



# *Geometry & Topology*

Volume 29 (2025)

**Relations in singular instanton homology**

PETER B KRONHEIMER

TOMASZ S MROWKA

# Relations in singular instanton homology

PETER B KRONHEIMER

TOMASZ S MROWKA

We calculate the singular instanton homology with local coefficients for the simplest  $n$ -strand braids in  $S^1 \times S^2$  for all odd  $n$ , describing these homology groups and their module structures in terms of the coordinate rings of explicit algebraic curves. The calculation is expected to be equivalent to computing the quantum cohomology ring of a certain Fano variety, namely a moduli space of stable parabolic bundles on a sphere with  $n$  marked points.

57R58; 14H60

1. Introduction	1975
2. A version of singular instanton homology	1979
3. Torus braids in $S^1 \times S^2$	1984
4. Relations in ordinary cohomology	1997
5. Relations in instanton homology	2007
6. Calculation of the ideals	2020
7. Further remarks	2038
References	2044

## 1 Introduction

### 1.1 Background

A pair  $(Y, K)$ , consisting of a closed, oriented 3-manifold and an embedded link, gives rise to a 3-dimensional orbifold  $Z = Z(Y, K)$  whose underlying topology is that of  $Y$  and whose singular locus consists of the locus  $K$  where the orbifold structure has local stabilizers of order 2. The pair  $(Y, K)$ , or the orbifold  $Z$ , is *admissible* if  $[K]$  has odd pairing with some integer homology class. To an admissible orbifold  $Z$ , there is associated its *singular instanton homology* (Kronheimer and Mrowka [20]), constructed from the Morse theory of the Chern–Simons functional on the space of  $\mathrm{SO}(3)$  orbifold connections modulo a determinant-1 gauge group. With rational coefficients, we denote the singular instanton homology by  $I(Z; \mathbb{Q})$ .

A deformation of this instanton homology is described in [21]. It can be viewed as an instanton homology group with values in a local coefficient system on the space of connections modulo gauge, and it appears in this paper as  $I(Z; \Gamma)$ , where  $\Gamma$  denotes a local system of free rank-1 modules over the ring of Laurent polynomials

$$\mathcal{R} = \mathbb{Q}[\tau^{\pm 1}].$$

The variable  $\tau$  should be seen as a deformation parameter, with the specialization  $\tau = 1$  recovering the original case of  $\mathbb{Q}$  coefficients. (See Section 2.2.)

A choice of a 2-dimensional homology class in  $Z$  gives rise to an operator  $\alpha$ , on both  $I(Z; \mathbb{Q})$  and  $I(Z; \Gamma)$ . For each choice of basepoint  $p \in K$ , there is also an operator  $\delta_p$ , depending on the connected component of  $K$  on which  $p$  lies and a choice of local orientation at  $p$ . These operators commute, and make  $I(Z; \mathbb{Q})$  and  $I(Z; \Gamma)$  into modules over the rings  $\mathbb{Q}[\alpha, \delta_1, \dots, \delta_n]$  and  $\mathcal{R}[\alpha, \delta_1, \dots, \delta_n]$  respectively, where  $n$  is the number of connected components of  $K$ .

In [32], Street completely described the instanton homology  $I(Z; \mathbb{Q})$  and its module structure in the case that  $Z$  is the product

$$Z_n = S^1 \times S_n^2.$$

Here  $S_n^2$  denotes the 2-sphere with  $n$  orbifold points. An extension of Street's result to the case of  $S^1 \times \Sigma_{g,n}$  was obtained by Xie and Zhang [36], and an earlier model for both of these calculations is the work of Muñoz [28; 27] on the case of  $S^1 \times \Sigma_g$  (where the orbifold locus is empty).

The purpose of this paper is to extend Street's calculation to the case of instanton homology with local coefficients  $\Gamma$ . Alongside  $Z_n$ , a closely related calculation is for the instanton homology of an orbifold we call  $Z_{n,1}$ . If the  $n$  orbifold points in  $S_n^2$  are arranged symmetrically around a circle, then a rotation  $h$  through  $2\pi/n$  is an automorphism of  $S_n^2$  which permutes the orbifold points, and we write  $Z_{n,1}$  for its mapping torus:

$$Z_{n,1} = M_h, \quad h: S_n^2 \rightarrow S_n^2.$$

Since the orbifold locus in  $Z_{n,1}$  is connected, there is only one operator  $\delta = \delta_p$  in this case, and  $I(Z_{n,1}; \Gamma)$  is a module for an algebra  $\mathcal{R}[\alpha, \delta]$ , where  $\mathcal{R}$  is again a ring of Laurent polynomials. We can summarize the main theme of this paper as the solution to the following.

**Problem** ( $\star$ ) Describe  $I(Z_n; \Gamma)$  and  $I(Z_{n,1}; \Gamma)$  explicitly as modules for the algebras  $\mathcal{R}[\alpha, \delta_1, \dots, \delta_n]$  and  $\mathcal{R}[\alpha, \delta]$  respectively.

The motivation for studying this question came from a desire to calculate a variant of the singular instanton homology of torus knots,  $I^{\natural}(T_{n,q}; \Gamma)$ , as studied in our paper [24], and the related knot concordance invariants of these. In [24], the base ring always had characteristic 2, as necessitated by the construction there. An alternative formulation allows characteristic 0, and the results of this paper are a main step. We return to this discussion briefly in Section 7.

### 1.2 Statement of the result

We shall give a complete answer to  $(\star)$ , and to give a flavor of the result here, we describe  $I(Z_{n,1}; \Gamma)$ . First, there is an involution on the configuration space of connections on both of these orbifolds, defined by multiplying the holonomy on the  $S^1$  factor in  $S^1 \times S^2$  by  $-1 \in \text{SU}(2)$ . This gives rise to an operator  $\epsilon$  on instanton homology, and there is therefore a decomposition

$$I(Z_{n,1}; \Gamma) = I(Z_{n,1}; \Gamma)^+ \oplus I(Z_{n,1}; \Gamma)^-$$

into the eigenspaces of  $\epsilon$ . As modules, these two are related by changing the variable  $\tau \in \mathcal{R}$  to  $-\tau$ . Each of the two summands is a cyclic module for  $\mathcal{R}[\alpha, \delta]$  and they are therefore characterized by their ideals of relations,  $J_{n,1}^\pm$  in the algebra:

$$I(Z_{n,1}; \Gamma)^+ \cong \mathcal{R}[\alpha, \delta]/J_{n,1}^+, \quad I(Z_{n,1}; \Gamma)^- \cong \mathcal{R}[\alpha, \delta]/J_{n,1}^-.$$

Over the field  $\mathbb{C}$ , we can regard  $J_{n,1}^+$  and  $J_{n,1}^-$  as the defining ideals of possibly nonreduced curves

$$D_n^+, D_n^- \subset \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$$

with coordinates  $(\tau, \alpha, \delta)$ . Our final description of these curves is as determinantal varieties: they are the loci of points where particular  $m \times (m + 1)$  matrices  $S^+$  and  $S^-$  with entries in  $\mathcal{R}[\alpha, \delta]$  fail to have full rank. Here  $m = \frac{1}{2}(n - 1)$ . Equivalently,  $J_{n,1}^\pm$  is the ideal generated by the  $m \times m$  minors of  $S^\pm$ . Explicitly when  $n = 11$  and  $m = 5$ , the matrix  $S^\pm$  is given by  $S_0 \pm S_1$ , where  $S_0$  is the matrix

$$\begin{pmatrix} -\alpha - \delta/2 & \alpha - 19\delta/2 & 0 & 0 & 0 & 0 \\ 0 & -\alpha - 5\delta/2 & \alpha - 15\delta/2 & 0 & 0 & 0 \\ 0 & 0 & -\alpha - 9\delta/2 & \alpha - 11\delta/2 & 0 & 0 \\ 0 & 0 & 0 & -\alpha - 13\delta/2 & \alpha - 7\delta/2 & 0 \\ 0 & 0 & 0 & 0 & -\alpha - 17\delta/2 & \alpha - 3\delta/2 \end{pmatrix}$$

and  $S_1$  is the matrix

$$\begin{pmatrix} \tau^7 & 0 & 0 & 0 & 0 \\ 0 & \tau^3 & 0 & 0 & 0 \\ 0 & 0 & 1/\tau & 0 & 0 \\ 0 & 0 & 0 & 1/\tau^5 & 0 \\ 0 & 0 & 0 & 0 & 1/\tau^9 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & -9 & 5\tau^4 + 4 \\ 0 & 0 & 0 & -7 & 3\tau^4 + 2 & 2\tau^4 \\ 0 & 0 & -5 & \tau^4 & 4\tau^4 & 0 \\ 0 & -3 & -\tau^4 - 2 & 6\tau^4 & 0 & 0 \\ -1 & -3\tau^4 - 4 & 8\tau^4 & 0 & 0 & 0 \end{pmatrix}.$$

Although the matrices may look elaborate at first glance, they follow a fairly simple pattern that is readily described for general  $n$ . (See [Section 6.3](#).) Note in particular that  $S_0$  is a 2-band matrix with entries that are linear forms in  $(\alpha, \delta)$ , while the entries of  $S_1$  depend only on  $\tau$ . On setting  $\tau = 1$  in  $S_0$  above, one recovers generators for the ideal that is identified by Street in [\[32\]](#). For a general fixed value of  $\tau$ , the corresponding locus is a subscheme of the  $(\alpha, \delta)$ -plane of length  $m(m + 1)$ . A picture of the real locus of  $D_n^\pm$  for  $n = 7$  is given in [Figure 1](#), together with the set of points on  $D_n^\pm$  where  $\tau = 0.6$ .

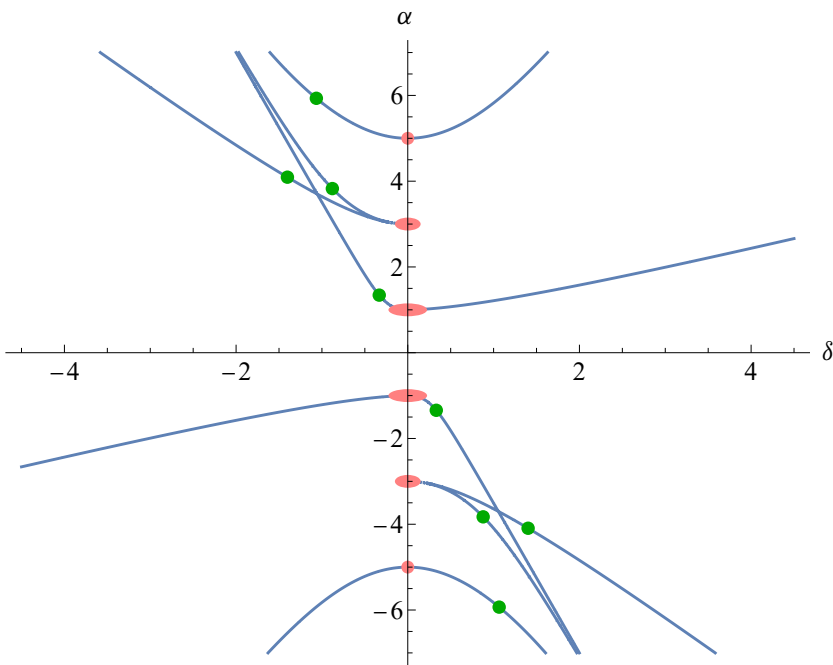


Figure 1: The blue curve is the projection of the real locus of  $D_n^\pm$  to the  $(\delta, \alpha)$  plane for  $n = 7$ . The green points are the points where  $\tau = 0.6$ , showing the simultaneous eigenvalues of the operators  $\delta$  and  $\alpha$  for this value of  $\tau$ . There are 12 of these, only 8 of which are real. The pink points indicate the subscheme of total length 12 defined by the minors of  $S_0 \pm S_1$  when  $\tau = 1$ , which is the case described by Street [32]. Although the real curve looks rather smooth at  $\alpha = \pm 1$ , it has a uni-branch triple point there: in local analytic coordinates, the equation of the curve has the form  $y^3 = x^7$ .

**Remark** This description of  $D_n^\pm$  as a determinantal variety means that the corresponding ideal  $J_{n,1}^\pm$  is generated by  $m + 1$  elements, for this is the number of  $m \times m$  minors. We shall see in fact that each of these ideals can be generated by just two of the minors.

As in Muñoz [28; 27], Street [32] and Xie and Zhang [36], the starting point for the calculation is an explicit generating set for the ideal of relations in the ordinary cohomology of a representation variety: in our case, as in [36], these are the “Mumford relations” in the cohomology of the representation variety associated to  $S_n^2$ . (See Earl and Kirwan [8] for example.) We obtain simple explicit formulae for these relations as products of linear forms in the variables  $\alpha$  and  $\delta_i$ . The matrix  $S_0$  above arises as a matrix of syzygies for the Mumford relations. To compute the deforming term  $S_1$ , it is only necessary to understand the contributions of moduli spaces of instantons on  $\mathbb{R} \times Z_n$  of smallest nonzero action (action  $\frac{1}{4}$  in the normalization where the standard instanton on  $\mathbb{R}^4$  has action 1). The contributions of these moduli spaces can be understood quite explicitly by a wall-crossing argument. A closely related phenomenon is present in [27].

### 1.3 Outline

In [Section 2](#) we recall the definition of singular instanton homology with local coefficients and the construction of the operators that act on it in general. (Note that from [Section 2](#) onwards, we simply write  $I(Z)$  for the homology group referred to as  $I(Z; \Gamma)$  above, without explicit mention of the local coefficients.) In [Section 3](#), we introduce  $Z_n$  and  $Z_{n, \pm 1}$  and study the ordinary cohomology of the relevant representation varieties and instanton homologies, enough to show that these can be described as cyclic modules for the algebra of operators which act on them. This material is quite standard.

In [Section 4](#), we describe the Mumford relations in the ordinary cohomology of the representation variety of  $Z_n$ . We derive a very explicit formula for generators of the ideal of relations in these cohomology groups. The relations in the ordinary cohomology ring of the representation variety of  $Z_n$  admit a deformation which yields relations in the instanton homology  $I(Z_n)$ . The existence of this deformation is established in [Section 5](#) together with a calculation of the subleading term using a wall-crossing calculation rather as in [\[27\]](#).

Knowledge of the subleading term turns out to be sufficient to obtain a complete answer, and the description of  $I(Z_{n,1})$  (or equivalently  $I(Z_{n,-1})$ ) that is outlined earlier in this introduction is derived in [Section 6](#). Some further remarks are contained in [Section 7](#) at the end of the paper.

**Acknowledgements** The work of Kronheimer was supported by the National Science Foundation through NSF grants DMS-1707924 and DMS-2005310. The work of Mrowka was supported by NSF grants DMS-1808794 and DMS-2105512. Both authors were supported by a Simons Foundation Award #994330 (Simons Collaboration on *New structures in low-dimensional topology*). This paper was completed while Mrowka was in residence at the Simons Laufer Mathematical Sciences Institute as a Clay Senior Scholar and supported by NSF grant DMS-1928930. The authors are grateful to the referee for improvements and corrections.

## 2 A version of singular instanton homology

In this section we review the construction of instanton homology with local coefficients, for admissible bifolds. General references include [\[20\]](#) and [\[23\]](#).

### 2.1 Bifolds and their Floer homology

For economy of notation, we will typically write simply  $Z$  for a pair consisting of a connected, oriented 3-manifold  $Y$  and an embedded (unoriented) link  $K = K(Z) \subset Y$ . Following [\[20\]](#) and [\[19\]](#), we will regard  $Z$  as determining an orbifold (a *bifold* in the terminology of [\[22\]](#)) whose underlying topological space is  $Y$  and whose singular set is  $K(Z)$ . The local stabilizer of the orbifold geometry at points of  $K(Z)$  is of order 2. When talking of (for example) Riemannian metrics on  $Z$ , we will always mean

orbifold Riemannian metrics. A bifold  $Z$  is *admissible* if there is an element of  $H^1(Y; \mathbb{Z})$  which has nonzero mod-2 pairing with the class  $[K(Z)] \in H_1(Y; \mathbb{Z}/2)$ .

Associated to a 3-dimensional bifold  $Z$ , we have a space of bifold connections  $\mathcal{B}(Z)$ . In this paper,  $\mathcal{B}(Z)$  will always consist of the bifold  $\mathrm{SO}(3)$  connections with  $w_2 = 0$  modulo the determinant-1 gauge group. In the language of [23, Section 2], this is the space of marked bifold connections in which the marking region is the complement of the singular set  $K(Z)$  and the bundle has  $w_2 = 0$  on the marking region.

**Remark** The space  $\mathcal{B}(Z)$  can be identified with the space of gauge equivalence classes of  $\mathrm{SU}(2)$  connections on the complement of the singular set  $K(Z)$  such that the associated  $\mathrm{SO}(3)$  bundle extends to an orbifold  $\mathrm{SO}(3)$  bundle on  $Z$  with nontrivial monodromy (of order 2) at the singular points. When interpreted as  $\mathrm{SU}(2)$  connections in this way, the limiting holonomy of the  $\mathrm{SU}(2)$  connections on small loops linking the singular locus has order 4. This is the viewpoint adopted, for example, in [17; 18].

**Definition 2.1** We write  $\mathrm{Rep}(Z) \subset \mathcal{B}(Z)$  for the space of flat bifold connections modulo the determinant-1 gauge group. If  $Z$  is admissible, then  $\mathrm{Rep}(Z)$  consists only of irreducible connections.

## 2.2 A local coefficient system

For each component  $K^i \subset K(Z)$ , after choosing a framing, we obtain a map to  $S^1$ ,

$$h_i: \mathcal{B}(Z) \rightarrow S^1,$$

as in [20] and [23, Section 2.2]. Specifically, following [20], given  $[A] \in \mathcal{B}(Z)$ , we may restrict the connection  $[A]$  to the boundary of the framed  $\epsilon$ -tubular neighborhood of  $K^i$  and obtain, in the limit as  $\epsilon \rightarrow 0$ , a flat  $\mathrm{SO}(3)$  connection on the torus, whose structure group reduces to  $\mathrm{SO}(2)$ . The holonomy of the  $\mathrm{SO}(2)$  connection along the longitude defines  $h_i([A])$ .

An orientation of  $K^i$  is not needed here, because the orientation of the  $\mathrm{SO}(2)$  bundle also depends on an orientation of  $K^i$ . (That is, the orientation of  $K^i$  is used twice in this construction.) The framing is also inessential, as a change of framing will change  $h_i$  by a half-period.

Taking the product over the set of all components of  $K$ , we define a single map  $h: \mathcal{B}(Z) \rightarrow S^1$  by

$$h = \times_i h_i.$$

Over the circle  $S^1$ , there is a standard local system with fiber the ring of finite Laurent series

$$(1) \quad \mathcal{R} = \mathbb{Q}[\tau^{\pm 1}]$$

such that the monodromy of the local system around the positive generator of  $S^1$  is multiplication by  $\tau$ . Then by pulling back this local system by the map  $h$ , we obtain a local system  $\Gamma$  on  $\mathcal{B}(Z)$ . We summarize this construction with a definition.

**Definition 2.2** Unless otherwise stated, the notation  $\mathcal{R}$  will denote the ring  $\mathbb{Q}[\tau^{\pm 1}]$ , and  $\Gamma$  will denote the corresponding local system of free rank-1  $\mathcal{R}$ -modules over  $\mathcal{B}(Z)$ , for any 3-dimensional bifold  $Z$ .

If  $Z$  is admissible, then by the standard construction (see [20; 21]), we obtain an instanton homology group for admissible bifolds:

**Definition 2.3** Let  $Z$  be an admissible bifold of dimension 3. After choosing a Riemannian metric and perturbation to achieve a Morse–Smale condition for the gradient flow of the Chern–Simons functional on  $\mathcal{B}(Z)$ , we obtain an instanton Floer complex  $CI(Z; \Gamma)$  of free  $\mathcal{R}$ -modules whose homology  $I(Z; \Gamma)$  is the instanton homology of  $Z$ . We will generally write  $I(Z)$  and omit  $\Gamma$  from the notation, unless the context demands otherwise. This is a  $\mathbb{Z}/4$  graded module.

### 2.3 Functoriality and operators

We consider 4-dimensional bifolds  $W$  as cobordisms between 3-dimensional bifolds. In the context of this paper, the singular locus  $\Sigma = \Sigma(W)$  of the orbifold  $W$  will always be an embedded surface (not necessarily orientable). In particular, we do not consider foams — singular surfaces — as in [22]. The Floer homology groups  $I(Z)$  are functorial in the sense that a bifold cobordism  $W$  from  $Z^0$  to  $Z^1$  gives rise to a map

$$I(W): I(Z^0) \rightarrow I(Z^1)$$

compatible with compositions.

The map  $I(W)$  is obtained from suitable weighted counts of solutions to the perturbed anti-self-duality equations on the bifold  $W$ , after attaching cylindrical ends. This construction initially gives rise only to a projective functor, in that the overall sign of  $I(W)$  is ambiguous. When  $\Sigma(W)$  is oriented, the sign ambiguity can be resolved by choosing a homology orientation for  $W$  in the sense of [20]. In the case that  $\Sigma(W)$  is not necessarily orientable, an appropriate substitute is the notion of an  $\iota$ -orientation introduced in [19]. (The sign ambiguity in the nonorientable case will not particularly concern us in this paper.)

Recall that in the present context  $I(Z)$  denotes the instanton homology with coefficients in the local system  $\Gamma$ . That being so, the solutions  $A$  to the perturbed anti-self-duality equations on  $W$  are counted not just with signs  $\pm 1$ , but with weights that are units in the ring  $\mathcal{R}$ . More precisely, if  $\rho_0$  and  $\rho_1$  are critical points of the perturbed Chern–Simons functional in  $\mathcal{B}(Z^0)$  and  $\mathcal{B}(Z^1)$ , and if  $[A]$  is a solution of the perturbed equations on  $W$  with cylindrical ends, asymptotic to  $\rho_0$  and  $\rho_1$ , then  $[A]$  contributes to the matrix entry of the map  $I(W)$  at the chain level with a contribution  $\pm \Gamma(A)$ , where  $\Gamma(A): \Gamma(\rho_0) \rightarrow \Gamma(\rho_1)$  is given by

$$(2) \quad \Gamma(A) = \tau^{\nu(A) + \frac{1}{2}(\Sigma \cdot \Sigma)}.$$

Here  $\nu$  is obtained from a curvature integral on the 2-dimensional singular set  $\Sigma = \Sigma(W)$ , and the self-intersection number  $\Sigma \cdot \Sigma$  is computed relative to chosen framings of the singular sets  $K(Z^0)$  and  $K(Z^1)$ . The expression on the right-hand side of (2) is not an element of  $\mathcal{R}$  itself, because the exponent is not generally an integer. It is, however, a homomorphism between the rank-1  $\mathcal{R}$ -modules  $\Gamma(\rho_0) \rightarrow \Gamma(\rho_1)$  in a natural way. For details of this construction see, for example, [20, Section 3.9] and [23]. As explained

there, the choice of framings is essentially immaterial. Consistent with our notation  $I(Z)$  in which the local coefficient system  $\Gamma$  is implied, we will continue to write simply  $I(W)$  for the  $\mathcal{R}$ -module homomorphism between these instanton homology groups.

As well as the map  $I(W)$  above, we have the generalizations obtained by cutting down the moduli spaces on  $W$  by cohomology classes in the configuration space of bifold connections  $\mathcal{B}(W)$ . Here  $\mathcal{B}(W)$  is a space of  $SO(3)$  bifold connections modulo the determinant-1 gauge group, and in the language of [23], this is the space of marked bifold connections in which the marking region is the complement of the singular set  $\Sigma(W)$  and the bundle has  $w_2 = 0$  on the marking region.

To describe the relevant cohomology classes more specifically, and to fix conventions, there is a universal orbifold  $SO(3)$  bundle,

$$\mathbb{E} \rightarrow \mathcal{B}^*(W) \times W,$$

which has an orbifold Pontryagin class,

$$p_1^{\text{orb}}(\mathbb{E}) \in H^4(\mathcal{B}^*(W) \times W; \mathbb{Q}).$$

We adopt the convention that our preferred 4-dimensional characteristic class is  $-\frac{1}{4}p_1^{\text{orb}}(\mathbb{E})$ , which coincides with  $c_2^{\text{orb}}(\tilde{\mathbb{E}})$  in the case that there is a lift to an  $SU(2)$  bundle  $\tilde{\mathbb{E}}$ . Given a class  $\gamma$  in  $H^2(W; \mathbb{Q})$  or  $H^0(W; \mathbb{Q})$ , we obtain classes

$$(3) \quad -\frac{1}{4}p_1^{\text{orb}}(\mathbb{E})/[\gamma]$$

in  $H^2(\mathcal{B}^*(W); \mathbb{Q})$  or  $H^4(\mathcal{B}^*(W); \mathbb{Q})$  respectively.

In addition to the classes (3), if  $p$  is a point of the orbifold locus  $\Sigma(W)$ , then the restriction of  $\mathbb{E}$  to  $\mathcal{B}^*(W) \times \{p\}$  has a decomposition

$$\mathbb{E}_p = \mathbb{R} \oplus \mathbb{V}_p,$$

where  $\mathbb{V}_p$  is a 2-plane bundle. An orientation of  $\mathbb{V}_p$  depends on a choice of normal orientation to the orbifold locus at  $p$ . Having chosen such an orientation, a class  $\delta_p \in H^2(\mathcal{B}^*(W); \mathbb{Q})$  is then defined as

$$(4) \quad \delta_p = \frac{1}{2}e(\mathbb{V}_p).$$

We can regard  $\delta$  here as depending on a choice of an element in  $H_0(\Sigma(W); O)$ , where  $O$  is the orientation bundle of  $\Sigma(W)$  with rational coefficients.

Combining the classes (3) for  $\gamma \in H^i(W; \mathbb{Q})$  and the classes  $\delta_p$ , we obtain homomorphisms of  $\mathcal{R}$ -modules

$$(5) \quad I(W, a): I(Z^0) \rightarrow I(Z^1)$$

depending linearly on

$$(6) \quad a \in \text{Sym}_*(H_2(W; \mathbb{Q}) \oplus H_0(W; \mathbb{Q}) \oplus H_0(\Sigma(W); O)).$$

Since  $I(Z^0)$  and  $I(Z^1)$  are  $\mathcal{R}$ -modules, we may extend linearly over  $\mathcal{R}$  to allow also

$$(7) \quad a \in \text{Sym}_*(H_2(W; \mathbb{Q}) \oplus H_0(W; \mathbb{Q}) \oplus H_0(\Sigma(W); O)) \otimes \mathcal{R}.$$

The construction of the operators  $I(W, a)$  is suitably functorial. In particular, this means for us that, in the case that  $W$  is a cylinder  $[0, 1] \times Z$ , we have

$$I(W, a_1 a_2) = I(W, a_1) I(W, a_2).$$

We will always be dealing with the case that  $W$  is connected, so there is only one class  $[w]$  in  $H_0(W; \mathbb{Q})$ . From [16; 20], we note the following relation among the homomorphisms  $I(W, a)$ .

**Proposition 2.4** *Let  $p$  a point in  $\Sigma(W)$  with a chosen orientation of  $T_p \Sigma(W)$ , representing a class in  $H_0(\Sigma(W); O)$  in the algebra (6). Let  $w$  be a point in  $W$ , representing a class in  $H_0(W; \mathbb{Q})$ . Then we have a relation*

$$I(W, (p^2 + w - \tau^2 - \tau^{-2})b) = 0$$

for any  $b$  in the algebra (6).

**Corollary 2.5** *The map  $I(W, p^2 b)$  is independent of the choice of oriented point  $p \in \Sigma(W)$ .*

**Remark** The relation in Proposition 2.4 reflects (in part) a relation in the cohomology ring of  $\mathcal{B}^*(W)$ , where we have a 2-dimensional class  $\delta_p$  and a 4-dimensional class  $-\frac{1}{4} p_1^{\text{orb}}(\mathbb{E})/[w]$ . From their construction as characteristic classes, these satisfy

(8)

The extra terms  $\tau^2 + \tau^{-2}$  in the proposition arise from instanton bubbling contributions [16].

Proposition 2.4 also tells that the generator corresponding to  $[w] \in H_0(W; \mathbb{Q})$  is redundant. We obtain the most general homomorphism  $I(W, a)$  if we only take  $a$  in the smaller algebra

$$(9) \quad \text{Sym}_*(H_2(W; \mathbb{Q}) \oplus H_0(\Sigma(W); O)).$$

There is an additional construction we can make if we are given a distinguished class  $e \in H_2(W; \mathbb{Z})$ . We consider the space  $\mathcal{B}(W)^e$  of marked bifold  $\text{SO}(3)$  connections on  $W$  where the marking region is again the complement of  $\Sigma(W)$  and where the marking data has

$$w_2 = \text{PD}(e)|_{W \setminus \Sigma(W)} \pmod{2}.$$

After attaching cylindrical ends, the instantons in  $\mathcal{B}(W)^e$  provide us with maps

$$(10) \quad I(W, a)^e : I(Z^0) \rightarrow I(Z^1).$$

When the singular set  $\Sigma(W)$  is oriented, the integer lift  $e$  in homology, together with the homology-orientation of  $W$ , is used to orient the moduli spaces and determines the overall sign of the map  $I(W, a)^e$ . If  $e - e' = 2v$ , so that  $e$  and  $e'$  define the same mod 2 class, then (as in [6]) we have

$$(11) \quad I(W, a)^{e'} = (-1)^{v \cdot v} I(W, a)^e.$$

**Remark** As discussed for example in [19], one can more generally consider the case that  $e$  is a relative class so that  $\partial e \in H_1(\Sigma(W))$ , but the more restrictive version here is required because we wish to use the local coefficient system  $\Gamma$ , which is otherwise not defined. See also [23, Section 2.2].

### 3 Torus braids in $S^1 \times S^2$

#### 3.1 The torus braids

The following examples play an important role for us.

**Definition 3.1** Let  $\pi = \{p_1, \dots, p_n\}$  be  $n$  points arranged symmetrically around the equator of  $S^2$ . We write  $Z_n$  for the bifold whose underlying 3-manifold  $Y$  is the product  $S^1 \times S^2$  and whose singular locus  $K$  is the  $n$ -component link

$$K_n = S^1 \times \pi \subset S^1 \times S^2.$$

**Definition 3.2** For any  $q \in \mathbb{Z}$ , we define a bifold  $Z_{n,q}$  as follows. The 3-manifold  $Y$  is again  $S^1 \times S^2$ . If  $\varphi \in \mathbb{R}/(2\pi\mathbb{Z})$  denotes an angular coordinate on the equator of  $S^2$ , and  $\theta$  a coordinate on the  $S^1$  factor, then  $K = K_{n,q}$  will be the link determined by  $n\varphi = q\theta \pmod{2\pi}$ .

The bifold  $Z_{n,q}$  is admissible when  $n$  is odd. The link  $K_{n,q} \subset S^1 \times S^2$  is connected (a knot) when  $n$  and  $q$  are coprime. When  $q = 0$ , the orbifold  $Z_{n,0}$  coincides with  $Z_n$  above.

When needed, we orient the singular set  $K_n \subset Z_n$  as the boundary of  $n$  disks in the product 4-manifold  $D^2 \times S^2$ , and we orient  $K_{n,q}$  similarly using the fact that they have the same infinite cyclic cover.

It is evident from the definitions that the orbifold  $Z_{n,q}$  is isomorphic to  $Z_{n,-q}$  by an orientation-reversing map. With a little more thought, one can see that there is also an orientation-preserving isomorphism:

**Lemma 3.3** *The link  $K_{n,q}$  is isotopic in  $S^1 \times S^2$  to the link  $K_{n,-q}$ . As a consequence, there is an orientation-preserving isomorphism of bifolds from  $Z_{n,q}$  to  $Z_{n,-q}$ , preserving the orientation of the singular set.*

**Proof** Let  $L$  be an oriented axis in  $\mathbb{R}^3$  passing through two points of the equatorial circle in the above description of  $K_{n,q}$ . Let  $\rho_t$  be the rotation of  $S^2$  about this axis through angle  $2\pi t$ , and let  $1 \times \rho_t$  be the resulting map  $S^1 \times S^2 \rightarrow S^1 \times S^2$ . Then the link

$$K_t = (1 \times \rho_t)(K_{n,-q}) \subset S^1 \times S^2$$

coincides with  $K_{n,-q}$  when  $t = 0$  and with  $K_{n,q}$  when  $t = \frac{1}{2}$ . □

We aim to give a description of  $I(Z_n)$  (the instanton homology with local coefficients) as an  $\mathcal{R}$ -module, together with a description of the operators

$$I([0, 1] \times Z_n, a): I(Z_n) \rightarrow I(Z_n) \quad \text{and} \quad I([0, 1] \times Z_n, a)^e: I(Z_n) \rightarrow I(Z_n)$$

arising from classes  $a$  by the general construction (5) and (10), where  $e$  is the 2-dimensional class in  $H_2(Z_n; \mathbb{Q})$ .

