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We say that a tropical subvariety $V \subset \mathbb{R}^n$ is B-realizable if it can be lifted to an analytic subset of $(\Lambda^*)^n$. When V is a smooth curve or hypersurface, there always exists a Lagrangian submanifold lift $L_V \subset (\mathbb{C}^*)^n$. We prove that whenever L_V has well-defined Floer cohomology, we can find for each point of V a Lagrangian torus brane whose Lagrangian intersection Floer cohomology with L_V is nonvanishing. Assuming an appropriate homological mirror symmetry result holds for toric varieties, it follows that whenever L_V is a Lagrangian submanifold that can be made unobstructed by a bounding cochain, the tropical subvariety V is B-realizable.

As an application, we show that the Lagrangian lift of a genus-0 tropical curve is unobstructed, thereby giving a purely symplectic argument for Nishinou and Siebert's proof that genus-0 tropical curves are B-realizable. We also prove that tropical curves inside tropical abelian surfaces are B-realizable.

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1 Introduction

Mirror symmetry is a collection of equivalences between symplectic geometry (A-model) and algebraic geometry (B-model) on a pair of mirror spaces. A general proposal for constructing mirror pairs of a symplectic space X_A and algebraic space X_B comes from Strominger, Yau, and Zaslow [56], who conjectured that mirror pairs can be presented as dual torus fibrations over an integral affine manifold Q. One relation between these spaces arises in the form of Kontsevich's homological mirror symmetry (HMS) conjecture [36], which predicts an equivalence between the Fukaya category of X_A and the category of coherent sheaves on a mirror manifold X_B. Roughly, the objects of the Fukaya category of X_A are Lagrangian submanifolds $L \subset X_A$. A blueprint for mirror symmetry is that Lagrangian submanifolds of X_A relate to sheaves supported on a subvariety of X_B via mutual comparison to tropical subvarieties on the base Q.

We consider the relatively well-understood example of $X_A = T^*\mathbb{R}^n / T^*_{\mathbb{Z}}\mathbb{R}^n$, $X_B = (\Lambda^*)^n$, and $Q = \mathbb{R}^n$. On the A side, it will be convenient for us to identify X_A with $(\mathbb{C}^*)^n$, which has holomorphic coordinates

$x_i = e^{q_i + i\theta_i}$ and standard symplectic form $\sum_{i=1}^n dq_i \wedge d\theta_i$. Note that X_A does not naturally come with a complex structure. On the B-side, we take Λ to be the Novikov field

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\, \lambda \in \mathbb{R},\, \lim_{i\to\infty} \lambda_i = \infty \right\},$$

whose valuation map $\operatorname{val}\colon \Lambda \to \mathbb{R} \cup \{\infty\}$ is the smallest exponent appearing the expansion of $\sum_{i=0}^{\infty} a_i T^{\lambda_i}$. On the A-side, the torus fibration is given by

$$\pi_A\colon X_A \to Q, \quad (x_1, \dots, x_n) \mapsto (\log|x_1|, \dots, \log|x_n|) = (q_1, \dots, q_n),$$

whose fibers are Lagrangian tori. The dual fibration $\operatorname{TropB}\colon X_B \to Q$ is given by taking coordinatewise valuation

$$\operatorname{TropB}\colon X_B \to Q, \quad (z_1, \dots, z_n) \mapsto (\operatorname{val}(z_1), \dots, \operatorname{val}(z_n)).$$

Instead of using tropical geometry as an intuition for HMS, we use HMS and our understanding of the tropical-to-A correspondence to study the tropical-to-B correspondence. We now review these correspondences before stating our results.

Tropical-to-*B* correspondence The tropical-to-complex correspondence and its applications to enumerative geometry have been a particularly rich field of study since the pioneering work of Mikhalkin [41], which related counts of tropical curves in $\mathbb{R}^2$ to counts of curves in the *complex* algebraic torus (as opposed to the Λ analytic torus we study). This relation consists of two parts: *tropicalization*, which associates to a holomorphic curve in $(\mathbb{C}^*)^2$ a tropical curve in $\mathbb{R}^2$, and *realization*, which lifts every tropical curve $V \subset \mathbb{R}^2$ to a holomorphic curve in $(\mathbb{C}^*)^2$. Both of these constructions have been extended to greater generality; we provide a coarse overview of the constructions here:

- ***B*-Tropicalization** The tropicalization map associates to a closed analytic subset $Y \subset X_B$ its tropicalization $\operatorname{TropB}(Y) \subset Q$. The expectation (which holds for algebraic subvarieties by work of Bieri and Groves [11]) is that the tropicalization is a *tropical subvariety* (Definition 2.2.1).
- ***B*-Realization** Starting with $V \subset Q$ a tropical subvariety, we say that V is *B-realizable* if there exists a closed analytic subset $Y \subset X_B$ with $\operatorname{TropB}(Y) = V$.

One goal of tropical geometry is to determine which tropical subvarieties $V \subset Q$ are B-realizable. Examples such as that of Mikhalkin [40, Example 5.12] show that there exist tropical curves $V \subset \mathbb{R}^n$ for $n > 2$ that are nonrealizable. In some cases, there are criteria determining if a tropical subvariety is B-realizable.

For example, if $V \subset Q$ is a tropical hypersurface, then there exists a tropical polynomial (piecewise integral affine convex function) $f\colon Q \to \mathbb{R}$ such that V is the locus of points where f is nonaffine. The function f is called a tropical polynomial as it can be written using the tropical sum and product operations:

$$\oplus\colon (\mathbb{R} \cup \{\infty\}) \times (\mathbb{R} \cup \{\infty\}) \to (\mathbb{R} \cup \{\infty\}), \quad q_1 \oplus q_2 = \min(q_1, q_2),$$
$$\odot\colon (\mathbb{R} \cup \{\infty\}) \times (\mathbb{R} \cup \{\infty\}) \to (\mathbb{R} \cup \{\infty\}), \quad q_1 \odot q_2 = q_1 + q_2.$$

Let $f = \bigoplus_{\alpha\in\mathbb{N}^n} a_\alpha \odot q^{\odot\alpha}$ be a tropical polynomial whose nonaffine locus is V. Let Λ be a complete non-Archimedean valued field, and let $X_B = (\Lambda^*)^n$ be the algebraic torus. For each a_α, select a coefficient $c_\alpha \in \Lambda$ whose valuation is $\operatorname{val}(c_\alpha) = a_\alpha$. Then the zero set of the polynomial $\sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha$ defines a subvariety of X_B which is the B-realization of V. Note that this construction does not produce a unique lift.

The other examples where we have B-realization criteria are tropical curves. In [45], Nishinou and Siebert showed that if $V \subset Q$ is a trivalent tropical curve of genus 0, then V is realizable. This was extended to all balanced maps from trees by Ranganathan [50]. In higher genus, the space of deformations of a tropical curve may have higher dimension than the expected dimension of a possible B-realization. In this case, we say that the tropical curve is superabundant; see Mikhalkin [41]. We expect that a generically chosen superabundant curve is not B-realizable. It is known that all 3-valent nonsuperabundant curves are realizable; see Cheung, Fantini, Park, and Ulirsch [14]. In the superabundant setting, Speyer [55, Theorem 3.4] established that if $V \subset Q$ is a tropical curve of genus 1 and satisfies a condition called *well-spacedness*, then V is realizable.

Tropical-to-A correspondence The tropical-to-Lagrangian correspondence is a more recent construction, independently arrived at by Hicks [30], Mak and Ruddat [38], Matessi [39], and Mikhalkin [42]. Each of the papers associates to a (certain type of) tropical subvariety $V \subset Q$ a Lagrangian submanifold $L_V^\varepsilon \subset X_A$ whose projection to the base of the Lagrangian torus fibration $\pi_A(L_V^\varepsilon)$ is contained within an ϵ-neighborhood of the tropical subvariety V. We call this a geometric Lagrangian lift of V. When V is a hypersurface, [30] proves that under homological mirror symmetry $L_V^\varepsilon \subset (\mathbb{C}^*)^n$ is identified with a sheaf whose support is a hypersurface $Y \subset (\Lambda^*)^n$.

In contrast to B-realization, the constructions in [30; 38; 39; 42] can construct a geometric Lagrangian lift L_V of *any* smooth tropical curve $V \subset Q$. This difference occurs because the map $\pi_A\colon X_A \to Q$ does not provide a good tropicalization map. For example, for any subset $U \subset Q$ and $\varepsilon > 0$ there exists a Lagrangian submanifold $L \subset X_A$ with the property that the Hausdorff distance between $\pi_A(L)$ and U is less than ε. Additionally, it would be desirable to have a tropicalization map that only depends on the Hamiltonian isotopy class of the Lagrangian submanifold — and $\pi_A(L)$ can change substantially when we apply a Hamiltonian isotopy to L.

To obtain a correspondence from Lagrangian submanifolds to tropical subsets of Q, and justify why the Lagrangian L_V is the "correct" A-model realization of a tropical curve V, one needs to employ techniques from Floer theory. Not all Lagrangian submanifolds are amenable to such analysis. We call a Lagrangian submanifold unobstructed if its filtered A_∞ algebra $\operatorname{CF}^\bullet(L)$ admits a bounding cochain. The pair (L, b) of Lagrangian submanifold equipped with a bounding cochain is called a Lagrangian brane. Examples of unobstructed Lagrangian submanifolds include those which bound no pseudoholomorphic disks for a given choice of almost complex structure. In particular, if L is exact, it is unobstructed.

If (L_1, b_1) and (L_2, b_2) are Lagrangian branes then there exists a cochain complex $\operatorname{CF}^\bullet((L_1, b_1), (L_2, b_2))$ generated by the intersections of L_1 and L_2, and whose cohomology groups $\operatorname{HF}^\bullet((L_1, b_1), (L_2, b_2))$ are invariant under Hamiltonian isotopies of either L_1 or L_2. We can use this to define A-tropicalization and A-realization.

- **A-Tropicalization** Starting with the fibration $X_A \to Q$ and a Lagrangian brane $(L, b) \subset X_A$, we define the A-tropicalization
$$\mathrm{TropA}(L, b) := \{q \in Q \mid \exists (F_q, \nabla) \text{ with } \mathrm{HF}^\bullet((L, b), (F_q, \nabla)) \neq 0\},$$
where $F_q = \pi_A^{-1}(q)$ is equipped with a unitary local system ∇, and $\mathrm{HF}^\bullet((L, b), (F_q, \nabla))$ is the Lagrangian intersection Floer cohomology of (L, b) with F_q deformed by the local system ∇. An advantage of $\mathrm{TropA}(L, b)$ over $\pi_A(L)$ is that the former depends only on the Hamiltonian isotopy class of L.

- **A-Realizability** In light of the definition of A-tropicalization, we say that $V \subset Q$ is A-realizable if there exists a Lagrangian brane $(L, b) \subset X_A$ with $\mathrm{TropA}(L) = V$.

The Lagrangian submanifold L_V^ε associated to V provides a geometric candidate for an A-realization of V. However, to verify A-realizability, one still needs to check that L_V^ε is unobstructed with bounding cochain b and that $\mathrm{TropA}(L_V, b) = V$. We call this last condition *faithfulness*.

1.1 Results

The three components (geometric realizability, unobstructedness, and faithfulness) of the A-realizability problem and its implications for the B-realizability problem in $Q = \mathbb{R}^n$ are summarized in Figure 1.

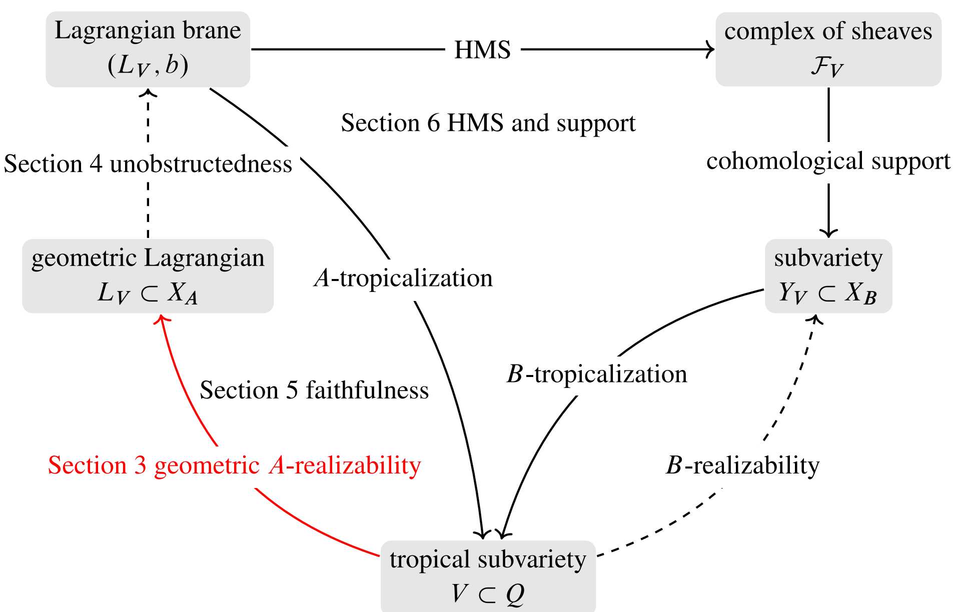

Figure 1

The correspondences given by solid black lines always exist. Geometric A-realizability (the solid red arrow) is only known to exist for certain examples of tropical subvarieties of Q. For the applications we consider (smooth tropical hypersurfaces and curves) we always have geometric A-realizability. We conjecture that every tropical subvariety of Q is geometrically A-realizable. For any given tropical subvariety $V \subset Q$, there is no reason for either of the dashed arrows to hold. However, the following conjecture seems natural:

Conjecture 1.1.1 *Let $V \subset \mathbb{R}^n$ be a tropical curve. Then a geometric Lagrangian lift L_V is unobstructed if and only if V is B-realizable.*

The main step in proving the forward direction of the conjecture is to establish the faithfulness of the Lagrangian brane lift, that is showing that $\mathrm{TropA}((L_V, b)) = V$. Our primary result is to prove faithfulness (for *all* tropical subvarieties admitting unobstructed Lagrangian lifts).

Theorem A (restatement of Lemma 5.2.2) *Let $V \subset Q$ be a tropical subvariety. Let (L_V^ε, b) be a Lagrangian brane lift of V. Then $\mathrm{TropA}(L_V^\varepsilon, b) = V$.*

When we can apply homological mirror symmetry, we obtain the forward direction of Conjecture 1.1.1. Depending on the affine manifold Q and Lagrangian L_V, we may require Assumption 6.1.2, which states that the family Floer construction of Abouzaid [6] extends to the noncompact and unobstructed setting.

Theorem B (restatement of Corollary 6.2.1) *Suppose Assumption 6.1.2. Let $V \subset \mathbb{R}^n$ be a tropical subvariety. Suppose there exists $(L_V, b) \subset (\mathbb{C}^*)^n$ a Lagrangian brane lift of V. Then V is B-realizable.*

Our second goal is to show that this can be used to produce realizability criteria. We first recover a theorem of Nishinou and Siebert [45]:

Corollary C (restatement of Corollary 4.3.3) *Suppose Assumption 6.1.2. Every smooth genus-0 tropical curve $V \subset \mathbb{R}^n$ has a Lagrangian brane lift (L_V, b) and is, therefore, B-realizable.*

The results of Nishinou [44] give necessary and sufficient conditions for when a tropical curve can be realized by a family of algebraic curves in a degenerating family of complex tori. In contrast to those results, our results show that every 3-valent tropical curve can be realized by a closed analytic subset. The following result *does not* assume Assumption 6.1.2:

Corollary D (restatement of Corollary 6.2.4) *Let $Q = T^2$ be a tropical abelian surface. Let $V \subset Q$ be a 3-valent tropical curve. V has a Lagrangian brane lift $(L_V, 0)$, and is, therefore, B-realizable.*

In summary, we can recover B-realizability results using unobstructedness for the first five cases in Table 1. We also provide some insight into the existence of holomorphic curves with boundary on tropical Lagrangian submanifolds.

V and Q	A-model (unobstructedness)	B-model (realizability)	HMS status
curves in abelian surfaces	Corollary 6.2.4	[44][1]	✓
curves in $\mathbb{R}^2$	[31]	[41]	$(*)$
hypersurfaces of $\mathbb{R}^n$	[31]	folklore	$(*)+(**)$
hypersurfaces in abelian varieties	Corollary 6.2.3	—	$(**)$
genus-0 curves in $\mathbb{R}^n$	Theorem C	[45]	$(*)+(**)$
compact genus-0 curves in $\dim(Q)=3$	[38]	[45]	—
well-spaced genus-1 curves	spec. in Section 6.4	[55]	$(*)+(**)$

Table 1: Relating A-unobstructedness to B-realizability. Here $(*)$ and $(**)$ refer to the needed extensions of family Floer cohomology (Assumption 6.1.2) to the noncompact and nontautologically unobstructed settings.

Example E (restatement of Example 6.3.2) Let $V_c \subset \mathbb{R}^3$ be a generic tropical line. The Lagrangian L_{V_c} is unobstructed, but not tautologically unobstructed.

Outline In Section 2, we give a toy computation that explores the entire roadmap above for a simple example, $V_{\text{pants}} \subset \mathbb{R}^2$, the tropical pair of pants. In addition to providing context for the remainder of the paper, the computation reviews some background for tropical geometry and symplectic geometry. We also use this section to fix notation. It is our hope that this section will be accessible to both tropical and symplectic geometers.

Section 3 discusses the geometric lifting problem. Definition 3.0.2 specifies when a family of Lagrangian submanifolds L_V^ε is a geometric Lagrangian lift of a tropical subvariety V. We show that Definition 3.0.2 distinguishes tropical subvarieties among all polyhedral complexes as the ones which permit geometric Lagrangian lifts. Definition 3.0.2 requires that geometric Lagrangian lifts are monomially admissible, graded, and spin. In Sections 3.1–3.3, we show that known constructions of geometric Lagrangian lifts of tropical subvarieties of Hicks [30], Matessi [39], and Mikhalkin [42] satisfy these conditions. We also prove Lemma 3.3.1, which shows that for smooth genus-0 tropical curves, the map $H^2(L_V) \to H^2(\partial L_V)$ is an injection.

Section 4 investigates Lagrangian submanifolds which can be unobstructed by a bounding cochain (Lagrangian branes). We provide a brief overview of the pearly model for Lagrangian submanifolds in Section 4.1. This is followed by examples of unobstructed geometric Lagrangian lifts (Lagrangian brane lifts) of tropical subvarieties in Section 4.2 (summarized in Table 1). Section 4.3 gives a new method for checking unobstructedness of Lagrangian submanifolds inside noncompact symplectic spaces which have a potential function $W : X_A \to \mathbb{C}$ (see Definition A.0.1).

[1]The realization result of Nishinou and Siebert [45] considers B-tropicalizations coming from degenerating families of abelian surfaces so that a tropical curve is realized by a parametrized algebraic curve. The B-realization we take is by closed analytic subsets. In the setting of genus-0 stable tropical curves in toric varieties, these tropicalizations can be related by Ranganathan [50].

Theorem F (restatement of Theorem 4.3.1) *Let $W: X_A \to \mathbb{C}$ be a symplectic fibration outside of a compact set of $\mathbb{C}$. Let $L \subset X_A$ be a W-admissible Lagrangian submanifold with boundary $M \subset W^{-1}(t)$ for $t \in \mathbb{R}_{\gg 0}$. Suppose M is a tautologically unobstructed Lagrangian submanifold of $W^{-1}(t)$, and the connecting map $H^1(M) \to H^2(L, M)$ is surjective. Then there exists a bounding cochain b such that (L, b) is a Lagrangian brane.*

The proof uses a lemma on filtered A_∞ algebras (Lemma B.2.8). Since we have previously proven in Lemma 3.3.1 that the geometric Lagrangian lifts L_V of smooth genus-0 tropical curves satisfy the criterion of Theorem 4.3.1, we obtain that such L_V are unobstructed (Corollary 4.3.3).

In Section 5, we prove faithfulness (Lemma 5.2.2), which shows that the A-tropicalization (Floer-theoretic support) of a Lagrangian brane lift L_V is V. The proof uses that the Lagrangian intersection Floer cohomology between (L_V, b) and F_q is a deformation of the cohomology of a subtorus of F_q. An application of Lemma B.3.1 shows that this can be "undeformed" by a bounding cochain, so that $\mathrm{HF}^0((L, \nabla_0, b_0), (F_q, \nabla, b)) = \Lambda$.

Section 6 applies the previous constructions to address questions of realizability for tropical subvarieties. Abouzaid [5, Remark 1.1] states that we expect that the family Floer functor can be adapted to include unobstructed Lagrangians. We instead use Assumption 6.1.2 — the weaker assumption that the family Floer construction of Abouzaid [6] can be employed for unobstructed Lagrangian submanifolds in the Lagrangian torus fibration $(\mathbb{C}^*)^n \to \mathbb{R}^n$ to construct a sheaf on the mirror space. We give a brief outline of the modifications to [6] which would be required to prove Assumption 6.1.2. With this assumption, we prove the forward direction of Conjecture 1.1.1 in Corollary 6.2.1. We also discuss the first five cases in Table 1.

Finally, we discuss evidence towards the reverse direction of Conjecture 1.1.1. This requires us to understand some of the holomorphic disks which appear on Lagrangian lifts of tropical subvarieties. In Example 6.3.2, we show that the lift of the tropical line in $\mathbb{R}^3$ bounds a holomorphic disk whose symplectic area is dictated by the internal edge length on the tropical line. We also discuss applications of B nonrealizability to obstructedness in Section 6.4. We consider the superabundant tropical elliptic curve $V \subset \mathbb{R}^3$ of Mikhalkin [40, Example 5.12] and provide a sketch for how Speyer's well-spacedness criterion might be recovered from holomorphic disk counts on L_V. Section 6.5 looks at how to relate tropical line bundles on tropical curves to Lagrangian isotopies of their geometric Lagrangian lifts. We conjecture a relation between superabundance of a tropical curve V and the relative ranks of $\mathrm{HF}(L_V, b)$ and $H(L_V)$ (wide versus nonwide).

We provide some auxiliary results in the appendices. Appendix A discusses how to adapt the pearly model of Floer cohomology of Charest and Woodward [12] to the setting of noncompact spaces equipped with a potential $W: X \to \mathbb{C}$. In Appendix B, we prove the results on filtered A_∞ algebras used in this paper.

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2 A guided calculation to the support of the Lagrangian pair of pants

This section contains an expository computation that is designed to frame the main ideas of the paper, provide background, and fix notation. The exposition here is not intended to be comprehensive, although we hope that through explicit examples, direct computations, and additional references, we've made this section accessible to both the tropical and symplectic geometry communities. As a result, the materials outside of Sections 2.5 and 2.5 are expository. As we will frequently use notation from Examples 2.4.3 and 2.4.4, we suggest that the readers take a look at these computations of Lagrangian intersection Floer cohomology for conormal bundles in the cotangent bundle of the torus.

2.1 A-model, B-model, and Lagrangian torus fibrations

We provide a high-level overview of the viewpoint of [26; 56] on mirror symmetry. Let Q be an integral affine manifold, that is, a manifold equipped with a choice of integrable full-rank lattice $T_\mathbb{Z} Q \subset TQ$. This identifies a dual lattice $T^*_\mathbb{Z} Q \subset T^*Q$, and also a flat connection on TQ. There are three kinds of geometries that we may associate with Q: symplectic geometry, complex geometry, and tropical geometry.

A-model A symplectic manifold is a $2n$-manifold X_A with a choice of 2-form $\omega \in \Omega^2(X_A)$ which is closed ($d\omega = 0$) and nondegenerate ($\omega^n \neq 0$). The submanifolds of interest for us in X_A are Lagrangian submanifolds $L \subset X_A$, which are n-dimensional submanifolds on which the symplectic form vanishes ($\omega|_L = 0$). For any manifold Q, the cotangent bundle T^*Q (whose local coordinates are (q, p)) carries a canonical symplectic form $\sum_{i=1}^n dq_i \wedge dp_i$. This descends to a symplectic form on the quotient $X_A := T^*Q/T^*_\mathbb{Z} Q$.

Given an integral affine submanifold $\underline{V} \subset Q$ such that $T_\mathbb{Z} \underline{V} \subset T_\mathbb{Z} Q$, the periodized conormal bundle $L_{\underline{V}} := N^*\underline{V}/N^*_\mathbb{Z}\underline{V} \subset X_A$ is an example of a Lagrangian submanifold. The simplest example is when we pick a point $q \in Q$ such that

$$L_{\underline{q}} = N^*q/N^*_\mathbb{Z} q = T^*_q Q/T^*_{q,\mathbb{Z}} Q \tag{1}$$

is a Lagrangian torus of X_A. We will call this Lagrangian torus $F_q \subset X_A$. For this reason, we call the projection $\pi_A\colon X_A \to Q$ a Lagrangian torus fibration.

B-model We can also build an almost-complex manifold from the data of Q. An almost complex structure on $X_B^{\mathbb{C}}$ is an endomorphism $J\colon TX_B^{\mathbb{C}} \to TX_B^{\mathbb{C}}$ which squares to $-\mathrm{id}$. The submanifolds of interest in the B-model are the *almost-complex submanifolds* $Y^{\mathbb{C}} \subset X_B^{\mathbb{C}}$ whose tangent spaces are fixed under the almost complex structure, so that $J(T_yY^{\mathbb{C}}) = T_yY^{\mathbb{C}}$.

As Q is integral affine, there exists a connection on TQ whose flat sections are locally constant sections of $T_{\mathbb{Z}}Q$. This provides a splitting $T(TQ) = T_qQ \oplus \ker(\pi)$. We define an almost complex structure on TQ which interchanges the components of this splitting with a sign:

$$J := \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}.$$

The almost complex structure on TQ descends to an almost complex structure on $X_B^{\mathbb{C}} := T^*Q/T^*_{\mathbb{Z}}Q$; the fibers of $\pi_B\colon X_B^{\mathbb{C}} \to Q$ are real tori.

Given an integral affine submanifold $\underline{V} \subset Q$, the periodized tangent bundle

$$Y_{\underline{V}}^{\mathbb{C}} := T\underline{V}/T_{\mathbb{Z}}\underline{V} \subset X_B^{\mathbb{C}}$$

is an example of an almost-complex submanifold. If we start with $q \subset Q$ a point, we see that $Y_q \subset X_B^{\mathbb{C}}$ is a point of $X_B^{\mathbb{C}}$.

Mirror symmetry from Lagrangian torus fibrations We now describe in more detail the relationship between the Lagrangian tori of X_A and the points of $X_B^{\mathbb{C}}$. First, we note that for fixed $q \in Q$, there are a torus worth of points z in $X_B^{\mathbb{C}}$ with the property that $\pi_B(z) = q$.

