilde{\Psi}_{-t}(x),\widetilde{\Psi}_{-t}(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\widetilde{\Psi}_{-t}(x),\widetilde{\Psi}_{-t}(y)}(\tau))\,d(m\times m)(x,y)\,d\tau.\end{aligned} \tag{95}$$

For any $t\in[0,s]$, $\omega\in[0,t]$ and $(x,y)\in U_3^t$,

(I) $d(x,y) < 5r$ since $\widetilde{\Psi}_s(D_2)\subseteq B_{4r}(\Psi_s(z'))$ by (90),

(II) $d(\widetilde{\Psi}_{-\omega}(x),\Psi_{s-\omega}(z)) = d(\widetilde{\Psi}_{s-\omega}(x'),\Psi_{s-\omega}(z)) < 6r$ for some $x'\in D_2$ by (90) and (92),

(III) $dt_3(t)(x,y) < r$ by definition of U_3^t (94).

Hence,

$$\widetilde{\Psi}_{-\omega}(y) \in B_{12r}(\Psi_{s-\omega}(z)) \tag{96}$$

by the triangle inequality. Therefore $(\widetilde{\Psi}_{-\omega},\widetilde{\Psi}_{-\omega})(U_3^t)\subseteq B_{(c_1/2)r}(\Psi_{s-\omega}(z))\times B_{(c_1/2)r}(\Psi_{s-\omega}(z))$ for all $\omega\in[0,t]$ since $c_1>100$. For any $(x,y)\in U_3^t$,

$$\begin{aligned}\Delta h^-(\widetilde{\Psi}_{-\omega}(x)) &= \Delta h^+(\widetilde{\Psi}_{-\omega}(x)) + \Delta\hat{e}(\widetilde{\Psi}_{-\omega}(x))\\ &\ge -c(N,\delta) && \text{by Lemmas 4.2 and 4.3(III), using } e(z)\le c_1^2r^2,\end{aligned} \tag{97}$$

where h^+ and $\hat{e}$ are heat flow approximations of h_0^+ and e_0, respectively, up to the same time as h^-. We have the same bound for $\Delta h^-(\widetilde{\Psi}_{-\omega}(y))$. Therefore,

$$\begin{aligned}(98)\quad &\int_0^1\int_{U_3^t} d(\widetilde{\Psi}_{-t}(x),\widetilde{\Psi}_{-t}(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\widetilde{\Psi}_{-t}(x),\widetilde{\Psi}_{-t}(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\\ &\le c(N,\delta)\int_0^1\int_{(\widetilde{\Psi}_{-t},\widetilde{\Psi}_{-t})(U_3^t)} d(x,y)|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\,d(m\times m)(x,y)\,d\tau\\ &\le c(N,\delta)rm(B_{(c_1/2)r}(\Psi_{s-t}(z)))\int_{B_{c_1r}(\Psi_{s-t}(z))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm\\ &\le c(N,\delta)rm(B_r(z))^2 ⨍_{B_{c_1r}(\Psi_{s-t}(z))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm,\end{aligned}$$

where the first inequality follows by (97) and Remark 3.5, the second by the segment inequality (Theorem 3.22), and the third by Bishop–Gromov and property (I) of z. Integrating in $t\in[0,s]$,

$$\begin{aligned}(99)\quad &\int_0^s\Big(\int_0^1\int_{U_3^t} d(\widetilde{\Psi}_{-t}(x),\widetilde{\Psi}_{-t}(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\widetilde{\Psi}_{-t}(x),\widetilde{\Psi}_{-t}(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\Big)\,dt\\ &\le crm(B_r(z))^2\int_0^s ⨍_{B_{c_1r}(\Psi_{s-t}(z))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm\,ds\\ &\le c(N,\delta)rm(B_r(z))^2\sqrt{s},\end{aligned}$$

where the last line follows from the definition of h^-, Theorem 4.12(IV) and Cauchy–Schwarz. Therefore,

$$\begin{aligned}(100)\quad &\int_{\widetilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(s)(x,y)\,d(m\times m)(x,y)\\ &=\int_0^s\Big[\frac{d}{dt}\int_{\widetilde{\Psi}_s(D_2)\times B_r(\Psi_s(z'))} dt_3(t)(x,y)\,d(m\times m)(x,y)\Big]\,dt\\ &\le c(N,\delta)rm(B_r(z))^2\sqrt{s}.\end{aligned}$$

We previously computed that $\widetilde{\Psi}_s(D_2)$ is nontrivial in measure compared to $B_r(z)$ in (91), and so there exists $z_2\in\widetilde{\Psi}_s(D_2)$ with

$$\int_{B_r(\Psi_s(z'))} dt_3(s)(z_2,y)\,dm(y)\le c(N,\delta)rm(B_r(z))\sqrt{s}.$$

By Bishop–Gromov, $m(B_r(\Psi_s(z')))/m(B_r(\Psi_s(z)))\ge c(N)$ and so by property (I) of z,

$$⨍_{B_r(\Psi_s(z'))} dt_3(s)(z_2,y)\,dm(y)\le c(N,\delta)r\sqrt{s}.$$

Using this, we bound ϵ_1 sufficiently small depending on N and δ so there exists $D_3\subseteq B_r(\Psi_s(z'))$ with

$$(101)\quad \frac{m(D_3)}{m(B_r(\Psi_s(z')))}\ge 1-V(1,10)\qquad\text{and}\qquad dt_3(s)(z_2,y)\le\tfrac{1}{2}r\quad\text{for all } y\in D_3.$$

For each $y\in D_3$,

- (I) $d(z_2,y)<5r$ by (90),
- (II) $d(\widetilde{\Psi}_{-s}(z_2),z')=d(z_2',z')<r$, where $z_2'\in D_2\subseteq B_r(z')$ is such that $\widetilde{\Psi}_s(z_2')=z_2$,
- (III) $dt_3(s)(z_2,y)\le\frac{1}{2}r$,

and so $\widetilde{\Psi}_{-s}(D_3) \subseteq B_{7r}(z')$. Notice also for the next calculation that $\widetilde{\Psi}_{-t}(D_3) \subseteq B_{12r}(z)$ for any $t \in [0,s]$ by the calculations of (96). Therefore one has the same lower bound for the Laplacian of h^- on $\widetilde{\Psi}_{-t}(D_3)$ as in (97).

We estimate

$$\begin{aligned}
\frac{m(B_r(\Psi_s(z')))}{m(B_r(z'))} &\le \frac{1}{V(1,7)}\frac{m(B_r(\Psi_s(z')))}{m(B_{7r}(z'))} && \text{by Bishop–Gromov} \\
&\le \frac{1}{V(1,7)}\frac{1}{1-V(1,10)}\frac{m(D_3)}{m(B_{7r}(z'))} && \text{by property (101) of } D_3 \\
&\le \frac{1}{V(1,7)}\frac{1}{1-V(1,10)}e^{c(N,\delta)s}\frac{m(\widetilde{\Psi}_{-s}(D_3))}{m(B_{7r}(z'))} && \text{by Remark 3.5} \\
&\le \frac{1}{V(1,7)}\frac{1}{1-V(1,10)}e^{cs}.
\end{aligned}$$

We bound ϵ_1 sufficiently small depending on N and δ so that for $s \le \epsilon \le \epsilon_1$, the last line is less than $1/(2V(1,100))$.

All this implies that if (I)–(III) hold for $z \in B_r(\gamma(t_0))$ and $\epsilon \le \epsilon_1(N,\delta)$, then there exists $z' \in B_r(\gamma(t_0))$ satisfying (I)–(III) for the same ϵ. By the Bishop–Gromov volume comparison and the fact that Ψ is defined using unit-speed geodesics, there is some $T(N,\delta,r) > 0$ such that (I) and (II) hold for z', A' and $0 \le s \le \epsilon + T$. Combining this with the existence of some z and ϵ depending on N, δ, and r satisfying (I)–(III) mentioned at the beginning of the proof, we conclude there exists $z \in B_r(\gamma(t_0))$ such that (I)–(III) hold for $\epsilon = \epsilon_1$. □

5.2 Construction of limit geodesics

In this subsection we construct a geodesic between p and q that satisfies Lemma 5.1(I)–(II) on the geodesic itself. Roughly, this means we construct $\bar{\gamma}$ so that small balls centered on $\bar{\gamma}$ between δ and $1-\delta$ stay close to the geodesic itself for a short amount of time under the flows Ψ and Φ. The main idea is that for any z as in the main lemma from the previous section, the good behavior on the scale of r (in the sense that Lemma 5.1(I)–(II) hold on the scale of r) can be bootstrapped to scales $r' > r$ for any r' less than some uniform $\bar{r}_3$ depending only on structural constants. This means that each z will have good behavior from scales r to $\bar{r}_3$. Taking a sequence of $z_i \to \gamma(t_0)$ corresponding to $r_i \to 0$ and so that sequence of geodesics from z_i to p converges gives a geodesic from $\gamma(t_0)$ to p which has good behavior for any scale $r \in (0, \bar{r}_3]$.

We start by showing that for any fixed scale r we can find points z arbitrarily close to $\gamma(t_0)$ which satisfy Lemma 5.1(I)–(III). In order to do this we will prove the following lemma, which will form our induction step:

Lemma 5.2 *There exists $\epsilon_2(N,\delta) > 0$ and $\bar{r}_2(N,\delta) > 0$ such that for any $r \le \bar{r}_2$ and $\delta \le t_0 \le 1-\delta$, if there exist $z \in B_r(\gamma(t_0))$ and $\epsilon \le \epsilon_2$ such that*

(I) $V(1,100) \le m(B_r(\Psi_s(z)))/m(B_r(z)) \le 1/V(1,100)$ *for any* $s \le \epsilon$,

(II) *there exists* $A_r \subseteq B_r(z)$ *with* $m(A_r) \geq (1 - V(1,10))m(B_r(z))$ *and* $\Psi_s(A_r) \subseteq B_{2r}(\Psi_s(z))$ *for any* $s \leq \epsilon$,

(III) $e(z) \leq c_1^2 r^2$,

then for the same z *and any* $r' \in [4r, 16r]$,

(i) $V(1,100) \leq m(B_{r'}(\Psi_s(z)))/m(B_{r'}(z)) \leq 1/V(1,100)$ *for any* $s \leq \epsilon$,

(ii) *There exists* $A_{r'} \subseteq B_{r'}(z)$ *with* $m(A_{r'}) \geq (1 - V(1,10))m(B_{r'}(z))$ *and* $\Psi_s(A_{r'}) \subseteq B_{2r'}(\Psi_s(z))$ *for any* $s \leq \epsilon$,

(iii) $e(z) \leq c_1^2 r^2 < c_1^2 (r')^2$.

Proof Fix $\delta \leq t_0 \leq 1-\delta$. We assume $\bar{r}_2 \leq \frac{1}{1000}\delta$ and $\epsilon_2 \leq \frac{1}{1000}\delta$ to begin with, but will impose more bounds on both depending on N and δ as the proof continues. We will not keep track of $\bar{r}_2$ for the sake of brevity. Fix a scale $r \leq \bar{r}_2$ and $r' \in [4r, 16r]$. Fix z, A_r, and $\epsilon \leq \epsilon_2$ so that (I)–(III) hold.

