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Holonomic Poisson geometry of Hilbert schemes

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We undertake a detailed study of the geometry of Bottacin’s Poisson structures on Hilbert schemes of points in Poisson surfaces, ie smooth complex surfaces equipped with an effective anticanonical divisor. We focus on three themes that, while logically independent, are linked by the interplay between (characteristic) symplectic leaves and deformation theory. Firstly, we construct the symplectic groupoids of the Hilbert schemes and develop the classification of their symplectic leaves, using the methods of derived symplectic geometry. Secondly, we establish local normal forms for the Poisson brackets, and combine them with a toric degeneration argument to verify that Hilbert schemes satisfy our recent conjecture characterizing holonomic Poisson manifolds in terms of the geometry of the modular vector field. Finally, using constructible sheaf methods, we compute the space of first-order Poisson deformations when the anticanonical divisor is reduced and has only quasihomogeneous singularities. (The latter is automatic if the surface is projective.) Along the way, we find a tight connection between the Poisson geometry of the Hilbert schemes and the finite-dimensional Lie algebras of affine transformations, which is mediated by syzygies. In particular, we find that the Hilbert scheme has a natural subvariety that serves as a global counterpart of the nilpotent cone, and we prove that the Lie algebras of affine transformations have holonomic dual spaces — the first such series of Lie algebras to be discovered.

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1. Introduction	2047
2. Hilbert groupoids of curves in surfaces	2052
3. Symplectic and Poisson structures	2068
4. Local models and holonomicity	2075
5. Deformation theory	2095
References	2101

1 Introduction

Let X be a Poisson surface, ie a smooth connected complex variety or analytic space of dimension two, equipped with a holomorphic Poisson bivector (a section of the anticanonical line bundle). In [5], Bottacin constructed a natural Poisson structure on the Hilbert scheme $\mathcal{X}^{[n]}$ parametrizing zero-dimensional

subschemes of X , generalizing the celebrated symplectic structure on the Hilbert schemes of K3 and abelian surfaces due to Beauville [2, Section 6] and Mukai [28]. Concretely, this Poisson manifold is obtained by resolving the singularities of the symmetric power $X^{(n)} = X^n/S_n$, but it also has a natural interpretation as a moduli space. Our goal in this paper is to develop some of the beautiful geometry of Bottacin's Poisson structures, focusing on the following three themes:

- (1) symplectic leaves and symplectic groupoids,
- (2) local normal forms and holonomicity,
- (3) deformation theory.

In so doing, we encounter connections with a range of topics, from combinatorial linear algebra and Lie theory, to D-modules and derived algebraic geometry.

As we shall see, the three themes listed above are closely intertwined on a conceptual level. However, the results we establish for each of them (summarized in Sections 1.1 through 1.3 below, respectively) are based on different techniques and are essentially logically independent. Thus, for instance, while we make use of the formalism of derived geometry and shifted symplectic structures for the first theme, and this informs our intuition throughout the paper, the results in the second theme are proven using classical techniques of Poisson geometry and toric degeneration, and those in the third theme use constructible sheaves.

1.1 Symplectic leaves and the Hilbert groupoid

Bottacin's description of $X^{[n]}$ as a moduli space gives rise to an explicit description of the Poisson bivector as a pairing on Ext groups, which when combined with a deformation-theoretic argument, leads to a modular description of its symplectic leaves. Namely, the vanishing locus of the Poisson structure on X is a curve $D \subset X$, and two elements $Z_1, Z_2 \in X^{[n]}$ lie on the same symplectic leaf if and only if the restrictions of their ideal sheaves to D are isomorphic as \mathcal{O}_D -modules; see eg Pym and Rains [36] and Rains [38; 39], where such results are established in the more general context of moduli spaces of perfect complexes on noncommutative surfaces, although for Hilbert schemes we obtain this by different means below, as a direct consequence of [Theorem 1.1](#).

It is natural to ask for a geometric (rather than module-theoretic) interpretation of the leaves. For instance, one can check that the map sending an element $Z \in X^{[n]}$ to the intersection $Z \cap D$ is invariant under Hamiltonian flow, and hence if two elements Z_1, Z_2 lie on the same symplectic leaf, we must have $Z_1 \cap D = Z_2 \cap D$. Over the open set of $X^{[n]}$ corresponding to collections of n distinct points of X , this characterizes the symplectic leaves completely, but the situation in general is not so simple. Indeed, even for a smooth curve D , there exist pairs $Z_1, Z_2 \in X^{[n]}$ that have the same scheme-theoretic intersection with D , but lie on different symplectic leaves.

The reason for this subtlety is that the intersection $Z \cap D$, if nonempty, is never transverse. Hence, it is natural to consider the derived intersection $Z \cap^h D$, which enhances the naive algebra of functions $\mathcal{O}_{Z \cap D} \cong \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_D$ to the dg algebra $\mathcal{O}_{Z \cap^h D} := \mathcal{O}_Z \otimes_{\mathcal{O}_X}^L \mathcal{O}_D$, whose cohomology gives the Tor modules $\text{Tor}_{\bullet}^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_D)$. An important feature is that the derived intersection $Z \cap^h D$ can have nontrivial automorphisms that preserve the embedding $Z \cap^h D \hookrightarrow D$, so that the notion of the equality $Z_1 \cap D = Z_2 \cap D$ of ordinary scheme-theoretic intersections is enhanced to a whole space of equivalences $Z_1 \cap^h D \sim Z_2 \cap^h D$ of derived subschemes of D , which we call *D-equivalences* of the pair (Z_1, Z_2) ; these turn out to be the same as determinant-one \mathcal{O}_D -module isomorphisms between the restrictions of their ideal sheaves to D . These equivalences can be composed, giving a groupoid $(X, D)^{[n]}$ whose objects are elements of $X^{[n]}$ and whose morphisms are homotopy classes of *D-equivalences*. We call this groupoid the *Hilbert groupoid of the pair* (X, D) , and develop its structure in Sections 2 and 3 below. We summarize these results as follows:

Theorem 1.1 *Let X be a smooth complex surface, and let $D \subset X$ be any curve (which may be singular or even nonreduced). Then $(X, D)^{[n]}$ naturally has the structure of a smooth groupoid (ie a complex Lie groupoid), which is birationally equivalent to the product $X^{[n]} \times X^{[n]}$. If, in addition, the curve D is anticanonical, then a choice of Poisson structure on X whose vanishing locus is D endows $(X, D)^{[n]}$ with a natural symplectic structure, making it into a symplectic groupoid that integrates Bottacin’s Poisson structure on $X^{[n]}$. In particular, the symplectic leaves are the connected components of the *D-equivalence classes*, and these are locally closed subvarieties of $X^{[n]}$.*

We remark that although the output of the construction is an ordinary smooth variety, the proof we present takes full advantage of the derived geometry toolbox, particularly foundational results on moduli stacks of sheaves and shifted symplectic structures from Brav and Dyckerhoff [6], Calaque [9], Pantev, Toën, Vaquié and Vezzosi [34], Toën and Vaquié [46] and Schürg, Toën and Vezzosi [43]. Such an approach naturally (and rather easily) constructs $(X, D)^{[n]}$ as a derived stack; our main new addition is a representability statement, which shows that this derived stack is in fact a smooth variety. Combining this with normal forms for sheaves on smooth and nodal curves, we obtain a classification of symplectic leaves in the case that D has only nodal singularities. For smooth curves, this problem was also treated in [38, Section 11.2]; we expand on the results in [38], emphasizing connections with the combinatorics of Young diagrams and monomial ideals.

1.2 Local models and holonomicity

In Section 4 we turn to our second theme, namely the construction of explicit local models for the Poisson manifold $X^{[n]}$ and its symplectic groupoid.

The guiding principle here is that the derived intersection $Z \cap^h D$ can be described in purely classical terms using the Hilbert–Burch resolution of the structure sheaf \mathcal{O}_Z , which expresses the ideal defining Z locally as the maximal minors of a $(k + 1) \times k$ matrix with entries in \mathcal{O}_X (the “syzygy matrix”). As

explained by Ellingsrud and Strømme [12] one can construct natural coordinates on $\mathcal{X}^{[n]}$ by choosing suitable normal forms for the syzygy matrices. We use these coordinates to establish a tight connection between the local structure of the Poisson manifolds $\mathcal{X}^{[n]}$ and the Lie algebras $\mathfrak{aff}_k(\mathbb{C}) \cong \mathfrak{gl}_k(\mathbb{C}) \ltimes \mathbb{C}^k$ of affine transformations of \mathbb{C}^k , for $k \geq 1$. Indeed, by passing to open subsets and taking products, one can reduce the problem of understanding the local structure of the Hilbert schemes to the case of open balls around the elements $Z = p(k) \in \mathcal{X}^{[n]}$ that correspond to the k^{th} order neighbourhoods of points $p \in X$, where $k \geq 1$, for which we establish the following description.

Theorem 1.2 *A choice of log Darboux coordinates on X at p induces an analytic Poisson isomorphism from a neighbourhood of $p(k)$ in $\mathcal{X}^{[k(k+1)/2]}$ to a neighbourhood of the origin in $\mathfrak{aff}_k(\mathbb{C})^\vee$, where the latter is equipped with the following Poisson structure:*

- a constant symplectic structure with explicit Darboux coordinates, if $p \in X \setminus D$,
- the Lie–Poisson structure, if p is a smooth point of D , or
- a Lie-compatible quadratic Poisson structure, if p is a node of D .

As a consequence, when D is smooth, the symplectic groupoid $\mathfrak{aff}_k(\mathbb{C})^\vee \rtimes \text{Aff}_k(\mathbb{C})$ of $\mathfrak{aff}_k(\mathbb{C})^\vee$ locally integrates the Poisson structure; in fact, we show that this is a local model for the groupoid $(X, D)^{[n]}$ constructed in Theorem 1.1, when $n = \frac{1}{2}k(k+1)$. Another interesting aspect of this case is that we can identify a natural subvariety in $\mathcal{X}^{[k(k+1)/2]}$ that globalizes the embedding $\mathfrak{gl}_k(\mathbb{C})^\vee \subseteq \mathfrak{aff}_k(\mathbb{C})^\vee$, namely the locus W of schemes whose intersection with D has length k . The resulting map $W \rightarrow \text{Sym}^k D$ locally encodes the universal Poisson deformation of the nilpotent cone, giving a link to Springer theory—in particular, the fibre over $k \cdot p$ is a global version of the nilpotent cone; see Remark 4.9.

Of the three cases in Theorem 1.2, the third one is the most subtle. The quadratic Poisson structure in question seems to be new, and has the remarkable property that it admits a canonical isotrivial toric degeneration. The construction of this degeneration involves a careful analysis of the combinatorics of the weights of a natural torus action on $\mathfrak{aff}_k(\mathbb{C})$, which we recast visually as a sort of “game of dominoes”; see Section 4.6.

As an application of our local models, we show that the Hilbert schemes give new examples of Poisson manifolds satisfying a natural and subtle nondegeneracy condition, called holonomicity. (This was the problem that motivated the broader investigation we present in this paper.) The notion of holonomicity was introduced by the second- and third-named authors in [37], motivated by deformation theory; it is roughly equivalent to the statement that for each point in the Poisson manifold, the space of derived Poisson deformations of its germ is finite-dimensional. This notion was further developed and applied in our joint work [27], where we emphasized the importance of a certain collection of symplectic leaves, which we call “characteristic”: these are the symplectic leaves that are preserved by the flow of the modular vector field. In particular, while the original definition of holonomicity involves \mathcal{D} -modules, we conjectured that it is equivalent to a simple geometric criterion:

Conjecture 1.3 [27] *A Poisson manifold is holonomic if and only if its germ at any point has only finitely many characteristic symplectic leaves.*

In the present work, we show that the symplectic leaf through a point $Z \in \mathcal{X}^{[n]}$ is characteristic if and only if the corresponding derived intersection $Z \cap^h D$ is preserved by the flow of the modular vector field on D . Once again, this can be turned into a concrete local computation in examples without relying on derived techniques. Combining such concrete calculations with the results in [27; 37], the aforementioned toric degenerations, and some subtle combinatorial linear algebra, we establish our conjecture in (almost) complete generality for Hilbert schemes:

Theorem 1.4 *For a Poisson surface X whose vanishing locus is the anticanonical divisor $D \subset X$, the following statements are equivalent:*

- (1) *For every $n \geq 0$, the induced Poisson structure on the Hilbert scheme $\mathcal{X}^{[n]}$ is holonomic.*
- (2) *For every $n \geq 0$, the germ of $\mathcal{X}^{[n]}$ at any point has only finitely many characteristic symplectic leaves.*
- (3) *The only singularities of D are nodes.*

Note that the order of the quantifiers in statements (1) and (2) is important: for $n < 6$ there exist some exceptions for which $\mathcal{X}^{[n]}$ has finitely many characteristic leaves, but D has nonnodal singularities. However, we can treat most of these cases directly, leaving only one case left to determine whether [Conjecture 1.3](#) holds for all Hilbert schemes, namely the case when $n = 5$ and D has only A_2 -singularities (cusps).

On the other hand, combining [Theorem 1.4](#) with [Theorem 1.2](#), we obtain the first (and so far only known) infinite series of holonomic Lie algebras:

Corollary 1.5 *The Lie–Poisson structure on $\text{aff}_k(\mathbb{C})^\vee$ is holonomic for any $k \geq 1$, and in particular, its Poisson cohomology is finite-dimensional.*

It would be interesting to know if there are other series of Lie algebras with holonomic duals. Note that such Lie algebras must have an open coadjoint orbit, ie they are Frobenius. However, many of the Frobenius Lie algebras known to us are not holonomic.

1.3 Deformation theory

We close the paper in [Section 5](#) by discussing the Poisson deformations of $\mathcal{X}^{[n]}$. Such deformations are governed by the Poisson cohomology $H_{\sigma^{[n]}}^\bullet(\mathcal{X}^{[n]})$, which in turn is controlled by the geometry of the open symplectic leaf and the codimension-two characteristic leaves. We compute the Poisson cohomology in low degrees under the additional assumption that D has only quasihomogeneous singularities (which includes all cases where X is projective and D is reduced). This extends earlier work of Ran [40], who treated the case where D is smooth. In particular, we determine space of first-order deformations (the second Poisson cohomology):

Theorem 1.6 Assume that X is connected, and that the anticanonical divisor D is reduced and has only quasihomogeneous singularities. Then the second Poisson cohomology of $X^{[n]}$, is given, for all $n \geq 2$, by

$$H_{\sigma}^2(X^{[n]}) \cong H^2((X \setminus D)^{[n]}; \mathbb{C}) \oplus H^0(\mathcal{K}_X^{\vee}|_{D_{\text{sing}}}) \cong H_{\sigma}^2(X) \oplus \wedge^2 H^1(X \setminus D; \mathbb{C}) \oplus \mathbb{C} \cdot [E],$$

where \mathcal{K}_X^{\vee} is the anticanonical bundle and $[E]$ is the Poisson Chern class of the exceptional divisor of the Hilbert–Chow morphism.

This decomposition of the deformation space has the following interpretation. The summand $H_{\sigma}^2(X)$ corresponds to deformations of $X^{[n]}$ induced by applying Bottacin’s construction to a deformation of X itself. Meanwhile the summand $\wedge^2 H^1(X \setminus D; \mathbb{C})$ gives deformations in which the symplectic form on the open symplectic leaf $(X \setminus D)^{[n]}$ is deformed by the pullback of a closed two-form along the Albanese map $(X \setminus D)^{[n]} \rightarrow H^1(X \setminus D; \mathbb{C})^{\vee}/H_1(X \setminus D; \mathbb{Z})$. Finally, the deformations in the direction $\mathbb{C} \cdot [E]$ should correspond to Hilbert schemes of *noncommutative* surfaces given by deformation quantization of (X, σ) as in [38; 39]. In the case $X = \mathbb{P}^2$ and D is an elliptic curve, this interpretation was established by Hitchin [21], who proved that the resulting deformations are exactly the Hilbert schemes of Sklyanin algebras introduced by Nevins and Stafford [32].

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2 Hilbert groupoids of curves in surfaces

2.1 Hilbert schemes of surfaces

Throughout this paper, by a \mathbb{C} -scheme we mean a separated scheme of finite type over \mathbb{C} or a complex analytic space. By a *smooth (complex) surface*, we mean a smooth \mathbb{C} -scheme of dimension two.

Let X be a smooth complex surface and suppose that $n \in \mathbb{Z}_{>0}$ is a positive integer. We denote by $X^{[n]}$ the Hilbert scheme (or Douady space) parametrizing zero-dimensional subschemes of X of total length n . There are many equivalent presentations of this space, each of which will be useful in what follows:

- As a functor of points: if S is a \mathbb{C} -scheme, the set of maps from S to $X^{[n]}$ is given by

$$X^{[n]}(S) := \{\text{subschemes of } S \times X \text{ that are finite and flat of length } n \text{ over } S\}.$$

It is a theorem of Fogarty [13] that this functor is representable by a smooth scheme.

- As a resolution of singularities: let $X^{(n)}$ be the n^{th} symmetric power of X , ie the quotient of X^n by the permutations of the factors; it is a singular variety when $n \geq 2$. There is a natural map $X^{[n]} \rightarrow X^{(n)}$, called the Hilbert–Chow morphism, which sends a length- n subscheme $Z \subset X$ to its support (the collection of points in Z , counted with multiplicity). This map is a resolution of singularities that is crepant [2, proof of Proposition 5] and strictly semismall [22, Theorem 2.12].
- As a moduli space of sheaves: a point $Z \in X^{[n]}$ is equivalent to the data of its ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$. We may view \mathcal{I}_Z as a perfect complex of coherent sheaves on X . This complex is canonically isomorphic to \mathcal{O}_X away from Z , and this induces a canonical trivialization of the determinant $\det \mathcal{I}_Z$ away from Z , which extends to all of X since X is smooth and Z has codimension two. In this way, we identify $X^{[n]}$ with an open substack of the derived stack $\text{Perf}_0(X)$ of perfect complexes with trivialized determinant [43]. Note that the tangent complex of $X^{[n]}$ at Z is then given by

$$\mathbb{T}_Z X^{[n]} \cong \mathbb{T}_{\mathcal{I}_Z} \text{Perf}_0(X) \cong \text{R}\Gamma(\mathcal{R}\text{End}(\mathcal{I}_Z)_0)[1],$$

where $\mathcal{R}\text{End}(\mathcal{I}_Z)_0$ is the homotopy fibre of the Illusie trace map $\mathcal{R}\text{End}(\mathcal{I}_Z) \rightarrow \mathcal{O}_X$. The cohomology of this complex is concentrated in degree zero, giving the tangent space of $X^{[n]}$ at Z as

$$(2-1) \quad \mathbb{T}_Z X^{[n]} \cong H^1(\text{R}\Gamma(\mathcal{R}\text{End}(\mathcal{I}_Z)_0)) =: \text{Ext}_X^1(\mathcal{I}_Z, \mathcal{I}_Z)_0 \cong \text{Hom}_X(\mathcal{I}_Z, \mathcal{O}_Z).$$

2.2 Derived intersection of subschemes with curves

If X is a smooth surface, a *curve in X* is an effective divisor $D \subset X$; unless otherwise specified, we allow the possibility that D is singular, or even nonreduced.

Suppose that $i : D \hookrightarrow X$ is a curve. If $Z \subset X$ is a length- n subscheme, then the scheme-theoretic intersection $Z \cap D \subset D$ is a closed subscheme of D of length at most n , with structure sheaf $\mathcal{O}_{Z \cap D} \cong \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_D$. The *derived intersection* $Z \cap^h D$, is the derived subscheme of D obtained as the spectrum of the sheaf of commutative dg \mathcal{O}_D -algebras $\mathcal{O}_{Z \cap^h D} := \mathcal{O}_Z \otimes_{\mathcal{O}_X}^L \mathcal{O}_D$, where \otimes^L denotes the derived tensor product. In other words, we consider the dg algebra $Li^* \mathcal{O}_Z$, where

$$Li^* : \text{Perf}(X) \rightarrow \text{Perf}(D), \quad Li^* \mathcal{E} := i^{-1} \mathcal{E} \otimes_{i^{-1} \mathcal{O}_X}^L \mathcal{O}_D,$$

is the derived restriction of perfect complexes. Resolving \mathcal{O}_D by the Koszul complex

$$\mathcal{O}_D \cong (\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X)$$

we have concretely that

$$(2-2) \quad \mathcal{O}_{Z \cap^h D} \cong (\mathcal{O}_X(-D)|_Z \rightarrow \mathcal{O}_Z),$$

with \mathcal{O}_Z placed in cohomological degree zero.

Remark 2.1 The number of points of intersection, counted with multiplicity, is equal to

$$(2-3) \quad \#(Z \cap D) := \dim_{\mathbb{C}} H^0(\mathcal{O}_{Z \cap D})$$

by definition. For the derived intersection, it is natural to consider the virtual count of points, defined by the Euler characteristic of $\mathcal{O}_{Z \cap^h D}$, rather than its zeroth cohomology. Using the Koszul resolution (2-2), we find

$$\#_{\text{vir}}(Z \cap^h D) := \chi(H^\bullet(\mathcal{O}_{Z \cap^h D})) = \dim_{\mathbb{C}} H^0(\mathcal{O}_Z) - \dim_{\mathbb{C}} H^0(\mathcal{O}_X(-D)|_Z) = 0$$

for all Z , since Z is zero-dimensional and $\mathcal{O}_X(-D)|_Z$ is an invertible \mathcal{O}_Z -module, in accordance with the general principle that derived intersection numbers are deformation-invariant. \diamond

We now determine when two elements $Z_1, Z_2 \in X^{[n]}$ have equivalent derived intersection with D . Note that since the ideal sheaf \mathcal{I}_Z is torsion-free, we have $Tor_j(\mathcal{I}_Z, \mathcal{O}_D) = 0$ for $j > 0$, so the natural map $Li^*\mathcal{I}_Z \rightarrow i^*\mathcal{I}_Z$ comparing the derived and ordinary pullbacks is a quasi-isomorphism. Hence we obtain an exact triangle

$$(2-4) \quad i^*\mathcal{I}_Z \rightarrow \mathcal{O}_D \rightarrow Li^*\mathcal{O}_Z$$

of perfect complexes on D , corresponding to the exact sequence

$$(2-5) \quad 0 \rightarrow Tor_1^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_D) \rightarrow i^*\mathcal{I}_Z \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_{Z \cap D} \rightarrow 0$$

of coherent sheaves. Note that $Tor_1(\mathcal{O}_Z, \mathcal{O}_D)$ is a torsion module on D . Since \mathcal{O}_D is torsion-free (by definition), the image

$$i^*\mathcal{I}_Z^{\text{tf}} := \text{image}(i^*\mathcal{I}_Z \rightarrow \mathcal{O}_D) \cong i^*\mathcal{I}_Z / Tor_1^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_D)$$

is the maximal torsion-free quotient of $i^*\mathcal{I}_Z$. We then have the following.

