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A cubical model for (∞, n) -categories

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We propose a new model for the theory of (∞, n) -categories (including the case $n = \infty$) in the category of marked cubical sets with connections, similar in flavor to complicial sets of Verity. The model structure characterizing our model is shown to be monoidal with respect to suitably defined (lax and pseudo) Gray tensor products; in particular, these tensor products are both associative and biclosed. Furthermore, we show that the triangulation functor to precomplicial sets is a left Quillen functor and is strong monoidal with respect to both Gray tensor products.

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Introduction

The theory of (∞, n) -categories is becoming an important tool in a number of areas of mathematics, including manifold topology, where it is used in the definition and classification of extended topological quantum field theories [Lurie 2009], and in derived algebraic geometry, where it is used to capture certain properties of the “category” of correspondences [Gaitsgory and Rozenblyum 2017a, 2017b]. There are several equivalent models for this theory, including: n -trivial saturated complicial sets [Verity 2008b, 2007; Riehl and Verity 2020; Loubaton 2022], n -quasicategories [Ara 2014], Θ_n -spaces [Rezk 2010] and n -fold complete Segal spaces [Barwick 2005].

We propose a new model for the theory of (∞, n) -categories, using *comical* (composition + cubical) sets. Comical sets are certain marked cubical sets (having marked n -cubes for all values $n \geq 1$), just like complicial sets are certain marked simplicial sets. Our model allows for a particularly elegant and simple treatment of the (lax and pseudo) Gray tensor products since they are inherently cubical in nature. One can find drawings of cubes in Gray's book [1974], and the simplest ways of defining the lax Gray tensor product of strict ω -categories are via cubical sets [Crans 1995; Al-Agl et al. 2002].

Because of the obvious similarities with complicial sets, there is a natural comparison functor to marked simplicial sets. To obtain it, we extend the usual triangulation functor $T: \text{cSet} \rightarrow \text{sSet}$ from cubical sets to simplicial sets to a marked version $T: \text{cSet}^+ \rightarrow \text{PreComp}$. Here T is valued not in the whole category sSet^+ of marked simplicial sets but in the reflective subcategory PreComp of precomplicial sets, so that our results hold up to isomorphism rather than homotopy. PreComp supports a model structure that is Quillen equivalent to the complicial model structure on sSet^+ and the lax Gray tensor product on sSet^+ is more well-behaved when restricted to PreComp .

With that, our main results (see Theorems 3.3, 2.16, 6.5, 6.10 and 7.1) can be summarized as follows:

Theorem *The category cSet^+ of marked cubical sets carries a model structure whose cofibrations are the monomorphisms and whose fibrant objects are the comical sets. This model structure is monoidal with respect to both the lax and pseudo Gray tensor products, which are simultaneously associative and biclosed.*

Furthermore, the triangulation functor $T: \text{cSet}^+ \rightarrow \text{PreComp}$ is left Quillen and strong monoidal with respect to both Gray tensor products.

Since this paper was first made available in 2020, a slight adaptation of our model was proven to be Quillen equivalent via the triangulation functor to n -trivial saturated complicial sets in [Doherty et al. 2023]. The “special cases” of this result had previously been known for $(\infty, 0)$ -categories (ie ∞ -groupoids) [Cisinski 2006] and $(\infty, 1)$ -categories [Doherty et al. 2024], although these papers consider slightly different versions of the cubical site from us.

In particular, our model validates the assertions [Gaitsgory and Rozenblyum 2017a, Propositions 10.3.2.6 and 10.3.2.9], given there without a proof or a reference. They are essentially the desiderata of a convenient model of $(\infty, 2)$ -categories used throughout [Gaitsgory and Rozenblyum 2017a, 2017b], and in that sense our model in cSet^+ is a convenient such model. We should note however that these assertions were previously proven in [Verity 2008b] and [Maehara 2021] in the contexts of complicial sets and 2-quasicategories, respectively.

Finally, our work owes a great deal to [Steiner 2006], where the (semi)cubical nerves of strict ω -categories are analyzed. In particular, our definition of comical open boxes in Section 3 follows [Steiner 2006, Example 2.9].

Organization of the paper We begin in Section 1 by reviewing the necessary background on model categories, cubical sets and comical sets. In Section 2, we introduce marked cubical sets, study their basic properties, and construct both the lax and the pseudo Gray tensor products. In Section 3, we define comical sets and construct the model structure for them. As a proof of concept, we define in Section 4 the homotopy 1-category of a comical set and show that it has expected properties. We then turn our attention to the comparison between the cubical and the simplicial approaches. We extend the triangulation functor to marked cubical sets in Section 5, show that it is strong monoidal with respect to both the lax and the pseudo Gray tensor products in Section 6, and that it is a left Quillen functor in Section 7.

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1 Background

In this section we introduce the notation and collect preliminary results to be used later in the paper.

1.1 Model categories

In this subsection, we review (a special case of) the theory of Olschok [2009] for constructing combinatorial model structures with all objects cofibrant, which generalizes the theory of Cisinski [2006] for constructing combinatorial model structures on presheaf categories with cofibrations the monomorphisms. This theory will be used to construct the model structures for comical sets.

Definition 1.1 [Simpson 2012] We say a set Λ of trivial cofibrations in a model category \mathcal{M} is *pseudogenerating* if and only if any map f that has a fibrant codomain and the right lifting property with respect to Λ is a fibration.

Now fix a locally presentable category \mathcal{K} .

Given a bifunctor $\odot: \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ and maps $f: A \rightarrow A'$ and $g: B \rightarrow B'$ in \mathcal{K} , we denote by

$$f \hat{\odot} g: (A' \odot B) \amalg_{A \odot B} (A \odot B') \rightarrow A' \odot B'$$

the *Leibniz \odot -product* of f and g .

Similarly, for any natural transformation $\phi: F \Rightarrow G$ between endofunctors $F, G: \mathcal{K} \rightarrow \mathcal{K}$ and for any $f: A \rightarrow A'$ in \mathcal{K} , we denote by

$$\widehat{\phi}_f: G(A) \amalg_{F(A)} F(A') \rightarrow G(A')$$

the *Leibniz product* of ϕ and f .

By the *cellular closure* of a set S of maps in \mathcal{K} , we mean the closure of S under pushouts along arbitrary maps and transfinite composition. In the rest of this subsection, assume that we are given a small set I of maps in \mathcal{K} whose cellular closure is precisely the monomorphisms.

Definition 1.2 A *functorial cylinder* on \mathcal{K} is a functor $C: \mathcal{K} \rightarrow \mathcal{K}$ equipped with natural transformations $\partial^0, \partial^1: \text{Id} \Rightarrow C$ and $\sigma: C \rightarrow \text{Id}$ such that $\sigma \partial^0 = \sigma \partial^1 = \text{id}$. We also write $\partial_X = [\partial_X^0, \partial_X^1]: X + X \rightarrow CX$. We say that C is a *cartesian cylinder* if the functor C preserves colimits and moreover ∂_X is a monomorphism for all X .

Definition 1.3 Suppose that \mathcal{K} admits a cartesian functorial cylinder $C = (C, \partial^0, \partial^1, \sigma)$. Let S be a set of morphisms in \mathcal{K} . We define $\Lambda(\mathcal{K}, I, C, S) \subseteq \text{Mor } \mathcal{K}$ to be the smallest class of morphisms containing

$$S \cup \{\widehat{\partial}_i^0 \mid i \in I\} \cup \{\widehat{\partial}_i^1 \mid i \in I\},$$

closed under the operation $f \mapsto \widehat{\partial}_f$.

Theorem 1.4 [Olschok 2009, Theorem 2.2.5, Lemma 2.2.20] *Let \mathcal{K} and I be as above. Suppose we are given a cartesian functorial cylinder C on \mathcal{K} and a set S of monomorphisms in \mathcal{K} . Then there exists a model structure on \mathcal{K} uniquely characterized by the following properties:*

- *The cofibrations are the monomorphisms.*
- *The set $\Lambda(\mathcal{K}, I, C, S)$ is a pseudogenerating set of trivial cofibrations.*

This model structure is combinatorial and left proper.

Proposition 1.5 *Let \mathcal{K} and I be as above. Suppose \mathcal{K} admits a model structure whose cofibrations are the monomorphisms, and a pseudogenerating set Λ of trivial cofibrations. Suppose further that \mathcal{K} is equipped with a tensor product $\odot: \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ that forms part of a biclosed monoidal structure. Then these data form a monoidal model structure if and only if*

- *$f \widehat{\odot} g$ is a cofibration whenever $f, g \in I$,*
- *$f \widehat{\odot} g$ is a trivial cofibration whenever $f \in \Lambda$ and $g \in I$, and*
- *$f \widehat{\odot} g$ is a trivial cofibration whenever $f \in I$ and $g \in \Lambda$.*

Proof This is an instance of [Maehara 2021, Proposition A.4]. See also [Henry 2020, Appendix B]. \square

1.2 Cubical sets

We will define cubical sets as presheaves on the *box category*, denoted by \square . The category \square is the (nonfull) subcategory of the category of posets whose objects are the posets of the form $[1]^n := \{0 \leq 1\}^n$, and whose maps are generated by the cubical operators

- *faces* $\partial_{i,\varepsilon}^n : [1]^{n-1} \rightarrow [1]^n$ for $i = 1, \dots, n$ and $\varepsilon = 0, 1$ given by

$$\partial_{i,\varepsilon}^n(x_1, x_2, \dots, x_{n-1}) = (x_1, x_2, \dots, x_{i-1}, \varepsilon, x_i, \dots, x_{n-1}),$$

- *degeneracies* $\sigma_i^n : [1]^n \rightarrow [1]^{n-1}$ for $i = 1, 2, \dots, n$ given by

$$\sigma_i^n(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n),$$

- *max-connections* $\gamma_{i,0}^n : [1]^n \rightarrow [1]^{n-1}$ for $i = 1, 2, \dots, n-1$ given by

$$\gamma_{i,0}^n(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{i-1}, \max\{x_i, x_{i+1}\}, x_{i+2}, \dots, x_n),$$

- *min-connections* $\gamma_{i,1}^n : [1]^n \rightarrow [1]^{n-1}$ for $i = 1, 2, \dots, n-1$ given by

$$\gamma_{i,1}^n(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{i-1}, \min\{x_i, x_{i+1}\}, x_{i+2}, \dots, x_n).$$

We will omit the superscript n when no confusion is possible.

A straightforward computation shows that cubical operators satisfy the following *cubical identities*. These maps obey the following *cubical identities*:

$$\begin{aligned} \partial_{j,\varepsilon'} \partial_{i,\varepsilon} &= \partial_{i+1,\varepsilon} \partial_{j,\varepsilon'} & \text{for } j \leq i, \\ \sigma_i \sigma_j &= \sigma_j \sigma_{i+1} & \text{for } j \leq i, \\ \sigma_j \partial_{i,\varepsilon} &= \begin{cases} \partial_{i-1,\varepsilon} \sigma_j & \text{for } j < i, \\ \text{id} & \text{for } j = i, \\ \partial_{i,\varepsilon} \sigma_{j-1} & \text{for } j > i, \end{cases} \\ \gamma_{j,\varepsilon'} \gamma_{i,\varepsilon} &= \begin{cases} \gamma_{i,\varepsilon} \gamma_{j+1,\varepsilon'} & \text{for } j > i, \\ \gamma_{i,\varepsilon} \gamma_{i+1,\varepsilon} & \text{for } j = i, \varepsilon' = \varepsilon, \end{cases} \end{aligned} \quad \gamma_{j,\varepsilon'} \partial_{i,\varepsilon} = \begin{cases} \partial_{i-1,\varepsilon} \gamma_{j,\varepsilon'} & \text{for } j < i-1, \\ \text{id} & \text{for } j \in \{i-1, i\}, \varepsilon = \varepsilon', \\ \partial_{i,\varepsilon} \sigma_i & \text{for } j \in \{i-1, i\}, \varepsilon = 1-\varepsilon', \\ \partial_{i,\varepsilon} \gamma_{j-1,\varepsilon'} & \text{for } j > i, \end{cases}$$

$$\sigma_j \gamma_{i,\varepsilon} = \begin{cases} \gamma_{i-1,\varepsilon} \sigma_j & \text{for } j < i, \\ \sigma_i \sigma_i & \text{for } j = i, \\ \gamma_{i,\varepsilon} \sigma_{j+1} & \text{for } j > i. \end{cases}$$

Let us point out that this is only one of the many choices of the box category that appear in the literature. References such as [Maltsiniotis 2009; Kapulkin et al. 2019] consider a box category that is spanned by faces, degeneracies and one of the connections, specifically the max-connection (although dual arguments can be used to work with min-connections as well). In [Cisinski 2006; Jardine 2006], a subcategory of our \square is considered that is generated by the face and degeneracy maps, but no connections; and in [Steiner 2006], an even smaller subcategory is considered, as it is spanned by the face maps alone. At the other extreme, [Kapulkin and Voevodsky 2020] works with \square as the full subcategory of Cat .

Our choice is intentional. Since our (marked) cubical sets will be used to model (∞, n) -categories, all of our cubes need to have an orientation, and hence the symmetry and diagonal maps appearing in the

choices strictly larger than ours are undesirable. On the other hand, the box category with at least one connection is known to have better categorical properties than the ones without connections; see [Tonks 1992; Maltiniotis 2009]. Finally, allowing for at least one connection, we choose to work with both to allow for a convenient description of the opposite (∞, n) -category.

Given our choice of the box category, we have the following *normal form* of morphisms in \square .

Theorem 1.6 [Grandis and Mauri 2003, Theorem 5.1] *Every map in the category \square can be factored uniquely as a composite*

$$(\partial_{c_1, \varepsilon'_1} \cdots \partial_{c_r, \varepsilon'_r})(\gamma_{b_1, \varepsilon_1} \cdots \gamma_{b_q, \varepsilon_q})(\sigma_{a_1} \cdots \sigma_{a_p}),$$

where $1 \leq a_1 < \cdots < a_p$, $1 \leq b_1 \leq \cdots \leq b_q$, $b_i < b_{i+1}$ if $\varepsilon_i = \varepsilon_{i+1}$ and $c_1 > \cdots > c_r \geq 1$. \square

With this, one can describe \square as the category generated by the cubical operators, subject to the cubical identities.

Remark 1.7 In particular, any composite face map can be written uniquely as $\alpha = \partial_{k_1, \varepsilon_1} \cdots \partial_{k_t, \varepsilon_t}$ with $k_1 > \cdots > k_t$. Geometrically, such an α is the intersection of all the $\partial_{k_s, \varepsilon_s}$.

This theorem allows us to prove the following key property of \square :

Theorem 1.8 *The category \square is an EZ Reedy category with the structure defined as*

- $\deg[1]^n = n$,
- \square_- is generated under composition by degeneracies and (both kinds of) connections,
- \square_+ is generated under composition by face maps.

The key difficulty in proving this theorem lies in showing that each map in \square_- is determined by its sections. This is done by induction on the length of the decomposition of such a map given in Theorem 1.6. Before proceeding with the proof, we state two lemmas. The first of these contains the base case of induction, whereas the second contains the technical heart of the proof—a case analysis allowing us to complete the inductive step.

Lemma 1.9 (1) *The sections of σ_i are $\partial_{i,0}$ and $\partial_{i,1}$.*

(2) *The sections of $\gamma_{i,0}$ are $\partial_{i,0}$ and $\partial_{i+1,0}$.*

(3) *The sections of $\gamma_{i,1}$ are $\partial_{i,1}$ and $\partial_{i+1,1}$.* \square

Lemma 1.10 *Given two distinct maps $p, p': [1]^n \rightarrow [1]^{n-k}$ in \square_- , there is a face map*

$$\partial_{i,\varepsilon}: [1]^{n-k} \rightarrow [1]^{n-(k-1)}$$

such that $p\partial_{i,\varepsilon} \neq p'\partial_{i,\varepsilon}$ and at least one of $p\partial_{i,\varepsilon}$ and $p'\partial_{i,\varepsilon}$ is in \square_- .

Proof We proceed by induction with respect to k with the base case of $k = 1$ handled by Lemma 1.9.

For the inductive step, we may use Theorem 1.6 to write

$$p = \gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_l, \varepsilon_l} \sigma_{j_1} \cdots \sigma_{j_m} \quad \text{and} \quad p' = \gamma_{i'_1, \varepsilon'_1} \cdots \gamma_{i'_{l'}, \varepsilon'_{l'}} \sigma_{j'_1} \cdots \sigma_{j'_{m'}},$$

and without loss of generality we may assume that $m \leq m'$.

We first suppose that there is an index j_i that does not appear in the set $j'_1, \dots, j'_{m'}$, ie there is a degeneracy in the decomposition of p that is not present in the decomposition of p' . Then we may note that the normal form of $p\partial_{j_i, 0}$ is obtained by removing σ_{j_i} from the normal form of p , and hence the resulting map is in \square_- . On the other hand, the normal form $p'\partial_{j_i, 0}$ will contain more degeneracy maps than that of $p\partial_{j_i, 0}$, since $\partial_{j_i, 0}$ will not cancel with any of the degeneracy maps present in the normal form of p' and we assumed $m \leq m'$.

If there is no such j_i , then the string $\sigma_{j_1} \cdots \sigma_{j_m}$ is a substring of $\sigma_{j'_1} \cdots \sigma_{j'_{m'}}$. By precomposing with different face maps, we may assume that $m = 0$. We proceed by case analysis, addressing $m' \geq 2$, $m' = 1$ and $m' = 0$ in order.

For $m' \geq 2$, we can write

$$p = \gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_\ell, \varepsilon_\ell} \quad \text{and} \quad p' = \gamma_{i'_1, \varepsilon'_1} \cdots \gamma_{i'_{\ell'}, \varepsilon'_{\ell'}} \sigma_{j'_1} \cdots \sigma_{j'_{m'}}.$$

Now observe that $p\partial_{i_\ell, \varepsilon_\ell} = \gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_{\ell-1}, \varepsilon_{\ell-1}}$ is in the normal form (and belongs to \square_-), but the normal form of $p\partial_{i_\ell, \varepsilon_\ell}$ must end with at least one degeneracy.

For $m' = 1$, we can write

$$p = \gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_\ell, \varepsilon_\ell} \quad \text{and} \quad p' = \gamma_{i'_1, \varepsilon'_1} \cdots \gamma_{i'_{\ell-1}, \varepsilon'_{\ell-1}} \sigma_{j'}.$$

To treat this case, we will precompose both p and p' with $\partial_{j', \varepsilon}$ to cancel the degeneracy appearing in the normal form of p' , yielding a normal form of $p'\partial_{j', \varepsilon}$, which then clearly belongs to \square_- . However, some care is needed to choose the correct ε in order to ensure that the normal form of $p\partial_{j', \varepsilon}$ is different from that of $p'\partial_{j', \varepsilon}$. If j' appears in the sequence: i_1, \dots, i_ℓ , then we pick $\varepsilon = 1 - \varepsilon_{j'}$. With this choice, the normal form of $p\partial_{j', \varepsilon}$ will end with a degeneracy, making it distinct from $p'\partial_{j', \varepsilon}$. If on the other hand j' does not appear on the list of indices: i_1, \dots, i_ℓ , we first need to determine whether when using cubical identities to write $p\partial_{j', \varepsilon}$ in normal form, we will encounter a connection with which our face map will cancel: if not, then we can pick either ε ; otherwise, we pick ε in such a way as to ensure that as a result of commuting the face and the connection, we obtain a degeneracy map.

At this point, it remains to treat the case when $m' = 0$, ie the normal forms of p and p' consist solely in connections:

$$p = \gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_k, \varepsilon_k} \quad \text{and} \quad p' = \gamma_{i'_1, \varepsilon'_1} \cdots \gamma_{i'_k, \varepsilon'_k}.$$

Note that the two decompositions have the same length, since both p and p' are maps $[1]^n \rightarrow [1]^{n-k}$. Without loss of generality, we may assume that $i_k \leq i'_k$, and we proceed by case analysis based on $i'_k - i_k$, considering three cases: $i'_k = i_k$, $i'_k = i_k + 1$ and $i'_k \geq i_k + 2$.

If $i_k = i'_k$, we precompose p and p' with $\partial_{i_k, \varepsilon_k}$. In the case of $\varepsilon_k = \varepsilon'_k$, this reduces us to the inductive hypothesis. If however $\varepsilon_k \neq \varepsilon'_k$, then the normal form of $p\partial_{i_k, \varepsilon_k}$ will be $\gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_{k-1}, \varepsilon_{k-1}}$, making it an element of \square_- , whereas the normal form of $p'\partial_{i_k, \varepsilon_k}$ will end in a degeneracy.