Since $e(z) \leq c_1^2 r^2$, by integral Abresch–Gromoll there exists $B_{r'}(z)' \subseteq B_{r'}(z)$ such that

$$(102)\qquad e(x) \leq c(N,\delta) r^2 \quad \text{for all } x \in B_{r'}(z)' \qquad \text{and} \qquad \frac{m(B_{r'}(z)')}{m(B_{r'}(z))} \geq 1 - \tfrac{1}{2}V(1,10).$$

Similarly, there exists $A' \subseteq A_r$ such that

$$(103)\qquad e(x) \leq c(N,\delta) r^2 \quad \text{for all } x \in A' \qquad \text{and} \qquad \frac{m(A')}{m(B_r(z))} \geq 1 - 2V(1,10).$$

As in the previous lemma, the curve traversing $\gamma_{z,p}$ in reverse and then $\gamma_{z,q}$ is a $c_1 r$-geodesic from p to q. Fix $h^- \equiv h^-_{\rho(c_1 r)^2}$ satisfying Theorem 4.12(IV) for the balls of radius $c_1 r$ along this curve, where $\rho \in \left[\frac{1}{2}, 2\right]$.

For all $s \in [0,\epsilon]$ and $(x,y) \in X \times X$, define

$$(104)\qquad dt_1(s)(x,y) := \min\Big\{r, \max_{0 \leq \tau \leq s} |d(x,y) - d(\Psi_\tau(x), \Psi_\tau(y))|\Big\}$$

and

$$(105)\qquad U_1^s := \{(x,y) \in A' \times B_{r'}(z)' : dt_1(s)(x,y) < r\}.$$

Consider $\int_{A' \times B_{r'}(z)'} dt_1(s)(x,y)\, d(m \times m)(x,y)$ for $0 \leq s \leq \epsilon$. For any $s \in [0,\epsilon]$ and $(x,y) \in U_1^s$,

(I) $d(x,y) < r' + r$,

(II) $d(\Psi_s(x), \Psi_s(z)) < 2r$ by definition of $A' \subseteq A_r$,

(III) $dt_1(s)(x,y) < r$.

Therefore $\Psi_s(y) \in B_{r'+4r}(\Psi_s(z)) \subseteq B_{(c_1/2)r}(\Psi_s(z))$ since $c_1 > 100$.

Using exactly the same type of computation as the first part of the proof of the main lemma, by interpolating between the two local flows of Ψ_s from A' and $B_{r'}(z)'$ with ∇h^-, we obtain

$$\int_{A'\times B_{r'}(z)'} dt_1(\epsilon)(x,y)\,d(m\times m)(x,y) \le c(N,\delta) r m(B_r(z))^2\sqrt{\epsilon}. \tag{106}$$

Since A' takes a significant portion of the measure of $B_r(z)$ by (103),

$$⨍_{A'}\int_{B_{r'}(z)'} dt_1(\epsilon)(x,y)\,dm(y) \le c(N,\delta) r m(B_r(z))\sqrt{\epsilon},$$

and so there exists $z'\in A'$ such that

$$\int_{B_{r'}(z)'} dt_1(\epsilon)(z',y)\,dm(y) \le c r m(B_r(z))\sqrt{\epsilon}.$$

Therefore, by the fact that $m(B_{r'}(z)')/m(B_{r'}(z)) \ge 1-\frac{1}{2}V(1,10)$ and Bishop–Gromov,

$$⨍_{B_{r'}(z)'} dt_1(\epsilon)(z',y)\,dm(y) \le c(N,\delta) r\sqrt{\epsilon}.$$

Using this, we bound ϵ_2 sufficiently small depending on N and δ so that there exists $A_{r'}\subseteq B_{r'}(z)'$ with

$$\frac{m(A_{r'})}{m(B_{r'}(z))} \ge 1-V(1,10) \qquad \text{and} \qquad dt_1(\epsilon)(z',y)\le \tfrac{1}{2}r \quad \text{for all } y\in A_{r'}. \tag{107}$$

For each $y\in A_{r'}$ and $s\in[0,\epsilon]$,

(I) $d(z',y) < r'+r$,

(II) $d(\Psi_s(z'),\Psi_s(z)) < 2r$,

(III) $dt_1(\epsilon)(z',y)\le \frac{1}{2}r$,

and so

$$\Psi_s(A_{r'}) \subseteq B_{r'+7r/2}(\Psi_s(z)) \subseteq B_{2r'}(\Psi_s(z)). \tag{108}$$

This proves (ii).

This also gives one direction of the bound in (i). For each $s\le\epsilon$,

$$\begin{aligned}
\frac{m(B_{r'}(\Psi_s(z)))}{m(B_{r'}(z))} &\ge V(1,2)\frac{m(B_{2r'}(\Psi_s(z)))}{m(B_{r'}(z'))} && \text{by Bishop–Gromov}\\
&\ge V(1,2)\frac{m(\Psi_s(A_{r'}))}{m(B_{r'}(z'))}\\
&\ge V(1,2)(1+c(N,\delta)s)^{-N}\frac{m(A_{r'})}{m(B_{r'}(z'))} && \text{by Theorem 3.15(II)}\\
&\ge V(1,2)(1+c(N,\delta)s)^{-N}(1-V(1,10)).
\end{aligned}$$

We bound ϵ_2 sufficiently small depending on N and δ so that for $s\le\epsilon\le\epsilon_2$, the last line is greater than $V(1,100)$.

To obtain the other direction of the bound in (i), we employ the same strategy as the proof of the main lemma as well. Let $(\widetilde{\Psi}_t)_{t\in[0,1]}$ and $(\widetilde{\Psi}_{-t})_{t\in[0,1]}$ be the RLFs of the time-independent vector fields $-\nabla h^-$ and ∇h^-, respectively, as before.

For all $s \in [0,\epsilon]$ and $(x,y) \in X \times X$, define

$$dt_2(s)(x,y) := \min\bigl\{r, \max_{0\le\tau\le s} |d(x,y) - d(\Psi_\tau(x), \widetilde{\Psi}_\tau(y))|\bigr\} \tag{109}$$

and

$$U_2^s := \{(x,y) \in A' \times B_r(z) : dt_2(s)(x,y) < r\}. \tag{110}$$

Consider $\int_{A'\times B_r(z)} dt_2(s)(x,y)\, d(m\times m)(x,y)$ for $0 \le s \le \epsilon$. For any $s \in [0,\epsilon]$ and $(x,y) \in U_2^s$,

(I) $d(x,y) < 2r$,

(II) $d(\Psi_s(x), \Psi_s(z)) < 2r$ by definition of $A' \subseteq A_r$,

(III) $dt_2(s)(x,y) < r$.

Therefore $\widetilde{\Psi}_s(y) \in B_{5r}(\Psi_s(z)) \subseteq B_{(c_1/2)r}(\Psi_s(z))$.

Using exactly the same type of computation as the second part of the proof of the main lemma,

$$\int_{A'\times B_r(z)} dt_2(\epsilon)(x,y)\, d(m\times m)(x,y) \le c(N,\delta) r m(B_r(z))^2 \sqrt{\epsilon}. \tag{111}$$

A' takes a significant portion of the measure of $B_r(z)$ by (103). The same considerations as before gives the existence of $D_1 \subseteq B_r(z)$ such that

$$\frac{m(D_1)}{m(B_r(z))} \ge 1 - V(1,10) \quad \text{and} \quad \widetilde{\Psi}_s(D_1) \subseteq B_{5r}(\Psi_s(z)), \tag{112}$$

for any $s \in [0,\epsilon]$ after we bound ϵ_2 sufficiently small depending only on N and δ. Moreover, $\widetilde{\Psi}_s(D_1)$ is nontrivial in measure compared to $B_r(z)$.

$$\begin{aligned} \frac{m(\widetilde{\Psi}_s(D_1))}{m(B_r(z))} &\ge e^{-c(N,\delta)(\delta/1000)} \frac{m(D_1)}{m(B_r(z))} && \text{by Lemma 4.2, (18) in Theorem 3.4 and } \epsilon_2 \le \tfrac{1}{1000}\delta \\ &\ge c(N,\delta) && \text{by definition of } D_1. \end{aligned} \tag{113}$$

Fix $s \in [0,\epsilon]$. By Proposition 3.12, we may assume, up to choosing a full-measure subset, that D_1 satisfies

$$\widetilde{\Psi}_{-t}(\widetilde{\Psi}_s(x)) = \widetilde{\Psi}_{s-t}(x) \quad \text{for all } t \in [0,s] \text{ and for all } x \in D_1. \tag{114}$$

For all $t \in [0,s]$ and $(x,y) \in X \times X$, define

$$dt_3(t)(x,y) := \min\bigl\{r, \max_{0\le\tau\le t} |d(x,y) - d(\widetilde{\Psi}_{-\tau}(x), \widetilde{\Psi}_{-\tau}(y))|\bigr\} \tag{115}$$

and

$$U_3^t := \{(x,y) \in \widetilde{\Psi}_s(D_1) \times B_{r'}(\Psi_s(z)) : dt_3(t)(x,y) < r\}. \tag{116}$$

Consider $\int_{\widetilde{\Psi}_s(D_1)\times B_{r'}(\Psi_s(z))} dt_3(t)(x,y)\,d(m\times m)(x,y)$ for $0\le t\le s$. For any $t\in[0,s]$, $\omega\in[0,t]$ and $(x,y)\in U_3^t$,

(I) $d(x,y)<r'+5r$ by (112),

(II) $d(\widetilde{\Psi}_{-\omega}(x),\Psi_{s-\omega}(z))=d(\widetilde{\Psi}_{s-\omega}(x'),\Psi_{s-\omega}(z))<5r$ for some $x'\in D_1$ by (112) and (114),

(III) $dt_3(t)(x,y)<r$.

Hence

$$\widetilde{\Psi}_{-\omega}(y)\in B_{r'+11r}(\Psi_{s-\omega}(z)) \tag{117}$$

by the triangle inequality. Therefore $(\widetilde{\Psi}_{-\omega},\widetilde{\Psi}_{-\omega})(U_3^t)\subseteq B_{(c_1/2)r}(\Psi_{s-\omega}(z))\times B_{(c_1/2)r}(\Psi_{s-\omega}(z))$ for any $\omega\in[0,t]$ since $c_1>100$.

Using exactly the same type of computation as the third part of the proof of the main lemma,

$$\int_{\widetilde{\Psi}_s(D_1)\times B_{r'}(\Psi_s(z))} dt_3(\epsilon)(x,y)\,d(m\times m)(x,y)\le c(N,\delta)rm(B_r(z))^2\sqrt{\epsilon}. \tag{118}$$

By (113), $\widetilde{\Psi}_s(D_1)$ is nontrivial in measure compared to $B_r(z)$. By Bishop–Gromov and property (I) of z, $B_{r'}(\Psi_s(z))$ is also nontrivial in measure compared to $B_r(z)$. The same considerations as before gives the existence of $D_2\subseteq B_{r'}(\Psi_s(z))$ such that

$$\frac{m(D_2)}{m(B_{r'}(\Psi_s(z)))}\ge 1-V(1,10)\quad\text{and}\quad \widetilde{\Psi}_{-s}(D_2)\subseteq B_{r'+7r}(z)\subseteq B_{3r'}(z), \tag{119}$$

for any $s\in[0,\epsilon]$ after we bound ϵ_2 sufficiently small depending only on N and δ.

