Geometry & Topology

Volume 29 (2025)

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Geometry & Topology **29**:3 (2025) 1571–1618
DOI: 10.2140/gt.2025.29.1571
Published: 31 May 2025

Exotic Dehn twists on sums of two contact 3-manifolds

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We exhibit the first examples of exotic contactomorphisms with infinite order as elements of the contact mapping class group. These are given by certain Dehn twists on the separating sphere in a connected sum of two closed contact 3-manifolds. We detect these by a combination of hard and soft techniques. We make essential use of an invariant for families of contact structures which generalizes the Kronheimer–Mrowka contact invariant in monopole Floer homology. We then exploit an h-principle for families of convex spheres in tight contact 3-manifolds, from which we establish a parametric version of Colin's decomposition theorem. As a further application, we exhibit new exotic 1-parametric phenomena in overtwisted contact 3-manifolds.

53D35, 57K33, 57R58

1 Introduction

Throughout this article all 3-manifolds are closed, oriented and connected unless otherwise noted, and all contact structures on 3-manifolds are co-oriented and positive.

1.1 Main result

A fundamental problem in contact topology is to understand the isotopy classes of contact diffeomorphisms, usually called "contactomorphisms", of a contact manifold. The following is a longstanding open question in all dimensions:

Question 1.1 *Do there exist exotic contactomorphisms with infinite order as elements in the contact mapping class group?*

In this article we answer this question in the *affirmative* in dimension *three*. Here, and throughout the article, by *exotic* we will mean *nontrivial in the contact category* but *formally trivial* (and, in particular, *trivial in the smooth category*). See Section 2.3 and below for further details. We consider a contact 3-manifold given by the connected sum of two contact 3-manifolds $(Y_\#, \xi_\#) := (Y_-, \xi_-) \# (Y_+, \xi_+)$. Recall that the connected sum is built by removing Darboux balls $B_\pm \subset Y_\pm$ and gluing the complements $Y \setminus B_\pm$ by an orientation-reversing diffeomorphism of their boundary spheres which preserves their characteristic foliations. Reparametrization of one of the spheres provides a $U(1)$ worth of choices for gluing, and thus $(Y_\#, \xi_\#)$ naturally belongs in a *family* of contact 3-manifolds

$$(Y_\#, \xi_\#) \hookrightarrow \mathcal{Y}_\# \to U(1).$$

The monodromy of this family is realized by a contactomorphism of $(Y_\#, \xi_\#)$, well-defined up to contact isotopy. Its underlying diffeomorphism is the *Dehn twist* on the separating sphere $S_\#$ in the neck of the connected sum $Y_\# = Y_- \# Y_+$. We denote this contactomorphism by $\tau_{S_\#}$ and call it the *contact Dehn twist* on $S_\#$. Unlike previous constructions of contactomorphisms, the contact Dehn twist is a *local symmetry* of an arbitrarily small neighborhood of a 2-sphere (see Section 3 for further details). As a diffeomorphism, the Dehn twist can be isotoped so that it is supported on a neighborhood $[0,1] \times S^2$ of $S_\# \simeq S^2$ on which it acts as $[0,1] \times S^2 \ni (t,p) \mapsto (t, R_{\theta(t)}(p))$, where R_φ denotes the rotation of angle φ along the z axis in $\mathbb{R}^3$, and $\theta : [0,1] \to [0, 2\pi]$ is a smooth function with $\theta \equiv 0$ near $t = 0$ and $\theta \equiv 2\pi$ near $t = 1$. Because $\pi_1 \mathrm{SO}(3) = \mathbb{Z}/2$, the 2-fold iterate $\tau^2_{S_\#}$ is *smoothly* isotopic to the identity, but it remains to be understood whether:

Question 1.2 *Is $\tau^2_{S_\#}$ contact isotopic to the identity?*

Associated to the contact structures $\xi_\pm$ we have their Kronheimer–Mrokwa contact invariants $\boldsymbol{c}(\xi_\pm) \in \widecheck{\mathrm{HM}}(-Y_\pm)$; see [Kronheimer and Mrowka 1997; Kronheimer et al. 2007]. These are canonical elements (defined up to sign) in the "to" flavor of the monopole Floer homology of $-Y_\pm$. The contact invariant was also defined in the setting of Heegaard–Floer homology by Ozsváth and Szabó [2005]. Under the isomorphism between the monopole and Heegaard–Floer homologies [Kutluhan et al. 2020; Colin et al. 2011] the contact invariants agree. Throughout this article we only consider monopole Floer homology and the contact invariant with *coefficients* in $\mathbb{Q}$, for simplicity. Our main result is the following:

Theorem 1.3 *Let $(Y_\pm, \xi_\pm)$ be* ***irreducible*** *contact 3-manifolds. Suppose that the Kronheimer–Mrowka contact invariants $\boldsymbol{c}(\xi_\pm)$ do not lie in the image of the U-map*

$$U : \widecheck{\mathrm{HM}}(-Y_\pm) \to \widecheck{\mathrm{HM}}(-Y_\pm).$$

Then:

(A) *The k-fold iterates $\tau^k_{S_\#}$ for $k \geq 1$ of the contact Dehn twist are not contact isotopic to the identity.*

(B) *If the Euler classes of $\xi_\pm$ vanish, then $\tau^2_{S_\#}$ is formally contact isotopic to the identity.*

We now explain the meaning of the assertion in Theorem 1.3(B). Given a contact 3-manifold (Y,ξ), a *formal contactomorphism* of (Y,ξ) consists of a pair (f,F) where f is a diffeomorphism of Y and $F=(F_s)$ is a homotopy through vector bundle isomorphisms $F_s\colon TY\to f^*TY$ such that $F_0=df$ and F_1 preserves ξ. Any contactomorphism f yields a formal contactomorphism, and one says that f is *formally trivial* if f can be deformed to the identity through formal contactomorphisms. A contactomorphism f of (Y,ξ) will be called *exotic* if it is formally contact isotopic to the identity but is not contact isotopic to the identity. Thus exotic contactomorphisms are those which are "geometrically" nontrivial, and not for reasons having to do with the underlying smooth or tangential structures. See Section 2.3 for further context. Thus Theorem 1.3 asserts that $\tau^2_{S_\#}$ and all its iterates are exotic.

Remark 1.4 In fact, we will establish more: the contactomorphism $\tau^2_{S_\#}$ from Theorem 1.3 has infinite order as an element in the *abelianization* of the group

(1) $$\ker(\pi_0\operatorname{Cont}(Y,\xi)\to\pi_0\operatorname{Diff}(Y)).$$

Remark 1.5 For comparison with Theorem 1.3, whenever either of $(Y_\pm,\xi_\pm)$ is the tight $S^1\times S^2$ or a quotient of tight (S^3,ξ) — eg the lens spaces $L(p,q)$ or the Poincaré sphere $\Sigma(2,3,5)$ — then the squared contact Dehn twist $\tau^2_{S_\#}$ of $(Y_\#,\xi_\#)$ is contact isotopic to the identity; see Lemmas 3.13–3.15.

We also establish an analogous result for connected sums with multiple summands. Let (Y,ξ) be a *tight* 3-manifold. By the prime decomposition theorem combined with Colin's decomposition theorem [1997] (see also [Honda 2002; Ding and Geiges 2007]) we have a unique connected sum decomposition

$$(Y,\xi)\cong(Y_0,\xi_0)\#\cdots\#(Y_N,\xi_N)$$

into tight contact 3-manifolds (Y_j,ξ_j), where each piece Y_j is a prime 3-manifold. Let $n+1\le N$ be the number of prime summands (Y_j,ξ_j) such that $\boldsymbol{c}(\xi_j)\notin\operatorname{Im}U$ and the Euler class of ξ_j vanishes. Let $\mathcal{C}(Y,\xi)$ (resp. $\Xi(Y,\xi)$) be the space of contact structures (resp. co-oriented 2-plane fields) on Y in the path-component of ξ.

Theorem 1.6 *With (Y,ξ) as above, when $n\ge 1$ there is a $\mathbb{Z}^n$ subgroup in the kernel of*

$$\pi_1\mathcal{C}(Y,\xi)\to\pi_1\Xi(Y,\xi)$$

which induces a $\mathbb{Z}^n$ subgroup in the first singular homology $H_1(\mathcal{C}(Y,\xi);\mathbb{Z})$.

In particular, the exotic subgroup $\mathbb{Z}^n$ exhibited in Theorem 1.6 can be arbitrarily large in the following sense: for every $n\ge 1$ there exists a tight contact 3-manifold, in fact infinitely many, such that the kernel of the previous homomorphism contains a subgroup isomorphic to $\mathbb{Z}^n$.

Remark 1.7 The n homologically independent loops of contact structures that we detect in Theorem 1.6 yield, under the natural map

$$\pi_1\mathcal{C}(Y,\xi)\to\pi_0\operatorname{Cont}(Y,\xi),$$

the squared contact Dehn twists on each of the n spheres which separate the $n+1$ prime summands (Y_j,ξ_j). However, we are unable to establish that the corresponding squared contact Dehn twists are nontrivial or that they yield a subgroup $\mathbb{Z}^n\subset\pi_0\operatorname{Cont}(Y,\xi)$ when $n\geq 2$, but we conjecture that this should be true. See Remark 6.6.

The proofs of Theorems 1.3 and 1.6 combine rigid obstructions arising from Floer homology with flexibility results. An essential ingredient is a families generalization of the Kronheimer–Mrowka contact invariant in monopole Floer homology, introduced by the second author [Muñoz-Echániz 2024]. This obstructs the existence of sections of a natural fibration given by the *evaluation map* $\mathrm{ev}\colon\mathcal{C}(Y,\xi)\to S^2$ which sends a contact structure to its plane at p, where $p\in Y$ is some fixed point. We combine this machinery with the multiparametric convex surface theory techniques introduced by the first author together with J Martínez-Aguinaga and F Presas [Fernández et al. 2020]. In particular, we use these techniques to establish the following generalization of the much-celebrated decomposition theorem of Colin [1997], which could be of independent interest for contact topologists, and which will be crucial to the proof of Theorem 1.6.

We consider two tight contact 3-manifolds $(Y_\pm,\xi_\pm)$ equipped with Darboux balls $B_\pm\subset(Y_\pm,\xi_\pm)$. Let $\mathcal{C}(Y_\pm,\xi_\pm,B_\pm)\subset\mathcal{C}(Y_\pm,\xi_\pm)$ denote the subspace of contact structures on $Y_\pm$ that coincide with $\xi_\pm$ over $B_\pm$. We consider the evaluation maps $\mathrm{ev}_\pm\colon\mathcal{C}(Y_\pm,\xi_\pm)\to S^2$ which send a contact structure to its plane at the point $p_\pm$ given by the center of $B_\pm$. These maps are fibrations, and the inclusion of $\mathcal{C}(Y_\pm,\xi_\pm,B_\pm)$ into the fiber of $\mathrm{ev}_\pm$ induces a homotopy equivalence. We form the connected sum $(Y_\#,\xi_\#)=(Y_-,\xi_-)\#(Y_+,\xi_+)$ by carving out the balls $B_\pm$ and gluing together the boundary components thus created. Consider the evaluation map $\mathrm{ev}_\#\colon\mathcal{C}(Y_\#,\xi_\#)\to S^2$ at a point on the "neck" region. We establish the following h-principle type result, which should be regarded as a parametric version of Colin's theorem:

Theorem 1.8 *The inclusion of $\mathcal{C}(Y_-,\xi_-,B_-)\times\mathcal{C}(Y_+,\xi_+,B_+)$ into the fiber of $\mathrm{ev}_\#$ induces a homotopy equivalence. Thus there is a fibration sequence*

$$\mathcal{C}(Y_-,\xi_-,B_-)\times\mathcal{C}(Y_+,\xi_+,B_+)\hookrightarrow\mathcal{C}(Y_\#,\xi_\#)\xrightarrow{\mathrm{ev}_\#}S^2.$$

We refer to Theorem 6.1 for a more general version.

1.2 Examples

We now give examples of irreducible contact 3-manifolds (Y,ξ) such that $\boldsymbol{c}(\xi)\notin\operatorname{Im}U$, many of which also have vanishing Euler class.

Example 1.9 (links of singularities) The simplest example is the Brieskorn sphere

$$\Sigma(p,q,r)=\{(x,y,z)\in\mathbb{C}^3\mid x^p+y^q+z^r=0\text{ and }|x|^2+|y|^2+|z|^2=\epsilon\},$$

where $\epsilon\in\mathbb{R}_{>0}$ is small and $p,q,r\geq 1$ are integers with $1/p+1/q+1/r<1$, equipped with the contact structure ξ_{sing} induced from the Brieskorn singularity. More generally, we could take any isolated normal surface singularity germ (X, o) and let (Y, ξ_{sing}) be the contact manifold arising as the *link* of the singularity. Neumann [1981] proved that the 3-manifold Y is irreducible. Provided that Y is also a rational homology sphere, then the following are equivalent statements, as proved by Bodnár and Plamenevskaya [2021] and Némethi [2017]:

(a) $\boldsymbol{c}(\xi_{\text{sing}}) \notin \operatorname{Im} U$.

(b) Y is not an L-space.

(c) (X, o) is not a rational singularity.

For instance, all Seifert fibered integral homology spheres excluding S^3 or the Poincaré sphere carry a contact structure ξ_{sing} with the above properties.

Example 1.10 Several surgeries on the figure eight knot are hyperbolic (and hence irreducible) and support contact structures with $c(\xi) \notin \operatorname{Im} U$. Contact structures on these manifolds have been classified by Conway and Min [2020].

Example 1.11 All but one of the $\frac{1}{2}n(n-1)$ tight contact structures supported on $-\Sigma(2, 3, 6n-1)$, up to isotopy, were classified by Ghiggini and Van Horn-Morris [2016].

1.3 Exotic overtwisted phenomena

Let (Y, ξ) be such that $\boldsymbol{c}(\xi) \notin \operatorname{Im} U$ and ξ has vanishing Euler class. Let $B \subset (Y, \xi)$ be a Darboux ball. From this, one can produce overtwisted contact manifolds by modifying (Y, ξ) by a Lutz twist inside B, or by taking the connected sum (using B) with an overtwisted contact manifold (M, ξ_{ot}). In either case, the squared contact Dehn twist on the boundary of B becomes isotopic to the identity in this new overtwisted manifold, by an application of Eliashberg's h-principle [1989] for overtwisted contact structures. However, this has surprising implications (see Section 7 for the precise statement).

Proposition 1.12 (A) *There exist overtwisted contact 3-manifolds that have an exotic loop of Lutz twist embeddings.*

(B) *There exist overtwisted contact 3-manifolds that have an exotic loop of standard sphere embeddings.*

In other words, (A) says that the h-principle for codimension-0 isocontact embeddings of embedded S^1-families of overtwisted disks fails in 1-parametric families; see [Gromov 1986; Eliashberg and Mishachev 2002]. To the best of our knowledge this is the first example of this nature. On the other hand, (B) says that the h-principle for standard spheres [Fernández et al. 2020] in tight contact 3-manifolds fails in the overtwisted case.

The first known exotic phenomena regarding overtwisted disks in overtwisted contact 3-manifolds are due to Vogel [2018]. He has proved that the space of overtwisted disks in certain overtwisted 3-sphere is disconnected and used this to construct an exotic loop of overtwisted contact structures. By Eliashberg's

h-principle [1989], understanding the homotopy type of the space of overtwisted disks is the only obstacle remaining in order to completely understand the homotopy type of the space of overtwisted contact structures on a 3-manifold. Thus understanding families of overtwisted disks or overtwisted objects bears special importance in 3-dimensional contact topology.

1.4 Context

1.4.1 *h*-principles As with symplectic topology, an ubiquitous theme of contact topology is the contrast between two types of behaviors: flexible (similar to differential topology) and rigid (similar to algebraic geometry). Beyond the tight–overtwisted dichotomy, 3-dimensional contact topology would seem to be dominated by *flexibility*, due to the following h-principle of Eliashberg and Mishachev:

Theorem 1.13 [Eliashberg and Mishachev 2021] *Let $(\mathbb{B}^3, \xi_{\mathrm{st}} = \ker(dz - y\,dx))$ be the standard contact unit 3-ball. Then the inclusion $\mathrm{Cont}(\mathbb{B}^3, \xi_{\mathrm{st}}) \to \mathrm{Diff}(\mathbb{B}^3)$ is a homotopy equivalence.*

Here $\mathrm{Cont}(\mathbb{B}^3, \xi)$ is the group of contactomorphisms of Y fixing a neighborhood of $\partial\mathbb{B}^3$, and likewise for the group of diffeomorphisms $\mathrm{Diff}(\mathbb{B}^3)$. This result was claimed, without a complete proof, by Eliashberg [1992], treating the 0-1 parametric case. The complete proof recently appeared in [Eliashberg and Mishachev 2021]. To give some context, the analogous statement that $\mathrm{Diff}(\mathbb{B}^3) \to \mathrm{Homeo}(\mathbb{B}^3)$ is a homotopy equivalence is equivalent to the Smale conjecture in dimension 3, a deep result proved by Hatcher [1983]. Then an argument due to Cerf [1968] shows that the Smale conjecture implies that $\mathrm{Diff}(Y) \to \mathrm{Homeo}(Y)$ is a homotopy equivalence for all 3-manifolds. Thus the exotic phenomena at the π_0-level which are exhibited in Theorems 1.3–1.6 are in sharp contrast with the above, and unexpected.

Remark 1.14 In 4-dimensional symplectic topology, the statement analogous to the h-principle of Eliashberg and Mishachev is false: for the standard symplectic $(\mathbb{R}^4, \omega = dx \wedge dy + dz \wedge dw)$, the inclusion

$$\mathrm{Symp}_c(\mathbb{R}^4, \omega) \to \mathrm{Diff}_c(\mathbb{R}^4)$$

is not a homotopy equivalence. This follows from M Gromov's [1985] result on the contractibility of $\mathrm{Symp}_c(\mathbb{R}^4, \omega)$ combined with Watanabe's recent disproof of the 4-dimensional Smale conjecture [Watanabe 2018].

1.4.2 Gompf's contact Dehn twist We will see (Section 3) that the contact Dehn twist is well defined on a (co-oriented) sphere $S \subset (Y, \xi)$ with a *tight neighborhood.* To the authors' knowledge, this contactomorphism was first considered by Gompf [1998] on the nontrivial sphere in the tight $S^1 \times S^2$. Gompf observed that τ_S and its iterates are not contact isotopic to the identity. Ding and Geiges [2010] later established that τ_S^2 generates all smoothly trivial contact mapping classes; see also [Min 2024]. Gironella [2021] has recently studied higher-dimensional analogues of Gompf's contactomorphism. However, all iterates of Gompf's τ_S and Gironella's generalizations happen to be *formally nontrivial* already, and hence *not* exotic.

1.4.3 Finite-order exotic contactomorphisms The previously known exotic 3-dimensional contactomorphisms have *finite* order, and the underlying 3-manifolds have $b_1 \geq 3$. These were detected on torus bundles by Geiges and Gonzalo [2004], who used an essentially elementary argument to reduce the problem to the Giroux–Kanda classification of tight contact structures on T^3. This was reproved using contact homology by Bourgeois [2006], who also found more exotic contactomorphisms in Legendrian circle bundles over surfaces of positive genus. In the latter case, those contactomorphisms have been shown to generate the group (1) by Geiges and Klukas [2014] and Giroux and Massot [2017]. Unlike the squared Dehn twists, these exotic contactomorphisms are all given by global symmetries. The paradigmatic example is the following:

Example 1.15 [Geiges and Gonzalo Perez 2004; Bourgeois 2006] Consider the 3-torus T^3 with the fillable contact structure $\xi_1 = \ker(\cos\theta\, dx - \sin\theta\, dy)$. By passing to n-fold covers $T^3 \to T^3$ given by $(\theta, x, y) \mapsto (n\theta, x, y)$, we obtain contact structures ξ_n on T^3. By a classical result of Giroux [1999] and Kanda [1997], the contact structures ξ_n (for $n \geq 1$) are pairwise not contactomorphic and give all the tight contact structures on T^3. When $n \geq 2$ the deck transformations of the n-fold cover $T^3 \to T^3$ generate all the exotic contactomorphisms of (T^3, ξ_n).

1.4.4 Other exotic Dehn twists Dehn twists have been a common source of exotic phenomena in topology:

(a) Let $Y_\# = Y_- \# Y_+$ be the sum of two aspherical 3-manifolds $Y_\pm$. By a result of McCullough [1990] (see also [Hatcher and Wahl 2010]) it follows that the kernel of $\pi_0 \operatorname{Diff}(Y_\#) \to \operatorname{Out}(\pi_1 Y_\#)$ is $\cong \mathbb{Z}_2$, generated by the smooth Dehn twist on the separating sphere.

(b) Seidel [1999] used Lagrangian Floer homology to detect exotic 4-dimensional symplectomorphisms with infinite order in the symplectic mapping class group, given by squared Dehn twists on Lagrangian spheres. He later generalized these results to higher dimensions [Seidel 2000; 2003]. See also the recent work of Smirnov [2020; 2022] using Seiberg–Witten gauge theory.

(c) Kronheimer and Mrowka [2020] have proved that the smooth Dehn twist on the separating sphere in the connected sum of two copies of the smooth 4-manifold underlying a $K3$ surface is not smoothly isotopic to the identity, even if it is topologically. For this they employ the Bauer–Furuta homotopical refinement of the Seiberg–Witten invariants of 4-manifolds. See also [Lin 2023].

1.5 Sketch of the proof of Theorem 1.3(A)

We outline here a proof of Theorem 1.3(A) which is simpler than the one we give later. In particular, the proof that we present now does not yield the stronger conclusion that the class of $\tau_{S_\#}^2$ is nontrivial in the abelianization of (1). We will need a stronger argument, which uses Theorem 1.8, in order to deduce both this and Theorem 1.6.

The main ideas go as follows. First, we have a *relative* version of the problem. Given a Darboux ball B in a contact 3-manifold (Y, ξ) we have a contactomorphism given by a Dehn twist $\tau_{\partial B}$ performed on an exterior sphere parallel to ∂B. This contactomorphism fixes the ball B and need not be contact isotopic to the identity *relative* to B, even if it always is globally (not fixing the ball). The problem of whether the squared Dehn twist $\tau_{\partial B}^2$ is isotopic rel B to the identity can be essentially recast as a lifting problem involving families of contact structures: if Y is aspherical (irreducible and with infinite fundamental group) then $\tau_{\partial B}^2$ is isotopic to the identity rel B precisely when the fibration given by the evaluation map $\mathrm{ev}\colon \mathcal{C}(Y,\xi) \to S^2$ admits a (homotopy) section (see Corollary 3.7). We recall that ev is defined by evaluating contact structures at a point. The key point that we exploit is that this fibration resembles a corresponding "evaluation map" pertaining to the Seiberg–Witten gauge theory of the manifold Y, and which is closely related to the U map in monopole Floer homology. As a result, an obstruction to the existence of a section was given by the second author in [Muñoz-Echániz 2024]: if $\boldsymbol{c}(\xi) \notin \operatorname{Im} U$ then no (homotopy) section exists, and thus $\tau_{\partial B}^2$ isn't isotopic to the identity rel B.