Next, suppose that $i'_k = i_k + 1$. Then the cases of $\varepsilon_k = \varepsilon'_k$ and $\varepsilon_k \neq \varepsilon'_k$ need to be treated separately. In the former, we precompose with $\partial_{i_k, \varepsilon_k}$. Then $p\partial_{i_k, \varepsilon_k} = \gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_{k-1}, \varepsilon_{k-1}}$ is the normal form, making it an element of \square_- , but the normal form of $p'\partial_{i_k, \varepsilon_k}$ ends with one of: a degeneracy, a connection of first index greater than i_k , or a connection $\gamma_{i_k, \varepsilon'_k}$, making it distinct from $p\partial_{i_k, \varepsilon_k}$. In the latter case, we see that the normal form of $p\partial_{i_{k+1}, \varepsilon'_k}$ ends with a degeneracy, but the normal form of $p'\partial_{i_{k+1}, \varepsilon'_k}$ ends with $\gamma_{i'_{k-1}, \varepsilon'_{k-1}}$ and this element belongs to \square_- .

Finally, if $i'_k \geq i_k + 2$, then we precompose both p and p' with $\partial_{i_k, \varepsilon_k}$. This gives the normal form of $p\partial_{i_k, \varepsilon_k}$ as $\gamma_{i_1, \varepsilon_1} \cdots \gamma_{i_{k-1}, \varepsilon_{k-1}}$, making it an element of \square_- . But the normal form of $p'\partial_{i_k, \varepsilon_k}$ ends with either a degeneracy or the connection $\gamma_{i'_k, \varepsilon'_k}$, making it distinct. □

Proof of Theorem 1.8 The Reedy part follows immediately from Theorem 1.6. Every morphism in \square_- is a split epimorphism by Lemma 1.9.

It remains to show that maps in \square_- are determined by their sections. To do this, we pick two such maps $p, p': [1]^n \rightarrow [1]^{n-k}$ and proceed by induction with respect to k . The base case of $k = 1$ is handled by Lemma 1.9, whereas the inductive step is handled by Lemma 1.10. □

The category \square carries a canonical strict monoidal product \otimes given by $[1]^m \otimes [1]^n := [1]^{m+n}$ with unit given by $[1]^0$. Note that this product is not cartesian since, for instance, there is no “diagonal” map $[1]^1 \rightarrow [1]^2$ in \square . This monoidal structure leads to another characterization of our box category, due to Grandis and Mauri [Grandis and Mauri 2003, Section 5], as a certain kind of a free monoidal category.

A *cubical monoid* in a monoidal category $(\mathcal{C}, \otimes, I)$ is an object X equipped with maps

$$\partial_0, \partial_1: I \rightarrow X, \quad \sigma: X \rightarrow I, \quad \gamma_0, \gamma_1: X \otimes X \rightarrow X,$$

subject to the axioms

$$\begin{aligned} \sigma \partial_\varepsilon &= \text{id} \quad \text{for } \varepsilon = 0, 1, \\ \sigma \gamma_\varepsilon &= \sigma(\sigma \otimes \text{id}_X) = \sigma(\text{id}_X \otimes \sigma) \quad \text{for } \varepsilon = 0, 1, \\ \gamma_\varepsilon(\gamma_\varepsilon \otimes \text{id}_X) &= \gamma_\varepsilon(\text{id}_X \otimes \gamma_\varepsilon) \quad \text{for } \varepsilon = 0, 1, \\ \gamma_\varepsilon(\partial_\varepsilon \otimes \text{id}_X) &= \text{id}_X = \gamma_\varepsilon(\text{id}_X \otimes \partial_\varepsilon) \quad \text{for } \varepsilon = 0, 1, \\ \gamma_\varepsilon(\partial_\delta \otimes \text{id}_X) &= \partial_\delta \sigma = \gamma_\varepsilon(\text{id}_X \otimes \partial_\delta) \quad \text{for } \delta \neq \varepsilon. \end{aligned}$$

Theorem 1.11 [Grandis and Mauri 2003, Theorem 5.2(d)] *The box category \square is the free strict monoidal category equipped with a cubical monoid.* \square

Having established basic properties of the box category, we can now define cubical sets and fundamental constructions on them.

Definition 1.12 *A cubical set is a presheaf $X : \square^{\text{op}} \rightarrow \text{Set}$. A cubical map is a natural transformation of such presheaves. The category of cubical sets and cubical maps will be denoted by cSet .*

We write \square^n for the cubical set represented by $[1]^n$ and call it the (*generic*) n -cube. The *boundary* of the n -cube, denoted by $\partial \square^n \rightarrow \square^n$, is the maximal proper subobject of the representable \square^n , ie the union of all of its faces. The subobject of \square^n given by the union of all faces except the $(i, \varepsilon)^{\text{th}}$ one is called the (i, ε) -open box and denoted by $\square^n_{i, \varepsilon} \rightarrow \square^n$.

Proposition 1.13 *The monomorphisms of cSet are the cellular closure of the set*

$$\{\partial \square^n \hookrightarrow \square^n \mid n \geq 0\}.$$

Proof This follows from Theorem 1.6. \square

The monoidal product \otimes can be extended via Day convolution from \square to cSet , making $(\text{cSet}, \otimes, \square^0)$ a biclosed monoidal category. We refer to this monoidal product as the *geometric product* of cubical sets.

We adopt the convention of writing the action of cubical operators on the right, eg the $(1, 0)$ -face of an n -cube $x : \square^n \rightarrow X$ will be denoted by $x \partial_{1,0}$.

Proposition 1.14 *The geometric product $X \otimes Y$ of cubical sets X and Y admits the following description.*

- For $n \geq 0$, the n -cubes in $X \otimes Y$ are the formal products $x \otimes y$ of pairs $x \in X_k$ and $y \in Y_\ell$ such that $k + \ell = n$, subject to the identification $(x \sigma_{k+1}) \otimes y = x \otimes (y \sigma_1)$.
- For $x \in X_k$ and $y \in Y_\ell$, the faces, degeneracies and connections of the $(k + \ell)$ -cube $x \otimes y$ are computed as follows:

$$\begin{aligned} (x \otimes y) \partial_{i, \varepsilon} &= \begin{cases} (x \partial_{i, \varepsilon}) \otimes y & \text{if } 1 \leq i \leq k, \\ x \otimes (y \partial_{i-k, \varepsilon}) & \text{if } k + 1 \leq i \leq k + \ell; \end{cases} \\ (x \otimes y) \sigma_i &= \begin{cases} (x \sigma_i) \otimes y & \text{if } 1 \leq i \leq k + 1, \\ x \otimes (y \sigma_{i-k}) & \text{if } k + 1 \leq i \leq k + \ell + 1; \end{cases} \\ (x \otimes y) \gamma_{i, \varepsilon} &= \begin{cases} (x \gamma_{i, \varepsilon}) \otimes y & \text{if } 1 \leq i \leq k, \\ x \otimes (y \gamma_{i-k, \varepsilon}) & \text{if } k + 1 \leq i \leq k + \ell. \end{cases} \end{aligned}$$

Proof This is proven in [Doherty et al. 2024, Proposition 1.20] in the case of cubical sets with one kind of connection. The proof given there works almost verbatim in our case. \square

Given cubes $x \in X_k$ and $y \in Y_\ell$, we may regard them as cubical maps $x: \square^k \rightarrow X$ and $y: \square^\ell \rightarrow Y$. Then applying the geometric product to these maps yields a map $x \otimes y: \square^{k+\ell} \rightarrow X \otimes Y$, which corresponds precisely to the $(k+\ell)$ -cube with the same name. Moreover, every n -cube of $X \otimes Y$ arises via this construction for some, perhaps nonunique, pair of cubes $(x: \square^k \rightarrow X, y: \square^\ell \rightarrow Y)$ for $k + \ell = n$.

Since the identification in Proposition 1.14 only concerns degenerate cubes, we obtain the following corollary.

Corollary 1.15 *A pair of nondegenerate cubes $x \in X_k, y \in Y_\ell$ yields a nondegenerate $(k+\ell)$ -cube $x \otimes y$ in $X \otimes Y$. Conversely, every nondegenerate cube in $X \otimes Y$ arises this way from a unique pair of nondegenerate cubes.* □

Remark 1.16 In particular, when $X = \square^m$ and $Y = \square^n$ are representable, this pairing is given by the formula

$$(\partial_{k_1, \varepsilon_1} \cdots \partial_{k_t, \varepsilon_t}) \otimes (\partial_{\ell_1, \eta_1} \cdots \partial_{\ell_s, \eta_s}) = \partial_{m+\ell_1, \eta_1} \cdots \partial_{m+\ell_s, \eta_s} \partial_{k_1, \varepsilon_1} \cdots \partial_{k_t, \varepsilon_t},$$

where all strings of ∂ 's are in the normal form specified by Theorem 1.6. The factors are permuted because the geometric product lists cubes in order (in the sense that x in $x \otimes y$ corresponds to smaller values of i) whereas the normal form lists faces in reverse order.

Proposition 1.17 *For natural numbers k, m and n , and $\varepsilon = 0, 1$, we have natural isomorphisms*

$$\begin{aligned} (\partial \square^m \hookrightarrow \square^m) \widehat{\otimes} (\partial \square^n \hookrightarrow \square^n) &\cong (\partial \square^{m+n} \hookrightarrow \square^{m+n}), \\ (\Gamma_{k, \varepsilon}^m \hookrightarrow \square^m) \widehat{\otimes} (\partial \square^n \hookrightarrow \square^n) &\cong (\Gamma_{k, \varepsilon}^{m+n} \hookrightarrow \square^{m+n}), \\ (\partial \square^m \hookrightarrow \square^m) \widehat{\otimes} (\Gamma_{k, \varepsilon}^n \hookrightarrow \square^n) &\cong (\Gamma_{m+k, \varepsilon}^{m+n} \hookrightarrow \square^{m+n}). \end{aligned}$$

Proof This follows from [Doherty et al. 2024, Lemma 1.26] and the associativity of $\widehat{\otimes}$. □

Using the above proposition and the fact that \square is an elegant Reedy category, we obtain:

Corollary 1.18 *If f and g are monomorphisms in cSet , then $f \widehat{\otimes} g$ is again a monomorphism.* □

The category \square admits two canonical identity-on-objects automorphisms $(-)^{\text{co}}, (-)^{\text{co-op}}: \square \rightarrow \square$. The first one takes $\partial_{i, \varepsilon}^n$ to $\partial_{n+1-i, \varepsilon}^n$, σ_i^n to σ_{n+1-i}^n , and $\gamma_{i, \varepsilon}^n$ to $\gamma_{(n-1)+1-i, \varepsilon}^n$. The second one takes $\partial_{i, \varepsilon}^n$ to $\partial_{i, 1-\varepsilon}^n$, σ_i^n to σ_i^n and $\gamma_{i, \varepsilon}^n$ to $\gamma_{i, 1-\varepsilon}^n$. (Their names are motivated by the fact that, according to the source/target distinction described in Section 3 below, $(-)^{\text{co}}$ reverses the direction of even-dimensional cubes and $(-)^{\text{co-op}}$ reverses the direction of all cubes.) Precomposition with these automorphisms induces functors also denoted by $(-)^{\text{co}}, (-)^{\text{co-op}}: \text{cSet} \rightarrow \text{cSet}$. Moreover, $(-)^{\text{co}} \circ (-)^{\text{co-op}} = (-)^{\text{co-op}} \circ (-)^{\text{co}}$, yielding a third automorphism $(-)^{\text{op}} := (-)^{\text{co}} \circ (-)^{\text{co-op}}$.

The ‘‘contravariant’’ behavior of these automorphisms with respect to the cubical structure can be seen via their interaction with the geometric product.

- Proposition 1.19** (1) The functor $(-)^{\text{co}}: \text{cSet} \rightarrow \text{cSet}$ is strong antimonoidal, ie $(X \otimes Y)^{\text{co}} \cong Y^{\text{co}} \otimes X^{\text{co}}$, naturally in X and Y .
- (2) The functor $(-)^{\text{co-op}}: \text{cSet} \rightarrow \text{cSet}$ is strong monoidal, ie $(X \otimes Y)^{\text{co-op}} \cong X^{\text{co-op}} \otimes Y^{\text{co-op}}$, naturally in X and Y .
- (3) The functor $(-)^{\text{op}}: \text{cSet} \rightarrow \text{cSet}$ is strong antimonoidal, ie $(X \otimes Y)^{\text{op}} \cong Y^{\text{op}} \otimes X^{\text{op}}$, naturally in X and Y . \square

Finally, the composite $\square \rightarrow \text{Cat} \rightarrow \text{sSet}$ given by $\square^n \mapsto (\Delta^1)^n$ defines a cocubical object in the category of simplicial sets. Taking the Yoneda extension, we obtain an adjoint pair

$$T: \text{cSet} \rightleftarrows \text{sSet} : U.$$

We will call $T: \text{cSet} \rightarrow \text{sSet}$ the *triangulation* functor.

The triangulation functor can also be seen through the lenses of Theorem 1.11. The simplicial faces $\partial_1, \partial_0: [0] \rightarrow [1]$ and degeneracy $\sigma_0: [1] \rightarrow [0]$ maps, along with $\max, \min: [1]^2 \rightarrow [1]$, equip $[1]$ with the structure of a cubical monoid. Since the nerve functor preserves products, this gives a structure of a cubical monoid on Δ^1 in sSet . The triangulation functor $T: \text{cSet} \rightarrow \text{sSet}$ arises from this cubical monoid via Theorem 1.11.

We conclude this section by recording some properties of the triangulation functor.

- Proposition 1.20** (1) T is strong monoidal.
- (2) T preserves monomorphisms.

Proof The first statement follows by the fact that T preserves colimits and sSet is cartesian closed.

The second statement follows from first, since T takes boundary inclusions, ie the elements of the cellular model, to monomorphisms. \square

1.3 Complicial sets

In this section, we recall marked simplicial sets and model structures for (n -trivial) complicial sets from [Verity 2008a, 2008b]. The reader is referred to those papers for more detail on the subject. The theory developed in Section 3 draws great insight from this simplicial precursor.

Just as in the case of cubical sets, when working with simplicial sets, we will write the action of simplicial operators on the right.

Definition 1.21 A *marked simplicial set* is a simplicial set X equipped with a subset eX of its simplices called the *marked* simplices such that

- no 0-simplex is marked, and
- every degenerate simplex is marked.

A *map* of marked simplicial sets $f: X \rightarrow Y$ is a map of simplicial sets which carries marked simplices to marked simplices. We denote sSet^+ for the category of marked simplicial sets with maps for morphisms.

Marked simplicial sets used to be called *stratified simplicial sets* (see eg [Verity 2008a]), but the name “marked” is more descriptive and has since become more popular.

There is a natural forgetful functor $\mathbf{sSet}^+ \rightarrow \mathbf{sSet}$, which has both left and right adjoints. The left adjoint $X \mapsto X^b$ endows a simplicial set X with the *minimal marking*, marking only the degenerate simplices. The right adjoint $X \mapsto X^\#$ endows a simplicial set X with the *maximal marking*, marking all simplices. If X is a simplicial set, we will by default consider it as a marked simplicial set *with its minimal marking* X^b .

Definition 1.22 We say that $X \in \mathbf{sSet}^+$ is *n-trivial* if every simplex of dimension $\geq n + 1$ is marked.

Given a marked simplicial set X , we will write $\text{core}_n X$ for its maximal n -trivial subset. In other words, the k -simplices of $\text{core}_n X$ are precisely those k -simplices x in X such that $x\alpha$ is marked in X for any $\alpha: [m] \rightarrow [k]$ with $m > n$. This assignment extends to a functor $\text{core}_n: \mathbf{sSet}^+ \rightarrow \mathbf{sSet}^+$, which admits a left adjoint $\tau_n: \mathbf{sSet}^+ \rightarrow \mathbf{sSet}^+$. Explicitly, τ_n acts as the identity on the underlying simplicial set and a k -simplex is marked in $\tau_n X$ if either $k \leq n$ and x is marked in X or $k \geq n + 1$.

Definition 1.23 A map $X \rightarrow Y$ of marked simplicial sets is

- *regular* if it creates markings, ie for an n -simplex x of X we have that $x \in eX_n$ if and only if $f(x) \in eY_n$, and
- *entire* if the induced map between the underlying simplicial sets is invertible.

We now define several distinguished objects and maps in \mathbf{sSet}^+ . These will be essential to the description of various model structures we will be considering.

We denote by $\tilde{\Delta}^n = \tau_{n-1}(\Delta^n)$ the n -simplex with the nondegenerate n -simplex marked and no other nondegenerate simplices marked. We call the canonical map $\Delta^n \rightarrow \tilde{\Delta}^n$ the *n-marker*.

For $n \geq 1$ and $0 \leq k \leq n$, we denote by Δ_k^n the n -simplex with the following marking: a nondegenerate simplex is marked if and only if it contains all of the points $\{k - 1, k, k + 1\} \cap [n]$ among its vertices. We call Δ_k^n the *k-complicial n-simplex*. We denote by $\Lambda_k^n \subset \Delta_k^n$ the k -horn of dimension n (ie the simplicial subset missing the nondegenerate n -simplex and the k^{th} $(n-1)$ -face) endowed with the marking making it a regular subset of Δ_k^n . We call Λ_k^n the *complicial k-horn of dimension n*. We call the inclusion $\Lambda_k^n \rightarrow \Delta_k^n$ the *k-complicial horn inclusion of dimension n*. We write $\Delta_k^{n''} = \tau_{n-2}\Delta_k^n$, and we write $\Delta_k^{n'} = \Delta_k^n \amalg_{\Lambda_k^n} \tau_{n-2}\Delta_k^n$. The canonical inclusion $\Delta_k^{n'} \rightarrow \Delta_k^{n''}$ is called the *elementary k-complicial marking extension of dimension n*.

Let Δ_{eq}^3 denote the marked simplicial set obtained from Δ^3 by marking the 1-simplices $\{0, 2\}$, $\{1, 3\}$ and all 2- and 3-simplices. By a *saturation map*, we mean a map of the form $\Delta^m \star \Delta_{\text{eq}}^3 \rightarrow \Delta^m \star (\Delta^3)^\#$ for $m \geq -1$ (where Δ^{-1} is interpreted as \emptyset).

There are two standard model structures on marked simplicial sets:

Theorem 1.24 The category \mathbf{sSet}^+ carries two model structures:

- (1) The **complicial model structure** characterized by the following properties:
 - The cofibrations are the monomorphisms.
 - The set of
 - complicial horn inclusions, and
 - elementary complicial marking extensions
 forms a pseudogenerating set of trivial cofibrations.
- (2) The **saturated complicial model structure** characterized by the following properties:
 - The cofibrations are the monomorphisms.
 - The set of
 - complicial horn inclusions,
 - elementary complicial marking extensions, and
 - saturation maps
 forms a pseudogenerating set of trivial cofibrations.

Both of these model structures are cartesian.

Proof This is a combination of [Verity 2008b, Lemma 72, Theorem 100 and Lemma 105] and [Ozornova and Rovelli 2020, Appendix B]. \square

Note that since the terminal object is always fibrant, this includes a characterization of the fibrant objects of the model structures, which are called (*saturated*) *complicial sets*.

Definition 1.25 A map of marked simplicial sets $X \rightarrow Y$ is

- a *complicial marking extension* if it is in the cellular closure of the elementary complicial marking extensions, and
- *complicial* if it is in the cellular closure of the complicial horn inclusions and the elementary complicial marking extensions.

There is also the n -trivial version of the (saturated) complicial model structure.

Theorem 1.26 The category \mathbf{sSet}^+ carries two model structures:

- (1) The **n -trivial complicial model structure** characterized by the following properties:
 - The cofibrations are the monomorphisms.
 - The set of
 - complicial horn inclusions,
 - elementary complicial marking extensions of dimension $\leq n + 1$, and
 - markers of dimension $> n$
 forms a pseudogenerating set of trivial cofibrations.

(2) The **saturated n -trivial complicial model structure** characterized by the following properties:

- The cofibrations are the monomorphisms.
- The set of
 - complicial horn inclusions,
 - elementary complicial marking extensions of dimension $\leq n + 1$,
 - markers of dimension $> n$, and
 - saturation maps

forms a pseudogenerating set of trivial cofibrations.

Proof Essentially the proof of Theorem 1.24, but combined with [Verity 2008b, Example 104]. □

In sSet^+ , the pseudo Gray tensor product is modeled by the cartesian product. We will adopt the following notation from [Verity 2008b], which emphasizes this view.

Notation 1.27 The cartesian product on sSet^+ (and its reflective subcategory PreComp described below) is denoted by \otimes .

Thus Theorem 1.24 in particular says that Verity’s model structure is monoidal with respect to the pseudo Gray tensor product.

The following proposition will be useful later.

Proposition 1.28 Let $f : A \rightarrow X$ and $g : B \rightarrow Y$ be entire maps in sSet^+ . Then their Leibniz pseudo Gray tensor $f \hat{\otimes} g$ is a complicial marking extension.