We estimate

$$\begin{aligned}
\frac{m(B_{r'}(\Psi_s(z)))}{m(B_{r'}(z))} &\le \frac{1}{V(1,3)}\frac{m(B_{r'}(\Psi_s(z)))}{m(B_{3r'}(z))} && \text{by Bishop–Gromov}\\
&\le \frac{1}{V(1,3)}\frac{1}{1-V(1,10)}\frac{m(D_2)}{m(B_{3r'}(z))} && \text{by property (119) of } D_2\\
&\le \frac{1}{V(1,3)}\frac{1}{1-V(1,10)}e^{c(N,\delta)s}\frac{m(\widetilde{\Psi}_{-s}(D_2))}{m(B_{3r'}(z))} && \text{by Remark 3.5}\\
&\le \frac{1}{V(1,7)}\frac{1}{1-V(1,10)}e^{cs},
\end{aligned}$$

where for the third inequality we used the fact that $\widetilde{\Psi}_{-t}(D_2)\subseteq B_{27r}(\Psi_{s-t}(z))$ for $0\le t\le s$ by the calculations of (117). On these sets Δh^- is bounded below by $-c(N,\delta)$ by the same argument as (97). Using this, we bound ϵ_2 sufficiently small depending on N and δ so that for $s\le\epsilon\le\epsilon_2$, the last line is less than $V(1,100)$. This shows the other half of the bound in (i). Property (iii) is obvious. □

Combined with the main lemma, this gives:

Lemma 5.3 *There exists $\epsilon_3(N,\delta) > 0$ and $\bar{r}_3(N,\delta) > 0$ such that for any $r \le \bar{r}_3$ and $\delta \le t_0 \le 1-\delta$, there exists $z \in B_r(\gamma(t_0))$ such that for any $r \le r' \le \bar{r}_3$,*

(I) $V(1,100) \le m(B_{r'}(\Psi_s(z)))/m(B_{r'}(z)) \le 1/V(1,100)$ *for any $s \le \epsilon_3$,*

(II) *there exists $A \subseteq B_{r'}(z)$ with $m(A) \ge (1-V(1,10))m(B_{r'}(z))$ and $\Psi_s(A) \subseteq B_{2r'}(\Psi_s(z))$ for any $s \le \epsilon_3$,*

(III) $e(z) \le c_1^2 r^2$.

Proof Choose $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ and $\bar{r}_3 := \min\{\bar{r}_1, \bar{r}_2\}$. Apply Lemma 5.1 to find some $z \in B_{r/4}(\gamma(t_0))$ which satisfies (I)–(III) on the scale of $\frac{1}{4}r$ and then use Lemma 5.2 repeatedly to conclude. □

The following corollary immediately follows from Lemma 5.3 by a limiting argument:

Corollary 5.4 *There exist $\epsilon_3(N,\delta) > 0$ and $\bar{r}_3(N,\delta) > 0$ such that if γ is the unique geodesic between p and q, then for all $r \le \bar{r}_3$ and $\delta \le t_0 \le 1-\delta$,*

(I) $V(1,100) \le m(B_r(\gamma(t_0-s)))/m(B_r(\gamma(t_0))) \le 1/V(1,100)$ *for any $0 \le s \le \epsilon_3$,*

(II) *there exists $A \subseteq B_r(\gamma(t_0))$ with $m(A) \ge (1-V(1,10))m(B_r(\gamma(t_0)))$ and $\Psi_s(A) \subseteq B_{2r}(\gamma(t_0-s))$ for any $0 \le s \le \epsilon_3$.*

Indeed, by taking a sequence $z_i \in B_{1/i}(\gamma(t_0))$ as in Lemma 5.3, we see that the $\Psi_\cdot(z_i)$ necessarily converge to $\gamma(t_0 - \cdot)$ since γ is by assumption a unique geodesic. The properties for $\Psi_\cdot(z_i)$ then pass over to the desired properties for γ as in Corollary 5.4.

We can use the corollary directly in the proof of Theorem 5.10 to prove the main result in the case where there is a unique geodesic between p and q. Taking Theorem 2.25 and Remark 2.26 into account, this is already enough for several applications. Nevertheless, we will next prove the existence of a geodesic between any $p, q \in X$ with Hölder continuity on the geometry of small-radius balls in its interior, which is the full result of [30]. The desired geodesic will be constructed using multiple limiting and gluing arguments.

Lemma 5.5 *There exist $\epsilon_4(N,\delta) > 0$ and $\bar{r}_4(N,\delta) > 0$ such that for any unit-speed geodesic γ from p to q, there exists a unit-speed geodesic γ^δ from p to q with $\gamma^\delta \equiv \gamma$ on $[1-\delta, 1]$ such that for all $r \le \bar{r}_4$ and $\delta \le t_0 \le t_1 \le 1-\delta$, if $t_1 - t_0 \le \epsilon_4$, then*

(I) $V(1,100)^4 \le m(B_r(\gamma^\delta(t_1)))/m(B_r(\gamma^\delta(t_0))) \le 1/V(1,100)^4$,

(II) *there exists $A \subseteq B_r(\gamma^\delta(t_1))$ such that $m(A) \ge (1-V(1,10))m(B_r(\gamma^\delta(t_1)))$ and $\Psi_s(A) \subseteq B_{2r}(\gamma^\delta(t_1 - s))$ for all $s \in [0, t_1 - t_0]$.*

Proof Let ϵ_3 and $\bar{r}_3$ be from Lemma 5.3. We begin by assuming $\bar{r}_4 \le \bar{r}_3$ and $\epsilon_4 \le \frac{1}{2}\epsilon_3$, but will impose more bounds on both as the proof continues. We will not keep track of $\bar{r}_4$ for the sake of brevity.

Partition $[\delta, 1-\delta]$ by $\{\sigma_0 = \delta, \sigma_1, \ldots, \sigma_k = 1-\delta\}$ for some $k \in \mathbb{N}$ so that all subintervals have the same width equal to some $\omega \in \left[\frac{1}{2}\epsilon_3, \epsilon_3\right]$. This is always possible since the width of the original interval is $1-2\delta > 0.8$ and ϵ_3 is much smaller than 0.4. We construct γ^δ inductively as follows:

- Define $\gamma^\delta \equiv \gamma$ for $t \in [1-\delta, 1]$.

- Let $1 \le i \le k$. Assume γ^δ has been constructed on the interval $[\sigma_i, 1]$. Fix a sequence of $r_j \to 0$. For each j, choose $z_j \in B_{r_j}(\gamma^\delta(\sigma_i))$ as in the statement of Lemma 5.3. Use the Arzelà–Ascoli theorem to take a limit, after passing to a subsequence, of the unit-speed geodesics from z_j to p. This limit $\gamma^{\delta,i}$ is a unit-speed geodesic from $\gamma^\delta(\sigma_i)$ to p. For $t \in [\sigma_{i-1}, \sigma_i]$, define $\gamma^\delta(t) = \gamma^{\delta,i}(\sigma_i - t)$.

- For $t \in [0, \delta]$, define γ^δ to be any geodesic from p to $\gamma^\delta(\sigma_0)$.

For any $1 \le i \le k$ we have $\sigma_i - \sigma_{i-1} \le \epsilon_3$ so it follows from the construction that γ^δ satisfies, for any $r \le \bar{r}_4$,

(i) $V(1,100) \le m(B_r(\gamma^\delta(\sigma_i - s)))/m(B_r(\gamma^\delta(\sigma_i))) \le 1/V(1,100)$ for any $0 \le s \le \omega$,

(ii) there exists $A \subseteq B_r(\gamma^\delta(\sigma_i))$ such that $m(A) \ge (1 - V(1,10))m(B_r(\gamma^\delta(\sigma_i)))$ and $\Psi_s(A) \subseteq B_{2r}(\gamma^\delta(\sigma_i - s))$ for any $0 \le s \le \omega$.

Fix $\delta \le t_0 \le t_1 \le 1-\delta$ with $t_1 - t_0 \le \epsilon_4$ and a scale $r \le \bar{r}_4$. Since $\epsilon_4 \le \frac{1}{2}\epsilon_3$ is no greater than the widths of the subintervals of the partition $\omega \ge \frac{1}{2}\epsilon_3$, t_0 and t_1 must be contained in $[\sigma_{i-2}, \sigma_i]$ for some $2 \le i \le k$. Statement (I) of the lemma then follows trivially from property (i) of γ^δ.

Then t_0 and t_1 must then be either contained in a single subinterval or two neighboring subintervals of the partition. We will assume the second case; the first case follows from a similar and simpler argument. Let $t_1 \in (\sigma_{i-1}, \sigma_i]$ and $t_0 \in (\sigma_{i-2}, \sigma_{i-1}]$ for some i. By property (ii) of γ^δ and Abresch–Gromoll, there exists $A_1 \subseteq B_{r/16}(\gamma^\delta(\sigma_i))$ such that

(I) $m(A_1)/m(B_{r/16}(\gamma^\delta(\sigma_i))) \ge 1 - 2V(1,10)$,

(II) $\Psi_s(A_1) \subseteq B_{r/8}(\gamma^\delta(\sigma_i - s))$ for all $s \in [0, \omega]$,

(III) $e(x) \le c(N,\delta)r^2$ for all $x \in A_1$.

Similarly, there exists $A_2 \subseteq B_{r/16}(\gamma^\delta(\sigma_{i-1}))$ such that

(I) $m(A_2)/m(B_{r/16}(\gamma^\delta(\sigma_{i-1}))) \ge 1 - 2V(1,10)$,

(II) $\Psi_s(A_2) \subseteq B_{r/8}(\gamma^\delta(\sigma_{i-1} - s))$ for all $s \in [0, \omega]$,

(III) $e(x) \le c(N,\delta)r^2$ for all $x \in A_2$.

The plan is as follows: first we show a significant portion of A_1 can be flowed by Ψ a nontrivial amount of time past $\gamma^\delta(\sigma_{i-1})$ while staying close γ^δ, then we use the flow of A_1 under Ψ to control the flow of $B_r(\gamma^\delta(t_1))$ under Ψ.

Fix $h^- \equiv h^-_{\rho(8r)^2}$ satisfying Theorem 4.12(IV) for the balls of radius $8r$ along γ^δ, where $\rho \in \left[\frac{1}{2}, 2\right]$. For all $s \in [0, \omega]$ and $(x, y) \in X \times X$, define

$$dt(s)(x, y) := \min\left\{r, \max_{0 \le \tau \le s} |d(x, y) - d(\Psi_\tau(x), \Psi_\tau(y))|\right\} \tag{120}$$

and

$$U_1^s := \{(x, y) \in A_2 \times A_1 : dt(s)(x, \Psi_\omega(y)) < r\}. \tag{121}$$

Consider $\int_{A_2 \times A_1} dt(s)(x, \Psi_\omega(y))\, d(m \times m)(x, y)$ for $0 \le s \le \omega$.

For any $s \in [0, \omega]$ and $(x, y) \in U_1^s$,

(I) $d(x, \Psi_\omega(y)) < \frac{3}{16} r$,

(II) $d(\Psi_s(x), \gamma^\delta(\sigma_{i-1} - s)) < \frac{1}{8} r$,

(III) $dt(s)(x, \Psi_\omega(y)) < r$.

Therefore $\Psi_s(\Psi_\omega(y)) \in B_{r+5r/16}(\gamma^\delta(\sigma_{i-1} - s))$ and so

$$(\Psi_s, \Psi_s \circ \Psi_\omega)(U_1^s) \subseteq B_{4r}(\gamma^\delta(\sigma_{i-1} - s)) \times B_{4r}(\gamma^\delta(\sigma_{i-1} - s)).$$

By Remark 2.26, $\Psi_s \circ \Psi_\omega = \Psi_{s+\omega}$ m-a.e. Since we may always choose subsets of full measure where the equality is satisfied, we will replace the former with the latter freely.

