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Mitul Islam

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Rank-one Hilbert geometries

MITUL ISLAM

We develop a notion of rank-one properly convex domains (or Hilbert geometries) in real projective space. This is in the spirit of rank-one nonpositively curved Riemannian manifolds and CAT(0) spaces. We define rank-one isometries for Hilbert geometries and characterize them as being equivalent to contracting elements (in the sense of geometric group theory). We prove that if a discrete subgroup of automorphisms of a Hilbert geometry contains a rank-one isometry, then the subgroup is either virtually cyclic or acylindrically hyperbolic. This leads to several applications like infinite dimensionality of the space of quasimorphisms, counting results for conjugacy classes and genericity results for rank-one isometries.

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1 Introduction

A *properly convex domain* in $\mathbb{P}(\mathbb{R}^{d+1})$ is an open subset $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ such that $\overline{\Omega}$ is a bounded convex domain in an affine chart. Any such domain Ω carries a canonical distance function d_Ω, called the *Hilbert metric* on Ω, defined using projective cross-ratios; see Section 3. Then Ω equipped with its Hilbert metric constitutes a *Hilbert geometry*. A motivating example is given by the open projective disk $\Omega_2 := \{[x : y : 1] \in \mathbb{P}(\mathbb{R}^3) \mid x^2 + y^2 < 1\}$, a properly convex domain in $\mathbb{P}(\mathbb{R}^3)$. In fact, $(\Omega_2, \mathrm{d}_{\Omega_2})$ is the projective model of the 2-dimensional real hyperbolic space $\mathbb{H}^2$.

For a properly convex domain Ω, the group $\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}_{d+1}(\mathbb{R}) \mid g\Omega = \Omega\}$ acts properly and isometrically on $(\Omega, \mathrm{d}_\Omega)$. If $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup, then the quotient space Ω/Γ is "locally modeled" on $(\Omega, \mathrm{d}_\Omega)$. These are the main objects that we study in this paper. We make the following definition.

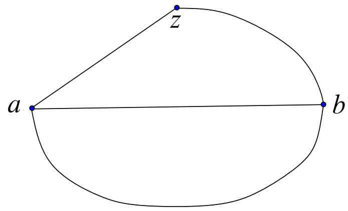
Definition 1.1 We say that Ω is a *Hilbert geometry* if $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a properly convex domain. Further, we say that a pair (Ω, Γ) is a *Hilbert geometry* if $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup. A Hilbert geometry (Ω, Γ) is *divisible* if $\Gamma \leq \mathrm{Aut}(\Omega)$ acts cocompactly on Ω.

Example Consider the projective model Ω_2 of $\mathbb{H}^2$. Here $\mathrm{Aut}(\Omega_2) = \mathrm{PO}(2,1)$. If $\Gamma \leq \mathrm{PO}(2,1)$ is any discrete subgroup, then (Ω_2, Γ) is a Hilbert geometry and (Ω_2, Γ) is divisible when Γ is a uniform lattice.

The boundary of a Hilbert geometry Ω, denoted by $\partial\Omega$, is the topological boundary of Ω as a subset of $\mathbb{P}(\mathbb{R}^{d+1})$. The regularity of $\partial\Omega$ strongly influences the geometric properties of $(\Omega, \mathrm{d}_\Omega)$. For instance, consider the class of *strictly convex* Hilbert geometries, ie Hilbert geometries Ω such that $\partial\Omega$ does not contain any nontrivial projective line segments. Benoist [8] showed that strictly convex divisible Hilbert geometries (Ω, Γ) have C^1 boundaries and behave like compact Riemannian manifolds of negative curvature (more precisely, Γ is Gromov hyperbolic and the geodesic flow is Anosov). This analogy between strictly convex Hilbert geometries and Riemannian negative curvature was subsequently studied by many authors with much success; see Benoist [12] or Marquis [43] for a survey.

On the other hand, the *nonstrictly convex* Hilbert geometries (ie when $\partial\Omega$ contains nontrivial projective line segments) have remained elusive. There are only a few examples (see Section 3.4) and, until recently, only a limited number of results. Taking a cue from the strictly convex case, one hopes to liken nonstrictly convex Hilbert geometries to Riemannian nonpositive curvature, or more generally, CAT(0) spaces. This will be our guiding principle in this paper. But we remark that the similarity with CAT(0) geometry is superficial. In fact, an old theorem of Kelly and Straus [41] states that Ω is CAT(0) if and only if Ω is the projective model of the real hyperbolic space. Thus, one needs to use very different tools and techniques for working with Hilbert geometries as compared to CAT(0) spaces.

Our target in this paper is to classify Hilbert geometries into two broad classes: "rank one" and "higher rank". The motivation for this classification comes from the success of the rank rigidity theorem for



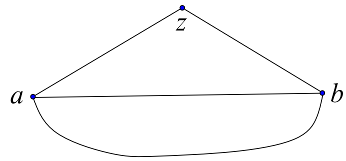


Figure 1: In the left figure, (a,b) is a *rank-one geodesic* while in the right figure, (a,b) is contained in a *half triangle in* Ω. However, Proposition 6.5 will show that neither of these can be a closed rank-one geodesic (ie a *rank-one axis*) in Ω/Γ.

nonpositively curved Riemannian manifolds; see Ballmann [3] and Burns and Spatzier [22]. Roughly, this theorem states that there is a dichotomy for irreducible compact Riemannian manifolds of nonpositive curvature: either the manifold is "rank one", or it is a higher rank Riemannian locally symmetric space. Similar rank rigidity theorems have been proven in other "nonpositive curvature" settings (see Caprace and Sageev [23] and Ricks [47]) and conjectured for CAT(0) spaces. We remark that the usual definition of rank for Riemannian manifolds uses Jacobi fields and will not be useful for Hilbert geometries. This is because the geodesic flow on a generic nonstrictly convex Hilbert geometry is only C^0.

We introduce a notion of rank-one geodesics in $(\Omega, \mathrm{d}_\Omega)$ using projective geometry. Consider an open projective line segment $(a,b) \subset \Omega$ with $a, b \in \partial\Omega$. Then (a,b) is a bi-infinite geodesic for the Hilbert metric d_Ω. We will say that (a,b) is a *rank-one geodesic* provided it is not contained in a *half triangle in* Ω, ie either $(a,c) \subset \Omega$ or $(c,b) \subset \Omega$ for any $c \in \partial\Omega$; see Figure 1 and Definitions 6.1 and 6.2. The notion of a half triangle in Hilbert geometry is analogous to the notion of a half flat in CAT(0) geometry; see Ballmann [4, Section III.3]. Our above definition of a rank-one geodesic is motivated by an analogous characterization of rank-one geodesics in CAT(0) geometry. In a CAT(0) geodesic metric space, a rank-one geodesic does not bound a half flat.

We will say that an isometry $\gamma \in \mathrm{Aut}(\Omega)$ is a *rank-one isometry* if γ acts by a translation along a rank-one geodesic $\ell \subset \Omega$; see Definition 6.3. We remark that acting by a translation along a rank-one geodesic (ie having a *rank-one axis*) is much more special than simply translating along any projective geodesic (ie having an *axis*); see Remark 6.4. Our definition of rank-one isometry is again analogous to a characterization of rank-one isometries in CAT(0) geometry; see Ballmann [2] and Ballmann and Brin [5]. A rank-one isometry $\gamma \in \mathrm{Aut}(\Omega)$ has several properties reminiscent of hyperbolic isometries in $\mathrm{Isom}(\mathbb{H}^2)$: γ is biproximal, has exactly two fixed points $\gamma^{\pm}$ in $\overline{\Omega}$, has a unique axis $(\gamma^+, \gamma^-) \subset \Omega$, both fixed points are "visible" (ie $(\gamma^+, z) \cup (z, \gamma^-)$ for any $z \in \partial\Omega - \{\gamma^+, \gamma^-\}$), and γ has the so-called north–south dynamics on $\partial\Omega$; see Proposition 6.5 and Corollary 6.7. In the case where (Ω, Γ) is divisible, it is quite easy to detect a rank-one isometry: if $\gamma \in \mathrm{Aut}(\Omega)$ has an axis and is biproximal, then γ is a rank-one isometry; see Proposition 6.8.

We further the analogy between rank-one isometries in Hilbert geometry and CAT(0) geometry by proving that rank-one isometries (in our sense above) are contracting elements in the sense of Sisto [49]. Sisto introduced the notion of contracting elements to capture the essence of "negative curvature" in groups; see Section 9. He proved in [49, Proposition 3.14] that if Λ acts properly by isometries on a proper CAT(0) space X, then an element of Λ is contracting if and only if it is rank one (in the sense of CAT(0) geometry). Our first main result in the paper is an analogue of this result for Hilbert geometries. If Ω is a Hilbert geometry, let $\mathcal{PS}^\Omega := \{[x,y] \mid x, y \in \Omega\}$, where $[x,y]$ is a projective line segment joining x and y.

Theorem 1.2 (see Part III) *If Ω is a Hilbert geometry, then $\gamma \in \mathrm{Aut}(\Omega)$ is a contracting element for $(\Omega, \mathcal{PS}^\Omega)$ if and only if γ is a rank-one isometry.*

In the light of these analogies, one naturally expects that the presence of many rank-one isometries would induce interesting "negative curvature"-like behavior. To formalize this, we now introduce the notion of rank-one Hilbert geometries. An example to keep in mind is (Ω_2, Γ) where $\Omega_2 \subset \mathbb{P}(\mathbb{R}^3)$ is the projective model of $\mathbb{H}^2$ and $\Gamma \le \mathrm{PO}(2,1)$ is an infinite discrete subgroup.

Definition 1.3 A *rank-one Hilbert geometry* is a pair (Ω, Γ) where $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry and Γ is a discrete subgroup of $\mathrm{Aut}(\Omega)$ that contains a rank-one isometry.

Morally, if a rank-one group Γ as in Definition 1.3 is not virtually cyclic (ie does not contain a finite-index cyclic subgroup), then it contains many rank-one isometries and we expect the group Γ to appear quite "hyperbolic". But of course we cannot expect such a group Γ to always be Gromov hyperbolic — there are many examples to the contrary; see Section 3.4. The main result of this paper is to identify the notion of hyperbolicity that rank-one groups satisfy. We prove that a rank-one group is either virtually cyclic or an acylindrically hyperbolic group; see Theorem 1.4 below.

The notion of acylindrically hyperbolic groups, introduced by Osin in [45], is a generalization of the notion of nonelementary Gromov hyperbolic groups. Roughly speaking, a group is acylindrically hyperbolic if it admits a nonelementary action on a (possibly nonproper) Gromov hyperbolic metric space with all but finitely many elements acting "hyperbolically"; see Definition 12.1. This family includes many important classes of groups: mapping class groups of most finite-type surfaces, rank-one CAT(0) groups that are not virtually abelian, relatively hyperbolic groups that are not virtually cyclic and have proper peripheral subgroups, and outer automorphism groups of free groups on at least two generators; see [45, Appendix]. We prove the following.

Theorem 1.4 (see Section 12) *If (Ω, Γ) is a rank-one Hilbert geometry, then either Γ is virtually cyclic or Γ is an acylindrically hyperbolic group.*

The acylindrical hyperbolicity of rank-one Hilbert geometries (Ω, Γ) have several applications. We defer this discussion until the section Applications below. Instead we first mention some of the precursors to our above result. In all those previous results, the conclusion is that the group under consideration is either Gromov hyperbolic or relatively hyperbolic — both of which are acylindrically hyperbolic groups; see the previous paragraph. Benoist [8] showed that if (Ω, Γ) is divisible and $\partial\Omega$ is strictly convex, then

Γ is Gromov hyperbolic. If instead Ω/Γ is noncompact but has finite volume, then Cooper, Long and Tillmann [25, Theorem 0.15] showed that Γ is relatively hyperbolic (with respect to the cusp subgroups). More generally, if Ω/Γ is geometrically finite, then Crampon and Marquis [27, Theorem 1.8] proved that Γ is relatively hyperbolic. However, they require that $\partial\Omega$ is C^1, and not just strictly convex.

Outside the strictly convex setting, Islam and Zimmer [39] have recently shown that if Γ acts cocompactly on Ω, then Γ is relatively hyperbolic (with peripheral subgroups free abelian of rank at least two) if and only if the set of properly embedded simplices in Ω of dimension at least two (see Section 3.3) forms an "isolated family". The results in [39] hold in greater generality — whenever Γ acts convex cocompactly on Ω; see Definition A.1 and Section A.6 for further discussion. We can interpret the above Theorem 1.4 as a generalization of these aforementioned results in the general setup of (possibly nonstrictly convex) Hilbert geometries. Theorem 1.4 characterizes the existence of rank-one isometries in Γ as a key factor that underpins the presence of these various weak forms of hyperbolicity for the group Γ.

Zariski density and rank one

Before moving on to contrasting rank one against "higher rank" Hilbert geometries, we indulge in a short discussion about Hilbert geometries (Ω, Γ) where $\Gamma \leq \mathrm{Aut}(\Omega)$ is Zariski dense in $\mathrm{SL}_{d+1}(\mathbb{R})$, ie it is "large" in an algebraic sense. For such groups Γ, one can define a notion of proximal limit set $\Lambda_\Gamma^{G/Q} \subset \mathbb{P}(\mathbb{R}^{d+1})$ (see Definition 8.3 and Remark 8.4) that is independent of the properly convex domain Ω. In Section 8, we prove that if $x, y \in \Lambda_\Gamma^{G/Q} \cap \partial\Omega$ are such that $(x, y) \subset \Omega$ is a rank-one geodesic, then the set of rank-one isometries in Γ form a Zariski dense set in $\mathrm{SL}_{d+1}(\mathbb{R})$ and (x, y) can be approximated by the axes of rank-one isometries (Lemma 8.10). In particular, a Hilbert geometry (Ω, Γ) with Γ Zariski dense in $\mathrm{SL}_{d+1}(\mathbb{R})$ is rank one if and only if Ω contains a rank-one geodesic (x, y) with $x, y \in \Lambda_\Gamma^{G/Q} \cap \partial\Omega$; see the answer to Question 8.1.

Rank one versus higher rank

The class of rank-one Hilbert geometries is quite rich. Besides the strictly convex Hilbert geometries, there are several examples of nonstrictly convex divisible Hilbert geometries which are rank one, eg the 3-manifold groups constructed in Benoist [10] from projective reflection groups. For more examples, see Section 3.4. In Appendix A, we discuss this further and also generalize the notion of rank one for convex cocompact actions.

On the other hand, there are several examples of Hilbert geometries that are not rank one, or in other words, have "higher rank". Projective simplices of dimension at least two and symmetric domains of real rank at least two are the key examples of "higher rank" Hilbert geometries; see Sections 3.3 and 3.4. The former are examples of reducible "higher rank" while the latter are examples of irreducible "higher rank" domains; see Definition 3.6. At this point, it is natural to ask whether these are all the "higher rank" divisible Hilbert geometries, akin to the case of Riemannian nonpositive curvature. Recently, A Zimmer [50] has proven that this is indeed the case.

We will now briefly discuss Zimmer's result for context. Zimmer calls Ω a *higher rank Hilbert geometry* if any $(p,q) \subset \Omega$ is contained in a properly embedded projective simplex S in Ω of dimension at least two; see also Section 3.4. Under some assumptions, he proves that his notion of higher rank is exactly complementary to our notion of rank one. We remark that Zimmer does not develop a theory of rank-one geometries. He focuses only on higher rank geometries and proves that an irreducible divisible Hilbert geometry (Ω,Γ) is higher rank (in his sense) if and only if it does not satisfy the notion of rank one (in the sense introduced in this paper).

Theorem 1.5 (part of [50, Theorem 1.4]) *Suppose* (Ω,Γ) *is a divisible Hilbert geometry and* Ω *is irreducible. Then the following are equivalent*:

(i) Ω *has higher rank* (*in the sense of Zimmer* [50, Definition 1.1]).

(ii) Ω *is a symmetric domain of real rank at least two.*

(iii) Ω *does not contain any rank-one geodesics* (*in the sense of this paper*, *Definition 6.2*).

(iv) Γ *does not contain any rank-one isometries* (*in the sense of this paper*, *Definition 6.3*).

Applications

We now return to our discussion about rank-one geometries. There is a sizeable literature exploring different properties of acylindrically hyperbolic groups. By virtue of Theorem 1.4, we can use these to establish several interesting results about rank-one Hilbert geometries. We remark that in the ensuing discussion, we usually do not require the additional assumption of divisibility.

1.1 Second bounded cohomology and quasimorphisms

A quasimorphism of a group G is a function $f: G \to \mathbb{R}$ such that $\sup_{g,h\in G} |f(gh)-f(g)-f(h)|$ is finite. We say that two quasimorphisms are equivalent if they differ by a bounded function or a homomorphism of G into $\mathbb{R}$. The set of all equivalence classes of quasimorphisms of G constitute $\widetilde{\mathrm{QH}}(G)$, which is a $\mathbb{R}$-vector space. More generally, if $\rho\colon G \to \mathcal{U}(E)$ is a unitary representation of G on a complete normed $\mathbb{R}$-vector space $(E,\|\cdot\|)$, then we can define $\widetilde{\mathrm{QC}}(G;\rho)$; see Section 13.

Bestvina and Fujiwara proved in [17] that if M is a compact nonpositively curved Riemannian manifold, then — under some mild assumptions — $\widetilde{\mathrm{QH}}(\pi_1(M))$ is infinite-dimensional if and only if M is a rank-one Riemannian manifold. We prove a similar cohomological characterization of rank-one Hilbert geometries.

Theorem 1.6 (see Section 13) *If* (Ω,Γ) *is a rank-one Hilbert geometry*, Γ *is torsion-free and* Γ *is not virtually cyclic*, *then*

(i) $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = \infty$, *and*

(ii) *if* $p \in (1,\infty)$ *and* $\rho^p_{\mathrm{reg}}\colon \Gamma \to \mathcal{U}(\ell^p(\Gamma))$ *is the regular representation*, *then* $\dim(\widetilde{\mathrm{QC}}(\Gamma;\rho^p_{\mathrm{reg}})) = \infty$.

We prove a more general Theorem 13.1. On the other hand, if $\Gamma \leq G$ is a lattice in a higher-rank simple Lie group G, then a result of Burger and Monod [21, Theorem 21] implies that $\widetilde{\mathrm{QH}}(\Gamma) = 0$. Then Theorem 1.6 and the rank rigidity result (Theorem 1.5) implies:

Corollary 1.7 (see Section 13) *If* (Ω, Γ) *is a divisible Hilbert geometry and* Ω *is irreducible, then* $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = \infty$ *if and only if* (Ω, Γ) *is a rank-one Hilbert geometry. Otherwise* $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = 0$.

1.2 Counting of conjugacy classes

For $g \in \mathrm{Aut}(\Omega)$, define the translation length $\tau_\Omega(g) := \inf_{x\in\Omega} \mathrm{d}_\Omega(x, gx)$ (see also Section 3.8) and the stable translation length

$$\tau_\Omega^{\mathrm{stable}}(g) := \lim_{n\to\infty} \frac{\mathrm{d}_\Omega(x, g^n x)}{n}.$$

Note that $\tau_\Omega^{\mathrm{stable}}(g)$ is independent of the basepoint $x \in \Omega$. Now suppose that (Ω, Γ) is a rank-one Hilbert geometry. For $g \in \Gamma$, let $[c_g]$ denote the conjugacy class of g in Γ. Both τ_Ω and $\tau_\Omega^{\mathrm{stable}}$ are well-defined on the set of conjugacy classes in Γ. Then for $t > 0$, define

$$\mathscr{C}(t) := \#\{[c_g] \mid g \in \Gamma,\ \tau_\Omega([c_g]) \leq t\} \quad \text{and} \quad \mathscr{C}^{\mathrm{stable}}(t) := \#\{[c_g] \mid g \in \Gamma, \tau_\Omega^{\mathrm{stable}}([c_g]) \leq t\}.$$

Here $\mathscr{C}(t)$ (resp. $\mathscr{C}^{\mathrm{stable}}(t)$) counts the number of conjugacy classes in Γ whose translation length (resp. stable translation length) is at most t. For divisible rank-one Hilbert geometries, we prove an asymptotic growth formula for $\mathscr{C}(t)$ and $\mathscr{C}^{\mathrm{stable}}(t)$. To state our result, we will require the critical exponent of Γ, which is defined as

$$\omega_\Gamma := \limsup_{n\to\infty} \frac{\log \#\{g \in \Gamma \mid \mathrm{d}_\Omega(x, gx) \leq n\}}{n}$$

for some (and hence any) basepoint $x \in \Omega$.

Theorem 1.8 (see Section 14) *Suppose* (Ω, Γ) *is a divisible rank-one Hilbert geometry and* Γ *is not virtually cyclic. Then there exists a constant* D' *such that if* $t \geq 1$,

$$\frac{1}{D'} \frac{\exp(t\omega_\Gamma)}{t} \ \leq\ \mathscr{C}(t) \ \leq\ D' \frac{\exp(t\omega_\Gamma)}{t}.$$

The function $\mathscr{C}^{\mathrm{stable}}(t)$ *also satisfies a similar growth formula as above.*

Remark 1.9 (i) An element $g \in \Gamma$ is called primitive if $g \neq h^n$ for any $h \in \Gamma$ and $|n| \geq 2$. If $\mathscr{C}_{\mathrm{Prim}}(t)$ is the number of conjugacy classes $[c_g]$ of primitive elements in Γ such that $\tau_\Omega([c_g]) \leq t$, then $\mathscr{C}_{\mathrm{Prim}}(t)$ satisfies a similar growth formula as $\mathscr{C}(t)$.

(ii) In [18, Proposition 1.5], Blayac establishes finer counting results for (a related notion of) rank-one Hilbert geometries using very different techniques.

Counting of conjugacy classes is often connected to counting of closed geodesics. However, this connection is subtle for Hilbert geometries, since there could be elements in Γ that do not act by a translation along any projective line in Ω (ie do not have an axis, see Example 5.11(B)).

1.3 Genericity and random walks

Suppose (Ω, Γ) is a rank-one Hilbert geometry, Γ is not virtually cyclic and Γ is finitely generated. If S is a finite symmetric generating set of Γ, let $W_n(S)$ be the set of words of length n in the elements of S. A simple random walk on Γ (with support S) is a sequence of Γ-valued random variables $\{X_n\}_{n \in \mathbb{N}}$ with laws μ_n defined by: if $n \geq 1$ and $g \in \Gamma$,

$$\mu_n(\{g\}) = \frac{\#\{w \in W_n(S) \mid w \text{ represents } g\}}{\# W_n(S)}.$$

Using results in [49], we show that rank-one isometries in Γ are exponentially generic from the viewpoint of simple random walks. This roughly means that the probability that a long word, written down by randomly choosing generators of Γ, is not a rank-one isometry is small. In particular, this probability decays exponentially in the length of the word.

Proposition 1.10 (see Section 15) *Suppose (Ω, Γ) is a rank-one Hilbert geometry, Γ is not virtually cyclic and Γ is finitely generated. Then the rank-one isometries in Γ are exponentially generic: if $\{X_n\}_{n \in \mathbb{N}}$ is a simple random walk on Γ, then there exists a constant $C \geq 1$ such that for all $n \geq 1$,*

$$\mathbb{P}[X_n \text{ is not a rank-one isometry}] \leq C e^{-n/C}.$$

1.4 More consequences of acylindrical hyperbolicity

Proposition 1.11 (see Section 15) *If (Ω, Γ) is a rank-one Hilbert geometry and Γ is not virtually cyclic, then:*

(i) *Γ is SQ-universal, ie every countable group embeds in a quotient of Γ.*

(ii) *If Γ is the Baumslag–Solitar group $\mathrm{BS}(m, n)$, then $m = n = 0$ and Γ is the free group on two generators.*

1.5 Morse geodesics and Morse boundary

Roughly speaking, the Morse geodesics [26] in a geodesic metric space identify the "hyperbolic directions". As a corollary to Theorem 1.2, we prove that the axis of a rank-one isometry is a Morse geodesic.

Proposition 1.12 (see Section 15) *If Ω is a Hilbert geometry and $\gamma \in \mathrm{Aut}(\Omega)$ is a rank-one isometry, then the axis ℓ_γ of γ is $\mathcal{K}$-Morse for some Morse gauge $\mathcal{K}\colon [1, \infty) \times [0, \infty) \to [0, \infty)$, ie if α is a (λ, ε)-quasigeodesic with endpoints on ℓ_γ, then $\alpha \subset \mathcal{N}_{\mathcal{K}(\lambda, \varepsilon)}(\ell_\gamma)$.*

In [26], Cordes introduced a notion of Morse boundary for proper geodesic metric spaces. In the cases of proper CAT(0) spaces and hyperbolic metric spaces, the Morse boundary coincides with the contracting boundary and the Gromov boundary respectively. Theorem 1.2 and Proposition 1.12 implies that the Morse boundary $\partial_M \Omega$ of a rank-one Hilbert geometry (Ω, Γ) is nonempty. This inspires the following question (that we will not answer in this paper).

Question 1.13 *Describe the Morse boundary $\partial_M \Omega$ of a rank-one Hilbert geometry (Ω, Γ).*

Recent developments

Since this paper first appeared on arXiv, there have been tremendous new developments in the field. We mention a few of them. Blayac [18] has developed the Patterson–Sullivan theory for rank-one Hilbert geometries. In [20], Blayac and Viaggi constructed examples of divisible rank-one Hilbert geometries (Ω, Γ) in every dimension $d \geq 3$ where Γ is not Gromov hyperbolic. In their examples, Γ is Zariski dense in $\mathrm{SL}_{d+1}(\mathbb{R})$ and relatively hyperbolic with peripheral subgroups isomorphic to $\mathbb{Z} \times H$, where H is possibly nonabelian [20, Theorem 1.3].

Outline of the paper

We discuss the preliminaries in Part I. Section 4 of Part I is of particular interest as it discusses the geometry of ω-limit sets of automorphisms in $\mathrm{Aut}(\Omega)$. Part II develops the notion of rank-one Hilbert geometries. We define rank-one isometries and study their geometric properties in Sections 6 and 7. Our main tools here are the lemmas proven in Section 5. In Section 8, we study rank-one groups which are Zariski dense.

In Part III, we prove Theorem 1.2 (in Sections 10 and 11) and Theorem 1.4 (in Section 12). Part IV discusses several applications of our results, like computing the dimension of the space of quasimorphisms, counting of conjugacy classes and genericity from the viewpoint of random walks. We discuss generalizations, examples and nonexamples of rank-one Hilbert geometries in Appendix A. In Appendix B, we discuss the equivalence of two notions of contracting elements.

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Part I Preliminaries

2 Notation

We set the following notation as standard for the rest of the paper.

(i) If $v \in \mathbb{R}^{d+1} \setminus \{0\}$, let $[v]$ or $\pi(v)$ denote its image in $\mathbb{P}(\mathbb{R}^{d+1})$. Conversely, if $u \in \mathbb{P}(\mathbb{R}^{d+1})$, we will use $\widetilde{u}$ to denote a lift of u (ie $\pi(\widetilde{u}) = u$).

(ii) If $A \in \mathrm{GL}_{d+1}(\mathbb{R})$, let $[A]$ denote its image in $\mathrm{PGL}_{d+1}(\mathbb{R})$, while $\widetilde{B} \in \mathrm{GL}_{d+1}(\mathbb{R})$ will denote a lift of $B \in \mathrm{PGL}_{d+1}(\mathbb{R})$.

(iii) If $W \le \mathbb{R}^{d+1}$ is a nonzero linear subspace, $\mathbb{P}(W)$ denotes its projectivization.

(iv) If $g \in \mathrm{GL}_{d+1}(\mathbb{R})$, the eigenvalues of g (over $\mathbb{C}$) are denoted by $\lambda_1(g), \dots, \lambda_{d+1}(g)$. We index them in the nonincreasing order of their absolute values, ie $|\lambda_1(g)| \ge \cdots \ge |\lambda_{d+1}(g)|$. Let $\lambda_{\max}(g) := |\lambda_1(g)|$ and $\lambda_{\min}(g) := |\lambda_{d+1}(g)|$.

(v) If $g \in \mathrm{PGL}_{d+1}(\mathbb{R})$ and $1 \le i \ne j \le d+1$, define $\left|\frac{\lambda_i}{\lambda_j}(g)\right| := \left|\frac{\lambda_i(\tilde{g})}{\lambda_j(\tilde{g})}\right|$ for some (hence any) lift $\tilde{g} \in \mathrm{GL}_{d+1}(\mathbb{R})$ of g.

3 Hilbert geometries

3.1 Properly convex domains

An open set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is called a *properly convex domain* if there exists a codimension one subspace $H \le \mathbb{R}^{d+1}$ such that $\overline{\Omega}$ is a bounded (Euclidean) convex domain in the affine chart $\mathbb{A} := \mathbb{P}(\mathbb{R}^{d+1}) \setminus \mathbb{P}(H)$.

Remark 3.1 If $L \le \mathbb{R}^{d+1}$ is a 2-dimensional linear subspace, then $\mathbb{P}(L) \not\subset \overline{\Omega}$ for any properly convex domain Ω. This elementary observation that a properly convex domain cannot contain an entire projective line will be used in many of the proofs in this paper.

For a nonempty set $X \subset \mathbb{R}^{d+1}$, let $\mathrm{Span}(X)$ denote the linear span of X. If $X' \subset \mathbb{P}(\mathbb{R}^{d+1})$ is nonempty, define

$$\mathrm{Span}(X') := \mathrm{Span}(\{\tilde{x} \in \mathbb{R}^{d+1} \mid \pi(\tilde{x}) \in X'\}).$$

Suppose Ω is a properly convex domain. If $x, y \in \overline{\Omega}$, let $[x, y]$ denote the unique closed connected subset of $\mathbb{P}(\mathrm{Span}\{x, y\}) \cap \overline{\Omega}$ that joins x and y. We will call $[x, y]$ *the projective line segment* between x and y (note that the notion of a projective line segment depends on Ω, but we assume that Ω will be clear from context and suppress it for brevity). We introduce the notation $(x, y) := [x, y] \setminus \{x, y\}$, $[x, y) := [x, y] \setminus \{y\}$ and $(x, y] := [x, y] \setminus \{x\}$. We will call (x, y) an *open projective line segment*. When we say that $[x, y]$ (or (x, y)) is *nontrivial*, we mean $x \ne y$.

