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Closed geodesics in dilation surfaces

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We prove that directions of closed geodesics in every dilation surface form a dense subset of the circle. The proof draws on a study of the degenerations of the Delaunay triangulation of dilation surfaces under the action of Teichmüller flow in the moduli space.

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1 Introduction

We consider the problem of the existence of regular closed geodesics in dilation surfaces. Our main theorem is the following:

Theorem 1.1 *For any closed dilation surface Σ, there is a dense set of directions θ such that the directional foliation $\mathcal{F}_\theta$ has a periodic leaf. Equivalently, the set of directions covered by a cylinder is dense in $\mathbb{RP}^1$.*

In particular, any dilation surface Σ carries at least one closed geodesic. This generalizes to the context of dilation surfaces a celebrated theorem of Masur [4] for translation surfaces.

As the two equivalent formulations of Theorem 1.1 suggest, it can be viewed from either a dynamical or geometric perspective. From the geometric point of view, it guarantees that every dilation surface contains the simplest building block that can be imagined, a cylinder, thus giving valuable insight into the geometric structure of the arbitrary dilation surface.

On the dynamical side, this theorem guarantees the ubiquity of periodic orbits in some particular (but very natural) one-parameter families of one-dimensional dynamical systems, in the form of the following corollary:

Corollary 1.2 *For every affine interval exchange transformation* $T_0 : [0, 1] \to [0, 1]$, *the set of parameters* t *such that the map* $x \mapsto T_0(x) + t \mod 1$ *has a periodic orbit is dense in* $\mathbb{R}$.

Results about particular families of dynamical systems of this type are usually difficult to prove; a result analogous to Corollary 1.2 where T_0 is an arbitrary generalized interval exchange map seems out of reach of current methods.

1.1 Affine structures on surfaces

The question of the existence of closed geodesics can be considered in the wider context of affine (complex or real) structures on surfaces.[1] For Riemannian structures, the existence of closed geodesics has been known for a long time (see for example Gromoll and Meyer [2]). The case of translation surfaces, which lies in the intersection of the affine and Riemannian world, is now very well understood. On the contrary, for general affine structures very little is known. We therefore pose the following problem.

Problem 1.3 Characterize the affine structures on closed surfaces which carry a regular[2] closed geodesic.

Note that a complete solution to this problem is likely to be very difficult, as it contains as a particular case the notoriously hard question of determining whether the billiard flow of every polygonal table has a periodic orbit.

1.2 Dilation surfaces vs general affine surfaces

Dilation surfaces are particular complex affine surfaces whose structural group is the set of transformations of the form $z \mapsto az + b$ where a is a positive real number and $b \in \mathbb{C}$. Although it is expected that generic complex affine surfaces do not have any closed geodesics, our main theorem predicts that any dilation surface does.

We explain what the condition on the structural group defining dilation surfaces implies at the dynamical level. Every (complex or real) affine structure induces a geodesic foliation on $T^1\Sigma$ the unit tangent bundle of the surface. $T^1\Sigma$ is a three-dimensional manifold, thus the dynamical system induced by the foliation is essentially two-dimensional. Indeed, for a given Poincaré section, the first return map may change both the direction and the position of the intersection of the leaf with the interval.

[1] A real (resp. complex) affine structure on a surface is an atlas of charts taking values in $\mathbb{R}^2$ (resp. $\mathbb{C}$) such that transition maps lie in the group of real affine transformations $\mathrm{GL}^+(2, \mathbb{R}) \ltimes \mathbb{R}^2$ (resp. complex affine transformations $\mathbb{C}^* \ltimes \mathbb{C}$), with possibly finitely many cone-type singularities.

[2] A regular closed geodesic is a closed geodesic that does not contain any singularity of the affine structure.

In the particular case of dilation surfaces, $T^1\Sigma$ decomposes into a one-parameter family of invariant surfaces for the foliation. While this gives no indication as to which affine structures always have periodic leaves, it explains why dilation surfaces are essentially different from the general case:

- the problem for dilation surfaces is about finding periodic orbits in *one-parameter families* of one-dimensional dynamical systems,
- the problem for the generic affine surface is about finding a periodic orbit for a *given* two-dimensional dynamical system.