In contrast to the complex lift, there is only one Lagrangian torus $F_q \subset X_A$ with $\pi_A(F_q) = \{q\}$. To get a matching family of Lagrangian lifts to our complex lift, we consider Lagrangian tori equipped with the additional data of a local system. Let (F_q, ∇) be a pair consisting of a Lagrangian torus F_q and a choice of $U(1)$ local system on F_q. Then there is a bijection between pairs $(F_q, \nabla) \subset X_A$ and points $z \in X_B^{\mathbb{C}}$. A similar story holds for the Lagrangian and complex lifts of integral affine subspace $\underline{V} \subset Q$.

To generalize beyond the submanifolds $\underline{V}$, $L_{\underline{V}}$, and $Y_{\underline{V}}$ discussed above, we need to look at tropical geometry.

Notation 2.1.1 Unless otherwise stated, we only consider $Q = \mathbb{R}^n$, so that $X_A = (\mathbb{C}^*)^n$ and $X_B^{\mathbb{C}} = (\mathbb{C}^*)^n$.

2.2 A quick introduction to tropical geometry and B-tropicalization

A convex polyhedral domain is the intersection of finitely many closed half-spaces in $\mathbb{R}^n$,

$$\underline{V} = \{q \in Q \mid \langle q, \vec{v}_i \rangle \geq \lambda_i\},$$

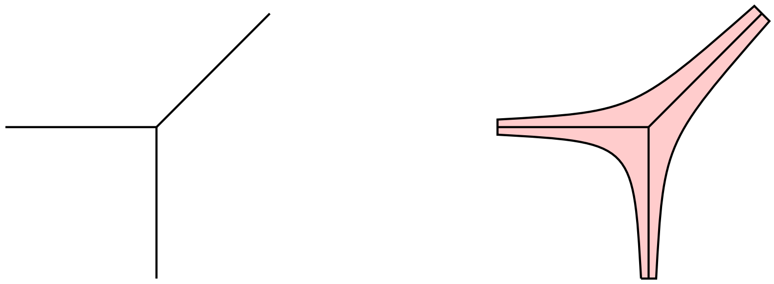

Figure 2: The tropical pair of pants (left) approximates the Amoeba of a curve (sketched on the right).

where $\vec{v}_i$ is a collection of vectors in $\mathbb{R}^n$, and λ_i is some set of constants in $\mathbb{R}$. We say that this is a rational convex domain if $\vec{v}_i \in \mathbb{Z}^n$ for all i, equivalently if there is a full lattice $T_{\mathbb{Z}}\underline{V} \subset T\underline{V}$ which is a sublattice of $T_{\mathbb{Z}}Q$. A tropical subvariety is built out of these pieces.

Definition 2.2.1 A k-dimensional tropical subvariety $V \subset Q$ is a collection of k-dimensional rational convex polyhedral domains $\{\underline{V}_s \subset Q\}$ and weights $\{w_s \in \mathbb{N}\}$ which are required to satisfy the following conditions:

- **Polyhedral complex condition** At each pair of rational convex polyhedral domains, the intersection $\underline{V}_s \cap \underline{V}_t$ is either empty, or a boundary facet of both $\underline{V}_s$ and $\underline{V}_t$.
- **Balancing condition** At facets $\underline{W} \subset \underline{V}_s$, let $\underline{V}_1, \dots, \underline{V}_k$ be the rational polyhedral domains containing $\underline{W}$. Consider lattices $T_{\mathbb{Z}}\underline{W}$, each of which is a sublattice of $T_{\mathbb{Z}}\underline{V}_i$ for each $i \in \{1, \dots, k\}$. Select for each i a vector $\vec{v}_i \in T_{\mathbb{Z}}\underline{V}_i$ such that $T_{\mathbb{Z}}\underline{V}_i = T_{\mathbb{Z}}\underline{W} \oplus \langle \vec{v}_i \rangle$ as oriented lattices. We require that
$$\sum_i w_i \vec{v}_i \equiv 0 \in T_{\mathbb{Z}}Q / T_{\mathbb{Z}}\underline{W}.$$

Example 2.2.2 Consider the polyhedral domains in $Q = \mathbb{R}^2$
$$\underline{V}_1 = \{(-t, 0) \mid t \in \mathbb{R}_{\geq 0}\}, \quad \underline{V}_2 = \{(0, -t) \mid t \in \mathbb{R}_{\geq 0}\}, \quad \underline{V}_3 = \{(t, t) \mid t \in \mathbb{R}_{\geq 0}\}.$$
As the directions $\langle -1, 0\rangle$, $\langle 0, -1\rangle$, and $\langle 1, 1\rangle$ sum to zero this is balanced and gives us a tropical curve. The collection of these three polyhedral domains is called the *standard tropical pair of pants*. The curve $V_{\text{pants}} \subset \mathbb{R}^2$ is drawn in Figure 2, left.

We say that a tropical curve $V \subset \mathbb{R}^n$ is *smooth* if every 0-dimensional stratum is locally modeled after the pair of pants.

Notation 2.2.3 Given $V \subset Q$ a tropical subvariety, we will use $V^{(0)}$ to denote the union of the interiors of the $\underline{V}_s$, and $V^{(1)}$ to denote the union of the interiors of the boundaries of the $\underline{V}_s$; more generally we will use $V^{(i)}$ to denote the codimension-i linearity strata of V. For any $\underline{W} \subset V^{(i)}$, let $\operatorname{star}(\underline{W})$ be the set of all strata which contain $\underline{W}$. If V is a tropical curve, we will usually call the strata vertices and edges, and use v and w for vertices and e and f for edges.

2.3 B-tropicalization

B-tropicalization is the process of taking a subvariety of $X_B^{\mathbb{C}}$ and obtaining a tropical subvariety of Q. The first approach one considers is the image of $Y^{\mathbb{C}} \subset X_B^{\mathbb{C}}$ under the B-torus fibration

$$\pi_B \colon X_B^{\mathbb{C}} \to Q.$$

Under good conditions, $\pi_B(Y^{\mathbb{C}}) \subset Q$ approximates a tropical subvariety of Y; see for instance [40]. The image $\pi_B(Y^{\mathbb{C}})$ is called the *amoeba* of $Y^{\mathbb{C}}$, which computationally can be checked to approach the tropical curve (see Figure 2, right).

To obtain a theory where the tropicalization of a subvariety is a tropical subvariety, we look to non-Archimedean geometry. Let Λ be the Novikov field. Given M a rank-n lattice, denote by X_B the torus $\operatorname{Spec} \Lambda[M]$. The points of X_B can be identified with n-tuples of invertible elements of Λ, so we will frequently write

$$X_B = \{(z_1, \dots, z_n) \mid z_i \in \Lambda^*\}.$$

We build a tropicalization map by taking the valuation coordinatewise:

$$\operatorname{TropB} \colon X_B \to M \otimes \mathbb{R} = Q, \quad (z_1, \dots, z_n) \mapsto (\operatorname{val}(z_1), \dots, \operatorname{val}(z_n)).$$

Given a $Y \subset X_B$ a closed analytic subset, we call the image $\operatorname{TropA}(Y) \subset Q$ its tropicalization.

Example 2.3.1 Consider $M = \mathbb{R}^2$, and the closed analytic subset $Y \subset (\Lambda^*)^2$ given by

$$Y = \{(z_1, z_2) \mid 1 + z_1 + z_2 = 0\}.$$

We compute the valuation of such a point $(z_1, z_2) \in Y$. Since

$$\operatorname{val}(1 + z_1 + z_2) \geq \min(\operatorname{val}(1), \operatorname{val}(z_1), \operatorname{val}(z_2)),$$

with equality holding whenever the valuations differ, we obtain that for all $(z_1, z_2) \in Y$ at least one of the following equalities hold:

$$\operatorname{val}(z_1) = \operatorname{val}(z_2), \quad \operatorname{val}(z_1) = \operatorname{val}(1), \quad \operatorname{val}(z_2) = \operatorname{val}(1).$$

This means that the image of $\operatorname{TropB}(Y)$ agrees with $V_{\text{pants}} \subset \mathbb{R}^2$ from Example 2.2.2. It follows that V is B-realizable.

This phenomenon holds much more broadly:

Theorem 2.3.2 [11; 27] *Let $Y \subset X_B$ be an irreducible k-dimensional analytic subset. Then* $\operatorname{TropB}(Y)$ *is a k-dimensional polyhedral complex.*

It is expected that when Y is an irreducible k-dimensional analytic subset, $\operatorname{TropB}(Y)$ is a k-dimensional tropical subvariety. To our knowledge, this result has not appeared in the literature. A discussion on the current status of tropicalization for analytic subsets is included in [53, Section 5.3].

2.4 Floer cohomology and A-tropicalization

The definition of the A-tropicalization of a Lagrangian submanifold requires a little more exposition because we wish to do some computations of the A-tropicalization. Our goal is to replace the Lagrangian torus fibration map $\pi_A\colon X_A \to Q$ with a correspondence of subsets

$$\text{TropA}\colon \{\text{Lagrangian branes}\} \to \{\text{subsets of } Q\}$$

which only depends on the Hamiltonian isotopy class of the Lagrangian brane.

2.4.1 Lagrangian intersection Floer cohomology Our main computational tool will be Lagrangian intersection Floer cohomology. We first equip a symplectic manifold (X, ω) with an ω-compatible choice of almost complex structure J.

Definition 2.4.1 [19] Suppose we have a pair of transversely intersecting Lagrangian submanifolds $L_0, L_1 \subset X$ and choice of almost complex structure J such that

(i) X, L_1, and L_2 are compact,

(ii) the symplectic area of all disks with boundary on L_i vanish $\omega(\pi_2(X, L_i)) = 0$,

(iii) the Lagrangians L_i are equipped with spin structures,

(iv) the Lagrangians L_i are graded (in the sense of [52]),

(v) the moduli spaces of J-holomorphic strips in (2) are regular.

Then the Lagrangian intersection Floer cohomology is a chain complex where:

- The generators are the points of intersection between L_0 and L_1, so that as a vector space

$$\text{CF}^\bullet(L_0, L_1) := \bigoplus_{x \in L_0 \cap L_1} \Lambda_x,$$

where Λ is the *Novikov field*. The grading $\deg(x)$ of an intersection point $x \in L_0 \cap L_1$ is determined by the Maslov index.

- The differential on this complex is defined by a count of holomorphic strips with boundary on $L_0 \cup L_1$ and ends limiting to the intersection points. Namely, let $x_\pm \in L_0 \cap L_1$ be two intersection points, and $\beta \in H^2(X, L_0 \cup L_1)$. Let $\mathcal{M}_\beta(L_0, L_1, x_+, x_-)$ denote the moduli space

$$\left\{ u\colon \mathbb{R}_s \times [0,1]_t \to X_A \;\middle|\; \begin{matrix} u(s,0) \in L_0,\ u(s,1) \in L_1,\ \lim_{s \to \pm\infty} u(s,t) = x_\pm, \\ \bar\partial_J u = 0,\ [u] = \beta \in H_2(X_A, L_0 \cup L_1) \end{matrix} \right\} \Big/ (s \mapsto s + c) \tag{2}$$

of holomorphic strips with ends limiting to $x^\pm$ in the relative homology class β, up to reparametrization of the strip along the s-coordinate. Using the grading data on L_0 and L_1, one can compute that

$$\dim(\mathcal{M}_\beta(L_0, L_1, x_+, x_-)) = \deg(x_-) - \deg(x_+) - 1.$$

The spin structures on L_0 and L_1 provide orientations for the spaces $\mathcal{M}_\beta(L_0, L_1, x_+, x_-)$; in particular if $\deg(x_+) + 1 = \deg(x_-)$, then $\dim(\mathcal{M}_\beta(L_0, L_1, x_+, x_-)) = 0$ and we can count the points in this moduli space with signs. The structure coefficients of the differential $d\colon \mathrm{CF}^\bullet(L_0, L_1) \to \mathrm{CF}^\bullet(L_0, L_1)$ are obtained by counting the elements in $\mathcal{M}_\beta(L_0, L_1, x_+, x_-)$,

$$\langle d(x_+), x_-\rangle = \sum_{\beta\in H^2(X, L_0\cup L_1)} T^{\omega(\beta)} \#\mathcal{M}_\beta(L_0, L_1, x_+, x_-),$$

where # is the signed count of points with orientation and $T^{\omega(\beta)}$ records the symplectic area of the strip u whose homology class is β.

The proof that this is a chain complex proceeds in a similar method to Morse theory. Because of (i), one can use Gromov compactness to prove that the 1-dimensional moduli spaces of strips have compactifications whose boundaries are given by products of the 0-dimensional moduli spaces of strips

$$\partial\mathcal{M}_\beta(L_0, L_1, x_+, x_-) = \bigsqcup_{x_0\in L_0\cap L_1} \mathcal{M}_\beta(L_0, L_1, x_+, x_0) \times \mathcal{M}_\beta(L_0, L_1, x_0, x_-).$$

To ensure that the only broken configurations which show up in the compactification are given by strips breaking (as opposed to disk bubbling), we use (ii), which states that there are no holomorphic disks with boundary on either L_0 or L_1. The compactification is compatible with the orientations given to the moduli spaces of holomorphic strips. Since the signed count of boundary components of a 1-dimensional manifold is zero, $\langle d^2(x_+), x_-\rangle = 0$. Unless otherwise stated, all Lagrangians we consider will be $\mathbb{Z}$-graded and spin. A major feature of Lagrangian intersection Floer cohomology is its invariance under Hamiltonian isotopy.

Theorem 2.4.2 [19] *Let L_0 and L_1 be Lagrangian submanifolds of (X, ω) satisfying (i)–(v). Let $\phi\colon X \to X$ be a Hamiltonian isotopy. Suppose that L_0 and L_1 intersect transversely and we've picked ϕ in such a way that $\phi(L_0)$ and L_1 intersect transversely. Then* $\mathrm{HF}^\bullet(L_0, L_1) = \mathrm{HF}^\bullet(\phi(L_0), L_1)$.

For this reason, whenever L_0 and L_1 do not intersect transversely, we can compute their Floer cohomology by taking a Hamiltonian perturbation which makes their intersection transverse; the resulting cohomology groups are independent of the choice of perturbation taken. One can similarly show that it does not depend on the choice of an almost complex structure.

The conditions (i) and (ii) can be weakened. For example, (i) — which is required to prove that the moduli spaces of strips admit compactifications — can be replaced with the weaker condition of monomial admissibility (Definition 3.1.1) or W-admissibility (Definition A.0.1). Later we will look at weakening (ii) to *unobstructedness* (Section 4). We now drop (i) and compute the Lagrangian intersection Floer cohomology between two Lagrangians in a cotangent bundle. The computation we give is a direct generalization of [54, Example 3.1].

Example 2.4.3 (running example) Let $F_0 = T^n$ be the n-dimensional torus. Let $T^{n-k} \subset F_0$ be the subtorus spanning the first $n-k$ coordinates on T^n. Then T^*F_0 is an example of an exact symplectic manifold. The zero section F_0 and the conormal bundle N^*T^{n-k} are examples of exact Lagrangian submanifolds. Lagrangian intersection Floer cohomology requires that our Lagrangians intersect transversely, so we will apply a Hamiltonian perturbation to one of the Lagrangians to achieve transverse intersections. Pick $\lambda_0 \in \mathbb{R}_{>0}$. Consider the Hamiltonian function

$$H = \sum_{i=1}^{n-k} \lambda_0 \cos(\theta_i) \tag{3}$$

on T^*F_0. Let $\phi\colon T^*F_0 \to T^*F_0$ be the time-1 Hamiltonian flow of H. The resulting intersections of $\phi(N^*T^{n-k})$ with F_0 are the points

$$\phi(N^*T^{n-k}) \cap F_0 = \{(a_1\pi, \dots, a_{n-k}\pi, 0, \dots, 0) \mid a_i \in \{0,1\}\},$$

and the index of each intersection point x is given by $\deg(x) = \sum_{i=1}^{n-k} a_i$. We will call the corresponding generators of Floer cohomology x_I, where $a_i = 1$ whenever $i \in I \subset \{1, \dots, n-k\}$. Write $I \lessdot J$ if $I = J \cup \{x_i\}$ for some i. As a vector space $\mathrm{CF}^\bullet(\phi(N^*T^{n-k}), F_0)$ matches $\mathrm{CM}^\bullet(T^{n-k})$ for the Morse function H.

The differential on $\mathrm{CF}^\bullet(\phi(N^*T^{n-k}), F_q)$ is related to the Morse differential. Let $|I| + 1 = |J|$, meaning that $\deg(x_I)$ and $\deg(x_J)$ differ by one. If I and J differ at more than two elements, then $\mathcal{M}_\beta(\phi(N^*T^{n-k}), F_0, x_I, x_J)$ has nonzero dimension. If $I \lessdot J$ differ at a single element j, then there are exactly two holomorphic strips traveling between x_I and x_J,

$$\mathcal{M}(\phi(N^*T^{n-k}), F_0, x_I, x_J) = \{u^+_{I \lessdot J}, u^-_{I \lessdot J}\},$$

which as points receive opposite orientations. By our choice of perturbation, the symplectic areas of the strips $u^-_{I \lessdot J}$ and $u^-_{I \lessdot J}$ agree (and are exactly λ_0). Therefore

$$\langle d(x_I), x_J \rangle = \begin{cases} T^{\omega(u^+_{I \lessdot J})} - T^{\omega(u^-_{I \lessdot J})} = 0 & \text{if } I \lessdot J, \\ 0 & \text{otherwise,} \end{cases}$$

and we conclude that

$$\mathrm{HF}^\bullet(\phi(N^*T^{n-k}), F_q) = \Lambda\langle x_I \rangle = \bigwedge_{i \in \{1,\dots,n-k\}} \Lambda\langle x_i \rangle.$$

The example relates to the discussion of tropicalization as T^*F_0 can be identified with $X_A = (\mathbb{C}^*)^n = T^*Q/T^*_{\mathbb{Z}}Q$. If T^{n-k} is a linear subtorus of F_q, it corresponds to an $(n-k)$-dimensional subspace of $\tilde{T}^{n-k} \subset T^*_0 Q$; let $\underline{V} \subset T_0 Q$ correspond to the set of vectors which are annihilated by $\tilde{T}^{n-k}$. By abuse of notation, we use $\underline{V}$ to denote the integral affine subspace of Q with prescribed tangent space at 0. Under this identification, $N^*T^{n-k} \subset T^*F_0$ is $L_{\underline{V}} \subset X_A$. Using that Lagrangian intersection Floer cohomology is invariant under symplectomorphisms, and noting that if $q \notin \underline{V}$ then $F_q \cap L_{\underline{V}} = \varnothing$, we have computed

$$\mathrm{HF}^\bullet(L_{\underline{V}}, F_q) = \begin{cases} \bigwedge_{i \in \{1,\dots,n-k\}} \Lambda\langle x_i \rangle & \text{if } q \in \underline{V}, \\ 0 & \text{if } q \notin \underline{V}. \end{cases}$$

2.4.2 Local systems Recall that the points of X_B are in bijection with pairs (F_q, ∇) of Lagrangian torus fibers equipped with local systems. We now discuss how to incorporate this data into Lagrangian intersection Floer cohomology. The *unitary Novikov elements*

$$U_\Lambda := \left\{ a_0 + \sum_{i=1}^{\infty} a_i T^{\lambda_i} \;\middle|\; \lim_{i\to\infty} \lambda_i = \infty, \lambda_i > 0, a_0 \in \mathbb{C}^*, a_i \in \mathbb{C} \right\}$$

are those elements whose nonzero lowest-order term is a constant. We now consider (L_i, ∇_i), which are Lagrangian submanifolds with the additional choice of a trivial Λ-line bundle E_i and a U_Λ local system ∇_i. Given L_0 and L_1 which intersect transversely satisfying (i)–(v) we define $\mathrm{CF}^\bullet((L_0, \nabla_0), (L_1, \nabla_1))$ to be the chain complex where:

- The underlying vector space is $\bigoplus_{x \in L_0 \cap L_1} \hom((E_0)_x, (E_1)_x)$.
- The differential is given by taking a ∇_i-weighted count of the holomorphic strips with boundary in $L_0 \cup L_1$. More precisely, let $\partial_i u$ be the boundary of u contained in L_i, and let $P_\gamma^{\nabla_i} : (E_i)_{\gamma(0)} \to (E_i)_{\gamma(1)}$ be the parallel transport induced by the local system along a path $\gamma\colon [0,1] \to L_i$.

As in the definition of Lagrangian intersection Floer cohomology without local systems, let $x_+, x_- \in L_0 \cap L_1$ be intersection points with $\deg(x_+) + 1 = \deg(x_-)$. Given $\phi_x \in \hom((E_0)_{x_+}, (E_1)_{x_+})$ and a holomorphic strip $u \in \mathcal{M}_\beta(L_0, L_1, x_+, x_-)$ we obtain a map between the fibers above x_0,

$$P^{\nabla_1}_{(\partial^1 u)} \circ \phi_{x_+} \circ P^{\nabla_0}_{(\partial^0 u)^{-1}} \in \hom((E_0)_{x_-}, (E_1)_{x_-}).$$

The differential on $\mathrm{CF}^\bullet(L_0, L_1)$ is defined by taking the contributions $u \cdot \phi_{x_+}$ over all holomorphic strips between x_+ and x_-, weighted by the symplectic area,

$$d_{\nabla_0,\nabla_1}(\phi_{x_+}) := \sum_{x_- \mid \deg(x_-) = \deg(x_+)+1} \; \sum_{u \in M_\beta(L_0, L_1, x_+, x_-)} \pm T^{\omega(\beta)} P^{\nabla_1}_{(\partial^1 u)} \circ \phi_{x_+} \circ P^{\nabla_0}_{(\partial^0 u)^{-1}},$$

where the sign is determined by the orientation of the moduli space.

When ∇_i are the trivial local systems, this recovers $\mathrm{CF}^\bullet(L_0, L_1)$.

Example 2.4.4 (running example, continued) We now return to Example 2.4.3. Fix coordinates on F_q, and let $\{c_1, \dots, c_n\}$ be generators of $H^1(F_q, \mathbb{Z})$ associated to the coordinate directions. A local system on F_q is determined completely by its monodromy on the c_i. Given a Λ-unitary local system ∇_1 on F_q, we write $z_i = P^{\nabla_1}_{c_i}$. Let ∇_0 be the trivial local system on $L_{\underline{V}}$. We now compute the differential on $\mathrm{CF}^\bullet((L_{\underline{V}}, \nabla_0), (F_q, \nabla_1))$. Given $\phi_{x_I} \in \hom((E_0)_{x_I}, (E_1)_{x_I})$, and $I \lessdot J$ an index which differs at one spot j, we have

$$d_{\nabla_0,\nabla_1}(\phi_I) = \Big(T^{\omega(u^+_{I\lessdot J})} P^{\nabla_1}_{(\partial^1 u^+_{I\lessdot J})} \circ \phi_I \circ P^{\nabla_0}_{(\partial^0 u^+_{I\lessdot J})^{-1}}\Big) - \Big(T^{\omega(u^-_{I\lessdot J})} P^{\nabla_1}_{(\partial^1 u^-_{I\lessdot J})} \circ \phi_I \circ P^{\nabla_0}_{(\partial^0 u^-_{I\lessdot J})^{-1}}\Big).$$

Recall that all of the holomorphic strips between intersection points differing in index by 1 have the same area $\lambda_0 = \omega(u^+_{I\lessdot J}) = \omega(u^-_{I\lessdot J})$. Using that ∇_0 is the trivial local system, we get

$$T^{\lambda_0} P^{\nabla_1}_{(\partial^1 u^+_{I\lessdot J})} (\mathrm{id} - P^{\nabla_1}_{c_j}) \circ \phi_I \circ P^{\mathrm{id}}_{\partial^0 u^+_{I\lessdot J}}.$$

This vanishes if and only if $P_{c_j}^{\nabla_1} = z_j = 1$ for all $1 \leq j \leq n-k$. We conclude that

$$\mathrm{HF}^\bullet((L_{\underline{V}}, \mathrm{id}), (F_q, \nabla_1)) = \begin{cases} H^\bullet(T^{n-k}) z_j = 1 & \text{for all } 1 \leq j \leq n-k, \\ 0 & \text{otherwise.} \end{cases}$$

Notation 2.4.5 Given two Lagrangians L_0 and L_1 which intersect transversely, we will pick at each intersection point $x \in L_0 \cap L_1$ an isomorphism in $\hom((E_0)_x, (E_1)_x)$; by abuse of notation, we will denote this isomorphism also by $x \in \hom((E_0)_x, (E_1)_x)$. We can in this way write

$$\mathrm{CF}^\bullet(L_0, L_1) = \Lambda\langle x\rangle,$$

and the differential on this complex will be given by the structure coefficients

$$\langle d(x), y\rangle = \sum_{u \in \mathcal{M}_\beta(L_0, L_1, x, y)} T^{\omega(\beta)} P_{\partial u}^{\nabla_1, \nabla_2},$$

where $P_{\partial u}^{\nabla_1,\nabla_2} \in U_\Lambda$ is a unitary element determined by $P_{\partial u}^{\nabla_1,\nabla_2} \cdot y = P_{(\partial^1 u)}^{\nabla_1} \circ x \circ P_{(\partial^0 u)^{-1}}^{\nabla_0}$. This allows us to use the simpler (and more commonly employed) notation from Definition 2.4.1.

2.4.3 A-tropicalization When considering a complex space $X_B^{\mathbb{C}}$ on the B side, we used the projection $\pi_B\colon X_B^{\mathbb{C}} \to Q$ to obtain from each subvariety of $X_B^{\mathbb{C}}$ an amoeba which approximated the tropical subvariety. Just as with the B-tropicalization, given a Lagrangian submanifold $L \subset Q$ we could consider the Lagrangian torus fibration image of a Lagrangian submanifold $\pi_A(L) \subset Q$. However, since even Hamiltonian isotopic Lagrangian submanifolds can have different projections to the base of the Lagrangian torus fibration, this does not provide a very good definition of A-tropicalization. Instead, we use Lagrangian intersection Floer theory to define the A-tropicalization.

Definition 2.4.6 (preliminary) Let $L \subset X_A$ be a Lagrangian submanifold satisfying the conditions of Definition 2.4.1. We define the *A-tropicalization* or Floer-theoretic support of L to be the set

$$\mathrm{TropA}(L) := \{q \in Q \mid \text{there exists a Lagrangian brane } (F_q, \nabla) \text{ with } \mathrm{HF}^\bullet(L, (F_q, \nabla)) \neq 0\}.$$

The A-tropicalization is a decategorification of a much more powerful invariant captured by family Floer theory due to [5; 20]. From this viewpoint, the chain complexes $\mathrm{CF}^\bullet(L, (F_q, \nabla))$ should be considered as the stalks of a sheaf which are appropriately bundled together into a sheaf on X_B. This viewpoint on tropicalization is also employed in [53]. The A-tropicalization is a refinement of projection to the base of the Lagrangian torus fibration in the following sense:

Proposition 2.4.7 *Let $L \subset X_A$ be a Lagrangian brane. Then* $\mathrm{TropA}(L) \subset \pi_A(L)$.

