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The systole of large genus minimal surfaces in positive Ricci curvature

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We use Colding–Minicozzi lamination theory to show that the systole, and more generally any homology systole, of a sequence of embedded minimal surfaces in an ambient three-manifold of positive Ricci curvature tends to zero as the genus becomes unbounded.

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1 Introduction

In this paper we study properties of compact, embedded minimal surfaces in a closed (ie compact without boundary) ambient three-manifold M of positive Ricci curvature as their genus becomes unbounded. This complements the celebrated theorem of Choi and Schoen [1985]. Recall that this states that for a three-manifold M with positive Ricci curvature, the space of compact, embedded minimal surfaces in M with bounded genus is compact in the C^ℓ-topology for any $\ell \geq 2$.

Our main result shows that the systole of such a sequence of minimal surfaces tends to zero. Recall that the *systole* of a closed surface $\Sigma \subset M$ is defined to be

$$\operatorname{sys}(\Sigma) := \inf\{\operatorname{length}(c) \mid c\colon S^1 \to \Sigma \text{ noncontractible}\}.$$

Note that this takes into account all curves that do not bound a disk in Σ. Similarly, the *homology systole* is given by

$$\operatorname{sys}^h(\Sigma) := \inf\{\operatorname{length}(c) \mid 0 \neq [c] \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})\},$$

taking into account only curves that are not a boundary in Σ. Clearly, we have

$$\operatorname{sys}(\Sigma) \leq \operatorname{sys}^h(\Sigma).$$

More generally, for $k \in \mathbb{N}^*$, let us define the k^{th} *homology systole* by

$$\operatorname{sys}^h_k(\Sigma) := \inf\Big\{ \max_{i=1,\dots,k} \operatorname{length}(c_i) \mid \operatorname{rank}(\langle c_1, \dots c_k \rangle) = k \Big\},$$

where the span $\langle c_1, \dots, c_k \rangle$ is taken in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$.

We use $\mathbb{Z}/2\mathbb{Z}$-coefficients here to deal with orientable and nonorientable surfaces simultaneously. Of course, for orientable surfaces we can equivalently use $\mathbb{Z}$-coefficients.

We can now state our main result.

Theorem 1.1 *Assume that (M,g) is a three-manifold with positive Ricci curvature. Let $k \in \mathbb{N}^*$ and consider a sequence $(\Sigma_j)_{j\in\mathbb{N}}$ of closed, embedded minimal surfaces in M with $\chi(\Sigma_j) \to -\infty$ as $j \to \infty$. Then the k^{th} homology systole satisfies*

$$\mathrm{sys}_k^h(\Sigma_j) \to 0 \quad \text{as } j \to \infty.$$

Before putting this result into context we briefly discuss the different assumptions that we make.

The reader might wonder if the assumption of the surfaces being minimal is really required. This is because we need to squish the large genus surfaces Σ_j into the compact manifold M, which may force the systole to tend to zero anyway.

Remark 1.2 There is a sequence $(S_j)_{j\in\mathbb{N}^*}$ of embedded (and unknotted) surfaces in S^3 such that $\mathrm{genus}(S_j) \to \infty$ and $\mathrm{sys}(S_j) \geq c_0 > 0$ for some constant c_0.

This can be constructed as follows: Take a surface R_j of genus j with systole $\mathrm{sys}(R_j) \geq 2c_0 > 0$. By the Nash–Kuiper theorem, there is a $C^{1,\alpha}$-isometric embedding of R_j in an arbitrarily small ball $B_\delta \subset \mathbb{R}^3$, where here and below we denote by B_r the Euclidean ball of radius $r > 0$ centered at the origin. After smoothing this and applying stereographic projection, we get a surface $S_j \subset S^3$ of genus j that is closed, unknotted, embedded and has $\mathrm{sys}(S_j) \geq c_0$.

Also the assumption on the surfaces being embedded is crucial, since high-degree covers of a given minimal surface of positive genus provide trivial counterexamples to the immersed version.

Finally, the following example shows that Theorem 1.1 does not hold without any assumptions on the ambient geometry.

Example 1.3 Denote by Σ_γ a closed surface of genus γ for $\gamma \geq 2$. It is shown in [Tollefson 1969] (see also [Neumann 1976] for a generalization) that the three-manifold $M = S^1 \times \Sigma_\gamma$ admits fiber bundles

$$\Sigma_\delta \to M \to S^1 \tag{1.4}$$

for $\delta = \gamma + n(\gamma - 1)$ and $n \in \mathbb{N}$. Since $\pi_2(S^1) = 0$, the long exact sequence for homotopy groups associated to these fibrations implies that $\Sigma_\delta \to M$ is incompressible, ie the induced map $\pi_1(\Sigma_\delta) \to \pi_1(M)$ is injective. It follows from [Schoen and Yau 1979, Theorem 3.1] that there are immersed minimal surfaces S_δ in M which are diffeomorphic to Σ_δ and the induced map on π_1 is given by the inclusion of the fibers from (1.4). Moreover, [Freedman et al. 1983, Theorem 5.1] implies that these are not only immersions but even embeddings. Since $\pi_1(S_\delta) \to \pi_1(M)$ is injective, we have in particular that

$$\mathrm{sys}(S_\delta) \geq \mathrm{sys}(M) > 0.$$

This shows that Theorem 1.1 cannot hold for M.

Remark 1.5 It follows from [Schoen and Yau 1979, Theorem 5.2] that M does not admit any metric of positive scalar curvature, which leaves open the possibility to replace Ricci by scalar curvature in Theorem 1.1.

Let us now put Theorem 1.1 into some more context.

Balacheff, Parlier and Sabourau [2012] provided general results on systoles of surfaces; see Section 2.1 for more details. As a consequence thereof, for any sequence (Σ_j) of surfaces with $\chi(\Sigma_j) \to -\infty$ and area growth in the genus γ bounded by $|\gamma|^\alpha$ for some $\alpha < 1$, and for any $k \in \mathbb{N}$, the k^{th} homology systole tends to zero, ie $\operatorname{sys}_k^h(\Sigma_j) \to 0$ as $j \to \infty$. This motivates the following discussion about area bounds in the genus.

Thanks to the recent work of Chodosh and Mantoulidis [2020] on the Allen–Cahn equation, any closed three-manifold with positive Ricci curvature contains a sequence of embedded minimal surfaces with unbounded genus; see also the related earlier work by Marques and Neves [2017] and Aiex [2018]. Their construction gives a sequence of minimal surfaces $(\Sigma_p)_{p\in\mathbb{N}}$ with

$$\operatorname{area}(\Sigma_p) \sim \operatorname{genus}(\Sigma_p)^{1/3} \sim p^{1/3},$$

ie area growing sublinearly in the genus. In fact, the same result has now also been established using Almgren–Pitts min-max theory through the works of Marques and Neves [2021] and Zhu [2020]. At this point we also would like to point out Song's work [2023] settling the general case of Yau's conjecture and the papers by Irie, Marques and Neves [Irie et al. 2018], Marques, Neves and Song [Marques et al. 2019], and Liokumovich, Marques and Neves [Liokumovich et al. 2018] giving information on the distribution of min-max minimal hypersurfaces in the ambient manifold for generic metrics. In a similar direction Theorem 1.1 provides information on embedded minimal surfaces of high complexity. Because of the sublinear growth of the area, Theorem 1.1 is automatically true for min-max minimal surfaces. However, Theorem 1.1 applies to any family of minimal surfaces, not only those arising from min-max methods.

The best known bound for the area of embedded minimal surfaces in an ambient three manifold of positive Ricci curvature is linear in the genus [Choi and Wang 1983]. More precisely, we have that

$$\operatorname{area}(\Sigma) \leq C(\operatorname{genus}(\Sigma) + 1)$$

for a constant C depending on the topology of M and the lower bound on the Ricci curvature. It is by no means clear if this bound is sharp and Theorem 1.1 could be considered as some hint towards the nonsharpness of the linear bound. It appears to be an interesting question to understand the maximal possible area growth of a sequence of embedded, minimal surfaces with genus tending to infinity. To the best knowledge of the authors, among all known families of minimal surfaces in $\mathbb{S}^3$ the Lawson surfaces $\xi_{m,m}$, see [Lawson 1970], exhibit the fastest area growth in terms of the genus. More precisely, $\operatorname{genus}(\xi_{m,m}) = m^2$ while $\operatorname{area}(\xi_{m,m}) \sim m$.

Main problems and strategy

Let us for simplicity focus on the case of M being simply connected, $k = 1$ and the systole instead of the homology systole.

We want to argue by contradiction and consider a sequence of minimal surfaces $\Sigma_j \subset M$ with $\mathrm{sys}(\Sigma_j) \geq l_0 > 0$ and $\mathrm{genus}(\Sigma_j) \to \infty$. In general, we would like to pass to a limit $\Sigma_j \to \mathscr{L}$ in the class of minimal laminations (see eg Definition 3.1) and argue that $\mathscr{L}$ has a stable leaf, which would easily lead to a contradiction since M has positive Ricci curvature.

The problem about this is that we can only do this outside the closed set at which $|A^{\Sigma_j}|^2$ blows-up, where A^{Σ_j} denotes the second fundamental form of Σ_j. A priori, the blow-up set could even be all of M. Work of Colding and Minicozzi gives strong structural information about the blow-up set if the surfaces in question have bounded genus. The main step of our proof is to show that the sequence Σ_j as above can locally be dealt with in this framework.

The reason why this is not obvious is that we do not have $-\Delta_{\Sigma_j} d^2(x,\cdot) \leq 0$ globally (as it is the case for minimal surfaces in $\mathbb{R}^3$). Therefore, the assumption on $\mathrm{sys}(\Sigma_j)$ does not directly imply that there is $R_0 = R_0(l_0)$ such that the intrinsic balls $B^{\Sigma_j}(x, R_0)$ are contained in (intrinsic) *disks* in the intersection $B(x, R_0) \cap \Sigma_j$ with an extrinsic ball. Instead, $B^{\Sigma_j}(x, R_0)$ is contained in some disk $D_x^j \subset \Sigma_j$ but D_x^j could leave any mean-convex ball $B(x, r)$ centered at x. The main step is to show that this is impossible after going to a (potentially much) smaller scale. The proof of this is of global nature and also relies on the positivity of the Ricci curvature of M. It also proceeds by contradiction and follows broadly the same strategy. Given a contradicting sequence we try to find a stable minimal surface in M. The key step to achieve this is to show that Σ_j serves as a good barrier for a minimization problem in M. This in turn is shown by promoting information about singularities of Σ_j for $j \to \infty$ across scales using the maximum principle.

The general case of the theorem follows similar steps but is technically more involved. This requires for instances a more careful blow-up argument in the case $k \geq 2$ and also makes use of some additional elementary topological arguments.

Organization

In Section 2 we give some rather elementary preliminaries on surfaces and topology and recall a fundamental result from systolic geometry which are needed in our arguments. In Section 3 we provide necessary background from [Colding and Minicozzi 2015] on Colding–Minicozzi lamination theory of minimal surfaces with some control on the topology. Section 4 contains two weak chord-arc properties for minimal surfaces contained in small extrinsic balls of an ambient three-manifold. Our main result, Theorem 1.1, is proved first in the case $k = 1$ in Section 5, and then in Section 6 in the general case.