We have a notion of convexity and convex hull in a properly convex domain Ω. A set $Y \subset \overline{\Omega}$ is *convex* if $[y_1, y_2] \subset Y$ for all $y_1, y_2 \in Y$. If $X \subset \overline{\Omega}$ is a nonempty set, then $\mathrm{ConvHull}_{\overline{\Omega}}(X)$ is the smallest closed convex subset of $\overline{\Omega}$ that contains X. We define

$$\mathrm{ConvHull}_{\Omega}(X) := \Omega \cap \mathrm{ConvHull}_{\overline{\Omega}}(X).$$

3.2 Hilbert metric

Suppose Ω is a properly convex domain and $\mathbb{A}$ is an affine chart that contains $\overline{\Omega}$ as a compact subset. We equip $\mathbb{A}$ with the Euclidean norm $|\cdot|$. If $x, y \in \Omega$, then there exist $a, b \in \partial\Omega$ such that

$$\mathbb{P}(\mathrm{Span}(\{x, y\})) \cap \overline{\Omega} = [a, b],$$

where the four points appear in the order a, x, y, b. The cross-ratio of these four points is given by

$$[a, x, y, b] := \frac{|b-x||y-a|}{|b-y||x-a|}.$$

The *Hilbert metric* on Ω is defined by

$$\mathrm{d}_\Omega(x, y) := \tfrac{1}{2}\log([a, x, y, b]).$$

Observation 3.2 *If $\Omega' \subset \Omega$ are properly convex domains and $x, y \in \Omega'$, then $\mathrm{d}_\Omega(x, y) \leq \mathrm{d}_{\Omega'}(x, y)$.*

If $x, y \in \Omega$, then $[x, y]$ is a geodesic in $(\Omega, \mathrm{d}_\Omega)$ joining x and y. In order to emphasize this fact, we will often refer to the projective line segment $[x, y]$ as the *projective geodesic segment* between x and y. If $(x, y) \subset \Omega$ with $x, y \in \partial\Omega$, then (x, y) is a bi-infinite geodesic in $(\Omega, \mathrm{d}_\Omega)$ and we will call it a (*bi-infinite*) *projective geodesic*.

The space $(\Omega, \mathrm{d}_\Omega)$ is a proper, complete and geodesic metric space and we will call Ω a *Hilbert geometry*, see Definition 1.1. The group $\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}_{d+1}(\mathbb{R}) \mid g\Omega = \Omega\}$ acts properly and isometrically on $(\Omega, \mathrm{d}_\Omega)$. However, the projective geodesic may not be the unique geodesic between points in $(\Omega, \mathrm{d}_\Omega)$. Consider, for example, the two-dimensional simplex $T_2 := \mathbb{P}(\mathbb{R}^+e_1 \oplus \mathbb{R}^+e_2 \oplus \mathbb{R}^+e_3)$ with its Hilbert metric d_{T_2}. Then generic points $x \neq y \in T_2$ have uncountably many geodesics (for the Hilbert metric d_{T_2}) joining them [36, Proposition 2].

Definition 3.3 For a Hilbert geometry Ω, the preimage $\pi^{-1}(\Omega) := \{v \in \mathbb{R}^{d+1} \mid \pi(v) \in \Omega\}$ has two connected components. The *cone above* (*or over*) Ω, denoted by $\widetilde{\Omega}$, is a connected component of $\pi^{-1}(\Omega)$.

Then $\pi^{-1}(\Omega) = \widetilde{\Omega} \sqcup (-\widetilde{\Omega})$. If $g \in \mathrm{Aut}(\Omega)$, then there is a lift $\widetilde{g} \in \mathrm{GL}_{d+1}(\mathbb{R})$ of g that preserves $\widetilde{\Omega}$, ie $\widetilde{g} \cdot \widetilde{\Omega} = \widetilde{\Omega}$. Indeed, if a lift $\widetilde{g}$ does not preserve $\widetilde{\Omega}$, then $\widetilde{g} \cdot \widetilde{\Omega} = -\widetilde{\Omega}$ and hence $-\widetilde{g}$ preserves $\widetilde{\Omega}$. We have the following elementary observation about such lifts of automorphisms.

Observation 3.4 *Suppose $\widetilde{\Omega}$ is a cone above Ω and $\widetilde{g} \in \mathrm{GL}_{d+1}(\mathbb{R})$ preserves $\widetilde{\Omega}$. If $\widetilde{a} \in \widetilde{\Omega}$ satisfies $\widetilde{g} \cdot \widetilde{a} = \lambda \cdot \widetilde{a}$, then $\lambda > 0$.*

Proof Clearly $\lambda \neq 0$. As $\widetilde{g}$ preserves $\widetilde{\Omega}$, $\widetilde{g} \cdot \widetilde{a} \in \widetilde{\Omega}$ which implies that $\lambda \cdot \widetilde{a} \in \widetilde{\Omega}$. Now if $\lambda < 0$, then $\widetilde{a} \in (-\widetilde{\Omega})$. But then $\widetilde{a} \in \widetilde{\Omega} \cap (-\widetilde{\Omega}) = \varnothing$, a contradiction. □

3.3 Projective simplices

The standard k-dimensional projective simplex in $\mathbb{P}(\mathbb{R}^{d+1})$ is

$$T_k := \{[x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^{d+1}) \mid x_1, \ldots, x_{k+1} > 0\}.$$

A k-dimensional projective simplex is a subset of $\mathbb{P}(\mathbb{R}^{d+1})$ of the form gT_k for some $g \in \mathrm{PGL}_{d+1}(\mathbb{R})$. If $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a properly convex domain and $S \subset \Omega$ is a projective simplex, then we say that S is a *properly embedded simplex in* Ω if and only if $\partial S \subset \partial\Omega$.

The Hilbert metric d_{T_k} on T_k is given by

$$\mathrm{d}_{T_k}([x_1:\cdots:x_{k+1}:0:\cdots:0],[y_1:\cdots:y_{k+1}:0:\cdots:0]) = \max_{1\le i,j\le k+1}\frac{1}{2}\left|\log\frac{x_i\,y_j}{x_j\,y_i}\right|.$$

Then (T_k,d_{T_k}) is quasi-isometric to the real Euclidean space of dimension k. For a more elaborate discussion, see [39, Section 5], [44] or [36]. The group $\mathrm{Aut}(T_d)$ is generated by the group of permutation matrices in $\mathrm{PGL}_{d+1}(\mathbb{R})$ and the group $\{[\mathrm{diag}(\lambda_1,\dots,\lambda_{d+1})]\in \mathrm{PGL}_{d+1}(\mathbb{R}) \mid \lambda_1,\dots,\lambda_{d+1}>0\}$.

Lemma 3.5 *Suppose $g := [\mathrm{diag}(\lambda_1,\dots,\lambda_{d+1})]\in \mathrm{Aut}(T_d)$ where $\lambda_i>0$ for all $i=1,\dots,d+1$. Let $\lambda_{\max} := \max_{1\le i\le d+1}\lambda_i$ and $\lambda_{\min} := \min_{1\le i\le d+1}\lambda_i$. Then $\mathrm{d}_{T_d}(x,gx) = \frac{1}{2}\log(\lambda_{\max}/\lambda_{\min})$ for any $x\in T_d$.*

Proof Fix $x=[x_1:\cdots:x_{d+1}]\in T_d$. Using the formula for d_{T_d},

$$\mathrm{d}_{T_d}(x,gx) = \max_{1\le i,j\le d+1}\frac{1}{2}\left|\log\frac{x_i\,\lambda_j\,x_j}{x_j\,\lambda_i\,x_i}\right| = \max_{1\le i,j\le d+1}\frac{1}{2}\left|\log\frac{\lambda_j}{\lambda_i}\right| = \frac{1}{2}\log\frac{\lambda_{\max}}{\lambda_{\min}}. \qquad \square$$

3.4 Examples of Hilbert geometries

The projective open ball $\Omega_d := \{[x_1:\cdots:x_d:1] \mid \sum_{i=1}^d x_i^2 < 1\}$ in $\mathbb{P}(\mathbb{R}^{d+1})$ is the simplest example of a divisible strictly convex Hilbert geometry. In fact Ω_d with its Hilbert metric is isometric to $\mathbb{H}^d$ and is well-known as the Beltrami–Klein model of real hyperbolic space. Moreover, $\mathrm{Aut}(\Omega_d)=\mathrm{PO}(d,1)$. There are several examples of divisible strictly convex Hilbert geometries that are not isometric to $\mathbb{H}^d$: in dimension 4, there is a construction due to Benoist [11, Proposition 3.1], while Kapovich [40] constructed examples in all dimensions above 4.

Among nonstrictly convex (divisible) Hilbert geometries, the simplest example is the standard d-simplex T_d; see Section 3.3. But this example is reducible, a term which we now define. Recall that a convex cone in $\mathbb{R}^{d+1}$ is a set $C\subset\mathbb{R}^{d+1}$ such that $r_1v_1+r_2v_2\in C$ whenever $v_1,v_2\in C$ and $r_1,r_2>0$.

Definition 3.6 A properly convex domain $\Omega\subset\mathbb{P}(\mathbb{R}^{d+1})$ is *reducible* if there exist convex cones $C_1\subset\mathbb{R}^{d_1}$ and $C_2\subset\mathbb{R}^{d_2}$ with $d_1,d_2\ge 1$ such that $\Omega=\mathbb{P}(C_1\oplus C_2)$. Otherwise, Ω is *irreducible*.

An irreducible nonstrictly convex (divisible) example is Pos_d with $d\ge 3$, the set of positive-definite real symmetric $d\times d$ matrices of unit trace. It is a properly convex domain in $\mathbb{R}^{d(d+1)/2}$ and is a projective model for the symmetric space of $\mathrm{SL}_d(\mathbb{R})$. The notion of symmetric domains generalize Pos_d. A symmetric domain Ω is a properly convex domain such that: for each $x\in\Omega$, there exists an order two isometry $s_x\in\mathrm{Aut}(\Omega)$ where x is the unique fixed point of s_x in Ω. Symmetric domains of real rank at least two are real projective analogues of higher rank Riemannian symmetric spaces of nonpositive curvature; see [12; 50] for details. As one might expect, symmetric domains are very special in the theory of properly convex domains. The rank rigidity theorem (Theorem 1.5) mentioned in the introduction is also a result in this spirit. Benoist proved the following result.

Theorem 3.7 [12, Theorem 5.2] *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is an irreducible properly convex domain that is not a symmetric domain. If $\Gamma \leq \mathrm{SL}_{d+1}(\mathbb{R})$ is a discrete subgroup that acts cocompactly on Ω, then Γ is Zariski dense in $\mathrm{SL}_{d+1}(\mathbb{R})$.*

Besides simplices and the symmetric domains of real rank at least two, only a few examples of divisible nonstrictly convex Hilbert geometries are known. These are low-dimensional examples; see for instance [10; 24], which rely on Coxeter group constructions, or [1], which uses "cusp-doubling" construction for certain three manifolds.

3.5 Closest-point projection for the Hilbert metric

Suppose Ω is a Hilbert geometry. If $r > 0$, we set

$$\mathcal{B}_\Omega(x,r) := \{y \in \Omega \mid \mathrm{d}_\Omega(x,y) < r\}.$$

Lemma 3.8 [25, Lemma 1.7] *$\mathcal{B}_\Omega(x,r)$ is a relatively compact and convex set.*

If $C \subset \Omega$ is a closed convex set, we define the closest-point projection on C as: if $x \in \Omega$,

$$\Pi_C(x) := C \cap \overline{\mathcal{B}_\Omega(x, \mathrm{d}_\Omega(x,C))}.$$

As the intersection of two closed convex sets is again a closed convex set, Lemma 3.8 immediately implies the following.

Observation 3.9 *Suppose Ω is a Hilbert geometry, C is a closed convex set and $x \in \Omega$. Then $\Pi_C(x)$ is a compact convex set.*

Corollary 3.10 *Suppose $\sigma : \mathbb{R} \to (\Omega, \mathrm{d}_\Omega)$ is a unit-speed parametrization of the bi-infinite projective geodesic $\sigma(\mathbb{R})$ with $\sigma(\pm\infty) \in \partial\Omega$. If $x \in \Omega$, then there exist $T_x^-, T_x^+ \in \mathbb{R}$ with $T_x^- \leq T_x^+$ such that*

$$\Pi_{\sigma(\mathbb{R})}(x) = [\sigma(T_x^-), \sigma(T_x^+)].$$

Proof Any compact convex subset of the bi-infinite projective geodesic $\sigma(\mathbb{R})$ is of the form $[\sigma(T), \sigma(T')]$ with $T \leq T'$. □

3.6 Faces of properly convex domains

Suppose Ω is a Hilbert geometry. We define the relation $\sim_\Omega$: if $p, q \in \overline{\Omega}$, then $p \sim_\Omega q$ if and only if there exists an open projective line segment in $\overline{\Omega}$ that contains both p and q. The relation $\sim_\Omega$ is an equivalence relation; see [27, Section 3.3]. The equivalence class of $p \in \overline{\Omega}$ is called the (open) face of p and is denoted by $F_\Omega(p)$.

Proposition 3.11 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry.*

(i) *If $x \in \partial\Omega$, then $F_\Omega(x) \subset \partial\Omega$.*

(ii) *$F_\Omega(x) = \Omega$ if and only if $x \in \Omega$.*

(iii) *If $x, y \in \partial\Omega$, then either $[x, y] \subset \partial\Omega$ or $(x, y) \subset \Omega$.*

(iv) *Suppose $[x, y] \subset \partial\Omega$, $a \in F_\Omega(x)$ and $b \in F_\Omega(y)$. Then $[a, b] \subset \partial\Omega$.*

Proof For part (i), note that if $y \in \Omega$, then $[y, x]$ cannot be extended beyond x in $\overline{\Omega}$. Thus $F_\Omega(x) \subset \overline{\Omega} - \Omega = \partial\Omega$. Part (ii) follows from part (i).

(iii) If $[x, y] \not\subset \partial\Omega$, choose any $z \in (x, y) \cap \Omega$. So $F_\Omega(z) = \Omega$. Then $(x, y) \subset F_\Omega(z) \subset \Omega$.

(iv) It suffices to prove this for $b = y$, ie to prove that $[a, y] \subset \partial\Omega$. Suppose, for a contradiction, that $(a, y) \subset \Omega$. Then $a \neq x$. Pick $a' \in F_\Omega(x)$ such that $x \in (a, a')$. As $(a, y) \subset \Omega$, pick $w \in (a, y)$. Then $(w, a') \subset \Omega$. Thus $\mathrm{ConvHull}_\Omega\{a', y, a\}$ is a nonempty set, as it contains $(w, a') \subset \Omega$. Hence, $\mathrm{ConvHull}_\Omega\{a', y, a\} \subset \Omega$. This implies that a, a' and y span a 2-simplex in $\overline{\Omega}$ and the interior of this 2-simplex is contained in Ω. As $x \in (a, a')$, (x, y) is contained in the interior of this 2-simplex. Thus $(x, y) \subset \Omega$, a contradiction. □

3.7 Distance estimates

Proposition 3.12 [38, Proposition 5.2] *Suppose Ω is a Hilbert geometry and $\{x_n\}$ and $\{y_n\}$ are sequences in Ω such that $x := \lim_{n\to\infty} x_n$ and $y := \lim_{n\to\infty} y_n$ exist in $\overline{\Omega}$. If*

$$\liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n) < \infty,$$

then $y \in F_\Omega(x)$ and

$$\mathrm{d}_{F_\Omega(x)}(y, x) \le \liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n).$$

Note that if $\liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n) = 0$, then the above proposition implies that $y = x$.

Next we need the notion of Hausdorff distance. If (X, d) is a metric space, then the Hausdorff distance between $A, B \subset X$ is defined by

$$d^{\mathrm{Haus}}(A, B) = \max\Bigl\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\Bigr\}.$$

Proposition 3.13 [38, Proposition 5.3], [25, Lemma 1.8] *Suppose Ω is a Hilbert geometry and $x_1, x_2, y_1, y_2 \in \overline{\Omega}$ satisfy $F_\Omega(x_1) = F_\Omega(x_2)$ and $F_\Omega(y_1) = F_\Omega(y_2)$. If $(x_1, y_1) \subset \Omega$, then*

$$\mathrm{d}_\Omega^{\mathrm{Haus}}((x_1, y_1), (x_2, y_2)) \le \max\{\mathrm{d}_{F_\Omega(x_1)}(x_1, x_2), \mathrm{d}_{F_\Omega(x_2)}(y_1, y_2)\}.$$

In particular, if $x_i, y_i \in \Omega$, then $\mathrm{d}_\Omega^{\mathrm{Haus}}([x_1, y_1], [x_2, y_2]) \le \max\{\mathrm{d}_\Omega(x_1, x_2), \mathrm{d}_\Omega(y_1, y_2)\}$.

3.8 Translation length

Suppose Ω is a Hilbert geometry and let $g \in \mathrm{Aut}(\Omega)$. Its *translation length* is defined by $\tau_\Omega(g) := \inf_{x\in\Omega} \mathrm{d}_\Omega(x, gx)$.

Remark 3.14 [38, Observation 7.2] Suppose $g \in \mathrm{Aut}(\Omega)$ and $W \leq \mathbb{R}^{d+1}$ is a g-invariant subspace of dimension ≥ 2 such that $\Omega \cap \mathbb{P}(W)$ is nonempty. Then $\tau_\Omega(g) \leq \tau_{\Omega\cap\mathbb{P}(W)}(g|_W)$.

Proposition 3.15 [25, Proposition 2.1] *If* $g \in \mathrm{Aut}(\Omega)$, *then* $\tau_\Omega(g) = \dfrac{1}{2}\log\left|\dfrac{\lambda_1}{\lambda_{d+1}}(g)\right|$.

This differs from the formula in [25] by a factor of $\frac{1}{2}$. This is because our definition of Hilbert metric has the factor of $\frac{1}{2}$. We further remark that if $\tilde{g} \in \mathrm{GL}_{d+1}(\mathbb{R})$ is any lift of g, then $\tau_\Omega(g) = \frac{1}{2}\log(\lambda_{\max}(\tilde{g})/\lambda_{\min}(\tilde{g}))$.

3.9 Minimal translation sets

Suppose Ω is a Hilbert geometry and $\Gamma \leq \mathrm{Aut}(\Omega)$. If $H \leq \Gamma$ is a subgroup, then the minimal translation set of H in Ω is

$$\mathrm{Min}_\Omega(H) := \bigcap_{h\in H} \{x \in \Omega \mid \mathrm{d}_\Omega(x, h\cdot x) = \tau_\Omega(h)\}.$$

Example 3.16 (i) If $g = [\mathrm{diag}(\lambda_1, \dots, \lambda_{d+1})] \in \mathrm{Aut}(T_d)$ with $\lambda_i > 0$ for all $i = 1, \dots, d+1$, then Lemma 3.5 implies that $T_d = \mathrm{Min}_{T_d}(\langle g\rangle)$.

(ii) The minimal translation set could be empty; eg if u is a parabolic isometry in $\mathrm{PO}(2,1)$, then $\tau_{\mathbb{H}^2}(u) = 0$ and the minimal translation set of $\langle u\rangle$ is empty.

We will need the following result connecting eigenspaces with minimal translation sets.

Lemma 3.17 *Suppose* $a, b, c \in \partial\Omega$ *are three distinct fixed points of* $g \in \mathrm{Aut}(\Omega)$. *Then*

$$\mathrm{ConvHull}_\Omega\{a, b, c\} \subset \mathrm{Min}_\Omega(\langle g\rangle).$$

Proof Without loss of generality, we can assume that $\mathrm{ConvHull}_\Omega\{a, b, c\}$ is nonempty since the result is obviously true otherwise. Let $T := \mathrm{ConvHull}_\Omega\{a, b, c\}$. Fix a cone $\tilde{\Omega}$ over Ω and a lift $\tilde{g}$ of g that preserves $\tilde{\Omega}$. Let $V = \mathrm{Span}\{a, b, c\}$ and $g_0 := \tilde{g}|_V$. Let $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{\Omega}$ be lifts of a, b and c respectively, and fix the basis $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ of V. In this basis, $g_0 = \mathrm{diag}(t_1, t_2, t_3)$. By Observation 3.4, we can assume that $t_1, t_2, t_3 > 0$. Since T is a 2-simplex in $\mathbb{P}(V)$, Lemma 3.5 implies that

$$d_T(x, g_0x) = \frac{1}{2}\log\frac{\max\{t_1, t_2, t_3\}}{\min\{t_1, t_2, t_3\}}$$

for any $x \in T$.

Suppose $\lambda_1(\widetilde{g}), \dots, \lambda_{d+1}(\widetilde{g})$ are the eigenvalues of $\widetilde{g}$, indexed in the nonincreasing order of their modulus. Then

$$|\lambda_1(\widetilde{g})| \geq \max\{t_1, t_2, t_3\} \geq \min\{t_1, t_2, t_3\} \geq |\lambda_{d+1}(\widetilde{g})|.$$

By Proposition 3.15,

$$\tau_\Omega(g) = \frac{1}{2} \log \left| \frac{\lambda_1(\widetilde{g})}{\lambda_{d+1}(\widetilde{g})} \right|.$$

Then $d_T(x, g_0 x) \leq \tau_\Omega(g)$ for any $x \in T$.

As $\Omega \cap \mathbb{P}(V) \supset T$, Observation 3.2 implies that $d_T(y', y) \geq d_{\Omega \cap \mathbb{P}(V)}(y', y)$ for any $y', y \in T$. Then $d_T(x, g_0 x) \geq \tau_{\Omega \cap \mathbb{P}(V)}(g_0)$ for any $x \in T$. Then Remark 3.14 implies that $d_T(x, g_0 x) \geq \tau_\Omega(g)$. Thus $d_T(x, g_0 x) = \tau_\Omega(g)$ for any $x \in T$. Hence $T \subset \mathrm{Min}_\Omega(g) = \mathrm{Min}_\Omega(\langle g \rangle)$. □

The next result concerns translation length and minimal translation sets of compact subgroups. This is essentially a restatement of [43, Lemma 2.1], which shows that every compact subgroup of $\mathrm{Aut}(\Omega)$ has a fixed point in Ω.

Lemma 3.18 [43, Lemma 2.1] *Suppose Ω is a Hilbert geometry and $H \leq \mathrm{Aut}(\Omega)$ is a compact subgroup. Then $\tau_\Omega(h) = 0$ for all $h \in H$ and $\mathrm{Min}_\Omega(H) = \{x \in \Omega \mid H \cdot x = x\} \neq \varnothing$.*

3.10 Centralizers

Suppose Ω is a Hilbert geometry and $\Gamma \leq \mathrm{Aut}(\Omega)$. If $H \leq \Gamma$ is a subgroup, the centralizer of H in Γ is

$$C_\Gamma(H) := \bigcap_{h \in H} \{g \in \Gamma \mid g h g^{-1} = h\}.$$

We will need to following result on cocompactness of centralizer subgroups.

Theorem 3.19 [38, Theorem 1.10] *Suppose Ω is a Hilbert geometry, $\mathscr{C} \subset \Omega$ is a closed convex subset, and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup that acts cocompactly on $\mathscr{C}$. If $A \leq \Gamma$ is an abelian subgroup, then $C_\Gamma(A)$ acts cocompactly on $\mathrm{ConvHull}_\Omega(\mathrm{Min}_{\mathscr{C}}(A))$, where*

$$\mathrm{Min}_{\mathscr{C}}(A) := \mathscr{C} \cap \mathrm{Min}_\Omega(A).$$

3.11 Proximality

We call $g \in \mathrm{GL}_{d+1}(\mathbb{R})$ proximal if g has a unique eigenvalue of maximum modulus and the multiplicity of this eigenvalue is 1, or equivalently if

$$|\lambda_1(g)| > |\lambda_2(g)|.$$

We will say that $g \in \mathrm{GL}_{d+1}(\mathbb{R})$ is biproximal if both g and g^{-1} are proximal, ie $|\lambda_1(g)| > |\lambda_2(g)|$ and $|\lambda_d(g)| > |\lambda_{d+1}(g)|$. Note that the notion of proximality is invariant under scaling a matrix by nonzero real numbers. Then $\gamma \in \mathrm{PGL}_{d+1}(\mathbb{R})$ is proximal (resp. biproximal) if some (hence any) lift of γ is proximal (resp. biproximal).

4 Dynamics of automorphisms

4.1 ω-limit sets of automorphisms

Let $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ be a Hilbert geometry and let $\gamma \in \mathrm{Aut}(\Omega)$ with

$$\tau_\Omega(\gamma) = \frac{1}{2}\log\left|\frac{\lambda_1}{\lambda_{d+1}}(\gamma)\right| > 0.$$

Recall that for any $X \subset \Omega$, $\overline{X}$ denotes the closure of X in $\overline{\Omega}$. We define the ω-limit set of γ as

$$\omega(\gamma, \Omega) := \bigcup_{x \in \Omega} (\overline{\{\gamma^n x \mid n \in \mathbb{N}\}} \cap \partial\Omega).$$

Thus, $\omega(\gamma, \Omega)$ is the union of all accumulation points in $\partial\Omega$ of all $\{\gamma^n \mid n \in \mathbb{N}\}$-orbits in Ω.

Example 4.1 Let $\Omega = T_2$ and $\gamma = [\mathrm{diag}(1, 2, 2)]$. Then for any $x = [x_1 : x_2 : x_3] \in T_2$, $\lim_{n\to\infty} \gamma^n x = [0 : x_2 : x_3]$. Thus

$$\omega(\gamma, T_2) = \{[0 : x_2 : x_3] \in \mathbb{P}(\mathbb{R}^3) \mid x_2, x_3 > 0\}.$$

Thus $\omega(\gamma, T_2)$ is the open projective line segment $(\pi(e_2), \pi(e_3)) \subset \partial T_2$. Also note that $\omega(\gamma, T_2) = E_\gamma^+ - \{\pi(e_2), \pi(e_3)\}$, where $E_\gamma^+ = \mathbb{P}(\mathrm{Span}\{e_2, e_3\}) \cap \partial\Omega$. Here $\mathbb{P}(\mathrm{Span}\{e_2, e_3\})$ is the projectivization of the direct sum of the eigenspace of γ that correspond to eigenvalues of maximum modulus. This observation that $\omega(\gamma, T_2) \subset E_\gamma^+$ holds more generally, as we will see in Proposition 4.9.

Remark 4.2 We now compare the notion of ω-limit set with that of the full orbital limit set introduced in [30]. Given an infinite discrete subgroup $H \leq \mathrm{Aut}(\Omega)$, the full orbital limit set of H is defined in [30] as

$$\mathscr{L}_\Omega^{\mathrm{orb}}(H) := \bigcup_{x \in \Omega} (\overline{H \cdot x} \cap \partial\Omega).$$

If $\gamma \in \mathrm{Aut}(\Omega)$ and $\tau_\Omega(\gamma) > 0$, then $\{\gamma^n \mid n \in \mathbb{N}\}$ is an infinite discrete subsemigroup of $\mathrm{Aut}(\Omega)$. Then $\omega(\gamma, \Omega)$ can be interpreted as the full orbital limit set of the subsemigroup $\{\gamma^n \mid n \in \mathbb{N}\}$.

4.2 Geometry of ω-limit sets of automorphisms

For the rest of this subsection, fix a Hilbert geometry $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ and $\gamma \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(\gamma) > 0$. Fix a lift $\tilde{\gamma}$ of γ. Our goal here is Proposition 4.9 — a description of $\omega(\gamma, \Omega)$ using the real Jordan decomposition of $\tilde{\gamma}$. We first give an intuitive idea. Suppose $c_1, \dots, c_q$ are all the eigenvalues (with repetitions) of $\tilde{\gamma}$ of modulus $\lambda_{\max}(\tilde{\gamma})$. If $c_1, \dots, c_q \in \mathbb{R}$, then $\omega(\gamma, \Omega)$ is contained in the projective subspace spanned by the eigenvectors corresponding to the eigenvalues of modulus $\lambda_{\max}(\tilde{\gamma})$. In the notation of Definition 4.3 below, this subspace is precisely $\mathbb{P}(E_{\tilde{\gamma}})$. Now suppose that among the c_i, there is a complex conjugate pair of eigenvalues $\mu, \overline{\mu} \in \mathbb{C} - \mathbb{R}$. Then, in the above subspace, we need to replace the eigenvectors for μ and $\overline{\mu}$ with a 2-dimensional γ-invariant projective real subspace on which γ acts by a rotation (E_μ in the notation of Definition 4.3). This is the key intuition behind Proposition 4.9. The references for this section are [42, II.1] and [25, Section 2]).

Now we start the formal discussion. First we introduce some notation. If $\mu \in \mathbb{R}$, define

$$J_\mu := \begin{pmatrix} \mu & 1 & 0 & \dots & 0 \\ 0 & \mu & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 1 \\ 0 & \dots & 0 & \dots & \mu \end{pmatrix}.$$

If $\mu = \alpha + i\beta \in \mathbb{C} - \mathbb{R}$, define

$$J_\mu := \begin{pmatrix} R(\mu) & \mathrm{Id}_2 & 0 & \dots & 0 \\ 0 & R(\mu) & \mathrm{Id}_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & \mathrm{Id}_2 \\ 0 & \dots & 0 & \dots & R(\mu) \end{pmatrix},$$

where Id_2 is the 2×2 identity matrix and

$$R(\mu) := \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Consider the real Jordan decomposition of $\widetilde{\gamma}$. This says that there is a $\widetilde{\gamma}$ invariant decomposition $\mathbb{R}^{d+1} = V_{\mu_1} \oplus \dots \oplus V_{\mu_n}$ into real vector subspaces and, with an appropriate choice of basis for V_{μ_j},

$$\widetilde{\gamma} = \begin{pmatrix} J_{\mu_1} & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & J_{\mu_n} \end{pmatrix}.$$

We remark that $V_\mu = V_{\bar{\mu}}$, as conjugate pairs of eigenvalues correspond to the same invariant subspace in the real Jordan decomposition. Without loss of generality, we can assume that $\mu_1, \dots, \mu_l \in \mathbb{R}$ and $\mu_{l+1}, \dots, \mu_n \in \mathbb{C} - \mathbb{R}$. Then $\mu_1, \dots, \mu_l, \mu_{l+1}, \bar{\mu}_{l+1}, \dots, \mu_n, \bar{\mu}_n$ are eigenvalues (possibly with repetitions) of $\widetilde{\gamma}$ over $\mathbb{C}$ and the multiplicity of μ_i is determined by the Jordan block J_{μ_i}.