Going back to the original problem, consider two *tight* irreducible contact manifolds $(Y_\pm, \xi_\pm)$ and their sum $(Y_\#, \xi_\#)$. Let $\mathrm{CEmb}(S^2, (Y_\#, \xi_\#))_{S_\#}$ be the space of co-oriented *convex* embeddings $S^2 \hookrightarrow (Y_\#, \xi_\#)$ with standard characteristic foliation, in the isotopy class of the separating sphere $S_\#$. The group of contactomorphisms of $(Y_\#, \xi_\#)$ acts transitively on this space and yields a fibration[1]

$$\mathrm{Cont}(Y_\#, \xi_\#, S_\#) \to \mathrm{Cont}(Y_\#, \xi_\#) \to \mathrm{CEmb}(S^2, (Y_\#, \xi_\#))_{S_\#}, \quad f \mapsto f(S_\#). \tag{2}$$

From the long exact sequence of homotopy groups, a contactomorphism f of $(Y_\#, \xi_\#)$ fixing the sphere $S_\#$ is contact isotopic to the identity (not necessarily fixing $S_\#$) precisely when it arises as the monodromy in (2) of a loop of sphere embeddings. It thus becomes essential to understand the topology of the sphere embedding space. This brings us to the following h-principle-type result, which asserts that the topological complexity of this space only comes from reparametrizations of the source:

Theorem 1.16 *If $(Y_\pm, \xi_\pm)$ are irreducible and tight then the reparametrization map provides a homotopy equivalence* $U(1) \xrightarrow{\simeq} \mathrm{CEmb}(S^2, (Y_\#, \xi_\#))_{S_\#}$.

In the smooth case, the result analogous to the above was proved by Hatcher [1981]. The proof of Theorem 1.16 rests on the h-principle for standard convex spheres established by the first author together with Martínez-Aguinaga and Presas [Fernández et al. 2020], and should be regarded as an application of the h-principle of Eliashberg and Mishachev [2021].

With these ingredients in place, the proof of Theorem 1.3(A) goes as follows. The monodromy in (2) over the standard loop in $U(1)$ is given by the product of Dehn twists $\tau_{\partial B_-}\tau_{\partial B_+}$ (see Lemma 3.5). The contact Dehn twist $\tau_{S_\#}$ agrees with the image of $\tau_{\partial B_-}$ in $\pi_0 \mathrm{Cont}(Y_\#, \xi_\#)$. Because the manifolds $(Y_\pm, \xi_\pm)$ have infinite-order contact Dehn twists $\tau_{\partial B_\pm}$ rel $B_\pm$, for all $k \geq 1$ the class $\tau_{\partial B_-}^k \in \pi_0 \mathrm{Cont}(Y_\#, \xi_\#, S_\#)$ is not an iterate of $\tau_{\partial B_-}\tau_{\partial B_+}$ or its inverse. It follows that $\tau_{S_\#}$ and its iterates are not contact isotopic to the identity in $(Y_\#, \xi_\#)$.

[1] Strictly speaking, we should replace $\mathrm{Cont}(Y_\#, \xi_\#)$ with the subgroup consisting of contactomorphisms which preserve the isotopy class of the co-oriented sphere $S_\#$.

Outline The structure of the article is as follows. In Section 2 we introduce notation and present background material. In Section 3 we define the contact Dehn twist, establish various key properties and present examples where it is isotopic to the identity. In Section 4 we provide background on the families version of the Kronheimer–Mrowka contact invariant introduced in [Muñoz-Echániz 2024], which will be one of the main ingredients in the proofs of our main results. In Section 5 we review the h-principle for families of convex spheres in tight contact 3-manifolds established in [Fernández et al. 2020]. In Section 6 we use this h-principle to establish Theorem 1.8. We then complete the proofs of Theorems 1.3 and 1.6. In Section 7 we deduce exotic 1-parametric phenomena in overtwisted contact 3-manifolds.

Acknowledgements Fernández would like to acknowledge his advisor Francisco Presas for valuable conversations. Muñoz-Echániz thanks his advisor Francesco Lin for his support and encouragement, together with Hyunki Min for useful conversations. We would also like to thank the referee for the careful suggestions from which this manuscript has greatly benefited. Muñoz-Echániz was partially supported by NSF grant DMS-2203498.

2 Background

This section introduces the main players in this article: spaces of contact structures, contactomorphisms, embeddings, etc.

Remark 2.1 For convenience, throughout this article by a "fibration" we will mean a "Serre fibration". By a "homotopy equivalence" we will mean a "weak homotopy equivalence". However, the latter distinction isn't important: the various infinite-dimensional spaces that we consider are Fréchet manifolds, and hence they have the homotopy type of countable CW complexes [Palais 1966; Milnor 1959] and Whitehead's theorem applies.

2.1 Notation

Let (Y, ξ) be a closed contact 3-manifold. We always assume Y is connected and oriented, and ξ cooriented and positive. Occasionally we will allow Y to be compact with nonempty boundary, in which case we assume that ∂Y is *convex* for the contact structure ξ and we fix a collar neighborhood $C = (-1, 0] \times \partial Y$ of ∂Y. We quickly introduce here some of the spaces that will be relevant, all of which are equipped with the Whitney C^∞ topology:

- We denote by $\mathrm{Emb}(\mathbb{B}^3, Y)$ the space of orientation-preserving smooth embeddings $\phi\colon \mathbb{B}^3 \hookrightarrow Y$ of the closed unit ball (avoiding the closure of C, if $\partial Y \neq \varnothing$). Let $\mathrm{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi))$ be the subspace consisting of contact embeddings of the standard contact unit ball. Such embeddings will be referred to as *Darboux balls* in (Y, ξ). Darboux's theorem asserts that for any interior point p of a contact manifold we may find such ϕ with $\phi(0) = p$. We will often abuse notation by referring to a Darboux ball only by its image $B := \phi(\mathbb{B}^3)$.

- We denote by $\mathrm{Diff}(Y)$ the group of orientation-preserving diffeomorphisms, and by $\mathrm{Diff}(Y,B)$ the subgroup consisting of those which fix a Darboux ball B pointwise. By $\mathrm{Diff}_0(Y)$ and $\mathrm{Diff}_0(Y,B)$ we denote the subgroups consisting of those which are smoothly isotopic to the identity (rel B in the second case). We denote by $\mathrm{Cont}(Y)\subset \mathrm{Diff}$ the subgroup of co-orientation-preserving contactomorphisms of (Y,ξ), and by $\mathrm{Cont}(Y,B)$ the subgroup consisting of those which fix a Darboux ball B pointwise. By $\mathrm{Cont}_0(Y)$ and $\mathrm{Cont}_0(Y,B)$ we denote the subgroups consisting of those which are *smoothly* isotopic to the identity (rel B in the second case).

- We denote by $\mathcal{C}(Y,\xi)$ the space of contact structures on Y in the path-component of ξ. When $\partial Y\neq\varnothing$ we also require that they agree with ξ over C. Given a Darboux ball B in (Y,ξ) we denote by $\mathcal{C}(Y,\xi,B)$ the subspace consisting of contact structures ξ' for which the coordinate ball B is a Darboux ball for (Y,ξ') — ie $\xi=\xi'$ over B.

- We denote by $\mathrm{Fr}(Y)$ the principal $(\mathrm{SO}(3)\simeq)\,\mathrm{GL}^+(3)$-bundle over Y of oriented frames in TY, and by $\mathrm{CFr}(Y)$ the principal $(U(1)\simeq)\,\mathrm{CSp}^+(2,\mathbb{R})$-bundle over Y of co-oriented frames in ξ. Here $\mathrm{CSp}^+(2,\mathbb{R})$ denotes the linear conformal-symplectomorphism group. By the smooth and contact versions of the disk theorem[2] we have homotopy equivalences

$$\begin{aligned}\mathrm{Emb}(\mathbb{B}^3,Y)&\xrightarrow{\simeq}\mathrm{Fr}(Y), &\quad \phi&\mapsto (d\phi)_0(e_1,e_2,e_3),\\ \mathrm{Emb}((\mathbb{B}^3,\xi_{\mathrm{st}}),(Y,\xi))&\xrightarrow{\simeq}\mathrm{CFr}(Y,\xi), &\quad \phi&\mapsto (d\phi)_0(e_1,e_2).\end{aligned}\tag{3}$$

Notice that $\mathrm{Fr}(Y)\simeq Y\times \mathrm{SO}(3)$ and, when the Euler class of ξ vanishes, $\mathrm{CFr}(Y,\xi)\simeq Y\times U(1)$.

- We denote by $\mathrm{Emb}(S^2,Y)$ the space of co-oriented embeddings of 2-spheres. By $\mathrm{CEmb}(S^2,(Y,\xi))$ we denote the subspace consisting of *convex* embeddings with *standard characteristic foliation* ("standard convex spheres" in short). Recall that a surface $\Sigma\subset(Y,\xi)$ is convex [Giroux 1991; Geiges 2008] if there exists a contact vector field on a neighborhood which is transverse to Σ. The standard characteristic foliation on S^2 is that induced from its embedding as the boundary of the Darboux ball.

- We denote by $\mathrm{Cont}(Y,\xi,S)$ the subgroup of contactomorphisms which fix a standard convex sphere S pointwise, and likewise for $\mathrm{Diff}(Y,S)$.

2.2 Standard fibrations

Next, we review how the spaces introduced above relate to each other through various natural fibrations. Some of the material from this section is treated in [Giroux and Massot 2017] in greater detail.

2.2.1 Diffeomorphisms acting on contact structures By an application of Gray's stability theorem (Moser's argument) [Geiges 2008] with parameters, one can show:

[2] The key point in the contact case is that $\varphi_t(x,y,z):=(tx,ty,t^2z)$ is a contactomorphism of $(\mathbb{R}^3,\xi_{\mathrm{st}})$ for every $t>0$, so the proof in the contact case follows along the same lines as in the smooth case; see [Geiges 2008, Theorem 2.6.7].

Lemma 2.2 *The action $f \mapsto f_*\xi$ of the group of diffeomorphisms on a fixed contact structure ξ gives a fibration*

$$\mathrm{Cont}_0(Y,\xi) \to \mathrm{Diff}_0(Y) \to \mathcal{C}(Y,\xi). \tag{4}$$

Similarly, there is fibration

$$\mathrm{Cont}_0(Y,\xi,B) \to \mathrm{Diff}_0(Y,B) \to \mathcal{C}(Y,\xi,B). \tag{5}$$

By (4), understanding the homotopy type of the space of contact structures $\mathcal{C}(Y,\xi)$ and the group of contactomorphisms $\mathrm{Cont}_0(Y,\xi)$ is essentially equivalent, since the homotopy type of $\mathrm{Diff}_0(Y)$ is often well understood (eg for all prime 3-manifolds by now).

2.2.2 Contactomorphisms acting on Darboux balls By an application of the contact isotopy extension theorem [Geiges 2008] with parameters, we have:

Lemma 2.3 *The action $f \mapsto f(B)$ of the group of contactomorphisms on a fixed Darboux ball $B \subset Y$ gives a fibration*

$$\mathrm{Cont}(Y,\xi,B) \to \mathrm{Cont}(Y,\xi) \to \mathrm{Emb}((\mathbb{B}^3,\xi_{\mathrm{st}}),(Y,\xi)). \tag{6}$$

Similarly, there is a fibration

$$\mathrm{Diff}(Y,B) \to \mathrm{Diff}(Y) \to \mathrm{Emb}(\mathbb{B}^3,Y). \tag{7}$$

2.2.3 Evaluation of contact structures at a point Fix a Darboux ball $B \subset Y$ with center $0 \in Y$. By regarding the 2-sphere S^2 as the space of co-oriented planes in the tangent space T_0B, we obtain the *evaluation map*

$$\mathrm{ev}_B\colon \mathcal{C}(Y,\xi) \to S^2, \quad \xi' \mapsto \xi'(0). \tag{8}$$

The following result is well known, but we provide a proof:

Lemma 2.4 *The evaluation map* (8) *is a fibration. The inclusion $\mathcal{C}(Y,\xi,B) \to (\mathrm{ev}_B)^{-1}(\xi(0))$ is a homotopy equivalence.*

Proof Let $\mathbb{B}^j$ be the unit j-disk and consider a homotopy $[0,1]\times\mathbb{B}^j \to S^2$, $(t,u) \mapsto \sigma_{t,u}$, together with a lift of the time zero map $\{0\}\times\mathbb{B}^j \to \mathcal{C}(Y,\xi)$ given by $u \mapsto \xi_u$, ie at the point $0 \in B$ we have $\xi_u(0) = \sigma_{0,u}$. We must find a family of contact structures $\xi_{t,u}$ with $\xi_{t,u}(0) = \sigma_{t,u}$ and $\xi_{0,u} = \xi_u$.

Let $v_{t,u} \in S(T_0B) = S^2$ be the unit normal (with respect to the standard flat metric on B) to the plane $\sigma_{t,u}$. Since the action of SO(3) on S^2 gives a fibration $\mathrm{SO}(3) \to S^2$ given by $A \mapsto Ae_3$, we may find $A_{t,u} \in \mathrm{SO}(3)$ such that $A_{t,u}e_3 = v_{t,u}$. Differentiating $A_{t,u}$ in t we get a vector field $V_{t,u}$ on $\mathbb{R}^3$. After cutting off $V_{t,u}$ outside the unit ball $B \subset Y$ we regard $V_{t,u}$ as a u-family of t-dependent vector fields on Y whose associated flows (starting at time $t = 0$) we denote by ϕ_u^t. We obtain contact structures $\xi_{t,u} := (\phi_u^t)_*\xi_u$ with the desired property, which in fact agree with ξ outside $B \subset Y$.

For the second part, let $\xi_u = \ker \alpha_u$ be a family of contact structures parametrized by a disk $\mathbb{B}^j \ni u$ so that $\xi_u(0) = \xi(0)$ for all $u \in \mathbb{B}^j$ and $\xi_u(p) = \xi(p)$ for all $(u, p) \in \partial\mathbb{B}^j \times B$. We must deform relative to $\partial\mathbb{B}^j$ this family of contact structures to another family which agrees with ξ over the Darboux ball B. Denote by $i : \mathbb{B}^3 \hookrightarrow Y$ the inclusion of $B = i(\mathbb{B}^3)$. By the parametric version of Darboux's theorem we obtain a family of disk embeddings $\phi_u : \mathbb{B}^3 \hookrightarrow Y$, which are Darboux balls for ξ_u, such that $\phi_u(0) = 0 \in B$, $(d\phi_u)_0 = \mathrm{id}$ for all $u \in \mathbb{B}^j$, and $\phi_u = i$ for all $u \in \partial\mathbb{B}^j$. By (3) we may deform the family of embeddings ϕ_u to the inclusion i relative to $\partial\mathbb{B}^j$, and this deformation may be followed by an isotopy $f_{u,t} \in \mathrm{Diff}(Y)$ for $(u, t) \in \mathbb{B}^j \times [0, 1]$, with $f_{u,t} = \mathrm{id}$ for all $(u, t) \in \mathbb{B}^j \times \{0\} \cup \partial\mathbb{B}^j \times [0, 1]$. The homotopy of contact structures $(f_{u,t})_* \xi_u$ for $(u, t) \in \mathbb{B}^j \times [0, 1]$ solves the problem since the $(f_{u,1})_* \xi_u$ agree with ξ over B for all $u \in \mathbb{B}^j$. □

2.2.4 Contactomorphisms act on standard convex spheres Again, an application of the contact isotopy extension theorem gives:

Lemma 2.5 *The action $f \mapsto f(S)$ of the group of contactomorphisms on a fixed standard convex sphere $S \subset Y$ gives a fibration*

$$\mathrm{Cont}(Y, \xi, S) \to \mathrm{Cont}(Y, \xi) \to \mathrm{CEmb}(S^2, (Y, \xi)). \tag{9}$$

Similarly, there is a fibration

$$\mathrm{Diff}(Y, S) \to \mathrm{Diff}(Y) \to \mathrm{Emb}(S^2, Y). \tag{10}$$

Remark 2.6 The above statement isn't quite precise. For either (9) or (10), the downstairs projection is not surjective in general, so strictly speaking we only have a fibration over a union of connected components of the right-hand side. We will make no further comment on this point from now on.

2.3 Formal triviality and exoticness

Here we collect basic material that we need related to the notion of a formal contactomorphism. The material in this section should be well known to experts, but we did not find a convenient reference.

2.3.1 Formal contact structures and contactomorphisms For a 3-manifold Y, the flexible analogue[3] of a contact structure is a 2-*plane field*, ie a codimension-1 distribution $\xi \subset TY$. Henceforth all 2-planes in a 3-manifold are assumed to be co-oriented, as we've been assuming with contact structures. Let $\Xi(Y, \xi)$ denote the path-component of a fixed 2-plane field ξ in the space of all such. If ξ is a contact structure we have a natural inclusion map $\mathcal{C}(Y, \xi) \to \Xi(Y, \xi)$. The correct flexible analogue of a contactomorphism is:

Definition 2.7 A *formal contactomorphism* of (Y, ξ), where ξ is a 2-plane field, is a pair $(f, \{\phi^s\}_{0\le s\le 1})$ such that $f \in \mathrm{Diff}(Y)$ and $\{\phi^s\}_{0\le s\le 1}$ is a homotopy through vector bundle isomorphisms

$$\phi^s : TY \xrightarrow{\cong} f^* TY$$

such that $\phi^0 = df$ and ϕ^1 preserves the 2-plane field ξ.

[3] In general, if Y has dimension $2n + 1 \ge 3$, one should define $\Xi(Y, \xi)$ as the space of codimension-1 hyperplane fields in TY *equipped with* a $U(n)$ structure.

Of course, the above notion can be generalized to an arbitrary n-manifold equipped with a reduction of structure group to a subgroup $G \subset \mathrm{GL}(n,\mathbb{R})$. The group of formal contactomorphisms of (Y,ξ) is denoted by $\mathrm{FCont}(Y,\xi)$. When ξ is a contact structure there is the obvious inclusion map $\mathrm{Cont}(Y,\xi) \to \mathrm{FCont}(Y,\xi)$ given by $f \mapsto (f, df)$, where df denotes the constant homotopy at df.

A homotopy class in $\pi_j \mathrm{Cont}(Y,\xi)$ is said to be *formally trivial* if it lies in the kernel of $\pi_j \mathrm{Cont}(Y,\xi) \to \pi_j \mathrm{FCont}(Y,\xi)$. If, in addition, such a homotopy class is nontrivial in $\pi_j \mathrm{Cont}(Y,\xi)$ then we call it *exotic*. Similar terminology applies for families of contact structures.

2.3.2 A flexible analogue of (4) We introduce a flexible counterpart of the fibration (4). This is done via fibrant replacement of the map $\mathrm{Diff}_0(Y) \to \Xi(Y,\xi)$ given by $f \mapsto f^*\xi$. That is, we decompose this map as the composite of a homotopy equivalence $\mathrm{Diff}_0(Y) \xrightarrow{\simeq} \mathrm{FDiff}_0(Y)$ and a fibration $\mathrm{FDiff}_0(Y) \to \Xi(Y,\xi)$. Here $\mathrm{FDiff}(Y)$ is the topological group which consists of pairs $(f, \{\phi^t\}_{0\le t\le 1})$ where $f \in \mathrm{Diff}(Y)$ and $\{\phi^t\}_{0\le t\le 1}$ is a homotopy of vector bundle isomorphisms $\phi^t : TY \xrightarrow{\cong} f^*TY$ such that $\phi^0 = df$. By $\mathrm{FDiff}_0(Y)$ we denote the identity component. Clearly the inclusion induces a homotopy equivalence $\mathrm{Diff}(Y) \simeq \mathrm{FDiff}(Y)$. Define a mapping

$$\mathrm{FDiff}_0(Y) \to \Xi(Y,\xi), \quad (f, \{\phi^t\}) \mapsto \phi^1(\xi). \tag{11}$$

Lemma 2.8 *Let ξ be a 2-plane field on a compact oriented 3-manifold Y. Then the mapping* (11) *is a fibration with fiber* $\mathrm{FCont}_0(Y,\xi)$. *Thus, for a contact structure ξ, we have a commuting diagram of fibrations inducing a homotopy equivalence of total spaces*

$$\begin{array}{ccccc} \mathrm{FCont}_0(Y,\xi) & \longrightarrow & \mathrm{FDiff}_0(Y) & \longrightarrow & \Xi(Y,\xi) \\ \uparrow & & \simeq\uparrow & & \uparrow \\ \mathrm{Cont}_0(Y,\xi) & \longrightarrow & \mathrm{Diff}_0(Y) & \longrightarrow & \mathcal{C}(Y,\xi) \end{array}$$

We leave the proof of this lemma as an exercise.

Corollary 2.9 *Let (Y,ξ) be a contact 3-manifold. If $\beta \in \pi_j \mathcal{C}(Y,\xi)$ is formally trivial, then so is its image in $\pi_{j-1} \mathrm{Cont}_0(Y,\xi)$ under the connecting map of the fibration* (4).

The homotopy type of the space $\Xi(Y,\xi)$ is often easy to understand, unlike that of $\mathcal{C}(Y,\xi)$.

Example 2.10 Let Y be any integral homology 3-sphere, and ξ a 2-plane field on Y. Let ξ_{st} be any contact structure on S^3 (say, the tight one). By a result of Hansen [1978] there is a homotopy equivalence $\Xi(S^3,\xi_{\mathrm{st}}) \simeq \Xi(Y,\xi)$. From this one easily calculates

$$\pi_j \Xi(Y,\xi) \approx \pi_j S^2 \times \pi_{j+3} S^2.$$

3 Contact Dehn twists on spheres

In this section we define the contact Dehn twist on a sphere in several equivalent ways, establish some key properties and discuss some examples when its square is isotopic to the identity.

3.1 The contact Dehn twist

Let (Y,ξ) be a contact 3-manifold, and $S\subset Y$ be a co-oriented embedded sphere. Provided S has a tight neighborhood, we can associate to S a contactomorphism τ_S well defined in $\pi_0\operatorname{Cont}(Y,\xi)$. We discuss this construction now.

3.1.1 Local model We start by discussing the local picture. Consider the contact 3-manifold $Y_0=[-1,1]\times S^2$ with the tight contact structure $\xi_0=\ker(\alpha_0)$ where $\alpha_0=z\,ds+\frac{1}{2}x\,dy-\frac{1}{2}y\,dx$. Here s is the standard coordinate on $[-1,1]$, and x, y and z are coordinates on $\mathbb{R}^3$ restricted onto the unit sphere S^2. Consider the sphere $S_0=\{0\}\times S^2\subset Y_0$. We now describe the contact Dehn twist τ_{S_0} on the sphere S_0.