### 3.2 The representation variety of $S_n^2$

Let us assume henceforth that  $n$  is odd, so that the orbifold  $Z_n$  described above is admissible. We may describe  $Z_n$  as a product  $S^1 \times S_n^2$ , where  $S_n^2$  is a 2-dimensional bifold of genus 0, and we begin with some observations about the representation variety  $\text{Rep}(S_n^2)$ , drawn from [3; 35; 32]. Note that we can identify  $\text{Rep}(S_n^2)$  with the space of flat  $\text{SU}(2)$  connections on the complement of the  $n$  singular points such that the monodromy at each puncture has order 4. (See the remark in Section 2.1.)

First, as  $n$  is odd, the variety  $\text{Rep}(S_n^2)$  consists entirely of irreducible connections. It is a smooth, compact, connected manifold of dimension  $2n - 6$  for  $n \geq 3$ , and is empty for  $n = 1$ . We have no need for a detailed description of their topology, but we record the fact that  $\text{Rep}(S_3^2)$  is a single point and  $\text{Rep}(S_5^2)$  is diffeomorphic to the blow up of  $\mathbb{C}\mathbb{P}^2$  at 5 points. It will be convenient to make use of the following result, which the authors believe has the status of folklore. The statement and proof are very minor adaptations of the main result of [13]. See also [33].

**Lemma 3.4** *For any odd  $n$ , the manifold  $\text{Rep}(S_n^2)$  admits a Morse function with critical points only in even index.*

**Proof** Following [13], we present a proof by induction on  $n$ . So assume the result is true for a particular  $n$ , and consider  $\text{Rep}(S_{n+2}^2)$ . Let  $C \subset \text{SU}(2)$  be the subset of elements of order 4, ie the unit sphere of imaginary quaternions. Let  $\tilde{R} \subset C^{n+2}$  be the locus

$$\{(i_1, \dots, i_{n+2}) \in C^{n+2} \mid i_1 i_2 \cdots i_{n+2} = 1\},$$

so that the representation variety  $\text{Rep}(S_{n+1}^2)$  is the quotient of  $\tilde{R}$  by conjugation. For  $\mathbf{i} \in \tilde{R}$ , there is a unique  $\theta \in [0, \pi]$  such that

$$i_{n+1} i_{n+2} \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

and we have a smooth function

$$h = \cos(\theta) = \frac{1}{2} \text{tr}(i_{n+1} i_{n+2}),$$

which descends to a smooth function

$$h: \text{Rep}(S_{n+2}^2) \rightarrow [-1, 1].$$

We consider separately the loci  $h^{-1}(1)$ ,  $h^{-1}(-1)$  and  $h^{-1}((-1, 1))$ .

If  $\mathbf{i} \in h^{-1}(1)$ , then  $i_{n+1} i_{n+2} = 1$ , and it follows that  $i_1 i_2 \cdots i_n = 1$ . So these remaining  $n$  points define a point in  $\text{Rep}(S_n^2)$ . The remaining choice of  $i_{n+1}$  exhibits  $h^{-1}(1)$  as a 2-sphere bundle over  $\text{Rep}(S_n^2)$ . As in [13], we may use the induction hypothesis to show that a perturbation of  $h$  has critical points only of even index near  $h = 1$ . The situation at  $h^{-1}(-1)$  is essentially the same: multiplying  $i_1$  and  $i_{n+2}$  by  $-1$  interchanges these two loci.

On the locus  $h^{-1}((-1, 1))$ , the function  $h$  itself is Morse and its critical points can be described as follows. Let  $i, j, k$  in  $C$  be the standard unit quaternions with  $ijk = -1$ . Given any element of  $h^{-1}((-1, 1))$  we can use the action of conjugation to uniquely put in standard form with  $i_{n+1} = i$  and  $i_{n+2}$  lying in the interior of the semicircle  $\gamma$  which joins  $i$  to  $-i$  and passes through  $j$ . In this standard form, there is a circle action on  $h^{-1}((-1, 1))$  which fixes  $i_{n+1}$  and  $i_{n+2}$  and rotates the points  $i_1, \dots, i_n$  about the axis through  $k$ . The function  $\theta = \cos^{-1}(h)$  is smooth on this locus and is the moment map of the circle action. The critical points of  $h$  are therefore precisely the fixed points of this circle action. These fixed points are the points which in standard form have  $i_{n+1} = i, i_{n+2} = j$  and  $i_m = \pm k$  for all other  $m$ . The constraint  $i_1 i_2 \cdots i_{n+1} = 1$  means that  $i_m = -k$  for an even number of indices  $m$  in the range  $1, \dots, n$ . As a general property of moment maps, because these fixed points are isolated, they are Morse critical points for  $h$ , of even index. □

**Remark** The proof of the lemma above gives a little bit more, for we can easily identify the indices of the critical points, and hence establish the recursive formula for the Poincaré polynomial of  $\text{Rep}(S_n^2)$  which is given in [32]. The loci  $h^{-1}(1)$  and  $h^{-1}(-1)$ , which are the 2-sphere bundles over  $\text{Rep}(S_n^2)$  inside  $\text{Rep}(S_{n+2}^2)$ , are the minima and maxima of  $h$  and together make a contribution

$$(1 + t^2)^2 P_n(t)$$

to the Poincaré polynomial  $P_{n+2}$  for  $\text{Rep}(S_{n+2}^2)$ . Using the symmetries of  $\text{Rep}(S_{n+2}^2)$  obtained by multiplying an even number of the  $i_l$  by  $-1$ , it is easy to see that the remaining critical points in  $h^{-1}((-1, 1))$  all have the same index and that this index is the middle dimension  $(n - 1)$ . There are  $2^{n-1}$  of these critical points, so we recover the recursive formula from [32],

$$(12) \quad P_{n+2}(t) = (1 + t^2)^2 P_n(t) + (2t)^{n-1}.$$

Atiyah and Bott [1] described standard generators for the cohomology ring of representation varieties of surfaces in the nonorbifold case (a smooth surface of genus  $g$ ), and there is an extension of those techniques for the orbifold case, developed in [2]. For the specific case of  $S_n^2$ , the results are given in [32].

In this description, the generators of the cohomology ring  $H^*(\text{Rep}(S_n^2; \mathbb{Q}))$  are classes

$$(13) \quad \alpha \in H^2(\text{Rep}(S_n^2); \mathbb{Q}), \quad \beta \in H^4(\text{Rep}(S_n^2); \mathbb{Q}), \quad \delta_p \in H^2(\text{Rep}(S_n^2); \mathbb{Q}) \quad \text{for } p \in \pi,$$

which are the restrictions to  $\text{Rep}(S_n^2)$  of classes defined on the space of irreducible bifold connections,  $B^*(S_n^2)$ , arising from the slant product construction (3). More specifically, the classes  $\alpha$  and  $\beta$  arise from the fundamental 2-dimensional class  $[S_n^2] \in H_2(S_n^2)$  and the point class  $[w] \in H_0(S_n^2)$  respectively, while  $\delta_p$  is defined as in (4):

$$(14) \quad \alpha = -\frac{1}{4} p_1^{\text{orb}}(\mathbb{E})/[S_n^2], \quad \beta = -\frac{1}{4} p_1^{\text{orb}}(\mathbb{E})/[w], \quad \delta_p = \frac{1}{2} e(\nabla_p).$$

We will sometimes write

$$\delta_1, \dots, \delta_n$$

for the classes  $\delta_{p_i}$  as  $p_i$  runs through  $\pi$ .

The classes  $\alpha$  and  $\beta$  can also be seen as arising from the Künneth decomposition in  $H^4(\mathcal{B}^*(S_n^2) \times S_n^2; \mathbb{Q})$ ,

$$-\frac{1}{4}p_1^{\text{orb}}(\mathbb{E}) = \beta \times 1 + \alpha \times v,$$

where  $v$  is the generator of  $H^2(S_n^2; \mathbb{Q})$ . The generator  $\beta$  is redundant, because of the relation

$$\delta_p^2 = -\beta \quad \text{for all } p \in \pi,$$

which is a restatement of (8) in the current situation.

In the rational cohomology ring of  $\mathcal{B}^*(S_n^2)$ , there are no further relations: the cohomology ring is the algebra

$$(15) \quad H^*(\mathcal{B}^*(S_n^2); \mathbb{Q}) = \mathbb{Q}[\alpha, \delta_1, \dots, \delta_n] / \langle \delta_k^2 - \delta_l^2 \rangle_{k,l}.$$

We have a surjective homomorphism

$$(16) \quad \varphi: H^*(\mathcal{B}^*(S_n^2); \mathbb{Q}) \rightarrow H^*(\text{Rep}(S_n^2); \mathbb{Q}).$$

**Definition 3.5** We write  $A_n$  for the algebra

$$A_n = H^*(\mathcal{B}^*(S_n^2); \mathbb{Q}) = \mathbb{Q}[\alpha, \delta_1, \dots, \delta_n] / \langle \delta_k^2 - \delta_l^2 \rangle_{k,l},$$

and we write

$$j_n \subset A_n$$

for the kernel of the surjective homomorphism  $\varphi$ .

Generators for the ideal  $j_n$  are described in detail in [32], which leads to a complete description of the cohomology ring,

$$(17) \quad H^*(\text{Rep}(S_n^2); \mathbb{Q}) = A_n / j_n.$$

See also Proposition 4.8.

### 3.3 The representation variety of $Z_n$

The flat bifold connections on  $Z_n$  are of two sorts, which we call the “plus” and “minus” components, which can be distinguished by examining the holonomy of the flat connection along the  $S^1$  factor in  $Z_n = S^1 \times S_n^2$ . The representations in the plus component are pulled back from  $S_n^2$ . The representations in the minus component are obtained from these by multiplication by a flat real line bundle with holonomy  $-1$  on the  $S^1$  factor. Thus we have

$$(18) \quad \text{Rep}(Z_n) = \text{Rep}(Z_n)_+ \cup \text{Rep}(Z_n)_- = \text{Rep}(S_n^2) \cup \text{Rep}(S_n^2).$$

Because of this, the description (17) of the cohomology ring of  $\text{Rep}(S_n^2)$  leads immediately to a description of the cohomology of  $\text{Rep}(Z_n)$ . We are also eventually interested in the cohomology of the representation variety with constant coefficients  $\mathcal{R}$  rather than  $\mathbb{Q}$  (because of our interest in instanton homology with local coefficients  $\Gamma$ ). With this in mind, let

$$\epsilon: H^*(\text{Rep}(Z_n); \mathcal{R}) \rightarrow H^*(\text{Rep}(Z_n); \mathcal{R})$$

be the map obtained from interchanging the two copies, so that  $\epsilon^2 = 1$ . We write  $\mathcal{A}_n$  for the algebra

$$(19) \quad \mathcal{A}_n = \mathcal{R}[\alpha, \delta_1, \dots, \delta_n, \epsilon] / \langle \epsilon^2 - 1, \delta_k^2 - \delta_l^2 \rangle_{k,l}.$$

That is, we extend the coefficient ring of the algebra (17) from  $\mathbb{Q}$  to  $\mathcal{R}$ , and we adjoin the element  $\epsilon$  with square 1. This provides us with the following description. In the statement below, we write

$$1_+ \in H^0(\text{Rep}(Z_n))$$

for the element Poincaré dual to the fundamental class of the component  $\text{Rep}(Z_n)_+$ .

**Proposition 3.6** *The cohomology of the representation variety  $\text{Rep}(Z_n)$  with coefficients in  $\mathcal{R}$  is a cyclic module for the algebra  $\mathcal{A}_n$  with generator the element  $1 \in H^0(\text{Rep}(Z_n); \mathcal{R})$ . We have*

$$(20) \quad H^*(\text{Rep}(Z_n); \mathcal{R}) \cong \mathcal{A}_n / J_n, \quad \text{where } J_n = (j_n + \epsilon j_n) \otimes \mathcal{R},$$

and  $j_n$  is the ideal in (17). Using Poincaré duality, the homology  $H_*(\text{Rep}(Z_n); \mathcal{R})$  can equivalently be described as a cyclic  $\mathcal{A}_n$ -module with generator the class  $[\text{Rep}(Z_n)_+]$ , with the classes  $\alpha$  and  $\delta_k$  acting by cap product.

We regard  $\mathcal{A}_n$  as a graded algebra with the generators  $\alpha$  and  $\delta_k$  in grading 1 (not 2) and  $\epsilon$  in grading 0. From the grading,  $\mathcal{A}_n$  obtains an increasing filtration, which for future reference we record as

$$(21) \quad \mathcal{A}_n^{(0)} \subset \mathcal{A}_n^{(1)} \subset \mathcal{A}_n^{(2)} \subset \dots \subset \mathcal{A}_n,$$

where  $\mathcal{A}_n^{(s)}$  is the  $\mathcal{R}$ -submodule generated by elements in grading less than or equal to  $s$ .

From the explicit description of the generators of  $j_n$  given in [32] (for rational coefficients), we can read off that there are no relations between the generators up to the middle dimension of  $\text{Rep}(Z_n)$ :

**Proposition 3.7** *For  $s \leq (n - 3)/2$ , we have  $J_n \cap \mathcal{A}_n^{(s)} = \{0\}$ . □*

### 3.4 The instanton homology of $Z_n$

The instanton homology  $I(Z_n; \mathbb{Q})$  with rational coefficients was described, together with its ring structure, by Street [32] drawing on work of Boden [3] and Weitsman [35]. We summarize part of these results here, adapted to the case of  $I(Z_n)$  (by which we continue to mean the instanton homology with local coefficients).

The representation variety  $\text{Rep}(Z_n)$  is a Morse–Bott critical locus for the Chern–Simons functional. By Lemma 3.4, there is a Morse function on  $\text{Rep}(Z_n)$  with critical points only in even index. The proof of that lemma allows one to construct such a Morse function as a linear combination of traces of holonomies around loops in  $Z_n$ . We may use such a Morse function as a holonomy perturbation for the Chern–Simons functional, so that the critical points of the perturbed Chern–Simons functional correspond to the critical points of the Morse function on  $\text{Rep}(Z_n)$ . After making such a perturbation, the set of

critical points forms a natural basis both for the ordinary homology of  $\text{Rep}(Z_n)$  as a  $\mathbb{Q}$ -vector space, and for the instanton homology  $I(Z_n)$  as an  $\mathcal{R}$ -module. We therefore obtain an isomorphism

$$I(Z_n) = H_*(\text{Rep}(Z_n)) \otimes \mathcal{R}.$$

In the  $\mathbb{Z}/4$  grading of the instanton homology, the minus component  $\text{Rep}(S_n^2)_-$  is shifted by 2 relative to the plus component. This is established in [32] for rational coefficients, but the argument extends to any coefficients, including our local coefficient system  $\Gamma$ . We record this in the following proposition.

**Proposition 3.8** *As  $\mathcal{R}$ -modules with  $\mathbb{Z}/4$  grading, we have an isomorphism,*

$$\Lambda : I_*(Z_n) = H_*(\text{Rep}(S_n^2); \mathcal{R}) \oplus H_*(\text{Rep}(S_n^2); \mathcal{R})[2]$$

for all odd  $n \geq 1$ . In particular, the instanton homology is a free  $\mathcal{R}$ -module and is nonzero only in even degrees mod 4.

The isomorphism  $\Lambda$  in the above proposition depends on the choice of perturbation (at least a priori), because the isomorphism goes by identifying both sides with the free  $\mathcal{R}$ -module generated by the critical points. The following two propositions add some additional structure. In the statement of the first proposition below, we write  $\mathbb{1}_+ \in I(Z_n)$  for the relative invariant of the 4-dimensional orbifold  $D^2 \times S_n^2$  with boundary  $Z^n$ :

$$\mathbb{1}_+ = I(D^2 \times S_n^2).$$

**Proposition 3.9** *The instanton homology  $I(Z_n)$  is a cyclic module for the filtered algebra  $\mathcal{A}_n$  in (19), with cyclic generator the element  $\mathbb{1}_+$ .*

This proposition (whose proof is given below) prompts the following definition.

**Definition 3.10** We write  $\mathcal{J}_n \subset \mathcal{A}_n$  for the annihilator of the cyclic module  $I(Z_n)$ , so that

$$I(Z_n) \cong \mathcal{A}_n / \mathcal{J}_n.$$

From this description, the instanton homology  $I(Z_n)$  inherits an increasing filtration from the filtration of  $\mathcal{A}_n$ :

$$I(Z_n)^{(m)} = (\mathcal{A}_n^{(m)} + \mathcal{J}_n) / \mathcal{J}_n.$$

**Proposition 3.11** *The isomorphism  $\Lambda$  of Proposition 3.8 respects the filtrations, and the isomorphism on the associated graded is an isomorphism of  $\mathcal{A}_n$ -modules, independent of the choice of perturbations.*

We begin the proof of the two propositions above by describing the  $\mathcal{A}_n$ -module structure of  $I(Z_n)$ . Recall that the  $\mathcal{A}_n$ -module structure of  $H_*(\text{Rep}(Z_n); \mathcal{R})$  arises from operators  $\alpha, \delta_1, \dots, \delta_n$  (acting by cap product) and  $\epsilon$ . The instanton homology  $I(Z_n)$  carries parallel operators which we now make explicit.

First, the classes  $\alpha$ ,  $\beta$  and  $\delta_p$  in  $H^*(\mathcal{B}^*(Z_n); \mathbb{Q})$  correspond to operators on the Floer homology  $I(Z_n)$  by the general construction (5). We write these operators as

$$(22) \quad \tilde{\alpha}: I_*(Z_n) \rightarrow I_{*-2}(Z_n), \quad \tilde{\beta}: I_*(Z_n) \rightarrow I_{*-4}(Z_n) = I_*(Z_n), \quad \tilde{\delta}_p: I_*(Z_n) \rightarrow I_{*-2}(Z_n),$$

where the subscripts denote the mod 4 grading. In the notation of (5), these are the operators

$$\begin{aligned} \tilde{\alpha} &= I([0, 1] \times Z_n, [S_n^2]), \\ \tilde{\beta} &= I([0, 1] \times Z_n, [w]) \quad \text{for } [w] \in H_0([0, 1] \times Z_n), \\ \tilde{\delta}_p &= I([0, 1] \times Z_n, [p]) \quad \text{for } [p] \in H_0([0, 1] \times K_n). \end{aligned}$$

**Remark** According to the results of [16], the operator  $2\tilde{\delta}_p$  can be realized as the map corresponding to a cobordism  $W_1$  from  $Z$  to  $Z$ , derived from the product cobordism  $I \times Z$  by summing a standard torus to  $I \times K$  at the point  $(\frac{1}{2}, p)$ . The local orientation of  $K$  is used to fix a homology orientation of the torus.

The counterpart of the operator  $\epsilon$  is a special case of the construction of  $I(W, a)^e$ . Specifically, following Street [32], it is the map (10) in the special case that  $W$  is the cylindrical cobordism, the element  $a$  is 1, and  $e$  is the class  $[\{\text{point}\} \times S_n^2]$ :

$$\tilde{\epsilon} = I([0, 1] \times S_n^2)^e.$$

In order for the operators  $\tilde{\alpha}$ ,  $\tilde{\delta}_p$  and  $\tilde{\epsilon}$  to make the instanton homology  $I(Z_n)$  into a module over the algebra  $\mathcal{A}_n$ , we need to see that they satisfy the relations that are baked into the definition of  $\mathcal{A}_n$ . We turn to this next. The relation in Proposition 2.4 specializes to the following:

**Lemma 3.12** *With  $\mathcal{R} = \mathbb{Q}[\tau^{\pm 1}]$  as usual, the actions of the operators  $\tilde{\delta}_p$  and  $\tilde{\beta}$  on the  $\mathcal{R}$ -module  $I(Z_n)$  are related by*

$$\tilde{\delta}_p^2 = -\tilde{\beta} + \tau^2 + \tau^{-2}.$$

*In particular,  $\tilde{\delta}_p^2$  is independent of the chosen point  $p$  on the singular set of  $Z_n$ .* □

The element  $\epsilon$  in  $\mathcal{A}_n$  has square 1 by definition, so we need the following lemma also.

**Lemma 3.13** *The operator  $\tilde{\epsilon}: I(Z_n) \rightarrow I(Z_n)$  has square 1, and under the isomorphism of Proposition 3.8 it corresponds to the interchange of the two summands.*

**Proof** This is proved in [32] for rational coefficients, except that an ambiguity in the orientation of the moduli spaces left the sign of  $\tilde{\epsilon}^2$  unresolved there. (See also the proof of Proposition 3.14 below.) In our present context we have

$$\tilde{\epsilon}^2 = I([0, 1] \times S_n^2)^e \circ I([0, 1] \times S_n^2)^e = I([0, 1] \times S_n^2)^{2e} = (-1)^{e \cdot e} I([0, 1] \times S_n^2) = 1,$$

where the second equality is by functoriality and the third equality is from (11). □

The relations in Lemmas 3.12 and 3.13 are the same relations satisfied by the elements  $\epsilon$  and  $\delta_k$  in the algebra  $\mathcal{A}_n$ , so we can indeed use these operators to define an  $\mathcal{A}_n$ -module structure on  $I(Z_n)$  by

$$(23) \quad \alpha \mapsto \tilde{\alpha}, \quad \delta_i \mapsto \tilde{\delta}_i \quad \text{for } i = 1, \dots, n, \quad \epsilon \mapsto \tilde{\epsilon}.$$

Having described the module structure of  $I(Z_n)$ , the fact that it is a cyclic module generated by  $\mathbb{1}_+$  (Proposition 3.9) and the assertions of Proposition 3.11 are both consequences of the fact that, under the isomorphism of Proposition 3.8, the operators  $\tilde{\alpha}$ ,  $\tilde{\delta}_p$  and  $\tilde{\epsilon}$  agree with the operators  $\alpha$ ,  $\delta_p$  and  $\epsilon$  on  $H_*(\text{Rep}(S_n^2))$  in their leading terms. This is the assertion of the proposition below, which is the final proposition of this subsection.

**Proposition 3.14** *Let  $\Lambda$  be the isomorphism of Proposition 3.8. Then for any  $\xi \in I(Z_n)^{(m)}$  and  $u \in \mathcal{A}_n^{(k)}$ , we have*

$$(24) \quad \Lambda(u\xi) = u\Lambda(\xi) \pmod{I(Z_n)^{(m+k-1)},}$$

and  $\Lambda(\mathbb{1}_+) = 1_+$ .

**Proof** It is enough to verify (24) in the case that  $u$  is one of the generators,  $\alpha$ ,  $\delta_p$  or  $\epsilon$ . The essential point is that  $u\xi$  is defined using instantons on the cylinder  $\mathbb{R} \times Z_n$  and that the leading term is defined by (perturbations of) the flat connections, while the nonleading terms are defined by instantons with positive action.

In more detail, let us write  $\text{Rep}(Z_n) = R_+ \cup R_-$ , as an abbreviation for the components  $\text{Rep}(Z_n)_\pm$ . Before any perturbations are made, we have seen that the two components  $R_+ \cup R_-$  are copies of the representation variety  $\text{Rep}(S_n^2)$  of the orbifold sphere (equation (18)). For each  $\kappa > 0$ , let us write

$$M_\kappa(R_\pm, R_\pm)$$

for the moduli space of (unperturbed) instanton trajectories from one component of  $\text{Rep}(Z_n)$  to another, with action  $\kappa$ .

- Lemma 3.15**
- (i) *The moduli spaces  $M_\kappa(R_+, R_+)$  and  $M_\kappa(R_-, R_-)$  are nonempty only for  $\kappa \in \frac{1}{2}\mathbb{Z}$ .*
  - (ii) *The moduli spaces  $M_\kappa(R_+, R_-)$  and  $M_\kappa(R_-, R_+)$  are nonempty only for  $\kappa \in \frac{1}{2}\mathbb{Z} + \frac{1}{4}$ .*
  - (iii) *The formal dimension of the moduli space, in every case, is  $8\kappa + (2n - 6)$ .*

**Proof** The moduli spaces  $M_\kappa(R_+, R_+)$  and  $M_\kappa(R_-, R_-)$  are nonempty when  $\kappa = 0$ , consisting then of constant trajectories on the cylinder and forming a regular moduli space of dimension  $2n - 6$  (the dimension of the representation variety). For other values of  $\kappa$ , these moduli spaces are related to each other by gluing in instantons and monopoles, which will change  $\kappa$  by multiples of  $\frac{1}{2}$  while always changing the formal dimension by  $8\kappa$ ; see [17; 20].

The formal dimension and action  $\kappa$  for the moduli spaces  $M_\kappa(R_+, R_-)$  and  $M_\kappa(R_-, R_+)$  are the same as for moduli spaces on the closed bifold  $S^1 \times Z_n = T^2 \times S_n^2$  for a bundle with marking data where  $w_2(E)$  is dual to the class  $T^2 \times \{\text{point}\}$ . The action in this case is equal to  $\frac{1}{4}n$  modulo  $\frac{1}{2}$ , or in other words belongs to  $\frac{1}{4} + \frac{1}{2}\mathbb{Z}$  since  $n$  is odd. (In the language of [17], the monopole number on each of the  $n$  components of the singular set is a half-integer.) The formula for the formal dimension in terms of the action  $\kappa$  is unchanged. □

After perturbation of the Chern–Simons functional, the manifolds  $R_+$  and  $R_-$  each become a finite set of nondegenerate critical points,  $\mathfrak{C}_+$  and  $\mathfrak{C}_-$ . The action of the perturbed instantons will be close to integer multiples of  $\frac{1}{4}$  if the perturbation is small, so for critical points  $c$  and  $c'$  and  $\kappa \in \frac{1}{4}\mathbb{Z}$  we continue to write  $M_\kappa(c, c')$  for the perturbed moduli spaces. We have the dimension formula

$$\dim M_\kappa(c, c') = 8\kappa + \text{index}(c) - \text{index}(c'),$$

where  $\text{index}$  denotes the ordinary Morse index for the Morse function on  $R_\pm$ . Furthermore, the moduli space is nonempty only if  $\kappa \in \frac{1}{2}\mathbb{Z}$  in the case that  $c, c'$  both belong to  $\mathfrak{C}_+$  or to  $\mathfrak{C}_-$ , and only if  $\kappa \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$  otherwise.

Consider now the operator  $\tilde{\alpha}$  for example. (The case of  $\tilde{\delta}_p$  is no different.) When  $\kappa = 0$ , the moduli space  $M_0(c, c')$  between critical points  $c, c' \in \mathfrak{C}_+$  or  $c, c' \in \mathfrak{C}_-$  coincides with a perturbation of the space of ordinary Morse trajectories between the critical points in  $R_\pm$ . The construction of  $\tilde{\alpha}$  means that we can write it as a sum

$$(25) \quad \tilde{\alpha} = \sum_{\kappa \in \frac{1}{4}\mathbb{Z}, \kappa \geq 0} \tilde{\alpha}_{(\kappa)}$$

according to the contributions of the different moduli spaces  $M_\kappa$ . The matrix entry of  $\tilde{\alpha}_{(0)}$  is the evaluation of the cohomology class  $\alpha$  on the Morse trajectory space  $M_0(c, c')$  between critical points on  $R_+$  or  $R_-$  with  $\text{index}(c) - \text{index}(c') = 2$ . This is the cap product by the class  $\alpha$ , under the isomorphism between Morse homology and singular homology. Thus we have

$$\Lambda(\tilde{\alpha}_{(0)}\xi) = \alpha\Lambda(\xi),$$

where  $\xi$  is the class corresponding to the critical point  $c$ . The dimension formula shows that the remaining terms  $\Lambda(\tilde{\alpha}_{(\kappa)}\xi)$  for positive  $\kappa$  correspond to 2-dimensional moduli spaces  $M_\kappa(c, c'')$  where the index difference  $\text{index}(c) - \text{index}(c'')$  is 4 or more.

In the case of  $\tilde{\tau}$ , the equality (24) holds exactly. This is the content of Lemma 3.13. In the present context it can be understood by the same argument as applies to  $\tilde{\alpha}$  and  $\tilde{\delta}_p$ , but with the additional observation that the moduli spaces of positive action contribute zero because of the action of translation on these moduli spaces. □

If we keep track of the difference between  $R_+$  and  $R_-$  which is highlighted in part (ii) of Lemma 3.15, then we can extract a slightly more detailed statement from the proof of the proposition above. Recall that  $J_n \subset \mathcal{A}_n$  is the annihilator of  $H_*(\text{Rep}(Z_n))$ . (See Proposition 3.6.) In the following corollary, we also write

$$\mathcal{A}_n^+ \subset \mathcal{A}_n$$

for the subalgebra generated over  $\mathcal{R}$  by  $\alpha$  and  $\delta_1, \dots, \delta_n$ , so that

$$\mathcal{A}_n = \mathcal{A}_n^+ + \epsilon\mathcal{A}_n^+.$$

**Corollary 3.16** For any element  $w \in \mathcal{J}_n \cap \mathcal{A}_n^{(m)}$ , there exists  $\omega \in \mathcal{J}_n \cap \mathcal{A}_n^{(m)}$  with

$$\omega - w \in \mathcal{A}_n^{(m-1)}.$$

More particularly, if  $w$  is a homogeneous element of degree  $m$  in the graded algebra  $\mathcal{A}_n$ , then  $\omega$  can be taken to have the form

$$\omega = w(0) + w(2) + w(4) + \cdots + \epsilon(w(1) + w(3) + \cdots),$$

where  $w(0) = w$  and  $w(i) \in \mathcal{A}_n^{(m-i)} \cap \mathcal{A}_n^+$  is homogeneous of degree  $m - i$  for all  $i$ . Furthermore, if  $m \leq \frac{1}{2}(n - 1)$ , then  $\omega$  is uniquely determined by  $w$ .

**Proof** This follows from the proposition above and Proposition 3.7. □

### 3.5 The instanton homology of $Z_{n,-1}$

We now examine the bifold  $Z_{n,-1}$ ; see Definition 3.2. The singular locus  $K(Z_{n,-1})$  in this case is a knot in  $S^1 \times S^2$ , with winding number  $n$ . We still require  $n$  to be odd, so that this is an admissible bifold.

**Proposition 3.17** The representation variety of  $Z_{n,-1}$  is nondegenerate and consists of  $\frac{1}{4}(n^2 - 1)$  points.

**Proof** The orbifold  $Z_{n,-1}$  is a fiber bundle over the circle, with fiber the orbifold sphere  $S_n^2$ . The restriction map to the fiber,

$$\text{Rep}(Z_{n,-1}) \rightarrow \text{Rep}(S_n^2),$$

has image the set of representations in  $\text{Rep}(S_n^2)$  which are invariant under the action  $h_*$  of the monodromy of the circle bundle,  $h: S_n^2 \rightarrow S_n^2$ . The latter is the map which rotates the sphere through  $2\pi/n$ . The restriction map is two-to-one, just as it is for  $Z_n$ , and for the same reason.

The fixed points of  $h_*$  are representations of the orbifold fundamental group of the quotient  $\Sigma = S_n^2 / \langle h \rangle$ . This orbifold surface has one orbifold point of order 2 and two orbifold points of order  $n$ . For a spherical orbifold with three singular points, the representation variety consists of isolated points, and this is essentially the situation considered in [10] (for example). The enumeration of representations, as in [10], becomes an enumeration of lattice points in a region. (The same conclusion can also be reached by identifying the representations with stable parabolic bundles on a curve of genus 0 with appropriate parabolic structure at the orbifold points. See Section 4.1.) In this particular case, the number of representations of the orbifold fundamental group of  $S_n^2 / \langle h \rangle$  is  $\frac{1}{8}(n^2 - 1)$ , and  $\text{Rep}(Z_{n,-1})$  therefore consists of  $\frac{1}{4}(n^2 - 1)$  points. The nondegeneracy of the former leads to the nondegeneracy of the latter. □

We can view  $K(Z_{n,-1})$  as the closure of a braid in  $S^1 \times D^2 \subset S^1 \times S^2$  whose braid diagram has  $n - 1$  negative crossings. There is therefore a cobordism  $W$  of bifolds, from  $Z_{n,-1}$  to  $Z_n$ , obtained by smoothing each of the crossings. We can write  $W$  as a composite of  $n - 1$  cobordisms,  $W_1, \dots, W_{n-1}$ ,

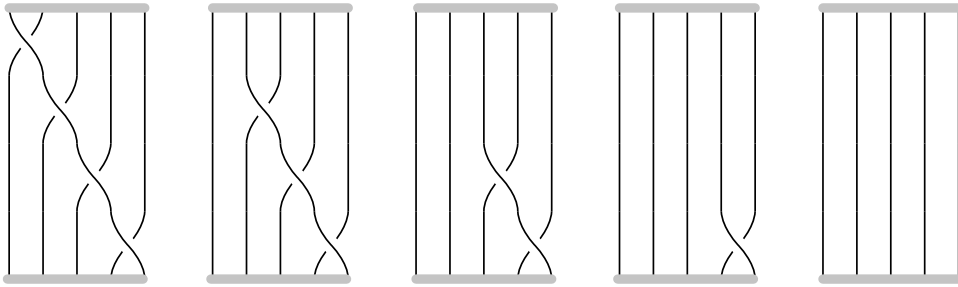


Figure 2: The composite cobordism from  $Z_{n,-1}$  to  $Z_n$ , illustrated for  $n = 5$ .

in the order illustrated in Figure 2. The intermediate bifolds each correspond to braids with  $k$  “straight” strands and  $n - k$  braided strands: a side-by-side juxtaposition of  $Z_k$  and  $Z_{n-k,-1}$ , which we temporarily denote by  $Z_k * Z_{n-k,-1}$  (with the understanding that  $Z_0$  is  $S^1 \times S^2$  with an empty link). So we have

$$I(W_k): I(Z_{k-1} * Z_{n-k+1,-1}) \rightarrow I(Z_k * Z_{n-k,-1}) \quad \text{for } k = 1, \dots, n - 1.$$

(Note that, when  $k = n - 1$ , we have  $Z_k * Z_{n-k,-1} \cong Z_n$ .)