Proof Suppose that $q \notin \pi_A(L)$. Then $F_q = \pi_A^{-1}(q)$ is disjoint from L. As the Floer intersection complex is generated on the intersection points, $\mathrm{CF}^\bullet(L, (F_q, \nabla)) = 0$. □

While $\mathrm{TropA}(L) \subset \pi_A(L)$ always holds, it will rarely be the case that $\pi_A(L) \subset \mathrm{TropA}(L)$. By invariance of $\mathrm{TropA}(L)$ under Hamiltonian isotopies, we obtain that

$$\mathrm{TropA}(L) \subset \bigcap_{\phi \in \mathrm{Ham}(X_A)} (\pi_A(\phi(L))),$$

where $\mathrm{Ham}(X_A)$ is the set of Hamiltonian isotopies of X_A. However, there is no reason to expect even this to be equality. Section 5 proves that when L is a Lagrangian constructed from the data of a tropical subvariety of Q, the above inclusion becomes equality. We see a toy version of this statement below.

Example 2.4.8 (running example, continued) We now are able to compute the A-tropicalization of a Lagrangian submanifold. Let $\underline{V} \subset Q$ be an integral affine k-subspace, so that $L_{\underline{V}} \subset X_A$ is a $T^{n-k} \times \mathbb{R}^k$ Lagrangian submanifold. We now compute the A-tropicalization of q. Since $\pi_A(L_{\underline{V}}) = \underline{V}$, by Proposition 2.4.7 $\mathrm{TropA}(L_{\underline{V}}) \subset \underline{V}$. By Example 2.4.4, whenever $q \in \underline{V}$ there exists a local system such that $\mathrm{HF}^\bullet(L_{\underline{V}}, (F_q, \nabla_1)) \neq 0$. Therefore $\mathrm{TropA}(L_{\underline{V}}) = \underline{V}$.

In this example, we see there are three steps of the A-realizability problem.

(i) First, we constructed a *geometric* lift $L_{\underline{V}}$ of $\underline{V}$.

(ii) The second step is to show that we have well-defined Floer cohomology groups. In the example above, this follows from $\pi_2(X_A, L_{\underline{V}}) = 0$, but more generally amounts to showing that the Lagrangian $L_{\underline{V}}$ is *unobstructed*.

(iii) Finally, the computation of support from Example 2.4.4 proves that this is a *faithful* lift of $\underline{V}$.

In the example of the lift of $\underline{V}$, we can do slightly more than compute the tropicalization of $L_{\underline{V}}$. We compute the *A-support*, which is the set of pairs (F_q, ∇) which have nontrivial pairing with $L_{\underline{V}}$:

$$\mathrm{SuppA}(L_{\underline{V}}) = \{(F_q, \nabla) \mid q \in \underline{V},\ P_c^\nabla = 0 \text{ for } c \cdot \underline{V} = 0\}. \tag{4}$$

Here we identify $H_1(F_q, \mathbb{Z})$ with $T^*_{\mathbb{Z}}(Q)$. At each point $q \in \underline{V}$, there is a $(U_\Lambda)^k$ choice of local systems satisfying the above criteria. The support can be identified with the set $\mathrm{SuppA}(L_{\underline{V}}) = \underline{V} \times (U_\Lambda)^k = (\Lambda^*)^k \subset X_B$.

2.5 A-tropicalization for the pair of pants

In this subsection, we carry out the entire A-realizability process with the tropical curve V_{pants} from Example 2.2.2. This computation first appeared in unpublished work from [30, Section 4.3], and stems from a discussion with Diego Matessi. We use this example computation to outline the remainder of the paper.

Geometric realizability: Section 3 We first discuss the process of building a Lagrangian submanifold which geometrically is a lift of V in the sense that $\pi_A(L_{V_{\mathrm{pants}}})$ approximates V_{pants}. In dimension 2, one can obtain Lagrangian submanifolds in $(\mathbb{C}^*)^2$ by hyper-Kähler rotation of complex curves.[2] We therefore can build a Lagrangian lift of V_{pants} by starting with the holomorphic lift $\{(z_1, z_2) \mid 1 + z_1 + z_2 = 0\} \subset (\mathbb{C}^*)^2$

[2]We only use this construction for the ease by which it builds a Lagrangian pair of pants in dimension 2; we emphasize at this juncture that hyper-Kähler rotation is *not* mirror symmetry.

and applying hyper-Kähler rotation. For every $\epsilon > 0$, we can find a Lagrangian submanifold $L^{\varepsilon}_{V_{\text{pants}}} \subset X_A$ Hamiltonian isotopic to our hyper-Kähler rotation with the following properties:

- When restricted to the complement of a neighborhood of $0 \in Q$, we have
$$L^{\varepsilon}_{V_{\text{pants}}} \setminus \pi_A^{-1}(B_\varepsilon(0)) = L_{\underline{V}_1} \cup L_{\underline{V}_2} \cup L_{\underline{V}_3} \setminus \pi_A^{-1}(B_\varepsilon(0)).$$
This is one of the properties which characterizes a Lagrangian lift of a tropical curve.
- Furthermore, we can construct this Lagrangian so that it is symmetric under the permutation of coordinates (z_1, z_2) on X_A.

Unobstructedness: Section 4 The next step to the A-realization process is to show that the Lagrangian submanifold one builds can be analyzed with Floer theory. In this example, $L_{V_{\text{pants}}}$ is exact and so $\omega(\pi_2(X_A, L_{V_{\text{pants}}}))$ vanishes. It follows that $\mathrm{HF}^\bullet(L_{V_{\text{pants}}}, F_q)$ will be well defined.

Faithfulness: Section 5 We now compute $\mathrm{TropA}(L_{V_{\text{pants}}})$. Consider the Lagrangian pair of pants $L_{V_{\text{pants}}}$, the Lagrangian fiber F_q, and the holomorphic cylinder $z_1 = z_2$ as drawn in Figure 3, left. We take Hamiltonian perturbations so that the Lagrangian submanifolds intersect transversely. Nearby the point q, the Lagrangian $L_{V_{\text{pants}}}$ agrees with $L_{\underline{V}_3}$; therefore $F_q \cap L_{V_{\text{pants}}} = F_q \cap L_{\underline{V}_3}$. Following the notation from Example 2.4.4, we call the degree-0 intersection point $x_\varnothing$, and the degree-1 intersection point x_1. In addition to the agreement of intersection points, there are two "small strips" contributing to the differential on $\mathrm{CF}^\bullet(L_{V_{\text{pants}}}, F_q)$ which match the strips in the differential of $\mathrm{CF}^\bullet(L_{\underline{V}_3}, F_q)$. We call these holomorphic strips $u^+_{x_\varnothing < x_1}$ and $u^-_{x_\varnothing < x_1}$.

From the symmetry of our setup, the Lagrangian $L_{V_{\text{pants}}}$ intersects the complex plane $z_1 = z_2$ cleanly along a curve. Furthermore, the holomorphic cylinder $z_1 = z_2$ intersects F_q along a circle; therefore the portion of $z_1 = z_2$ bounded by $L_{V_{\text{pants}}}$ and F_q gives an example of a holomorphic strip with boundary on $L_{V_{\text{pants}}}$ and F_q. The ends of this holomorphic strip limit toward $x_\varnothing$ and x_1. The valuation projection of this strip

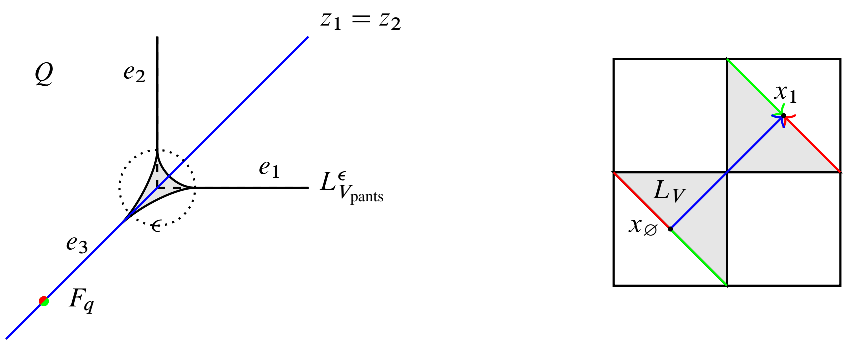

Figure 3: Left: the intersection of the blue holomorphic cylinder and the tropical Lagrangian pair of pants is clean, and gives a holomorphic strip with boundary on $L_{V_{\text{pants}}}$ and F_q. Right: the argument projection of $L_{V_{\text{pants}}}$ to F_q. The intersection points are labeled. The three holomorphic strips are denoted by the arrows, with u^{qv} drawn in blue.

is a line segment connecting the point q with the vertex of the tropical pair of pants. For this reason, we will call this holomorphic strip u^{qv}. The area of this strip is the length of the line segment corresponding to $\pi_A(u^{qv})$. The three holomorphic strips are more readily seen by considering the argument projection of $L_{V_{\text{pants}}}$ to F_q as in Figure 3, right.

We will think of u^{qv} as being a "big strip" as we can choose λ_0 small enough that $\lambda_0 = \omega(u^+_{x_\varnothing<x_1}) = \omega(u^-_{x_\varnothing<x_1}) \ll \omega(u^{qv})$. If no local systems are used, the differential on $\mathrm{CF}^\bullet(L_{V_{\text{pants}}}, F_q)$ is

$$d(x_\varnothing) = (T^{\omega(u^+_{x_\varnothing<x_1})} - T^{\omega(u^-_{x_\varnothing<x_1})} + T^{\omega(u_{qv})}) \cdot x.$$

This does not vanish, so $\mathrm{HF}^\bullet(L_{V_{\text{pants}}}, F_q) = 0$.

However, to compute the A-support we must compute Lagrangian intersection Floer cohomology where we equip F_q with a local system. We characterize the local system ∇ on F_q in terms of its holonomy along the $\arg(z_1)$ and $\arg(z_2)$ loops of F_q, giving us quantities $(\exp(b_1), \exp(b_2)) \in (U_\Lambda)^2$. We'll denote this nonunitary local system by ∇_{b_1,b_2}. Given a point $q = (-a,-a) \in \underline{V}_3$, we compute the quantities

$$\begin{aligned} &\omega(u^+_{x_\varnothing<x_1}) = \lambda_0, && \omega(u^-_{x_\varnothing<x_1}) = \lambda_0, && \omega(u^{qv}) = -a + \lambda_0, \\ &P^{\nabla_{b_1,b_2}}_{\partial u^+_{x_\varnothing<x_1}} = \exp\big(\tfrac{1}{2}(b_1-b_2)\big), && P^{\nabla_{b_1,b_2}}_{\partial u^-_{x_\varnothing<x_1}} = \exp\big(\tfrac{1}{2}(b_2-b_1)\big), && P^{\nabla}_{\partial u^{qv}} = \exp(\tfrac{1}{2}(b_1+b_2)). \end{aligned}$$

The weights given by the local system are determined by the paths drawn in Figure 3, right, from which we obtain the differential on the $\mathrm{CF}^\bullet(L_{V_{\text{pants}}}, (F_q, \nabla_{b_1,b_2}))$:

$$\begin{aligned} \langle d_{\nabla_{b_1,b_2}}(x_\varnothing), x_1\rangle &= \overbrace{(P^{\nabla_{b_1,b_2}}_{\partial u^+_{x_\varnothing<x_1}} \cdot T^{\omega(u^+_{x_\varnothing<x_1})} - P^{\nabla_{b_1,b_2}}_{\partial u^-_{x_\varnothing<x_1}} \cdot T^{\omega(\partial u^-_{x_\varnothing<x_1})})}^{\text{small strips near } q} + \overbrace{P^{\nabla}_{\partial u^{qv}} \cdot T^{\omega(u^{qv})}}^{\text{large strips}} \\ &= T^{\lambda}_0\big(\exp\big(\tfrac{1}{2}(b_1-b_2)\big) - \exp\big(\tfrac{1}{2}(b_2-b_1)\big) + \exp\big(\tfrac{1}{2}(b_1+b_2)\big)\cdot T^{-a}\big) \\ &= T^{-a+\lambda_0}\exp\big(-\tfrac{1}{2}(-b_1-b_2)\big)(T^a\exp(b_1) - T^a\exp(b_2) + 1). \end{aligned}$$

This always has a U_Λ-worth of solutions obtained by setting $b_1 = \log(T^{-a}(T^a\exp(b_2)-1))$. Therefore $(-a,-a) \in \mathrm{TropA}(L_{V_{\text{pants}}})$. From this we conclude that $\mathrm{TropA}(L_{V_{\text{pants}}}) = V_{\text{pants}}$.

This is one of the rare situations where we can compute the Floer-theoretic support explicitly: under the substitution $z_1 = T^{a_1}\exp(b_1)$, $z_2 = T^{a_2}\exp(b_2)$, the Lagrangian tori $(F_{a_1,a_2}, \nabla_{b_1,b_2})$ belong to the support of $L_{V_{\text{pants}}}$ if and only if $z_1 - z_2 + 1 = 0$. This should be compared with the computation of the B-realization of V_{pants} from Example 2.2.2.

***B*-realizability: Section 6** The matching of the supports of the A- and B-realizations of V_{pants} can be captured in the language of homological mirror symmetry. This requires a description of the *Fukaya category* of a symplectic manifold. We define the *Fukaya precategory* of a compact symplectic manifold (X,ω):

- Objects are given by mutually transverse Lagrangian submanifolds $L \subset X$ which are graded, spin, and tautologically unobstructed (Section 4).

- For $L_0 \neq L_1$, the morphisms $\hom(L_0, L_1)$ are given by Lagrangian intersection Floer cochains $\mathrm{CF}^\bullet(L_0, L_1)$.
- k-compositions of morphisms
$$m^k : \bigotimes_{i=0}^{k-1} \hom^{g_i}(L_i, L_{i+1}) = \hom^{2-k+\sum g_i}(L_0, L_k)$$
are given by counts of holomorphic polygons with boundary on the L_k.

This is an A_∞ *precategory*, meaning that for every collection of objects $L_0, \dots, L_k$, the filtered A_∞ relations hold:
$$\sum_{j_1+j+j_2=k} (-1)^{\clubsuit} m^{j_1+1+j_2}(\mathrm{id}^{\otimes j_1} \otimes m^j \otimes \mathrm{id}^{\otimes j_2}) = 0.$$
Here $\clubsuit = j_1 + \sum_1^{j_1} g_i$, and $k \geq 1$.

The precategory can be appropriately completed to give a triangulated A_∞ category, the Fukaya category $\mathrm{Fuk}(X_A)$. Some of the hypotheses of the construction can be dropped or modified: for example, if X_A is a cotangent bundle (and not compact) there is a version of the Fukaya category (the wrapped Fukaya category, $\mathcal{W}(X_A)$) which can be defined with appropriate Lagrangian submanifolds. $X_A = (\mathbb{C}^*)^n = T^*F_0$ is one of these cases.

The homological mirror symmetry conjecture predicts that on mirror spaces the Fukaya category and derived category of coherent sheaves are derived equivalent.

Theorem *Let $X_A = (\mathbb{C}^*)^n$ and $X_B^{\mathbb{C}} = (\mathbb{C}^*)^n$. There is an equivalence of derived categories*
$$\mathcal{F} \colon \mathcal{W}(X_A) \to D^b_{dg} \operatorname{Coh}(X_B^{\mathbb{C}})$$
between the wrapped Fukaya category of exact admissible Lagrangian submanifolds of X_A and the bounded derived category of coherent sheaves on $X_B^{\mathbb{C}}$.

The proof of the theorem first shows that the zero section $L(0)$ of $\pi_A \colon X_A \to Q$ is a Lagrangian submanifold that generates $\mathrm{Fuk}(X_A)$. Then $\mathrm{HF}^\bullet(L(0), L(0))$ is shown to be the algebra $\mathbb{C}[(\mathbb{Z})^n] = \hom(\mathcal{O}_{(\mathbb{C}^*)^n}, \mathcal{O}_{(\mathbb{C}^*)^n})$. Since this generates $D^b_{dg} \operatorname{Coh}(X_B^{\mathbb{C}})$, these two categories are equivalent. However, this proof is nonconstructive: given an arbitrary exact Lagrangian submanifold $L \subset X_A$, there is no immediate way of determining the corresponding mirror sheaf in $D^b_{dg} \operatorname{Coh}(X_B^{\mathbb{C}})$. There are a few objects which we can match up under this functor. Let (F_q, ∇) be an exact fiber of the Lagrangian torus fibration. Then $\mathcal{F}(F_q, \nabla) \simeq \mathcal{O}_z$ for some $z \in X_B^{\mathbb{C}}$. From here, we obtain the following toy result, whose extension to the general V is the objective of the remainder of this paper.

Theorem 2.5.1 *$V_{\text{pants}} \subset \mathbb{R}^2$ is B-realizable.*

Proof From Section 2.5, we proved that V_{pants} is A-realizable by a Lagrangian $L_{V_{\text{pants}}}$. The support of the mirror sheaf $\mathcal{F}(L_{V_{\text{pants}}})$ is a B-realization of V_{pants}. □

Remark 2.5.2 There are other approaches to homological mirror symmetry which would yield the same theorem. A stronger result than what is given here would be to show that V_{pants} is realizable, and its realization compactifies to a subvariety (a line) in the projective plane. To prove this result, one would first show that L_V belongs to an appropriate "partially wrapped Fukaya category", and apply homological mirror symmetry theorems for toric varieties for the appropriate partially wrapped Fukaya category; see [37; 23] or [2; 28; 29]. Then one would need a mirror symmetry statement for the exact Lagrangian torus fiber equipped with nonunitary local systems, and replicate the argument of Section 2.5.

3 Geometric realization

The flexibility of Lagrangian submanifolds both complicates and simplifies the construction of a Lagrangian lift of a tropical subvariety. The additional flexibility means that we have a lot of wiggle room to construct a potential lift; however, identifying a Lagrangian submanifold as "the" lift of a tropical subvariety becomes impossible. For example, given any candidate lift L_V of a tropical subvariety V, one could apply a Hamiltonian isotopy to V to obtain a new Lagrangian submanifold. More generally, each potential Lagrangian lift L_V of V is supposed to represent the data of a sheaf on X_B whose support has tropicalization V; there are many such sheaves!

Despite all of this flexibility, we already have a good idea of what the Lagrangian lift L_V of V should look like from (1). Recall that $V^{(0)}$ is the union of the interiors of the top-dimensional polyhedral domains $\underline{V}$ defining V. At each component we can take the conormal torus construction to obtain a Lagrangian chain:

$$L_{V^{(0)}} := \bigcup_{\underline{V}\subset V^{(0)}} L_{\underline{V}}.$$

Intuitively, a geometric Lagrangian lift of V should approximate the chain $L_{V^{(0)}}$.

Remark 3.0.1 Fix an orientation on F_q, a fiber of the SYZ fibration. Then $L_{\underline{V}}$ inherits an orientation (which in local coordinates comes from $dq_1 \wedge \cdots \wedge dq_k \wedge dp_{k+1} \wedge \cdots \wedge dp_n$). We will assume that we have fixed an orientation on F_q in advance so that $L_{\underline{V}}$ is equipped with a standard orientation.

We propose the following definition for a geometric Lagrangian lift of a tropical subvariety (which is similar to that proposed in [42, Definition 2.1]):

Definition 3.0.2 A family of oriented Lagrangian submanifolds L^{ε}_V for $\varepsilon > 0$ is a *geometric* Lagrangian lift of a weight-1 polyhedral complex $V \subset Q$ if the following conditions hold:

(i) The Lagrangians L^{ε}_V are all Hamiltonian isotopic,

(ii) Let $V^{(i)}$ be the collection of codimension-i strata of V. We require that

$$L^{\varepsilon}_V \setminus \pi_A^{-1}(B_\varepsilon(V^{(1)})) = L_{\underline{V}^{(0)}} \setminus \pi_A^{-1}(B_\varepsilon(V^{(1)})) \tag{5}$$

away from the codimension-1 strata, as oriented submanifolds.

(iii) The Lagrangians L^{ε}_V are embedded, graded, spin, and admissible (in the sense of Definition 3.1.1).

Remark 3.0.3 Definition 3.0.2 has two simplifying requirements; one is included due to current technical limitations in the definition of Floer cohomology, and the second is for convenience.

The requirement that L_V^ϵ is embedded is a technically needed assumption; we believe that this condition can be dropped without modifying our main results. Our reason for restricting ourselves to the embedded setting is that the Charest–Woodward pearly model as written does not include a description of Floer cohomology for immersed Lagrangian submanifolds.

While Definition 3.0.2 looks only at weight-1 polyhedral complexes, one can extend the story to weighted polyhedral complexes by asking that at each top-dimensional stratum $\underline{V} \subset \underline{V}^{(0)}$ with weight m, the realization $L_{\underline{V}}$ is m-disjoint copies of $N^*\underline{V}/N^*_{\mathbb{Z}}\underline{V}$. All results in this paper can be extended to the weighted setting.

The constructions from [30; 38; 39; 42] all satisfy Definition 3.0.2(i)–(ii). To prove that the previous definitions give examples of geometric Lagrangian lifts, we need to additionally show that they are admissible, graded, and spin. We prove these properties for certain examples of Lagrangian lifts in Sections 3.1–3.3.

While Definition 3.0.2 only asks that we take the lift of a weight-1 polyhedral complex, the only polyhedral complexes which admit such lifts are tropical ones.

Proposition 3.0.4 *Let V be a weight-*1 *rational polyhedral complex, and suppose that it has a Lagrangian lift L_V^ε satisfying Definition 3.0.2*(i)–(ii)*. Then V is a tropical subvariety.*

Proof Select an interior point $r \in \underline{W} \subset V^{(1)}$ of the codimension-1 stratum of V. Pick $U_r \subset T_rQ$ a rational subspace such that $U_r \oplus T_rW = T_rQ$. Let $R \subset Q$ be a small polyhedral domain passing through r with tangent space U_r. Then $V|_R$ is a weight-1 rational polyhedral curve. By taking R small enough, $V|_R$ has a single vertex and edges pointing in directions $v_1, \dots, v_k$ corresponding to facets $F_1, \dots, F_k$ containing W. We need to prove that $\sum_{i=1}^k v_i = 0$; see Figure 4.

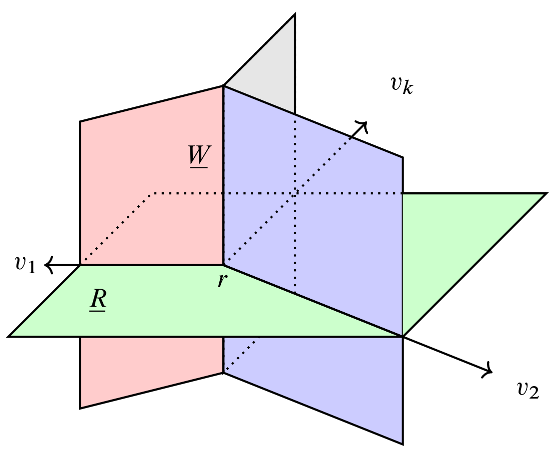

Figure 4: The Polyhedral complexes discussed in Proposition 3.0.4.

Consider the symplectic manifold $Y_A := T^*R/T^*_{\mathbb{Z}}R \subset T^*Q/T^*_{\mathbb{Z}}Q$, with the Lagrangian torus fibration $\pi_{Y_A}\colon Y_A \to R$. Let $i\colon R \to Q$ be the inclusion. Select ε small enough that $B^\varepsilon(W) \cap R$ is an interior set of R. Given a Lagrangian submanifold $L \subset T^*Q/T^*_{\mathbb{Z}}Q$, we can take a Hamiltonian perturbation of L so that

$$L_{i^*} \circ L := \{(r, i^*(p)) \mid (r, p) \in L, r \in R\}$$

is a Lagrangian submanifold of Y_A. See [29, Section 5.2] for a more general discussion of this construction from the perspective of Lagrangian correspondences. By definition $\pi_{Y_A}(L_{i^*} \circ L) = \pi_{Y_A}(L) \cap R$, so $L^\varepsilon_{V|_R} := L_{i^*} \circ L^\varepsilon_V$ is a geometric realization of $V|_R \subset R$. We therefore have reduced to the setting which is the lift of a tropical curve with a single vertex.

Given a tropical curve $V|_R \subset R$ with a single vertex v, the Lagrangian $L^\varepsilon_{V|_R}$ is a manifold with boundary. Consider the projection $\arg_R\colon Y_A \to F_r = T^*_r R/T^*_{\mathbb{Z},r}R$. Considering $\arg_R(L^\varepsilon_{V|_R})$ as a $\dim(F_r)-1$ chain, we obtain the relation in homology

$$0 = [\arg_R(\partial(L^\varepsilon_{V|_R}))] \in H_{\dim_{F_r}-1}(F_r).$$

There is an identification (as vector spaces) that sends an integral basis $e_1, \dots, e_n$ to the class of the perpendicular subtorus

$$T_r R \to H_{\dim_{F_r}-1}(F_r), \quad e_i \mapsto [\{\eta \in T^*_r R \mid \eta(e_i) = 0\}].$$

Since the boundary of $L^\varepsilon_{V|_R}$ lies in the region where (5) holds and we have an agreement of oriented submanifolds, we can therefore compute

$$[\arg_R(\partial(L^\varepsilon_{V|_R}))] = \sum_{i=1}^{k} [\{\eta \in T^*_r R \mid \eta(e_i) = 0\}],$$

proving that $\sum e_i = 0$. □

Notation 3.0.5 From here on, we will drop the ε in L^ε_V and simply write L_V for a Lagrangian which belongs to such a family.

3.1 Geometric Lagrangian lifts: admissibility

When Lagrangian submanifolds are noncompact, we need to place taming conditions on them so that they are Floer-theoretically well-behaved.

Definition 3.1.1 [28] Let $W_\Sigma\colon X_A \to \mathbb{C}$ be a Laurent polynomial whose monomials are indexed by A, the set of rays of a fan Σ. A *monomial division* Δ_Σ for $W_\Sigma = \sum_{\alpha \in A} c_\alpha z^\alpha$ is an assignment of a closed set $U_\alpha \subset Q$ to each monomial $\alpha \in A$ so that the following conditions hold:

- The sets U_α cover the complement of a compact subset of $Q = \mathbb{R}^n$.

- There exist constants $k_\alpha \in \mathbb{R}_{>0}$ such that for all z with $\operatorname{val}(z) \in U_\alpha$ the expression
$$\max_{\alpha \in A}(|c_\alpha z^\alpha|^{k_\alpha})$$
is always achieved by $|c_\alpha z^\alpha|^{k_\alpha}$.
- U_α is a subset of the open star of the ray α in the fan Σ.

A Lagrangian $L \subset X_A$ is Δ_Σ*-monomially admissible* if over $\pi_A^{-1}(U_\alpha)$ the argument of $c_\alpha z^\alpha$ restricted to L is zero outside of a compact set.