Proof Since the forgetful functor $\text{sSet}^+ \rightarrow \text{sSet}$ preserves colimits and products, $f \hat{\otimes} g$ is entire. We assume for the sake of simplicity that $f \hat{\otimes} g$ is an inclusion. Let (x, y) be an n -simplex that is marked in $X \otimes Y$ but not in $\text{dom}(f \hat{\otimes} g) = (A \otimes Y) \cup (X \otimes B)$. Equivalently, x is marked in X but not in A , and y is marked in Y but not in B . Then we must have $n \geq 1$, so the $(n+1)$ -simplex $z = (x\sigma_0, y\sigma_1)$ is well-defined. We claim that this simplex z extends as indicated below:

$$\begin{array}{ccc}
 \Delta^{n+1} & \xrightarrow{z} & (A \otimes Y) \cup (X \otimes B) \\
 \downarrow & \nearrow \exists & \\
 \Delta_1^{n+1} & &
 \end{array}$$

To see that z at least extends to Δ_1^{n+1} , let $\alpha : [m] \rightarrow [n + 1]$ be a simplicial operator with $0, 1, 2 \in \text{im } \alpha$. Then both $x(\sigma_0 \circ \alpha)$ and $y(\sigma_1 \circ \alpha)$ are degenerate, so $z\alpha$ is marked in $\text{dom}(f \hat{\otimes} g)$. Since the face $z\partial_0 = (x, y(\partial_0 \circ \sigma_0))$ is marked in $X \otimes B$ and the face $z\partial_2 = (x(\partial_1 \circ \sigma_0), y)$ is marked in $A \otimes Y$, we

indeed have an extension as indicated. Therefore we have a pushout square

$$\begin{array}{ccc} \coprod \Delta_1^{n+1'} & \longrightarrow & (A \otimes Y) \cup (X \otimes B) \\ \downarrow & & \downarrow \\ \coprod \Delta_1^{n+1''} & \longrightarrow & X \otimes Y \end{array}$$

where the coproducts are taken over all n -simplices (x, y) that are marked in $X \otimes Y$ but not in $\text{dom}(f \hat{\otimes} g)$, and both horizontal maps are induced by the simplices of the form $(x\sigma_0, y\sigma_1)$. \square

Definition 1.29 Let $[n] \in \Delta$ and let $0 \leq p, q \leq n$ be such that $p + q = n$. Then we write $\perp_1^{p,q} : [p] \rightarrow [n]$ for the simplicial operator $i \mapsto i$, and $\perp_2^{p,q} : [q] \rightarrow [n]$ for the operator $i \mapsto p + i$.

In the following definition, we use slightly different terminology from Verity’s original one [2008a, Definitions 127 and 128].

Definition 1.30 Let $X, Y \in \text{sSet}^+$, let $(x, y) \in X_n \times Y_n$ be a simplex of $X \times Y$, and let $0 \leq i \leq n$. We say that (x, y) is *i -cloven* if either $x \perp_1^{i, n-i}$ is marked in X or $y \perp_2^{i, n-i}$ is marked in Y . We say that (x, y) is *fully cloven* if it is i -cloven for all $0 \leq i \leq n$.

The *Gray tensor product* of X and Y , denoted by $X \otimes Y$, is defined to be the marked simplicial set with underlying simplicial set $X \times Y$, where a simplex $(x, y) \in X_n \times Y_n$ is marked if and only if it is fully cloven.

Theorem 1.31 [Verity 2008a, Lemma 131] *The Gray tensor product endows the category sSet^+ with a (nonsymmetric) monoidal structure such that the forgetful functor $(\text{sSet}^+, \otimes) \rightarrow (\text{sSet}, \times)$ is strict monoidal.*

Definition 1.32 A *precomplicial set* is a marked simplicial set X with the right lifting property with respect to the complicial marking extensions. These form a reflective subcategory of sSet^+ , which we will denote by PreComp . We will denote the localization functor by $X \mapsto X^{\text{pre}}$.

Proposition 1.33 *The unit of the reflection $X \rightarrow X^{\text{pre}}$ is a complicial marking extension for any $X \in \text{sSet}^+$.*

Proof Obtain a complicial marking extension $f : X \rightarrow Y$ into a precomplicial set Y by applying the small object argument to the unique map $X \rightarrow 1$ with respect to the elementary complicial marking extensions. Then any map $X \rightarrow Z$ into a precomplicial set Z factors through f . Moreover, since f is an epimorphism, such a factorization is necessarily unique. In other words, f has the universal property of the unit $X \rightarrow X^{\text{pre}}$. \square

Theorem 1.34 *For each of the four model structures in Theorems 1.24 and 1.26, the category PreComp carries an analogous model structure characterized by the following conditions:*

- *The cofibrations are the monomorphisms.*
- *A pseudogenerating set of trivial cofibrations can be obtained by taking that for the corresponding model structure on sSet^+ (described in Theorem 3.3 or Theorem 1.26), removing the complicial marking extensions, and then applying the precomplicial reflection.*

The localization $(-)^{\text{pre}}$ is a left Quillen equivalence between the complicial model structures (resp. the n -trivial complicial model structures).

Proof Fix one of the four model structures on sSet^+ . Observe that if we factor a map between precomplicial sets into a cofibration followed by a fibration (one of which is trivial) with respect to that model structure, then the middle object must also be precomplicial. Thus we obtain a restricted model structure on PreComp .

We obtain the characterization of its cofibrations by observing that the reflective inclusion $\text{PreComp} \hookrightarrow \text{sSet}^+$ preserves and reflects monomorphisms. It is straightforward to check that the precomplicial reflection preserves pseudogenerating sets, and moreover it inverts all (elementary) complicial marking extensions.

It follows that the adjunction $\text{sSet}^+ \rightleftarrows \text{PreComp}$ is a Quillen adjunction. Finally, since the unit is a natural weak equivalence this is in fact a Quillen equivalence. \square

Remark 1.35 We believe that the precomplicial reflection does not actually affect the remaining members of the pseudogenerating set. However we do not provide a proof as it is not essential.

Now we analyze the precomplicial reflection of the Gray tensor product on PreComp .

Definition 1.36 We write $\otimes^{\text{pre}}: \text{PreComp} \times \text{PreComp} \rightarrow \text{PreComp}$ for the precomplicial Gray tensor product functor $(X, Y) \mapsto (X \otimes Y)^{\text{pre}}$.

Theorem 1.37 *The bifunctor \otimes^{pre} is part of a biclosed monoidal structure on PreComp . Moreover each model structure on PreComp described in Theorem 1.34 is monoidal with respect to \otimes^{pre} .*

Proof The first assertion is [Verity 2008a, Theorem 148]. It is straightforward to check that the Leibniz Gray tensor product preserves monomorphisms. Since the complicial model structure on sSet^+ is monoidal with respect to the Gray tensor product [Verity 2008b, Theorem 109] (although it is not biclosed on sSet^+) and the unit of the precomplicial reflection is a levelwise complicial marking extension, it follows that the complicial model structure on PreComp is monoidal. The n -trivial and saturated versions follow from [Ozornova and Rovelli 2020, Section 2]. \square

2 Marked cubical sets and Gray tensor products

In this section, we introduce marked cubical sets and define their Gray tensor product.

2.1 Marked cubical sets

In order to define marked cubical sets, we need to introduce certain enlargement \square^+ of the box category. The objects of \square^+ consist of: $[1]^n$ for every $n \geq 0$ and $[1]_e^n$ for every $n \geq 1$. The morphisms of \square^+ are generated by the maps

$$\begin{aligned} \partial_{i,\varepsilon}^n &: [1]^{n-1} \rightarrow [1]^n && \text{for every } n \geq 1, i = 1, \dots, n \text{ and } \varepsilon = 0, 1, \\ \sigma_i^n &: [1]^n \rightarrow [1]^{n-1} && \text{for } n \geq 1 \text{ and } i = 1, \dots, n, \\ \gamma_{i,\varepsilon}^n &: [1]^n \rightarrow [1]^{n-1} && \text{for } n \geq 2, i = 1, \dots, n-1 \text{ and } \varepsilon = 0, 1, \\ \varphi^n &: [1]^n \rightarrow [1]_e^n && \text{for } n \geq 1, \\ \zeta_i^n &: [1]_e^n \rightarrow [1]^{n-1} && \text{for } n \geq 1 \text{ and } i = 1, \dots, n, \\ \xi_{i,\varepsilon}^n &: [1]_e^n \rightarrow [1]^{n-1} && \text{for } n \geq 1, i = 1, \dots, n \text{ and } \varepsilon = 0, 1, \end{aligned}$$

subject to the usual cubical identities and the following additional relations:

$$\begin{aligned} \zeta_i \varphi &= \sigma_i, & \xi_{i,\varepsilon} \varphi &= \gamma_{i,\varepsilon}, & \sigma_i \zeta_j &= \sigma_j \zeta_{i+1} & \text{for } j \leq i, \\ \gamma_{j,\varepsilon} \xi_{i,\delta} &= \begin{cases} \gamma_{i,\delta} \xi_{j+1,\varepsilon} & \text{for } j > i, \\ \gamma_{i,\delta} \xi_{i+1,\delta} & \text{for } j = i \text{ and } \delta = \varepsilon, \end{cases} & \sigma_j \xi_{i,\delta} &= \begin{cases} \gamma_{i-1,\delta} \zeta_j & \text{for } j < i, \\ \sigma_i \zeta_i & \text{for } j = i, \\ \gamma_{i,\delta} \zeta_{j+1} & \text{for } j > i. \end{cases} \end{aligned}$$

Proposition 2.1 *The category \square^+ is an EZ Reedy category with the following Reedy structure:*

- $\deg[1]^0 = 0$, $\deg[1]^n = 2n - 1$ for $n \geq 1$, and $\deg[1]_e^n = 2n$ for $n \geq 1$.
- \square^+ is generated by the maps σ_i^n , $\gamma_{i,\varepsilon}^n$, ζ_i^n and $\xi_{i,\varepsilon}^n$.
- \square^+ is generated by the maps $\partial_{i,\varepsilon}^n$ and φ^n .

The proof of this fact follows the one in [Ozornova and Rovelli 2020, Appendix C]. We begin by noting the following simple lemma:

Lemma 2.2 (1) *The are no nonidentity maps in \square^+ whose target is in $\square^+ \setminus \square$.*

(2) *The are no nonidentity maps in \square^+ whose source is in $\square^+ \setminus \square$.* □

Proof of Proposition 2.1 We first note that the sections of ζ_i are $\varphi \partial_{i,0}$ and $\varphi \partial_{i,1}$, the sections of $\xi_{i,1}$ are $\varphi \partial_{i,0}$ and $\varphi \partial_{i+1,0}$, and the sections of $\xi_{i,0}$ are $\varphi \partial_{i,1}$ and $\varphi \partial_{i+1,1}$. Thus all maps in \square^+ have sections.

Using the techniques of [Grandis and Mauri 2003, Theorem 5.1], we can then extend Theorem 1.6 to write normal forms for maps in \square^+ . These are established separately for the four cases:

- (1) The normal form of a map of the form $[1]^m \rightarrow [1]^n$ is given by its normal form in \square . If such a form were to be nonunique, we would need to have $[1]^m \rightarrow [1]_e^k \rightarrow [1]^n$ with $[1]_e^k \rightarrow [1]^n \in \square_+^+$, which is impossible by Lemma 2.2(2).
- (2) The normal form of a map $[1]^m \rightarrow [1]_e^n$ is obtained by observing that it is necessarily a composite $[1]^m \rightarrow [1]^n \xrightarrow{\varphi} [1]_e^n$ and taking the normal form of the first map in \square . Again, Lemma 2.2(2) implies uniqueness.
- (3) The normal form of a map of the form $[1]_e^m \rightarrow [1]^n$ is obtained by factoring it as $[1]_e^m \rightarrow [1]_e^{m-1} \rightarrow [1]^n$, where the first map is either ζ_i or $\xi_{i,\varepsilon}$, and taking the normal form of Theorem 1.6 of $[1]_e^{m-1} \rightarrow [1]^n$ in \square . Note that the choice of ζ_i or $\xi_{i,\varepsilon}$ as the first map may not be unique, but it can be made so by imposing the additional compatibility requirement with the factorization of Theorem 1.6 — this is because of the additional relations relating the ζ_i to the σ_i , and the $\xi_{i,\varepsilon}$ to the $\gamma_{i,\varepsilon}$. Put differently, we may precompose $[1]_e^m \rightarrow [1]^n$ with φ , use the normal form in \square , and replace the last element by ζ_i or $\xi_{i,\varepsilon}$ as appropriate.
- (4) $[1]_e^m \rightarrow [1]_e^n$. In this case, we obtain the normal form by combining the techniques from the previous two cases, namely factoring

$$[1]_e^m \rightarrow [1]_e^{m-1} \rightarrow [1]^n \xrightarrow{\varphi} [1]_e^n,$$

where again the first map is one of ζ_i or $\xi_{i,\varepsilon}$, and the composite $[1]_e^{m-1} \rightarrow [1]^k \rightarrow [1]^n$ is obtained in \square .

Having established the normal forms, we proceed in a manner analogous to the proof of Theorem 1.8. \square

Definition 2.3 A *structurally marked cubical set* is a presheaf $X : (\square^+)^{\text{op}} \rightarrow \text{Set}$. A *map of structurally marked cubical sets* is a natural transformation of such presheaves.

Given a structurally marked cubical set X , we will write X_n for $X([1]^n)$ and eX_n for $X([1]_e^n)$. Just as in the case of cubical sets, we adopt the convention of writing cubical operators on the right, eg for $x \in eX_1$, we write $x\varphi$ for the resulting element of X_1 .

Definition 2.4 A *marked cubical set* is a structurally marked cubical set $X : (\square^+)^{\text{op}} \rightarrow \text{Set}$ for which the map $X\varphi : eX_n \rightarrow X_n$ is a monomorphism for all $n \geq 1$. We write cSet^+ for the full subcategory of $\text{Set}^{(\square^+)^{\text{op}}}$ spanned by the marked cubical sets.

We think of a marked cubical set X as a cubical set in which certain n -cubes have been designated as *equivalences*, ie those in $eX_n \subseteq X_n$. The maps ζ_i and $\xi_{i,\varepsilon}$ ensure that every degenerate cube is marked.

We may apply the same intuition to structurally marked cubical sets. However, failure of the maps $X\varphi$ to be monomorphisms (in an arbitrary structurally marked cubical set X) means that being an equivalence is not a property of an n -cube of X , but a structure on it, as there can be multiple markings on a single cube.

Every (structurally) marked cubical set has an underlying cubical set, defining a functor $v : \text{cSet}^+ \rightarrow \text{cSet}$. Given a cubical set X , we can form a marked cubical set in two ways:

- The *minimal marking* functor takes a cubical set X to a marked cubical set X^b , where only degenerate n -cubes are marked.
- The *maximal marking functor* assigns to X the marked cubical set $X^\#$ in which all cubes are marked (ie all maps $X\varphi^n$ are identities).

This gives two functors $(-)^b, (-)^\#: \mathbf{cSet} \rightarrow \mathbf{cSet}^+$. A straightforward verification shows:

Proposition 2.5 We have a string of adjoint functors $(-)^b \dashv \nu \dashv (-)^\#$. □

Remark 2.6 (limits and colimits of marked cubical sets) The proposition above gives a recipe for computing limits and colimits of diagrams $F: \mathcal{I} \rightarrow \mathbf{cSet}^+$. In both cases, we first compute the underlying cubical set by taking the (co)limit of νF in \mathbf{cSet} , and then equipping it with the minimal marking making the colimit inclusions maps of marked cubical sets, or the maximal marking making the limit projections maps of marked cubical sets. It follows, for example, that a cube in a colimit is marked if and only if it is in the image of a marked cube under one of the colimit inclusions.

Furthermore, the canonical embedding $\mathbf{cSet}^+ \hookrightarrow \mathbf{Set}^{(\square^+)^{\text{op}}}$ of marked cubical sets into structurally marked cubical sets admits a left adjoint, denoted by $\text{Im}: \mathbf{Set}^{(\square^+)^{\text{op}}} \rightarrow \mathbf{cSet}^+$. Explicitly, $\text{Im} X$ is obtained by factoring all the φ_n via their image $eX_n \rightarrow (eX_n)\varphi_n \rightarrow X_n$ and taking the resulting object as a new set of marked n -cubes. We may summarize it with the following statement:

Proposition 2.7 Marked cubical sets form a reflective subcategory of the structurally marked cubical sets with the reflector given by $\text{Im}: \mathbf{Set}^{(\square^+)^{\text{op}}} \rightarrow \mathbf{cSet}^+$. □

Corollary 2.8 The category \mathbf{cSet}^+ of marked cubical sets is locally presentable. □

Definition 2.9 A map $f: X \rightarrow Y$ of marked cubical sets is

- *regular* if it creates markings, ie for an n -cube x of X we have: $x \in eX_n$ if and only if $f(x) \in eY_n$,
- *entire* if the induced map between the underlying cubical sets is invertible.

Definition 2.10 We say that $X \in \mathbf{cSet}^+$ is *n-trivial* if every cube of dimension $\geq n + 1$ is marked.

Given a marked cubical set X , we will write $\text{core}_n X$ for its maximal n -trivial subset. In other words, the k -cubes of $\text{core}_n X$ are precisely those k -cubes x such that $x\alpha$ is marked for all $\alpha: [1]^m \rightarrow [1]^k$ with $m > n$. This assignment extends to a functor $\text{core}_n: \mathbf{cSet}^+ \rightarrow \mathbf{cSet}^+$, which admits a left adjoint $\tau_n: \mathbf{cSet}^+ \rightarrow \mathbf{cSet}^+$. Explicitly, τ_n acts as the identity on the underlying cubical set and a k -cube is marked in $\tau_n X$ if either $k \leq n$ and x is marked in X or $k \geq n + 1$.

When a cubical set is considered as a marked cubical set, it will *almost* always be considered with its minimal marking. The only exception is the open boxes; see Section 3. We denote by $\square^n = (\square^n)^b$ the

n -cube regarded as a marked cubical set and likewise $\partial\Box^n = (\partial\Box^n)^b$. Just as in the case of cubical sets, we call the inclusion map $\partial\Box^n \rightarrow \Box^n$ the *boundary inclusion*. We denote by $\tilde{\Box}^n = \tau_{n-1}(\Box^n)$ the n -cube with the nondegenerate n -cube marked and no other nondegenerate cubes marked. We call the canonical map $\Box^n \rightarrow \tilde{\Box}^n$ the *n -marker*.

Proposition 2.11 *The monomorphisms of cSet^+ (and $\text{Set}^{(\Box^+)^{\text{op}}}$) are the cellular closure of the set*

$$\{\partial\Box^n \hookrightarrow \Box^n \mid n \geq 0\} \cup \{\Box^n \hookrightarrow \tilde{\Box}^n \mid n \geq 1\}.$$

The functors $(-)^{\text{co}}, (-)^{\text{co-op}}, (-)^{\text{op}}: \text{cSet} \rightarrow \text{cSet}$ generalize to the marked setting in the straightforward manner. For $(-)^{\text{co}}$ we send φ^n to itself, ζ_i^n to ζ_{n+1-i}^n , and $\xi_{i,\varepsilon}^n$ to $\xi_{n+1-i,\varepsilon}^n$. For $(-)^{\text{co-op}}$, we send φ^n and ζ_i^n to themselves and $\xi_{i,\varepsilon}^n$ to $\xi_{i,1-\varepsilon}^n$. These then induce functors by precomposition $(-)^{\text{co}}, (-)^{\text{co-op}}, (-)^{\text{op}}: \text{cSet}^+ \rightarrow \text{cSet}^+$.

2.2 Gray tensor products

The following definition makes use of Corollary 1.15.

Definition 2.12 *The (lax) Gray tensor product $X \otimes Y$ of two marked cubical sets $X, Y \in \text{cSet}^+$ is the geometric product $\nu X \otimes \nu Y$, wherein a nondegenerate cube $x \otimes y$ is marked if and only if either x is marked in X or y is marked in Y . This extends to a functor $\otimes: \text{cSet}^+ \times \text{cSet}^+ \rightarrow \text{cSet}^+$ in the obvious way.*

Definition 2.13 *The pseudo Gray tensor product $X \circledast Y$ is the geometric product $\nu X \otimes \nu Y$, wherein a nondegenerate cube $x \otimes y$ is unmarked if and only if*

- x is a 0-cube and y is unmarked in Y , or
- x is unmarked in X and y is a 0-cube.

This extends to a functor $\circledast: \text{cSet}^+ \times \text{cSet}^+ \rightarrow \text{cSet}^+$ in the obvious way.