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2 Some preliminaries on surfaces and topology

In this section we recall some elementary and well-known facts about the topology of surfaces. We also recall some results from systolic geometry.

2.1 A result from systolic geometry

We will use the following result from systolic geometry, that relates the area and the k^{th} homology systole.

Theorem 2.1 ([Balacheff et al. 2012, Theorem 1.2]; see also [Gromov 1996]) *Let $\eta\colon \mathbb{N} \to \mathbb{N}$ be a function such that*

$$\lambda := \sup_{\gamma} \frac{\eta(\gamma)}{\gamma} < 1.$$

Then there exists a constant C_λ such that for every closed, orientable Riemannian surface Σ of genus γ, we have

$$\operatorname{sys}^h_{\eta(\gamma)}(\Sigma) \le C_\lambda \frac{\log(\gamma+1)}{\sqrt{\gamma}} \sqrt{\operatorname{area}(\Sigma)}.$$

Recall that a nonorientable surface Σ can be written as a connected sum $\Sigma = \Sigma_1 \# \Sigma_2$, with Σ_1 closed, orientable and Σ_2 diffeomorphic to $\mathbb{RP}^2$ or $\mathbb{RP}^2 \# \mathbb{RP}^2$. If we replace Σ_2 by a disk, Theorem 2.1 easily implies the following for nonorientable surfaces.

Corollary 2.2 *Let η and λ be as above. Then there is a constant C_λ such that for every closed, nonorientable surface of nonorientable genus δ, we have*

$$\operatorname{sys}^h_{\eta(\gamma_\delta)}(\Sigma) \le C_\lambda \frac{\log(\gamma_\delta+1)}{\sqrt{\gamma_\delta}} \sqrt{\operatorname{area}(\Sigma)},$$

where $\gamma_\delta = \lfloor(\delta-1)/2\rfloor$.

We will only use the following consequence of these results.

Corollary 2.3 *Let (Σ_j) be a sequence of surfaces with $-\chi(\Sigma_j) \to \infty$. If $\operatorname{area}(\Sigma_j) = O((-\chi(\Sigma_j))^\alpha)$ for some $0 \le \alpha < 1$, then, for any $k \in \mathbb{N}$,*

$$\operatorname{sys}^h_k(\Sigma_j) \to 0 \quad \text{as } j \to \infty.$$

To put this into context, notice that the Choi–Wang bound [Choi and Wang 1983] implies, for a closed, embedded, orientable, minimal surface Σ, that

$$\operatorname{area}(\Sigma) \le C(\operatorname{genus}(\Sigma)+1),$$

where $C = C(k)$, if $\operatorname{Ric}(M) \ge k > 0$.

2.2 Some elementary facts about the topology of surfaces

Lemma 2.4 *Let Σ be a closed surface and $c \subset \Sigma$ a simple closed curve. Then $[c] \ne 0 \in H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$ if and only if c is nonseparating.*

Proof If c is separating, then $[c] = 0$ in $H_1(\Sigma,\mathbb{Z}/2\mathbb{Z})$. On the other hand, if c is nonseparating, there is a curve d such that $|c \cap d| = 1$. In particular, from the intersection pairing, $[c] \ne 0 \in H_1(\Sigma,\mathbb{Z}/2\mathbb{Z})$. □

Lemma 2.5 *Let Σ be a closed surface and let $c_1,\dots,c_k \subset \Sigma$ be simple closed curves. Assume that $c_i \subset \bigcup_{i=1}^k B_i$ for pairwise disjoint balls B_i. Then, for a simple closed curve $d \subset \Sigma$ such that $0 \ne [d] \in \langle [c_1],\dots,[c_k]\rangle \subseteq H_1(\Sigma,\mathbb{Z}/2\mathbb{Z})$ and $d \subset M \setminus \bigcup_{i=1}^k B_i$, we have that d is separating in $\Sigma \cap \bigl(M \setminus \bigcup_{i=1}^k B_i\bigr)$.*

Remark 2.6 In the case $k = 1$, this states that if c is nonseparating in Σ and $c \subset B$, then any curve in $M \setminus B$ that is homologous to c separates in $M \setminus B$.

Proof Write $B = \bigcup_{i=1}^k B_i$. If d is nonseparating in $\Sigma \setminus B$ we can find a closed curve $e \subset \Sigma \setminus B$ that intersects d exactly once. On the other hand, $c_i \cap e = \varnothing$ for any e, but this is impossible since d is in the span of the c_i. □

Definition 2.7 Let $\Sigma \subset M$ be an embedded surface, $x \in M$ and $r > 0$. We say that $c\colon S^1 \to \Sigma$ is *contractible on scale r at x* if there is a disk $\phi\colon D \to B(x,r) \cap \Sigma$ with $\phi|_{S^1} = c$. If there is some $x \in M$ such that c is contractible on scale r at x, we say that *c is contractible on scale r*.

At this point it is worth recalling the following version of the maximum principle for minimal surfaces.

Theorem 2.8 *Let N be a compact manifold with mean-convex boundary and $\Sigma \subset N$ be a minimal surface (possibly with boundary). Then $(\Sigma \setminus \partial\Sigma) \cap \partial N = \varnothing$ or $\Sigma \subseteq \partial N$.*

In the context of Definition 2.7 this has the following consequence, that we will make use of frequently.

Lemma 2.9 *Choose $r > 0$ such that any ball $B(x,s) \subset M$ with $s \le r$ and $x \in M$ is mean convex. If $\Sigma \subset M$ is a complete minimal surface and $c \subset \Sigma$ is a simple closed curve that is contractible on scale r (at x), then c is contractible on scale t (at x) for any $t \le r$ such that $c \subset B(x,t)$.*

Lemma 2.10 *Let $\Sigma \subset M$ be a surface, $x \in \Sigma$ and $R > 0$. If any curve $d\colon S^1 \to B^{\Sigma}(x,R)$ with $\operatorname{length}(d) \le 3R$ is contractible on scale r at x, then any curve $c\colon S^1 \to B^{\Sigma}(x,R)$ is contractible on scale r at x.*

Proof Let $c\colon S^1 \to B^\Sigma(x, R)$ be a loop. Choose a subdivision

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$$

of $[0, 1]$ such that

$$\operatorname{length}(c_{|[t_i,t_{i+1}]}) \leq R.$$

Fix curves $d_i\colon I \to B^\Sigma(x, R)$ with $d_i(0) = x$ and $d_i(1) = c(t_i)$ and such that

$$\operatorname{length}(d_i) \leq R.$$

We can then write

$$c = (c_{|[t_{k-1},t_k]} * d_{k-1}) * (\bar{d}_{k-1} * c_{|[t_{k-2},t_{k-1}]} * d_{k-2}) * \cdots * (\bar{d}_2 * c_{|[t_1,t_2]} * d_1) * (\bar{d}_1 * c_{|[t_0,t_1]}),$$

which implies the assertion. □

Since the Hurewicz homomorphism $\pi_1(B^\Sigma(x, R), x) \to H_1(B^\Sigma(x, R); \mathbb{Z})$ as well as the map

$$H_1(B^\Sigma(x, R); \mathbb{Z}) \to H_1(B^\Sigma(x, R); \mathbb{Z}/2\mathbb{Z})$$

are surjective, we immediately get the following corollary.

Corollary 2.11 *Let Σ be a surface, $x \in \Sigma$ and $R > 0$. Then the group $H_1(B^\Sigma(x, R); \mathbb{Z}/2\mathbb{Z})$ is generated by curves of length at most $3R$.*

Lemma 2.12 *Let Σ be a closed surface and $\pi\colon \hat{\Sigma} \to \Sigma$ a covering. Consider a simple closed curve $c \subset \Sigma$ and its preimage $\hat{c} = \pi^{-1}(c) \subset \hat{\Sigma}$. If c is separating, then also $\hat{c}$ is separating.*

Proof If c is separating, we can write $\Sigma \setminus c = \Sigma_+ \cup \Sigma_-$ with connected surfaces $\Sigma_\pm$. Moreover, there is a function $f\colon \Sigma \to [-1, 1]$ such that $\{f = 0\} = c$ and $\Sigma_\pm = \{f \gtrless 0\}$. We can then consider the lifted function $\hat{f} = f \circ \pi$, which clearly satisfies $\{\hat{f} = 0\} = \hat{c}$. Therefore, $\hat{c}$ separates $\hat{\Sigma}$ into $\hat{\Sigma}_- = \{\hat{f} < 0\}$ and $\hat{\Sigma}_+ = \{\hat{f} > 0\}$. □

It will be important to keep in mind that the domains $\hat{\Sigma}_\pm$ might be disconnected and $\hat{c}$ is potentially not the boundary of a compact subsurface.

3 Background on Colding–Minicozzi lamination theory

Colding and Minicozzi developed a theory that describes how minimal surfaces of uniformly bounded genus in an ambient three-manifold can degenerate in the absence of curvature bounds. We use this section to provide a very brief introduction to those parts of their theory that will be relevant in the present paper. We will focus here on the case of planar domains, since this is sufficient for our purposes.

We start by recalling the definition of a lamination.

Definition 3.1 [Colding and Minicozzi 2004e, Appendix B] (1) A codimension-one lamination on a three-manifold M is a collection $\mathscr{L}$ of smooth disjoint surfaces $\Gamma \subset M$, the so-called leaves, such that $\bigcup_{\Gamma \in \mathscr{L}} \Gamma$ is closed. Furthermore, for each point $x \in M$, there exists an open neighborhood U of x and a coordinate chart (U, Φ) with $\Phi(U) \subset \mathbb{R}^3$ so that in these coordinates the leaves in $\mathscr{L}$ pass through $\Phi(U)$ in slices of the form $(\mathbb{R}^2 \times \{t\}) \cap \Phi(U)$.

(2) A foliation is a lamination for which one has $M = \bigcup_{\Gamma \in \mathscr{L}} \Gamma$, ie the union of the leaves is all of M.

(3) A minimal lamination is a lamination whose leaves are minimal.

(4) A Lipschitz lamination is a lamination for which the chart maps Φ are Lipschitz.

Given any sequence of minimal surfaces $\Sigma_j \subset M$, we consider the *singular* or *blow-up set*

$$\mathscr{S} = \left\{ z \in M \;\middle|\; \inf_{\delta>0} \sup_j \sup_{B(z,\delta)} |A^{\Sigma_j}| = \infty \right\},$$

ie the points z where the curvature blows up. Up to taking a subsequence one can always pass to a limit

$$\Sigma_j \to \mathscr{L} \text{ in } M \setminus \mathscr{S},$$

where the convergence is in $C^{0,\alpha}$ and the limit lamination is a minimal Lipschitz lamination.

In the case of minimal surfaces $\Sigma_j \subset B(0, R_j) \subset \mathbb{R}^3$ with bounded genus and $\partial\Sigma_j \subset \partial B(0, R_j)$ one can always extract a subsequence such that either $R_j \to \infty$ or with R_j bounded. In the former case one can reach much stronger conclusions on the structure of the limit lamination, see eg the example in [Colding and Minicozzi 2004a]. Since we only deal with the local case, ie $R_j = R$ is fixed, which in general only allows us to draw significantly weaker conclusions about the structure of the limit lamination, we do not discuss stronger conclusion valid in the global case.