We have

$$\begin{aligned} &\int_0^1 \int_{U_1^s} d(\Psi_s(x), \Psi_{s+\omega}(y)))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x), \Psi_{s+\omega}(y)}(\tau))\, d(m \times m)(x, y)\, d\tau \\ &\quad \le c(N, \delta) \int_0^1 \int_{(\Psi_s, \Psi_{s+\omega})(U_1^s)} d(x, y) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\, d(m \times m)(x, y)\, d\tau \\ &\quad \le c(N, \delta) r m(B_{4r}(\gamma^\delta(\sigma_{i-1} - s))) \int_{B_{8r}(\gamma^\delta(\sigma_{i-1} - s))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm \\ &\quad \le c(N, \delta) r m(B_r(\gamma^\delta(\sigma_i)))^2 ⨍_{B_{8r}(\gamma^\delta(\sigma_{i-1} - s))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm, \end{aligned} \tag{122}$$

where the first inequality follows by Theorem 3.15(II), the second by the segment inequality (Theorem 3.22), and the third by Bishop–Gromov and property (i) of γ^δ. Integrating in $s \in [0, \omega']$ for some $\omega' \in (0, \omega]$ to be fixed later, we have

$$\begin{aligned} &\int_0^{\omega'} \left(\int_0^1 \int_{U_1^s} d(\Psi_s(x), \Psi_{s+\omega}(y))) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x), \Psi_{s+\omega}(y)}(\tau))\, d(m \times m)(x, y)\, d\tau \right) ds \\ &\quad \le c r m(B_r(\gamma^\delta(\sigma_i)))^2 \int_0^{\omega'} ⨍_{B_{8r}(\gamma^\delta(\sigma_{i-1} - s))} |\mathrm{Hess}\, h^-|_{\mathrm{HS}}\, dm\, ds \\ &\quad \le c(N, \delta) r m(B_r(\gamma^\delta(\sigma_i)))^2 \sqrt{\omega'}, \end{aligned} \tag{123}$$

where the last line follows from the definition of h^-, Theorem 4.12(IV) and Cauchy–Schwarz.

By Lemma 4.13(III), the excess bound on the elements of A_2 and property (i) of γ^δ,

$$\int_0^{\omega'} \int_{U_1^s} |\nabla h^- - \nabla d_p|(\Psi_s(x))\, d(m \times m)(x, y)\, ds \le c(N, \delta) r m(B_r(\gamma^\delta(\sigma_i)))^2 \sqrt{\omega'}. \tag{124}$$

Similarly by the excess bounds on the elements of A_1,

$$\int_0^{\omega'}\int_{U_1^s} |\nabla h^- - \nabla d_p|(\Psi_{s+\omega}(y))\, d(m\times m)(x,y)\, ds \le c(N,\delta) m(B_r(\gamma^\delta(\sigma_i)))^2 r\sqrt{\omega'}. \tag{125}$$

By Proposition 3.27,

$$\int_{A_2\times A_1} dt(\omega')(x,\Psi_\tau(y)) \le c(N,\delta) m(B_r(\gamma^\delta(\sigma_i)))^2 r\sqrt{\omega'}. \tag{126}$$

Arguing as in the proof of Lemma 5.1 and using property (i) of γ^δ, we can then fix ω' sufficiently small depending only on N and δ so that there exist $z\in A_2$ and $A_1'\subseteq A_1$ with

$$\frac{m(A_1')}{m(B_{r/16}(\gamma^\delta(\sigma_i)))} \ge 1-3V(1,10) \qquad\text{and}\qquad dt(\omega')(z,\Psi_\omega(y)) \le \tfrac{1}{16}r \quad\text{for all } y\in A_1'. \tag{127}$$

The latter implies

$$\Psi_{s+\omega}(A_1') \subseteq B_{3r/8}(\gamma^\delta(\sigma_{i-1}-s)) \quad\text{for all } s\in[0,\omega']. \tag{128}$$

Notice by definition of A_1, $\Psi_s(A_1')\subseteq B_{r/8}(\gamma^\delta(\sigma_i-s))$ for any $s\in[0,\omega]$.

We now compare the flow of $B_r(\gamma^\delta(t_1))$ to that of $\Psi_{\sigma_i-t_1}(A_1')$ under Ψ_s for $s\in[0,\omega']$. By integral Abresch–Gromoll there exists $B_r(\gamma^\delta(t_1))'\subseteq B_r(\gamma^\delta(t_1))$ such that

$$e(x)\le c(N,\delta)r^2 \quad\text{for all } x\in B_r(\gamma^\delta(t_1))' \qquad\text{and}\qquad \frac{m(B_r(\gamma^\delta(t_1))')}{m(B_r(\gamma^\delta(t_1)))} \ge 1-\tfrac12 V(1,10). \tag{129}$$

For all $s\in[0,\omega']$, define

$$U_2^s := \{(x,y)\in A_1'\times B_r(\gamma^\delta(t_1))' : dt(s)(\Psi_{\sigma_i-t_1}(x),y) < r\}. \tag{130}$$

Consider $\int_{A_1'\times B_r(\gamma^\delta(t_1))'} dt(s)(\Psi_{\sigma_i-t_1}(x),y)\, d(m\times m)(x,y)$ for $0\le s\le\omega'$. For any $s\in[0,\omega']$ and $(x,y)\in U_2^s$,

- (I) $d(\Psi_{\sigma_i-t_1}(x),y) < r+\frac18 r$ by definition of A_1,
- (II) $d(\Psi_s(\Psi_{\sigma_i-t_1}(x)),\gamma^\delta(t_1-s)) < \frac38 r$ by (128) and the line below it,
- (III) $dt(s)(\Psi_{\sigma_i-t_1}(x),y) < r$.

Therefore $\Psi_s(y)\in B_{5r/2}(\gamma^\delta(t_1-s))$ and so $(\Psi_s\circ\Psi_{\sigma_i-t_1},\Psi_s)(U_2^s)\subseteq B_{4r}(\gamma^\delta(t_1-s))\times B_{4r}(\gamma^\delta(t_1-s))$.

By the same type of computations as the first part of this proof, for some $\omega''\in(0,\omega']$ to be fixed later,

$$\int_{A_1'\times B_r(\gamma(t_1))'} dt(\omega'')(\Psi_{\sigma_i-t_1}(x),y) \le c(N,\delta) m(B_r(\gamma^\delta(\sigma_i)))^2 r\sqrt{\omega''}. \tag{131}$$

Arguing as in the proof of Lemma 5.1 and using property (i) of γ^δ, we can then fix ω'' sufficiently small depending only on N and δ so that there exist $z'\in A_1'$ and $A\subseteq B_r(\gamma(t_1))'$ with

$$\frac{m(A)}{m(B_r(\gamma^\delta(t_1)))} \ge 1-V(1,10) \qquad\text{and}\qquad dt(\omega'')(\Psi_{\sigma_i-t_1}(z'),y) \le \tfrac12 r \quad\text{for all } y\in A. \tag{132}$$

The latter implies

$$\Psi_s(A) \subseteq B_{2r}(\gamma^\delta(t_1 - s)) \quad \text{for all } s \in [0, \omega'']. \tag{133}$$

We bound $\epsilon_4 \le \omega''$ and so (I) is proved. □

We may also apply Lemma 5.5 in the other direction of γ towards q. However, there is no guarantee the two geodesics we end up with in the two applications of the lemma are the same. Therefore we will show that a geodesic which has the properties of Lemma 5.5 necessarily has the same properties going in the other direction. The reason for this is Lemma 4.3, which roughly implies the local flows of h^+ and h^- are close to each other near a geodesic on the scale of r.

Lemma 5.6 *There exist $\epsilon_5(N,\delta) > 0$ and $\bar{r}_5(N,\delta) > 0$ such that if there exist a unit-speed geodesic γ from p to q and $\epsilon \le \epsilon_5$ and $\bar{r} \le \bar{r}_5$ which satisfy, for all $r \le \bar{r}$ and $\delta \le t_0 \le t_1 \le 1-\delta$ with $t_1 - t_0 \le \epsilon$,*

- (I) $V(1,100)^4 \le m(B_r(\gamma(t_1)))/m(B_r(\gamma(t_0))) \le 1/V(1,100)^4$,
- (II) *there exists $A_1 \subseteq B_r(\gamma(t_1))$ such that $m(A_1) \ge (1 - V(1,10))m(B_r(\gamma(t_1)))$ and $\Psi_s(A_1) \subseteq B_{2r}(\gamma(t_1 - s))$ for all $s \in [0, t_1 - t_0]$,*

then for the same geodesic γ, ϵ and $\bar{r}$, for all $r \le \bar{r}$ and $\delta \le t_0 \le t_1 \le 1-\delta$ with $t_1 - t_0 \le \epsilon$, there exists $A_2 \subseteq B_r(\gamma(t_0))$ such that

$$\frac{m(A_2)}{m(B_r(\gamma(t_0)))} \ge 1 - V(1,10) \qquad \text{and} \qquad \Phi_s(A_2) \subseteq B_{2r}(\gamma(t_0 + s)) \quad \text{for all } s \in [0, t_1 - t_0]. \tag{134}$$

Proof As a reminder, Φ is defined by (64) and is the local flow of $-\nabla d_q$, at least from the sets and for the time interval we are concerned with.

We assume $\bar{r}_5 \le \frac{1}{10}\delta$ and $\epsilon_5 \le \frac{1}{10}\delta$ to begin with, but will impose more bounds on both as the proof continues. We will not keep track of $\bar{r}_5$ for the sake of brevity. Fix γ, $\epsilon \le \epsilon_5$ and $\bar{r} \le \bar{r}_5$ which satisfy (I) and (II). Fix $r \le \bar{r}$ and $\delta \le t_0 \le t_1 \le 1-\delta$ with $t_1 - t_0 \le \epsilon$.

Fix $h^- \equiv h^-_{\rho(8r)^2}$ satisfying Theorem 4.12(IV) for the balls of radius $8r$ along γ, where $\rho \in \left[\frac{1}{2}, 2\right]$. Let $(\widetilde{\Psi}_t)_{t\in[0,1]}$ and $(\widetilde{\Psi}_{-t})_{t\in[0,1]}$ be the RLFs of the time-independent vector fields $-\nabla h^-$ and ∇h^-, respectively, as before.

The plan is as follows: first we use (II) to make sure a significant portion of $B_{r/16}(\gamma(t_1))$ stays close to γ from t_1 to t_0 under the flow of $\widetilde{\Psi}_s$; then we reverse flow the image of this portion under $\widetilde{\Psi}_{-s}$ and use it to make sure a significant portion of $B_r(\gamma(t_0))$ stays close to γ from t_0 to t_1 under Φ_s.

By (II) and integral Abresch–Gromoll, there exists $A_1 \subseteq B_{r/16}(\gamma(t_1))$ such that

- (I) $m(A_1)/m(B_{r/16}(\gamma(t_1))) \ge 1 - 2V(1,10)$,
- (II) $\Psi_s(A_1) \subseteq B_{r/8}(\gamma(t_1 - s))$ for all $s \in [0, t_1 - t_0]$,
- (III) $e(x) \le c(N,\delta)r^2$ for all $x \in A_1$.