Lemma 2.2 For any $(Z_1, Z_2) \in X^{[n]}$, the following spaces are canonically homotopy equivalent:

- (1) The space of equivalences $Z_1 \cap^h D \xrightarrow{\sim} Z_2 \cap^h D$ of derived subschemes of D .
- (2) The space of quasi-isomorphisms $Li^*\mathcal{O}_{Z_1} \xrightarrow{\sim} Li^*\mathcal{O}_{Z_2}$ of commutative differential graded \mathcal{O}_D -algebras.
- (3) The space of quasi-isomorphisms $Li^*\mathcal{O}_{Z_1} \xrightarrow{\sim} Li^*\mathcal{O}_{Z_2}$ of perfect complexes of \mathcal{O}_D -modules, commuting up to coherent homotopy with the natural maps $Li^*\mathcal{O}_{Z_1} \leftarrow \mathcal{O}_D \rightarrow Li^*\mathcal{O}_{Z_2}$.
- (4) The space of equivalences $i^*\mathcal{I}_{Z_1} \xrightarrow{\sim} i^*\mathcal{I}_{Z_2}$ in $\text{Perf}_0(D)$.
- (5) The discrete set of isomorphisms $i^*\mathcal{I}_{Z_1} \xrightarrow{\sim} i^*\mathcal{I}_{Z_2}$ commuting with the natural maps $i^*\mathcal{I}_{Z_1} \rightarrow \mathcal{O}_D \leftarrow i^*\mathcal{I}_{Z_2}$.
- (6) The discrete set of isomorphisms $i^*\mathcal{I}_{Z_1} \xrightarrow{\sim} i^*\mathcal{I}_{Z_2}$ that induce the identity map on the maximal torsion-free quotients $i^*\mathcal{I}_{Z_j}^{\text{tf}} \subset \mathcal{O}_D$.

In particular, all of these spaces are equivalent to their discrete sets of connected components.

Proof The spaces in (1) and (2) are equivalent by definition of the derived intersection.

For the equivalence of (2) and (3), choose an arbitrary cofibrant resolution $\mathcal{E}^\bullet \rightarrow \mathcal{O}_{Z_1}$ of \mathcal{O}_{Z_1} as a sheaf of \mathcal{O}_X -dg algebras concentrated in degrees ≤ 0 , with degree zero term $\mathcal{E}^0 = \mathcal{O}_X$. The dg \mathcal{O}_D -algebra $i^*\mathcal{E} \cong \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_D$ is then also cofibrant. Meanwhile, we may resolve \mathcal{O}_D by the Koszul complex $\mathcal{B}_D := (\mathcal{O}_X(-D) \rightarrow \mathcal{O}_X)$. The space of equivalences $Li^*\mathcal{O}_{Z_1} \cong Li^*\mathcal{O}_{Z_2}$ of dg \mathcal{O}_D -algebras is then equivalent to the space of dg \mathcal{O}_D -algebra maps $i^*\mathcal{E}^\bullet \rightarrow i^*(\mathcal{O}_{Z_2} \otimes \mathcal{B}_D)$. By degree considerations, such a dg algebra map is equivalent to a map of complexes whose degree zero part is the natural projection $\mathcal{O}_D \rightarrow \mathcal{O}_{Z_2}$, as desired. Note further that the full space of equivalences is modelled by the simplicial set of dg algebra maps

$$(2-6) \quad i^*\mathcal{E} \rightarrow i^*(\mathcal{O}_{Z_2} \otimes \mathcal{B}_D) \otimes \Omega_\Delta^\bullet,$$

where Ω_Δ^\bullet is the simplicial dg algebra of polynomial differential forms on simplices, and the degree-zero part is the canonical map $\mathcal{O}_D \rightarrow \mathcal{O}_{Z_2}$ (constant in the simplex direction). Similar considerations now show that the sets of higher simplices of spaces (2) and (3) are the same, and agree with the set of 0-simplices, so that these spaces are equivalent and both discrete. Namely, examining the degrees and using the compatibility with the differentials, we see that if a map (2-6) has degree-zero part the map $\mathcal{O}_D \rightarrow \mathcal{O}_{Z_2}$, it must take values in the subspace $i^*(\mathcal{O}_{Z_2} \otimes \mathcal{B}_D) \otimes \Omega_\Delta^{0,cl} \cong i^*(\mathcal{O}_{Z_2} \otimes \mathcal{B}_D)$, so that it reduces to a 0-simplex, as desired.

The equivalence of (3) and (4) follows from the exact triangle (2-4), using the fact that the trivialization of the determinant of \mathcal{I}_Z is defined by the inclusion $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X$. Note that it is also clear that the space (4) is discrete: since $i^*\mathcal{I}_{Z_1}, i^*\mathcal{I}_{Z_2}$ are sheaves concentrated in degree zero, the complex $\text{RHom}(i^*\mathcal{I}_{Z_1}, i^*\mathcal{I}_{Z_2})$ is concentrated in nonnegative degrees, so there are no nontrivial homotopies between morphisms, and similarly for the determinants.

Finally, (5) and (6) are clearly equivalent by definition of $i^*\mathcal{I}_Z^{\text{tf}}$, and they are equivalent to (4) because for an arbitrary morphism $\phi: \mathcal{I}_{Z_1} \rightarrow \mathcal{I}_{Z_2}$, the diagram

$$\begin{array}{ccc} \mathcal{I}_{Z_1} & \xrightarrow{\phi} & \mathcal{I}_{Z_2} \\ \downarrow & & \downarrow \\ \mathcal{O}_D & \xrightarrow{\det \phi} & \mathcal{O}_D \end{array}$$

is commutative. □

Definition 2.3 We refer to any of the equivalent discrete sets in Lemma 2.2 as the *set of D-equivalences from Z_1 to Z_2* . We say that Z_1 and Z_2 are *D-equivalent* if there exists a D-equivalence from Z_1 to Z_2 .

Evidently D-equivalence defines an equivalence relation on $\mathcal{X}^{[n]}$. If we are just interested in the equivalence classes, an a priori weaker notion will suffice:

Lemma 2.4 *Two elements $Z_1, Z_2 \in \mathcal{X}^{[n]}$ are D-equivalent if and only if the \mathcal{O}_D -modules $i^*\mathcal{I}_{Z_1}$ and $i^*\mathcal{I}_{Z_2}$ are isomorphic.*

Proof By part (4) of Lemma 2.2, it suffices to show that $i^*\mathcal{I}_{Z_1}$ and $i^*\mathcal{I}_{Z_2}$ are isomorphic as \mathcal{O}_D -modules if and only if they are isomorphic by a map with trivial determinant. But if $\phi: i^*\mathcal{I}_{Z_1} \rightarrow i^*\mathcal{I}_{Z_2}$ is any isomorphism, then its determinant gives an isomorphism

$$\mathcal{O}_D \cong \det \mathcal{I}_{Z_1} \xrightarrow{\sim} \det \mathcal{I}_{Z_2} \cong \mathcal{O}_D,$$

or equivalently an invertible element $g := \det \phi \in H^0(\mathcal{O}_D^\times)$. Since the rank of an ideal sheaf is equal to one, the determinant is a linear functional on endomorphisms and we have

$$\det(g^{-1}\phi) = g^{-1} \det(\phi) = \det(\phi)^{-1} \det(\phi) = 1.$$

Therefore $g^{-1}\phi: \mathcal{I}_{Z_1} \rightarrow \mathcal{I}_{Z_2}$ is an isomorphism with trivial determinant, as desired. □

Remark 2.5 For the rest of the paper, we work almost exclusively with the spaces 4–6 in Lemma 2.2; in those cases, as explained in the proof, the discreteness is immediate because the objects involved are sheaves concentrated in degree zero, so there is no room for higher homotopies. ◇

Remark 2.6 Lemma 2.2 holds in much more general circumstances, with essentially the same proof. Namely, statements (1), (2), (3), (5) and (6) are equivalent for any pair of subschemes Z_1, Z_2 in a scheme X with an effective Cartier divisor D . For the equivalence with (4), we need to speak about trivialization of the determinant being determined away from codimension two subsets, and for this we need to assume in addition that X is S_2 (eg smooth or more generally normal) and that the subschemes $Z_1, Z_2 \subset X$ have codimension at least two. ◇

Remark 2.7 In the proof, the discreteness of the space (1) is deduced from the equivalence with the other spaces. We can also argue directly using standard results on mapping spaces for derived schemes, as follows.

Let $U = Z_1 \cap^h D$ and let $V = Z_2 \cap^h D$. By definition, space (1) is the space of equivalences $U \xrightarrow{\sim} V$ commuting with the inclusions into D , ie it is the homotopy fibre of the map $\text{Maps}(U, V) \rightarrow \text{Maps}(U, D)$ induced by composition with the inclusion $V \hookrightarrow D$. Now observe that

$$\text{Maps}(U, V) \cong \text{Maps}\left(U, Z_j \underset{X}{\overset{h}{\times}} D\right) \cong \text{Maps}(U, Z_2) \underset{\text{Maps}(U, X)}{\overset{h}{\times}} \text{Maps}(U, D);$$

hence the homotopy fibre in question is equivalent to the homotopy fibre W of the map $\text{Maps}(U, Z_2) \rightarrow \text{Maps}(U, X)$. It thus suffices to show that the homotopy groups $\pi_i(W)$ are trivial for $i > 0$. By the long exact sequence of homotopy, it suffices to show that

- (a) $\pi_i(\text{Maps}(U, Z_2)) = \pi_i(\text{Maps}(U, X)) = 0$ for $i > 1$, and
- (b) the map $\pi_1(\text{Maps}(U, Z_2)) \rightarrow \pi_1(\text{Maps}(U, X))$ is injective.

For this, we use that U is affine with $\pi_i(\mathcal{O}(U)) := H^{-i}(\mathcal{O}(U)) = 0$ for $i > 1$, and that Z_2 and X are ordinary schemes. Moreover, without loss of generality, the latter are affine, hence (-1) -geometric as

derived stacks. Thus (a) follows immediately from [46, Corollary 2.2.4.6]. Furthermore from the proof of [loc. cit.] using the Postnikov system for \mathcal{U} , one sees that the map on fundamental groups in (b) is identified with the canonical map

$$(2-7) \quad \text{Hom}(\mathbb{L}_{Z_2}, \mathcal{F}) \rightarrow \text{Hom}(\mathbb{L}_X, \mathcal{F}),$$

where $\mathcal{F} := H^{-1}(\mathcal{O}_{\mathcal{U}})$. Since Z_2 and X are schemes, their cotangent complexes are concentrated in nonpositive degrees, with zeroth cohomology the Kähler differentials Ω^1 . Hence (2-7) is, in turn, identified with the map $\text{Hom}(\Omega_{Z_2}^1, \mathcal{F}) \rightarrow \text{Hom}(\Omega_X^1, \mathcal{F})$ induced by the quotient $\Omega_X^1 \rightarrow \Omega_{Z_2}^1$, giving the desired injectivity. \diamond

2.3 The Hilbert groupoid

Evidently D -equivalences between points of $\mathcal{X}^{[n]}$ can be composed, so that they form the arrows of a groupoid whose objects are the points of $\mathcal{X}^{[n]}$. We will now show that this defines a smooth groupoid scheme (or in the analytic context, a Lie groupoid).

More precisely, let

$$(X, D)^{[n]} := \mathcal{X}^{[n]} \times_{\text{Perf}_0(D)} \mathcal{X}^{[n]} \subset \text{Perf}_0(X) \times_{\text{Perf}_0(D)} \text{Perf}_0(X)$$

be the derived stack defined as the homotopy fibre product of $\mathcal{X}^{[n]}$ with itself along the map sending a subscheme Z to $i^* \mathcal{I}_Z$, where \mathcal{I}_Z is the ideal sheaf of Z equipped with the canonical trivialization of $\det \mathcal{I}_Z$. By definition of the homotopy fibre product, points of $(X, D)^{[n]}$ consist of triples (Z_1, Z_2, λ) , where Z_1, Z_2 are points of $\mathcal{X}^{[n]}$, and $\lambda: Z_1 \cap^h D \xrightarrow{\sim} Z_2 \cap^h D$ is a D -equivalence. Composing the D -equivalences gives $(X, D)^{[n]}$ the structure of a groupoid in the category of derived stacks.

Definition 2.8 The groupoid $(X, D)^{[n]}$ over $\mathcal{X}^{[n]}$ is the n^{th} Hilbert groupoid of (X, D) .

In fact, this object is a classical smooth variety:

Theorem 2.9 The derived stack $(X, D)^{[n]}$ is a smooth classical scheme. The projections $s, t: (X, D)^{[n]} \rightarrow \mathcal{X}^{[n]}$ are smooth morphisms, so that $(X, D)^{[n]}$ defines a smooth groupoid scheme over $\mathcal{X}^{[n]}$ whose orbits are the D -equivalence classes. Moreover, the induced map $(s, t): (X, D)^{[n]} \rightarrow \mathcal{X}^{[n]} \times \mathcal{X}^{[n]}$ restricts to an isomorphism over the Zariski open set $(X \setminus D)^{[n]} \times (X \setminus D)^{[n]}$, and is therefore birational.

Proof To prove representability, we will use the interpretation of D -equivalences from part (5) of Lemma 2.2: the fibre of $(X, D)^{[n]}$ over a point $(Z_1, Z_2) \in \mathcal{X}^{[n]} \times \mathcal{X}^{[n]}$ is the set of isomorphisms $\mathcal{I}_{Z_1} \otimes_{\mathcal{O}_D} \rightarrow \mathcal{I}_{Z_2} \otimes_{\mathcal{O}_D}$ commuting with the natural maps to \mathcal{O}_D . Recall that there is a universal ideal sheaf \mathcal{I} on $\mathcal{X}^{[n]} \times \mathcal{X}$ whose restriction to any slice $\{Z\} \times \mathcal{X} \cong \mathcal{X}$ is the ideal defining Z . Let $\tilde{\mathcal{X}} := \mathcal{X}^{[n]} \times \mathcal{X}^{[n]} \times \mathcal{X}$ and let $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}_{\tilde{\mathcal{X}}}$ be the pullbacks of \mathcal{I} along the two projections to $\mathcal{X}^{[n]} \times \mathcal{X}$. Let $\tilde{D} := \mathcal{X}^{[n]} \times \mathcal{X}^{[n]} \times D \subset \tilde{\mathcal{X}}$. Finally, let $p: \tilde{\mathcal{X}} \rightarrow \mathcal{X}^{[n]} \times \mathcal{X}^{[n]}$ be the projection. Observe that composition with the map $\mathcal{I}_2 \rightarrow \mathcal{O}_{\tilde{\mathcal{X}}} \rightarrow \mathcal{O}_{\tilde{D}}$ gives a natural map $p_* \text{Hom}(\mathcal{I}_1 \otimes_{\mathcal{O}_{\tilde{D}}}, \mathcal{I}_2 \otimes_{\mathcal{O}_{\tilde{D}}}) \rightarrow p_* \text{Hom}(\mathcal{I}_1, \mathcal{O}_{\tilde{D}})$ of quasicoherent sheaves on $\mathcal{X}^{[n]} \times \mathcal{X}^{[n]}$,

and that the morphisms $\mathcal{I}_1 \otimes \mathcal{O}_{\tilde{D}} \rightarrow \mathcal{I}_2 \otimes \mathcal{O}_{\tilde{D}}$ compatible with the maps to $\mathcal{O}_{\tilde{D}}$ form a torsor for the quasicoherent sheaf

$$\mathcal{E} := \ker(p_*\mathcal{H}om(\mathcal{I}_1 \otimes \mathcal{O}_{\tilde{D}}, \mathcal{I}_2 \otimes \mathcal{O}_{\tilde{D}}) \rightarrow p_*\mathcal{H}om(\mathcal{I}_1, \mathcal{O}_{\tilde{D}})).$$

We claim that \mathcal{E} is, in fact, a coherent sheaf. Indeed, applying the exact sequence (2-5) to the universal scheme $\mathcal{Z}_2 \subset \tilde{X}$ given by the vanishing of \mathcal{I}_2 , we see that \mathcal{E} is identified with the image of the canonical map

$$p_*\mathcal{H}om(\mathcal{I}_1 \otimes \mathcal{O}_{\tilde{D}}, \mathcal{T}or_1(\mathcal{O}_{\mathcal{Z}_2}, \mathcal{O}_{\tilde{D}})) \rightarrow p_*\mathcal{H}om(\mathcal{I}_1 \otimes \mathcal{O}_{\tilde{D}}, \mathcal{I}_2 \otimes \mathcal{O}_{\tilde{D}}).$$

Since $\mathcal{T}or_1(\mathcal{O}_{\mathcal{Z}_2}, \mathcal{O}_{\tilde{D}})$ is supported on \mathcal{Z}_2 , and the latter is proper over $X^{[n]} \times X^{[n]}$, it follows that $p_*\mathcal{H}om(\mathcal{I}_1 \otimes \mathcal{O}_{\tilde{D}}, \mathcal{T}or_1(\mathcal{O}_{\mathcal{Z}_2}, \mathcal{O}_{\tilde{D}}))$ is coherent, and hence so is \mathcal{E} , as claimed. It now follows that the total space $\text{Tot}(\mathcal{E})$ is a scheme that is relatively affine over $X^{[n]} \times X^{[n]}$, and hence so is the torsor in question. The invertible morphisms then form an open subscheme of the torsor, as desired.

To establish the smoothness of s, t , it is now enough to check that their relative tangent complexes $\mathbb{T}s, \mathbb{T}t$ at any closed point of $(X, D)^{[n]}$ are concentrated in degree zero; note that this statement is independent from the details of the explicit construction of the scheme above, and is indeed proved most easily from the abstract definition as a derived stack. We will prove the smoothness for s ; the argument for t is identical.

We have the commutative diagram of tangent complexes

$$\begin{array}{ccccc} \mathbb{T}s & \longrightarrow & \mathbb{T}(X, D)^{[n]} & \longrightarrow & s^*\mathbb{T}X^{[n]} \\ \downarrow & & \downarrow & & \parallel \\ t^*\mathbb{T}X^{[n]} & \longrightarrow & s^*\mathbb{T}X^{[n]} \oplus t^*\mathbb{T}X^{[n]} & \longrightarrow & s^*\mathbb{T}X^{[n]} \\ \downarrow & & \downarrow & & \downarrow \\ (Li^*)^*\mathbb{T}\text{Perf}_0(D) & \xlongequal{\quad} & (Li^*)^*\mathbb{T}\text{Perf}_0(D) & \longrightarrow & 0 \end{array}$$

where $\mathbb{T}s$ denotes the relative tangent complex of the map $s: (X, D)^{[n]} \rightarrow X^{[n]}$. Since homotopy fibre products of stacks give homotopy fibre sequences of tangent complexes, all three rows and the right two columns are fibre sequences, from which we deduce that

$$(2-8) \quad \mathbb{T}s \cong \text{fibre}(t^*\mathbb{T}X^{[n]} \rightarrow (Li^*)^*\mathbb{T}\text{Perf}_0(D)).$$

Suppose that $(Z_1, Z_2, \lambda) \in (X, D)^{[n]}$ is a closed point. Let $\mathcal{I} = \mathcal{I}_{Z_2}$ be the ideal defining Z_2 . Identifying the tangent complexes of $X^{[n]} \subset \text{Perf}_0(X)$ and $\text{Perf}_0(D)$ in (2-8) with the complexes of traceless derived endomorphisms of \mathcal{I} and $Li^*\mathcal{I}$ respectively, we have

$$\begin{aligned} \mathbb{T}_{(Z_1, Z_2, \lambda)}s &\cong \text{fibre}(\text{R}\Gamma(\mathcal{R}\text{End}_X(\mathcal{I})_0) \rightarrow \text{R}\Gamma(\mathcal{R}\text{End}_D(Li^*\mathcal{I})_0)[1]) \\ &\cong \text{R}\Gamma(\mathcal{R}\text{End}_X(\mathcal{I})_0(-D)[1]), \end{aligned}$$

where the second equivalence is induced by tensoring the complex $\mathcal{R}End_X(\mathcal{I})_0$ with the exact sequence for the quotient $\mathcal{O}_D \cong \mathcal{O}_X/\mathcal{O}_X(-D)$. As explained in [5, Corollary 2.6], we have

$$\mathcal{R}End_X(\mathcal{I})_0[1] \cong \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_{Z_2}),$$

so that

$$\mathcal{R}End_X(\mathcal{I})_0(-D)[1] \cong \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_{Z_2}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$$

is a skyscraper sheaf concentrated on the zero-dimensional scheme Z_2 . Therefore

$$\begin{aligned} \mathbb{T}_{(Z_1, Z_2, \lambda)}^s &\cong \mathbf{R}\Gamma(\mathcal{H}om(\mathcal{I}, \mathcal{O}_{Z_2}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)) \\ &\cong \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_{Z_2}(-D)) \end{aligned}$$

is concentrated in degree zero, as desired. □

Example 2.10 The construction of the Hilbert groupoid is already interesting in the case when D is smooth and $n = 1$, where it recovers a known construction of Lie groupoids associated with curves in surfaces [11; 18], or more precisely a groupoid integrating the Lie algebroid $\mathcal{T}_X(-D)$ of vector fields vanishing on D , via blowing up. Namely, a length-one subscheme is simply a point of X with the reduced scheme structure, so that $X^{[1]} = X$, and hence by Theorem 2.9, we have a natural birational map

$$(s, t): (X, D)^{[1]} \rightarrow X \times X.$$

A pair of points in $X^{[1]}$ are D -equivalent if and only if one of two things occurs: either they both lie in $X \setminus D$, or they are the same point of D . It follows that the isotropy groups of points $p \in D$ are two-dimensional, and so the preimage of the diagonal embedding of D under the map (s, t) is a hypersurface in $(X, D)^{[1]}$. Hence (s, t) factors through the blowup of $X \times X$ along the diagonal copy of D . One can check (eg using Theorem 4.7 below) that this identifies $(X, D)^{[1]}$ with the complement of the strict transforms of $D \times X$ and $X \times D$ in the blowup. It would be interesting to give a similarly explicit description of the birational map $(X, D)^{[n]} \rightarrow X \times X$ when $n > 1$ or D is singular. ◇

2.4 Stabilizer groups

Definition 2.11 We denote by $G_{Z,D}$ the stabilizer group algebra of $Z \in X^{[n]}$ in the groupoid $(X, D)^{[n]}$, ie the group of homotopy classes of infinitesimal self D -equivalences of Z , and by $\mathfrak{g}_{Z,D}$ its Lie algebra.