We first consider the case when the Σ_j are disks. Colding and Minicozzi [2004b; 2004c; 2004d; 2004e] proved that every embedded minimal disk is either a graph of a function or is a double spiral staircase where each staircase is a multivalued graph. More precisely, they show that if the curvature blows up at some point (and thus the surface is not a graph), then the surface is a double spiral staircase like the helicoid; see also [Colding and Minicozzi 2004e, Theorem 0.1].

Below we also want to deal with the case where Σ_j are more general domains than disks, namely, so-called uniformly locally simply connected (in short: ULSC) domains.

A sequence of minimal surfaces $\Sigma_j \subset M$ is called *uniformly locally simply connected*[1] if given any compact $K \subset M$ there is some $r > 0$ such that

$$\Sigma_j \cap B(x, r) \text{ consists of disks for any } x \in K.$$

[1]We remark that this is stronger than the definition of Colding–Minicozzi in the case of nonplanar domains.

Moreover, we define

$$\mathcal{S}_{\mathrm{ulsc}} := \{z \in \mathcal{S} \mid \Sigma_j \text{ is ULSC near } z\}.$$

The main local structural result we need for (not necessarily globally planar or bounded genus) ULSC sequences concerns so-called collapsed leaves, whose existence is described in the next lemma. We assume that $\Sigma_j \to \mathcal{L}'$ in $M \setminus \mathcal{S}$, where Σ_j is a ULSC sequence.

Lemma 3.2 [Colding and Minicozzi 2015, Lemma II.2.3] *Given a point $x \in \mathcal{S}_{\mathrm{ulsc}}$, there exists $r_0 > 0$ such that $B(x, r_0) \cap \mathcal{L}'$ has a component Γ_x whose closure $\overline{\Gamma}_x$ is a smooth minimal graph containing x and with boundary in $\partial B(x, r_0)$ (so x is a removable singularity for Γ_x).*

We want to emphasize that while [Colding and Minicozzi 2015] starting at the end of Section II.1 makes the general assumption to be in the global case $R_j \to \infty$ this does not apply to everything contained in the following sections. In particular a look at the proof of Lemma II.2.3 show that this does not make use of this assumption. Similarly, an inspection of the arguments shows the statements from Proposition 3.3 below are valid without this assumption.

The leaves of the limit lamination $\mathcal{L}'$ may not be complete. A special type of incomplete leaves are collapsed leaves. A leaf Γ of $\mathcal{L}'$ is *collapsed* if there exists some $x \in \mathcal{S}_{\mathrm{ulsc}}$ so that Γ contains the local leaf Γ_x given by Lemma 3.2; see Definition II.2.9 in [Colding and Minicozzi 2015].

Until the end of the section, we assume that the ambient manifold is given as $N = M \setminus \{x_1, \dots, x_k\}$, where M is complete and $x_i \in M$. In order to state the key structural results on collapsed leave we need to introduce some notation. Given a leaf $\Gamma \subset \mathcal{L}'$ we fix a point $x \in \Gamma$ and write

$$\Gamma_{\mathrm{clos}} = \bigcup_{R>0} \overline{B^{\Sigma}(x, R)},$$

where the closure is taken in N.

Proposition 3.3 [Colding and Minicozzi 2015, Section II.3] *Each collapsed leaf Γ of $\mathcal{L}'$ has the following properties*:

(1) *Given any $y \in \Gamma_{\mathrm{clos}} \cap \mathcal{S}_{\mathrm{ulsc}}$, there exists $r_0 > 0$ such that the closure in M of each component of $\Gamma \cap B(y, r_0)$ is a compact embedded disk with boundary in $\partial B(y, r_0)$. Furthermore, $\Gamma \cap B(y, r_0)$ must contain the component Γ_y given by Lemma 3.2, and Γ_y is the only component of $\Gamma \cap B(y, r_0)$ with y in its closure.*

(2) Γ *is a limit leaf.*

(3) Γ *extends to a complete minimal surface away from $\{x_1, \dots, x_k\}$.*[2]

[2]In other words, there is a Γ' containing Γ such that if a geodesic in Γ' cannot be extended, it limits to some x_i.

The sequences Σ_j appearing in this manuscript will essentially all be ULSC. This is equivalent to the fact that the singular set $\mathcal{S}$ is given by $\mathcal{S}_{\mathrm{ulsc}}$, ie $\mathcal{S} = \mathcal{S}_{\mathrm{ulsc}}$. Although we will not directly apply the results for non-ULSC surfaces here, some of our arguments (in particular the proof of Lemma 5.12) are inspired by those in [Colding and Minicozzi 2015] for this case.

4 Chord arc properties

We need two weak chord-arc properties for minimal surfaces contained in small extrinsic balls of an ambient three-manifold. Given $x \in M$ and $r > 0$, we write $B(x, r)$ for the metric ball in (M, g). If $z \in \Sigma$ and $r > 0$, we denote by $B^\Sigma(z, r)$ the metric ball of radius r in Σ with respect to the induced Riemannian metric.

Let (M, g) be a closed Riemannian three-manifold. Let $R_0 > 0$ be small, so that the metric in all balls of radius R_0 in M is sufficiently close to the Euclidean metric after rescaling to unit size. We will indicate in the proof when making specific assumptions on how small R_0 has to be but point out that all of this will be only dependent on the geometry of M. A first assumption on R_0 is that all balls in M of radius $r \leq R_0$ are mean convex so that Lemma 2.9 will be useful.

We consider minimal embedded disks Σ in $B(x_0, R_0)$ for some $x_0 \in M$. We write

$$\Sigma_{x_0,r} \subseteq \Sigma \cap B(x_0, r)$$

for the connected component of $\Sigma \cap B(x_0, r)$ that contains x_0.

Theorem 4.1 *There are $R_0 > 0$ sufficiently small and $\alpha > 0$ (both depending only on M) such that for any embedded minimal disk $\Sigma \subset B(x_0, R_0) \subset M$ with $x_0 \in \Sigma$, the following holds. For any $R > 0$ with $B^\Sigma(x_0, R) \subset \Sigma \setminus \partial\Sigma$, we have $\Sigma_{x_0,\alpha R} \subset B^\Sigma(x_0, R/2)$.*

Remark 4.2 (1) This result is proven in [Colding and Minicozzi 2008, Proposition 1.1] for minimal disks in $\mathbb{R}^3$. The proof applies here as well with one minor modification in [loc. cit., Proposition 3.4] that we explain below.

(2) The following property of minimal surfaces in $\mathbb{R}^3$ was used in [loc. cit., Proposition 3.1]. Since minimal surfaces in $\mathbb{R}^3$ have nonpositive curvature it follows that any intrinsic ball B in a minimal disk $D \subset \mathbb{R}^3$ with $B \cap \partial D = \varnothing$ is itself a topological disk. This may not apply in our setting since a minimal surface $\Sigma \subset M$ can have points of positive curvature provided M has such points.

Proof The proof of [Colding and Minicozzi 2008, Proposition 1.1] applies to this setting as well with a few minor modifications. We will provide an outline of the overall proof of [loc. cit., Proposition 1.1] and indicate necessary alterations for the proof of Theorem 4.1.

The proof of [loc. cit., Proposition 1.1] consists of the following two main steps.

Step 1 Colding and Minicozzi first provide this result under the additional assumptions that Σ is compact and that $\partial\Sigma$ is contained in the boundary of an extrinsic ball:

Proposition 4.3 [Colding and Minicozzi 2008, Proposition 2.1] *Let $\Sigma \subset \mathbb{R}^3$ be a compact embedded minimal disk. There exists a constant $\delta_2 > 0$ independent of Σ such that if $x \in \Sigma$ and $\Sigma \subset B_R(x)$ with $\partial\Sigma \subset \partial B_R(x)$, then the component $\Sigma_{x,\delta_2 R}$ of $B_{\delta_2 R}(x) \cap \Sigma$ containing x satisfies*

$$\Sigma_{x,\delta_2 R} \subset B^{\Sigma}\left(x, \tfrac{1}{2}R\right).$$

The benefit of the above-mentioned additional assumptions is that the authors can directly apply previous results which were provided by Colding and Minicozzi [2004b; 2004c; 2004d; 2004e]. Since this step applies to our setting as well for R_0 chosen sufficiently small depending only on the geometry of M, we do not provide more details, but refer the interested reader to Chapter 2 of [Colding and Minicozzi 2008], which consists of the proof of Proposition 2.1.

Step 2 Colding and Minicozzi [2008, Chapter 3] remove the additional assumptions from Step 1, ie that Σ is compact and that $\partial\Sigma$ is contained in the boundary of an extrinsic ball. In order to formulate the key ingredient for Step 2, the authors define a weak chord arc property for intrinsic balls:

Definition 4.4 An intrinsic ball $B^{\Sigma}(x,s) \subset \Sigma \setminus \partial\Sigma$ is said to be δ-weakly chord arc for some $\delta > 0$ if we have $\Sigma_{x,\delta s} \subset B^{\Sigma}(x, s/2)$.

Furthermore, they need the following result.

Lemma 4.5 [Colding and Minicozzi 2008, Lemma 3.6] *There exists $C_0 > 1$ such that for every $C_a > 0$, there exists $\tau > 0$ such that if $B^{\Sigma}(x_1, C_0)$ and $B^{\Sigma}(x_2, C_0)$ are disjoint intrinsic balls in $\Sigma \setminus \partial\Sigma$ with*

$$\sup_{B^{\Sigma}(x_1,C_0) \cup B^{\Sigma}(x_2,C_0)} |A|^2 \le C_a \quad \text{and} \quad |x_1 - x_2| < \tau,$$

then for $i = 1, 2$ we have

$$B_{10}(x_i) \cap \partial B^{\Sigma}(x_i, 11) = \varnothing.$$

The key result is as follows, where δ_2 is the constant given in [loc. cit., Proposition 2.1], see Proposition 4.3 above:

Proposition 4.6 [Colding and Minicozzi 2008, Proposition 3.4] *Assume that g is a metric that is sufficiently close (depending only on M) to the Euclidean metric on B_2 and let $\Sigma \subset (B_1(0), g)$ be an embedded minimal disk. There exists a constant $C_b > 1$ independent of Σ such that if $B^{\Sigma}(y, C_b\, R_0) \subset \Sigma \setminus \partial\Sigma$ is an intrinsic ball and every intrinsic subball $B^{\Sigma}(z, R_0) \subset B^{\Sigma}(y, C_b\, R_0)$ is δ_2-weakly chord arc, then, for every $s \le 5\, R_0$, the intrinsic ball $B^{\Sigma}(y,s)$ is δ_2-weakly chord arc.*

Proof The detailed proof can be found on pages 229–231 of [loc. cit.]. For convenience of the reader we outline the main steps and emphasize where attention is required to apply the arguments in our setting.

After rescaling and translating Σ, we can assume that $R_0 = 1$ and $y = 0$. It suffices to prove the following claim, since applying [loc. cit., Proposition 2.1] to $\Sigma_{0,5}$ then establishes [loc. cit., Proposition 3.4].