For all $s \in [0, t_1 - t_0]$ and $(x, y) \in X \times X$, define

$$dt_1(s)(x, y) := \min\{r, \max_{0 \le \tau \le s} |d(x, y) - d(\Psi_\tau(x), \widetilde{\Psi}_\tau(y))|\} \tag{135}$$

and

$$U_1^s := \{(x, y) \in A_1 \times B_{r/16}(\gamma(t_1)) : dt_1(s)(x, y) < r\}. \tag{136}$$

Consider $\int_{A_1 \times B_{r/16}(\gamma(t_1))} dt_1(s)(x, y)\, d(m \times m)(x, y)$ for $0 \le s \le t_1 - t_0$. For any $s \in [0, t_1 - t_0]$ and $(x, y) \in U_1^s$,

(I) $d(x, y) < \frac{1}{8} r$,

(II) $d(\Psi_s(x), \gamma(t_1 - s)) < \frac{1}{8} r$ by definition of A_1,

(III) $dt_1(s)(x, y) < r$.

Hence $\widetilde{\Psi}_s(y) \in B_{5r/4}\gamma(t_1 - s)$ by the triangle inequality, and so $(\Psi_s, \widetilde{\Psi}_s)(U_1^s) \subseteq B_{4r}(\Psi_s(z)) \times B_{4r}(\Psi_s(z))$.

Using exactly the same type of computation as the second part of the proof of the main lemma,

$$\int_{A_1 \times B_{r/16}(\gamma(t_1))} dt_1(t_1 - t_0)(x, y)\, d(m \times m)(x, y) \le c(N, \delta) r m(B_r(\gamma(t_1)))^2 \sqrt{t_1 - t_0}. \tag{137}$$

A_1 takes a significant portion of the measure of $B_r(\gamma(t_1))$ by definition. The same considerations as in the proof of Lemma 5.1 gives the existence of $z \in A_1$ and $D_1 \subseteq B_r(\gamma(t_1))$ such that

$$\frac{m(D_1)}{m(B_{r/16}(\gamma(t_1)))} \ge 1 - V(1, 10) \qquad \text{and} \qquad dt_1(t_1 - t_0)(z, y) \le \tfrac{1}{16} r \quad \text{for all } y \in D_1, \tag{138}$$

after we bound ϵ_5 sufficiently small depending only on N and δ. The latter implies

$$\widetilde{\Psi}_s(D_1) \subseteq B_{5r/16}(\gamma(t_1 - s)) \quad \text{for all } s \in [0, t_1 - t_0]. \tag{139}$$

By Proposition 3.12, we may assume in addition that D_1 satisfies

$$\widetilde{\Psi}_{-s}(\widetilde{\Psi}_{t_1 - t_0}(x)) = \widetilde{\Psi}_{t_1 - t_0 - s}(x) \quad \text{for all } s \in [0, t_1 - t_0] \text{ and for all } x \in D_1. \tag{140}$$

Furthermore, $\widetilde{\Psi}_s(D_1)$ is nontrivial in measure compared to $B_r(\gamma(t_1))$ for all $s \in [0, t_1 - t_0]$.

$$\begin{aligned} \frac{m(\widetilde{\Psi}_s(D_1))}{m(B_r(\gamma(t_1)))} &\ge e^{-c(N,\delta)(\delta/10)} \frac{m(D_1)}{m(B_r(z))} && \text{by Lemma 4.2, (18) in Theorem 3.4 and } \epsilon_5 \le \tfrac{1}{10}\delta \\ &\ge c(N, \delta) && \text{by definition of } D_1 \text{ and Bishop–Gromov.} \end{aligned} \tag{141}$$

By integral Abresch–Gromoll, there exists $B_r(\gamma(t_0))' \subseteq B_r(\gamma(t_0))$ such that

$$e(x) \le c(N, \delta) r^2 \quad \text{for all } x \in B_r(\gamma(t_0))' \qquad \text{and} \qquad \frac{m(B_r(\gamma(t_0))')}{m(B_r(\gamma(t_0)))} \ge 1 - \tfrac{1}{2} V(1, 10). \tag{142}$$

For all $s \in [0, t_1 - t_0]$ and $(x, y) \in X \times X$, define

$$dt_2(s)(x, y) := \min\{r, \max_{0 \le \tau \le s} |d(x, y) - d(\widetilde{\Psi}_{-\tau}(x), \Phi_\tau(y))|\} \tag{143}$$

and

$$U_2^s := \{(x, y) \in \widetilde{\Psi}_{t_1 - t_0}(D_1) \times B_r(\gamma(t_0))' : dt_2(s)(x, y) < r\}. \tag{144}$$

Consider $\int_{\widetilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} dt_2(s)(x,y)\,d(m\times m)(x,y)$ for $0\le s\le t_1-t_0$. By Proposition 3.27, for a.e. $s\in[0,t_1-t_0]$,

$$\begin{aligned}(145)\quad \frac{d}{ds}\int_{\widetilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} & dt_2(s)(x,y)\,d(m\times m)(x,y)\\ &\le \int_{U_2^s}|\nabla h^- + \nabla d_q|(\Phi_s(y))\,d(m\times m)(x,y)\\ &\quad + \int_0^1\int_{U_2^s} d(\widetilde{\Psi}_s(x),\Phi_s(y))|\operatorname{Hess} h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\widetilde{\Psi}_s(x),\Phi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau.\end{aligned}$$

For any $s\in[0,t_1-t_0]$, $s'\in[0,s]$ and $(x,y)\in U_2^s$,

(I) $d(x,y)<r+\frac{5}{16}r$ by (139),

(II) $d(\widetilde{\Psi}_{-s'}(x),\gamma(t_0+s')) = d(\widetilde{\Psi}_{t_1-t_0-s'}(x'),\gamma(t_0+s'))<\frac{5}{16}r$ for some $x'\in D_1$ by (139) and (140),

(III) $dt_2(s)(x,y)<r$ by definition of U_2^s (144).

Hence

$$(146)\qquad \Phi_{s'}(y)\in B_{2r+5r/8}(\gamma(t_0+s'))$$

by the triangle inequality. Therefore $(\widetilde{\Psi}_{-s'},\Phi_{s'})(U_2^s)\in B_{4r}(\gamma(t_0+s'))\times B_{4r}(\gamma(t_0+s'))$ for all $s'\in[0,s]$. For any $(x,y)\in U_2^s$,

$$\begin{aligned}(147)\qquad \Delta h^-(\widetilde{\Psi}_{-s'}(x)) &= \Delta h^+(\widetilde{\Psi}_{-s'}(x)) + \Delta\hat{e}(\widetilde{\Psi}_{-s'}(x))\\ &\ge -c(N,\delta) \qquad \text{by Lemmas 4.2 and 4.3(III)},\end{aligned}$$

where h^+ and $\hat{e}$ are heat flow approximations of h_0^+ and e_0, respectively, up to the same time as h^-. Therefore,

$$\begin{aligned}(148)\quad \int_0^1\int_{U_2^s} & d(\widetilde{\Psi}_{-s}(x),\Phi_s(y))|\operatorname{Hess} h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\widetilde{\Psi}_{-s}(x),\Phi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\\ &\le c(N,\delta)\int_0^1\int_{(\widetilde{\Psi}_{-s},\Phi_s)(U_2^s)} d(x,y)|\operatorname{Hess} h^-|_{\mathrm{HS}}(\tilde{\gamma}_{x,y}(\tau))\,d(m\times m)(x,y)\,d\tau\\ &\le c(N,\delta)rm(B_{4r}(\gamma(t_0+s)))\int_{B_{8r}(\gamma(t_0+s))}|\operatorname{Hess} h^-|_{\mathrm{HS}}\,dm\\ &\le c(N,\delta)rm(B_r(\gamma(t_1)))^2 \fint_{B_{8r}(\gamma(t_0+s))}|\operatorname{Hess} h^-|_{\mathrm{HS}}\,dm,\end{aligned}$$

where the first inequality follows by Theorem 3.15(II), (147) and Remark 3.5, the second by the segment inequality (Theorem 3.22), and the third by Bishop–Gromov and property (I) of γ. Integrating in $s\in[0,t_1-t_0]$,

$$\begin{aligned}(149)\quad \int_0^{t_1-t_0}\Big(\int_0^1\int_{U_2^s} & d(\widetilde{\Psi}_{-s}(x),\Phi_s(y))|\operatorname{Hess} h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\widetilde{\Psi}_{-s}(x),\Phi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\Big)\,dt\\ &\le crm(B_r(\gamma(t_1)))^2\int_0^{t_1-t_0}\fint_{B_{8r}(\gamma(t_0+s))}|\operatorname{Hess} h^-|_{\mathrm{HS}}\,dm\,ds\\ &\le c(N,\delta)rm(B_r(\gamma(t_1)))^2\sqrt{t_1-t_0},\end{aligned}$$

where the last line follows from the definition of h^-, Theorem 4.12(IV) and Cauchy–Schwarz.

We have

$$\int_0^{t_1-t_0}\int_{U_2^s}|\nabla h^+ + \nabla d_q|(\Phi_s(y))\,d(m\times m)(x,y)\,ds \le c(N,\delta)rm(B_r(\gamma(t_1)))^2\sqrt{t_1-t_0}, \tag{150}$$

where the first inequality is from Lemma 4.13(III) and the excess bound on the elements of $B_r(\gamma(t_0))'$ (142), and the second inequality is from property (I) of γ and Bishop–Gromov. Similarly, using Lemma 4.3(III) with (146),

$$\begin{aligned}\int_0^{t_1-t_0}\int_{U_2^s}|\nabla h^- - \nabla h^+|(\Phi_s(y))\,d(m\times m)(x,y)\,ds &\le c(N,\delta)rm(B_r(\gamma(t_1)))^2(t_1-t_0)\\ &\le crm(B_r(\gamma(t_1)))^2\sqrt{t_1-t_0}.\end{aligned} \tag{151}$$

Combining (148)–(150) with the bound (145) on $(d/ds)\int_{\widetilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} dt_2(s)(x,y)$, we obtain

$$\begin{aligned}&\int_{\widetilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} dt_2(t_1-t_0)(x,y)\,d(m\times m)(x,y)\\ &\qquad= \int_0^{t_1-t_0}\Big[\frac{d}{ds}\int_{\widetilde{\Psi}_{t_1-t_0}(D_1)\times B_r(\gamma(t_0))'} dt_2(s)(x,y)\,d(m\times m)(x,y)\Big]\,ds\\ &\qquad\le c(N,\delta)rm(B_r(\gamma(t_1)))^2\sqrt{t_1-t_0}.\end{aligned} \tag{152}$$

Both $\widetilde{\Psi}_{t_1-t_0}(D_1)$ and $B_r(\gamma(t_0))'$ are nontrivial in measure compared to $B_r(\gamma(t_1))$ by (141) and property (I), respectively. The same considerations as in the proof of Lemma 5.1 give the existence of $z'\in\widetilde{\Psi}_{t_1-t_0}(D_1)$ and $A_2\subseteq B_r(\gamma(t_0))'$ such that

$$\frac{m(A_2)}{m(B_r(\gamma(t_0)))}\ge 1-V(1,10) \qquad\text{and}\qquad dt_2(t_1-t_0)(z',y)\le \tfrac{3}{8}r \quad\text{for all } y\in A_2, \tag{153}$$

after we bound ϵ_5 sufficiently small depending only on N and δ. The latter implies

$$\Phi_s(A_2)\subseteq B_{2r}(\gamma(t_0+s)) \quad\text{for all } s\in[0,t_1-t_0]. \tag{154}$$