By Lemma 2.2, $G_{Z,D}$ can be identified with the group of automorphisms of $i^*\mathcal{I}_Z$ that act as the identity on the maximal torsion-free quotient $i^*\mathcal{I}_Z^{\text{tf}}$. Correspondingly, $\mathfrak{g}_{Z,D}$ is identified with the endomorphisms of $i^*\mathcal{I}_Z$ that act by zero on $i^*\mathcal{I}_Z^{\text{tf}}$. Since the maximal torsion subsheaf

$$\tau_{Z,D} := \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_D) = \mathcal{H}^{-1}(\mathcal{O}_{Z \cap h_D})$$

of $i^*\mathcal{I}_Z$ is preserved by any endomorphism, we immediately deduce the following:

Lemma 2.12 *There is a canonical exact sequence of groups*

$$0 \rightarrow \text{Hom}_{\mathbb{D}}(i^* \mathcal{I}_Z^{\text{tf}}, \tau_{Z, \mathbb{D}}) \rightarrow G_{Z, \mathbb{D}} \rightarrow \text{Aut}_{\mathbb{D}}(\tau_{Z, \mathbb{D}}),$$

where the group law on $\text{Hom}_{\mathbb{D}}(i^* \mathcal{I}_Z^{\text{tf}}, \tau_{Z, \mathbb{D}})$ is addition. Correspondingly we have an exact sequence of Lie algebras

$$0 \rightarrow \text{Hom}_{\mathbb{D}}(i^* \mathcal{I}_Z^{\text{tf}}, \tau_{Z, \mathbb{D}}) \rightarrow \mathfrak{g}_{Z, \mathbb{D}} \rightarrow \text{End}_{\mathbb{D}}(\tau_{Z, \mathbb{D}}),$$

where $\text{Hom}_{\mathbb{D}}(i^* \mathcal{I}_Z^{\text{tf}}, \tau_{Z, \mathbb{D}})$ is abelian.

2.5 Syzygy matrices and orbit data

We now turn to the classification of the orbits of $(X, D)^{[n]}$, ie the enumeration of the D-equivalence classes. In light of Lemma 2.4, this is equivalent to the classification of $\mathcal{O}_{\mathbb{D}}$ -modules \mathcal{F} such that $\mathcal{F} \cong i^* \mathcal{I}_Z$ for some $Z \in X^{[n]}$.

In order to characterize such modules, note that since X is smooth of dimension two the stalk of \mathcal{I}_Z at any point $p \in X$ is an $\mathcal{O}_{X, p}$ -module of projective dimension at most two; indeed, if $p \notin Z$ then $\mathcal{I}_{Z, p} = \mathcal{O}_{X, p}$, while if $p \in Z$, the classical Hilbert–Burch theorem states that the minimal resolution has the form

$$(2-9) \quad 0 \rightarrow \mathcal{O}_{X, p}^{\oplus k} \xrightarrow{S} \mathcal{O}_X^{\oplus(k+1)} \xrightarrow{M(S)} \mathcal{I}_{Z, p} \rightarrow 0,$$

where S is a $(k + 1) \times k$ -matrix of elements of the maximal ideal $\mathfrak{m}_{X, p} < \mathcal{O}_{X, p}$, called the *syzygy matrix*, and $M(S)$ is the row vector formed from the maximal minors of S with alternating signs, ie

$$M(S) = (\det(S_0) \quad -\det(S_1) \quad \dots \quad (-1)^k \det(S_k)),$$

where S_j is the $k \times k$ submatrix obtained by removing the $(j + 1)$ st row from S . The matrix S is determined by Z and p up to multiplication on the left and right by elements in $\text{GL}_{k+1}(\mathcal{O}_{X, p})$ and $\text{GL}_k(\mathcal{O}_{X, p})$, respectively.

Applying the functor Li^* to (2-9), we obtain a minimal resolution

$$(2-10) \quad 0 \rightarrow \mathcal{O}_{\mathbb{D}, p}^{\oplus k} \xrightarrow{S|_{\mathbb{D}}} \mathcal{O}_{\mathbb{D}, p}^{\oplus(k+1)} \xrightarrow{M(S|_{\mathbb{D}})} i^* \mathcal{I}_{Z, p} \rightarrow 0$$

for any $p \in \mathbb{D}$, so that the D-equivalence class of Z near p is controlled by the equivalence class of the matrix $S|_{\mathbb{D}}$. For instance, $S|_{\mathbb{D}}$ can be used to construct an explicit model for $\mathcal{O}_{Z \cap \mathbb{D}}$ as a dg $\mathcal{O}_{\mathbb{D}}$ -algebra. Namely, by extending (2-9) one step to the right we obtain a minimal resolution of $\mathcal{O}_{Z, p}$, which carries a canonical dg algebra structure due to Herzog [20] (see also [3, Section 6]), whose structure constants are determined by S . Tensoring the resolution with $\mathcal{O}_{\mathbb{D}, p}$, we obtain an explicit model for $\mathcal{O}_{Z \cap \mathbb{D}, p}$ as an $\mathcal{O}_{\mathbb{D}, p}$ -algebra.

This motivates the following definition. Note that in this definition, there is a unique local orbit datum of size $k = 0$, corresponding to the case in which the point $p \in \mathbb{D}$ does not lie in Z and $S|_{\mathbb{D}}$ is the zero map $0 \rightarrow \mathcal{O}_{\mathbb{D}, p}$, which we think of as a matrix of size 1×0 .

Definition 2.13 A local orbit datum supported at $p \in D$ of size $k \geq 0$ is an element of the double quotient

$$(2-11) \quad \mathrm{GL}_{k+1}(\mathcal{O}_{D,p}) \backslash \mathfrak{m}_{D,p}^{(k+1) \times k} / \mathrm{GL}_k(\mathcal{O}_{D,p}),$$

which can be realized as the restriction to D of the syzygy matrix of a zero-dimensional subscheme $Z \subset X$.

We denote the set of local orbit data of size k by $\mathrm{OrbDat}_k(D, p)$, and set

$$\mathrm{OrbDat}(D, p) := \bigsqcup_{k \geq 0} \mathrm{OrbDat}_k(D, p).$$

A (global) orbit datum is an assignment $\phi: p \mapsto \phi(p)$ of a local orbit datum $\phi(p) \in \mathrm{OrbDat}_k(D, p)$ to every point $p \in D$ such that there are only finitely many p with $\phi(p)$ of positive size.

Note that every global orbit datum can be realized by a zero-dimensional subscheme, simply by taking the disjoint union of the zero-dimensional subschemes at each p where $\phi(p)$ has positive size.

If $Z \in X^{[n]}$, we denote by

$$[Z]_p \in \mathrm{OrbDat}(D, p)$$

the image of the syzygy matrix of Z , which is well-defined. If ϕ is a global orbit datum, we define a subset

$$X_\phi^{[n]} := \{Z \in X^{[n]} \mid [Z]_p = \phi(p) \text{ for all } p \in D\}.$$

Then clearly $X_\phi^{[n]}$, if nonempty, is a D -equivalence class, ie an orbit of $(X, D)^{[n]}$, and every D -equivalence class arises in this way.

2.6 Classification of orbits

2.6.1 Smooth points Let $p \in D$ be a smooth point of D and suppose that $Z \in X^{[n]}$. Then the ring $\mathcal{O}_{D,p}$ of analytic germs at p is a principal ideal domain, isomorphic to the ring $\mathbb{C}\{x\}$ of convergent power series in a coordinate x centred at p . Then since $i^* \mathcal{I}_{Z,p}$ is a finitely generated module whose free part has rank one, it must have the form

$$(2-12) \quad i^* \mathcal{I}_{Z,p} \cong \mathcal{O}_{D,p} \oplus \bigoplus_{j=1}^k \mathcal{O}_{D,p} / \mathfrak{m}_{D,p}^{\mu_j} \cong \mathbb{C}\{x\} \oplus \bigoplus_{j=1}^k \mathbb{C}[x] / x^{\mu_j}$$

for a unique collection $\mu = (\mu_1 \geq \mu_2 \geq \dots)$ of positive integers. Correspondingly, the syzygy matrix can be put in the Smith normal form

$$(2-13) \quad S|_D \sim S_\mu(x) := \begin{pmatrix} x^{\mu_1} & 0 & 0 & \dots & 0 \\ 0 & x^{\mu_2} & 0 & \dots & 0 \\ 0 & 0 & x^{\mu_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x^{\mu_k} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

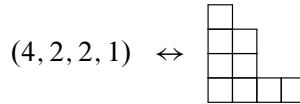
Recall the following definition.

Definition 2.14 A Young diagram (of length k) is a decreasing sequence $\mu = (\mu_1 \geq \dots \geq \mu_k)$ of positive integers. Its *size* is its sum:

$$|\mu| = \mu_1 + \dots + \mu_k.$$

By convention, we set $\mu_j = 0$ for $j > k$.

We depict a Young diagram λ by placing λ_j boxes in the j^{th} row, starting the numbering of rows from 1, for example



is a Young diagram of length four and size nine.

Rephrasing the above, we have the following characterization of the local orbit data at p :

Lemma 2.15 If $p \in D$ is a smooth point, the map sending a Young diagram μ to the equivalence class of the corresponding Smith normal form matrix gives a canonical bijection

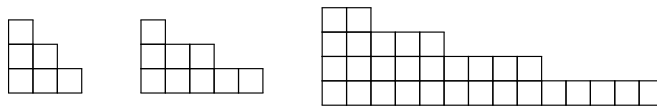
$$\text{OrbDat}_k(D, p) \cong \{\text{Young diagrams of length } k\}.$$

In particular, if D is smooth, then the global orbit data are in bijection with functions from D to the set of Young diagrams of arbitrary size.

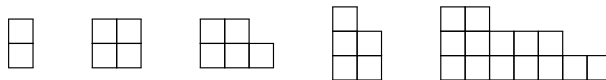
We now describe the basic characteristics of the orbit datum corresponding to a Young diagram μ . In order to do so, we introduce the following auxiliary notion:

Definition 2.16 We say that a Young diagram $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ is *horizontally convex* if $\lambda_i - \lambda_{i+1} \geq \lambda_{i+1} - \lambda_{i+2}$ for every $i > 0$.

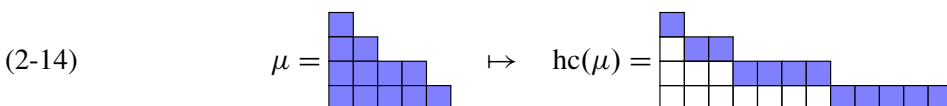
For instance, the following Young diagrams are horizontally convex:



whereas the following are *not* horizontally convex:



For a Young diagram μ , let $\text{hc}(\mu)$ be the unique Young diagram λ such that $\lambda_j - \lambda_{j+1} = \mu_j$ for all $j > 0$, or equivalently $\lambda_j = \sum_{l \geq j} \mu_l$. Then $\text{hc}(\mu)$ is horizontally convex, and evidently every horizontally convex diagram can be obtained in this way. For example,



Note that the number of boxes of $hc(\mu)$ is given by

$$(2-15) \quad |hc(\mu)| = \sum_{j \geq 1} j\mu_j,$$

which will be useful below.

Given a Young diagram μ and a smooth point $p \in D$, we define a zero-dimensional subscheme

$$Z_p^\mu \subset X$$

supported at p , as follows. First, choose coordinates (x, y) centred at p , such that D is given locally by the equation $y = 0$, and let $\lambda = hc(\mu)$. Then set

$$(2-16) \quad Z_p^\mu := \text{the vanishing locus of the monomials } x^j y^l \text{ for } j \geq \lambda_{l+1}.$$

Using the horizontal convexity of λ , it is straightforward to verify that this definition is independent of the choice of coordinates (x, y) .

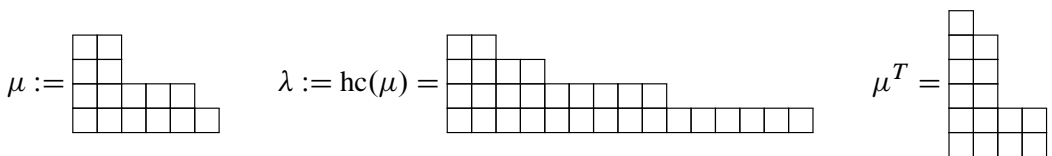
Evidently the algebra $\mathcal{O}_{Z_p^\mu} \cong \mathbb{C}[x, y]/(x^j y^l)_{j < \lambda_{l+1}}$ has a basis given by the monomials $x^j y^l$, where $j < \lambda_{l+1}$ for $l = 0, \dots, k - 1$. The following is therefore immediate:

Lemma 2.17 *The algebra $\mathcal{O}_{Z_p^\mu}$ has a basis indexed by the boxes of the Young diagram $hc(\mu)$. In particular, the length of the scheme Z_p^μ is given by*

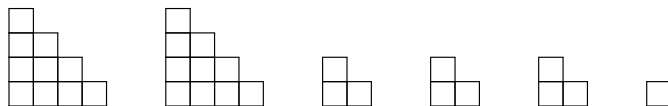
$$\#Z_p^\mu = |hc(\mu)|.$$

Remark 2.18 Z_p^μ can be defined in a coordinate-free fashion as follows. Let μ^T be the transpose of μ , defined by $\mu_j^T = \#\{l > 0 \mid \mu_l \geq j\}$, and let $k^T = \mu_1$ be its length. For a collection p_1, \dots, p_{k^T} of pairwise distinct points in D , consider the subscheme given by the disjoint union of $(\mu_j^T)^{\text{th}}$ order neighbourhoods of p_j for $1 \leq j \leq k^T$. Taking the limit as the points $p_1, \dots, p_{k^T} \rightarrow p$ we obtain the subscheme Z_p^μ .

For example, consider the Young diagrams



Then Z_p^μ is obtained as the limit of the schemes defined by the intersection of the ideals $m_{X,p_1}^4, m_{X,p_2}^4, m_{X,p_3}^2, m_{X,p_4}^2, m_{X,p_5}^2, m_{X,p_6}$ when all points $p_i \in D$ tend to p while remaining distinct. Pictorially, this corresponds to the fact that the Young diagram λ can be obtained by taking the Young diagrams



and piling their boxes onto one another horizontally. ◇

The importance of the subscheme Z_p^μ is that it gives a canonical representative of the corresponding orbit datum:

Proposition 2.19 *Let μ be a Young diagram. Then $[Z_p^\mu] = \mu \in \text{OrbDat}(D, p)$.*

Proof Let $Z = Z_p^\mu$. In light of the above, $i^*\mathcal{I}_{Z,p}$ is determined up to isomorphism by its torsion subsheaf, which is isomorphic as an \mathcal{O}_D -module to $\mathcal{T}or_1(\mathcal{O}_Z, \mathcal{O}_D)$. In local coordinates (x, y) such that D is the vanishing locus of y , we can use the Koszul resolution of \mathcal{O}_D to compute

$$\mathcal{T}or_1(\mathcal{O}_Z, \mathcal{O}_D) \cong \ker(\mathcal{O}_Z \xrightarrow{y} \mathcal{O}_Z).$$

Using the basis of \mathcal{O}_Z indexed by the boxes of $\lambda := \text{hc}(\mu)$ as above, the operator y corresponds to a translation upwards by one step. Therefore $\mathcal{T}or_1(\mathcal{O}_Z, \mathcal{O}_D)$ has a \mathbb{C} -basis indexed by the boxes of λ that sit at the top of the columns, and these are in bijection with the boxes of μ , as illustrated in (2-14). Moreover, x restricts to a coordinate on D , identifying $\mathcal{O}_{D,p} \cong \mathbb{C}\{x\}$. The action of x on the basis corresponds to translation rightwards by one step in the Young diagram, and in this way we deduce that $\mathcal{T}or_1(\mathcal{O}_Z, \mathcal{O}_D)$ is isomorphic to the direct sum of the modules $\mathbb{C}\{x\}/(x^{\mu_j})$ for $j = 1, \dots, k$, as desired. \square

Proposition 2.20 *If $Z \in \mathcal{X}^{[n]}$ is any element whose orbit datum at p is the Young diagram μ , then Z_p^μ is a subscheme of Z . Moreover, the orbit of Z contains as a Zariski open dense subset the locus of elements of the form $Z' \sqcup Z_p^\mu$ such that $p \notin Z'$.*

Proof Choose local coordinates (x, y) centred at p for which $D = \{y = 0\}$. By assumption, the syzygy matrix for Z is equivalent to one of the form

$$S_Z = S_\mu(x) + yM = \sum_{j=1}^k x^{\mu_j} E_j + yM,$$

where E_j is whose (j, j) -entry is 1 and whose other entries are zero, M is in $\mathcal{O}_{X,p}^{(k+1) \times k}$ and $S_\mu(x)$ is the Smith normal form matrix (2-13). Moreover Z is given locally by the vanishing of the maximal minors of S_Z . Using the skew-multilinearity of the minors of a matrix, we deduce that each maximal minor is a sum of terms the form $x^{\mu_{j_1}} \dots x^{\mu_{j_l}} y^{k-l} \det(M_{j_1, \dots, j_l})$ for a $(k-l) \times (k-l)$ submatrix M_{j_1, \dots, j_l} of M , where the indices j_1, \dots, j_l are pairwise distinct. Since $\mu_k \leq \mu_{k-1} \leq \dots$, each such term is divisible by $x^{\mu_k + \dots + \mu_{k-l+1}} y^{k-l} = x^{\text{hc}(\mu)_{l-1}} y^{k-l}$ and hence \mathcal{I}_Z is contained in the ideal defining Z_p^μ , or equivalently $Z_p^\mu \subset Z$, as desired.

For the second statement, first note that the Zariski open property follows by upper semicontinuity of the multiplicity of the point p ; this upper semicontinuity follows from the continuity of the Hilbert–Chow morphism, and the fact that multiplicity is clearly upper semicontinuous on the base. So we only need to prove that the given locus is dense. The problem localizes around the points of Z , so we may assume without loss of generality that $X = \mathbb{C}^2$; in this case D is anticanonical so by Corollary 3.5 below, the orbits

are disjoint unions of symplectic leaves of one of Bottacin’s Poisson structures on $X^{[n]}$. Furthermore, we may assume that Z is supported at a point $p \in D$, with $Z \neq Z_p^\mu$. Hence $\#Z > \#Z_p^\mu = |\text{hc}(\mu)|$, so by Lemma 2.17 above and Lemma 2.22 below, the orbit of Z has positive dimension. Since the monomial ideals give isolated points in $X^{[n]}$, we may assume in addition that the ideal defining Z is not generated by monomials. Then the vector field on $X^{[n]}$ induced by the Euler vector field $E = \sum_{i=1}^n y_i \partial_{y_i}$ generating the standard \mathbb{C}^* action in coordinates is nonzero at Z ; see eg [29, Proposition 7.4]. Pick a locally defined function $g \in \mathcal{O}_{X^{[n]}}$ such that $E(g) = 1$. Then we have $\{g, h\} = 1$, where h is the locally defined function on $X^{[n]}$ sending Z to the sum of the x -coordinates of its support. Therefore, the Hamiltonian vector field of g pushes Z away from the locus where $\sum_{i=1}^n x_i = 0$. In particular, it pushes Z away from the set of elements of $(\mathbb{C}^2)^{[n]}$ supported at p . Since the flow cannot change the intersection with D , we deduce that Z is equivalent to a scheme of the form $Z' \sqcup Z''$, where $Z' \neq \emptyset$ is disjoint from p and $Z'' \supset Z_p^\mu$. The result follows by downward induction on the length of Z . □

Remark 2.21 In the above proof, we cite Corollary 3.5 below. There is no circular logic because the above proposition is not used after the present subsection. Moreover Corollary 3.5 (and all of Section 3.2) do not require anything from the present Section 2.6. ◇

In light of (2-12) we have a (noncanonical) splitting $i^* \mathcal{I}_{Z,p} \cong i^* \mathcal{I}_{Z,p}^{\text{tf}} \oplus \tau_{Z,D}$, where

$$i^* \mathcal{I}_{Z,p}^{\text{tf}} \cong \mathcal{O}_{D,p} \cong \mathbb{C}\{x\} \quad \text{and} \quad \tau_{Z,D,p} \cong \bigoplus_{j=1}^k \mathcal{O}_{D,p}/\mathfrak{m}_{D,p}^{\mu_j} \cong \bigoplus_{j=1}^k \mathbb{C}[x]/(x^{\mu_j})$$

are the torsion-free and torsion parts, respectively. It follows immediately that the natural map $G_{Z,D} \rightarrow \text{Aut}_D(\tau_{Z,D})$ is a split surjection, with kernel $\text{Hom}(i^* \mathcal{I}_{Z,p}^{\text{tf}}, \tau_{Z,D}) \cong H^0(\tau_{Z,D})$, giving the following description of the stabilizer group:

Lemma 2.22 *Suppose $Z \cap D$ is supported in the smooth locus of D . Then the stabilizer group of Z in $(X, D)^{[n]}$ is given by*

$$G_{Z,D} \cong \text{Aut}_D(\tau_{Z,D}) \ltimes H^0(\tau_{Z,D}),$$

where $H^0(\tau_{Z,D})$ is viewed as an abelian group under addition. In particular, the orbit of Z has codimension

$$\text{codim}((X, D)^{[n]} \cdot Z) = 2 \sum_{p \in Z} |\text{hc}(\mu(p))|,$$

where $\mu(p)$ is the Young diagram representing the orbit datum $[Z]_p$ for all $p \in D$.

Proof It remains to establish the formula for the codimension of the orbit, or equivalently the dimension of the Lie algebra $\mathfrak{g}_{Z,D}$ of $G_{Z,D}$. But the latter is the sum of the dimensions of the vector spaces

$$H^0(\tau_{Z,D}) \cong \bigoplus_{j=1}^k \mathbb{C}[x]/(x^{\mu_j}) \quad \text{and} \quad \text{End}_D(\tau_{Z,D}) \cong \bigoplus_{j,l=1}^k \text{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x]/(x^{\mu_j}), \mathbb{C}[x]/(x^{\mu_l})).$$

Note that $\dim \mathbb{C}[x]/(x^{\mu_j}) = \mu_j$, while $\dim \operatorname{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x]/(x^{\mu_j}), \mathbb{C}[x]/(x^{\mu_l})) = \min\{\mu_j, \mu_l\}$.