Note that $\lambda_{\max}(\widetilde{\gamma})$ and $\lambda_{\min}(\widetilde{\gamma})$ are the maximum and the minimum of the set $\{|\mu_i| \mid 1 \le i \le n\}$. Now we focus on the eigenvalues of maximum modulus. By reindexing the μ_i, we now assume that $\mu_1, \dots, \mu_m$ are precisely those μ_i that satisfy $|\mu_i| = \lambda_{\max}(\widetilde{\gamma})$. We further assume that among them, $\mu_1, \dots, \mu_k \in \mathbb{R}$ and $\mu_{k+1}, \dots, \mu_m \in \mathbb{C} - \mathbb{R}$. Then $\mu_1, \dots, \mu_k, \mu_{k+1}, \bar{\mu}_{k+1}, \dots, \mu_m, \bar{\mu}_m$ are eigenvalues (possibly with repetitions) of $\widetilde{\gamma}$ of modulus $\lambda_{\max}(\widetilde{\gamma})$ and their multiplicities are determined by the Jordan block structure of $\widetilde{\gamma}$.

Definition 4.3 If $\mu_j \in \mathbb{R}$, let E_{μ_j} be the eigenvector for $\widetilde{\gamma}$ in V_{μ_j} with eigenvalue μ_j. If $\mu_j \in \mathbb{C} - \mathbb{R}$, let E_{μ_j} be the two-dimensional $\widetilde{\gamma}$-invariant subspace of V_{μ_j} such that $\widetilde{\gamma}|_{E_{\mu_j}}$ is conjugated to $R(\mu_j)$. Define

$$E_{\widetilde{\gamma}} := \bigoplus_{1 \le j \le m} E_{\mu_j} = \bigoplus_{|\mu_j| = \lambda_{\max}(\widetilde{\gamma})} E_{\mu_j}.$$

We also define

$$L_{\widetilde{\gamma}} := \bigoplus_{|\mu_j| = \lambda_{\max}(\widetilde{\gamma})} V_{\mu_j} \quad \text{and} \quad K_{\widetilde{\gamma}} := \bigoplus_{|\mu_j| < \lambda_{\max}(\widetilde{\gamma})} V_{\mu_j}.$$

Then $\widetilde{\gamma}|_{E_{\widetilde{\gamma}}}$ is conjugated in $\mathrm{GL}(E_{\widetilde{\gamma}})$ to

$$\lambda_{\max}(\widetilde{\gamma}) \cdot \begin{pmatrix} M_k & 0 & \dots & 0 \\ 0 & R\left(\frac{\mu_{k+1}}{\lambda_{\max}(\widetilde{\gamma})}\right) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & R\left(\frac{\mu_m}{\lambda_{\max}(\widetilde{\gamma})}\right) \end{pmatrix}, \tag{1}$$

where M_k is a $k \times k$ diagonal matrix with each diagonal entry ± 1. Thus $\langle \widetilde{\gamma}|_{E_{\widetilde{\gamma}}} \rangle$ is conjugated in $\mathrm{GL}(E_{\widetilde{\gamma}})$ to a cyclic subgroup of $\{\pm \mathrm{Id}\}^k \times O(E_{\mu_{k+1}}) \times \cdots \times O(E_{\mu_m}) < O(E_{\widetilde{\gamma}})$. Here, $O(W)$ denotes the group of orthogonal transformations preserving a linear subspace $W \subset \mathbb{R}^d$.

Claim 4.3.1 *There exists a sequence $\{m_k\}$ in $\mathbb{N}$ with $m_k \to \infty$ such that*

$$\lim_{k \to \infty} \frac{1}{\lambda_{\max}(\widetilde{\gamma})^{m_k}} (\widetilde{\gamma}|_{E_{\widetilde{\gamma}}})^{m_k} = \mathrm{Id}|_{E_{\widetilde{\gamma}}}.$$

Proof Let $k_\gamma := (1/\lambda_{\max}(\widetilde{\gamma}))\widetilde{\gamma}|_{E_{\widetilde{\gamma}}}$ and $\mathcal{K} := \overline{\langle k_\gamma \rangle}$. By equation (1), there exists $h \in \mathrm{GL}(E_{\widetilde{\gamma}})$ such that $h \cdot \mathcal{K} \cdot h^{-1}$ is a compact subgroup of $\{\pm \mathrm{Id}\}^k \times O(E_{\mu_{k+1}}) \times \cdots \times O(E_{\mu_m})$. Thus $\mathcal{K}' := h \cdot \mathcal{K} \cdot h^{-1}$ is a Lie subgroup of $O(E_{\widetilde{\gamma}})$. Hence either Id is an isolated point of $\mathcal{K}'$ or there exists a neighborhood U of Id in $O(E_{\widetilde{\gamma}})$ such that $U \cap \mathcal{K}' \subset \mathcal{K}'$.

In the latter case, it is obvious that there exists a monotonic sequence of integers $\{m_p\}$ such that $\lim_{p \to \infty} k_\gamma^{m_p} = \mathrm{Id}|_{E_{\widetilde{\gamma}}}$. Up to passing to a subsequence, we can assume that $m_p \to \infty$ or $m_p \to -\infty$. If $m_p \to \infty$, the claim is proved. Otherwise, choose the sequence $-m_p$.

In the former case (ie when Id is an isolated point of $\mathcal{K}'$), it implies $(k_\gamma)^s = \mathrm{Id}|_{E_{\widetilde{\gamma}}}$ for some $s \in \mathbb{N}$. Then $m_p := sp$ proves the claim. □

We will now discuss the dynamics of $(\widetilde{\gamma})^n$ on $\mathbb{P}(\mathbb{R}^{d+1})$. The results are quite standard and the proofs are fairly elementary computations using Jordan blocks; see [42, II.1] or [25, Lemma 2.5] for instance.

Observation 4.4 (i) *For a generic point $v \in V_\mu$, all accumulation points of*

$$\left\{ \frac{1}{|\mu|^n} (\widetilde{\gamma}|_{V_\mu})^n v \;\middle|\; n \in \mathbb{N} \right\}$$

lie in E_μ.

(ii) *Let* $W = V_{\mu_1} \oplus V_{\mu_2}$ *and* $|\mu_1| > |\mu_2|$. *Then, for any* $w \in W - V_{\mu_2}$, *all accumulation points of* $\{(1/|\mu_1|^n)(\tilde{\gamma}|_W)^n w \mid n \in \mathbb{N}\}$ *lie in* E_{μ_1}.

(iii) *Let* $W' = V_\mu \oplus V_{\mu'}$ *with* $|\mu| = |\mu'|$. *Then, for a generic point* $w' \in W'$, *all accumulation points of* $\{(1/|\mu|^n)(\tilde{\gamma}|_W)^n w' \mid n \in \mathbb{N}\}$ *lie in* E_μ *if* $\dim V_\mu > \dim V_{\mu'}$. *If* $\dim V_\mu = \dim V_{\mu'}$, *then the accumulation points lie in* $E_\mu \oplus E_{\mu'}$.

Recall the notation from Definition 4.3. Then the above observations imply the following result.

Fact 4.5 *If* $w \in \mathbb{P}(\mathbb{R}^{d+1}) \setminus \mathbb{P}(K_{\tilde{\gamma}})$, *then the accumulation points of* $\{\gamma^n w \mid n > 0\}$ *lie in* $\mathbb{P}(E_{\tilde{\gamma}})$. *In particular, if* $w' \in \mathbb{P}(L_{\tilde{\gamma}})$, *then all accumulation points of* $\{\gamma^n w' \mid n > 0\}$ *also lie in* $\mathbb{P}(E_{\tilde{\gamma}})$.

Remark 4.6 In fact, a finer conclusion is possible in Fact 4.5. Following [25], call a real Jordan subspace V_{μ_i} *most powerful* if $|\mu_i| = \lambda_{\max}(\tilde{\gamma})$ and $\dim(V_{\mu_i}) = \max\{\dim(V_{\mu_j}) \mid |\mu_j| = \lambda_{\max}(\tilde{\gamma})\}$. Let $F_{\tilde{\gamma}}$ be the direct sum of the E_{μ_j} that correspond to the most powerful Jordan subspaces V_{μ_j}. Then, $F_{\tilde{\gamma}} \subset E_{\tilde{\gamma}}$. For any $w \in \mathbb{P}(\mathbb{R}^{d+1}) \setminus \mathbb{P}(K_{\tilde{\gamma}})$ as above, the accumulation points of $\{\gamma^n w \mid n > 0\}$ actually lie in $\mathbb{P}(F_{\tilde{\gamma}})$; see part (iii) of Observation 4.4 above or [25, Proposition 2.5]. We record this finer conclusion for completeness, but we will not need it in this paper.

Claim 4.6.1 $\Omega \cap \mathbb{P}(K_{\tilde{\gamma}}) = \varnothing$, $\quad \mathbb{P}(E_{\tilde{\gamma}}) \cap \overline{\Omega} \subset \partial\Omega$ *and* $\omega(\gamma, \Omega) \subset \mathbb{P}(E_{\tilde{\gamma}}) \cap \partial\Omega$.

Proof We first note that $\Omega \cap \mathbb{P}(K_{\tilde{\gamma}}) = \varnothing$. Otherwise, Remark 3.14 implies that

$$\tau_{\Omega\cap\mathbb{P}(K_{\tilde{\gamma}})}(\gamma) = \log\left(\frac{\lambda_{\max}(\tilde{\gamma}|_{K_{\tilde{\gamma}}})}{\lambda_{\min}(\tilde{\gamma}|_{K_{\tilde{\gamma}}})}\right) < \log\left(\frac{\lambda_{\max}(\tilde{\gamma})}{\lambda_{\min}(\tilde{\gamma})}\right) = \tau_\Omega(\gamma),$$

a contradiction. Suppose, if possible, that $\mathbb{P}(E_{\tilde{\gamma}}) \cap \Omega$ is nonempty. Then $\tau_\Omega(\gamma) \le \tau_{\mathbb{P}(E_{\tilde{\gamma}})\cap\Omega}(\gamma|_{E_{\tilde{\gamma}}}) = 0$ by Remark 3.14, a contradiction. Finally, $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1}) \setminus \mathbb{P}(K_{\tilde{\gamma}})$ since $\Omega \cap \mathbb{P}(K_{\tilde{\gamma}}) = \varnothing$. Then Fact 4.5 implies that $\omega(\gamma, \Omega) \subset \mathbb{P}(E_{\tilde{\gamma}})$. Moreover, $\omega(\gamma, \Omega) \subset \partial\Omega$ by definition. □

Note that these subspaces in Definition 4.3 as well as the discussion above are independent of the lift $\tilde{\gamma}$ of γ that we fix. Thus we introduce the following definitions.

Definition 4.7 If $\gamma \in \mathrm{Aut}(\Omega)$, fix some (hence any) lift $\tilde{\gamma} \in \mathrm{GL}_{d+1}(\mathbb{R})$ of γ that preserves the cone $\tilde{\Omega}$ above Ω, and define

$$E_\gamma^+ := \mathbb{P}(E_{\tilde{\gamma}}), \quad L_\gamma^+ := \mathbb{P}(L_{\tilde{\gamma}}) \quad \text{and} \quad K_\gamma^+ := \mathbb{P}(K_{\tilde{\gamma}}),$$

where the subspaces $E_{\tilde{\gamma}}$, $L_{\tilde{\gamma}}$ and $K_{\tilde{\gamma}}$ are as in Definition 4.3. We also define

$$E_\gamma^- := E_{\gamma^{-1}}^+, \quad L_\gamma^- := L_{\gamma^{-1}}^+ \quad \text{and} \quad K_\gamma^- := K_{\gamma^{-1}}^+.$$

Remark 4.8 A linear subspace $V \subset \mathbb{R}^{d+1}$ is a real Jordan subspace for $\widetilde{\gamma}$ with eigenvalue μ if and only if V is a real Jordan subspace for $\widetilde{\gamma}^{-1}$ with eigenvalue μ^{-1}. Indeed, this follows because $\ker(\widetilde{\gamma}-\mu\,\mathrm{Id})^k = \ker(\widetilde{\gamma}^{-1}-\mu^{-1}\,\mathrm{Id})^k$ for any $k \in \mathbb{N}$. Thus, if the V_μ are the real Jordan subspaces for $\widetilde{\gamma}$ as above, then

$$E_{\widetilde{\gamma}^{-1}} = \bigoplus_{|\mu|=\lambda_{\min}(\widetilde{\gamma})} E_\mu, \quad L_{\widetilde{\gamma}^{-1}} = \bigoplus_{|\mu|=\lambda_{\min}(\widetilde{g})} V_\mu \quad \text{and} \quad K_{\widetilde{\gamma}^{-1}} = \bigoplus_{|\mu|>\lambda_{\min}(\widetilde{g})} V_\mu.$$

The key upshot of the discussion in this subsection is the following proposition.

Proposition 4.9 *If Ω is a Hilbert geometry, $\gamma \in \mathrm{Aut}(\Omega)$ and $\tau_\Omega(\gamma) > 0$, then*

(i) $\omega(\gamma, \Omega) \subset E_\gamma^+$,

(ii) *the action of γ on E_γ^+ is conjugated into the projective orthogonal group $\mathrm{PO}(E_\gamma^+)$, and*

(iii) *there exists a sequence of positive integers $\{m_k\}$ with $m_k \to \infty$ such that*

$$\lim_{k\to\infty} (\gamma|_{E_\gamma^+})^{m_k} = \mathrm{Id}|_{E_\gamma^+}.$$

Remark 4.10 A similar proposition is true if we replace γ by γ^{-1} and E_γ^+ by E_γ^-. Moreover, it is possible that $\omega(\gamma, \Omega) \subsetneq E_\gamma^+ \subset \partial\Omega$; see Example 4.1. We finally remark that a finer conclusion is possible here: $\omega(\gamma, \Omega) \subset \mathbb{P}(F_{\widetilde{\gamma}}) \subset E_\gamma^+$, where $F_{\widetilde{\gamma}}$ is as defined in Remark 4.6. We will not need this finer conclusion, but we record it for completeness.

4.3 ω-limit sets and faces in a properly convex domain

We continue our discussion about ω-limit sets from the previous subsection. Our goal now is to prove a result about the faces $F_\Omega(x)$ for $x \in E_\gamma^\pm$. This result will be used in Section 11. Before formulating the precise result, we give an illustrative example.

Example 4.11 Let $g = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1 > \lambda_2 > \lambda_3 > 0$, and let g preserve a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^3)$. Suppose $\pi(e_3) \in \partial\Omega$ and let $F := F_\Omega(\pi(e_3))$. We will show that $\pi(e_2) \notin F$. Suppose, on the contrary, that $\pi(e_2) \in F$. Then $I_t := [\pi(e_3 - te_2), \pi(e_3 + te_2)] \subset F$ for some $t > 0$. It is an elementary observation that I_t gets expanded by the action of g and $\overline{\bigcup_{k=1}^\infty g^k I_t} = \mathbb{P}(\mathrm{Span}\{e_2, e_3\})$. Thus $\mathbb{P}(\mathrm{Span}\{e_2, e_3\}) \subset \overline{F} \subset \overline{\Omega}$, which contradicts that Ω is a properly convex domain. Thus $\pi(e_2) \notin F$. By similar reasoning $\pi(e_1) \notin F$.

The takeaway from this example should be the following: since $\pi(e_3)$ is an eigenvector corresponding to an eigenvalue of modulus $\lambda_{\min}(g)$, the corresponding face $F_\Omega(\pi(e_3))$ cannot intersect any eigenspace whose eigenvalue has modulus greater than $\lambda_{\min}(g)$. The above philosophy works even if we replace eigenspaces by Jordan blocks, and is the key idea behind the next result.

We now state the precise version of the result. Recall the notation L_γ^- from the previous section (see Definition 4.7 and Remark 4.8): for any $\gamma \in \mathrm{Aut}(\Omega)$,

$$L_\gamma^- = \mathbb{P}\Big(\bigoplus_{|\mu|=\lambda_{\min}(\tilde{\gamma})} V_\mu\Big).$$

As in the previous subsection, $\tilde{\gamma}$ is some (hence any) lift of γ and V_μ is the real Jordan subspace of $\tilde{\gamma}$ for the eigenvalue μ. Thus L_γ^- is the direct sum of all the Jordan subspaces corresponding to the eigenvalues of $\tilde{\gamma}$ of minimum absolute value.

Lemma 4.12 *Suppose Ω is a Hilbert geometry and $\gamma \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(\gamma) > 0$. If $y \in E_\gamma^-$, then $F_\Omega(y) \subset L_\gamma^-$.*

Proof Suppose, for contradiction, that $v \in F_\Omega(y) - L_\gamma^-$. Fix a lift $\tilde{\gamma}$ of γ. As $y \in E_\gamma^-$, Proposition 4.9(iii) implies we can find a sequence $\{d_k\}$ of positive integers with $d_k \to \infty$ such that $(\gamma|_{E_\gamma^-})^{d_k} \to \mathrm{Id}|_{E_\gamma^-}$.

Up to passing to a subsequence of $\{d_k\}$, we can assume that $\gamma^{d_k} v \to v_\infty \in \overline{\Omega}$. As $v \notin L_\gamma^-$, Observation 4.4 part (ii) implies that there exists $c > \lambda_{\min}(\tilde{\gamma})$ such that the accumulation points of $\{(\tilde{\gamma}/c)^{d_k} v \mid k \geq 1\}$ do not lie in L_γ^-. Thus $v_\infty \notin L_\gamma^-$ and $\lim_{k\to\infty}(c/\lambda_{\min})^{d_k} = \infty$. We can then fix lifts $\tilde{y}$, $\tilde{v}$ and $\tilde{v}_\infty$ such that

$$\left(\frac{\tilde{\gamma}}{\lambda_{\min}(\tilde{\gamma})}\right)^{d_k} \tilde{y} \to \tilde{y} \quad \text{and} \quad \left(\frac{\tilde{\gamma}}{c}\right)^{d_k} \tilde{v} \to \tilde{v}_\infty.$$

We claim that

$$\mathbb{P}(\mathrm{Span}\{y, v_\infty\}) \subset \overline{\Omega}.$$

To prove this claim, it suffices to show that $\pi(\tilde{y} + t\tilde{v}_\infty) \in \overline{\Omega}$ for any real number $t \neq 0$. Fix $0 \neq t \in \mathbb{R}$. Define

$$s_k := t \cdot \frac{\lambda_{\min}^{d_k}}{c^{d_k} + t\lambda_{\min}^{d_k}}.$$

Then $s_k \to 0$ as $k \to \infty$. In fact, for k large enough, s_k belongs to $(0,1)$ or $(-1,0)$ accordingly as $t > 0$ or $t < 0$. Set

$$w_k := \pi((1-s_k)\tilde{y} + s_k\tilde{v}) = \pi\left(\tilde{y} + \frac{s_k}{1-s_k}\tilde{v}\right) = \pi\left(\tilde{y} + t\frac{\lambda_{\min}^{d_k}}{c^{d_k}}\tilde{v}\right),$$

since $s_k/(1-s_k) = t(\lambda_{\min}^{d_k}/c^{d_k})$. Then $w_k \in \mathbb{P}(\mathrm{Span}\{y, v\})$ and $\lim_{k\to\infty} w_k = y$. Thus, for k large enough, $w_k \in F_\Omega(y) \cap \mathbb{P}(\mathrm{Span}\{y, v\})$ because $v \in F_\Omega(y)$. Moreover, w_k lies on opposite sides of y in $F_\Omega(y) \cap \mathbb{P}(\mathrm{Span}\{y, v\})$ accordingly as $t > 0$ or $t < 0$. Thus the following computation will show that γ^{d_k} expands small neighborhoods of y in $\mathbb{P}(\mathrm{Span}\{y, v\}) \cap F_\Omega(y)$ to large subintervals of the projective line $\mathbb{P}(\mathrm{Span}\{y, v_\infty\})$. More precisely, we observe that

$$\begin{aligned}\lim_{k\to\infty} \gamma^{d_k} w_k &= \lim_{k\to\infty} \pi\left((1-s_k)\frac{\tilde{\gamma}^{d_k}}{\lambda_{\min}(\tilde{\gamma})^{d_k}}\tilde{y} + s_k\frac{\tilde{\gamma}^{d_k}}{\lambda_{\min}^{d_k}}\tilde{v}\right) = \lim_{k\to\infty} \pi\left(\frac{\tilde{\gamma}^{d_k}}{\lambda_{\min}(\tilde{\gamma})^{d_k}}\tilde{y} + \frac{s_k}{1-s_k}\frac{c^{d_k}}{\lambda_{\min}^{d_k}}\frac{\tilde{\gamma}^{d_k}}{c^{d_k}}\tilde{v}\right)\\ &= \lim_{k\to\infty} \pi\left(\frac{\tilde{\gamma}^{d_k}}{\lambda_{\min}(\tilde{\gamma})^{d_k}}\tilde{y} + t\frac{\tilde{\gamma}^{d_k}}{c^{d_k}}\tilde{v}\right) = \pi(\tilde{y} + t\tilde{v}_\infty).\end{aligned}$$

Thus $\pi(\tilde{y}+t\tilde{v}_\infty)\in\overline{\Omega}$, since $w_k\in F_\Omega(y)$ for k large enough. Since $t\neq 0$ is arbitrary, $\mathbb{P}(\mathrm{Span}\{y,v_\infty\})\subset\overline{\Omega}$. This proves the claim.

However, if $\overline{\Omega}$ contains the nontrivial projective line $\mathbb{P}(\mathrm{Span}\{y,v_\infty\})$, then Ω cannot be properly convex. This is a contradiction. □

Corollary 4.13 *Suppose $\Omega\subset\mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry and $\gamma\in\mathrm{Aut}(\Omega)$ with $\tau_\Omega(\gamma)>0$.*

(i) *If $y\in E^-_\gamma$, then $F_\Omega(y)\cap E^+_\gamma=\varnothing$.*

(ii) *If $y\in E^-_\gamma$, $z\in F_\Omega(y)$ and $\{i_k\}$ is a sequence in $\mathbb{Z}$ such that $z_\infty:=\lim_{k\to\infty}\gamma^{i_k}z$ exists, then $z_\infty\in E^-_\gamma$.*

Proof By Lemma 4.12, $F_\Omega(y)\subset L^-_\gamma$. Since $\tau_\Omega(\gamma)>0$, $L^-_\gamma\cap E^+_\gamma$ is empty by definition and this proves the first part. For the second part, note that $z\in F_\Omega(y)$ implies that $z\in L^-_\gamma$. On $\mathrm{Span}(L^-_\gamma)$, all eigenvalues of $\tilde{\gamma}$ have the same modulus $\lambda_{\min}(\tilde{\gamma})$. Then Observation 4.4(iii) implies that all accumulation points of $\{\gamma^n z\mid n\in\mathbb{N}\}$ lie in E^-_γ. By similar reasoning, all accumulation points of $\{\gamma^{-n}z\mid n\in\mathbb{N}\}$ also lie in E^-_γ. This proves the second part. □

Remark 4.14 Analogues of Lemma 4.12 and Corollary 4.13 hold for $F_\Omega(x)$ where $x\in E^+_\gamma$. One has to replace γ with γ^{-1} to obtain the analogous results.

Part II Rank-one Hilbert geometries

5 Axis of isometries

Definition 5.1 Suppose $\Omega\subset\mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry and $g\in\mathrm{Aut}(\Omega)$. An *axis* of g is a nontrivial projective line segment $\ell_g:=\mathbb{P}(V_g)\cap\Omega$ where $V_g\le\mathbb{R}^{d+1}$ is a two-dimensional g-invariant linear subspace.

We will show that if g has an axis and $\tau_\Omega(g)>0$, then g acts by a translation along its axis ℓ_g and the endpoints of ℓ_g correspond to eigenvectors with eigenvalues of maximum and minimum modulus respectively. Recall the notation $E^+_g, E^-_g\subset\mathbb{P}(\mathbb{R}^{d+1})$ from Definition 4.7.

Lemma 5.2 *Suppose that $\Omega\subset\mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry, and that $g\in\mathrm{Aut}(\Omega)$ with $\tau_\Omega(g)>0$ and g has an axis $\ell_g=\mathbb{P}(V_g)\cap\Omega$. If $\tilde{g}$ is a lift of g in $\mathrm{GL}_{d+1}(\mathbb{R})$, then*

(i) *$\tilde{g}|_{V_g}$ has two distinct eigenvalues $\lambda_+>\lambda_-$,*

(ii) *there exist $\tilde{g_+},\tilde{g_-}\in\mathbb{R}^{d+1}$ such that $\tilde{g}\cdot\tilde{g_\pm}=\lambda_\pm\cdot\tilde{g_\pm}$ and $\ell_g=(g_+,g_-)$, where $g_\pm=\pi(\tilde{g_\pm})$,*

(iii) *$|\lambda_+|=\lambda_{\max}(\tilde{g})$, $|\lambda_-|=\lambda_{\min}(\tilde{g})$ and $\tau_\Omega(g)=\log(|\lambda_+/\lambda_-|)>0$,*

(iv) *$g_+\in E^+_g$ and $g_-\in E^-_g$.*

Remark 5.3 If the lift $\widetilde{g}$ preserves the cone $\widetilde{\Omega}$ above Ω and $\widetilde{g_\pm} \in \widetilde{\Omega}$, then $\lambda_\pm > 0$; see Observation 3.4. Then $\lambda_+ = \lambda_{\max}(\widetilde{g})$ and $\lambda_- = \lambda_{\min}(\widetilde{g})$.

Proof Let $\ell_g = (a,b)$. Note that g preserves the set $\{a,b\} = \ell_g \cap \partial\Omega$. Fix any lift $\widetilde{g}$ of g. In the basis $\{\mathbb{R}\cdot a, \mathbb{R}\cdot b\}$ of V_g, there exist $c_1, c_2 \neq 0$ such that $\widetilde{g}|_{V_g}$ is either

$$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}.$$

In the latter case, both eigenvalues of $\widetilde{g}|_{V_g}$ have the same modulus and $\tau_{\Omega\cap\mathbb{P}(V_g)}(g|_{\mathbb{P}(V_g)}) = 0$, by Proposition 3.15. But then Remark 3.14 implies

$$0 \le \tau_\Omega(g) \le \tau_{\Omega\cap\mathbb{P}(V_g)}(g|_{\mathbb{P}(V_g)}) = 0,$$

a contradiction. Thus we are in the former case and g is diagonalizable with eigenvalues c_1 and c_2. Note that $c_1 \neq c_2$, since otherwise the same reasoning as above implies that $\tau_\Omega(g) = 0$. Then set $\lambda_+ := \max\{c_1, c_2\}$ and $\lambda_- := \min\{c_1, c_2\}$ and this proves part (i). For part (ii), let $\widetilde{g_\pm}$ be the eigenvectors of $\widetilde{g}$ in V_g with eigenvalues $\lambda_\pm$. Then note that by previous discussion, the set $\{\pi(\widetilde{g_+}), \pi(\widetilde{g_-})\}$ equals the set $\{a,b\}$. Thus $\ell_g = (\pi(\widetilde{g_+}), \pi(\widetilde{g_-}))$ and $\widetilde{g}|_{V_g} = \operatorname{diag}(\lambda_+, \lambda_-)$ in this basis.

For part (iii), first note that Remark 3.14 implies $\tau_\Omega(g) \le \tau_{\Omega\cap\mathbb{P}(V_g)}(g|_{\Omega\cap\mathbb{P}(V_g)})$. Proposition 3.15 then implies that $\log(\lambda_{\max}/\lambda_{\min})(\widetilde{g}) \le \log|\lambda_+/\lambda_-|$. Since $|\lambda_+| \le \lambda_{\max}(\widetilde{g})$ and $|\lambda_-| \ge \lambda_{\min}(\widetilde{g})$, we get $|\lambda_+/\lambda_-| \le (\lambda_{\max}/\lambda_{\min})(\widetilde{g})$. Thus

$$\left|\frac{\lambda_+}{\lambda_-}\right| = \frac{\lambda_{\max}}{\lambda_{\min}}(\widetilde{g}).$$

Then $|\lambda_+| = |\lambda_-|\cdot(\lambda_{\max}/\lambda_{\min})(\widetilde{g}) \ge \lambda_{\max}(\widetilde{g})$, implying $|\lambda_+| = \lambda_{\max}(\widetilde{g})$. Similarly, $|\lambda_-| = \lambda_{\min}(\widetilde{g})$. This proves part (iii). Then part (iv) follows by definition of E_g^+ and E_g^-. □

Corollary 5.4 *Suppose $g \in \operatorname{Aut}(\Omega)$ with $\tau_\Omega(g) > 0$ and g has an axis. If $\#(E_g^+) = \#(E_g^-) = 1$, then g has a unique axis given by $(E_g^+, E_g^-) \subset \Omega$. In particular, if g is biproximal (see Section 3.11) and has an axis, then the axis of g is unique.*

Proof Immediate from Lemma 5.2 parts (ii) and (iv), and the hypothesis that $\#(E_g^+) = \#(E_g^-) = 1$. For the "In particular" part, it suffices to note that if g is biproximal, then $\#(E_g^+) = \#(E_g^-) = 1$. □

Remark 5.5 Although biproximality of g implies $\#(E_g^+) = \#(E_g^-) = 1$, its converse fails in general. For example, consider

$$g = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

However, we will show in Lemma 5.17 that if g has an axis, then g is biproximal if and only if $\#(E_g^+) = \#(E_g^-) = 1$.