We choose a smooth function $\theta\colon[-1,1]\to[0,2\pi]$ with $\theta(s)\equiv 0$ near $s=-1$ and $\theta(s)=2\pi$ near $s=1$. Let R_φ be the counterclockwise rotation in the xy plane with angle φ. Consider the diffeomorphism $\tilde\tau_{S_0}$ of Y_0 given by a smooth Dehn twist along S_0

$$\tilde\tau_{S_0}(s,x,y,z)=(s,R_{\theta(s)}(x,y),z).$$

Since $\pi_1\operatorname{SO}(3)=\mathbb{Z}/2$ it follows that the squared Dehn twist $\tilde\tau_{S_0}^2$ is smoothly isotopic to the identity rel ∂Y_0. We don't quite have a contactomorphism of (Y_0,ξ_0), since

$$\tilde\tau_{S_0}^*\alpha_0=\alpha_0+\tfrac{1}{2}\theta'(s)(x^2+y^2)\,ds.$$

However, consider the naive interpolation from α_0 to $\tilde\tau_{S_0}^*\alpha_0$

$$\alpha_t=\alpha_0+t\tfrac{1}{2}\theta'(s)(x^2+y^2)\,ds,$$

and observe that:

Lemma 3.1 *For any $t\in[0,1]$ the form α_t is a contact form.*

Proof A straightforward calculation shows $\alpha_t\wedge d\alpha_t=\alpha_0\wedge d\alpha_0>0$. □

Thus, by Gray stability (Moser's argument) [Geiges 2008], the deformation of contact structures $\xi_t=\ker(\alpha_t)$ is realized by an isotopy f_t, ie $f_0=\mathrm{id}$ and $(f_t)^*\xi_t=\xi_0$. Since the forms α_t don't depend on t near ∂Y_0 we may further assume that $f_t=\mathrm{id}$ near ∂Y_0. We then replace $\tilde\tau_{S_0}$ with $\tau_{S_0}:=\tilde\tau_{S_0}\circ f_1$, and the latter is a contactomorphism of (Y_0,ξ_0). Also, the support of τ_{S_0} can be made arbitrarily close to the sphere S_0 by choosing $\theta(s)$ appropriately. Then for any $\epsilon\in(0,1]$ we have a well-defined isotopy class of contact Dehn twist

$$\tau_{S_0}\in\pi_0\operatorname{Cont}([-\epsilon,\epsilon]\times S^2,\xi_0).$$

It is worth pointing out the following:

Lemma 3.2 *The group $\operatorname{Cont}(Y_0,\xi_0)$ is homotopy equivalent to $\Omega U(1)\simeq\mathbb{Z}$. Its π_0 is generated by the contact Dehn twist τ_{S_0}.*

Proof Gluing a Darboux ball B to (Y_0, ξ_0) gives back the standard contact ball $(\mathbb{B}^3, \xi_{\mathrm{st}})$. Thus, from the fibration (6), we have a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{Cont}(Y_0,\xi_0) & \longrightarrow & \mathrm{Cont}(\mathbb{B}^3,\xi_{\mathrm{st}}) & \longrightarrow & \mathrm{Emb}((\mathbb{B}^3,\xi_{\mathrm{st}}),(\mathbb{B}^3,\xi_{\mathrm{st}})) \\ \uparrow & & \simeq\uparrow & & \simeq\uparrow \\ \Omega U(1) & \longrightarrow & \{*\} & \longrightarrow & U(1) \end{array}$$

where the middle homotopy equivalence follows from Theorem 1.13 combined with Hatcher's theorem [1983]. The first assertion now follows. For the second assertion, we need to show that the generator $1 \in \pi_1 U(1)$ maps to the class of the contact Dehn twist τ_{S_0} under the connecting map.

We first describe the contact Dehn twist on S_0 more conveniently in terms of the coordinates on the ball $\mathbb{B}^3 = B \cup Y_0$. Recall that the standard contact structure on $\mathbb{B}^3$ is $\xi_{\mathrm{st}} = \ker \alpha_{\mathrm{st}}$ where $\alpha_{\mathrm{st}} = dz + \frac{1}{2} x\, dy - \frac{1}{2} y\, dx$. Choose a smooth function $\theta\colon [0,1] \to [0, 2\pi]$ with $\theta = 0$ near 0 and $\theta = 2\pi$ near 1. Let $r^2 := x^2 + y^2 + z^2$ be the radius squared function on $\mathbb{B}^3$. Then the diffeomorphism of $\mathbb{B}^3$ given by

$$\tilde{\tau}(x, y, z) := (R_{\theta(r^2)}(x, y), z)$$

does not quite preserve the contact structure, but

$$(\tilde{\tau})^* \alpha_{\mathrm{st}} = \alpha_{\mathrm{st}} + \tfrac{1}{2}(x^2 + y^2)\theta'(r^2)\, d(r^2).$$

As in Lemma 3.1, the obvious interpolation that takes the second term in the above identity to zero gives a path of *contact* forms, and as in Section 3.1.2 we may canonically deform $\tilde{\tau}$ to a contactomorphism τ_{S_0} in the isotopy class of the contact Dehn twist on S_0.

Consider now a homotopy of maps $\theta_t\colon [0,1] \to [0, 2\pi]$ with θ_t constant near 1 (with value 2π), such that $\theta_0 = \theta$ and θ_1 is the constant function with value 2π. We obtain an isotopy through diffeomorphisms of $\mathbb{B}^3$ (fixing a neighborhood of the boundary $\partial \mathbb{B}^3$, but not the smaller ball B!) given by

$$\tilde{\tau}_t(x, y, z) := (R_{\theta_t(r^2)}(x, y), z)$$

such that $\tilde{\tau}_0 = \tilde{\tau}$ and $\tilde{\tau}_1 = \mathrm{id}$. Again, by observing that for each t the obvious interpolation from $(\tilde{\tau}_t)^* \alpha_{\mathrm{st}}$ and α_{st} gives a path of contact forms, we may canonically deform the isotopy $\tilde{\tau}_t$ to a *contact* isotopy τ_t with $\tau_0 = \tau_{S_0}$ and $\tau_1 = \mathrm{id}$.

Now, the path of contactomorphisms τ_{1-t} from the identity to $\tau_{\partial B}$ induces a *loop* of Darboux balls $(\tau_{1-t})(B)$ in the class of the generator $1 \in \mathbb{Z} = \pi_1 \mathrm{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (\mathbb{B}^3, \xi_{\mathrm{st}}))$. □

Likewise, we have a firm hold on the topology of the space of standard spheres in our local model. Let $S_\pm = \{\pm\frac{1}{2}\} \times S^2 \subset Y_0$, and denote by $e_0\colon S^2 \hookrightarrow Y_0$ the embedding of $S_0 \subset Y_0$.

Lemma 3.3 *The map induced by reparametrization of* e_0

$$U(1) \to \mathrm{CEmb}(S^2, (Y_0, \xi_0)), \quad \theta \mapsto e_0 \circ r_\theta,$$

is a homotopy equivalence. Here $r_\theta(x,y,z) = (R_\theta(x,y),z)$. *Under the connecting homomorphism of the fibration* (9), *the generator of* $\pi_1 U(1) = \mathbb{Z}$ *maps to the class*

$$(\tau_{S_-})^{-1}\tau_{S_+} \in \pi_0 \operatorname{Cont}(Y_0, \xi_0, S_0).$$

Proof We have the following map of fiber sequences, with homotopy equivalences on the fiber and total space by Lemma 3.2:

$$\begin{array}{ccccc} \operatorname{Cont}(Y_0,\xi_0,S_0) & \longrightarrow & \operatorname{Cont}(Y_0,\xi_0) & \longrightarrow & \operatorname{CEmb}(S^2,(Y_0,\xi_0)) \\ \simeq\uparrow & & \simeq\uparrow & & \uparrow \\ \Omega U(1)\times\Omega U(1) & \longrightarrow & \Omega U(1) & \longrightarrow & U(1) \end{array}$$

□

3.1.2 General case The robustness of our local picture allows us to consider contact Dehn twists in more general settings. We fix a 3-manifold (Y,ξ) together with a co-oriented *standard convex sphere* $S \subset Y$, ie an embedded sphere whose characteristic foliation agrees with that of $S_0 \subset Y_0$ in the local model. It follows that neighborhoods of $S \subset Y$ and $S_0 \subset Y_0$ are contactomorphic in a (homotopically) canonical fashion [Giroux 1991; Geiges 2008], and by making the support of τ_{S_0} sufficiently close to S_0 we may therefore implant τ_{S_0} into (Y,ξ) as a compactly supported contactomorphism τ_S, which we refer to as the *contact Dehn twist* on the co-oriented standard convex sphere $S \subset Y$. The class of τ_S in $\pi_0 \operatorname{Cont}(Y,\xi)$ only depends on the isotopy class of S in the space of co-oriented standard convex spheres, defining a map of sets

$$\pi_0 \operatorname{CEmb}(S^2,(Y,\xi)) \to \pi_0 \operatorname{Cont}(Y,\xi), \quad S \mapsto \tau_S.$$

The contactomorphism τ_S makes sense more generally whenever $S \subset Y$ is a just a convex co-oriented sphere with a *tight neighborhood* U (but not necessarily having standard characteristic foliation). Indeed, by Giroux's criterion [2001] the dividing set of S is connected. Then by Giroux's realization theorem, we may find a smooth isotopy of sphere embeddings S_t whose image lies in the tight neighborhood U, $S_0 = S$ and S_1 is a *standard* convex sphere, to which we associate the Dehn twist τ_{S_1} by the previous construction. A different choice of isotopy S'_t may yield a different standard convex sphere S'_1. The two spheres (S_1 and S'_1) are isotopic within U as *standard* convex spheres by a result of Colin [1997, Proposition 10], so the contact Dehn twists τ_{S_1} and $\tau_{S'_1}$ are contact isotopic. Therefore we have a well-defined contact Dehn twist $\tau_S \in \pi_0 \operatorname{Cont}(Y,\xi)$ associated to the convex sphere S with tight neighborhood U. In fact, since any *smooth* sphere can be made convex by a small isotopy [Giroux 1991], this construction defines a map

$$\pi_0 \operatorname{Emb}_{\mathrm{tight}}(S^2,(Y,\xi)) \to \pi_0 \operatorname{Cont}(Y,\xi), \quad S \mapsto \tau_S,$$

where $\operatorname{Emb}_{\mathrm{tight}}(S^2,(Y,\xi))$ stands for the space of *smooth* co-oriented embeddings $S^2 \subset Y$ which admit a tight neighborhood. In particular, if (Y,ξ) is tight (globally) then τ_S only depends up to contact isotopy on the *smooth* isotopy class of the co-oriented sphere S.

The following particular case will play an essential role in this article, so we emphasize it now. Consider a Darboux ball $B = \phi(\mathbb{B}^3)$ in a contact manifold (Y,ξ). Associated to an exterior sphere (a sphere contained in the complement $Y \setminus B$) parallel to ∂B we have a well-defined contact Dehn twist which fixes B pointwise. By abuse of notation and for convenience we denote this contactomorphism by $\tau_{\partial B}$ even if the Dehn twist is not on the sphere ∂B. This defines a map of sets

$$\pi_0 \operatorname{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)) \to \pi_0 \operatorname{Cont}(Y, \xi, B), \quad B \mapsto \tau_{\partial B}.$$

The following convenient description of $\tau_{\partial B}$ follows from the local calculation in the proof of Lemma 3.2.

Lemma 3.4 *The Dehn twist $\tau_{\partial B} \in \pi_0 \operatorname{Cont}(Y, \xi, B)$ agrees with the image of $1 \in \mathbb{Z}$ under the map*

$$\mathbb{Z} = \pi_1 U(1) \to \pi_1 \operatorname{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)) \to \pi_0 \operatorname{Cont}(Y, \xi, B),$$

where the first map is induced by the reparametrization map

$$U(1) \to \operatorname{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (Y, \xi)), \quad \theta \mapsto \phi \circ r_\theta,$$

and the second map is the connecting map in the long exact sequence of the fibration (6).

Let $e \in \operatorname{CEmb}(S^2, (Y, \xi))$ be an embedding of a standard convex sphere. Thus the image $S = e(S^2) \subset (Y, \xi)$ is a co-oriented standard convex sphere. Let $S_\pm$ be two parallel copies of S given by pushing S forward and backward. By the local calculation in Lemma 3.3 we have:

Lemma 3.5 *The product of Dehn twists $(\tau_{S_-})^{-1} \tau_{S_+} \in \pi_0 \operatorname{Cont}(Y, \xi, S)$ agrees with the image of $1 \in \mathbb{Z}$ under the map*

$$\mathbb{Z} = \pi_1 U(1) \to \pi_1 \operatorname{CEmb}(S^2, (Y, \xi)) \to \pi_0 \operatorname{Cont}(Y, \xi, S),$$

where the first map is induced by the reparametrization map

$$U(1) \to \operatorname{CEmb}(S^2, (Y, \xi)), \quad \theta \mapsto e \circ r_\theta,$$

and the second map is the connecting map in the long exact sequence of the fibration (9).

3.2 The Dehn twist and the evaluation map

We move on to study a *relative* version of the isotopy problem for the Dehn twist. Consider the Dehn twist $\tau_{\partial B}$ on (an exterior sphere parallel to) the boundary ∂B of a Darboux ball, as in the previous section. We will now rephrase the problem of whether $\tau_{\partial B}^2$ defines the trivial class in $\pi_0 \operatorname{Cont}_0(Y, \xi, B)$ as a *lifting* problem.

3.2.1 The obstruction class The main player is the evaluation mapping $\mathrm{ev}_B \colon \mathcal{C}(Y, \xi) \to S^2$ defined by (8), which is a fibration (Lemma 2.4). If $\delta \colon \pi_2 S^2 \to \pi_1 \mathcal{C}(Y, \xi, B)$ is the connecting map in the homotopy long exact sequence, then we have a distinguished class

$$\mathcal{O}_\xi := \delta(1) \in \pi_1 \mathcal{C}(Y, \xi, B), \tag{12}$$

which, by construction, is the *obstruction class* to finding a homotopy section of ev_B (a map $s \colon S^2 \to \mathcal{C}(Y, \xi)$ such that $\mathrm{ev}_B \circ s \colon S^2 \to S^2$ has degree 1):

$$\mathrm{ev}_B \text{ admits a homotopy section if and only if } \mathcal{O}_\xi = 0.$$

Later in this section we will explicitly describe a loop of contact structures that represents the obstruction class $\mathcal{O}_\xi \in \pi_1\mathcal{C}(Y,\xi,B)$.

We now relate the problem of finding a section of ev_B to the triviality of the Dehn twist $\tau^2_{\partial B}$ as follows. Consider the connecting map $\delta'\colon \pi_1\mathcal{C}(Y,\xi,B) \to \pi_0 \mathrm{Cont}_0(Y,\xi,B)$ of the fibration (5). The key observation is the following:

Proposition 3.6 *The class $\delta'(\mathcal{O}_\xi) \in \pi_0 \mathrm{Cont}_0(Y,\xi,B)$ agrees with the **squared** contact Dehn twist $\tau^2_{\partial B}$.*

Proof Consider first the case when (Y,ξ) is the contact unit ball $\bigl(\mathbb{B}^3, \xi_{\mathrm{st}} = \ker\bigl(dz + \tfrac{1}{2}x\,dy - \tfrac{1}{2}y\,dx\bigr)\bigr)$ and $B \subset \mathbb{B}^3$ is a subball of smaller radius with center at 0. The fibrations from Section 2.2 fit into a commuting diagram

$$\begin{array}{ccccc}
\mathcal{C}(\mathbb{B}^3,\xi_{\mathrm{st}},B) & \longrightarrow & \mathcal{C}(\mathbb{B}^3,\xi_{\mathrm{st}}) & \xrightarrow{\mathrm{ev}_B} & S^2 \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Diff}_0(\mathbb{B}^3,B) & \longrightarrow & \mathrm{Diff}_0(\mathbb{B}^3) & \longrightarrow & \mathrm{Emb}(\mathbb{B}^3,\mathbb{B}^3) \simeq \mathrm{SO}(3) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B) & \longrightarrow & \mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}}) & \longrightarrow & \mathrm{Emb}((\mathbb{B}^3,\xi_{\mathrm{st}}),(\mathbb{B}^3,\xi_{\mathrm{st}})) \simeq U(1)
\end{array}$$

In the third vertical fiber sequence the map $\pi_2 S^2 = \mathbb{Z} \to \pi_1 U(1) = \mathbb{Z}$ is multiplication by 2. From the diagram we see that the image of $\mathcal{O}_{\xi_{\mathrm{st}}} \in \pi_1\mathcal{C}(\mathbb{B}^3,\xi_{\mathrm{st}},B)$ in $\pi_0\mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B)$ can be alternatively calculated as the image of $2 \in \mathbb{Z} = \pi_1 U(1)$ in $\pi_0 \mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B)$. From Lemma 3.4 this is the class of $\tau^2_{\partial B}$.

For an arbitrary (Y,ξ) and a Darboux ball, $B \subset Y$ the result then follows from the previous local calculation by extending the contact embedding $B \hookrightarrow Y$ to a contact embedding $B \subset \mathbb{B}^3 \hookrightarrow Y$, and considering the commuting diagram

$$\begin{array}{ccccc}
\pi_2 S^2 & \longrightarrow & \pi_1\mathcal{C}(\mathbb{B}^3,\xi_{\mathrm{st}},B) & \longrightarrow & \pi_0\mathrm{Cont}_0(\mathbb{B}^3,\xi_{\mathrm{st}},B) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_2 S^2 & \longrightarrow & \pi_1\mathcal{C}(Y,\xi,B) & \longrightarrow & \pi_0\mathrm{Cont}_0(Y,\xi,B)
\end{array}$$

□

Corollary 3.7 *Suppose Y is aspherical (irreducible and with infinite fundamental group). Then $\tau^2_{\partial B}$ is isotopic to the identity rel B if and only if the evaluation mapping (8) admits a homotopy section (the obstruction class $\mathcal{O}_\xi$ vanishes).*

Proof By the fibration (5) we have the exact sequence

$$\pi_1 \mathrm{Diff}_0(Y,B) \to \pi_1\mathcal{C}(Y,\xi,B) \to \pi_0\mathrm{Cont}_0(Y,\xi,B),$$

so by Proposition 3.6 the result will follow from $\pi_1\mathrm{Diff}_0(Y,B) = 0$. Let us now explain why this group vanishes. By the fibration (7) we have an exact sequence

$$1 \to \pi_1\mathrm{Diff}_0(Y,B) \to \pi_1\mathrm{Diff}_0(Y) \to \pi_1\mathrm{Fr}(Y) \cong \pi_1 Y \times \mathbb{Z}_2.$$

Here, to have a 1 on the left we use $\pi_2 Y = 0$ (which follows from Y being aspherical). Since the homomorphism $\pi_1 \operatorname{Diff}_0(Y) \to \pi_1 Y$, and hence $\pi_1 \operatorname{Diff}_0(Y) \to \operatorname{Fr}(Y)$, is injective, it follows by exactness that $\pi_1 \operatorname{Diff}_0(Y, B) = 0$ as required. The fact that $\pi_1 \operatorname{Diff}_0(Y) \to \pi_1 Y$ is injective follows from the calculation of the homotopy type of the group $\operatorname{Diff}_0(Y)$ for all aspherical[4] 3-manifolds. More precisely, the papers [Hatcher 1976, 1981; 1983, Gabai 2001; Ivanov 1976; McCullough and Soma 2013] cover all aspherical 3-manifolds with the exception of the non-Haken infranilmanifold (see [McCullough and Soma 2013] for a nice summary). The latter consist of the nontrivial S^1-bundles over T^2, which are covered by [Bamler and Kleiner 2024]. In all these cases $\operatorname{Diff}_0(Y)$ has the homotopy type of $(S^1)^k$, where k is the rank of the center of $\pi_1 Y$ and $\pi_1 \operatorname{Diff}_0(Y) \to \pi_1 Y$ is the inclusion of the center. □

In the local model $(Y,\xi) = \big(\mathbb{B}^3, \xi_{\mathrm{st}} = \ker\big(dz + \tfrac{1}{2}x\,dy - \tfrac{1}{2}y\,dx\big)\big)$, and letting $B \subset \mathbb{B}^3$ be any concentric subball, we have the following unique characterization of the obstruction class:

Lemma 3.8 *The evaluation of contact structures on $\mathbb{B}^3$ along the radial line $\{(0,0,z) \mid z \in [0,1]\} \subset \mathbb{B}^3$ identifies the evaluation fibration on $\mathcal{C}(\mathbb{B}^3, \xi_{\mathrm{st}})$ with the path fibration on S^2:*

$$\begin{array}{ccccc} \mathcal{C}(\mathbb{B}^3, \xi_{\mathrm{st}}, B) & \longrightarrow & \mathcal{C}(\mathbb{B}^3, \xi_{\mathrm{st}}) & \xrightarrow{\mathrm{ev}_B} & S^2 \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow= \\ \Omega S^2 & \longrightarrow & PS^2 & \longrightarrow & S^2. \end{array}$$

Thus the obstruction class $\mathcal{O}_{\xi_{\mathrm{st}}} \in \pi_1\mathcal{C}(\mathbb{B}^3, \xi_{\mathrm{st}}, B)$ corresponds to the standard generator of $\pi_1\Omega S^2 = \pi_2 S^2$.

Proof By the Eliashberg–Mishachev theorem [2021], the space $\mathcal{C}(\mathbb{B}^3, \xi_{\mathrm{st}})$ is contractible. So is the path space PS^2, so the desired result follows. □

3.2.2 Geometric description of the obstruction class It is instructive to describe an explicit loop $(\xi_\varphi)_{\varphi\in S^1}$ of contact structures on Y fixed over a Darboux ball $B \subset Y$ which represents the obstruction class $\mathcal{O}_\xi \in \pi_1\mathcal{C}(Y,\xi,B)$; we do this now, but, won't use this construction in the remainder of the article.

By definition of the connecting map $\delta\colon \pi_2 S^2 \to \pi_1\mathcal{C}(Y,\xi,B)$ associated to the fibration (8), a based loop $\xi_\varphi \in \mathcal{C}(Y,\xi,B)$ represents the obstruction class $\mathcal{O}_\xi$ precisely when there exists a $\mathbb{B}^2$-family of contact structures $\xi_{r,\varphi} \in \mathcal{C}(Y,\xi)$ — here, the unit 2-ball $\mathbb{B}^2$ is parametrized using polar coordinates (r,φ) — with $\xi_{r,0} = \xi$ such that $\xi_{1,\varphi} = \xi_\varphi$ and the mapping $\mathbb{B}^2 \ni (r,\varphi) \mapsto \mathrm{ev}_B(\xi_{r,\varphi}) = \xi_{r,\varphi}(p) \in S^2$ induces a *degree*-1 mapping $\mathbb{B}^2/\partial\mathbb{B}^2 \to S^2$ (note that the map out of $\mathbb{B}^2/\partial\mathbb{B}^2$ is well defined because $\xi_{1,\varphi}(p)$ is constant in φ).