Therefore,

$$\dim G_{G,Z} = \sum_{j \geq 1} \mu_j + \sum_{j,l \geq 1} \min\{\mu_j, \mu_l\} = \sum_{j \geq 1} \mu_j + \sum_{j \geq 1} (2j - 1)\mu_j = 2 \sum_{j \geq 1} j\mu_j = 2|\operatorname{hc}(\mu)|,$$

by (2-15). □

Combined with Lemma 2.17, we deduce the following:

Corollary 2.23 *The zero-dimensional orbits of $(X, D)^{[n]}$ are exactly the elements of the form $Z = \bigsqcup_{p \in D} Z_p^{\mu(p)}$, such that $\sum_{p \in D} |\operatorname{hc}(\mu(p))| = n$.*

Combining these results, we have obtained the following description of the orbits in the case where D is smooth:

Theorem 2.24 *Suppose that $D \subset X$ is smooth. Then the orbits of $(X, D)^{[n]}$ are in bijection with functions μ from D to the set of all Young diagrams, such that*

$$\sum_{p \in D} |\operatorname{hc}(\mu(p))| \leq n.$$

Moreover, for any such μ , the set

$$\left\{ Z' \sqcup \bigsqcup_{p \in Z} Z_p^{\mu(p)} \mid Z' \subset X \setminus D \right\} \cong (X \setminus D)^{[n - \sum_p |\operatorname{hc}(\mu(p))|]}$$

is open and dense in the corresponding orbit $X_{\mu}^{[n]}$. In particular, if X is connected, then all orbits are connected.

Remark 2.25 In addition, one can show that the orbit $X_{\mu}^{[n]}$ lies in the closure of the orbit $X_{\tilde{\mu}}^{[n]}$ if and only if $\tilde{\mu} \geq \mu$ in the dominance order on Young diagrams. However the proof uses more refined information about the local structure of $X^{[n]}$, so we postpone it; see Proposition 4.10. ◇

2.6.2 Nodes We now briefly describe the classification of the orbit data at a nodal singularity $p \in D$.

Choose coordinates (x, y) centred at p , so that $D = \{xy = 0\}$, giving an isomorphism

$$\mathcal{O}_{D,p} \cong \mathbb{C}\{x, y\}/(xy).$$

The classification of finitely generated modules over this ring is known: the torsion modules were classified in [14] (see also [4]), and the classification of all finitely generated modules over $\mathbb{C}[x, y]/(xy)$ was done in [25; 26], based on earlier works [30; 31]. In particular, every finitely generated module decomposes as a direct sum of indecomposable ones, and this decomposition can be determined algorithmically [24]. Furthermore, the minimal resolutions of the indecomposables are presented in [8, Appendix A]. Of these, only two families have projective dimension at most one, namely those listed as (1) and (3) on pages 224–225 of [8]:

- A “continuous series” of torsion modules, expressed as the cokernels of the square matrices

$$M(\mathbf{d}, k, u) := \begin{pmatrix} x^{\nu_1} I_k & 0 & 0 & \cdots & y^{\mu_l} J_k(u) \\ y^{\mu_1} I_k & x^{\nu_2} I_k & 0 & \cdots & 0 \\ 0 & y^{\mu_2} I_k & x^{\nu_3} I_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y^{\mu_{l-1}} I_k & x^{\nu_l} I_k \end{pmatrix}$$

of size $k \cdot l$, where $\mathbf{d} = (\nu_1, \mu_1), (\nu_2, \mu_2), \dots, (\nu_l, \mu_l)$ is a nonperiodic sequence of pairs of positive integers, I_k denotes the identity matrix of size k and $J_k(u)$ is a maximal Jordan block of size k with eigenvalue $u \neq 0$. We include here the degenerate case $l = 1$, in which case $M(\mathbf{d}, k, u) = x^{\nu_1} I_k - y^{\mu_1} J_k(u)$.

- A “discrete series” of rank-one modules, expressed as the cokernels of the $(k + 1) \times k$ matrices

$$\mathcal{D}(\mathbf{d}) := \begin{pmatrix} x^{\nu_1} & 0 & 0 & \cdots & 0 \\ y^{\mu_1} & x^{\nu_2} & 0 & \cdots & 0 \\ 0 & y^{\mu_2} & x^{\nu_3} & \cdots & 0 \\ 0 & 0 & y^{\mu_3} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & x^{\nu_k} \\ 0 & 0 & 0 & \cdots & y^{\mu_k} \end{pmatrix},$$

where $\mathbf{d} = (\mu_1, \nu_1), \dots, (\mu_k, \nu_k)$ is a sequence of pairs of positive integers. We include here the degenerate case $k = 0$, which corresponds to the trivial module.

Since for any $Z \in \mathcal{X}^{[n]}$, the module $i^* \mathcal{I}_{Z,p}$ has rank one and projective dimension at most one, we have the following classification of the orbit data:

Proposition 2.26 *If p is a node, then each local orbit datum is presented as the direct sum of a single matrix of the form $\mathcal{D}(\mathbf{d})$ and at most finitely many matrices of the form $M(\mathbf{d}, k, u)$. This presentation is unique up to permutation of the summands.*

Remark 2.27 By choosing suitable lifts of the direct sums of the matrices $\mathcal{D}(\mathbf{d})$ and $M(\mathbf{d}, k, u)$ to matrices valued in $\mathcal{O}_{\mathcal{X},p}$, one can show that all direct sums as in Proposition 2.26 are realized as orbit data for sufficiently large n . It would be interesting to know the minimal n for which a given direct sum can occur. ◇

Remark 2.28 It is straightforward to check that the indecomposable rank-one modules in the discrete series above are determined up to isomorphism by their torsion submodules, which have infinite projective dimension and form the discrete series described in item (2) on page 225 of [8]. Hence, if the singularities of D are at worst nodal, the orbit type of an element $Z \in \mathcal{X}^{[n]}$ is completely determined by the isomorphism class of the torsion submodule $\tau_{Z,D}$ of \mathcal{I}_Z . ◇

3 Symplectic and Poisson structures

3.1 Bottacin’s Poisson structure

From now on, consider the particular case in which the divisor $D \subset X$ is anticanonical, and fix a nonzero section

$$\sigma \in H^0(\mathcal{K}_X^\vee)$$

vanishing on D . Thus σ is a Poisson structure on X having the open set $X \setminus D$ as a two-dimensional symplectic leaf, and the points of D as zero-dimensional symplectic leaves. In [5], Bottacin shows that for every $n \geq 0$, the induced Poisson structure $\sigma^{(n)}$ on the symmetric power $X^{(n)}$ lifts along the Hilbert–Chow morphism to a Poisson structure on the Hilbert scheme $X^{[n]}$, and that the corresponding bivector

$$\sigma^{[n]} \in H^0(\wedge^2 \mathcal{T}_{X^{[n]}})$$

has the following description: using the various identifications (2-1) of the tangent space at $Z \in X^{[n]}$, and the corresponding Serre dual descriptions of the cotangent space, the anchor map $(\sigma^{[n]})^\# : T_Z^* X^{[n]} \rightarrow T_Z X^{[n]}$ is determined by the following commutative diagram:

$$\begin{array}{ccccc} T_Z^* X^{[n]} & \xrightarrow{\sim} & \text{Ext}_X^1(\mathcal{I}_Z, \mathcal{I}_Z \otimes \mathcal{K}_X)_0 & \xrightarrow{\sim} & \text{Hom}_X(\mathcal{I}_Z, \mathcal{O}_Z \otimes \mathcal{K}_X) \\ \downarrow (\sigma^{[n]})^\# & & \downarrow \sigma & & \downarrow \sigma \\ T_Z X^{[n]} & \xleftarrow{\sim} & \text{Ext}_1^X(\mathcal{I}_Z, \mathcal{I}_Z)_0 & \xleftarrow{\sim} & \text{Hom}_X(\mathcal{I}_Z, \mathcal{O}_Z) \end{array}$$

3.2 Symplectic groupoid and symplectic leaves

We now sketch an alternative approach to the construction of Bottacin’s Poisson structure $\sigma^{[n]}$, based on the formalism of shifted symplectic structures from [34], and the now standard correspondence between Poisson structures, symplectic groupoids and shifted Lagrangian morphisms. This conceptual approach has the advantage of immediately yielding a description of the symplectic groupoid and symplectic leaves of $\sigma^{[n]}$, and is indeed how we arrived at the definition of the groupoid $(X, D)^{[n]}$. However, a posteriori one can construct the symplectic structure on the groupoid without invoking the theory of shifted symplectic structures; see Remark 3.4 below.

We will assume here that X is algebraic, since the cited references make this assumption. However, the constructions are effectively local in X , so that the result holds also in the nonalgebraic setting with essentially the same arguments.

First, we observe that if $D \subset X$ is an anticanonical divisor, the map $Li^* : \text{Perf}(X) \rightarrow \text{Perf}(D)$ of derived stacks carries a one-shifted Lagrangian structure in a neighbourhood of any perfect complex on X with proper support. If X itself is proper, this can be obtained by viewing the anticanonical section σ as a relative orientation on the inclusion $D \rightarrow X$, in the sense of [9], and combining the main result of [9] with

the 2-shifted symplectic structure on the stack $\text{Perf}(-)$ of perfect complexes from [34]. More generally, one can invoke the results of [6]. We may then pull this Lagrangian structure back along the square

$$\begin{array}{ccc} \text{Perf}_0(X) & \longrightarrow & \text{Perf}(X) \\ \downarrow & & \downarrow \\ \text{Perf}_0(D) & \longrightarrow & \text{Perf}(D) \end{array}$$

thus obtaining a canonical isotropic structure on the map $\text{Perf}_0(X) \rightarrow \text{Perf}_0(D)$ over the locus of complexes with proper support.

Lemma 3.1 *The induced isotropic structure on $\text{Perf}_0(X) \rightarrow \text{Perf}_0(D)$ is nondegenerate, ie a Lagrangian structure.*

Since the construction of Lagrangian structures on mapping stacks is functorial for maps of the shifted symplectic structures on the target, the lemma is an immediate consequence of the definitions and the following basic fact:

Proposition 3.2 *The pullback of the 2-shifted symplectic structure on $\text{Perf}(-)$ along the canonical map $\text{Perf}_0(-) \rightarrow \text{Perf}(-)$ is nondegenerate, and thus defines a 2-shifted symplectic structure on $\text{Perf}_0(-)$.*

Proof Let $\text{Pic}(-) = B\mathbb{G}_m$ be the Picard stack. Then by definition

$$\text{Perf}_0(-) \cong \text{Perf}(-) \times_{\text{Pic}(-)} *$$

is the fibre of the derived determinant map $\det: \text{Perf}(-) \rightarrow \text{Pic}(-)$ from [43] at the point $* \rightarrow \text{Pic}(-)$ corresponding to the universal trivial line bundle. Let $x: \text{Spec}(A) \rightarrow \text{Perf}_0(-)$ be the point classifying a perfect complex \mathcal{E} on $\text{Spec}(A)$ with trivialized determinant. By [43, Proposition 3.2], we have a homotopy fibre sequence

$$x^* \mathbb{T}\text{Perf}_0(-) \rightarrow \mathcal{R}\text{End}_A(\mathcal{E}) \xrightarrow{\text{tr}} A$$

of complexes on X , where tr is the Illusie trace map, so that

$$x^* \mathbb{T}\text{Perf}_0(-) \cong \mathcal{R}\text{End}_A(\mathcal{E})_0 := \text{fibre}(\mathcal{R}\text{End}_A(\mathcal{E}) \xrightarrow{\text{tr}} A)$$

is the complex of “traceless” derived endomorphisms. Meanwhile, as constructed in [34, Section 2.3], the two-form underlying the 2-shifted symplectic structure on $\text{Perf}(-)$ is given by the trace pairing

$$\omega: \mathcal{R}\text{End}_A(\mathcal{E}) \overset{L}{\otimes}_A \mathcal{R}\text{End}_A(\mathcal{E}) \rightarrow \mathcal{R}\text{End}_A(\mathcal{E}) \xrightarrow{\text{tr}} A.$$

Hence the induced two-form on $\text{Perf}_0(-)$ is given by the restriction of the pairing ω to the traceless endomorphisms $\mathcal{R}\text{End}_A(\mathcal{E})_0$. But by definition, every element of $\text{Perf}(A)$ is equivalent to a strictly perfect one, ie a dg module \mathcal{E} whose underlying module is projective of finite type, ie a graded vector bundle. Hence the statement reduces to the classical nondegeneracy of the trace pairing on traceless matrices. \square

Restricting to the open substack $\mathcal{X}^{[n]} \subset \text{Perf}_0(X)$, we deduce that the map $\mathcal{X}^{[n]} \rightarrow \text{Perf}_0(D)$ is one-shifted Lagrangian, so that our groupoid

$$(\mathcal{X}, D)^{[n]} \cong \mathcal{X}^{[n]} \times_{\text{Perf}_0(D)} \mathcal{X}^{[n]}$$

carries a zero-shifted symplectic structure by the Lagrangian intersection theorem [34, Theorem 2.9]. But by Theorem 2.9, the derived stack $(\mathcal{X}, D)^{[n]}$ is a smooth classical scheme, so by [34, pages 297–298] this 0-shifted symplectic structure is an ordinary symplectic structure in the classical sense. Moreover, by definition, this symplectic structure on $(\mathcal{X}, D)^{[n]}$ is the difference of the pullbacks of the isotropic structure on $\mathcal{X}^{[n]}$ along the two maps $s, t: (\mathcal{X}, D)^{[n]} \rightarrow \mathcal{X}^{[n]}$. It is therefore automatically multiplicative, so that $(\mathcal{X}, D)^{[n]}$ has the structure of a symplectic groupoid. As for any symplectic groupoid, the embedding of the identity arrows $\mathcal{X}^{[n]} \hookrightarrow (\mathcal{X}, D)^{[n]}$ is Lagrangian, giving an identification $\mathbb{L}\mathcal{X}^{[n]}|_{\mathcal{Z}} \cong \mathbb{T}s|_{(\mathcal{Z}, \mathcal{Z}, \text{id})}$. The induced Poisson bivector on $\mathcal{X}^{[n]}$ is the composition of this isomorphism with the differential $\mathbb{T}s|_{(\mathcal{Z}, \mathcal{Z}, \text{id})} \rightarrow \mathbb{T}\mathcal{X}^{[n]}|_{\mathcal{Z}}$ of the target map. Considering the definition of the symplectic structure in terms of Serre duality and using the identifications of the (relative) tangent spaces from the proof of Theorem 2.9, we deduce that this Poisson structure is exactly the one defined by Bottacin.

In summary, we have the following:

Theorem 3.3 *Let (X, σ) be a Poisson surface with degeneracy curve $D = \sigma^{-1}(0) \subset X$. Then $(\mathcal{X}, D)^{[n]}$ carries a canonical symplectic structure, making it a symplectic groupoid over $\mathcal{X}^{[n]}$ that integrates Bottacin’s Poisson structure $\sigma^{[n]}$ on $\mathcal{X}^{[n]}$.*

Remark 3.4 The symplectic structure on $(\mathcal{X}, D)^{[n]}$ constructed above is the unique two-form for which the map $(s, t): (\mathcal{X}, D)^{[n]} \rightarrow \mathcal{X}^{[n]} \times \mathcal{X}^{[n]}$ is a symplectomorphism over the open dense set $(\mathcal{X} \setminus D)^{[n]} \times (\mathcal{X} \setminus D)^{[n]}$, where the latter is equipped with the symplectic form $\Omega := \text{pr}_1^* \omega - \text{pr}_2^* \omega$ obtained by pulling back the symplectic form ω on $\mathcal{X} \setminus D^{[n]}$ along the two projections pr_1, pr_2 . Thus, to prove that $(\mathcal{X}, D)^{[n]}$ is symplectic, one simply needs to check that the pullback of Ω to $(\mathcal{X}, D)^{[n]}$ has no poles, and similarly that the pullback of Ω^n has no zeroes.

These properties can be verified directly in at least two ways, giving a construction of the symplectic form that does not rely on the theory of shifted symplectic structures. The first way is to explicitly write down the induced nondegenerate pairing on the tangent space at any point of $(\mathcal{X}, D)^{[n]}$ using Grothendieck duality for coherent sheaves, and see directly that it agrees with the pullback of Ω . The second way begins with the observation that the desired properties are local and can be checked in codimension one. Thus, using Weinstein’s splitting theorem, one can reduce the problem to the case in which $n = 1$, and the reduced curve underlying D is smooth. In this case, the desired properties can be obtained by direct calculation using the explicit description of $(\mathcal{X}, D)^{[1]} \rightarrow \mathcal{X} \times \mathcal{X}$ in terms of blowups from Example 2.10, in the spirit of [18, Section 2.4], which treats the reduced case. \diamond

Since the connected components of the orbits of any symplectic groupoid are exactly the symplectic leaves of the corresponding Poisson structure, and moreover the orbits of any smooth algebraic groupoid are locally closed algebraic subvarieties, Theorem 3.3 has the following immediate consequences.

Corollary 3.5 *The symplectic leaves of $(X^{[n]}, \sigma^{[n]})$ are the connected components of the D-equivalence classes. In particular, they are locally closed in the Zariski topology when (X, σ) is algebraic, and if D is smooth they are in bijection with Young-diagram-valued functions on D as in [Theorem 2.24](#).*

Note that the stabilizer groups in $(X, D)^{[n]}$ are exactly the self D-equivalences, and meanwhile, as for any symplectic groupoid, the Lie algebras of these groups are exactly the conormal Lie algebras of the Poisson structure (ie the linearization of the Poisson structure in the direction transverse to the symplectic leaves) so we have the following:

Corollary 3.6 *If $Z \in X^{[n]}$ is any point, then its conormal Lie algebra*

$$\ker((\sigma^{[n]})^\# : T_Z^*X^{[n]} \rightarrow T_ZX^{[n]})$$

is canonically identified with the Lie algebra $\mathfrak{g}_{Z,D}$ of infinitesimal self-D-equivalences defined in [Definition 2.11](#) and described in [Lemma 2.12](#).

3.3 Characteristic leaves

Of particular interest to us are the symplectic leaves that are “characteristic” in the sense of [\[27\]](#); let us recall the definition. First, recall from [\[7; 35; 47\]](#) the notion of the modular vector field of a Poisson structure σ on a manifold W with respect to a volume form μ : it is the derivation of \mathcal{O}_W sending a function f to the μ -divergence of the Hamiltonian vector field of f . The modular vector field is an infinitesimal symmetry of the Poisson structure, and is independent of μ up to the addition of a Hamiltonian vector field, so that its projection to the normal space of any symplectic leaf is unambiguously defined. We then have the following.

Definition 3.7 [\[27\]](#) *A symplectic leaf of a Poisson manifold is *characteristic* if it is preserved by the flow of the modular vector field, ie the projection of the modular vector field to the normal bundle of the leaf is identically zero.*

Now suppose that (X, σ) is a Poisson surface, and let $D \subset X$ be the anticanonical divisor on which σ vanishes. Then the restriction of the modular vector field to D is independent of the choice of volume form, giving a canonically defined global vector field

$$\zeta \in H^0(\mathcal{T}_D).$$

The characteristic leaves in the Hilbert scheme then have the following description.

Proposition 3.8 *The symplectic leaf of $\sigma^{[n]}$ through a point $Z \in X^{[n]}$ is characteristic if and only if the derived subscheme $Z \cap^h D \subset D$ is preserved (up to equivalence) by the flow of ζ .*

Proof The problem is local in X around Z , so we may assume without loss of generality that X admits a global volume form μ and let $\tilde{\zeta}$ be the corresponding modular vector field, so that $\tilde{\zeta}|_D = \zeta$ by definition of ζ . Consider the product X^n and let $p_i : X^n \rightarrow X$ be the i^{th} projection. For a tensor ξ

on X let $\xi^n = p_1^* \xi + \cdots + p_n^* \xi$ denote its symmetric lift to X^n . Then σ^n , μ^n and $\tilde{\zeta}^n$ are, respectively, a Poisson structure, a volume form, and the corresponding modular vector field on X^n . These are invariant under permutation, and hence descend to the symmetric power $X^{(n)}$, giving a Poisson structure $\sigma^{(n)}$, a trivialization $\mu^{(n)}$ of the canonical sheaf, and a vector field $\tilde{\zeta}^{(n)}$, which agrees with the modular vector field over the smooth locus of $X^{(n)}$. We claim that these structures lift to corresponding structures on the Hilbert scheme under the Hilbert–Chow morphism $X^{[n]} \rightarrow X^{(n)}$. Indeed, $\sigma^{(n)}$ lifts to Bottacin’s structure $\sigma^{[n]}$ by definition, and $\mu^{(n)}$ lifts to a volume form $\mu^{[n]}$ since the Hilbert–Chow morphism is crepant. We may then form the modular vector field $\tilde{\zeta}^{(n)}$ of $\sigma^{[n]}$ with respect to $\mu^{[n]}$, which must agree with $\tilde{\zeta}^{(n)}$ over the smooth locus of $X^{(n)}$, and hence it must be a lift of $\tilde{\zeta}^{(n)}$ (which is necessarily unique). It follows that the modular flow on $X^{[n]}$ is given by pulling back subschemes $Z \subset X$ along the flow of $\tilde{\zeta}$ on X . The result now follows immediately, since $\tilde{\zeta}|_D = \zeta$. \square

Remark 3.9 Let us sketch an interpretation of this result in terms of derived symplectic geometry. We first remark, following a discussion with P Safronov, that the modular vector field of a Poisson manifold can be thought of as the “Hamiltonian vector field of the first Chern class of the leaf space”, in the following sense. If Y is a one-shifted symplectic stack, then the symplectic form induces an isomorphism $\mathbb{T}_Y \cong \mathbb{L}_Y[1]$. Under the induced isomorphism $H^1(\mathbb{L}_Y) \cong H^0(\mathbb{L}_Y[1]) \cong H^0(\mathbb{T}_Y)$, the first Chern class $c_1(Y) \in H^1(\mathbb{L}_Y)$ corresponds to a canonical vector field $\zeta \in H^0(\mathbb{T}_Y)$ (its “Hamiltonian vector field”). If $p: W \rightarrow Y$ is a Lagrangian morphism, then W inherits a 0-shifted Poisson structure whose symplectic leaves are the fibres of p , and the vector field ζ agrees up to a constant factor with the projection of the modular vector field to the leaf space. In particular, the characteristic symplectic leaves in W correspond to fixed points of ζ in Y .