The analysis of two-dimensional dynamical systems is far more intricate than that of their one-dimensional counterparts; furthermore with dilation surfaces we have an entire one-parameter family of one-dimensional dynamical systems (which are easier to analyze) to find a periodic orbit. This discussion also explains why, despite the fact that in principle it is plausible that a lot of real affine surfaces carry closed geodesics, the dilation case is of a different nature and probably easier to analyze.

1.3 The action of $\mathrm{SL}(2,\mathbb{R})$ and strategy of proof

We now explain the ideas behind the proof of Theorem 1.1. It is very much inspired by the translation case, and we remind the reader of the general structure of its proof. We refer to Masur [4] for the original proof in the translation case.

Both moduli spaces of dilation and translation surfaces carry an action of the group $\mathrm{SL}(2,\mathbb{R})$. This action is naturally defined by the postcomposition of the charts defining the dilation/translation structure. It has the following important property: two surfaces are on the same $\mathrm{SL}(2,\mathbb{R})$-orbit if and only if they define the same underlying real affine structure. In particular, if a surface has a closed geodesic, it is the case for every surface in its $\mathrm{SL}(2,\mathbb{R})$-orbit.

In the translation case, the proof goes by induction on the combinatorial complexity of the surface.[3]

(1) It is easy to check that translation surfaces of lowest complexity (flat tori) always carry closed geodesics.

(2) Assume that we know that all surfaces of complexity less than k do carry closed geodesics, and consider a translation surface Σ of complexity k. It is not hard to find a sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of translation surfaces in the $\mathrm{SL}(2,\mathbb{R})$-orbit of Σ which diverges, ie leaves any compact subset in the moduli space of surfaces of complexity k.

(3) Geometric tools building on the Riemannian structure of translation surfaces allow us to show the following dichotomy: either $(\Sigma_n)_{n\in\mathbb{N}}$ Gromov–Hausdorff converges (up to passing to a subsequence) towards a translation surface of less complexity, or the Riemannian diameter of Σ_n tends to infinity.

[3] We define the complexity to be the number of triangles in a triangulation whose set of vertices is the set of singular points of the surface.

(4) In the first case, having a cylinder is a property that is open in parameter space, and by the induction hypothesis, for n large enough, Σ_n has a closed geodesic. Since Σ_n has the same real affine structure as Σ, so does Σ.

(5) In the second case, an elegant lemma due to Masur and Smillie [5, Corollary 5.5] ensures that a translation surface of large diameter contains a long flat cylinder and thus contains a closed geodesic, which concludes the proof.

This strategy relies heavily on the Riemannian nature of translation surfaces to get a rather simple analysis of the ways a sequence of translation surfaces can degenerate; this part of the proof breaks down when trying to generalize it to the case of dilation surfaces. Most of the work done here is to replace the last three points of the strategy outlined above by a suitable analysis of the different ways a sequence of dilation surfaces can degenerate. We will give a precise roadmap of the proof in Section 4. The three key technical steps of the proof (Propositions 4.1, 4.3 and 4.4) are proved respectively in Sections 5, 6 and 7.

1.4 An important shortcoming and an open problem

We prove that every dilation surface contains a closed geodesic, but unfortunately we were not able to infer anything concerning the nature of the cylinder carrying this closed geodesic. In particular, our proof does not preclude the existence of a dilation surface which is not a translation surface all of whose cylinders are flat (although the existence of such a surface seems highly unlikely).

Problem 1.4 Show that a dilation surface whose cylinders are all flat is a translation surface.

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2 Dilation surfaces

The symbol Σ will always stand for a compact surface of genus $g \geq 0$ with a finite number of boundary components.

2.1 Dilation cones

Singularities of dilation surfaces are modeled on singularities of dilation cones.

For any $k \in \mathbb{N}^*$, a *flat cone* of angle $2k\pi$ is obtained as the cyclic cover of $\mathbb{C}$ of degree k ramified at 0. The vertex 0 is a *cone point* of angle $2k\pi$ in the flat cone.

For any $k \in \mathbb{N}^*$ and any $\lambda \in \mathbb{R}^*$, a *dilation cone* of angle $2k\pi$ and multiplier λ is obtained from a flat cone of angle $2k\pi$ by cutting a slit along a half-line starting from the vertex 0 and identifying the left side with the right side by a homothety of multiplier λ. The vertex 0 is then a *cone point* of angle $2k\pi$ and dilation multiplier λ.

In particular, for the affine structure induced by the gluing, the holonomy of any closed simple loop around the vertex is a homothety of dilation multiplier λ.