Such a family $\xi_{r,\varphi}$ can be constructed as follows. First, it suffices to consider the case $(Y,\xi) = (\mathbb{B}^3, \xi_{\mathrm{st}})$, and construct a family $\xi_{r,\varphi} \in \mathcal{C}(\mathbb{B}^3, \xi, B)$ as above (and with $\xi_{r,\varphi} = \xi_{\mathrm{st}}$ near $\partial\mathbb{B}^3$). For this, choose any smooth mapping $q\colon [0,1] \times S^1 \to \mathrm{SO}(3)/U(1)$ with

$$q(0,\varphi) = q(1,\varphi) = [\mathrm{id}], \quad q(r,0) = q(r,2\pi) = [\mathrm{id}],$$

[4] For the irreducible 3-manifolds with finite fundamental group, the calculation of the homotopy type of $\operatorname{Diff}_0(Y)$ has also been completed [Hong et al. 2012; Bamler and Kleiner 2019, 2023]. Thus the homotopy type of $\operatorname{Diff}_0(Y)$ is known for all prime 3-manifolds.

such that the induced map $\Sigma S^1 \to \mathrm{SO}(3)/U(1)$ has degree 1. Here $\Sigma S^1 \approx S^2$ is the reduced suspension of S^1, and in what follows we regard S^1 as $\mathbb{R}/2\pi\mathbb{Z}$. Next, regarding q as a homotopy of based maps $S^1 \to \mathrm{SO}(3)/U(1)$, we may lift it along the fibration $\mathrm{SO}(3) \to \mathrm{SO}(3)/U(1)$ to produce a family of matrices $A_{r,\varphi} \in \mathrm{SO}(3)$ parametrized by $(r,\varphi) \in [0,1] \times S^1$ such that

$$[A_{r,\varphi}] = q(r,\varphi), \quad A_{0,\varphi} = \mathrm{id}, \quad A_{r,0} = A_{r,2\pi} = \mathrm{id}. \tag{13}$$

Because of the second condition in (13), $A_{r,\varphi}$ is, in fact, a family of matrices parametrized by the unit 2-ball $\mathbb{B}^2 \cong [0,1] \times S^1/0 \times S^1$. At this point, we would like to take the $\mathbb{B}^2$-family of contact structures on $\mathbb{B}^3$ given by $\xi_{r,\varphi} = (A_{r,\varphi})_* \xi_{\mathrm{st}}$. By the first condition in (13) $A_{1,\varphi} \in U(1)$, and from this it follows that $\xi_{1,\varphi}$ is a loop of contact structures on $\mathbb{B}^3$ fixed over the ball $B \subset \mathbb{B}^3$ (in fact this loop is constant everywhere on $\mathbb{B}^3$, $\xi_{1,\varphi} = \xi_{\mathrm{st}}$) and the induced map $\mathbb{B}^2/\partial\mathbb{B}^2 \to S^2$ given by $(r,\varphi) \mapsto \xi_{r,\varphi}(0)$ has degree 1, as required. However, the $\mathbb{B}^2$-family $\xi_{r,\varphi}$ is not constant near the boundary of $\mathbb{B}^3$, so we must appropriately "cut off" this family near the boundary.

We can do this as follows. Introduce a smooth cutoff function β on $\mathbb{B}^3$ which is identically 1 over $B \subset \mathbb{B}^3$ and vanishes near $\partial\mathbb{B}^3$. For each $(r,\varphi) \in [0,1] \times S^1$ we consider the following vector field supported in the interior of $\mathbb{B}^3$:

$$V_{r,\varphi}(x) = \beta(x) \frac{\partial}{\partial r} A_{r,\varphi} \cdot x.$$

We regard $V_{r,\varphi}$ as a φ-family of r-dependent vector fields on $\mathbb{B}^3$, and consider the associated φ-family of flows $\Phi^r_\varphi \colon \mathbb{B}^3 \to \mathbb{B}^3$ starting at time $r = 0$, namely

$$\frac{\partial}{\partial r} \Phi^r_\varphi(x) = V_{r,\varphi}(\Phi^r_\varphi(x)), \quad \Phi^0_\varphi(x) = x.$$

Over the ball $B \subset \mathbb{B}^3$ we have, by construction, that $\Phi^r_\varphi(x) = A_{r,\varphi} \cdot x$, and near the boundary of $\mathbb{B}^3$ we have $\Phi_{r,\varphi} = \mathrm{id}$. Hence the $\mathbb{B}^2$-family of contact structures defined by $\xi_{r,\varphi} := (\Phi^r_\varphi)_* \xi_{\mathrm{st}}$ is now constant near the boundary of $\mathbb{B}^3$, and still has the required properties.

The loop $\xi_{1,\varphi}$ of contact structures in $\mathcal{C}(\mathbb{B}^3, \xi_{\mathrm{st}}, B)$ thus constructed is an explicit representative of the obstruction class $\mathcal{O}_\xi \in \pi_1 \mathcal{C}(\mathbb{B}^3, \xi_{\mathrm{st}}, B)$. This loop can then be implanted into an arbitrary contact 3-manifold (Y,ξ) along a Darboux chart $(\mathbb{B}^3, \xi_{\mathrm{st}}) \subset (Y,\xi)$ to give a representative of the obstruction class for arbitrary (Y,ξ).

3.3 Formal triviality of $\tau^2_{\partial B}$

We continue in the setting of the previous section, and we show:

Lemma 3.9 *Suppose the Euler class of ξ vanishes. Then both the loop of contact structures given by the obstruction class $\mathcal{O}_\xi \in \pi_1 \mathcal{C}(Y,\xi,B)$ and the squared Dehn twist $\tau^2_{\partial B} \in \pi_0 \mathrm{Cont}_0(Y,\xi,B)$ are formally trivial rel B.*

Proof On the space of co-oriented plane fields we have an analogous evaluation mapping (a fibration also, in fact)

$$\Xi(Y,\xi) \to S^2, \quad \xi' \mapsto \xi'(0).$$

When the Euler class of ξ vanishes we may identify $\Xi(Y,\xi)$ with the space $\mathrm{Map}_0(Y,S^2)$ of nullhomotopic smooth maps $Y \to S^2$. The evaluation mapping becomes identified with the obvious evaluation mapping on this latter space. Clearly this fibration admits a section given by the constant maps $Y \to S^2$. Hence the corresponding obstruction class vanishes, and hence

$$\mathcal{O}_\xi \in \ker(\pi_1 \mathcal{C}(Y,\xi,B) \to \pi_1 \Xi(Y,\xi,B)),$$

so $\mathcal{O}_\xi$ is formally trivial. From the rel B analogue of Corollary 2.9 it follows that $\tau^2_{\partial B}$ is formally trivial also. □

3.4 Behavior of $\mathcal{O}_\xi$ under summation

We proceed by discussing how the obstruction class $\mathcal{O}_\xi$ from (12) interacts with the formation of connected sums.

First, we briefly review a convenient model for the contact connected sum [Colin 1997; Geiges 2008]. We write $(Y_0,\xi_0) = \big([-1,1]\times S^2, \ker\big(z\,ds + \frac{1}{2}x\,dy - \frac{1}{2}y\,dx\big)\big)$. Let $(Y_\pm,\xi_\pm)$ be two contact 3-manifolds with Darboux balls $B_\pm \subset Y_\pm$, and coordinates $\phi_\pm\colon (Y_0,\xi_0) \hookrightarrow (Y_\pm,\xi_\pm)$ around $\partial B_\pm$ such that $\phi_\pm(\{0\}\times S^2) = \partial B_\pm$ and $\phi_\pm(Y_0) \cap Y_\pm \backslash B_\pm = \phi_\pm((0,1]\times S^2)$. Consider the smaller ball $B^0_\pm = B_\pm \backslash \phi_\pm(Y_0) \subset B_\pm$. To define the contact connected sum we use the gluing contactomorphism G of (Y_0,ξ_0) given by $G(s,x,y,z) = (-s,-x,-y,-z)$.

Definition 3.10 The *connected sum* of contact manifolds

$$(Y_\#,\xi_\#) = (Y_-,\xi_-) \,\#\, (Y_+,\xi_+)$$

is defined to be $(Y_- \backslash B^0_-, \xi_-) \cup_G (Y_+ \backslash B^0_+, \xi_+)$.

The connected sum of contact manifolds is well defined and independent of choices up to contactomorphism [Colin 1997].

We will fix a Darboux ball $B_\# \subset Y_\#$ inside the neck region $[-1,1]\times S^2 = \phi_-(Y_0) = \phi_+(Y_0) \subset Y_\#$. We also have natural inclusions $\mathcal{C}(Y_\pm,\xi_\pm,B_\pm) \subset \mathcal{C}(Y_\#,\xi_\#,B_\#)$. We consider their induced maps on π_1

$$(-)\,\#\,\xi_+ \colon \pi_1\mathcal{C}(Y_-,\xi_-,B_-) \to \pi_1\mathcal{C}(Y_\#,\xi_\#,B_\#),$$
$$\xi_-\,\#\,(-) \colon \pi_1\mathcal{C}(Y_+,\xi_+,B_+) \to \pi_1\mathcal{C}(Y_\#,\xi_\#,B_\#).$$

Proposition 3.11 *The obstruction class $\mathcal{O}_{\xi_\#} \in \pi_1\mathcal{C}(Y_\#,\xi_\#,B_\#)$ is given by*

$$\mathcal{O}_{\xi_\#} = (\mathcal{O}_{\xi_-}\,\#\,\xi_+)\cdot(\xi_-\,\#\,\mathcal{O}_{\xi_+}).$$

Proof It suffices to prove the corresponding statement in the local model where $(Y_\pm,\xi_\pm)=(\mathbb{B}^3,\xi_{\mathrm{st}})$, $B_\pm\subset Y_\pm$ are smaller concentric subballs and $(Y_\#,\xi_\#)=(Y_-\setminus B_-,\xi_-)\cup_{\partial B_-=-\partial B_+}(Y_+,\xi_+)$. We let $B_\#\subset Y_\#$ be a Darboux chart contained in the interior.

Consider the map $j:\mathcal{C}(Y_-,\xi_-,B_-)\times\mathcal{C}(Y_+,\xi_+,B_+)\to\mathcal{C}(Y_\#,\xi_\#,B_\#)$ given by gluing together the contact structures on the pieces $Y_\pm\setminus B_\pm$ to produce a contact structure on $Y_\#$. Choose paths $\gamma_\pm\subset Y_\#$ which connect the ball $B_\#$ to the component $\partial Y_\pm$ of the boundary of $Y_\#$. There is a commuting diagram

$$\begin{array}{ccc}
\mathcal{C}(Y_\#,\xi_\#,B_\#) & \xrightarrow{\mathrm{ev}_{\gamma_-}\times\mathrm{ev}_{\gamma_+}} & (\Omega S^2)^2 \\
{\scriptstyle j}\uparrow & & \uparrow{\scriptstyle =} \\
\mathcal{C}(Y_-,\xi_-,B_-)\times\mathcal{C}(Y_+,\xi_+,B_+) & \xrightarrow{\simeq} & (\Omega S^2)^2
\end{array}$$

where the bottom horizontal map is given by the homotopy equivalences of Lemma 3.8. By a similar argument to that in the proof of Lemma 3.8, one shows that the top horizontal map is a homotopy equivalence. Thus j is also a homotopy equivalence.

Next, consider the following diagram of spaces and maps, where the bottom row is two copies of the path fibration over S^2, and Δ is the diagonal map:

$$\begin{array}{ccccc}
\mathcal{C}(Y_\#,\xi_\#,B_\#) & \longrightarrow & \mathcal{C}(Y_\#,\xi_\#) & \xrightarrow{\mathrm{ev}_{B_\#}} & S^2 \\
\downarrow{\scriptstyle \mathrm{ev}_{\gamma_-}\times\mathrm{ev}_{\gamma_+}} & & \downarrow{\scriptstyle \mathrm{ev}_{\gamma_-}\times\mathrm{ev}_{\gamma_+}} & & \downarrow{\scriptstyle \Delta} \\
(\Omega S^2)^2 & \longrightarrow & (PS^2)^2 & \longrightarrow & (S^2)^2
\end{array}$$

The path fibration on S^2 can be identified with the evaluation fibration on $\mathcal{C}(Y_\pm,\xi_\pm)$ at the center of $B_\pm$, by Lemma 3.8. Under this identification, the leftmost vertical map in the second diagram becomes the homotopy inverse of the map j (which follows from the first diagram). Combining these observations and passing to the long exact sequence in homotopy groups in the second diagram yields a commutative square

$$\begin{array}{ccc}
\pi_2 S^2 & \xrightarrow{\delta} & \pi_1\mathcal{C}(Y_\#,\xi_\#,B_\#) \\
\downarrow & & \uparrow{\scriptstyle j} \\
\pi_2 S^2\times\pi_2 S^2 & \xrightarrow{\delta\times\delta} & \pi_1\mathcal{C}(Y_-\setminus B_-,\xi_-)\times\pi_1\mathcal{C}(Y_+\setminus B_+,\xi_+)
\end{array}$$

from which the desired result follows. □

Remark 3.12 In particular, it follows from Propositions 3.11 and 3.6 that we have the relation $\tau^2_{\partial B_+}\tau^2_{\partial B_-}=\tau^2_{B_\#}$ in $\pi_0\,\mathrm{Cont}_0(Y_\#,\xi_\#,\partial B_\#)$.

3.5 Examples: trivial Dehn twists

For comparison with Theorem 1.3 we now exhibit examples where the squared Dehn twist on a connected sum becomes trivial as a contactomorphism.

3.5.1 Quotients of S^3 Let Γ be a finite subgroup of $U(2)$. Then Γ preserves the standard contact structure $\xi_{\mathrm{st}} = \ker\bigl(\sum_{j=1,2} x_j\, dy_j - y_j\, dx_j\bigr)$ on the unit 3-sphere S^3, so it descends onto the quotient $M_\Gamma = S^3/\Gamma$. The M_Γ are the spherical 3-manifolds and include, among others, the lens spaces $L(p,q)$ and the Poincaré sphere $\Sigma(2,3,5)$.

Lemma 3.13 *The squared Dehn twist $\tau^2_{\partial B}$ on the boundary of a Darboux ball $B \subset M_\Gamma$ is contact isotopic to the identity rel B. Hence the squared Dehn twist $\tau^2_{S_\#}$ on the separating sphere $S_\#$ in $(Y,\xi)\,\#\,(M_\Gamma,\xi_{\mathrm{st}})$ is contact isotopic to the identity.*

Proof The center of $U(2)$ is given by the subgroup $\cong U(1)$ of diagonal matrices with diagonal (λ,λ) for some $\lambda \in U(1)$. This subgroup acts on M_Γ by contactomorphisms and thus also on the space of Darboux balls, which is homotopy equivalent to $M_\Gamma \times U(1)$ by (3). This gives a map $\pi_1 U(1) = \mathbb{Z} \to \pi_1(M_\Gamma \times U(1)) = \Gamma \times \mathbb{Z}$ which we assert is given by $1 \mapsto (e,2)$ where $e \in \Gamma$ is the identity element. From Lemma 3.4 and this assertion, the result would follow.

That the component $\mathbb{Z} \to \Gamma$ is trivial follows from $U(1)$ being the center of $U(2)$. To verify that $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 2 we need to calculate the change in contact framing under the action of $U(1)$. We view S^3 as the unit sphere in the quaternions $\mathbb{H} = \mathbb{R}\langle 1,i,j,k\rangle$, so the tangent space at $q \in S^3$ is given by $T_qS^3 = \mathbb{R}\langle iq, jq, kq\rangle$ and the standard contact structure is $\xi_{\mathrm{st}}(q) = \mathbb{R}\langle jq, kq\rangle = \mathbb{C}\langle jq\rangle$. Thus the frame jq trivializes $\xi_{\mathrm{st}} \cong \mathbb{C}$ as a complex line bundle. The center subgroup $U(1) \subset U(2)$ acts on S^3 by $(\lambda,q) \mapsto \lambda q$, and the action of $U(1)$ on the frame jq is

$$\lambda \cdot jq = j\bar{\lambda} q = \lambda^2 \cdot j(\lambda q).$$

Thus the action on $\xi_{\mathrm{st}} \cong \mathbb{C}$ is by multiplication by λ^2 on the fibers. This establishes our assertion, and hence the proof is complete. □

Remark 3.14 When $\Gamma \subset \mathrm{SU}(2)$, an alternative proof of Lemma 3.13 can be obtained by instead exhibiting a section of $\mathrm{ev}_B : \mathcal{C}(M_\Gamma,\xi_{\mathrm{st}}) \to S^2$. The point is that the radial vector field $x\partial_x + y\partial_y + z\partial_z + w\partial_w$ is a Liouville vector field for each of the symplectic forms ω_u, for $u \in S^2$, in the flat hyperkähler structure of $\mathbb{R}^4$. The induced S^2-family of contact structures ξ_u on S^3 descends to the quotients M_Γ (with $\Gamma \subset \mathrm{SU}(2)$) and provides a section of ev_B.

3.5.2 $S^1 \times S^2$ Consider the unique tight contact structure on $S^1 \times S^2$, given by

$$\xi_0 = \ker\bigl(z\, d\theta + \tfrac{1}{2} x\, dy - \tfrac{1}{2} y\, dx\bigr).$$

Lemma 3.15 *The squared Dehn twist $\tau^2_{\partial B}$ on the boundary of a Darboux ball $B \subset S^1 \times S^2$ is contact isotopic to the identity rel B. Hence the squared Dehn twist $\tau^2_{S_\#}$ on the separating sphere $S_\#$ in any contact connected sum of the form $(Y,\xi)\,\#\,(S^1\times S^2,\xi_0)$ is contact isotopic to the identity.*

Proof Let R_φ be the counterclockwise rotation in the xy plane of angle φ. By considering the subgroup $\{F_\varphi \mid \varphi \in S^1\} \simeq U(1) \subset \mathrm{Cont}(S^1 \times S^2, \xi_0)$, given by $F_\varphi(\theta,x,y,z) := (\theta, R_\varphi(x,y), z)$, one easily checks that $\pi_1(\mathrm{Cont}(S^1 \times S^2, \xi_0) \to \pi_1 \mathrm{Emb}((\mathbb{B}^3, \xi_{\mathrm{st}}), (S^1 \times S^2, \xi_0)) \to \pi_1 U(1)$ is surjective, so the result follows. □

Remark 3.16 In turn, the contact Dehn twist on the nontrivial sphere in $(S^1 \times S^2, \xi_0)$ is nontrivial (and has infinite order). However, it is formally nontrivial already and therefore not exotic; see Section 3.6.

3.5.3 Sum with an overtwisted contact 3-manifold Let $(r, \theta, z) \in \mathbb{R}^3$ be cylindrical coordinates. Consider the contact structure ξ_{ot} in $\mathbb{R}^3$ defined by the kernel of

$$\alpha_{\mathrm{ot}} = \cos r\, dz + r \sin r\, d\theta.$$

The disk $\Delta_{\mathrm{ot}} = \{(r, \theta, z) \in \mathbb{R}^3 \mid z = 0,\ r \le \pi\}$ is an *overtwisted disk*.

Definition 3.17 [Eliashberg 1989] An overtwisted contact 3-manifold is a contact 3-manifold that contains an embedded overtwisted disk.

Let $\mathcal{C}(Y, \Delta_{\mathrm{ot}})$ be the space of contact structures in Y with a fixed overtwisted disk $\Delta_{\mathrm{ot}} \subset Y$. Let $\Xi(Y, \Delta_{\mathrm{ot}})$ be the space of co-oriented plane fields in Y tangent to Δ_{ot} at the point $0 \in \Delta_{\mathrm{ot}}$. A foundational result of Eliashberg, generalized in higher dimensions by Borman, Eliashberg and Murphy, is:

Theorem 3.18 [Eliashberg 1989; Borman et al. 2015] *The inclusion*

$$\mathcal{C}(Y, \Delta_{\mathrm{ot}}) \to \Xi(Y, \Delta_{\mathrm{ot}})$$

is a homotopy equivalence.

Remark 3.19 A relative version Eliashberg's h-principle is available. Suppose $A \subseteq Y \setminus \Delta_{\mathrm{ot}}$ is compact and $Y \setminus A$ is connected. Given a family of co-oriented plane fields $\xi^k \in \Xi(Y, \Delta_{\mathrm{ot}})$ that is contact over an open neighborhood of A, there exists a homotopy rel A from ξ^k to a family of contact structures.

Using Eliashberg's h-principle we obtain:

Lemma 3.20 *Let (Y, ξ) be a contact 3-manifold with vanishing Euler class. Then, for every overtwisted contact 3-manifold (M, ξ_{ot}), the squared contact Dehn twist $\tau^2_{S_\#}$ in $(Y, \xi) \# (M, \xi_{\mathrm{ot}})$ is contact isotopic to the identity.*

Proof Let $B \subset (Y, \xi)$ be a Darboux ball that we remove when performing the connected sum. By Lemma 3.9 $\tau^2_{\partial B}$ is formally contact isotopic to the identity rel B. It follows that $\tau^2_{S_\#}$ is formally contact isotopic to the identity on $Y \# M$, in fact relative to a small ball B_{ot} containing an overtwisted disk $\Delta_{\mathrm{ot}} \subset M$. At this point, by Eliashberg's Theorem 3.18 and Lemma 2.8 applied to the contact 3-manifold with convex boundary $(Y \# (M \setminus B_{\mathrm{ot}}), \xi \# \xi_{\mathrm{ot}})$, the group of contactomorphisms fixing Δ_{ot} is homotopy equivalent to the corresponding space of formal contactomorphisms. □

In Section 7 we will see that Lemma 3.20 implies exotic 1-parametric phenomena in overtwisted contact 3-manifolds.

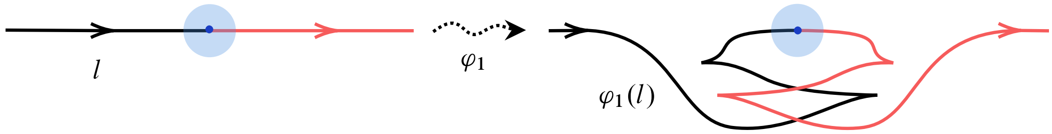

Figure 1: Front projection of l and $\hat{l}$. The shaded ball represents the small ball $B_\varepsilon \subset \mathbb{B}^3$.