An isometry $g \in \mathrm{Aut}(\Omega)$ may not have an axis; see Example 5.11 part (B) below. Hence we introduce the notion of a pseudoaxis.

Definition 5.6 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry and $g \in \mathrm{Aut}(\Omega)$. A *pseudoaxis* of g is a nontrivial projective line segment $\sigma_g := \mathbb{P}(W_g) \cap \overline{\Omega}$, where $W_g \leq \mathbb{R}^{d+1}$ is a two-dimensional g-invariant linear subspace such that $\mathbb{P}(W_g) \cap \Omega = \varnothing$.

Observation 5.7 *If $\tau_\Omega(g) > 0$, then g has either an axis or a pseudoaxis.*

This observation is immediate from the following result of Benoist (also see [43, Proposition 2.2]). Here $\widetilde{\Omega}$ denotes a cone above Ω; see Definition 3.3 and the remark that follows.

Proposition 5.8 [9, Lemma 3.2] *Suppose Ω is a Hilbert geometry, $g \in \mathrm{Aut}(\Omega)$ and $\tau_\Omega(g) > 0$. Let $\widetilde{g}$ be a lift of g that preserves $\widetilde{\Omega}$. Then $\widetilde{g}$ has a real positive eigenvalue that equals $\lambda_{\max}(\widetilde{g})$ and there exists v such that $\widetilde{g} \cdot v = \lambda_{\max}(\widetilde{g}) \cdot v$ and $\pi(v) \in \overline{\Omega}$. A similar result holds if we replace $\lambda_{\max}(\widetilde{g})$ by $\lambda_{\min}(\widetilde{g})$.*

Remark 5.9 If $\widetilde{g}$ is an arbitrary element of $\mathrm{GL}_{d+1}(\mathbb{R})$, then $\lambda_{\max}(\widetilde{g})$ doesn't have to be an eigenvalue of $\widetilde{g}$. In fact, $\widetilde{g}$ may only have complex nonreal eigenvalues of modulus $\lambda_{\max}(\widetilde{g})$. So the key point of the above proposition is that preserving the cone $\widetilde{\Omega}$ above Ω imposes a strong restriction, namely that $\widetilde{g}$ has a positive real eigenvalue that equals $\lambda_{\max}(\widetilde{g})$.

However, the proposition does not imply anything about the number (or nature) of the other eigenvalues whose modulus is $\lambda_{\max}(\widetilde{g})$. In Example 5.11 part (A), the matrix g_2^{-1} has a repeated eigenvalue $1/\lambda_2$ of maximum modulus. Moreover, $\widetilde{g}$ can have complex eigenvalues of modulus $\lambda_{\max}(\widetilde{g})$; see Example 5.12.

We will now discuss a few examples to illustrate the notions introduced. An isometry may have a unique axis, infinitely many axes, or no axes at all. An isometry can have pseudoaxes without having an axis, and vice versa.

Example 5.10 (unique axis, no pseudoaxes) Consider the Hilbert geometry $\Omega := \{[x : y : 1] \mid x^2 + y^2 < 1\}$ in $\mathbb{P}(\mathbb{R}^3)$. It is the projective model of $\mathbb{H}^2$ and $\mathrm{Aut}(\Omega) = \mathrm{PO}(2,1)$. If $g \in \mathrm{SO}(2,1)$ has $\tau_\Omega([g]) > 0$ (ie g is a hyperbolic isometry in $\mathrm{Isom}(\mathbb{H}^2)$), then $[g]$ has a unique axis.

Example 5.11 Consider the two-dimensional simplex $T_2 := \{[x_1 : x_2 : x_3] \mid x_1, x_2, x_3 > 0\}$.

- (A) (uncountably many axes, several pseudoaxes) Let $g_2 := [\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)]$, where $\lambda_1 > \lambda_2 > 0$. For $0 < t < 1$, let $Q_t := ([e_1], [te_2 + (1-t)e_3])$. Then $\{Q_t\}_{t \in (0,1)}$ is an uncountable family of axes of g_2. There are three pseudoaxes: $[e_1, e_2]$, $[e_2, e_3]$ and $[e_1, e_3]$.
- (B) (several pseudoaxes, no axis) Let $g_1 := [\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)]$, where $\lambda_1 > \lambda_2 > \lambda_3 > 0$. The pseudoaxes of g_1 are $[e_1, e_2]$, $[e_2, e_3]$ and $[e_1, e_3]$. But g_1 does not have an axis.

Example 5.12 Let $\Omega_2 \subset \mathbb{P}(\mathbb{R}^3)$ be the projective disk model of $\mathbb{H}^2$ and fix a cone $\widetilde{\Omega}_2$ over Ω_2. Define $\Omega_* := \{[v : x] \in \mathbb{P}(\mathbb{R}^4) \mid v \in \widetilde{\Omega}_2, x > 0\}$, ie $\Omega_* \subset \mathbb{P}(\mathbb{R}^4)$ is the properly convex domain obtained by the join of Ω_2 with a point. Let

$$g := \begin{bmatrix} \lambda A & 0 \\ 0 & \dfrac{1}{\lambda^3} \end{bmatrix} \in \mathrm{Aut}(\Omega_*), \quad \text{where } \lambda > 1 \text{ and } A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{SO}(2,1).$$

Then g has three eigenvalues $\lambda, \lambda e^{\pm i\theta}$ of maximum modulus.

Note that g has an axis $\ell_g := (\pi(e_3), \pi(e_4)) \subset \Omega_*$. The action of g is by a translation along ℓ_g and a rotation (by angle θ) around ℓ_g. The axis ℓ_g is contained in properly embedded triangles in Ω_*.

5.1 Three key lemmas

We conclude this section by establishing three lemmas that will be used in the next section. The first one is a consequence of Lemma 5.2.

Lemma 5.13 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry, $g \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(g) > 0$, and a, b are fixed points of g with $a \in E_g^+$ and $b \in E_g^-$. If c is a fixed point of g such that $c \in \overline{\Omega} - (E_g^+ \cup E_g^-)$, then $[a, c] \cup [b, c] \subset \partial\Omega$.*

Proof First observe that $c \in \partial\Omega$. Otherwise, $\tau_\Omega(g) = d_\Omega(c, gc) = 0$, a contradiction. Suppose $(a, c) \subset \Omega$. Then (a, c) is an axis of g with $a \in E_g^+$. Lemma 5.2 then implies that $c \in E_g^-$, a contradiction. Thus $[a, c] \subset \partial\Omega$. Similar reasoning implies that $[c, b] \subset \partial\Omega$. □

The next lemma shows that if $g \in \mathrm{Aut}(\Omega)$, $\tau_\Omega(g) > 0$, g has an axis (a, b) and $\#(E_g^+) > 1$, then $F_\Omega(a)$ contains a nontrivial projective line segment in $\partial\Omega$. Before formulating the precise result and its proof, let us give an intuitive explanation of the main idea. Suppose $u \neq a \in E_g^+$ and let ξ be a point in $(a, b) \subset \Omega$. As Ω is open, we can find a point $\xi' \in \Omega \cap \mathbb{P}(\mathrm{Span}\{\xi, u\})$ that is distinct from ξ. Then, up to extracting a suitable subsequence of $\{g^n\}$, $g^{n_k}\xi \to a$ while $g^{n_k}\xi' \to a'$. As $\xi \neq \xi'$ and $u, a \in E_g^+$, one can check that $a' \neq a$; see the proof below. Then a property of the Hilbert metric (Proposition 3.12) implies that $a' \in F_\Omega(a)$. This is the gist of the proof below.

Lemma 5.14 *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry, $g \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(g) > 0$, and g has an axis (a, b), where $a \in E_g^+$ and $b \in E_g^-$. If $u \in E_g^+ \setminus \{a\}$, then there exist $x_u^- \neq x_u^+ \in \partial\Omega$ such that $a \in (x_u^-, x_u^+)$ and*

$$F_\Omega(a) \cap \mathbb{P}(\mathrm{Span}\{a, u\}) = (x_u^-, x_u^+).$$

Remark 5.15 Suppose we have the same setup as Lemma 5.14. By similar reasoning, if $v \in E_g^- \setminus \{b\}$, then $F_\Omega(b) \cap \mathbb{P}(\mathrm{Span}\{b, v\}) = (x_v^-, x_v^+)$, where $x_v^- \neq x_v^+$.

Proof Let us fix a cone $\widetilde{\Omega}$ over Ω. Then, we fix lifts $\widetilde{g}, \widetilde{a}, \widetilde{b}, \widetilde{u}$ of g, a, b, u such that $\widetilde{a}, \widetilde{b}, \widetilde{u} \in \widetilde{\Omega}$ and $\widetilde{g} \cdot \widetilde{\Omega} = \widetilde{\Omega}$. Note that $\widetilde{a}$ is an eigenvector of $\widetilde{g}$ corresponding to the eigenvalue $\lambda_{\max}(\widetilde{g})$ or $-\lambda_{\max}(\widetilde{g})$. Since $\widetilde{g}$ preserves $\widetilde{\Omega}$, Observation 3.4 implies that $\widetilde{g} \cdot \widetilde{a} = \lambda_{\max}(\widetilde{g}) \cdot \widetilde{a}$. Similarly, $\widetilde{g} \cdot \widetilde{b} = \lambda_{\min}(\widetilde{g}) \cdot \widetilde{b}$. Since $u \in E_g^+$, Proposition 4.9(iii) implies that there exists an unbounded sequence of positive integers $\{m_k\}$ such that

$$\left(\frac{\widetilde{g}}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \widetilde{u} = \widetilde{u}.$$

For $t \in \mathbb{R}$, let $\widetilde{p}_t := \frac{1}{2}(\widetilde{a} + \widetilde{b}) + t\widetilde{u}$ and $p_t := \pi(\widetilde{p}_t)$. Since (a, b) is an axis, $p_0 \in \Omega$. Then, as Ω is an open set, there exists $\varepsilon_0 > 0$ such that $\widetilde{p}_t \in \widetilde{\Omega}$ for all $t \in (-\varepsilon_0, \varepsilon_0)$. Fix $t \in (-\varepsilon_0, \varepsilon_0)$. Then

$$\begin{aligned}\lim_{k\to\infty} g^{m_k} p_t &= \lim_{k\to\infty} \pi\left(\left(\frac{\widetilde{g}}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \widetilde{p}_t\right) = \lim_{k\to\infty} \pi\left(\frac{\widetilde{a}}{2} + \left(\frac{\lambda_{\min}(\widetilde{g})}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \frac{\widetilde{b}}{2} + t\left(\frac{\widetilde{g}}{\lambda_{\max}(\widetilde{g})}\right)^{m_k} \widetilde{u}\right) \\ &= \pi(\widetilde{a} + 2t\widetilde{u}) \in \overline{\Omega}.\end{aligned}$$

Then, $\lim_{k\to\infty} g^{m_k} p_0 = a$, and $\lim_{k\to\infty} g^{m_k} p_t \neq a$ whenever $t \neq 0$. By Proposition 3.12,

$$\lim_{k\to\infty} g^{m_k} p_t \in F_\Omega(a)$$

because $\lim_{k\to\infty} \mathrm{d}_\Omega(g^{m_k} p_0, g^{m_k} p_t) = \mathrm{d}_\Omega(p_0, p_t)$. Thus there exist $x_u^+ \neq x_u^- \in \partial\Omega$ such that

$$F_\Omega(a) \cap \mathbb{P}(\mathrm{Span}\{a, u\})) = (x_u^-, x_u^+). \qquad \square$$

The next lemma shows that if $\gamma \in \mathrm{Aut}(\Omega)$ has an axis and $\#(E_\gamma^-) = 1$, then γ^{-1} is a proximal element in $\mathrm{PGL}_{d+1}(\mathbb{R})$; see Section 3.11. Before stating the precise version of the result, we give an illustrative example to explain the main idea behind it.

Example 5.16 Let $\mu > \lambda > 0$. Suppose that

$$g = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

preserves $\Omega \subset \mathbb{P}(\mathbb{R}^3)$ and $\pi(e_1), \pi(e_2) \in \partial\Omega$. Here g satisfies $\#(E_g^-) = 1$ but g^{-1} is not proximal. The main takeaway from this example will be that such a matrix g cannot have an axis in Ω, ie the only candidate for an axis, namely $(\pi(e_1), \pi(e_2))$, cannot lie in Ω.

To proceed, we will first explain that $\pi(e_3)$ cannot lie in $\overline{\Omega}$. For this, first note that

$$g^{\pm k}\pi(e_3) = \pi(k\lambda^{k-1}e_2 + \lambda^k e_3).$$

Hence $g^{\pm k}\pi(e_3) \to \pi(e_2)$ as $k \to \infty$, but they approach $\pi(e_2)$ from "opposite directions" in the projective line $\mathbb{P}(\mathrm{Span}\{e_2, e_3\})$. That is, $g^{\pm k}$ "wraps" $[g^{-1}\pi(e_3), g\pi(e_3)]$ around $\mathbb{P}(\mathrm{Span}\{e_2, e_3\})$. Then, $\pi(e_3) \in \overline{\Omega}$ will imply that $\mathbb{P}(\mathrm{Span}\{e_2, e_3\}) \subset \overline{\Omega}$, which is a contradiction as Ω is a properly convex domain. Thus $\pi(e_3) \notin \overline{\Omega}$.

Now we revisit our basic proposition: that $(\pi(e_1),\pi(e_2))$ cannot lie in Ω. Suppose this is false and $(\pi(e_1),\pi(e_2))\subset\Omega$. Since $g^k(\pi(e_1),\pi(e_3))\to(\pi(e_1),\pi(e_2))$, we can find $\pi(y_k)\in\Omega\cap(\pi(e_1),\pi(e_3))$ such that $g^k\pi(y_k)$ converges to the midpoint of $(\pi(e_1),\pi(e_2))$. Now unless $\pi(y_k)\to\pi(e_3)$, one can use the action of g to show that $g^k\pi(y_k)\to\pi(e_1)$, a contradiction; see the computation in equation (5). Thus $\pi(y_k)\to\pi(e_3)$ and hence $\pi(e_3)\in\overline{\Omega}$. This contradicts the previous paragraph.

The argument discussed above is the gist of the proof below. We now precisely formulate and prove our result.

Lemma 5.17 *Suppose $\Omega\subset\mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry, $\gamma\in\mathrm{Aut}(\Omega)$ with $\tau_\Omega(\gamma)>0$ and γ has an axis. If $\#(E_\gamma^-)=1$, then*

$$\left|\frac{\lambda_d}{\lambda_{d+1}}(\gamma)\right|>1.$$

Remark 5.18 Similar reasoning with γ replaced by γ^{-1} implies that if $\#(E_\gamma^+)=1$, then

$$\left|\frac{\lambda_1}{\lambda_2}(\gamma)\right|>1.$$

Proof Suppose the axis of γ is (a,b) with $a\in E_\gamma^+$ and $b\in E_\gamma^-$. Let us fix $\widetilde{\Omega}$, a cone above Ω. Fix lifts $\widetilde{\gamma}$, $\widetilde{a}$ and $\widetilde{b}$ where $\widetilde{a},\widetilde{b}\in\widetilde{\Omega}$ and $\widetilde{\gamma}\cdot\widetilde{\Omega}=\widetilde{\Omega}$. Set $\lambda_{\max}:=\lambda_{\max}(\widetilde{\gamma})$ and $\lambda_{\min}:=\lambda_{\min}(\widetilde{\gamma})$. Since $b\in E_\gamma^-$ is a fixed point and $\widetilde{b}\in\widetilde{\Omega}$, Observation 3.4 implies that $\widetilde{\gamma}\cdot\widetilde{b}=\lambda_{\min}\cdot\widetilde{b}$. Similarly, $\widetilde{\gamma}\cdot\widetilde{a}=\lambda_{\max}\cdot\widetilde{a}$.

Since $\#(E_\gamma^-)=1$, there is a one-dimensional eigenspace of $\widetilde{\gamma}$ (namely $\mathbb{R}\widetilde{b}$) and a single Jordan block $J_{\min}$ corresponding to eigenvalues of modulus $\lambda_{\min}$ (immediate from the definition, see Definition 4.7). Thus, in order to prove $|(\lambda_d/\lambda_{d+1})(\gamma)|>1$, it is enough to show that the Jordan block $J_{\min}$ has size 1. Suppose this is false. Then there exists $\widetilde{w}\in\mathbb{R}^{d+1}$ such that if $k\in\mathbb{Z}$, then

$$\widetilde{\gamma}^k\widetilde{w}=k\lambda_{\min}^{k-1}\widetilde{b}+\lambda_{\min}^k\widetilde{w}. \tag{2}$$

Setting $w:=\pi(\widetilde{w})$, $\lim_{k\to\infty}\gamma^k w=b$. Since $\gamma^k a=a$ for all k, $\lim_{k\to\infty}\gamma^k[a,w]=[a,b]$. Fix $p\in(a,b)\subset\Omega$. Then there exist $y_k\in(a,w)$ such that

$$\lim_{k\to\infty}\gamma^k y_k=p. \tag{3}$$

Since $p\in\Omega$ and Ω is open, $\gamma^k y_k\in\Omega$ for k large enough. Thus, up to truncating finitely many terms of the sequence $\{y_k\}$, we can assume that for $k\geq 1$,

$$y_k\in(a,w)\cap\Omega.$$

We can fix lifts $\widetilde{y}_k$ of y_k in $\widetilde{\Omega}$ such that

$$\widetilde{y}_k=c_k\widetilde{a}+d_k\widetilde{w}, \tag{4}$$

where $c_k,d_k\in[0,1]$. Thus, up to passing to a subsequence, we can assume that $c_\infty:=\lim_{k\to\infty}c_k$ and $d_\infty:=\lim_{k\to\infty}d_k$ exist. Then $\widetilde{y}_\infty:=\lim_{k\to\infty}\widetilde{y}_k$ exists and we set

$$y_\infty:=\pi(\widetilde{y}_\infty)=\pi(c_\infty\widetilde{a}+d_\infty\widetilde{w}).$$

We now claim that $y_\infty = \pi(\widetilde{w}) = w$. If this is not true, then $c_\infty \neq 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $c_k > c_\infty/2$ for all $k > k_0$, and $\lim_{k\to\infty}(d_k/c_k) = d_\infty/c_\infty$ exists in $\mathbb{R}$. Then using equation (3) followed by (4) and (2),

$$(5)\quad p = \lim_{k\to\infty} \gamma^k y_k = \lim_{k\to\infty} \pi\left(\frac{\widetilde{\gamma}^k \widetilde{y}_k}{c_k \lambda_{\max}^k}\right) = \lim_{k\to\infty} \pi\left(\widetilde{a} + \frac{d_k}{c_k}\left(\frac{k}{\lambda_{\max}}\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{k-1} \widetilde{b} + \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^k \widetilde{w}\right)\right) = \pi(\widetilde{a}) = a.$$

This is a contradiction since $p \in \Omega$ while $a \in \partial\Omega$. Thus $y_\infty = w$.

Since $y_k \in \Omega$ for $k \geq 1$, $w = y_\infty$ implies that $w \in \overline{\Omega}$. Then for all $k \in \mathbb{Z}$,

$$(6)\qquad [w, \gamma^k w] \subset \overline{\Omega}.$$

We now show that this implies $\mathbb{P}(\mathrm{Span}\{w, b\}) \subset \overline{\Omega}$. For $t > 0$, let

$$\mathcal{H}_t := \{\pi(\widetilde{w} + r\widetilde{b}) \mid -t \leq r \leq t\}.$$

Then $\overline{\bigcup_{t>0} \mathcal{H}_t} = \mathbb{P}(\mathrm{Span}\{b, w\})$. Now observe that if $k \in \mathbb{Z}$, then equation (2) implies that

$$\gamma^k w = \pi\left(\frac{\widetilde{\gamma}^k \widetilde{w}}{\lambda_{\min}^k}\right) = \pi\left(\widetilde{w} + \frac{k}{\lambda_{\min}}\widetilde{b}\right).$$

Then, for every $t > 0$, there exists $k_t \in \mathbb{N}$ such that $\mathcal{H}_t \subset [\gamma^{-(k_t-1)} w, w] \cup [w, \gamma^{k_t} w]$. Then, by equation (6), $\mathcal{H}_t \subset \overline{\Omega}$ for any $t > 0$. Thus $\mathbb{P}(\mathrm{Span}\{w, b\}) = \overline{\bigcup_{t>0} \mathcal{H}_t} \subset \overline{\Omega}$. This is a contradiction as Ω is a properly convex domain and hence $\overline{\Omega}$ cannot contain a projective line. □

6 Rank-one isometries: definition and properties

In this section, we introduce the notion of rank-one isometries for Hilbert geometries. Our definition is analogous to the definition of rank-one isometries for CAT(0) spaces [2; 5]. The notion of half triangles that we introduce is analogous to the notion of half flats used in the CAT(0) setting. Refer to Figure 1 for the next two definitions.

Definition 6.1 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry. Then the points $x, z, y \in \partial\Omega$ form a *half triangle in* Ω if

$$[x, z] \cup [y, z] \subset \partial\Omega \quad \text{and} \quad (x, y) \subset \Omega.$$

For $x, y \in \partial\Omega$, we will say that the projective geodesic $(x, y) \subset \Omega$ is *not contained in any half triangle in* Ω if for any $z \in \partial\Omega$, either $(x, z) \subset \Omega$ or $(z, y) \subset \Omega$.

Definition 6.2 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry and $a, b \in \partial\Omega$. The projective geodesic (a, b) is a *rank-one geodesic* provided $(a, b) \subset \Omega$ is not contained in any half triangle in Ω.

We now define rank-one isometries for Hilbert geometries. An isometry in $\mathrm{Aut}(\Omega)$ is rank one if it acts by a translation along a rank-one geodesic; see Figure 1.

Definition 6.3 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry.

(i) An element $\gamma \in \mathrm{Aut}(\Omega)$ is a *rank-one isometry* if
   (a) $\tau_\Omega(\gamma) = \log\left|\frac{\lambda_1}{\lambda_{d+1}}(\gamma)\right| > 0$,
   (b) γ has an axis,
   (c) none of the axes ℓ_γ of γ are contained in a half triangle in Ω.

(ii) A bi-infinite projective geodesic $\ell \subset \Omega$ is a *rank-one axis* if ℓ is the axis of a rank-one isometry in $\mathrm{Aut}(\Omega)$.

Remark 6.4 The prototypical example of a rank-one isometry is a hyperbolic isometry $[\mathrm{diag}(\lambda, 1, 1/\lambda)]$ with $\lambda > 1$ in $\mathrm{Isom}(\mathbb{H}^2)$; see Example 5.10. On the other hand, any element in $\mathrm{Aut}(T_d)$, where T_d is a d-dimensional simplex, is a nonexample. In fact, this nonexample highlights the necessity of the half triangle condition in the definition of a rank-one isometry, as we now explain. Recall Example 5.11 part (A). In that example, $g_2 = [\mathrm{diag}(\lambda_1, \lambda_2, \lambda_2)]$ has an axis Q_t for each $0 < t < 1$ and $\tau_{T_2}(g_2) > 0$. But all of these axes are contained in the projective triangle T_2 (and hence a half triangle). For another nonexample, see Example 5.12.

Recall Definition 1.3: a *rank-one Hilbert geometry* is a pair (Ω, Γ) where Ω is a Hilbert geometry and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup that contains a rank-one isometry. In Appendix A, we discuss examples and also a generalization for convex cocompact groups.

We will now establish some key geometric and dynamical properties of rank-one isometries. The essence here is that translating along a rank-one axis is much more special than translating along any axis, and Proposition 6.5 could be interpreted as strengthening Lemma 5.2 under the rank one assumption. Our results are reminiscent of Ballmann's results in rank-one Riemannian nonpositive curvature [2; 4]. Recall the notation $E_g^{\pm}$ from Definition 4.7 and the notion of proximality from Section 3.11.

Proposition 6.5 *Suppose Ω is a Hilbert geometry and $\gamma \in \mathrm{Aut}(\Omega)$ is a rank-one isometry with an axis $\ell_\gamma = (a, b)$, where $a \in E_\gamma^+$ and $b \in E_\gamma^-$. Then*

(i) *γ is biproximal,*
(ii) *ℓ_γ is the unique axis of γ in Ω,*
(iii) *the only fixed points of γ in $\overline{\Omega}$ are a and b,*
(iv) *if $z' \in \partial\Omega \setminus \{a, b\}$, then $(a, z') \cup (b, z') \subset \Omega$ (see Figure 1),*
(v) *if $z \in \partial\Omega \setminus \{a, b\}$, then neither (a, z) nor (b, z) is contained in a half triangle in Ω.*

Remark 6.6 If γ is a rank-one isometry, then the above proposition shows that $\#(E_\gamma^\pm) = 1$ and we will henceforth use the notation $\gamma^\pm := E_\gamma^\pm$. We will call γ^+ the *attracting fixed point of* γ and γ^- the *repelling fixed point of* γ. We choose this terminology because γ has *north–south dynamics* on $\partial\Omega$; see Corollary 6.7.

Proof Let us fix $\widetilde{\Omega}$, a cone over Ω. For the rest of this proof, fix lifts $\widetilde{\gamma}$, $\widetilde{a}$ and $\widetilde{b}$, where $\widetilde{a}, \widetilde{b} \in \widetilde{\Omega}$ and $\widetilde{\gamma} \cdot \widetilde{\Omega} = \widetilde{\Omega}$. Set $\lambda_{\max} := \lambda_{\max}(\widetilde{\gamma})$ and $\lambda_{\min} := \lambda_{\min}(\widetilde{\gamma})$. Since $a \in E_\gamma^+$ is a fixed point of γ, the lift $\widetilde{a}$ is an eigenvector of $\widetilde{\gamma}$ corresponding to the eigenvalue $\lambda_{\max}$ or $-\lambda_{\max}$. By Observation 3.4,

$$\widetilde{\gamma} \cdot \widetilde{a} = \lambda_{\max} \cdot \widetilde{a}.$$

Similarly, $\widetilde{\gamma} \cdot \widetilde{b} = \lambda_{\min} \cdot \widetilde{b}$.

(i) By the hypothesis, $\#(E_\gamma^\pm) \geq 1$. In order to prove that γ is biproximal, we first prove that:

Claim 6.6.1
$$\#(E_\gamma^+) = \#(E_\gamma^-) = 1.$$

Proof It suffices to prove the claim for E_γ^+, since the same arguments with γ replaced by γ^{-1} implies the result for E_γ^-. Now suppose the claim is false and there exists $u \in E_\gamma^+ \setminus \{a\}$. Then Lemma 5.14 implies that there exist $z^-, z^+ \in \partial\Omega$ such that $a \in (z^-, z^+)$ and

$$F_\Omega(a) \cap \mathbb{P}(\mathrm{Span}\{a, u\}) = (z^-, z^+).$$

Then, $\mathcal{I}_z := [z_-, z_+]$ is the maximal projective line segment in $\partial\Omega$ containing both z_- and z_+.

Since γ is a rank-one isometry, its axis (a, b) cannot be contained in a half triangle in Ω. But $[a, z_+] \subset \partial\Omega$, which implies that $(z_+, b) \subset \Omega$. Similarly, $(z_-, b) \subset \Omega$. Choose $x_\pm \in (z_\pm, b) \cap \Omega$. By Proposition 4.9 part (iii), there exists a sequence $\{m_k\}$ of positive integers with $m_k \to \infty$ such that

$$\lim_{k\to\infty} (\gamma|_{E_\gamma^+})^{m_k} = \mathrm{Id}_{E_\gamma^+}.$$

Since $z_+ \in \mathbb{P}(\mathrm{Span}\{a, u\})$, it follows that $z_+ \in E_\gamma^+$. Fix a lift $\widetilde{z_+} \in \widetilde{\Omega}$ of z_+. Then

$$\lim_{k\to\infty} \left(\frac{\widetilde{\gamma}}{\lambda_{\max}}\right)^{m_k} \widetilde{z_+} = \widetilde{z_+}.$$

On the other hand,

$$\lim_{k\to\infty} \left(\frac{\widetilde{\gamma}}{\lambda_{\max}}\right)^{m_k} \widetilde{b} = \lim_{k\to\infty} \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{m_k} \widetilde{b} = 0,$$

as $\lambda_{\max} > \lambda_{\min}$. Then, since $x_+ \in (z_+, b)$,

$$\lim_{k\to\infty} \gamma^{m_k} x_+ = z_+.$$

Similarly,

$$\lim_{k\to\infty} \gamma^{m_k} x_- = z_-.$$

Since $\lim_{k\to\infty} \mathrm{d}_\Omega(\gamma^{m_k} x_+, \gamma^{m_k} x_-) = \mathrm{d}_\Omega(x_+, x_-)$, Proposition 3.12 implies that $z_+ \in F_\Omega(z_-)$. Thus there is an open projective line segment in $\partial\Omega$ containing both z_+ and z_-. This contradicts the maximality of $\mathcal{I}_z$ and finishes the proof of Claim 6.6.1. □

By the above claim, $\#(E_\gamma^+) = \#(E_\gamma^-) = 1$, where $\tau_\Omega(\gamma) > 0$ and γ has an axis (a,b). Then Lemma 5.17 implies that γ is biproximal.

(ii) This follows from biproximality of γ and Corollary 5.4.

(iii) Suppose c is a fixed point of γ in $\partial\Omega$ that is distinct from both a and b. By part (i) of this proposition, γ is biproximal. Thus $c \notin E_\gamma^+ \cup E_{\gamma^-}$. Then, by Lemma 5.13, $[a,c] \subset \partial\Omega$ and $[b,c] \subset \partial\Omega$. Thus, the axis $\ell_\gamma = (a,b)$ of γ is contained in a half triangle, contradicting that γ is a rank-one isometry.