**Proposition 3.18** For each odd  $n$  and each  $k \leq n - 1$ , the induced map  $I(W_k)$  is an inclusion of one free  $\mathcal{R}$ -module in another, as a direct summand.

**Proof** The cobordism  $W_k$  is one map in a skein exact triangle [23; 19], in which the third instanton homology group is  $I(X_{n,k})$ , where  $X_{n,k}$  is a braid as shown in Figure 3. Thus,

$$(26) \quad \dots \xrightarrow{c} I(Z_{k-1} * Z_{n-k+1,-1}) \rightarrow I(Z_k * Z_{n-k,-1}) \rightarrow I(X_{n,k}) \xrightarrow{c} \dots$$

is a long exact sequence.

After an isotopy, we have, for  $k \leq n - 2$ ,

$$(27) \quad X_{n,k} = Z_{k-1} * Z_{n-2-k+1,-1}.$$

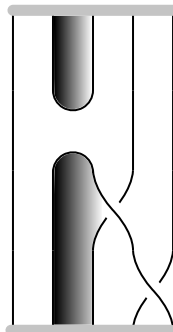


Figure 3: The third braid  $X_{n,k}$  in the exact triangle, illustrated in the case  $n = 5$  and  $k = 2$ . The shaded region (which is connected in a projection of  $S^1 \times S^2$ ) can be eliminated by a Reidemeister I move.

The case  $k = n - 1$  is slightly different: in this case  $X_{n,n-1}$  is the connected sum of  $Z_{n-2}$  and the bifold obtained from an unknot in  $S^3$ . (See Figure 3 again.) From another application of the skein triangle, we have an exact sequence

$$\cdots \rightarrow I(Z_{n-2}) \rightarrow I(X_{n,n-1}) \rightarrow I(Z_{n-2}) \rightarrow \cdots .$$

The connecting homomorphism  $I(Z_{n-2}) \rightarrow I(Z_{n-2})$  has odd degree in the mod 2 grading, while  $I(Z_{n-2})$  is supported in even gradings only. So the connecting homomorphism is zero and so

$$\text{rank}_{\mathcal{R}} I(X_{n,n-1}) = 2 \text{rank}_{\mathcal{R}} I(Z_{n-2}).$$

For brevity, let us write

$$f(n, k) = \text{rank}_{\mathcal{R}} I(Z_k * Z_{n-k,-1}) \quad \text{and} \quad x(n, k) = \text{rank}_{\mathcal{R}} I(X_{n-k}).$$

From the long exact sequence (26), we obtain

$$f(n, k) \leq f(n, k - 1) + x(n, k)$$

with equality if and only if the connecting homomorphism  $c$  has rank zero. We have also seen that

$$x(n, k) = \begin{cases} f(n - 2, k - 1) & \text{if } k \leq n - 2, \\ 2f(n - 2, n - 2) & \text{if } k = n - 1. \end{cases}$$

From these we inductively obtain

$$(28) \quad f(n, n) \leq \sum_{p=0}^{(n-1)/2} \binom{n}{p} f(n - 2p, 0).$$

The quantities  $f(n - 2p, 0)$  are the ranks of the instanton homologies of  $Z_{n-2p,-1}$ , which are bounded above by the number of generators, which in turn can be read off from Proposition 3.17:

$$(29) \quad f(n - 2p, 0) \leq \frac{1}{4}((n - 2p)^2 - 1).$$

Combining this with the previous inequality we have

$$(30) \quad f(n, n) \leq \sum_{p=0}^{(n-1)/2} \binom{n}{p} \cdot \frac{1}{4}((n - 2p)^2 - 1).$$

On the other hand, we know what  $f(n, n)$  is: it is twice  $P(1)$ , where  $P$  is the Poincaré polynomial of the representation variety of  $S_n^2$ , given by the recursive formula (12). From that recursion, we can verify the closed formula

$$(31) \quad f(n, n) = 2^{n-3}(n - 1).$$

But the sum on the right-hand side of (30) is also  $2^{n-3}(n - 1)$ , as is easily verified by comparing it to the second derivative of  $(t + t^{-1})^n$  at  $t = 1$ . It follows that the inequalities above are all equalities. In particular, from the equality in (30), we learn that  $I(Z_{n,-1})$  is a free module of rank  $(n^2 - 1)/4$ . It follows that the connecting homomorphisms  $c$  in the exact sequences (26) all have rank zero. An inductive

argument now shows the modules in the exact sequences are all free  $\mathcal{R}$ -modules and the connecting homomorphisms are all zero. The proposition follows.  $\square$

The following corollary summarizes the conclusions of the previous propositions.

**Corollary 3.19** *The instanton homology  $I(Z_{n,-1})$  with local coefficients is a free  $\mathcal{R}$ -module of rank  $(n^2 - 1)/4$ , supported in even degrees mod 4. The cobordism  $W : Z_{n,-1} \rightarrow Z_n$  induces a map  $I(W)$  on instanton homology with local coefficients,*

$$I(W) : I(Z_{n,-1}) \rightarrow I(Z_n),$$

*which is an inclusion of this free  $\mathcal{R}$ -module as a direct summand.*  $\square$

The bifold obtained from  $Z_{n,-k}$  by reversing the orientation is  $Z_{n,k}$ , and by dualizing the above corollary we obtain:

**Corollary 3.20** *The instanton homology  $I(Z_{n,1})$  with local coefficients is also a free  $\mathcal{R}$ -module of rank  $(n^2 - 1)/4$ . The cobordism  $W^\dagger : Z_n \rightarrow Z_{n,1}$  induces a surjective map  $I(W^\dagger)$  on these free modules.*  $\square$

On the other hand, we have [Lemma 3.3](#) which identifies  $Z_{n,-1}$  and  $Z_{n,1}$  in an orientation-preserving manner by an isotopy. So we have another variant of the corollary:

**Corollary 3.21** *There is a surjective homomorphism of free  $\mathcal{R}$ -modules from  $I(Z_n)$  to  $I(Z_{n,-1})$  obtained from a cobordism between the links  $K(Z_n)$  and  $K(Z_{n,-1})$  inside  $[0, 1] \times S^1 \times S^2$ .*  $\square$

Like  $Z_n$ , the bifold  $Z_{n,-1}$  contains a copy  $S$  of the orbifold sphere  $S_n^2$  intersecting the singular locus in  $n$  points. By the general constructions of [Section 2.3](#), this gives rise to operators  $\tilde{\alpha}, \tilde{\delta}_1, \dots, \tilde{\delta}_n$  and  $\tilde{\epsilon}$ , acting on  $I(Z_{n,-1})$  just as in the case of  $I(Z_n)$ , making  $I(Z_{n,-1})$  also an  $\mathcal{A}_n$ -module. Note that the  $n$  points of intersection with  $S$  all lie on the same component of the singular locus  $K(Z_{n,-1})$  (which is now a knot, not a link). The operators  $\tilde{\delta}_p$  are therefore all equal on  $I(Z_{n,-1})$ , and we will sometimes write this operator as  $\tilde{\delta}$ .

**Proposition 3.22** *With the instanton module structure in which  $\alpha, \delta_i, \epsilon \in \mathcal{A}_n$  act by the operators  $\tilde{\alpha}, \tilde{\delta}$  and  $\tilde{\epsilon}$ , the instanton homology  $I(Z_{n,-1})$  is a cyclic module for the algebra  $\mathcal{A}_n$  and can therefore be described as a quotient,*

$$I(Z_{n,-1}) \cong \mathcal{A}_n / \mathcal{J}_{n,-1}.$$

*The ideal  $\mathcal{J}_{n,-1}$  contains the ideal  $\mathcal{J}_n$  as well as the elements  $\delta_i - \delta_j$ .*

**Proof** We have seen that there is a cobordism from  $Z_n$  to  $Z_{n,-1}$  inducing a surjection on instanton homology ([Corollary 3.21](#)). The proposition follows from this and the above remark that the actions of the  $\tilde{\delta}_i$  are all equal.  $\square$

It is helpful here to introduce the smaller algebra

$$\bar{\mathcal{A}} = \mathcal{A}_n / \langle \delta_i - \delta_j \rangle_{i,j},$$

which we can write simply as

$$(32) \quad \bar{\mathcal{A}} = \mathcal{R}[\alpha, \delta, \epsilon] / \langle \epsilon^2 - 1 \rangle,$$

where  $\delta$  denotes the image of the  $\delta_i$  in the quotient ring. The algebra  $\bar{\mathcal{A}}$  described this way is independent of  $n$ . The above proposition then can be recast as

$$(33) \quad I(Z_{n,-1}) \cong \bar{\mathcal{A}} / \bar{\mathcal{J}}_{n,-1},$$

where  $\bar{\mathcal{J}}_{n,-1}$  is the image of  $\mathcal{J}_{n,-1}$  in  $\bar{\mathcal{A}}$ .

Our main goal in this paper is to identify  $I(Z_n)$  and  $I(Z_{n,-1})$  completely, by describing the ideals  $\mathcal{J}_n \subset \mathcal{A}_n$  and  $\bar{\mathcal{J}}_{n,-1} \subset \bar{\mathcal{A}}$ . In particular, as described in the introduction, we will eventually provide a set of generators of  $\bar{\mathcal{J}}_{n,-1}$  in closed form, as minors of an explicit matrix.

## 4 Relations in ordinary cohomology

### 4.1 Loci in families of parabolic bundles on $S_n^2$

Recall from [Proposition 3.6](#) the description of the cohomology ring of the representation variety

$$\text{Rep}(Z_n) = \text{Rep}(S_n^2) \cup \text{Rep}(S_n^2)$$

as a quotient  $\mathcal{A}_n / J_n$ , where  $J_n$  is an ideal. (The coefficient ring here, as in [Proposition 3.6](#), is  $\mathcal{R}$ , though at this point our calculations will involve only  $\mathbb{Q}$ , so rational coefficients would suffice.) The Betti numbers of  $\text{Rep}(S_n^2)$  were calculated recursively by Boden [\[3\]](#), and a full presentation of the cohomology ring (in a more general case) is described in [\[8\]](#). Generators for the ideal of relations in the specific case of  $\text{Rep}(S_n^2)$  are given by Street [\[32\]](#). We shall describe a particular source of such relations, arising from a mechanism first pointed out by Mumford in the smooth case [\[1\]](#). (In [\[8\]](#) it is shown that essentially the same mechanism gives rise to a complete set of relations in the orbifold case.)

As stated earlier, although we have taken  $\text{SO}(3)$  connections as our starting point, the representation variety  $\text{Rep}(S_n^2)$  can be identified with the space of flat  $\text{SU}(2)$  connections having monodromy of order 4 at each of the  $n$  punctures. In turn, this representation variety can be identified with a moduli space of stable parabolic bundles by the results of [\[25\]](#). We adopt the following conventions to make this more specific in the rank-2 case, following [\[17; 18\]](#).

We consider a compact Riemann surface  $S$  equipped with a set of distinguished points  $\pi = \{p_1, \dots, p_n\}$ , and a parameter  $\alpha \in (0, \frac{1}{2})$ . Given a fixed holomorphic line bundle  $\Theta \rightarrow S$  (usually trivial in our case), we study rank-2 holomorphic bundles  $\mathcal{E} \rightarrow S$  with  $\Lambda^2 \mathcal{E} = \Theta$ , together with a filtration of the rank-2

fiber at each  $p \in \pi$  determined by a choice of a one-dimensional subspace (a line)  $\mathcal{L}_p \subset \mathcal{E}_p$ . The data  $(\mathcal{E}, \mathcal{L}_{p_1}, \dots, \mathcal{L}_{p_n}, \alpha)$  is a bundle with parabolic structure. Given a line subbundle  $\mathcal{F} \subset \mathcal{E}$ , the *parabolic degree* of  $\mathcal{F}$  is defined by

$$(34) \quad \text{par-deg } \mathcal{F} = c_1(\mathcal{F})[S] + \sum_{\pi} \pm \alpha,$$

where we take  $+\alpha$  in the sum when  $\mathcal{F}$  contains  $\mathcal{L}_p$  at  $p$  and  $-\alpha$  when it does not. The parabolic bundle is *semistable* if

$$\text{par-deg } \mathcal{F} \leq \frac{1}{2} \text{deg } \Theta$$

for every line subbundle  $\mathcal{F}$ , and is *stable* if strict inequality holds. At present we will take  $\Theta$  to be trivial and *we are only concerned with the special case*  $\alpha = \frac{1}{4}$ . In this case, when  $n$  is odd, all semistable bundles are strictly stable, and the moduli space of stable parabolic bundles is a projective variety of complex dimension  $3g - 3 + n$ . In the case of genus 0, we write  $\mathcal{M}(S_n^2)$  for this projective variety: the moduli space of stable parabolic bundles, with parabolic structure at the  $n$  marked points and  $\alpha = \frac{1}{4}$ .

With this notation understood, the theorem of [25] identifies the representation variety  $\text{Rep}(S_n^2)$  for odd  $n$  with the moduli space of stable parabolic bundles:

$$\text{Rep}(S_n^2) \cong \mathcal{M}(S_n^2).$$

Suppose now that we have a family of parabolic bundles on  $S_n^2$  parametrized by a space  $T$ . This means that we have a rank-2 bundle,

$$\mathcal{E} \rightarrow T \times S^2,$$

with  $\Lambda^2 \mathcal{E} \cong \Phi \boxtimes \Theta$  (with  $\Theta$  still trivial on  $S^2$  at the moment, but  $\Phi$  a nontrivial line bundle on the base  $T$ ), together with line subbundles

$$\mathcal{L}_p \subset \mathcal{E}|_{T \times p} \quad \text{for } p \in \pi.$$

The bundle  $\mathcal{E}$  is equipped with a holomorphic structure on each  $\{t\} \times S^2$ , giving rise to parabolic bundles  $\mathcal{E}_t$ .

In such a family over  $T$ , we can consider the locus of those  $t \in T$  where the parabolic bundle  $\mathcal{E}_t$  is unstable (for  $\alpha = \frac{1}{4}$ ). From the definition at (34), being unstable means the following.

- (i) We have a holomorphic line bundle  $\mathcal{F} \rightarrow S^2$ , of degree  $f$  say, necessarily the bundle  $\mathcal{O}(f)$ .
- (ii) We have a subset  $\eta \subset \pi$ , whose cardinality we denote by  $h$ .
- (iii) There is a nonzero holomorphic map  $\iota: \mathcal{F} \rightarrow \mathcal{E}_t$  such that  $\iota(\mathcal{F}|_p) \subset \mathcal{L}_t|_p$  for all  $p \in \eta$ .
- (iv) We have  $f + \frac{1}{4}(2h - n) > 0$ .

Altering this slightly, given any  $\lambda \in \mathbb{R}$ , we make the following definition.

**Definition 4.1** Let  $\eta \subset \pi = \{p_1, \dots, p_n\}$  be any subset, and write  $h = |\eta|$  for its cardinality. Let  $\lambda$  be an odd multiple of  $\frac{1}{4}$  satisfying the additional constraint that

$$(35) \quad h = \frac{1}{2}(n - 4\lambda) \pmod{2}.$$

This being so, there is  $f \in \mathbb{Z}$  such that

$$(36) \quad f + \frac{1}{4}(2h - n) = -\lambda.$$

Let  $\mathcal{F} \rightarrow S^2$  be the line bundle  $\mathcal{O}(f)$ . Given a family of parabolic bundles on  $S_n^2$  parametrized by  $T$  as above, we define

$$(37) \quad T_\lambda^\eta \subset T$$

to be the locus of points  $t \in T$  such that there is a nonzero holomorphic map  $\iota: \mathcal{F} \rightarrow \mathcal{E}_t$  with  $\iota(\mathcal{F}|_p) \subset \mathcal{L}_t|_p$  for all  $p \in \eta$ .

This definition is set up so that the unstable locus is the union

$$\bigcup_{\eta; \lambda \leq -\frac{1}{4}} T_\lambda^\eta.$$

The definition of the locus  $T_\lambda^\eta$  is readily rephrased as the statement that a certain Fredholm operator  $P_t$  (defined below at (41), and determined by the parabolic bundle  $\mathcal{E}_t$  and the choice of  $\lambda$  and  $\eta$ ) has nonzero kernel. If we suppose that the resulting map

$$P: T \rightarrow \text{Fred}$$

is transverse to the stratification of the space of Fredholm operators by the dimension of the kernel, then the locus  $T_\lambda^\eta \subset T$  will itself be a stratified space whose Poincaré dual is a cohomology class that one can calculate using the index theorem for families. With slight abuse of notation, we write (37) as

$$T_\lambda^\eta = T \cap U_\lambda^\eta,$$

where  $U_\lambda^\eta$  denotes the locus where the Fredholm operator has kernel. It will also be useful to group together the different subsets  $\eta$  according to their size  $h = |\eta|$ , so that we write (with a slight further abuse of notation),

$$U_\lambda^h = \bigcup_{|\eta|=h} U_\lambda^\eta \quad \text{and} \quad T_\lambda^h = T \cap U_\lambda^h.$$

Again, this locus is nonempty only if  $h$  satisfies the parity condition (35).

We now compute the Chern classes of the index of the family of operators  $P$  in order to derive a formula for the class dual to the stratum  $T_\lambda^\eta$ . Note that if  $P$  is a family of complex Fredholm operators of index  $-k + 1$ , then (assuming transversality) the locus where  $P_t$  has kernel is dual to

$$(38) \quad c_k(-\text{index}(P)) \in H^{2k}(T).$$

(This is the first case of Porteous’s formula in the case of Fredholm maps [29; 15].)

It is evident from the definition that the locus  $T_\lambda^\eta$  is unchanged if the family of bundles  $\mathcal{E}$  is modified by tensoring with a line bundle pulled back from the base  $T$ . Recall that we have written  $\Lambda^2 \mathcal{E} = \Phi \boxtimes \Theta$ , where  $\Phi \rightarrow T$  is a line bundle and  $\Theta$  is taken to be trivial. If  $\Phi$  has a square root, we may tensor by  $\Phi^{-1/2}$  to make  $c_1(\mathcal{E}) = 0$ . Although a square root will not exist in general, the calculation below is not invalidated by assuming that  $c_1(\mathcal{E}) = 0$ , and we will make this simplification from here on. This means in particular that  $c_2(\mathcal{E}) = -p_1(\text{ad } \mathcal{E})/4$ . Let us then write

$$c_2(\mathcal{E}) = \beta \times 1 + \hat{\alpha} \times v \in H^4(T \times S^2),$$

where  $v$  is the unit volume form on  $S^2$ . From the binomial theorem, we have

$$(39) \quad c_2(\mathcal{E})^r = \beta^r \times 1 + r\hat{\alpha}\beta^{r-1} \times v.$$

The class  $\hat{\alpha}$  here does not quite correspond to the class  $\alpha$  in (14), because the latter was defined using the orbifold Pontryagin class. The relation between the two is:

$$(40) \quad \hat{\alpha} = \alpha - \frac{1}{2} \sum_{p \in \pi} \delta_p.$$

For each  $p \in \pi$  we also have the line subbundle  $\mathcal{L}_p$  and the quotient line bundle  $\mathcal{Q}_p = (\mathcal{E}|_{T \times p})/\mathcal{L}_p$ , and from these we obtain the cohomology class

$$\delta_p = \frac{1}{2}(c_1(\mathcal{Q}_p) - c_1(\mathcal{L}_p)).$$

The definition is set up so that  $2\delta_p$  coincides with the Euler class of the oriented rank-2 subbundle of  $\text{ad}(\mathcal{E}|_{T \times p})$  determined by  $\mathcal{L}_p$ .

Fix a holomorphic line bundle  $\mathcal{F} \cong \mathcal{O}(f)$  on  $S^2$ . We are seeking a nonzero holomorphic map  $\iota: \mathcal{F} \rightarrow \mathcal{E}_t$  such that the composite with the quotient map,

$$\mathcal{F} \rightarrow \mathcal{E}_t \xrightarrow{\pi_p} \mathcal{Q}_{(t,p)},$$

vanishes for all  $p \in \eta$ . That is,  $\iota \in \Omega^{0,0}(\mathcal{F}^* \otimes \mathcal{E}_t)$  lies in the kernel of the map

$$(41) \quad P_t: \Omega^{0,0}(\mathcal{F}^* \otimes \mathcal{E}_t) \rightarrow \Omega^{0,1}(\mathcal{F}^* \otimes \mathcal{E}_t) \oplus \left( \bigoplus_{p \in \eta} \mathcal{Q}_{(t,p)} \right)$$

given by  $\iota \mapsto (\bar{\partial}\iota, \sum_{p \in \eta} \pi_p \circ \iota(p))$ . So, for the family of Fredholm operators  $P$  that we are interested in,

$$\text{index}(P) = \text{index}(\bar{\partial}_{\mathcal{F}^* \otimes \mathcal{E}}) - \sum_{p \in \eta} [\mathcal{Q}_p],$$

where the first part is the ordinary family  $\bar{\partial}$  operators. From the index theorem for families, we have

$$(42) \quad \text{ch}(\text{index}(P)) = ((\text{Todd}(S^2) \cup \text{ch}(\mathcal{F}^* \otimes \mathcal{E})) / [S^2]) - \sum_{p \in \eta} \text{ch}(\mathcal{Q}_p).$$

To compute the Chern characters that appear on the right-hand side of this formula, we introduce formal Chern roots  $\pm\rho \in H^2(T \times S^2; \mathbb{Q})$  so that  $c_2(\mathcal{E}) = -\rho^2$ . Then we can write

$$\text{ch}(\mathcal{E}) = e^{-\rho} + e^{\rho} = 2 \cosh(\sqrt{-c_2(\mathcal{E})}),$$

and a short calculation using (39) yields

$$\text{ch}(\mathcal{E}) = 2 \cosh(\sqrt{-\beta}) - v \frac{\sinh(\sqrt{-\beta})}{\sqrt{-\beta}} \hat{\alpha}.$$

We also have

$$\text{ch}(\mathcal{F}^*) = 1 - fv \quad \text{and} \quad \text{ch}(\mathcal{Q}_p) = e^{\delta_p}.$$

Finally, on the right-hand side of (42) we have  $\text{Todd}(S^2) = 1 + v$ . Assembling these and calculating the slant product by  $[S^2]$ , we find

$$\text{ch}(\text{index}(P)) = (2 - 2f - h) \cosh(\sqrt{-\beta}) - \frac{\sinh(\sqrt{-\beta})}{\sqrt{-\beta}} \left( \hat{\alpha} + \sum_{p \in \eta} \delta_p \right),$$

where  $h$  is the number of elements of  $\eta$ . If we use the equality of (36), and if we substitute  $\alpha$  for  $\hat{\alpha}$  using the relation (40), we obtain:

$$(43) \quad \text{ch}(-\text{index}(P)) = \left(\frac{1}{2}n - 2\lambda - 2\right) \cosh(\sqrt{-\beta}) + \frac{\sinh(\sqrt{-\beta})}{\sqrt{-\beta}} \left( \alpha + \frac{1}{2} \sum_{p \in \eta} \delta_p - \frac{1}{2} \sum_{p \notin \eta} \delta_p \right).$$

If we recall that  $\delta_p^2 = -\beta$  for all  $p$ , then we can equivalently write this formula as

$$(44) \quad \text{ch}(-\text{index}(P)) = \left(\frac{1}{2}n - 2\lambda - 2\right) \cosh(\delta_1) + \frac{\sinh(\delta_1)}{\delta_1} \left( \alpha + \frac{1}{2} \sum_{p \in \eta} \delta_p - \frac{1}{2} \sum_{p \notin \eta} \delta_p \right),$$

or in abbreviated form as

$$(45) \quad \text{ch}(-\text{index}(P)) = i_\lambda \cosh(\delta_1) + \frac{\sinh(\delta_1)}{\delta_1} B_\eta,$$

where  $i_\lambda$  and  $B_\eta$  are the indicated subexpressions of (44). Note that  $i_\lambda$  is minus the numerical index of  $P$ .

The above formula defines a graded infinite sum of elements of the algebra

$$A_n = \mathbb{Q}[\alpha, \delta_1, \dots, \delta_n] / \langle \delta_i^2 - \delta_j^2 \rangle_{i,j} = H^*(\mathcal{B}^*(S_n^2); \mathbb{Q})$$

(see Definition 3.5), thus an element of the formal completion

$$\widehat{H}^*(\mathcal{B}^*(S_n^2); \mathbb{Q}) \supset H^*(\mathcal{B}^*(S_n^2); \mathbb{Q}).$$

By the usual formulae expressing elementary symmetric polynomials in terms of power sums, there is a map

$$c_k : \widehat{H}^*(\mathcal{B}^*(S_n^2); \mathbb{Q}) \rightarrow H^{2k}(\mathcal{B}^*(S_n^2); \mathbb{Q})$$

such that  $c_k(\text{ch}(V)) = c_k(V)$  for any  $V$ , and so we have explicit formulae for

$$c_k(-\text{index}(P)) \in H^*(\mathcal{B}^*(S_n^2); \mathbb{Q}),$$

given as  $c_k(r)$ , where  $r$  is the right-hand side of (44). The case we are interested in from (38) is the Chern class  $c_k$ , where  $-k + 1$  is the numerical index of  $P$ . From the constant term in the formula for the Chern character above, we read

$$(46) \quad k = \frac{1}{2}n - 2\lambda - 1.$$

So we make the following definition.

**Definition 4.2** Given  $\lambda$  an odd multiple of  $\frac{1}{4}$  and given a subset  $\eta \subset \pi = \{p_1, \dots, p_n\}$  of size  $h$ , where  $h$  satisfies the parity condition (35), let  $k$  be the integer given by (46), and denote by

$$w_{n,\eta}^k \in H^*(\mathcal{B}^*(S_n^2); \mathbb{Q}) \subset \mathcal{A}_n$$

the element  $c_k(r)$ , where  $r$  is the right-hand side of (44).

To illustrate the general shape of the answers here, we take  $n = 5$ . When  $\lambda = -\frac{1}{4}$ , the value of  $k$  is 2. The parity condition allows the size of  $\eta$  to be 1, 3 or 5, and we have

$$w_{5,\eta}^2 = \frac{1}{2}((\alpha + \frac{1}{2}(\pm\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4 \pm \delta_5))^2 - \delta_1^2),$$

where the sign is  $+$  when  $p_i \in \eta$  and  $-$  otherwise. When  $\lambda = \frac{1}{4}$ , the value of  $k$  is 1, and the parity condition allows the size of  $\eta$  to be 0, 2 or 4. We have

$$w_{5,\eta}^1 = \alpha + \frac{1}{2}(\pm\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4 \pm \delta_5).$$

Our definition means that, in  $H^*(T; \mathbb{Q})$ , we have  $c_k(-\text{index } P) = \varphi(w_{n,\eta}^k)$ , where  $\varphi: \mathcal{A}_n \rightarrow H^*(T; \mathbb{Q})$  is the natural map (given, with slight abuse of notation, by  $\alpha \mapsto \alpha$  and  $\delta_p \mapsto \delta_p$ ).

**Corollary 4.3** Let  $(\mathcal{E}, \mathcal{L}) \rightarrow T \times S^2$  be a family of parabolic bundles on  $S_n^2$  parametrized by  $T$ . Let  $\lambda$  and  $\eta$  be given, satisfying the conditions in Definition 4.2, and let  $T_\lambda^\eta \subset T$  be the locus defined by (37). Assume that the corresponding family of Fredholm operators  $P$  is transverse to the stratification by the dimension of the kernel. Then the cohomology class dual to this stratum is given by

$$\text{PD}[T_\lambda^\eta] = \varphi(w_{n,\eta}^k),$$

where  $\varphi$  is the natural linear map  $\mathcal{A}_n \rightarrow H^*(T; \mathbb{Q})$ , and  $k$  is given in terms of  $n$  and  $\lambda$  by (46).

**Remarks** In Definition 4.1, the loci  $T_\lambda^\eta$  are characterized by the existence of a holomorphic map  $\iota: \mathcal{F} \rightarrow \mathcal{E}$  satisfying additional constraints at the distinguished points  $\eta \subset \pi$ . In the language of parabolic bundles, we can regard  $\mathcal{F}$  as a line bundle with parabolic structure described by a subsheaf  $\mathcal{F}_1 \subset \mathcal{F}$  such that in a neighborhood  $\mathcal{U}_p$  of each  $p \in \pi$  we have

$$\begin{aligned} \mathcal{F}_1|_{\mathcal{U}_p} &= \mathcal{F}|_{\mathcal{U}_p} && \text{if } p \in \eta, \\ \mathcal{F}_1|_{\mathcal{U}_p} &= (\mathcal{F} \otimes \mathcal{O}[-p])|_{\mathcal{U}_p} && \text{if } p \notin \eta. \end{aligned}$$

In these terms, what  $T_\lambda^\eta$  describes is the existence of a map  $\mathcal{F} \rightarrow \mathcal{E}$  of parabolic bundles: ie a map which respects the filtrations. When regarded as a line bundle with parabolic structure in this way, we shall call  $\eta \subset \pi$  the set of ‘‘hits’’ for  $\mathcal{F}$ .

### 4.2 The Mumford relations

As a consequence of Corollary 4.3, we have the following statement, which is the essential mechanism in Mumford’s relations. (See the discussion in [1] for the earlier history of such relations.)

**Proposition 4.4** *Let  $(\mathcal{E}, \mathcal{L})$  be a family of parabolic bundles on  $S_n^2$  parametrized by a space  $T$  as in the previous subsection. Suppose that for every  $t \in T$  the parabolic bundle  $(\mathcal{E}_t, \mathcal{L}_t)$  on  $S_n^2$  is stable (with  $\alpha = \frac{1}{4}$  as always). Then for any  $\lambda$  and  $\eta$  satisfying the conditions in Definition 4.2, with  $\lambda < 0$ , we have*

$$\varphi(w_{n,\eta}^k) = 0 \quad \text{in } H^{2k}(T; \mathbb{Q}),$$

where  $k = \frac{1}{2}n - 2\lambda - 1$  and  $\varphi: H^*(\mathcal{B}^*(S_n^2); \mathbb{Q}) \rightarrow H^*(T; \mathbb{Q})$  is the natural map determined by the characteristic classes of  $\mathcal{E}$  and  $\mathcal{L}$ .

**Proof** When  $\lambda < 0$ , the stratum  $T_\lambda^\eta$  consists of unstable parabolic bundles, so the hypothesis of the proposition means that such strata are empty. The transversality condition is then vacuously satisfied and the result follows from Corollary 4.3. □

**Proposition 4.5** *Let  $\lambda = -\frac{1}{4}$  and let  $\eta \subset \pi = \{p_1, \dots, p_n\}$  be a subset whose size  $h$  satisfies*

$$(47) \quad h = \frac{1}{2}(n + 1) \bmod 2 \quad \text{and} \quad 0 \leq h \leq n.$$

(The first condition is the parity condition (35) for  $\lambda = -\frac{1}{4}$ .) As in Definition 3.5, let  $j_n$  be the kernel of the restriction map to cohomology of the representation variety,  $H^*(\text{Rep}(S_n^2); \mathbb{Q})$ . Then we have

$$w_{n,\eta}^m \in j_n$$

for  $m = \frac{1}{2}(n - 1)$ . That is,  $w_{n,\eta}^m$  is a relation in the cohomology ring of  $\text{Rep}(S_n^2)$ .

**Proof** This follows from the previous proposition by specializing to the case  $\lambda = -\frac{1}{4}$ , because  $\text{Rep}(S_n^2) \cong \mathcal{M}(S_n^2)$  parametrizes a family of stable parabolic bundles. □

**Definition 4.6** Let  $j_n \subset A_n$  be again the ideal of relations in the cohomology of  $\text{Rep}(S_n^2)$ . With  $m = \frac{1}{2}(n - 1)$  and  $\eta \subset \pi$  a subset whose size  $h$  satisfies the parity condition (47), we refer to the relation  $w_{n,\eta}^m \in j_n$  as a *Mumford relation*. The collection of all these, as  $\eta$  varies, are the *Mumford relations* in the cohomology ring of  $\text{Rep}(S_n^2)$ .