□

Lemmas 5.5 and 5.6 give:

Corollary 5.7 *For any $\delta\in(0,0.1)$, there exist $\epsilon_6(N,\delta)>0$ and $\bar r_6(N,\delta)>0$ such that for any unit-speed geodesic γ from p to q, there exists a unit-speed geodesic γ^δ from p to q with $\gamma^\delta\equiv\gamma$ on $[1-\delta,1]$ such that for all $r\le\bar r_6$ and $\delta\le t_0\le t_1\le 1-\delta$, if $t_1-t_0\le\epsilon_6$, then*

- (I) $V(1,100)^4\le m(B_r(\gamma^\delta(t_1)))/m(B_r(\gamma^\delta(t_0)))\le 1/V(1,100)^4$,
- (II) *there exists $A_1\subseteq B_r(\gamma^\delta(t_1))$ such that $m(A_1)\ge(1-V(1,10))m(B_r(\gamma^\delta(t_1)))$ and $\Psi_s(A_1)\subseteq B_{2r}(\gamma^\delta(t_1-s))$ for all $s\in[0,t_1-t_0]$,*
- (III) *there exists $A_2\subseteq B_r(\gamma^\delta(t_0))$ such that $m(A_2)\ge(1-V(1,10))m(B_r(\gamma^\delta(t_0)))$ and $\Phi_s(A_2)\subseteq B_{2r}(\gamma^\delta(t_0+s))$ for all $s\in[0,t_1-t_0]$.*

We would like to now construct a geodesic that has this behavior for all δ by taking a limit of γ^δ as $\delta \to 0$. To have the properties pass over to the limit, we need to make sure each γ^δ also satisfies the above properties for any $\delta' > \delta$. For all $\delta \in (0, 0.1)$, we fix the constants $\epsilon_6(N, \delta)$ and $\bar{r}_6(N, \delta)$ which come from taking the minima of their respective counterparts in Lemmas 5.5 and 5.6.

Lemma 5.8 *Let $\delta \in (0, 0.1)$ and assume some unit-speed geodesic γ^δ from p to q satisfies properties* (I)–(III) *for δ, $\epsilon_6(N, \delta)$ and $\bar{r}_6(N, \delta)$. Then for any $\delta' \in (\delta, 0.1)$, γ^δ also satisfies properties* (I)–(III) *for δ', $\epsilon_6(N, \delta')$ and $\bar{r}_6(N, \delta')$*

Proof Fix $\delta' \in (\delta, 0.1)$. Use γ^δ to construct some $\gamma^{\delta'}$ with the construction of Lemma 5.5. Then $\gamma^{\delta'}$ satisfies properties (I)–(III) for δ', $\epsilon_6(N, \delta')$ and $\bar{r}_6(N, \delta')$ by Lemmas 5.5 and 5.6. Furthermore, $\gamma^\delta(t) = \gamma^{\delta'}(t)$ for all $t \in [1-\delta', 1]$ by construction. We will show $\gamma^\delta(t) = \gamma^{\delta'}(t)$ for all $t \in [\delta', 1]$, which will allow us to conclude.

Assume this is not the case. Define $s_0 := \min\{s \in [\delta', 1-\delta'] : \gamma^\delta(t) = \gamma^{\delta'}(t)\, \forall t \in [s, 1]\}$. By assumption, $\delta' < s_0 \le 1-\delta'$. Therefore there exists $t_0 \in [\delta', s_0)$ such that $t_1 - t_0 \le \min\{\epsilon_6(N, \delta), \epsilon_6(N, \delta')\}$ and $\gamma^\delta(t_0) \neq \gamma^{\delta'}(t_0)$. Choose any $r < \min\bigl\{\bar{r}_6(N, \delta), \bar{r}_6(N, \delta'), \frac{1}{4} d(\gamma^\delta(t_0), \gamma^{\delta'}(t_0))\bigr\}$ and consider $B_r(\gamma^\delta(t_1))$. On one hand, most of $B_r(\gamma^\delta(t_1))$ needs to end up in $B_{2r}(\gamma^\delta(t_0))$ under $\Psi_{t_1-t_0}$ by property (II) for γ^δ. On the other hand, most of $B_r(\gamma^\delta(t_1))$ needs to end up in $B_{2r}(\gamma^{\delta'}(t_0))$ under $\Psi_{t_1-t_0}$ by property (II) for $\gamma^{\delta'}$. These two balls are disjoint and so we have a contradiction. □

This immediately gives the existence of a geodesic with the desired properties for any $\delta \in (0, 0.1)$.

Theorem 5.9 *There exists a unit-speed geodesic $\bar{\gamma}$ from p to q such that for any $\delta \in (0, 0.1)$, there exist $\epsilon_6(N, \delta) > 0$ and $\bar{r}_6(N, \delta) > 0$ such that for all $r \le \bar{r}_6$ and $\delta \le t_0 \le t_1 \le 1-\delta$, if $t_1 - t_0 \le \epsilon_6$, then*

- (I) $V(1, 100)^4 \le m(B_r(\bar{\gamma}(t_1)))/m(B_r(\bar{\gamma}(t_0))) \le 1/V(1, 100)^4$,
- (II) *there exists $A_1 \subseteq B_r(\bar{\gamma}(t_1))$ such that $m(A_1) \ge (1 - V(1, 10)) m(B_r(\bar{\gamma}(t_1)))$ and $\Psi_s(A_1) \subseteq B_{2r}(\bar{\gamma}(t_1 - s))$ for all $s \in [0, t_1 - t_0]$,*
- (III) *there exists $A_2 \subseteq B_r(\bar{\gamma}(t_0))$ such that $m(A_2) \ge (1 - V(1, 10)) m(B_r(\bar{\gamma}(t_0)))$ and $\Phi_s(A_2) \subseteq B_{2r}(\bar{\gamma}(t_0 + s))$ for all $s \in [0, t_1 - t_0]$.*

Proof Fix any unit-speed geodesic γ from p to q and then use the construction of Lemma 5.5 to obtain a γ^δ for each $\delta \in (0, 0.1)$. By the Arzelà–Ascoli theorem, we can take a limit $\bar{\gamma}$ of γ^δ as $\delta \to 0$ after passing to a subsequence. By Lemma 5.8 and Bishop–Gromov, $\bar{\gamma}$ will have the desired properties. □

5.3 Proof of the main theorem

We now prove the Hölder continuity in pointed Gromov–Hausdorff distance of small balls along the interior of any geodesic between p and q constructed in Theorem 5.9, using essentially the same argument as in [30].

Theorem 5.10 *There exists a unit-speed geodesic γ between p and q such that for any $\delta \in (0, 0.1)$, there exist $\epsilon(N,\delta) > 0$, $\bar{r}(N,\delta) > 0$ and $C(N,\delta)$ such that for any $r \le \bar{r}$ and $t_0, t_1 \in [\delta, 1-\delta]$, if $|t_1 - t_0| \le \epsilon$ then*[1]

$$d_{pGH}\big((B_r(\gamma(t_0)), \gamma(t_0)), (B_r(\gamma(t_1)), \gamma(t_1))\big) \le Cr|t_1 - t_0|^{1/(2N(1+2N))}.$$

Proof Let $\bar{\gamma}$, ϵ_6 and $\bar{r}_6$ be as in Theorem 5.9 and let $\gamma = \bar{\gamma}$. Fix $\delta \in (0, 0.1)$. We begin by assuming $\epsilon \le \epsilon_6$ and $\bar{r} \le \bar{r}_6$, but will impose more bounds as the proof continues. Fix $r \le \bar{r}$ and $\delta \le t_0 \le t_1 \le 1-\delta$ with $\omega := t_1 - t_0 \le \epsilon$.

We begin by showing that a large portion of $B_r(\gamma(t_1))$ is mapped by Ψ_ω close (on the scale of r) to $\gamma(t_0)$, where the closeness and the relative size of the portion are both Hölder dependent on ω. This also shows the measure of $B_r(\gamma)$ is Hölder along $\gamma|_{[\delta, 1-\delta]}$ as a consequence.

Define $\eta := \omega^{N/(2(1+2N))}$ and $\mu := \eta^{1/N} = \omega^{1/(2(1+2N))}$. By property (II) of γ from Theorem 5.9 and integral Abresch–Gromoll, there exists $B_{\mu r}(\gamma(t_1))' \subseteq B_{\mu r}(\gamma(t_1))$ such that

(I) $m(B_{\mu r}(\gamma(t_1))')/m(B_{\mu r}(\gamma(t_1))) \le 1 - 2V(1, 10)$,

(II) $\Psi_s(B_{\mu r}(\gamma(t_1))') \subseteq B_{2\mu r}(\gamma(t_1 - s))$ for all $s \in [0, \omega]$,

(III) $e(x) \le c(N,\delta)\mu^2 r^2 \le cr^2$ for all $x \in B_{\mu r}(\gamma(t_1))'$.

By integral Abresch–Gromoll, there exists $B_r(\gamma(t_1))' \subseteq B_r(\gamma(t_1))$ such that

$$e(x) \le c(N,\delta)\frac{1}{\eta}r^2 \quad \text{for all } x \in B_r(\gamma(t_1))' \qquad \text{and} \qquad \frac{m(B_r(\gamma(t_1))')}{m(B_r(\gamma(t_1)))} \ge 1 - \eta. \tag{155}$$

Fix $h^- \equiv h^-_{\rho(4r)^2}$ satisfying Theorem 4.12(IV) for the balls of radius $4r$ along γ, where $\rho \in \left[\frac{1}{2}, 2\right]$. For all $s \in [0, \omega]$ and $(x, y) \in X \times X$, define

$$dt(s)(x,y) := \min\Big\{4r, \max_{0 \le \tau \le s} |d(x,y) - d(\Psi_\tau(x), \Psi_\tau(y))|\Big\}. \tag{156}$$

Note for any $(x,y) \in B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'$ we have $\Psi_s(x), \Psi_s(y) \in B_{2r}(\gamma(t_1 - s))$, and therefore $dt(s)(x,y) < 4r$.

By Proposition 3.27, for a.e. $s \in [0, \omega]$,

$$\begin{aligned}
&\frac{d}{ds}\int_{B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'} dt(s)(x,y)\, d(m \times m)(x,y) \\
&\quad \le \int_{B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'} \big(|\nabla h^- - \nabla d_p|(\Psi_s(x)) + |\nabla h^- - \nabla d_p|(\Psi_s(y))\big)\, d(m \times m)(x,y) \\
&\qquad + \int_0^1 \int_{B_{\mu r}(\gamma(t_1))' \times B_r(\gamma(t_1))'} d(\Psi_s(x), \Psi_s(y)) |\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x), \Psi_s(y)}(\tau))\, d(m \times m)(x,y)\, d\tau.
\end{aligned} \tag{157}$$

[1] We note the Hölder exponent is slightly different to that of [30] due to a minor error in the first line of (3.38).