(iv) Let $v \in \partial\Omega \setminus \{a,b\}$. Then $v \notin \mathbb{P}(\mathrm{Span}\{a,b\})$ as $(a,b) \subset \Omega$. Suppose $[a,v] \subset \partial\Omega$. Since γ is biproximal, there exists a γ-invariant decomposition of $\mathbb{R}^{d+1}$ given by

$$\mathbb{R}^{d+1} = \mathbb{R}\tilde{a} \oplus \mathbb{R}\tilde{b} \oplus \tilde{E}.$$

Choose any lift $\tilde{v}$ of v in $\tilde{\Omega}$. Then $\tilde{v}$ decomposes as

$$\tilde{v} = c_1\tilde{a} + c_2\tilde{b} + \tilde{v_0},$$

where $c_1, c_2 \in \mathbb{R}$ and $\tilde{v_0} \neq 0$. If $c_2 \neq 0$, then $\lim_{n\to\infty} \gamma^{-n} v = b$, that is, $\lim_{n\to\infty} \gamma^{-n}[a,v] = [a,b]$. Since $[a,v] \subset \partial\Omega$ by assumption, $[a,b] \subset \partial\Omega$. This is a contradiction since $(a,b) \subset \Omega$. Thus, $c_2 = 0$.

Set $\lambda_{\tilde{E}} := \lambda_{\max}(\tilde{\gamma}|_{\tilde{E}})$. Since γ is biproximal, $\lambda_{\tilde{E}} < \lambda_{\max}$. Then, for every $n > 0$,

$$\left(\frac{\tilde{\gamma}}{\lambda_{\tilde{E}}}\right)^{-n} \tilde{v} = c_1 \left(\frac{\lambda_{\max}}{\lambda_{\tilde{E}}}\right)^{-n} \tilde{a} + \left(\frac{\tilde{\gamma}|_{\tilde{E}}}{\lambda_{\tilde{E}}}\right)^{-n} \tilde{v_0}.$$

Up to passing to a subsequence, we can assume that $v_\infty := \lim_{n\to\infty} \gamma^{-n} v$ exists. Note that $v_\infty \in \overline{\Omega} \cap \mathbb{P}(\tilde{E})$. But $\overline{\Omega} \cap \mathbb{P}(\tilde{E})$ is a γ-invariant nonempty convex compact subset of $\mathbb{R}^{d-1}$ and Brouwer's fixed point theorem implies that γ has a fixed point in $\overline{\Omega} \cap \mathbb{P}(\tilde{E})$. But $\overline{\Omega} \cap \mathbb{P}(\tilde{E}) \cap \{a,b\} = \varnothing$. This contradicts part (iii). Hence, $(a,v) \subset \Omega$. Similarly we can show that $(b,v) \subset \Omega$.

(v) This is a consequence of part (iv). □

Corollary 6.7 *Suppose $\gamma \in \mathrm{Aut}(\Omega)$ is a rank-one isometry. Then γ has* ***north–south dynamics*** *on $\partial\Omega$, that is,*

$$(\gamma|_{\overline{\Omega} - \{\gamma^\mp\}})^{\pm n} \to \gamma^\pm \quad \text{as } n \to \infty,$$

and the convergence is locally uniform on compact subsets of $\overline{\Omega} - \{\gamma^\mp\}$.

Proof The proof is very similar to part (iv) of Proposition 6.5. By the above proposition, γ is biproximal. Thus there exists a γ-invariant decomposition $\mathbb{R}^{d+1} = \mathbb{R}\gamma^+ \oplus H_\gamma \oplus \mathbb{R}\gamma^-$, where $\gamma^\pm = E_\gamma^\pm$. Moreover, γ^n converges to the constant map γ^+ locally uniformly on compact subsets of $\mathbb{P}(\mathbb{R}^{d+1}) - \mathbb{P}(H_\gamma \oplus \mathbb{R}\gamma^-)$ as $n \to \infty$.

We claim that $\mathbb{P}(H_\gamma \oplus \mathbb{R} \cdot \gamma^-) \cap \overline{\Omega} = \{\gamma^-\}$. If the claim is false, pick $v \in \mathbb{P}(H_\gamma \oplus \mathbb{R} \cdot \gamma^-) \cap \overline{\Omega}$ such that $v \neq \gamma^-$. Up to passing to a subsequence, we can assume that $v_\infty = \lim_{n\to\infty} \gamma^n v$ exists in $\overline{\Omega}$.

Since $v \in \mathbb{P}(H_\gamma \oplus \mathbb{R}\gamma^-) - \{\gamma^-\}$, similar reasoning as in part (iv) implies that $v_\infty \in \mathbb{P}(H_\gamma)$. Thus $v_\infty \in \overline{\Omega} \cap \mathbb{P}(H_\gamma)$. Again, as in part (iv), Brouwer's fixed point theorem will imply the existence of a fixed point of γ in $\overline{\Omega} \cap \mathbb{P}(H_\gamma)$ which is distinct from $\gamma^\pm$. This contradicts Proposition 6.5 part (iii). This finishes the proof of the claim.

By the claim and the first paragraph of the proof, γ^n converges to the constant map γ^+ locally uniformly on compact subsets of $\overline{\Omega} - \{\gamma^-\}$ as $n \to \infty$. The proof for γ^{-n} follows by similar reasoning. □

Now we prove a simpler characterization of rank-one isometries for cocompact actions.

Proposition 6.8 *Suppose Ω is a Hilbert geometry, $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup that acts cocompactly on Ω and $\gamma \in \Gamma$, where $\tau_\Omega(\gamma) > 0$. If $\gamma \in \Gamma$ has an axis, then the following are equivalent*:

(i) *γ is biproximal.*

(ii) *None of the axes of γ are contained in a half triangle in Ω.*

(iii) *γ is a rank-one isometry.*

Proof Note that (ii) $\iff$ (iii) is by definition (see Definition 6.3), and (iii) $\implies$ (i) is Proposition 6.5 part (i). We will prove (i) $\implies$ (ii) under the assumption that Ω/Γ is compact.

Let (a,b) be the axis of γ with $a \in E_\gamma^+$ and $b \in E_\gamma^-$. We first show that γ has no other fixed points in $\partial\Omega$ except a and b. If this is not true, let v be such a fixed point of γ. Since γ is biproximal, $v \notin E_\gamma^+ \cup E_\gamma^-$. Then Lemma 5.13 implies that

$$[a,v] \cup [v,b] \subset \partial\Omega. \tag{7}$$

Since $(a,b) \subset \Omega$, $\mathrm{ConvHull}_\Omega\{a,v,b\}$ is a nonempty set.

Let $A_\gamma := \langle\gamma\rangle$. Recall the notation $\mathrm{Min}_\Omega(A_\gamma) = \bigcap_{h \in A_\gamma}\{x \in \Omega \mid \mathrm{d}_\Omega(x, h\cdot x) = \tau_\Omega(h)\}$ from Section 3.9. Lemma 3.17 implies that

$$\mathrm{ConvHull}_\Omega\{a,v,b\} \subset \mathrm{Min}_\Omega(A_\gamma). \tag{8}$$

The group Γ acts cocompactly on Ω. Then, Theorem 3.19 implies that $C_\Gamma(A_\gamma)$ acts cocompactly on $\mathrm{ConvHull}_\Omega(\mathrm{Min}_\Omega(A_\gamma))$. Fix $p \in (a,b)$ and choose $v_n \in [p,v)$ such that $\lim_{n\to\infty} v_n = v$. By equation (8), $v_n \in \mathrm{Min}_\Omega(A_\gamma)$. Then there exists $h_n \in C_\Gamma(A_\gamma)$ such that $q := \lim_{n\to\infty} h_n v_n$ exists in Ω. Thus $\lim_{n\to\infty} \mathrm{d}_\Omega(h_n^{-1}q, v_n) = 0$. Then Proposition 3.12 implies that, up to passing to a subsequence,

$$\lim_{n\to\infty} h_n^{-1}q = \lim_{n\to\infty} v_n = v.$$

Pick a point $q' \in (a,b)$. Up to passing to a subsequence, $v' := \lim_{n\to\infty} h_n^{-1}q'$ exists in $\overline{\Omega}$. Since $\lim_{n\to\infty} \mathrm{d}_\Omega(h_n^{-1}q, h_n^{-1}q') = \mathrm{d}_\Omega(q,q')$, Proposition 3.12 implies that $v \in F_\Omega(v')$. Now we show that $v' \in \{a,b\}$. Since $h_n \in C_\Gamma(A_\gamma)$, $h_n(a,b)$ is an axis of γ. As γ is biproximal and has an axis, Corollary 5.4 implies that $h_n(a,b) = (a,b)$. Then, since $q' \in (a,b)$, we get $v' = \lim_{n\to\infty} h_n^{-1}q' \in \{a,b\}$. Hence

$$v \in F_\Omega(a) \cup F_\Omega(b).$$

If possible, let $v \in F_\Omega(a)$. By equation (7), $[a,v] \cup [v,b] \subset \partial\Omega$. Now, by Proposition 3.11 part (iv), $v \in F_\Omega(a)$ and $[v,b] \subset \partial\Omega$ implies that $[a,b] \subset \partial\Omega$. This is a contradiction as $(a,b) \subset \Omega$. Thus, $v \notin F_\Omega(a)$. So v must be in $F_\Omega(b)$. By a similar reasoning, we now observe that $v \notin F_\Omega(b)$. Thus we have a contradiction.

So we have shown that if γ has an axis (a,b) and is biproximal, then γ has no fixed points in $\partial\Omega$ other than a and b. Then the proof of part (iv) of Proposition 6.5 goes through verbatim. Thus $(a,z) \cup (z,b) \subset \Omega$ for all $z \in \partial\Omega \setminus \{a,b\}$, that is, the axis (a,b) is not contained in any half triangle in $\partial\Omega$. This finishes the proof. □

7 Rank-one axis and thin triangles

In this section, we prove that any projective geodesic triangle in Ω with one of its sides on a rank-one axis ℓ is $\mathfrak{D}_\ell$-thin for some constant $\mathfrak{D}_\ell$.

Proposition 7.1 *Suppose Ω is a Hilbert geometry. If $\ell \subset \Omega$ is a rank-one axis, then there exists a constant $\mathfrak{D}_\ell \geq 0$ such that if $\Delta(x,y,z) := [x,y] \cup [y,z] \cup [z,x]$ is a projective geodesic triangle in Ω with $[y,z] \subset \ell$, then $\Delta(x,y,z)$ is $\mathfrak{D}_\ell$-thin.*

Remark 7.2 The thinness constant $\mathfrak{D}_\ell$ in the above theorem depends only on the axis ℓ (and not on the rank-one isometry that has ℓ as its axis).

But first let us introduce some relevant definitions and technical results that we will need.

7.1 Thin triangles

Definition 7.3 Suppose (X,d) is a geodesic metric space.

(i) A geodesic triangle T with vertices y_1, y_2, y_3 is a union of geodesics $\sigma_1 \cup \sigma_2 \cup \sigma_3$ where σ_i is a geodesic joining y_i and y_{i+1}, where the indices $i \in \{1,2,3\}$ are counted modulo 3.

(ii) A geodesic triangle $T := \sigma_1 \cup \sigma_2 \cup \sigma_3$ is called D-thin for some $D \geq 0$ if

$$\sigma_i \subset \{x \in X \mid d(x, \sigma_{i-1} \cup \sigma_{i+1}) < D\},$$

where the indices $i \in \{1,2,3\}$ are counted modulo 3.

The following is an elementary observation about thin triangles that we use later in the paper.

Observation 7.4 *Suppose (X,d) is a geodesic metric space and $T := \sigma_1 \cup \sigma_2 \cup \sigma_3$ is a geodesic triangle with vertices y_1, y_2, y_3, and each σ_i is a continuous geodesic path joining y_i and y_{i+1} (the indices $i \in \{1,2,3\}$ are counted modulo 3). If T is D-thin, then there exist $x_i \in \sigma_i$ for $i = 1,2,3$ such that* $\max\{d(x_1,x_2), d(x_1,x_3)\} \leq D$.

Proof By slight abuse of notation, let $\sigma_1\colon [0,b] \to X$ denote the continuous parametrization of the geodesic path σ_1 for some $b \geq 0$. Without loss of generality, we assume that $\sigma_1(0) = y_1$. Since T is D-thin,

$$\sigma_1([0,b]) \subset \{x \in X \mid d(x, \sigma_2 \cup \sigma_3) < D\}. \tag{9}$$

Note that $d(\sigma_1(0), \sigma_3) = 0$ as $y_1 \in \sigma_1 \cap \sigma_3$. Let $E := \{t \in [0,b] \mid d(\sigma_1(t), \sigma_3) < D\}$. Then $0 \in E$ and $s_0 := \sup E$ exists. We can find a sequence $\{t_n\}$ in E such that $t_n \to s_0$. Then, by continuity of σ_1,

$$d(\sigma_1(s_0), \sigma_3) = \lim_{t_n \to s_0} d(\sigma_1(t_n), \sigma_3) \leq D.$$

Now note that $d(\sigma_1(s_0), \sigma_2) \leq D$. Indeed, if $t > s_0$, then $d(\sigma_1(t), \sigma_3) \geq D$ by definition of s_0. Then equation (9) implies that $d(\sigma_1(t), \sigma_2) < D$. By continuity of σ_1,

$$d(\sigma_1(s_0), \sigma_2) = \lim_{t \to s_0^+} d(\sigma_1(t), \sigma_2) \leq D.$$

Then set $x_1 := \sigma_1(s_0)$ and for $i = 2, 3$, let $x_i \in \sigma_i$ be such that $d(x_1, x_i) = d(x_1, \sigma_i)$. □

Suppose (Ω, d_Ω) is a Hilbert geometry. Then there are some special geodesic triangles in Ω, namely the ones whose edges are projective geodesic segments.

Definition 7.5 If $v_1, v_2, v_3 \in \Omega$, a projective geodesic triangle (with vertices v_1, v_2 and v_3) is

$$\Delta(v_1, v_2, v_3) := [v_1, v_2] \cup [v_2, v_3] \cup [v_3, v_1].$$

We will say that $\Delta(v_1, v_2, v_3)$ is D-thin if it is D-thin in the sense of Definition 7.3. There is a simple criterion to determine whether a projective geodesic triangle is D-thin. This proof comes from [39], and we include it here for the convenience of the reader.

Lemma 7.6 *Suppose Ω is a Hilbert geometry, $R \geq 0$ and $\Delta(x, y, z)$ is projective geodesic triangle such that $[y, z] \subset \mathcal{N}_R([x, y] \cup [x, z])$. Then $\Delta(x, y, z)$ is $(2R)$-thin.*

Proof Since $[y, z] \subset \mathcal{N}_R([x, y] \cup [x, z])$, there exist $m_{yz} \in [y, z]$, $m_{xy} \in [x, y]$ and $m_{xz} \in [x, z]$ such that $d_\Omega(m_{yz}, m_{xy}) \leq R$ and $d_\Omega(m_{yz}, m_{xz}) \leq R$. Indeed, the existence of three such points follows by a similar reasoning as in the proof of Observation 7.4. Then, by Proposition 3.13,

$$d_\Omega^{\mathrm{Haus}}([y, m_{yz}], [y, m_{xy}]) \leq R,$$
$$d_\Omega^{\mathrm{Haus}}([z, m_{yz}], [z, m_{xz}]) \leq R,$$
$$d_\Omega^{\mathrm{Haus}}([x, m_{xy}], [x, m_{xz}]) \leq 2R.$$

Hence, $\Delta(x, y, z)$ is $(2R)$-thin. □

7.2 Proof of Proposition 7.1

Now we prove Proposition 7.1. Fix a Hilbert geometry Ω and a rank-one axis $\ell \subset \Omega$. The remark following Proposition 7.1 will be a consequence of the proof — the proof only uses the fact that there is some rank-one isometry γ that acts along ℓ by a translation; it does not rely on γ in any other manner. Lemma 7.6 reduces Proposition 7.1 to the following.

Proposition 7.7 *If $\ell \subset \Omega$ is a rank-one axis, then there exists a constant $\mathcal{B}_\ell$ with the following property: if $\Delta(x,y,z)$ is an projective geodesic triangle in Ω with $[y,z] \subset \ell$, then $[y,z] \subset \mathcal{N}_{\mathcal{B}_\ell}([x,y] \cup [x,z])$. Moreover, this constant $\mathcal{B}_\ell$ depends only on the rank-one axis ℓ (and not on the rank-one isometry whose axis is ℓ).*

Proof The "moreover" statement will again follow from the proof since the proof is independent of the choice of the rank-one isometry which has ℓ as its axis. Now we begin the proof of the first part.

If the claim is false, then for each $n \geq 0$, there exists a projective geodesic triangle $\Delta(a_n, b_n, c_n) \subset \Omega$ with $[a_n, b_n] \subset \ell$, $c_n \in \Omega$ and $e_n \in (a_n, b_n)$ such that

$$d_\Omega(e_n, [c_n, a_n] \cup [c_n, b_n]) \geq n.$$

Since ℓ is a rank-one axis, there exists a rank-one isometry γ' whose axis is ℓ. Thus, translating $\Delta(a_n, b_n, c_n)$ by elements in $\langle \gamma' \rangle$ and passing to a subsequence, we can assume that $e := \lim_{n\to\infty} e_n$ exists and $e \in \ell$. Up to passing to a subsequence, we can assume that $a := \lim_{n\to\infty} a_n$, $b := \lim_{n\to\infty} b_n$ and $c := \lim_{n\to\infty} c_n$ exist. Observe that

$$d_\Omega(e, [a,c] \cup [c,b]) = \lim_{n\to\infty} d_\Omega(e_n, [a_n, c_n] \cup [c_n, b_n]) \geq \lim_{n\to\infty} n = \infty.$$

Thus $[a,c] \cup [c,b] \subset \partial\Omega$. But $(a,b) \subset \Omega$ since $e \in (a,b) \cap \Omega$. Thus a, c and b form a half triangle in Ω. But since $[a_n, b_n] \subset \ell$, $[a,b] \subset \overline{\ell}$. Since $a, b \in \partial\Omega$ and $\ell \subset \Omega$, we get $\overline{\ell} = [a,b]$. Thus $\ell = (a,b)$. So the rank-one axis ℓ is contained in a half triangle in Ω, a contradiction. □

8 Rank-one Hilbert geometry: Zariski density and limit sets

Recall the definition of rank-one geodesics from Definition 6.2. In this section we would like to address the following question.

Question 8.1 *Suppose (Ω, Γ) is a Hilbert geometry and Ω contains a rank-one geodesic. When does this imply that (Ω, Γ) is a rank-one Hilbert geometry?*

It is a natural question that aims to understand how the geometry of a properly convex domain influences the algebraic properties of a "large" group acting on it. Under certain assumptions, Zimmer answers this question in [50].

Proposition 8.2 *Suppose Ω is an irreducible Hilbert geometry and $\Gamma \leq \mathrm{Aut}(\Omega)$ acts cocompactly on Ω. Then (Ω, Γ) is a rank-one Hilbert geometry if and only if Ω contains a rank-one geodesic.*

This is immediate from Theorem 1.5. So our main goal in this section is to answer Question 8.1 without the assumptions of irreducibility or cocompactness as above. Instead, we will work with groups that satisfy the following assumption.

Assumption $\Gamma \leq \mathrm{SL}_{d+1}(\mathbb{R})$ is a discrete Zariski dense subgroup of $\mathrm{SL}_{d+1}(\mathbb{R})$ and there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ such that $\Gamma \cdot \Omega = \Omega$.

In this assumption, Zariski density may be interpreted as an assurance that the group Γ is "large". We will work with $\mathrm{SL}_{d+1}(\mathbb{R})$ in this section instead of $\mathrm{PGL}_{d+1}(\mathbb{R})$. Indeed, given $\Gamma \leq \mathrm{PGL}_{d+1}(\mathbb{R})$, we can pass to a subgroup of index at most 2 and assume that $\Gamma \leq \mathrm{SL}_{d+1}(\mathbb{R})$. In Section 8.2, we will formulate a hypothesis on the proximal limit set $\Lambda_\Gamma^{G/Q}$ (see Definition 8.3), that we call Hypothesis ($\star$), and use it to provide an answer to Question 8.1.

Notation For the rest of this section, let $G := \mathrm{SL}_{d+1}(\mathbb{R})$, let $P \leq G$ be the subgroup of upper-triangular matrices and Q be the stabilizer in G of $[e_1] = [1 : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^{d+1})$. Fix the standard inner product on $\mathbb{R}^{d+1}$ and let $K := \mathrm{SO}(d+1)$.

Let ε_i be the evaluation of the i^{th} diagonal entry of a diagonal matrix. Take $\Delta := \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq d\}$ to be the set of positive simple roots. For any $\theta = \{\varepsilon_{i_1} - \varepsilon_{i_1+1}, \dots, \varepsilon_{i_k} - \varepsilon_{i_k+1}\} \subset \Delta$, let P_θ denote the subgroup of block upper-triangular matrices in G with square diagonal blocks of sizes $i_1, i_2 - i_1, \dots, i_k - i_{k-1}$, $d - i_k$, respectively. In particular, $P_\Delta = P$ and G/P is the full flag variety, while $P_{\{\varepsilon_1 - \varepsilon_2\}} = Q$ and $G/Q \cong \mathbb{P}(\mathbb{R}^{d+1})$.

8.1 Limit sets in flag varieties

We will require the notion of limit sets of discrete subgroups of G in flag varieties, in particular G/P and G/Q. This has been defined and studied by various authors in different degrees of generality: Guivarch [35] (for subgroups of $\mathrm{SL}_{d+1}(\mathbb{R})$ acting proximally and strongly irreducibly on $\mathbb{R}^d$), Benoist [6] (for Zariski dense subgroups of reductive groups) and Guéritaud, Guichard, Kassel and Wienhard [34] (for arbitrary subgroups of reductive groups). We use the definition from [34, Section 5.1]

First recall the notion of singular value decomposition (or more generally, Cartan decomposition in G): for any $g \in G$, there exist $k_1, k_2 \in \mathrm{SO}(d+1)$ and $A_g = \mathrm{diag}(a_1, \dots, a_{d+1})$ with $a_1 \geq \cdots \geq a_{d+1} > 0$ such that

$$g = k_1 A_g k_2.$$

The Cartan decomposition defines the Cartan projection $\mu(g) := \mathrm{diag}(\log(a_1), \dots, \log(a_{d+1}))$. It maps G into the space of trace-zero diagonal matrices of size $(d+1) \times (d+1)$.

Let $\theta \subset \Delta$. If $g \in G$ has a singular value decomposition $g = k_1 A_g k_2$, define $E_\theta \colon G \to G/P_\theta$ by

$$E_\theta(g) := k_1 \cdot eP_\theta.$$

The map E_θ does not depend on the choices of k_1 and k_2, provided $\alpha(\mu(g)) > 0$ for all $\alpha \in \theta$; see [34, Section 5.1].

Definition 8.3 [34, Definition 5.1] Suppose Γ_0 is a discrete subgroup of G. The limit set $\Lambda_{\Gamma_0}^{G/P_\theta}$ of Γ_0 in G/P_θ is defined to be the set of all accumulation points of sequences $\{E_\theta(\gamma_n)\}_{n\in\mathbb{N}}$ where $\{\gamma_n\}_{n\in\mathbb{N}}$ is any sequence in Γ_0 such that $\alpha(\mu(\gamma_n)) \to \infty$ for all $\alpha \in \theta$.

Remark 8.4 Suppose Γ_0 is Zariski dense in G.

(i) Then $\Lambda_{\Gamma_0}^{G/P_\theta}$ is nonempty and is the closure of the set of attracting fixed points of proximal elements in G/P_θ; see [6] and [34, Section 5.1]. Here, an element $g \in G$ is called *proximal*[1] in G/P_θ provided $\alpha(\mu(g)) > 0$ for all $\alpha \in \theta$. Moreover, g is proximal in G/P_θ if and only if g has a unique attracting fixed point[2] in G/P_θ; see [34, Definition 2.25].

(ii) Suppose $\theta = \{\varepsilon_1 - \varepsilon_2\}$ so that $P_\theta = Q$. Then $\Lambda_{\Gamma_0}^{G/Q}$ is the unique minimal closed Γ-invariant subset of G/Q; see [13, Lemma 4.2]. This may not be true for arbitrary choices of θ; see [13, Remark 4.4].

Lemma 8.5 *Suppose $\Gamma \le \mathrm{SL}_{d+1}(\mathbb{R})$ satisfies the assumption. Then $\Lambda_\Gamma^{G/Q} \neq \varnothing$ and $\Lambda_\Gamma^{G/Q} \subset \partial\Omega$ is the unique minimal closed Γ-invariant subset of $\partial\Omega$.*

Proof Note that $\partial\Omega$ is a closed Γ-invariant set and the unique attracting fixed point of any proximal element (in G/Q) of Γ lies in $\partial\Omega$. The lemma then follows from Remark 8.4 above. □

If we do not assume Zariski density, then we may still have nonempty limit set (in an appropriate G/P_θ) but with some unusual properties. The following is such an example.

Example 8.6 Consider the discrete subgroup $\Gamma' := \{\mathrm{diag}(2^{m_1}, \dots, 2^{m_{d+1}}) \mid \sum_{i=1}^{d+1} m_i = 0\}$ of $\mathrm{Aut}(T_d)$ and d-dimensional torus T_d/Γ'. Although Γ' is not Zariski dense in $\mathrm{SL}_{d+1}(\mathbb{R})$, the proximal limit set in $\mathbb{P}(\mathbb{R}^{d+1})$ is nonempty and in fact $\Lambda_{\Gamma'}^{G/Q} = \{[e_1], \dots, [e_{d+1}]\}$. Thus $\Lambda_{\Gamma'}^{G/Q}$ is a proper subset of ∂T_d. Note that (T_d, Γ') is not a rank-one Hilbert geometry; see Remark 6.4.

In the light of Lemma 8.5 and this example, it is natural to ask when does $\Lambda_\Gamma^{G/Q}$ equal $\partial\Omega$.

Remark 8.7 In general, if Γ only satisfies the assumption, then $\Lambda_\Gamma^{G/Q}$ can be a proper subset of $\partial\Omega$. For example, let $\Gamma \le \mathrm{PO}(2,1)$ be a Zariski dense convex cocompact Kleinian group. Then $\Lambda_\Gamma^{G/Q} = \overline{\Gamma \cdot x} \cap \partial\mathbb{H}^2$, where $x \in \mathbb{H}^2$. Unless Γ is cocompact, $\Lambda_\Gamma^{G/Q} \neq \partial\mathbb{H}^2$. However, under the additional cocompactness assumption, we often have equality. Blayac [19, Theorem 1.3] has recently shown that if (Ω, Γ) is a divisible rank-one Hilbert geometry, then $\Lambda_\Gamma^{G/Q} = \partial\Omega$.

[1] This coincides with the notion of proximality discussed in Section 3.11 when $\theta = \{\varepsilon_1 - \varepsilon_2\}$.

[2] A fixed point $x \in X$ of a smooth map $f : X \to X$ is attracting if $\|Df_x\| < 1$.

8.2 Hypothesis ($\star$) and an answer to Question 8.1

We now introduce a special hypothesis under which we can answer Question 8.1.

Hypothesis ($\star$) *Suppose* $\Gamma \leq G$ *is a discrete subgroup and* $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ *is a properly convex domain such that* $\Gamma \cdot \Omega = \Omega$*. We will say that* (Ω, Γ) *satisfies Hypothesis* ($\star$) *if there exists a rank-one geodesic* $(a', b') \subset \Omega$ *with its endpoints* $a', b' \in \Lambda_\Gamma^{G/Q} \cap \partial\Omega$*.*

We will show that for any Zariski dense discrete subgroup Γ, this hypothesis is equivalent to the rank one property. One implication is easy and does not require Zariski density.

Lemma 8.8 *Suppose* $\Gamma \leq G$ *is a discrete subgroup that preserves a properly convex domain* Ω *and* (Ω, Γ) *is a rank-one Hilbert geometry. Then* (Ω, Γ) *satisfies Hypothesis* ($\star$)*.*

Remark 8.9 In this lemma, we do not assume that Γ is Zariski dense in G.

Proof Since (Ω, Γ) is a rank-one Hilbert geometry, we can find a rank-one isometry $\gamma \in \Gamma$. Let $\gamma^\pm \in \partial\Omega$ be the attracting and the repelling fixed points of γ. Then $\gamma^\pm \in \Lambda_\Gamma^{G/Q}$ by definition of $\Lambda_\Gamma^{G/Q}$. Also (γ^+, γ^-) is the axis of γ and hence a rank-one geodesic; see Definition 6.3 and Proposition 6.5. □

Next we will seek a converse of the above lemma and this will require the Zariski density assumption on Γ. But first we recall the notion of loxodromic elements. We will call $g \in G$ *loxodromic* if

$$|\lambda_1(g)| > \cdots > |\lambda_{d+1}(g)|.$$

If g is loxodromic, then it has unique attracting fixed points in both G/Q and G/P. We will denote by $a_g^\pm \in G/Q$ (resp. $\mathbb{X}_g^\pm \in G/P$) the unique attracting fixed point of $g^{\pm 1}$ in G/Q (resp. G/P). With this notation, $\Pi_{PQ}(\mathbb{X}_g^\pm) = a_g^\pm$, where

$$\Pi_{PQ} \colon G/P \to G/Q$$

is the natural smooth projection map. Also recall that if $g \in \mathrm{Aut}(\Omega)$ is a rank-one isometry, then we denote by $g^\pm$ the attracting and the repelling fixed points of g; see Remark 6.6.