3.6 The Reidemeister I move and Gompf's contactomorphism

We now describe the contact Dehn twist diagrammatically by means of front projections of Legendrian arcs. This approach is in the spirit of Gompf's description [1998] of the contact Dehn twist. For convenience we consider the unit ball $(\mathbb{B}^3, \xi = \ker(dz - y\,dx))$. Let $Y_0 = [-1,1] \times S^2$ be the complement in $\mathbb{B}^3$ of a small open ball B_ε around the origin. Consider the standard Legendrian arc $l\colon [-1,1] \to \mathbb{B}^3$ given by $t \mapsto (t,0,0)$. Perform two Reidemeister I moves to the Legendrian l to obtain a second Legendrian arc $\hat{l}$. We may assume that $\hat{l}$ coincides with l over the B_ε. The fronts of these arcs are depicted in Figure 1.

These arcs are Legendrian isotopic, so there exists a contact isotopy $\varphi_t \in \mathrm{Cont}(\mathbb{B}^3, \xi)$ with $\varphi_0 = \mathrm{id}$ and $\varphi_1 \circ l = \hat{l}$. Moreover, φ_1 can be taken to be the identity over B_ε. Therefore φ_1 gives a contactomorphism τ of the contact manifold with convex boundary (Y_0, ξ). From now on, we will denote the restrictions of l and $\hat{l}$ to the red segments in Figure 1 by the same letters for convenience. We have $\tau(l) = \hat{l}$ and the arc $\hat{l}$ is obtained in (Y_0, ξ) from l by a positive stabilization; see Figure 2. In particular,

$$\mathrm{rot}(\tau(l)) = \mathrm{rot}(l) + 1.$$

It follows that τ is not (formally) contact isotopic to the identity as a contactomorphism of (Y_0, ξ) rel ∂Y_0. This contactomorphism is contact isotopic to the contact Dehn twist as we have defined it in this section. In fact, as we will see in Lemma 3.22, since the complement of l is a tight 3-ball, any contactomorphism of (Y_0, ξ) can be described, up to contact isotopy, just in terms of its effect on l and, therefore, just by means of front projections of Legendrian arcs. First, we observe that the path-connected components of the space $\mathrm{Leg}(Y_0, \xi)$ of Legendrian embeddings of arcs that coincide with l at the endpoints can be easily understood:

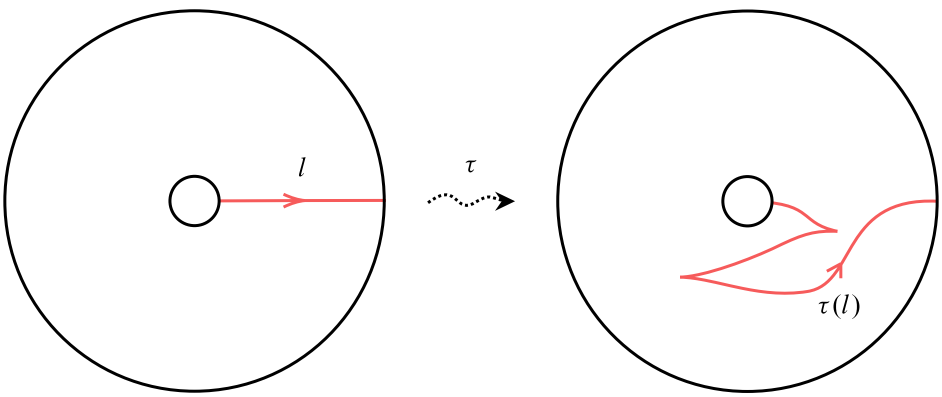

Figure 2: The image of l under τ.

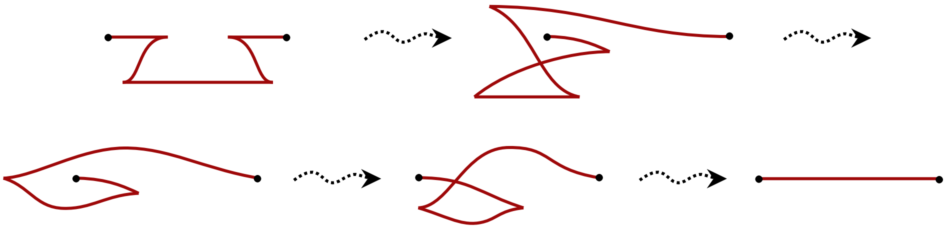

Figure 3: Legendrian isotopy from a double stabilization of l to l in (Y_0, ξ).

Lemma 3.21 *The map* $\mathrm{rot}\colon \pi_0 \operatorname{Leg}(Y_0, \xi) \to \mathbb{Z}$ *given by* $L \mapsto \mathrm{rot}(L)$ *is an isomorphism.*

Proof Two smoothly isotopic Legendrian arcs with the same rotation number are Legendrian isotopic after adding a finite number of double stabilizations (pairs of positive and negative stabilizations) because of the Fuchs–Tabachnikov theorem [1997]. As depicted in Figure 3, this can done by a Legendrian isotopy in (Y_0, ξ). Therefore the proof follows from the 3-dimensional light bulb theorem. □

We conclude the following:

Lemma 3.22 *The map* $\operatorname{Cont}(Y_0, \xi) \to \operatorname{Leg}(Y_0, \xi)$ *given by* $f \mapsto f \circ l$ *is a homotopy equivalence. In particular,*

$$\pi_0 \operatorname{Cont}(Y_0, \xi) \to \mathbb{Z}, \quad f \mapsto \mathrm{rot}(f \circ l),$$

is an isomorphism. Moreover, the contact Dehn twist is characterized, up to contact isotopy, by the relation

$$\mathrm{rot}(f(l)) = \mathrm{rot}(l) + 1.$$

Proof This follows by the previous lemma, the Eliashberg–Mishachev theorem (Theorem 1.13) and Hatcher's theorem [1983], since the fiber of $\operatorname{Cont}(Y_0, \xi) \to \operatorname{Leg}(Y_0, \xi)$ can be identified with the contactomorphism group of the complement of a neighborhood of l, and the latter is a tight 3-ball. □

4 Monopole Floer homology and families of contact structures

In this section we provide the necessary background on the Floer-theoretic ingredients that come into the proof of Theorem 1.3. Henceforth all homology groups are taken with $\mathbb{Q}$ coefficients for simplicity.

4.1 Monopole Floer homology and the contact invariant

For a quick introduction to Kronheimer and Mrowka's monopole Floer homology groups we recommend [Lin 2016; Kronheimer et al. 2007] and for a detailed treatment the monograph [Kronheimer and Mrowka 2007]. Here we just comment briefly on a few formal aspects.

Consider a 3-manifold Y together with spin-c structure $\mathfrak{s}$ (here the only spin-c structure that will be relevant is that induced by a contact structure ξ, denoted by $\mathfrak{s}_\xi$). Associated to it there are various monopole Floer homology groups ($\mathbb{Q}$-vector spaces in this article). The ones relevant to us are the "to" and "tilde" flavors: $\widecheck{\mathrm{HM}}(Y,\mathfrak{s})$ and $\widetilde{\mathrm{HM}}(Y,\mathfrak{s})$. The former arises "formally" as the S^1-equivariant Morse homology of the Chern–Simons–Dirac functional. An algebraic manifestation of this equivariant nature is that $\widecheck{\mathrm{HM}}(Y,\mathfrak{s})$ carries a module structure over the polynomial algebra $\mathbb{Q}[U]$ (the S^1-equivariant cohomology of a point, $H^\bullet_{S^1}(\text{point}) = \mathbb{Q}[U]$) and U decreases grading by two. In turn, the "tilde" flavor should be regarded as the (nonequivariant) Morse homology, and thus is an $H_\bullet(S^1) = \mathbb{Q}[\chi]/(\chi^2)$-module, with χ raising the degree by one. A standard *Gysin sequence* relates the two groups:

$$\cdots \xrightarrow{p} \widecheck{\mathrm{HM}}_\bullet(Y,\mathfrak{s}) \xrightarrow{U} \widecheck{\mathrm{HM}}_{\bullet-2}(-Y,\mathfrak{s}) \xrightarrow{j} \widetilde{\mathrm{HM}}_{\bullet-1}(-Y,\mathfrak{s}) \xrightarrow{p} \cdots.$$

The map χ is recovered from this by $\chi = jp$. A common feature of all flavors of the monopole groups is a canonical grading by the set of homotopy classes of plane fields $\pi_0\Xi(Y)$, which carries a natural $\mathbb{Z}$-action.

The *contact invariant* $\boldsymbol{c}(\xi)$ is an element of $\widecheck{\mathrm{HM}}_{[\xi]}(-Y,\mathfrak{s}_\xi)$ which is well defined up to a sign, and is canonically attached to a contact structure ξ on Y. It was defined by Kronheimer, Mrowka, Ozsváth and Szabó in [Kronheimer et al. 2007], but its definition goes back essentially to the earlier paper [Kronheimer and Mrowka 1997]. Ozsváth and Szabó [2005] gave a definition of $\boldsymbol{c}(\xi)$ in Heegaard–Floer homology. Under the isomorphism between the monopole and Heegaard–Floer groups [Kutluhan et al. 2020; Colin et al. 2011], the contact invariants are shown to agree. Some of the basic properties of $\boldsymbol{c}(\xi)$ are:

- [Mrowka and Rollin 2006] $\boldsymbol{c}(\xi) = 0$ if (Y,ξ) is overtwisted.
- [Echeverria 2020] $\boldsymbol{c}(\xi) \neq 0$ if (Y,ξ) admits a strong symplectic filling.
- [Echeverria 2020] $\boldsymbol{c}(\xi)$ is natural under symplectic cobordisms: if (W,ω) is a symplectic cobordism $(Y_1,\xi_1) \rightsquigarrow (Y_2,\xi_2)$ — here the convex end is (Y_2,ξ_2) — then
$$\widecheck{\mathrm{HM}}(-W,\mathfrak{s}_\omega)\boldsymbol{c}(\xi_2) = \boldsymbol{c}(\xi_1).$$
- $U \cdot \boldsymbol{c}(\xi) = 0$ (this is clear from the Heegaard–Floer point of view; in the monopole case this follows from Theorem 4.2).

4.2 The families contact invariant

Remark 4.1 Throughout this section we assume that $\boldsymbol{c}(\xi) \neq 0$ because it simplifies a little the exposition that follows (otherwise one should consider homologies with twisted coefficients; see [Muñoz-Echániz 2024]). We also resolve the sign ambiguity of $\boldsymbol{c}(\xi)$ by fixing one of the two. All homologies are taken with $\mathbb{Q}$ coefficients.

A version of the contact invariant for a family of contact structures was introduced by the second author in [Muñoz-Echániz 2024]. We summarize now some of those results. We have homomorphisms

$$\boldsymbol{Fc}_\bullet\colon H_\bullet(\mathcal{C}(Y,\xi)) \to \widecheck{\mathrm{HM}}_{[\xi]+\bullet}(-Y,\mathfrak{s}_\xi), \tag{14}$$

$$\widetilde{\boldsymbol{Fc}}_\bullet\colon H_\bullet(\mathcal{C}(Y,\xi,B)) \to \widetilde{\mathrm{HM}}_{[\xi]+\bullet}(-Y,\mathfrak{s}_\xi). \tag{15}$$

The invariant $\boldsymbol{Fc}_\bullet$ recovers the usual contact invariant: $H_0(\mathcal{C}(Y,\xi)) = \mathbb{Q}$ and then $\boldsymbol{Fc}_0(1) = \boldsymbol{c}(\xi)$. Their main property we exploit is the following. Associated to the fibration $\mathrm{ev}_{\boldsymbol{B}}\colon \mathcal{C}(Y,\xi) \to S^2$ there is the Serre spectral sequence in homology. The latter collapses on the E^3 page and assembles into the *Wang* long exact sequence,

$$\cdots \to H_\bullet(\mathcal{C}(Y,\xi)) \xrightarrow{U_B} H_{\bullet-2}(\mathcal{C}(Y,\xi,B)) \to H_{\bullet-1}(\mathcal{C}(Y,\xi,B)) \xrightarrow{\iota_*} \cdots,$$

where $U_{\boldsymbol{B}}$ takes the intersection of cycles in the total space of the fibration with the fiber, and ι_* is the map induced by inclusion $\iota\colon \mathcal{C}(Y,\xi,B) \to \mathcal{C}(Y,\xi)$. We note that the obstruction class $\mathcal{O}_\xi$ for $\mathrm{ev}_{\boldsymbol{B}}$ to admit a homotopy section arises here homologically as the image of 1 under $\mathbb{Q} = H_0(\mathcal{C}(Y,\xi,B)) \to H_1(\mathcal{C}(Y,\xi,B))$.

Theorem 4.2 [Muñoz-Echániz 2024] *There is a commutative diagram* (*up to signs*)

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{p} & \widetilde{\mathrm{HM}}_{[\xi]+\bullet}(-Y,\mathfrak{s}_\xi) & \xrightarrow{U} & \widetilde{\mathrm{HM}}_{[\xi]+\bullet-2}(-Y,\mathfrak{s}_\xi) & \xrightarrow{j} & \widetilde{\mathrm{HM}}_{[\xi]+\bullet-1}(-Y,\mathfrak{s}_\xi) & \xrightarrow{p} & \cdots \\
 & & \uparrow{\scriptstyle \boldsymbol{Fc}_\bullet} & & \uparrow{\scriptstyle (\boldsymbol{Fc}_{\bullet-2})\circ\iota_*} & & \uparrow{\scriptstyle \widetilde{\boldsymbol{Fc}}_{\bullet-1}} & & \\
\cdots & \longrightarrow & H_\bullet(\mathcal{C}(Y,\xi)) & \xrightarrow{U_B} & H_{\bullet-2}(\mathcal{C}(Y,\xi,B)) & \longrightarrow & H_{\bullet-1}(\mathcal{C}(Y,\xi,B)) & \xrightarrow{\iota_*} & \cdots
\end{array}$$

Some observations are in order:

- As a particular case, Theorem 4.2 recovers a property about the contact invariant $\boldsymbol{c}(\xi)$ which is well known from the Heegaard–Floer point of view: that $U\cdot \boldsymbol{c}(\xi) = 0$ and we have a canonical element $\tilde{\boldsymbol{c}}(\xi) := \widetilde{\boldsymbol{Fc}}_0(1) \in \widetilde{\mathrm{HM}}_{[\xi]}(-Y,\mathfrak{s}_\xi)$ such that $p\tilde{\boldsymbol{c}}(\xi) = \boldsymbol{c}(\xi)$. Conjecturally, the invariant $\boldsymbol{c}(\xi)$ corresponds to the Heegaard–Floer contact invariant that takes values in $\widehat{\mathrm{HF}}(-Y,\mathfrak{s}_\xi)$, which is defined in [Ozsváth and Szabó 2005].

- For 2-dimensional families, Theorem 4.2 gives us the simple formula

$$U\cdot \boldsymbol{Fc}_2(\beta) = \deg(\beta)\boldsymbol{c}(\xi),$$

where $\deg(\beta) = (\mathrm{ev}_{\boldsymbol{B}})_*\beta \in H_2(S^2) = \mathbb{Q}$ is the *degree* of the family $\beta \in H_2(\mathcal{C}(Y,\xi,B))$. In particular, by Theorem 4.2 we have the following:

Corollary 4.3 [Muñoz-Echániz 2024] *If* $\boldsymbol{c}(\xi) \notin \operatorname{Im} U$ *then the fibration* $\mathrm{ev}_{\boldsymbol{B}}$ *does not admit a homotopy section and thus the obstruction class* $\mathcal{O}_\xi$ *is nonvanishing homologically.*

- Other statements that are easily derived from Theorem 1.3 are:

$$\boldsymbol{c}(\xi) \notin \operatorname{Im} U \text{ if and only if } \widetilde{\boldsymbol{Fc}}_1(\mathcal{O}_\xi) \neq 0, \quad \widetilde{\boldsymbol{Fc}}_1(\mathcal{O}_\xi) = \chi\tilde{\boldsymbol{c}}(\xi).$$

- If we define a $\mathbb{Q}[U]$-module structure on $H_\bullet(\mathcal{C}(Y,\xi))$ by setting $U := \iota_*\circ U_{\boldsymbol{B}}$, then Theorem 4.2 asserts, in particular, that the homomorphism $\boldsymbol{Fc}_\bullet\colon H_\bullet(\mathcal{C}(Y,\xi,B)) \to \widetilde{\mathrm{HM}}_{[\xi]+\bullet}(-Y,\mathfrak{s}_\xi)$ is a map of $\mathbb{Q}[U]$-modules. Notice that we have, in fact, a $\mathbb{Q}[U]/(U^2)$-module structure on $H_2(\mathcal{C}(Y,\xi))$, ie the action of U^2 on $H_\bullet(\mathcal{C}(Y,\xi))$ vanishes. This can be regarded as a manifestation of the following geometric fact, that we have already encountered in Section 3. Consider two disjoint Darboux balls $B, B' \subset Y$. Whereas the spaces $\mathcal{C}(Y,\xi)$ and $\mathcal{C}(Y,\xi,B)$ are related in a possibly nontrivial way by the fibration ev_B, the spaces $\mathcal{C}(Y,\xi,B)$ and $\mathcal{C}(Y,\xi,B\cup B')$ are related in a straightforward way:

$$\mathcal{C}(Y,\xi,B\cup B') \simeq \Omega S^2 \times \mathcal{C}(Y,\xi,B).$$

Indeed, the evaluation map corresponding to the ball B' gives a fibration

$$\mathcal{C}(Y,\xi,B\cup B') \to \mathcal{C}(Y,\xi,B) \xrightarrow{\mathrm{ev}_{B'}} S^2,$$

but now the map $\mathrm{ev}_{B'}$ is nullhomotopic, as can be seen by dragging the evaluation point (the center of B') into the first ball B.

4.3 Summary of the construction of the families invariants

We summarize in this section the construction of the invariants $\boldsymbol{Fc}$ and $\widetilde{\boldsymbol{Fc}}$, carried out in detail by the second author in [Muñoz-Echániz 2024]. This is included here for background purposes, but will not be used.

4.3.1 The invariant $\boldsymbol{Fc}$ We begin with some general observations. Let X be a 4-manifold together with a nondegenerate 2-form ω, ie ω^2 is a volume form. We use ω^2 to orient X. Choose an almost-complex structure J compatible with ω, which by definition gives a metric $g=\omega(\cdot\,,J\cdot)$. The space of choices of J is contractible. The structure J equips X with a spin-c structure, ie a lift of the SO(4)-frame bundle of X along the map $\mathrm{Spin}^c(4)\to \mathrm{SO}(4)$. In differential-geometric terms this yields rank-2 complex hermitian bundles $S^{\pm}\to X$ and Clifford multiplication $\rho\colon TX\to \mathrm{Hom}(S^+,S^-)$ satisfying the "Clifford identity" $\rho(v)^*\rho(v)=g(v,v)\,\mathrm{id}$. We follow the notation and conventions from [Kronheimer and Mrowka 2007, Section 1] and we assume the reader is familiar with these.

The Clifford action of the 2-form ω on S^+ splits the bundle S^+ into $\mp 2i$ eigensubbundles of rank 1. These are given by $S^+=E\oplus EK_J^{-1}$, where K_J is the canonical bundle of (X,J) and E is a complex line bundle which is easily verified to be trivial. Choose a unit-length section Φ_0 of E. A simple calculation shows that there is a unique spin-c connection A_0 on S^+ such that $\nabla_{A_0}\Phi_0$ is a 1-form with values in the $+2i$ eigenspace EK_J^{-1}. At this point, the symplectic condition comes in through the following calculation involving the coupled Dirac operator $D_{A_0}\colon \Gamma(S^+)\to\Gamma(S^-)$:

Lemma 4.4 [Taubes 1994] *The nondegenerate* 2*-form* ω *is symplectic* ($d\omega=0$) *if and only if* $D_{A_0}\Phi_0=0$.

We now bring in a smoothly varying family of symplectic structures ω_u parametrized by a smooth manifold $U\ni u$, with each ω_u in the same deformation class as ω. Again, we equip the ω_u with compatible almost-complex structures J_u varying smoothly, which provide us with a family of metrics g_u. From our original Clifford bundle $(S^{\pm}, \rho)$ we canonically obtain new ones as follows. The bundles $S^{\pm}$ remain the same, but new Clifford structures ρ_u are obtained by setting $\rho_u = \rho \circ b_u$, where b_u is the canonical isometry $(TX, g_u) \xrightarrow{\cong} (TX, g)$ — the unique isometry which is positive and symmetric with respect to g_u. The Clifford action of ω_u again decomposes S^+ into eigenspaces $S^+ = E_u \oplus E_u K_{J_u}^{-1}$. Each E_u is trivializable individually, but the family $(E_u)_{u \in U}$ might give a nontrivial line bundle over $U \times X$. When U is contractible we may choose a family of trivializing sections Φ_u of E_u with unit length, and as before these determine unique spin-c connections A_u with $D_{A_u}\Phi_u = 0$. Then, associated to our family (ω_u, J_u) and the choices of Φ_u, we have a family of "deformed" Seiberg–Witten equations on X given by

$$\tfrac{1}{2}\rho_u(F_A^+) - (\Phi\Phi^*)_0 = \tfrac{1}{2}\rho_u(F_{A_u}^+) - (\Phi_u\Phi_u^*)_0, \quad D_A\Phi = D_{A_u}\Phi_u.$$

For each $u \in U$ this is an equation on the pair (A, Φ), where A is a connection on $\Lambda^2 S^+$ and Φ is a section of S^+. In this "deformed" version of the equations, the configurations (A_u, Φ_u) solve the equation for u.

We apply now the above considerations to a special case. Let (Y, ξ) be a closed contact 3-manifold with a contact form α, and let (X, ω) be the *symplectization* $X = [1, +\infty) \times Y$, with the exact symplectic form $\omega = d\left(\frac{1}{2}t^2\alpha\right)$. The structure J is chosen to be invariant under the Liouville flow, and the associated Riemannian metric on X is conical. We now bring into the picture a family of contact structures ξ_u parametrized by $U = \Delta^n$, to which we would like to associate an element in the Floer chain complex of $-Y = \partial X$. Here Δ^n is the standard n-simplex. We equip our family ξ_u with corresponding contact forms α_u. This gives a family ω_u of symplectic structures on X.

The construction now proceeds by forming a manifold Z^+ by gluing the cylinder $Z = (-\infty, 0] \times Y$ with the symplectic manifold X. We extend all metrics g_u over to Z^+ in such a way that they all agree with a fixed translation-invariant metric on the cylinder Z. Then the bundle S^+, together with its splitting $S^+ = E \oplus E K_J^{-1}$, extends over Z^+ naturally in a translation-invariant manner. The U-family of metrics and spin-c structures thus constructed on Z^+ are independent of u over Z, so we have effectively trivialized our data over the cylinder end $Z \subset Z^+$. In order to extend the Seiberg–Witten equations over Z^+ we cut off the perturbation term on the right-hand side of the equations so that it vanishes on the cylinder end Z. This way, we have a U-parametric family of Seiberg–Witten equations over Z^+, and natural boundary conditions for these equations (modulo gauge) are:

- On the cylinder Z, solutions should approach a translation-invariant solution $\mathfrak{a}$ (a generator of the "to" Floer complex $\check{C}(-Y, \mathfrak{s}_\xi)$, ie $\mathfrak{a}$ is an irreducible or boundary-stable monopole on $-Y$).
- On the symplectic end X solutions should approach the configuration (A_u, Φ_u).