2.2 Generalities

The main objects we will deal with are dilation structures, defined as follows:

Definition 2.1 A *marked topological surface* is a topological surface Σ — possibly with boundary — with a nonempty finite set $S \subset \Sigma$ of *marked points* such that each boundary component contains an element of S.

A *dilation structure* on a marked topological surface (Σ, S) is an atlas of charts $\mathcal{A} = (U_i, \varphi_i)_{i \in I}$ on $\Sigma \setminus S$ such that

- the transition maps are locally restrictions of elements of $\mathrm{Aff}_{\mathbb{R}_+^*}(\mathbb{C}) = \{z \mapsto az + b \mid a \in \mathbb{R}_+^*, b \in \mathbb{C}\}$,
- each marked point in the interior of Σ has a punctured neighborhood which is affinely equivalent to a punctured neighborhood of the cone point of a dilation cone,
- each marked point on the boundary of Σ has a punctured neighborhood which is affinely equivalent to a neighborhood of the center of a Euclidean angular sector of arbitrary angle,
- unless it is a marked point, each point of the boundary of Σ has a punctured neighborhood which is affinely equivalent to a neighborhood of the center of a Euclidean angular sector of angle π.

Elements of S are the *singularities* of the dilation structure.

A particularly simple way of constructing a dilation surface is to glue planar polygons together by using translations and dilations as illustrated in Figure 1. We will see that, up to addition of finitely many singularities with an angle of 2π and a trivial dilation multiplier, every dilation surface can be constructed in this way.

Note that the notion of a straight line on the surface is well defined, since changes of coordinates are affine maps. Moreover, in any direction $\theta \in \mathbb{RP}^1$, the foliation by straight lines of $\mathbb{C}$ in the direction defined by θ being invariant by dilation maps, it gives rise to a well-defined oriented foliation $\mathcal{F}_\theta$ on any dilation surface. Such a foliation is called a *directional foliation*. We call the resulting family of foliations the *directional foliations*; it is indexed by $\mathbb{RP}^1$ and denoted by $(\mathcal{F}_\theta)_{\theta \in \mathbb{RP}^1}$. We shall call any oriented leaf of one of these foliations a *trajectory*.

Definition 2.2 Let Σ be a dilation surface.

- A *closed geodesic* in Σ is a periodic leaf of a directional foliation.

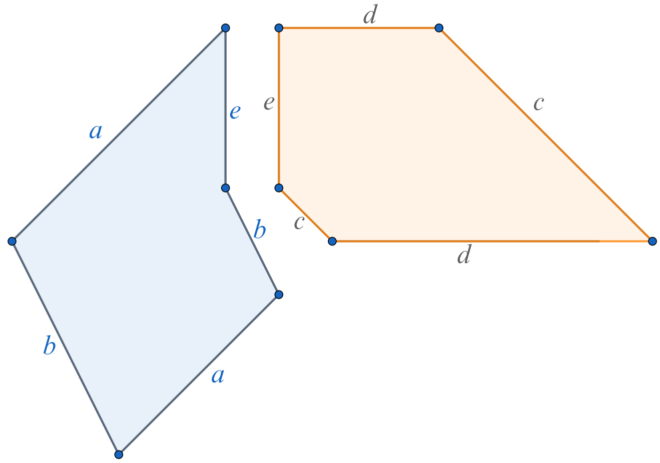

Figure 1: The sides of the two polygons are glued according to their names. Topologically, the resulting surface has genus two and has only one singularity, which corresponds to the extremal points of these two polygons.

- A *saddle connection* is a topological segment on the surface $\Sigma \setminus S$ which is also a straight line (a piece of leaf of a directional foliation) and whose boundary consists of two singularities (possibly identical).

We conclude this subsection with the following definition, which we will use to measure the complexity of a dilation surface:

Lemma 2.3 *We consider a compact topological surface X of genus g with b boundary components, n_i marked points in its interior and n_b marked points on its boundary.*

Assuming $n_b + n_i \geq 1$ and that every boundary component contains at least one marked point, any topological triangulation of X whose set of vertices coincides with the marked points of X is formed by exactly $4g + 2n_i + 2b + n_b - 4$ topological triangles.

Proof The Euler characteristic $\chi(X)$ of surface X is $2-2g-b$. For any such topological triangulation, the number of vertices is n_i+n_b. Thus $2-2g-b = T-A+n_i+n_b$, where T is the number of triangles in the triangulation and A is the number of arcs.