Lemma 8.10 *Suppose* $\Gamma \leq G$ *satisfies the assumption. If there exists a rank-one geodesic* $(a', b') \subset \Omega$ *with its endpoints* $a', b' \in \Lambda_\Gamma^{G/Q} \cap \partial\Omega$*, then*

(i) *there exist rank-one isometries* $\{g_n\}$ *in* Γ *such that* $\lim_{n\to\infty} g_n^+ = a'$ *and* $\lim_{n\to\infty} g_n^- = b'$*,*

(ii) (Ω, Γ) *is a rank-one Hilbert geometry,*

(iii) *the set of rank-one isometries in* Γ *is Zariski dense in* G*,*

(iv) $\Lambda_\Gamma^{G/Q} = \overline{\{\gamma^+ \mid \gamma \text{ is a rank-one isometry}\}}$*.*

Corollary 8.11 *Suppose $\Gamma \le G$ satisfies the assumption, and (Ω,Γ) satisfies Hypothesis $(\star)$. Then (Ω,Γ) is a rank-one Hilbert geometry.*

Proof of Lemma 8.10 The key idea of this proof is in [46] and it relies on results of Benoist [7]. Before starting the proof, we informally outline the main idea. The key technical point is to find a sequence $\{g_n\}$ of biproximal elements in Γ such that $a_{g_n^+} \to a'$ and $a^-_{g_n} \to b'$. A direct way to find such a $\{g_n\}$ is: using Zariski density, find a pair $g,h \in \Gamma$ of transversally biproximal elements [13, Chapter 7] such that a^+_g and a^-_h are arbitrarily close to a' and b', respectively. Then, for large enough n, $g^n h^n$ is a biproximal element whose attracting and repelling fixed points are close to a' and b'. However, in this proof, we will take a more indirect approach by passing to the limit set in G/P and using a result of Benoist. We rely on [7, Lemma 2.6(c)]: given two distinct points $\mathbb{X}_+,\mathbb{X}_- \in \Lambda^{G/P}_\Gamma$, there exist loxodromic elements $g_n \in \Gamma$ such that $\mathbb{X}^{\pm}_{g_n} \to \mathbb{X}_{\pm}$. Once we have this sequence $\{g_n\}$, Claim 8.11.1 implies that all but finitely many of them are rank-one isometries.

Now we begin the formal proof. Equip G/P and G/Q with K-invariant Riemannian metrics and denote the corresponding Riemannian distance functions by d_P and d_Q respectively. We remark that this specific choice of Riemannian metrics will be insignificant as G/P and G/Q are compact manifolds. Let Γ_{lox} be the set of loxodromic elements in Γ. Since Γ is Zariski dense in G, Remark 8.4 implies that $\Pi_{PQ}(\Lambda^{G/P}_\Gamma) = \Lambda^{G/Q}_\Gamma$. Then pick $\mathbb{X}_a,\mathbb{X}_b \in \Lambda^{G/P}_\Gamma$ such that $\Pi_{PQ}(\mathbb{X}_a) = a'$ and $\Pi_{PQ}(\mathbb{X}_b) = b'$. For any $\varepsilon > 0$,

$$\Gamma_\varepsilon := \{g \in \Gamma_{\mathrm{lox}} \mid \mathrm{d}_P(\mathbb{X}^+_g,\mathbb{X}_a) < \varepsilon, \mathrm{d}_P(\mathbb{X}^-_g,\mathbb{X}_b) < \varepsilon\}$$

is Zariski dense in G; see [7, Lemma 2.6(c)].

For any $g \in \Gamma_{\mathrm{lox}}$, $a^{\pm}_g = \Pi_{PQ}(\mathbb{X}^{\pm}_g)$ and $a^{\pm}_g \in \partial\Omega$. Moreover, Π_{PQ} is continuous and $(a',b') \subset \Omega$. Thus there exists ε' such that if $\varepsilon \in (0,\varepsilon')$, then $(a^+_g,a^-_g) \subset \Omega$ for any $g \in \Gamma_\varepsilon$. In fact $(a^+_g,a^-_g) \subset \Omega$ is the unique axis in Ω for any such $g \in \Gamma_\varepsilon$. Indeed, the uniqueness follows from Corollary 5.4 because g has an axis $(a^+_g,a^-_g) \subset \Omega$, g is loxodromic and $\tau_\Omega(g) > 0$. We now claim that:

Claim 8.11.1 *If $\varepsilon \in (0,\varepsilon')$ is small enough, then g is a rank-one isometry for all $g \in \Gamma_\varepsilon$.*

Proof Suppose the claim is false. Then there exist a sequence $\{\varepsilon_n\}$ in $(0,\varepsilon')$ with $\varepsilon_n \to 0$ and $g_n \in \Gamma_{\varepsilon_n}$ such that g_n is not a rank-one isometry. Then $\mathbb{X}^+_{g_n} \to \mathbb{X}_a$ and $\mathbb{X}^-_{g_n} \to \mathbb{X}_b$. Since Π_{PQ} is continuous, $a^+_{g_n} \to a'$ and $a^-_{g_n} \to b'$.

By the paragraph before the claim, each g_n has a unique axis $(a^+_{g_n},a^-_{g_n}) \subset \Omega$. Moreover, $(a^+_{g_n},a^-_{g_n}) \to (a',b')$. But since g_n is not a rank-one isometry by assumption, this implies that there exists $\{c_n\}$ with $c_n \in \partial\Omega - \{a^+_{g_n},a^-_{g_n}\}$ such that

$$[a^+_{g_n},c_n] \cup [c_n,a^-_{g_n}] \subset \partial\Omega.$$

Up to passing to a subsequence, we can assume that $c_n \to c$ in $\partial\Omega$. Then $[a',c] \cup [c,b'] \subset \partial\Omega$ while $(a',b') \subset \Omega$. Thus (a',b') cannot be a rank-one geodesic and we have a contradiction. This finishes the proof of this claim. □

Now we finish the proof of the lemma. Let us choose an $\varepsilon \in (0, \varepsilon')$ as in the above claim.

(i) The result follows by choosing $g_n \in \Gamma_{\varepsilon/n}$ for all $n \geq 1$.

(ii) This follows from (i), since there is at least one rank-one isometry in Γ.

(iii) The set Γ_ε is a subset of the set of rank-one isometries of Γ and Γ_ε is Zariski dense.

(iv) By Lemma 8.5, $\Lambda_\Gamma^{G/Q} \subset \partial\Omega$ is a minimal, closed Γ-invariant set which contains the unique attracting fixed points of all proximal elements. Since a rank-one isometry is necessarily proximal,

$$\overline{\{\gamma^+ \mid \gamma \text{ is a rank-one isometry}\}} \subset \Lambda_\Gamma^{G/Q}.$$

Since $\overline{\{\gamma^+ \mid \gamma \text{ is a rank-one isometry}\}}$ is a closed Γ-invariant set, the equality then follows from minimality of $\Lambda_\Gamma^{G/Q}$. □

We now observe that Hypothesis ($\star$) gives:

Answer to Question 8.1 (see Lemma 8.8 and Corollary 8.11) If $\Gamma \leq \mathrm{SL}_{d+1}(\mathbb{R})$ is a discrete Zariski dense subgroup of $\mathrm{SL}_{d+1}(\mathbb{R})$ that preserves a properly convex domain Ω, then (Ω, Γ) *is a rank-one Hilbert geometry* if and only if Ω *contains a rank-one geodesic* $(a', b') \subset \Omega$ *with* $a', b' \in \Lambda_\Gamma^{G/Q} \cap \partial\Omega$.

We finish the section with an example where Hypothesis ($\star$) fails. Recall Example 8.6. In that case, Γ' preserves the standard d-simplex T_d, T_d/Γ' is homeomorphic to a d-torus, and T_d does not contain any rank-one geodesics. Thus (T_d, Γ') does not satisfy Hypothesis ($\star$). However, in this example, the group Γ' is not Zariski dense in $\mathrm{SL}_{d+1}(\mathbb{R})$, and one may wonder if that is the reason why Hypothesis ($\star$) fails. So we ask the following question.

Question 8.12 *Suppose $\Gamma \leq G$ is a discrete subgroup that preserves a properly convex domain Ω. If Γ is Zariski dense in G, then does (Ω, Γ) satisfy Hypothesis ($\star$)?*

To the best of the author's knowledge, the answer to this question is not known unless one makes other assumptions, eg say Ω/Γ is compact and Ω is irreducible. Under these assumptions, Remark 8.7 and Theorem 1.5 together provide an answer.

Part III Contracting elements in Hilbert geometry

In this part of the paper, we prove our main results: Theorems 1.2 and 1.4. The outline of this part of the paper is as follows. We recall the notion of contracting elements in Section 9. The proof of Theorem 1.2 is split into two sections: Sections 10 and 11. In Section 12, we introduce the notion of acylindrically hyperbolic groups and prove Theorem 1.4.

9 Contracting elements: definition and properties

Suppose $K \geq 1$ and $C \geq 0$. A function $F\colon (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is called a (K, C)-quasi-isometric embedding if for any $x_1, x_2 \in X$,

$$\frac{1}{K}\,\mathrm{d}_X(x_1, x_2) - C \leq \mathrm{d}_Y(F(x_1), F(x_2)) \leq K\,\mathrm{d}_X(x_1, x_2) + C.$$

Fix a proper geodesic metric space (X, d) and a group G that acts properly and by isometries on X. If $K \geq 1$ and $C \geq 0$, then a (K, C)-path in (X, d) is a set $F(\mathbb{R})$ where $F\colon (\mathbb{R}, |\cdot|) \to (X, \mathrm{d})$ is a (K, C)-quasi-isometric embedding. A subpath of the path $F(\mathbb{R})$ is $F(I)$ where $I \subset \mathbb{R}$ is an interval, possibly unbounded.

Definition 9.1 Let $K \geq 1$ and $C \geq 0$. Let $\mathcal{PS}$ be a collection of (K, C)-paths in X. Then:

(i) $\mathcal{PS}$ is called a *path system* on X if
   (a) any subpath of a path in $\mathcal{PS}$ is also in $\mathcal{PS}$, and
   (b) any pair of points in X can be connected by a path in $\mathcal{PS}$.

(ii) $\mathcal{PS}$ is called a *geodesic path system* if all paths in $\mathcal{PS}$ are geodesics in (X, d).

(iii) If G preserves $\mathcal{PS}$, then $(X, \mathcal{PS})$ is called a *path system for the group G*.

Definition 9.2 (contracting subsets [49]) If $\mathcal{PS}$ is a path system on X, then $\mathcal{A} \subset X$ is said to be *$\mathcal{PS}$-contracting* (with constant C) if there exists a map $\pi_{\mathcal{A}}\colon X \to \mathcal{A}$ such that

(i) if $x \in \mathcal{A}$, then $\mathrm{d}(x, \pi_{\mathcal{A}}(x)) \leq C$,

(ii) if $x, y \in X$ and $\mathrm{d}(\pi_{\mathcal{A}}(x), \pi_{\mathcal{A}}(y)) \geq C$, then for any path $\sigma \in \mathcal{PS}$ joining x and y,

$$\mathrm{d}(\sigma, \pi_{\mathcal{A}}(x)) \leq C \quad \text{and} \quad \mathrm{d}(\sigma, \pi_{\mathcal{A}}(y)) \leq C.$$

A prototypical example of a contracting subset is a bi-infinite geodesic in $\mathbb{H}^2$ (with the map $\pi_{\mathcal{A}}$ being the closest point projection on the geodesic). Generally speaking, one should think of the projection map $\pi_{\mathcal{A}}$ as an analogue of the closest-point projection. In fact, the following lemma makes this analogy concrete in the context of geodesic path systems. We will use the notation

$$\rho_{\mathcal{A}}(x) := \{a \in \mathcal{A} \mid \mathrm{d}(x, a) = \mathrm{d}(x, \mathcal{A})\}$$

for the set-valued closest-point projection map on $\mathcal{A}$.

Lemma 9.3 *Suppose $\mathcal{PS}$ is a geodesic path system and $\mathcal{A} \subset X$ is $\mathcal{PS}$-contracting (with constant C) with the projection map $\pi_{\mathcal{A}}\colon X \to \mathcal{A}$. Then $\sup_{a \in \rho_{\mathcal{A}}(x)} \mathrm{d}(\pi_{\mathcal{A}}(x), a) \leq 2C$ for all $x \in X$.*

Proof Suppose there exist $x \in X$ and $a \in \rho_{\mathcal{A}}(x)$ such that

$$\mathrm{d}(\pi_{\mathcal{A}}(x), a) > 2C.$$

Since $\mathcal{A}$ is $\mathcal{PS}$-contracting and $a \in \mathcal{A}$, one gets that $\mathrm{d}(\pi_{\mathcal{A}}(a), a) \leq C$. Then

$$\mathrm{d}(\pi_{\mathcal{A}}(x), \pi_{\mathcal{A}}(a) \geq \mathrm{d}(\pi_{\mathcal{A}}(x), a) - \mathrm{d}(\pi_{\mathcal{A}}(a), a) > C.$$

Let $\sigma_{x,a}$ be a geodesic path in $\mathcal{PS}$ joining x and a. Since $\mathcal{A}$ is $\mathcal{PS}$-contracting, there exists $z \in \sigma_{x,a}$ such that $\mathrm{d}(z, \pi_{\mathcal{A}}(x)) \leq C$. As $z \in \sigma_{x,a}$, $\mathrm{d}(a, z) = \mathrm{d}(a, x) - \mathrm{d}(z, x)$. As $a \in \rho_{\mathcal{A}}(x)$, $\mathrm{d}(x, a) \leq \mathrm{d}(\pi_{\mathcal{A}}(x), x)$. Then

$$\mathrm{d}(a, z) \leq \mathrm{d}(\pi_{\mathcal{A}}(x), x) - \mathrm{d}(x, z) \leq \mathrm{d}(\pi_{\mathcal{A}}(x), z) + \mathrm{d}(z, x) - \mathrm{d}(z, x) \leq C.$$

Then $\mathrm{d}(\pi_{\mathcal{A}}(x), a) \leq \mathrm{d}(\pi_{\mathcal{A}}(x), z) + \mathrm{d}(z, a) \leq 2C$, a contradiction. □

Using the notion of contracting subsets, one introduces the notion of contracting group elements. A prototypical example of a contracting element is

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

for some $\lambda > 1$, that acts on $\mathbb{H}^2$ by a translation along a bi-infinite geodesic in $\mathbb{H}^2$.

Definition 9.4 (contracting elements [49]) If $(X, \mathcal{PS})$ is a path system for G, then $g \in G$ is a *contracting element* for $(X, \mathcal{PS})$ provided for some (hence any) $x_0 \in X$,

- (i) g has infinite order and $\langle g \rangle \cdot x_0$ is a quasi-isometric embedding of $\mathbb{Z}$ in X, and
- (ii) there exists $\mathcal{A} \subset X$ containing x_0 that is $\langle g \rangle$-invariant, $\mathcal{PS}$-contracting and has cobounded $\langle g \rangle$-action.

Remark 9.5 If $g \in G$ is a contracting element and $\mathcal{PS}$ is a geodesic path system, then $\pi_{\mathcal{A}}$ is coarsely $\langle g \rangle$-equivariant. This is immediate from Lemma 9.3 since $\pi_{\mathcal{A}}$ is coarsely equivalent to $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{A}}$ is clearly $\langle g \rangle$-equivariant.

In the definition of a contracting element, the set $\mathcal{A}$ is not necessarily a $\langle g \rangle$-orbit in X. We will now explain that we can always replace $\mathcal{A}$ by a $\langle g \rangle$-orbit. Moreover, we also show g has positive translation length for its action on X. We remark that the following observation does not require that $\mathcal{PS}$ is a geodesic path system.

Observation 9.6 *Suppose $(X, \mathcal{PS})$ is a path system for G and $g \in G$ is a contracting element for $(X, \mathcal{PS})$. Then*

- (i) $\tau_X(g) := \inf_{x \in X} \mathrm{d}(x, gx)$ *is positive, and*
- (ii) *for any $x_0 \in \mathcal{A}$, $\mathcal{A}_{\min}(x_0) := \langle g \rangle x_0$ is the minimal $\mathcal{PS}$-contracting, $\langle g \rangle$-invariant subset of X containing x_0 with a cobounded $\langle g \rangle$-action.*

Proof (i) Recall the definition of stable translation length

$$\tau_X^{\text{stable}}(g) := \lim_{n\to\infty} \frac{\mathrm{d}(x, g^n x)}{n}.$$

Then $\tau_X(g) \ge \tau_X^{\text{stable}}(g)$ and it suffices to show $\tau_X^{\text{stable}}(g) > 0$. Fix any $x_0 \in X$. Since g is contracting, $\langle g\rangle x_0$ is a quasigeodesic, that is, there exists $K \ge 1$ and $C \ge 0$ such that $\mathrm{d}(x_0, g^n x_0) \ge (1/K)|n| - C$ for every $n \in \mathbb{Z}$. Then, $\tau_X^{\text{stable}}(g) \ge 1/K > 0$.

(ii) Let $\mathscr{A}$ be $\mathscr{PS}$-contracting with constant $C_{\mathscr{A}}$ and the map $\pi_{\mathscr{A}}\colon X \to \mathscr{A}$. Fix any $x_0 \in \mathscr{A}$ and set $R_{\mathscr{A}} := \mathrm{diam}(\mathscr{A}/\langle g\rangle)$, $C_0 := C_{\mathscr{A}} + 2R_{\mathscr{A}}$ and $\mathscr{A}_{\min}(x_0) := \langle g\rangle x_0$.

Since $\mathscr{A}_{\min}(x_0) \subset \mathscr{A}$, if $x \in X$, then there exists $m \in \mathbb{Z}$ such that $\mathrm{d}(\pi_{\mathscr{A}}(x), g^m x_0) \le R_{\mathscr{A}}$. Define $\pi_{\min}\colon X \to \mathscr{A}_{\min}(x_0)$ by setting $\pi_{\min}(x) = g^m x_0$. Then, if $x \in \mathscr{A}_{\min}(x_0)$, $\pi_{\min}(x) = x$. If $x, y \in X$ and $\mathrm{d}(\pi_{\min}(x), \pi_{\min}(y)) \ge C_0$, then $\mathrm{d}(\pi_{\mathscr{A}}(x), \pi_{\mathscr{A}}(y)) \ge C_{\mathscr{A}}$. Thus, if $\sigma \in \mathscr{PS}$ is a path from x to y, $\mathrm{d}(\pi_{\mathscr{A}}(x), \sigma) \le C_{\mathscr{A}}$ and $\mathrm{d}(\pi_{\mathscr{A}}(y), \sigma) \le C_{\mathscr{A}}$. Hence,

$$\mathrm{d}(\pi_{\min}(x), \sigma) \le C_0 \quad \text{and} \quad \mathrm{d}(\pi_{\min}(y), \sigma) \le C_0. \qquad \square$$

There are many other notions of contracting subsets in geometric group theory. We will require one such notion in Section 14 for proving our Theorem 1.8. We will call this notion *contraction in the sense of BF* — it was introduced by Bestvina and Fujiwara for CAT(0) spaces [17] and by Gekhtman and Yang [32] in general. We defer all further discussion about this to Appendix B and only remark that in our case, this notion of contraction will be equivalent to Definition 9.2.

Remark 9.7 If Ω is a Hilbert geometry, we use the geodesic path system $\mathscr{PS}^{\Omega} := \{[x, y] \mid x, y \in \Omega\}$ given by projective geodesics. We use the $\mathscr{PS}$-contracting notion everywhere in the paper except in Section 14 (where we use *contraction in the sense of BF*, see Definition B.2). Proposition 9.8 below implies that these two notions of contraction are equivalent in our setup. Hence, in the rest of the paper, we will use the term contracting subset (and element) without additional labels.

Proposition 9.8 *Suppose $(X, \mathscr{PS})$ is a geodesic path system. Then*:

(i) *$\mathscr{A} \subset X$ is $\mathscr{PS}$-contracting if and only if $\mathscr{A}$ is contracting in the sense of BF.*

(ii) *If G preserves $\mathscr{PS}$, then $g \in G$ is a contracting element for $(X, \mathscr{PS})$ if and only if $g \in G$ is a contracting element in the sense of BF.*

Proof See Appendix B. $\square$

10 Rank-one isometries are contracting

In this section, we prove one implication in Theorem 1.2. Fix a Hilbert geometry Ω and let $\mathscr{PS}^{\Omega} := \{[x, y] \mid x, y \in \Omega\}$.

Theorem 10.1 *If $\gamma \in \mathrm{Aut}(\Omega)$ is a rank-one isometry, then γ is a contracting element for $(\Omega, \mathcal{PS}^{\Omega})$.*

The key step will be part (ii) of Lemma 10.3, which shows that a rank-one axis is $\mathcal{PS}^{\Omega}$-contracting. First, we construct suitable projection maps on a rank-one axis. Recall the notion of closest-point projection on closed convex subsets, particularly Corollary 3.10.

Definition 10.2 Suppose Ω is a Hilbert geometry, ℓ is a bi-infinite projective geodesic in Ω and $\sigma : \mathbb{R} \to \ell$ is its unit-speed parametrization. Then $\Pi_\ell(x) = [\sigma(T_x^-), \sigma(T_x^+)]$ for $T_x^-, T_x^+ \in \mathbb{R}$. We define the projection map $\pi_\ell : \Omega \to \ell$ as

$$\pi_\ell(x) := \sigma\left(\frac{T_x^- + T_x^+}{2}\right).$$

We now establish some properties of the map π_ℓ when ℓ is a rank-one axis.

Lemma 10.3 *If $\ell \subset \Omega$ is a rank-one axis, then there exists $\mathscr{C}_\ell \geq 0$ such that:*

(i) *If $x \in \Omega$ and $z \in \ell$, then there exists $p_{xz} \in [x, z]$ such that*

$$d_\Omega(\pi_\ell(x), p_{xz}) \leq 3\mathscr{C}_\ell.$$

(ii) *The geodesic ℓ is $\mathcal{PS}^{\Omega}$-contracting with constant $\mathscr{C}_\ell$ and the map π_ℓ.*

Proof (i) Let $x \in \Omega$ and $z \in \ell$. Choose any $C_\ell \geq D_\ell$, where D_ℓ is the constant from Proposition 7.1. Proposition 7.1 implies that $\Delta(x, \pi_\ell(x), z)$ is $\mathscr{D}_\ell$-thin. By Observation 7.4, there exists $p \in [x, \pi_\ell(x)]$, $q \in [\pi_\ell(x), z]$ and $r \in [z, x]$ such that

$$d_\Omega(q, p) \leq \mathscr{D}_\ell \quad \text{and} \quad d_\Omega(q, r) \leq \mathscr{D}_\ell.$$

Then

$$d_\Omega(\pi_\ell(x), p) = d_\Omega(\pi_\ell(x), x) - d_\Omega(p, x) \leq d_\Omega(q, x) - d_\Omega(p, x) \leq d_\Omega(p, q) \leq \mathscr{D}_\ell.$$

Thus

$$d_\Omega(\pi_\ell(x), q) \leq d_\Omega(\pi_\ell(x), p) + d_\Omega(q, p) \leq 2\mathscr{D}_\ell.$$

Set $p_{xz} := r$. Then

$$d_\Omega(\pi_\ell(x), p_{xz}) \leq d_\Omega(\pi_\ell(x), q) + d_\Omega(q, r) \leq 3\mathscr{D}_\ell \leq 3C_\ell.$$

(ii) Set $\pi := \pi_\ell$ for ease of notation. Let us label the endpoints of ℓ so that $\ell := (a, b)$. Observe that we only need to verify (ii) in Definition 9.2. Suppose, for a contradiction, that it is not satisfied. Then, for every $n \in \mathbb{N}$, there exist $x_n, y_n \in \Omega$ such that

$$d_\Omega(\pi(x_n), \pi(y_n)) \geq n \quad \text{and} \quad d_\Omega([x_n, y_n], \pi(x_n)) \geq n.$$

Since ℓ is a rank-one axis, fix a rank-one isometry γ whose axis is ℓ. Then $\gamma \circ \pi = \pi \circ \gamma$. Hence, up to translating x_n and y_n using elements in $\langle\gamma\rangle$, we can assume that $\alpha := \lim_{n\to\infty} \pi(x_n)$ exists in $\ell \subset \Omega$.

Up to passing to a subsequence, we can further assume that the following limits exist in $\overline{\Omega}$:

$$x := \lim_{n\to\infty} x_n, \quad y := \lim_{n\to\infty} y_n, \quad \beta := \lim_{n\to\infty} \pi(y_n).$$

Then $\lim_{n\to\infty}[x_n, y_n] = [x, y]$. We will now show that

$$[x, y] \subset \partial\Omega. \tag{10}$$

This follows from the following estimate:

$$\begin{aligned} d_\Omega(\alpha, [x, y]) = \lim_{n\to\infty} d_\Omega(\alpha, [x_n, y_n]) &\geq \lim_{n\to\infty} \big(d_\Omega(\pi(x_n), [x_n, y_n]) - d_\Omega(\pi(x_n), \alpha)\big) \\ &\geq \lim_{n\to\infty} \big(n - d_\Omega(\pi(x_n), \alpha)\big) = \infty. \end{aligned}$$

We also observe that

$$\begin{aligned} d_\Omega(\alpha, \beta) = \lim_{n\to\infty} d_\Omega(\alpha, \pi(y_n)) &\geq \lim_{n\to\infty} \big(d_\Omega(\pi(x_n), \pi(y_n)) - d_\Omega(\pi(x_n), \alpha)\big) \\ &\geq \lim_{n\to\infty} \big(n - d_\Omega(\pi(x_n), \alpha)\big) = \infty. \end{aligned}$$

Thus $\beta \in \partial\Omega$. However, since $\beta \in \overline{\ell} = [a, b]$, $\beta \in \{a, b\}$. Thus, up to switching the labels of the endpoints of ℓ, we can assume that

$$\beta = b. \tag{11}$$

Claim 10.3.1 $\quad x = y = b.$

Proof We first show that $y = b$. Since $y_n \in \Omega$ and $\alpha \in \ell$, part (i) of Lemma 10.3 implies that there exists $p_n \in [y_n, \alpha]$ such that

$$d_\Omega(p_n, \pi(y_n)) \leq 3\mathscr{C}_\ell.$$

Up to passing to a subsequence, we can assume that $p := \lim_{n\to\infty} p_n$ exists in $\overline{\Omega}$. Then by Proposition 3.12, $p \in F_\Omega(\beta)$. By equation (11), $\beta = b$, which implies $p \in F_\Omega(b)$. Since b is an endpoint of the rank-one axis ℓ, part (iv) of Proposition 6.5 implies that $F_\Omega(b) = b$. Thus $p = b$. On the other hand, since $p_n \in [y_n, \alpha]$, we have $p \in [y, \alpha]$. Since $p = b$, we have $p \in \partial\Omega$. Thus,

$$p \in [\alpha, y] \cap \partial\Omega = \{y\}.$$

Hence,

$$y = p = b.$$

We now show that $x = b$. By equation (10), $[x, y] \subset \partial\Omega$. But since $y = b$, this contradicts part (iv) of Proposition 6.5 unless $x = y$. Hence $x = y = b$. This concludes the proof of Claim 10.3.1. □

Consider the points $x_n \in \Omega$ and $\pi(y_n) \in \ell$. By part (i) of Lemma 10.3, there exists $q_n \in [x_n, \pi(y_n)]$ such that $d_\Omega(\pi(x_n), q_n) \leq 3\mathscr{C}_\ell$. Up to passing to a subsequence, we can assume that $q := \lim_{n\to\infty} q_n$ exists in $\overline{\Omega}$. Then by Proposition 3.12, $q \in F_\Omega(\alpha) = \Omega$. Thus $\lim_{n\to\infty}[x_n, \pi(y_n)]$ is a projective line segment containing q and hence intersects Ω. However, $\lim_{n\to\infty}[x_n, \pi(y_n)] = [x, \beta] = \{b\} \subset \partial\Omega$. This is a contradiction. □

We will now apply Lemma 10.3 to prove Theorem 10.1. Suppose $\gamma \in \mathrm{Aut}(\Omega)$ is a rank-one isometry. Then $\tau_\Omega(\gamma) > 0$, which implies that γ has infinite order. By part (ii) of Proposition 6.5, γ has a unique axis ℓ_γ along which γ acts by a translation. Fix $x_0 \in \ell_\gamma$. As $\langle\gamma\rangle$ acts cocompactly on ℓ_γ, $\langle\gamma\rangle \cdot x_0$ is a quasi-isometric embedding of $\mathbb{Z}$ in Ω. Part (ii) of Lemma 10.3 implies that ℓ_γ is a $\mathcal{PS}^\Omega$-contracting set. Thus γ is a contracting element for $(\Omega, \mathcal{PS}^\Omega)$; see Definition 9.4.

11 Contracting isometries are rank one

In this section, we prove the other implication of Theorem 1.2. Fix a Hilbert geometry Ω and let $\mathcal{PS}^\Omega := \{[x,y] \mid x, y \in \Omega\}$.

Theorem 11.1 *If $\gamma \in \mathrm{Aut}(\Omega)$ is a contracting element for $(\Omega, \mathcal{PS}^\Omega)$, then γ is a rank-one isometry.*

We begin by recalling a result of Sisto which says that contracting elements are "Morse" in the following sense.

Proposition 11.2 [49, Lemma 2.8] *If $\mathcal{PS}$ is a path system on (X, d) and $\mathcal{A} \subset X$ is $\mathcal{PS}$-contracting with constant C, then there exists a constant $M = M(C)$ such that if θ is a (C,C)-quasigeodesic with endpoints in $\mathcal{A}$, then $\theta \subset \mathcal{N}_M(\mathcal{A}) := \{x \in X \mid \mathrm{d}(x, \mathcal{A}) < M\}$.*

We use this Morse property to show that a contracting element has at least one axis and none of its axes are contained in half triangles in Ω. The first step is the following lemma. Recall the notation E_γ^+, E_γ^- from Definition 4.7.

Lemma 11.3 *Suppose Ω is a Hilbert geometry and $\gamma \in \mathrm{Aut}(\Omega)$ is a contracting element for $(\Omega, \mathcal{PS}^\Omega)$. If there exist $x_0 \in \Omega$ and two unbounded sequences of positive integers $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ such that*

$$p := \lim_{k\to\infty} \gamma^{n_k} x_0 \quad \text{belongs to } E_\gamma^+ \qquad \text{and} \qquad q := \lim_{k\to\infty} \gamma^{-m_k} x_0 \quad \text{belongs to } E_\gamma^-,$$

then

(i) $(p,q) \subset \Omega$, *and*

(ii) (p,q) *is not contained in any half triangle in* Ω.

Proof Since γ is a contracting element, Observation 9.6 implies that $\tau_\Omega(\gamma) > 0$. Thus $p \neq q$.

(i) Suppose this is false. Then $[p,q] \subset \partial\Omega$. Choose any $r \in (p,q)$. Set $L_k := [\gamma^{-m_k} x_0, \gamma^{n_k} x_0]$. Then $L_\infty := \lim_{k\to\infty} L_k = [q,p]$. Thus we can choose $r_k \in L_k$ such that $\lim_{k\to\infty} r_k = r$.