In the case at hand, we have $Y = \text{Perf}_0(D)$, and \mathbb{T}_Y is the complex of traceless derived endomorphisms of the universal sheaf. One can then compute the first Chern class of Y using the Grothendieck–Riemann–Roch theorem, and show that the corresponding vector field ζ generates the one-parameter group of automorphisms of Y given by pulling back complexes along the modular vector field on the curve D as above, recovering [Proposition 3.8](#). \diamond

Corollary 3.10 *Let X be a Poisson surface with anticanonical divisor D . If $Z \in X^{[n]}$ lies on a characteristic symplectic leaf, then $Z \cap D$ is contained in the singular locus of D .*

Proof The modular vector field is nonzero at any smooth point of D . Hence if the support of $Z \cap D$ contains a smooth point, it will not be fixed by the flow of the modular vector field. \square

Lemma 3.11 *If X is connected, the codimension-two characteristic leaves are exactly the subvarieties of the form $\{Z \in X^{[n]} \mid Z \cap D = \{p\} \text{ as schemes}\}$, where p is a singular point of D .*

Proof If Z lies on a characteristic leaf of codimension two, then by [Corollary 3.10](#), $Z \cap D$ must be contained in the singular locus of D . If $\#(Z \cap D) > 1$, then the symplectic leaf through Z has codimension

greater than two, so can assume that $Z \cap D = \{p\}$ for some singular point $p \in D$. Since $p \in D$ is a singular point, it follows that Z itself is reduced at p , and hence $Z = \{p\} \sqcup Z'$, where $Z' \subset X \setminus D$. Hence the leaf is exactly an embedded copy of the symplectic variety $(X \setminus D)^{[n-1]}$. The transverse germ is isomorphic to the germ of X at p , and hence any such subvariety is a characteristic leaf. \square

Proposition 3.12 *If D has only nodal singularities, then each germ of $X^{[n]}$ has only finitely many characteristic symplectic leaves.*

Proof The problem is local and invariant under rescaling the Poisson structure by a constant, so by the log Darboux theorem it is enough to treat the case in which $X = \mathbb{C}^2$ and $\sigma = xy \partial_x \wedge \partial_y$ (eg see [1]), in which case the modular vector field is given by $\zeta = y\partial_y - x\partial_x$ and its flow integrates to the \mathbb{C}^* -action on X given by $t \cdot (x, y) = (tx, t^{-1}y)$. It is straightforward to verify that this action preserves the equivalence class of the matrices $\mathcal{D}(\mathbf{d})$, and changes the equivalence classes of the matrices $M(\mathbf{d}, k, u)$ by a suitable rescaling of u . Hence according to Proposition 2.26, the orbit data for $X^{[n]}$ that are fixed by the modular flow are exactly those given by a single matrix from the series $\mathcal{D}(\mathbf{d})$, of which only finitely many can occur in $X^{[n]}$. Hence there are only finitely many characteristic leaves globally in $X^{[n]}$. But the leaves are algebraic subvarieties (locally closed in the Zariski topology), so the analytic germ at every point can have only finitely many connected components (bounded by the number of irreducible components in the analytic or formal germ of the leaf closure), as desired. \square

For $n = 1$, the characteristic leaves are given by the open leaf $X \setminus D$ and the singular points of D , and hence are locally finite as long as D is reduced. However if D has nonnodal singularities and n is sufficiently large, one finds the existence of infinitely many characteristic symplectic leaves in $X^{[n]}$. Recall that a point $p \in D$ is a *double point* if the multiplicity of D at p is exactly two, ie the defining equation vanishes to order exactly two, in which case it is an A_k singularity for some $k \geq 1$, meaning that there exists coordinates such that D is given locally by the equation

$$y^2 - x^{k+1} = 0.$$

We have the following:

Proposition 3.13 *Let X be a Poisson surface with anticanonical divisor D . Then the following statements hold:*

- (1) $X^{[2]}$ has finitely many characteristic symplectic leaves if and only if the only singularities of D are double points.
- (2) If D has a nonnodal singularity, then $X^{[n]}$ has infinitely many characteristic leaves for every $n \geq n(D)$, where

$$n(D) := \begin{cases} 2 & \text{if } D \text{ has a triple point,} \\ 3 & \text{if } D \text{ has a singularity of type } A_k \text{ for some } k \geq 3, \\ 6 & \text{if } D \text{ has an } A_2 \text{ singularity.} \end{cases}$$

Proof Suppose $Z \in \mathcal{X}^{[2]}$. If Z is reduced, then $Z = \{p_1, p_2\}$ is a pair of distinct points and $\mathcal{X}^{[2]}$ is locally isomorphic to a neighbourhood of (p_1, p_2) in the product $X \times X$, and hence there are only finitely many characteristic leaves in a neighbourhood of Z .

If Z is not reduced, then it is the image of an embedding $\text{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \rightarrow X$ and is therefore classified by a pair (p, L) where $p \in X$ and $L < T_p X$ is a line. By [Corollary 3.10](#) we need only consider the case in which p is a singular point of D , in which case Z is contained entirely in D , so that σ vanishes at Z . Let us choose local coordinates x, y on X centred at p so that the bivector on X has the form $\sigma = f(x, y) \partial_x \wedge \partial_y$. The modular vector field is then given in terms of the partial derivatives f_x, f_y by the formula

$$\zeta = f_x \partial_y - f_y \partial_x$$

and hence the linearization of ζ at p is given by the matrix

$$\begin{pmatrix} -f_{xy}(0) & -f_{yy}(0) \\ f_{xx}(0) & f_{xy}(0) \end{pmatrix}$$

If f vanishes to order three or more, then this matrix is zero. Hence every line $L < T_p X$ is preserved by the flow of ζ , and we obtain infinitely many characteristic leaves in $\mathcal{X}^{[2]}$, and hence also in $\mathcal{X}^{[n]}$ for $n > 2$. On the other hand, if f vanishes to order two, this matrix is nonzero. Since it is traceless, it preserves at most two lines in $T_p X$ and so the set of characteristic leaves with support at p is finite.

Now suppose that D has an A_k -singularity at $p \in X$. Then we may choose coordinates (x, y) centred at p such that $f(x, y) = h(x, y)(y^2 - x^{k+1})$ and $h(0, 0) \neq 0$. Then the modular vector field of π is given by

$$\zeta = h(x, y)(-2y \partial_x - (k+1)x^k \partial_y) + (y^2 - x^{k+1})(h_x(x, y) \partial_y - h_y(x, y) \partial_x).$$

Suppose first that $k \geq 3$, and consider the ideal $\mathcal{I}_a := (y + ax^2, x^3)$, where $a \in \mathbb{C}$. Then each \mathcal{I}_a defines a point in $\mathcal{X}^{[3]}$ and contains the ideal $\mathcal{I}_D = (y^2 - x^{k+1})$ defining D . Therefore, multiplication by f annihilates $\text{Hom}(\mathcal{I}_a, \mathcal{O}_X/\mathcal{I}_a)$, and so $\sigma^{[3]}$ vanishes at \mathcal{I}_a . Moreover, the modular vector field ζ sends each generator of \mathcal{I}_a to another element of \mathcal{I}_a . Therefore, the vector field $\zeta^{[3]}$ on $\mathcal{X}^{[3]}$ vanishes at \mathcal{I}_a , so that the singleton $\{\mathcal{I}_a\}$ is a characteristic leaf for all $a \in \mathbb{C}$.

Finally, assume $k = 2$. Then we can assume $h(x, y) = 1$ in the expression for σ above by a theorem of Arnol'd [\[1\]](#). For $a \in \mathbb{C}$, consider the ideal $\mathcal{J}_a = (y^2 + ax^3, x^2y, xy^2)$, which defines an element of $\mathcal{X}^{[6]}$. We claim that multiplication by $f = y^2 - x^3$ annihilates each element $\varphi \in \text{Hom}(\mathcal{J}_a, \mathcal{O}_X/\mathcal{J}_a)$ for each $a \in \mathbb{C}$. Denoting the generators by $g_1 = y^2 + ax^3$, $g_2 = x^2y$, $g_3 = xy^2$, we have

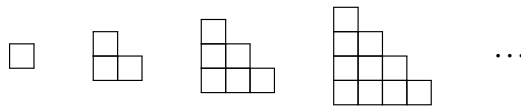
$$y\varphi(g_2) - x\varphi(g_3) = 0 \quad \text{and} \quad -xy\varphi(g_1) + ax^2\varphi(g_2) + y\varphi(g_3) = 0.$$

It is easy to deduce from these equations that the image of any such φ lies inside $(x, y)\mathcal{O}_X/\mathcal{J}_a$, which is clearly annihilated by multiplication by f . Therefore, the Poisson tensor $\sigma^{[6]}$ vanishes at \mathcal{J}_a . Moreover, the vector field $\zeta = -2y \partial_x - 3x^2 \partial_y$ sends each generator of \mathcal{J}_a to another element of \mathcal{J}_a . Therefore, $\zeta^{[6]}$ vanishes at \mathcal{J}_a , so that the singleton $\{\mathcal{J}_a\}$ is a characteristic symplectic leaf for each $a \in \mathbb{C}$. \square

4 Local models and holonomicity

4.1 Coordinates on the Hilbert scheme

There are two well-known methods for constructing coordinate charts on the Hilbert schemes, indexed by Young diagrams: the Haiman coordinates [19], and the Ellingsrud–Strømme coordinates [12]. In this section we use these coordinate systems to describe the local behaviour of Bottacin’s Poisson structures. Of particular relevance to us are the charts associated to the “triangular” Young diagrams



which are defined as follows.

Choose a point $p \in X$ and an integer $k > 0$, and let $p(k)$ be the k^{th} order neighbourhood of p , ie the vanishing locus of the ideal $\mathfrak{m}_p^{k+1} \subset \mathcal{O}_X$ where \mathfrak{m}_p is the maximal ideal corresponding to p . In local coordinates (x, y) on an open set U centred at p , the scheme $p(k)$ is expressed as the vanishing locus of the monomials $x^j y^l$ where $j + l \geq k$, so that $\mathcal{O}_{p(k), p}$ has a basis given by the monomials $x^j y^l$ with $j + l < k$. The latter are in bijection with the boxes of the triangular Young diagram

$$\triangle := (k > k - 1 > k - 2 > \dots)$$

so that

$$p(k) \in X^{[n]},$$

where

$$n := |\triangle| = \frac{1}{2}k(k + 1)$$

is the size of \triangle . The point $p(k) \in X^{[n]}$ then has an open neighbourhood $U_\triangle \subset U^{[n]} \subset X^{[n]}$ given by

$$U_\triangle := \{Z \in U^{[n]} \mid \text{the monomials } (x^j y^l)|_Z \text{ for } j + l < k \text{ form a basis for } H^0(\mathcal{O}_Z)\} \subset X^{[n]},$$

and the two coordinate systems on these charts are defined as follows.

- **Ellingsrud–Strømme coordinates** If $E \in \mathbb{C}^{(k+1) \times k}$ is a $(k + 1) \times k$ matrix, let

$$(4-1) \quad S_E(x, y) := E + \begin{pmatrix} -x & 0 & 0 & \dots & 0 \\ y & -x & 0 & \dots & 0 \\ 0 & y & -x & \dots & 0 \\ 0 & 0 & y & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -x \\ 0 & 0 & 0 & \dots & y \end{pmatrix} = E - x \begin{pmatrix} I_k \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ I_k \end{pmatrix},$$

and define a subscheme $Z(E) \subset X$ by the formula

$$Z(E) := \text{vanishing locus of the maximal minors of } S_E(x, y).$$

Then the map $E \mapsto Z(E)$ identifies a neighbourhood of the origin in $\mathbb{C}^{(k+1) \times k}$ with U_{Δ} . Note that $S_E(x, y)$ is a Hilbert–Burch syzygy matrix for $Z(E)$. The functions

$$E_i^j : U_{\Delta} \rightarrow \mathbb{C} \quad \text{for } 0 \leq i \leq k - 1 \text{ and } 0 \leq j \leq k$$

sending $Z(E)$ to the (j, i) -entry of the corresponding matrix E are the Ellingsrud–Strømme coordinates on U_{Δ} . Note that here we start numbering of rows and columns from 0.

- **Haiman coordinates** For every $Z \in U_{\Delta}$, the ideal \mathcal{I}_Z is generated by functions of the form

$$(4-2) \quad f_{Z,j} = x^j y^{k-j} - \sum_{a+b \leq k-1} c_{ab}^j(Z) x^a y^b \quad \text{for } 0 \leq j \leq k,$$

where c_{ab}^j are certain regular functions on U_{Δ} called Haiman’s functions. Haiman’s coordinates are then given by the collection

$$C_i^j := c_{i,k-1-i}^j \quad \text{for } 0 \leq i \leq k - 1 \text{ and } 0 \leq j \leq k.$$

These coordinate systems are related by the linear transformations

$$E_i^j = C_j^{i+1} - C_{j-1}^i \quad \text{and} \quad C_i^j = \begin{cases} E_{j-1}^i + E_{j-2}^{i-1} + \dots + E_{j-i-1}^0 & \text{if } j \geq i + 1, \\ -E_j^{i+1} - E_{j+1}^{i+2} - \dots - E_{j+k-i-1}^k & \text{if } j \leq i, \end{cases}$$

where we use the convention that $C_{-1}^j = C_k^j = 0$ for any j . One way to explicitly verify the second transformation is by expanding the maximal minors of $S_E(x, y)$, considering only the linear term in E (equivalently, total degree $k - 1$ in x, y).

Remark 4.1 Each set of coordinates E_i^j and C_i^j splits into two halves: $j \leq i$ and $j \geq i + 1$. In the formula above, the E -coordinates in one half are expressed in terms of the C -coordinates in the other half, and vice versa. ◇

Remark 4.2 In the particular case where $X = \mathbb{C}^2$ and (x, y) are the standard coordinates, these charts are defined on the whole space of $(k + 1) \times k$ -matrices, giving open embeddings $\mathbb{C}^{(k+1) \times k} \hookrightarrow (\mathbb{C}^2)^{[n]}$ for all $k > 0$. ◇

An important feature of these coordinate charts is that their construction is \mathbb{C}^\times -equivariant, in the following sense. Define an action of \mathbb{C}^\times on the germ of X at p by dilation of the coordinates we chose:

$$u \cdot (x, y) = (ux, uy) \quad \text{for } u \in \mathbb{C}^\times.$$

By functoriality, we have an induced action on the germ of $X^{[n]}$ at $p(k)$.

Lemma 4.3 *The Ellingsrud–Strømme and Haiman coordinates have weight one with respect to the induced \mathbb{C}^\times -action, ie*

$$u \cdot E_i^j = u E_i^j \quad \text{and} \quad u \cdot C_i^j = u C_i^j$$

for all indices i, j and all $u \in \mathbb{C}^\times$.

Proof Since the two coordinate systems are related by a linear transformation, it suffices to prove the statement for one of them; we do so for the Ellingsrud–Strømme coordinates. For this, note that

$$S_E(ux, uy) = E - ux \begin{pmatrix} I_k \\ 0 \end{pmatrix} + uy \begin{pmatrix} 0 \\ I_k \end{pmatrix} = u \left(u^{-1}E - x \begin{pmatrix} I_k \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ I_k \end{pmatrix} \right) = uS_{u^{-1}E}(x, y).$$

Hence the ideals generated by the minors of the matrices $S_E(ux, uy)$ and $S_{u^{-1}E}(x, y)$ are the same, which implies that the Ellingsrud–Strømme coordinates scale linearly, as desired. \square

4.2 Expression for the Poisson structure

Our interest in the triangular charts stems, in part, from the fact that they exhibit all possible local behaviours of Bottacin’s Poisson structures. Assume as before that σ is a nonzero Poisson structure on a surface X , vanishing on the anticanonical divisor D . Recall from [27] that two germs W_1 and W_2 of Poisson manifolds are *stably equivalent* if there exist symplectic germs S_1 and S_2 such that $W_1 \times S_1 \cong W_2 \times S_2$. Then we have the following:

Lemma 4.4 *Let (X, σ) be a Poisson surface as above and suppose $Z \in X^{[m]}$ for some $m \geq 0$. Then the germ of $(X^{[m]}, \sigma^{[m]})$ at Z is stably equivalent to a product of germs of points in the triangular charts of Hilbert schemes $X^{[k(k+1)/2]}$ for $k \geq 0$.*

Proof Any element $Z \in X^{[m]}$ has an analytic neighbourhood consisting of products of neighbourhoods of its connected components, so we may assume without loss of generality that Z is supported at a single point $p \in D$. It suffices to show that there exists an integer $k > 0$ and a subscheme $Z' \subset X \setminus D$ such that $Z \sqcup Z'$ lies in the triangular chart $U_{\triangle} \subset X^{[k(k+1)/2]}$, since in this case the germ at Z' is symplectic and hence the germs at Z and at $Z \sqcup Z'$ are stably equivalent.

To see this choose local coordinates on an open set U centred at p and choose k large enough that $H^0(\mathcal{O}_Z)$ is spanned by the images of the monomials $A = \{x^j y^l \mid j + l < k\} \subset H^0(\mathcal{O}_U)$. Now choose a collection of distinct points $p_1, p_2, \dots \in U \setminus (U \cap D)$ such that the evaluation maps $ev_{p_j} : H^0(\mathcal{O}_U) \rightarrow \mathbb{C}$ form a basis for $\text{span}(A)^\vee$. After reordering, we can assume by linear independence that $ev_{p_1}, \dots, ev_{p_{k-m}}$ restrict to a basis of $(\ker(\text{span}(A) \rightarrow H^0(\mathcal{O}_Z)))^\vee = \text{coker}(H^0(\mathcal{O}_Z)^\vee \rightarrow \text{span}(A)^\vee)$, and then $Z' = \{p_1, \dots, p_{k-m}\}$ is the desired subscheme. \square

An explicit expression for the Poisson bracket can be obtained via the following algorithm:

- (1) First obtain a formula in the case of the Poisson structure $\sigma_0 := \partial_x \wedge \partial_y$; we do so in [Section 4.3](#) below.
- (2) Define the *recursion operators* J_x, J_y to be the endomorphisms of the tangent bundle given by the action of x, y on the tangent spaces under the identification

$$T_Z U_{\triangle} \cong \text{Hom}_U(\mathcal{I}_Z, \mathcal{O}_Z).$$

Note that these operators have weight one, which implies they are linear in Ellingsrud–Strømme or Haiman coordinates. Explicitly, they are given as follows (we postpone their derivation to Section 4.4):

$$(4-3) \quad \begin{aligned} J_x \cdot \partial_{C_i^j} &= \sum_{b=0}^k E_i^b \partial_{E_{j-1}^b} - \sum_{a=0}^{k-1} E_a^j \partial_{E_a^{i+1}}, \\ J_y \cdot \partial_{C_i^j} &= \sum_{b=0}^k E_i^b \partial_{E_j^b} - \sum_{a=0}^{k-1} E_a^j \partial_{E_a^i}. \end{aligned}$$

(3) Now given an arbitrary Poisson structure

$$\sigma = f(x, y) \partial_x \wedge \partial_y = f(x, y) \sigma_0$$

on U , it follows from Bottacin’s formula that $\sigma^{[n]}$ is the bivector corresponding to the linear map $(\sigma^{[n]})^\# : T^*U_\triangleleft \rightarrow TU_\triangleleft$ given by the formula

$$(\sigma^{[n]})^\# = f(J_x, J_y) (\sigma_0^{[n]})^\#.$$

Note that the pair $(\sigma^{[n]}, \sigma_0^{[n]})$ is a nondegenerate bi-Hamiltonian structure and $f(J_x, J_y)$ is its associated recursion operator in the sense of integrable systems; this is the source of our name for J_x and J_y .

In the following sections we describe the local structure in the cases in which D has at worst nodal singularities; by the Darboux theorem and its generalizations, this corresponds to the case in which σ is homogeneous of nonpositive weight in a suitable chart (x, y) .

4.3 Darboux coordinates

Suppose that $p \in X \setminus D$. Then σ is nondegenerate at p , so by the Darboux theorem there exists coordinates (x, y) centred at p such that

$$\sigma = \partial_x \wedge \partial_y.$$

In particular σ has weight -2 , ie is constant, and hence the corresponding Poisson structure $\sigma^{[n]}$ is induced by a constant symplectic structure $\omega^{[n]}$ in the chart U_\triangleleft . Natural Darboux coordinates for $\omega^{[n]}$ are then given by the following.

Proposition 4.5 *The Ellingsrud–Strømme and Haiman coordinates on U_\triangleleft are symplectically dual, in the sense that*

$$\{E_i^j, C_{i'}^{j'}\} = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j', \\ 0 & \text{otherwise.} \end{cases}$$

Hence E_i^j and C_i^j for $j \leq i$ form Darboux coordinates, ie

$$\omega^{[n]} = \sum_{0 \leq j \leq i \leq k-1} dC_i^j \wedge dE_i^j.$$

Proof U_{\triangleleft} has an open dense set consisting of reduced schemes Z lying in U . Near such an element we have Darboux coordinates $x_1, y_1, \dots, x_n, y_n$ indicating the x and y coordinates of the support the subscheme Z' near Z . Thus

$$\omega^{[n]} = \sum_{i=1}^n dy_i \wedge dx_i = \text{Tr}(dM_y \wedge dM_x),$$

where M_x and M_y are the matrices of functions representing the operators of multiplication by x and y on $H^0(\mathcal{O}_Z)$ in the basis of indicator functions of the points of Z . By Lemma 4.6 below, the right-hand side may in fact be computed using the same formula with respect to *any* basis for $H^0(\mathcal{O}_Z)$. In particular, in U_{\triangleleft} we may choose the basis of monomials $e_{ij} := x^i y^j$, with $i + j \leq k - 1$, and compute the actions x and y using Haiman’s defining relations (4-2) for \mathcal{O}_Z :

$$\begin{aligned} x \cdot e_{ij} &= \begin{cases} e_{i+1,j} & \text{if } i + j \leq k - 2, \\ \sum_{a+b \leq k-1} c_{ab}^k e_{ab} & \text{if } i + j = k - 1, \end{cases} \\ y \cdot e_{ij} &= \begin{cases} e_{i,j+1} & \text{if } i + j \leq k - 2, \\ \sum_{a+b \leq k-1} c_{ab}^k e_{ab} & \text{if } i + j = k - 1, \end{cases} \end{aligned}$$

from which we deduce that

$$\omega^{[n]} = \text{Tr}(dM_y \wedge dM_x) = \sum_{i+j, a+b \leq k-1} d\langle y \cdot e_{ij}, e_{ab}^* \rangle \wedge d\langle x \cdot e_{ab}, e_{ij}^* \rangle = \sum_{i,a=0}^{k-1} dC_a^i \wedge dC_i^{a+1}.$$

Therefore

$$(4-4) \quad \omega^{[n]}(\partial_{C_i^j}, -) = dC_j^{i+1} - dC_{j-1}^i = dE_i^j,$$

so that the Hamiltonian vector field of E_i^j is $\partial_{C_i^j}$, as claimed.