### 4.3 Explicit formulae

The elements  $w_{n,\eta}^m \in A_n$  appearing as the Mumford relations, and more generally the cohomology classes  $w_{n,\eta}^k$ , have been described using a characterization that does not immediately yield explicit formulae. In particular,  $w_{n,\eta}^k$  is defined in terms of a Chern class of an index element, while the explicit formula (44) provides the Chern character in closed form.

As a first step towards a closed formula for  $w_{n,\eta}^k$ , as in [36] for example, and following [37], a formula for the total Chern class can be derived as the formal series

$$(48) \quad \sum_{k=0}^{\infty} c_k(-\text{index}(P)) = (1 + \beta)^{i_\lambda/2} \left( \frac{1 + \delta_1}{1 - \delta_1} \right)^{B_n/(2\delta_1)},$$

where  $i_\lambda$  and  $B_\eta$  are as in (44):

$$(49) \quad i_\lambda = \left(\frac{1}{2}n - 2\lambda - 2\right) \quad \text{and} \quad B_\eta = \alpha + \frac{1}{2} \sum_{p \in \eta} \delta_p - \frac{1}{2} \sum_{p \notin \eta} \delta_p.$$

(See [37] for the interpretation of the right-hand side of this formula.) We can therefore write

$$(50) \quad w_{n,\eta}^k = \frac{1}{k!} \left( \frac{d^k}{dt^k} \right) \Big|_{t=0} \left( (1 + t^2\beta)^{i_\lambda/2} \left( \frac{1 + t\delta_1}{1 - t\delta_1} \right)^{B_\eta/(2\delta_1)} \right).$$

Note here that  $i_\lambda$  is minus the numerical index of  $P$ , and that in the definition of  $w_{n,\eta}^k$  the integer  $k$  is  $-\text{index}(P) + 1$ , so we can write

$$(51) \quad w_{n,\eta}^k = \frac{1}{k!} \left( \frac{d^k}{dt^k} \right) \Big|_{t=0} \left( (1 + t^2\beta)^{(k-1)/2} \left( \frac{1 + t\delta_1}{1 - t\delta_1} \right)^{B_\eta/(2\delta_1)} \right).$$

The following proposition gives a closed formula for this  $k^{\text{th}}$  term in the power series.

**Proposition 4.7** 
$$k! w_{n,\eta}^k = \prod_{\substack{j=-k+1 \\ j=-k+1 \pmod 2}}^{k-1} (B_\eta + j\delta_1).$$

**Proof** Let us write

$$C_k = k! w_{n,\eta}^k = \left( \frac{d^k}{dt^k} \right) \Big|_{t=0} G_{k-1}(t), \quad \text{where } G_{k-1}(t) = (1 + t^2\beta)^{(k-1)/2} \left( \frac{1 + t\delta}{1 - t\delta} \right)^{B/(2\delta)},$$

and we have abbreviated

$$B = B_\eta \quad \text{and} \quad \delta = \delta_1$$

to streamline the notation.

Let  $\hat{C}_k$  denote the right-hand side in the proposition,

$$\hat{C}_k = \prod_{\substack{j=-k+1 \\ j=-k+1 \pmod 2}}^{k-1} (B + j\delta),$$

so that what we aim to prove is that  $C_k$  and  $\hat{C}_k$  are equal. We shall prove in fact that

$$(52) \quad \frac{d^k}{dt^k} G_{k-1}(t) = \hat{C}_k G_{-k-1}(t),$$

which yields the desired equality  $C_k = \hat{C}_k$  on substituting  $t = 0$ , since  $G_l(0) = 1$  for all  $l$ .

We prove (52) by induction on  $k$ : specifically, assuming that (52) holds for  $k$ , we prove the result for  $k + 2$ . The seed cases,  $k = 0, 1$ , are clear. Note first that  $\hat{C}_k$  satisfies a recurrence relation

$$(53) \quad \hat{C}_{k+2} = (B^2 + (k + 1)^2\beta)\hat{C}_k = (B^2 - (k + 1)^2\delta^2)\hat{C}_k.$$

Next we examine the first two derivatives of  $G_k(t)$ : by induction on  $k$  and using the fact that  $G_k(t) = (1 + t^2\beta)G_{k-2}$ , we obtain the following identity for the first derivative:

$$(54) \quad \frac{d}{dt}G_k(t) = (B - k\delta^2t)G_{k-2}(t).$$

Applying this twice, we obtain an identity for the second derivative:

$$(55) \quad \frac{d^2}{dt^2}G_k(t) = (B^2 - k\delta^2 - 2(k-1)B\delta^2t + k(k-1)\delta^4t^2)G_{k-4}(t).$$

Using these identities for the first two derivatives, together with the induction hypothesis (52) and the recurrence relation (53), we compute

$$\begin{aligned} & \frac{d^{k+2}}{dt^{k+2}}G_{k+1}(t) \\ &= \frac{d^{k+2}(1-\delta^2t^2)G_{k-1}(t)}{dt^{k+2}} \\ &= (1-\delta^2t^2)\frac{d^{k+2}G_{k-1}(t)}{dt^{k+2}} - 2(k+2)\delta^2t\frac{d^{k+1}G_{k-1}(t)}{dt^{k+1}} - (k+2)(k+1)\delta^2\frac{d^kG_{k-1}(t)}{dt^k} \\ &= \left( (1-\delta^2t^2)\frac{d^2}{dt^2} - 2(k+2)\delta^2t\frac{d}{dt} - (k+2)(k+1)\delta^2 \right) \frac{d^kG_{k-1}(t)}{dt^k} \\ &= \left( (1-\delta^2t^2)\frac{d^2}{dt^2} - 2(k+2)\delta^2t\frac{d}{dt} - (k+2)(k+1)\delta^2 \right) \widehat{C}_k G_{-k-1}(t) \\ &= ((B^2 + \delta^2(k+1) + 2(k+2)B\delta^2t + (k+1)(k+2)\delta^4t^2) - 2(k+2)\delta^2t(B + (k+1)\delta^2t) \\ & \quad - (k+2)(k+1)\delta^2(1-\delta^2t^2)) \widehat{C}_k G_{-k-3}(t) \\ &= (B^2 - (k+1)^2\delta^2) \widehat{C}_k G_{-k-3}(t) = \widehat{C}_{k+2} G_{-k-3}(t), \end{aligned}$$

as required. □

### 4.4 The Mumford relations as generators of the ideal

In [32], a presentation of the cohomology ring of  $\text{Rep}(S_n^2)$  is given by exhibiting a complete set of generators for the ideal of relations  $j_n \subset A_n$ , all of which have degree  $m = \frac{1}{2}(n-1)$ . We now show that the elements  $w_{n,\eta}^m$  also generate the ideal, by relating them to the generators in [32].

**Remark** The statement that the elements  $w_{n,\eta}^s$ , for  $s \geq m$ , generate the ideal is a counterpart of Kirwan’s result [14] in the case of a (nonorbifold) surface of genus  $g$ . Kirwan’s result was strengthened by Kiem [12], who showed that the relations in the middle dimension (ie the case  $s = m$  in our context) are sufficient. The results of [14] were extended to the case of parabolic bundles on surfaces of genus  $g \geq 2$  with one marked point by Earl and Kirwan [8].

**Proposition 4.8** Fix  $n \geq 3$  odd, and write  $n = 2m + 1$ . Then as  $\eta$  runs through all subsets of  $\pi$  whose size  $h = |\eta|$  satisfies the parity condition (47), the elements  $w_{n,\eta}^m \in A_n$  form a set of generators of the ideal  $j_n$ . That is, the elements  $w_{n,\eta}^m$  form a complete set of relations for the cohomology of  $\text{Rep}(S_n^2)$  as a quotient of the algebra  $A_n$ .

**Proof** From the results of [32], in the ideal  $j_n$ , there is an element  $r^m$  that has degree  $m$  and belongs to the subalgebra  $\mathbb{Q}[\alpha, \beta] \subset A_n$ , where  $\beta = -\delta_p^2$ . The element  $r^m$  is unique up to scale. According to [32, Corollary 1.6.2], the ideal  $j_n$  is generated by the elements

$$R_\zeta^m = r^{m-|\zeta|}(\alpha, \beta) \prod_{p \in \zeta} \delta_p,$$

where  $\zeta$  runs through all subsets of  $\pi$  of size  $0 \leq |\zeta| \leq m$ . These elements all have degree  $m$ .

As we vary  $\eta$ , we obtain  $2^{n-1}$  expressions  $w_{n,\eta}^m$ , all of which are elements of  $j_n$  of degree  $m$ . Because  $m$  is the lowest degree in which relations exist, each  $w_{n,\eta}^m$  is a  $\mathbb{Q}$ -linear combination of the generators  $R_\zeta^m$ . The number of generators  $R_\zeta^m$  is also  $2^{n-1}$ ; so in order to see that the elements  $w_{n,\eta}^m$  generate the ideal  $j_n$ , it will be enough if we show that they are linearly independent over  $\mathbb{Q}$ .

The fact that the elements  $w_{n,\eta}^m$  are linearly independent can be deduced through a direct examination of the formulae which define it, as follows. Let us specialize the formulae by setting  $\beta = 0$ , in which case the expression (48) for the total Chern class of  $-\text{index}(P)$  simplifies to

$$(1 + 2\delta_1)^{B_\eta/(2\delta_1)} = \exp B_\eta$$

because  $\delta_1^2 = 0$ . The element  $w_{n,\eta}^m$  therefore specializes to  $B_\eta^m/m!$ , or if we further specialize by setting  $\alpha = 1$ , to

$$\frac{1}{m!} \left( 1 + \sum_p \eta_p \delta_p \right)^m,$$

where  $\eta_p = 1$  for  $p \in \eta$  and  $\eta_p = -1$  otherwise. We can expand this as

$$\sum_{|\zeta| \leq m} C_{\eta,\zeta} \left( \prod_{p \in \zeta} \delta_p \right),$$

where the rational coefficient  $C_{\eta,\zeta}$  is given by

$$C_{\eta,\zeta} = \frac{1}{(m - |\zeta|)!} \left( \prod_{p \in \zeta} \eta_p \right).$$

We wish to see that the matrix  $C = (C_{\eta,\zeta})$ , which is square of size  $2^{n-1}$ , is nonsingular. To do this, we compute the dot product of the columns of  $C$  corresponding to different subsets  $\zeta_1$  and  $\zeta_2$ . For fixed  $\eta$  we have

$$C_{\eta,\zeta_1} C_{\eta,\zeta_2} = \frac{1}{(m - |\zeta_1|)! (m - |\zeta_2|)!} \times \begin{cases} +1 & \text{if } |\eta \cap (\zeta_1 \ominus \zeta_2)^c| \text{ is even,} \\ -1 & \text{if } |\eta \cap (\zeta_1 \ominus \zeta_2)^c| \text{ is odd,} \end{cases}$$

where the superscript  $c$  denotes the complement and  $\ominus$  means the symmetric difference. Since  $\zeta_1$  and  $\zeta_2$

are distinct subsets of size strictly less than  $n/2$ , their symmetric difference is a nonempty proper subset of  $\pi$ . The number of subsets  $\eta$  of a given parity for which the intersection is even and the number for which it is odd are therefore equal, and we see that

$$\sum_{\eta} C_{\eta, \xi_1} C_{\eta, \xi_2} = 0.$$

The columns are therefore orthogonal, showing that the square matrix  $C$  is nonsingular, as required.  $\square$

**Remarks** An alternative verification of the linear independence of the elements  $w_{n,\eta}^m$ , not depending on an examination of the formula, will be seen later, in the remarks at the end of [Section 5.4](#).

## 5 Relations in instanton homology

### 5.1 Deforming the Mumford relation with instanton terms

The element  $w_{n,\eta}^m \in j_n$  in [Proposition 4.5](#) is a relation in the ordinary cohomology ring  $H^*(\text{Rep}(S_n^2); \mathbb{Q})$ . Via its description in terms of the multiplicative generators  $\alpha$  and  $\delta_p$ , as an explicit element of the ring

$$\mathbb{Q}[\alpha, \delta_1, \dots, \delta_n] / \langle \delta_i^2 - \delta_j^2 \rangle_{i,j},$$

we may regard  $w_{n,\eta}^m$  also as an element of the ideal  $J_n \subset \mathcal{A}_n$  of [Proposition 3.6](#), where it is a relation in the ordinary cohomology ring  $H^*(\text{Rep}(Z_n); \mathcal{R})$ . As  $\eta$  varies over all subsets of  $\pi$  satisfying the parity condition, the elements  $w_{n,\eta}^m \in J_n$  form a set of generators of the ideal, as follows immediately from the corresponding statement for  $\text{Rep}(S_n^2)$  ([Proposition 4.8](#)).

The following proposition promotes  $w_{n,\eta}^m$  to a relation between the corresponding operators on the instanton homology  $I(Z_n)$  by adding terms of lower degree. Recall that  $\mathcal{J}_n \subset \mathcal{A}_n$  is the annihilator of  $I(Z_n)$  as an  $\mathcal{A}_n$ -module.

**Proposition 5.1** *Let  $n$  be odd and let  $\eta \subset \pi$  be a subset whose size  $h$  satisfies the parity condition (47). Write  $m = \frac{1}{2}(n - 1)$  and let  $w_{n,\eta}^m \in j_n \subset J_n$  be as in [Proposition 4.5](#), regarded as a relation in the ordinary cohomology of the representation variety  $\text{Rep}(Z_n)$ . Then there is a unique element  $W_{\eta}^m \in \mathcal{J}_n \subset \mathcal{A}_n$  in filtration degree  $m$  whose leading term is  $w_{n,\eta}^m$ :*

$$W_{\eta}^m = w_{n,\eta}^m \pmod{\mathcal{A}_n^{(m-1)}}.$$

*As  $\eta$  varies over all subsets satisfying the parity condition, these elements  $W_{\eta}^m$  form a set of generators for the ideal of relations  $\mathcal{J}_n$ .*

**Remark** Our notation for  $W_{\eta}^m$  does not include  $n$ , since  $n$  is always related to  $m$  in this context by  $n = 2m + 1$ .

**Proof of Proposition 5.1** Corollary 3.16 gives the existence of  $W_\eta^m \in \mathcal{J}_n$  with leading term  $w_{n,\eta}^m$ . The uniqueness assertion is a consequence of Proposition 3.7. The fact that these are a complete set of generators for the ideal follows from the corresponding statement for the elements  $w_{n,\eta}^m \in J_n$  together with the fact that  $\mathcal{A}_n/J_n$  and  $\mathcal{A}_n/\mathcal{J}_n$  are free modules of the same rank, because they are respectively the ordinary homology of  $\text{Rep}(Z_n)$  and the instanton homology of  $Z_n$  (Proposition 3.8).  $\square$

We aim to give an algorithm for computing  $W_\eta^m$  as a deformation of  $w_{n,\eta}^m$ , and our first main step will be to determine the subleading term. That is, Corollary 3.16 provides the existence of  $w(1)$  with

$$W_\eta^m = w_{n,\eta}^m + \epsilon w(1) \pmod{\mathcal{A}_n^{(m-2)}},$$

and we wish to determine  $w(1)$ .

**Proposition 5.2** *The subleading term of  $W_\eta^m$  is given by  $\epsilon \tau^{n-2h} w_{n,\eta'}^{m-1}$ , where  $\eta'$  is the complement  $\pi \setminus \eta$  and  $h = |\eta|$ , so*

$$W_\eta^m = w_{n,\eta}^m + \epsilon \tau^{n-2h} w_{n,\eta'}^{m-1} \pmod{\mathcal{A}_n^{(m-2)}}.$$

The proof of this proposition will require some preparation. To understand how to characterize the subleading term  $\epsilon w(1)$ , we draw on the mechanism behind Proposition 3.14 and Corollary 3.16. Let  $\mathbb{1}_+$  again be the standard cyclic generator of  $I(Z_n)$  from Proposition 3.9, and let  $\mathbb{1}_- = \tilde{\epsilon} \mathbb{1}_+$ . Let  $\Lambda$  be the  $\mathcal{R}$ -module isomorphism in Proposition 3.14, and let  $1_\pm = \Lambda(\mathbb{1}_\pm) \in H_*(\text{Rep}(Z_n); \mathcal{R})$ . Then  $w(1)$  is a homogeneous polynomial of degree  $m - 1$  in  $\alpha$  and the  $\delta_p$ , with coefficients in  $\mathcal{R}$ , such that

$$\Lambda(w_{n,\eta}^m \mathbb{1}_-) - w_{n,\eta}^m 1_- = w(1) 1_+ \pmod{\bigoplus_{k \leq 2(m-2)} H^k(\text{Rep}(Z_n); \mathcal{R})}.$$

(The right-hand side is the  $(m - 2)^{\text{th}}$  step of the increasing filtration of  $H^*(\text{Rep}(Z_n); \mathcal{R})$ .)

Recall next we have an expansion of the operator  $\tilde{\alpha}$  according to the action  $\kappa \in \frac{1}{4}\mathbb{Z}$ , as in (25) and Proposition 3.14. There is a similar expansion of each  $\tilde{\delta}_p$ . This gives a  $\kappa$ -expansion of any monomial in  $\tilde{\alpha}$  and the  $\tilde{\delta}_p$ , and therefore of the multiplication operator of any  $u \in \mathcal{A}_n$  acting on  $I(Z_n)$ . That is, we may write

$$u\xi = \sum_{\kappa \in \frac{1}{4}\mathbb{Z}} u *_\kappa \xi.$$

This description is set up so that if  $u \in \mathcal{A}_n$  is in grading  $k$  and  $\Lambda(\xi) \in H^{2l}(\text{Rep}(Z_n); \mathcal{R})$ , then

$$\Lambda(u *_\kappa \xi) \in H^{2(l+k)-8\kappa}(\text{Rep}(Z_n); \mathcal{R}).$$

The description of  $w(1)$  then becomes

$$(56) \quad w(1) 1_+ = \Lambda(w_{n,\eta}^m *_{1/4} \mathbb{1}_-).$$

Computation of  $w(1)$  in this form therefore depends directly on understanding the instantons on the cylinder  $\mathbb{R} \times Z_n$  with action  $\frac{1}{4}$ . We address this calculation in the following subsection, where the proof of Proposition 5.2 will be completed.

### 5.2 Characterizing the subleading term

From the discussion above, we are interested in the moduli space  $M_\kappa(\mathbb{R} \times Z_n)$  of anti-self-dual bifold  $SU(2)$  connections on the cylinder, particularly for  $\kappa = \frac{1}{4}$ . By attaching a copy of the bifold  $D^2 \times S_n^2$  to each of the two ends, we form from the cylinder a compact bifold

$$X = S^2 \times S_n^2.$$

For clarity in distinguishing the two factors here, we will write

$$X = B \times C,$$

where  $B$  is  $S^2$  and  $C$  is the bifold  $S_n^2$ . We write  $M_\kappa(X)$  for the moduli space of anti-self-dual  $SU(2)$  connections on the bifold  $X$ , with action  $\kappa$ , and we write  $M_\kappa^e(X)$  for the moduli space corresponding to  $w_2 = [e]$ , where  $[e] = \{b\} \times C$ . The moduli spaces depend, of course, on a choice of conformal structure on  $X$ . The moduli space  $M_\kappa(X)$  is nonempty only if  $\kappa \in \frac{1}{2}\mathbb{Z}$ , while  $M_\kappa^e(X)$  is nonempty only if  $\kappa \in \frac{1}{2}\mathbb{Z} + \frac{1}{4}$ . The moduli spaces have formal dimension

$$d(\kappa) = 8\kappa + 2n - 6.$$

For any element  $u \in \mathcal{A}_n$  of degree  $d(\kappa)/2$  in the variables  $\alpha$  and  $\delta_i$ , we can seek to evaluate a Donaldson polynomial invariant by evaluating the corresponding cohomology class on  $M_\kappa(X)$  or  $M_\kappa^e(X)$ . Because we are working with local coefficients  $\Gamma$ , our Donaldson invariants should also involve  $\mathcal{R}$ -valued weights. By the formula (2), the local system  $\Gamma$  defines a locally constant function

$$(57) \quad \Gamma: M_\kappa(X) \rightarrow \mathcal{R}^\times,$$

and so the moduli spaces are a collection of oriented, weighted manifolds.

However, the bifold  $X$  has  $b_2^+ = 1$ , so the appearance of reducibles in one-parameter families means that the Donaldson invariant depends on a choice of chamber in the space of Riemannian metrics on  $X$ . We consider a product metric in which the area of  $B$  is very large compared to the area of  $C$ , and we call this the *B-chamber*. (This means that the self-dual 2-form for the Riemannian metric on  $X$  is nearly Poincaré dual to a multiple of  $PD[C]$ .) Similarly there is a distinguished chamber, the *C-chamber*, in which the area of  $C$  is very large compared to  $B$ . There is then a well-defined Donaldson invariant  $q_\kappa^B$  in the *B-chamber*,

$$u \mapsto q_\kappa^B(X; u), \quad \mathcal{A}_n \rightarrow \mathcal{R},$$

calculated using either the moduli space  $M_\kappa(X)$  or the moduli space  $M_\kappa^e(X)$ , depending on whether  $4\kappa$  is even or odd respectively. Our notation again makes no explicit mention of the local coefficient system, but the contributions of the various components of the moduli spaces are to be weighted by the locally constant function (57).

These Donaldson invariants of  $X$  are related to the action of  $u$  on  $I(Z_n)$  by a gluing argument, because of the description of  $X$  as the union of the cylinder  $[-1, 1] \times Z_n$  and the two copies of  $D^2 \times S_n^2$ .

More specifically, let  $\mathbb{1}_+ \in I(Z_n)$  be once more the cyclic generator obtained as the relative invariant of the manifold  $D^2 \times S_n^2$ , and let  $\mathbb{1}_+^\dagger$  be the element of the instanton cohomology group  $I^*(Z_n)$  obtained by regarding  $D^2 \times Z_n$  as a manifold with boundary  $-Z_n$ . Then for  $\kappa \in \frac{1}{2}\mathbb{Z}$  and  $u \in \mathcal{A}_n^+$ , we can write

$$q_\kappa^B(X; u) = \langle u *_\kappa \mathbb{1}_+, \mathbb{1}_+^\dagger \rangle,$$

where the pairing on the right is the  $\mathcal{R}$ -valued pairing between  $I(Z_n)$  and  $I^*(Z_n)$ . For  $\kappa \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ , we have

$$q_\kappa^B(X; \epsilon u) = \langle u *_\kappa \mathbb{1}_+, \mathbb{1}_-^\dagger \rangle.$$

From this relationship and Poincaré duality, it follows that (56) is equivalent to

$$(58) \quad q_{1/4}^B(w(0)v) = q_0^B(\epsilon w(1)v)$$

for all  $v \in \mathcal{A}_n$  of degree

$$\deg(v) = \frac{1}{2}d\left(\frac{1}{4}\right) - \deg(w(0)) = n - 2 - m = m - 1,$$

where  $n = 2m + 1$  as usual.

The situation is somewhat simplified now because the moduli spaces  $M_0(X)$  and  $M_{1/4}^e(X)$  are compact. This is because noncompactness of the moduli space arises only from bubbling, and bubbles decrease  $\kappa$  by multiples of  $\frac{1}{2}$ . So for  $\kappa \leq \frac{1}{4}$ , the Donaldson invariants are simply evaluations on  $[M_\kappa(X)]$  or  $[M_\kappa^e(X)]$  of ordinary cohomology classes in  $H^*(\mathcal{B}^*(X); \mathcal{R})$ , weighted by the function locally constant (57). We will write  $[\Gamma \cdot M_\kappa(X)]$  and  $[\Gamma \cdot M_\kappa^e(X)]$  for these weighted fundamental classes, as elements of the ordinary homology  $H_*(\mathcal{B}^*(X); \mathcal{R})$ .

Via the relationship between  $\mathcal{A}_n$  and  $H^*(\mathcal{B}^*(Z_n); \mathcal{R})$ , we have an inclusion

$$\mathcal{A}_n \hookrightarrow H^*(\mathcal{B}^*(X); \mathcal{R}).$$

The relation (58) can therefore be stated in terms of ordinary pairings, between these cohomology classes and the fundamental classes of the moduli spaces:

$$\langle w(0)v, [\Gamma \cdot M_{1/4}^e(X)] \rangle = \langle w(1)v, [\Gamma \cdot M_0(X)] \rangle.$$

The assertion in Proposition 5.2 concerning the value of the subleading term  $w(1)$  can therefore be restated as the following proposition.

**Proposition 5.3** *Let  $n = 2m + 1$  as usual let  $v \in \mathcal{A}_n$  be any element of degree  $m - 1$ . Let  $w_{n,\eta}^k \in \mathcal{A}_n$  be the explicit polynomials described in Definition 4.2. Then we have*

$$\langle w_{n,\eta}^m v, [\Gamma \cdot M_{1/4}^e(X)] \rangle = \langle \tau^{n-2|\eta|} w_{n,\eta'}^{m-1} v, [\Gamma \cdot M_0(X)] \rangle,$$

where the (compact) moduli space  $M_{1/4}^e(X)$  is computed using a metric on  $X$  in the  $B$ -chamber, and  $M_0(X)$  is the moduli space of flat bifold connections, a copy of  $\text{Rep}(S_n^2)$ .

The proof of Proposition 5.3 is given in Section 5.4, after a digression on the wall-crossing behavior of moduli spaces on  $X$ .

### 5.3 A wall-crossing argument

The structure of our argument up to this point is closely related to the work of Muñoz [27], in which a key step is the calculation of the contribution of the first nonflat moduli space (our  $M_{1/4}^e(X)$  in the present context). In [27], the relevant moduli space was of the form  $M_{1/2}^e(S^2 \times \Sigma_g)$  for a smooth surface  $\Sigma_g$ , and the key observation is that this moduli space is empty in one chamber (when the area of the  $S^2$  factor is small, corresponding to the  $C$ -chamber in our notation) and undergoes a single wall-crossing where the metric passes to the  $B$ -chamber. (See [27, Proposition 2].) The description of the wall-crossing for  $S^2 \times \Sigma_g$  leads to a description of the moduli space on the  $B$  side of the wall as a bundle over the Jacobian  $J(\Sigma_g)$  with fiber a complex projective space.

Such a description has an exact parallel in our orbifold context, with the Jacobian  $J(\Sigma_g)$  in Muñoz’s situation replaced now by the finite set of bifold line bundles on  $S_n^2$  of a fixed bifold degree. That is, the wall-crossing contributes to  $M_{1/4}^e(X)$  a finite number of copies of a complex projective space, where an explicit understanding of the cohomology classes allows a calculation of the Donaldson invariant. We now turn to the details of this calculation.

**Lemma 5.4** *In the  $C$ -chamber, the Donaldson invariants  $q_\kappa^C(u)$  are zero when  $\kappa$  is in  $\frac{1}{4} + \frac{1}{2}\mathbb{Z}$ .*

**Proof** The bifold  $X$  decomposes into two parts along a copy of  $B \times S^1 \subset B \times C$ , ie an  $S^2 \times S^1$ . The bundle has  $w_2$  nonzero on this  $S^2 \times S^1$  when  $\kappa$  is in  $\frac{1}{2}\mathbb{Z} + \frac{1}{4}$ , so there are no flat connections on  $B \times S^1$ . A stretching argument therefore shows that the anti-self-dual moduli space is empty when the metric on  $X$  contains a long neck  $[-T, T] \times B \times S^1$ . A metric with such a long neck lies in the  $C$ -chamber, so the invariant in this chamber is zero. □

**Lemma 5.5** *For the moduli spaces  $M_\kappa^e(X)$  with  $\kappa \leq \frac{1}{4}$ , in a 1-parameter family of product metrics on  $X = B \times C$  passing from the  $C$ -chamber to the  $B$ -chamber, exactly one wall is crossed.*

**Proof** The only nonempty moduli space  $M_\kappa^e(X)$  with  $\kappa \leq \frac{1}{4}$  is the moduli space  $M_{1/4}^e(X)$ , and a wall is crossed when the Riemannian metric allows the existence of a reducible anti-self-dual connection in this moduli space. We are therefore looking for a reduction of the bifold adjoint  $SO(3)$  bundle as  $\mathbb{R} \oplus K$ , where  $K$  is a bifold 2-plane bundle. Let us write the bifold Euler class  $\text{eul}(K)$  as

$$\text{PD eul}(K) = x[B] + y[C].$$

Here  $y$  is an odd integer because  $\text{eul}(K)[B]$  is odd. On the curve  $C$ , the bundle  $K$  has  $n$  bifold points, and  $n$  is odd; so  $2x$  is also an odd integer. For a given Riemannian metric, let us write the class of the self-dual 2-form as

$$\text{PD}[\omega^+] = [B] + t[C],$$

suitably normalized. The condition that the curvature of  $K$  is anti-self-dual imposes the constraint that  $\text{eul}(K)$  and  $[\omega^+]$  are orthogonal, which is to say

$$y = -tx.$$

The action  $\kappa$  is  $-\text{eul}(K)^2/4$  which is  $-xy/2$ . Using the orthogonality condition, we write this as  $\kappa = tx^2/2$ . With  $\kappa = \frac{1}{4}$ , our constraints therefore become

- (i)  $tx$  and  $2x$  are odd integers, and
- (ii)  $tx^2 = \frac{1}{2}$ .

These constraints force  $x = \pm\frac{1}{2}$  and  $t = 2$ . The orientation of  $K$  is indeterminate, and the sign of  $x$  can therefore be taken to be positive. A path of Riemannian metrics passing from the  $C$  chamber to the  $B$  chamber is a path in which  $t$  begins close to 0 and ends close to  $+\infty$ , and the wall is crossed at  $t = 2$ .  $\square$

The proof the lemma shows that the wall-crossing occurs when there is an orbifold 2-plane bundle  $K$  with

$$\text{PD eul}(K) = \frac{1}{2}[B] - [C].$$

The degree of  $K$  on  $C = S_n^2$  is thus  $\frac{1}{2}$ . In terms of an  $\text{SU}(2)$  lift on the curve  $\{b\} \times C$  then, we can write the bundle as

$$F \oplus F^{-1},$$

where  $F$  is a complex line bundle with limiting holonomy  $\pm i$  on the linking circles at the  $n$  singular points. We orient  $K$  as  $F^{-2}$ . The Chern–Weil integral for the first Chern class of the singular connection on  $F$  is  $-\frac{1}{4}$ . As a parabolic bundle on  $S_n^2$  we can write the underlying rank-2 vector bundle as  $\mathcal{E} = \mathcal{F} \oplus \mathcal{F}^{-1}$ , and for each  $p \in \pi$  the distinguished line  $\mathcal{L}_p \subset \mathcal{E}_p$  is the summand  $\mathcal{F}_p$  if the limiting holonomy is  $-i$ , and  $\mathcal{F}_p^{-1}$  otherwise. Write  $\xi \subset \pi$  for the set where the holonomy is  $-i$ . Then

$$c_1(\mathcal{F})[C] + \frac{1}{4}|\xi| - \frac{1}{4}(n - |\xi|) = -\frac{1}{4}.$$

This constraint imposes the parity condition  $|\xi| = \frac{1}{2}(n - 1) \pmod{2}$ , which allows  $2^{n-1}$  possibilities for  $\xi$ . We summarize this with another lemma.

**Lemma 5.6** *When the Riemannian metric on  $X = B \times C$  lies on the wall between the two chambers, the moduli space  $M_\kappa^e(X)$  consists of  $2^{n-1}$  reducible anti-self-dual connections, corresponding to the subsets  $\xi \subset \pi$  whose size  $|\xi|$  has the same parity as  $\frac{1}{2}(n - 1)$ .*

Let  $A_0$  denote any one of the reducible connections described in the lemma. The formal dimension of the moduli space  $M_{1/4}^e(X)$  is  $2n - 4$ . If we write the orbifold adjoint bundle as  $\mathbb{R} \oplus K$  now on the whole of  $X$ , then in the deformation theory of  $A_0$  we have a contribution of 1 to the dimension of  $H_{A_0}^0$  coming from the  $\mathbb{R}$  summand because  $A_0$  is reducible, and there is a similar contribution of 1 to the dimension of  $H_{A_0}^2$  from the  $\mathbb{R}$  summand because  $b_2^+ = 1$ . If we assume that the deformation theory is otherwise unobstructed (an assumption which we will see later is justified for product metrics on  $B \times C$ , without the need for perturbing the equations), then it follows that  $H_{A_0}^1$  has dimension  $2n - 2$  and that this comes from the  $K$  summand of the adjoint bundle. With this in place, the standard model for wall-crossing describes the moduli space  $M_{1/4}^e(X; g_t)$  for a Riemannian metric  $g_t$  whose conformal parameter  $t$  is  $2 + \epsilon$  for small  $\epsilon$  as a copy of  $\mathbb{C}\mathbb{P}^{n-2}$  in a neighborhood of each reducible  $A_0$ . We therefore have the following proposition.