We estimate

$$\int_0^1\int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau \tag{158}$$
$$\begin{aligned}
&\le c(N,\delta)\int_0^1\int_{(\Psi_s,\Psi_s)(B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))')} d(x,y)|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde\gamma_{x,y}(\tau))\,d(m\times m)(x,y)\,d\tau\\
&\le c(N,\delta)rm(B_{2r}(\gamma(t_1-s)))\int_{B_{4r}(\gamma(t_1-s))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm\\
&\le c(N,\delta)rm(B_r(\gamma(t_1)))^2 ⨍_{B_{4r}(\gamma(t_1-s))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm,
\end{aligned}$$

where the first inequality follows by Theorem 3.15(II), the second by the segment inequality (Theorem 3.22), and the third by Bishop–Gromov and property (I) of γ. Integrating in $s\in[0,\omega]$,

$$\int_0^\omega\Big(\int_0^1\int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\,h^-|_{\mathrm{HS}}(\tilde\gamma_{\Psi_s(x),\Psi_s(y)}(\tau))\,d(m\times m)(x,y)\,d\tau\Big)\,ds \tag{159}$$
$$\begin{aligned}
&\le crm(B_r(\gamma(t_1)))^2\int_0^\omega ⨍_{B_{4r}(\gamma(t_1-s))}|\mathrm{Hess}\,h^-|_{\mathrm{HS}}\,dm\,ds\\
&\le c(N,\delta)m(B_r(\gamma(t_1)))^2\sqrt{\omega}r,
\end{aligned}$$

where the last line follows from the definition of h^-, Theorem 4.12(IV) and Cauchy–Schwarz.

By Lemma 4.13(III) and the excess bound on the elements of $B_{\mu r}(\gamma(t_1))'$,

$$\int_0^\omega\int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'}|\nabla h^- - \nabla d_p|(\Psi_s(x))\,d(m\times m)(x,y)\,ds\le c(N,\delta)m(B_r(\gamma(t_1)))^2\sqrt{\omega}r. \tag{160}$$

Similarly, by the excess bound on the elements of $B_r(\gamma(t_1))'$ (155),

$$\int_0^\omega\int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'}|\nabla h^- - \nabla d_p|(\Psi_s(y))\,d(m\times m)(x,y)\,ds \le c(N,\delta)\frac{1}{\sqrt{\eta}}m(B_r(\gamma(t_1)))m(B_{\mu r}(\gamma(t_1)))\sqrt{\omega}r. \tag{161}$$

Combining (159)–(161) with (157) we immediately obtain

$$\int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} dt(\omega)(x,y)\,d(m\times m)(x,y) \tag{162}$$
$$\begin{aligned}
&=\int_0^\omega\Big[\frac{d}{ds}\int_{B_{\mu r}(\gamma(t_1))'\times B_r(\gamma(t_1))'} dt(s)(x,y)\,d(m\times m)(x,y)\Big]\,ds\\
&\le c(N,\delta)m(B_r(\gamma(t_1)))\Big(m(B_r(\gamma(t_1)))+\frac{1}{\sqrt{\eta}}m(B_{\mu r}(\gamma(t_1)))\Big)\sqrt{\omega}r.
\end{aligned}$$

By Bishop–Gromov, $m(B_{\mu r}(\gamma(t_1)))/m(B_r(\gamma(t_1)))\ge C(N)\mu^N = c\eta$. Therefore, there exists $z\in B_{\mu r}(\gamma(t_1))'$ such that

$$\int_{B_r(\gamma(t_1))'} dt(\omega)(z,y)\,dm(y)\le c(N,\delta)m(B_r(\gamma(t_1)))\Big(\frac{1}{\eta}+\frac{1}{\sqrt{\eta}}\Big)\sqrt{\omega}r\le c(N,\delta)m(B_r(\gamma(t_1)))\frac{1}{\eta}\sqrt{\omega}r.$$

This means there exists $D \subseteq B_r(\gamma(t_1))'$ such that

$$\frac{m(D)}{m(B_r(\gamma(t_1)))} \geq 1-2\eta \qquad \text{and} \qquad dt(\omega)(z,y) \leq c(N,\delta)\frac{1}{\eta^2}\sqrt{\omega}r \quad \text{for all } y \in D. \tag{163}$$

Since $\omega = (\eta^2\mu)^2$, the latter combined with $z \in B_{\mu r}(\gamma(t_1))'$ implies

$$\Psi_\omega(D) \subseteq B_{(1+c(N,\delta)\mu)r}(\gamma(t_0)). \tag{164}$$

We have the following estimate on the volume of $B_r(\gamma(t_1))$ compared to volume of $B_r(\gamma(t_0))$ after possibly constraining ϵ further depending on N and δ:

$$\begin{aligned}\frac{m(B_r(\gamma(t_1)))}{m(B_r(\gamma(t_0)))} &\leq (1+c\eta)\frac{m(D)}{m(B_r(\gamma(t_0)))}\\ &\leq (1+c\eta)(1+c(N,\delta)\omega)^N\frac{m(\Psi_\omega(D))}{m(B_r(\gamma(t_0)))}\\ &\leq (1+c\eta)(1+c\omega)^N(1+c(N,\delta)\mu)^N\frac{m(\Psi_\omega(D))}{m(B_{(1+c\mu)r}(\gamma(t_0)))}\\ &\leq 1+c(N,\delta)\mu = 1+c\omega^{1/(2(1+2N))}.\end{aligned} \tag{165}$$

Here the first inequality follows by (163), the second by Theorem 3.15(II), and the third by Bishop–Gromov. Making the same calculation with Φ in the other direction as well, we obtain the following Hölder estimate on volume:

$$\left|\frac{m(B_r(\gamma(t_1)))}{m(B_r(\gamma(t_0)))}-1\right| \leq c(N,\delta)|t_1-t_0|^{1/(2(1+2N))}. \tag{166}$$

We now show the required Gromov–Hausdorff approximation can be constructed by using Ψ_ω on a $c\mu r$-dense subset of $B_r(\gamma(t_1))$.

Fix representatives for $|\mathrm{Hess}\, h^-|_{\mathrm{HS}}$ and $|\nabla h^- - \nabla d_p|$. Using the same calculation as before,

$$\int_0^\omega\left(\int_0^1\int_{D\times D} d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_s(y)}(\tau))\, d(m\times m)(x,y)\, d\tau\right) ds \leq c(N,\delta)m(B_r(\gamma(t_1)))^2\sqrt{\omega}r, \tag{167}$$

and so by Fubini's theorem, there exists $A \subseteq D$ such that

(I) $m(A)/m(B_r(\gamma(t_1))) \geq 1-3\eta$,

(II) for all $x \in A$,

$$\int_0^\omega\left(\int_0^1\int_D d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_s(y)}(\tau))\, dm(y)\, d\tau\right) ds \leq c(1/\eta)m(B_r(\gamma(t_1)))\sqrt{\omega}r.$$

For each $x \in A$, there exists $A_x \subseteq D$ such that

(I) $m(A_x)/m(B_r(\gamma(t_1))) \geq 1-3\eta$,

(II) $\int_0^\omega\left(\int_0^1 d(\Psi_s(x),\Psi_s(y))|\mathrm{Hess}\, h^-|_{\mathrm{HS}}(\tilde{\gamma}_{\Psi_s(x),\Psi_s(y)}(\tau))\, d\tau\right) ds \leq c(1/\eta^2)\sqrt{\omega}r$ for all $y \in A_x$.

Since A and each A_x are contained in $B_r(\gamma(t_1))'$, their elements have an excess bound of $c(1/\eta)r^2$ by (155). By Lemma 4.13(III), for m-a.e. $x \in A$,

$$\int_0^\omega |\nabla h^- - \nabla d_p|(\Psi_s(x))\,ds \le c(N,\delta)\frac{1}{\sqrt{\eta}}\sqrt{\omega}r. \tag{168}$$

Similarly, for each $x \in A$ and m-a.e. $y \in A$,

$$\int_0^\omega |\nabla h^- - \nabla d_p|(\Psi_s(y))\,ds \le c(N,\delta)\frac{1}{\sqrt{\eta}}\sqrt{\omega}r. \tag{169}$$

By a Fubini's theorem argument, it is clear that the inequality in Proposition 3.27 holds pointwise for $(m\times m)$-a.e. (x,y). We first replace A with a full-measure subset so that in addition the inequality in Proposition 3.27 holds for all $x \in A$ and m-a.e. $y \in A$. We then replace each A_x with a full-measure subset so that the same inequality holds for all $x \in A$ and all $y \in A_x$. Therefore, for all $x \in A$ and $y \in A_x$,

$$dt(\omega)(x,y) \le c(N,\delta)\left(\frac{1}{\eta^2}+\frac{1}{\sqrt{\eta}}\right)\sqrt{\omega}r \le c(N,\delta)\frac{1}{\eta^2}\sqrt{\omega}r \le c\mu r.$$

For any $x, y \in A$, $A_x \cap A_y$ is $c(N)\eta^{1/n}r$-dense in $B_r(\gamma(t_1))$ and so there exists some $z \in A_x \cap A_y$ where $d(x,z) < c\mu r$. Therefore

$$\begin{aligned} &|d(\Psi_\omega(x),\Psi_\omega(y)) - d(x,y))| \\ &\quad\le |d(\Psi_\omega(x),\Psi_\omega(y)) - d(\Psi_\omega(z),\Psi_\omega(y)| + |d(\Psi_\omega(z),\Psi_\omega(y)) - d(z,y)| + |d(z,y) - d(x,y)| \\ &\quad\le c(N,\delta)\mu r. \end{aligned} \tag{170}$$

Moreover, we have the following estimate on the volume of $\Psi_\omega(A) \subseteq B_{(1+c\mu)r}(\gamma(t_0))$ after possibly constraining ϵ further depending on N and δ:

$$\begin{aligned} \frac{m(\Psi_\omega(A))}{m(B_{1+c\mu r}(\gamma(t_0)))} &\ge \frac{1}{(1+c(N,\delta)\omega)^N}\frac{m(A)}{m(B_{(1+c\mu)r}(\gamma(t_0)))} && \text{by Theorem 3.15(II)} \\ &\ge \frac{1}{(1+c\omega)^N}\frac{1}{(1+c(N,\delta)\mu)^N}\frac{m(A)}{m(B_r(\gamma(t_0)))} && \text{by Bishop–Gromov} \\ &\ge \frac{1}{(1+c\omega)^N}\frac{1}{(1+c\mu)^N}\frac{1}{1+c(N,\delta)\mu}\frac{m(A)}{m(B_r(\gamma(t_1)))} && \text{by (165)} \\ &\ge \frac{1}{(1+c\omega)^N}\frac{1}{(1+c\mu)^N}\frac{1}{1+c\mu}(1-3\eta) \ge 1-c(N,\delta)\mu. \end{aligned} \tag{171}$$

To summarize, $A \in B_r(\gamma(t_1))$ is such that

- (I) $\Psi_\omega(A) \subseteq B_{(1+c(N,\delta)\mu)r}(\gamma(t_0))$,
- (II) for all $x, y \in A$, $|d(\Psi_\omega(x),\Psi_\omega(y)) - d(x,y)| \le c(N,\delta)\mu r$,
- (III) A is $c(N)\mu r$-dense in $B_r(\gamma(t_1))$,
- (IV) $\Psi_\omega(A)$ is $c(N,\delta)\mu^{1/N}r$-dense in $B_{(1+c(N,\delta)\mu)r}(\gamma(t_0))$.