To prove the claim about the Darboux coordinates, note that due to Remark 4.1 we have $\{E_i^j, E_{i'}^{j'}\} = \{C_i^j, C_{i'}^{j'}\} = 0$ for all $j \leq i, j' \leq i'$. □

Lemma 4.6 *Let A and B be a pair of commuting square matrices with functional entries. Then the two-form $\text{Tr}(dA \wedge dB)$ is invariant under simultaneous conjugation of A and B via an invertible matrix with functional entries.*

Proof We learned this from the preprint version of [45, proof of Proposition C.1]. Since the published version does not contain the proof, we include it here for completeness. Let $\tilde{A} = gAg^{-1}, \tilde{B} = gBg^{-1}$ for some invertible matrix of functions g . Then we have

$$\text{Tr}(d\tilde{A} \wedge d\tilde{B}) = \text{Tr}(dA \wedge dB) - \text{Tr} d([A, B] g^{-1} dg).$$

which gives the result. □

4.4 Recursion operators

In this subsection, we derive the formulae for the recursion operators (4-3) in the triangular chart of the Hilbert scheme of $X = \mathbb{C}^2$. Note that since x and y have weight one with respect to the natural \mathbb{C}^\times -action, the operators J_x and J_y also have weight one, and in particular they send the basis vector fields $\partial_{C_i^j}$ in Haiman coordinates to linear vector fields. By applying the automorphism that interchanges x and y (which has the form $E_i^j \mapsto -E_{k-i-1}^{k-j}$ in Ellingsrud–Strømme coordinates) the calculation of J_x reduces to that of J_y , so we will just explain how to do the latter.

The tangent vector $\partial_{C_i^j}$, with $0 \leq i \leq k - 1$ and $0 \leq j \leq k$, corresponds to an element $\varphi_i^j \in \text{Hom}(\mathcal{I}, \mathcal{O}_Z)$ of the form

$$\varphi_i^j(f_r) = \sum_{a+b \leq k-1} \gamma_{abi}^{rj} x^a y^b \pmod{\mathcal{I}},$$

where $f_r = f_{z,r}$ for $r = 0, 1, \dots, k$ are the generators (4-2) of \mathcal{I} . Note that each f_r is a homogeneous polynomial of degree k in variables x, y and the Haiman coordinates C_i^j . Therefore, each γ_{abi}^{rj} is a homogeneous polynomial of degree $k - 1 - a - b$ in Haiman coordinates. By definition of the Haiman coordinates, we have

$$(4-5) \quad \varphi_i^j(f_r) = \delta_{jr} x^i y^{k-1-i} + \sum_{a=0}^{k-2} \theta_{ai}^{rj} x^a y^{k-2-a} \pmod{V_3},$$

where δ_{jr} is the Kronecker delta symbol, θ_{ai}^{rj} are linear functions in Haiman coordinates, and for $l \geq 0$ we denote by

$$V_l := \mathcal{I} \oplus \bigoplus_{c+d \leq k-l} \mathbb{C} \cdot x^c y^d \subset \mathbb{C}[x, y]$$

the linear subspace generated by \mathcal{I} and the monomials of degree at most $k - l$.

By using the equalities $\varphi_i^j(xf_r - yf_{r+1}) = x\varphi_i^j(f_r) - y\varphi_i^j(f_{r+1})$ for $r = 0, 1, \dots, k - 1$, and looking the terms modulo V_2 , we obtain the following linear equations on γ :

$$\theta_{ai}^{r+1,j} - \theta_{a-1,i}^{rj} = \delta_{rj} C_a^{i+1} - \delta_{r+1,j} C_a^i - \delta_{ai} (C_j^{r+1} - C_{j-1}^r) \quad \text{for } 0 \leq i, r \leq k - 1 \text{ and } 0 \leq j, a \leq k.$$

Here and below we adopt the convention that if an index of C or θ falls outside the allowed range, then we declare the value of such C or θ to be zero.

Finally, to compute $J_y \cdot \partial_{C_i^j}$, we need to calculate $y\varphi_i^j$. By (4-5), one has

$$\begin{aligned} y\varphi_i^j(f_b) &= \delta_{jb} x^i y^{k-i} + \sum_{a=0}^{k-2} \theta_{ai}^{bj} x^a y^{k-1-a} = \delta_{bj} \sum_{a=0}^{k-1} C_a^i x^a y^{k-1-a} + \sum_{a=0}^{k-2} \theta_{ai}^{bj} x^a y^{k-1-a} \\ &= \sum_{a=0}^{k-1} (\delta_{bj} C_a^i + \theta_{ai}^{bj}) x^a y^{k-1-a} \pmod{V_2}. \end{aligned}$$

This implies that

$$J_y \cdot \partial C_i^j = \sum_{b=0}^k \sum_{a=0}^{k-1} (\delta_{bj} C_a^i + \theta_{ai}^{bj}) \frac{\partial}{\partial C_a^b},$$

and therefore

$$\begin{aligned} \langle J_y \cdot \partial C_i^j, dE_a^b \rangle &= \left\langle \sum_{\beta=0}^k \sum_{\alpha=0}^{k-1} (\delta_{\beta j} C_\alpha^i + \theta_{\alpha i}^{\beta j}) \frac{\partial}{\partial C_\alpha^\beta}, dC_b^{a+1} - dC_{b-1}^a \right\rangle \\ &= \delta_{a+1,j} C_b^i - \delta_{aj} C_{b-1}^i + \theta_{b,i}^{a+1,j} - \theta_{b-1,i}^{a,j} \\ &= \delta_{a+1,j} C_b^i - \delta_{aj} C_{b-1}^i + \delta_{aj} C_b^{i+1} - \delta_{a+1,j} C_b^i - \delta_{bi} (C_j^{a+1} - C_{j-1}^a) \\ &= \delta_{aj} (C_b^{i+1} - C_{b-1}^i) - \delta_{bi} (C_j^{a+1} - C_{j-1}^a) \\ &= \delta_{aj} E_i^b - \delta_{bi} E_a^j, \end{aligned}$$

as desired.

4.5 Smooth curves and the Lie algebra $\text{aff}_k(\mathbb{C})$

Now suppose that $p \in D$ is a smooth point. Then by the log Darboux theorem there exist coordinates (x, y) centred at p such that D is given by $y = 0$ and

$$\sigma = y \partial_x \wedge \partial_y$$

In particular σ is linear in these coordinates, thus $\sigma^{[n]}$ is linear in the corresponding Haiman/Ellingsrud–Strømme coordinates, so that it corresponds dually to a Lie algebra structure on the space of $(k + 1) \times k$ matrices.

We claim that this is precisely the Lie algebra $\text{aff}_k(\mathbb{C}) \cong \mathfrak{gl}_k(\mathbb{C}) \times \mathbb{C}^k$ of the group $\text{Aff}_k(\mathbb{C}) \cong \text{GL}_k(\mathbb{C}) \times \mathbb{C}^k$ of affine transformations of \mathbb{C}^k . To see this, we use the natural embedding

$$\psi : \text{Aff}_k(\mathbb{C}) \hookrightarrow \text{GL}_{k+1}(\mathbb{C}), \quad \psi(g, v) = \begin{pmatrix} g & v \\ 0 & 1 \end{pmatrix},$$

where $g \in \text{GL}_k(\mathbb{C})$ and $v \in \mathbb{C}^k$, which identifies $\text{aff}_k(\mathbb{C})$ with $\mathbb{C}^{(k+1) \times k}$. Identifying the duals via the trace pairing, the coadjoint action of $\text{Aff}_k(\mathbb{C})$ on the vector space $\text{aff}_k(\mathbb{C})^\vee \cong \mathbb{C}^{(k+1) \times k}$ is given by the formula

$$(g, v) \cdot E := \psi(g, v) E g^{-1}.$$

Now if $E \in \mathbb{C}^{(k+1) \times k}$, the syzygy matrix of the scheme $Z(E)$ is the matrix $S_E(x, y)$ from (4-1), and hence $i^* \mathcal{I}_{Z(E)}$ is presented as the cokernel of the matrix

$$S_E(x, y)|_D = S_E(x, 0) = E - x \begin{pmatrix} I_k \\ 0 \end{pmatrix},$$

and direct computation shows that

$$\psi(g, v)S_E(x, 0)g^{-1} = S_{(g,v)\cdot E}(x, 0),$$

so that (g, v) induces a D-equivalence from $Z(E)$ to $Z((g, v) \cdot E)$. This defines a map of groupoids between $\text{Aff}_k(\mathbb{C}) \times \text{aff}_k(\mathbb{C})^\vee$ and $(X, D)^{[n]}$ over the open sets corresponding to U_\triangleleft , giving a local model for the symplectic groupoid $(X, D)^{[n]}$:

Theorem 4.7 *The map $E \rightarrow Z(E)$ gives a Poisson isomorphism from an open set in $\text{aff}_k(\mathbb{C})^\vee$ to U_\triangleleft , and the corresponding map of groupoids is a symplectomorphism onto its image.*

Proof We first prove that the map $E \mapsto Z(E)$ is Poisson. For this we compute the Poisson bivector on U_\triangleleft following the algorithm of Section 4.2. From the formulae in Proposition 4.5 and (4-3) we have $\{E_i^j, E_a^b\} = \langle J_y \cdot \partial_{C^j}, dE_a^b \rangle = \delta_{aj} E_i^b - \delta_{bi} E_a^j$, which is exactly the formula for the bracket on $\text{aff}_k(\mathbb{C})^\vee$ in the given basis, as desired. Hence the map $E \mapsto Z(E)$ is Poisson.

This implies immediately that the map of groupoids is compatible with symplectic structures, and hence it is étale onto its image. It remains to check that it is injective. That the map is injective on objects of the groupoid is simply the statement that U_\triangleleft is a chart, so we only need to show injectivity on the target (or source) fibres. Thus, it suffices to fix $E \in \text{aff}_k(\mathbb{C})^\vee$ and show that we obtain an injective map from $\text{Aff}_k(\mathbb{C}) \times \{E\}$ to D-equivalences. For this, we note that any element $(g, v) \in \text{Aff}_k(\mathbb{C})$ can be recovered from the corresponding D-equivalence, where the latter is viewed as an automorphism of the first two terms of the resolution (2-9). Indeed, evaluating the aforementioned complex at p and considering the endomorphism of the second term, $\mathcal{O}_{\{p\}}^{\oplus(k+1)}$, we obtain a $(k + 1) \times (k + 1)$ -matrix lying in $\text{Aff}_k(\mathbb{C})$. This evaluation is well-defined modulo isomorphisms of D-equivalences, ie, depends only on the homotopy class of the map of cochain complexes, because the nontrivial differential in the two-term complex becomes zero after restriction to p . □

Corollary 4.8 *For $X = U = \mathbb{C}^2$ and $\sigma = y\partial_x \wedge \partial_y$, we have $U_\triangleleft \cong \text{aff}_k(\mathbb{C})^\vee$ as Poisson manifolds. Under this identification:*

- (1) *The symplectic leaves in U_\triangleleft are identified with the coadjoint orbits of $\text{aff}_k(\mathbb{C})$.*
- (2) *The linear subspace $\mathfrak{gl}_k(\mathbb{C})^\vee \subset \text{aff}_k(\mathbb{C})^\vee$ corresponds to the set \mathcal{A}_k of elements $Z \in U_\triangleleft$ such that $\#(Z \cap D) = k$.*
- (3) *The nilpotent cone in $\mathfrak{gl}_k(\mathbb{C})^\vee$ is identified with the set of elements $Z \in U_\triangleleft$ such that $\mathcal{I}_{Z \cap D} = (x^k, y)$.*

Proof It remains to establish points (2) and (3) above. For this, note that the subspace $\mathfrak{gl}_k(\mathbb{C})^\vee$ is cut out by the k equations $E_i^k = 0$ for $i = 0, 1, \dots, k - 1$. If $E_i^k \neq 0$ for some $0 \leq i \leq k - 1$, then the $(k - i)$ th minor of $S_E(x, 0)$ is a polynomial whose highest-degree term is $\pm E_i^k x^{k-1}$. Since this function vanishes on $Z \cap D$, we deduce that $\#Z \cap D < k$. Conversely, if $E_i^k = 0$ for all i , then all maximal minors

of $S_E(x, 0)$ are zero, except the last one, which is equal to the characteristic polynomial of the square matrix $\tilde{E} := (E_i^j)_{i,j=0}^{k-1}$, which has degree exactly k . Hence $\#(Z \cap D) = k$, which establishes statement (2). Statement (3) follows immediately since \tilde{E} lies in the nilpotent cone if and only if its characteristic polynomial is x^k . □

Remark 4.9 More generally, if (X, σ) is a Poisson surface with smooth zero divisor D , then for any $k \leq n$ one can define a locally closed Poisson submanifold $W \subset X^{[n]}$ as the locus of elements $Z \in X^{[n]}$ such that $\#(Z \cap D) = k$, and one has a natural projection $W \rightarrow D^{(k)}$, whose fibres are Poisson subvarieties of $X^{[n]}$. Away from the big diagonal in $D^{(k)}$, the fibres are smooth and symplectic, but in general they can be singular. Corollary 4.8 implies that the singularities are products of slices of nilpotent orbits in $\mathfrak{sl}_m(\mathbb{C})$ for $m \leq k$. These singularities are all symplectic. Near a most singular fibre $(Z \cap D) = k \cdot p$ with $n \geq \frac{1}{2}k(k + 1)$, the family restricts to the universal Poisson deformation of the nilpotent cone (times a parameter giving the centre of mass of Z , which makes sense in local coordinates). We expect it to have a unique simultaneous resolution $\tilde{W} \rightarrow W$, giving a global analogue of the Grothendieck–Springer resolution, which restricted to the fibre over $k \cdot p$ is a global analogue of the Springer resolution. It would be interesting to further study these objects. For example, are all Poisson deformations of W, \tilde{W} given by deformations of the Hilbert scheme? And are all Poisson deformations of the fibre over $k \cdot p \in D$ given by deforming the Hilbert scheme and moving the point in $D^{(k)}$ (at least infinitesimally)? ◇

Using this result, we obtain a complete characterization of the closure relation between symplectic leaves as a condition on the corresponding Young diagrams. Note that the necessity of this condition was proven by Rains in [38, Section 11.2], but our proof of sufficiency uses our local normal form in an essential way.

Proposition 4.10 *Let (X, σ) be a Poisson surface with a smooth zero divisor D , and let $\mu, \tilde{\mu}$ be global orbit data as in Lemma 2.15. Then the closure of the symplectic leaf $X_\mu^{[n]}$ contains $X_{\tilde{\mu}}^{[n]}$ if and only if for every $p \in D$, we have the inclusion $\text{hc}(\mu(p)) \subseteq \text{hc}(\tilde{\mu}(p))$, or equivalently, $\mu(p) \leq \tilde{\mu}(p)$ in the dominance order on Young diagrams.*

Proof The second equivalence is a simple consequence of the definitions. Let us prove the first equivalence.

Let us start by remarking for any ideal $Z \in X^{[m]}$, with $m < n$, the set

$$\{Z' \in X^{[n]} \mid Z' \supseteq Z\}$$

is a closed subvariety. By Theorem 2.24, every element $Z \in X_\mu^{[n]}$ satisfies $Z \supset \bigsqcup_{p \in D} Z_p^{\mu(p)}$. Therefore, if $X_{\tilde{\mu}}^{[n]}$ lies in the closure of $X_\mu^{[n]}$, then we have $Z_p^{\tilde{\mu}(p)} \supseteq Z_p^{\mu(p)}$ for each $p \in D$.

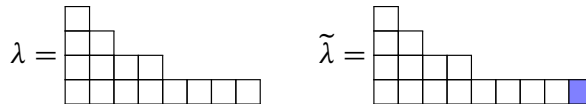
For the converse, suppose that $\mu \leq \tilde{\mu}$. We wish to show that $X_\mu^{[n]} \subset \overline{X_{\tilde{\mu}}^{[n]}}$. It suffices to prove this result in the complex topology, since it is finer.

It is enough to consider the case when both μ and $\tilde{\mu}$ consist of one nontrivial Young diagram each, concentrated at the same point $p \in D$. By abuse of notation, we denote these diagrams simply by μ and $\tilde{\mu}$.

We can also, without loss of generality, take $X = \mathbb{C}^2$ equipped with the Poisson bivector, $\sigma = y \partial_x \wedge \partial_y$, and $p = (0, 0)$. Furthermore, note that by [Theorem 2.24](#) it suffices to prove the closure relation for the open dense sets consisting of subschemes of the form $Z_p^{\mu(p)} \sqcup Z'$, where $Z' \subset X \setminus D$ is reduced. Such an element has an analytic neighbourhood of the form $U^{[k]} \times V^{[n-k]}$, where U is an analytic neighbourhood of p and V is an analytic neighbourhood of Z' such that $\bar{V} \subset X \setminus D$ and $\bar{U} \cap \bar{V} = \emptyset$. Hence by adding or removing points from Z' , we see that it is enough to prove the statement for any single value of $n \geq |\text{hc}(\tilde{\mu})|$.

Let $\lambda = \text{hc}(\mu)$ and $\tilde{\lambda} = \text{hc}(\tilde{\mu})$. We consider two special cases for the pair $(\lambda, \tilde{\lambda})$ below, and then explain how the general case follows from these.

Case 1 ($\lambda_1 < \tilde{\lambda}_1$, but $\lambda_i = \tilde{\lambda}_i$, for $i > 1$) In this case, by induction on the difference $\tilde{\lambda}_1 - \lambda_1$, we may assume without loss of generality that $\lambda_1 + 1 = \tilde{\lambda}_1$. In other words, the Young diagram $\tilde{\lambda}$ is obtained from λ by adding one box to the first row. For instance:



We may assume without loss of generality that $n = |\lambda| + 1 = |\tilde{\lambda}|$. Then $X_{\mu}^{[n]}$ consists of only one point given by $Z_p^{\tilde{\mu}}$, whereas $X_{\mu}^{[n]}$ is two-dimensional and its generic point is of the form $Z_p^{\mu} \sqcup \{(x_1, y_1)\}$, $y_1 \neq 0$. The idea is to send the point (x_1, y_1) to the origin along a curve that is tangent to the divisor $\{y = 0\}$ to a high enough order, so that the limiting ideal will be $Z_p^{\tilde{\mu}}$. Here is the calculation that makes this heuristic precise.

Let us write

$$\mathcal{I}_{Z_p^{\mu}} = (x^{a_j} y^j, j = 0, 1, \dots, \ell),$$

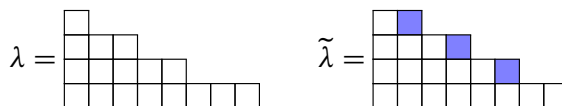
where $a_j = \lambda_{j+1}$ for $j = 0, 1, \dots, \ell$ and $\ell = \lambda_1^T$. Let $x_1 = \varepsilon$, and $y_1 = \varepsilon^N$, where $\varepsilon \in \mathbb{C}^*$, and N is a large positive integer to be determined in a moment. Then

$$\begin{aligned} \mathcal{I}_{Z_p^{\mu} \sqcup \{(x_1, y_1)\}} &= (x^{a_j} y^j, j = 0, 1, \dots, \ell) \cap (x - \varepsilon, y - \varepsilon^N) \\ &= (x^{a_0+1} - \varepsilon x^{a_0}, x^{a_j} y^j - \varepsilon^{b_j} x^{a_0}, j = 1, 2, \dots, \ell), \end{aligned}$$

where $b_j = jN + a_j - a_0$ for $j = 1, 2, \dots, \ell$. Now choose N so that $b_j > 0$ for all j . Then

$$Z_p^{\mu} \sqcup \{(x_1, y_1)\} \rightarrow Z_p^{\tilde{\mu}} \quad \text{as } \varepsilon \rightarrow 0.$$

Case 2 ($\lambda_1 = \tilde{\lambda}_1$) In this case, we cannot assume that $|\tilde{\lambda}| = |\lambda| + 1$, because we must maintain the condition that the Young diagrams involved are horizontally convex. For instance, in the following example there are no intermediate horizontally convex diagrams between λ and $\tilde{\lambda}$, even though $|\tilde{\lambda}| = |\lambda| + 3$:



We may assume without loss of generality that $n = \frac{1}{2}k(k + 1)$, where k is the common value of λ_1 and $\tilde{\lambda}_1$. It is then enough to show that

$$X_{\mu}^{[n]} \cap \mathcal{U}_{\Delta} \subset \overline{X_{\tilde{\mu}}^{[n]}} \cap \mathcal{U}_{\Delta},$$

where \mathcal{U}_{Δ} is the triangular chart in $(\mathbb{C}^2)^{[n]}$. But under the isomorphism $\mathcal{A}_k \cong \mathfrak{gl}_k(\mathbb{C})^{\vee}$, described in Corollary 4.8(2), the symplectic leaves $X_{\mu}^{[n]} \cap \mathcal{U}_{\Delta}$ and $X_{\tilde{\mu}}^{[n]} \cap \mathcal{U}_{\Delta}$ correspond to the conjugacy classes of nilpotent Jordan type μ and $\tilde{\mu}$, respectively. Recall that the condition $\lambda \subseteq \tilde{\lambda}$ implies that $\mu \leq \tilde{\mu}$ in the dominance order. Hence the result follows from the classical Gerstenhaber–Hesselink theorem; see eg [33].

General case Note that for any pair $\lambda \subseteq \tilde{\lambda}$, we can find an intermediate diagram $\hat{\lambda}$ such that $\lambda \subseteq \hat{\lambda}$ falls into Case 1 and $\hat{\lambda} \subseteq \tilde{\lambda}$ falls into Case 2. This completes the proof. □

4.6 Nodal points and toric degenerations

Now suppose that $p \in D$ is a nodal singularity. Then by [1], there exist coordinates (x, y) such that, after rescaling σ by a nonzero constant, we have

$$\sigma = xy \partial_x \wedge \partial_y.$$

Applying the algorithm from Section 4.2 to compute the bracket $\sigma^{[n]} = J_x J_y \sigma_0^{[n]}$, we find after a straightforward but tedious calculation that

$$\begin{aligned} (4-6) \quad \{E_i^j, E_a^b\} &= \delta_{a \geq j} \sum_{p=0}^a E_i^{p+j-a} E_p^b - \sum_{p=i}^a E_{a+i-p}^j E_p^b \\ &- \delta_{a+1 \leq j} \sum_{p=a+1}^{k-1} E_i^{p+j-a} E_p^b + \sum_{p=a+1}^{i-1} E_{a+i-p}^j E_p^b + \sum_{q=0}^{\min(j,b-1)} E_i^{b+j-q} E_a^q \\ &- \delta_{b \leq i} \sum_{q=0}^{b-1} E_{q+i-b}^j E_a^q - \sum_{q=\max(j+1,b)}^k E_i^{b+j-q} E_a^q + \delta_{b-1 \geq i} \sum_{q=b}^k E_{q+i-b}^j E_a^q. \end{aligned}$$

A remarkable feature of this quadratic Poisson structure is that it admits a canonical toric degeneration, as we now explain.

4.6.1 The torus-invariant part Consider the action of the torus $(\mathbb{C}^{\times})^{2n} = (\mathbb{C}^{\times})^{k(k+1)}$ on the germ of \mathcal{U}_{Δ} by independent dilation of the Ellingsrud–Strømme coordinates. The induced action on the space of germs of bivectors preserves the space of quadratic bivectors, and hence we may project $\sigma^{[n]}$ onto its torus-invariant part, which is also quadratic.