**Proposition 5.7** For a product metric on  $X$  which lies in the  $B$ -chamber and is close to the wall, the moduli space  $M_{1/4}^e(X)$  consists of  $2^{n-1}$  copies of  $\mathbb{C}\mathbb{P}^{n-2}$ .

As mentioned earlier, this is a close counterpart to the result [27, Proposition 2], where the corresponding description of the moduli space of smallest positive action is a bundle of projective spaces over the Jacobian of a smooth curve.

### 5.4 A proof of Proposition 5.3

From their definition,  $w_{n,\eta}^m$  and  $w_{n,\eta'}^{m-1}$  represent cohomology classes dual to loci  $U_{-1/4}^\eta$  and  $U_{1/4}^{\eta'}$  in the space of bifold connections  $\mathcal{B}^*(S_n^2)$ . If we select a fiber

$$\{b_0\} \times S_n^2 \subset B \times S_n^2 = X,$$

then we obtain by restriction corresponding loci in the spaces of bifold connections on  $X$ :

$$U_{-1/4}^\eta(b_0) \subset \mathcal{B}^*(X)^e, \quad U_{1/4}^{\eta'}(b_0) \subset \mathcal{B}^*(X).$$

In this way we can interpret the equality to be proved in Proposition 5.3 as

$$(59) \quad \langle v, [\Gamma \cdot M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)] \rangle = \tau^{n-2|\eta|} \langle v, [\Gamma \cdot M_0(X) \cap U_{1/4}^{\eta'}(b_0)] \rangle,$$

provided that the loci are transverse to the filtration of the space of Fredholm operators by the dimension of the kernel. The moduli spaces on  $X$  should be obtained from metrics in the  $B$ -chamber as always.

We can obtain more information about  $M_{1/4}^e(X)$  and the loci on both sides of (59) by interpreting the moduli space of bifold anti-self-dual connections as a moduli space of stable parabolic bundles on the pair  $(X, \Sigma)$  where  $\Sigma$  is the singular locus  $B \times \pi \subset X$ . To this end, we adopt the notation and results of [18] to identify  $M_{1/4}^e(X)$  with the moduli space of parabolic bundles  $(\mathcal{E}, \mathcal{L})$  with  $\kappa = \frac{1}{4}$  satisfying the parabolic stability condition with parameter  $\alpha = \frac{1}{4}$ . Here we can write  $\kappa$  as  $k + l/2$  following [17; 18], where in this case

$$(60) \quad k = (c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2)[X], \quad l = (\frac{1}{2}c_1(\mathcal{E}) - c_1(\mathcal{L}))[\Sigma].$$

(The quantities  $k$  and  $l$  are the “instanton number” and “monopole number” in the notation of [17].) The rank-2 bundle  $\mathcal{E}$  should have  $c_1(\mathcal{E})[B]$  odd, so we take

$$\Lambda^2(\mathcal{E}) = \mathcal{O}(1, 0),$$

by which we mean the holomorphic line bundle with degree 1 on  $B$ . The moduli space  $M_0(X)$  is similarly a moduli space of stable parabolic bundles on  $X$ , now with  $\Lambda^2(\mathcal{E}) = \mathcal{O}$  and  $\kappa = 0$ . These bundles are the pullbacks of the stable parabolic bundles on the curve  $C = S_n^2$ .

The loci on either side of (59) have the following interpretations. Let  $\mathcal{F} \rightarrow C$  be the parabolic line bundle whose set of hits is  $\eta$  and whose parabolic degree is  $\text{par-deg } \mathcal{F} = \frac{1}{4}$ . (See the remarks at the end of Section 4.1.) The dual parabolic bundle  $\mathcal{F}^*$  has parabolic degree  $-\frac{1}{4}$  and its set of hits is  $\eta' = \pi \setminus \eta$ . Given a stable parabolic bundle  $\mathcal{E}$  on  $X$ , let  $\mathcal{E}_b$  be the parabolic bundle obtained by restriction to  $\{b\} \times C$ .

**Lemma 5.8** *Let  $\mathcal{F}$  be the parabolic line bundle described above and  $\mathcal{F}^*$  its dual. Then:*

- (i) *The locus  $M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)$  is the locus of stable parabolic bundles  $\mathcal{E} \in M_{1/4}^e(X)$  such that there exists a nonzero holomorphic map of parabolic bundles*

$$\mathcal{F} \rightarrow \mathcal{E}_{b_0}.$$

- (ii) *The locus  $M_0(X) \cap U_{1/4}^{\eta'}(b_0)$  is the locus of stable parabolic bundles  $\mathcal{E} \in M_0(X)$  such that there exists a nonzero holomorphic map of parabolic bundles*

$$\mathcal{F}^* \rightarrow \mathcal{E}_{b_0}.$$

**Proof** These statements follow directly from the definitions. □

Going beyond the above lemma, we have the following constructions for the relevant bundles.

**Lemma 5.9** (i) *The locus  $M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)$  consists of parabolic bundles  $\mathcal{E} \rightarrow X = B \times C$  which are nonsplit extensions*

$$\mathcal{O}(1) \boxtimes \mathcal{F}^* \rightarrow \mathcal{E} \rightarrow \mathcal{F}$$

*such that the extension class vanishes on  $\{b_0\} \times C$ .*

- (ii) *The locus  $M_0(X) \cap U_{1/4}^{\eta'}(b_0)$  is the locus of parabolic bundles  $\mathcal{E} \in M_0(X)$  which are nonsplit extensions*

$$\mathcal{F}^* \rightarrow \mathcal{E} \rightarrow \mathcal{F}.$$

*In both cases, all bundles obtained as such extensions are stable in the  $B$ -chamber on  $X$ .*

**Proof** In (ii), the bundles in  $M_0(X)$  are pulled from the stable parabolic bundles on  $C$ , and the existence of a nonzero map of parabolic bundles  $\iota: \mathcal{F}^* \rightarrow \mathcal{E}$  is the definition of the locus  $U_{1/4}^{\eta'}$ . The map  $\iota$  must be an inclusion of a parabolic line subbundle, for otherwise this map would destabilize  $\mathcal{E}$ . So  $\mathcal{E}$  is an extension of parabolic line bundles as described. The extension must be nonsplit, for otherwise  $\mathcal{E}$  is destabilized by  $\iota$ .

For (i), the first task is to verify that every stable parabolic bundle in  $M_{1/4}^e(X)$  in the  $B$ -chamber is a nonsplit extension

$$(61) \quad \mathcal{O}(1) \boxtimes \mathcal{G}^* \rightarrow \mathcal{E} \rightarrow \mathcal{G},$$

where  $\text{par-deg } \mathcal{G} = -\frac{1}{4}$  and the set of hits for  $\mathcal{G}$  is a subset  $\xi \subset \pi$  which is arbitrary, except for the parity constraint (35). There are  $2^{n-1}$  choices for  $\xi$ , and once  $\xi$  is given, the nonsplit extensions are parametrized by a projective space, in this case of dimension  $n-2$ . In this way we find  $2^{n-1}$  copies of  $\mathbb{C}\mathbb{P}^{n-2}$  in  $M_{1/4}^e$ , and it is straightforward to see that these are disjoint, because a given bundle  $\mathcal{E}$  cannot be presented as an extension of this sort in two different ways. The verification that these  $2^{n-1}$  copies of  $\mathbb{C}\mathbb{P}^{n-2}$  comprise the entire moduli space  $M_{1/4}^e(X)$  in the  $B$ -chamber is the holomorphic analog of wall-crossing result described in Proposition 5.7, and it is proved in essentially the same way. This is also the content of [27, Proposition 2] in the slightly different context of that paper, which serves the same purpose there.

For an extension such as (61), the restriction to  $\{b_0\} \times C$  is an extension of parabolic line bundles on  $C$ ,

$$\mathcal{G}^* \rightarrow \mathcal{E}_{b_0} \rightarrow \mathcal{G},$$

and because  $\text{par-deg}(\mathcal{F}) = \text{par-deg}(\mathcal{G}) > \text{par-deg}(\mathcal{G}^*)$ , there can be a nonzero map  $\mathcal{F} \rightarrow \mathcal{E}_{b_0}$  only if  $\mathcal{F} = \mathcal{G}$  and the extension class is zero on  $\{b_0\} \times C$ . □

The extensions that arise in (ii) are parametrized by the projective space

$$(62) \quad \mathbb{P}(H^1(C; (\mathcal{F}^*)^{\otimes 2})),$$

where the cohomology group is interpreted as the cohomology of a sheaf on a bifold. The extensions that arise in (i) are parametrized by the subset of the projective space

$$\mathbb{P}(H^0(B; \mathcal{O}(1)) \otimes H^1(C; (\mathcal{F}^*)^{\otimes 2}))$$

corresponding to elements vanishing at  $b_0$ . If  $Z_{b_0} \subset H^0(B; \mathcal{O}(1))$  is the one-dimensional space of sections vanishing at  $b_0$ , then this is the space

$$\mathbb{P}(Z_{b_0} \otimes H^1(C; (\mathcal{F}^*)^{\otimes 2})),$$

which is canonically identified with (62). Both spaces of extensions are copies of  $\mathbb{C}\mathbb{P}^{m-1}$ .

We have now seen that there is a canonical identification of the two loci,

$$M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0) = M_0(X) \cap U_{1/4}^{\eta'}(b_0),$$

both of which are projective spaces. Furthermore, for any  $b \neq b_0$  in  $B$ , the restrictions of the corresponding bundles in these loci to  $\{b\} \times C$  agree. Indeed they are the same family of nonsplit extensions of  $\mathcal{F}$  by  $\mathcal{F}^*$  on  $C$ . The cohomology classes  $v$  arising from elements of the algebra  $\mathcal{A}_n$  can be regarded as being pulled back via the restriction to  $\{b\} \times C$ , so it follows that the evaluation of such a class  $v$  is the same in the two cases.

Before accounting for the weights arising from the local system  $\Gamma$ , we therefore have an equality

$$(63) \quad \langle v, [M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)] \rangle = \langle v, [M_0(X) \cap U_{1/4}^{\eta'}(b_0)] \rangle.$$

However, while  $M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)$  and  $M_0(X) \cap U_{1/4}^{\eta'}(b_0)$  are both copies of  $\mathbb{C}\mathbb{P}^{m-1}$  and are canonically identified, the (constant) functions

$$\Gamma: M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0) \rightarrow \mathcal{R} \quad \text{and} \quad \Gamma: M_0(X) \cap U_{1/4}^{\eta'}(b_0) \rightarrow \mathcal{R}$$

are different. The next lemma provides these values.

**Lemma 5.10** (i) On  $M_0(X) \cap U_{1/4}^{\eta'}(b_0)$ , the value of  $\Gamma$  is 1.

(ii) On  $M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)$ , the value of  $\Gamma$  is  $\tau^{n-2|\eta|}$ .

**Proof** The singular set  $\Sigma \subset X$  is a collection of spheres with trivial normal bundle, so there is no self-intersection term in the formula (2), and we simply have

$$\Gamma(A) = \tau^{\nu(A)},$$

where  $\nu(A)$  is a 2-dimensional Chern–Weil integral on  $\Sigma$ . In the case of  $M_0(X)$ , the connections are flat and  $\nu(A) = 0$ . So  $\Gamma = 1$  in this case, as stated in the first item of the lemma.

In the case of a closed manifold, the value  $\nu(A)$  is  $-2l$ , where  $l$  is the “monopole number” of the bundle (60). The bundles that contribute to the moduli space  $M_{1/4}(X)^e \cap U_{-1/4}^\eta(b_0)$  are described in Lemma 5.9. From there we read off that  $c_1(\mathcal{E})[\Sigma_p] = 1$  for each of the  $n$  components  $\Sigma_p \subset \Sigma$ , so that  $c_1(\mathcal{E})[\Sigma] = n$ . For  $p \in \eta'$ , the distinguished line subbundle  $\mathcal{L} \subset \mathcal{E}|_{\Sigma_p}$  coincides with the image of the subbundle  $\mathcal{O}(1) \boxtimes \mathcal{F}^*$  on  $\Sigma_p$ , which has degree 1. For  $p \in \eta$ , the distinguished line subbundle  $\mathcal{L}$  on  $\Sigma_p$  maps isomorphically to the restriction of  $\mathcal{F}$  in the extension in Lemma 5.9, so has degree 0. In all then,

$$c_1(\mathcal{L})[\Sigma] = |\eta'|.$$

The formula for the monopole number  $l$  in (60) therefore gives  $(n/2) - |\eta'|$ , which is  $|\eta| - (n/2)$ . Since  $\nu(A) = -2l$ , we have  $\nu(A) = n - 2|\eta|$ , as the lemma claims. □

From the lemma, we see that

$$[\Gamma \cdot M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)] = \tau^{n-2h} [M_{1/4}^e(X) \cap U_{-1/4}^\eta(b_0)],$$

while

$$[\Gamma \cdot M_0(X) \cap U_{1/4}^{\eta'}(b_0)] = [M_0(X) \cap U_{1/4}^{\eta'}(b_0)].$$

The equality to be proved in Proposition 5.3 now follows from the unweighted equality (63), and this completes the proof of the proposition.

**Remark** In the course of these arguments, we have seen first that  $M_{1/4}^e(X)$  is a disjoint union of  $2^{n-1}$  copies of  $\mathbb{C}\mathbb{P}^{n-2}$  and second that the class  $w_{n,\eta}^m$  restricts to be nonzero on exactly one of them, being dual to a  $\mathbb{C}\mathbb{P}^{m-1}$  in exactly one of the copies of  $\mathbb{C}\mathbb{P}^{n-1}$ . The components  $\mathbb{C}\mathbb{P}^{n-2}$  of  $M_{1/4}^e(X)$  are in one-to-one correspondence with the subsets  $\eta \subset \pi$  of the correct parity, so let us write them as  $\mathbb{C}\mathbb{P}_\eta^{n-2}$ . If we choose a class  $v$  which has nonzero pairing (say 1) with each  $\mathbb{C}\mathbb{P}^{m-1} \subset \mathbb{C}\mathbb{P}_\eta^{n-2}$ , then we have

$$\langle w_{n,\eta}^m \cup v, [\mathbb{C}\mathbb{P}_\xi^{n-2}] \rangle = \begin{cases} 1 & \text{if } \eta = \xi, \\ 0 & \text{otherwise,} \end{cases}$$

from which it follows that the classes  $w_{n,\eta}^m$  are linearly independent in  $\mathcal{A}_n$ . This provides an alternative verification of the result used in the proof of Proposition 4.8.

### 5.5 Changing the orientation of the singular set

Recall that in defining the bifold  $Z_n$  we gave a standard orientation to the  $n$  circles comprising the singular set  $K_n$ . Let  $Z_n^*$  denote the same bifold but with some of the circles of  $K_n$  equipped with the opposite orientation. Let  $f$  be the number of search circles. The construction of the operators  $\delta_p$  depends on an orientation of the singular set at  $p$ , so in a straightforward way the corresponding operators  $\delta_p^*$  on  $I(Z_n^*)$  differ in sign from the operators  $\delta_p$  if we define  $\delta_p^*$  using the new orientations. But there is also a more subtle way in which the module structures differ.

Both  $I(Z_n)$  and  $I(Z_n^*)$  are  $\mathcal{A}_n$ -modules by our constructions. Let us continue to denote by  $\alpha$ ,  $\delta_p$  and  $\epsilon$  the operators on  $I(Z_n)$ , and let us denote by  $\alpha^*$ ,  $\delta_p^*$  and  $\epsilon^*$  the operators which define the  $\mathcal{A}_n$ -module structure of  $I(Z_n^*)$ .

**Proposition 5.11** *There is an identification of the  $\mathcal{R}$ -modules  $I(Z_n^*)$  and  $I(Z_n)$  which is canonical up to overall sign-change. Under either one of this canonical pairs of identifications, the operators  $\alpha^*$  etc on  $I(Z_n^*)$  are related to the operators on  $I(Z_n)$  by:*

- (i)  $\alpha^* = \alpha$ ;
- (ii)  $\delta_p^* = \pm \delta_p$ , according to whether or not the corresponding circles of  $K_n$  have the standard orientation in  $Z_n^*$ ;
- (iii)  $\epsilon^* = (-1)^f \epsilon$ , where  $f$  as above is the number of circles which have the nonstandard orientation.

**Proof** First let us recall that the  $SU(2)$  instanton moduli spaces  $M_k(X)$  for a closed Riemannian manifold  $X$  are orientable and are oriented by a choice of an element from a 2-element set  $\Lambda(X)$ , which can be identified with the set of homology orientations of  $X$ . In the case of a closed bifold with orientable singular set, if we regard the moduli space as the space of singular  $SU(2)$  connections  $M_{k,l}(X, \Sigma)$  in the sense of [17; 18] and [20], then an element of the 2-element orientation set  $\Lambda(X, \Sigma)$  can be specified by a choice of homology orientation of  $X$  together with an orientation of  $\Sigma$ . Changing the orientation of  $\Sigma$  changes the sign of the element of  $\Lambda(X, \Sigma)$  by  $(-1)^{\chi/2}$ , where  $\chi$  is the Euler number. (See [18].) To briefly explain why this is so, the conventions of [18] identify the difference between  $\Lambda(X)$  and  $\Lambda(X, \Sigma)$  as the set  $\Lambda(\Sigma)$  of orientations of the real determinant line of the index of the  $\bar{\partial}$  operator on  $\Sigma$  coupled to a line bundle of degree  $2l$ . The index of the  $\bar{\partial}$  operator is  $2l - \chi(\Sigma)/2$ . Changing the orientation of  $\Sigma$  changes the index element to its complex conjugate and therefore changes the orientation of the real determinant line by  $(-1)^{2l - \chi/2} = (-1)^{\chi/2}$ . A similar formula holds if the orientation of  $\Sigma$  is changed only on certain components.

In the case that  $(X, \Sigma)$  is a product  $S^1 \times (Y, K)$ , there is a canonical homology orientation for  $X$  and the components of  $\Sigma$  are tori; so there is a canonical element of  $\Lambda(X, \Sigma)$  in this case, independent of the orientation of the components of  $\Sigma$ .

Continuing with the closed case, we consider next the moduli space  $M_{k,l}(X, \Sigma)^e$  corresponding to an  $SO(3)$  bundle whose  $w_2$  has an integer lift  $e$ . As usual the gauge group is the determinant-1 gauge group. In the absence of  $\Sigma$ , the orientation set  $\Lambda(X)^e$  is still canonically identified with the set of homology orientations of  $X$ , as in [6]. The difference between  $\Lambda(X)^e$  and  $\Lambda(X, \Sigma)^e$  is again identified with the real determinant line of the same  $\bar{\partial}$  operator. The difference now however is that the monopole number  $l$  is in  $\frac{1}{2} + \mathbb{Z}$  on any component of  $\Sigma$  having odd pairing with  $e$ . Changing the orientation of a component  $\Sigma_1 \subset \Sigma$  therefore changes the orientation element in  $\Lambda(X, \Sigma)^e$  by  $(-1)^{g(\Sigma_1) - 1}$  if  $e$  has even pairing with  $\Sigma_1$  and by  $(-1)^{g(\Sigma_1)}$  if the pairing is odd. In the special case that  $(X, \Sigma) = S^1 \times (Y, K)$ , an orientation of the moduli spaces  $M(X, \Sigma)^e$  therefore depends on the orientation of the components of the singular

set if and only if they have odd pairing with  $e$ . If the pairings are even, all orientations of moduli spaces are canonical and do not depend on the orientation of the singular set.

To apply these observations to the instanton homology, we recall the standard approach to orientations in Floer homology, described for example in [20, Section 3.6]. Let  $\mathcal{B}^*$  denote the space of irreducible singular  $SU(2)$  connections on the bifold  $(Y, K)$ . To each pair of points  $a, b \in \mathcal{B}^*$  and each path  $\zeta$  joining them, we may associate a 2-element set  $\Lambda_\zeta(a, b)$  as the set of orientations of the determinant line of a Fredholm operator  $P(a, b)$ , in such a way that  $\Lambda_\zeta(a, b)$  orients the moduli space of trajectories if  $a$  and  $b$  are nondegenerate critical points. If  $\zeta_1$  and  $\zeta_2$  are two different paths, then  $\Lambda_{\zeta_1}(a, b)$  and  $\Lambda_{\zeta_2}(a, b)$  are canonically identified, because their difference can be identified with  $\Lambda(X, \Sigma)$ , where  $(X, \Sigma) = S^1 \times (Y, K)$ . We can therefore define  $\Lambda(a, b)$  without issue. No orientation of  $K$  is needed here.

Given a basepoint  $\theta$  in  $\mathcal{B}^*$ , one may then define  $\Lambda(a) = \Lambda(a, \theta)$  for all  $a$ . The chain complex for the singular instanton homology with local coefficients  $\Gamma$  is then

$$\bigoplus_a (\mathbb{Z}\Lambda(a)) \otimes \Gamma_a,$$

where the sum is over perturbed flat connections in a Morse perturbation of the Chern–Simons functional. Two different choices of basepoints  $\theta$  and  $\theta'$  will give rise to complexes which are identified up to an overall factor of  $-1$ : that ambiguity is a choice of element from  $\Lambda(\theta, \theta')$ . A canonical choice of basepoint is possible when  $K$  is oriented, as described in [20], making  $I(Y, K)$  well-defined up to canonical isomorphism. The modules  $I(Z_n)$  and  $I(Z'_n)$  are identified only up to overall sign, because the basepoints are different. The remaining interesting point is the final assertion of the proposition. To determine the sign of the matrix entries of the endomorphism  $\epsilon$  between critical points  $a$  and  $b$ , one uses the canonical orientation of the product

$$\mathbb{Z}\Lambda(a, b)^e \otimes \mathbb{Z}\Lambda(a) \otimes \mathbb{Z}\Lambda(b).$$

Orienting this product is equivalent to orienting the moduli spaces  $M(X, \Sigma)^e$  for the product  $(X, \Sigma) = S^1 \times (Y, K)$ . We have described above how these moduli spaces are canonically oriented once one has an orientation of the components of  $\Sigma$  (which are tori). Here the class  $e$  has pairing 1 with each of the components. So changing the orientation of any component changes the canonical orientation of  $M(X, \Sigma)^e$  and changes the sign of all the matrix entries of  $\epsilon$ . □

### 5.6 Passing to $Z_{n,-1}$

Recall that the algebra  $\bar{\mathcal{A}}$  is defined as the quotient of  $\mathcal{A}_n$  in which all the  $\delta_i$  are equal (see equation (32)), and let  $w_{n,\eta}^k \in \mathcal{A}_n$  be the elements from Definition 4.2. The image of  $w_{n,\eta}^k$  in  $\bar{\mathcal{A}}$  depends only on the cardinality of the subset  $\eta \subset \pi$ , not otherwise on its elements, and we write this element of  $\bar{\mathcal{A}}$  as

$$(64) \quad \bar{w}_{n,h}^k = w_{n,\eta}^k + \langle \delta_i - \delta_j \rangle_{i,j} \in \bar{\mathcal{A}}$$

when  $|\eta| = h$ . Recall from (33) that we can write  $I(Z_{n,-1})$  as  $\mathcal{A}_n/\mathcal{J}_{n,-1}$  or as  $\bar{\mathcal{A}}/\bar{\mathcal{J}}_{n,-1}$  and that  $\mathcal{J}_{n,-1}$  contains  $\mathcal{J}_n$  (Proposition 3.22). Propositions 5.1 and 5.2 therefore yield the following version for  $Z_{n,-1}$ .

**Proposition 5.12** Write  $n = 2m + 1$ , let  $h$  be an integer satisfying the conditions (47), and let  $\bar{w}_{n,h}^m$  be defined as above. Then there is an element  $\bar{W}_h^m \in \bar{\mathcal{J}}_{n,-1}$  of the filtered algebra  $\bar{\mathcal{A}}$  in filtration degree  $m$  whose leading term is  $\bar{w}_{n,h}^m$ . The subleading term of  $\bar{W}_h^m$  is given by  $\epsilon \tau^{n-2h} \bar{w}_{n,h'}^{m-1}$ , where  $h' = n - h$ . Thus

$$\bar{W}_h^m = \bar{w}_{n,h}^m + \epsilon \tau^{n-2h} \bar{w}_{n,h'}^{m-1} \pmod{\bar{\mathcal{A}}^{(m-2)}}.$$

The element  $\bar{W}_h^m$  in  $\bar{\mathcal{A}}$  is the image of  $W_\eta^m \in \mathcal{J}_n$  under the quotient map  $\mathcal{A}_n \rightarrow \bar{\mathcal{A}}$ . □

We have not yet established that  $\bar{\mathcal{J}}_{n,-1}$  is the image of  $\mathcal{J}_n$ , so we do not know yet that the elements  $\bar{W}_h^m$  generate the ideal of relations  $\bar{\mathcal{J}}_{n,-1}$  for  $I(Z_{n,-1})$ . We turn to this next.

**Proposition 5.13** When  $n = 2m + 1$ , the elements  $\bar{W}_h^m$  for  $h$  in the range  $0 \leq h \leq n$  with  $h = \frac{1}{2}(n + 1) \pmod{2}$  are a set of generators for the ideal  $\bar{\mathcal{J}}_{n,-1} \subset \bar{\mathcal{A}}$ . In particular,  $\bar{\mathcal{J}}_{n,-1}$  is the image of  $\mathcal{J}_n$  in  $\bar{\mathcal{A}}$ .

**Proof** The quotient  $\bar{\mathcal{A}}/\bar{\mathcal{J}}_{n,-1}$  is  $I(Z_{n,-1})$  which we know to be a free  $\mathcal{R}$ -module of rank  $\frac{1}{4}(n^2 - 1)$  by Corollary 3.19. If  $\mathcal{J}' \subset \bar{\mathcal{J}}_{n,-1}$  denotes the ideal generated by the elements  $\bar{W}_h^m$ , then the desired equality  $\mathcal{J}' = \bar{\mathcal{J}}_{n,-1}$  will follow if we can prove that  $\bar{\mathcal{A}}/\mathcal{J}'$  has the same rank. The leading  $m^{\text{th}}$ -degree terms of the elements  $\bar{W}_h^m$  are the elements  $\bar{w}_{n,h}^m$ , so let us denote by  $\bar{\mathcal{J}}_n \subset \bar{\mathcal{A}}$  the ideal generated by these leading terms. (This is the image in  $\bar{\mathcal{A}}$  of the ideal of relations  $\mathcal{J}_n \subset \mathcal{A}_n$  for the ordinary cohomology ring  $H^*(\text{Rep}(Z_n); \mathcal{R})$  in (20).) It will therefore suffice to show that  $\bar{\mathcal{A}}/\bar{\mathcal{J}}_n$  has rank  $\frac{1}{4}(n^2 - 1)$ , and this is the content of the lemma below, which completes the proof. □

**Lemma 5.14** Write  $n = 2m + 1$  again and let  $\bar{\mathcal{J}}_n \subset \bar{\mathcal{A}}$  be as above, generated by the elements  $\bar{w}_{n,h}^m$ . Then  $\bar{\mathcal{J}}_n$  is the  $m^{\text{th}}$  power  $\langle \alpha, \delta \rangle^m$  of the ideal  $\langle \alpha, \delta \rangle$ . In particular, the rank of  $\bar{\mathcal{A}}/\bar{\mathcal{J}}_n$  is  $m(m + 1)$ , which is also equal to  $\frac{1}{4}(n^2 - 1)$ .

**Remark** The quotient of a polynomial algebra in two variables by the  $m^{\text{th}}$  power of the maximal ideal at 0 has rank  $\frac{1}{2}m(m + 1)$ . The extra factor of two in the lemma arises because of the extra generator  $\epsilon$  in the algebra  $\bar{\mathcal{A}}$ .

**Proof** Recall that  $w_{n,\eta}^m$  arises from the formal computation of  $c_m(-\text{index}(P))$ , where  $P$  is a family of Fredholm operators, Definition 4.2. The formula (44) for the Chern character of  $-\text{index}(P)$  becomes the following, after passing to the formal completion of the quotient ring  $\bar{\mathcal{A}}$  in which all the  $\delta_i$  are equal:

$$(65) \quad (m - 1) \cosh(\delta) + \frac{\sinh(\delta)}{\delta} (\alpha + (h - \frac{1}{2}n)\delta).$$

Passing from the Chern character to the  $m^{\text{th}}$  Chern class, we find that the image of  $c_m(-\text{index}(P))$  in  $\bar{\mathcal{A}}$  has the form

$$V_m(B_h, \delta),$$

where  $V_m(x, y)$  is a homogeneous polynomial of degree  $m$  in two variables and  $B_h = \alpha + (-h + \frac{1}{2}n)\delta$ . Furthermore the coefficient of  $x^m$  in  $V_m$  is  $1/m!$ .

Thus  $\bar{J}_n$  is generated by the elements  $V_m(B_h, \delta)$ , for  $h$  in the range  $0 \leq h \leq n$  with  $h \equiv \frac{1}{2}(n + 1) \pmod{2}$ . The assertion of the lemma is equivalent to the statement that the homogeneous polynomials  $V_m(x + (h - \frac{1}{2}n)y, y)$  in  $\mathbb{Q}[x, y]$  span the space of homogeneous degree- $m$  polynomials. This in turn is true because  $h - \frac{1}{2}n$  runs through  $m + 1$  distinct values in  $\mathbb{Q}$  as  $h$  runs through its allowed range. (This is the same assertion as the statement that any  $m + 1$  distinct translates of a polynomial  $f(x)$  of degree  $m$  are necessarily independent.)  $\square$

## 6 Calculation of the ideals

### 6.1 Hilbert schemes of points in the plane

We present here and in Section 6.2 below some material on Hilbert schemes of points in the plane, specialized to the particular situation for which we have application. General references are [26] for Section 6.1 and [9] for Section 6.2.

Let  $A$  be the algebra  $k[x, y]$ , with  $k$  a field, which we may take to be  $\mathbb{C}$ . Let  $A_n \subset A$  be the subspace of homogeneous polynomials of degree  $n$ , and let  $A^{(n)} = \bigoplus_{k \leq n} A_k$ . Let  $\mathfrak{m} \subset A$  be the maximal ideal  $\langle x, y \rangle$ , and consider the  $m^{\text{th}}$  power  $\mathfrak{m}^m$ , which has generators

$$(66) \quad \mathfrak{m}^m = \langle x^m, x^{m-1}y, \dots, y^m \rangle.$$

The colength of  $\mathfrak{m}^m$  (the dimension of the quotient  $A/\mathfrak{m}^m$  as a  $k$ -vector space) is  $N = \frac{1}{2}m(m + 1)$ , and a vector space complement is the linear subspace  $A^{(m-1)}$ :

$$A = \mathfrak{m}^m \oplus A^{(m-1)}.$$

We can consider  $\mathfrak{m}^m$  as defining a point in the Hilbert scheme  $\mathcal{H}^N$  which parametrizes all ideals of colength  $N$  in  $A$ . In the Hilbert scheme, there is an open neighborhood  $U \ni \mathfrak{m}^m$  defined as

$$(67) \quad U = \{I \in \mathcal{H}^N \mid A = I \oplus A^{(m-1)}\}.$$

For  $I \in U$ , there is the projection to the second factor,  $A \rightarrow A^{(m-1)}$  with kernel  $I$ :

$$\varphi_I: A \rightarrow A^{(m-1)}.$$

It is an elementary matter to check that the restriction of  $\varphi_I$  to  $A_m$  completely determines  $I$ , and that  $I$  is in fact generated by

$$I = \langle a - \varphi_I(a) \mid a \in A_m \rangle.$$

We have in particular  $a = \varphi_I(a) \pmod{I}$  for all  $a \in A_m$ .

The map  $\varphi = \varphi_I$  is constrained by the condition that its kernel is an ideal rather than just a codimension- $N$  linear subspace in  $A$ . To see how, consider elements  $a, a' \in A_m$  with

$$xa = ya'.$$

We have  $a = \varphi(a) \pmod I$ , and therefore  $xa = x\varphi(a)$ , and applying  $\varphi$  again

$$xa = \varphi(x\varphi(a)) \pmod I.$$

Similarly with  $ya'$  so  $\varphi(y\varphi(a')) = \varphi(x\varphi(a)) \pmod I$ . However both sides of the last equality lie in the complementary subspace  $A^{(m-1)}$ , so in fact

$$(68) \quad \varphi(y\varphi(a')) = \varphi(x\varphi(a)).$$

Conversely, if we are given a linear map  $\psi: A_m \rightarrow A^{(m-1)}$  satisfying the constraint (68), then there exists a unique (well-defined) extension to a linear map  $\varphi: A \rightarrow A^{(m-1)}$  characterized by  $\varphi(x^i y^j a) = \varphi(x^i y^j \varphi(a))$ , and the kernel of  $\varphi$  is then an ideal  $I$  belonging to  $U \subset \mathcal{H}^N$ .