Remark 2.14 *Since no 0-cubes are marked, one can easily check that $X \circledast Y$ may be obtained from $X \otimes Y$ by marking those nondegenerate $x \otimes y$ such that $x \in X_m, y \in Y_n$ with $m, n \geq 1$. Thus the identity at $\nu X \otimes \nu Y$ lifts to an entire map $\mu_{X,Y}: X \otimes Y \rightarrow X \circledast Y$. This map is clearly natural in X and Y , and moreover $\mu_{X,Y}$ is invertible if either X or Y is 0-trivial.*

Remark 2.15 *The Gray tensor products \otimes and \circledast share many properties, and often a statement or a proof applies equally well to both tensor products. In such situations, we write \odot to mean either. Of course the interpretation of \odot is to be kept consistent within each statement and its proof.*

- Theorem 2.16** (1) *The Gray tensor product \odot forms part of a biclosed monoidal structure on \mathbf{cSet}^+ such that the forgetful functor $v: (\mathbf{cSet}^+, \odot) \rightarrow (\mathbf{cSet}, \otimes)$ is strict monoidal.*
- (2) *The entire inclusions $\mu_{X,Y}: X \otimes Y \rightarrow X \otimes Y$ together with $\mu_0 = \text{id}_{\square_0}$ equip the identity functor with a monoidal structure $(\text{id}_{\mathbf{cSet}^+}, \mu): (\mathbf{cSet}^+, \otimes) \rightarrow (\mathbf{cSet}^+, \otimes)$.*
- (3) *The minimal marking functor $(-)^b: (\mathbf{cSet}, \otimes) \rightarrow (\mathbf{cSet}^+, \otimes)$ is strict monoidal.*
- (4) *The maximal marking functor $(-)^\# : (\mathbf{cSet}, \otimes) \rightarrow (\mathbf{cSet}^+, \odot)$ is strict monoidal.*

Proof We first check the associativity of the tensor product. Suppose we are given nondegenerate cubes $x \in X_m, y \in Y_n, z \in Z_k$ in $X, Y, Z \in \mathbf{cSet}^+$. Then the $(m+n+k)$ -cube $(x \otimes y) \otimes z$ in $(X \odot Y) \odot Z$ is unmarked if and only if

- (\otimes) none of x, y, z is marked, or
- (\otimes) (at least) two of x, y, z are 0-cubes and the last is unmarked.

One can give a similar characterization of when $x \otimes (y \otimes z)$ is unmarked, and it follows that the associativity isomorphism $(vX \otimes vY) \otimes vZ \cong vX \otimes (vY \otimes vZ)$ in \mathbf{cSet} lifts to an isomorphism $(X \odot Y) \odot Z \cong X \odot (Y \odot Z)$ in \mathbf{cSet}^+ . The unit isomorphisms can be lifted similarly, and moreover these lifted isomorphisms are suitably natural and coherent. Thus we indeed obtain a monoidal structure on \mathbf{cSet}^+ such that v is strict monoidal. The clauses (2)–(4) are then obvious from the definitions of the tensor products.

It remains to show that this monoidal structure is biclosed. Equivalently, we must show that \odot preserves colimits in each variable separately. So let $F: \mathcal{I} \rightarrow \mathbf{cSet}^+$ and $X \in \mathbf{cSet}^+$. Since the geometric product is cocontinuous in each variable and v is cocontinuous and strict monoidal, the canonical comparison map

$$\text{colim}(X \odot F) \rightarrow X \odot \text{colim } F$$

is v -invertible. Moreover one can check using Remark 2.6 that a nondegenerate cube in either side is marked if and only if it is the image of some marked cube under the canonical map from $X \odot Fi$ for some $i \in \mathcal{I}$. It follows that this comparison map is invertible. Dually, $(-) \odot X$ preserves colimits. Since \mathbf{cSet}^+ is locally finitely presentable, the existence of the desired biclosed structure now follows. \square

Lemma 2.17 *Let $f: A \rightarrow X$ and $g: B \rightarrow Y$ be monomorphisms in \mathbf{cSet}^+ . Then $f \hat{\odot} g$ is again a monomorphism. Moreover:*

- (1) *If both f and g are regular, then so is $f \hat{\odot} g$.*
- (2) *If either f or g is entire, then so is $f \hat{\odot} g$.*
- (3) *If both f and g are entire, then $f \hat{\odot} g$ is invertible.*

(4) If either f or g is entire, then the square

$$\begin{array}{ccc}
 (X \otimes B) \cup (A \otimes Y) & \longrightarrow & (X \otimes B) \cup (A \otimes Y) \\
 f \hat{\otimes} g \downarrow & & \downarrow f \hat{\otimes} g \\
 X \otimes Y & \xrightarrow{\mu_{X,Y}} & X \otimes Y
 \end{array}$$

is a pushout in cSet^+ , where the upper horizontal map is induced by μ .

Proof Since a map in cSet^+ is a monomorphism if and only if its underlying map in cSet is a monomorphism, the first (unnumbered) assertion follows from Corollary 1.18, Theorem 2.16(1) and the cocontinuity of ν . We will assume for the sake of simplicity that $f \hat{\otimes} g$ is an inclusion.

(1), case (\otimes) Let $x \otimes y$ be a nondegenerate cube in $\text{dom}(f \hat{\otimes} g)$. By duality, we may assume that x is in A . If $x \otimes y$ is marked in $X \otimes Y$, then either x is marked in X or y is marked in Y . It follows (by the regularity of f in the former case) that $x \otimes y$ is marked in $A \otimes Y$. This shows that $f \hat{\otimes} g$ is regular.

(1), case (\oplus) Let $x \otimes y$ be a marked nondegenerate cube in $X \otimes Y$. Suppose that $x \otimes y$ is in the image of $f \hat{\otimes} g$. The case $(1 \otimes)$ combined with the commutativity of the square in (4) imply that if $x \otimes y$ is marked in $X \otimes Y$ then it is also marked in $\text{dom}(f \hat{\otimes} g)$. Thus by Remark 2.14, it suffices to consider the case where $x \in X_m$ and $y \in Y_n$ for some $m, n \geq 1$. But in this case $x \otimes y$ is marked in $\text{dom}(f \hat{\otimes} g)$ by the definition of \oplus .

(2) Since ν preserves colimits, we have $\nu(f \hat{\otimes} g) \cong \nu f \hat{\otimes} \nu g$. Thus this assertion follows from the fact that the pushout of an isomorphism along any map is itself an isomorphism.

(3), case (\otimes) We know from (2) that $f \hat{\otimes} g$ is entire, so it suffices to show that this map is also regular. Let $x \otimes y$ be a marked nondegenerate cube in $X \otimes Y$. Then either x is marked in X or y is marked in Y . The cube $x \otimes y$ is then marked in $X \otimes B$ in the first subcase and it is marked in $A \otimes Y$ in the second subcase. Thus $f \hat{\otimes} g$ is indeed regular.

(4) By (2), each map in this square is entire. Thus its image under ν is trivially a pushout in cSet . Moreover, for each of the horizontal maps, Remark 2.14 implies that the codomain is obtained from the domain by marking those cubes $x \otimes y$ such that $x \in X_m$ and $y \in Y_n$ with $m, n \geq 1$. Now the assertion follows by Remark 2.6.

(3), case (\oplus) This case follows from (3), case (\otimes) , and (4). □

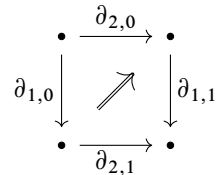
Proposition 2.18 For any $m, n \geq 0$, the Leibniz Gray tensor product $(\partial \square^m \hookrightarrow \square^m) \hat{\otimes} (\partial \square^n \hookrightarrow \square^n)$ of boundary inclusions in cSet^+ is isomorphic to $\partial \square^{m+n} \hookrightarrow \square^{m+n}$.

Proof This is a straightforward consequence of Proposition 1.17, Theorem 2.16(3) and the cocontinuity of $(-)^b$. □

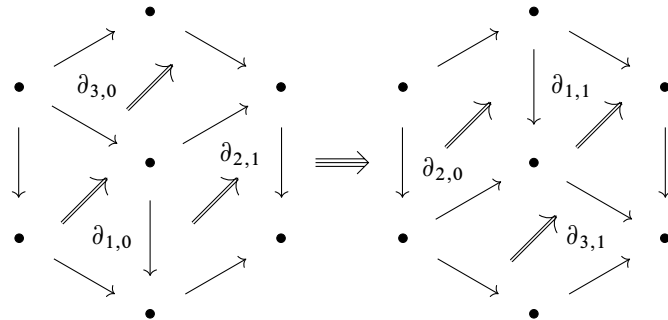
3 Model structure for comical sets

In this section, we construct two families of model structures on the category cSet^+ of marked cubical sets. The former of those has as its fibrant objects (*saturated*) *comical sets*, which we will define, and it is our tentative model for the theory of weak ω -categories. The fibrant objects of the latter are the n -trivial comical sets, and it is our tentative model for the theory of (∞, n) -categories.

A comical set is to be thought of as a kind of weak ω -category, and an n -cube therein represents an n -dimensional morphism. The $(n-1)$ -source of such an n -cube is the “composite” of the faces $\partial_{k,\varepsilon}$ with $k + \varepsilon$ odd, and similarly the $(n-1)$ -target is given by the even faces. (This idea of parity-based decomposition into source and target goes back to [Street 1987], where Street considers the free ω -categories on simplices. In the case of cubes, see eg [Aitchison 1986; Street 1991; Steiner 1993; Al-Agl et al. 2002]). For instance, a 2-cube can be seen as a morphism of the form



and a 3-cube represents a morphism between the composites

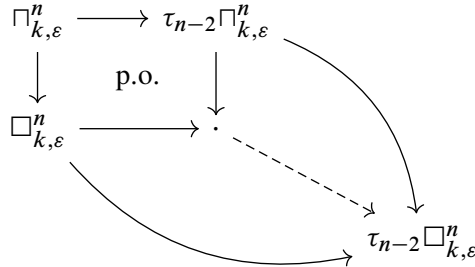


Marked n -cubes are to be thought of as being (weakly) invertible, although not every invertible cube is marked unless the comical set is saturated.

Before defining comical sets, we will need a few auxiliary definitions.

For $n \geq 1$, $1 \leq k \leq n$ and $\varepsilon \in \{0, 1\}$, we denote by $\square_{k,\varepsilon}^n$ the n -cube with the following marking: a nondegenerate cube $\partial_{k_1,\varepsilon_1} \cdots \partial_{k_t,\varepsilon_t}$, written in the form specified by Theorem 1.6, is marked whenever this string does not contain $\partial_{k-1,\varepsilon}$, $\partial_{k,\varepsilon}$, $\partial_{k,1-\varepsilon}$ or $\partial_{k+1,\varepsilon}$. (This is exactly the marking described in [Steiner 2006, Example 2.9].) We call this the (k, ε) -comical n -cube. We denote by $\square_{k,\varepsilon}^n \subset \square_{k,\varepsilon}^n$ the (k, ε) -open box of dimension n (ie the cubical subset missing the nondegenerate n -cube and the $(k, \varepsilon)^{\text{th}}$ $(n-1)$ -face) endowed with the marking making it a regular subset of $\square_{k,\varepsilon}^n$. We call $\square_{k,\varepsilon}^n$ the *comical (k, ε) -open box of dimension n* . We call the inclusion $\square_{k,\varepsilon}^n \rightarrow \square_{k,\varepsilon}^n$ the *(k, ε) -comical open box inclusion of dimension n* .

The elementary (k, ε) -comical marking extension of dimension n , denoted by $\square_{k,\varepsilon}^n \rightarrow \square_{k,\varepsilon}^{n'}$, is the Leibniz product of the unit $\text{id} \rightarrow \tau_{n-2}$ and the comical box inclusion $\square_{k,\varepsilon}^n \rightarrow \square_{k,\varepsilon}^n$, ie the dashed map in



For each $x, y \in \{\nearrow, \swarrow\}$, we define the basic Rezk map $L_{xy} \hookrightarrow L'_{xy}$ as the entire inclusion depicted below:

$$\begin{aligned}
 L_{\nearrow\nearrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \nearrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}, & L'_{\nearrow\nearrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \nearrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}, \\
 L_{\nearrow\swarrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \nearrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}, & L'_{\nearrow\swarrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \nearrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}, \\
 L_{\swarrow\nearrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}, & L'_{\swarrow\nearrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}, \\
 L_{\swarrow\swarrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}, & L'_{\swarrow\swarrow} &= \left\{ \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right\}.
 \end{aligned}$$

Here thick arrows indicate marked cubes. More precisely, $L_{\nearrow\nearrow}$ is the pushout of the span

$$X \xleftarrow{\partial_{1,1}} \square^1 \xrightarrow{\partial_{1,0}} Y,$$

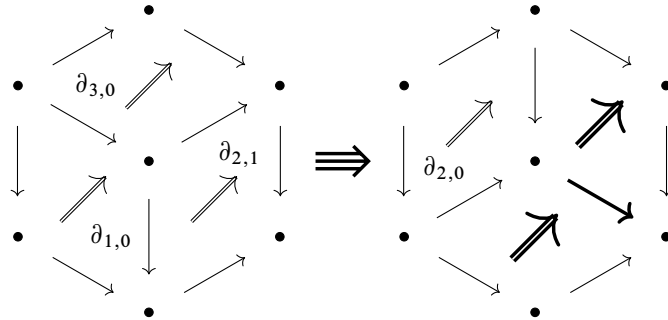
where X is obtained from $\tilde{\square}^2$ by marking $\partial_{1,0}$ and $\partial_{2,1}$, and Y is obtained from $\tilde{\square}^2$ by marking $\partial_{1,1}$ and $\partial_{2,0}$. The codomain $L'_{\nearrow\nearrow}$ is the 0-trivialization $\tau_0(L_{\nearrow\nearrow})$. The marked cubical sets L_{xy}, L'_{xy} are defined similarly for other choices of $x, y \in \{\nearrow, \swarrow\}$. By a Rezk map we mean any map of the form

$$(\partial \square^m \hookrightarrow \square^m) \hat{\otimes} (L_{xy} \hookrightarrow L'_{xy}) \hat{\otimes} (\partial \square^n \hookrightarrow \square^n).$$

- Definition 3.1**
- (1) A *comical set* is a marked cubical set with the right lifting property with respect to the comical open box inclusions and the elementary comical marking extensions.
 - (2) A *saturated comical set* is a marked cubical set with the right lifting property with respect to the comical open box inclusions, the elementary comical marking extensions and the Rezk maps.

Remark 3.2 We briefly explain how the definition of (saturated) comical set should be interpreted. In the comical n -cube $\square_{k,\varepsilon}^n$, any subcube not contained in $\partial_{k-1,\varepsilon}, \partial_{k,\varepsilon}, \partial_{k,1-\varepsilon}$ or $\partial_{k+1,\varepsilon}$ is marked. In

particular, the unique nondegenerate n -cube is marked, so it can be thought of as an equivalence between the composite of its odd faces and the composite of even faces. In other words, the comical n -cube $\square_{k,\varepsilon}^n$ exhibits $\partial_{k,\varepsilon}$ as a composite of $\partial_{k-1,\varepsilon}$, $\partial_{k,1-\varepsilon}$ and $\partial_{k+1,\varepsilon}$. For example, $\square_{2,0}^3$ looks like



One can thus interpret the right lifting property with respect to the comical box inclusions and the comical marking extensions respectively as the existence of composites and the closure of marked cubes under composition. In Section 4, we show how these conditions additionally encode such expected properties of composition as the unit and associative laws, at least for 1-cubes.

There are two standard model structures on marked cubical sets:

Theorem 3.3 (model structure for comical sets) *The category cSet^+ carries two model structures:*

(1) *The **comical model structure**, characterized by the following properties:*

- *The cofibrations are the monomorphisms.*
- *The set of*
 - *comical open box inclusions, and*
 - *elementary comical marking extensions**forms a pseudogenerating set of trivial cofibrations.*

(2) *The **saturated comical model structure**, characterized by the following properties:*

- *The cofibrations are the monomorphisms.*
- *The set of*
 - *comical open box inclusions,*
 - *elementary comical marking extensions, and*
 - *Rezk maps**forms a pseudogenerating set of trivial cofibrations.*

Both of these model structures are combinatorial, left proper, monoidal with respect to either of the Gray tensor products, and have all objects cofibrant.

The proof of this theorem is an application of the Cisinski–Olschok theory and verification of the closure of anodyne maps under pushout-product. The latter part is contained in Lemma 3.5 below.

Definition 3.4 We say that a map of marked cubical sets $X \rightarrow Y$ is

- (1) a *comical marking extension* if it is in the cellular closure of the elementary comical marking extensions, and
- (2) *comical* if it is in the cellular closure of the comical open box inclusions and the elementary comical marking extensions.

Lemma 3.5 For any $1 \leq k \leq m$, $\varepsilon \in \{0, 1\}$ and $n \geq 0$ (or $n \geq 1$ for g), the Leibniz Gray tensor products

$$\begin{aligned} f &= (\sqcap_{k,\varepsilon}^m \hookrightarrow \square_{k,\varepsilon}^m) \hat{\circ} (\partial \square^n \hookrightarrow \square^n), \\ g &= (\sqcap_{k,\varepsilon}^m \hookrightarrow \square_{k,\varepsilon}^m) \hat{\circ} (\square^n \hookrightarrow \tilde{\square}^n), \\ h &= (\square_{k,\varepsilon}^{m'} \hookrightarrow \square_{k,\varepsilon}^{m''}) \hat{\circ} (\partial \square^n \hookrightarrow \square^n) \end{aligned}$$

are all comical.

Proof Since the case $n = 0$ is trivial, we will assume otherwise.

Consider a face of \square^{m+n} whose normal form $\partial_{k_1, \varepsilon_1} \cdots \partial_{k_c, \varepsilon_c}$ does not involve $\partial_{k-1, \varepsilon}$, $\partial_{k, 0}$, $\partial_{k, 1}$ or $\partial_{k+1, \varepsilon}$. Then clearly any terminal segment of this normal form does not involve any of these four ∂ 's. This observation implies that the second isomorphism of Proposition 1.17 may be lifted to the following commutative square:

$$\begin{array}{ccc} \sqcap_{k,\varepsilon}^{m+n} & \longrightarrow & (\sqcap_{k,\varepsilon}^m \odot \partial \square^n) \cup (\sqcap_{k,\varepsilon}^m \odot \square^n) \\ \downarrow & & \downarrow f \\ \square_{k,\varepsilon}^{m+n} & \longrightarrow & \square_{k,\varepsilon}^m \odot \square^n \end{array}$$

Observe that this is a pushout square on the underlying cubical set level.

In the case $\odot = \otimes$, it is in fact a pushout square in cSet^+ . To see this, it suffices to check that the marking on $\square_{k,\varepsilon}^m \otimes \square^n$ agrees with that described in Remark 2.6. This is indeed the case since f is regular by Lemma 2.17(1) and the only marked nondegenerate cube in $\text{cod}(f) \setminus \text{dom}(f)$ is the $(m+n)$ -cube, which is the image of a marked cube under the lower horizontal map. Thus f is indeed comical.

Now we consider the case $\odot = \otimes$. If $m = 1$ then we can simply repeat the above argument since we have natural isomorphisms

$$\sqcap_{1,\varepsilon}^1 \otimes (-) \cong \sqcap_{1,\varepsilon}^1 \otimes (-) \quad \text{and} \quad \square_{1,\varepsilon}^1 \otimes (-) \cong \square_{1,\varepsilon}^1 \otimes (-)$$

by Remark 2.14. So assume $m \geq 2$. Then there is an extra marked nondegenerate cube in $\text{cod}(f) \setminus \text{dom}(f)$, namely $\partial_{k,\varepsilon}$. But then it is straightforward to check that $\partial_{k-1, \varepsilon}$, $\partial_{k, 1-\varepsilon}$ and $\partial_{k+1, \varepsilon}$ are also marked (whenever they exist). So f can be written as a pushout of the open box inclusion $\sqcap_{k,\varepsilon}^{m+n} \hookrightarrow \square_{k,\varepsilon}^{m+n}$ followed by a pushout of the comical marking extension $\square_{k,\varepsilon}^{m+n'} \hookrightarrow \square_{k,\varepsilon}^{m+n''}$.

The map g is entire by Lemma 2.17(2). Similarly to the above argument, one can deduce the existence of the following commutative square of entire monomorphisms:

$$\begin{array}{ccc} \square_{k,\varepsilon}^{m+n'} & \longrightarrow & (\square_{k,\varepsilon}^m \odot \square^n) \cup (\square_{k,\varepsilon}^m \odot \tilde{\square}^n) \\ \downarrow & & \downarrow g \\ \square_{k,\varepsilon}^{m+n''} & \longrightarrow & \square_{k,\varepsilon}^m \odot \tilde{\square}^n \end{array}$$

One can check that, in the case where $\odot = \otimes$ and $m \geq 2$, the map g is in fact invertible. Otherwise, the only cube in $\text{cod}(g)$ that is not marked in $\text{dom}(g)$ is $\partial_{k,\varepsilon}$ and it is the image of a marked cube under the lower horizontal map. This shows that the above square is a pushout. Hence g is a comical marking extension.

Similarly, one can check that the following square is a pushout:

$$\begin{array}{ccc} \square_{k,\varepsilon}^{m+n'} & \longrightarrow & (\square_{k,\varepsilon}^{m''} \odot \partial \square^n) \cup (\square_{k,\varepsilon}^{m'} \odot \square^n) \\ \downarrow & & \downarrow h \\ \square_{k,\varepsilon}^{m+n''} & \longrightarrow & \square_{k,\varepsilon}^{m''} \odot \square^n \end{array}$$

Therefore h is a comical marking extension. □

Proof of Theorem 3.3 We apply the Cisinski–Olschok theory, ie Theorem 1.4 with $\mathcal{K} = \text{cSet}^+$ and I the set of boundary inclusions and markers. The set S consists of the comical open box inclusions and the comical marking extensions in (1), and it additionally contains all Rezk maps in (2). For our cylinder functor C , we can use either $\tilde{\square}^1 \otimes (-)$ or $\tilde{\square}^1 \otimes (-)$ as they are equal by Remark 2.14. This produces a model structure on cSet^+ in which the cofibrations are the monomorphisms and $\Lambda(\text{cSet}^+, I, C, S)$ is a pseudogenerating set of trivial cofibrations.