We will always assume that $\arg(c_\alpha) = 0$. An advantage of using the monomial admissibility condition for Lagrangian submanifolds is that it is a relatively simple check to see if a Lagrangian submanifold satisfies the condition.

Theorem [31, Theorem 3.1.7] *Suppose that V is the tropicalization of a hypersurface whose Newton polytope has dual fan Σ. Then the construction of L_V from* [30] *is Δ_Σ-monomially admissible.*

Let $V \subset Q$ be a tropical curve. We say that V is adapted to Σ if each semi-infinite edge of V points in the direction of a ray of Σ.

Claim 3.1.2 *Suppose that $V \subset \mathbb{R}^n$ is a weight-1 tropical curve adapted to Σ. Any Lagrangian lift L_V is Δ_Σ-monomially admissible.*

Proof Let $V_\infty^{(0)} = \{e_i\}_{i=1}^k$ denote the semi-infinite edges of V. We note that there exists a compact set $K \subset Q$ such that $L_C \setminus \pi_A^{-1}(K) = \bigsqcup_{e \in V_\infty^{(0)}} L_{\underline{e}} \setminus \pi_A^{-1}(K)$. Furthermore, K can be chosen so that $e \setminus K \subset U_\alpha$ if and only if e points in the direction $\alpha \in \Sigma$. Over this region, we observe that $\arg(z^\alpha)|_{N^*e/N^*_{\mathbb{Z}}e} = 0$. □

Remark 3.1.3 If some of the semi-infinite edges of V are weighted, we must replace the last condition in monomially admissible with "there exists a discrete set of values $\{\theta_i\}$ such that the argument of $c_\alpha z^\alpha|_{(L\cap C)\alpha}$ is a subset of $\{\theta_i\}$". The Floer-theoretic arguments in [28] can be applied to this setting as well (simply by letting θ_i be k-roots of unity, and replacing α with $k\alpha$).

3.2 Geometric Lagrangian lifts: homologically minimal and graded

The additional amount of flexibility that symplectic geometry affords us means that there are many geometric Lagrangian lifts of a single tropical subvariety. Some of these lifts differ for unimportant reasons: for instance, we could have included some extra topology in our Lagrangian by attaching a Lagrangian with vanishing Floer cohomology to a previously constructed lift. The following condition is imposed to weed out some of these worst offenders:

Definition 3.2.1 Let $j : L_{V^{(0)}} \setminus \pi_A^{-1}(B_\epsilon(V^{(1)})) \hookrightarrow L_V$ be the inclusion that is induced from the inclusion of the codimension-0 strata of V into V. We say that a lifting is *homologically minimal* if there exists a section $i : V \to L_V \subset X_A$ such that $H_1(L_V)$ is generated by the images of
$$(i)_* : H_1(V) \to H_1(L_V), \quad (j)_* : H_1(L_{V^{(0)}} \setminus \pi_A^{-1} B_\epsilon(V^{(1)})) \to H_1(L_V).$$

Let $i_{L_V}\colon L_V \to X_A$ be the inclusion of our Lagrangian submanifold. We say that L_V is an untwisted realization of V if the composition

$$V \xrightarrow{(i_{L_V}\circ i)} X_A \xrightarrow{\arg} F_q$$

is nullhomologous (for any choice of $q \in Q$).

Remark 3.2.2 For a fixed tropical subvariety V, there can be several geometric Lagrangian lifts of V which are meaningfully different. We expand on how these different choices of lifts correspond to tropical line bundles of V in Section 6.5.

The homologically minimal condition places some constraints on our Lagrangian submanifolds.

Lemma 3.2.3 *If L_V is homologically minimal and untwisted, then L_V is graded.*

Proof We recall the definition of graded from [52, Example 2.9]. Since $c_1(X_A) = 0$, we can take a section $\bigwedge_{i=1}^{n}(dq_i + i\,d\theta_i)^{\otimes 2}$ of $\Lambda^n(TX_A, J)^{\otimes 2}$. This determines a map

$$\det{}^2 \circ s_L\colon L \to S^1, \quad x \mapsto \Big(\bigwedge (dq_i + i\,d\theta_i)(T_x L)\Big)^2$$

A Lagrangian is $\mathbb{Z}$ graded if this map can be lifted to $\mathbb{R}$.

Consider a homologically minimal Lagrangian and untwisted Lagrangian L_V. There exist generators $\{[\alpha_k], [\beta_l]\}$ for $H_1(L_V)$ such that α_k is in the image of i and β_l is in the image of j. Since the compositions

$$\det{}^2 \circ s_{L_V} \circ i\colon V \to S^1, \quad \det{}^2 \circ s_{L_V} \circ j\colon (L_{V^{(0)}}) \to S^1$$

are constantly 0, it follows that there is no obstruction to lifting $\det^2 \circ s_{L_V}\colon L_V \to S^1$ to $\mathbb{R}$. □

Proposition 3.2.4 *Suppose that $V \subset \mathbb{R}^n$ is either a smooth tropical curve or a smooth tropical hypersurface. Then the construction of L_V given by* [30; 39; 42] *produces a homologically minimal Lagrangian lift L_V. The lifts are therefore graded.*

Proof In the cases of tropical curves, this follows from computing the homology of L_V from a cover given by $L_{\mathrm{star}(v)}$. For hypersurfaces, this is proven in [31, Proposition 3.18]. □

Unless otherwise specified, the lift of a smooth tropical curve or hypersurface will always be the one given by [30; 39; 42].

3.3 Geometric Lagrangian lifts: spin

We start with a lemma on the topology of lifts of smooth genus-0 tropical curves.

Lemma 3.3.1 *Let $V \subset \mathbb{R}^n$ be a smooth genus-0 tropical curve.*

(i) *For any semi-infinite edge $f \in V_\infty^{(0)}$, the restriction map $\mathrm{res}_f^V\colon H^1(L_V) \to H^1(L_f)$ is a surjection.*

(ii) *For any semi-infinite edge f, the restriction map $\mathrm{res}_{V_\infty \setminus f}^V\colon H^2(L_V) \to \bigoplus_{g \neq f} H^2(L_g)$ is an injection.*

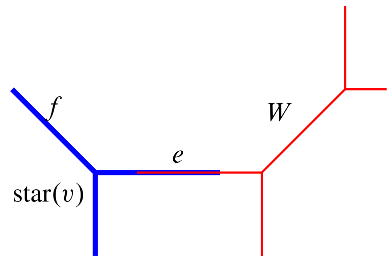

Figure 5: Covering our tropical curve V with two charts: W and a pair of pants $\mathrm{star}(v)$ centered at v.

Proof We prove (i) and (ii) by induction on the number of vertices in V.

Base case Suppose that V has one vertex. Then V is planar, and there exists a splitting of $(\mathbb{C}^*)^n = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{n-2}$ such that $L_V = L_{\mathrm{pants}} \times T^{n-2}$, where $L_{\mathrm{pants}} \subset (\mathbb{C}^*)^2$ is the standard pair of pants. The boundary of the pair of pants is $S^1_{e_1} \cup S^1_{e_2} \cup S^1_{e_3}$, where e_1, e_2, and e_3 label the three edges of the pair of pants. A direct computation shows that

$$H^0(L_{\mathrm{pants}}) \to H^0(S^1_{e_1}), \quad H^1(L_{\mathrm{pants}}) \to H^1(S^1_{e_1}),$$

surjects, and that

$$H^0(L_{\mathrm{pants}}) \to H^0(S^1_{e_1} \cup S^1_{e_2}), \quad H^1(L_{\mathrm{pants}}) \to H^1(S^1_{e_1} \cup S^1_{e_2}),$$

injects. An application of the Künneth formula gives (i) and (ii) for L_V.

Inductive step Let $f \in V^{(0)}_\infty$ be any semi-infinite edge and let v be the vertex of V belonging to that edge. Let W be the tropical curve given by vertices not equal to v, so that $L_{\mathrm{star}(v)}$ and L_W cover L_V with intersection $L_{\mathrm{star}(v)} \cap L_W = L_e = T^{n-1} \times e$, as in Figure 5. This can be done because V is a tree.

(i) We use $L_{\mathrm{star}(v)}$ and L_W to compute the first cohomology of L_V using Mayer–Vietoris, and show that the red arrow in the diagram below is a surjection:

$$\begin{array}{ccccc}
H^1(L_V) & \xrightarrow{\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W} & H^1(L_{\mathrm{star}(v)}) \oplus H^1(L_W) & \xrightarrow{\mathrm{res}^{\mathrm{star}(v)}_e - \mathrm{res}^W_e} & H^1(L_e) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{res}^{\mathrm{star}(v)}_f \oplus 0} & & \\
H^1(L_f) & = & H^1(L_f) \oplus 0 & &
\end{array}$$

From the base case, given $\alpha \in H^1(L_f)$, there exists $\alpha' \in H^1(L_{\mathrm{star}(v)})$ with $\mathrm{res}^{\mathrm{star}(v)}_f(\alpha') = \alpha$. From the induction hypothesis, there exists $\beta' \in H^1(L_W)$ with $\mathrm{res}^W_e(\beta') = \mathrm{res}^{\mathrm{star}(v)}_e(\alpha')$. Therefore $(\alpha', \beta') \in \ker(\mathrm{res}^{\mathrm{star}(v)}_e - \mathrm{res}^W_e)$, and by exactness of the rows is in the image of $\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W$. Let α'' be in the preimage of (α', β'). By commutativity of the below diagram, we conclude $\mathrm{res}^V_f(\alpha'') = \alpha$.

(ii) We compute $H^2(L_V)$ using Mayer–Vietoris, and show that the blue arrow of the following diagram is injective:

$$\begin{array}{ccccccc}
H^1(L_{\mathrm{star}(v)}) \oplus H^1(L_W) & \twoheadrightarrow & H^1(L_e) & \xrightarrow{0} & H^2(L_V) & \xhookrightarrow{\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W} & H^2(L_{\mathrm{star}(v)}) \oplus H^2(L_W) \\
 & & & & \Big\downarrow {\scriptstyle \bigoplus_{g \neq f} \mathrm{res}^V_g} & & \Big\downarrow {\scriptstyle C \oplus D} \\
 & & & & \bigoplus_{\substack{g \in V^{(0)}_\infty \\ g \neq f}} H^2(L_g) & \hookrightarrow & \bigoplus_{\substack{g \in \mathrm{star}(v)^{(0)}_\infty \\ g \neq e,f}} H^2(L_g) \oplus \bigoplus_{\substack{g \in W^{(0)}_\infty \\ g \neq e}} H^2(L_g)
\end{array}$$

By (i), the leftmost arrow is surjective. By the exactness of the sequence, $\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W$ is injective on the second cohomology groups. Let $C = \bigoplus_{g \in \mathrm{star}(v)^{(0)}_\infty, g \neq e, f} \mathrm{res}^{\mathrm{star}(v)}_g$ and $D = \bigoplus_{g \in W^{(0)}_\infty, g \neq e} \mathrm{res}^W_g$. Now consider a class $\alpha \in H^2(L_V)$. Suppose that $\bigoplus_{g \neq f} \mathrm{res}^V_g(\alpha) = 0$. We will show that $\alpha = 0$. By commutativity of the diagram, $(C \oplus D) \circ (\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W) = 0$. Because D is injective and $(\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W)$ is injective, $C \circ \mathrm{res}^V_{\mathrm{star}(v)}(\alpha) = 0$ and $\mathrm{res}^V_W(\alpha) = 0$. We now break into two cases:

Case I $\boldsymbol{\mathrm{res}^V_{\mathrm{star}(v)}(\alpha) = 0}$ This implies $(\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W)(\alpha) = 0$, which by injectivity of $\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W$ tells us that $\alpha = 0$.

Case II $\boldsymbol{\mathrm{res}^V_{\mathrm{star}(v)}(\alpha) \neq 0}$ Observe that $(C \oplus \mathrm{res}^{\mathrm{star}(v)}_e) \circ \mathrm{res}^V_{\mathrm{star}(v)}$ is injective from the base case, so $\mathrm{res}^{\mathrm{star}(v)}_e \circ \mathrm{res}^V_{\mathrm{star}(v)}(\alpha) \neq 0$. Since $\mathrm{res}^V_W(\alpha) = 0$, we obtain that

$$(\mathrm{res}^{\mathrm{star}(v)}_e \oplus \mathrm{res}^W_e) \circ (\mathrm{res}^V_{\mathrm{star}(v)} \oplus \mathrm{res}^V_W)(\alpha) \neq 0.$$

This violates the exactness of the top row, so Case II cannot occur. □

Proposition 3.3.2 *In the setting where $V \subset \mathbb{R}^n$ has genus 0, the constructions of [30; 39; 42] give homologically minimal untwisted geometric Lagrangian lifts L_V of V.*

Proof We prove that this Lagrangian submanifold is homologically minimal because the homology of the pair of pants is generated by the homology of the legs. If $n = 2$, then L_V is a surface, and therefore spin.

To prove that the $n \geq 3$ cases are spin, we induct on the number of vertices in V. For the 1-vertex case, $L_{\mathrm{star}(v)} \simeq L_{\mathrm{pants}} \times T^{n-2}$. The manifolds $L_{\mathrm{pants},v} \times T^{n-2}$ have trivializations given by embedding $L_{\mathrm{pants},v}$ into $\mathbb{R}^2$, and is therefore spin.

As in the proof of Lemma 3.3.1, write $V = L_W \cup L_{\mathrm{star}(v)}$, where e is the common edge $L_W \cap L_{\mathrm{star}(v)}$. By the induction hypothesis we have a spin structure on L_W. By pullback, this gives a spin structure over L_e. Since $H^1(L_{\mathrm{star}(v)}, \mathbb{Z}/2\mathbb{Z}) \to H^1(L_e)$ surjects, there is no obstruction to picking a spin structure on $L_{\mathrm{star}(v)}$ agreeing with the prescribed spin structure on L_e. □

This method of proof can be extended to a slightly larger set of examples. We say that a smooth tropical curve V has planar genus if there exist cycles $c_1, \dots, c_k \subset V$ such that $\{[c_1], \dots, [c_k]\}$ generate $H_1(V)$, and there exist 2-dimensional planes $\underline{V}_k \subset \mathbb{R}^n$ such that $c_i \subset \underline{V}_k$.

Corollary 3.3.3 *If $V \subset \mathbb{R}^n$ is a smooth tropical curve V with planar genus, then L_V is spin.*

The other setting where tropical Lagrangian lifts have been studied is the setting of hypersurfaces.

Lemma 3.3.4 *If $V \subset \mathbb{R}^n$ is a smooth tropical hypersurface, the construction of* [31; 39] *of L_V is spin.*

Proof We break into several cases.

- If $n = 2$, then L_V is a punctured surface (and therefore spin).
- If $n = 3$, then L_V is an orientable 3-manifold (and therefore spin).
- If $n \geq 4$, then by [31] the Lagrangian L_V is the connected sum of two copies of $\mathbb{R}^n$ at several contractible regions U_α indexed by Δ, the Newton polytope of the defining tropical polynomial for V. Assume that $\dim(\Delta) = n \geq 4$ (as otherwise, we may reduce to one of the previous cases). Following [31, Proposition 3.18], we take two charts $L_r, L_s \simeq \mathbb{R}^n \setminus \bigcup_{\alpha\in\Delta} U_\alpha$ such that $L_V = L_r \cup L_s$. The L_r and L_s are homotopic to V. Then $L_r \cap L_s \simeq \bigcup_{\alpha\in\Delta} \partial U_\alpha$, where each ∂U_α is homotopic to either S^{n-1} or D^{n-1}. By Mayer–Vietoris, we compute

$$\bigoplus_{\alpha\in\Delta} H^1(\partial U_\alpha) \to H^2(L_V, \mathbb{Z}/2\mathbb{Z}) \to H^2(L_r, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(L_r \mathbb{Z}/2\mathbb{Z}).$$

The left and right terms are zero when $n \geq 4$, so $H^2(L_V, \mathbb{Z}/2\mathbb{Z}) = 0$ and our Lagrangian is spin. □

4 Unobstructed Lagrangian lifts of tropical subvarieties

Since the geometric Lagrangian lifts L_V we construct will not be exact, to obtain a Lagrangian Floer *cohomology* theory we need to show that these Lagrangian submanifolds have Λ-filtered A_∞ algebra which can be unobstructed.

4.1 Pearly model for Floer cohomology

We will adopt the model employed in [12] to define $\mathrm{CF}^\bullet(L)$.

Theorem [12] *Let $L \subset X$ be a compact relative spin and graded Lagrangian submanifold inside a rational compact symplectic manifold X. Pick $h\colon L \to \mathbb{R}$ a Morse function, and $D \subset X \setminus L$ a stabilizing divisor. There exists a choice of perturbation datum $\mathcal{P}$ which defines a filtered A_∞ algebra $\mathrm{CF}^\bullet(L, h, \mathcal{P}, D)$ where:*

- *Chains are given by the Morse cochains of L, so that $\mathrm{CF}^\bullet(L, h, \mathcal{P}, D) = \Lambda\langle \mathrm{Crit}(h)\rangle$.*
- *Product structures come from counting configurations of treed disks. More precisely, given a collection of critical points $\underline{x} = (x_1, \dots, x_k)$, we define the structure coefficients*

$$\langle m^k(x_1 \otimes \cdots \otimes x_k), x_0\rangle = \sum_{\beta\in H_2(X,L)} (-1)^{\heartsuit}(\sigma(u)!)^{-1} T^{\omega(\beta)} \cdot \#\mathcal{M}_{\mathcal{P}}(X, L, D, \underline{x}, \beta)$$

which determine the A_∞ product structure. Here $\#\mathcal{M}_{\mathcal{P}}(X, L, D, \underline{x}, \beta)$ is the count of points in the moduli space of $\mathcal{P}$-perturbed pseudoholomorphic treed disks, $\sigma(u)$ denotes the number of stabilizing points on each of these treed disks, and $\heartsuit = \sum_{i=1}^{k} i\,|x_i|$.

The Λ-filtered A_∞ homotopy class does not depend on the choices of perturbation, divisor, and Morse function taken in the construction.

When the choices of $h, \mathcal{P}$, and D are unimportant, we will write $\mathrm{CF}^\bullet(L)$ instead of $\mathrm{CF}^\bullet(L, h, \mathcal{P}, D)$. The most visible difference between the tautologically unobstructed setting and this more general definition is that there now exists a curvature term $m^0\colon \Lambda \to \mathrm{CF}^\bullet(L)$, which obstructs the squaring of the differential to zero. We say that L is *unobstructed* if $\mathrm{CF}^\bullet(L)$ has a bounding cochain $b \in \mathrm{CF}^\bullet(L)$; see Section B.1. When L is unobstructed, the deformed A_∞ structure on $\mathrm{CF}^\bullet(L, b)$ is a chain complex.

In this section we discuss whether a geometric Lagrangian lift L_V of a tropical subvariety is an unobstructed Lagrangian submanifold. We give an example computation in the pearly disk model to fix notation.

Example 4.1.1 (running example, continued) Returning to Example 2.4.4, we first examine $\mathrm{CF}^\bullet(F_q, \nabla_1)$. Since F_q bounds no topological disks, it does not bound any holomorphic disks. Therefore the Floer complex is the Morse-tree algebra of F_q. Give F_q the Morse function

$$f = \sum_{i=1}^{n} \cos(\theta_i). \tag{6}$$

We label the generators of

$$\mathrm{CF}^\bullet(F_q, \nabla_1) = \Lambda\langle y_I^1\rangle,$$

where $I \subset \{1, \dots, n\}$. The differential is given by $m^1(y_I^1) = 0$, and for a particular set of perturbations the product structure is

$$m^2(y_I^1 \otimes y_J^1) = \begin{cases} \pm y_{I\cup J}^1 & \text{if } I \cap J = \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

where the sign is determined by the number of transpositions required to reorder $I \cup J$.

Remark 4.1.2 To our knowledge, it is unknown if there exists a perturbation scheme for Morse flow trees such that all higher products $m^k\colon \mathrm{CM}^\bullet(S^1)^{\otimes k} \to \mathrm{CM}^\bullet(S^1)[2-k]$ vanish.

To work in the setting where X_A is noncompact, we need to place restrictions on the noncompact behavior of the Lagrangian L to ensure that the moduli spaces of pseudoholomorphic treed-disks considered by [12] remain compact. A natural condition to impose is that $L \subset X_A$ is admissible (Definition A.0.1) with respect to a potential function $W\colon X_A \to \mathbb{C}$, so that the projection $W(L)$ fibers over the real axis $\mathbb{R}_{>0}$ outside of a compact set. Let $Y_A = W^{-1}(t)$ for $t \in \mathbb{R}_{\gg 0}$. Choices of different sufficiently large t yield fibers which are symplectomorphic. The restriction of L to a large fiber will be called $M := L|_{Y_A}$; this is a Lagrangian submanifold of Y_A. By Theorem A.0.2 there exists a treed-disk model for Lagrangian

Floer cohomology $\mathrm{CF}^\bullet(L)$ for W-admissible Lagrangians L. Furthermore, there exist compatible choices of perturbation data such the standard projection

$$\mathrm{CF}^\bullet(L) \to \mathrm{CF}^\bullet(M)$$

is a Λ-filtered map of A_∞ algebras.

A useful lemma of [28] states that when we have a monomially admissible Lagrangian L, there exists a potential function W such that L is W-admissible. From the data of a fan and $t \in \mathbb{R}$, [3] constructs *tropicalized potential*, which is a symplectic fibration $W_\Sigma^{t,1} \colon (\mathbb{C}^*)^n \to \mathbb{C}$ outside of a compact set.

Lemma 4.1.3 [28, Section 4.4; 29, Remark 2.10] *Suppose that L is a Lagrangian submanifold that is monomially admissible with respect to a monomial division adapted to Σ (in the sense of Definition 3.1.1). Then L can be made admissible for the tropicalized potential.*

4.2 Geometric Lagrangians versus Lagrangian branes

Definition 4.2.1 We say that an unobstructed Lagrangian submanifold (L_V, b) is a *Lagrangian brane lift* of V if L_V is a geometric Lagrangian lift of V.

Before developing constructions of bounding cochains for geometric Lagrangian lifts, we give some examples of geometric Lagrangian lifts which are known to be unobstructed (or tautologically unobstructed) Lagrangian submanifolds.

Example 4.2.2 (Lagrangian pair of pants) In [39], it was shown that the tropical pair of pants centered at the origin is an exact Lagrangian submanifold; a similar proof was given in [33], which showed that all tropical Lagrangian submanifolds constructed from the data of a dimer are exact.

Claim 4.2.3 *Let $V \subset \mathbb{R}^n$ be a tropical variety such that $0 \in \underline{V}_i$ for all facets $\underline{V}_i \subset V$. Let L_V^ε be a homologically minimal lift of V. Then L_V^ε is exact.*

Proof Let $\eta = p\,dq$ be the primitive for ω on $X_A = T^*T^n$. We need to show that η is exact on L_V^ε; equivalently we show that $\eta(\gamma) = 0$ for all $[\gamma] \in H_1(L_V^\varepsilon)$. Observe that L_V^ε retracts onto $L_V^\varepsilon \cap \pi_A^{-1}(B_\epsilon(0))$. Therefore, for every loop $\gamma \in H_1(L_V^\varepsilon)$, there exists γ' which is homotopic to γ and lives within in $\pi_A^{-1}(B_\epsilon(0))$; by letting $\epsilon \to 0$ we obtain $[\gamma'] = [\gamma]$ and $\gamma' \subset F_0$. As F_0 is exact, $\eta(\gamma') = 0$. □

Since these Lagrangians are exact, they are tautologically unobstructed and we can conclude that L_V^ε is a tropical Lagrangian brane.

In some cases, one obtains tautological unobstructedness (or unobstructedness) of the Lagrangian submanifold for free.

Example 4.2.4 We can obtain tautological unobstructedness for curves $V \subset \mathbb{R}^2$. Since L_V is a graded Lagrangian submanifold, the only holomorphic curves which might cause us difficulty are Maslov index 0 curves. However, the expected dimension of Maslov index 0 disks with boundary on a 2-dimensional Lagrangian is negative, therefore for a generic choice of almost complex structure these disks disappear and L_V^ε is tautologically unobstructed.

It is possible for nonregular Maslov index 0 disks to appear with boundary on L_V^ε, even in simple examples (see Example 4.2.11). More generally, [33] shows that there exists a "wall-crossing" phenomenon which occurs for isotopies between tropical Lagrangian submanifolds, and that the count of these Maslov index 0 holomorphic disks play a crucial role in understanding coordinates on the moduli space of tropical Lagrangian submanifolds.

Example 4.2.5 We now examine a setting outside of the mirrors to toric varieties. Let Q be any tropical abelian surface; then $X_A := T^*Q/T_{\mathbb{Z}}Q$ is a symplectic 4-torus. Given any tropical curve $V \subset Q$, there is a Lagrangian surface $L_V^\varepsilon \subset X_A$. By the same reasoning as above, L_V is tautologically unobstructed for generic choice of almost complex structure.

Example 4.2.6 We also know unobstructedness for geometric Lagrangian lifts is [38]. In that setting, the base of the Lagrangian torus fibration has nontrivial discriminant locus, and the tropical Lagrangians constructed are lifts of compact genus-0 tropical curves in the base. Mak and Ruddat show that the associated tropical Lagrangians are homology spheres and therefore are always unobstructed by a choice of bounding cochain [22].

In general, other techniques are required to prove that a geometric Lagrangian lift of a tropical subvariety is unobstructed.

Example 4.2.7 Given any smooth tropical hypersurface $V \subset \mathbb{R}^n$, [30] shows that the tropical Lagrangian lift can be equipped with a bounding cochain so that (L_V, b) is an unobstructed Lagrangian submanifold of $(\mathbb{C}^*)^n$. The proof uses that L_V can be constructed as a mapping cone of two Lagrangian sections in the Fukaya category; as these sections bound no holomorphic strips or disks, one expects that their Lagrangian connected sum can be equipped with a bounding cochain. In practice, the process of constructing the bounding cochain is delicate.

We furthermore expect that similar methods should show that given $V = V_1 \cap \cdots \cap V_k$ a transverse intersection of tropical hypersurfaces V_i, there exists L_V an unobstructed Lagrangian lift of V. The Lagrangian L_V is constructed by using the fiberwise sum of the lifts [29; 57], so that

$$L_V = L_{V_1} +_Q \cdots +_Q L_{V_k}.$$

While the resulting Lagrangian submanifold L_V may be immersed, over the top-dimensional stratum of V the Lagrangian submanifold L_V satisfies Definition 3.0.2. This provides the geometric realization. To obtain unobstructedness, we can also write L_V as the geometric composition of unobstructed Lagrangian correspondences (each giving the fiberwise sum with L_{V_i}). It is expected (from [21; 60]) that the geometric composition of unobstructed Lagrangian correspondences is unobstructed in this setting, from which it follows that L_V is unobstructed by the pushforward bounding cochain.