This way we obtain parametrized moduli spaces of solutions

$$\pi : M([\mathfrak{a}], \Delta^n) \to \Delta^n.$$

By introducing suitable perturbations we may achieve the necessary transversality [Muñoz-Echániz 2024] and $M([\mathfrak{a}], \Delta^n)$ will be C^1-manifolds of finite dimension. At this point we note that, because of the gauge invariance of the equations, a different choice of trivializations Φ_u would yield diffeomorphic moduli spaces. The connected components of $M([\mathfrak{a}], \Delta^n)$ where the index of π is $-n$ consist of a finite number of isolated points lying over values in the interior of Δ^n, and a signed count of these points gives an integer $\#M([\mathfrak{a}], \Delta^n) \in \mathbb{Z}$. We organize these counts into a Floer chain $\psi(\Delta^n)$:

$$\psi(\Delta^n) = \sum_{[\mathfrak{a}]} \#M([\mathfrak{a}], \Delta^n) \cdot [\mathfrak{a}] \in \check{\mathrm{C}}(-Y, \mathfrak{s}_\xi).$$

The assignment $\Delta^n \mapsto \psi(\Delta^n)$ can be made into a chain map

$$\psi \colon C_\bullet(\mathcal{C}(Y,\xi)) \to \check{\mathrm{C}}_\bullet(-Y, \mathfrak{s}_\xi)$$

from the complex of singular chains on $\mathcal{C}(Y,\xi)$. Taking homology yields the families invariant (14). The analytic underpinnings that make all the above rigorous are discussed in [Muñoz-Echániz 2024], and are essentially no different than those of [Kronheimer and Mrowka 1997; Taubes 2000].

4.3.2 The invariant $\widetilde{\boldsymbol{Fc}}$ In terms of the "to" Floer complex $\widetilde{C}_\bullet$, the "tilde" Floer complex can be defined by taking the mapping cone of (a suitable chain level version of) the U map. We have $\widetilde{C}_\bullet(Y,\mathfrak{s}) = \check{C}_\bullet(Y,\mathfrak{s}) \oplus \check{C}_{\bullet-1}(Y,\mathfrak{s})$ with differential given by the matrix (ignoring signs)

$$\tilde{\partial} = \begin{pmatrix} \check{\partial} & 0 \\ U & \check{\partial} \end{pmatrix}.$$

If a family $\beta \in H_n(\mathcal{C}(Y,\xi))$ is in the image of $\iota_* \colon H_n(\mathcal{C}(Y,\xi,B)) \to H_n(\mathcal{C}(Y,\xi))$ then it is proved in [Muñoz-Echániz 2024] that $U \cdot \boldsymbol{Fc}(\beta) = 0$. At the chain level this is witnessed by a canonical chain homotopy θ:

$$U \cdot \psi \circ \iota_* = \check{\partial}\theta + \theta\partial. \tag{16}$$

From this we build the chain map

$$\tilde{\psi} = (\psi \circ \iota_*, \theta) \colon C_\bullet(C(Y,\xi,B)) \to \widetilde{C}_\bullet(-Y, \mathfrak{s}_\xi),$$

which, upon taking homology, gives the definition of (15). The chain homotopy θ is roughly constructed as follows. We introduce a new parameter $t \in \mathbb{R}$ and let $0 \in Y$ be the center of the ball B. Consider the moduli space

$$\mathcal{M}([\mathfrak{a}], \Delta^n) \to \mathbb{R} \times \Delta^n$$

consisting of quadruples (A, Φ, u, t) such that (A, Φ, u) solve the previous set of equations and boundary conditions subject to the further constraint that at the point $(t, 0) \in \mathbb{R} \times Y \cong Z^+$ the spinor Φ lies in the second component of the splitting $S^+ = E \oplus EK_J^{-1}$. By a simple modification of this construction one can again achieve transversality and ensure that the $\mathcal{M}([\mathfrak{a}], \Delta^n)$ are C^1-manifolds of finite dimension. Then we set

$$\theta(\Delta^n) = \sum_{[\mathfrak{a}]} \#\mathcal{M}([\mathfrak{a}], \Delta^n) \cdot [\mathfrak{a}].$$

Theorem 4.2 is established by carefully analyzing the "boundary at infinity" of the 1-dimensional components of the moduli $\mathcal{M}([\mathfrak{a}], \Delta^n)$; see [Muñoz-Echániz 2024].

5 The space of standard convex spheres in a tight contact 3-manifold

In this section we provide background on an h-principle for standard convex embeddings in tight contact 3-manifolds which was established in work of the first author with Martínez-Aguinaga and Presas [Fernández et al. 2020]. For the sake of completeness, we will provide here a detailed account which isn't quite the same as in [loc. cit.].

Throughout this section (Y, ξ) will be a tight contact 3-manifold. Recall that given a contact 3-manifold (Y, ξ), by a standard convex embedding of S^2 we mean a convex embedding $e\colon S^2 \hookrightarrow (Y, \xi)$ such that its oriented characteristic foliation $(e^*\xi) \cap TS^2$ coincides with the characteristic foliation of the sphere

$$e_0\colon S^2 \hookrightarrow \{0\} \times S^2 \subset (Y_0, \xi_0) = \big([-1, 1] \times S^2, \ker\big(z\,ds + \tfrac{1}{2}x\,dy - \tfrac{1}{2}y\,dx\big)\big).$$

In fact, by this property we obtain a (homotopically) unique contact embedding of a neighborhood of $e_0(S^2) \subset Y_0$ inside Y such that e_0 is identified with e. We recall that the north pole of e is then a positive elliptic point and the south pole a negative elliptic point. See Figure 4.

All of our arguments below work well for any other foliation of a convex sphere. The key fact is that the space of tight convex spheres with fixed characteristic foliation is C^0-dense inside the space of smooth spheres when the contact 3-manifold is tight, because of Giroux's genericity and realization theorems and Giroux's tightness criterion [1991; 2001].

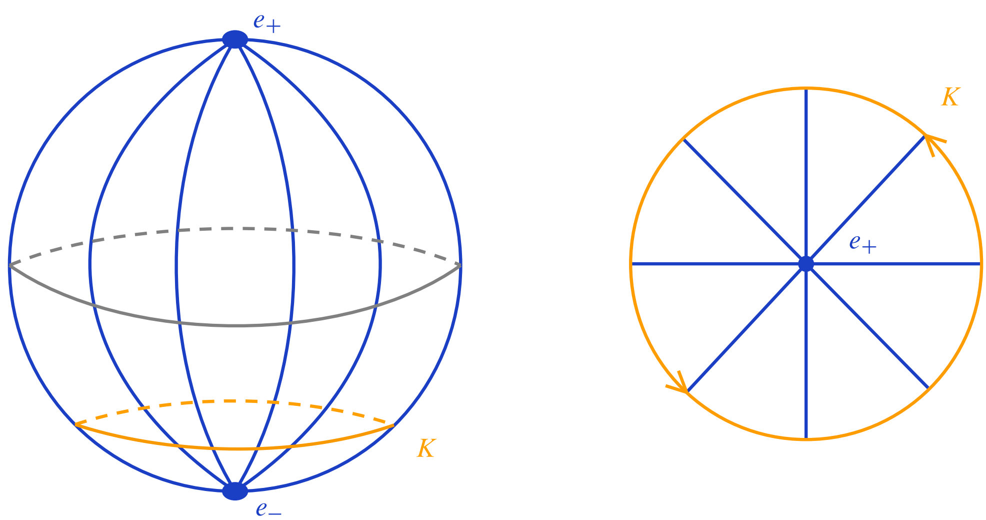

Figure 4: Schematic depiction of the standard sphere and the transverse curve K on the left. The mini-disk is depicted on the right.

Remark 5.1 The tightness condition is just required "locally", and therefore the results described in this section hold in overtwisted contact 3-manifolds if one replaces the space of smooth spheres by the space of smooth spheres with a tight neighborhood.

The main goal of this section is Theorem 5.8, which states that the space of standard embeddings of spheres into (Y,ξ) fixed near the south pole is homotopy equivalent to the corresponding space of smooth embeddings. In order to prove this result we will first study the closely related space of "mini-disks".

5.1 Mini-disks in a tight 3-manifold

Pick a small positively transverse curve $K \subseteq e(S^2)$ surrounding the negative elliptic point $e(s)$. The curve K divides the standard embedded sphere $e(S^2)$ in two disjoint disks $e(S^2)\backslash K = D^2_+ \cup D^2_-$. Here D^2_+ contains the positive elliptic point and D^2_- the negative one. In particular, we observe that the self-linking number of K is -1. The curve K is oriented as the boundary of D^2_+. Each disk $D^2_\pm$ is equipped with a natural parametrization induced by e. In particular, we will still denote by $e\colon \mathbb{D}^2 \to (Y,\xi)$ the parametrization of D^2_+. A smooth embedding of a disk with positive transverse boundary with self-linking number -1, which is convex and induces the same characteristic foliation as e, is called a *mini-disk*.[5]

We will denote by $\mathrm{CEmb}(\mathbb{D}^2,(Y,\xi))$ the space of embeddings of mini-disks which coincide with e over an open neighborhood of the boundary $\partial\mathbb{D}^2 \subset \mathbb{D}^2$. Define the space of smooth embeddings $\mathrm{Emb}(\mathbb{D}^2,Y)$ analogously. A consequence of Giroux's elimination theorem and the tightness of (Y,ξ) is the following result, which will be crucial to us:

Lemma 5.2 [Eliashberg 1993; Giroux 1991] *Let $f \in \mathrm{Emb}(\mathbb{D}^2,Y)$ be a smooth embedding. Then there exists a C^0-small isotopy of f, relative to an open neighborhood of the boundary, that makes f standard.*

This result is also explained in [Eliashberg 1993]; see also [Colin 1997]. Here the tightness condition is crucial.

We will prove the following h-principle:

Theorem 5.3 [Fernández et al. 2020] *The inclusion $\mathrm{CEmb}(\mathbb{D}^2,(Y,\xi)) \hookrightarrow \mathrm{Emb}(\mathbb{D}^2,Y)$ is a homotopy equivalence whenever (Y,ξ) is tight.*

Remark 5.4

- The π_0-surjectivity of the previous map follows from the previous lemma.
- The π_0-injectivity and also the π_1-surjectivity follow from Colin [1997], who proved this by applying his discretization trick. However, this does not quite work parametrically due to the fact that convexity is not generic among k-parametric families for $k>0$.

[5]The terminology is due to Presas; see Figure 4. Observe that a mini-disk can be contracted inside small neighborhoods of the positive elliptic point.

Here we will use the approach of [Fernández et al. 2020] based on the notion of a *microfibration*, introduced by Gromov [1986]. We will apply the following "microfibration trick", which can also be applied to an arbitrary space of convex embeddings whenever this space is dense inside the space of smooth embeddings (Lemma 5.2) and we are able to establish a corresponding local version of the h-principle (ie in a neighborhood of a smooth embedding). These ingredients are the same as those required to effectively apply Colin's trick. Our advantage with respect to Colin is that our techniques work parametrically. However, we lose control of the geometric picture by using an algebraic construction.

Definition 5.5 A map $p\colon Y \to X$ of topological spaces is a *Serre microfibration* if for every homotopy $H\colon \mathbb{D}^k \times [0,1] \to X$ with a lift $h_0\colon \mathbb{D}^k \times \{0\} \to Y$ along p at time $t=0$ there exists a positive real number $\varepsilon > 0$ together with an extension $h\colon \mathbb{D}^k \times [0,\varepsilon] \to Y$ of h_0 such that $p \circ h = H|_{\mathbb{D}^k \times [0,\varepsilon]}$.

A key property about microfibrations that we will use is:

Lemma 5.6 [Weiss 2005] *Every microfibration $p\colon Y \to X$ with* ***nonempty*** *and contractible fiber is a Serre fibration and, therefore, a weak homotopy equivalence.*

Proof of Theorem 5.3 Let K be a compact parameter space and $G \subset K$ a subspace. Consider $e^k \in \mathrm{Emb}(\mathbb{D}^2, Y)$ for $k \in K$ a family of smooth embeddings such that $e^k \in \mathrm{CEmb}(\mathbb{D}^2, (Y,\xi))$ for every $k \in G$. It is enough to establish the existence of a homotopy $e^k_t \in \mathrm{Emb}(\mathbb{D}^2, Y)$ such that

- $e^k_0 = e^k$,
- $e^k_t = e^k_0$ for $k \in G$,
- $e^k_1 \in \mathrm{CEmb}(\mathbb{D}^2, (Y,\xi))$.

Consider any extension of the embeddings e^k into a family of closed tubular neighborhood embeddings

$$E^k\colon \mathbb{D}^2 \times [-1,1] \hookrightarrow Y$$

such that

$$E^k|_{\mathbb{D}^2 \times \{0\}} = e^k.$$

Consider the space $\mathcal{B}$ of embeddings $E\colon \mathbb{D}^2 \times [-1,1] \hookrightarrow Y$ such that $E|_{\mathbb{D}^2 \times \{0\}}$ coincides with e over an open neighborhood of the boundary of $\mathbb{D}^2 = \mathbb{D}^2 \times \{0\}$. This space is the base of a microfibration

$$p\colon \mathcal{X} \to \mathcal{B}, \quad (E, p_t) \mapsto E,$$

where $\mathcal{X}$ is the space consisting of pairs (E, p_t) such that

- $E \in \mathcal{B}$,
- $p_t \in \mathrm{Emb}(\mathbb{D}^2, E(\mathbb{D}^2 \times [-1,1]))$ for $t \in [0,1]$ is a homotopy of proper embeddings of disks into the closed ball $E(\mathbb{D}^2 \times [-1,1])$, agreeing with the fixed embedding e near the boundary, and joining $p_0 = E|_{\mathbb{D}^2 \times \{0\}}$ with a mini-disk embedding $p_1 \in \mathrm{CEmb}(\mathbb{D}^2, (E(\mathbb{D}^2 \times [-1,1]), \xi))$.

The microfibration property is obviously satisfied. We will use Lemma 5.6 to conclude that p is, in fact, a fibration. Observe that the fiber $\mathcal{F}_E$ of p is nonempty because of Lemma 5.2. We claim that the fiber is also contractible. This is equivalent to the fact that the space of mini-disk embeddings, fixed near the boundary, into a tight 3-ball is homotopy equivalent to the space of smooth embeddings, fixed near the boundary, which is a combination of Eliashberg and Mishachev's theorem [2021] and Hatcher's theorem [1983]. Indeed, let $(\mathbb{B}^3,\xi)=(E(\mathbb{D}^2\times[-1,1]),\xi)$ and consider any mini-disk embedding $f\colon \mathbb{D}^2\to(\mathbb{B}^3,\xi)$ which coincides near the boundary with $E|_{\mathbb{D}^2\times\{0\}}$. The complement of $f(\mathbb{D}^2)$ in $\mathbb{B}^3$ is given by two tight balls $\mathbb{B}^3_\pm$. Denote by $\mathrm{CEmb}(\mathbb{D}^2,(\mathbb{B}^3,\xi))$ the corresponding space of mini-disk embeddings and by $\mathrm{Emb}(\mathbb{D}^2,\mathbb{B}^3)$ the smooth analogue. There is a map between fibrations

$$\begin{array}{ccccc}
\mathrm{Diff}(\mathbb{B}^3_+)\times\mathrm{Diff}(\mathbb{B}^3_-) & \longrightarrow & \mathrm{Diff}(\mathbb{B}^3) & \longrightarrow & \mathrm{Emb}(\mathbb{D}^2,\mathbb{B}^3)\\
\uparrow & & \uparrow & & \uparrow\\
\mathrm{Cont}(\mathbb{B}^3_+,\xi)\times\mathrm{Cont}(\mathbb{B}^3_-,\xi) & \longrightarrow & \mathrm{Cont}(\mathbb{B}^3,\xi) & \longrightarrow & \mathrm{CEmb}(\mathbb{D}^2,(\mathbb{B}^3,\xi))
\end{array}$$

inducing homotopy equivalences of total spaces and fibers, and thus the claim follows. Then from Lemma 5.6 we have that the map $p\colon\mathcal{X}\to\mathcal{B}$ is a Serre fibration with contractible fibers. This is enough to conclude the proof. Indeed, recall that we have built a map $j: K\to\mathcal{B}$ given by $k\mapsto E^k$, so $j^*\mathcal{X}\to K$ is a fibration with contractible nonempty fibers. We have a natural section over $G\subset K$ given by the constant homotopy:

$$s\colon G\to\mathcal{X},\quad k\mapsto(E^k,p^k_t=e^k).$$

Hence, by the contractibility of the fibers, we may extend this section over to K and obtain $\hat{s}\colon K\to\mathcal{X}$ given by $k\mapsto(E^k,p^k_t)$. The homotopy

$$e^k_t=p^k_t$$

solves our problem. □

We will need the following generalization. Let $\mathrm{CEmb}\big(\bigsqcup_j\mathbb{D}^2,(Y,\xi)\big)$ be the space of embeddings $e\colon\bigsqcup_j\mathbb{D}^2\hookrightarrow(Y,\xi)$ of n mini-disks, all of them fixed at an open neighborhood of $\bigsqcup_j\partial\mathbb{D}^2$. Denote also by $\mathrm{Emb}\big(\bigsqcup_j\mathbb{D}^2,Y\big)$ the corresponding space of smooth embeddings.

Theorem 5.7 *The natural inclusion* $\mathrm{CEmb}\big(\bigsqcup_j\mathbb{D}^2,(Y,\xi)\big)\hookrightarrow\mathrm{Emb}\big(\bigsqcup_j\mathbb{D}^2,Y\big)$ *is a homotopy equivalence whenever* (Y,ξ) *is tight.*

Proof The proof follows word by word the proof of Theorem 5.3. In this case the microfibration built is going to have as fiber the space of isotopies of n 2-disks into n disjoint tubular neighborhoods $\cong\mathbb{D}^2\times[-1,1]$. □

5.2 The space of standard spheres

As a consequence of our previous discussion we may compare the homotopy types of the space of standard spheres and the space of smooth spheres in a tight contact 3-manifold (Y,ξ). For this, consider the space of smooth embeddings $\mathrm{Emb}\big(\bigsqcup_j S^2,Y\big)$ of n-disjoint spheres and the corresponding subspace of standard spheres $\mathrm{CEmb}\big(\bigsqcup_j S^2,(Y,\xi)\big)$. Fix also an arbitrary standard embedding $e\colon\bigsqcup S^2\to(Y,\xi)$ and consider the subspaces $\mathrm{Emb}\big(\bigsqcup_j S^2, Y, \bigsqcup_j s_j\big)$ of embeddings that agree with e on an open neighborhood $\bigsqcup_j U_j$ of the south pole s_j of each sphere. Here we assume that the boundary $e|_{\bigsqcup_j \partial U_j}$ parametrizes n disjoint positively transverse knots K_j as in the previous section. Similarly, consider the analogous subspace of standard embeddings $\mathrm{CEmb}\big(\bigsqcup_j S^2, (Y,\xi), \bigsqcup_j s_j\big)$. Observe that the space $\mathrm{CEmb}\big(\bigsqcup_j S^2, (Y,\xi), \bigsqcup_j s_j\big)$ is homotopy equivalent to the space of n mini-disk embeddings into the tight contact manifold with convex boundary obtained from (Y,ξ) by removing an open neighborhood of $e\big(\bigsqcup_j U_j\big)$ whose boundary parametrizes K_j. The same observation applies to the space $\mathrm{Emb}\big(\bigsqcup_j S^2, Y, \bigsqcup_j s_j\big)$. We obtain:

Theorem 5.8 *Assume that (Y,ξ) is tight. Then*:

- *The inclusion* $\mathrm{CEmb}\big(\bigsqcup_j S^2, (Y,\xi), \bigsqcup_j s\big) \hookrightarrow \mathrm{Emb}\big(\bigsqcup_j S^2, Y, \bigsqcup_j s_j\big)$ *is a homotopy equivalence.*
- *For every $k \geq 1$ the natural homomorphism*

$$\pi_k(\mathrm{SO}(3)^n, U(1)^n) \to \pi_k\left(\mathrm{Emb}\left(\bigsqcup_j S^2, Y\right), \mathrm{CEmb}\left(\bigsqcup_j S^2, (Y,\xi)\right)\right)$$

induced by reparametrization on the source is an isomorphism.

Proof As explained above, the proof of the first assertion follows from Theorem 5.7. For the second assertion note that there is a natural map of fibrations given by the evaluation at the n south poles

$$\begin{array}{ccccc} \mathrm{Emb}\big(\bigsqcup_j S^2, Y, \bigsqcup_j s_j\big) & \longrightarrow & \mathrm{Emb}\big(\bigsqcup_j S^2, Y\big) & \longrightarrow & \mathrm{Fr}_n(Y) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{CEmb}\big(\bigsqcup_j S^2, (Y,\xi), \bigsqcup_j s_j\big) & \longrightarrow & \mathrm{CEmb}\big(\bigsqcup_j S^2, (Y,\xi)\big) & \longrightarrow & \mathrm{CFr}_n(Y,\xi) \end{array}$$

in which the vertical maps are inclusions. Here the base $\mathrm{Fr}_n(Y)$ is the space of framings over n different points of M, that is, the total space of a fiber bundle over the configuration space $\mathrm{Conf}_n(Y)$ with fiber $\approx \mathrm{GL}^+(3)^n$, and likewise for $\mathrm{CFr}_n(Y,\xi)$ but with contact frames. Observe that the map between the fibers is a homotopy equivalence because of the first assertion, so that the homomorphism induced by the evaluation map

$$\pi_k\left(\mathrm{Emb}\left(\bigsqcup_j S^2, Y\right), \mathrm{CEmb}\left(\bigsqcup_j S^2, (Y,\xi)\right)\right) \to \pi_k(\mathrm{Fr}_n(Y), \mathrm{CFr}_n(Y,\xi)) \cong \pi_k(\mathrm{SO}(3)^n, U(1)^n)$$

is an isomorphism and defines an inverse to the reparametrization map. □

5.3 Standard spheres in sums of two irreducible 3-manifolds

In this section we establish Theorem 1.16. We first discuss its smooth counterpart. The relevant reference on this topic is Hatcher's work [1981]. Let $Y_\# = Y_- \# Y_+$ with $Y_\pm$ now *irreducible*. Let $\mathrm{Emb}(S^2, Y_\#)_{S_\#} \subset \mathrm{Emb}(S^2, Y_\#)$ be the subspace of smooth co-oriented embeddings $S^2 \hookrightarrow Y_\#$ isotopic to a fixed given one $S_\#$, and let

$$\mathcal{S} = \mathrm{Emb}(S^2, Y_\#)_{S_\#} / \mathrm{Diff}(S^2)$$

be the space of *unparametrized* co-oriented nontrivial spheres. Hatcher [1981] proved that $\mathcal{S}$ is contractible. We also have a fibration

$$\mathrm{SO}(3) \simeq \mathrm{Diff}(S^2) \to \mathrm{Emb}(S^2, Y_\#)_{S_\#} \to \mathcal{S},$$

and hence

$$\mathrm{Emb}(S^2, Y_\#)_{S_\#} \simeq \mathrm{SO}(3).$$

Proof of Theorem 1.16 This is immediate from the long exact sequence of pairs associated to the horizontal maps in the commutative diagram

$$\begin{array}{ccc} \mathrm{CEmb}(S^2, (Y_\#, \xi_\#))_{S_\#} & \longrightarrow & \mathrm{Emb}(S^2, Y_\#)_{S_\#} \\ \uparrow & & \simeq\uparrow \\ U(1) & \longrightarrow & \mathrm{SO}(3) \end{array}$$

combined with Theorem 5.8. □

6 Families of contact structures on sums of contact 3-manifolds

In this section we establish our main results, Theorems 1.3, 1.6 and 1.8, by combining the tools discussed in Sections 4 and 5.