To expand on the constraint (68), write

$$\varphi|_{A_m} = \varphi_1 + \varphi_2 + \dots + \varphi_m,$$

where  $\varphi_r: A_m \rightarrow A_{m-r}$ , and use the fact that  $\varphi|_{A_k} = 1$  for  $k < m$  to obtain

$$\varphi(y\varphi_1(a')) + y\varphi_2(a') + \dots + y\varphi_m(a') = \varphi(x\varphi_1(a)) + x\varphi_2(a) + \dots + x\varphi_m(a).$$

Finally compare terms of like degree to see that

$$(69) \quad y\varphi_{r+1}(a') - x\varphi_{r+1}(a) = -\varphi_r(y\varphi_1(a')) + \varphi_r(x\varphi_1(a))$$

for all  $r \geq 1$  and all  $a, a' \in A_m$  with  $ya' = xa$ . If we write  $a' = xb$  and  $a = yb$  for  $b \in A_{m-1}$ , the constraint becomes

$$y\varphi_{r+1}(xb) - x\varphi_{r+1}(yb) = -\varphi_r(y\varphi_1(xb)) + \varphi_r(x\varphi_1(yb)),$$

which we can express as

$$(70) \quad L_r(\varphi_{r+1}) = Q_r(\varphi_1, \varphi_r),$$

where  $L_r: \text{Hom}(A_m, A_{m-r-1}) \rightarrow \text{Hom}(A_{m-1}, A_{m-r})$  is a linear map and  $Q_r$  is a bilinear expression. It is easy to verify that the operator  $L_r$  is injective (see below), so the constraints determine  $\varphi_{r+1}$  once  $\varphi_r$  and  $\varphi_1$  are known.

We have shown:

**Lemma 6.1** *Given a  $k$ -linear map  $\varphi_1: A_m \rightarrow A_{m-1}$ , there exists at most one linear map  $\varphi = \varphi_1 + \varphi_2 + \dots + \varphi_m$ , with  $\varphi_r: A_m \rightarrow A_{m-r}$ , such that constraints (69) hold. The ideal  $I$  generated by the elements  $\{a - \varphi(a) \mid a \in A_m\}$  then belongs to the open set  $U \subset \mathcal{H}^N$ . Every ideal in  $U$  arises in this way.  $\square$*

The lemma exhibits  $U$  as a closed subset of the vector space  $\text{Hom}_k(A_m, A_{m-1})$ , which has dimension  $m(m+1) = 2N$ . This subset is also invariant under the action by scalars. It will follow that  $U \cong \text{Hom}_k(A_m, A_{m-1})$  if it can be shown that  $U$  has dimension  $2N$ . To do this, one can show that  $U$  contains

an ideal  $I$  whose zero set consists of  $N$  distinct points in the plane  $k^2$ . Such an ideal can be realized as the “distracted” of  $m^m$ . This is the ideal  $I$  generated by the elements

$$u_h = \left( \prod_{0 \leq j < h} (x - j) \right) \left( \prod_{0 \leq l < m-h} (y - l) \right) \quad \text{for } h = 0, \dots, m,$$

(allowing that one of the products may be empty). Its zero-set is the set of lattice points  $(j, l)$  in the first quadrant with  $j + l < m$ .

**Proposition 6.2** *Given a  $k$ -linear map  $\varphi_1: A_m \rightarrow A_{m-1}$  there exists exactly one linear map  $\varphi = \varphi_1 + \varphi_2 + \dots + \varphi_m$ , with  $\varphi_r: A_m \rightarrow A_{m-r}$ , such that the ideal  $I$  generated by the elements  $\{a - \varphi(a) \mid a \in A_m\}$  has colength  $N$ . The matrix entries of  $\varphi_r$  for  $r \geq 2$  can be expressed as polynomials in the matrix entries of  $\varphi_1$ . □*

The proposition tells us that, at each stage  $r$  in the equations (70), the right-hand side  $Q_r(\varphi_1, \varphi_r)$  is in the image of the linear operator  $L_r$ . If we choose a right-inverse  $P_r$  for  $L_r$ , then we can express the iterative solution as

$$(71) \quad \varphi_{r+1} = P_r Q_r(\varphi_1, \varphi_r).$$

To give  $P_r$  explicitly, let us temporarily make our polynomials inhomogeneous by setting  $y = 1$ , so identifying  $A_m$  with the polynomials in  $x$  of degree at most  $m$ , and let us write

$$u_k = \varphi_{r+1}(x^k)$$

as a polynomial of degree at most  $m - r - 1$  in  $x$ . Then the equations (70) take the form

$$u_{k+1} - x u_k = v_k$$

for  $k = 0, \dots, m - 1$ , where  $v_k$  is a given polynomial in  $x$  of degree at most  $m - r$  and the equations are to be solved for  $u_k$  of degree at most  $m - r - 1$ . If a solution exists, then

$$u_m = v_{m-1} + x v_{m-2} + \dots + x^{m-1} v_0 + x^m u_0.$$

Since all polynomials  $u_k$  and  $v_k$  here have degree less than  $m$ , this equation determines the coefficients of  $u_0$  as linear combinations of the coefficients of the  $v_k$ :

$$u_0 = -(x^{-m} v_{m-1} + x^{-m+1} v_{m-2} + \dots + x^{-1} v_0)_+,$$

where the subscript  $+$  means to discard the negative powers of  $x$ . Having found  $u_0$ , we can express the complete solution, if it exists, by the recurrence

$$u_{k+1} = \text{trunc}_{m-r-1}(v_k + x u_k),$$

where  $\text{trunc}_{m-r-1}$  is the truncation of the polynomial to the given degree. Whether or not a solution exists, this process defines  $u_k$  as a linear function of the  $v$ 's, and so defines a right inverse  $P_r$  for the linear map  $L_r$ . In this form, the coefficients of  $P_r$  are integers, and this allows us to pass to any ring. These leads to the following version.

**Proposition 6.3** Let  $R$  be a Noetherian ring, let  $A = R[x, y]$  and let  $I \subset A$  be an ideal such that

- $A/I$  is a free  $R$ -module of rank  $N = \frac{1}{2}m(m + 1)$ ;
- there is an  $R$ -module homomorphism  $\varphi: A_m \rightarrow A^{(m-1)}$  such that  $a - \varphi(a) \in I$  for all  $a \in A_m$ .

Then  $I$  is generated by the elements  $a - \varphi(a)$  for  $a \in A_m$ . Furthermore, if we write

$$\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_m,$$

with  $\varphi_r: A_m \rightarrow A_{m-r}$ , then  $\varphi_r$  for  $r \geq 2$  is determined by  $\varphi_1$  through the iterative solution (71). This establishes a bijection between ideals  $I$  satisfying the above two constraints and module homomorphisms  $\varphi_1: A_m \rightarrow A_{m-1}$ .

**Proof** If  $I$  satisfies the second condition, the relations  $a = \varphi(a) \text{ mod } I$  show that the map  $A^{(m-1)} \rightarrow A/I$  is surjective. The first of the two conditions tells us that these are free  $R$ -modules of equal rank, and it follows that the map is an isomorphism because  $R$  is Noetherian. Thus we have a direct sum decomposition  $A = I \oplus A^{(m-1)}$ . As before, the constraints then lead to the relations (71), which determine  $\varphi_r$  for  $r \geq 2$ .  $\square$

## 6.2 Syzygies

Proposition 6.3, which determines  $\varphi$  entirely in terms of  $\varphi_1$ , will be applied in Section 6.3 to see that the generators  $\overline{W}_h^m$  of the ideal  $\overline{\mathcal{J}}_{n,-1}$  can be determined completely in terms of the leading and subleading terms. (The subleading terms are already supplied by Proposition 5.12.) This will provide a complete description of the instanton homology  $I(Z_{n,-1})$ . First, however, we pursue further our discussion of the Hilbert scheme of points in the plane, to explain that the way in which  $\varphi_1$  determines  $\varphi$  can be packaged by considering the syzygies of the module  $A/I$ . This will lead to quite explicit formulae for the generators.

We return temporarily to the case  $A = k[x, y]$  as above, and we take  $k = \mathbb{C}$ . Fix  $m$  again and write  $N = \frac{1}{2}m(m + 1)$ . Let  $U \subset \mathcal{H}^N$  be as before (67). An ideal  $I \in U$  contains no nonzero polynomials of degree less than  $m$  and is generated by  $m + 1$  elements whose leading terms are a basis for  $A_m$ . Choose a basis for  $A_m$  so as to identify  $A_m = A^{\oplus(m+1)}$ , say the monomial basis (66). We then have generators for  $I$  in the form

$$g_i = x^{m-i} y^i - \varphi(x^{m-i} y^i).$$

Because  $A$  has dimension 2, a resolution of  $A/I$  has only one more step, and we therefore have a presentation of the ideal  $I$  in the form

$$(72) \quad 0 \rightarrow A^{\oplus m} \xrightarrow{S} A^{\oplus(m+1)} \xrightarrow{g} I \rightarrow 0.$$

Here  $g = (g_i)$  is given by the generators (the relations in  $A/I$ ) and  $S$  is the matrix of syzygies.

In the special case that  $I = \mathfrak{m}^m$  and  $g_i = x^{m-i}y^i$  the syzygy matrix can be taken to be

$$(73) \quad S_0 = \begin{pmatrix} -y & 0 & 0 & \dots & 0 \\ x & -y & 0 & \dots & 0 \\ 0 & x & -y & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -y \\ 0 & 0 & 0 & \dots & x \end{pmatrix}.$$

**Lemma 6.4** For a general  $I \in U$ , the syzygy matrix  $S$  has the form  $S = S_0 + S_1$ , where  $S_0$  is as above and  $S_1$  is a matrix of scalars (polynomials of degree 0).

**Proof** Write  $g = g(0) + g(1) + \dots + g(m)$ , where  $g(r)$  is a vector of homogeneous polynomials of degree  $m - r$  and  $g(0)$  is the basis of monomials of degree  $m$ . (So the entries of  $g(r)$  are the polynomials  $-\varphi_r(x^{m-i}y^i)$ .) Let

$$g^t = g(0) + tg(1) + t^2g(2) + \dots,$$

and let  $I^t$  be the ideal generated by the entries of  $g^t$ . Because the colength of  $I = I^1$  is the same as that of  $I^0$ , this is a flat family, and the syzygy matrix  $S^0$  for  $g^0$  therefore lifts to a syzygy matrix  $S^t$ , whose entries are polynomials in  $(x, y, t)$  and which coincides with  $S^0$  at  $t = 0$ . Because the entries of  $g^t$  are homogeneous (of degree  $m$ ) in  $(t, x, y)$ , we may assume that  $S^t$  is also homogeneous. Since  $S_0$  has homogeneous degree 1, so too does  $S^t$ , and it follows that

$$S^t = S_0 + tS_1,$$

where the entries of  $S_1$  have degree 0 in  $(x, y)$ . □

Note that in the above lemma, the matrix  $S_1$  is entirely determined by the leading term  $g(0)$  and the subleading term  $g(1)$  (or equivalently by  $\varphi_1 : A_m \rightarrow A_{m-1}$ ) via the condition

$$(74) \quad g(0) \cdot S_1 + g(1) \cdot S_0 = 0.$$

Quite concretely, taking  $g(0)$  to be again the standard monomial basis, taking  $S_0$  as above, and writing the subleading terms  $g_i(1)$  as

$$g_i(1) = \sum_{j=0}^{m-1} G_{ij}x^{m-1-j}y^j,$$

then

$$(75) \quad S_1 = \begin{pmatrix} -G_{1,0} & -G_{2,0} & \dots & -G_{m,0} \\ G_{0,0} - G_{1,1} & G_{1,0} - G_{2,1} & \dots & G_{m-1,0} - G_{m,1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{0,m-2} - G_{1,m-1} & G_{1,m-2} - G_{2,m-1} & \dots & G_{m-1,m-2} - G_{m,m-1} \\ G_{0,m-1} & G_{1,m-1} & \dots & G_{m-1,m-1} \end{pmatrix}.$$

**Proposition 6.5** Let  $S = S_0 + S_1$  be the syzygy matrix as above, so that  $S_0$  is the matrix of syzygies of the standard monomial ideal  $\mathfrak{m}^m$  and  $S_1$  is determined by the subleading terms  $g_i(1)$  by (75). Then the generators  $g_0, \dots, g_m$  of the ideal  $I$  are precisely the  $m \times m$  minors of the  $(m + 1) \times m$  matrix  $S$  (ie the determinants of the matrices obtained by deleting a single row of  $S$ , with alternating sign).

**Proof** Let  $h = (h_0, h_1, \dots, h_m)$  be the minors. We have both  $h \cdot S = 0$  (by standard properties of determinants) and  $g \cdot S = 0$  (by construction), and it follows that  $ah = bg$  for some  $a$  and  $b$  in  $A$ , because the rank of the kernel of  $S^T$  is 1. On the other hand, by inspection, the leading term of  $h_i$  is the same as that of  $g_i$ , namely  $x^{m-i}y^i$ . So  $h = g$ . □

Finally, we can pass from the case of  $k[x, y]$  to more general coefficients without difficulty. The next proposition summarizes the situation.

**Proposition 6.6** As in Proposition 6.3, let  $R$  be a Noetherian ring, let  $A = R[x, y]$  and let  $I \subset A$  be an ideal such that

- $A/I$  is a free  $R$ -module of rank  $N = \frac{1}{2}m(m + 1)$ ;
- there is an  $R$ -module homomorphism  $\varphi: A_m \rightarrow A^{(m-1)}$  such that  $a - \varphi(a) \in I$  for all  $a \in A_m$ .

Let  $(g_0(0), \dots, g_m(0))$  be a basis for  $A_m \cong A^{\oplus(m+1)}$  and let

$$g_i = g_i(0) - \varphi(g_i(0)) = g_i(0) + g_i(1) + g_i(2) + \dots + g_i(m),$$

where  $g_i(j)$  is homogeneous of degree  $m - j$ . Then the elements  $(g_0, \dots, g_m)$  are generators of the ideal  $I$ . Furthermore, let  $S_0$  be a matrix of syzygies for the leading parts  $g_i(0)$ , with entries which are homogeneous of degree 1, and let  $S_1$  be the matrix of scalars determined by the subleading parts  $g_i(1)$  via equation (74). Then:

- (i) The matrix  $S = S_0 + S_1$  is the matrix of syzygies for the generators  $(g_0, g_1, \dots, g_m)$  of the ideal  $I$ .
- (ii) If  $h_0, \dots, h_m$  are the  $m \times m$  minors of the matrix  $S$ , then  $(h_0, h_1, \dots, h_m)$  is a set of generators for  $I$ .
- (iii) If  $S_0$  is chosen so that its minors are the leading terms  $(g_0(0), \dots, g_m(0))$ , then the generators  $g_i$  for  $I$  are equal to the minors  $h_i$  of  $S$ .

In this way, the generators  $g$  are determined by their leading and subleading terms,  $g(0)$  and  $g(1)$ .

**Proof** We may take it that  $g(0)$  is the standard monomial basis and that  $S_0$  is given (73). The matrix  $S_1$  is then given by (75) where the terms  $G_{i,j}$  are the coefficients of the subleading terms  $g(1)$ . According to Proposition 6.3, the lower terms in the entries of  $g$  are expressible as universal polynomials in the coefficients of  $g(1)$ . On the other hand, the recipe in terms of the minors of  $S$  expresses the lower terms of  $g$  as polynomials in the coefficients of  $g(1)$ , at least when  $R$  is a field  $k$ . The polynomials occurring in the minors have integer coefficients, and must coincide with the polynomials in Proposition 6.3. □

### 6.3 Equations for the curve $D_n$

Let  $\mathcal{R} = \mathbb{Q}[\tau, \tau^{-1}]$ . Let  $R$  temporarily denote the ring

$$R = \mathcal{R}[\epsilon]/\langle \epsilon^2 - 1 \rangle.$$

The algebra  $\bar{\mathcal{A}}$  in (32) is  $R[\alpha, \delta]$  and the instanton homology  $I(Z_{n,-1})$  is described as a quotient  $\bar{\mathcal{A}}/\bar{\mathcal{J}}_{n,-1}$  in (33). We know that  $I(Z_{n,-1})$  is a free  $\mathcal{R}$ -module of rank  $\frac{1}{4}(n^2 - 1) = m(m + 1)$  from Corollary 3.19, and it is a free  $R$ -module of rank  $\frac{1}{2}m(m + 1)$ . We know that there are elements  $\bar{W}_h^m$  in  $\bar{\mathcal{J}}_{n,-1}$  of degree  $m$  in  $(\alpha, \delta)$  having the form

$$(76) \quad \bar{W}_h^m = w(0)_h + \epsilon w(1)_h + \dots = \bar{w}_{n,h}^m + \epsilon \bar{w}_{n,h'}^{m-1} + \dots$$

(see Proposition 5.12). The leading and subleading terms  $w(0)$  and  $\epsilon w(1)$  are known from Proposition 5.12 and Definition 4.2. We also know that the leading terms  $w(0)_h$  are a basis for the  $m^{\text{th}}$  power of the maximal ideal,  $\langle \alpha, \delta \rangle^m$ , by Lemma 5.14.

The ideal  $\bar{\mathcal{J}}_{n,-1} \subset R[\alpha, \delta]$  therefore satisfies the hypotheses of Propositions 6.3 and 6.6. In the notation of Proposition 6.6, we know  $\varphi_1$  explicitly, as it is determined by the subleading terms  $\epsilon w(1)_h$ . We therefore have the following result as a corollary. In this statement, we write  $n = 2m + 1$  as usual.

**Theorem 6.7** *Let  $\bar{\mathcal{J}}_{n,-1}$  be the ideal of relations for the instanton homology  $I(Z_{n,-1})$  with local coefficients, and let*

$$\bar{W}_h^m = w(0)_h + w(1)_h + \dots + w(m)_h, \quad \text{with } 0 \leq h \leq n \text{ and } h = m + 1 \pmod{2},$$

*be the generators for this ideal, as in (76). There are explicit polynomial formulae which express the coefficients of all the lower terms  $w(r)_h$  for  $r \geq 2$  in terms of the leading and subleading terms*

$$w(0)_h = \bar{w}_{n,h}^m \quad \text{and} \quad w(1)_h = \epsilon \bar{w}_{n,n-h}^{m-1}$$

*in Proposition 5.12. If the syzygy matrix*

$$S = S_0 + S_1$$

*is constructed as in Proposition 6.6, as a matrix whose entries are inhomogeneous linear forms in  $(\alpha, \delta)$  with coefficients in  $R = \mathcal{R}[\epsilon]/\langle \epsilon^2 - 1 \rangle$ , then the generators  $\bar{W}_h^m$  are the  $m \times m$  minors of  $S$ .  $\square$*

To obtain a final form for the generators, we now need to find an explicit formula for the syzygy matrix  $S$ , starting from our formulae for  $w(0)_h$  and  $w(1)_h$ . In Section 6.2 above, we illustrated the calculation when the leading terms of the generators were the standard monomial basis in the polynomials in two variables, so that the term  $S_0$  was the standard syzygy matrix (73). The leading terms  $w(0)_h$  are not monomials in our case, so we must first write down a suitable matrix of syzygies  $S_0$  for these.

From Proposition 4.7, on setting all  $\delta_i$  equal to  $\delta$  to pass from the ring  $\mathcal{A}_j$  to  $\bar{\mathcal{A}}$ , we obtain an expression for  $w(0)_h = \bar{w}_{n,h}^m$  as a product of linear factors. It is convenient to remove the combinatorial factor of  $1/m!$  and write

$$g(0)_h = m!w(0)_h = m!\bar{w}_{n,h}^m,$$

for which Proposition 4.7 yields the formula

$$\begin{aligned} g(0)_h &= \prod_{\substack{j=-m+1 \\ j=-m+1 \pmod 2}}^{m-1} (\alpha + \frac{1}{2}(2h - n - 2j)\delta), \\ &= (\alpha + \frac{1}{2}(2h - 3)\delta)(\alpha + \frac{1}{2}(2h - 7)\delta) \cdots (\alpha + \frac{1}{2}(2h - 4m + 1)\delta). \end{aligned}$$

We introduce some abbreviated notation, setting

$$L(k) = (\alpha + k\delta/2) \quad \text{and} \quad P(k, l) = L(k)L(k + 4)L(k + 8) \cdots L(l).$$

(The latter notation will be used only when  $k = l \pmod 4$ .) Then we can write

$$g(0)_h = P(2h - 4m + 1, 2h - 3).$$

If we compare  $g(0)_h$  to  $g(0)_{h+2}$ , only the first and last factors in this product differ, so we have a relation

$$-L(2h + 1)g(0)_h + L(2h - 4m + 1)g(0)_{h+2} = 0.$$

That is, for  $h'$  in the range  $0 \leq h' \leq n - 2$  with  $h' = m + 1 \pmod 2$ , we have

$$\sum_h S_0^{h'h} g(0)_h = 0,$$

where

$$(77) \quad S_0^{h'h} = \begin{cases} -L(2h' + 1) & \text{if } h = h', \\ L(2h' - 4m + 1) & \text{if } h = h' + 2, \\ 0 & \text{otherwise.} \end{cases}$$

This is therefore the leading part  $S_0$  of the required syzygy matrix  $S = S_0 + S_1$ . It is straightforward to verify that the minors of  $S_0^{h'h}$  are the terms  $g(0)_h$ , as required.

We normalize the subleading terms just as we did the leading terms, so that

$$g(1)_h = m!w(1)_h = m!\epsilon\tau^{n-2h}\bar{w}_{n,n-h}^{m-1},$$

from Proposition 5.2. We then have the explicit formulae again from Proposition 4.7 (noting that  $|\eta'| = n - h$ ),

$$\begin{aligned} g(1)_h &= m\epsilon\tau^{n-2h} \prod_{\substack{j=-m+2 \\ j=m \pmod 2}}^{m-2} (\alpha + \frac{1}{2}(n - 2h - 2j)\delta) \\ &= m\epsilon\tau^{n-2h} P(-2h + 5, -2h + 4m - 3). \end{aligned}$$

To obtain the other term  $S_1$  in the syzygy matrix, we need to solve the following equations for  $S_1^{h'h}$ :

$$\sum_h S_1^{h'h} g(0)_h + \sum_h S_0^{h'h} g(1)_h = 0,$$

where  $h, h' = m + 1 \pmod 2$ , with  $0 \leq h \leq n$  and  $0 \leq h' \leq n - 2$ . Using the formulae for  $g(0)_h, g(1)_h$  and  $S_0^{h'h}$ , we write this out as

$$0 = \sum_h S_1^{h'h} P(2h - 4m + 1, 2h - 3) - m\epsilon\tau^{n-2h'} L(2h' + 1)P(-2h' + 5, -2h' + 4m - 3) + m\epsilon\tau^{n-2h'-4} L(2h' - 4m + 1)P(-2h' + 1, -2h' + 4m - 7).$$

The solution  $S_1^{h'h}$  consisting of scalars in  $R$  is unique, because the terms  $g(0)_h$  are a basis for the homogeneous polynomials of degree  $m$  in  $(\alpha, \delta)$ .

The last two of the three terms above have at least  $m - 2$  common linear factors  $L(k)$ , and have  $m - 1$  common factors in two edge cases. The  $m - 2$  factors are the expression

$$Q(h') = P(-2h' + 5, -2h' + 4m - 7).$$

The edge cases are  $h' = 0$  (which only occurs when  $m$  is odd), and  $h' = n - 2$  (which occurs only when  $m$  is even). In these two edge cases the  $m - 1$  common factors are respectively,

$$Q_+ = L(1)Q(0) = P(1, 4m - 7) \quad \text{and} \quad Q_- = L(-1)Q(n - 2) = P(-4m + 7, -1).$$

We seek a solution  $S_1^{h'h}$  to the above equations in the special form where, for each  $h'$ , the coefficients  $S_1^{h'h}$  are nonzero only for those values of  $h$  for which  $g(0)_h$  is divisible by  $Q(h')$  (respectively  $Q_+$  or  $Q_-$  in the edge cases). Excluding the edge cases, there are three such values of  $h$ , namely

$$(78) \quad h \in \{n - h' - 3, n - h' - 1, n - h' + 1\}, \quad \text{where } 0 < h' < n - 2.$$

In each of the edge cases, there are two such values of  $h$ :

$$(79) \quad h \in \begin{cases} \{n - 3, n - 1\} & \text{if } h' = 0, \\ \{1, 3\} & \text{if } h' = n - 2. \end{cases}$$

In the nonedge cases, the equations for the nonzero coefficients  $S_1^{h'h}$  then take the general shape

$$(80) \quad S_1^{h', n-h'-3} A + S_1^{h', n-h'-1} B + S_1^{h', n-h'+1} C + D = 0,$$

where  $A, B$  and  $C$  are the homogeneous quadratic polynomials in  $(\alpha, \delta)$  given by

$$g(0)_h / Q(h'), \quad \text{where } h \in \{n - h' - 3, n - h' - 1, n - h' + 1\},$$

and  $D$  is a quadratic polynomial

$$D = (S_0^{h', h'} g(1)_{h'} + S_0^{h', h'+2} g(1)_{h'+2}) / Q(h').$$

Explicitly,

$$\begin{aligned}
 A &= L(-2h' - 3)L(-2h' + 1), \\
 B &= L(-2h' + 1)L(-2h' + 4m - 3), \\
 C &= L(-2h' + 4m - 3)L(-2h' + 4m + 1), \\
 D &= m\epsilon\tau^{n-2h'}(-L(2h' + 1)L(-2h' + 4m - 3) + \tau^{-4}L(2h' - 4m + 1)L(-2h' + 1)).
 \end{aligned}$$

The three polynomials  $A$ ,  $B$  and  $C$  are independent, and we know there to be a unique solution, which we can now find by equating coefficients of  $\alpha^2$ ,  $\alpha\delta$  and  $\delta^2$  in (80). The two edge cases are similar. Thus in the case  $h' = 0$ , the equations for the two unknown coefficients of  $S_1$  take the form

$$(81) \quad S_1^{0, n-3} X + S_1^{0, n-1} Y = Z,$$

where  $X$ ,  $Y$  and  $Z$  are homogeneous linear forms in  $(\alpha, \delta)$ , while in the case  $h' = n - 2$  we have similar equations

$$(82) \quad S_1^{n-2, 1} X' + S_1^{n-2, 3} Y' = Z'.$$

Solving the equations (80)–(82) for the coefficients  $S_1^{h'h}$  leads to the following answer, valid for all  $h'$ , whether or not we are in an edge case. We find

$$(83) \quad S_1^{h'h} = \begin{cases} \epsilon\tau^{n-4-2h'}(-n + 2 + h') & \text{if } h = n - h' - 3, \\ \epsilon\tau^{n-4-2h'}(m - h' - 1 + (m - h')\tau^4) & \text{if } h = n - h' - 1, \\ \epsilon\tau^{n-2h'}h' & \text{if } h = n - h' + 1, \\ 0 & \text{otherwise,} \end{cases}$$

for all  $h'$ ,  $h$  in the range  $0 \leq h \leq n$  and  $0 \leq h' \leq n - 2$  with the parity constraint  $h = h' = m + 1 \pmod{2}$ . So we have obtained the desired closed form for the generators of the ideal  $\bar{\mathcal{J}}_{n,-1}$  for the instanton homology  $I(Z_{n,-1})$ :

**Theorem 6.8** *Let  $S = S_0 + S_1$  be an  $m \times (m + 1)$  with rows indexed by  $h'$  and columns indexed by  $h$  in the range  $0 \leq h \leq n$  and  $0 \leq h' \leq n - 2$  with the parity constraint  $h = h' = m + 1 \pmod{2}$ . Let the entries of  $S_0$  be given by (77) and the entries of  $S_1$  be given by (83), so that the entries of  $S$  belong to the ring  $\bar{\mathcal{A}} = \mathbb{Q}[\tau, \tau^{-1}, \epsilon, \alpha, \delta]/\langle \epsilon^2 = 1 \rangle$ . Then the normalized generators  $m! \bar{W}_h^m$  of the ideal  $\bar{\mathcal{J}}_{n,-1}$  are given by the  $m \times m$  minors of  $S$ .*

**Remark** The matrix the matrix  $S$  has  $m + 1$  different  $m \times m$  minors, and explicitly the generators of the ideal can be expressed as

$$m! \bar{W}_h^m = \pm \det S[h],$$

where  $S[h]$  is obtained from  $S$  by deleting the column indexed by  $h$ . (Recall again that the indexing of the columns is by only those integers  $h$  with  $h = m + 1 \pmod{2}$ .) The signs alternate as usual. Although there are  $m + 1$  generators in this description, in fact only two generators suffice, as the following proposition states.

**Proposition 6.9** *The ideal  $\bar{\mathcal{J}}_{n,-1}$  is generated by the two elements  $\bar{W}_{m-1}^m$  and  $\bar{W}_{m+1}^m$ , or equivalently by the two determinants*

$$G_1(n) = \det S[m-1] \quad \text{and} \quad G_2(n) = \det S[m+1].$$

**Proof** It is sufficient to show that the matrix  $S[m-1, m+1]$  obtained by deleting both columns  $h = m-1$  and  $h = m+1$  has full rank  $m-1$ . To do this, let us examine the  $(m-1) \times (m-1)$  matrix  $T$  obtained from  $S[m-1, m+1]$  by deleting either the first or last row, according as  $m$  is odd or even respectively. An inspection of the entries of  $S$  reveals first that the entries of  $T$  on the contra-diagonal are all units in  $\bar{\mathcal{A}}$ : they are nonzero integers times powers of  $\tau$ . Furthermore, a reordering of the rows and columns makes  $T$  triangular, with these same units on the diagonal. The determinant of  $T$  is therefore nonzero, which shows that  $S[m-1, m+1]$  indeed has full rank as desired.  $\square$

As illustration, when  $m = 3$  (ie  $n = 7$ ) the two generators  $G_1(7)$  and  $G_2(7)$  are

$$\begin{aligned} \frac{1}{48} (8\alpha^3 + 36\alpha^2\delta + 22\alpha\delta^2 - 21\delta^3 + 24\epsilon\tau^3\alpha^2 - 72\epsilon\tau^3\alpha\delta + 30\epsilon\tau^3\delta^2 - (88\tau^2 + 16\tau^{-2})\alpha \\ - (52\tau^2 + 56\tau^{-2})\delta - 24\epsilon\tau^5 - 96\epsilon\tau) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{48} (8\alpha^3 - 12\alpha^2\delta - 26\alpha\delta^2 + 15\delta^3 + 24\epsilon\tau^{-1}\alpha^2 + 24\epsilon\tau^{-1}\alpha\delta - 18\epsilon\tau^{-1}\delta^2 - (40\tau^2 + 64\tau^{-2})\alpha \\ + (68\tau^2 - 32\tau^{-2})\delta - 72\epsilon\tau - 48\epsilon\tau^{-3}). \end{aligned}$$

### 6.4 Relating different values of $n$

**Theorem 6.8** provides a complete description of the instanton homology of  $Z_{n,-1}$  with local coefficients, but we have not yet presented a full description for the case of  $Z_n$ . As preliminary material for this, we describe how the functoriality of instanton homology can be used to obtain relations between the ideal of relations in  $Z_n$  for different values of  $n$ .

The fact that the ideal  $\mathcal{J}_n$  annihilates  $I(Z_n)$  leads, via a standard approach, to the interpretation of the elements of  $\mathcal{J}_n$  as universal relations that hold for the maps defined by general bifold cobordisms. To spell this out, let  $W$  be a homology-oriented bifold cobordism from  $Z^0$  to  $Z^1$ , both of which are admissible. We have seen in [Section 2.3](#) that  $W$  gives rise to homomorphisms of  $\mathcal{R}$ -modules

$$I(W, a): I(Z^0) \rightarrow I(Z^1)$$

depending linearly on

$$a \in \text{Sym}_*(H_2(W; \mathbb{Q}) \oplus H_0(\Sigma(W); O)) \otimes \mathcal{R},$$

where  $O$  is the orientation bundle of the singular set  $\Sigma(W)$  with coefficients  $\mathbb{Q}$ . Further, given a distinguished 2-dimensional class  $e$  we can use marked connections with nonzero  $w_2$  to define maps

$$I(W, a)^e: I(Z^0) \rightarrow I(Z^1).$$

Using  $\delta_p$  to denote the generator of the symmetric algebra corresponding the homology class of a point  $p \in \Sigma(W)$  with local orientation, let us imitate the definition of  $\mathcal{A}_n$  and write

$$\mathcal{A}(W) = (\text{Sym}_*(H_2(W; \mathbb{Q}) \oplus H_0(\Sigma(W); O)) \otimes \mathcal{R}[\epsilon]) / \langle \epsilon^2 - 1, \delta_p^2 - \delta_q^2 \rangle_{p,q}$$

where the indexing in the ideal runs through all pairs of points  $p, q$  in  $\Sigma(W)$ . We obtain a linear map

$$(84) \quad \Psi: \mathcal{A}(W) \rightarrow \text{Hom}(I(Z^0), I(Z^1)) \quad \text{by} \quad a_1 + \epsilon a_2 \mapsto I(W, a_1) + I(W, a_2)^e.$$

This construction has been phrased so that, in the special case that  $W$  is the product cobordism from  $Z_n$  to itself and  $e$  is the generator of  $H_2$ , the algebra  $\mathcal{A}(W)$  coincides with  $\mathcal{A}_n$  as defined above, and the map  $\Psi$  is the action of the algebra  $\mathcal{A}_n$  on the module  $I(Z_n)$  via the instanton module structure.