Example 4.2.8 Given a smooth tropical hypersurface V of a tropical abelian variety $Q = \mathbb{R}^n / M_{\mathbb{Z}}$, [30, Example 5.2.0.7] constructs an unobstructed Lagrangian lift (L_V, b) inside the symplectic torus $T^*Q/T^*_{\mathbb{Z}}Q$. The proof of unobstructedness is easier than the hypersurface setting (as one does not need to worry about issues of compactness).

The next two examples were suggested by Dhruv Ranganathan.

Example 4.2.9 Let $L_1 \subset X_1$ and $L_2 \subset X_2$ be tautologically unobstructed Lagrangian submanifolds. Then $L_1 \times L_2 \subset X_1 \times X_2$ is again a tautologically unobstructed Lagrangian submanifold. Furthermore, if the methods in [7] can be adapted to the Charest–Woodward model of Floer cohomology that we use, then the product of unobstructed Lagrangians is unobstructed. It follows that when $V_i \subset Q_i$ have Lagrangian brane lifts, then so does $V_1 \times V_2 \subset Q_1 \times Q_2$.

Example 4.2.10 Suppose for $t \in [0, 1]$ we have geometric Lagrangian lifts L_{V_t} of a family of tropical subvarieties V_t. Furthermore, suppose that for $t \in [0, 1)$ the lift is a Lagrangian brane lift. Then L_{V_1} is a Lagrangian brane lift of V_1. The proof uses Fukaya's trick to choose perturbation data so that for t close to 1, the L_{V_t} all have the same moduli spaces of pseudoholomorphic disks. Then there exists a subsequence of bounding cochains for the L_t which converge to a bounding cochain on L_1.

There are few general criteria for determining if a Lagrangian submanifold is unobstructed. To highlight some of the subtlety of the problems, we exhibit a tropical Lagrangian which bounds a nonregular Maslov index 0 disk.

Example 4.2.11 Consider the projection to the base of the Lagrangian torus fibration of the tropical Lagrangian submanifold L_{V_4} drawn in Figure 6. Let ℓ be the dashed red line. Take the standard metric on $\mathbb{R}^2$ so we may identify $T\mathbb{R}^2$ with $T^*\mathbb{R}^2$. Provided that one takes a symmetric construction of the Lagrangian pairs of pants (for example, using the construction of [39]), the holomorphic cylinder $T\ell/T_{\mathbb{Z}}\ell$ tropicalizing to the line ℓ intersects the Lagrangian L_{V_4} cleanly along an S^1. This yields an isolated holomorphic disk with boundary on L_{V_4}. This is *not* a regular holomorphic disk.

Further examples of nonregular Maslov index 0 disks are given in [32]. In Section 6.3, we give examples of geometric Lagrangians lifts which are unobstructed, but not tautologically unobstructed for any choice of admissible almost complex structure on $(\mathbb{C}^*)^n$. In Section 6.4, we show that there exists V such that L_V is obstructed.

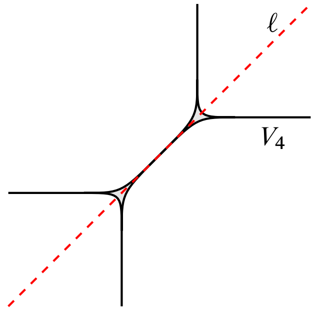

Figure 6: The projection to the base of the Lagrangian torus fibration of a tropical Lagrangian. The holomorphic cylinder above the red dashed line cleanly intersects the tropical Lagrangian.

4.3 Unobstructedness at the boundary

We now give a method for constructing a bounding cochain for a Lagrangian which is W-admissible. By Lemma 4.1.3, for every L a Δ_Σ-monomially admissible Lagrangian there exists a tropicalized potential $W_\Sigma^{t,1} : (\mathbb{C}^*)^n \to \mathbb{C}$ such that L is $W_\Sigma^{t,1}$-admissible. We state our results for W-admissible Lagrangians as the methods may be of interest beyond the monomially admissible setting.

Theorem 4.3.1 *Let $W : X \to \mathbb{C}$ be a potential function, and suppose that L is a W-admissible Lagrangian submanifold whose restriction to a large fiber is $M = L \cap (W^{-1}(t))$, where $t \in \mathbb{R}_{\gg 0}$. Suppose that there exists $M_0 \subset M$ a union of connected components of M with the property that*

(i) *the Lagrangian M_0 bounds no holomorphic disks, and*

(ii) *the map $H^1(M_0) \to H^2(L, M_0)$ is surjective.*

Then L is unobstructed.

The idea of the proof is to construct the bounding cochain for L by "lifting the curvature term of L to the boundary M_0". The condition that $H^1(M_0) \to H^2(L, M_0)$ shows that the curvature term (which takes values in the subcomplex $H^2(L, M)$) is the coboundary of something coming from the boundary M_0 of L. The algebraic content of this statement is Lemma B.2.8.

Proof We show that the A_∞ algebras $A = \mathrm{CF}^\bullet(L, M_0)$, $B = \mathrm{CF}^\bullet(L)$, and $C = \mathrm{CF}^\bullet(M_0)$ satisfy Lemma B.2.8(i)–(iii). From Theorem A.0.2, the sequence $A \to B \to C$ is exact, and A is an A_∞ ideal. Since M_0 bounds no holomorphic disks, C is tautologically unobstructed and A is a strong ideal, giving us Lemma B.2.8(i). Because M_0 bounds no holomorphic disks, $\mathrm{CF}^\bullet(M_0) = \mathrm{CM}^\bullet(M_0)$, which is quasi-isomorphic to $\Omega^\bullet(M)$. Thus we have Lemma B.2.8(ii). Finally, the hypothesis that $H^1(M_0) \to H^2(L, M_0)$ surjects is exactly Lemma B.2.8(iii). □

We give an example that relates to the discussion in [18, Section 5.2]:

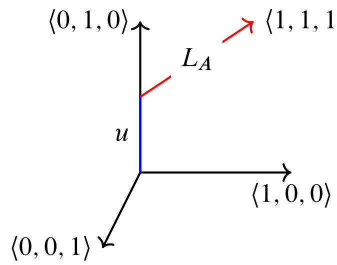

Figure 7: The projection of the AV Lagrangian L_A to the base $(\mathbb{R}_{\geq 0})^3$ of the Lagrangian torus fibration of the Aganagic–Vafa Lagrangian $L_A \subset \mathbb{C}^3$. We also draw the projection of the single simple holomorphic disk which contributes to a bounding cochain for L_A.

Example 4.3.2 (Aganagic–Vafa brane) Let $A \in \mathbb{R}_{>0}$ be some constant. The Aganagic–Vafa (AV) brane is a Lagrangian submanifold $L_A \subset \mathbb{C}^3$ parametrized by

$$D^2 \times S^1 \to \mathbb{C}^3, \quad (r_1, \theta_1, \theta_2) \mapsto ((\sqrt{A^2 + r^2})e^{-i(\theta_1+\theta_2)}, re^{i\theta_1}, re^{i\theta_1}).$$

The Lagrangian L_A is admissible for the potential function $W(z_1, z_2, z_3) = z_1 z_2 z_3$. The restriction to the fiber $M_A \subset W^{-1}(s) = (\mathbb{C}^*) \times (\mathbb{C}^*)$ is a product-type torus, so it bounds no holomorphic disks, and we may apply Theorem 4.3.1 to conclude that this Lagrangian is unobstructed by a bounding cochain.

The bounding cochain corrects this Lagrangian submanifold so that it agrees with predictions from mirror symmetry. By application of the open mapping theorem, the only holomorphic disks with boundary on L_A for the standard complex structure must lie in the fiber $W^{-1}(0)$; in fact, the only simple holomorphic disk with boundary on L_A is parametrized by

$$u\colon (D^2, \partial D^2) \to (\mathbb{C}^2, L_A), \quad z \mapsto (Az, 0, 0).$$

A computation shows that the partial Maslov indices of this disk are $(2, -1, -1)$ and therefore this is a regular Maslov index 0 disk by [46]. This shows that the bounding cochain constructed by Theorem 4.3.1 is nontrivial.

Under an additional assumption [34, Assumption 5.2.3] one can compute the m^0-term, which counts the multiple covers of the disk u with an appropriate weight. The bounding cochain is $\sum_{k=1}^{\infty}(1/k)T^{k\omega(u)}x$, where $x \in \mathrm{CM}^\bullet(M_A)$ is a meridional class of the torus.

We remark that the Lagrangian L_A is an example of a tropical Lagrangian submanifold considered in [42], and the projection of L_A under the moment map $\mathbb{C}^3 \to Q = \mathbb{R}^3_{\geq 0}$ is the ray $(|A|^2, 0, 0) + t\langle 1, 1, 1\rangle$.

Corollary 4.3.3 *Let $V \subset Q$ be a genus-0 smooth tropical curve. Let L_V be a homologically minimal geometric Lagrangian lift of V. Then L_V is unobstructed, so there exists (L_V, b) a Lagrangian brane lift of V.*

Proof We show that the Lagrangian L_V satisfies the criteria of Theorem 4.3.1. Let $V_\infty^{(0)} \subset V$ be the set of semi-infinite edges of V. The boundary of this tropical Lagrangian realization $M \subset Y_A$ is contained within the lift of the semi-infinite edges $\bigsqcup_{e\in V_\infty^{(0)}} L_e = T_e^{n-1}\times e$. Therefore M is the disjoint union of tori indexed by the semi-infinite edges of V, $\bigcup_{e\in V_\infty^{(0)}} T_e^{n-1}$. At each edge, we see that $\pi_2(X_A, L_e)=0$. It follows that $M\subset Y_A$ bounds no holomorphic disks, so we satisfy Theorem 4.3.1(i). Select $f\in V_\infty^{(0)}$ any edge, and let $M_0=\bigcup_{g\in V_\infty^{(0)}, g\neq f} T^{n-1}$.

It remains to prove Theorem 4.3.1(ii), that the image of $H^1(M_0)$ generates $H^2(L_V, M_0)$. From Lemma 3.3.1, for any semi-infinite edge f of V, $\mathrm{res}^V_{V_\infty\setminus f}\colon H^2(L_V)\to\bigoplus_{g\in V_\infty^{(0)}, g\neq f} H^2(L_g)$ is an injection. From the long exact sequence for relative cohomology,

$$\bigoplus_{\substack{g\in V_\infty^{(0)}\\ g\neq f}} H^1(L_g)\to H^2(L_V, M_0)\xrightarrow{0} H^2(L_V)\hookrightarrow \bigoplus_{\substack{g\in V_\infty^{(0)}\\ g\neq f}} H^2(L_g),$$

the leftmost arrow surjects. □

5 Faithfulness: unobstructed lifts as A-realizations

Given a Lagrangian torus fibration $X_A\to Q$, the *A-tropicalization* of a tautologically unobstructed Lagrangian submanifold L is the set of points $q\in Q$ such that $\mathrm{HF}^\bullet(L,(F_q,\nabla))\neq 0$ for some choice of local system ∇ on F_q; see (4). We now describe the A-tropicalization when L is unobstructed by a bounding cochain. Because we again work in the scenario where the space X is noncompact, we must apply a taming condition at infinity to study Floer cohomology. By the same arguments as for Theorem A.0.2, whenever L_0 is admissible and L_1 is compact for a potential $W\colon X_A\to\mathbb{C}$, there exists a well-defined $\mathrm{CF}^\bullet(L_0,\nabla_0)-\mathrm{CF}^\bullet(L_1,\nabla_1)$ bimodule $\mathrm{CF}^\bullet((L_0,\nabla_0),(L_1,\nabla_1))$ given by [12]. As in the setting of Definition 2.4.1, $\mathrm{CF}^\bullet((L_0,\nabla_0),(L_1,\nabla_1))$ is generated on the transverse intersections between L_0 and L_1. The A_∞ bimodule structure comes from counting pseudoholomorphic treed strips. We require L_1 to be compact to avoid issues of determining how to apply wrapping Hamiltonians in the definition.

Example 5.0.1 (running example, continued) We return to Example 4.1.1. Now we bring in the second Lagrangian $(L_{\underline{V}},\nabla_0)$, which we give the trivial local system. The bimodule $\mathrm{CF}^\bullet((L_{\underline{V}},\nabla_0),(F_q,\nabla_1))$ has the same generators as Example 2.4.4; following Notation 2.4.5, we call these generators x_J^{01}, where $J\subset\{0,\dots,n-k\}$. Since neither F_q nor $L_{\underline{V}}$ bounds a holomorphic disk, the differential agrees with Example 2.4.4,

$$m^1(x_I^{01})=\sum_{I\lessdot J}\pm T^{\lambda_0}(\mathrm{id}-P_{c_j}^{\nabla_1})x_J.$$

where λ_0 is the area of the small holomorphic strips.

We now describe the module product structure. This is given by counts of configurations of a Morse flow-line on F_q which are incident to a strip with boundary on $F_q\cup L_{\underline{V}}$. Recall that the Hamiltonian pushoff for $L_{\underline{V}}$ is given by (3) while the Morse function for F_q is given by (6). As before, we use

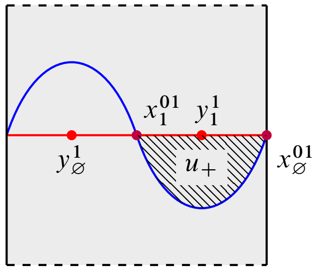

Figure 8: The treed strip contributing to the bimodule product $m^2(x_\varnothing^{01} \otimes y_1^1) = x_1^{01}$.

$\{y_I^1\}_{I \subset \{1,\dots,n\}}$ to label the critical points of $f: F_q \to \mathbb{R}$. The moduli space of strips from $\underline{x}_I^{01}$ and $\underline{x}_J$ is nonempty when $I < J$; the boundary of the strips sweep out the subtorus spanned by the indices of $J \setminus I$. The downward flow space of y_K is the subtorus spanned by indices $\{1,\dots,n\} \setminus K$. These two subtori intersect transversely only when $(J \setminus I) \sqcup (\{1,\dots,n\} \setminus K) = \{1,\dots,n\}$, which can be rephrased as

$$J = K \cup I, \quad K \cap I = \varnothing.$$

See Figure 8 for a treed strip that contributes to the product. From this, it follows that the module product structure is given by

$$m^2(x_I^{01} \otimes y_J^1) = \begin{cases} P_{\partial u^+}^{\nabla_1} T^{|J|\lambda_0} x_{I \cup J}^{01} & \text{if } I \cap J = \varnothing \text{ and } I \cup J \subset \{0,\dots,n-k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $|J|\lambda_0$ is the area of a holomorphic strip from x_i^{01} to $x_{I \cup J}^{01}$, and $P_{\partial u}^{\nabla_1}$ is the holonomy of the local system along the F_q boundary of the strip. We remark that when $J = \varnothing$, the same formula holds (simply that u^+ is regarded as the constant strip at x_I^{01}). The map

$$m^2(x_\varnothing^{01}, -): \mathrm{HF}^1((F_q, \nabla_0)) \to T^{\lambda_0}\, \mathrm{HF}^1((L_{\underline{V}}, \nabla_0), (F_q, \nabla_1))$$

surjects whenever the local system ∇_1 has holonomy of the form $\mathrm{id} + T^{\lambda_1} A$ along all the F_q boundary of all strips.

5.1 Definition of support

When Lagrangians (L_0, ∇_0) and (L_1, ∇_1) are unobstructed by bounding cochains b_0 and b_1, we can deform the Lagrangian intersection Floer cohomology $\mathrm{CF}^\bullet((L_0, \nabla_0), (L_1, \nabla_1))$ by these bounding cochains to obtain $\mathrm{CF}^\bullet((L_0, \nabla_0, b_0), (L_1, \nabla_1, b_1))$, a $\mathrm{CF}^\bullet(L_0, \nabla_0, b_0)$–$\mathrm{CF}^\bullet(L_1, \nabla_1, b_1)$ bimodule. Since $\mathrm{CF}^\bullet(L_j, \nabla_0, b_i)$ has no curvature, the differential

$$m^1: \mathrm{CF}^\bullet((L_0, \nabla_0, b_0), (L_1, \nabla_1, b_1)) \to \mathrm{CF}^\bullet((L_0, \nabla_0, b_0), (L_1, \nabla_1, b_1))$$

squares to zero, giving us cohomology groups which we can study.

Definition 5.1.1 Let $(L, \nabla, b) \subset X_A$ be an admissible Lagrangian brane. The *A-tropicalization* of (L, ∇, b) is the set

$$\mathrm{TropA}(L, \nabla, b) := \{q \mid \exists (F_q, \nabla') \text{ such that } \mathrm{HF}^\bullet((L, \nabla, b), (F_q, \nabla')) \neq 0\}.$$

Remark 5.1.2 Suppose that there is a bounding cochain b' for F_q such that

$$\mathrm{HF}^\bullet((L,\nabla,b),(F_q,\nabla',b') \neq 0.$$

As F_q is tautologically unobstructed, a general principle of Lagrangian Floer cohomology (the divisor axiom) states that there exists a local system called ∇'' such that

$$\mathrm{HF}^\bullet((L,\nabla,b),(F_q,\nabla'')) = \mathrm{HF}^\bullet((L,\nabla,b),(F_q,\nabla',b')). \tag{7}$$

The local system ∇'' is usually denoted by $\exp(b')$. To our knowledge, the divisor axiom has not been proven for the [12] model of Lagrangian intersection Floer cohomology. In [9] a proof of the divisor axiom was given for the de Rham version of open Gromov–Witten invariants. The central idea of the proof is that the coefficients in the exponential function make an appearance through the application of the "forgetting boundary points" relation between moduli spaces of holomorphic disks. The coefficients $1/k!$ in the expansion of the exponential function show up via the number of ways one can forget boundary marked points. Under the assumptions that Auroux uses, the forgetful axiom for pseudoholomorphic disks holds. In the Charest–Woodward model for $\mathrm{CF}^\bullet(L_0,L_1)$ we do not expect that perturbations for Morse theory admit a "forgetting marked point" axiom. In our setting (where $\omega(\pi_2(X_A,F_q))=0$) the arguments used in Lemma 5.2.2 show that for all (F_q,∇',b') there exists (F_q,∇'') such that the identity on (F_q,∇',b') factors through (F_q,∇'') and vice-versa. Provided that a Charest–Woodward model of the Fukaya *category* with homotopy unit exists, this would prove (7) (although not give the closed-form expression for ∇' as the exponential of the bounding cochain, as in the de Rham version). From the divisor axiom, it follows that the A-tropicalization can be rewritten as

$$\mathrm{TropA}(L,\nabla,b) = \{q \mid \exists (F_q,\nabla',b') \text{ with } \mathrm{HF}^\bullet((L,\nabla,b),(F_q,\nabla',b')) \neq 0\}. \tag{8}$$

There remain some subtle differences between bounding cochains and local systems in general. It is clear that we can only expect to replace bounding cochains with local systems in the setting where L is tautologically unobstructed. Furthermore, we do not expect that when L is tautologically unobstructed that we can replace (L,∇) with (L,b). This is because $\mathrm{val}(b)>0$, so it can only be expected to represent local systems whose holonomy is of the form $\mathrm{id}+T^\lambda A$ where $A\in U_\Lambda$ and $\lambda>0$. In the specialization to Lagrangian tori in a Lagrangian torus fibration, we believe that the requirement that $\mathrm{val}(b)>0$ may be loosened to $\mathrm{val}(b)\geq 0$ by application of the reverse isoperimetric inequality (in the same way that the reverse isoperimetric inequality is used to prove that the family Floer sheaf has structure coefficients defined over an affinoid algebra).

5.2 A-tropicalization of tropical Lagrangian lifts

In general, it is difficult to compute $\mathrm{TropA}(L,b)$, as it requires having a very good understanding of the differential on $\mathrm{CF}^\bullet((L,b),(F_q,b'))$. In Section 2.5 we performed this computation for the pair of pants $V\subset\mathbb{R}^2$. Computation of the A-tropicalization is more tractable when the Lagrangian L_V is a geometric lift of a tropical subvariety because we have good control of leading-order contributions to the differential.

The main tool that we use to compute the A-tropicalization is the following lemma:

Lemma 5.2.1 *Let L_V be a Lagrangian lift of a tropical curve, and let $U \subset Q$ be an open set such that $L|_{\pi_A^{-1}(U)} = L_{\underline{V}}|_{\pi_A^{-1}(U)}$. For $q \in U$, let $R(q)$ be the distance from q to $Q \setminus U$. There exists a function $A_{L_V,U} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that every holomorphic strip u with boundary on $L \cup F_q$ with $q \in U$ is either*

- *"small" and has image contained within $\pi_A^{-1}(U)$, and therefore describes a holomorphic strip with boundary on $L_{\underline{V}} \cup F_q$, or*
- *"large" and has symplectic energy greater than $A_{L_V,U}(R(q))$.*

Furthermore, there exists a constant $C_{L_V,U}$ such that

$$\lim_{R\to\infty} \frac{A_{L_V,U}(R)}{R} = 2C_{L_V,U}.$$

Additionally, we may replace F_q with a small Hamiltonian pushoff of F_q while preserving the bound.

Proof The lemma is an application of the reverse isoperimetric inequality from [25]. We use the proof for holomorphic strips which is employed by [13] following [6; 16]. Recall that the reverse isoperimetric inequality states that given a Lagrangian L and choice of almost complex structure J there exists a constant $A_{L,J}$ such that we can lower-bound the energy of pseudoholomorphic disks u with boundary on L by

$$A_{L,J}\ell(\partial u) \leq \int_u \omega, \tag{9}$$

where ℓ is the length as computed by the metric determined by J and ω. The reverse isoperimetric inequality for pseudoholomorphic strips requires the intersections of our Lagrangians L_V and F_q to be "locally standard" [13, Definition II.1]. Any F_q and $L_{\underline{V}}$ satisfy this criterion, so whenever $q \in U$, the Lagrangian submanifolds L_V and F_q have locally standard intersection. The reverse isoperimetric inequality from [13] can be stated as

$$sA_{L_V,F_q}\ell(\partial u \cap B(C)^c) \leq \omega(u \cap \widetilde{U}_s), \tag{10}$$

where C is chosen so that the radius-s normal neighborhoods $N_s(L_V)$ and $N_s(F_q)$ are defined for all $s < C$, and

$$\widetilde{U}_s = N_s(L_V) \cup N_s(F_q), \quad B(C) = N_C(L_V) \cap N_C(F_q).$$

We obtain a weaker but more applicable bound by making the replacement $B(C) := \pi_A^{-1}(B_C(q))$, for which F_q is a subset. With this substitution, the left-hand side of (10) only depends on the length of the boundary of u in L_V. As the excluded neighborhood $B(C)$ is monotonic in C, if we choose $C(R) < \min\left(\frac{1}{4}R, C_{L_V,U}\right)$ where $C_{L_V,U}$ is the injectivity radius of L_V we can impose the additional condition that $\pi_A(B(C(R))) \subset U$. We now bound the constant A_{L_V,F_q}, called K in [13, Corollary II.11]. It is the product of the constants

- C_1 and C_2^{-1} from [13, Proposition II.8], which provide constants of domination between the pseudometric given by a particular plurisubharmonic function h and the standard metric, and
- A, which provides a bound for $|\mathrm{grad}\, h|$ over $\widetilde{U}_s$.

A special feature of tropical Lagrangian submanifolds is that over the region $\pi_A^{-1}(U) \setminus B(C)$, the function h agrees with the distance to $L_{\underline{V}}$. As L_V is totally geodesic over $\pi_A^{-1}(U)$, we obtain that $dd^c h(-, \sqrt{-1}-)$ agrees with the metric induced by the standard metric over this chart. Therefore the constants C_1, C_2^{-1}, and A are all 1 over this region. By restricting the integral on the penultimate line of [13, (9)] to the region $\pi_A^{-1}(U)$, we may replace A_{L_V,F_q} with 1 to obtain the bound $C(R)\ell(\partial u \cap B(C(R))^c \cap \pi_A^{-1}(U)) \leq \omega(u)$.

We now show that every strip is either "small" or "large":

- Suppose that $\partial u \subset \pi_A^{-1}(U)$. Then u describes a strip with boundary on $L_{\underline{V}} \cup F_0$; we know that all such strips are contained within $\pi_A^{-1}(U)$ and have an upper bound for their energy.

- Otherwise $\partial u \not\subset \pi_A^{-1}(U)$. Let ℓ_Q be distance as measured on Q. Observe that for any path γ with one endpoint in F_q and another endpoint in $X \setminus \pi_A^{-1}(U)$ we have the bound

$$R - C(R) \leq \ell_Q(\pi_A(\gamma \cap B(C)^c)) \leq \ell(\gamma \cap B(C)^c).$$

Since the boundary of u must have at least two such paths,

$$A_{L_V,U}(R) := 2C(R)(R - C(R)) < \omega(u).$$

As $R \to \infty$, we have that $C(R) \to C_{L_V,U}$, from which we obtain the asymptotic behavior of $A_{L_V,U}$. □

The constant $C_{L_V,U}$ giving the injectivity radius of $L_{\underline{V}}$ can be computed from the tropical data of $\underline{V}$. In the 2-dimensional setting, we obtain the following nice relation. At a top-dimensional stratum (edge) e with integral primitive direction $\vec{v}$, the constant $C_{L_V,U}$ in Lemma 5.2.1 is $1/(2|\vec{v}|)$. The bound for the holomorphic energy of the strips becomes

$$2C(R - C_{L_V,U}) = \frac{R - C_{L_V,U}}{|\vec{v}|}.$$

We observe that $R/|\vec{v}|$ is the *affine radius* of the neighborhood around the point q. This can be observed in Section 2.5 and Example 6.3.2, where the affine lengths of edges in tropical curves govern the areas of holomorphic disks and strips which appear in those computations.

Lemma 5.2.2 *Let $U \subset Q$ be a neighborhood of q. Suppose that $(L, \nabla, b_0) \subset X_A$ is a Lagrangian brane whose restriction to $\pi_A^{-1}(U)$ is*

$$L|_{\pi_A^{-1}(U)} = L_{\underline{V},m}|_{\pi_A^{-1}(U)},$$

where $\underline{V} \subset U$ is a k-dimensional linear subspace, and m the multiplicity. Then there exists a choice of bounding cochain and local system on F_q such that

$$\mathrm{HF}^0((L, \nabla_0, b_0), (F_q, \nabla, b)) = \Lambda.$$

Proof To reduce notation in the proof, we will take the same simplifying assumptions as in Lemma 5.2.1. Additionally, we assume that the local system ∇_0 and bounding cochain b_0 on L are trivial.