Connected components of the boundary are loops. Thus the number of boundary arcs is exactly n_b. Every arc has two sides (excepted the boundary arcs). Thus $3T = 2A - n_b$. We have $4-4g-2b = 2T-2A+2n_i+2n_b$. It follows that $4-4g-2b = -T+2n_i+n_b$ and thus $T = 4g+2n_i+2b+n_b-4$. □

Definition 2.4 The *complexity* of a marked topological surface is the number of triangles of any topological triangulation whose set of vertices is exactly the set of marked points. By convention, we define the complexity of the empty set to be zero.

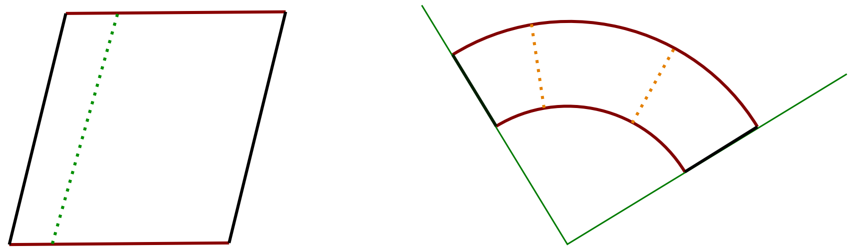

Figure 2: Left: a flat cylinder with a closed geodesic represented by the dashed line (corresponding to the only direction in which there is a closed geodesic). Right: a dilation cylinder with two closed geodesics of two different directions.

We define the complexity of a dilation surface as the complexity of the underlying marked topological surface.

2.3 Cylinders

Cylinders are the geometric counterpart of the periodic leaves of the directional foliations as, in particular, each cylinder contains a closed geodesic. Conversely, any neighborhood of a closed geodesic contains a portion of cylinder. Here we always understand cylinders as maximal: we say that a cylinder is maximal if it is not included in any cylinder but itself.

A *flat cylinder* is a dilation surface with boundary obtained by gluing a pair of opposite sides of a parallelogram embedded in $\mathbb{R}^2$.

A *dilation cylinder* is a dilation surface (with boundary) obtained by cutting a sector C_θ of angle θ in the universal cover of $\mathbb{C}^*$. The quotient of C_θ by the dilation $z \mapsto \lambda z$ with $\lambda > 1$ real is called a *dilation cylinder* (see Figure 2).

2.4 Moduli of cylinders

In this subsection we give an interpretation of conformal moduli of cylinders in dilation structures.

- Recall that the modulus of a flat cylinder obtained from a rectangle of base $(z_1, z_2) \in \mathbb{C}^2$ where the sides glued together are those corresponding to z_2 is by definition $|z_2|/|z_1|$.
- A dilation cylinder of angle θ and dilation multiplier $\lambda > 1$ is biholomorphic (using the exponential map) to the flat cylinder obtained from a rectangle of base $(\ln(\lambda), i\theta)$. Its conformal modulus is thus $\theta/\ln(\lambda)$.

We call a closed geodesic within a cylinder a waist curve of this cylinder.

Lemma 2.5 *There is an absolute constant $M > 0$ such that for any pair of cylinders C_1 and C_2 of conformal modulus at least M in a dilation surface Σ, either C_1 and C_2 are disjoint, or their waist curves are in the same homotopy class.*

Proof A consequence of the Margulis lemma is the existence of a universal constant ϵ such that in any hyperbolic surface of unit area, closed geodesics of length smaller than ϵ are automatically disjoint (see [3, Section 4.2.4] for a reference).

In the conformal class of Σ (punctured at the singularities), we consider the unique hyperbolic metric of unit area. In the homotopy class of waist curves of cylinder C_1 (resp. C_2), there is a unique simple closed geodesic γ_1 (resp. γ_2). We denote by $l(\gamma_1)$ and $l(\gamma_2)$ the lengths of geodesics γ_1 and γ_2. Assuming that waist curves of C_1 and C_2 do not belong to the same homotopy class, γ_1 and γ_2 are distinct.