Continuing with the case of a general cobordism  $W$ , we suppose now that we have an embedded orbifold sphere  $S \subset W$  meeting the singular set in  $n$  orbifold points  $\{p_1, \dots, p_n\}$ . Choose an orientation for  $S$  and define local orientations for the singular set in the neighborhood of the  $n$  points of intersection in such a way that the intersections are all positive. In this way we obtain elements  $\delta_{p_k} \in \mathcal{A}(W)$ , where for the class  $e$  in the definition of  $\mathcal{A}(W)$  we take the fundamental class  $[S]$ . Let the singular set of  $W$  also be oriented globally, and let the operators  $\delta_{p_k}^*$  be defined using this global orientation of the singular set. We then have

$$\delta_{p_k}^* = \nu_k \delta_{p_k},$$

where  $\nu_k = \pm 1$  according to whether the orientations agree or not.

Let us suppose that the normal bundle of  $S$  is trivial so that the boundary of the tubular neighborhood of  $S$  is a copy of  $Z_n$ . From the definitions, there is a natural map

$$i_*: \mathcal{A}_n \rightarrow \mathcal{A}(W)$$

arising from the inclusion, which we define so that  $i_*(\delta_{p_k}) = \nu_k \delta_{p_k}^*$  for all  $k$ , while  $i_*(\alpha) = [S] \in H_2(W)$  and  $i_*(\epsilon) = (-1)^f \epsilon$ , where  $f$  is the number of signs  $\nu_k$  which are  $-1$ .

**Proposition 6.10** *For an embedded orbifold sphere  $S \subset W$  as above, the ideal  $\mathcal{J}_n$  lies in the kernel of the map  $\Psi$  defined at (84). That is, for  $a = a_1 + \epsilon a_2 \in \mathcal{J}_n \subset \mathcal{A}_n$ , we have*

$$I(W, i_*(a_1)) + (-1)^f I(W, i_*(a_2))^e = 0.$$

*More generally, if  $b$  is another class in  $\mathcal{A}(W)$  which can be expressed as a polynomial in cycles disjoint from  $S$ , then we have*

$$I(W, i_*(a_1)b) + (-1)^f I(W, i_*(a_2)b)^e = 0.$$

**Proof** In its structure, this is a standard argument based on the observation that we can factor the cobordism  $W$  as a composite cobordism in which the first factor is the cobordism from  $Z^0$  to  $Z^0 \amalg Z_n$ . For the disjoint union, we can construct the instanton homology as a tensor product, and then we apply functoriality. See [22] and [32], for example, for similar arguments. The details of the signs, in particular the sign  $(-1)^f$  come from Proposition 5.11. □

Our application of [Proposition 6.10](#) is equivalent to [\[32, Corollary 2.6.8\]](#). (A closely related result appears in [\[28\]](#).) Suppose that

$$n = n' + 2f, \quad \text{where } f \geq 0.$$

Consider an embedding of the orbifold sphere  $S = S_n^2$  in the trivial cobordism  $W = [0, 1] \times Z_{n'}$ , representing the generator in homology. This means that  $S$  meets the singular locus  $[0, 1] \times K(Z_{n'})$  geometrically in  $n' + 2f$  points, while the algebraic intersection number is  $n'$ . There are therefore  $2f$  signed intersection points that cancel in pairs. Such a sphere  $S \subset [0, 1] \times Z_{n'}$  can be constructed by taking the standard generating sphere  $S' \subset Z_{n'}$  and introducing  $2f$  extra intersection points by doing  $f$  “finger moves” to the sphere  $S'$ . We take these extra intersection points to be the orbifold points numbered  $n' + 1, \dots, n' + 2f$  in  $S \cong S_n^2$ , and we suppose that they all lie on the component  $[0, 1] \times K^{n'} \subset [0, 1] \times K(Z_{n'})$ . Among these  $2f$  points, there are  $f$  of them that have negative intersection number, and we can take it that these are the points numbered  $n' + f + 1, \dots, n' + 2f$  in  $S_n^2$ . There is a corresponding map

$$i_*^{n,n'} : \mathcal{A}_n \rightarrow \mathcal{A}_{n'}, \quad \text{where } n = n' + 2f,$$

and our choice of numbering means that it is given by

$$i_*^{n,n'}(\alpha) = \alpha, \quad i_*^{n,n'}(\epsilon) = (-1)^f \epsilon \quad \text{and} \quad i_*^{n,n'}(\delta_k) = \begin{cases} \delta_k & \text{if } 1 \leq k \leq n', \\ \delta_{n'} & \text{if } n' + 1 \leq k \leq n' + f, \\ -\delta_{n'} & \text{if } n' + f + 1 \leq k \leq n' + 2f. \end{cases}$$

[Proposition 6.10](#) now yields the following.

**Corollary 6.11** [\[32, Corollary 2.6.8\]](#) *When  $n = n' + 2f$  and  $i_*^{n,n'} : \mathcal{A}_n \rightarrow \mathcal{A}_{n'}$  is defined as above, we have an inclusion of ideals,*

$$i_*^{n,n'} \mathcal{J}_n \subset \mathcal{J}_{n'}. \quad \square$$

With a little more work and an examination of the explicit formulae for the leading and subleading terms of the generators of  $\mathcal{J}_n$  ([Proposition 4.7](#)), we can strengthen the above corollary as follows.

**Proposition 6.12** *In the situation of [Corollary 6.11](#) above, we have inclusions*

$$(\tau^4 - 1)^f \mathcal{J}_{n'} \subset i_*^{n,n'} \mathcal{J}_n \subset \mathcal{J}_{n'}.$$

*In particular, the ideals  $i_*^{n,n'} \mathcal{J}_n$  and  $\mathcal{J}_{n'}$  become equal after tensoring with the field of fractions of the ring  $\mathcal{R} = \mathbb{Q}[\tau, \tau^{-1}]$ .*

**Proof** It suffices to treat the case  $f = 1$ , so  $n' = n - 2$ . Let  $\eta_0 \subset 1, \dots, n - 2$ , and let  $\eta_1, \eta_2 \subset \{1, \dots, n\}$  be respectively the same as  $\eta_0$  and  $\eta_0 \cup \{n - 1, n - 2\}$ . From the explicit formulae, we see

$$i_*^{n,n-2}(w_{n,\eta_1}^m) = i_*^{n,n-2}(w_{n,\eta_2}^m),$$

because  $i_*^{n,n-2}(B_{\eta_1}) = i_*^{n,n-2}(B_{\eta_2})$ . Similarly

$$i_*^{n,n-2}(w_{n,\eta'_1}^{m-1}) = i_*^{n,n-2}(w_{n,\eta'_2}^{m-1}).$$

We therefore have (using the general shape of the subleading term)

$$\begin{aligned} i_*^{n,n-2}(W_{\eta_1}^m - W_{\eta_2}^m) &= (-1)^f \epsilon (\tau^{n-2h} - \tau^{n-2h-4}) i_*^{n,n-2}(w_{n,\eta'_1}^{m-1}) + \text{lower terms} \\ &= u(\tau^4 - 1) i_*^{n,n-2}(w_{n,\eta'_1}^{m-1}) + \text{lower terms,} \end{aligned}$$

where  $u$  is a unit in  $\mathbb{Q}[\tau, \tau^{-1}]$ . By the previous corollary, these belong to  $\mathcal{J}_{n-2}$ . It is now enough to show that the elements  $i_*^{n,n-2}(w_{n,\eta'_1}^{m-1})$  generate the ideal  $j_{n-2}$  of relations in the ordinary cohomology of  $\text{Rep}(S_{n-2}^2)$ , because the statement about instanton homology will follow as before. From the formulae in Proposition 4.7, we see that this is the same as showing that the elements  $w_{n-2,\eta'_0}^{m-1}$  generate the ideal  $j_{n-2}$ , which has already been established (as the case  $n - 2$ ) in Proposition 4.8.  $\square$

The homomorphism  $i_*^{n,n'}$  does not pass to a homomorphism between the quotient rings  $\bar{\mathcal{A}}$ . But we can at least compose with the quotient map  $\mathcal{A}_{n'} \rightarrow \bar{\mathcal{A}}$  to get the following immediate corollary. In the statement of the corollary, we note that the choices of sign in the definition of  $i_*^{n,n'}$  are arbitrary and can be replaced by a more general phrasing.

**Corollary 6.13** *Let  $v \in \{\pm 1\}^n$  be any choice of signs. Write  $n' = \sum v_i$  and assume  $n' \geq 1$ . Consider the homomorphism  $\bar{v}: \mathcal{A}_n \rightarrow \bar{\mathcal{A}}$  defined by  $\bar{v}(\delta_i) = v_i \delta$  for all  $i$ , and  $\bar{v}(\epsilon) = (-1)^{(n-n')/2} \epsilon$ . Then we have an inclusions of ideals in  $\bar{\mathcal{A}} = \mathcal{R}[\delta, \alpha, \epsilon]/(\epsilon^2 - 1)$ ,*

$$(\tau^4 - 1)^{(n-n')/2} \bar{\mathcal{J}}_{n',-1} \subset \bar{v}(\mathcal{J}_n) \subset \bar{\mathcal{J}}_{n',-1}. \quad \square$$

We refer to the relations between the ideals in Corollaries 6.11 and 6.13 as “finger-move relations”, because of the interpretation of the sphere  $S$  as having been obtained from the standard sphere  $S' \subset W$  by finger moves.

**Remark** A second application of Proposition 6.10 will be given in the proof of Proposition 7.1 later in this paper.

### 6.5 Decomposition of the instanton curve

We are now ready to harness our understanding of  $I(Z_{n,-1})$  from Theorem 6.8 to obtain a description of  $I(Z_n)$ . Write

$$V_n = \text{Spec } \mathbb{Q}[\tau, \tau^{-1}, \alpha, \delta_1, \dots, \delta_n, \epsilon].$$

The set of complex-valued points  $V_n(\mathbb{C})$  is  $\mathbb{C}^\times \times \mathbb{C}^{n+2}$ , with  $\tau$  a coordinate on the first factor. We can describe the  $\mathcal{A}_n$ -module  $I(Z_n)$  geometrically as the coordinate ring of the closed subscheme

$$C_n \subset V_n$$

defined by the vanishing of the elements of the ideal  $\mathcal{J}_n$  together with the additional relations that define the algebra  $\mathcal{A}_n$ , namely the vanishing of  $\delta_i^2 - \delta_j^2$  and  $\epsilon^2 - 1$ . We can write  $C_n = \text{Spec}(I(Z_n))$ , where  $I(Z_n)$  is considered as a quotient ring of the algebra  $\mathcal{A}_n$ . To describe  $I(Z_n)$  as an  $\mathcal{A}_n$ -module, we can

therefore use geometrical language to describe the subscheme  $C_n$ . Note that the relation  $\epsilon^2 = 1$  means that  $C_n$  is contained in the union of the two hyperplanes  $\epsilon = 1$  and  $\epsilon = -1$ , so we may write

$$C_n = C_n^+ \cup C_n^-.$$

In a similar way, let us write

$$\bar{V} = \text{Spec } \mathbb{Q}[\tau, \tau^{-1}, \alpha, \delta, \epsilon],$$

so that the instanton homology group  $I(Z_{n,-1})$  defines, (via its ideal of relations  $\bar{\mathcal{J}}_{n,-1}$  and the relation  $\epsilon^2 = 1$ ), a subscheme  $D_n = \text{Spec}(I(Z_{n,-1}))$ , which is a closed subscheme of  $\bar{V}$ :

$$(85) \quad D_n = D_n^+ \cup D_n^- \subset \bar{V}.$$

We can interpret [Corollary 6.13](#) as describing a relation between the curves  $C_n$  for  $I(Z_n)$  and  $D_n$  for  $I(Z_{n,-1})$ . First, given any choice of signs  $v \in \{\pm 1\}^n$ , write  $n' = \sum v_i$ , and suppose henceforth that this odd integer  $n'$  is positive. Write  $f = (n - n')/2$ . Define a morphism

$$\bar{i}_v^* : \bar{V} \rightarrow V_n$$

by  $\delta_i \mapsto v_i \delta$  and  $\epsilon \mapsto (-1)^f \epsilon$ . Write

$$V_{n,v} \subset V_n$$

for the image of  $\bar{i}_v^*$ . This is the linear subvariety cut out by the linear conditions  $v_i \delta_i = v_j \delta_j$ . Their union is the subvariety defined by  $\delta_i^2 = \delta_j^2$  for all  $i, j$ ; so we have

$$C_n \subset \bigcup_v V_{n,v}.$$

We have an isomorphic copy of the affine scheme  $D_{n'}$  as the image of  $D_{n'}$  under the embedding  $\bar{i}_v^*$ :

$$(86) \quad \bar{i}_v^*(D_{n'}) \subset V_{n,v}.$$

**Proposition 6.14** *The subscheme  $C_n \subset V_n$  is the union of the subschemes (86) as  $v$  runs through all choices of sign  $\{\pm 1\}^n$  with  $n'(v) > 0$ :*

$$(87) \quad C_n = \bigcup_{v; n'=n'(v)>0} \bar{i}_v^*(D_{n'}).$$

The curves  $D_{n'}$  are completely known via their defining equations from [Theorem 6.8](#), so the proposition above is a complete characterization of the curve  $C_n$  for  $I(Z_n)$ . In the language of the defining ideals, this proposition is a converse to [Corollary 6.13](#). In other words, we have the following:

**Corollary 6.15** *In the notation of [Corollary 6.13](#), the defining ideal  $\mathcal{J}_n$  for  $I(Z_n)$  can be characterized as*

$$\mathcal{J}_n = \{w \in \mathcal{A}_n \mid \bar{i}_v(w) \in \bar{\mathcal{J}}_{n'(v),-1} \text{ for all } v\}.$$

Thus  $I(Z_n)$  is determined as an  $\mathcal{A}_n$ -module by the finger-move constraints, once  $I(Z_{n',-1})$  is known for all odd  $n' \leq n$ .

**Proof of Proposition 6.14** Let us write  $C'$  for the union on the right-hand side of (87). The inclusion of ideals  $\bar{\tau}_v(\mathcal{J}_n) \subset \bar{\mathcal{J}}_{n',-1}$  in Corollary 6.13 says that the curve  $C_n$  contains  $C'$ .

The coordinate ring of the scheme on the left-hand side of (87) is  $I(Z_n)$ , and if we temporarily write  $I'$  for the coordinate ring of the affine scheme  $C'$ , then the inclusion of schemes means that we have a surjection of rings,

$$I(Z_n) \rightarrow I'.$$

We know that  $I(Z_n)$  is a free  $\mathcal{R}$ -module of finite rank, where  $\mathcal{R} = \mathbb{Q}[\tau, \tau^{-1}]$ . So to prove that the rings are isomorphic, and to complete the proof of the proposition, it will suffice to prove that these two  $\mathcal{R}$ -modules have the same rank, or in geometrical language,

$$\deg C_n = \deg C',$$

where  $\deg$  denotes the degree of the projection to the  $\tau$  coordinate. (The inclusion one way means that we already have  $\deg C_n \geq \deg C'$ .)

To prove this last equality we note that

$$(88) \quad \deg C_n \leq \sum_{\nu; n'(\nu) > 0} \deg(C_n \cap V_{n,\nu}),$$

with equality if and only if the schemes  $C_n \cap V_{n,\nu}$  for different  $\nu$  have no common component of positive degree. The two-way inclusions of Corollary 6.13 tell us that  $C_n \cap V_{n,\nu}$  and  $i_\nu^*(\bar{C}_{n'})$  coincide over the locus where  $\tau^4 - 1$  is nonzero. In particular,

$$\deg(C_n \cap V_{n,\nu}) = \deg(i_\nu^*(D_{n'})),$$

and if the schemes on the left have no common component of positive degree for different  $\nu$ , then the same is true of the schemes on the right. From (88) we therefore obtain

$$(89) \quad \deg C_n \leq \sum_{\nu; n'(\nu) > 0} \deg D_{n'},$$

with equality if and only if the schemes on the right-hand side of (87) have no common component of nonzero degree.

In terms of instanton homology, the inequality (88) can be restated as

$$(90) \quad \text{rank}_{\mathcal{R}} I(Z_n) \leq \sum_{\nu; n'(\nu) > 0} \text{rank}_{\mathcal{R}} I(Z_{n',-1}).$$

On the other hand we can verify directly that we have equality here:

$$(91) \quad \text{rank}_{\mathcal{R}} I(Z_n) = \sum_{\nu; n'(\nu) > 0} \text{rank}_{\mathcal{R}} I(Z_{n',-1}).$$

Indeed, the right-hand side can be calculated by Corollary 3.19, and is

$$\sum_{f=0}^{(n-1)/2} \binom{n}{f} \cdot \frac{1}{4}((n-2f)^2 - 1).$$

The left-hand side of (91) is twice the rank of the ordinary cohomology of the representation variety  $\text{Rep}(S_n^2)$  calculated by Boden [3], and can be expressed as

$$\text{rank}_{\mathcal{R}} I(Z_n) = 2^{n-3}(n-1) = \frac{1}{8}(F''(1) - F(1)),$$

where  $F(t) = (t + t^{-1})^n$ . Equality with the right-hand side of (91) can be seen easily from the binomial expansion of  $F(t)$ .

It follows that the parts making up the union  $C'$  on the right-hand side of (87) have no common components of positive degree, and we therefore have, as required,

$$\text{deg } C' = \sum_v \text{deg } D_{n'(v)} = \text{deg } C_n. \quad \square$$

**Remark** In the course of the proof, we have seen that  $C_n$  has pure dimension 1, and we refer to it as the instanton curve for  $Z_n$ . Although it has no embedded points, we have not shown that the curve  $C_n$  is reduced: it may perhaps have components with multiplicity larger than 1, but the authors have not seen this arise in calculations.

### 6.6 Equations for the curve $C_n$

We now have a geometric description of  $I(Z_n)$  as a module, namely as the coordinate ring of an affine curve  $C_n$ . The curve  $C_n$  is a union of curves each of which is isomorphic to some  $D_{n'}$ . However, although we have an explicit description of the defining relations for the  $D_{n'}$ , the resulting description of  $C_n$  does not immediately provide explicit generators for the corresponding ideal  $\mathcal{J}_n \subset \mathcal{A}_n$ . Instead, it describes the ideal  $\mathcal{J}_n$  as an intersection of known ideals (expressed essentially in Corollary 6.15).

To practically compute the intersection of the ideals in this particular context, we can leverage what we know about  $\mathcal{J}_n$ . From Propositions 5.1 and 5.2, we know the ideal  $\mathcal{J}_n$  is generated by elements  $W_\eta^m$  which can be written in the form

$$(92) \quad W_\eta^m = w(0) + \epsilon w(1) + w(2) + \epsilon w(3) + \dots,$$

where  $w(i)$  is a homogeneous polynomial of degree  $m - i$  in  $(\alpha, \delta_1, \dots, \delta_n)$ , and furthermore

$$w(0) = w_{n,\eta}^m \quad \text{and} \quad w(1) = w_{n,n-\eta}^{m-1}.$$

Furthermore, the element  $W_\eta^m$  is the unique element of the ideal having leading term  $w(0)$ . The lower terms in  $W_\eta^m$  are therefore uniquely characterized by the linear constraints of Corollary 6.15, namely that  $\bar{v}(W_\eta^m)$  belongs to the known ideal  $\bar{\mathcal{J}}_{n'(v),-1}$ , for all  $v$ . Solving this large linear system provides the generators.

There is an alternative way to package the calculation of  $W_\eta^m$ , which does not explicitly pass through a determination of the ideals  $\bar{\mathcal{J}}_{n,-1}$ , albeit the same ingredients are used. To set this up, the terms in (92) which are as yet *unknown* are the terms which belong to a lower part of the increasing filtration of  $\mathcal{A}_n$ , and with this in mind we write

$$L_\eta^m = w(2) + \epsilon w(3) + \dots,$$

so that

$$(93) \quad W_h^m = w(0) + \epsilon w(1) + L_\eta^m, \quad \text{where } L_\eta^m \in \mathcal{A}_n^{(m-2)}.$$

There is some symmetry that can be usefully exploited. The braid group  $B_\pi$  for the  $n$ -element subset  $\pi \subset S^2$  acts on  $I(Z_n)$  because of its interpretation as a mapping class group. This action factors through the symmetric group  $S_\pi$ , as one can see from the description of  $I(Z_n)$  as a cyclic module for the algebra  $\mathcal{A}_n$ . Indeed, given a permutation  $\sigma \in S_\pi$ , we obtain an automorphism  $\sigma_* : \mathcal{A}_n \rightarrow \mathcal{A}_n$  permuting the generators  $\delta_p$  and preserving the ideal  $\mathcal{J}_n \subset \mathcal{A}_n$ , so establishing the automorphism  $\sigma_* : I(Z_n) \rightarrow I(Z_n)$ . From this, we can see that

$$\sigma_*(W_\eta^m) = W_{\sigma(\eta)}^m.$$

In particular, the element  $W_\eta^m \in \mathcal{A}_n$  is invariant under the action of group of permutations  $S_\eta \times S_{\eta'} \subset S_\pi$ . The lower terms  $L_\eta^m$  therefore have the same symmetry. Furthermore, it will be enough if we determine  $L_\eta^m$  for just one subset  $\eta \subset \pi$  of each cardinality  $h$  satisfying the parity condition (35). Note also that the expression  $L_\eta^m$  is empty unless  $m$  is at least 2 (ie  $n$  is at least 5).

The proposed recursive procedure for identifying the lower terms  $L_\eta^m$  is to again use Corollary 6.11, which gives us the finger-move relation

$$(94) \quad i_*^{n,n-2}(W_\eta^m) \in \mathcal{J}_{n-2}.$$

We would like to see that, if the ideal  $\mathcal{J}_{n-2}$  is already known, then the constraint (94) will be sufficient to determine the lower terms. In line with the remarks above, since either  $\eta$  or  $\eta'$  can be assumed to have at least  $m + 1$  elements (ie more than half), we will assume that the indices  $\{m, m + 1, \dots, n\}$  all belong either to  $\eta$  or to  $\eta'$ . In particular this means that  $W_\eta^m$  and its lower terms  $L_\eta^m$  are invariant under the symmetric group  $S_{m+1}$  acting by permutation of the variables  $\{\delta_m, \delta_{m+1}, \dots, \delta_n\}$ . (These indices include the three indices  $\{n - 2, n - 1, n\}$ , which are involved in the definition of the finger move  $i_*^{n,n-2}$ .)

**Lemma 6.16** Write  $n = 2m + 1$  and let  $L \in \mathcal{A}_n^{(m-2)}$  be an element that is symmetric in the variables  $\delta_{m+1}, \dots, \delta_{n-1}, \delta_n$  (ie more than half of the variables). Suppose  $L$  satisfies

$$(95) \quad i_*^{n,n-2}(L) \in \mathcal{J}_{n-2}.$$

Then  $L = 0$ .

**Proof** Let  $\sigma_k$  be the  $k^{\text{th}}$  symmetric polynomial in  $\delta_{m+1}, \dots, \delta_n$ , and let  $\sigma'_k$  be the symmetric polynomial in  $\delta_{m+1}, \dots, \delta_{n-2}$ , regarded as elements of  $\mathcal{A}_n$  and  $\mathcal{A}_{n-2}$  respectively. From Proposition 3.7, we know that  $\mathcal{J}_{n-2} \cap \mathcal{A}_{n-2}^{m-2} = 0$ , so the hypothesis  $i_*^{n,n-2}(L) \in \mathcal{J}_{n-2}$  actually means that  $i_*^{n,n-2}(L)$  is zero. We compute what  $i_*^{n,n-2}$  does to  $\sigma_k$ , and we find

$$i_*^{n,n-2}(\sigma_k) = \begin{cases} \sigma'_k & \text{if } k = 0, 1, \\ \sigma'_k + \beta\sigma'_{k-2} & \text{if } 2 \leq k \leq m - 1, \\ \beta\sigma'_{k-2} & \text{if } k = m, m + 1, \end{cases}$$

where  $\beta = -\delta_p^2$  (independent of  $p$ ).

Because  $L$  has degree at most  $m - 2$ , we can write it as

$$L = \sum_{k=0}^{m-2} P_k \sigma_k,$$

where each  $P_k$  is an expression in  $\mathcal{A}_m$ , ie involving only  $\delta_1, \dots, \delta_m$ . Thus

$$i_*^{n,n-2}(L) = \sum_{k=0}^{m-2} (P_k + \beta P_{k+2}) \sigma'(k),$$

where we set  $P_j = 0$  for  $j > m - 2$ . The injectivity of  $i_*^{n,n-2}$  is now clear from the upper triangular nature of this linear transformation, because the symmetric functions  $\sigma'(k)$  are nonzero in this range.  $\square$

The lemma tells us that the finger-move constraint can be used to determine the lower terms  $L_\eta^m$  uniquely. So we obtain a procedure which determines the ideals  $\mathcal{J}_n$  recursively for all odd  $n$ , as follows.

- (i) In the base case  $n = 1$ , the ideal  $\mathcal{J}_1$  is  $\langle 1 \rangle$ .
- (ii) For general  $n \geq 3$  (and  $n$  odd as always), assume that the ideal  $\mathcal{J}_{n'}$  is already known for  $n' < n$ .
- (iii) Write  $m = \frac{1}{2}(n - 1)$ . According to Propositions 5.1 and 5.2, for each  $\eta$  satisfying the parity condition (47), there exists an element  $W_\eta^m \in \mathcal{J}_n$  which can be written in the form (93):

$$W_h^m = w(0) + \epsilon w(1) + w(2) + \epsilon w(3) + \dots = w(0) + \epsilon w(1) + L_\eta^m, \quad \text{where } L_\eta^m \in \mathcal{A}_n^{(m-2)}.$$

The first terms  $w(0) + \epsilon w(1)$  are known because  $w(0)$  is the Mumford relation and Proposition 5.2 provides the term  $w(1)$ .

- (iv) According to Lemma 6.16, the unknown terms  $L_\eta^m$  in  $W_\eta^m$  are uniquely determined by the finger-move relations (94), which impose linear conditions on the coefficients of  $L_\eta^m$ . Solving these linear equations determines  $L_\eta^m$  and hence determines  $W_\eta^m \in \mathcal{A}_n$ .
- (v) As  $\eta$  runs through the subsets satisfying (47), the elements  $W_\eta^m$  generate the ideal  $\mathcal{J}_n \subset \mathcal{A}_n$  according to Proposition 5.1. So we have a known set of generators for  $\mathcal{J}_n$ . This determines  $\mathcal{J}_n$  and completes the inductive step.

## 7 Further remarks

### 7.1 Singularities of the instanton curve

When the local coefficient system  $\Gamma$  is replaced by constant coefficients  $\mathbb{Q}$ , we obtain a description of the instanton homology  $I(Z_n; \mathbb{Q})$  which was earlier completely determined by Street [32]. Those results therefore provide a description of the scheme-theoretic intersection of the curve  $C_n$  with the hyperplane  $\tau = 1$ . It is shown in [32] that the simultaneous eigenvalues of the pair of operators  $(\alpha, \delta)$  on  $I(Z_n; \mathbb{Q})$  are of the form  $(\lambda, \delta)$ , where  $\lambda$  runs through the odd integers in the range  $|\lambda| < n$ . The multiplicities of the eigenspaces are also computed.

We can apply these results to learn that the curve  $D_n$  corresponding to  $I(Z_{n,-1}; \Gamma)$  intersects the plane  $\tau = 1$  in the points  $x_\lambda$  with coordinates

$$(\tau, \alpha, \delta, \epsilon) = (1, \lambda, 0, \pm 1),$$

where  $\lambda$  runs through the same odd integers, and the sign of  $\epsilon$  is  $(-1)^{(\lambda+1)/2}$ . We also learn that the intersection multiplicity at  $x_\lambda$  is  $\mu_\lambda = \frac{1}{2}(n - |\lambda|)$ .

Knowing the intersection multiplicity puts an upper bound on the order of a possible singular point of the curve at  $x_\lambda$ . In particular, it means that  $D_n$  is smooth at the points  $x_\lambda$  for the two extreme values of  $\lambda$ , namely  $\lambda = \pm(n - 2)$ , because the intersection multiplicity is 1 at those points.

A little experimentation suggests that equality holds at all the points  $x_\lambda$  where  $D_n$  meets  $\tau = 1$ : that is,

$$(96) \quad \text{ord}(D_n, x_\lambda) = \mu_\lambda = \frac{1}{2}(n - |\lambda|).$$

With the understanding that these results have been verified only experimentally for modest values of  $n$ , one can describe the singularity of  $D_n$  at  $x_\lambda$  in greater detail. First of all, we have seen that the ideal  $\bar{J}_{n,-1}$  which defines  $D_n$  has just two generators  $G_1(n)$  and  $G_2(n)$  (Proposition 6.9), and it follows that the singularity of  $D_n$  at  $x_\lambda$  is a local complete intersection. Indeed, each of  $D_n^+$  and  $D_n^-$  is cut out as a global complete intersection inside the variety defined by  $\epsilon = \pm 1$  and  $\tau \neq 0$ . Experiment also indicates that the surfaces defined by the vanishing of  $G_1(n)$  and  $G_2(n)$  are both smooth at  $x_\lambda$ . Indeed, the  $\alpha$ -derivative of both is nonzero. By the implicit function theorem, the zero-sets of  $G_1(n)$  and  $G_2(n)$  are therefore described in a local analytic neighborhood of  $x_\lambda$  by

$$\alpha = \lambda + f_{n,\lambda,1}(\delta, \tau) \quad \text{and} \quad \alpha = \lambda + f_{n,\lambda,2}(\delta, \tau)$$

for two analytic functions  $f_{n,\lambda,1}$  and  $f_{n,\lambda,2}$ . At the singular points (that is, when  $|\lambda| < n - 2$ ), the derivatives of both  $f_{n,\lambda,1}$  and  $f_{n,\lambda,2}$  vanish at  $(\delta, \tau) = (0, 1)$ . The singular germ  $(D_n, x_\lambda)$  is therefore analytically isomorphic to the germ of the analytic plane singularity

$$g_{n,\lambda}(\delta, \tau) = 0, \quad g_{n,\lambda} = f_{n,\lambda,1} - f_{n,\lambda,2},$$

at  $(\delta, \tau) = (0, 1)$ .

In computations up to  $n = 31$ , the function  $g_{n,\lambda}$  vanishes to order  $\mu_\lambda$  at  $(0, 1)$ , verifying that  $\mu_\lambda$  is indeed the order of the singular point. Furthermore we find

$$g_{n,\lambda}(\delta, \tau) = \text{const} \cdot (\delta \pm 2(\tau - 1))^{\mu_\lambda} + O(\delta, \tau - 1)^{\mu_\lambda + 1},$$

where the sign depends on  $\epsilon$  and  $\lambda$ . This means that the tangent cone to the singular point is the line  $\delta \pm 2(\tau - 1) = 0$ , with multiplicity  $\mu_\lambda$ .

The highest-order singular points on the curve are the points  $x_\lambda$  with  $\lambda = \pm 1$ , where the order of the singularity is  $m = \frac{1}{2}(n - 1)$ . At these points, the analytic form of the singularity is  $x^m = y^{m+1}$ , where  $x = \delta \pm 2(\tau - 1)$ . In particular the singularity is unibranch. The authors have not determined (even experimentally) whether the singularity is unibranch at other singular points. Note, however, that the entire

curves  $D_n^\pm$  are reducible when  $n$  is composite (as discussed below) and it follows that the singularities are not unibranch when  $\lambda$  and  $n$  have a common factor.