It remains to prove that the set S is in fact pseudogenerating, and moreover the model structure is monoidal with respect to either of the Gray tensor products. By duality and Proposition 1.5 it suffices to show that

- $f \hat{\odot} g$ is in the cellular closure of I whenever $f, g \in I$, and
- $f \hat{\odot} g$ is in the cellular closure of S whenever $f \in S$ and $g \in I$.

The first clause essentially follows from the unmarked version (Corollary 1.18). We now treat the second clause.

There are three kinds of maps in S , namely

- (A) comical box inclusions,
- (B) elementary comical marking extensions, and
- (C) Rezk maps,

and two kinds of maps in I , namely

- (a) boundary inclusions, and
- (b) markers.

The case $(Ca\otimes)$ is a straightforward consequence of the associativity of $\hat{\otimes}$ and Proposition 2.18. The case $(Ca\otimes)$ then follows by Lemma 2.17(4). In the cases (Bb) and (Cb) , the map $f \hat{\otimes} g$ is invertible by Lemma 2.17(3). The remaining cases are treated in Lemma 3.5. \square

There are also n -trivial versions of these model structures.

Theorem 3.6 (model structure for n -trivial comical sets) *The category cSet^+ carries two families of model structures:*

- (1) *The **n -trivial comical model structure** characterized by the following properties:*

- *The cofibrations are the monomorphisms.*
- *The set of*
 - *comical open box inclusions,*
 - *elementary comical marking extensions of dimension $\leq n + 1$, and*
 - *markers of dimension $> n$**forms a pseudogenerating set of trivial cofibrations.*

- (2) *The **saturated n -trivial comical model structure** characterized by the following properties:*

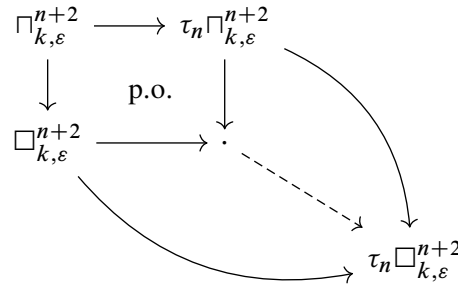
- *The cofibrations are the monomorphisms.*
- *The set of*
 - *comical open box inclusions,*
 - *elementary comical marking extensions of dimension $\leq n + 1$,*
 - *markers of dimension $> n$, and*
 - *Rezk maps**form a pseudogenerating set of trivial cofibrations.*

Proof Analogous to the proof of Theorem 3.3. Note that the Leibniz Gray tensor product of the m -marker with any monomorphism is in the cellular closure of the m' -markers with $m' \geq m$. \square

Proposition 3.7 *The functor $\tau_n: \text{cSet}^+ \rightarrow \text{cSet}^+$ is a left Quillen functor from the n -trivial comical model structure (resp. saturated n -trivial comical model structure) to the comical model structure (resp. saturated comical model structure).*

Lemma 3.8 *For any $n \geq 0$, $1 \leq k \leq n + 2$ and $\varepsilon \in \{0, 1\}$, the map $\tau_n \square_{k,\varepsilon}^{n+2} \hookrightarrow \tau_n \square_{k,\varepsilon}^{n+2} = \square_{k,\varepsilon}^{n+2''}$ is comical.*

Proof Recall the defining pushout square of the comical marking extension:



This diagram exhibits the desired result. □

Proof of Proposition 3.7 That τ_n preserves cofibrations is obvious. Thus it suffices to check (by [Joyal and Tierney 2007, Lemma 7.14]) that τ_n sends each member $f: X \rightarrow Y$ of the pseudogenerating set to a trivial cofibration.

Unless f is the open box inclusion $\square_{k,\varepsilon}^m \hookrightarrow \square_{k,\varepsilon}^m$ with $m \geq n + 2$, any marked cube in Y of dimension $> n$ admits a (not necessarily marked) preimage in X . In these cases, the naturality square for the unit

$$\begin{array}{ccc}
 X & \longrightarrow & \tau_n X \\
 f \downarrow & & \downarrow \tau_n f \\
 Y & \longrightarrow & \tau_n Y
 \end{array}$$

is a pushout in cSet^+ by Remark 2.6, so $\tau_n f$ is a trivial cofibration.

So assume that f is the open box inclusion $\square_{k,\varepsilon}^m \hookrightarrow \square_{k,\varepsilon}^m$ with $m \geq n + 2$. Then we have the following commutative square:

$$\begin{array}{ccc}
 \tau_{m-2} \square_{k,\varepsilon}^m & \longrightarrow & \tau_n \square_{k,\varepsilon}^m \\
 \tau_{m-2} f \downarrow & & \downarrow \tau_n f = \tau_n(\tau_{m-2} f) \\
 \tau_{m-2} \square_{k,\varepsilon}^m & \longrightarrow & \tau_n \square_{k,\varepsilon}^m
 \end{array}$$

Observe that the left vertical map is comical by Lemma 3.8 (with suitable substitution), and moreover it satisfies the condition on f described in the previous paragraph. Thus this square is a pushout, exhibiting $\tau_n f$ as a comical map. This completes the proof. □

As we mentioned earlier, our definition of comical box inclusion uses the marking described in [Steiner 2006], where Steiner characterizes the nerves of strict ω -categories. Phrased in the language of comical sets, his characterization implies the following result:

Theorem 3.9 (cf [Steiner 2006, Theorem 3.16]) *The cubical nerve of a globular ω -category, or equivalently the underlying cubical set of a cubical ω -category with connections, is a comical set.* □

This is analogous to the statement that the simplicial nerve of a strict ω -category is a complicial set [Verity 2008a].

We conclude this section with the following observation, which will be useful in Section 7.

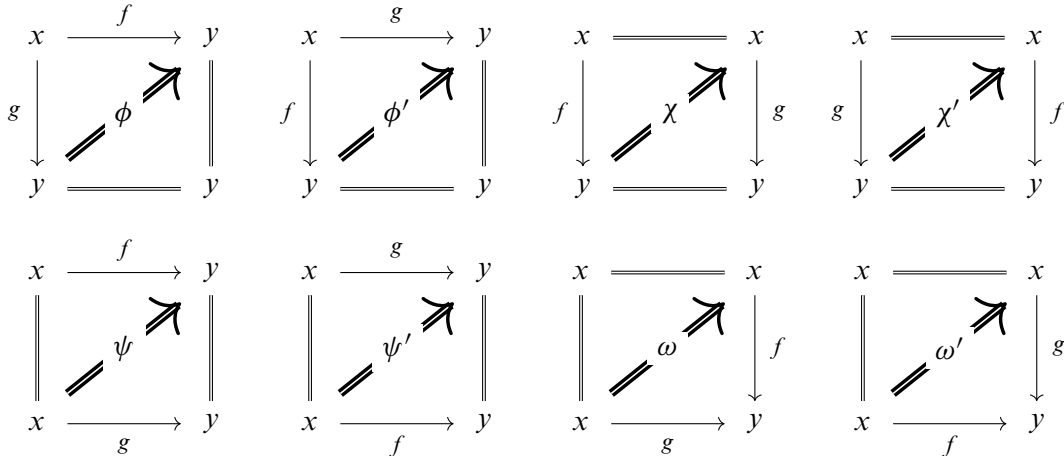
Proposition 3.10 *For any $n \geq 2$, $1 \leq k \leq n$ and $\varepsilon \in \{0, 1\}$, the comical box inclusion $\square_{k,\varepsilon}^n \hookrightarrow \square_{k,\varepsilon}^n$ may be written as*

$$\begin{aligned} (\square_{1,\varepsilon}^2 \hookrightarrow \square_{1,\varepsilon}^2) \widehat{\otimes} (\partial \square^{n-2} \hookrightarrow \square^{n-2}) & \quad \text{if } k = 1, \\ (\partial \square^{k-2} \hookrightarrow \square^{k-2}) \widehat{\otimes} (\square_{2,\varepsilon}^3 \hookrightarrow \square_{2,\varepsilon}^3) \widehat{\otimes} (\partial \square^{n-k-1} \hookrightarrow \square^{n-k-1}) & \quad \text{if } 1 < k < n, \\ (\partial \square^{n-2} \hookrightarrow \square^{n-2}) \widehat{\otimes} (\square_{2,\varepsilon}^2 \hookrightarrow \square_{2,\varepsilon}^2) & \quad \text{if } k = n. \end{aligned}$$

Proof It is easy to check that the underlying cubical maps match, and also the markings on the codomains match. Now observe that these Leibniz Gray tensor products are regular by Lemma 2.17(1). \square

4 Homotopy 1-categories of comical sets

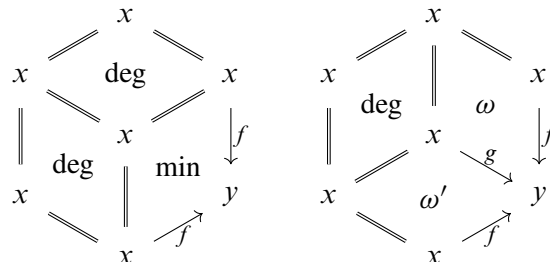
Suppose we are given two 1-cubes $f, g : x \rightarrow y$ in a comical set X . Then a marked 2-cube satisfying any one of the following boundary conditions may be reasonably regarded as a homotopy $f \sim g$:



Here equalities indicate degenerate (and hence marked) 1-cubes.

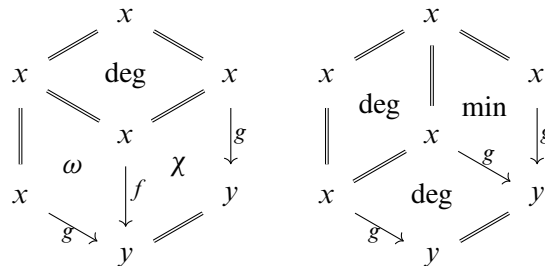
Proposition 4.1 *If any one of the above boundary conditions admits a marked solution in the comical set X then so do the others.*

Proof Consider the following picture:



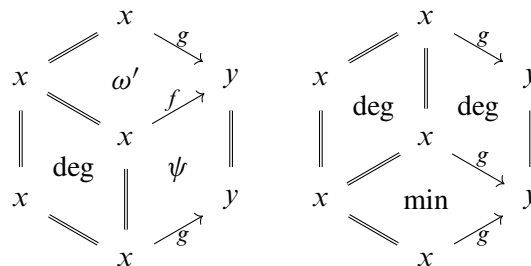
(Here the faces labeled “deg” are fully degenerate on the 0-cube x , and the face labeled “min” is the min-connection on f .) If we have a marked 2-cube ω satisfying the boundary condition specified above, then this picture may be interpreted as a map $\tau_1 \square_{3,1}^3 \rightarrow X$, which may be extended to $\square_{3,1}^3$ by Lemma 3.8, yielding a marked 2-cube ω' . Conversely, if we are given ω' then this picture specifies a map $\tau_1 \square_{1,1}^3 \rightarrow X$ and extending it to $\square_{1,1}^3$ yields a marked 2-cube ω .

Similarly, the following picture shows that χ exists if and only if ω does:

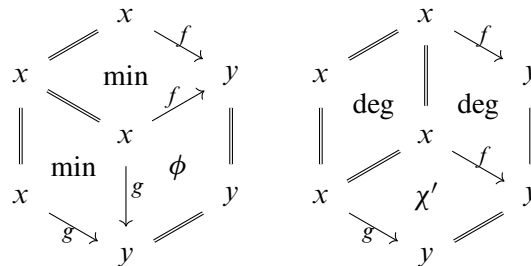


Dually, χ' exists if and only if ω' does.

The following picture shows that ψ exists if and only if ω' does (dually, ψ' exists if and only if ω does):



Finally the following picture shows that ϕ exists if and only if χ' does (dually, ϕ' exists if and only if χ does):

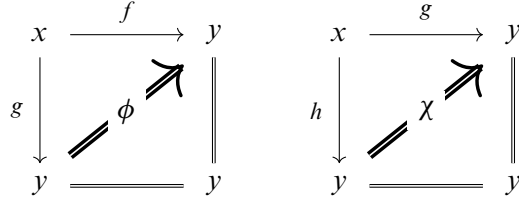


This completes the proof. □

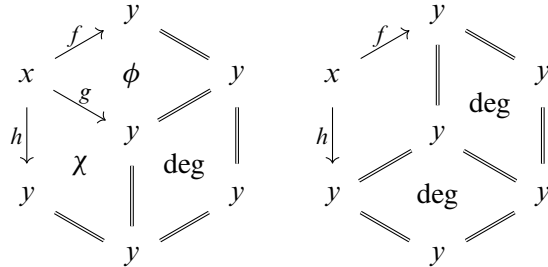
Definition 4.2 We say two 1-cubes f, g in a comical set X are *homotopic*, and write $f \sim g$ if any one of the above marked 2-cubes exists in X .

Proposition 4.3 For any pair of 0-cubes x, y in a comical set X , the homotopy relation is an equivalence relation on the set of all 1-cubes $x \rightarrow y$.

Proof The reflexivity of \sim is obvious, and its symmetry follows from Proposition 4.1. For transitivity, suppose we are given two homotopies:

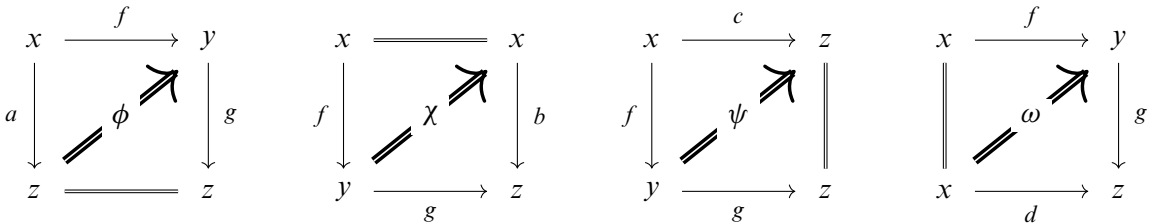


Then the following picture specifies a map $\tau_1 \square_{2,0}^3 \rightarrow X$:



This map extends to $\square_{2,0}^3$, which in particular yields a homotopy $f \sim h$. □

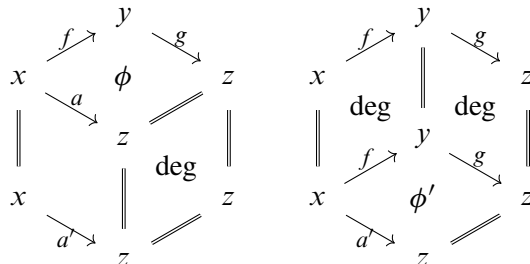
Now consider a “composable” pair of 1-cubes $f: x \rightarrow y$ and $g: y \rightarrow z$ in a conical set X . We may “compose” f and g by filling any one of the open boxes $\square_{1,0}^2, \square_{1,1}^2, \square_{2,0}^2, \square_{2,1}^2$ as follows:



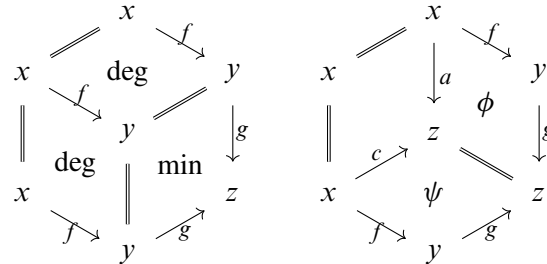
We will temporarily call such a a $(1, 0)$ -composite of f and g , and similarly call b, c and d $(1, 1)$ -, $(2, 0)$ - and $(2, 1)$ -composites of f and g respectively.

Proposition 4.4 Any two composites of f and g are homotopic to each other.

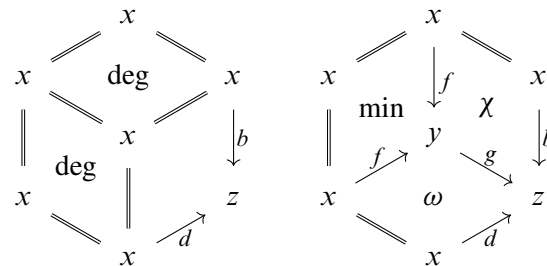
Proof First, consider two $(1, 0)$ -composites a and a' , witnessed by 2-cubes ϕ and ϕ' , respectively. Then the following picture specifies a map $\tau_1 \square_{1,0}^3 \rightarrow X$:



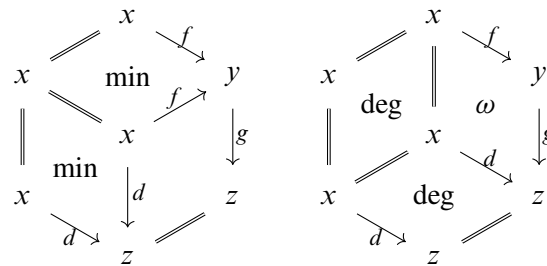
This map extends to $\square_{2,0}^3$, which yields a homotopy $a \sim a'$. Similarly, for any given a, b, c, d as above, a homotopy $a \sim c$ can be obtained by filling the following open box:



and a homotopy $b \sim d$ can be obtained using



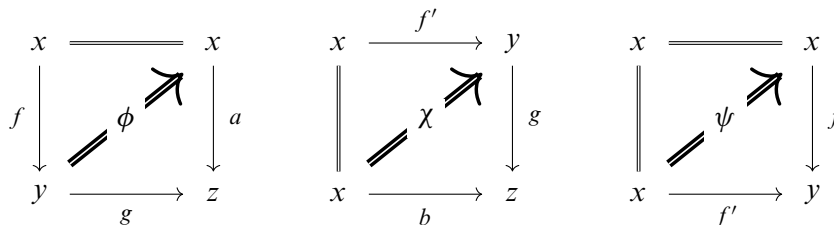
Finally, we can turn the $(2, 1)$ -composite d into a $(1, 0)$ -composite using the following open box:



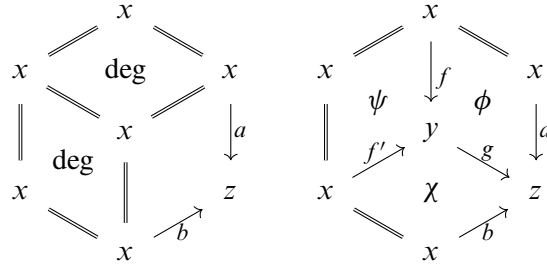
This completes the proof. □

Proposition 4.5 Let $f, f': x \rightarrow y$ and $g, g': y \rightarrow z$ be 1-cubes in a comical set X such that $f \sim f'$ and $g \sim g'$. Then any composite of f and g is homotopic to any composite of f' and g' .

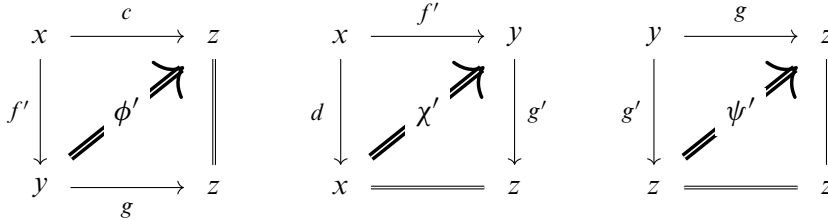
Proof Choose witnesses of the following forms for compositions and a homotopy $f \sim f'$:



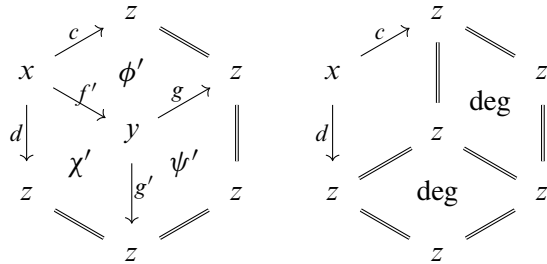
Then extending the following map $\tau_1 \square_{2,1}^3 \rightarrow X$ to $\square_{2,1}^3$ yields a homotopy $a \sim b$:



Similarly, we may combine marked 2-cubes of the forms



into a map $\tau_1 \square_{2,0}^3 \rightarrow X$ as follows:

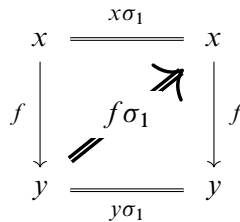


Extending this map to $\square_{2,0}^3$ yields a homotopy $c \sim d$. The desired result now follows by Propositions 4.3 and 4.4. □

Definition 4.6 We define the *homotopy 1-category* $\text{ho}_1 X$ of a comical set X to be the category of 0-cubes and homotopy classes of 1-cubes in X .