Following the interpretation of conformal modulus in terms of extremal length, if the conformal modulus of C_1 is strictly bigger than $M = \epsilon^{-2}$, then $l(\gamma_1) < \epsilon$. The same holds for C_2 and $l(\gamma_2)$. Thus γ_1 and γ_2 are disjoint, and waist curves of C_1 and C_2 do not intersect. □

Corollary 2.6 *For any dilation surface Σ, there exists a constant $M(\Sigma) > 0$ such that any cylinder in Σ has conformal modulus smaller than $M(\Sigma)$.*

Proof We assume for contradiction that Σ contains an infinite family of cylinders of arbitrarily large moduli. Since two different cylinders always define two different free homotopy classes, we can always find intersecting cylinders with arbitrarily large moduli. This contradicts Lemma 2.5. □

2.5 Pencils

Pencils were studied in [7] to make explicit the geometric properties of strict dilation surfaces in comparison with translation surfaces. We gather the needed results in this section.

Definition 2.7 A *pencil* is a continuous family of oriented trajectories starting from the same point. Let x be a (possibly singular) point of a dilation surface Σ, and I an open interval of $\mathbb{RP}^1$. The notation $P(x, I)$ will refer to a pencil of trajectories starting at x and covering directions of I.

It should be noted that there are usually several pencils for a given pair (x, I).

The following statement provides a geometric criterion for the existence of dilation cylinders:

Lemma 2.8 [7, Lemma 3.3] *Let x be a point in a dilation surface Σ (possibly with boundary) and I be an open interval of $\mathbb{RP}^1$. For a given pencil $P(x, I)$, at least one of the following statements must hold*:

(1) *a trajectory of $P(x, I)$ hits a singularity*,

(2) *there exists a closed geodesic whose direction belongs to the interval I*,

(3) *there is an open subset $J \subset I$ such that trajectories of the restricted pencil $P(x, J)$ cross the interior of a boundary component of Σ.*

Note that in the case where Σ is without boundary then only the two first items can hold.

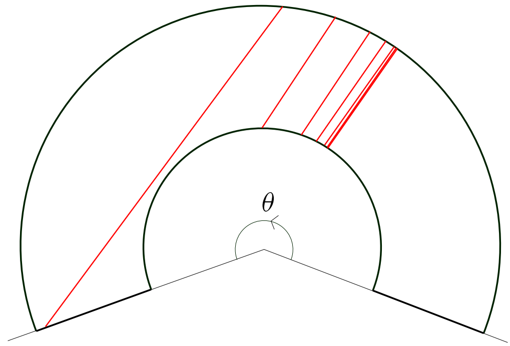

Figure 3: A fundamental domain for the action of $z \mapsto 2z$ on a cone of angle $\theta > \pi$. Any trajectory entering the cylinder is trapped within it forever regardless of the direction of the trajectory, as the one represented here. This property prevents a polygonation from "connecting" both sides of the cylinder.

A second result proves the existence of dilation cylinders in dilation surfaces with nonempty boundary where trajectories of a pencil avoid the boundary:

Proposition 2.9 [7, Corollary 4.6] *Let Σ be a connected dilation surface with a nonempty boundary, a point $x \in \Sigma$ and an open interval I in $\mathbb{RP}^1$. Then at least one of following statements holds:*

(1) *there is an open interval $J \subset I$ such that every trajectory of the restricted pencil $P(x, J)$ accumulates on a closed geodesic of a dilation cylinder of Σ,*
(2) *there is an open interval $J \subset I$ such that every trajectory of $P(x, J)$ crosses the interior of a boundary saddle connection of Σ.*

2.6 Nonpolygonable surfaces

Definition 2.10 A polygonation of a dilation surface Σ is family of saddle connections $\gamma_1, \dots, \gamma_k$ such that

(i) interiors of saddle connections $\gamma_1, \dots, \gamma_k$ are disjoint,
(ii) connected components of $\Sigma \setminus \bigcup_{i=1}^{k} \gamma_i$ are flat polygons without any interior singularity.

A surface Σ is polygonable if it admits a polygonation.

Veech's criterion provides a geometric characterization of polygonable surfaces. This theorem is optimal since cylinders of angle at least π are not polygonable, as shown in Figure 3.

Theorem 2.11 (Veech's criterion [1]) *For a closed dilation surface Σ containing at least one singularity, the three following propositions are equivalent:*

- *Σ is polygonable,*
- *Σ does not contain a dilation cylinder of angle at least π,*
- *every affine immersion of the open unit disk $\mathbb{D} \subset \mathbb{C}$ in Σ extends continuously to its closure $\overline{\mathbb{D}}$.*

Remark 2.12 Up to adding enough singularities of angle 2π and trivial dilation multiplier, we can nevertheless decompose cylinders of angle at least π into smaller cylinders and then into polygons.