6.1 The space of tight contact structures on a sum

Consider $n+1$ tight contact 3-manifolds (Y_j, ξ_j) for $j = 0, \dots, n$ with $n \geq 1$. Let $(Y_\#, \xi_\#)$ be their connected sum, which we build as follows. We fix Darboux balls $B_{0-} \subset Y_0$, $B_{n+} \subset Y_n$ and for each $0 < j < n$ we fix two disjoint Darboux balls $B_{j\pm} \subset Y_j$. Then the connected sum $(Y_\#, \xi_\#)$ is formed by gluing in the order

$$(Y_0 \setminus B_{0-}) \bigcup_{\partial B_{0-} = -\partial B_{1+}} (Y_1 \setminus (B_{1+} \cup B_{1-})) \cdots \bigcup_{\partial B_{(n-1)-} = -\partial B_{n+}} (Y_n \setminus B_{n+}).$$

We will denote by $e_j \colon S^2 \hookrightarrow (Y_\#, \xi_\#)$, for $j = 1, \dots, n$, the embedding of the j^{th} separating standard sphere given by $\partial B_{(j-1)-} = -\partial B_{j+}$ in the connected sum $(Y_\#, \xi_\#)$. Denote by s_j the south pole on the j^{th} sphere, regarded as a point in $e_j(S^2) \subset Y_\#$.

We will denote by $\mathrm{Tight}(Y, B)$ the space of tight contact structures on Y that are fixed on a Darboux ball B, and by $\mathrm{Tight}(Y, B, B')$ the subspace of $\mathrm{Tight}(Y, B)$ given by contact structures that are fixed on a second Darboux ball B' disjoint from B. A classical result of Colin [1997] asserts that the contact manifold $(Y_\#, \xi_\#)$ is tight, and we have a well-defined map

$$\#_{n+1} \colon \mathrm{Tight}(Y_0, B_{0-}) \times \prod_{j=1}^{n-1} \mathrm{Tight}(Y_j, B_{j+}, B_{j-}) \times \mathrm{Tight}(Y_n, B_{n+}) \to \mathrm{Tight}(Y_\#). \tag{17}$$

On the other hand, the evaluation map of each tight contact structure on Y at the south poles s_j defines a fibration

$$\mathrm{ev}_{n+1}\colon \mathrm{Tight}(Y_\#) \to (S^2)^n. \tag{18}$$

The fiber $\mathcal{F}$ of ev_{n+1} over $(\xi_\#(s_j))$ has the homotopy type of the space of tight contact structures on $Y_\#$ that agree with $\xi_\#$ over n disjoint Darboux balls $B_{\#j}$ around s_j. Therefore there is a natural inclusion

$$i_\#\colon \mathrm{Tight}(Y_0, B_{0-}) \times \prod_{j=1}^{n-1} \mathrm{Tight}(Y_j, B_{j+}, B_{j-}) \times \mathrm{Tight}(Y_n, B_{n+}) \hookrightarrow \mathcal{F}.$$

We establish the following stronger version of Theorem 1.8:

Theorem 6.1 *The inclusion $i_\#$ is a homotopy equivalence.*

Remark 6.2 Since S^2 is simply connected, we deduce from the long exact sequence in homotopy groups of (18) that

$$\pi_0(\mathrm{Tight}(Y_\#)) \cong \prod_{j=0}^{n} \pi_0(\mathrm{Tight}(Y_j)),$$

which is the classical result of Colin [1997].

Proof Let K be a compact parameter space and $G \subseteq K$ a subspace. It is enough to prove that if $\xi^k \in \mathcal{F}$ is a K-family of tight contact structures on $Y_\#$ that coincide with $\xi_\#$ over the n Darboux balls $B_{\#j}$ and such that $\xi^k \in \mathrm{Im}(i_\#)$ for $k \in G$, then there exists a homotopy of tight contact structures ξ^k_t for $t \in [0,1]$ such that

- $\xi^k_0 = \xi^k$,
- $\xi^k_t = \xi^k$ for $k \in G$, and
- $\xi^k_1 \in \mathrm{Im}(i_\#)$.

The key point is to observe that $\xi^k \in \mathrm{Im}(i_\#)$ if and only if the embeddings $e_j : S^2 \hookrightarrow (Y_\#, \xi^k)$ are standard for $j = 1, \dots, n$. For a given tight contact structure ξ, denote by

$$\mathcal{CE}_\xi := \mathrm{CEmb}\biggl(\bigsqcup_{j=1}^{n} S^2, (Y_\#, \xi), \bigsqcup_{j=1}^{n} s_j\biggr)$$

the space of standard embeddings of n disjoint spheres that coincide with (e_j) over a neighborhood of the south poles (s_j), and by

$$\mathcal{E} := \mathrm{Emb}\biggl(\bigsqcup_{j=1}^{n} S^2, Y_\#, \bigsqcup_{j=1}^{n} s_j\biggr)$$

the analogous space of smooth embeddings. Consider the space $\mathcal{X}$ of pairs (ξ, e_t) where $\xi \in \mathcal{F}$ and $e_t \in \mathcal{E}$, with $t \in [0,1]$, is a homotopy of embeddings with $e_0 = e$ and $e_1 \in \mathcal{CE}_\xi$. There is a natural forgetful map

$$p\colon \mathcal{X} \to \mathcal{F}, \quad (\xi, e_t) \mapsto \xi,$$

which is in fact a fibration because of Lemma 2.2. By Theorem 5.8 the inclusion $\mathcal{CE}_\xi \to \mathcal{E}$ is a homotopy equivalence. Therefore the fibers of the previous fibration are contractible.

This is enough to conclude the proof. Indeed, our initial family ξ^k is given by a map $j : K \to \mathcal{F}$ and the pullback fibration $j^*\mathcal{X} \to K$ has a well-defined section over $G \subseteq K$ given by the constant isotopy $e_t^k = e$ for $(k,t) \in G \times [0,1]$. Since the fiber of this fibration is contractible we can extend this section over K, obtaining a section e_t^k for $(k,t) \in K \times [0,1]$. Then we apply the smooth isotopy extension theorem to this family of embeddings to find an isotopy $\varphi_t^k \in \mathrm{Diff}(Y_\#)$ for $(k,t) \in K \times [0,1]$ such that

- $\varphi_0^k = \mathrm{id}$,
- φ_t^k is the identity over a neighborhood of the south poles (s_j),
- $\varphi_t^k \circ e = e_t^k$,
- $\varphi_t^k = \mathrm{id}$ for $(k,t) \in G \times [0,1]$.

The homotopy of contact structures $\xi_t^k = (\varphi_t^k)^*\xi^k$ solves the problem since now $e = (\varphi_1^k)^{-1} \circ e_1^k$ is standard for $(\varphi_t^k)^*\xi^k$ because e_1^k is standard for ξ^k. □

6.2 Diffeomorphisms of connected sums of two irreducible 3-manifolds

Consider $Y_\# = Y_- \# Y_+$ with $Y_\pm$ *irreducible*. Recall from Section 5.3 that Hatcher [1981] proved

$$\mathrm{Emb}(S^2, Y_\#)_{S_\#} \simeq \mathrm{SO}(3).$$

This has the following useful consequence:

Lemma 6.3 *Suppose that $Y_\pm$ are aspherical (irreducible and with infinite fundamental group). Then* $\pi_1 \mathrm{Diff}(Y_\#) = 0$.

Proof From the fibration (10) we have an exact sequence

$$\pi_1 \mathrm{Diff}(Y_-, B_-) \times \pi_1 \mathrm{Diff}(Y_+, B_+) \longrightarrow \pi_1 \mathrm{Diff}(Y_\#) \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_0 \mathrm{Diff}(Y_-, B_-) \times \pi_0 \mathrm{Diff}(Y_+, B_+)$$

Under the connecting map, the nontrivial element in $\mathbb{Z}/2$ maps to

$$\tau_{\partial B_-}\tau_{\partial B_+} \in \pi_0 \mathrm{Diff}(Y_-, B_-) \times \pi_0 \mathrm{Diff}(Y_+, B_+).$$

We saw in the proof of Corollary 3.7 that the Dehn twists $\tau_{\partial B_\pm} \in \pi_0 \mathrm{Diff}(Y_\pm, B_\pm)$ are nontrivial and $\pi_1 \mathrm{Diff}(Y_\pm, B_\pm) = 0$. From this and the exact sequence above it now follows that $\pi_1 \mathrm{Diff}(Y_\#) = 0$. □

6.3 Proof of Theorem 1.3

As we've been doing so far, all homologies considered below are taken with $\mathbb{Q}$ coefficients, unless otherwise noted.

By Theorem 6.1 we have

$$\mathcal{C}(Y_\#, \xi_\#, B_\#) \simeq \mathcal{C}(Y_-, \xi_-, B_-) \times \mathcal{C}(Y_+, \xi_+, B_+),$$

and then by Proposition 3.11 the obstruction class $\mathcal{O}_{\xi_\#} \in \pi_1 \mathcal{C}(Y_\#, \xi_\#, B_\#)$ to finding a homotopy section of $\mathrm{ev}_\# \colon \mathcal{C}(Y_\#, \xi_\#) \to S^2$ corresponds to

$$\mathcal{O}_{\xi_\#} \cong (\mathcal{O}_{\xi_-}, \mathcal{O}_{\xi_+}) \in \pi_1 \mathcal{C}(Y_-, \xi_-, B_-) \times \pi_1 \mathcal{C}(Y_+, \xi_+, B_+).$$

We recall that all homologies are taken with $\mathbb{Q}$ coefficients. A portion of the Wang long exact sequence for the fibration $\mathrm{ev}_{B_\#}$ is

$$\mathbb{Q} \xrightarrow{\delta} H_1(\mathcal{C}(Y_-, \xi_-, B_-)) \oplus H_1(\mathcal{C}(Y_+, \xi_+, B_+)) \to H_1(\mathcal{C}(Y_\#, \xi_\#)) \to 0,$$

where $\delta(1) = \mathcal{O}_{\xi_\#} = (\mathcal{O}_{\xi_-}, \mathcal{O}_{\xi_+})$. In the latter formula, and in what follows, we will incur a small abuse of notation by denoting the obstruction class and its image in homology by the same symbol.

The nontrivial input from Floer theory appears now. Because $\boldsymbol{c}(\xi_\pm) \notin \operatorname{Im} U$, by Corollary 4.3 to Theorem 4.2 the classes $\mathcal{O}_{\xi_\pm}$ are nontrivial in $H_1(\mathcal{C}(Y_\pm, \xi, B_\pm))$. It then follows from the Wang exact sequence that the class $(\mathcal{O}_{\xi_-}, 0)$ is not in the image of δ; thus the image of $(\mathcal{O}_{\xi_-}, 0)$ in $H_1(\mathcal{C}(Y_\#, \xi_\#))$ is nontrivial.

By Lemma 6.3, $\pi_1 \operatorname{Diff}(Y_\#) = 0$. Note that the hypotheses of that lemma indeed apply, because the manifolds $Y_\pm$ are aspherical. To see this, recall that an irreducible 3-manifold is aspherical precisely when it is not one of the quotients M_Γ of S^3 by a finite subgroup Γ of $\mathrm{SO}(4)$. The manifolds M_Γ have $\widecheck{\mathrm{HM}}(-M_\Gamma, \mathfrak{s}) = \mathbb{Q}[U, U^{-1}]$ in every spin-c structure $\mathfrak{s}$ because M_Γ is a rational homology sphere with a positive scalar curvature metric; see [Kronheimer and Mrowka 2007, Proposition 36.1.3]. In particular, the U map is surjective on $\widecheck{\mathrm{HM}}(-M_\Gamma, \mathfrak{s})$. The manifolds $Y_\pm$ are irreducible but can't be of the form M_Γ since $\boldsymbol{c}(\xi_\pm) \notin \operatorname{Im} U$.

Then, by the long exact sequence in homotopy groups of (4), it follows that

$$H_1(\mathcal{C}(Y_\#, \xi_\#); \mathbb{Z}) \cong \mathrm{Ab}(\pi_0 \operatorname{Cont}_0(Y_\#, \xi_\#)).$$

Under this isomorphism, the nontrivial class $(\mathcal{O}_{\xi_-}, 0)$ corresponds to the class of the squared Dehn twist $\tau_{S_\#}^2$ by Proposition 3.6. This proves that $\tau_{S_\#}^2$ is not contact isotopic to the identity. Since we've shown that $(\mathcal{O}_{\xi_-}, 0)$ is nontrivial *rationally*, it follows that all the even powers of $\tau_{S_\#}$ (and therefore all the powers) are also not contact isotopic to the identity. This completes the proof of Theorem 1.3(A).

By Lemma 3.9, the image of $\tau_{\partial B_\pm}$ in $\pi_0 \operatorname{FCont}_0(Y_\pm, \xi_\pm, B_\pm)$ is trivial. Hence so is the image of $\tau_{S_\#}^2$ in $\pi_0 \operatorname{FCont}_0(Y_\#, \xi_\#)$, proving Theorem 1.3(B). □

Remark 6.4 Working with $\mathbb{Z}$ coefficients rather than $\mathbb{Q}$, we can establish the following analogue of Theorem 1.3(A) by the same argument. If $2\boldsymbol{c}(\xi_\pm; \mathbb{Z}) \neq 0$ in $\widecheck{\mathrm{HM}}(-Y_\pm)$ and $0 \neq k \in \mathbb{Z}$ satisfies $k\boldsymbol{c}(\xi_\pm; \mathbb{Z}) \notin \operatorname{Im} U$, then the $2k$-fold iterate $\tau_{S_\#}^{2k}$ is not contact isotopic to the identity. All examples known to the authors where the latter hypothesis is satisfied for some $k \neq 0$ also satisfy the stronger $\mathbb{Q}$ version of the hypothesis. The assumption $2\boldsymbol{c}(\xi_\pm;\mathbb{Z}) \neq 0$ guarantees the orientability of the moduli spaces involved in the construction of the families contact invariant; see [Muñoz-Echániz 2024, Corollary 1.8].

6.4 Proof of Theorem 1.6

We write $(Y,\xi) = (Y_0,\xi_0)\,\#\cdots\#\,(Y_n,\xi_n)\,\#\,(Y_{n+1},\xi_{n+1})$, where $(Y_0,\xi_1),\dots,(Y_n,\xi_n)$ are those prime summands of (Y,ξ) such that $\boldsymbol{c}(\xi_j) \notin \operatorname{Im} U$ and the Euler class of ξ_j vanishes, and (Y_{n+1},ξ_{n+1}) is the sum of the remaining prime summands. We take the latter to be (S^3,ξ_{st}) if there are no prime summands remaining. We choose Darboux balls $B_{0-} \subset Y_0$ and $B_{n+1,+} \subset Y_{n+1}$, and for $j = 1,\dots,n$ we choose two Darboux balls $B_{j\pm} \subset Y_j$ disjoint from each other. We may take the connected sum (Y,ξ) to be built by gluing in the order

$$(Y_0 \setminus B_{0-}) \bigcup_{\partial B_{0-}=-\partial B_{1+}} (Y_1 \setminus (B_{1+}\cup B_{1-}))\cdots(Y_n \setminus (B_{n+}\cup B_{n-})) \bigcup_{\partial B_{n-}=-\partial B_{n+1,+}} (Y_{n+1}\setminus B_{n+1,+}),$$

with $n+1$ separating spheres. We fix $n+1$ Darboux balls $B_{\#j}$ for $j = 1,\dots,n+1$ centered at the south poles of the separating spheres ($B_{\#j}$ is centered at the south pole of the sphere which separates the pieces $Y_{j-1}\setminus B_{j-1,-}$ and $Y_{j+}\setminus B_{j+}$) and which are disjoint from each other.

Consider the evaluation map at the $n+1$ south poles of the spheres, which provides a fibration

$$\mathcal{F} \to \mathcal{C}(Y,\xi) \to (S^2)^{n+1}. \tag{19}$$

Theorem 6.1 identifies the fiber as

$$\mathcal{F} \simeq \mathcal{C}(Y_0,B_{0-}) \times \Bigl(\prod_{j=1,\dots,n+1} \mathcal{C}(Y_j, B_{j+}\cup B_{j-})\Bigr) \times \mathcal{C}(Y_{n+1},B_{n+1,+}).$$

Recall that we have a homotopy equivalence

$$\mathcal{C}(Y_j, B_{j+}\cup B_{j-}) \simeq \Omega S^2 \times \mathcal{C}(Y_j,B_{j-}),$$

since the evaluation map $\mathrm{ev}_{B_{j+}}\colon \mathcal{C}(Y_j,B_{j-}) \to S^2$ is nullhomotopic (a nullhomotopy is obtained by dragging the evaluation point from B_{j+} into B_{j-}, and this yields the required homotopy equivalence). Thus

$$\mathcal{F} \simeq \mathcal{C}(Y_0,B_{0-}) \times \Omega S^2 \times \mathcal{C}(Y_1,B_{1-}) \times\cdots\times \Omega S^2 \times \mathcal{C}(Y_n,B_{n-}) \times \mathcal{C}(Y_{n+1},B_{n+1,+}).$$

The connecting map in the long exact sequence in homotopy groups of the fibration (19) yields a homomorphism

$$\delta\colon \mathbb{Z}^{n+1} \to \pi_1\mathcal{C}(Y_0,B_{0-}) \times \mathbb{Z} \times \pi_1\mathcal{C}(Y_1,B_{1-}) \times\cdots\times \mathbb{Z} \times \pi_1\mathcal{C}(Y_n,B_{n-}) \times \pi_1\mathcal{C}(Y_{n+1},B_{n+1,+}),$$

which we now calculate.

Lemma 6.5 *For $(a_1,\dots,a_{n+1}) \in \mathbb{Z}^{n+1}$ we have*

$$\delta(a_1,\dots,a_{n+1}) = (a_1\cdot\mathcal{O}_{\xi_0}, a_1, a_2\cdot\mathcal{O}_{\xi_1},\cdots,a_n, a_{n+1}\cdot\mathcal{O}_{\xi_n}, a_{n+1}\cdot\mathcal{O}_{\xi_{n+1}}).$$

Proof Our argument is modeled on the proof of Proposition 3.11. It suffices to work in the local model where $(Y_j,\xi_j)=(\mathbb{B}^3,\xi_{\mathrm{st}})$ for all $j=0,\dots,n+1$. We choose $2n+2$ paths $\gamma_{0-},\gamma_{1\pm},\dots,\gamma_{n\pm},\gamma_{n+1,+}$ in Y, where each γ_{j+} goes from $B_{\#j}$ to $\partial Y_j\subset\partial Y$, and each γ_{j-} goes from $B_{\#j+1}$ to $\partial Y_j\subset\partial Y$. We consider the following commutative diagram of maps and spaces:

$$\begin{array}{ccccc} \mathcal{C}(Y,\cup_{j=1}^{n+1}B_{\#j}) & \longrightarrow & \mathcal{C}(Y) & \xrightarrow{\mathrm{ev}_{B_\#}} & (S^2)^{n+1} \\ \downarrow{\scriptstyle \mathrm{ev}_\gamma} & & \downarrow{\scriptstyle \mathrm{ev}_\gamma} & & \downarrow{\scriptstyle \Delta^{n+1}} \\ (\Omega S^2)^{2n+2} & \longrightarrow & (PS^2)^{2n+2} & \longrightarrow & (S^2)^{2n+2} \end{array}$$

Here $\mathrm{ev}_{\#B}$ stands for the evaluation map at the centers of the $n+1$ balls $B_{\#j}$, and the bottom row is given by $2n+2$ product of the path fibration on S^2. In particular, both rows are fibration sequences. The maps denoted by ev_γ stand for evaluation of contact structures along the $2n+2$ paths chosen above, and $\Delta\colon S^2\to(S^2)^2$ is the diagonal map.

Each of the two fibrations $\mathrm{ev}_{B_{0-}}\colon\mathcal{C}(Y_0)\to S^2$ and $\mathrm{ev}_{B_{n+1,+}}\colon\mathcal{C}(Y_{n+1})\to S^2$ is identified with the path fibration on S^2, by Lemma 3.8. Similarly, each of the n fibrations $\mathrm{ev}_{B_{j+}}\times\mathrm{ev}_{B_{j-}}\colon\mathcal{C}(Y_j)\to(S^2)^2$ with $j=1,\dots,n$ is identified with two copies of the path fibration on S^2. Using these, we identify the bottom row of the first diagram with the product of these $n+2$ fibrations.