Claim *There exists n such that*

$$\Sigma_{0,5} \subset B^{\Sigma}(0, (6n+3)\, C_0), \tag{4.7}$$

where $C_0 > 1$ is given by Lemma 4.5.

Colding and Minicozzi prove the claim, ie (4.7), by arguing by contradiction. So suppose that (4.7) fails for some large n. Consequently, there exists a curve

$$\sigma \subset \Sigma_{0,5} \subset B_5 \tag{4.8}$$

from 0 to a point in $\partial B^{\Sigma}(0, (6n+3)\, C_0)$. For $i = 1, \dots, n$, fix points

$$z_i \in \partial B^{\Sigma}(0, 6i\, C_0) \cap \sigma.$$

It follows that the intrinsic balls $B^{\Sigma}(z_i, 3\, C_0)$

- are disjoint, and
- have centers in $B_5 \subset \mathbb{R}^3$.

Note, however, that these are not guaranteed to be topological disks in the presence of some positive ambient curvature. We will return to this issue in a moment.

Since the n points $\{z_i\}$ are all in the Euclidean ball $B_5 \subset \mathbb{R}^3$, there exist integers i_1 and i_2 with

$$0 < |z_{i_1} - z_{i_2}| < C'\, n^{-1/3}. \tag{4.9}$$

Now we use that each intrinsic ball of radius one about any z_i is δ-weakly chord arc by the assumption that every intrinsic subball $B^{\Sigma}(z, R_0) \subset B^{\Sigma}(y, C_b\, R_0)$ is δ_2-weakly chord arc. Recall that this means that

$$\Sigma_{z_{i_j},\delta} \subseteq B^{\Sigma}\big(z_{i_j}, \tfrac{1}{2}\big) \quad \text{for} \quad j = 1, 2. \tag{4.10}$$

By construction, the two intrinsic balls $B^{\Sigma}\big(z_{i_j}, \tfrac{1}{2}\big)$ for $j = 1, 2$ are disjoint, which implies that also the surfaces $\Sigma_{z_{i_j},\delta}$ for $j = 1, 2$ need to be disjoint.

We now return to the issue of intrinsic balls not necessarily being disks. Let Σ' be a component of $B(z_{i_j}, \delta) \cap \Sigma$ with $\Sigma' \cap \partial\Sigma = \varnothing$. Consider any simple closed curve $c \subset \Sigma'$. Since Σ is a disk, we know that c is contractible within Σ. But in this scenario, the maximum principle, cf Lemma 2.9, implies that c is contractible on scale δ. This in turn implies that c is contractible within Σ'. Since this applies to any simple closed curve in Σ', we find that Σ' must be a disk.

Thanks to (4.10) this applies to the components $\Sigma_{z_{i_j},\delta}$, proving that these are in fact topological disks with

$$\partial\Sigma_{z_{i_j},\delta} \subset \partial B_{\delta}(z_{i_j}).$$

Consequently, for n large enough, (4.9) implies that the components Σ_1 and Σ_2 of

$$B_{\delta/2}(z_{i_1}) \cap \Sigma$$

containing z_{i_1} and z_{i_2}, respectively, are compact and have

$$\partial\Sigma_i \subset \partial B_{\delta/2}(z_{i_1}). \tag{4.11}$$

The rest of the proof will remain unchanged in our setting. Therefore we just outline the strategy; for details we refer the reader to [Colding and Minicozzi 2008].

From (4.11) Colding and Minicozzi deduce a curvature bound on the intrinsic balls $B^{\Sigma}(z_{i_j}, \delta(2c))$; namely they have previously shown that if two disjoint embedded minimal disks with boundary in the boundary of a ball both come close to the center, then each has an interior curvature estimate. This curvature bound in turn implies that there exists a constant $r' = r'(\delta, c)$ such that for n sufficiently large, the intrinsic ball $B^{\Sigma}(z_{i_2}, 3\,r')$ can be written as a normal exponential graph of a function u over a domain Ω. Applying the Harnack inequality to u we obtain

$$\sup_{B^{\Sigma}(z_{i_1}, 3\,r')} u \le \widetilde{C}' n^{-1/3}.$$

For n large enough, this inequality guarantees that we can repeat the previous argument with z_{i_1} substituted by a point in $\partial B^{\Sigma}(z_{i_1}, r')$. Thus, by repeatedly combining the above curvature bound and the Harnack inequality, one can extend the curvature bound to larger intrinsic balls. Applying Lemma 4.5 then yields a contradiction. □

Thus the argument from (4.8) till (4.11) also works in our framework, whence the proof. □

We also need a related chord arc property for uniformly locally simply connected surfaces.

Theorem 4.12 *Let $\Sigma \subset B(x_0, R) \subset M$ be a minimal surface with $x_0 \in \Sigma$. Assume that there is an $r > 0$ such that $\Sigma \cap B(y, r)$ consists only of proper disks for any $y \in B(x_0, R - r)$. Then, given $k \in \mathbb{N}$ such that $k\,r \le R$, there is a $\beta_k > 0$ depending only on M such that if $B^{\Sigma}(x_0, \beta_k r) \cap \partial\Sigma = \varnothing$, then $\partial(\Sigma_{x_0, k\,r}) \subseteq \partial B(x_0, k\,r)$.*

Remark 4.13 (1) This result is stated in [Colding and Minicozzi 2015, Appendix B.1] with the uniformly locally simply connected assumption for intrinsic rather than extrinsic balls, ie it is assumed that all intrinsic balls of a fixed radius are disks.

(2) As already mentioned in Remark 4.2, in our setting intrinsic balls that are contained in a disk may not be disks themselves, which is why we use extrinsic balls in the uniformly locally simply connected assumption.

Proof The argument is analogous to the proof of [Colding and Minicozzi 2008, Proposition 3.4] with some changes that we now explain; compare also the proof of Theorem 4.1.

For simplicity we scale everything so that $r = 1$.

We can follow the argument in [loc. cit., Proposition 3.4] up to (3.20) and consider two disjoint intrinsic balls $B^{\Sigma}(z_{i_j}, 3C_0) \subset \Sigma \setminus \partial\Sigma$. We first consider the surfaces $\Sigma_{z_{i_j},1}$, which are disks by assumption. Note that clearly $B^{\Sigma}\bigl(z_{i_j}, \frac{1}{2}\bigr) \cap \partial\Sigma_{z_{i_j},1} = \varnothing$. Now we apply Theorem 4.1 with $R = \frac{1}{2}$. This gives that the surfaces

$$\Sigma_{z_{i_j},\alpha/2} \subset B^{\Sigma}\bigl(z_{i_j}, \tfrac{3}{2}C_0\bigr)$$

are disjoint, proper disks, where α is given by Theorem 4.1.

From here on we can again follow the argument in [loc. cit., Proposition 3.4]. □

5 Existence of one short curve

Throughout this section let (M, g) be a closed three-manifold with positive Ricci curvature. In order to prove Theorem 1.1, we want to argue by contradiction. Therefore, we study properties of a sequence $\Sigma_j \subset (M, g)$ of closed, embedded minimal surfaces with $\operatorname{sys}_k^h(\Sigma_j) \geq l_0 > 0$. More precisely, we will be concerned with a limit lamination

$$\Sigma_j \to \mathscr{L} \quad \text{in } M \setminus \mathscr{S}$$

of such a sequence. For the sake of clarity, and since we need the corresponding arguments anyways, we will focus first on the case $k = 1$, ie the first homology systole, and explain the necessary extensions to handle the general case afterwards.

5.1 The singular set is nonempty

We start with a simple observation concerning the maximum of the curvature of a sequence of minimal surfaces in M with unbounded genus. It says, that for a sequence of minimal surfaces of unbounded genus $\Sigma_j \subset M$, we necessarily have $\mathscr{S} \neq \varnothing$. This works without any assumption on the systole.

Lemma 5.1 *Let $\Sigma_j \subset (M, g)$ be a sequence of closed, embedded minimal surfaces with $\chi(\Sigma_j) \to -\infty$. Then there is a sequence of points $z_j \in \Sigma_j$ such that $|A^{\Sigma_j}|^2(z_j) \to \infty$.*

Proof Assume that there is a constant $C > 0$, such that

$$\sup_j \sup_{\Sigma_j} |A^{\Sigma_j}|^2 \leq C. \tag{5.2}$$

By scaling we may for simplicity assume that the sectional curvature satisfies $|\sec(M)| \leq 1$. Thus, by minimality and the theorem of Gauss–Bonnet, the total curvature satisfies

$$\int_{\Sigma_j} |A^{\Sigma_j}|^2 \, d\mu_{\Sigma_j} = -2 \int_{\Sigma_j} (K^{\Sigma_j} - \sec(T_x\Sigma_j)) \, d\mu_{\Sigma_j}(x) \geq 4\pi|\chi(\Sigma_j)| - 2\operatorname{area}(\Sigma_j). \tag{5.3}$$

On the other hand we have

$$\int_{\Sigma_j} |A^{\Sigma_j}|^2 d\mu_{\Sigma_j} \le C\,\mathrm{area}(\Sigma_j) \tag{5.4}$$

by assumption. Combining (5.3) and (5.4), we obtain

$$4\pi|\chi(\Sigma_j)| \le (C+2)\,\mathrm{area}(\Sigma_j).$$

By assumption the left-hand side tends to infinity, therefore we find that

$$\mathrm{area}(\Sigma_j) \to \infty \quad \text{as } j \to \infty.$$

We consider the universal covering $\pi\colon \widetilde{M} \to M$, where $\widetilde{M}$ is compact by the Bonnet–Myers theorem. Clearly, the minimal surfaces

$$\hat{\Sigma}_j := \pi^{-1}(\Sigma_j)$$

also satisfy the pointwise curvature bound (5.2) and have diverging area,

$$\mathrm{area}(\hat{\Sigma}_j) \to \infty. \tag{5.5}$$

The pointwise curvature bound (5.2) allows us to pass to a subsequence (not relabeled) such that

$$\hat{\Sigma}_j \to \mathscr{L} \text{ in } C^{0,\alpha}(\widetilde{M}),$$

where $\mathscr{L}$ is a Lipschitz lamination, whose leaves are smooth, complete minimal surfaces. Moreover, since $\mathrm{area}(\Sigma_j) \to \infty$, we can conclude that there needs to be at least one leaf Γ with stable universal cover, which also implies that Γ is compact, hence diffeomorphic to S^2 thanks to [Fischer-Colbrie and Schoen 1980] and [Schoen and Yau 1983]. For the convenience of the reader we include the argument here following the proof of [Chodosh et al. 2017, Theorem 1.3].