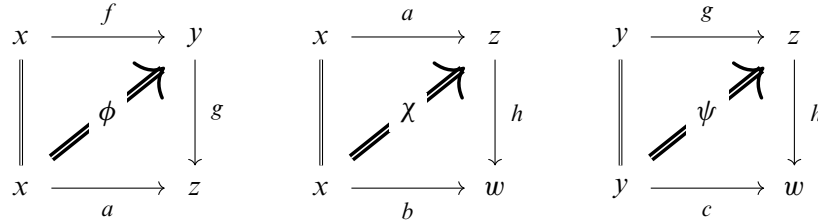
Proposition 4.7 For a comical set X , $\text{ho}_1 X$ is indeed a 1-category.

Proof Proposition 4.5 implies that we have a well-defined composition operation on $\text{ho}_1 X$. For any 0-cube x in X , we claim that the homotopy class containing $x\sigma_1$ is the identity at x . Indeed for any $f: x \rightarrow y$, the degenerate 2-cube

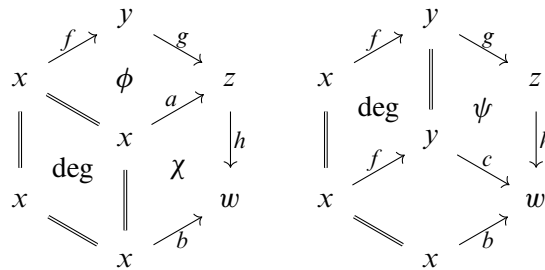


exhibits f as a $(1, 0)$ -composite of $x\sigma_1$ and f , and also as a $(1, 1)$ -composite of f and $y\sigma_1$.

For associativity, suppose we are given 1-cubes $f: x \rightarrow y$, $g: y \rightarrow z$ and $h: z \rightarrow w$ in X . Compose these 1-cubes as follows:



Then we may combine them into a map $\tau_1 \square_{3,1}^3 \rightarrow X$:



Extending this map to $\square_{3,1}^3$ then yields a marked 2-cube that witnesses the desired associativity. □

The following proposition is straightforward to verify.

Proposition 4.8 *The assignment $X \mapsto \text{ho}_1 X$ extends to a functor from the category of comical sets to Cat . Moreover there is a natural isomorphism $\text{ho}_1 (X^{\text{op}}) \cong (\text{ho}_1 X)^{\text{op}}$.* □

5 Triangulation

In this section, we upgrade the triangulation adjunction described in Section 1.2 to a marked version. We start by recalling the basic combinatorics of simplicial cubes, which can be found in [Verity 2007, Section 5]. (Note however that our indexing is reversed from Verity's.)

Given an r -simplex ϕ in the simplicial set $(\Delta^1)^n$, we can define a function

$$\{1, \dots, n\} \rightarrow \{1, \dots, r, \pm\infty\}$$

by declaring

$$i \mapsto \begin{cases} +\infty & \text{if } \pi_i \circ \phi(r) = 0, \\ p & \text{if } \pi_i \circ \phi(p-1) = 0 \text{ and } \pi_i \circ \phi(p) = 1, \\ -\infty & \text{if } \pi_i \circ \phi(0) = 1. \end{cases}$$

If we regard ϕ as an r -step walk on the n -cube with the p^{th} step connecting $\phi(p-1)$ to $\phi(p)$, the above function takes i to the unique p such that the p^{th} step moves in the i^{th} direction; it takes the value $+\infty$ if we never move in the i^{th} direction, and the value $-\infty$ if we have already moved in that direction before we start.

Conversely, any function $\{1, \dots, n\} \rightarrow \{1, \dots, r, \pm\infty\}$ determines a unique r -simplex in $(\Delta^1)^n$, so we will identify the r -simplices and these functions.

Remark 5.1 In what follows, we sometimes write such expressions as $p \pm k$ for $p \in \{1, \dots, r, \pm\infty\}$ and finite k . These expressions are to be interpreted as p when $p \in \{\pm\infty\}$. We will never consider expressions involving more than one $\pm\infty$.

Definition 5.2 We will write $\iota = \iota_n: \{1, \dots, n\} \rightarrow \{1, \dots, n, \pm\infty\}$ for the inclusion regarded as an n -simplex in $(\Delta^1)^n$.

We will think of any set of the form $\{1, \dots, r, \pm\infty\}$ as a linearly ordered set

$$-\infty < 1 < \dots < r < +\infty.$$

Note however that simplices $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, r, \pm\infty\}$ are not necessarily order-preserving.

Proposition 5.3 Under this identification, a simplicial operator $\alpha: [q] \rightarrow [r]$ sends an r -simplex ϕ to the q -simplex $\phi\alpha$ given by

$$(\phi\alpha)(i) = \begin{cases} +\infty & \text{if } \phi(i) > \alpha(q), \\ p & \text{if } \alpha(p-1) < \phi(i) \leq \alpha(p), \\ -\infty & \text{if } \phi(i) \leq \alpha(0). \end{cases}$$

Example 5.4 Again, let us think of an r -simplex ϕ as an r -step walk. Then the last face $\phi\partial_r$ of ϕ moves in the i^{th} direction at exactly the same step as ϕ does except that it does not have an r^{th} step. This agrees with the following formula obtained using Proposition 5.3:

$$(\phi\partial_r)(i) = \begin{cases} +\infty & \text{if } \phi(i) = r, \\ \phi(i) & \text{otherwise.} \end{cases}$$

On the other hand, taking the 0^{th} face decreases the index of each step by 1, so we have

$$(\phi\partial_0)(i) = \begin{cases} -\infty & \text{if } \phi(i) = 1, \\ \phi(i) - 1 & \text{otherwise.} \end{cases}$$

For $0 < k < n$, taking the k^{th} step merges the k^{th} and the $(k + 1)^{\text{st}}$ steps, so it does not affect the endpoints of the whole walk. When regarded as a function, this means that $\phi\partial_k$ takes the values $\pm\infty$ on exactly the same inputs as ϕ does. However, some of the indices are shifted:

$$(\phi\partial_k)(i) = \begin{cases} \phi(i) - 1 & \text{if } k < \phi(i) \leq r, \\ \phi(i) & \text{otherwise.} \end{cases}$$

For any $0 \leq m \leq r$, taking the last face $(r - m)$ times yields $\perp_1^{m, r-m}$, so we have

$$(\phi \perp_1^{m, r-m})(i) = \begin{cases} +\infty & \text{if } \phi(i) > m, \\ \phi(i) & \text{if } \phi(i) \leq m. \end{cases}$$

Similarly, since taking the 0th face m times yields $\mathbb{L}_2^{m, r-m}$, we have

$$(\phi \mathbb{L}_2^{m, r-m})(i) = \begin{cases} \phi(i) - m & \text{if } \phi(i) > m, \\ -\infty & \text{if } \phi(i) \leq m. \end{cases}$$

It is easy to verify the following proposition using Proposition 5.3.

Proposition 5.5 *An r -simplex ϕ in $(\Delta^1)^n$ is nondegenerate if and only if $\phi^{-1}(p) \neq \emptyset$ for each $1 \leq p \leq r$. More precisely, ϕ is degenerate at $p - 1$ if and only if $\phi^{-1}(p) = \emptyset$.*

Now we upgrade the *codomain* of the triangulation functor to a marked version. More precisely, we first consider the functor $\square \rightarrow \text{PreComp}$ associated (in the sense of Theorem 1.11) to the cubical monoid Δ^1 , where PreComp is considered to be monoidal with respect to the Gray tensor product \otimes^{pre} . Its object part is thus given by $[1]^n \mapsto (\Delta^1)^{\otimes^{\text{pre}} n}$. This functor induces a strong monoidal left adjoint $T : \text{cSet} \rightarrow \text{PreComp}$ with right adjoint U . We first show that $(\Delta^1)^{\otimes^{\text{pre}} n} = (\Delta^1)^{\otimes n}$.

Proposition 5.6 *An r -simplex $\phi : \{1, \dots, n\} \rightarrow \{1, \dots, r, \pm\infty\}$ in $(\Delta^1)^{\otimes n}$ is unmarked if and only if there exist*

$$1 \leq i_1 < \dots < i_r \leq n$$

such that $\phi(i_p) = p$ for all $1 \leq p \leq r$. In particular, the only unmarked n -simplex in $(\Delta^1)^{\otimes n}$ is ι_n .

Proof This is proved in [Verity 2007, Observation 27] (with opposite indexing from ours). □

Using the above characterization, we can indeed prove the following.

Proposition 5.7 *The marked simplicial set $(\Delta^1)^{\otimes n}$ is precomplicial for any $n \geq 0$.*

Proof Suppose for the sake of contradiction that there exists a map $\Delta_k^{r'} \rightarrow (\Delta^1)^{\otimes n}$ that cannot be extended to $\Delta_k^{r''}$. Regard this map as an r -simplex $\phi : \{1, \dots, n\} \rightarrow \{1, \dots, r, \pm\infty\}$. Then $\phi \partial_k$ is unmarked, so Proposition 5.6 implies that there exist

$$1 \leq i_1 \leq \dots \leq i_{r-1} \leq n$$

such that

- $\phi(i_p) = p$ for $1 \leq p \leq k - 1$,
- $\phi(i_k) = k$ or $\phi(i_k) = k + 1$, and
- $\phi(i_p) = p + 1$ for $k + 1 \leq p \leq r - 1$.

But if $\phi(i_k) = k$ then the same sequence i_1, \dots, i_{r-1} witnesses that $\phi \partial_{k+1}$ is unmarked, and similarly if $\phi(i_k) = k + 1$ then $\phi \partial_{k-1}$ is unmarked. In either case, it contradicts with our assumption that ϕ corresponds to a map $\Delta_k^{r'} \rightarrow (\Delta^1)^{\otimes n}$. □

Next we would like to upgrade the *domain* of T to a marked version too, by sending the marked n -cube to $\tau_{n-1}((\Delta^1)^{\otimes n})$. Proposition 5.6 implies that this marked simplicial set may be obtained from $(\Delta^1)^{\otimes n}$ by marking ι_n . The following lemma shows that it is indeed an object in PreComp .

Lemma 5.8 *The marked simplicial set $\tau_{n-1}((\Delta^1)^{\otimes n})$ is precomplicial for any $n \geq 1$.*

Proof Suppose for the sake of contradiction that we are given a map $\phi : \Delta_k^{m'} \rightarrow \tau_{n-1}((\Delta^1)^{\otimes n})$ that cannot be extended to $\Delta_k^{m''}$. Then ϕ must not factor through the precomplicial set $(\Delta^1)^{\otimes n}$, so ϕ sends at least one of the marked, nondegenerate simplices in $\Delta_k^{m'}$ to ι_n . Since all simplices in $\Delta_k^{m'}$ of dimension $> m$ are degenerate, it follows that $m \geq n$. On the other hand, we cannot have $m > n$ since $\phi \partial_k$ is unmarked in the $(n-1)$ -trivial marked simplicial set $\tau_{n-1}((\Delta^1)^{\otimes n})$. Thus we must have $m = n$ and $\phi = \iota_n$. But at least one of ∂_{k-1} and ∂_{k+1} is a well-defined face of $\Delta_k^{m'}$, and it can be easily checked using Proposition 5.6 that ϕ sends this face to an unmarked simplex. This is the desired contradiction. \square

Thus we have defined the object part of $T : \square^+ \rightarrow \text{PreComp}$, but we still need to define its value on the generating morphisms φ^n , ζ_i^n and $\xi_{i,\varepsilon}^n$, and verify the cocubical identities. The maps $T\varphi^n : T[1]^n \rightarrow T[1]_e^n$ are identity on the underlying simplicial sets and add the additional marking on ι_n . To define $T\zeta_i^n$ (resp. $T\xi_{i,\varepsilon}^n$), notice that we must have $T\zeta_i^n T\varphi^n = T\sigma_i^n$ (resp. $T\xi_{i,\varepsilon}^n T\varphi^n = T\gamma_{i,\varepsilon}^n$). Since $T\sigma_i^n$ (resp. $T\gamma_{i,\varepsilon}^n$) sends the n -simplex ι_n to a degenerate (and hence marked) one, it must factor through $T\varphi^n$. Moreover, since $T\varphi^n$ is (entire and hence) an epimorphism, this factorization is unique, yielding a unique possible choice for $T\zeta_i^n$ (resp. $T\xi_{i,\varepsilon}^n$). Finally, to see that this definition satisfies the additional identities, we note that these involving φ are clear, whereas the remaining ones can be reduced to the usual cubical identities by precomposing with φ and using the fact that it is an epimorphism.

Hence we obtain a left adjoint functor T from structurally marked cubical sets to precomplicial sets. Moreover, the right adjoint U takes values in marked cubical sets, because the map $\square^n \rightarrow \tilde{\square}^n$ is carried by T to an epimorphism. Thus by restricting the domain of T , we have constructed an adjunction $T \dashv U$ between marked cubical sets and precomplicial sets. In the remainder of the paper, we show that T is strong monoidal with respect to either version of the Gray tensor products and moreover left Quillen with respect to suitable model structures. We will make use of the following observation.

Proposition 5.9 *There are isomorphisms $T(X^{\text{op}}) \cong (TX)^{\text{op}}$ natural in $X \in \text{cSet}^+$.*

Proof Since both $X \mapsto T(X^{\text{op}})$ and $X \mapsto (TX)^{\text{op}}$ are cocontinuous, it suffices to verify the assertion for $X = \square^n$ for $n \geq 0$ and $X = \tilde{\square}^n$ for $n \geq 1$. The component at each \square^n is simply given by

$$T((\square^n)^{\text{op}}) = T(\square^n) = (\Delta^1)^{\otimes n} = ((\Delta^1)^{\text{op}})^{\otimes n} \cong ((\Delta^1)^{\otimes n})^{\text{op}} = T(\square^n)^{\text{op}},$$

where the isomorphism is induced by the antimonicity of $(-)^{\text{op}} : \text{sSet}^+ \rightarrow \text{sSet}^+$ [Verity 2008a, Lemma 131], and the component at each $\tilde{\square}^n$ is then obtained by applying τ_{n-1} .

It remains to check that these components are natural. Since the forgetful functor $\mathbf{sSet}^+ \rightarrow \mathbf{sSet}$ is faithful, we may instead check the naturality of the whiskering:

$$\begin{array}{ccc}
 & T((-)^{\text{op}}) & \\
 \square^+ & \begin{array}{c} \curvearrowright \\ \Downarrow \\ \curvearrowleft \end{array} & \mathbf{sSet}^+ \longrightarrow \mathbf{sSet} \\
 & T(-)^{\text{op}} &
 \end{array}$$

Now it is straightforward to check that this (potentially unnatural) transformation may also be written as

$$\begin{array}{ccc}
 & T((-)^{\text{op}}) & \\
 \square^+ \longrightarrow \square & \begin{array}{c} \curvearrowright \\ \Downarrow \\ \curvearrowleft \end{array} & \mathbf{sSet} \\
 & T(-)^{\text{op}} &
 \end{array}$$

where the first factor forgets the marking and the second factor (potentially unnatural transformation) is the unmarked analogue of our desired natural isomorphism. But in the unmarked case, we know that both $X \mapsto T(X^{\text{op}})$ and $X \mapsto T(X)^{\text{op}}$ are antimonoidal, and moreover it is straightforward to manually check that the naturality of its restriction to the full subcategory spanned by \square^0 , \square^1 and \square^2 . Thus the desired naturality follows from Theorem 1.11. □

6 Triangulating Gray tensor product

We now prove that the triangulation functor is strong monoidal with respect to either version of the Gray tensor product. We begin by describing a proof strategy that will be used in both the lax and the pseudo cases.

6.1 Proof strategy

The proof typically reduces to showing an entire inclusion $A \hookrightarrow B$ to be a complicial marking extension where A and B are certain entire supersets of $(\Delta^1)^{\otimes N}$. (The integer N will be of the form $N = m + n$ in the actual proofs, but this is irrelevant in this subsection.) There are three kinds of simplices of interest, namely those that are

- (i) marked in $(\Delta^1)^{\otimes N}$,
- (ii) marked in A but not in $(\Delta^1)^{\otimes N}$, and
- (iii) marked in B but not in A .

The simplices of type (i) are characterized by Proposition 5.6. The first step of the proof will be to (define suitable A, B and) characterize simplices of type (ii) and (iii).

Before proceeding, we need the following definitions.

Definition 6.1 For any r -simplex $\phi: \{1, \dots, N\} \rightarrow \{1, \dots, r, \pm\infty\}$ in $(\Delta^1)^N$, define

$$\mathcal{D}(\phi) = |\phi^{-1}(\{1, \dots, r\})| - r \quad \text{and} \quad \mathcal{O}(\phi) = \{(i, j) \in \{1, \dots, N\}^2 \mid i < j, \phi(i) < \phi(j)\}.$$

The integer $\mathcal{D}(\phi)$ measures how “diagonal” ϕ is, and the set $\mathcal{O}(\phi)$ measures how “in order” ϕ is.

We complicially extend the marking on A to those simplices ϕ of type (iii) by nested induction on $\mathcal{D}(\phi)$ and $|\mathcal{O}(\phi)|$. More precisely, consider the lexicographical ordering on $\mathbb{Z} \times \mathbb{N}$ so that $(u_1, v_1) \leq (u_2, v_2)$ if and only if

- $u_1 < u_2$, or
- $u_1 = u_2$ and $v_1 \leq v_2$.

For each $(u, v) \in \mathbb{Z} \times \mathbb{N}$, let $A(u, v)$ denote the marked simplicial set obtained from A by marking those simplices ϕ such that ϕ is marked in B and $(\mathcal{D}(\phi), |\mathcal{O}(\phi)|) < (u, v)$. Then

- $A(u_1, v_1)$ is an entire subset of $A(u_2, v_2)$ for any $(u_1, v_1) \leq (u_2, v_2)$,
- $\text{colim}_v A(u, v) = A(u + 1, 0)$ for any $u \in \mathbb{Z}$,
- $\text{colim}_{u,v} A(u, v) = B$, and
- $A(0, 0) = A$ (by Proposition 5.5).

Now we assume the following.

Assumption 1 Any marked simplex ϕ in B with $\mathcal{D}(\phi) = 0$ is marked in A .

Then we may upgrade the last bulleted item to $A(1, 0) = A$. Thus to prove that $A \rightarrow B$ is a complicial marking extension, it suffices to exhibit the map $A(u, v) \rightarrow A(u, v + 1)$ as a complicial marking extension for each $(u, v) \geq (1, 0)$.

So fix $(u, v) \geq (1, 0)$ and suppose that we are given an r -simplex ϕ of type (iii) with $(\mathcal{D}(\phi), |\mathcal{O}(\phi)|) = (u, v)$. Then in particular $\mathcal{D}(\phi) \geq 1$. So by the pigeonhole principle, we can choose $1 \leq p_\phi \leq r$ such that $|\phi^{-1}(p_\phi)| \geq 2$. Let $i_\phi = \min \phi^{-1}(p_\phi)$. Let $\tilde{\phi}$ be the $(r+1)$ -simplex given by

$$\tilde{\phi}(i) = \begin{cases} \phi(i) & \text{if } \phi(i) \leq p_\phi \text{ and } i \neq i_\phi, \\ \phi(i) + 1 & \text{if } \phi(i) > p_\phi \text{ or } i = i_\phi. \end{cases}$$

Observe that we have $\tilde{\phi} \partial_{p_\phi} \phi = \phi$. We wish to show that this simplex $\tilde{\phi}$ extends to $\Delta_{p_\phi}^{r+1}$:

$$\begin{array}{ccc} \Delta^{r+1} & \xrightarrow{\tilde{\phi}} & A(u, v) \\ \downarrow & \nearrow \exists & \\ \Delta_{p_\phi}^{r+1'} & & \end{array}$$

Assuming this fact, we can deduce that we have a pushout square

$$\begin{array}{ccc} \coprod \Delta_{p_\phi}^{r+1'} & \longrightarrow & A(u, v) \\ \downarrow & & \downarrow \\ \coprod \Delta_{p_\phi}^{r+1''} & \longrightarrow & A(u, v + 1) \end{array}$$

where the coproducts are taken over all r -simplices ϕ of type (iii) with $(\mathcal{D}(\phi), |\mathcal{C}(\phi)|) = (u, v)$ for various r , and the horizontal maps are induced by $\tilde{\phi}$.

The following lemma implies that $\tilde{\phi}$ at least extends to $\Delta_{p_\phi}^{r+1}$.