We see that $L|_{\pi_A^{-1}(U)} \cap F_q$ cleanly intersect along a $T^{n-k} \subset F_0$; morally we now apply the spectral sequence of [48; 51] to compute the Floer cohomology of $\mathrm{CF}^\bullet(L, F_0)$ as a deformation of $C^\bullet(T^{n-k})$. Because $L|_{\pi_A^{-1}(U)} = L_{\underline{V}}|_{\pi_A^{-1}(U)}$, we can apply Lemma 5.2.1. Following Example 5.0.1, apply a Hamiltonian isotopy to L so that L and F_q intersect transversely. Take the perturbation small enough that the area of the holomorphic strips λ_0 is less than the bound $\lambda_1 := A_{L_V,U}(R)$ provided by Lemma 5.2.1. By Lemma 5.2.1, the map $m^2\colon \mathrm{CF}^\bullet(L_V, F_q) \otimes \mathrm{CF}^\bullet(F_q) \to \mathrm{CF}^\bullet(L_V, F_q)$ agrees with Example 5.0.1 at valuation less than λ_1:

$$m^2(x_I^{01} \otimes x_J^1) \equiv \begin{cases} T^{\lambda_0} x_{I\cup J}^{01} & \text{if } I \cap J = \varnothing \text{ and } I \cup J \subset \{0, \dots, n-k\}, \\ 0 & \text{otherwise,} \end{cases} \quad \mod T^{\lambda_1}. \tag{11}$$

Let $\mathrm{CF}^\bullet(F_q, \Lambda_{\geq 0})$ and $\mathrm{CF}^\bullet(L_V, F_q, \Lambda_{\geq 0})$ be the filtered A_∞ algebra and bimodule where we use $\Lambda_{\geq 0}$ rather than Λ-coefficients. It follows that the map on chains

$$m^2\colon (x_\varnothing^{01}) \otimes \mathrm{CF}^1(F_q, \Lambda_{\geq 0}) \to \mathrm{CF}^1(L_V, F_q, \Lambda_{\geq \lambda_0})/\mathrm{CF}^1(L_V, F_q, \Lambda_{\geq \lambda_1})$$

surjects. Hence $\mathrm{CF}^\bullet(L_V, F_q, \Lambda_{\geq 0})$ as a right $\mathrm{CF}^\bullet(F_q, \Lambda_{\geq 0})$ module satisfies the criterion of Lemma B.3.1 and there exists $b \in \mathrm{CF}^\bullet(F_q)$ such that $\mathrm{HF}^0(L_V, (F_q, b)) \neq 0$.

To extend to the setting where L has a local system ∇_0, we simply require that F_q be equipped with a local system ∇_1 which agrees with ∇_0 on the torus spanned by the classes $\{c_1, \dots, c_{n-k}\}$. □

Remark 5.2.3 The constant λ_0 can be taken to zero provided that one works with a model of $\mathrm{CF}^\bullet(L_V, F_q)$ which allows for clean intersections between L_V and F_q; the proof of Lemma B.3.1 becomes slightly simpler in that setting. The pearly model developed by [12] allows for such configurations of Lagrangian submanifolds.

Corollary 5.2.4 *Let (L, ∇_0, b_0) and (F_q, ∇, b) be as above. Then*

$$\mathrm{HF}^\bullet((L, \nabla_0, b_0), (F_q, \nabla, b)) = \bigwedge_{i \in \{1,\dots,n-k\}} \Lambda\langle x_i\rangle.$$

Proof Again for expositional purposes, we assume that ∇_0 and ∇ are trivial local systems, assume that the multiplicity of the local model $\underline{V}$ is 1, and suppress the bounding cochain on L. On chains, the action of $\mathrm{CF}^\bullet(F_q, b)$ on $\mathrm{CF}^\bullet(L, (F_q, b))$ is a deformation of the action of $\mathrm{CF}^\bullet(F_q, b)$ on $\mathrm{CF}^\bullet(L, (F_q, b))$. By using an argument on filtration similar to the one above, the map

$$m^2(x_\varnothing, -)\colon \mathrm{CF}^\bullet((F_q, b)) \to \mathrm{CF}^\bullet(L, (F_q, b))$$

is a surjection. As every class in $\mathrm{CF}^\bullet((F_q, b))$ is closed and we've proven that $x_\varnothing$ is closed, every element in $\mathrm{CF}^\bullet(L, (F_q, b))$ is closed. This proves that $m^1_{\mathrm{CF}^\bullet(L,(F_q,b'))} = 0$, and that $\mathrm{HF}^\bullet(L, (F_q, b)) = \mathrm{CF}^\bullet(L, (F_q, b)) = \bigwedge_{i\in\{1,\dots,n-k\}} \Lambda\langle x_i\rangle$. □

Corollary 5.2.5 *Let (L_V, b) be an unobstructed geometric Lagrangian lift of V. Then $V^{(0)} \setminus V^{(1)} \subset \mathrm{TropA}(L_V, b)$.*

If we assume (8), this immediately follows.

Proof The proof of Lemma 5.2.2 can be modified to replace the bounding systems everywhere with local systems. The needed observation is that the map from the space of local systems

$$H^1(F_q, U_\Lambda) \to \mathrm{CF}^1(L_V, F_q, \Lambda_{\geq\lambda_0})/\mathrm{CF}^1(L_V, F_q, \Lambda_{\geq\lambda_1}), \quad \nabla \mapsto m^1_{(F_q,\nabla)}(x_\varnothing),$$

is surjective. The same argument as in Lemma B.3.1 can be used to construct a local system term by term so that $m^1_{(F_q,\nabla)}(x_\varnothing) = 0$. See also [53, Proposition 5.13], which proves a similar statement for tropical curves using the implicit function theorem [1, Section 10.8]. □

6 B-realizability and unobstructedness

6.1 HMS for $(\mathbb{C}^*)^n$

6.1.1 Construction of the mirror space Given $\pi_A\colon X_A \to Q$ a Lagrangian torus fibration, there is a rigid analytic space X_B with a tropicalization map $\mathrm{TropB}\colon X_B \to Q$. As a set, X_B is the set of Lagrangian torus fibers equipped with a U_Λ local system,

$$X_B := \{(F_q, \nabla)\}$$

which comes with a map $\pi_B\colon X_B \to Q$ given by $(F_q, \nabla) \mapsto q$. When $Q = \mathbb{R}^n$, the points of X_B are in bijection with $(\Lambda^*)^n$. We now describe, following [4; 17], how this can be realized as the set of points of a rigid analytic space. We also recommend the discussion in [53, Section 5.1].

The *Tate algebra* in n-variables over Λ is the set of formal power series

$$T_n := \left\{ \sum_{A\in\mathbb{Z}^n} f_A z^A \;\middle|\; f_A \in \Lambda,\ \mathrm{val}(f_A) \to \infty \text{ as } |A| \to \infty \right\},$$

which is equipped with the *sup-norm*

$$\left\| \sum_{A\in\mathbb{Z}^n} f_A z^A \right\| := \max_A |f_A| \geq 0.$$

We note that the maximal ideals of T_n are $\{(f_1, \ldots, f_n) \mid \mathrm{val}(f_i) \leq 1\}$.

To build our spaces we will glue together *affinoid algebras*, which are quotients of the Tate algebra. The affinoid algebras we will look at are the polytope algebras. Given a bounded rational polytope $P \subset \mathbb{R}^n$, define

$$\mathcal{O}_P := \left\{ \sum_{A\in\mathbb{Z}^n} f_A z^A \;\middle|\; \mathrm{val}(f_A) + Ap \to \infty \text{ as } \|A\| \to \infty \text{ for all } p \in P \right\}.$$

This is the affinoid algebra. The elements of this affinoid algebra have the property that they converge when evaluated on $z \in (\Lambda^*)^n$ with $\mathrm{val}(z) \in P$. Furthermore, the points of $\mathcal{O}_P$ are seen to be in bijection with the points of $\pi_B^{-1}(P)$. When Q is compact X_B can be covered by finitely many sets $\pi_B^{-1}(P)$, giving X_B the structure of a rigid analytic space.

6.1.2 From Lagrangians to coherent sheaves Due to the limitations on currently existing constructions for Fukaya categories, we do not have homological mirror symmetry for a category of nonexact Lagrangian submanifolds in $(\mathbb{C}^*)^n$. However, different aspects of this homological mirror symmetry statement exist in the literature with strengthened hypotheses.

- The family Floer functor associates to a compact Lagrangian torus fibration $\pi_A : X_A \to Q$ a rigid analytic space $X_B \to Q$ whose points are in bijection with Lagrangian tori $F_q \subset X_A$ equipped with a U_Λ local system. Furthermore, [5, Theorem 2.10] constructs a faithful A_∞ functor $\mathcal{F} \colon \mathrm{Fuk}^{\mathrm{taut}}(X_A) \to \mathrm{Perf}(X_B)$. Here $\mathrm{Fuk}^{\mathrm{taut}}(X_A)$ is the Fukaya category of tautologically unobstructed Lagrangian submanifolds.

- In the exact setting, we have a complete proof of homological mirror symmetry for $(\mathbb{C}^*)^n$. The proof comes from recasting a section $L(0)$ of the fibration $\pi_A \colon (\mathbb{C}^*)^n \to Q$ as a cotangent fiber in T^*T^n, which is known to generate the exact Fukaya category. A computation shows that the A_∞ algebra $\mathrm{CF}^\bullet(L(0), L(0))$ is homotopy equivalent to $\hom(\mathcal{O}_{\mathbb{C}^n}, \mathcal{O}_{\mathbb{C}^n})$. In fact, we have a little bit more: it is known that the partially wrapped Fukaya category is mirror to the derived category of coherent sheaves on a toric variety [2; 37].

For this paper, we will only compute $\mathrm{CF}^\bullet((L_V, b), (F_q, \nabla))$, which means that we need substantially less than an HMS functor of [5].

Theorem 6.1.1 [4] *Consider the Lagrangian torus fibration $\pi_A \colon X_A \to Q$, with Q **compact**. From this data we can construct a rigid analytic mirror space X_B whose points z are in bijection with pairs (F_q, ∇). For any **tautologically unobstructed** Lagrangian brane $L \subset X_A$, there exists a coherent sheaf $\mathcal{F}(L)$ on X_B such that*

$$\hom(\mathcal{F}(L), \mathcal{O}_z) = \mathrm{HF}^0(L, (F_q, \nabla)).$$

Assumption 6.1.2 Theorem 6.1.1 still holds under the following weakened assumptions:

(∗) The base is allowed to be $Q = \mathbb{R}^n$, and we additionally require that the Lagrangian L be monomially admissible.

(∗∗) The Lagrangian L is allowed to be unobstructed by bounding cochains, in which case there exists a coherent sheaf $\mathcal{F}(L, b)$ on X_B such that

$$\hom(\mathcal{F}(L, b), \mathcal{O}_z) = \mathrm{HF}^0((L, b), (F_q, \nabla)).$$

We now discuss the difficulties, expectations, and progress of proving the assumption. The primary difficulties arise from noncompactness and unobstructedness.

Noncompactness presents three immediate issues. The first is Gromov compactness. We expect that after one places appropriate taming conditions on our Lagrangian submanifolds (as in Appendix A) the moduli spaces needed to construct the family Floer functor can be given appropriate compactifications.

The second more difficult issue regards the role that wrapping plays in computing the Floer cohomology between two noncompact Lagrangians. In the exact setting, the morphism space between two Lagrangians is computed as the limit of $\mathrm{CF}^\bullet(\phi^i(L_0), L^i)$, where ϕ^i is a wrapping Hamiltonian, and the limit is taken over continuation maps. In the nonexact setting, these continuation maps have a nonzero valuation, and only have inverses defined over the Novikov field (with possibly negative valuation). To our knowledge, this version of the Fukaya category has not been constructed. However, since for our application we only need to compute Floer cohomology against Lagrangian torus fibers (which are compact), we can ignore the issues of the wrapping Hamiltonian.

Finally, there is the issue of coherence of $\mathcal{F}(L,b)$. Here we use the monomial admissibility condition. We recall the proof of coherence when Q is compact. The sheaf $\mathcal{F}(L,b)$ is constructed by defining it over affinoid domains on the mirror, which correspond to convex domains $U \subset Q$. The convex domain U is "small enough" if there exists a Hamiltonian isotopy of L such that it intersects all Lagrangian torus fibers F_q with $q \in U$ transversely. Over each small enough U, the sheaf is computed by $\mathrm{CF}^\bullet((L,b),(F_q,\nabla)) \otimes \mathcal{O}_U$, where $\mathcal{O}_U$ is the affinoid ring of the affinoid domain $X_{U,B}$ associated to the convex domain U. Since $\mathrm{CF}^\bullet((L,b),(F_q,\nabla))$ is finitely generated, and (in the compact setting) we can cover Q with finitely many such U, we obtain that the mirror sheaf is coherent. If we drop the condition of Q being compact, and impose the condition that L is monomially admissible, we can still cover Q with a finite set of convex (possibly noncompact) small enough domains $U \subset Q$ by using invariance of the Lagrangian submanifold under symplectic flow in the direction of the monomial ray over each monomial region.

We now remark upon the difficulty of unobstructedness. Remark 1.1 of [5] states that the "tautologically unobstructed" hypothesis for construction of the family Floer functor is technical in nature, and it is expected that the family Floer functor should carry through using unobstructed Lagrangian submanifolds. As we do not require functoriality, such an adaptation of family Floer cohomology to the Charest–Woodward model would not require studying moduli spaces beyond those already studied in [12]. We believe the main items left to prove for this construction are the following:

- It must be shown that "Fukaya's trick" for pulling back a perturbation datum between Lagrangian fibers over sufficiently small convex domains can be worked out in the more technically challenging setting of domain-dependent perturbations. This does not appear to present a problem when working with the setup of [12].

- It must be shown that "homotopies of continuation maps" exist in the version of Lagrangian intersection Floer cohomology one is working with. In [12], continuation maps are constructed using holomorphic quilts. There is also an additional challenge of showing that one can construct homotopies of continuation maps corresponding to changes in the choice of stabilizing divisor.

Finally, we note that the work in progress of Abouzaid, Groman, and Varolgunes generalizing [24; 59] to the Fukaya category will prove homological mirror symmetry for unobstructed Lagrangian submanifolds of $(\mathbb{C}^*)^n$, giving us Assumption 6.1.2.

Remark 6.1.3 A different approach that would bypass family Floer theory would be to expand homological mirror symmetry for toric varieties [2; 37] in the nonexact setting. This would involve developing [23] to the nonexact setting. While there is no clear obstruction to expanding the Liouville sector framework to include obstructed Lagrangian submanifolds that are geometrically bounded, there are at least two technical and challenging issues that would need to be overcome. Many of the arguments used in [23] would have to be carefully redone by replacing geometric bounds obtained by energy and exactness with other methods for bounding holomorphic disks. In fact, these techniques are already employed in a limited capacity in [23] for the proof of the Künneth formula (as products of cylindrical Lagrangians are usually not cylindrical). The second issue is understanding how to incorporate curvature into the homological algebra constructions employed by [23]. One possible workaround would be to first construct the partially wrapped precategory of Lagrangian branes that are equipped with bounding cochains (which is an uncurved filtered A_∞ precategory) and localize at continuation maps to construct the partially wrapped category. This already requires some care, as it is not immediately clear how the filtration would play a role in this localization (the continuation maps would have positive energy, so there may be convergence issues). The second, more ambitious approach would be to attempt to construct a "curved partially wrapped Fukaya category", by starting with a partially wrapped Fukaya precategory whose objects are (potentially obstructed) Lagrangian submanifolds. This second approach would require one to understand what a filtered A_∞ precategory is and also to construct localizations of these categories.

6.2 Unobstructed Lagrangian lift implies B-realizability

By employing [6] (with the possible extensions stated in Assumption 6.1.2) we can associate to each Lagrangian brane (L_V, b) a closed analytic subset of X_B:

$$Y(L_V, b) := \mathrm{Supp}(H^0(\mathcal{F}(L_V, b))).$$

Corollary 6.2.1 *Consider the Lagrangian torus fibration $\pi_A : (\mathbb{C}^*)^n = X_A \to Q$ and a tropical subvariety $V \subset Q$. Suppose that (L_V, b) is a Lagrangian brane lift of V. Then*

- *(L_V, b) is an A-realization of V in the sense that $\mathrm{TropA}(L_V, b) = V$,*
- *V is B-realizable.*

Proof By Assumption 6.1.2, $\mathrm{TropA}(L_V, b) = \mathrm{TropB}(Y(L_V, b))$. In Corollary 5.2.5, we proved that $V^{(0)} \subset \mathrm{TropA}(L_V, b) \subset V$. Since $Y(L_V, b)$ is a closed analytic subset, $\mathrm{TropB}(Y(L_V, b))$ is the union of closed rational polyhedra in $N_\mathbb{R}$ [27, Proposition 5.2]. As a result, TropB is closed and contains $\overline{V^{(0)}} = V$. It follows that $Y(L_V, b)$ is a closed analytic subset of X_B which realizes V. □

Corollary 6.2.2 *Assuming Assumption 6.1.2(∗)–(∗∗), let V be a smooth hypersurface or a smooth genus-0 tropical curve in $\mathbb{R}^n$. Then V is B-realizable.*

Corollary 6.2.3 *Assuming Assumption 6.1.2(∗∗), let V be a smooth tropical hypersurface of a tropical abelian variety $Q = \mathbb{R}^n / M_\mathbb{Z}$. Then V is B-realizable.*

Corollary 6.2.4 *Without assuming any portion of Assumption 6.1.2, let V be a 3-valent tropical curve in a tropical abelian surface Q. Then V is B-realizable.*

Proof The condition of 3-valency comes from using

- [35] to build an affine dimer model associated to each 3-valent vertex, and
- [33] to build tropical Lagrangian lifts from a dimer model.

We now address why Assumption 6.1.2 may be dropped. Since Q is a tropical abelian surface (and is therefore compact), the symplectic manifold X_A is compact. Since the Lagrangian lift L_V is graded of dimension 2, it is tautologically unobstructed for a generic choice of almost complex structure (as Maslov index 0 disks appear in expected dimension -1). □

6.3 Nonplanar tropical curves do not have tautologically unobstructed lifts

Even in the setting where V is a genus-0 tropical curve, it is rare for the Lagrangian lift L_V to be a tautologically unobstructed Lagrangian submanifold.

Before constructing an example, we observe that the valuations of the "big-strips" in Lemma 5.2.2 are dictated by the radius of the neighborhood U_q that we can construct around the point q which is disjoint from $V^{(1)}$. In particular, this can be applied to [53, Proposition 5.10] to show that tautologically unobstructed Lagrangian lifts of tropical curves have supports that extend to an appropriate toric compactification of the mirror algebraic torus.

Proposition 6.3.1 *Let Σ be a fan. Suppose that V is a tropical curve with semi-infinite edges in the directions of the rays of Σ. Suppose the fan of Σ has the additional property that $\langle \alpha, \beta\rangle \leq 0$ for all 1-dimensional cones $\alpha \neq \beta$ and $\langle -, -\rangle$ is the standard inner product. Then $Y((L_V, b), 0)$ compactifies to a rigid analytic space inside $X_B(\Sigma)$, the rigid analytic toric variety with fan Σ.*

Proof We first describe the rigid analytic structure on $X_B(\Sigma)$ given by [49]. From [47], the space $X_B(\Sigma)$ comes with a fibration $\mathrm{TropB}\colon X_B(\Sigma) \to Q(\Sigma)$, which is a partial compactification of Q; see [49, Definition 3.6]. Rabinoff then covers $X_B(\Sigma)$ with charts given by the max-spec of affinoid algebras.

Let $P_\sigma \subset Q$ denote a convex set which can be written as $P' + \sigma$ for $\sigma \in \Sigma$ and some convex compact polytope $P' \subset Q$. Associated to P_σ is a subset $\overline{P}_\sigma \subset Q(\Sigma)$, and an affinoid algebra

$$\mathcal{O}_{P_\sigma} := \left\{ \sum_{A \in (\sigma^\vee \cap \mathbb{Z}^n)} f_A z^A \;\middle|\; \mathrm{val}(f_a) + ap \to \infty \text{ as } \|A\| \to \infty \text{ for all } p \in P_\sigma \right\}.$$

We can cover $X_B(\Sigma)$ with charts given by the max-spec of $\mathcal{O}_{P_\sigma}$ (which covers $\mathrm{TropB}^{-1}(\overline{P}_\sigma)$).

We now unpack what it means for a Lagrangian submanifold (L, b) constructed via family Floer theory to give a coherent sheaf $\mathcal{F}((L, b))$ on the rigid analytic space $X_B(\Sigma)$. In the family Floer construction, for a sufficiently small convex polytope P in the base of Q, one takes a Hamiltonian perturbation L_P of L so that $L_P|_{\pi_A^{-1}(P)}$ is a disjoint set of flat sections of $\pi_A^{-1}(P) \to P$, and that the bounding cochain is similarly parallel to the flat section. As a result, we may identify the chains $\mathrm{CF}^\bullet((L_P, b), (F_q, \nabla))$ for all $q \in P$. Additionally, for $q \in P$, one can appropriately choose almost complex structures (using Fukaya's trick) so that the moduli spaces of strips contributing to the differential on $\mathrm{CF}^\bullet((L, b), (F_q, \nabla))$ does not depend on q. Because the bounding cochain on L is parallel to the flat section, the contribution of b to the differential on $\mathrm{CF}^\bullet((L, b), (F_q, \nabla))$ does not depend on $q \in P$. As a consequence, the dependence of the structure coefficients $\langle m^1_{(L,b),(F_q,\nabla)}(x), y\rangle$ on (F_q, ∇) factors through the flux homomorphism. Pick a basepoint x_0 on F_q and for each $x \in L \cap F_q$ a path γ_x from x_0 to x. By identifying (F_q, ∇) with a point $z \in \mathrm{TropB}^{-1}(P)$, we obtain that

$$\langle m^1_z(x), y\rangle = \sum_{a \in H_1(F_q)} c_a z^a,$$

where c_a is the area and local-system weighted count of pseudoholomorphic strips u such that

$$[\gamma_x \partial_{F_q} u \gamma_y^{-1}] = a.$$

The Lagrangian L defines a complex of sheaves $\mathcal{F}(L)$ over $(X_B|_P)$ if these structure coefficients belong to $\mathcal{O}_P$. The restriction maps compose up to homotopy of the chain complex. This is proven using the reverse isoperimetric inequality to bound the area of holomorphic strips u (which govern convergence) below by the winding of the F_q boundary component of U (which governs the exponent appearing in z^a). To obtain a coherent sheaf of complexes on X_B, one must be able to cover Q with finitely many sufficiently small sets P. When Q is compact, this is always possible. In the setting we study, we must take some of the sets P to be of the form P_σ in order to construct a finite cover.

We now perform this construction for L_V our tautologically unobstructed Lagrangian lift of a tropical curve V. Let e be a semi-infinite edge of V pointing in the α direction, where $\alpha \in \Sigma$ is a 1-dimensional cone. Then there exists a P_α such that $V|_{P_\alpha}$ is a 1-dimensional ray. Since $\langle \alpha, \beta\rangle < 0$ for all 1-dimensional rays $\beta \neq \alpha$, the projection $\chi^\alpha : X_A \to \mathbb{C}$ given by the α-monomial has the property that the $\chi^\alpha|_{L_V} : L_V \to \mathbb{C}$ fibers over a real ray outside of a compact set. This is the main input needed in [53, Proposition 5.10]

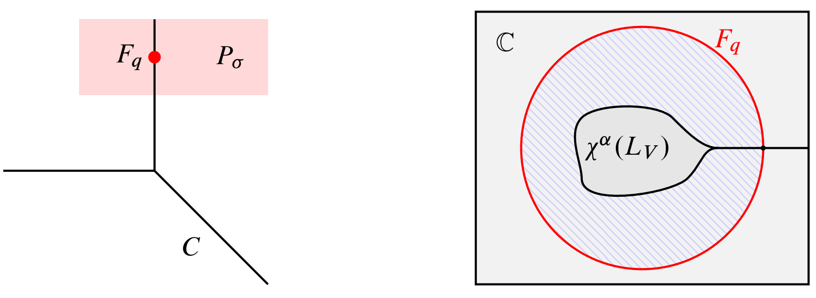

Figure 9: Monomial admissibility forces strips with $\langle \partial u, \alpha\rangle > 0$ to have large symplectic area. Left: condition on a tropical curve. Right: projection on $\chi^\alpha : X_A \to \mathbb{C}$.

to show that the differential on $\mathrm{CF}^\bullet(L_V, (F_q, \nabla))$ is of the form $\sum_{a \in H_1(F_q), \langle a, \alpha \rangle \geq 0} c_a z^a$, and that $\mathrm{val}(c_a) + ap \to \infty$ as $\|A\| \to \infty$ for all $p \in P_\alpha$. It follows that $\langle m_z^1(x), y \rangle \in \mathcal{O}_{P_\alpha}$.

We can choose a finite cover of Q by sets of the form P_σ so that $\pi_A(L_V) \subset P_\sigma$ if and only if $|\sigma| \leq 1$. It follows that $\mathcal{F}(L_V)$ defines a sheaf on $X_B(\Sigma)$. □

In the setting above (where L_V is tautologically unobstructed and equipped with the trivial local system), the above computation not only shows that $\mathcal{F}(L_V)$ extends to $X_B(\Sigma)$ but also shows that we can compute the points in the compactifying locus. For a semi-infinite edge e, let $P_\alpha = P + \langle \alpha \rangle$ be a convex polytope whose only intersection with V is along the edge e. Without loss of generality, we will assume that the edge e is of the form $(t, 0, \ldots, 0) \subset Q = \mathbb{R}^n$, with t tending to ∞. We can write the max-spec of P_α as

$$\{(z_1, \ldots, z_n) \in \Lambda \times (\Lambda^*)^{n-1} \mid \mathrm{val}(z_1, \ldots, z_n) \in \bar{P}_\alpha \subset (\mathbb{R} \cup \infty) \times \mathbb{R}^{n-1}\}.$$

We prove that $(0, 1, \ldots, 1) \in \mathrm{Supp}(\mathcal{F}(L_V))$. The $\langle a, \alpha \rangle = 0$ terms of $\langle m_z^1(x_\varnothing), x_I \rangle$ agree with holomorphic strips for the differential on $\mathrm{CF}^\bullet(L_{\underline{e}}, (F_q, \nabla))$, so we can write

$$\langle m_z^1(x_\varnothing), x_I \rangle = (1 - z^{\langle I, a \rangle}) + \sum_{a \in H_1(F_q), \langle a, \alpha \rangle > 0} c_a z^a.$$

When we have a sequence of points $\{z^k\}_{k \in \mathbb{N}}$ with the property that $m^1_{z^k}(x_\varnothing) = 0$ (ie $z^k \in \mathrm{Supp}(\mathcal{F}(L_V))$) and $\lim_{k \to \infty} \mathrm{val}(z_1^k) = \infty$ (so that the limit belongs to the compactifying toric divisor), the above equation states that $\lim_{k \to \infty} \mathrm{val}(z_i^k) = 1$ for all $i \neq 1$. We conclude that the closure of $\mathrm{Supp}(\mathcal{F}(L_V))$ inside of $X_B(\Sigma)$ contains the point $(0, 1, \ldots, 1)$.