By passing to another subsequence and using (5.5) we find that there has to be a point $p \in \Gamma$ such that

$$\liminf_{j\to\infty} \mathrm{area}(\hat{\Sigma}_j \cap B(p,r)) \to \infty \quad \text{for any } r > 0.$$

Since the $\hat{\Sigma}_j$ are embedded and by the curvature bound (5.2), this implies that for j sufficiently large and $r > 0$ sufficiently small (but only depending on the ambient geometry and the curvature bound), $\hat{\Sigma}_j \cap B(p,r)$ is given as the union of $n(j) \to +\infty$ graphical components over $\Gamma \cap B(p,r)$. Let $U \subset \widetilde{\Gamma}$ be a bounded and simply connected subset of the universal cover $\widetilde{\Gamma}$ of Γ with $\tilde{p} \in U$, where $\tilde{p}$ projects to p. Using the curvature bound, a covering argument and the standard elliptic theory we find that for j sufficiently large we can find at least two functions $v_{1,j}, v_{2,j}$ (out of the lifts of the $n(j)$ components above) defined on U such that the graphs define disjoint minimal surfaces over U and $\inf|v_{2,j}-v_{1,j}| \to 0$. Using the Harnack inequality we find that $w_j = \inf|v_{2,j}-v_{1,j}|^{-1}(v_{2,j}-v_{1,j})$ converges to a nontrivial signed solution of the Jacobi equation, hence U is stable. Since this applies to any such U it follows that $\widetilde{\Gamma}$ is stable.

It follows from [Fischer-Colbrie and Schoen 1980] and [Schoen and Yau 1983] that for any disk $D \subset \widetilde{\Gamma}$ and any $z \in D$ we have that

$$d(x, \partial D) \leq \frac{2\pi\sqrt{2}}{\sqrt{3\kappa_0}},$$

where $\mathrm{Scal} \geq \kappa_0 > 0$ on D. Since this applies to any such disk it follows that $\widetilde{\Gamma}$ is compact, hence a sphere. Since $\widetilde{M}$ is simply connected, it does not contain any embedded real projective plane. Therefore, we need to have $\widetilde{\Gamma} = \Gamma$. In particular, Γ is a closed, two-sided, stable minimal surface in $\widetilde{M}$, which gives the desired contradiction. □

Remark 5.6 Under the additional assumption that $\mathrm{sys}^h(\Sigma_j) \geq l_0 > 0$, we could have used Corollary 2.3 instead of the theorem of Gauss–Bonnet to obtain that $\mathrm{area}(\Sigma_j)$ has to be unbounded. However, this relies on the assumption on the systole and is less elementary. We will exploit such an argument below, in the proof of the existence of multiple pinching curves.

5.2 Localized systole and contractibility radius I

We now start to aim for Theorem 1.1 for $k = 1$, ie we show that there is at least one homologically nontrivial curve that becomes arbitrarily short. By Lemma 5.1, in order to prove Theorem 1.1 using a contradiction argument invoking a limit lamination, we are forced to study the structure of a limit lamination of $(\Sigma_j)_{j\in\mathbb{N}}$ in the presence of a nonempty singular set. In this subsection we use the global positivity of the Ricci curvature to rule out rather general neck-pinch singularities under appropriate assumptions.

We now fix $r_0 > 0$ sufficiently small such that, firstly, the results from Section 4 apply in any ball $B(x, r_0)$ and, secondly, all balls $B(x, r) \subset M$ with $r \leq r_0$ have strictly mean-convex boundary.

For an embedded closed surface $\Sigma \subset M$ and a point $x \in M$ we write

$$C(x,r) = C^\Sigma(x,r) = \{c\colon S^1 \to \Sigma \cap B(x,r) \mid 0 \neq [c] \in \pi_1(\Sigma \cap B(x,r))\}.$$

Note that $c \in C(x,r)$ could still be globally contractible in Σ. We also write

$$C(x) = C^\Sigma(x) = C(x, r_0).$$

At this point, recall that the maximum principle Lemma 2.9 says that if $C(x,r) = \varnothing$ for some $r \leq r_0$, then $C(x,s) = \varnothing$ for any $s \leq r$.

Definition 5.7 We call

$$c(\Sigma) = \inf_{x\in M} \sup\{r > 0 \mid \pi_1(\Sigma \cap B(x,r), x) = 0\}$$

the *contractibility radius* of Σ, and

$$\mathrm{sys}_r(\Sigma) = \inf_{x\in M} \inf_{c\in C(x,r)} \mathrm{length}(c)$$

the *r-local systole* of Σ.

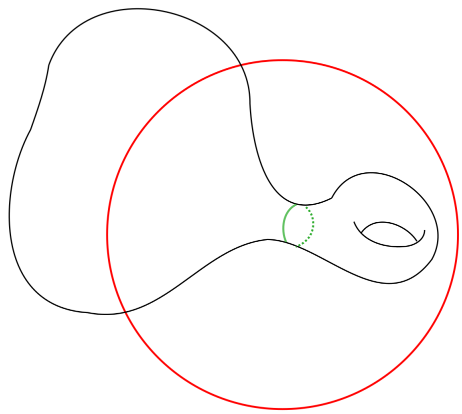

Figure 1: A surface with small r-local systole but not too small systole as the curve γ is globally contractible.

While the two definitions seem closely related, there is an important difference to be pointed out. Note that both of these are defined by looking at the intersection of Σ with extrinsic balls. However, the r-local systole still refers to intrinsic distances, ie we only localize extrinsically. Also observe that we have

$$\operatorname{sys}_r(\Sigma) = \inf_{x \in \Sigma} \operatorname{sys}(\Sigma \cap B(x,r)).$$

Note that both the contractibility radius and the r-local systole refer to extrinsic balls of radius r. One of the main challenges of our arguments is that in general there might be no strong connection between the systole (which is intrinsic) and the extrinsic r-local systole as indicated in Figure 1.

The goal of this section is to show that if $\Sigma \subset M$ is a minimal surface with homology systole bounded below by some constant l_0, then Σ is uniformly contractible on some potentially much smaller scale r_1 that depends on the ambient geometry and l_0 but not Σ otherwise; cf Proposition 5.21. Note that if $C(x) = \varnothing$ for any $x \in \Sigma$, then we automatically have that $c(\Sigma) \geq r_0$, where we recall that r_0 only depends on the ambient geometry. Similarly, in this case we have that $\operatorname{sys}_r(\Sigma) = \infty$ for any $r < r_0$. We are therefore mainly concerned with the case $C(x) \neq \varnothing$ for some $x \in M$.

The next lemma is our key scale-breaking argument indicated in Figure 2. Via the maximum principle we transfer some connectedness properties from the scale of certain singularities of a limit lamination to a definite scale. Given a very short and separating curve we show that both connected components of the complement have to extend a definite amount away.

Lemma 5.8 *Let $\Sigma \subset M$ be a closed minimal surface such that $\operatorname{sys}^h(\Sigma) \geq l_0$. There is an $l_1 = l_1(M, l_0) \leq \min(r_0/4, l_0/4)$ with the following property. Suppose that $\operatorname{sys}_{r_0}(\Sigma) \leq l_1$ and that $c \in C(x)$ is a simple closed curve for some $x \in M$ such that*

$$\operatorname{length}(c) \leq 2 \operatorname{sys}_{r_0}(\Sigma).$$

Then $\Sigma \setminus c$ has two connected components Σ_1 and Σ_2, and these satisfy

$$\Sigma_i \cap \partial B(x, r_0) \neq \varnothing \quad \text{for } i = 1, 2.$$

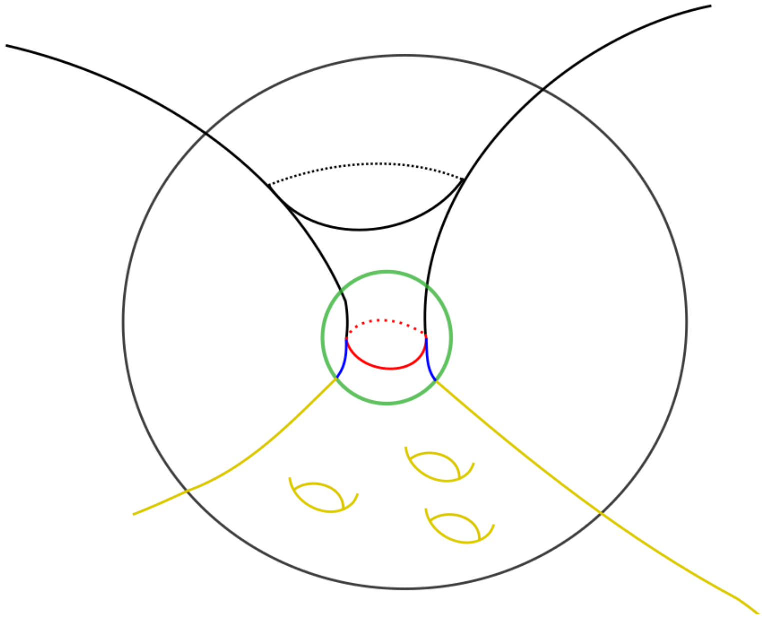

Figure 2: The proof of Lemma 5.8: Because $B(x,r)$ is mean convex for any $r \le r_0$ at least one component of $\Sigma \setminus c$ has to leave $B(x,r_0)$, say Σ_1. Once we can show that Σ_2 is forced to leave $B(x,9R_0)$ (still on the scale of c), the maximum principle gets us all the way to $\partial B(x,r_0)$.

Proof Write $R_0 = \text{length}(c)/8$ and also note that our assumptions on c, l_0 and l_1 imply that

$$\text{length}(c) \le 2l_1 \le \min\left(\tfrac{1}{2}r_0, \tfrac{1}{2}l_0\right)$$

thanks to our assumptions. Note the following two consequences of these choices.

Firstly, since the length of c is strictly below l_0, we find that c is homologically trivial. This means that $\Sigma \setminus c$ has two connected components, denoted by Σ_1 and Σ_2 with $\partial\Sigma_i = c$.

Secondly, note that since $x \in c$ we have that

(5.9) $$\partial\Sigma_i \subset B(x,4R_0) \subset B\left(x, \tfrac{1}{4}r_0\right).$$

We first show that these choices imply that there is no nontrivial topology on intrinsic scales below R_0. More precisely, we let $y \in \Sigma \cap B(x,r_0/2)$ and claim that there is a unique disk $D_y \subset \Sigma \cap B(x,r_0)$ with

$$B^{\Sigma}(y,R_0) \subset D_y \quad \text{and} \quad \partial D_y \subset \partial B^{\Sigma}(y,R_0).$$

This can be seen as follows. By Lemma 2.10, if there were a curve $\sigma \subset B^{\Sigma}(y,R_0)$ that is noncontractible on scale r_0 at x, we could find a simple closed curve $\sigma' \subset B^{\Sigma}(y,R_0)$ also noncontractible on scale r_0 at x, with

$$\text{length}(\sigma') \le 3R_0 < \tfrac{1}{2}\,\text{length}(c) \le \text{sys}_{r_0}(\Sigma)$$

by our choice of the curve c, but this is impossible by the definition of the r_0-local systole. We conclude that any simple closed curve contained in $B^{\Sigma}(y,R_0)$ admits a filling disk contained in $\Sigma \cap B(x,r_0)$, from which the existence of D_y follows. If Σ is not a sphere, it follows immediately that such a disk is unique. In the case of Σ being a sphere there are two such disks in Σ. However, by the choice of r_0 and the maximum principle, these disks cannot both be entirely contained in $B(x, r_0)$.