Lemma 6.2 *Let $\alpha: [q] \rightarrow [r + 1]$ be a face operator with $\{p_\phi, p_\phi \pm 1\} \subset \text{im } \alpha$. Then $\tilde{\phi}\alpha$ is marked in $(\Delta^1)^{\otimes N}$.*

Proof Let $p \in [q]$ be the necessarily unique element with $\alpha(p) = p_\phi$. Then we must have $\alpha(p-1) = p_\phi - 1$ and $\alpha(p+1) = p_\phi + 1$. Now one can check using Proposition 5.3 and the minimality of i_ϕ that $(\tilde{\phi}\alpha)^{-1}(p+1) = \{i_\phi\}$ and moreover any $1 \leq i \leq N$ satisfying $(\tilde{\phi}\alpha)(i) = p$ must also satisfy $i > i_\phi$. Thus $\tilde{\phi}\alpha$ is marked in $(\Delta^1)^{\otimes N}$ by Proposition 5.6. \square

Therefore it remains to prove that the faces $\chi = \tilde{\phi}\partial_{p_\phi-1}$ and $\psi = \tilde{\phi}\partial_{p_\phi+1}$ are marked in $A(u, v)$. First, we describe these simplices explicitly.

Lemma 6.3 *The simplex χ is given by*

$$\chi(i) = \begin{cases} -\infty & \text{if } \phi(i) = p_\phi \text{ and } i \neq i_\phi, \\ \phi(i) & \text{otherwise.} \end{cases}$$

if $p_\phi = 1$, and

$$\chi(i) = \begin{cases} p_\phi - 1 & \text{if } \phi(i) = p_\phi \text{ and } i \neq i_\phi, \\ \phi(i) & \text{otherwise} \end{cases}$$

if $p_\phi \geq 2$. The simplex ψ is given by

$$\psi(i) = \begin{cases} +\infty & \text{if } i = i_\phi, \\ \phi(i) & \text{otherwise} \end{cases}$$

if $p_\phi = r$ and

$$\psi(i) = \begin{cases} p_\phi + 1 & \text{if } i = i_\phi, \\ \phi(i) & \text{otherwise} \end{cases}$$

if $p_\phi < r$.

Proof This is a routine application of Proposition 5.3. \square

These explicit descriptions allow us to prove the following.

Lemma 6.4 *The simplices χ and ψ satisfy*

$$(\mathcal{D}(\chi), |\mathcal{O}(\chi)|) < (\mathcal{D}(\phi), |\mathcal{O}(\phi)|) \quad \text{and} \quad (\mathcal{D}(\psi), |\mathcal{O}(\psi)|) < (\mathcal{D}(\phi), |\mathcal{O}(\phi)|).$$

Proof If $p_\phi = 1$ then clearly $\mathcal{D}(\chi) < \mathcal{D}(\phi)$.

Suppose $p_\phi \geq 2$. Then we have $\mathcal{D}(\chi) = \mathcal{D}(\phi)$. We claim that $\mathcal{O}(\chi)$ is a proper subset of $\mathcal{O}(\phi)$. Indeed, it can be seen from Lemma 6.3 that if a pair (i, j) satisfies $\phi(i) \geq \phi(j)$ and $\chi(i) < \chi(j)$ then we must have $\phi(i) = \phi(j) = p_\phi$ and $i \neq i_\phi = j$. But then the minimality of i_ϕ implies $i > j$, and this shows that there is no pair (i, j) in $\mathcal{O}(\chi) \setminus \mathcal{O}(\phi)$. On the other hand, since ϕ is unmarked in $(\Delta^1)^{\otimes N}$, there exist $1 \leq i_1 \leq \dots \leq i_r \leq N$ such that $\phi(i_p) = p$ for $1 \leq p \leq r$. It is straightforward to check that the pair $(i_{p_\phi-1}, \max \phi^{-1}(p_\phi))$ is then in $\mathcal{O}(\phi) \setminus \mathcal{O}(\chi)$. Therefore $\mathcal{O}(\chi)$ is a proper subset of $\mathcal{O}(\phi)$, and this proves the lexicographical inequality concerning χ .

The simplex ψ can be treated dually. □

The last missing piece of the proof (that $A \rightarrow B$ is a complicial marking extension) is the following.

Assumption 2 *The simplices χ and ψ are marked in B .*

This completes the proof strategy. (Whether Assumptions 1 and 2 hold depends on the exact definitions of A and B , so there is no general strategy for verifying them.)

6.2 Triangulating the lax Gray tensor product

The goal of this subsection is to prove the following theorem.

Theorem 6.5 *The adjunction $T \dashv U$ is monoidal with respect to the lax Gray tensor products. Equivalently, $T : (\text{cSet}^+, \otimes) \rightarrow (\text{PreComp}, \otimes^{\text{pre}})$ is strong monoidal.*

Fix $m \geq 1$ and $n \geq 0$. Observe that

$$\begin{array}{ccc} \coprod \square^{m+k} & \longrightarrow & \square^{m+n} \\ \coprod \varphi \downarrow & & \downarrow \\ \coprod \tilde{\square}^{m+k} & \longrightarrow & \tilde{\square}^m \otimes \square^n \end{array}$$

is a pushout square in cSet^+ , where the coproducts are taken over all face maps $[1]^k \rightarrow [1]^n$. This pushout is preserved by T , so the right square in

$$\begin{array}{ccccc} \coprod \Delta^{m+k} & \xrightarrow{\coprod \iota_{m+k}} & \coprod (\Delta^1)^{\otimes(m+k)} & \longrightarrow & (\Delta^1)^{\otimes(m+n)} \\ \downarrow & & \downarrow & & \downarrow \\ \coprod \tilde{\Delta}^{m+k} & \longrightarrow & \coprod \tau_{m+k-1}((\Delta^1)^{\otimes(m+k)}) & \longrightarrow & T(\tilde{\square}^m \otimes \square^n) \end{array}$$

is a pushout square in PreComp , where the upper horizontal map is induced by $\text{id}_{(\Delta^1)^{\otimes m}} \otimes T(\phi)$ for various face maps $\phi: [1]^k \rightarrow [1]^n$. The left square is also a pushout by Proposition 5.6, so the pasted square is a pushout too. In this subsection, we define A to be the corresponding pushout in sSet^+ (and not in PreComp)

$$\begin{array}{ccc} \coprod \Delta^{m+k} & \longrightarrow & (\Delta^1)^{\otimes(m+n)} \\ \downarrow & \text{p.o.} & \downarrow \\ \coprod \tilde{\Delta}^{m+k} & \longrightarrow & A \end{array}$$

so that its precomplicial reflection A^{pre} is precisely $T(\tilde{\square}^m \otimes \square^n)$.

Now we give combinatorial characterizations of marked simplices in A and in $T(\tilde{\square}^m) \otimes T(\square^n)$.

Lemma 6.6 *An r -simplex ϕ is marked in A but not in $(\Delta^1)^{\otimes(m+n)}$ if and only if $r \geq m$, $\phi(i) = i$ for all $1 \leq i \leq m$ and the restriction*

$$\phi^{-1}(\{1, \dots, r\}) \rightarrow \{1, \dots, r\}$$

of ϕ is an isomorphism of linearly ordered sets.

Proof Compute the colimit. □

Lemma 6.7 *Let ϕ be an unmarked r -simplex in $(\Delta^1)^{\otimes(m+n)}$. Then ϕ is marked in $T(\tilde{\square}^m) \otimes T(\square^n)$ if and only if:*

- (1) $r \geq m$;
- (2) $\phi(i) = i$ for all $1 \leq i \leq m$; and
- (3) *there does NOT exist a sequence $m < i_m < i_{m+1} < \dots < i_r \leq m+n$ such that $\phi(i_p) = p$ for all $m \leq p \leq r$.*

Proof Write $\phi = (\phi_1, \phi_2)$ for ϕ regarded as a simplex in the product simplicial set $(\Delta^1)^m \times (\Delta^1)^n$. That is, $\phi_1: \{1, \dots, m\} \rightarrow \{1, \dots, r, \pm\infty\}$ and $\phi_2: \{1, \dots, n\} \rightarrow \{1, \dots, r, \pm\infty\}$ are respectively given by $\phi_1(i) = \phi(i)$ and $\phi_2(i) = \phi(m+i)$.

It follows from the definitions of \otimes and $T(\tilde{\square}^m)$ that ϕ (which we are assuming to be unmarked in $(\Delta^1)^{\otimes(m+n)}$) is marked in $T(\tilde{\square}^m) \otimes T(\square^n)$ if and only if

- (a) $r \geq m$,
- (b) (ϕ_1, ϕ_2) is q -cloven for all q except for $q = m$,
- (c) $\phi_1 \perp_1^{m, r-m} = \iota_m$, and
- (d) $\phi_2 \perp_2^{m, r-m}$ is unmarked in $(\Delta^1)^{\otimes n}$.

The clauses (a) and (c) here clearly correspond respectively to (1) and (2) in the lemma. Since we are assuming ϕ to be unmarked in $(\Delta^1)^{\otimes(m+n)}$, Proposition 5.6 implies that there exists a sequence $1 \leq j_1 < \dots < j_r \leq m+n$ such that $\phi(j_p) = p$ for all $1 \leq p \leq r$. Note that the strict inequalities imply $j_{m+1} > m$. One can now check using Example 5.4 that $(\phi_2 \ll_2^{m,r-m})(j_p - m) = p - m$ for all $m+1 \leq p \leq r$. Thus the clause (d) is in fact redundant by Proposition 5.6.

It remains to check that, assuming (a), (c) and (d), the clauses (3) and (b) are equivalent. Note that (ϕ_1, ϕ_2) is q -cloven for any $m < q \leq r$ since the q -simplex $\phi_1 \ll_1^{q,r-q}$ in the simplicial set $(\Delta^1)^m$ must be degenerate. For $0 \leq q < m$, since we are assuming (c), $\phi_1 \ll_1^{q,r-q} = \iota_q$ is unmarked in $T(\tilde{\square}^m)$. So (b) is equivalent to $\phi_2 \ll_2^{q,r-q}$ being marked in $(\Delta^1)^{\otimes n}$ for all $0 \leq q < m$. By Example 5.4 and Proposition 5.6, this latter condition for fixed q is equivalent to the NON-existence of a sequence

$$m < i_{q+1} < \dots < i_r \leq m+n$$

such that $\phi(i_p) = p$ for all $q+1 \leq p \leq r$. Clearly the nonexistence for $q = m-1$, which is precisely (3), implies the nonexistence for all other values of q . This completes the proof. \square

By combining Proposition 5.6 and Lemma 6.7, we obtain the following.

Lemma 6.8 *An r -simplex ϕ in $T(\tilde{\square}^m) \otimes T(\square^n)$ with $r \geq m$ is unmarked if and only if there exist*

$$1 \leq i_1 < \dots < i_r \leq m+n$$

such that $\phi(i_p) = p$ for all $1 \leq p \leq r$ and moreover $i_m > m$.

Proof Let ϕ be an unmarked r -simplex in $(\Delta^1)^{\otimes(m+n)}$ with $r \geq m$. Note that ϕ is unmarked in $T(\tilde{\square}^m) \otimes T(\square^n)$ if and only if it violates either Lemma 6.7(2) or (3).

The “if” direction is easy since the existence of a sequence satisfying the condition stated in the lemma would immediately contradict Lemma 6.7(3).

For the “only if” direction, assume that ϕ is unmarked in $T(\tilde{\square}^m) \otimes T(\square^n)$. Recall that by Proposition 5.6 there exist

$$1 \leq j_1 < \dots < j_r \leq m+n$$

such that $\phi(j_p) = p$ for all $1 \leq p \leq r$. If $j_m > m$, then simply taking $i_p = j_p$ for all p would yield the desired sequence. So assume $j_m = m$. Then since the inequalities $j_1 < \dots < j_m$ are strict, we must have $j_p = p$ for all $1 \leq p \leq m$. Thus ϕ cannot violate Lemma 6.7(2), so it must violate (3). That is, there exist $m < i_m < i_{m+1} < \dots < i_r \leq m+n$ such that $\phi(i_p) = p$ for all $m \leq p \leq r$. We then obtain the desired sequence by taking $i_p = j_p$ for $1 \leq p \leq m-1$. \square

Lemma 6.9 *There is a complicial marking extension $A \rightarrow T(\tilde{\square}^m) \otimes T(\square^n)$ that commutes with the evident inclusions of $(\Delta^1)^{\otimes(m+n)}$.*

Proof We apply the proof strategy from Section 6.1 with $B = T(\tilde{\square}^m) \otimes T(\square^n)$.

One can easily check using Lemmas 6.6 and 6.7 that any marked simplex ϕ in $T(\tilde{\square}^m) \otimes T(\square^n)$ with $\mathcal{D}(\phi) = 0$ must also be marked in A . This verifies Assumption 1.

To verify Assumption 2, let ϕ be an r -simplex that is marked in $T(\tilde{\square}^m) \otimes T(\square^n)$ but not in A . Then we necessarily have $r \geq m$ by Lemma 6.7.

Consider the simplex $\chi = \tilde{\phi} \partial_{p_\phi-1}$. Suppose for contradiction that χ is unmarked in $T(\tilde{\square}^m) \otimes T(\square^n)$. Then by Lemma 6.8 there exist $1 \leq i_1 < \dots < i_r \leq m+n$ such that $\chi(i_p) = p$ for all $1 \leq p \leq r$ and $i_m > m$.

- If $p_\phi = 1$, then we also have $\phi(i_p) = p$ for all p by Lemma 6.3, thus ϕ is unmarked in $T(\tilde{\square}^m) \otimes T(\square^n)$. This is the desired contradiction.
- Suppose $p_\phi \geq 2$. We claim that $\phi(i_p) = p$ holds for all p in this case too. According to Lemma 6.3, the only thing we must check is that $\phi(i_{p_\phi-1}) = p_\phi - 1$ holds (as opposed to $\phi(i_{p_\phi-1}) = p_\phi$). To see that this is indeed the case, observe that $\chi^{-1}(p_\phi) = \{i_\phi\}$ by Lemma 6.3. Thus we must have $i_{p_\phi} = i_\phi$. Since $i_{p_\phi-1} < i_{p_\phi}$, the minimality of i_ϕ implies that $i_{p_\phi-1} \notin \phi^{-1}(p_\phi)$. Therefore we have obtained the desired contradiction.

The simplex $\psi = \tilde{\phi} \partial_{p_\phi+1}$ can be similarly checked to be marked in $T(\tilde{\square}^m) \otimes T(\square^n)$. This completes the proof. □

Proof of Theorem 6.5 Since both $T(- \otimes -)$ and $T(-) \otimes^{\text{pre}} T(-)$ preserve colimits in each variable, it suffices to check the existence of natural isomorphisms $T(X \otimes Y) \cong T(X) \otimes^{\text{pre}} T(Y)$ for X, Y generic (possibly marked) cubes.

By construction of T , we have $T(\square^m \otimes \square^n) \cong T(\square^m) \otimes^{\text{pre}} T(\square^n)$ for any $m, n \geq 0$.

For any $m \geq 1$ and $n \geq 0$, we may obtain an isomorphism $T(\tilde{\square}^m \otimes \square^n) \cong T(\tilde{\square}^m) \otimes^{\text{pre}} T(\square^n)$ by reflecting the complicial marking extension of Lemma 6.9 into PreComp. Dually, we have $T(\square^m \otimes \tilde{\square}^n) \cong T(\square^m) \otimes^{\text{pre}} T(\tilde{\square}^n)$ for any $m \geq 0$ and $n \geq 1$.

Let $m, n \geq 1$. Observe that the left square below is a pushout in cSet^+ by Lemma 2.17(3):

$$\begin{array}{ccc}
 \square^m \otimes \square^n & \longrightarrow & \tilde{\square}^m \otimes \square^n \\
 \downarrow & & \downarrow \\
 \square^m \otimes \tilde{\square}^n & \longrightarrow & \tilde{\square}^m \otimes \tilde{\square}^n
 \end{array}
 \qquad
 \begin{array}{ccc}
 T(\square^m \otimes \square^n) & \longrightarrow & T(\tilde{\square}^m \otimes \square^n) \\
 \downarrow & & \downarrow \\
 T(\square^m \otimes \tilde{\square}^n) & \longrightarrow & T(\tilde{\square}^m \otimes \tilde{\square}^n)
 \end{array}$$

Since T is cocontinuous, it follows that the right square is a pushout in PreComp. On the other hand, since both $T(\square^m) \rightarrow T(\tilde{\square}^m)$ and $T(\square^n) \rightarrow T(\tilde{\square}^n)$ are entire, the square below is a pushout in PreComp

by [Verity 2008a, Lemma 140]:

$$\begin{array}{ccc}
 T(\square^m) \otimes^{\text{pre}} T(\square^n) & \longrightarrow & T(\tilde{\square}^m) \otimes^{\text{pre}} T(\square^n) \\
 \downarrow & & \downarrow \\
 T(\square^m) \otimes^{\text{pre}} T(\tilde{\square}^n) & \longrightarrow & T(\tilde{\square}^m) \otimes^{\text{pre}} T(\tilde{\square}^n)
 \end{array}$$

Thus by comparing the two pushout squares in PreComp , we obtain $T(\tilde{\square}^m \otimes \tilde{\square}^n) \cong T(\tilde{\square}^m) \otimes^{\text{pre}} T(\tilde{\square}^n)$. The naturality of these isomorphisms is evident, and this completes the proof. \square

6.3 Triangulating the pseudo Gray tensor product

The goal of this subsection is to prove the following theorem.

Theorem 6.10 *The adjunction $T \dashv U$ is monoidal with respect to the pseudo Gray tensor products. Equivalently, $T : (\text{cSet}^+, \otimes) \rightarrow (\text{PreComp}, \otimes)$ is strong monoidal.*

Fix $m, n \geq 1$. By Remark 2.14, the square

$$\begin{array}{ccc}
 \coprod \square^{k+\ell} & \longrightarrow & \square^{m+n} \\
 \downarrow & & \downarrow \\
 \coprod \tilde{\square}^{k+\ell} & \longrightarrow & \square^m \otimes \square^n
 \end{array}$$

is a pushout in cSet^+ , where the coproducts are taken over all pairs of face maps $\square^k \rightarrow \square^m$ and $\square^\ell \rightarrow \square^n$ such that $k, \ell \geq 1$. This pushout is preserved by T , so the right square in

$$\begin{array}{ccccc}
 \coprod \Delta^{k+\ell} & \xrightarrow{\coprod \iota_{k+\ell}} & \coprod (\Delta^1)^{\otimes(k+\ell)} & \longrightarrow & (\Delta^1)^{\otimes(m+n)} \\
 \downarrow & & \downarrow & & \downarrow \\
 \coprod \tilde{\Delta}^{k+\ell} & \longrightarrow & \coprod \tau_{k+\ell-1}((\Delta^1)^{\otimes(k+\ell)}) & \longrightarrow & T(\square^m \otimes \square^n)
 \end{array}$$

is a pushout square in PreComp . The left square is also a pushout by Proposition 5.6, so the pasted square is a pushout too. In this subsection, we define A to be the corresponding pushout in sSet^+ (and not in PreComp):

$$\begin{array}{ccc}
 \coprod \Delta^{k+\ell} & \longrightarrow & (\Delta^1)^{\otimes(m+n)} \\
 \downarrow & \text{p.o.} & \downarrow \\
 \coprod \tilde{\Delta}^{k+\ell} & \longrightarrow & A
 \end{array}$$

so that its precomplicial reflection A^{pre} is precisely $T(\square^m \otimes \square^n)$.

Lemma 6.11 An r -simplex ϕ is marked in A but not in $(\Delta^1)^{\otimes(m+n)}$ if and only if the restriction

$$\phi^{-1}(\{1, \dots, r\}) \rightarrow \{1, \dots, r\}$$

of ϕ is an isomorphism of linearly ordered sets and moreover $\phi^{-1}(\{1, \dots, r\})$ intersects both $\{1, \dots, m\}$ and $\{m + 1, \dots, m + n\}$.

Proof Compute the colimit. □

Lemma 6.12 An r -simplex ϕ in $T(\square^m) \otimes T(\square^n)$ is unmarked if and only if there exist either

$$1 \leq i_1 < \dots < i_r \leq m \quad \text{or} \quad m + 1 \leq i_1 < \dots < i_r \leq m + n$$

such that $\phi(i_p) = p$ for all $1 \leq p \leq r$.

Proof Since \otimes is the categorical product on PreComp, ϕ is marked if and only if both $\pi_1(\phi)$ and $\pi_2(\phi)$ are marked. Equivalently, ϕ is unmarked if and only if either $\pi_1(\phi)$ or $\pi_2(\phi)$ is unmarked. Thus the assertion follows from Proposition 5.6. □

Lemma 6.13 There is a complicial marking extension $A \rightarrow T(\square^m) \otimes T(\square^n)$ that commutes with the evident inclusions of $(\Delta^1)^{\otimes(m+n)}$.

Proof We apply the proof strategy from Section 6.1 with $B = T(\square^m) \otimes T(\square^n)$.

One can easily check using Proposition 5.6 and Lemmas 6.11 and 6.12 that any marked simplex ϕ in $T(\square^m) \otimes T(\square^n)$ with $\mathcal{D}(\phi) = 0$ must also be marked in A . This verifies Assumption 1.