We now construct an example of a Lagrangian brane lift of a tropical curve that is unobstructed, but not tautologically unobstructed.

Example 6.3.2 Consider the tropical line $V_c \in \mathbb{R}^3$ drawn in Figure 10. The tropical line V_c has two pants centered at the points $(0, 0, 0)$ and $(-c, -c, 0)$, whose legs at $(0, 0, 0)$ point in the directions

$$e_1 = \langle 1, 0, 0 \rangle, \quad e_2 = \langle 0, 1, 0 \rangle, \quad e_c = \langle -1, -1, 0 \rangle,$$

and whose legs at $(-c, -c, 0)$ point in the directions

$$e_3 = \langle 0, 0, 1 \rangle, \quad e_4 = \langle -1, -1, -1 \rangle, \quad -e_c = \langle 1, 1, 0 \rangle.$$

We prove that L_{V_c} bounds a holomorphic disk for all but at most 1 value of c.

Assume for contradiction that for all values of c the Lagrangian submanifold L_{V_c} is tautologically unobstructed, and requires no bounding cochain. Then the Lagrangians L_{V_c} satisfy the conditions of Proposition 6.3.1, so each $Y_{V_c} := \mathrm{Supp}(\mathcal{F}(L_{V_c}))$ compactifies to give a curve inside of $\mathbb{P}^3$. Since this curve intersects each of the toric divisors at a single point, we conclude that every Y_{V_c} is a line in $\mathbb{P}^3$. Furthermore, every one of these lines contains the points $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$ in $\mathbb{P}^3$. Since a line in $\mathbb{P}^3$ is determined by two points, this implies that $Y_{V_c} = Y_{V_{c'}}$. However, as $V_c \neq V_{c'}$, they cannot be realized by the same subvariety, a contradiction.

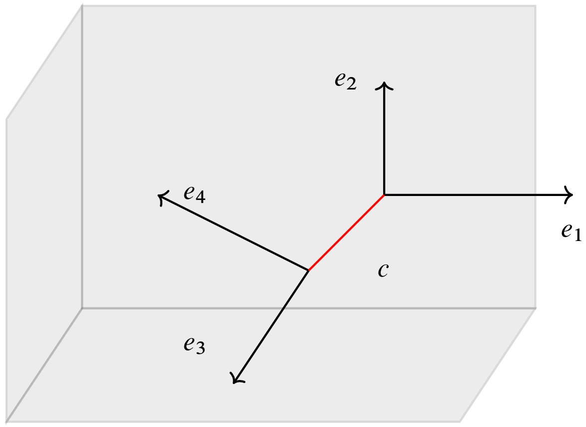

Figure 10: A tropical line V_c. The Lagrangian lift L_{V_c} necessarily bounds a holomorphic disk; we conjecture that the projection to the base of the Lagrangian torus fibration of this holomorphic disk lives over the red edge and has area controlled by the affine length c.

This doesn't contradict the realizability of V_c. Indeed, by Corollary 4.3.3, the bounding cochain on L_{V_c} need only be supported on three of the four legs of L_{V_c}. However, the above argument shows that one cannot construct a bounding cochain for L_{V_c} which restricts to zero on the two semi-infinite edges which share a vertex (which implies that the bounding cochain cannot be zero).

Using mirror symmetry, we can "back solve" for the valuation of the holomorphic disk, which necessitates the use of a bounding cochain on L_{V_c}. We may assume that the bounding cochain has trivial restriction to the e_1 edge. It follows that the tropical line Y_{V_c} may intersect toric divisors at the points $(0,1,1)$, $(1+\exp(b_1), 0, 1+\exp(b_3))$ and $(z^{-c}+\exp(c_1), z^{-c}+\exp(c_2), 0)$. Since these have to satisfy the equation of a line, there exists t such that

$$(1-t)(0,1,1)+t(\exp(b_1),0,\exp(b_2)) = (z^{-c}(\exp(c_1)), z^{-c}(\exp(c_2)), 0).$$

From examining the third term, $t=(1-\exp(b_2))^{-1}$, we already see that $b_2 \neq 0$. From examining the third term,

$$(1-\exp(b_2))^{-1}\exp(b_1) = z^c(\exp(c_1)),$$

from which we see that $\mathrm{val}(b_2)=c$. From this, we conclude that there exists a pseudoholomorphic disk of energy c on L_{V_c}.

6.4 Speculation on Speyer's well-spacedness criterion

Corollary 6.2.1 proves the forward direction of Conjecture 1.1.1. To investigate the reverse direction, we look at an example of a nonrealizable tropical curve. In [40] it was observed that every cubic curve in $\mathbb{CP}^3$ is planar (Figure 11, left). Consequently, the example drawn in Figure 11, right — a tropical cubic which is not contained within any tropical plane — cannot arise as the tropicalization of any curve in $\mathbb{CP}^3$.

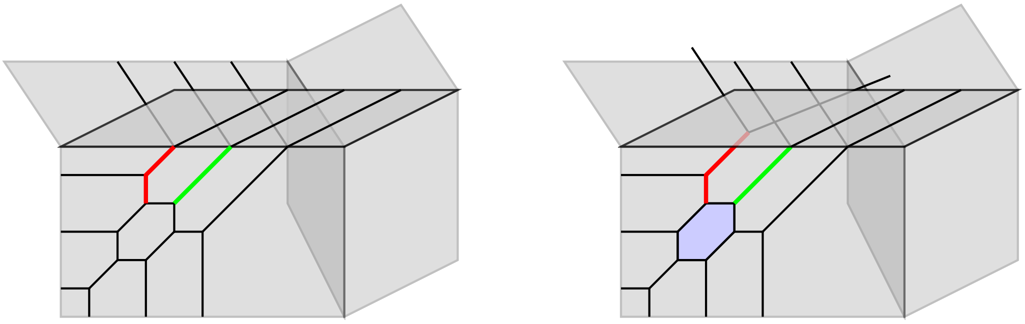

Figure 11: Left: A well-spaced tropical curve. The affine lengths of the red and green match. Right: A nonrealizable tropical curve. The affine length of the green segment is uniquely minimal among all paths from the cycle to the nonlinearity locus of the curve.

Corollary 6.4.1 *Let V be the tropical curve from* [40, Example 5.12]. *Then the standard lift of L_V is an obstructed Lagrangian.*

A general criterion for understanding this phenomenon was stated in [55]:

Theorem (Speyer's well-spacedness) *Let V be a genus-*1 *tropical curve whose cycle is contained within a linear subspace H. Let $d_1, \dots, d_k$ be the affine lengths of paths along the edges of V to the boundary of $V \cap H$. If the minimal distance occurs at least twice*, *the curve V is realizable.*

We now speculate on how Speyer's well-spacedness criterion can be understood in terms of holomorphic disks with boundary on L_V. For L_V to be unobstructed, it is necessary for the lowest-energy terms in m^0 to be nullhomologous. In particular, the set

$$\left\{u \;\middle|\; \omega(u) \leq \min_{0 \neq [\partial u'] \in H^2(L,M)} \omega(u')\right\}$$

of minimal-area nonnullhomologous disks must contain at least two elements. This matches the "two minimal distance" criterion of Speyer's well-spacedness theorem.

In [32], we saw that tropical cycles on $W \subset \mathbb{R}^2$ are related to nonregular Maslov index 0 disks with boundaries on the Lagrangian lifts L_W; it was speculated that these Maslov index 0 disks could appear regularly if they were glued onto a regular holomorphic disk or strip. In Example 6.3.2 we saw that the Lagrangian brane lift of a small neighborhood of the green segment in Figure 11, right, must have a regular disk with energy given by the affine length of the edge.

In the example given by Figure 11, right, we conjecture that there are regular holomorphic disks with boundaries on L_V whose projections under the moment map are:

- the union of the blue hexagon (a nonregular disk) and green path (a regular disk), call this speculative disk u_1, and
- the union of the blue hexagon (a nonregular disk) and red path (a regular disk), call this speculative disk u_2.

Using that the area of homology classes of disks with boundary on L_V correspond to affine length, the disks u_1 and u_2 have matching symplectic area if the affine lengths of the green and blue paths match. In this case, the homology class of $[\partial u_1]-[\partial u_2]$ doesn't wrap around the portion of the homology of L_V which arises from V, and by a similar argument to that used in Corollary 4.3.3 we see that $[\partial u_1]-[\partial u_2]\subset H_1(L_{V_\infty^{(0)}})$. We could then apply the methods used in the proof of Corollary 4.3.3 to conclude that L_V is unobstructed.

In the event that $\omega(u_2)$ is uniquely minimal, the boundary of $\partial(u_2)$ is a nontrivial homology class in $H_1(L_V)$, suggesting that the contribution to $m^0\in \mathrm{CF}^\bullet(L_V)$ is a nonremovable obstruction.

6.5 Deformations, superabundance, and not-wide

6.5.1 Geometric deformations of L and $(V,\mathcal{L})$ Given $V\subset\mathbb{R}^n$ a tropical subvariety, a Lagrangian L_V should correspond to a lift of V equipped with a line bundle. In this section, we examine how the deformations of L_V up to Hamiltonian isotopy match deformations of a tropical curve equipped with a line bundle $(V,\mathcal{L})$.

Given a fixed tropical line bundle $\mathcal{L}\to V$ we can identify deformations of $\mathcal{L}$ with $H^1(V,\mathbb{R})$; this is because deformations of invertible locally integral affine functions from U to $\mathbb{R}$ correspond to constant differences. Similarly, the deformations of $V\subset\mathbb{R}^n$ as a smooth tropical subvariety can be computed sheaf-theoretically. We choose a cover conducive to this computation. For each $v\in V$, let $\operatorname{star}(v)$ be the union of the edges that contain v. We allow v to be a leaf (at the end of a semi-infinite edge). Then the $\operatorname{star}(v)$ form a cover of V, with $\operatorname{star}(v)\cap\operatorname{star}(w)=\underline{V}_{vw}$ whenever vw is an edge. There are two types of vertices v that we must consider:

- If v is an internal vertex, then the deformations of $\operatorname{star}(v)$ are identified with the integral affine space $Nv=T_v\mathbb{R}^n=\mathbb{R}^n$.
- If v_∞ is a boundary vertex incident to edge e, then the deformations of $\operatorname{star}(v_\infty)$ are identified with the integral affine space $\mathbb{R}^{n-1}$ perpendicular to the semi-infinite edge attached to v_∞.

Over each edge e, the deformations of the tropical curve are given by the normal bundle to e. In summary, let Def_V be the sheaf of deformations of the tropical embedding of V, and let $\mathrm{Def}_{\mathcal{L}}$ be the deformations of a fixed line bundle $\mathcal{L}$ over V. We have:

$$\mathrm{Def}_V(\operatorname{star}(v))=\mathbb{R}^n,\quad \mathrm{Def}_V(\operatorname{star}(v_\infty))=e^\perp_{v_\infty},\quad \mathrm{Def}_V(\operatorname{star}(e))=e^\perp.$$

For compact Lagrangian L, the infinitesimal deformations of L up to Hamiltonian isotopy are described by classes in $H^1(L,\mathbb{R})$. Since L_V is noncompact, we only consider the *admissible* deformations of noncompact L_V which preserve the condition in Definition 3.1.1. Let $\Omega^1_{\mathrm{admis}}(L_V,\mathbb{R})$ be the 1-forms on L_V with the property that

- for each monomial region U_α, the 1-form $\eta|_{L_V\cap U_\alpha}$ is invariant under the flow in the α-direction,
- $\eta(\alpha)=0$.

We let $\Omega^0_{\mathrm{admis}}(L_V, \mathbb{R})$ be those functions which, outside of a compact set, are invariant under the flow in the α direction of the corresponding monomial region from Definition 3.1.1.

We can similarly decompose L_V into sets $L_{\mathrm{star}(v)}$, which we will take to be

- the standard Lagrangian pair of pants when v in an interior vertex such that $L_{\mathrm{star}(v)} \cap L_{\mathrm{star}(w)} = L_{\underline{V}_{vw}}$, in which case $\Omega^i_{\mathrm{admis}}(L_V, \mathbb{R}) = \Omega^i(L_V, \mathbb{R})$,
- a noncompact cylinder extending to the boundary whenever w is a vertex at a noncompact edge.

We then compute $H^1(\Omega^\bullet_{\mathrm{admis}}(L_V))$. The cohomology is the same as the first cohomology of the total complex; the first page in the spectral sequence is

$$\begin{array}{ccc} \bigoplus_{v\in V} H^0(\Omega^\bullet(L_v)) & \bigoplus_{v\in V} H^1(\Omega^\bullet(L_v)) & \cdots \\ \downarrow & \downarrow & \\ \bigoplus_{e\in V} H^0(\Omega^\bullet(L_e)) & \bigoplus_{e\in V} H^1(\Omega^\bullet(L_e)) & \cdots \\ \downarrow & \downarrow & \\ 0 & 0 & \end{array}$$

We now start to identify these with deformations of tropical curves,

$$\begin{aligned} \mathrm{Def}_V(\mathrm{star}(v)) &= H^1(\Omega^\bullet(L_v)), & \mathrm{Def}_V(\mathrm{star}(e)) &= H^1(\Omega^\bullet(L_e)), \\ \mathbb{R} &= H^0(\Omega^\bullet(L_v)), & \mathbb{R} &= H^0(\Omega^\bullet(L_e)), \end{aligned}$$

turning the first page of the spectral sequence into

$$\begin{array}{ccc} \bigoplus_{v\in V} \mathbb{R} & \bigoplus_{v\in V} \mathrm{Def}_V(e) & \cdots \\ \downarrow & \downarrow & \\ \bigoplus_{e\in V} \mathbb{R} & \bigoplus_{e\in V} \mathrm{Def}_V & \cdots \\ \downarrow & \downarrow & \\ 0 & 0 & \end{array}$$

The spectral sequence for $H^1(\Omega^\bullet(L))$ converges at the second page for this covering, so

$$H^1(\Omega^\bullet(L_V)) = H^0(V, \mathrm{Def}_V)) \oplus H^1(V, \mathbb{R}) = H^0(V, \mathrm{Def}_V) \oplus H^0(V, \mathrm{Def}_{\mathcal{L}}).$$

In general, understanding the moduli space of Lagrangian submanifolds isotopic to L_V modulo Hamiltonian isotopy is a difficult question. In the setting of Lagrangian torus fibrations, there is a smaller class of isotopies that we can hope to understand. We say that a Lagrangian isotopy $i_t \colon L_V \to X_A$ is a fiberwise isotopy if $\pi_A(i_t(q))$ is constant for all $q \in L_V$.

Conjecture 6.5.1 *Let L_V be a homologically minimal Lagrangian lift of a tropical curve V. Then the subspace of $H^1(L_V, \mathbb{R})$ arising from the flux classes of fiberwise Lagrangian isotopies is identified with $H^0(V, \mathrm{Def}_{\mathcal{L}})$. Additionally,*

$$\{\text{fiberwise isotopies}\}/\{\text{fiberwise Hamiltonian isotopies}\} \simeq H^1(V, \mathrm{Aff}^*_V)_0,$$

where $\mathrm{Aff}^*_V(U)$ *is the sheaf of invertible locally integral affine functions from* U *to* $\mathbb{R}$*, and* $H^1(V, \mathrm{Aff}^*_V)_0$ *is the connected component of the group which contains the identity.*

Previous work of Mikhalkin [43] identifies the tropical Picard group of V with $H^1(V, \mathrm{Aff}^*_V)$. Mirror symmetry, therefore, identifies fiberwise isotopies of a tropical Lagrangian L_V with modifications of the line bundle on the mirror curve Y_V.

6.5.2 Not-wide and superabundance A tropical curve is called *superabundant* if the space of deformations Def_V has a higher dimension than the expected dimension of deformations of the B-realization. Superabundance is a computable criterion that indicates that a curve may not be realizable. For example, the tropical curve examined in Section 6.4 is a superabundant curve. It is known in certain cases [14] that nonsuperabundant implies realizable.

In symplectic geometry there are two ways to make sense of deformations of Lagrangian submanifolds. The first is the deformation of geometric Lagrangian submanifolds up to Hamiltonian isotopy. The infinitesimal deformations of Lagrangian submanifolds modulo Hamiltonian isotopy are given by $H^1(L,\mathbb{R})$. The second is the component of the moduli space of objects at L [58]. The tangent space to this moduli space is $\mathrm{HF}^1(L)$. We note that as $\mathrm{CF}^\bullet(L)$ is a deformation of $C^\bullet(L)$, we have that $\dim \mathrm{HF}^1(L) \leq \mathrm{HF}^1(L)$. If $\dim \mathrm{HF}(L) = \dim H(L)$, then the Lagrangian L is called *wide*.

As the previous section identifies infinitesimal deformations of the pair $(V, \mathcal{L})$ with $H^1(L_V)$, we are led to conjecture:

Conjecture 6.5.2 *Let* V *be a smooth tropical curve, and let* L_V *be its Lagrangian lift. Then* V *is superabundant if and only if* L_V *is not wide.*

Appendix A The pearly model in symplectic fibrations

Given a compact, spin, and graded Lagrangian L inside of a rational compact symplectic manifold X, [12] constructs a filtered A_∞ algebra $\mathrm{CF}^\bullet(L, h, \mathcal{P}, D)$. In [12] it is assumed that the space X is compact. In this appendix, we outline how to extend [12] to the setting where X is noncompact and equipped with a potential function $W: X \to \mathbb{C}$, and L is a Lagrangian submanifold which is admissible with respect to W.

Definition A.0.1 Let X be a symplectic manifold, and let $W: X \to \mathbb{C}$ be a function. We say that W is a potential if there exists a compact subset $U \subset \mathbb{C}$ such that

- $W^{-1}(U)$ is compact, and
- the restriction $W: X \setminus W^{-1}(U) \to \mathbb{C} \setminus U$ is a symplectic fibration with compact fibers.

We say that a Lagrangian L is *W-admissible* if there exists $R \in \mathbb{R}$ such that $W(L) \cap \{z \mid |z| > R\} \subset \mathbb{R}_{>R}$.

Given a W-admissible L, we say that a Morse function $h\colon L \to \mathbb{R}$ is admissible if there exists $R' > R$ such that

$$W(\mathrm{Crit}(h)) \cap \{z \mid |z| > R\} \subset \{R'\},$$

and $\operatorname{grad} h$ points outwards from R' under the projection W.

Let $Y = W^{-1}(R')$. Given a W-admissible Lagrangian submanifold L, the restriction to the fiber $M := L \cap Y$ is a Lagrangian submanifold of Y. Because h points outwards along the collar $M \times \mathbb{R}_{>R'} \subset L$, the Morse complex $\mathrm{CM}^\bullet(L, h)$ is well defined. The compatibility of the Morse function with the potential function means that $h^+ := h|_M$ is a Morse function for M and that we have a map of A_∞ algebras

$$\underline{\pi}\colon \mathrm{CM}^\bullet(L, h) \to \mathrm{CM}^\bullet(M, h^+).$$

This should be interpreted as the pullback map of the inclusion of the boundary.

We show that [12] extends to the setting of W-admissible Lagrangian submanifolds:

Theorem A.0.2 *Let $W\colon X \to \mathbb{C}$ be a potential function. Let L be a W-admissible Lagrangian submanifold whose restriction to a large fiber is $M \subset Y = W^{-1}(t)$. Let $h\colon L \to \mathbb{R}$ and $h^+ := h|_M\colon M \to \mathbb{R}$ be admissible Morse functions. There exist*

- *stabilizing symplectic divisors $D_X \subset X$, $D_Y \subset Y$, and*
- *regular choices for perturbation systems $\mathcal{P}_L$ and $\mathcal{P}_M$ for L and M,*

such that the construction of [12] *can be applied to give a well-defined A_∞ algebra $\mathrm{CF}^\bullet(L, h, \mathcal{P}_L, D_X)$. Furthermore, the choices of perturbations and divisors can be taken so that the projection on chains*

$$\pi\colon \mathrm{CF}^\bullet(L, h, \mathcal{P}_L, D_X) \to \mathrm{CF}^\bullet(M, h^+, \mathcal{P}_M, D_Y)$$

is a Λ-filtered A_∞ algebra homomorphism.

The theorem consists of two statements: the construction of a pearly model of stabilized treed disks in the setting of Lagrangians which are admissible for a potential function, and the compatibility between the pearly model of total space of the fibration and the pearly model of the fiber. These are analogous to [34, Corollary C.4.2 and Theorem C.5.1] which handle the setting where $X = Y \times \mathbb{C}$ and $W\colon X \to \mathbb{C}$ is projection to the second factor. In this appendix, we prove that $\mathrm{CF}^\bullet(L, h, \mathcal{P}_L, D_X)$ is well defined; the existence of the projection $\pi\colon \mathrm{CF}^\bullet(L, h, \mathcal{P}_L, D_X) \to \mathrm{CF}^\bullet(M, h^+, \mathcal{P}_M, D_Y)$ is the same as the proof of [34, Theorem C.5.1].

To construct $\mathrm{CF}^\bullet(L, h, \mathcal{P}_L, D_X)$ one needs to

(1) construct a stabilizing divisor for X which is suitably compatible with the potential $W\colon X \to \mathbb{C}$,

(2) show that we can pick perturbations for the almost complex structure so that the map $W\colon X \to \mathbb{C}$ is holomorphic outside of a compact set, and

(3) prove that for such choices of perturbations the moduli spaces have appropriate Gromov compactifications.

Item (1): constructing a stabilizing divisor

Pick R sufficiently large so that outside of $U = B_R(0) \subset \mathbb{C}$ the Lagrangian submanifold L fibers over the positive real ray, and the map $X \setminus W^{-1}(U) \to \mathbb{C} \setminus U$ is a symplectic fibration. For $\theta \in [0, 2\pi]$ and $r \geq R$ we take a path

$$\gamma_{\theta,r}(t) = \begin{cases} Re^{i\theta(2t)} & t \in \left[0, \frac{1}{2}\right), \\ (R + (2t-1)(r-R))e^{i\theta} & t \in \left[\frac{1}{2}, 1\right), \end{cases}$$

which travels first in the angular, then radial direction from R to $re^{i\theta}$. For every path $\gamma(t)\colon I \to \mathbb{C}_{|z|>R}$ we have a symplectic parallel transport map $P_\gamma\colon Y_{\gamma(0)} \to Y_{\gamma(1)}$. Consider the monodromy $P_{\gamma_{2\pi,R}}\colon Y_R \to Y_R$ given by parallel transport around the loop $Re^{i\theta}$ in the positive direction. Pick a path of ω_Y-tamed almost complex structures $J_{Y_R,\theta}\colon [0, 2\pi] \to J_\tau(Y_R, \omega_Y)$ such that $P^*_{\gamma_{2\pi,R}} J_{2\pi} = J_0$. This gives us an endomorphism of the subbundle of the tangent spaces to the fibers

$$J_{re^{i\theta}}\colon TY_{re^{i\theta}} \to TY_{re^{i\theta}}, \quad J_{re^{i\theta}} = P^*_{\gamma_{\theta,r}} J_{Y_R,\theta}.$$

Since over every point with $|z| > R$ we have a splitting $T_{(y,z)}X = T_yY \oplus T_z\mathbb{C}$, we can give $T(X \setminus W^{-1}(U))$ the tame almost complex structure locally defined by $J_{re^{i\theta}} \oplus j_{\mathbb{C}}$.

Definition A.0.3 We say that an ω-tame almost complex structure on X is W-admissible if, when restricted to $W^{-1}(U)$, it can be written as $J_{re^{i\theta}} \oplus j_{\mathbb{C}}$ for some path of almost complex structures $J_{Y_R,\theta} \in \mathcal{J}_\tau(Y_R, \omega)$. We denote the space of such almost complex structures $\mathcal{J}_{\tau,W,R}(X, \omega_X)$.

The goal will to be to construct a stabilizing divisor $D_X \subset X$ in such a way that D_X is transverse to all $Y_{re^{i\theta}}$ with $r \geq R$, and subsequently show that there exists an open dense set of almost complex structures belonging to $\mathcal{J}_{\tau,W,R}(X, \omega_X)$ which are E-stabilized by D_X [12, Definition 4.24]. In the setting of Lagrangian cobordisms, the comparable statements are proven in [34, Appendix C.3 and Lemma C.1.3].

We first construct the divisor D_X. Take $E_X \to X$ a vector bundle whose first Chern class is $\frac{1}{2}\pi[\omega_X]$, so that the pullback $E_Y \to Y$ is a vector bundle whose first Chern class is $\frac{1}{2}\pi[\omega_Y]$. Pick a family of Hermitian structures on $E_{Y,\theta} \to Y_{Re^{i\theta}}$ depending on θ so that the curvature is $-i\omega_Y$ and so that $P^*_{\gamma_{2\pi,R}} E_{Y,2\pi} = E_{Y,0}$ as Hermitian line bundles. Let $i_{\theta_0}\colon Y_{Re^{i\theta_0}} \to X$ be the inclusion of the fiber over $Re^{i\theta_0}$. Take a Hermitian structure on $E_X \to X$ with curvature $-i\omega_X$ and the property that $i^*\theta_0 P^*_{\gamma_{\theta_0,r}|_{[1/2,1]}} E_X = E_{Y,\theta_0}$ as Hermitian line bundles.

We will construct the stabilizing divisor D_X as the zero locus of an asymptotically holomorphic section $s_{k,X}\colon X \to E_X^k$. First, using [10] we can pick asymptotically holomorphic sections $s_{k,Y}\colon Y \to E_Y^k$ with the property that $s_{k,Y}^{-1}(0)$ is disjoint from M. We obtain a second asymptotically holomorphic section by pullback $P^*_{\gamma_{2\pi,R}} s_{k,Y}$. By [8] we can find a family $s_{k,Y,\theta}$ of such sections satisfying $s_{k,Y,0} = s_{k,Y}$ and $s_{k,Y,2\pi} = P^*_{\gamma_{2\pi,R}} s_{k,Y}$.

Using this family of sections, we create an asymptotically holomorphic section $s_{k,X,\mathrm{out}}\colon X \to E_X^k$ which is given by

$$s_{k,X,\mathrm{out}} := \rho_{k,R+1}(|z|) P^*_{\gamma_{\theta,r}} s_{k,Y,\theta},$$

where $\rho_{k,R+1}(|z|)\colon \mathbb{C} \to \mathbb{R}$ is a function which is concentrated (in the sense of [10, Definition 2]) at the circle of radius $R+1$. The zero set $s^{-1}_{k,X,\mathrm{out}}(0)$ enjoys the properties that

- $s^{-1}_{k,X,\mathrm{out}}(0)$ is disjoint from L,
- for k sufficiently large, $s^{-1}_{k,X,\mathrm{out}}(0)$ is a symplectic divisor in $W^{-1}(\{z \mid |z| > R\})$, and
- $s^{-1}_{k,X,\mathrm{out}}(0)$ intersects $Y_{re^{i\theta}}$ transversely for all $r > R$.