It follows from Theorem 4.1 and the convex hull property, that we can find some small $\alpha > 0$ such that

$$\Sigma \cap B(y, \alpha R_0) \text{ consists of disks for any } y \in B\big(x, \tfrac{1}{2} r_0\big).$$

Now choose $k \in \mathbb{N}$ such that $k\alpha \geq 9$, and let $\beta_k > 1$ be given by Theorem 4.12. First assume that we can find $z \in \Sigma_i$ such that

$$B^{\Sigma}(z, \beta_k \alpha R_0) \cap \partial \Sigma_i = \varnothing. \tag{5.10}$$

Also assume that

$$B^{\Sigma}(z, \beta_k \alpha R_0) \subset B\big(x, \tfrac{1}{2} r_0\big), \tag{5.11}$$

since the conclusion otherwise follows from the maximum principle thanks to (5.9). Under these assumptions it follows from Theorem 4.12 that

$$\partial((B^{\Sigma}(z, \beta_k \alpha R_0))_{z, 9R_0}) \subset \partial B(z, 9R_0),$$

which clearly implies that

$$\Sigma_i \cap \partial B(x, 9R_0) \neq \varnothing,$$

since Σ_i is connected. Since on the other hand $\partial \Sigma_i \subset B(x, 4R_0)$, we then find from the maximum principle that

$$\Sigma_i \cap \partial B(x, r_0) \neq \varnothing.$$

Note that to go from one scale to the other scale the maximum principle is applied on all balls of radii between the two scales.

We still need to justify why we can assume (5.10). Take l_1 such that $\beta_k \alpha R_0 \leq 16 \beta_k \alpha l_1 \leq \frac{1}{12} r_0$. If with these choices (5.10) fails for any $z \in \Sigma_i$, we then have that

$$\operatorname{diam}(\Sigma_i) \leq 32 \beta_k \alpha l_1 \leq \tfrac{1}{6} r_0.$$

Suppose first that Σ_i is a disk. In this case the diameter estimate implies that c is contractible on scale r_0, contradicting our choice of c.

If Σ_i is not a disk it contains at least one nonseparating curve d, since $\partial \Sigma_i$ is connected. Thanks to the diameter estimate, Corollary 2.11 then implies that we can find a nonseparating curve d' having

$$\operatorname{length}(d') \leq 48 \beta_k \alpha R_0 < l_0,$$

contradicting the assumptions. □

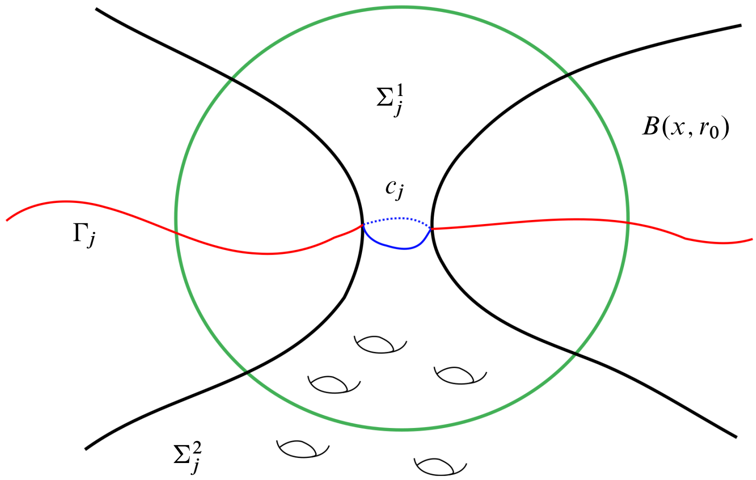

Figure 3: The construction in the proof of Lemma 5.12. The surface Σ_j is a good barrier for the Plateau problem: Both components of $\Sigma_j \setminus c_j$ extend out of $B(x, r_0)$ by Lemma 5.8.

Below, we solve a Plateau problem in $M \setminus \Sigma$ with boundary given by a curve c as above. In this situation, Lemma 5.8 implies that Σ is a useful barrier.

Lemma 5.12 *Given $l_0 > 0$ there is an $l_2 = l_2(M, l_0) > 0$ with the following property. Let $\Sigma \subset M$ be a closed minimal surface with $\mathrm{sys}^h(\Sigma) \geq l_0$. Then the r_0-local systole satisfies*

$$\mathrm{sys}_{r_0}(\Sigma) \geq l_2.$$

Note that this achieves two things simultaneously. Firstly, it shows that the systole is bounded away from zero if the homology systole is. Secondly, we also find that curves of controlled length which are very short, but potentially on a much smaller scale than the systole, can be contracted in an extrinsically controlled neighborhood.

This corresponds to the fact, already present in the argument for Lemma 5.8, that the proof handles two types of curves. On the one hand, applied to homologically trivial noncontractible curves, this implies that the homology systole of a sequence Σ_j tends to 0 if we can show that the systole does so. On the other hand, we will apply it to short curves bounding (large) disks in Σ_j in order to understand the convergence of Σ_j to a limit lamination.

Proof Let us first consider the case of M being simply connected. Afterwards we reduce the general case to this special case. We argue by contradiction and assume that we can find a sequence of minimal surfaces $(\Sigma_j)_{j \in \mathbb{N}}$ such that:

(1) All nonseparating curves in Σ_j have length at least l_0, ie $\mathrm{sys}^h(\Sigma_j) \geq l_0$.

(2) We have $\mathrm{sys}_{r_0}(\Sigma_j) \to 0$.

Up to taking a subsequence we then find $x \in M$, radii $r_j \to 0$, and simple closed curves $c_j \in C^{\Sigma_j}(x, r_j)$ such that

$$\mathrm{length}(c_j) \le 2\,\mathrm{sys}_{r_0}(\Sigma_j) \to 0.$$

Since M is simply connected, Σ_j separates M into two mean-convex connected components

$$M \setminus \Sigma_j = M_j^1 \cup M_j^2.$$

Clearly, once j is large enough such that $4\,\mathrm{sys}_{r_0}(\Sigma_j) \le l_0$, we have that c_j is null-homologous in the closure of both of these components.

In addition, we claim that at least one of M_j^1 and M_j^2 has the following property: If $\mathrm{length}(c_j) \le l_1$ from Lemma 5.8, then any minimal surface $S \subset M_j^i$ with $\partial S = c_j$ satisfies

$$S \cap \partial B(x, r_0) \ne \varnothing. \tag{5.13}$$

If this was not the case, we would find $S_j^1 \subset M_j^1 \cap B(x, r_0)$ and $S_j^2 \subset M_j^2 \cap B(x, r_0)$ such that $\partial S_j^i = c_j$. The surface $S_j = S_j^1 \cup S_j^2 \subset B(x, r_0)$ is a closed surface and separates $B(x, r_0)$ into two connected components. Moreover, (5.13) does not hold for S, so that one of these components is contained in $B(x, r_0 - \delta)$ for some small $\delta > 0$. By construction, this component contains a component of $\Sigma_j \setminus c_j$ contradicting Lemma 5.8.

Let M_j^1 be the component having property (5.13). By [Hardt and Simon 1979] we can find a stable minimal surface $\Gamma_j \subset M_j^1$ with $\partial\Gamma_j = c_j$ which minimizes area among all surfaces in M_j^1 which have boundary c_j. It follows from (5.13) that

$$\Gamma_j \cap \partial B(x, r_0) \ne \varnothing \tag{5.14}$$

for j sufficiently large. Moreover, by the curvature estimates [Schoen 1983], there is a constant C such that

$$\sup_j \sup_{\Gamma_j \cap (M \setminus B(x,r))} (r - r_j)^2 |A^{\Gamma_j}|^2 \le C$$

for any $r > r_j$, where r_j was defined such that $c_j \in C^{\Sigma_j}(x, r_j)$. In particular, we can pass to a subsequence such that

$$\Gamma_j \to \mathscr{L}$$

in $C^{0,\alpha}_{\mathrm{loc}}(M \setminus \{x\})$, where $\mathscr{L}$ is a minimal Lipschitz lamination. Since Γ_j is stable, the same argument as in [Chodosh et al. 2017, Lemma 4.1] implies[3] that the lamination $\mathscr{L}$ extends to a lamination $\widetilde{\mathscr{L}}$ across $\{x\}$.

We claim that also $\widetilde{\mathscr{L}}$ has stable leaves thanks to a standard argument using the log cut-off trick. The details are as follows. Let Γ be a leaf of $\widetilde{\mathscr{L}}$ passing through x. Let $r > 0$ such that on $B(x, r)$ we find

[3] If all leaves are two-sided this follows immediately from [Chodosh et al. 2017, Proposition D.3]. The argument in [Chodosh et al. 2017, Lemma 4.1] explains how this can be assumed by passing to the double cover.

Lipschitz coordinates for the lamination $\widetilde{\mathscr{L}}$. We write Γ_x for the component of $\Gamma \cap B(x,r)$ passing through x. Fix some $0 < R < \min(r,1)$ and denote by $\eta_R \colon \Gamma \to [0,1]$ the log cut-off function given by

$$\eta_R(y) = \begin{cases} 0 & \text{for } d(x,y) \leq R^2 \text{ and } y \in \Gamma_x, \\ 2 - \dfrac{\log(d(x,y))}{\log(R)} & \text{for } R^2 < d(x,y) \leq R \text{ and } y \in \Gamma_x, \\ 1 & \text{else,} \end{cases}$$

where d denotes the distance function in M. Moreover, for a smooth vector field X in the normal bundle of Γ, we write

$$L_\Gamma X = \Delta_\Gamma^N X + F(X)$$

for the stability operator of Γ. Here, F is a linear operator of order zero with smooth coefficients. The precise form of F is irrelevant to the argument.

Let X be a smooth vector field in the normal bundle of Γ with compact support and such that $|X|, |\nabla_\Gamma X| \in L^\infty(\Gamma)$. Note that Γ_x has bounded area by construction and that $\eta_R = 1$ on $\Gamma \setminus (\Gamma_x \cap B(x,R))$. Since X has compact support and the coefficients of F are locally bounded, it thus it follows from the dominated convergence theorem that

$$\textbf{(5.15)} \qquad \lim_{R \to 0} \int_\Gamma \langle F(\eta_R X), \eta_R X \rangle = \int_\Gamma \langle F(X), X \rangle.$$

We now turn to the gradient term. First, note that

$$\textbf{(5.16)} \qquad \int_\Gamma |\nabla_\Gamma X|^2 - \int_\Gamma |\nabla_\Gamma(\eta_R X)|^2 = \int_{\Gamma_x \cap B(x,R)} |\nabla_\Gamma X|^2 - \int_{\Gamma_x \cap B(x,R)} |\nabla_\Gamma(\eta_R X)|^2.$$

Since

$$\lim_{R \to 0} \int_{\Gamma_x \cap B(x,R^2)} |\nabla_\Gamma X|^2 = 0 = \int_{\Gamma_x \cap B(x,R^2)} |\nabla_\Gamma(\eta_R X)|^2,$$

thanks to dominated convergence once again, we are left with considering the contribution on the annulus $(B(x,R) \setminus B(x,R^2)) \cap \Gamma_x$.