Since γ is a contracting element, part (ii) of Observation 9.6 implies that $\mathcal{A}_{\min}(x_0) := \langle\gamma\rangle x_0$ is $\mathcal{PS}^\Omega$-contracting. Since the L_k are geodesics with endpoints in $\mathcal{A}_{\min}(x_0)$, Proposition 11.2 implies that there exists a constant M such that for all $k \geq 1$, $L_k \subset \mathcal{N}_M(\mathcal{A}_{\min}(x_0))$. Thus for every $k \geq 1$, there exists $\gamma^{t_k} x_0 \in \mathcal{A}_{\min}(x_0)$ such that

$$d_\Omega(r_k, \gamma^{t_k} x_0) \leq M. \tag{12}$$

Up to passing to a subsequence, we can assume that

$$t := \lim_{k \to \infty} \gamma^{t_k} x_0$$

exists in $\overline{\Omega}$. Since r_k leaves every compact subset of Ω, $\{t_k\}$ is an unbounded sequence. Then by Proposition 4.9 part (i), $t \in (E_\gamma^+ \sqcup E_\gamma^-)$. On the other hand, by Proposition 3.12 and equation (12),

$$t \in F_\Omega(r) \subset \partial\Omega. \tag{13}$$

We now analyze the two possibilities:

Case 1 If possible, suppose $t \in E_\gamma^-$. Then consider the sequence $\{\gamma^{n_k} r\}_{k \in \mathbb{N}}$. Up to passing to a subsequence, we can assume that $r_\infty := \lim_{k \to \infty} \gamma^{n_k} r$ exists in $\partial\Omega$. Since $p \in E_\gamma^+$, $q \in E_\gamma^-$ and $r \in (p, q)$ with $n_k > 0$, Observation 4.4 part (ii) implies that

$$r_\infty = \lim_{k \to \infty} \gamma^{n_k} r \in E_\gamma^+. \tag{14}$$

To sum up, we have $r \in F_\Omega(t)$, where $t \in E_\gamma^-$ and $r_\infty = \lim_{k \to \infty} \gamma^{n_k} r$; see (13) and (14). Now we apply part (ii) of Corollary 4.13 with t, r and $\{n_k\}$ taking the role of y, z and $\{i_k\}$ respectively. Then the conclusion is that $r_\infty \in E_\gamma^-$. This contradicts equation (14).

Case 2 If possible, suppose $t \in E_\gamma^+$. We can repeat the same arguments as in Case 1 by considering the sequence $\{\gamma^{-m_k} r\}_{k \in \mathbb{N}}$, and arrive at a contradiction — we need a version of Corollary 4.13 with γ replaced by γ^{-1}; see Remark 4.14.

The contradiction to both of these possibilities finishes the proof of (i).

(ii) By part (i), $(p, q) \subset \Omega$. Suppose there exists $z \in \partial\Omega$ such that p, z, q form a half triangle in Ω. Choose any sequence of points $z_k \in [x_0, z] \cap \Omega$ such that $\lim_{k \to \infty} z_k = z$. Since γ is contracting, part (ii) of Observation 9.6 implies that $\mathcal{A}_{\min}(x_0) = \langle\gamma\rangle x_0$ is $\mathcal{PS}^\Omega$-contracting (with constant, say C). Thus there exists a projection $\pi : \Omega \to \mathcal{A}_{\min}(x_0)$ that satisfies Definition 9.2. We will analyze the sequence $\pi(z_k)$. Since $\pi(z_k) \in \mathcal{A}_{\min}(x_0)$, there exists a sequence of integers $\{i_k\}$ such that $\pi(z_k) = \gamma^{i_k} x_0$. Up to passing to a subsequence, we can assume that the following limit exists in $\overline{\Omega}$:

$$w := \lim_{k \to \infty} \pi(z_k) = \lim_{k \to \infty} \gamma^{i_k} x_0. \tag{15}$$

Claim 11.3.1 *It holds that $w \in \partial\Omega$ and $w \in (E_\gamma^+ \sqcup E_\gamma^-)$.*

Proof Recall that Γ acts properly discontinuously on Ω. Moreover, $\omega(\gamma, \Omega) \cup \omega(\gamma^{-1}, \Omega) \subset E_\gamma^+ \cup E_\gamma^-$; see Proposition 4.9. Thus it suffices to show that $\{i_k\}$ is an unbounded sequence. Suppose, on the contrary, that $\{i_k\}$ is a bounded sequence. Then $w \in \Omega$ and $\lim_{k \to \infty} d_\Omega(w, \pi(z_k)) = 0$; see (15).

Recall that $\{n_k\}$ is the sequence such that $\gamma^{n_k} x_0 \to p \in \partial\Omega$. We will prove this claim by comparing $\gamma^{i_k} x_0 (= \pi(z_k))$ with $\gamma^{n_k} x_0$. We claim that $d_\Omega(\pi(z_k), \pi(\gamma^{n_k} x_0)) \to \infty$ as $k \to \infty$. To prove this subclaim, first note that (i) of Definition 9.2 implies that

$$d_\Omega(\gamma^{n_k} x_0, \pi(\gamma^{n_k} x_0)) \le C$$

because $\gamma^{n_k} x_0 \in \mathcal{A}_{\min}(x_0)$. The subclaim then follows from the equation

$$\begin{aligned}\lim_{k\to\infty} d_\Omega(\pi(z_k), \pi(\gamma^{n_k} x_0)) &\ge \lim_{k\to\infty} \big(d_\Omega(w, \gamma^{n_k} x_0) - d_\Omega(w, \pi(z_k)) - d_\Omega(\gamma^{n_k} x_0, \pi(\gamma^{n_k} x_0))\big) \\ &\ge \liminf_{k\to\infty} d_\Omega(w, \gamma^{n_k} x_0) - C = \infty.\end{aligned}$$

The above equation then implies that for k large enough, $d_\Omega(\pi(z_k), \pi(\gamma^{n_k} x_0)) \ge C$. Since π is a projection into a $\mathcal{PS}^\Omega$-contracting set, condition (ii) of Definition 9.2 implies that

$$d_\Omega(\pi(z_k), [z_k, \gamma^{n_k} x_0]) \le C.$$

Thus

$$d_\Omega(w, [z, p]) \le \lim_{k\to\infty} d_\Omega(\pi(z_k), [z_k, \gamma^{n_k} x_0]) \le C.$$

Then $[z, p] \cap \Omega \ne \varnothing$. But since p, z, q form a half triangle, $[z, p] \subset \partial\Omega$. This is a contradiction and it concludes the proof of this claim. □

Claim 11.3.2 $\quad w \in F_\Omega(z)$.

Proof First observe that for k large enough,

$$d_\Omega(\pi(z_k), \pi(x_0)) \ge C.$$

Indeed, this follows because $\pi(x_0) \in \Omega$ while $w = \lim_{k\to\infty} \pi(z_k) \in \partial\Omega$. Again, as π is a projection into a $\mathcal{PS}^\Omega$-contracting set, we have

$$d_\Omega(\pi(z_k), [x_0, z_k]) \le C.$$

Choose $\eta_k \in [x_0, z_k]$ such that $d_\Omega(\pi(z_k), \eta_k) \le C$. Up to passing to a subsequence, we can assume that $\eta := \lim_{k\to\infty} \eta_k$ exists. By Proposition 3.12, $\eta \in F_\Omega(w)$. Since $w \in \partial\Omega$, $\eta \in \partial\Omega$ (Proposition 3.11(i)). But $\eta \in [x_0, z]$, which intersects $\partial\Omega$ at exactly one point, namely z. Thus, $\eta = z$ implying $z \in F_\Omega(w)$, or equivalently, $w \in F_\Omega(z)$. This concludes the proof of Claim 11.3.2. □

Since p, z, q form a half triangle, $[p, z] \cup [z, q] \subset \partial\Omega$. By Claim 11.3.2, $w \in F_\Omega(z)$. Then part (iv) of Proposition 3.11 implies that

$$[p, w] \cup [q, w] \subset \partial\Omega. \tag{16}$$

Recall from Claim 11.3.1 that $\{i_k\}$ is an unbounded sequence and that $w = \lim_{k\to\infty} \gamma^{i_k} x_0$ lies in $E_\gamma^+ \sqcup E_\gamma^-$. We will now show that (16) contradicts this.

Suppose, up to passing to a subsequence, that $\{i_k\}$ is a sequence of positive integers. Then $w \in E_\gamma^+$. Since $\lim_{k\to\infty} \gamma^{i_k} x_0 = w \in E_\gamma^+$ and $\lim_{k\to\infty} \gamma^{-m_k} x_0 = q \in E_\gamma^-$, then part (i) of Lemma 11.3 implies that $(w, q) \subset \Omega$. This contradicts (16). On the other hand, if we suppose that $\{i_k\}$ is a sequence of negative integers, then $w \in E_\gamma^-$. Then, by a similar reasoning, $(p, w) \subset \Omega$ which again contradicts (16). These contradictions show that p, z, q cannot form a half triangle. □

We now prove Theorem 11.1 using the above lemma. Let $\gamma \in \mathrm{Aut}(\Omega)$ be a contracting element for $(\Omega, \mathcal{PS}^\Omega)$. By part (i) of Observation 9.6, $\tau_\Omega(\gamma) > 0$. The following will imply that γ is a rank-one isometry.

- **γ has an axis** By Proposition 5.8, there exists $(a, b) \subset \overline{\Omega}$ with a, b fixed points of γ such that $a \in E_\gamma^+$ and $b \in E_\gamma^-$. We will show that $(a, b) \subset \Omega$; hence it is an axis of γ.

Fix $x_0 \in \Omega$. Proposition 4.9 part (i) implies $\{\gamma^n x_0 \mid n \in \mathbb{N}\}$ has an accumulation point p in E_γ^+ and $\{\gamma^{-n} x_0 \mid n \in \mathbb{N}\}$ has accumulation point q in E_γ^-. By part (i) of Lemma 11.3, $(p, q) \subset \Omega$.

Note that $E_\gamma^+ \cap \overline{\Omega} \subset \partial\Omega$; see Claim 4.6.1. Thus $[a, p] \subset \partial\Omega$ as $a, p \in E_\gamma^+ \cap \partial\Omega$. Similarly, $[b, q] \subset \partial\Omega$. By part (ii) of Lemma 11.3, $(p, q) \subset \Omega$ is not contained in any half triangle in Ω. Since $[b, q] \subset \partial\Omega$, this implies that $(p, b) \subset \Omega$.

We will use $(p, b) \subset \Omega$ to derive that $(a, b) \subset \Omega$. First note that since $b \in E_\gamma^-$ is the endpoint of a pseudoaxis, b is a fixed point of γ. Thus $\lim_{k\to\infty} \gamma^{-k} y' = b \in E_\gamma^-$ for any $y' \in (p, b)$. We then note that p is an "almost-fixed" point of γ, ie there exists $\{n_k\}$ with $n_k \to \infty$ such that $\lim_{k\to\infty} \gamma^{n_k} p = p \in E_\gamma^+$. Indeed, Proposition 4.9 part (iii) implies that there exists a sequence of positive integers $\{n_k\}$ with $n_k \to \infty$ such that $\lim_{k\to\infty} \gamma|_{E_\gamma^+}^{n_k} = \mathrm{Id}_{E_\gamma^+}$, ie $\lim_{k\to\infty} \gamma^{n_k} p = p$. Now pick $y_0 \in (p, b) \in \Omega$. The above discussion implies that $\lim_{k\to\infty} \gamma^{n_k} y_0 = p \in E_\gamma^+$ while $\lim_{k\to\infty} \gamma^{-k} y_0 = b \in E_\gamma^-$. Then, by part (ii) of Lemma 11.3, $(p, b) \subset \Omega$ cannot be contained in a half triangle in Ω. But we know that $[a, p] \subset \partial\Omega$. Thus, $(a, b) \subset \Omega$.

- **None of the axes of γ are contained in a half triangle in Ω** Let $(a', b') \subset \Omega$ be any axis of γ with $a' \in E_\gamma^+$ and $b' \in E_\gamma^-$. If $z_0 \in (a', b')$, then $\lim_{k\to\infty} \gamma^k z_0 = a'$ and $\lim_{k\to\infty} \gamma^{-k} z_0 = b'$. Then, by part (ii) of Lemma 11.3, (a', b') cannot be contained in a half triangle in Ω.

12 Acylindrical hyperbolicity: proof of Theorem 1.4

Acylindrically hyperbolic groups are a generalization of nonelementary Gromov hyperbolic groups with many interesting examples, like mapping class groups of most finite-type surfaces, rank-one CAT(0) groups that are not virtually cyclic, outer automorphisms of free groups on at least two generators and relatively hyperbolic groups with proper peripheral subgroups that are not virtually cyclic [45, Appendix]. In this section, we will add a new class of examples by showing that discrete groups acting on Hilbert geometries with at least one rank-one isometry are either virtually cyclic or acylindrically hyperbolic.

12.1 Acylindrically hyperbolic groups

We first recall some basic definitions about Gromov hyperbolic metric spaces (not necessarily proper) and we refer to [33] for details. A geodesic metric space (Y, d_Y) is called *Gromov hyperbolic* if there exists $\delta \geq 0$ such that every geodesic triangle in Y is δ-thin (recall Definition 7.3). If (Y, d_Y) is Gromov hyperbolic, let ∂Y denote the *boundary of* Y defined via equivalence classes of sequences in Y "convergent at infinity"; see [33, Section 1.8]. We remark that this definition of ∂Y does not require that Y is a proper metric space.

If G acts isometrically on a Gromov hyperbolic space (Y, d_Y), let $\Lambda_G(Y) \subset \partial Y$ denote the *limit set* of the G-action (ie $\Lambda_G(Y)$ is the set of accumulation points in ∂Y of any G orbit in Y). The action is called *nonelementary* if $\#(\Lambda_G(Y)) = \infty$; see [45] for details.

Finally we define the notion of acylindrical actions on a metric space (not necessarily Gromov hyperbolic). An isometric action of a group G on a metric space (Y, d_Y) is called *acylindrical* if, for every $\varepsilon > 0$, there exists $R_\varepsilon, N_\varepsilon > 0$ such that if $x, y \in Y$ with $\mathrm{d}_Y(x, y) \geq R_\varepsilon$, then

$$\#\{g \in G \mid \mathrm{d}_Y(x, gx) \leq \varepsilon \text{ and } \mathrm{d}_Y(y, gy) \leq \varepsilon\} \leq N_\varepsilon.$$

Definition 12.1 A group G is called *acylindrically hyperbolic* if it admits an isometric nonelementary acylindrical action on a (possibly nonproper) Gromov hyperbolic metric space (Y, d_Y).

A motivating example of acylindrically hyperbolic groups is a nonelementary Gromov hyperbolic group. Indeed, if H is a finitely generated nonelementary Gromov hyperbolic group, then it has a nonelementary acylindrical action on its Cayley graph which is a Gromov hyperbolic metric space. More generally if H is a finitely generated nonelementary relatively hyperbolic group with proper peripheral subgroups, then H has an acylindrical action on its coned-off Cayley graph. Another interesting example is the mapping class group of a closed hyperbolic surface. It acts acylindrically and nonelementarily on the curve graph of the surface, which is a (nonproper) Gromov hyperbolic space.

Although Definition 12.1 of acylindrically hyperbolic groups is perhaps the cleanest to state, a characterization of acylindrically hyperbolic groups using contracting elements will be particularly well-suited for our purpose. We state such a characterization now, which follows directly from work of Osin and Sisto; a proof is included because we could not find a result stated in this form.

Theorem 12.2 [45; 49] *Suppose G has a proper isometric action on a geodesic metric space (X, d), and suppose that $(X, \mathcal{PS})$ is a path system for G and $g \in G$ is a contracting element for $(X, \mathcal{PS})$. Then either G is virtually cyclic, or G is acylindrically hyperbolic.*

Sketch of proof In [45], Osin introduces several characterizations of acylindrically hyperbolic groups that are equivalent to Definition 12.1. The one that we will use (Proposition 12.3) requires the notion of *hyperbolically embedded subgroups*. Results due to Osin and Sisto (see Propositions 12.3 and 12.4) will allow us to use this notion without defining it precisely. See [45, Definition 2.8] or [49, Definition 4.6].

Proposition 12.3 (Osin [45, Theorem 1.2 and Definition 1.3] and Remark 12.5) *A group G is acylindrically hyperbolic if G contains a proper infinite hyperbolically embedded subgroup.*

So in order to prove that a group is acylindrically hyperbolic, it suffices to produce a proper infinite hyperbolically embedded subgroup. For this, we rely on a result of Sisto.

Proposition 12.4 [49, Theorem 4.7] *Suppose $g \in G$ is a contracting element for $(X, \mathcal{PS})$ and $\mathcal{A} \subset X$ is $\langle g\rangle$-invariant, $\mathcal{PS}$-contracting and has cobounded $\langle g\rangle$-action. Then*

$$E(g) := \{h \in G \mid d^{\mathrm{Haus}}(\pi_{\mathcal{A}}(h\mathcal{A}), \mathcal{A}) < \infty\}$$

is a hyperbolically embedded subgroup of G which is infinite and contains $\langle g\rangle$ as a finite-index subgroup, ie $E(g)$ is virtually cyclic.

Now let us summarize how these results give us our desired conclusion. Suppose $g \in G$ is a contracting element. By Proposition 12.4, $E(g)$ is an infinite hyperbolically embedded subgroup of G which is virtually cyclic. Now note that if G is virtually cyclic, there is nothing to prove. So suppose that G is not virtually cyclic. Then $E(g) \subsetneq G$ as $E(g)$ is virtually cyclic. Thus $E(g)$ is a proper infinite hyperbolically embedded subgroup and Proposition 12.3 implies that G is an acylindrically hyperbolic group. See the following remark for further comments on the proof. □

Remark 12.5 Recall the alternate definition of an acylindrically hyperbolic group from Proposition 12.3. A subgroup $H \leq G$ is *proper infinite* if $H \subsetneq G$ and H is infinite. Such proper infinite hyperbolically embedded subgroups are sometimes called *nondegenerate* hyperbolically embedded subgroups in the terminology of [45; 29]. Notably, the existence of one such nondegenerate hyperbolically embedded subgroup $H \leq G$ implies the existence of nonabelian free subgroups in G; see [45, Lemma 5.12] or [29, Theorem 6.14]. Thus G is not virtually cyclic and contains infinitely many "independent loxodromic" elements [45; 29]. Roughly speaking, this is akin to producing nonabelian free subgroups in any nonelementary Gromov hyperbolic group.

12.2 Proof of Theorem 1.4

We first recall the theorem.

Theorem 1.4 *If (Ω, Γ) is a rank-one Hilbert geometry, then either Γ is virtually cyclic or Γ is an acylindrically hyperbolic group.*

The proof of Theorem 1.4 will be immediate from Theorem 12.2, thanks to the well-developed machinery of acylindrically hyperbolic groups due to the work of many authors; see for instance [45; 29; 49; 16]. In case the proof seems a bit opaque to a reader, we will first give an informal sketch of the underlying idea before providing a formal proof.

Our result Theorem 1.2 implies that rank-one isometries in $\mathrm{Aut}(\Omega)$ are contracting elements for $(\Omega, \mathcal{PS}^{\Omega})$. Thus a rank-one Hilbert geometry (Ω, Γ) contains contracting elements by definition. Now it is possible that Γ is virtually cyclic in which case Γ, up to passing to a finite-index subgroup, is generated by a single rank-one isometry. But if Γ is not virtually cyclic, then there will be infinitely many rank-one isometries $\gamma_1, \gamma_2, \ldots$ which are "independent loxodromics", ie there exists an abstract Gromov hyperbolic space X on which each γ_i acts "loxodromically" with exactly two distinct fixed points $\gamma_i^{\pm}$ and the sets $\{\gamma_i^{\pm}\}$ and $\{\gamma_j^{\pm}\}$ are pairwise disjoint whenever $i \neq j$. This last conclusion follows from results in [29] and [49] that we referred to in Remark 12.5. These infinitely many independent rank-one isometries γ_i generate nonabelian free subgroups of Γ, and the γ_i lie in distinct hyperbolically embedded subgroups $E(\gamma_i)$; see Proposition 12.4.

Now let us give the formal proof.

Proof of Theorem 1.4 Since (Ω, Γ) is a rank-one Hilbert geometry, Γ contains a rank-one isometry. Then Theorem 1.2 implies that Γ contains a contracting element for $(\Omega, \mathcal{PS}^{\Omega})$. The result follows from Theorem 12.2. □

Remark 12.6 By Theorem 1.4, a rank-one Hilbert geometry (Ω, Γ) where Γ is not virtually cyclic gives an example of an acylindrically hyperbolic group Γ. A natural question is: what is an example of a Gromov hyperbolic metric space X on which Γ acts acylindrically and nonelementarily? Is there a way to understand this space X in terms of the Hilbert geometry Ω?

It seems that one might be able to apply the projection complex construction in [16] (see also [14; 49]) to construct such a space X from the Hilbert geometry Ω. Roughly speaking, this will be a metric space obtained by collecting all rank-one axes in Ω and adding edges between them depending on diameters of images of some projection maps. We do not pursue this direction in this paper and this remark is mostly speculative in nature.

Part IV Applications

13 Second bounded cohomology and quasimorphisms

13.1 Definitions

We first introduce some definitions following [15, Section 1]. Suppose G is a group, $(E, \|\cdot\|)$ is a complete normed $\mathbb{R}$-vector space and $\rho\colon G \to \mathcal{U}(E)$ is a unitary representation. Let $C(G, E)$ be the space of all functions from G to E.

A function $F \in C(G, E)$ is called a quasicocycle if

$$\Delta(F) := \sup_{g, g' \in G} \|F(gg') - F(g) - \rho(g)F(g')\| < \infty.$$

Let V be the vector subspace of $C(G, E)$ that consists of all quasicocyles. Let V_0 be the subspace of V generated by bounded functions and the set

$$\{F : G \to E \mid F(gg') = F(g) + \rho(g)F(g') \text{ for all } g, g' \in G\}.$$

Define

$$\widetilde{\mathrm{QC}}(G;\rho) := V/V_0.$$

If ρ is the trivial representation $\rho_{\mathrm{triv}} : G \to \mathbb{R}$, then V is the space of quasimorphisms of G while V_0 is the space generated by bounded functions and group homomorphisms from G to $\mathbb{R}$. In this case, $\widetilde{\mathrm{QC}}(G;\rho_{\mathrm{triv}})$ recovers a classical object called the space of "nontrivial" quasimorphisms of G, usually denoted by $\widetilde{\mathrm{QH}}(G)$; see the definitions preceding Theorem 1.6.

Group cohomology of G (twisted by the representation ρ) affords an interesting interpretation of $\widetilde{\mathrm{QC}}(G;\rho)$. If F is a quasicocycle, then $dF(g, g') := F(gg') - F(g) - \rho(g)F(g')$ defines a class in the second bounded cohomology group $H^2_b(G;\rho)$. This class dF is trivial in the ordinary cohomology group $H^2(G;\rho)$. On the other hand, the class dF is nontrivial in $H^2_b(G;\rho)$ whenever F is nontrivial in V/V_0. Thus $\widetilde{\mathrm{QC}}(G;\rho)$ is the kernel of the comparison map $H^2_b(G,\rho) \to H^2(G;\rho)$. For a more detailed discussion, we refer the reader to [15, Section 1] or [31].

13.2 Results

Infinite dimensionality of $\widetilde{\mathrm{QH}}(G)$ and $\widetilde{\mathrm{QC}}(G;\rho)$ is often related to geometric phenomena. For example, [17] shows that a compact irreducible nonpositively curved Riemannian manifold M is (Riemannian) rank one if and only if $\dim(\widetilde{\mathrm{QH}}(\pi_1(M))) = \infty$. Now, in the same spirit as in Riemannian nonpositive curvature, we prove a cohomological characterization of rank-one Hilbert geometries. We will only consider unitary representations on uniformly convex Banach spaces,[3] eg $\mathbb{R}$ or $\ell^p(G)$, where G is a discrete group and $1 < p < \infty$.

Theorem 13.1 *Suppose that (Ω, Γ) is a rank-one Hilbert geometry, Γ is torsion-free and ρ is any unitary representation of Γ on a uniformly convex Banach space $E \neq 0$. Then either Γ is virtually cyclic or $\dim(\widetilde{\mathrm{QC}}(\Gamma;\rho)) = \infty$.*

The proof follows directly from the following general result about acylindrically hyperbolic groups.

Theorem 13.2 [15, Corollary 1.2] *If G is an acylindrically hyperbolic group, $E \neq 0$ is a uniformly convex Banach space, $\rho : G \to \mathcal{U}(E)$ is a unitary representation and any maximal finite normal subgroup of G has a nonzero fixed vector, then $\dim(\widetilde{\mathrm{QC}}(G;\rho)) = \infty$.* □

Proof of Theorem 13.1 If Γ is not virtually cyclic, then Theorem 1.4 implies that Γ is an acylindrically hyperbolic group. Since Γ is torsion-free, there are no finite normal subgroups. The claim then follows from Theorem 13.2. □

[3] A Banach space E is uniformly convex if for any $\varepsilon' > 0$, there exists $\delta' > 0$ such that if $u, v \in E$, $\|u\| \le 1$, $\|v\| \le 1$, $\|u - v\| \ge \varepsilon'$, then $\|(u+v)/2\| \le 1 - \delta'$.

We will now apply Theorem 13.1 to two specific choices of ρ and E to get Theorem 1.6. For the first, $\rho = \rho_{\mathrm{triv}}$ and $E = \mathbb{R}$, in which case $\widetilde{\mathrm{QC}}(\Gamma;\rho) = \widetilde{\mathrm{QH}}(\Gamma)$, the space of nontrivial quasimorphisms. For the second, $E = \ell^p(\Gamma)$ with $1 < p < \infty$ and $\rho = \rho_{\mathrm{reg}}^p$ is the regular representation, ie $\rho_{\mathrm{reg}}^p(\gamma) f(x) = f(\gamma^{-1}x)$ for any $f \in \ell^p(\Gamma)$ and $x \in \Gamma$.

Theorem 1.6 *If* (Ω,Γ) *is a rank-one Hilbert geometry*, Γ *is torsion-free and* Γ *is not virtually cyclic*, *then* $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = \infty$ *and* $\dim(\widetilde{\mathrm{QC}}(\Gamma;\rho_{\mathrm{reg}}^p)) = \infty$ *if* $1 < p < \infty$.

Proof Immediate from Theorem 13.1 and the fact that $\mathbb{R}$ and $\ell^p(\Gamma)$ with $1 < p < \infty$ are uniformly convex Banach spaces; see [15, Section 3]. □

Corollary 1.7 *If* (Ω,Γ) *is a divisible Hilbert geometry and* Ω *is irreducible*, *then* $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = \infty$ *if and only if* (Ω,Γ) *is a rank-one Hilbert geometry. Otherwise*, $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = 0$.

Proof If (Ω,Γ) is a rank-one Hilbert geometry, then Theorem 1.6 implies that $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = \infty$. If (Ω,Γ) is not rank one, then Theorem 1.5 implies that $\mathrm{Aut}(\Omega)$ is locally isomorphic to a simple Lie group of real rank at least two, ie Ω is an irreducible symmetric domain of rank at least two. Thus Γ is isomorphic to a uniform lattice in a higher-rank simple Lie group, which implies that $\dim(\widetilde{\mathrm{QH}}(\Gamma)) = 0$ [21, Theorem 21]. □

14 Counting of conjugacy classes

Suppose (Ω,Γ) is a rank-one Hilbert geometry. Recall the notions of translation length and stable translation length of a conjugacy class in Γ; see Section 1.2. We now introduce the notion of pointed length for a conjugacy class $[c_g]$ of $g \in \Gamma$; see [32]. Fix a basepoint $p \in \Omega$. The pointed length of $[c_g]$ is

$$\mathcal{L}_p([c_g]) := \inf_{g' \in [c_g]} \mathrm{d}_\Omega(p, g'p).$$

We first show that

$$\tau_\Omega([c_g]) = \tau_\Omega^{\mathrm{stable}}([c_g]).$$

Indeed, triangle inequality implies $\tau_\Omega^{\mathrm{stable}}(g) \le \tau_\Omega(g)$. On the other hand, by Proposition 3.15,

$$\tau_\Omega^{\mathrm{stable}}(g) \ge \lim_{n\to\infty} \frac{\tau_\Omega(g^n)}{n} = \frac{1}{n}\log\frac{\lambda_{\max}(\tilde{g}^n)}{\lambda_{\min}(\tilde{g}^n)} = \log\frac{\lambda_{\max}(\tilde{g})}{\lambda_{\min}(\tilde{g})} = \tau_\Omega(g).$$

Next, we show that if Ω/Γ is compact and $R := \mathrm{diam}(\Omega/\Gamma)$, then

$$\tau_\Omega([c_g]) \le \mathcal{L}_p([c_g]) \le \tau_\Omega([c_g]) + 2R.$$

Clearly $\tau_\Omega([c_g]) \le \mathcal{L}_p([c_g])$. On the other hand, if $x \in \Omega$ then there exists $h_x \in \Gamma$ such that $\mathrm{d}_\Omega(x, h_x p) \le R$. Then

$$\mathcal{L}_p([c_g]) \le \mathrm{d}_\Omega(p, h_x^{-1} g h_x p) \le 2\,\mathrm{d}_\Omega(h_x p, x) + \mathrm{d}_\Omega(x, gx) \le 2R + \mathrm{d}_\Omega(x, gx).$$

Thus, $\mathcal{L}_p([c_g]) \le \tau_\Omega([c_g]) + 2R$.