Definition 4.11 We denote by $\sigma_{\Delta}^{[n]}$ the torus-invariant part of $\sigma^{[n]}$.

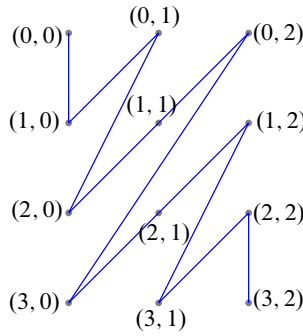


Figure 1: Ordering the indices of a matrix.

Explicitly, we have

$$\sigma_{\Delta}^{[n]} = \sum_{i,j,a,b} \Pi_{ia}^{jb} E_i^j E_a^b \partial_{E_i^j} \wedge \partial_{E_a^b},$$

where the coefficients are given by

$$(4-7) \quad \Pi_{ia}^{jb} = \delta_{a \geq j} - \delta_{a \geq i} - \delta_{bj} \delta_{a \geq i+1} + \delta_{b \geq j+1} \delta_{ai} - \delta_{b \geq j+1} + \delta_{b \geq i+1}.$$

This bivector is generically nondegenerate, and degenerates along the union of the coordinate hyperplanes. The inverse log symplectic form is then given by

$$\omega_{\Delta}^{[n]} = \sum_{i,j,a,b} B_{ia}^{jb} \frac{dE_i^j}{E_i^j} \wedge \frac{dE_a^b}{E_a^b},$$

where

$$(4-8) \quad B_{ia}^{jb} = -\delta_{a+b,i+j} \delta_{a \geq i+1} - \delta_{a+b,i+j+1} \delta_{a \leq i} + \delta_{a+b,i+j} \delta_{a \leq i-1} + \delta_{a+b,i+j-1} \delta_{a \geq i}$$

is the “biresidue” of $\omega_{\Delta}^{[n]}$ along the intersection of hyperplanes $E_i^j = E_a^b = 0$ in $\mathbb{C}^{k(k+1)}$. To understand more clearly the structure of the biresidues, it is helpful to order the coordinates according to their position in the syzygy matrix as follows: first order by the sum of the indices, and then order by the column index to break ties, ie

$$(4-9) \quad (0, 0) < (1, 0) < (0, 1) < (2, 0) < (1, 1) < (0, 2) < \dots$$

For instance, the case $k = 3$ is illustrated in Figure 1. With this ordering the biresidues B_{ia}^{jb} form a skew-symmetric matrix B of size $m := k(k + 1)$ with the following property, which will be useful in our study of holonomicity in Section 4.7:

Definition 4.12 A skew-symmetric $m \times m$ matrix $(b_{\alpha,\beta})_{\alpha,\beta=1}^m$ is *cyclically monotone* if, perhaps after multiplying all of its entries by the same nonzero constant, we have

$$b_{\alpha,\alpha+1} \geq b_{\alpha,\alpha+2} \geq \dots \geq b_{\alpha,m} \geq b_{\alpha,1} \geq b_{\alpha,2} \geq \dots \geq b_{\alpha,\alpha-1}$$

for each $1 \leq \alpha \leq n$.

For instance, for $k = 2$, we have

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

4.6.2 Toric degeneration: a game of dominoes We will prove the following:

Theorem 4.13 *There exists a rank-one subtorus $G \cong \mathbb{C}^\times \hookrightarrow (\mathbb{C}^\times)^{(k+1) \times k}$ such that*

$$\lim_{g \in G, g \rightarrow 0} g \cdot \sigma^{[n]} = \sigma_{\Delta}^{[n]}.$$

Remark 4.14 In light of this result, we can view $\sigma^{[n]}$ as a one-parameter deformation of the toric log symplectic structure $\sigma_{\Delta}^{[n]}$. As explained in [27], such deformations are obtained by smoothing out the nodal singularities along pairwise intersections of hyperplanes. The combinatorics of this process can be encoded in a “smoothing diagram” where we draw a vertex corresponding to each hyperplane, a coloured edge joining two vertices when the corresponding intersection is smoothed, and decorating triangle with angles that indicate the order to which the smoothing degenerates along triple intersections. In the case at hand, the hyperplanes are given by the equations $E_i^j = 0$ and are in bijection with the positions in a $(k + 1) \times k$ matrix; the corresponding smoothing diagram is shown in Figure 2. \diamond

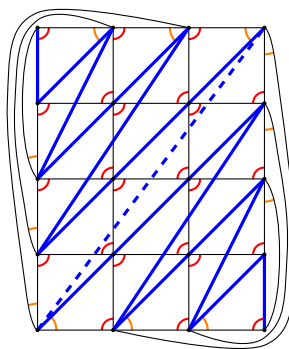


Figure 2: Smoothing diagram encoding the deformation from $\sigma_{\Delta}^{[n]}$ to $\sigma^{[n]}$ for a nodal curve in the triangular chart (with $k = 4$). The $k(k + 1)$ hyperplanes on which $\sigma_{\Delta}^{[n]}$ degenerates merge into two irreducible components corresponding to the two components of the node $xy = 0$; accordingly, the corresponding vertices are joined by two collections of solid blue lines. The solid dashed line corresponds to an additional possible deformation, which is induced by smoothing $xy = 0$ to $xy = \epsilon \neq 0$ in X , so that the divisor becomes irreducible.

We will prove [Theorem 4.13](#) by analyzing the decomposition of the space of quadratic bivectors into weight spaces for the torus action. Namely, the theorem is equivalent to the statement that $\sigma^{[n]} - \sigma_{\Delta}^{[n]}$ is a sum of weight vectors of the torus action, whose weights with respect to the subtorus G are strictly positive. Note that since the torus action on U_{Δ} is defined by rescaling matrix entries, it is natural to depict the weights of the action as matrices of the same size with integer coefficients. With this convention, the weight of a monomial bivector of the form

$$(4-10) \quad E_i^j E_k^l \partial_{E_a^b} \wedge \partial_{E_c^d}$$

is obtained by starting with the zero matrix, adding 1 to the positions (i, j) and (k, l) in turn, and then subtracting 1 from the positions (a, b) and (c, d) in turn.

In the Poisson bracket (4-6), only certain weights can appear. We will describe them in terms of the following objects.

Definition 4.15 A $(k + 1) \times k$ matrix is a *domino* if it has exactly two nonzero entries, one of which is $+1$ and the other of which is -1 , and they lie either on the same row or on the same column. We refer to the position of the entries $+1$ and -1 as the *head* and *tail* of the domino, respectively. We say that a domino is oriented to the *north* (resp. *south*, *east* or *west*) if its head is above (resp. below, right of, or left of) its tail. See [Figure 3](#).

This *size* of a domino is the distance between its head and tail, and the *valuation* is defined as follows:

- If the domino is oriented north or south, its valuation is the number of vertical translations required to move it so that its uppermost nonzero entry lies in the top column.
- If the domino is oriented east or west, its valuation is the number of horizontal translations required to move it so that its leftmost nonzero entry lies in the leftmost column, plus $\frac{1}{2}$.

Definition 4.16 A *rectangular weight* is a $(k + 1) \times k$ matrix given by a sum of two dominoes with opposite orientations, whose heads and tails form the vertices of a rectangle, with a head in the top left corner as in [Figure 4](#).

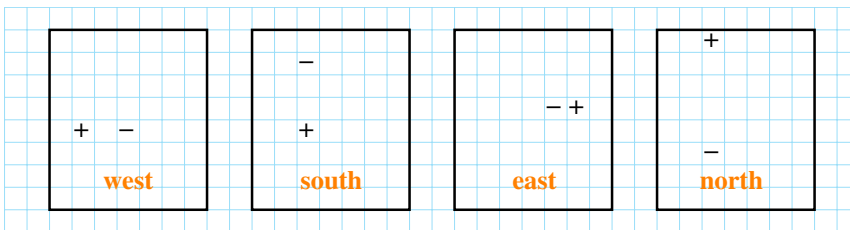


Figure 3: Four examples of dominoes, one from each possible orientation. Reading from left to right, the lengths are 2, 3, 1 and 5, and the valuations are $1 + \frac{1}{2}$, 1, $4 + \frac{1}{2}$ and 0, respectively.

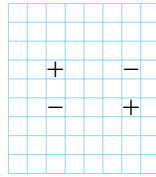


Figure 4: A rectangular weight.

Definition 4.17 An ordered pair of dominoes is *admissible* if both dominoes have the same length, the first domino is directed west or south, the second is directed east or north, and the valuation of the first is greater than the valuation of the second.

By direct inspection of the formula (4-6), we have the following:

Lemma 4.18 The weight of every monomial appearing in $\sigma^{[n]}$ is either rectangular, or the sum of an admissible pair of dominoes.

We will reduce such weights to sums of the following elementary ones, which correspond to the weights of the first-order deformations of $\sigma_{\Delta}^{[n]}$ that smooth a given pairwise intersection of divisor components (ie to the smoothable strata in the sense of [27]).

Definition 4.19 A *smoothable weight* is a $(k + 1) \times k$ matrix given by a sum of two dominoes of length one placed in one of the following configurations:

- **Type I** The two dominoes are adjacent, forming a square-shaped rectangular weight.
- **Type IIa** A south–east admissible pair of dominoes concentrated in the leftmost column and top row such that the tail of the first lies on the same diagonal as the head of the second.
- **Type IIb** A west–north admissible pair of dominoes concentrated in the bottom row and rightmost column such that the head of the first lies on the same diagonal as the tail of the second.

Such matrices are depicted in Figure 5. We remark that for each smoothable weight of type IIa or IIb, the valuation of the southwestern domino is only $\frac{1}{2}$ higher than the valuation of the northeastern domino, which is the minimum possible for the pair to be admissible.

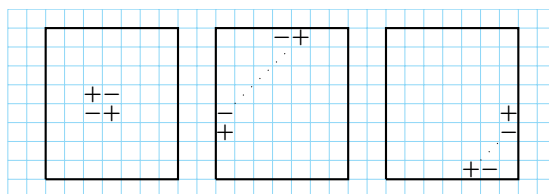


Figure 5: Smoothable weights; from left to right, the types are I, IIa and IIb.

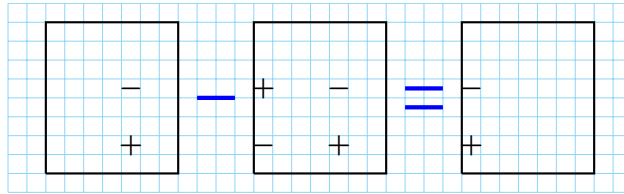


Figure 6: Pushing a southern domino to the leftmost column.

Lemma 4.20 *The set of smoothable weights is linearly independent.*

Proof Consider the square matrix of size $k(k + 1)$ formed from the biresidues B_{ip}^{jq} (4-8) of the toric log symplectic form with indices ordered as in (4-9). Then every smoothable weight can be expressed as the difference of a unique pair of consecutive rows of B_{ip}^{jq} . But this matrix is invertible, and hence the differences of its consecutive rows are linearly independent. \square

Lemma 4.21 *Any rectangular weight is a sum of smoothable weights of Type 1 with nonnegative integer coefficients.*

Proof Consider the sum of all smoothable weights of Type I that are contained inside the rectangle; this sum telescopes, so that only the vertices of the rectangle remain. \square

Proposition 4.22 *The sum of any admissible pair of dominoes is a linear combination of smoothable quadratic weights with nonnegative integer coefficients.*

Proof The proof is a sort of game, in which we translate dominoes in the plane by certain admissible operations, which correspond mathematically to subtracting collections of smoothable weights. We will show that by repeated application of such moves, every admissible pair may be reduced to zero.

Firstly, by subtracting a rectangular weight, we can push any southern domino all the way to the leftmost column, as illustrated in Figure 6. Therefore, whenever we have an admissible pair of dominoes, one of which is southern, we can assume without loss of generality that the southern domino is located in the leftmost column. Likewise, we can assume that every northern domino in an admissible pair is located in the rightmost column, every eastern domino in the top row, and every western domino in the bottom row (just as the dominoes in the smoothable weights of type II).

Secondly, we claim that by subtracting several rectangles and smoothable weights of types IIa and IIb, we can move any southern domino in the leftmost column one step to the north. For this, we proceed in three steps. In the first step, we subtract some smoothable weights of type IIa to turn the domino to the west, as in Figure 7. In the second step, we subtract a rectangular weight to push the obtained western domino to the bottom row as in Figure 8. Finally, in the third step, we subtract some smoothable weights of type IIb to turn the western domino back to the south, as in Figure 9. Because the matrix has one more row than it has columns, the domino is now one step northwards from where it started, as desired.

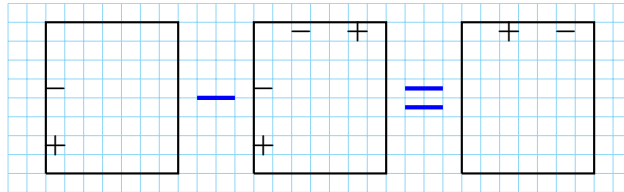


Figure 7: Pushing a southern domino to the north, step 1.

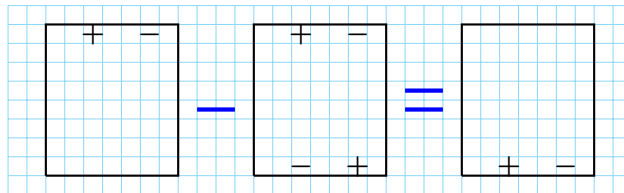


Figure 8: Pushing a southern domino to the north, step 2.

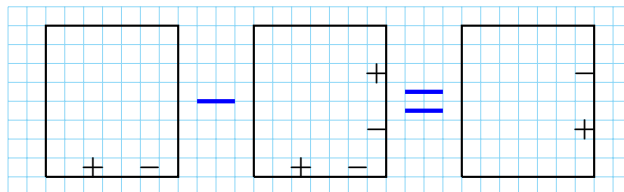


Figure 9: Pushing an eastern domino to the west, step 3.

Using these operations, we are free to push any southern domino in an admissible pair to the north, without changing the second domino in the pair, as long as the pair stays admissible. Similarly, we may push any western domino to the east, any northern domino to the south and any eastern domino to the west.

Now by definition, there are four types of admissible pairs of dominoes: south–east, south–north, west–north, and west–east. Note that by subtracting smoothable weights of type II as in Figure 7, we can convert any admissible south–north pair into an admissible west–north pair, and similarly we may convert any admissible west–east pair into an admissible south–east pair, so it suffices to treat the south–east and west–north cases.

To this end, consider a south–east pair and assume without loss of generality that the southern domino is in the left column, while the eastern one is in the top row. Pushing the southern one as far as possible to the north, we can reduce its valuation to the minimum value for which the pair remains admissible. In this way we obtain an admissible pair that is evidently a telescoping sum of smoothable weights of type IIa. Similarly, pushing a west–north pair to the bottom row and right column, and moving the northern domino south to maximize its valuation, we obtain a sum of smoothable weights of type IIb. \square

We can now complete the construction of the toric degeneration:

Proof of Theorem 4.13 By Lemma 4.20 there exists a coweight w of the torus $(\mathbb{C}^*)^{(k+1)k}$ that pairs with each smoothable weight to give a positive integer. By construction, the corresponding one-parameter subgroup $G \cong \mathbb{C}^\times$ then acts with positive weight on every smoothable weight space, and hence by Lemma 4.21 and Proposition 4.22 acts with positive weight on the weights spaces corresponding to rectangular weights or admissible pairs of dominoes. \square

4.7 Holonomicity

Recall from [37] that every holomorphic Poisson manifold (W, σ) has an associated *characteristic variety*

$$\text{Char}(\sigma) \subset T^*W$$

defined as the singular support of the complex of \mathcal{D}_W -modules associated to the deformation complex of (W, σ) . We say that (W, σ) is *holonomic* if its characteristic variety is Lagrangian. This is a local property that depends only on the stable equivalence classes of the germs of σ . It was shown in [37] that if (W, σ) is holonomic then every point has a neighbourhood with only finitely many characteristic symplectic leaves and that the converse holds when $\dim W = 2$. In [27], we conjectured that the converse holds in all dimensions, and we proved this when (W, σ) is generically symplectic and degenerates along a normal crossing divisor. In this section, we give further evidence for the conjecture.

Considering the invariance under stable equivalence and the results in Section 3.3, there are three cases one must consider in order to verify the conjecture for all Hilbert schemes: either

- (a) D has at worst nodal singularities and n is arbitrary,
- (b) D has only double points and $n = 2$, or
- (c) D has only A_2 singularities and $n = 5$.

We shall treat the cases (a) and (b); we do not know whether the conjecture holds in case (c) but have no reason to doubt it. Case (a) is the subject of the following theorem, whose proof will occupy most of this section. Note that it implies that $X^{[n]}$ is holonomic if D is smooth; a more direct proof is possible in this case, but we omit it. The smooth case allows us to immediately treat case (b) as a corollary.

Theorem 4.23 *For a Poisson surface X whose vanishing locus is the anticanonical divisor $D \subset X$, the following statements are equivalent:*

- (1) *For every $n \geq 0$, the induced Poisson structure on the Hilbert scheme $X^{[n]}$ is holonomic.*
- (2) *For every $n \geq 0$, the germ of $X^{[n]}$ at any point has only finitely many characteristic symplectic leaves.*
- (3) *The only singularities of D are nodes.*

Corollary 4.24 *Let X be a Poisson surface with reduced anticanonical divisor $D \subset X$. Then $X^{[2]}$ is holonomic if and only if every singular point of D is a double point.*

Proof of Corollary 4.24 Let $V \subset X^{[2]}$ be the set of elements set-theoretically supported at a single singular point of D . Then V is the disjoint union of several copies of \mathbb{P}^1 , one for each singular point, and $X^{[2]} \setminus V$ is holonomic by Theorem 4.23. It therefore suffices to show that $\text{Char}(\sigma^{[2]}) \cap (\mathbb{T}^*X^{[2]})|_V$ has pure dimension four. But since all singular points are double points, Proposition 3.8 implies that the zeros of the modular vector field in V are isolated, so it defines a function on the smooth five-dimension variety $\mathbb{T}^*X^{[2]}|_V$ that is nonzero on every connected component. Its vanishing locus contains $\text{Char}(\sigma^{[2]}) \cap \mathbb{T}^*X^{[2]}|_V$ by [37, Theorem 3.4, part 1], and hence the latter has dimension four, as desired. \square

We now proceed with the proof of Theorem 4.23. The equivalence of statements (2) and (3) was proven in Section 3.3, so in light of the above it remains only to prove that if D is nodal, then $X^{[n]}$ is holonomic for all n . The problem is local and invariant under stable equivalence and taking products, so by Lemma 4.4, it suffices to prove these statements for the triangular chart. Moreover, in light of Lemma 4.25 below and Theorem 4.13, the problem reduces to proving the statement for the toric degeneration $\sigma_{\Delta}^{[n]}$:

Lemma 4.25 *Let π_t for $t \in \mathbb{C}$ be a polynomial family of quadratic Poisson tensors on \mathbb{C}^{2n} such that π_0 is holonomic. Then π_t is holonomic for all but finitely many $t \neq 0$.*

Proof Let $V_t \subset \mathbb{T}\mathbb{C}^{2n} \cong \mathbb{C}^{4n}$ be the characteristic variety of the complex of \mathcal{D} -modules $\mathcal{M}_{\pi_t}^{\bullet}$, which is thus invariant under the action of \mathbb{C}^{\times} by dilation of the fibres. Additionally, since all Poisson structures π_t are quadratic, each V_t is also invariant under the dilation on the base \mathbb{C}^{2n} . Combining these two symmetries, we get that all V_t are invariant under the uniform dilation of all directions \mathbb{C}^{4n} . In other words, the projectivization $\mathbb{P}(V_t) \subset \mathbb{P}^{4n-1}$ is well-defined for each t . Since π_t is assumed holonomic, we have $\dim_{\mathbb{C}} \mathbb{P}(V_t) = 2n - 1$. Then by semicontinuity of dimension for proper maps, we have that $\dim_{\mathbb{C}} \mathbb{P}(V_t) \leq 2n - 1$ for t in a Zariski neighbourhood of the origin, as desired. \square

The toric degeneration $\sigma_{\Delta}^{[n]}$ is a log symplectic manifold with normal crossings boundary, so the results of our earlier work [27] apply. In particular, by condition (4) of [27, Theorem 1.5], and by the cyclic monotonicity property of the biresidues of $\sigma_{\Delta}^{[n]}$ (Definition 4.12), it suffices to prove the following linear-algebraic statement:

Lemma 4.26 *Let B be a cyclically monotone matrix of odd size, with only zeros and ones above the diagonal. Then the row span of B does not contain the constant vector $(1, 1, \dots, 1) \in \mathbb{C}^m$.*

Although the statement is elementary, we did not manage to find an easy direct proof. We will instead prove a more general statement, that is amenable to an inductive argument. For a tuple $J = (J_1, \dots, J_m)$ of nonempty open intervals $J_1 = (c_1, d_1), \dots, J_m = (c_m, d_m) \subset \mathbb{R}$, such that all the endpoints of J_i are distinct, let

$$\ell(J) := (d_1 - c_1, d_2 - c_2, \dots, d_m - c_m)$$

be the vector of their lengths. Let us say that J_α overlaps J_β on the left (resp. right) if $c_\alpha < c_\beta < d_\alpha < d_\beta$ (resp. $c_\beta < c_\alpha < d_\beta < d_\alpha$). Define an $m \times m$ skew-symmetric matrix $B(J)$ by the formula

$$(4-11) \quad B(J)_{\alpha\beta} = \begin{cases} 1 & \text{if } J_\alpha \text{ overlaps } J_\beta \text{ on the left,} \\ -1 & \text{if } J_\alpha \text{ overlaps } J_\beta \text{ on the right,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.27 Every matrix B as in Lemma 4.26 is of the form $B = B(J)$ for some collection J of intervals of lengths $\ell(J) = (1, \dots, 1)$.

Proof For $1 \leq \beta \leq m$, let k_β be the row index of the uppermost 1 in the β th column of B , with the convention that if there are no ones in the β th column, we define $k_\beta = \beta$. Set $J_1 = (0, 1)$, and choose intervals J_2, \dots, J_m of length one inductively by choosing the left endpoint $c_\beta > c_{\beta-1} + 1$ if $k_\beta = \beta$ and $c_\beta \in (\max\{c_{\beta-1}, c_{k_{\beta-1}} + 1\}, c_{k_\beta} + 1)$ if $k_\beta < \beta$. Then $\ell(J) = (1, \dots, 1)$ and $B(J) = B$, as desired. \square

Proposition 4.28 Let $J = (J_1, \dots, J_m)$ be a tuple of intervals, where m is odd and the endpoints of J_i are distinct. Then the vector $\ell(J) \in \mathbb{R}^m$ of their lengths does not lie in the row span of $B(J)$.