To see that Assumption 2 holds for χ , suppose for contradiction that χ is unmarked in $T(\square^m) \otimes T(\square^n)$. By Lemma 6.12, this unmarked-ness is witnessed by a sequence i_1, \dots, i_r , but then the same sequence can be checked to witness that ϕ is unmarked in $T(\square^m) \otimes T(\square^n)$. The details are similar to the corresponding part in the proof of Lemma 6.9. Assumption 2 for ψ can be checked similarly. □

Let $m \geq 1$ and $n \geq 0$. Observe that the square below is a pushout in cSet^+ :

$$\begin{array}{ccc} \coprod \square^m & \longrightarrow & \square^m \otimes \square^n \\ \downarrow & & \downarrow \\ \coprod \tilde{\square}^m & \longrightarrow & \tilde{\square}^m \otimes \square^n \end{array}$$

where the coproducts are taken over all $[1]^0 \rightarrow [1]^n$. This pushout is preserved by T , so the right square in

$$\begin{array}{ccccc} \coprod \Delta^m & \xrightarrow{\coprod \iota_m} & \coprod T(\square^m) & \longrightarrow & T(\square^m \otimes \square^n) \\ \downarrow & & \downarrow & & \downarrow \\ \coprod \tilde{\Delta}^m & \longrightarrow & \coprod T(\tilde{\square}^m) & \longrightarrow & T(\tilde{\square}^m \otimes \square^n) \end{array}$$

is a pushout in PreComp. The left square is also a pushout by Proposition 5.6, so the pasted square is a pushout too. Let A' denote the “corresponding” pushout in \mathbf{sSet}^+ (and not in PreComp):

$$\begin{array}{ccc} \coprod \Delta^m & \longrightarrow & A \\ \downarrow & \text{p.o.} & \downarrow \\ \coprod \tilde{\Delta}^m & \longrightarrow & A' \end{array}$$

so that its precomplicial reflection $(A')^{\text{pre}}$ is precisely $T(\tilde{\square}^m \otimes \square^n)$.

Lemma 6.14 *The marked simplicial set A' is obtained from A by marking those m -simplices ϕ such that $\phi(i) = i$ for $1 \leq i \leq m$ and $\phi(i) \in \{\pm\infty\}$ for $i > m$.*

The unmarked simplices in $B' = T(\tilde{\square}^m) \otimes T(\square^n)$ admit a characterization similar to Lemma 6.12.

Lemma 6.15 *An r -simplex in $T(\tilde{\square}^m) \otimes T(\square^n)$ with $r \neq m$ is unmarked if and only if it is unmarked in $T(\square^m) \otimes T(\square^n)$. An m -simplex ϕ in $T(\tilde{\square}^m) \otimes T(\square^n)$ is unmarked if and only if there exist*

$$m + 1 \leq i_1 < \dots < i_m \leq m + n$$

such that $\phi(i_p) = p$ for all $1 \leq p \leq m$.

Lemma 6.16 *There is a complicial marking extension $A' \rightarrow T(\tilde{\square}^m) \otimes T(\square^n)$ that commutes with the evident inclusions of $(\Delta^1)^{\otimes(m+n)}$.*

Proof We first check that Assumption 1 holds. Let ϕ be a marked r -simplex in $T(\tilde{\square}^m) \otimes T(\square^n)$ with $\mathcal{D}(\phi) = 0$. Note that, if ϕ is marked in $T(\square^m) \otimes T(\square^n)$ then we already know that it is marked in A (and so in A' too). So suppose otherwise. Then we can see from Lemmas 6.12 and 6.15 that we must have $r = m$ and a sequence

$$1 \leq i_1 < \dots < i_m \leq m$$

such that $\phi(i_p) = p$ for all $1 \leq p \leq m$. It is then easy to check that ϕ is one of the extra marked simplices described in Lemma 6.14.

For Assumption 2, we assume for contradiction that χ or ψ is unmarked, obtain a sequence using Lemma 6.15, and deduce that ϕ is unmarked. The details are similar to those in the proofs of Lemmas 6.9 and 6.13. □

Proof of Theorem 6.10 Since both $T(- \otimes -)$ and $T(-) \otimes T(-)$ preserve colimits in each variable, it suffices to check the existence of natural isomorphisms $T(X \otimes Y) \cong T(X) \otimes T(Y)$ for X, Y generic (possibly marked) cubes.

For the appropriate values of m and n , we may obtain isomorphisms

$$\begin{aligned} T(\square^m \otimes \square^n) &\cong T(\square^m) \otimes T(\square^n), \\ T(\tilde{\square}^m \otimes \square^n) &\cong T(\tilde{\square}^m) \otimes T(\square^n), \\ T(\square^m \otimes \tilde{\square}^n) &\cong T(\square^m) \otimes T(\tilde{\square}^n) \end{aligned}$$

by reflecting to PreComp the complicial marking extensions of Lemmas 6.13 and 6.16 and the dual of the latter respectively.

Let $m, n \geq 1$. Observe that the left square below is a pushout in cSet^+ by Lemma 2.17(3):

$$\begin{array}{ccc} \square^m \otimes \square^n & \longrightarrow & \tilde{\square}^m \otimes \square^n \\ \downarrow & & \downarrow \\ \square^m \otimes \tilde{\square}^n & \longrightarrow & \tilde{\square}^m \otimes \tilde{\square}^n \end{array} \quad \begin{array}{ccc} T(\square^m \otimes \square^n) & \longrightarrow & T(\tilde{\square}^m \otimes \square^n) \\ \downarrow & & \downarrow \\ T(\square^m \otimes \tilde{\square}^n) & \longrightarrow & T(\tilde{\square}^m \otimes \tilde{\square}^n) \end{array}$$

Since T is cocontinuous, it follows that the right square is a pushout in PreComp. On the other hand, since both $T(\square^m) \rightarrow T(\tilde{\square}^m)$ and $T(\square^n) \rightarrow T(\tilde{\square}^n)$ are entire, the square below is a pushout in PreComp by Proposition 1.28:

$$\begin{array}{ccc} T(\square^m) \otimes T(\square^n) & \longrightarrow & T(\tilde{\square}^m) \otimes T(\square^n) \\ \downarrow & & \downarrow \\ T(\square^m) \otimes T(\tilde{\square}^n) & \longrightarrow & T(\tilde{\square}^m) \otimes T(\tilde{\square}^n) \end{array}$$

Thus by comparing the two pushout squares in PreComp, we obtain $T(\tilde{\square}^m \otimes \tilde{\square}^n) \cong T(\tilde{\square}^m) \otimes T(\tilde{\square}^n)$. The naturality of these isomorphisms is evident, and this completes the proof. \square

7 Triangulating model structures

The main theorem of our final section is the following.

Theorem 7.1 *The adjunction $T \dashv U$ is a Quillen adjunction with respect to the comical model structure on cSet^+ and the complicial model structure on PreComp.*

In the following proof, we denote a nondegenerate r -simplex $\phi: \{1, 2, 3\} \rightarrow \{1, \dots, r, \pm\infty\}$ in the simplicial set $(\Delta^1)^{\times 3}$ by the sequence $\phi(1)\phi(2)\phi(3)$, omitting the letter ∞ . For instance, $21-$ denotes the 2-simplex ϕ given by $\phi(1) = 2$, $\phi(2) = 1$ and $\phi(3) = -\infty$. Note that since ϕ is assumed to be nondegenerate, the dimension of ϕ can be recovered as the maximum integer appearing in the sequence.

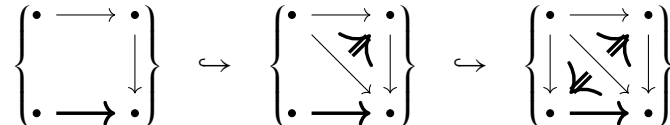
Proof We first show that T preserves cofibrations. It suffices to prove that T sends the boundary inclusions and the markers in cSet^+ to monomorphisms in sSet^+ . Clearly T sends the boundary inclusions $\partial\Box^0 \hookrightarrow \Box^0$ and $\partial\Box^1 \hookrightarrow \Box^1$ to (maps that are isomorphic to) the boundary inclusions $\partial\Delta^0 \hookrightarrow \Delta^0$ and $\partial\Delta^1 \hookrightarrow \Delta^1$ respectively. For any $n \geq 2$, we have

$$T(\partial\Box^n \hookrightarrow \Box^n) \cong T((\partial\Box^1 \hookrightarrow \Box^1)^{\widehat{\otimes} n}) \cong (T(\partial\Box^1 \hookrightarrow \Box^1))^{\widehat{\otimes} n} \cong (\partial\Delta^1 \hookrightarrow \Delta^1)^{\widehat{\otimes} n}$$

by Proposition 1.17 and Theorem 6.5, and the last map is clearly a monomorphism. Also, T sends the marker $\Box^n \hookrightarrow \widetilde{\Box}^n$ to the monomorphism $(\Delta^1)^{\otimes n} \rightarrow \tau_{n-1}((\Delta^1)^{\otimes n})$ by definition. This shows that T preserves cofibrations.

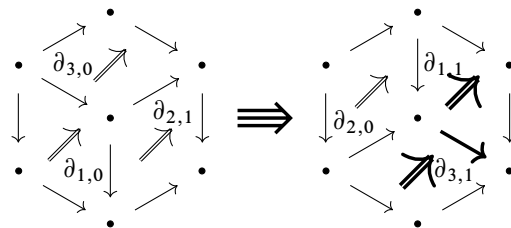
Next we show that T sends the open box inclusions to trivial cofibrations. We will check this “by hand” on the boxes of dimension ≤ 3 . This will imply the general case since the higher-dimensional box inclusions are generated by these low-dimensional ones in the sense of Proposition 3.10, T is strong monoidal with respect to the lax Gray tensor products (Theorem 6.5), and the Leibniz Gray tensor product of a complicial horn inclusion and a monomorphism (in PreComp) may be obtained as a composite of pushouts of complicial horn inclusions [Verity 2008b, Lemma 72].

Clearly T sends $\Box_{1,\varepsilon}^1 \hookrightarrow \Box_{1,\varepsilon}^1$ to the trivial cofibration $\Lambda_{1-\varepsilon}^1 \hookrightarrow \Delta_{1-\varepsilon}^1$. Consider the open box inclusion $\Box_{1,0}^2 \hookrightarrow \Box_{1,0}^2$. Its image under T may be written as a pushout of the horn inclusion $\Lambda_1^2 \hookrightarrow \Delta_1^2$ followed by a pushout of $\Lambda_2^2 \hookrightarrow \Delta_1^2$. The following pictures (in which thick arrows indicate marked simplices) depict this factorization:



The box inclusions $\Box_{k,\varepsilon}^2 \hookrightarrow \Box_{k,\varepsilon}^2$ for other values of k and/or ε can be treated similarly.

Now consider the open box inclusion $\Box_{2,0}^3 \hookrightarrow \Box_{2,0}^3$. Observe that the only marked, nondegenerate cubes in $\Box_{2,0}^3$ are id , $\partial_{1,1}$, $\partial_{3,1}$ and $\partial_{3,1}\partial_{1,1}$:



Let B' denote the marked simplicial set obtained from $(\Delta^1)^{\otimes 3}$ by marking the 3-simplex $\iota_3 = 123$, the 2-simplices $12-$ and -12 and the 1-simplex $-1-$. Then the precomplicial reflection $(B')^{\text{pre}}$ is precisely $T(\Box_{2,0}^3)$. Observe that the 3-simplex 231 specifies a map $\Delta_1^3 \rightarrow B'$. So if we mark its first face 121 , the resulting object B has the same precomplicial reflection as B' . We will adopt B (rather than B') as our

s	pushout of	interior	missing face
1	$\Lambda_1^2 \hookrightarrow \Delta_1^2$	211	111
2	$\Lambda_1^2 \hookrightarrow \Delta_1^2$	2+1	1+1
3	$\Lambda_2^3 \hookrightarrow \Delta_2^3$	312	212
4	$\Lambda_1^3 \hookrightarrow \Delta_1^3$	213	112
5	$\Lambda_2^3 \hookrightarrow \Delta_2^3$	123	122
6	$\tau_1 \Lambda_2^3 \hookrightarrow \Delta_2^{3''}$	321	221
7	$\tau_1 \Lambda_1^3 \hookrightarrow \Delta_1^{3''}$	231	121
8	$\Lambda_3^3 \hookrightarrow \Delta_3^3$	132	1+2

Table 1: Inclusions $A^{s-1} \hookrightarrow A^s$.

“model” for $T(\square_{2,0}^3)$. For the open box, we define A to be the regular subset of B (or B') consisting of those simplices ϕ such that:

- $\phi(1) \in \{\pm\infty\}$,
- $\phi(2) = -\infty$, or
- $\phi(3) \in \{\pm\infty\}$,

so that the precomplicial reflection A^{pre} is $T(\square_{2,0}^3)$. Then we have a sequence of inclusions

$$A = A^0 \hookrightarrow A^1 \hookrightarrow \dots \hookrightarrow A^8 = B,$$

where $A^{s-1} \hookrightarrow A^s$ is the pushout of a suitable trivial cofibration as indicated in Table 1. This table is to be interpreted as saying, for example, that the inclusion $A^0 \hookrightarrow A^1$ fits into the pushout square

$$\begin{array}{ccc} \Lambda_1^2 & \longrightarrow & A^0 \\ \downarrow & & \downarrow \\ \Delta_1^2 & \longrightarrow & A^1 \end{array}$$

in sSet^+ , where the composite $\Delta_1^2 \rightarrow A^1 \hookrightarrow B$ corresponds to the simplex $\phi = 211$, and the face $\phi\partial_1$ corresponding to the missing face in the horn is 111. One can check that every nondegenerate face in $B \setminus A$ appears exactly once in Table 1, and moreover it is marked if and only if it appears either in the “interior” column or in the sixth or seventh row. It is also straightforward to verify using Proposition 5.6 that, for each $1 \leq s \leq 8$, the marked simplicial set A^{s-1} indeed contains enough (marked) simplices to support a map from the domain in the “pushout of” column. By reflecting everything to PreComp , we can deduce that T sends the box inclusion $\square_{2,0}^3 \hookrightarrow \square_{2,0}^3$ to a trivial cofibration. The case $\square_{2,1}^3 \hookrightarrow \square_{2,1}^3$ is dual, and the other 3-dimensional boxes can be treated using Proposition 3.10.

It remains to prove that $T(\square_{k,\varepsilon}^{n\prime} \hookrightarrow \square_{k,\varepsilon}^{n\prime\prime})$ is a trivial cofibration for any n, k, ε . We show that this map is in fact invertible.

Consider the following commutative diagram in PreComp:

$$\begin{array}{ccccc} \Delta^{n-1} & \xrightarrow{\iota_{n-1}} & T(\square^{n-1}) & \xrightarrow{T(\partial_{k,\varepsilon})} & T(\square_{k,\varepsilon}^{n\prime}) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\Delta}^{n-1} & \xrightarrow{\iota_{n-1}} & T(\tilde{\square}^{n-1}) & \xrightarrow{T(\partial_{k,\varepsilon})} & T(\square_{k,\varepsilon}^{n\prime\prime}) \end{array}$$

The left square is a pushout by the definition of T , and the right square is a pushout because it is the image of a pushout square under T . Thus, to show that $T(\square_{k,\varepsilon}^{n\prime} \hookrightarrow \square_{k,\varepsilon}^{n\prime\prime})$ is invertible, it suffices to prove that the $(n-1)$ -simplex in $T(\square_{k,\varepsilon}^{n\prime})$ corresponding to the top row above is marked. Note that, by unwinding the above argument, one can express $T(\square_{k,\varepsilon}^{n\prime} \hookrightarrow \square_{k,\varepsilon}^{n\prime\prime})$ as a composite

$$T(\square_{k,\varepsilon}^{n\prime}) = X^0 \hookrightarrow X^1 \hookrightarrow \dots \hookrightarrow X^N = T(\square_{k,\varepsilon}^{n\prime\prime}),$$

where each map $X^{s-1} \hookrightarrow X^s$ is a pushout (in PreComp) of the precomplicial reflection of a complicial horn inclusion. We show by induction on s that all $(n-1)$ -simplices contained in X^s are marked in $T(\square_{k,\varepsilon}^{n\prime})$.

For the base case, write $\square_{k,\varepsilon}^{n\prime}$ as a pushout

$$\begin{array}{ccc} \coprod \square^m & \longrightarrow & \square^n \\ \downarrow & & \downarrow \\ \coprod \tilde{\square}^m & \longrightarrow & \square_{k,\varepsilon}^{n\prime} \end{array}$$

where the coproducts are taken over all marked faces of $\square_{k,\varepsilon}^{n\prime}$, which in particular include all faces $\partial_{\ell,\eta}$ of codimension 1 with $(\ell, \eta) \neq (k, \varepsilon)$. By applying T to this pushout square, we can deduce that any $(n-1)$ -simplex of the form

$$\Delta^{n-1} \xrightarrow{\iota_{n-1}} (\Delta^1)^{\otimes(n-1)} = T(\square^{n-1}) \xrightarrow{T(\partial_{\ell,\eta})} T(\square^n)$$

is marked in $T(\square_{k,\varepsilon}^{n\prime})$. By combining this observation with Proposition 5.6, one can deduce that any $(n-1)$ -simplex contained in X^0 is marked in $T(\square_{k,\varepsilon}^{n\prime})$. For the inductive step, suppose that all $(n-1)$ -simplices contained in X^{s-1} are marked in $T(\square_{k,\varepsilon}^{n\prime})$. Suppose further that X^s contains a nondegenerate $(n-1)$ -simplex ϕ that X^{s-1} does not contain (for otherwise we are done). Then $X^{s-1} \hookrightarrow X^s$ fits into either a pushout square of the form

$$\begin{array}{ccc} (\Delta_\ell^{n-1})^{\text{pre}} & \longrightarrow & X^{s-1} \\ \downarrow & & \downarrow \\ (\Delta_\ell^{n-1})^{\text{pre}} & \xrightarrow{\phi} & X^s \end{array}$$

or one of the form

$$\begin{array}{ccc} (\Delta_\ell^n)^{\text{pre}} & \longrightarrow & X^{s-1} \\ \downarrow & & \downarrow \\ (\Delta_\ell^n)^{\text{pre}} & \xrightarrow{\chi} & X^s \end{array}$$

with $\chi\partial_\ell = \phi$. In the former case, ϕ is marked in X^s and hence in $T(\square_{k,\varepsilon}^n)$ since the unique nondegenerate $(n-1)$ -simplex in Δ_ℓ^{n-1} is marked. In the latter case, the inductive hypothesis implies that χ extends to the marked simplicial set $\Delta_\ell^{n'}$. Since X^s is a precomplicial set, it follows that $\phi = \chi\partial_\ell$ is marked in X^s and hence in $T(\square_{k,\varepsilon}^n)$. This completes the proof. \square

The saturated and n -trivial versions can be proved analogously.

Theorem 7.2 *The adjunction $T \dashv U$ is a Quillen adjunction when cSet^+ and PreComp are respectively equipped with:*

- the saturated comical model structure and the saturated complicial model structure,
- the n -trivial comical model structure and the n -trivial complicial model structure for some $0 \leq n < \infty$, or
- the saturated n -trivial comical model structure and the saturated n -trivial complicial model structure for some $0 \leq n < \infty$.

Proof The proof is analogous to that of Theorem 7.1. For the n -trivial versions, observe that T sends the (cubical) m -marker to a pushout of the (simplicial) m -marker.

For the saturated versions, we only check that T sends the basic Rezk maps to trivial cofibrations. (That the higher Rezk maps are also sent to trivial cofibrations then follows from Theorems 1.37 and 6.5.) Note that T sends all four basic Rezk maps to the same map (up to isomorphism). This unique image, which we denote by $TL \hookrightarrow TL'$, may be visualized as

$$TL = \left\{ \begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array} \right\} \quad \text{and} \quad TL' = \left\{ \begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array} \right\}.$$

Let A (resp. A') be the regular subset of TL (resp. TL') consisting of the middle two nondegenerate 2-simplices so that they look like

$$A = \left\{ \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \nearrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \right\} \quad \text{and} \quad A' = \left\{ \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \nearrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \right\}.$$

Clearly $TL \hookrightarrow TL'$ is a pushout of its restriction $A \hookrightarrow A'$, so it suffices to show that the latter is a trivial cofibration.

Observe that A is isomorphic to the regular subset of Δ_{eq}^3 consisting of ∂_0 and ∂_3 . One can check that the inclusion $A \hookrightarrow \Delta_{\text{eq}}^3$ may be written as the composite of a pushout of $\Lambda_1^2 \hookrightarrow \Delta_1^2$ (attaching ∂_1) and a pushout of $\Lambda_2^3 \hookrightarrow \Delta_2^3$. Hence $A \hookrightarrow \Delta_{\text{eq}}^3$ is complicial. Similarly, $A' \hookrightarrow (\Delta^3)^\#$ is the composite of pushouts of two complicial horn inclusions and one elementary complicial marking extension (marking the 1-simplex $\{0, 3\}$), so it is complicial too. Since the square

$$\begin{array}{ccc} A & \hookrightarrow & A' \\ \downarrow & & \downarrow \\ \Delta_{\text{eq}}^3 & \hookrightarrow & (\Delta^3)^\# \end{array}$$

commutes, the desired conclusion now follows by the 2-out-of-3 property. \square

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
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