Now let us consider the following counting functions for conjugacy classes in Γ:

$$\mathscr{C}(t) := \#\{[c_g] \mid g \in \Gamma, \tau_\Omega([c_g]) \le t\},$$
$$\mathscr{C}^{\text{stable}}(t) := \#\{[c_g] \mid g \in \Gamma, \tau_\Omega^{\text{stable}}([c_g]) \le t\},$$
$$\mathscr{C}^{\mathscr{L}_p}(t) := \#\{[c_g] \mid g \in \Gamma, \mathscr{L}_p([c_g]) \le t\}.$$

Based on the above discussion,

$$\mathscr{C}(t) = \mathscr{C}^{\text{stable}}(t). \tag{17}$$

If Ω/Γ is compact and $R = \operatorname{diam}(\Omega/\Gamma)$, then

$$\mathscr{C}^{\mathscr{L}_p}(t) \le \mathscr{C}(t) \le \mathscr{C}^{\mathscr{L}_p}(t+2R). \tag{18}$$

We now prove asymptotic growth formula for these functions. It is a direct consequence of the Main Theorem in [32]. Recall that the critical exponent of Γ (see Section 1.2) is defined by

$$\omega_\Gamma := \limsup_{n\to\infty} \frac{\log \#\{g \in \Gamma \mid d_\Omega(x, gx) \le n\}}{n}.$$

Theorem 1.8 *Suppose (Ω, Γ) is a divisible rank-one Hilbert geometry and Γ is not virtually cyclic. Then there exists a constant D' such that for all $t \ge 1$,*

$$\frac{1}{D'}\frac{\exp(t\omega_\Gamma)}{t} \le \mathscr{C}(t) \le D'\frac{\exp(t\omega_\Gamma)}{t}. \tag{19}$$

The functions $\mathscr{C}^{\text{stable}}(t)$, $\mathscr{C}^{\mathscr{L}_p}(t)$ and $\mathscr{C}_{\text{Prim}}(t)$ (see Remark 1.9) also satisfy similar growth formulas.

Proof Part (1) of the Main Theorem in [32] implies that if Γ is a nonelementary group with a cocompact action (more generally, statistically convex cocompact action) on a geodesic metric space and Γ contains a contracting element (in the sense of BF, see Appendix B), then $\mathscr{C}^{\mathscr{L}_p}(t)$ satisfies the growth formula in (19). If (Ω, Γ) is as above, then it satisfies all of these conditions; see Theorem 1.2 and Remark 9.7. Then $\mathscr{C}^{\mathscr{L}_p}(t)$ satisfies equation (19). By equations (17) and (18), $\mathscr{C}(t)$ and $\mathscr{C}^{\text{stable}}(t)$ also satisfy equation (19).

For proving Remark 1.9 part (ii), set

$$\mathscr{C}^{\mathscr{L}_p}_{\text{Prim}}(t) := \#\{[c_g] \mid g \in \Gamma \text{ is primitive}, \mathscr{L}_p([c_g]) \le t\}.$$

Part (1) of the Main Theorem in [32] implies that the $\mathscr{C}^{\mathscr{L}_p}_{\text{Prim}}(t)$ satisfies a similar growth formula as (19). Since $\tau_\Omega([c_g]) \le \mathscr{L}_p([c_g]) \le \tau_\Omega([c_g]) + 2R$, this implies the result for $\mathscr{C}_{\text{Prim}}(t)$. □

15 Proofs of Propositions 1.10, 1.11 and 1.12

For the proofs in this section, recall the following implication of Theorem 1.4: if (Ω, Γ) is a rank-one Hilbert geometry and Γ is not virtually cyclic, then Γ is an acylindrically hyperbolic group.

Proposition 1.10 *If (Ω, Γ) is a rank-one Hilbert geometry, Γ is not virtually cyclic and Γ is finitely generated, then the rank-one isometries in Γ are exponentially generic: if $(X_n)_{n\in\mathbb{N}}$ is a simple random walk on Γ, then there exists a constant $C \geq 1$ such that for all $n \geq 1$,*

$$\mathbb{P}[X_n \text{ is not a rank-one isometry}] \leq Ce^{-n/C}.$$

Proof Under the hypotheses, Γ is an acylindrically hyperbolic group. The result then follows from [49, Theorem 1.6]. □

Proposition 1.11 *If (Ω, Γ) is a rank-one Hilbert geometry and Γ is not virtually cyclic, then:*

(i) *Γ is SQ-universal, ie every countable group embeds in a quotient of Γ.*

(ii) *If Γ is the Baumslag-Solitar group $\mathrm{BS}(m,n)$, then $m = n = 0$ and Γ is the free group on two generators.*

Proof Under the hypotheses, Γ is an acylindrically hyperbolic group. Then SQ-universality follows from [45, Theorem 8.1]. The second part follows from [45, Example 7.4], where Osin proves that $\mathrm{BS}(m,n)$ is acylindrically hyperbolic if and only if $m = n = 0$. But $\mathrm{BS}(0,0) = F_2$. □

Proposition 1.12 *If Ω is a Hilbert geometry and $\gamma \in \mathrm{Aut}(\Omega)$ is a rank-one isometry, then the axis ℓ_γ of γ is $\mathcal{K}$-Morse for some Morse gauge $\mathcal{K}\colon [1,\infty)\times[0,\infty)\to[0,\infty)$, ie if α is a (λ,ε)-quasigeodesic with endpoints on ℓ_γ, then $\alpha \subset \mathcal{N}_{\mathcal{K}(\lambda,\varepsilon)}(\ell_\gamma)$.*

Proof Since γ is a rank-one isometry, Theorem 1.2 implies that γ is a contracting element for $(\Omega, \mathcal{PS}^\Omega)$. Then the axis of ℓ_γ of γ is $\mathcal{PS}^\Omega$-contracting. Thus [49, Lemma 2.8] (Proposition 11.2 in this paper) implies that ℓ_γ is a Morse geodesic. □

Appendix A Rank-one Hilbert geometries: generalization, examples and nonexamples

This section is devoted to the discussion of examples and nonexamples of rank-one Hilbert geometries (see Definition 1.3) and generalizing the notion of rank one to convex cocompact actions.

A.1 Strictly convex examples

If Ω is a strictly convex Hilbert geometry, then $\partial\Omega$ does not contain any line segments. Thus, if $g \in \mathrm{Aut}(\Omega)$ with $\tau_\Omega(g) > 0$, then g is a rank-one isometry (as it has an axis and there are no half triangles in Ω). Then, all the strictly convex divisible examples in Section 3.4 are rank one.

A.2 Non-strictly-convex examples

Suppose (Ω, Γ) is a divisible Hilbert geometry where Γ is infinite and not virtually abelian. Assume that Γ is a relatively hyperbolic group with respect to a finite collection of free abelian subgroups of rank at least two. Then we claim that (Ω, Γ) *is a divisible rank-one Hilbert geometry*. The proof of this follows from Remark A.3(B) and Proposition A.4; see below. This claim implies that the divisible nonstrictly convex examples discussed in Section 3.4, that are neither simplices nor symmetric domains of rank at least two, are all examples of rank-one Hilbert geometries.

A.3 Nonexamples

The d-simplices T_d for $d \geq 2$ are clearly nonexamples of rank-one Hilbert geometries. If Ω is an irreducible symmetric domain of rank at least two and $\Gamma \leq \mathrm{Aut}(\Omega)$ acts cocompactly on Ω, then (Ω, Γ) *cannot be a rank-one Hilbert geometry*; see Theorem 1.5.

A.4 Generalization of rank one to convex cocompact actions

The notion of convex cocompact actions on Hilbert geometries [30] generalizes divisible Hilbert geometries. Suppose Ω is a Hilbert geometry and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete subgroup. The *full orbital limit set* is defined as $\mathcal{L}^{\mathrm{orb}}_\Omega(\Gamma) := \bigcup_{x \in \Omega}(\overline{\Gamma \cdot x} \cap \partial\Omega)$ and let $\mathcal{C}^c_\Omega(\Gamma) := \mathrm{ConvHull}_\Omega(\mathcal{L}^{\mathrm{orb}}_\Omega(\Gamma))$.

Definition A.1 An infinite discrete group $\Gamma \leq \mathrm{Aut}(\Omega)$ is *convex cocompact* if $\mathcal{C}^c_\Omega(\Gamma) \neq \varnothing$ and $\mathcal{C}^c_\Omega(\Gamma)/\Gamma$ is compact.

The *ideal boundary* of $\mathcal{C}^c_\Omega(\Gamma)$ is given by $\partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma) := \partial\Omega \cap \overline{\mathcal{C}^c_\Omega(\Gamma)}$. For convex cocompact groups, $\partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$ is the only part of $\partial\Omega$ "visible" to the group acting on Ω. Thus it is natural to modify the notion of rank-one isometries by considering half triangles in $\mathcal{C}^c_\Omega(\Gamma)$ instead of Ω. We say that the projective geodesic $(a, b) \subset \mathcal{C}^c_\Omega(\Gamma)$ is *not contained in any half triangle in* $\mathcal{C}^c_\Omega(\Gamma)$ if either $(a, z) \subset \mathcal{C}^c_\Omega(\Gamma)$ or $(z, b) \subset \mathcal{C}^c_\Omega(\Gamma)$ for any $z \in \partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$.

Definition A.2 Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ is a Hilbert geometry and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact group.

- (i) An element $\gamma \in \Gamma$ is a *convex cocompact rank-one isometry* if
  - (a) $\log|(\lambda_1/\lambda_{d+1})(\gamma)| > 0$ and γ has an axis (see Definition 5.1),
  - (b) none of the axes ℓ_γ of γ are contained in a half triangle in $\mathcal{C}^c_\Omega(\Gamma)$.
- (ii) We say that Γ is a *rank-one convex cocompact group* if Γ contains a convex cocompact rank-one isometry.

Remark A.3 (A) The notion of a *convex cocompact rank-one isometry* differs from the notion of a *rank-one isometry* (see Definition 6.3) only in condition (i)(b): for convex cocompact actions, we consider half triangles in $\mathcal{C}^c_\Omega(\Gamma)$ instead of Ω.

(B) If Γ acts cocompactly on Ω, then (Ω, Γ) is a *divisible rank-one Hilbert geometry* if and only if Γ is a *rank-one convex cocompact group*. This is because divisibility implies $\mathcal{C}^c_\Omega(\Gamma) = \Omega$.

If $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact group and $\gamma \in \Gamma$ is a convex cocompact rank-one isometry, then the analogues of Propositions 6.5, 6.7 and 6.8 hold. But now we need to replace Ω with $\mathcal{C}^c_\Omega(\Gamma)$ and $\partial\Omega$ with $\partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$. In particular, we have that $\gamma \in \Gamma$ is a convex cocompact rank-one isometry if and only if γ is biproximal and has an axis.

We sketch the proof ideas of these analogues; see [37] for details. Observe that if $|(\lambda_1/\lambda_{d+1})(\gamma)| > 0$, then $E^\pm_\gamma \cap \partial\Omega = E^\pm_\gamma \cap \partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$. Recall that if $x \in \partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$, then

$$F_{\mathcal{C}^c_\Omega(\Gamma)}(x) := \{x\} \cup \{y \in \overline{\mathcal{C}^c_\Omega(\Gamma)} \mid \text{an open projective line segment in } \overline{\mathcal{C}^c_\Omega(\Gamma)} \text{ contains } x \text{ and } y\}.$$

Convex cocompact groups have a special property: if $x \in \partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$, then $F_{\mathcal{C}^c_\Omega(\Gamma)}(x) = F_\Omega(x)$; see [30, Corollary 4.13]. Using these properties, one can now see that the proofs in Section 6 go through verbatim after replacing Ω by $\mathcal{C}^c_\Omega(\Gamma)$ and $\partial\Omega$ by $\partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$. Thus the analogues of Propositions 6.5, 6.7 and 6.8 hold; also see [37].

A.5 Convex cocompact examples: hyperbolic groups

Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact group that is word hyperbolic. We claim that Γ *is a rank-one convex cocompact group*. Indeed, [30, Theorem 1.15] implies that word hyperbolicity of Γ is equivalent to the property that $\partial_{\mathrm{i}}\mathcal{C}^c_\Omega(\Gamma)$ does not contain any nontrivial projective line segments. Then there are no half triangles in $\mathcal{C}^c_\Omega(\Gamma)$. Moreover, any infinite-order element γ has an axis [30, Corollary 7.4]. Thus every such γ is a convex cocompact rank-one isometry and the claim follows.

A.6 Convex cocompact examples: relatively hyperbolic groups

Proposition A.4 *Suppose $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact group that is relatively hyperbolic with respect to $\{A_1, A_2, \dots, A_m\}$, where each A_i is a virtually free abelian group of rank at least two. Then Γ is either a rank-one convex cocompact group or a virtually abelian group.*

This proposition shows that the divisible examples of Section A.2 and their convex cocompact deformations produce relatively hyperbolic examples that are rank-one convex cocompact. We will spend the rest of this subsection proving this proposition. We will rely on results from [39].

Proof Let $\mathcal{S}_\Gamma$ be the collection of all maximal properly embedded simplices in $\mathcal{C}^c_\Omega(\Gamma)$ of dimension at least two. Since Γ is relatively hyperbolic with respect to virtually abelian subgroups of rank at least two, [39, Theorem 1.7] implies that $(\mathcal{C}^c_\Omega(\Gamma), \mathrm{d}_\Omega)$ is a *Hilbert geometry with isolated simplices*, ie $\mathcal{S}_\Gamma$ is closed and discrete in the local Hausdorff topology induced by d_Ω. In this case, [39, Theorem 1.18] implies that for each $i \in \{1, \dots, m\}$, we can assume $A_i = \mathrm{Stab}_\Gamma(S_i)$, where S_i is a maximal properly embedded simplex in $\mathcal{C}^c_\Omega(\Gamma)$ of dimension ≥ 2 and $\mathcal{S}_\Gamma = \bigsqcup_{i=1}^m \Gamma \cdot S_i$. We will require the following result regarding simplices in $\mathcal{S}_\Gamma$.

Proposition A.5 [39, Theorem 1.8] *Suppose Γ and $\mathcal{S}_\Gamma$ are as above. Then*:

(i) *If $[x,y] \subset \partial_{\mathrm{i}}\mathcal{C}_\Omega^c(\Gamma)$ with $x \neq y$, then there exists $S \in \mathcal{S}_\Gamma$ such that $[x,y] \subset \partial S$.*

(ii) *If $S_1 \neq S_2 \in \mathcal{S}_\Gamma$, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \varnothing$.*

Since Γ is relatively hyperbolic with respect to $\{A_1, A_2, \dots, A_m\}$, [28, Lemma 2.3] implies either

- **Case 1** Γ is virtually $gA_i g^{-1}$ for some $g \in \Gamma$ and $1 \leq i \leq m$, or
- **Case 2** there exists $\gamma \in \Gamma$ such that $\gamma \notin \bigcup_{g\in\Gamma} \bigcup_{i=1}^m gA_i g^{-1} = \bigcup_{S\in\mathcal{S}_\Gamma} \mathrm{Stab}_\Gamma(S)$.

In Case 1, Γ is a virtually abelian group. So we can now assume that we are in Case 2.

Claim *If γ is as in Case 2, then γ is a convex cocompact rank-one isometry.*

From this claim, Proposition A.4 is immediate. □

All that remains is to prove this claim.

Proof of claim As Γ is a convex cocompact group, $\log|(\lambda_1/\lambda_{d+1})(\gamma)| = \tau_{\mathcal{C}_\Omega^c(\Gamma)}(\gamma) > 0$. We first show that γ has an axis in $\mathcal{C}_\Omega^c(\Gamma)$. Let

$$\mathcal{C}^+ := \overline{E_\gamma^+ \cap \mathcal{C}_\Omega^c(\Gamma)} \quad \text{and} \quad \mathcal{C}^- := \overline{E_\gamma^- \cap \mathcal{C}_\Omega^c(\Gamma)}.$$

Then $\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint, nonempty, compact, convex, γ-invariant subsets of $\mathbb{R}^d$. Then the Brouwer fixed point theorem implies the existence of distinct fixed points $\gamma^\pm$ of γ in $\mathcal{C}^\pm$. If $[\gamma^+,\gamma^-] \subset \partial_{\mathrm{i}}\mathcal{C}_\Omega^c(\Gamma)$, Proposition A.5 implies that there exists $S \in \mathcal{S}_\Gamma$ such that $[\gamma^+,\gamma^-] \subset \partial S$. Then $\partial(\gamma S) \cap \partial S \supset [\gamma^+,\gamma^-]$ and Proposition A.5 implies that $\gamma S = S$. Thus, $\gamma \in \mathrm{Stab}_\Gamma(S)$. This contradiction implies that $(\gamma^+,\gamma^-) \subset \mathcal{C}_\Omega^c(\Gamma)$ and is an axis of γ.

Suppose $A_\gamma := [A_\gamma^+, A_\gamma^-]$ is an axis of γ contained in a half triangle in $\mathcal{C}_\Omega^c(\Gamma)$: $[A_\gamma^+, z] \cup [z, A_\gamma^-] \subset \partial_{\mathrm{i}}\mathcal{C}_\Omega^c(\Gamma)$. Then, by Proposition A.5, there exist $S^\pm \in \mathcal{S}_\Gamma$ such that $[z, A_\gamma^\pm] \subset \partial S^\pm$. Since $z \in \partial S^+ \cap \partial S^-$, Proposition A.5 implies that $S := S^+ = S^-$ and $A_\gamma \subset S$. Since γ acts by a translation along A_γ, $\gamma S \cap S \supset A_\gamma$ which implies $\#(\gamma S \cap S) = \infty$. Then by Proposition A.5, $\gamma S = S$. Thus $\gamma \in \mathrm{Stab}_\Gamma(S)$, a contradiction. Thus A_γ is not contained in any half triangle in $\mathcal{C}_\Omega^c(\Gamma)$. This proves the claim. □

Appendix B Contracting elements

Fix a proper geodesic metric space (X, d) and a group G that acts properly isometrically on X. If $x \in X$ and $R > 0$, let $B(x,R) := \{y \in X \mid \mathrm{d}(x,y) < R\}$. If $\mathcal{A} \subset X$ and $x \in X$, let the closest-point projection onto $\mathcal{A}$ be defined by $\rho_{\mathcal{A}}(x) := \{y \in \mathcal{A} \mid \mathrm{d}(x,y) = \mathrm{d}(x,\mathcal{A})\}$. We let $\mathcal{N}_r(\mathcal{A}) := \{y \in X \mid \mathrm{d}(y,\mathcal{A}) < r\}$ and $\overline{\mathcal{N}_r(\mathcal{A})} := \{y \in X \mid \mathrm{d}(y,\mathcal{A}) \leq r\}$) denote the open and the closed r-neighborhoods of $\mathcal{A}$, respectively.

In [17], Bestvina and Fujiwara introduced the following notion of contracting subsets.

Definition B.1 A set $\mathcal{A} \subset X$ is B-contracting if there exists a constant B such that if $x \in X$, $R > 0$ and $B(x,R) \cap \mathcal{A} = \varnothing$, then $\mathrm{diam}\big(\rho_{\mathcal{A}}(B(x,R))\big) \leq B$.

We will, however, use a related but stronger notion of contracting subsets introduced in [32].

Definition B.2 [32] Fix a geodesic path system $\mathcal{PS}$ on X; see Definition 9.1. A set $\mathcal{A} \subset X$ is a *contracting subset in the sense of BF* if there exists a constant C such that if $\sigma \subset X$ is a geodesic in $\mathcal{PS}$ for which $\mathrm{d}(\sigma, \mathcal{A}) > C$, then

$$\operatorname{diam}(\rho_{\mathcal{A}}(\sigma)) \leq C.$$

Suppose G preserves $\mathcal{PS}$. Then $g \in G$ is a *contracting element in the sense of BF* if for any $x_0 \in X$,

(i) g has infinite order and $\langle g \rangle \cdot x_0$ is a quasi-isometric embedding of $\mathbb{Z}$ in X, and

(ii) $\langle g \rangle \cdot x_0$ is a contracting subset in the sense of BF.

Remark B.3 If X is a proper CAT(0) geodesic metric space, Bestvina and Fujiwara [17, Corollary 3.4] prove that Definitions B.1 and B.2 are equivalent. In fact, they prove this equivalence for any metric space that satisfies their axioms DD and FT; see [17]. However, it is unclear whether Definitions B.1 and B.2 are equivalent in complete generality. We will discuss this in Proposition B.6 below. Proposition B.6 suggests that it is unlikely that these definitions are equivalent in general.

We will now prove Proposition 9.8, that contraction in the sense of BF is equivalent to Sisto's notion (see Definitions 9.2 and 9.4), in the context of a geodesic path system. Before starting the proof, we record the following immediate consequence of Definition B.2.

Lemma B.4 *Suppose $A \subset X$ is* ***contracting in the sense of BF*** *with constant C. Let $\sigma_{x,x'} \in \mathcal{PS}$ be a geodesic joining x and x' such that $\sigma_{x,x'} \cap \overline{\mathcal{N}_{2C}(A)} = \{x'\}$. Then $\sup_{a \in \rho_A(x)} \mathrm{d}(a, x') \leq 3C$.*

Proof Since $\mathrm{d}(\sigma_{x,x'}, A) \geq 2C > C$, it follows that $\mathrm{d}(y, y') \leq C$ for any $y \in \rho_A(x)$ and any $y' \in \rho_A(x')$. But $\mathrm{d}(a', x') \leq 2C$ for any $a' \in \rho_A(x')$. Hence the conclusion. □

Proposition B.5 (Proposition 9.8) *Suppose $(X, \mathcal{PS})$ is a geodesic path system. Then*:

(i) *$\mathcal{A} \subset X$ is $\mathcal{PS}$-contracting if and only if $\mathcal{A}$ is contracting in the sense of BF.*

(ii) *If G preserves $\mathcal{PS}$, then $g \in G$ is a contracting element for $(X, \mathcal{PS})$ if and only if $g \in G$ is a contracting element in the sense of BF.*

Proof It suffices to prove only part (i), as (ii) then follows from definitions. We will use $\sigma_{p,q}$ to denote a geodesic path in $\mathcal{PS}$ joining p and q. We now start the proof of (i).

($\Longrightarrow$) Suppose $\mathcal{A}$ is contracting in the sense of BF. Define a projection map $\pi\colon X \to \mathcal{A}$ by choosing $\pi(x) \in \rho_{\mathcal{A}}(x)$ for each $x \in X$. We remark that such a map π is coarsely unique, ie any other such map π' has the property that $\mathrm{d}(\pi(x), \pi'(x)) \leq 2C$ for any $x \in X$. Indeed, for any x such that $\mathrm{d}(x, \mathcal{A}) > C$, we have that $\operatorname{diam}(\rho_{\mathcal{A}}(x)) \leq C$ as $\{x\}$ is a geodesic in $\mathcal{PS}$. On the other hand, if $\mathrm{d}(x, \mathcal{A}) \leq C$, then $\sup_{a \in \rho_{\mathcal{A}}(x)} \mathrm{d}(x, a) \leq C$ and hence $\operatorname{diam}(\rho_{\mathcal{A}}(x)) \leq 2C$. Hence the remark.

We now show that π satisfies Definition 9.2 with constant $3C$. Clearly, if $x \in \mathscr{A}$, then $\mathrm{d}(x, \pi(x)) = 0$. Now suppose that $x, y \in X$ is such that $\mathrm{d}(\pi(x), \pi(y)) \geq 3C$. Then $\mathrm{diam}(\rho_{\mathscr{A}}(\sigma_{x,y})) \geq 3C > C$. Then $d(\sigma_{x,y}, \mathscr{A}) \leq C$ and thus $\sigma_{x,y} \cap \overline{\mathcal{N}_{2C}(\mathscr{A})} \neq \varnothing$. Let x' be the first point along $\sigma_{x,y}$ that intersects $\overline{\mathcal{N}_{2C}(\mathscr{A})}$ (assume that $\sigma_{x,y}$ is continuously parametrized in the direction from x to y). If $x = x'$, then $\mathrm{d}(\pi(x), x') \leq 2C$. Otherwise apply Lemma B.4 to $\sigma_{x,x'} \subset \sigma_{x,y}$ to see that $\mathrm{d}(\pi(x), x') \leq 3C$. Similarly, if y' is the last point along $\sigma_{x,y}$ where $\sigma_{x,y}$ intersects $\overline{\mathcal{N}_{2C}(A)}$, then $\mathrm{d}(y', \pi(y)) \leq 3C$. Thus π is a contracting projection with constant $3C$.

($\Longleftarrow$) Suppose $\pi\colon X \to \mathscr{A}$ is a contracting projection with constant C. By Lemma 9.3, it follows that $\sup_{a \in \rho_{\mathscr{A}}(x)} \mathrm{d}(a, \pi(x)) \leq 2C$ for any $x \in X$. Let $\sigma_{x,y} \in \mathcal{PS}$ be such that $\mathrm{d}(\sigma_{x,y}, \mathscr{A}) > 5C$. If possible, let there exist $a_1 \in \rho_{\mathscr{A}}(x)$ and $b_1 \in \rho_{\mathscr{A}}(y)$ such that $\mathrm{d}(a_1, b_1) > 5C$. Then

$$\mathrm{d}(\pi(x), \pi(y)) \geq \mathrm{d}(a_1, b_1) - \mathrm{d}(a_1, \pi(x)) - \mathrm{d}(b_1, \pi(y)) > C.$$

Then, $\sigma_{x,y}$ must intersect $\mathcal{N}_C(\mathscr{A})$, a contradiction. □

B.1 Comparison between Definitions B.1 and B.2

To discuss the relationship between B.1 and B.2 for a general metric space, we need the following condition (♦). We will say that $A \subset X$ *satisfies* (♦) *if there exists a constant C such that for any $x \in X$, $z \in A$ and $a \in \rho_A(x)$,*

$$\mathrm{d}(x, z) \geq \mathrm{d}(x, a) + \mathrm{d}(a, z) - C.$$

We will now show:

Proposition B.6 *Fix a proper geodesic metric space X and a geodesic path system $\mathcal{PS}$ on X. Then $A \subset X$ is contracting in the sense of BF if and only if it satisfies (♦) and Definition B.1.*

The implication ($\Longrightarrow$) follows from [48, Lemma 2.10]. Note that in [48], condition (♦) is called (AP1) while Definition B.1 is called (AP2). This direction is then immediate from [48, Lemma 2.10]. The proof of the converse ($\Longleftarrow$) follows from the next two lemmas. For $p, q \in X$, we will denote by $\sigma_{p,q}$ a geodesic in $\mathcal{PS}$ joining p and q.

Lemma B.7 *Suppose $A \subset X$ satisfies Definition B.2. Then A satisfies (♦).*

Proof Fix any $x \in X$, $z \in A$ and $a \in \rho_A(x)$. It suffices to only consider the case when $\mathrm{d}(x, A) > 2C$. Let $x' \in \sigma_{x,z}$ be the first point along $\sigma_{x,z}$ that intersects $\overline{\mathcal{N}_{2C}(A)}$ (assume that $\sigma_{x,z}$ is continuously parametrized in the direction from x to z). By Lemma B.4, $\mathrm{d}(x', a) \leq 3C$. Then $\mathrm{d}(x, a) - \mathrm{d}(x, x') \leq \mathrm{d}(x', a) \leq 3C$ and $\mathrm{d}(z, a) - \mathrm{d}(z, x') \leq \mathrm{d}(x', a) \leq 3C$. Since $x' \in \sigma_{x,z}$, it follows that $\mathrm{d}(x, z) = \mathrm{d}(x, x') + \mathrm{d}(x', z)$. Thus

$$\mathrm{d}(x, z) - \mathrm{d}(x, a) - \mathrm{d}(a, z) = (\mathrm{d}(x, x') - \mathrm{d}(x, a)) + (\mathrm{d}(x', z) - \mathrm{d}(a, z)) \geq -6C.$$ □

Lemma B.8 *Suppose $\mathscr{A} \subset X$ satisfies Definition B.2. Then $\mathscr{A}$ also satisfies Definition B.1.*

Proof Proposition B.5 implies that $\mathscr{A}$ is $\mathcal{PS}$-contracting. Then there exists a projection map $\pi_{\mathscr{A}}\colon X \to \mathscr{A}$ with constant C satisfying Definition 9.2. Suppose $x \in X$ and $0 < R < d(x, \mathscr{A})$. We claim that

$\operatorname{diam}(\rho_{\mathscr{A}}(B(x,R))) \le 20C$. By Lemma 9.3, it suffices to prove that $\mathrm{d}(\pi_{\mathscr{A}}(x), \pi_{\mathscr{A}}(y)) \le 8C$ for any $y \in B(x,R)$.

Fix $y \in B(x,R)$ and let $\sigma_{x,y} \in \mathscr{PS}$. Without loss of generality, we can assume that $\mathrm{d}(\pi_{\mathscr{A}}(x), \pi_{\mathscr{A}}(y)) \ge C$. Then there exists $x_1 \in \sigma_{x,y}$ such that $\mathrm{d}(x_1, \pi_{\mathscr{A}}(x)) \le C$. Then

$$|\mathrm{d}(x,x_1) - \mathrm{d}(x,\mathscr{A})| \le \mathrm{d}(x_1,\pi_{\mathscr{A}}(x)) + \sup_{a \in \rho_{\mathscr{A}}(x)} \mathrm{d}(a, \pi_{\mathscr{A}}(x)) \le 3C.$$

Thus, $\mathrm{d}(y,x_1) = \mathrm{d}(y,x) - \mathrm{d}(x,x_1) \le \mathrm{d}(y,x) - \mathrm{d}(x,\mathscr{A}) + 3C$. As $\mathrm{d}(y,x) < \mathrm{d}(x,\mathscr{A})$, we get that $\mathrm{d}(y,x_1) \le 3C$. Then

$$\mathrm{d}(y,\pi_{\mathscr{A}}(y)) \le \mathrm{d}(y,\pi_{\mathscr{A}}(x)) \le \mathrm{d}(y,x_1) + \mathrm{d}(x_1,\pi_{\mathscr{A}}(x)) \le 4C,$$

which implies that

$$\mathrm{d}(\pi_{\mathscr{A}}(y),\pi_{\mathscr{A}}(x)) \le \mathrm{d}(\pi_{\mathscr{A}}(y),y) + \mathrm{d}(y,x_1) + \mathrm{d}(x_1,\pi_{\mathscr{A}}(x)) \le 8C. \qquad \square$$

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Department of Mathematics, University of Michigan
Ann Arbor, MI, United States

Current address: *Max Planck Institute for Mathematics in the Sciences*
Leipzig, Germany

`mitul.islam@gmail.com`

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