Proof We proceed by induction on the odd integer m . The base case $m = 1$ is obvious, so assume the proposition holds for some odd integer $m - 2$; we will prove that it holds for m .

Let view $B(J)$ as a bivector $B(J) \in \wedge^2 \mathbb{R}^m$ so that we have

$$B(J) := \sum_{1 \leq \alpha < \beta \leq m} B(J)_{\alpha,\beta} e_\alpha \wedge e_\beta \quad \text{and} \quad \ell(J) = \sum_{\alpha=1}^m \ell(J)_\alpha e_\alpha,$$

where e_1, e_2, \dots is the standard basis of \mathbb{R}^m .

By permuting the indices, we may assume without loss of generality that the sequence of left endpoints of the intervals is strictly increasing. Then if $\alpha < \beta$, the interval J_α can never overlap J_β on the right, and hence $B(J)_{\alpha,\beta} \in \{0, 1\}$ for $\alpha < \beta$.

If no interval J_γ with $\gamma < m$ overlaps J_m , then $B(J)_{\alpha,m} = 0$ for all α , and since $\ell(J)_m > 0$ we conclude that $\ell(J)$ is not in the row span of $B(J)$, as desired. Hence we may assume without loss of generality that there exists a maximal index $\gamma < m$ such that J_γ overlaps J_m on the left. Define a new basis $\tilde{e}_1, \dots, \tilde{e}_m$ for \mathbb{R}^m by the formula

$$\tilde{e}_\alpha := \begin{cases} e_\gamma - \sum_{\beta \neq \gamma, m} B(J)_{m,\beta} \cdot e_\beta & \text{if } \alpha = \gamma, \\ e_m + \sum_{\beta \neq \gamma, m} B(J)_{\gamma,\beta} \cdot e_\beta & \text{if } \alpha = m, \\ e_\alpha & \text{otherwise.} \end{cases}$$

It is tedious but straightforward to check that in this new basis, we have

$$B(J) = \tilde{e}_\gamma \wedge \tilde{e}_m + \sum_{\alpha, \beta, \gamma, m \text{ distinct}; \alpha < \beta} B(\tilde{J})_{\alpha,\beta} \tilde{e}_\alpha \wedge \tilde{e}_\beta \quad \text{and} \quad \ell = a_1 \tilde{e}_\gamma + a_2 \tilde{e}_m + \sum_{\alpha \neq \gamma, m} \ell(\tilde{J})_\alpha \tilde{e}_\alpha,$$

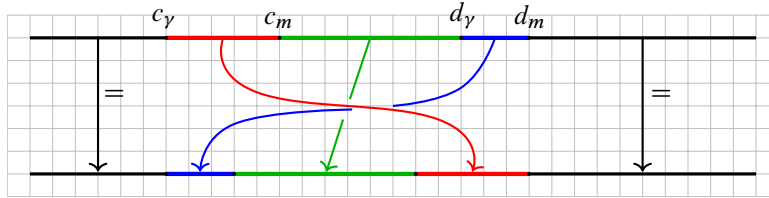


Figure 10: The interval exchange transformation T .

where $a_1, a_2 \in \mathbb{Z}$, and $\tilde{J} = (T \cdot J_1, \dots, \widehat{T \cdot J_\gamma}, \dots, T \cdot J_{m-1})$ is the collection of $m - 2$ intervals defined as follows. First, define the “interval exchange transformation” to be the bijection $T : \mathbb{R} \rightarrow \mathbb{R}$ which translates the intervals between the endpoints of $J_\gamma = (c_\gamma, d_\gamma)$ and $J_m = (c_m, d_m)$ as shown in Figure 10, and acts as the identity on the rest of the real line. Then $T \cdot J_\alpha = (T(c_\alpha), T(d_\alpha))$ is the interval spanned by the T -images of the endpoints of J_α . By induction, $\ell(\tilde{J})$ is not in the image of $B(\tilde{J})$, and hence $\ell(J)$ is not in the image of $B(J)$, as desired. \square

5 Deformation theory

In this section we address some aspects of the deformation theory of Bottacin’s Poisson structures. Recall that if (W, σ) is a holomorphic Poisson manifold, then the sheaf $\mathcal{X}_W^\bullet = \wedge^\bullet \mathcal{T}_W$ of polyvector fields comes equipped with a differential $d_\sigma := [\sigma, -]$. The hypercohomology $H_\sigma^\bullet(W)$ of this complex (called the Poisson cohomology) controls deformations of W as a holomorphic Poisson manifold, or more generally, as a generalized complex manifold.

We shall compute the Poisson cohomology of $X^{[n]}$ in low degrees, and in particular determine the deformation space, under the assumption that D is reduced, and has only quasihomogeneous singularities (ie is locally the zero set of a quasihomogeneous polynomial). As remarked in [37, Section 4.4], the quasihomogeneity is automatic if X is projective, thanks to the classification in [23, Section 7], or by analysis of the minimal models and their blowups.

The basic technique, developed in [27; 37], is to decompose the complex $\mathcal{X}_{X^{[n]}}^\bullet$ (in the derived category) into pieces supported on characteristic symplectic leaves; the only pieces that contribute to the cohomology in low degree are the ones supported on leaves of small codimension. We start with the surface case $n = 1$, and then bootstrap from there to treat the general case.

5.1 Surface case

Let us briefly recall the description of the Poisson cohomology in the case $n = 1$ from [37, Section 4.4] and [15]. Let us denote by $U := X \setminus D$ the open symplectic leaf, and by

$$j : U \hookrightarrow X$$

its open embedding. Since σ is symplectic over U , we have a canonical isomorphism

$$j^*(\mathcal{X}_X^\bullet, d_\sigma) \cong (\Omega_U^\bullet, d) \cong \mathbb{C}_U$$

in the constructible derived category of U , which gives, by adjunction, a natural map

$$(5-1) \quad (\mathcal{X}_X^\bullet, d_\sigma) \rightarrow Rj_*\mathbb{C}_U$$

in the constructible derived category of X .

On the other hand, let $D_{\text{sing}} \subset D$ denote the scheme-theoretic singular locus of D , cut out locally by the vanishing of a defining equation for D and its partial derivatives, and let

$$i : D_{\text{sing}} \hookrightarrow X$$

be the corresponding closed embedding. Its image is the union of all codimension-two characteristic symplectic leaves in X . One easily checks that the restriction of bivectors on X to D_{sing} annihilates the image of d_σ , and hence there is a canonical map

$$(5-2) \quad (\mathcal{X}_X^\bullet, d_\sigma) \rightarrow i_*i^*\mathcal{K}_X^\vee[-2],$$

where the right-hand side is a skyscraper sheaf supported on D_{sing} , viewed as a complex concentrated in degree two with trivial differential. Its stalk at a point $p \in D_{\text{sing}}$ is canonically identified with the tangent cohomology $H^1(\mathbb{T}_{D,p})$, ie the space of smoothings of the singularities of D at p .

The maps (5-1) and (5-2) determine the Poisson cohomology as follows:

Theorem 5.1 [37, Theorem 4.7] *If D is reduced with only quasihomogeneous singularities, then the canonical map*

$$(\mathcal{X}_X^\bullet, d_\sigma) \rightarrow Rj_*\mathbb{C}_U \oplus i_*i^*\mathcal{K}_X^\vee[-2]$$

is a quasi-isomorphism. In particular,

$$H_\sigma^\bullet(X) \cong H^\bullet(U; \mathbb{C}) \oplus H^0(D_{\text{sing}}, i^*\mathcal{K}_X^\vee)[-2].$$

as graded vector spaces.

5.2 Higher-dimensional case

We now use the results in the previous section to treat the case $n > 1$. Since we are interested in cohomology in degrees up to 2, we need to understand the contributions made by characteristic leaves of codimension ≤ 2 . According to Lemma 3.11, these are given by the open symplectic leaf $U^{[n]} \subset X^{[n]}$, and the elements $Z \subset X^{[n]}$ whose intersection with D is a single singular point of D . The latter give several copies $U^{[n-1]}$, one for each singular point. We treat the contributions from these leaves as follows.

- **The open leaf** We have the natural open embedding

$$j^{[n]}: U^{[n]} \hookrightarrow X^{[n]},$$

identifying $U^{[n]}$ with the open symplectic leaf in $X^{[n]}$. This gives a natural map

$$(5-3) \quad (\mathcal{X}_{X^{[n]}}^\bullet, d_{\sigma^{[n]}}) \rightarrow Rj_*\mathbb{C}_{U^{[n]}}.$$

- **The codimension-two leaves** We have a Poisson rational map

$$X^{[n-1]} \times X \dashrightarrow X^{[n]}$$

given by $(Z, p) \mapsto Z \sqcup \{p\}$ whenever $p \notin Z$. Since U and D_{sing} are disjoint by definition, this map restricts to an embedding

$$j^{[n]}: U^{[n-1]} \times D_{\text{sing}} \hookrightarrow X^{[n]}$$

with trivial normal bundle. By Lemma 3.11, the image of this embedding is the union of the codimension-two characteristic symplectic leaves of $X^{[n]}$. Since $U^{[n-1]}$ is symplectic, we have by Künneth decomposition and Theorem 5.1 that

$$\begin{aligned} (i^{[n]})^*(\mathcal{X}_{X^{[n]}}^\bullet, d_{\sigma^{[n]}}) &\cong \mathbb{C}_{U^{[n-1]}} \boxtimes i^*(Rj_*\mathbb{C}_U \oplus \mathcal{K}_X^\vee[-2]) \\ &\cong (i^{[n]})^*Rj_*\mathbb{C}_{U^{[n]}} \oplus (\mathbb{C}_{U^{[n-1]}} \boxtimes i^*\mathcal{K}_X^\vee[-2]), \end{aligned}$$

where \boxtimes denotes the external tensor product of sheaves on the product $U^{[n-1]} \times D_{\text{sing}}$. Projecting onto the second summand, we obtain a natural map

$$(5-4) \quad (\mathcal{X}_{X^{[n]}}^\bullet, d_{\sigma^{[n]}}) \rightarrow i_*^{[n]}(\mathbb{C}_{U^{[n-1]}} \boxtimes i^*\mathcal{K}_X^\vee[-2]).$$

Combining (5-3) and (5-4), we obtain a canonical map

$$(5-5) \quad \phi: (\mathcal{X}_{X^{[n]}}^\bullet, d_{\sigma^{[n]}}) \rightarrow Rj_*\mathbb{C}_{U^{[n]}} \oplus i_*^{[n]}(\mathbb{C}_{U^{[n-1]}} \boxtimes i^*\mathcal{K}_X^\vee[-2])$$

that encapsulates the contributions to $\mathcal{X}_{X^{[n]}}^\bullet$ made by characteristic leaves of codimension ≤ 2 .

Theorem 5.2 *If D is reduced and has only quasihomogeneous singularities, then the induced map*

$$H^j(\phi): H_{\sigma^{[n]}}^j(X^{[n]}) \rightarrow H^j(U^{[n]}; \mathbb{C}) \oplus (H^{j-2}(U^{[n-1]}; \mathbb{C}) \otimes H^0(i^*\mathcal{K}_X^\vee))$$

is an isomorphism for $j \leq 2$.

Before giving the proof, which will occupy the rest of this section, let us note that the cohomology $H^\bullet(U^{[n]}; \mathbb{C})$ can be computed from $H^\bullet(U; \mathbb{C})$ using the Göttsche–Soergel formula [16]. In particular, applying the theorem to the degree-two hypercohomology, we obtain the following.

Corollary 5.3 *If D is reduced and has only quasihomogeneous singularities, then the space of first-order deformations of $(X^{[n]}, \sigma^{[n]})$ is given, for all $n \geq 2$, by*

$$\begin{aligned} H^2_{\sigma^{[n]}}(X^{[n]}) &\cong H^2(U^{[n]}; \mathbb{C}) \oplus H^0(i^* \mathcal{K}_X^\vee) \\ &\cong (H^2(U; \mathbb{C}) \oplus \wedge^2 H^1(U; \mathbb{C}) \oplus \mathbb{C} \cdot [E]) \oplus H^0(i^* \mathcal{K}_X^\vee) \\ &\cong H^2_\sigma(X) \oplus \wedge^2 H^1(U; \mathbb{C}) \oplus \mathbb{C} \cdot [E], \end{aligned}$$

where $[E] \in H^2(U^{[n]}; \mathbb{C})$ is the first Chern class of the exceptional divisor of the Hilbert–Chow morphism.

Before proving [Theorem 5.2](#), we summarize the strategy as follows. The key observation is that, by a generalization of Hartogs’ principle, the Poisson cohomology in degree $\leq m$ is unchanged if we discard a subset of codimension $\geq m + 2$, and the restriction map is injective to the complement of a subset of codimension $\geq m + 1$. Thus, it is enough to show that

- (1) the isomorphism holds outside a subset of codimension three, and
- (2) ϕ is surjective in degree two.

For (1), the open subset we use is the locus of elements $Z \in X^{[n]}$ for which the intersection $Z \cap D$ consists of at most two reduced smooth points of D , or one reduced singular point of D . Both conditions are codimension at most two, so it is a natural subset to consider. Moreover, the case where Z does not touch the singular locus is normal crossings so that it is handled by the results of [\[27\]](#), and the case of a single singular point is immediate. For (2), we will explicitly exhibit the two summands of the codomain of ϕ as the image of elements of ϕ .

Proceeding with the proof, let

$$D^{[n]} := X^{[n]} \setminus U^{[n]} \cong \{Z \in X^{[n]} \mid Z \cap D \neq \emptyset\}$$

be the degeneracy divisor of the Poisson structure on $X^{[n]}$. (Note that despite the notation, it is *not* isomorphic to the Hilbert scheme of D .) Let $\Omega^\bullet_{X^{[n]}}(\log D^{[n]})$ be the sheaf of logarithmic differential forms in the sense of K Saito [\[42\]](#). Then, as for any log symplectic manifold, the natural map $\sigma^\sharp: \Omega^1_X \rightarrow \mathcal{T}_X$ induces a commutative triangle

$$(5-6) \quad \begin{array}{ccc} & (\Omega^\bullet_{X^{[n]}}(\log D^{[n]}), d) & \\ \swarrow & & \searrow \\ (\mathcal{R}^\bullet_X, d_\sigma) & \xrightarrow{\quad} & Rj_* \mathbb{C}_U \end{array}$$

where all maps are quasi-isomorphisms away from the singular locus $D^{[n]}_{\text{sing}}$ of $D^{[n]}$; see [\[37, page 695\]](#).

For our purposes, it suffices to understand these complexes along the generic part of the singular locus. To this end, define locally closed subvarieties $S_1, S_2 \subset D^{[n]}_{\text{sing}}$ by

$$S_1 := \{Z \in X^{[n]} \mid Z \cap D \text{ consists of two reduced points}\} \quad \text{and} \quad S_2 := i^{[n]}(U^{[n-1]} \times D_{\text{sing}}).$$

Evidently S_1 and S_2 are disjoint. Moreover, we have the following.

Lemma 5.4 *The subset*

$$W := X^{[n]} \setminus (U^{[n]} \sqcup D_{\text{reg}}^{[n]} \sqcup S_1 \sqcup S_2)$$

is a closed subvariety of codimension three.

Proof If $Z \in W$ then Z contains at least one of the following configurations: three distinct points of D ; a singular point and a smooth point; or a nonreduced point tangent to D . Each of these conditions defines a locally closed subvariety of $X^{[n]}$ that maps birationally onto its image in $X^{(n)}$, which clearly has codimension three. □

Lemma 5.5 *The morphism ϕ from (5-5) restricts to a quasi-isomorphism over $X^{[n]} \setminus W$.*

Proof It is already a quasi-isomorphism away from $D_{\text{sing}}^{[n]}$, and is furthermore a quasi-isomorphism over the image of $i^{[n]}$, so it suffices to show that it is an isomorphism over S_1 . Since S_1 is disjoint from the support of $i_*^{[n]}(\mathbb{C}_{U^{[n-1]}} \boxtimes i^* \mathcal{K}_X^\vee)$, this is equivalent to showing that the map (5-3) is a quasi-isomorphism there. But in a neighbourhood of this locus, $D^{[n]}$ has only normal crossings singularities, and the only characteristic leaf is the open one. Hence the result follows from our computation of the Poisson cohomology in the normal crossings case [27]; see also [41]. □

Since W has codimension three and \mathcal{X}_X^\bullet is locally free and concentrated in nonnegative degree, the restriction map from Poisson cohomology of $X^{[n]}$ to that of $X \setminus W^{[n]}$ is an isomorphism in degrees 0 and 1, and injective in degree 2, by a standard local cohomology argument (a generalization of Hartogs' principle). Since $U^{[n]}$ and $i^{[n]}(U^{[n-1]} \times D_{\text{sing}})$ are disjoint from W we have the following.

Corollary 5.6 *The map $H^j(\phi)$ on hypercohomology is an isomorphism in degree 0 and 1, and injective in degree 2.*

Hence to prove Theorem 5.2, it remains to show that the map $H^2(\phi)$ is surjective. To this end, we require a lemma:

Lemma 5.7 *If D has only quasihomogeneous singularities, then the natural map*

$$H^j(\Omega_{X^{[n]}}^\bullet(\log D^{[n]})) \rightarrow H^j(U^{[n]}; \mathbb{C})$$

is surjective for $j \leq 2$.

Proof The argument is a de Rham analogue of the approach of Göttsche and Soergel [16]. We have the natural maps

$$\begin{array}{ccccc} U^{[n]} & \longrightarrow & U^{(n)} & \longleftarrow & U^n \\ \downarrow j^{[n]} & & \downarrow j^{(n)} & & \downarrow j^n \\ X^{[n]} & \xrightarrow{r} & X^{(n)} & \xleftarrow{q} & X^n \end{array}$$

and corresponding divisors $D^{[n]}$, $D^{(n)}$ and D^n ; for instance, D^n is the set of n -tuples $(p_1, \dots, p_n) \in X^n$ such that $p_i \in D$ for some i .

Since $X^{(n)} = X^n/S_n$ is a finite quotient, it has only rational klt singularities, and hence by [44, Lemma 1.8] and [17, Theorem 1.4] we have

$$r_*\Omega_{X^{[n]}}^\bullet = (q_*\Omega_{X^n}^\bullet)^{S_n},$$

where on the right-hand side we have taken invariants with respect to the symmetric group action on X^n . An identical statement holds replacing X everywhere with U . Note that since the exceptional divisor $E \subset X^{[n]}$ is not contained in $D^{[n]}$, a form $\omega \in \Omega_{U^{[n]}}^\bullet$ extends to a logarithmic form on $X^{[n]}$ if and only if it extends to a logarithmic form on $X^{[n]} \setminus E$ that has no poles on E . Since the maps r and q are étale away from $r(E)$ and relate the corresponding divisors, we conclude that

$$\begin{aligned} (r_*\Omega_{X^{[n]}}^\bullet(\log D^{[n]}), d) &\cong (q_*\Omega_{X^n}^\bullet(\log D^n)^{S_n}, d) \\ &\cong (q_*R(j_*^n)\mathbb{C}_{U^n})^{S_n} \\ &\cong Rj_*^{(n)}\mathbb{C}_{U^{(n)}}, \end{aligned}$$

where the second isomorphism is a consequence of the Künneth decomposition and the results of [10] (using that D is quasihomogeneous).

Passing to hypercohomology, we obtain a commutative diagram

$$\begin{array}{ccccc} H^\bullet(r_*\Omega_{X^{[n]}}^\bullet(\log D^{[n]}), d) & \longrightarrow & H^\bullet(\Omega_{X^{[n]}}^\bullet(\log D^{[n]}), d) & \longleftarrow & H^\bullet(\Omega_{X^{[n]}}^\bullet) \\ \sim \downarrow & & \downarrow & & \downarrow \sim \\ H^\bullet(U^n; \mathbb{C})^{S_n} \cong H^\bullet(U^{(n)}; \mathbb{C}) & \longrightarrow & H^\bullet(U^{[n]}; \mathbb{C}) & \xleftarrow{(j^{[n]})^*} & H^\bullet(X^{[n]}; \mathbb{C}) \end{array}$$

where the bottom left arrow is the inclusion given by the codimension-zero contribution to the Göttsche–Soergel formula. Said map is an isomorphism in degrees zero and one, and in degree two it is an injection with complement spanned by the class of the exceptional divisor $E \subset U^{[n]}$. Since the latter extends canonically to X , it follows that the map $H^\bullet(\Omega_{X^{[n]}}^\bullet(\log D^{[n]}), d) \rightarrow H^\bullet(U^{[n]}; \mathbb{C})$ is surjective in degree two. □

Now let us observe that by applying Bottacin’s construction in families, any Poisson deformation of (X, σ) induces a Poisson deformation of $(X^{[n]}, \sigma^{[n]})$. Under this correspondence, deformations of the singularities of $D \subset X$ induce identical deformations of the singularities of $D^{[n]} \subset X^{[n]}$ transverse to the codimension-two characteristic leaves. Applied to first-order deformations, this means that we have a canonical morphism $H_\sigma^2(X) \rightarrow H_{\sigma^{[n]}}^2(X^{[n]})$ such that the composition

$$H^0(i^*\mathcal{K}_X^\vee) \hookrightarrow H_{\sigma^{[n]}}^2(X^{[n]}) \xrightarrow{\phi} H^2(U^{[n]}; \mathbb{C}) \oplus H^0(i^*\mathcal{K}_X^\vee) \rightarrow H^0(i^*\mathcal{K}_X^\vee)$$

is the identity map. Furthermore, the image of $H^0(i^*\mathcal{K}_X^\vee)$ in $H_{\sigma^{[n]}}^2(X^{[n]})$ is complementary to the image of $H^2(\Omega_{X^{[n]}}^\bullet(\log D^{[n]}))$, because the latter can only produce deformations in which the divisor $D^{[n]}$ deforms locally trivially. It follows that $H^2(\phi)$ is surjective, as desired. This completes the proof of [Theorem 5.2](#).

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The conjugacy problem for UPG elements of $\text{Out}(F_n)$	1693
MARK FEIGNH and MICHAEL HANDEL	
The systole of large genus minimal surfaces in positive Ricci curvature	1819
HENRIK MATTHIESEN and ANNA SIFFERT	
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MYKOLA MATVIICHUK, BRENT PYM and TRAVIS SCHEDLER	
Mutations and faces of the Thurston norm ball dynamically represented by multiple distinct flows	2105
ANNA PARLAK	
Unit inclusion in a (nonsemisimple) braided tensor category and (noncompact) relative TQFTs	2175
BENJAMIN HAÏOUN	
Closed geodesics in dilation surfaces	2217
ADRIEN BOULANGER, SELIM GHAZOUANI and GUILLAUME TAHAR	