Moreover, there exists $c(N)$ such that $m(B_{c(N)\mu r}(\gamma(t_1))) \geq 2\eta/(1-V(1,10))m(B_r(\gamma(t_1)))$ by Bishop–Gromov. By property (II) of γ, there exists $B_{c\mu r}(\gamma(t_1))' \subseteq B_{c\mu r}(\gamma(t_1))$ such that

$$\frac{m(B_{c\mu r}(\gamma(t_1))')}{m(B_{c\mu r}(\gamma(t_1)))} \geq 1-V(1,10) \quad \text{and} \quad \Psi_\omega(B_{c\mu r}(\gamma(t_1))') \subseteq B_{2c\mu r}(\gamma(t_0)).$$

Therefore $B_{c(N)\mu r}(\gamma(t_1))' \cap A$ is nonempty by measure considerations. In other words, there is an element in A which is $c(N)\mu r$ close to $\gamma(t_1)$ and is mapped $2c(N)\mu r$ close to $\gamma(t_0)$ under Ψ_ω.

These facts about A allow the construction of a $c(N,\delta)\mu^{1/N}r$-pointed Gromov–Hausdorff approximation, which finishes the proof. □

Before we prove Theorem 1.1, we will first prove RCD(K,N) spaces are nonbranching in the next section using the construction we have developed so far. A corollary then follows which immediately gives Theorem 1.1.

6 Applications

6.1 Nonbranching

In this subsection, we prove RCD(K,N) spaces are nonbranching. The use of the essentially nonbranching property of RCD(K,N) spaces in the proof was pointed out to the author by Vitali Kapovitch. We note that the proof works without using the essentially nonbranching property as well by using the MCP(K,N) property [62; 71] of RCD(K,N) spaces.

Proof of Theorem 1.3 Assume otherwise. By zooming in and cutting off geodesics if necessary, we may assume (X,d,m) is an RCD$(-(N-1),N)$ space for some $N \in (1,\infty)$ and we have two unit-speed geodesics $\gamma_1,\gamma_2\colon [0,1] \to X$ with

(I) $\gamma_1(0) = \gamma_2(0) = p$ for $p \in X$,

(II) $\gamma_1(1) = q_1$ and $\gamma_2(1) = q_2$ for $q_1, q_2 \in X$,

(III) $\max\{t \in [0,1] : \gamma_1(s) = \gamma_2(s)\ \forall s \in [0,t]\} = 0.5$.

Let $p' = \gamma_1(0.5)$ and Ψ_s be as in (63) towards p.

Since $(X,2d,m)$ is again an RCD$(-(N-1),N)$ space and $2d(p,p') = 1$, we may apply Theorem 5.9 to obtain a $2d$-unit-speed geodesic $\tilde\gamma\colon [0,1] \to X$ between p and p'. Reparametrize $\tilde\gamma$ to $\gamma\colon [0,0.5] \to X$ so that γ is a d-unit-speed geodesic.

Fix any $\delta \in (0,0.1)$ and use Corollary 5.7 to construct a unit-speed geodesic $\gamma_1^\delta\colon [0,1] \to X$ from p to q_1 with $\gamma_1^\delta(t) = \gamma(t)$ for all $t \in [0,\delta]$. Therefore the proof of Theorem 5.10 passes for γ_1^δ for the same δ, and in particular we have the estimates (163)–(166) for γ_1^δ. As a reminder, for $\delta \leq s_1 < s_2 \leq 1-\delta$ and sufficiently small r:

- (163) and (164) imply a portion of $B_r(\gamma_1^\delta(s_2))$ is sent to a ball of radius slightly larger than r around $\gamma_1^\delta(s_1)$ by $\Psi_{s_2-s_1}$, where the relative size of the portion and the increase in radius on the scale of r can be both made Hölder dependent on s_2-s_1 and go uniformly to 1 and 0, respectively, as $s_2-s_1\to 0$.
- (166) implies the ratio between the measures of $B_r(\gamma_1^\delta(s_1))$ and $B_r(\gamma_1^\delta(s_2))$ is Hölder dependent on s_2-s_1, and in particular goes uniformly to 1 as $s_2-s_1\to 0$.

We show $\gamma_1^\delta(t)=\gamma(t)$ for all $t\in[0,0.5]$. Suppose not; let $t_0:=\max\{t\in[0,0.5]:\gamma_1^\delta(s)=\gamma(s)\,\forall s\in[0,t]\}$ and so $t_0\in[\delta,0.5)$.

We claim there exist $t_1\in(t_0,0.5)$ and $\bar r>0$ such that for any $r\le\bar r$, there exists $A_1\subseteq B_r(\gamma_1^\delta(t_1))$ and $A_2\subseteq B_r(\gamma(t_1))$ such that

(I) $\gamma_1^\delta(t_1)\neq\gamma(t_1)$,

(II) $\Psi_{t_1-t_0}(A_1)\subseteq B_r(\gamma_1^\delta(t_0))$ and $\Psi_{t_1-t_0}(A_2)\subseteq B_r(\gamma(t_0))$,

(III) $m(\Psi_{t_1-t_0}(A_1))/m(B_r(\gamma_1^\delta(t_0)))>\frac{1}{2}$ and $m(\Psi_{t_1-t_0}(A_2))/m(B_r(\gamma(t_0)))>\frac{1}{2}$.

We can choose t_1 arbitrarily close to t_0 so that (I) holds by definition of t_0. Statements (II) and (III) then follow from (163)–(166) for γ_1^δ, the same for $\tilde\gamma$, Bishop–Gromov and Theorem 3.15(II) to control the volume distortion of Ψ, as soon as t_1 is chosen close enough to t_0. Choosing $r\le\min\{\bar r, d(\gamma_1^\delta(t_1),\gamma(t_1))/2\}$ then leads to a contradiction with Theorem 2.25. To be precise, Theorem 2.25 can be used to show the subset of points $x\in X$ where a geodesic from p to x can be extended to two branching geodesics must be measure 0 (see [20, Proposition 4.5]), which gives the contradiction.

Therefore $\gamma_1^\delta(t)=\gamma(t)$ for all $t\in[0,0.5]$. Since this is true for all $\delta\in(0,0.1)$, taking $\delta\to 0$ and using the Arzelà–Ascoli theorem, after possibly passing to a subsequence, we obtain a geodesic $\bar\gamma_1$ satisfying Theorem 5.9 with $\bar\gamma_1\equiv\gamma$ on $[0,0.5]$ and $\bar\gamma_1(1)=q_1$. The same construction for γ_2 gives $\bar\gamma_2$ satisfying Theorem 5.9 with $\bar\gamma_2\equiv\gamma$ on $[0,0.5]$ and $\bar\gamma_2(1)=q_2$. Applying the previous argument again for $\bar\gamma_1$ and $\bar\gamma_2$ shows they cannot split, which is a contradiction. □

As a corollary, we have the following improvement of Theorem 5.10:

Corollary 6.1 *Let (X,d,m) be an* RCD$(-(N-1),N)$ *space for some $N\in(1,\infty)$ and $p,q\in X$ with $d(p,q)=1$. For any $\delta\in(0,0.1)$, there exist $\epsilon(N,\delta)>0$, $\bar r(N,\delta)>0$ and $C(N,\delta)$ such that for any unit-speed geodesic γ between p and q, $r\le\bar r$ and $t_0,t_1\in[\delta,1-\delta]$, if $|t_1-t_0|\le\epsilon$ then*

$$d_{pGH}\big((B_r(\gamma(t_0)),\gamma(t_0)),(B_r(\gamma(t_1)),\gamma(t_1))\big)\le Cr|t_1-t_0|^{1/(2N(1+2N))}.$$

Proof Fix any $s\in[0,0.5]$. Since $(X,2d,m)$ is again an RCD$(-(N-1),N)$ space and

$$2d(\gamma(s),\gamma(s+0.5))=1,$$

we may use Theorem 5.10 to construct some $2d$-unit-speed geodesic γ^s between $\gamma(s)$ and $\gamma(s+0.5)$. Since X is nonbranching, γ^s and γ must coincide between $\gamma(s)$ and $\gamma(s+0.5)$. Since this is true for all

$s \in [0, 0.5]$ and all γ^s have the Hölder properties of Theorem 5.10 on $(X, 2d, m)$, the same is true for γ on (X, d, m) with slightly worse constants. □

Theorem 1.1 now follows immediately by rescaling.

6.2 Dimension and weak convexity of the regular set

In this subsection we will extend to the $\mathrm{RCD}(K, N)$ setting the results of [30] on regular sets. All proofs translate directly from [30]. We mention again that Theorem 6.2 has been proved through novel means in [15].

Theorem 6.2 (constancy of the dimension) *Let (X, d, m) be an $\mathrm{RCD}(K, N)$ metric measure space for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Assume X is not a point. There exists a unique $n \in \mathbb{N}$ with $1 \le n \le N$ such that $m(X \backslash \mathcal{R}_n) = 0$.*

Proof Let $A^1, A^2 \subseteq X \times X$ be the sets of $(x, y) \in X \times X$ such that geodesics from x to y are extendible past x and y, respectively. For each $x \in X$, let A_x be the set of $y \in X$ such that geodesics from x to y are extendible past y. Using the arguments of [20, Section 4], A^1 and A^2 are $(m \times m)$-measurable and A_x is m-measurable for all $x \in X$. By a standard argument using Bishop–Gromov, $m(X \backslash A_x) = 0$ for any $x \in X$. Let $A := A^1 \cap A^2$; Fubini's theorem then gives $(m \times m)((X \times X) \backslash A) = 0$.

Let $\gamma_{x,y} \colon [0, 1] \to X$ be a Borel selection (Remark 2.26) of constant-speed geodesics from any $x \in X$ to any $y \in X$. Since $m(\mathcal{S}) = 0$, it follows from applying the segment inequality to the characteristic function of $\mathcal{S}$ that for $(m \times m)$-a.e. $(x, y) \in X \times X$, $\gamma_{x,y} \cap \mathcal{R}_{\mathrm{reg}}$ has full measure, and therefore is also dense in $[0, 1]$. By Theorem 1.1, for any geodesic γ and k, $\gamma \cap \mathcal{R}_k$ is closed relative to the interior of γ. Combining these with the fact that almost every $\gamma_{x,y}$ is extendible, we obtain for $(m \times m)$-a.e. $(x, y) \in X \times X$ that there exists $k \in \mathbb{N}$ with $1 \le k \le N$ such that $\gamma_{x,y} \subseteq \mathcal{R}_k$. This leads to a contradiction if there are two regular sets of different dimension with positive measure. □

Definition 6.3 (m-a.e. convexity) Let (X, d, m) be a metric measure space. Let S be an m-measurable set in X. S is *m-a.e. convex* if and only if for $(m \times m)$-almost every pair $(x, y) \in S \times S$ there exists a minimizing geodesic $\gamma \subseteq S$ connecting x and y.

Definition 6.4 (weak convexity) Let (X, d) be a metric space. $S \subseteq X$ is *weakly convex* if and only if for all $(x, y) \in S \times S$ and $\epsilon > 0$ there exists an ϵ-geodesic (see Definition 4.5) $\gamma \subseteq S$ connecting x and y.

Theorem 6.5 (m-a.e. and weak convexity of the regular set) *Let $\mathcal{R}_n$ be as in Theorem 6.2. Then*

(I) *$\mathcal{R}_n$ is m-a.e. convex,*

(II) *$\mathcal{R}_n$ is weakly convex.*

In particular, $\mathcal{R}_n$ is connected.

Proof Statement (I) is contained in the proof of Theorem 6.2. The proof of (II) follows verbatim from [30, Theorem 1.20]. □

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