From the dominated convergence theorem we have that

$$\lim_{R \to 0} \int_{B(x,R) \cap \Gamma_x} |\nabla_\Gamma X^2| = 0,$$

which when combined with $|\eta_R| \leq 1$ and $|\nabla_\Gamma(\eta_R X)|^2 \leq 2\eta_R^2 |\nabla_\Gamma X|^2 + 2|\nabla_\Gamma \eta_R|^2 |X|^2$ implies that we only have to estimate

$$\int_{(B(x,R) \setminus B(x,R^2)) \cap \Gamma_x} |X|^2 |\nabla_\Gamma \eta_R|^2.$$

To this end, we decompose into dyadic annuli via

$$\big(B(x,R) \setminus B(x,R^2)\big) \cap \Gamma_x = \bigcup_{i=1}^{\lceil \log(1/R) \rceil} A_i, \quad \text{where } A_i \subseteq \big(\bar{B}(x, e^i R^2) \setminus B(x, e^{i-1} R^2)\big) \cap \Gamma_x.$$

Note that since Γ_x has bounded area, we find from the monotonicity formula that

$$\text{area}(A_i) \le \text{area}(B(x, e^i R^2) \cap \Gamma_x) \le C e^{2i} R^4 \,\text{area}(\Gamma_x), \tag{5.17}$$

where C is a constant that depends on M and r. Also note that

$$|\nabla_\Gamma \eta_R| \le \frac{1}{e^{i-1} R^2 \log(1/R)} \quad \text{on } A_i. \tag{5.18}$$

We thus find from (5.17) and (5.18), also using that X is bounded, that

$$\int_{(\bar{B}(x,e^i R^2)\setminus B(x,e^{i-1}R^2))\cap\Gamma_x} |X^2||\nabla_\Gamma \eta_R|^2 \le \frac{C}{\log(1/R)^2} \sum_{i=1}^{\lceil \log(1/R)\rceil} \frac{e^{2i} R^4}{e^{2i-2} R^4} \le \frac{C}{\log(1/R)}.$$

We have thus shown that

$$\lim_{R\to 0} \int_\Gamma |\nabla_\Gamma(\eta_R X)|^2 = \int_\Gamma |\nabla_\Gamma X|^2. \tag{5.19}$$

Combining (5.15) and (5.19) and integrating by parts gives that

$$-\int_\Gamma \langle L_\Gamma X, X\rangle = \int_\Gamma (|\nabla_\Gamma X|^2 - \langle F(X), X\rangle) = \lim_{R\to 0} \int_\Gamma \big(|\nabla_\Gamma(\eta_R X)|^2 - \langle F(\eta_R X), \eta_R X\rangle\big) \ge 0,$$

since $\eta_R X$ has compact support in $\Gamma \setminus \{x\}$ which is stable. This proves that Γ is stable.

From (5.14), we find that there is a leaf $\bar{\Gamma} \subset \widetilde{\mathscr{L}}$ with

$$\bar{\Gamma} \cap \partial B(x, r_0) \ne \varnothing.$$

In particular, $\bar{\Gamma}$ is nonempty. Moreover, invoking [Fischer-Colbrie and Schoen 1980] and [Schoen and Yau 1983] once again, $\bar{\Gamma}$ is closed. Thus, since M is simply connected, we find that $\bar{\Gamma}$ is two-sided. Since M has positive Ricci curvature, this is a contradiction since $\bar{\Gamma}$ is a nonempty, two-sided, closed, stable minimal surface in M.

We now consider the general case in which we can assume that M is not simply connected. We can pass to the universal covering $\pi : \widetilde{M} \to M$, which is compact by the Bonnet–Myers theorem. In particular, there is a finite group G acting freely on M such that $M = \widetilde{M}/G$. We obtain minimal surfaces

$$\widehat{\Sigma}_j = \pi^{-1}(\Sigma_j) \subset \widetilde{M}.$$

Since M has positive Ricci curvature, by the Frankel property, the surfaces $\widehat{\Sigma}_j$ are connected.

We may assume that r_0 is chosen sufficiently small that

$$g(B(x, r_0)) \cap B(x, r_0) = \varnothing$$

for any $g \in G \setminus \{e\}$. If there is a noncontractible curve $c_j \subset \Sigma_j \cap B(x, r_0)$, with

$$\text{length}(c_j) \le l_0,$$

we may again assume that c_j is chosen to have properties (1) and (2) from above. It follows from our assumption that c_j is separating. Therefore, by Lemma 2.12, also $\hat{c}_j := \pi^{-1}(c_j)$ is separating. Moreover, by the choice of r_0, and recalling $l_0 \le r_0$, we see that $\hat{c}_j$ consists of $|G|$ disjoint, closed curves. We can now argue exactly as above and minimize area in the correct component of $\widetilde{M} \setminus \widehat{\Sigma}_j$ relative to the boundary $\hat{c}_j$. Finally, by Lemma 5.8,[4] the limit lamination will be nonempty and we can conclude as in the first case. □

Remark 5.20 For curves that are noncontractible in $\Sigma \cap B(x,r)$ but contractible in Σ, it should be possible to extend Lemma 5.8 to bumpy metrics of positive scalar curvature. In this situation one component of $\Sigma_j \setminus c_j$ is a planar domain and one can write large parts of this component as a graph over Γ_j. This can then be used to construct a nontrivial Jacobi field on Γ.

Proposition 5.21 *For any $l_0 > 0$ there is an $r_1 > 0$ such that for any closed minimal surface $\Sigma \subset M$ with $\operatorname{sys}^h(\Sigma) \ge l_0$, we have for the contractibility radius that $c(\Sigma) \ge r_1$.*

Proof If we apply Lemma 5.12 to Σ we get some $l_2 > 0$ such that all curves in Σ of length at most l_2 are contractible in the intersection of Σ with some mean-convex ball $B(x, r_0)$. In particular, it follows from Lemma 2.10 that any intrinsic ball $B^{\Sigma}(z, l_2/3)$ is contained in some disk D_z with

$$B^{\Sigma}\big(z, \tfrac{1}{3}l_2\big) \subset D_z \subset \Sigma_j \cap B(z, r_0).$$

The claim now follows with $r_1 = \frac{1}{3}\alpha l_2$ from Theorem 4.1, where also $\alpha > 0$ is from Theorem 4.1. □

5.3 The first homology systole

At this stage we are in a position to prove the special case $k = 1$ of our main result.

Proof of Theorem 1.1 for $k = 1$ We argue by contradiction and assume that we have a sequence of minimal surfaces $\Sigma_j \subset M$ with $-\chi(\Sigma_j) \to \infty$ and

$$\operatorname{sys}^h(\Sigma_j) \ge l_0 > 0$$

for some positive constant l_0. Thanks to Proposition 5.21 we find that the sequence (Σ_j) is ULSC, ie

(5.22) $\Sigma_j \cap B(x, r_1)$ consists of disks for any $x \in M$.

Clearly, after potentially decreasing r_1, property (5.22) holds for the surfaces $\widehat{\Sigma}_j \subset \widetilde{M}$ as well. Therefore, it suffices to derive a contradiction from (5.22) if M is simply connected.

Thanks to (5.22) and [White 2015] (see also [Colding and Minicozzi 2015] which gives Lipschitz curves), we can pass to a subsequence such that

$$\Sigma_j \to \mathscr{L} \quad \text{in } M \setminus \mathscr{S}$$

outside the singular set $\mathscr{S}$ which is contained in a union of C^1-curves. It follows from Lemma 5.1, that $\mathscr{S} \neq \varnothing$. In particular, we can pick $x \in \mathscr{S}$ and the associated collapsed leaf Γ_x. Moreover, since Γ_x is

[4] We apply this to Σ_j and observe that this trivially implies (5.14) for $\widehat{\Sigma}_j$.

a limit leaf of $\mathscr{L}$ it is stable by [Meeks et al. 2010]. It follows from Proposition 3.3 that Γ_x extends to a complete minimal surface $\overline{\Gamma}$ in M and that $\mathscr{S} \cap \overline{\Gamma}$ is discrete. In particular, $\overline{\Gamma}$ is also stable and by [Fischer-Colbrie and Schoen 1980] and [Schoen and Yau 1983], its universal cover is diffeomorphic to S^2. Since M is simply connected, it does not contain any one-sided surfaces and we conclude that $\overline{\Gamma}$ is a two-sided, closed, stable minimal surface in M. This is clearly a contradiction, since M has positive Ricci curvature. □

6 Existence of multiple short curves

We now proceed to the proof of the general case of Theorem 1.1.

Recall that we assume M to be a closed three-manifold with positive Ricci curvature. Assume we have a sequence of minimal surfaces $(\Sigma_j)_{j\in\mathbb{N}}$ in M with the following properties. There is a natural number $k \geq 2$ and for each $j \in \mathbb{N}$ a set $\{c_j^1, \dots, c_j^{k-1}\}$ of simple closed curves in Σ_j such that

(1) $\operatorname{length}(c_j^i) \to 0$ for $i = 1, \dots, k-1$ as $j \to \infty$,

(2) $\operatorname{rank}\langle [c_j^1], \dots, [c_j^{k-1}] \rangle = k-1$ in $H_1(\Sigma_j; \mathbb{Z}/2\mathbb{Z})$,

(3) there is an $l_0 > 0$ such that if a closed curve $d_j \subset \Sigma_j$ has $\operatorname{length}(d_j) \leq l_0$, then $[d_j] \in \langle [c_j^1], \dots, [c_j^{k-1}] \rangle$.

Note that (3) allows for $[d_j] = 0$.

By taking a subsequence we may assume that $c_j^i \subset B(x_i, s_j)$ for a sequence of radii $s_j \to 0$.

We now follow the same steps that we used for the case of the first homology systole, but have to deal with several new difficulties.

6.1 Additional points in the singular set

In a first step we show that the singular points arising from the curves c_j^i do not comprise the entire singular set. This is the analogue of Lemma 5.1. In contrast to Lemma 5.1 the argument in this case relies on the assumption on the homology systole.

Lemma 6.1 *We have $\mathscr{S} \cap M \setminus \bigcup_{i=1}^{k-1} B(x_i, r_3) \neq \varnothing$ for some $r_3 > 0$.*

Proof Assume that $\mathscr{S} \subset \{x_1, \dots, x_{k-1}\}$. By Corollary 2.3, we can assume that $\operatorname{area}(\Sigma_j)$ is unbounded. For simplicity, let us scale M to have $|\sec| \leq 1$, and write $B_s = \bigcup_{i=1}^{k-1} B(x_i, s)$. The monotonicity formula then implies

$$\begin{aligned}\operatorname{area}(\Sigma_j \cap (B_{2r_3} \setminus B_{r_3})) &= \operatorname{area}(\Sigma_j \cap B_{2r_3}) - \operatorname{area}(\Sigma_j \cap B_{r_3}) \\ &\geq \left(\frac{4}{e^{2r_3}} - 1\right) \operatorname{area}(\Sigma_j \cap B_{r_3}) \geq \operatorname{area}(\Sigma_j \cap B_{r_3})\end{aligned}$$

if $r_3 \leq \log(2)/2$, which in turn implies

$$\textbf{(6.2)} \qquad 2\operatorname{area}(\Sigma_j \setminus B_{r_3}) \geq \operatorname{area}(\Sigma_j \setminus B_{r_3}) + \operatorname{area}(\Sigma_j \cap (B_{2r_3} \setminus B_{r_3})) \geq \operatorname{area}(\Sigma_j) \to \infty.$$

Now we can argue exactly as in Lemma 5.1 and obtain a limit lamination $\mathscr{L} \subset M \setminus \mathscr{S}$. Thanks to (6.2) we can conclude that $\mathscr{L}$ has a leaf with stable universal cover. We then use stability to extend it across the isolated singularities $\mathscr{S}$ and eventually use the log cut-off trick to conclude that this is still stable, which gives the desired contradiction. □

6.2 Localized systole and contractibility radius II

In a next step we prove that Σ_j is ULSC off the set $\{x_1, \dots x_{k-1}\}$.