This constructs the sections taking the place of $s_{k,X\times\mathbb{C},\mathrm{out}}$ in [34, Appendix C.3.2]. The remainder of the construction of D_X involves subsequently perturbing this section over the region $W^{-1}(U)$, which exactly follows [34, Appendix C.3.2].

Item (2): finding perturbations

The construction of an open dense set of E-stabilized almost complex structures proceeds in the same fashion as [34, Section C.3.3], which is itself based on the argument of [12, Section 4.5]. The main tool needed for the argument to run is to show that the space of almost complex structures regularizing holomorphic disks of energy up to E is dense in $\mathcal{J}_{\tau,W,R}(X,\omega_X)$. By application of the open mapping principle to W, every pseudoholomorphic disk in consideration must either

- pass through $W^{-1}(U)$, where they can be made regular through perturbations confined to the region $W^{-1}(U)$ by application of [15, Lemma 5.6], or
- be confined to a fiber $W^{-1}(t)$ with $t \in U$, in which case they can be made regular through perturbations constrained in the fiberwise direction, and since the fiber is compact, the set of such perturbations is open and dense.

Item (3): compactness of moduli spaces

The proof that the moduli spaces of pseudoholomorphic treed disks considered are compact uses that we may apply open-mapping principle type arguments for perturbations chosen from $\mathcal{J}_{\tau,W,R}(X,\omega_X)$, and that the Morse flow line components of treed disks point outwards at the boundary [34, Proposition C.4.1].

Remark A.0.4 In the examples we consider (potentials coming from tropicalized superpotentials associated to a monomial admissibility data), the fibers of the potential will in general not be compact. However, the monomially admissible condition ensures that the restriction of monomially admissible $L \subset X$ to $M \subset Y$ will be compact. As a result, all pseudoholomorphic disks contributing to treed disks will have boundary contained within a compact subset of X; we conclude that the moduli space of treed disks has compactification given by broken treed disks.

Appendix B Auxiliary results for filtered A_∞ algebras and modules

In this section, we give some background for filtered A_∞ algebras and bimodules, as well as provide some methods for constructing bounding cochains using the filtration on the A_∞ algebra.

B.1 A short review of bounding cochains

The *Novikov ring* with $\mathbb{C}$-coefficients is the ring of formal power series

$$\Lambda_{\geq 0} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\, \lambda \in \mathbb{R}_{\geq 0},\, \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

The field of fractions is the Novikov field Λ. A *filtered A_∞ algebra* (A, m_A^k) is a free graded $\Lambda_{\geq 0}$-module $A^\bullet$ equipped with $\Lambda_{\geq 0}$-linear products

$$m^k \colon (A^\bullet)^{\otimes k} \to (A^{\bullet+2-k})$$

for each $k \geq 0$. These are required to satisfy the axioms of [22, Definition 3.2.20]. Among these axioms are

- the quadratic filtered A_∞ relationship
$$0 = \sum_{k_1+k'+k_2=k} (-1)^{\clubsuit(\underline{x},k_1)} (m^{k_1+1+k_2}) \circ (\mathrm{id}^{\otimes k_1} \otimes m^{k'} \otimes id^{\otimes k_2})(x_1, \dots, x_k),$$
where the sign is determined by $\clubsuit(\underline{x}, k_1) := k_1 + \sum_{j=1}^{k_1} \deg(x_j)$,
- each A^i has a filtration $F^\lambda A^i$ respecting that of $\Lambda_{\geq 0}$, and a basis belonging to $F^0(A^i) \setminus \bigcup_{\lambda>0} F^\lambda A^i$.

Given a filtered A_∞ algebra, we can also consider the Λ-linear products on $A \otimes_{\Lambda_{\geq 0}} \Lambda$. We call $A \otimes_{\Lambda_{\geq 0}} \Lambda$ a Λ-filtered A_∞ algebra.

Let (A, m_A^k) and (B, m_B^k) be A_∞ algebras. A *filtered A_∞ homomorphism* from A to B is a sequence of filtered graded maps

$$f^k \colon A^{\otimes k} \to B$$

satisfying the *quadratic A_∞ homomorphism relations*

$$\sum_{k_1+k'+k_2=k} (-1)^{\clubsuit(\underline{x},k_1)} f^{k_1+1+k_2} \circ (\mathrm{id}^{\otimes k_1} \otimes m_A^{k'} \otimes \mathrm{id}^{k_2}) = \sum_{i_1+\cdots+i_j=k} m_B^j \circ (f^{i_1} \otimes \cdots \otimes f^{i_j}).$$

There similarly exists a notion of a homotopy between filtered A_∞ homomorphisms.

The main difficulty with filtered A_∞ algebras is that they do not have cohomology groups, as

$$(m_A^1)^2 = m_A^2(m_A^0 \otimes \mathrm{id}) \pm m_A^2(\mathrm{id} \otimes m_A^0).$$

When $m_A^0 = 0$, the right-hand side of the relation is zero and we say that A is a *tautologically unobstructed* A_∞ algebra.

It is desirable to work with tautologically unobstructed A_∞ algebras as they can be studied with the standard tools employed for cochain complexes. Therefore, one might restrict one's study to tautologically unobstructed filtered A_∞ algebras. Problematically, tautologically unobstructed filtered A_∞ algebras are not closed under the relation of filtered A_∞ homotopy equivalence. This can be remedied by considering filtered A_∞ algebras equipped with bounding cochains.

Let A be a filtered A_∞ algebra. A deforming cochain is an element $d \in A^1$ with $\operatorname{val}(d) > 0$. The d-deformation of A is the filtered A_∞ algebra (A, d) whose

- underlying chain groups agree with A, and
- composition maps are given by the d-deformed A_∞ products,

$$m^k_{(A,d)} = \sum_{l=0}^{\infty} \sum_{j_0+\cdots+j_k=l} m_A^{k+l}(d^{\otimes j_0} \otimes \mathrm{id} \otimes d^{\otimes j_1} \otimes \cdots \otimes \mathrm{id} \otimes d^{\otimes j_k}). \tag{12}$$

Definition B.1.1 When $m^0_{(A,b)} = 0$, we say that b is a *bounding cochain*, and we say that the algebra A is *unobstructed*.

Given $f: A \to B$ a filtered A_∞ homomorphism and $b \in A$ a bounding cochain, there is a pushforward bounding cochain $f_*(b) \in B$ such that $(B, f_*(b))$ is unobstructed. When $f^k = 0$ for $k \neq 1$, then $f_*(b) = f(b)$. The existence of a pushforward bounding cochain shows that unobstructedness is a property of filtered A_∞ algebras which is preserved under the equivalence relation of filtered A_∞ homotopy equivalence.

In applications, we use Λ-filtered A_∞ algebras as opposed to filtered A_∞ algebras.[3] However, the homological algebra of filtered A_∞ algebras is notationally easier to describe (as there exist elements living in a minimal filtration level). A computation allows us to understand deformations and bounding cochains for the former (defined using (12)) in terms of the latter.

Claim B.1.2 *Suppose that A is a filtered A_∞ algebra and b a bounding cochain for A. Then $b \otimes 1 \in A \otimes_{\Lambda_{\geq 0}} \Lambda$ is a bounding cochain for the Λ-filtered A_∞ algebra $A \otimes_{\Lambda_{\geq 0}} \Lambda$.*

B.2 Extending an unobstructed ideal

Following ideas from [22], we will provide a method for constructing bounding cochains by inducting on the valuation. In order to do this, we need a slight refinement of a filtered A_∞ algebra which states that the valuation of the structure coefficients is ordered by a monoid. A gapped A_∞ algebra is a filtered A_∞ algebra for which there exists a finitely generated monoid G and a monoid homomorphism $\omega: G \to \mathbb{R}_{\geq 0}$ such that $\omega(\beta) = 0$ implies that $\beta = 0$, and we have the decomposition

$$m^k = \sum_{\beta \in G} T^{\omega(\beta)} m^{k,\beta},$$

where $m^{k,\beta}$ are graded with respect to the filtration. We say that it satisfies the gapped A_∞ relations if for all $\beta \in G$,

$$\sum_{\beta_1+\beta_2=\beta} \sum_{j_1+j+j_2=k} (-1)^{\clubsuit} m^{j_1+1+j_2,\beta_1}(\mathrm{id}^{\otimes j_1} \otimes m^{j,\beta_2} \otimes \mathrm{id}^{\otimes j_2}) = 0.$$

[3]This is because the continuation maps in Lagrangian intersection Floer cohomology are usually only weakly filtered.

Given $b=\sum_{\beta\in G\setminus\{0\}} b_\beta$, we can deform the product structure by

$$m^{k,\beta}_{(B,b)}=\sum_{\substack{\beta_0+\cdots+\beta_k=\beta\\ \beta_i=\sum_{j=1}^{l_i}\beta_{i,j}}} m_B^{k+l}(\beta_{0,0}\otimes\cdots\otimes\beta_{0,l_0}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\beta_{k,0}\otimes\cdots\otimes\beta_{k,l_k}),$$

so that $m^k_{(B,b)}:=\sum_{\beta\in G}T^{\omega(\beta)}m_B^{k,\beta}$ gives a G-gapped A_∞ algebra satisfying the gapped A_∞ relations. There similarly exist G-gapped filtered A_∞ homomorphisms, which also contain the data of a morphism of monoids $\phi\colon G_A\to G_B$.

We will also need some basic statements about ideals in filtered A_∞ algebras:

Definition B.2.1 A subspace $A\subset B$ is a weak A_∞ ideal if for all $k=k_1+1+k_2>0$, the map

$$m^k\colon B^{\otimes k_1}\otimes A\otimes B^{\otimes k_2}\to B$$

has image contained in A.

Notably, we *do not* require that the curvature term m^0_A be an element of A. As a result, it is not necessarily the case that A is itself a filtered A_∞ algebra. We say that A is a *strong A_∞ ideal* if additionally $m^0_B\in A$.

Claim B.2.2 *Let $A\subset B$ be an A_∞ ideal. The quotient $C=A/B$ inherits a filtered A_∞ structure. A is a strong A_∞ ideal if and only if C is tautologically unobstructed.*

Proof The filtered A_∞ structure is the natural one:

$$m^k_C([x_1]\otimes\cdots\otimes[x_k]):=[m^k_B(x_1\otimes\cdots\otimes x_k)].$$

Because the m^k_B are multilinear, we see that if $[x_i]=[x_i']$, then $m^k_C([x_1]\otimes\cdots\otimes[x_i]\otimes\cdots\otimes[x_k])=m^k_C([x_1]\otimes\cdots\otimes[x_i']\otimes\cdots\otimes[x_k])$. A is a strong A_∞ ideal if and only if $m^0_C=[m^0_B]=[0]$. □

Example B.2.3 Given a formal filtered A_∞ morphism $f\colon B\to C$ (so that $f^k=0$ for all $k\neq 1$) the kernel of f is a weak A_∞ ideal.

Example B.2.4 Given a filtered A_∞ algebra A, the set $A_{>0}$ of positively filtered elements is an example of a strong A_∞ ideal. The quotient $\underline{A}:=A/A_{>0}$ is an example of a tautologically unobstructed A_∞ algebra. A relevant example comes from Lagrangian Floer cohomology, where $\underline{\mathrm{CF}^\bullet(L)}=\mathrm{CM}^\bullet(L)$.

Claim B.2.5 *Suppose that $A\subset B$ is an A_∞ ideal, and $d\in B$ is a deforming cochain. Then A is an A_∞ ideal of (B,d). If $A\subset B$ is a strong A_∞ ideal, and $m^0_d\in A$, then A is a strong A_∞ ideal of B. In particular, if $d\in A$ then A is a strong A_∞ ideal of (B,d).*

Proof Suppose that $a\in A$ is some element. Then

$$m^k_{(B,d)}(x_1\otimes\cdots\otimes a\otimes\cdots\otimes x_k)=\sum_{l=0}^{\infty}\sum_{j_0+\cdots+j_k=l} m_A^{k+l}(d^{\otimes j_1}\otimes\mathrm{id}\otimes d^{\otimes j_1}\otimes\cdots\otimes a\otimes\cdots\otimes\mathrm{id}\otimes d^{\otimes j_k})\in A,$$

proving that A is an A_∞ ideal of (B,d). □

The vector space $H^1(A)$ is a lowest-order approximation to the space of bounding cochains. When $\overline{C}$ is an anticommutative differential graded algebra, elements of $H^1(\overline{C})$ are bounding cochains.

Claim B.2.6 *Suppose that C is tautologically unobstructed. Suppose that $f: C \to \overline{C}$ is an A_∞ map with gapped A_∞ homotopy inverse $g: \overline{C} \to C$. Assume that $\overline{C}$ is an anticommutative differential graded algebra. Then for every class $[c] \in H^1(C)$ with $\operatorname{val}(c) > 0$, there exists a bounding cochain $c' \in C$ and $\lambda > \operatorname{val}(c')$ with $[c'] = [c] \in H^1(C/T^\lambda C)$.*

Proof Since C and $\overline{C}$ are gapped, we can select $\lambda > \operatorname{val}(c)$ such that $\omega(\beta) < \lambda$ implies $\omega(\beta) \leq \operatorname{val}(c)$. We observe that $f(c) \in \overline{C}$ is closed, and therefore provides a bounding cochain for $\overline{C}$, as

$$m^0_{(\overline{C}, f(c))} = m^1_{\overline{C}}(f(c)) + m^2_{\overline{C}}(f(c), f(c)) = 0.$$

We then take c' to be the pushforward bounding cochain

$$g_*(f(c)) = \sum_{k=1}^{\infty} g^k((f(c))^{\otimes k}).$$

Since $c' = (g \circ f)(c) \bmod T^\lambda$, we obtain that $[c] = [c'] \in H^1(C/T^\lambda C)$. □

Claim B.2.7 [34, Claim A.4.8] *Suppose that $A' = (A, a)$. Given a deforming cochain $a' \in A'$, the chain $a'' = a + a' \in A$ is a deforming cochain such that $(A', a') = (A, a'')$.*

We now come to the main lemma of this appendix. Suppose that we have an exact sequence (on the chain level) $A \to B \to C$. If A is a strong A_∞ ideal containing the curvature of B, then we prove that there is no obstruction to finding a bounding cochain for B. The argument is in the style of [22, Theorem 3.6.18].

Lemma B.2.8 *Consider a G-gapped A_∞ algebra B satisfying the gapped A_∞ relations. Suppose that:*

(i) *A is a strong A_∞ ideal of B and $C = B/A$, giving us an exact sequence $A \xrightarrow{i} B \xrightarrow{\pi} C$ of gapped A_∞ algebras,*

(ii) *there exists $\overline{C}$ which is A_∞ homotopic to C and is an anticommutative DGA,*

(iii) *the connecting map $\delta: H^1(\underline{C}) \to H^2(\underline{A})$ surjects.*

Then for every $\lambda > 0$ there exists a deforming cochain $b = \sum_{\beta \in G \setminus \{0\}} b_\beta$ for B such that for all β with $\omega(\beta) \leq \lambda$, we have $m^{0,\beta}_{(B,b)} = 0$.

Proof Because A,, B, and C are gapped A_∞ algebras, there exists $\{\lambda_i\}_{i=1}^n$ an ordering of the image $\omega(G) \in [0, \lambda]$.

We prove the statement by induction on λ_i. Suppose that $A' = (A, a_{i-1})$, $B' = (B, b_{i-1})$, and $C' = (C, c_{i-1})$ are G-gapped A_∞ algebras satisfying (i)–(iii) and additionally

(iv) the curvature has large valuation, $\operatorname{val}(m^0_{B'}) > \lambda_{i-1}$.

The inductive step will construct deforming cochains a', b', and c' such that the algebras (A', a'), (B', b'), and (C', c') satisfy (i)–(iv), where λ_{i-1} is replaced with λ_i. By Claim B.2.7, we can then construct the A_∞ algebras (A, a_i), (B, b_i), and (C, c_i).

Write $m^0_{B'} = \sum_{j=i}^{\infty} \sum_{\omega(\beta)=\lambda_j} \underline{b}_{j,\beta} T^{\lambda_j}$, where the $\underline{b}_{j,\beta}$ are elements of $\underline{B}' = \underline{B}$ of degree 2. Because A is a strong A_∞ ideal, we can find $\underline{a}_{i,\beta} \in \underline{A}$ with $i(\underline{a}_{i,\beta}) = \underline{b}_{i,\beta}$.

We examine the lowest-order terms of the A_∞ relation $m^1_{A'} \circ m^0_{A'} = 0$, and obtain

$$\underline{m}^1_{A'}(\underline{a}_{i,\beta}) = 0.$$

Since $[\underline{a}_{i,\beta}] \in H^2(\underline{A})$, by (iii) $[\underline{b}_{i,\beta}] = 0$. Therefore there exists $\hat{\underline{b}}_{i,\beta}$ such that $\underline{m}^1_{B'}(\hat{\underline{b}}_{i,\beta}) = \underline{b}_{i,\beta}$. The class $\underline{c}_{i,\beta} := \pi(\hat{\underline{b}}_{i,\beta})$ is closed. Using Claim B.2.6, we can find $\underline{c}'_{j,\beta}$ with $j \geq i$ such that $c' = \sum_{j=i}^{\infty} \sum_{\beta | \omega(\beta)=\lambda_i} \underline{c}'_{j,\beta} T^{\lambda_j}$ is a bounding cochain for C' with the property that $[c'_{i,\beta}] = [c_{i,\beta}] \in H^1(C'/T^{\lambda_{i+1}} C')$.

Because $\pi \colon B \to C$ surjects, we can find for all $j \geq i$ cochains $\underline{b}'_{\beta,j} \in \underline{B}$ with $\pi(\underline{b}'_{j,\beta}) = \underline{c}''_{j,\beta}$. Let

$$b' = -\sum_{j=i}^{\infty} \sum_{\beta | \omega(\beta)=\lambda_i} \underline{b}'_{i,\beta} T^{\lambda_i}.$$

This constructed b' satisfies the property

$$m^1_{B'}(b') \equiv -m^0_{B'} \mod T^{\lambda_{i+1}}.$$

Since π is a filtered A_∞ homomorphism without higher terms, the pushforward $\pi_*(b')$ equals c' and

$$\pi \circ m^0_{(B,b')} = m^0_{(C,c')} = 0.$$

Therefore $m^0_{(B',b')}$ is contained in A', and we write a' for the corresponding element in A'. Claim B.2.5 states that (A', a') is a strong A_∞ ideal of (B', b'), whose quotient is (C', c').

This gives us the G-gapped A_∞ algebras (A', b'), (B', b'), and (C', c'), which we've shown satisfy (i). We now show that these algebras satisfy (ii)–(iv). For (ii), observe that deformations by Maurer–Cartan classes preserve having an anticommutative model. Since the deformation occurs at valuation greater than 0, the map $H^1(\underline{C}) \to H^2(\underline{A})$ continues to surject, proving (iii).

To check (iv),

$$\operatorname{val}(m^0_{(B',b')}) = \operatorname{val}\left(\sum_{k=0}^{\infty} m^k_{B'}((b')^{\otimes k})\right) \geq \min\left(\operatorname{val}(m^0_{B'} + m^1_{B'}(b')), \sum_{k=2}^{\infty} m^k_{B'}((b')^{\otimes k})\right).$$

Given that $m^0_{B'} \equiv m^1_{B'}(b') \bmod T^{\lambda_i}$,

$$\operatorname{val}(m^0_{(B',b')}) \geq \lambda_{i+1}. \qquad \square$$

Corollary B.2.9 *Let A, B, and C be A_∞ algebras as in Lemma B.2.8. Then there exists a bounding cochain for B.*

Proof The deforming cochains constructed in the above proof satisfy the condition that

$$b_i \equiv b_{i+1} \mod T^{\lambda_i}.$$

It follows that if we use the inductive procedure to build a sequence of deforming cochains $\{b_i\}_{i=0}^{\infty}$ such that $\operatorname{val}(m^0_{(B,b_i)}) > \lambda_i$, the limit $\lim_{i\to\infty} b_i$ is a bounding cochain. □

B.3 A_∞ bimodules and bounding cochains

Let A and B be A_∞ algebras. An (A,B)- *bimodule* is a filtered graded $\Lambda_{\geq 0}$-module M, along with a set of maps for all $k_1, k_2 \geq 0$,

$$m^{k_1|1|k_2}_{A|M|B} \colon A^{\otimes k_1} \otimes M \otimes B^{\otimes k_2} \to M,$$

satisfying filtered quadratic A_∞ module relations for each triple $(k_1|1|k_2)$:

$$\begin{aligned} 0 = \sum_{\substack{j_1+j+j_2=k_1+1+k_2\\ j_1+j\leq k_1}} & m^{k_1-j+1|1|k_2}_{A|M|B} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m_A^j \otimes \mathrm{id}^{\otimes k_1-j_1-j} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2}) \\ &+ \sum_{\substack{j_1+j+j_2=k_1+1+k_2\\ j_1\leq k_1\leq j_1+j-1}} m^{j_1|1|j_2}_{A|M|B} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m^{k_1-j_1|1|k_2-j_2}_{A|M|B} \otimes \mathrm{id}_B^{\otimes j_2}) \\ &\quad + \sum_{\substack{j_1+j+j_2=k_1+1+k_2\\ k_1+1<j_1}} m^{k_1|1|k_2-j+1}_{A|M|B} \circ (\mathrm{id}_A^{\otimes k_1} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2-j_2-j} \otimes m_B^j \otimes \mathrm{id}_B^{\otimes j_2}). \end{aligned}$$

There is a G-gapped version of an A_∞ bimodule, where we have the data of a map of monoids $\omega \colon G_M \to \mathbb{R}_{\geq 0}$ and our A_∞ bimodule products can be decomposed as $m^{k_1|1|k_2,\beta}_{A|M|B}$; we also have morphisms $\phi_{A/B} \colon G_{A/B} \to G_M$ which intertwine with ω.

If M is a filtered (A,B) bimodule, and $a \in A$ and $b \in B$ are deforming cochains, then the filtered A_∞ bimodule products on M can be deformed to give it the structure of an $((A,a),(B,b))$ bimodule. As in the setting of Λ-filtered A_∞ algebras, we can define Λ-filtered A_∞ bimodules.

Lemma B.3.1 *Let M be a G-gapped (A,B) bimodule. Suppose that A and B are tautologically unobstructed and that A has an anticommutative DGA model $\bar{A}$ as in Claim B.2.6. Suppose that there exist $\lambda_0 < \lambda_1 \in \mathbb{R}$ such that*

(i) *the maps $m^{k|1|0}_{A|M|B} \colon A^{\otimes k} \otimes M \to M$ all have image contained within $T^{\lambda_0}M$, and*

(ii) *there exists $[\underline{e}] \in H^1(\underline{M})$ an element such that the map*

$$H^1(A) \to H^1(T^{\lambda_0}M/T^{\lambda_1}M), \quad [a] \mapsto [m^{1|1|0}_{A|M|B}(a \otimes \underline{e})]$$

is surjective.

Then there exists a choice of bounding cochain $a \in A^1$ and element $e \in M^0$ such that $m^{0|1|0}_{(A,a)|M|B}(e) = 0$.

Proof We again use the gapped structure and induct on valuations. For simplicity of exposition, we will assume that the monoid G is $\mathbb{N}$, so that $\omega(G) = \{n\lambda \mid n \in \mathbb{N}\} \subset \mathbb{R}$. We will construct a sequence of bounding cochains a_i and elements $e_i \in M^0$ with the property that

(iii) a_i are bounding cochains,

(iv) $m^{0|1|0}_{(A,a_i)|M|B}(e_i) \in T^{\lambda_0+i\lambda_1}M$, and

(v) for $i > 1$, $a_i - a_{i-1} \in T^{\lambda_0+i\lambda_1}A$ and $e_i - e_{i-1} \in T^{\lambda_0+i\lambda_1}M$.

Base case Let $a_0 = 0$ and $e_0 = e$. Items (i) and (ii) are given by the hypothesis, (iii) is trivial, (iv) follows from the gapped structure, and (v) has no content.

Inductive step Suppose we have constructed a_i and e_i satisfying the induction hypothesis. By (iv), we can write $m^{0|1|0}_{(A,a_i)|M|B}(e_i) \equiv c_i \mod T^{\lambda_0+(i+1)\lambda_1}$, where $c_i \in T^{\lambda_0+i\lambda_1}M$. At order $T^{\lambda_0+(i+1)\lambda_1}$,

$$\underline{m}^{0|1|0}_{A|M|B}(c_i) \equiv m^{0|1|0}_{(A,a_i)|M|B}(c_i) \equiv m^{0|1|0}_{(A,a_i)|M|B} \circ m^{0|1|0}_{(A,a_i)|M|B}(e_i) = 0 \mod T^{\lambda_0+(i+1)\lambda_1}.$$

We therefore obtain a class $[c_i] \in T^{\lambda_0+i\lambda_1}H^1(\underline{M})$. Using (ii), we have a homology class $\underline{a} \in T^{\lambda_0+i\lambda_1}A$ with

$$[m^{1|1|0}_{A|M|B}(\underline{a} \otimes \underline{e})] \equiv [c_i] \mod T^{\lambda_0+(i+1)\lambda_1}.$$

By Claim B.2.6, there exists a bounding cochain $a' \in T^{i\lambda_1}A^1$ for the product structures $m^k_{(A,a_i)}$ satisfying

$$a' \equiv \underline{a} \mod T^{\lambda_0+(i+1)\lambda_1}, \qquad [m^{1|1|0}_{A|M|B}(a' \otimes \underline{e})] = [c_i] \quad \text{in } H^1(T^{\lambda_0+i\lambda_1}M/T^{\lambda_0+(i+1)\lambda_1}M).$$

Write $m^{1|1|0}_{A|M|B}(a' \otimes \underline{e}) = c_i + m^{0|1|0}_{A|M|B}e'$, where $e' \in T^{\lambda_0+i\lambda_1}$. Then let $e_{i+1} = e_i + e'$ and let $a_{i+1} = a_i - a'$. By construction, we satisfy (v). By Claim B.2.7, a_{i+1} is a bounding cochain for A, and we therefore obtain (iii). Conditions (i) and (ii) are unchanged by deformations. It remains to prove (iv):

$$\begin{aligned} &m^{1|1|0}_{(A,a_{i+1})|M|B}(e_{i+1}) \\ &\qquad \equiv m^{0|1|0}_{(A,a_i)|M|B}(e_i) - m^{1|1|0}_{A|M|B}(a', \underline{e}) + m^{0|1|0}_{A|M|B}(e') - m^{1|1|0}_{A|M|B}(a', e') \mod T^{\lambda_0+(i+1)\lambda_1} \\ &\qquad \equiv 0 \mod T^{\lambda_0+(i+1)\lambda_1}. \end{aligned}$$

To complete the proof of the lemma, we can take the bounding cochain a and element e to be

$$a = \lim_{i\to\infty} a_i, \quad e = \lim_{i\to\infty} e_i. \qquad \square$$

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