One further experimental observation is that the local form of the surface  $G_i(n) = 0$ , given by  $\alpha = \lambda + f_{n,\lambda,i}(\delta, \tau)$  at  $x_\lambda$ , appears to approach a smooth limit as  $n$  increases with  $\lambda$  fixed. Indeed, after scaling by  $\lambda$ , we find that the limit is independent of  $\lambda$  also. That is, there is a convergent power series  $F(\delta, \tau)$ , independent of  $n, \lambda$  and  $i = 1, 2$ , such that

$$\lambda + f_{n,\lambda,i}(\delta, \tau) \rightarrow \lambda F(\delta, \tau).$$

The difference vanishes at  $(0, 1)$  to order  $(\delta, \tau - 1)^{O(n)}$ . Up to terms of degree 5, the series  $F$  is

$$F(\delta, 1 + \sigma) = 1 - \frac{1}{16}\delta^2 + \frac{31}{4}\delta\sigma + \frac{1}{4}\sigma^2 - \frac{31}{8}\delta\sigma^2 - \frac{1}{4}\sigma^3 - \frac{5}{1024}\delta^4 + \frac{31}{128}\delta^3\sigma + \frac{5}{128}\delta^2\sigma^2 + \frac{31}{32}\delta\sigma^3 + \frac{15}{64}\sigma^4 - \frac{31}{256}\delta^3\sigma^2 - \frac{5}{128}\delta^2\sigma^3 + \frac{31}{64}\delta\sigma^4 - \frac{7}{32}\sigma^5 + \dots$$

### 7.2 Reducibility when $n$ is composite

The curves  $D_n^+$  and  $D_n^-$  arising as  $\text{Spec}(I(Z_{n,-1}))$  are irreducible when  $n$  is prime in all cases that the authors have calculated. It seems to be an interesting conjecture whether this holds in general. For composite  $n$ , however, the curves  $D_n^+$  and  $D_n^-$  are reducible, as the following result implies.

**Proposition 7.1** *If  $n'$  divides the odd integer  $n$ , then the curves  $D_n^+$  and  $D_n^-$  contain  $\psi(D_{n'}^+)$  and  $\psi(D_{n'}^-)$  respectively, where  $\psi$  is the map on the ambient space  $\bar{V}$  given by*

$$\psi(\tau, \tau^{-1}, \delta, \alpha, \epsilon) = (\tau, \tau^{-1}, \delta, (n/n')\alpha, \epsilon).$$

**Proof** This is an application of the general principal described by Proposition 6.10. In the context of that proposition, take  $W$  to be the product cobordism  $[0, 1] \times Z_{n'}$ . Write  $l = n/n'$ . We can embed a sphere  $S \hookrightarrow W$  representing  $l$  times the generator of  $H_2(W)$  and meeting the singular set in  $ln'$  points, all with the same orientation. The relevant map  $\Psi$  in Proposition 6.10 is then the homomorphism of algebras

$$\Psi_l: \mathcal{A}_n \rightarrow \mathcal{A}_{n'}$$

which is given (with our standardly named generators, and suitably numbering the intersection points) by

$$\Psi_l(\alpha) = l\alpha, \quad \Psi_l(\delta_k) = \delta_{(k \bmod n')}.$$

The conclusion of Proposition 6.10 is that we have an inclusion of ideals  $\Psi_l(\mathcal{J}_n) \subset \mathcal{J}_{n'}$ .

Passing to the quotient rings  $\bar{\mathcal{A}}$  in which all the  $\delta_k$  are equal, and using the fact that  $\bar{\mathcal{J}}_{n,-1}$  is the image of  $\mathcal{J}_n$  in the quotient ring (Proposition 5.13), we obtain an inclusion of ideals  $\psi_l(\bar{\mathcal{J}}_{n,-1}) \subset \bar{\mathcal{J}}_{n',-1}$  when  $n = ln'$ , where  $\psi_l$  is algebra homomorphism the with  $\psi_l(\alpha) = l\alpha$  and  $\psi_l(\delta) = \delta$ . Proposition 7.1 is just a restatement of this inclusion of ideals, in the geometrical language of the subschemes that they define.  $\square$

### 7.3 Interpretation as the quantum cohomology ring

For every odd  $n$ , the representation variety  $M = \text{Rep}(S_n^2)$  is naturally a smooth symplectic manifold, by a standard construction [11]. If  $n$  points in  $\mathbb{C}\mathbb{P}^1$  are chosen, then  $M$  becomes also a smooth complex-algebraic variety of dimension  $n - 3$ , as a consequence of its interpretation as a moduli space of stable parabolic bundles. With the symplectic form, it is a Kähler manifold, and the cohomology class of the Kähler form is a negative multiple of the canonical class. The latter assertion is the statement of “monotonicity” for the symplectic structure. It can be deduced as a particularly simple case from [20], for example, or it can be deduced from the fact that there is only one class in  $H^2$  which is invariant under the “flip” symmetries [32]. This is therefore a Fano variety. (A concrete description is discussed in [4].)

The quantum cohomology ring of such a Fano variety is defined using a deformation of the usual triple intersection product. Given cycles  $A, B, C$ , the quantum intersection product is a scalar which is a weighted count of isolated pseudoholomorphic curves  $u: \mathbb{C}\mathbb{P}^1 \rightarrow M$ , with the constraint that  $u$  maps three marked points to  $A, B$  and  $C$ . For our purposes, the weight will be of the form  $\tau^{[u] \cdot T}$  for a suitable 2-dimensional cohomology class  $T = 2 \sum \delta_i$ . This leads to a quantum cohomology ring  $\text{QH}(M)$  which is a module over the ring of Laurent polynomials  $\mathcal{R}$ . In the spirit of results from [27] and [7], one should expect that the  $\epsilon = 1$  component of  $I(Z_n)$  is isomorphic to  $\text{QH}(M)$  as an algebra.

The special case  $n = 5$  in particular is discussed in [31], where the symplectic manifold  $M$  is the blow-up of  $\mathbb{C}\mathbb{P}^2$  at five points, and the quantum cup-product is computed. Also relevant from [30; 31] is Seidel’s long exact sequence [31, Proposition 3.5]. In the special case that  $M$  is  $\mathbb{C}\mathbb{P}^2 \# 5\overline{\mathbb{C}\mathbb{P}^2}$ , this long exact sequence essentially recovers the skein exact sequence in the proof of Proposition 3.18, involving the orbifold  $X_{5,4}$  from Figure 3, restricted to the  $+1$  eigenspace of  $\epsilon$ . The orbifold  $X_{5,4}$  plays the role of  $H_*(S^2; \Lambda)$  in [31, Proposition 3.5]. Seidel’s exact sequence is generalized by Wehrheim and Woodward in [34, Theorem 6.12], motivated by the application to skein triangles, and the generalization is relevant to the case of the skein triangle involving  $X_{n,n-1}$  for larger  $n$ .

### 7.4 General local coefficients

As an alternative to the local coefficient system  $\Gamma$  for  $I(Z_n)$ , there is a larger local coefficient system  $\Gamma_n$  that can be used. Rather than being a system of rank-1 modules over  $\mathcal{R} = \mathbb{Q}[\tau^{-1}, \tau]$ , the ground ring for  $\Gamma_n$  is the ring of finite Laurent series in  $n$  distinct variables  $\tau_1, \dots, \tau_n$  attached to the  $n$  components of the singular set of  $Z_n$ :

$$\mathcal{R}_n = \mathbb{Q}[\tau_1, \tau_1^{-1}, \dots, \tau_n, \tau_n^{-1}].$$

The instanton homology  $I(Z_n; \Gamma_n)$  is then a module over the ring

$$\mathcal{R}_n[\delta_1, \dots, \delta_n, \alpha, \epsilon].$$

It is no longer true that  $\delta_i^2 = \delta_j^2$ ; instead we have

$$\delta_i^2 - \tau_i^2 - \tau_i^{-2} = \delta_j^2 - \tau_j^2 - \tau_j^{-2} \quad \text{for all } i, j.$$

It should be possible to compute  $I(Z_n; \Gamma_n)$  by adapting the ideas of this paper. As the simplest example, our two generators for the relations in  $I(Z_3, -1)$ , where all  $\delta_i$  and all  $\tau_i$  are equal, were

$$\alpha + \frac{3}{2}\delta + \epsilon\tau^3 \quad \text{and} \quad \alpha - \frac{1}{2}\delta + \epsilon\tau^{-1}.$$

For  $I(Z_3; \Gamma_3)$  the corresponding relations are

$$\alpha + \frac{1}{2}(\delta_1 + \delta_2 + \delta_3) + \epsilon\tau_1\tau_2\tau_3 \quad \text{and} \quad \alpha + \frac{1}{2}(\delta_1 - \delta_2 - \delta_3) + \epsilon\tau_1\tau_2^{-1}\tau_3^{-1},$$

together with cyclic rotations of the second one. The instanton homology  $I(Z_3; \Gamma_3)$  is a free  $\mathcal{R}_3$ -module of rank 2.

There is an additional symmetry present when using  $\Gamma_n$ , which comes from the flip relation. So the ideal of generators is invariant under the symmetry which changes the sign of  $\delta_i$  and  $\delta_j$  for any two distinct indices while changing  $\tau_i$  and  $\tau_j$  to  $\tau_i^{-1}$  and  $\tau_j^{-1}$ . In the example of  $I(Z_3; \Gamma_3)$  there are four generators corresponding to the four subsets  $\eta \subset \{1, 2, 3\}$  of even parity, and the corresponding relations are all obtained from the first one (corresponding to  $\eta = \emptyset$ ) by applying flips. For larger  $n$ , the leading and subleading terms follow the same pattern. So the adaptation of [Proposition 5.2](#) to the case of  $\Gamma_n$  has the same leading term while the factor of  $\tau^{n-2h}$  in front of the subleading term is replaced by

$$\prod_{i \notin \eta} \tau_i \prod_{i \in \eta} \tau_i^{-1}.$$

## 7.5 Instanton homology for torus knots

As mentioned in the introduction, a motivation for this paper comes from wishing to calculate variants of framed instanton homology for torus knots. In [\[24\]](#), concordance invariants of knots were defined using a version of framed instanton homology  $I^\sharp$ . In that paper, for a knot  $K \subset Y$ , the framed instanton homology is defined using the connected sum  $(Y, K) \# (S^3, \Theta)$ , where  $\Theta$  is a theta-graph in  $S^3$ . A local coefficient system is used in [\[24\]](#), where the ground ring is the Laurent polynomials in three variables  $\tau_i$  corresponding to the three edges of  $\Theta$ . Because of the phenomenon of bubbling in codimension 2 which arises from the vertices of  $\Theta$ , it was necessary in [\[24\]](#) to use a ring of characteristic 2.

It is possible instead to work in characteristic zero by abandoning the pair  $(S^3, \Theta)$  and using the pair  $Z_3$  instead (as described just above). The local coefficient system comes from  $\Gamma_3$ . Because  $I(Z_3; \Gamma_3)$  has rank 2, one should take just the  $+1$  eigenspace of  $\epsilon$  to obtain a rank-1 module. Thus one can define  $I^\sharp(Z; \Gamma_3)$  for general bifolds  $Z$  as being  $I(Z \# Z_3; \Gamma_3)_+$ . The connected sum is of the 3-manifolds, not a connected sum of pairs. But a connected sum of pairs can be used instead to define a reduced version  $I^\natural(Z; \Gamma_3)$ .

A variant of the connected sum theorem from [\[5\]](#) allows one to pass to  $I^\natural(Z_{n,-1}; \Gamma_3)$  starting from the calculation of  $I(Z_{n,-1})$  in this paper. Using the surgery exact triangle for instanton homology, one can therefore take the calculation of  $I(Z_{n,-1})$  as a first step towards understanding the reduced instanton homology with local coefficients for torus knots in  $S^3$ . The authors hope to return to this in a future paper.

### 7.6 Universal relations

The relations in the instanton homology of  $Z_n$  and  $Z_{n,-1}$  give rise to universal relations for general admissible bifolds  $(Y, K)$  containing spheres. The following is an illustration.

**Proposition 7.2** *Let  $(Y, K)$  be a bifold and suppose that the singular set  $K$  is a knot meeting an embedded sphere  $S \subset Y$  transversely with odd geometric intersection number  $n$  and algebraic intersection number  $n'$ . Orient the sphere and  $K$  so that  $0 < n' \leq n$ . Let  $\alpha$  be the operator on  $I(Y, K)$  corresponding the sphere  $S$  and let  $\delta$  be the operator arising from a point on  $K$ . Let  $\epsilon^*$  be the involution on  $I(Y, K)$  arising from  $S$ , and let  $\epsilon = (-1)^{(n-n')/2} \epsilon^*$ . Let  $*$  denote the automorphism of the algebra  $\bar{\mathcal{A}}_n$  determined by  $*$ :  $\epsilon \mapsto \epsilon^*$ . Then the elements of the ideal*

$$*(\tau^4 - 1)^{(n-n')/2} \bar{\mathcal{J}}_{n',-1} \subset \mathcal{R}[\delta, \alpha, \epsilon]/\langle \epsilon^2 - 1 \rangle$$

annihilate  $I(Y, K)$ .

**Proof** Let  $\delta_1, \dots, \delta_n$  be the operators corresponding the intersection points of  $K$  with  $S$ , all oriented with the normal orientation to  $S$ . From an application of the general principle of Proposition 6.10, the instanton homology  $I(Y, K)$  is annihilated by the ideal  $\mathcal{J}_n$  in the algebra  $\mathcal{A}_n$ . On the other hand, because  $K$  is a knot, all the operators  $\delta_i$  are equal up to sign, so the action of the algebra  $\mathcal{A}_n$  factors through the quotient  $\bar{\mathcal{A}} = \mathcal{R}[\delta, \alpha, \epsilon]/\langle \epsilon^2 - 1 \rangle$  in which we set  $\delta_i = \pm\delta$  according to the sign of the corresponding intersection point of  $K$  with  $S$ . From Corollaries 6.11 and 6.13 the image of  $\mathcal{J}_n$  in the quotient contains the ideal described in the proposition. □

As a simplest example, if  $K$  is a knot in  $Y = S^1 \times S^2$  which has geometric intersection 3 and algebraic intersection 1 with  $S^2$ , then  $I(Y, K)$  is a torsion  $\mathcal{R}$ -module annihilated by  $\tau^4 - 1$ . In general, the proposition provides a lower bound on the geometric intersection number of  $K$  and  $S^2$ .

**Corollary 7.3** *Let  $Y$  contain an oriented 2-sphere  $S$ , and let  $K \subset Y$  be a knot having odd algebraic intersection number  $n' > 0$  with  $S$ . Then a lower bound for the transverse geometric intersection number  $K \cap S$  for any knot isotopic to  $K$  is  $n' + 2f$ , where*

$$f = \min\{F \geq 0 \mid (\tau^4 - 1)^F G_i(n') \text{ annihilates } I(Y, K) \text{ for } i = 1, 2\}.$$

Here  $G_1(n')$  and  $G_2(n')$  are the two generators in Proposition 6.9. □

In light of the results from [36] concerning higher-genus orbifolds, it is possible that the bound  $n' + 2f$  defined in the corollary is not particularly strong. It may be that  $n' + 2f$  is a lower bound for  $n_g + 2g$ , where  $n_g$  is the geometric intersection number with a surface  $S_g$  of genus  $g$  homologous to  $S$ . It is easy to visualize examples where  $n_1 + 2$  is much smaller than  $n_0$ , for example.

In the case that  $n = n'$  in Proposition 7.2 (ie when algebraic and geometric intersection numbers are equal), the  $\bar{\mathcal{A}}$ -module  $I(Y, K)$  is annihilated by the defining ideal of the curve  $D_n$ . This means that we can interpret  $I(Y, K)$  as a coherent sheaf on  $D_n$ .

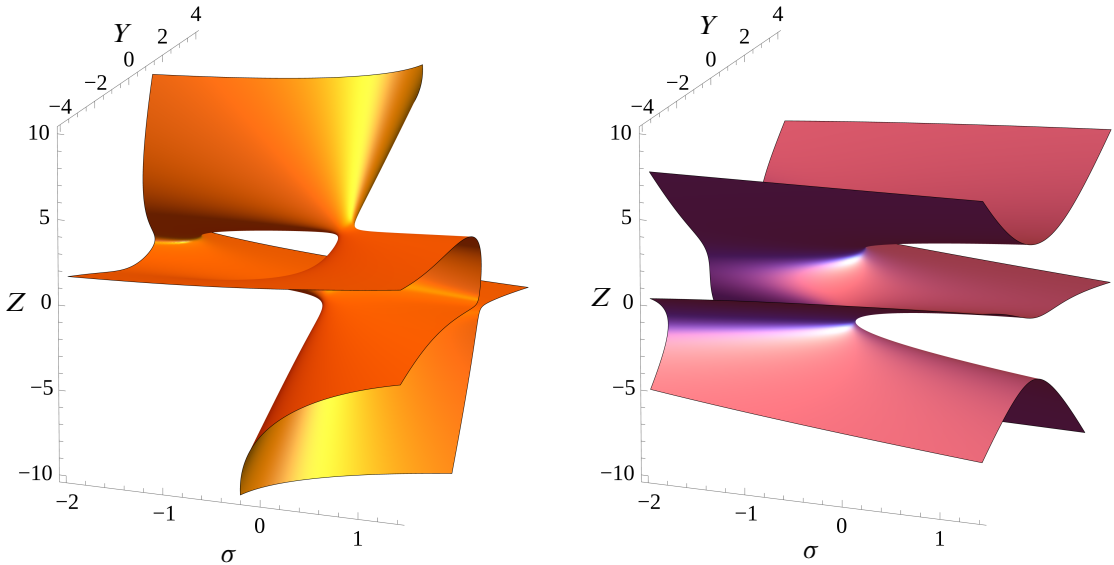


Figure 4: The real loci defined by the vanishing of the generators  $G_1(n)$ , left, and  $G_2(n)$ , right, for  $n = 7$  in the coordinates  $(\sigma, Y, Z)$ . Only the part with  $\epsilon = 1$  is shown. The part with  $\epsilon = -1$  is obtained by changing the sign of  $Y$  and  $Z$ . These are smooth affine cubic surfaces.

## 7.7 The degrees of the relations

The two generators  $G_1(n)$ ,  $G_2(n)$  for the ideal of relations for  $I(Z_{n,-1})$  both have total degree  $m = \frac{1}{2}(n-1)$  in  $(\alpha, \delta)$  but larger degree in  $\tau$ . However, a substitution simplifies the polynomials a little: if we substitute

$$Z = \tau\alpha \quad \text{and} \quad Y = \tau\delta$$

then, after clearing unnecessary powers of  $\tau$  from the denominator, we obtain a polynomial in  $Z$ ,  $Y$  and  $\tau^4$ . Writing  $\sigma = \tau^4$ , the total degree of the generators  $G_i(n)$  in  $(\sigma, Z, Y)$  is  $m$ . The real loci defined by the vanishing of these two polynomials in  $(\sigma, Y, Z)$  are shown in Figure 4 for  $n = 7$ .

## References

- [1] **MF Atiyah, R Bott**, *The Yang–Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. A 308 (1983) 523–615 [MR](#) [Zbl](#)
- [2] **I Biswas, N Raghavendra**, *Canonical generators of the cohomology of moduli of parabolic bundles on curves*, Math. Ann. 306 (1996) 1–14 [MR](#) [Zbl](#)
- [3] **HU Boden**, *Representations of orbifold groups and parabolic bundles*, PhD thesis, Brandeis University (1991) Available at <https://www.proquest.com/docview/303936590>
- [4] **C Casagrande**, *Rank 2 quasiparabolic vector bundles on  $\mathbb{P}^1$  and the variety of linear subspaces contained in two odd-dimensional quadrics*, Math. Z. 280 (2015) 981–988 [MR](#) [Zbl](#)
- [5] **A Daemi, C Scaduto**, *Equivariant aspects of singular instanton Floer homology*, preprint (2019) [arXiv 1912.08982](#)

- [6] **S K Donaldson**, *The orientation of Yang–Mills moduli spaces and 4-manifold topology*, J. Differential Geom. 26 (1987) 397–428 [MR](#) [Zbl](#)
- [7] **S Dostoglou, D A Salamon**, *Self-dual instantons and holomorphic curves*, Ann. of Math. 139 (1994) 581–640 [MR](#) [Zbl](#)
- [8] **R Earl, F Kirwan**, *Complete sets of relations in the cohomology rings of moduli spaces of holomorphic bundles and parabolic bundles over a Riemann surface*, Proc. Lond. Math. Soc. 89 (2004) 570–622 [MR](#) [Zbl](#)
- [9] **D Eisenbud**, *The geometry of syzygies: a second course in commutative algebra and algebraic geometry*, Grad. Texts in Math. 229, Springer (2005) [MR](#) [Zbl](#)
- [10] **R Fintushel, R J Stern**, *Instanton homology of Seifert fibred homology three spheres*, Proc. Lond. Math. Soc. 61 (1990) 109–137 [MR](#) [Zbl](#)
- [11] **W M Goldman**, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. 54 (1984) 200–225 [MR](#) [Zbl](#)
- [12] **Y-H Kiem**, *On the redundancy of the Mumford relations*, J. Algebraic Geom. 9 (2000) 395–402 [MR](#) [Zbl](#)
- [13] **P A Kirk, E P Klassen**, *Representation spaces of Seifert fibered homology spheres*, Topology 30 (1991) 77–95 [MR](#) [Zbl](#)
- [14] **F Kirwan**, *The cohomology rings of moduli spaces of bundles over Riemann surfaces*, J. Amer. Math. Soc. 5 (1992) 853–906 [MR](#) [Zbl](#)
- [15] **U Koschorke**, *Infinite-dimensional K-theory and characteristic classes of Fredholm bundle maps*, from “Global analysis” (S-s Chern, S Smale, editors), Proc. Sympos. Pure Math. 15, Amer. Math. Soc., Providence, RI (1970) 95–133 [MR](#) [Zbl](#)
- [16] **P B Kronheimer**, *An obstruction to removing intersection points in immersed surfaces*, Topology 36 (1997) 931–962 [MR](#) [Zbl](#)
- [17] **P B Kronheimer, T S Mrowka**, *Gauge theory for embedded surfaces, I*, Topology 32 (1993) 773–826 [MR](#) [Zbl](#)
- [18] **P B Kronheimer, T S Mrowka**, *Gauge theory for embedded surfaces, II*, Topology 34 (1995) 37–97 [MR](#) [Zbl](#)
- [19] **P B Kronheimer, T S Mrowka**, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. 113 (2011) 97–208 [MR](#) [Zbl](#)
- [20] **P B Kronheimer, T S Mrowka**, *Knot homology groups from instantons*, J. Topol. 4 (2011) 835–918 [MR](#) [Zbl](#)
- [21] **P B Kronheimer, T S Mrowka**, *A deformation of instanton homology for webs*, Geom. Topol. 23 (2019) 1491–1547 [MR](#) [Zbl](#)
- [22] **P B Kronheimer, T S Mrowka**, *Tait colorings, and an instanton homology for webs and foams*, J. Eur. Math. Soc. 21 (2019) 55–119 [MR](#) [Zbl](#)
- [23] **P B Kronheimer, T S Mrowka**, *Instantons and Bar-Natan homology*, Compos. Math. 157 (2021) 484–528 [MR](#) [Zbl](#)
- [24] **P B Kronheimer, T S Mrowka**, *Instantons and some concordance invariants of knots*, J. Lond. Math. Soc. 104 (2021) 541–571 [MR](#) [Zbl](#)

- [25] **V B Mehta, C S Seshadri**, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. 248 (1980) 205–239 [MR](#) [Zbl](#)
- [26] **E Miller, B Sturmfels**, *Combinatorial commutative algebra*, Grad. Texts in Math. 227, Springer (2005) [MR](#) [Zbl](#)
- [27] **V Muñoz**, *Quantum cohomology of the moduli space of stable bundles over a Riemann surface*, Duke Math. J. 98 (1999) 525–540 [MR](#) [Zbl](#)
- [28] **V Muñoz**, *Ring structure of the Floer cohomology of  $\Sigma \times \mathbb{S}^1$* , Topology 38 (1999) 517–528 [MR](#) [Zbl](#)
- [29] **I R Porteous**, *Simple singularities of maps*, from “Proceedings of Liverpool Singularities: symposium I” (C T C Wall, editor), Lecture Notes in Math. 192, Springer (1971) 286–307 [MR](#) [Zbl](#)
- [30] **P Seidel**, *A long exact sequence for symplectic Floer cohomology*, Topology 42 (2003) 1003–1063 [MR](#) [Zbl](#)
- [31] **P Seidel**, *Lectures on four-dimensional Dehn twists*, from “Symplectic 4-manifolds and algebraic surfaces” (F Catanese, G Tian, editors), Lecture Notes in Math. 1938, Springer (2008) 231–267 [MR](#) [Zbl](#)
- [32] **E J Street**, *Towards an instanton Floer homology for tangles*, PhD thesis, Harvard University (2012) Available at <https://www.proquest.com/docview/1027897859>
- [33] **M Thaddeus**, *Conformal field theory and the cohomology of the moduli space of stable bundles*, J. Differential Geom. 35 (1992) 131–149 [MR](#) [Zbl](#)
- [34] **K Wehrheim, C T Woodward**, *Exact triangle for fibered Dehn twists*, Res. Math. Sci. 3 (2016) art. id. 17 [MR](#) [Zbl](#)
- [35] **J Weitsman**, *Geometry of the intersection ring of the moduli space of flat connections and the conjectures of Newstead and Witten*, Topology 37 (1998) 115–132 [MR](#) [Zbl](#)
- [36] **Y Xie, B Zhang**, *Instanton Floer homology for sutured manifolds with tangles*, preprint (2019) [arXiv 1907.00547](#)
- [37] **D Zagier**, *On the cohomology of moduli spaces of rank two vector bundles over curves*, from “The moduli space of curves” (R Dijkgraaf, C Faber, G van der Geer, editors), Progr. Math. 129, Birkhäuser, Boston, MA (1995) 533–563 [MR](#) [Zbl](#)

Department of Mathematics, Harvard University  
Cambridge, MA, United States

Department of Mathematics, Massachusetts Institute of Technology  
Cambridge, MA, United States

[kronheim@math.harvard.edu](mailto:kronheim@math.harvard.edu), [mrowka@math.mit.edu](mailto:mrowka@math.mit.edu)

Proposed: Simon Donaldson

Seconded: Martin R Bridson, Ian Agol

Received: 29 November 2022

Revised: 2 November 2023

# GEOMETRY & TOPOLOGY

[msp.org/gt](http://msp.org/gt)

## MANAGING EDITORS

Robert Lipshitz University of Oregon  
[lipshitz@uoregon.edu](mailto:lipshitz@uoregon.edu)  
András I Stipsicz Alfréd Rényi Institute of Mathematics  
[stipsicz@renyi.hu](mailto:stipsicz@renyi.hu)

## BOARD OF EDITORS

Mohammed Abouzaid	Stanford University <a href="mailto:abouzaid@stanford.edu">abouzaid@stanford.edu</a>	Mark Gross	University of Cambridge <a href="mailto:mgross@dpms.cam.ac.uk">mgross@dpms.cam.ac.uk</a>
Dan Abramovich	Brown University <a href="mailto:dan_abramovich@brown.edu">dan_abramovich@brown.edu</a>	Rob Kirby	University of California, Berkeley <a href="mailto:kirby@math.berkeley.edu">kirby@math.berkeley.edu</a>
Ian Agol	University of California, Berkeley <a href="mailto:ianagol@math.berkeley.edu">ianagol@math.berkeley.edu</a>	Bruce Kleiner	NYU, Courant Institute <a href="mailto:bkleiner@cims.nyu.edu">bkleiner@cims.nyu.edu</a>
Arend Bayer	University of Edinburgh <a href="mailto:arend.bayer@ed.ac.uk">arend.bayer@ed.ac.uk</a>	Sándor Kovács	University of Washington <a href="mailto:skovacs@uw.edu">skovacs@uw.edu</a>
Mark Behrens	University of Notre Dame <a href="mailto:mbehren1@nd.edu">mbehren1@nd.edu</a>	Urs Lang	ETH Zürich <a href="mailto:urs.lang@math.ethz.ch">urs.lang@math.ethz.ch</a>
Mladen Bestvina	University of Utah <a href="mailto:bestvina@math.utah.edu">bestvina@math.utah.edu</a>	Marc Levine	Universität Duisburg-Essen <a href="mailto:marc.levine@uni-due.de">marc.levine@uni-due.de</a>
Martin R Bridson	University of Oxford <a href="mailto:bridson@maths.ox.ac.uk">bridson@maths.ox.ac.uk</a>	Ciprian Manolescu	University of California, Los Angeles <a href="mailto:cm@math.ucla.edu">cm@math.ucla.edu</a>
Jim Bryan	University of British Columbia <a href="mailto:jbryan@math.ubc.ca">jbryan@math.ubc.ca</a>	Haynes Miller	Massachusetts Institute of Technology <a href="mailto:hmr@math.mit.edu">hmr@math.mit.edu</a>
Dmitri Burago	Pennsylvania State University <a href="mailto:burago@math.psu.edu">burago@math.psu.edu</a>	Tomasz Mrowka	Massachusetts Institute of Technology <a href="mailto:mrowka@math.mit.edu">mrowka@math.mit.edu</a>
Tobias H Colding	Massachusetts Institute of Technology <a href="mailto:colding@math.mit.edu">colding@math.mit.edu</a>	Aaron Naber	Northwestern University <a href="mailto:anaber@math.northwestern.edu">anaber@math.northwestern.edu</a>
Simon Donaldson	Imperial College, London <a href="mailto:s.donaldson@ic.ac.uk">s.donaldson@ic.ac.uk</a>	Peter Ozsváth	Princeton University <a href="mailto:petero@math.princeton.edu">petero@math.princeton.edu</a>
Yasha Eliashberg	Stanford University <a href="mailto:eliash-gt@math.stanford.edu">eliash-gt@math.stanford.edu</a>	Leonid Polterovich	Tel Aviv University <a href="mailto:polterov@post.tau.ac.il">polterov@post.tau.ac.il</a>
Benson Farb	University of Chicago <a href="mailto:farb@math.uchicago.edu">farb@math.uchicago.edu</a>	Colin Rourke	University of Warwick <a href="mailto:gt@maths.warwick.ac.uk">gt@maths.warwick.ac.uk</a>
David M Fisher	Rice University <a href="mailto:davidfisher@rice.edu">davidfisher@rice.edu</a>	Roman Sauer	Karlsruhe Institute of Technology <a href="mailto:roman.sauer@kit.edu">roman.sauer@kit.edu</a>
Mike Freedman	Microsoft Research <a href="mailto:michaelf@microsoft.com">michaelf@microsoft.com</a>	Stefan Schwede	Universität Bonn <a href="mailto:schwede@math.uni-bonn.de">schwede@math.uni-bonn.de</a>
David Gabai	Princeton University <a href="mailto:gabai@princeton.edu">gabai@princeton.edu</a>	Natasa Sesum	Rutgers University <a href="mailto:natasas@math.rutgers.edu">natasas@math.rutgers.edu</a>
Stavros Garoufalidis	Southern U. of Sci. and Tech., China <a href="mailto:stavros@mpim-bonn.mpg.de">stavros@mpim-bonn.mpg.de</a>	Gang Tian	Massachusetts Institute of Technology <a href="mailto:tian@math.mit.edu">tian@math.mit.edu</a>
Cameron Gordon	University of Texas <a href="mailto:gordon@math.utexas.edu">gordon@math.utexas.edu</a>	Ulrike Tillmann	Oxford University <a href="mailto:tillmann@maths.ox.ac.uk">tillmann@maths.ox.ac.uk</a>
Jesper Grodal	University of Copenhagen <a href="mailto:jg@math.ku.dk">jg@math.ku.dk</a>	Nathalie Wahl	University of Copenhagen <a href="mailto:wahl@math.ku.dk">wahl@math.ku.dk</a>
Misha Gromov	IHÉS and NYU, Courant Institute <a href="mailto:gromov@ihes.fr">gromov@ihes.fr</a>	Anna Wienhard	Universität Heidelberg <a href="mailto:wienhard@mathi.uni-heidelberg.de">wienhard@mathi.uni-heidelberg.de</a>

---

See inside back cover or [msp.org/gt](http://msp.org/gt) for submission instructions.

The subscription price for 2025 is US \$865/year for the electronic version, and \$1210/year (+\$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Geometry & Topology is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

---

GT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing  
<http://msp.org/>

© 2025 Mathematical Sciences Publishers

# GEOMETRY & TOPOLOGY

Volume 29 Issue 4 (pages 1693–2250) 2025

---

- The conjugacy problem for UPG elements of  $\text{Out}(F_n)$  1693  
MARK FEIGNH and MICHAEL HANDEL
- The systole of large genus minimal surfaces in positive Ricci curvature 1819  
HENRIK MATTHIESEN and ANNA SIFFERT
- The Manhattan curve, ergodic theory of topological flows and rigidity 1851  
STEPHEN CANTRELL and RYOKICHI TANAKA
- Realizability in tropical geometry and unobstructedness of Lagrangian submanifolds 1909  
JEFFREY HICKS
- Relations in singular instanton homology 1975  
PETER B KRONHEIMER and TOMASZ S MROWKA
- Holonomic Poisson geometry of Hilbert schemes 2047  
MYKOLA MATVIICHUK, BRENT PYM and TRAVIS SCHEDLER
- Mutations and faces of the Thurston norm ball dynamically represented by multiple distinct flows 2105  
ANNA PARLAK
- Unit inclusion in a (nonsemisimple) braided tensor category and (noncompact) relative TQFTs 2175  
BENJAMIN HAÏOUN
- Closed geodesics in dilation surfaces 2217  
ADRIEN BOULANGER, SELIM GHAZOUANI and GUILLAUME TAHAR