Proposition 6.3 *Assume $(\Sigma_j)_{j\in\mathbb{N}}$ is as above. Given $r > 0$ there is an $r_2 = r_2(M, g, l_0, r)$ such that the contractibility radius satisfies $c\big(\Sigma_j \cap \big(M \setminus \bigcup_{i=1}^{k-1} B(x_i, 4r)\big)\big) \geq r_2$ for j sufficiently large.*

We want to follow the same strategy that we used to obtain Proposition 5.21, for which in turn Lemma 5.8 was the key input. Because of the short curves c_j^i, we need to be more careful in how we select the scale on which we work. Recall that in the case of Lemma 5.8 this was the smallest intrinsic scale of nontrivial topology. It turns out that there are two cases to consider in the more general case, depending on whether a potentially contradicting curve is separating or not. The instance of separating curves is more delicate, and we introduce some notation here related to this case. In order to find the correct scale, we define functions $l_j, f_j : \Sigma_j \to [0, \infty)$ as follows. For $x \in \Sigma_j$, we consider the set C_j' of curves in Σ_j given by

$$C_j'(x) := \{c : S_1 \to \Sigma_j \mid 0 \neq [c] \in \pi_1(\Sigma_j \cap B(x, r_0), x), 0 = [c] \in H_1(\Sigma_j; \mathbb{Z}/2\mathbb{Z})\}.$$

Note that we only take into account separating curves here. Then the first function is defined via

$$l_j(x) := \min\big\{1, \inf\{\text{length}(c) \mid c \in C_j'(x)\}\big\},$$

and f_j is a scale-invariant version of (the inverse of) this, incorporating the distance to the short curves c_j^i, given by

$$f_j(x) = l_j(x)^{-1} \operatorname{dist}(x, c_j^1 \cup \dots \cup c_j^{k-1}).$$

Proof of Proposition 6.3 We argue by contradiction and assume that we can find a simple closed curve $d_j \subset \Sigma_j$ such that

$$\text{length}(d_j) \to 0 \tag{6.4}$$

and

$$d_j \subset M \setminus \bigcup_{i=1}^{k-1} B(x_i, 2r), \tag{6.5}$$

but

$$d_j \text{ is noncontractible on scale } r_0. \tag{6.6}$$

If we cannot find such a curve, the assertion follows from Theorem 4.1 combined with Lemma 2.10 and the convex hull property exactly as in the proof of Proposition 5.21.

Up to taking a subsequence, by (6.4) and (6.5) we can assume that

$$d_j \to y \in M \setminus \bigcup_{i=1}^{k-1} B(x_i, 2r).$$

Observe that (6.4) combined with the assumption (3) implies that $[d_j] \in \langle [c_j^1], \dots, [c_j^{k-1}] \rangle$ for j sufficiently large, which we simply assume to be the case from here on.

We have to distinguish the following two cases:

(a) The curve d_j is nonseparating.

(b) The curve d_j is separating, ie $[d_j] = 0$.

We start with case (a). In this case it follows from Lemma 2.5 that

$$\Sigma_j \cap \left(M \setminus \bigcup_{i=1}^{k-1} B(x_i, s_j) \right) = \Sigma_j^1 \cup \Sigma_j^2,$$

where now Σ_j^i are connected, disjoint minimal surfaces with

$$\partial \Sigma_j^i \subset d_j \cup \bigcup_{i=1}^{k-1} \partial B(x_i, s_j).$$

Since d_j is nonseparating in Σ_j, it follows immediately that

$$\Sigma_j^i \cap \partial B(y, r_0) \neq \varnothing \tag{6.7}$$

holds for $i = 1, 2$ and for j sufficiently large. By the same arguments as in the proof of Lemma 5.12 we may assume that M is simply connected. We now want to minimize area with boundary d_j in $M \setminus \bigcup_{i=1}^{k-1} B(x_i, s_j)$ instead of all of M. In order to do so we first slightly modify the metric near $\bigcup_{i=1}^{k-1} \partial B(x_i, s_j)$ to obtain a mean-convex domain. Using a partition of unity we may simply choose a metric g_j on $M \setminus B(x_i, s_j)$ that agrees with the original metric outside of $\bigcup_{i=1}^{k-1} B(x_i, 2s_j)$ and has mean-convex boundary. We can now solve the Plateau problem as before in $\left(M \setminus \bigcup_{i=1}^{k-1} B(x_i, s_j), g_j\right)$ with prescribed boundary d_j. After passing to a subsequential limit we find a nonempty (thanks to (6.7)) limit lamination in $(M \setminus \{x_1, \dots, x_{k-1}\}, g)$. By stability, the limit lamination extends also across the set $\{x_1, \dots, x_{k-1}\}$ and we can argue as in the proof of Lemma 5.12.

For the remaining case (b), we prove the stronger assertion that f_j is uniformly bounded. This handles case (b) as follows. If $f_j \le C$, then for $x \in M \setminus \bigcup_{i=1}^{k-1} B(x_i, 2r)$, we find that

$$l_j(x) \ge C^{-1} \operatorname{dist}(x, c_j^1 \cup \dots \cup c_j^{k-1}) \ge C^{-1} r$$

for j sufficiently large, which contradicts (6.4)–(6.6).

In order to show that f_j is uniformly bounded, we argue by contradiction and assume that

$$\liminf_{j \to \infty} \sup_{\Sigma_j} f_j \to \infty, \tag{6.8}$$

which we simply assume to be a full limit after taking another subsequence. Note that $f_j \leq C_j$ for some constant $C_j > 0$, since Σ_j is a smooth and closed surface; therefore, we can pick[5] $z_j \in \Sigma_j$ such that

$$2 f_j(z_j) \geq \sup_{\Sigma_j} f_j .$$

The assumption (6.8) implies that there is a loop $e_j \in C'_j(z_j)$ based at z_j that is noncontractible on scale r_0 such that

$$\text{length}(e_j) \leq o(\text{dist}(z_j, c_j^1 \cup \cdots \cup c_j^{k-1})). \tag{6.9}$$

We can assume that any other loop $e'_j \in C'_j(z_j)$ has

$$\text{length}(e_j) \leq 2\,\text{length}(e'_j), \tag{6.10}$$

since we could otherwise replace our original loop e_j by an even shorter one satisfying all other assumptions.

For $z \in \Sigma_j \cap B(z_j, 2\,\text{length}(e_j))$, we find from (6.9) that

$$\text{dist}(z, c_j^1 \cup \cdots \cup c_j^{k-1}) \geq \text{dist}(z_j, c_j^1 \cup \cdots \cup c_j^{k-1}) - 2\,\text{length}(e_j) \geq \tfrac{1}{2}\,\text{dist}(z_j, c_j^1 \cup \cdots \cup c_j^{k-1})$$

for j sufficiently large. Therefore, by the choice of z_j, we have

$$4 l_j(z) \geq l_j(z_j) \tag{6.11}$$

for any $z \in \Sigma_j \cap B(z_j, 2\,\text{length}(e_j))$.

Since e_j is separating (recall that all curves in C'_j are separating by definition) we can write

$$\Sigma \setminus e_j = \Sigma_j^1 \cup \Sigma_j^2$$

for connected minimal surfaces Σ_j^i with boundary e_j. We claim that

$$\Sigma_j^i \cap \partial B(z_j, r_0) \neq \varnothing \quad \text{for } i = 1, 2. \tag{6.12}$$

For ease of notation, we prove (6.12) for Σ_j^1; the argument for Σ_j^2 is analogous.

We again distinguish two cases. In the first case we assume that there is a simple closed curve $g_j \subset \Sigma_j^1$ with $0 \neq [g_j] \in \langle [c_j^1], \dots, [c_j^{k-1}] \rangle$. This case follows for homological reasons: We can pick a closed curve $h_j \subset \Sigma_j$ that intersects g_j exactly once. Since e_j is separating and $g_j \subset \Sigma_j^1$, we can even choose h_j with $h_j \subset \Sigma_j^1$. But then h_j has to intersect at least one of the curves c_j^l, which implies that

$$\Sigma_j^1 \cap B(x_l, s_j) \neq \varnothing.$$

Thanks to (6.9) this implies that

$$\Sigma_j^1 \cap \partial B(z_j, \text{length}(e_j)) \neq \varnothing.$$

Since $\partial \Sigma_j^1 = e_j \subset B(z_j, \text{length}(e_j)/2)$, we obtain (6.12) from the convex hull property applied to Σ_j^1.

[5]This is the standard selection procedure for such scales adapted to our situation.

In the remaining case we have that if $g_j \subset \Sigma_j^1$ is a simple closed curve with $\mathrm{length}(g_j) \le l_0$, then $[g_j] = 0$. Moreover, the bound (6.11) combined with the choice (6.10) then implies that any simple closed curve $g_j \subset \Sigma_j^1 \cap B(z_j, 2\,\mathrm{length}(e_j))$ with $\mathrm{length}(g_j) \le \mathrm{length}(e_j)/4$ is contractible on scale $2\,\mathrm{length}(e_j)$ at z_j. At this point we are (for $\Sigma_j^1 \cap B(z_j, 2\,\mathrm{length}(e_j))$) in the setting of Lemma 5.8, and can simply repeat the same argument to obtain

$$\Sigma_j^1 \cap \partial B(z_j, 2\,\mathrm{length}(e_j)) \neq \varnothing,$$

which in turn implies (6.12) by the convex hull property using that $\partial \Sigma_j^1 \subset B(z_j, \mathrm{length}(e_j))$.

We can now once again argue as in the proof of Lemma 5.12 and conclude the proposition. □

6.3 Proof of the main result

We now give the proof for the general case of our main result.

Proof of Theorem 1.1 For $\pi : \widetilde{M} \to M$ the universal covering, consider the surfaces $\widehat{\Sigma}_j = \pi^{-1}(\Sigma_j)$ and write $\mathscr{X} = \pi^{-1}(\{x_1, \dots, x_{k-1}\})$. We can pass to a subsequential limit

$$\widehat{\Sigma}_j \to \mathscr{L} \quad \text{in } C^{0,\alpha}_{\mathrm{loc}}(\widetilde{M} \setminus \mathscr{S}),$$

where clearly $\mathscr{X} \subset \mathscr{S}$. It follows from Proposition 6.3 that the surfaces are ULSC away from $\mathscr{X}$. Moreover, thanks to Lemma 6.1, we can find a collapsed leaf $\Gamma \subset \mathscr{L}$, which extends across $\mathscr{S} \setminus \mathscr{X}$ by Proposition 3.3. Moreover, since this is stable, it also extends across the isolated points $\mathscr{X}$ to a complete, stable minimal surface, which implies a contradiction as in the case of the first homology systole. □

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