The leftmost vertical map in the first diagram is a homotopy equivalence, which follows by an argument similar to the proof of Lemma 3.8. Consider the inclusion map

$$j:\mathcal{C}(Y_0,B_{0-})\times\Bigl(\prod_{j=1}^{n}\mathcal{C}(Y_j,B_{j+}\cup B_{j-})\Bigr)\times\mathcal{C}(Y_{n+1},B_{n+1,+})\to\mathcal{C}\Bigl(Y,\bigcup_{j=1}^{n+1}B_{\#j}\Bigr).$$

Under the identification of the bottom row of the first diagram with the product of the $n+2$ fibrations from the previous paragraph, the map j becomes the homotopy inverse of the leftmost vertical map in the first diagram, as in the proof of Proposition 3.11. The required result follows now from the commuting square obtained from taking homotopy groups in the first diagram:

$$\begin{array}{ccc} (\pi_2S^2)^{n+1} & \longrightarrow & \pi_1\mathcal{C}(Y,\bigcup_{j=1}^{n+1}B_{\#j}) \\ \downarrow{\scriptstyle\Delta^{n+1}} & & \uparrow{\scriptstyle j} \\ (\pi_2S^2)^{2n+2} & \longrightarrow & \pi_1\mathcal{C}(Y_0,B_{0-})\times\bigl(\prod_{j=1}^{n}\pi_1\mathcal{C}(Y_j,B_{j+}\cup B_{j-})\bigr)\times\pi_1\mathcal{C}(Y_{n+1},B_{n+1,+}) \end{array}$$ □

With this in place, we now look at the Serre spectral sequence of the fibration (19). From it we can assemble an exact sequence

$$\mathbb{Q}^{n+1}\xrightarrow{\delta}H_1(\mathcal{F})\to H_1(\mathcal{C}(Y,\xi))\to 0,$$

where δ is given by the same formula as in Lemma 6.5. By $\boldsymbol{c}(\xi_j)\notin\operatorname{Im}U$ and Corollary 4.3 to Theorem 4.2 we again deduce that the classes $\mathcal{O}_{\xi_j}$ for $j=0,\dots,n$ are homologically nontrivial (over $\mathbb{Q}$). Hence the n-dimensional subspace of $H_1(\mathcal{F})$ given by the elements

$$(b_1\cdot\mathcal{O}_{\xi_0},0,b_2\cdot\mathcal{O}_{\xi_1},0,\dots,0,b_n\cdot\mathcal{O}_{\xi_{n-1}},0,0,0)\quad\text{for }(b_j)\in\mathbb{Q}^n$$

injects as a subspace of $H_1(\mathcal{C}(Y,\xi))$. The proof of the formal triviality assertion is similar to the one given for Theorem 1.3. □

Remark 6.6 When Y is the sum of two aspherical 3-manifolds we have $\pi_1 \operatorname{Diff}(Y) = 0$ (see Lemma 6.3). In the proof of Theorem 1.3 this allowed us to pass from a nontrivial element in $\pi_1\mathcal{C}(Y,\xi)$ to a nontrivial element in $\pi_0 \operatorname{Cont}_0(Y,\xi)$ via the fibration (4). This is a special situation. For instance, if Y is instead the sum of *at least three* aspherical 3-manifolds then it is known that $\pi_1 \operatorname{Diff}(Y)$ is not finitely generated [McCullough 1981]. A better control on $\pi_1 \operatorname{Diff}(Y)$ for general Y would allow us to understand whether the exotic loops of contact structures that we find in Theorem 1.6 yield nontrivial contactomorphisms.

7 Exotic phenomena in overtwisted contact 3-manifolds

In this final section we exhibit examples of 1-parametric exotic phenomena in *overtwisted* contact 3-manifolds.

On a heuristic level, Eliashberg's overtwisted h-principle [1989] is based on applying Gromov's h-principle for open manifolds to the complement of a 3-ball and using the overtwisted disk to fill in the ball. In the same spirit of this idea is what we call the "overtwisted escape principle", explained to us by Presas, which is a general strategy for proving an h-principle for a family of objects in a contact manifold (Y,ξ). First, perform the connected sum with an overtwisted manifold (M,ξ_{ot}), in order to apply the overtwisted h-principle [Eliashberg 1989; Borman et al. 2015] in the contact 3-manifold $(Y,\xi)\,\#\,(M,\xi_{\mathrm{ot}})$. This could be thought of as analogous to opening up the 3-manifold in the previous situation. Second, try to isotope the objects for which you want an h-principle so that they avoid ("escape") the overtwisted region $(M,\xi_{\mathrm{ot}})\setminus B$, where B is a Darboux ball. However, there could be obstructions to carrying out this second step. There are two scenarios: if these obstructions can be sorted out then our initial problem satisfies an h-principle; if not these obstructions should give rise to an exotic phenomenon in the overtwisted contact manifold $(Y,\xi)\,\#\,(M,\xi_{\mathrm{ot}})$. In [Casals et al. 2021] the authors successfully carry out this procedure to prove an existence h-principle for codimension-2 isocontact embeddings. Next, we will instead start out with a problem in (Y,ξ) which we know is geometrically obstructed a priori, and from this deduce an exotic overtwisted phenomenon.

Let $e\colon S^2 \to (Y,\xi)$ be a standard embedding into a contact manifold (Y,ξ). A *formal standard embedding* of a sphere into (Y,ξ) is a pair (f, F^s) for $s\in[0,1]$ such that $f\in \operatorname{Emb}(S^2,Y)$ is a smooth embedding and $F^s\colon TS^2 \to f^*TY$ is a homotopy of vector bundle injections with $F^0 = df$ and $(F^1)^*\xi = e^*\xi \subset TS^2$. We will denote by $\operatorname{FCEmb}(S^2,(Y,\xi))$ the space of formal standard embeddings and by $\operatorname{FCEmb}(S^2,(Y,\xi),s)$ the subspace of formal standard embedding that coincide with e over an open neighborhood U of the south pole $s\in S^2$.

Let (M,ξ_{ot}) be an overtwisted contact 3-manifold. Consider the overtwisted contact 3-manifold $(Y_\#,\xi_\#) = (Y,\xi)\#(M,\xi_{\mathrm{ot}})$. We will consider the spaces $\operatorname{CEmb}(S^2,(Y_\#,\xi_\#),s)$ and $\operatorname{FCEmb}(S^2,(Y_\#,\xi_\#),s)$ as pointed spaces with basepoint given by the separating sphere $e\colon S^2 \hookrightarrow (Y_\#, \xi_\#)$. We have a natural inclusion $\mathrm{CEmb}(S^2, (Y_\#, \xi_\#), s) \hookrightarrow \mathrm{FCEmb}(S^2, (Y_\#, \xi_\#), s)$. From our previous discussion and the theory developed in this article we deduce the following:

Corollary 7.1 *Assume that (Y, ξ) is irreducible, ξ has vanishing Euler class and $\boldsymbol{c}(\xi) \notin \mathrm{Im}\, U$. Then there exists an element with infinite order in*

$$\ker(\pi_1 \,\mathrm{CEmb}(S^2, (Y_\#, \xi_\#), s) \to \pi_1 \,\mathrm{FCEmb}(S^2, (Y_\#, \xi_\#), s)).$$

Remark 7.2

- This should be compared with Theorem 5.8, which in particular asserts that this type of phenomenon does not happen when the underlying contact manifold is tight.
- Under the same assumptions, our proof also yields an element with infinite order in

$$\ker(\pi_1 \,\mathrm{CEmb}(S^2, (Y_\#, \xi_\#)) \to \pi_1 \,\mathrm{FCEmb}(S^2, (Y_\#, \xi_\#))).$$

Proof Denote by $S_\# = e(S^2)$ the standard separating sphere. Consider the squared Dehn twist $\tau^2_{S^+_\#}$ along a parallel copy $S^+_\#$ of $S_\#$, where we assume that $S^+_\#$ is contained in $(Y, \xi)\backslash B$, where B is the Darboux ball used to perform the connected sum. By the vanishing of the Euler class of ξ there exists a homotopy through formal contactomorphisms joining the identity with $\tau^2_{S^+_\#}$ (Lemma 3.9). It follows from Eliashberg's Theorem 3.18 combined with Lemma 2.8 that we can deform this homotopy (through formal contactomorphisms) to a homotopy φ_t through contactomorphisms with $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \tau^2_{S^+_\#}$. This process can be done relative to an open neighborhood of the south pole $e(s) \in (Y \# M, \xi \# \xi_{\mathrm{ot}})$; see Remark 3.19. The loop of standard spheres $\varphi_t \circ e$ is formally trivial by construction but geometrically nontrivial. Indeed, by the contact isotopy extension theorem, the triviality of this loop would imply that $\tau^2_{S^+_\#}$, regarded as a contactomorphism of (Y, ξ), is contact isotopic to the identity rel B, which is in contradiction with Corollaries 3.7 and 4.3. □

Given a contact 3-manifold (Y, ξ) and a transverse knot $K \subset (Y, \xi)$, one can replace a small tubular neighborhood of K by a *Lutz twist* ($\mathrm{LT} = \mathbb{D}^2 \times S^1, \xi_{\mathrm{ot}}$) to obtain an *overtwisted* contact manifold (Y, ξ_K). Intuitively, the Lutz twist $(\mathrm{LT}, \xi_{\mathrm{ot}})$ is an *embedded* S^1-family of overtwisted disks; see [Geiges 2008] for precise definitions. We will denote by $\mathrm{LT}(Y, \xi_K)$ the space of contact embeddings $e\colon (\mathrm{LT}, \xi_{\mathrm{ot}}) \hookrightarrow (Y, \xi_K)$, regarded as a based space with basepoint the standard one, and by $\mathrm{FLT}(Y, \xi_K)$ the corresponding space of formal contact embeddings. As before, there is an inclusion map $\mathrm{LT}(Y, \xi_K) \to \mathrm{FLT}(Y, \xi_K)$. The following can be deduced using the same strategy as above:

Corollary 7.3 *Let (Y, ξ) be an irreducible contact 3-manifold with vanishing Euler class and such that $\boldsymbol{c}(\xi) \notin \mathrm{Im}\, U$. Consider a Darboux ball $B \subset (Y, \xi)$ and a transverse knot $K \subset B$. Then there exists an element with infinite order in*

$$\ker(\pi_1 \,\mathrm{LT}(Y, \xi_K) \to \pi_1 \,\mathrm{FLT}(Y, \xi_K)).$$

References

[Bamler and Kleiner 2019] **R H Bamler**, **B Kleiner**, *Ricci flow and contractibility of spaces of metrics*, preprint (2019) arXiv 1909.08710

[Bamler and Kleiner 2023] **R H Bamler**, **B Kleiner**, *Ricci flow and diffeomorphism groups of* 3*-manifolds*, J. Amer. Math. Soc. 36 (2023) 563–589 MR Zbl

[Bamler and Kleiner 2024] **R H Bamler**, **B Kleiner**, *Diffeomorphism groups of prime* 3*-manifolds*, J. Reine Angew. Math. 806 (2024) 23–35 MR Zbl

[Bodnár and Plamenevskaya 2021] **J Bodnár**, **O Plamenevskaya**, *Heegaard Floer invariants of contact structures on links of surface singularities*, Quantum Topol. 12 (2021) 411–437 MR Zbl

[Borman et al. 2015] **M S Borman**, **Y Eliashberg**, **E Murphy**, *Existence and classification of overtwisted contact structures in all dimensions*, Acta Math. 215 (2015) 281–361 MR Zbl

[Bourgeois 2006] **F Bourgeois**, *Contact homology and homotopy groups of the space of contact structures*, Math. Res. Lett. 13 (2006) 71–85 MR Zbl

[Casals et al. 2021] **R Casals**, **D M Pancholi**, **F Presas**, *The Legendrian Whitney trick*, Geom. Topol. 25 (2021) 3229–3256 MR Zbl

[Cerf 1968] **J Cerf**, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Math. 53, Springer (1968) MR Zbl

[Colin 1997] **V Colin**, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues*, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997) 659–663 MR Zbl

[Colin et al. 2011] **V Colin**, **P Ghiggini**, **K Honda**, *Equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions*, Proc. Natl. Acad. Sci. USA 108 (2011) 8100–8105 MR Zbl

[Conway and Min 2020] **J Conway**, **H Min**, *Classification of tight contact structures on surgeries on the figure-eight knot*, Geom. Topol. 24 (2020) 1457–1517 MR Zbl

[Ding and Geiges 2007] **F Ding**, **H Geiges**, *A unique decomposition theorem for tight contact* 3*-manifolds*, Enseign. Math. 53 (2007) 333–345 MR Zbl

[Ding and Geiges 2010] **F Ding**, **H Geiges**, *The diffeotopy group of $S^1 \times S^2$ via contact topology*, Compos. Math. 146 (2010) 1096–1112 MR Zbl

[Echeverria 2020] **M Echeverria**, *Naturality of the contact invariant in monopole Floer homology under strong symplectic cobordisms*, Algebr. Geom. Topol. 20 (2020) 1795–1875 MR Zbl

[Eliashberg 1989] **Y Eliashberg**, *Classification of overtwisted contact structures on* 3*-manifolds*, Invent. Math. 98 (1989) 623–637 MR Zbl

[Eliashberg 1992] **Y Eliashberg**, *Contact* 3*-manifolds twenty years since J Martinet's work*, Ann. Inst. Fourier (Grenoble) 42 (1992) 165–192 MR Zbl

[Eliashberg 1993] **Y Eliashberg**, *Legendrian and transversal knots in tight contact* 3*-manifolds*, from "Topological methods in modern mathematics" (L R Goldberg, A V Phillips, editors), Publish or Perish, Houston, TX (1993) 171–193 MR Zbl

[Eliashberg and Mishachev 2002] **Y Eliashberg**, **N Mishachev**, *Introduction to the h-principle*, Graduate Studies in Math. 48, Amer. Math. Soc., Providence, RI (2002) MR Zbl

[Eliashberg and Mishachev 2021] **Y Eliashberg**, **N Mishachev**, *The space of tight contact structures on $\mathbb{R}^3$ is contractible*, preprint (2021) arXiv 2108.09452

[Fernández et al. 2020] **E Fernández**, **J Martínez-Aguinaga**, **F Presas**, *The homotopy type of the contactomorphism groups of tight contact* 3*-manifolds*, *I*, preprint (2020) arXiv 2012.14948

[Fuchs and Tabachnikov 1997] **D Fuchs**, **S Tabachnikov**, *Invariants of Legendrian and transverse knots in the standard contact space*, Topology 36 (1997) 1025–1053 MR Zbl

[Gabai 2001] **D Gabai**, *The Smale conjecture for hyperbolic* 3*-manifolds:* $\mathrm{Isom}(M^3) \simeq \mathrm{Diff}(M^3)$, J. Differential Geom. 58 (2001) 113–149 MR Zbl

[Geiges 2008] **H Geiges**, *An introduction to contact topology*, Cambridge Stud. Adv. Math. 109, Cambridge Univ. Press (2008) MR Zbl

[Geiges and Gonzalo Perez 2004] **H Geiges**, **J Gonzalo Perez**, *On the topology of the space of contact structures on torus bundles*, Bull. London Math. Soc. 36 (2004) 640–646 MR Zbl

[Geiges and Klukas 2014] **H Geiges**, **M Klukas**, *The fundamental group of the space of contact structures on the 3-torus*, Math. Res. Lett. 21 (2014) 1257–1262 MR Zbl

[Ghiggini and Van Horn-Morris 2016] **P Ghiggini**, **J Van Horn-Morris**, *Tight contact structures on the Brieskorn spheres* $-\Sigma(2,3,6n-1)$ *and contact invariants*, J. Reine Angew. Math. 718 (2016) 1–24 MR Zbl

[Gironella 2021] **F Gironella**, *Examples of contact mapping classes of infinite order in all dimensions*, Math. Res. Lett. 28 (2021) 707–727 MR Zbl

[Giroux 1991] **E Giroux**, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991) 637–677 MR Zbl

[Giroux 1999] **E Giroux**, *Une infinité de structures de contact tendues sur une infinité de variétés*, Invent. Math. 135 (1999) 789–802 MR Zbl

[Giroux 2001] **E Giroux**, *Structures de contact sur les variétés fibrées en cercles audessus d'une surface*, Comment. Math. Helv. 76 (2001) 218–262 MR Zbl

[Giroux and Massot 2017] **E Giroux**, **P Massot**, *On the contact mapping class group of Legendrian circle bundles*, Compos. Math. 153 (2017) 294–312 MR Zbl

[Gompf 1998] **R E Gompf**, *Handlebody construction of Stein surfaces*, Ann. of Math. 148 (1998) 619–693 MR Zbl

[Gromov 1985] **M Gromov**, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985) 307–347 MR Zbl

[Gromov 1986] **M Gromov**, *Partial differential relations*, Ergebnisse der Math. (3) 9, Springer (1986) MR Zbl

[Hansen 1978] **V L Hansen**, *The homotopy type of the space of maps of a homology* 3*-sphere into the* 2*-sphere*, Pacific J. Math. 76 (1978) 43–49 MR Zbl

[Hatcher 1976] **A Hatcher**, *Homeomorphisms of sufficiently large* P^2*-irreducible* 3*-manifolds*, Topology 15 (1976) 343–347 MR Zbl

[Hatcher 1981] **A Hatcher**, *On the diffeomorphism group of* $S^1 \times S^2$, Proc. Amer. Math. Soc. 83 (1981) 427–430 MR Zbl

[Hatcher 1983] **A E Hatcher**, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. 117 (1983) 553–607 MR Zbl

[Hatcher and Wahl 2010] **A Hatcher**, **N Wahl**, *Stabilization for mapping class groups of* 3*-manifolds*, Duke Math. J. 155 (2010) 205–269 MR Zbl

[Honda 2002] **K Honda**, *Gluing tight contact structures*, Duke Math. J. 115 (2002) 435–478 MR Zbl

[Hong et al. 2012] **S Hong**, **J Kalliongis**, **D McCullough**, **J H Rubinstein**, *Diffeomorphisms of elliptic 3-manifolds*, Lecture Notes in Math. 2055, Springer (2012) MR Zbl

[Ivanov 1976] **N V Ivanov**, *Groups of diffeomorphisms of Waldhausen manifolds*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. 66 (1976) 172–176 MR Zbl In Russian; translated in J. Soviet Math. 12 (1979) 115–118

[Kanda 1997] **Y Kanda**, *The classification of tight contact structures on the 3-torus*, Comm. Anal. Geom. 5 (1997) 413–438 MR Zbl

[Kronheimer and Mrowka 1997] **P B Kronheimer**, **T S Mrowka**, *Monopoles and contact structures*, Invent. Math. 130 (1997) 209–255 MR Zbl

[Kronheimer and Mrowka 2007] **P Kronheimer**, **T Mrowka**, *Monopoles and three-manifolds*, New Math. Monogr. 10, Cambridge Univ. Press (2007) MR Zbl

[Kronheimer and Mrowka 2020] **P B Kronheimer**, **T S Mrowka**, *The Dehn twist on a sum of two $K3$ surfaces*, Math. Res. Lett. 27 (2020) 1767–1783 MR Zbl

[Kronheimer et al. 2007] **P Kronheimer**, **T Mrowka**, **P Ozsváth**, **Z Szabó**, *Monopoles and lens space surgeries*, Ann. of Math. 165 (2007) 457–546 MR Zbl

[Kutluhan et al. 2020] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, HF = HM*, I: Heegaard Floer homology and Seiberg–Witten Floer homology*, Geom. Topol. 24 (2020) 2829–2854 MR Zbl

[Lin 2016] **F Lin**, *Lectures on monopole Floer homology*, from "Proceedings of the Gökova geometry–topology conference 2015" (S Akbulut, D Auroux, T Önder, editors), GGT, Gökova (2016) 39–80 MR Zbl

[Lin 2023] **J Lin**, *Isotopy of the Dehn twist on $K3 \# K3$ after a single stabilization*, Geom. Topol. 27 (2023) 1987–2012 MR Zbl

[McCullough 1981] **D McCullough**, *Homotopy groups of the space of self-homotopy-equivalences*, Trans. Amer. Math. Soc. 264 (1981) 151–163 MR Zbl

[McCullough 1990] **D McCullough**, *Topological and algebraic automorphisms of 3-manifolds*, from "Groups of self-equivalences and related topics" (R A Piccinini, editor), Lecture Notes in Math. 1425, Springer (1990) 102–113 MR Zbl

[McCullough and Soma 2013] **D McCullough**, **T Soma**, *The Smale conjecture for Seifert fibered spaces with hyperbolic base orbifold*, J. Differential Geom. 93 (2013) 327–353 MR Zbl

[Milnor 1959] **J Milnor**, *On spaces having the homotopy type of a* CW*-complex*, Trans. Amer. Math. Soc. 90 (1959) 272–280 MR Zbl

[Min 2024] **H Min**, *The contact mapping class group and rational unknots in lens spaces*, Int. Math. Res. Not. 2024 (2024) 11315–11342 MR Zbl

[Mrowka and Rollin 2006] **T Mrowka**, **Y Rollin**, *Legendrian knots and monopoles*, Algebr. Geom. Topol. 6 (2006) 1–69 MR Zbl

[Muñoz-Echániz 2024] **J Muñoz-Echániz**, *A monopole invariant for families of contact structures*, Adv. Math. 439 (2024) art. id. 109483 MR Zbl

[Némethi 2017] **A Némethi**, *Links of rational singularities, L-spaces and* LO *fundamental groups*, Invent. Math. 210 (2017) 69–83 MR Zbl

[Neumann 1981] **W D Neumann**, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. 268 (1981) 299–344 MR Zbl

[Ozsváth and Szabó 2005] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and contact structures*, Duke Math. J. 129 (2005) 39–61 MR Zbl

[Palais 1966] **R S Palais**, *Homotopy theory of infinite dimensional manifolds*, Topology 5 (1966) 1–16 MR Zbl

[Seidel 1999] **P Seidel**, *Lagrangian two-spheres can be symplectically knotted*, J. Differential Geom. 52 (1999) 145–171 MR Zbl

[Seidel 2000] **P Seidel**, *Graded Lagrangian submanifolds*, Bull. Soc. Math. France 128 (2000) 103–149 MR Zbl

[Seidel 2003] **P Seidel**, *A long exact sequence for symplectic Floer cohomology*, Topology 42 (2003) 1003–1063 MR Zbl

[Smirnov 2020] **G Smirnov**, *Seidel's theorem via gauge theory*, preprint (2020) arXiv 2010.03361

[Smirnov 2022] **G Smirnov**, *Symplectic mapping class groups of $K3$ surfaces and Seiberg–Witten invariants*, Geom. Funct. Anal. 32 (2022) 280–301 MR Zbl

[Taubes 1994] **C H Taubes**, *The Seiberg–Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994) 809–822 MR Zbl

[Taubes 2000] **C H Taubes**, SW $\Rightarrow$ Gr: *from the Seiberg–Witten equations to pseudo-holomorphic curves*, from "Seiberg Witten and Gromov invariants for symplectic 4-manifolds" (R Wentworth, editor), First Int. Press Lect. Ser. 2, International, Somerville, MA (2000) 1–97 MR Zbl

[Vogel 2018] **T Vogel**, *Non-loose unknots, overtwisted discs, and the contact mapping class group of S^3*, Geom. Funct. Anal. 28 (2018) 228–288 MR Zbl

[Watanabe 2018] **T Watanabe**, *Some exotic nontrivial elements of the rational homotopy groups of* $\mathrm{Diff}(S^4)$, preprint (2018) arXiv 1812.02448

[Weiss 2005] **M Weiss**, *What does the classifying space of a category classify?*, Homology Homotopy Appl. 7 (2005) 185–195 MR Zbl

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Received: 3 July 2023
Revised: 9 April 2024

GEOMETRY & TOPOLOGY

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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Geometry & Topology

Volume 29 Issue 3 (pages 1115–1691) 2025