For our purpose, Theorem 2.11 proves in particular that every dilation surface that is not polygonable carries cylinders, and one can focus on polygonable surfaces.

2.7 The action of $\mathrm{SL}(2,\mathbb{R})$

We now define a natural action of $\mathrm{SL}(2,\mathbb{R})$ on the space of dilation surfaces.

Let Σ be a dilation surface and consider $A \in \mathrm{SL}(2,\mathbb{R})$. Let $(U_i,\varphi_i)_{i\in I}$ be a maximal atlas defining the dilation structure of Σ. Define $A\Sigma$ to be the dilation structure defined by the maximal atlas $(U_i, A\circ\varphi_i)_{i\in I}$ where A acts on $\mathbb{C}$ via the standard identification $\mathbb{C}\simeq\mathbb{R}^2$. This new atlas indeed defines a dilation structure, as $\mathrm{SL}(2,\mathbb{R})$ centralizes the group formed by maps $z\mapsto az+b$ where $a\in\mathbb{R}_+^*$ and $b\in\mathbb{C}$.

If the dilation surface was given by gluing a bunch of polygons together, the new surface is also polygonable. Indeed, the image of the initial set of polygons with edges identified is mapped by the linear action of the matrix A to another set of polygons. Since A is linear, the sides of the polygon that were parallel are still parallel after applying the matrix A, so that one can still glue them using dilations of the plane. The resulting dilation surface is the image of Σ under the matrix A.

A remarkable subgroup of $\mathrm{SL}(2,\mathbb{R})$ is formed by matrices

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

for $t\in\mathbb{R}$. The flow expands the horizontal direction and contracts the vertical direction.

3 Delaunay polygonations

3.1 Delaunay polygonation

The goal of this subsection is to define the Delaunay polygonation of a (polygonable) dilation surface. The construction we will give actually is Veech's proof of Theorem 2.11. To show that surfaces that do not carry cylinders of angle larger than π are polygonable, he proved that the following construction defines a polygonation. We refer to [1] for the full proof and will only describe it here.

The vertices of this polygonation are by definition the singularities of Σ. The edges of the polygonation are saddle connections: a given saddle connection between singularities s_1 and s_2 belongs to the edges of the Delaunay triangulation if there is a closed disk immersed in Σ such that s_1 and s_2 belong to the boundary circle of this disk and such that there are no other singularities in its interior. A disk in Σ is said to be Delaunay if it does not contain any singularities in its interior but at least three on its boundary.

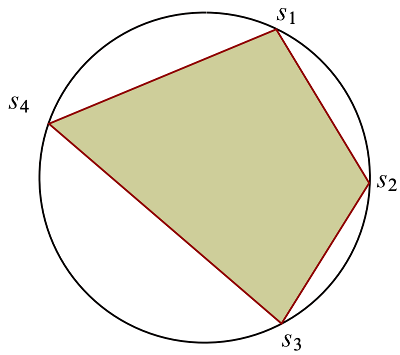

Figure 4: A Delaunay disk with four boundary singularities. Their convex hull in the disk is a face of the Delaunay polygonation.

The faces correspond to what is left after suppressing the edges and the vertices: they are convex polygons whose extremal points all belong to the same Delaunay disks. Figure 4 illustrates the construction: the disk is a Delaunay disk whose boundary contains singularities s_1, s_2, s_3 and s_4. The quadrilateral is one of the faces of the polygonation while its four sides are edges.

Note that the Delaunay polygonation gives you a way to recover from an "abstract" polygonable dilation surface a concrete set of polygons that defines it.

Remark 3.1 Here, even if surfaces with boundary may appear, we will only consider Delaunay polygonations of dilation surfaces without boundary.

3.2 Polygons up to dilation and their limits

In this subsection we consider the space of polygons with exactly $p \geq 3$ vertices arising from Delaunay polygonation. We consider these polygons *as marked and up to dilation*, which means that

- we think of Delaunay polygons as within the unit circle, as we can use a dilation to map the Delaunay circle to the unit one,
- we keep track of the role of each side and each vertex, which is what we mean by marked,
- a polygon and its image under a rotation are considered to be different (because polygons are considered up to dilation and not similarity).

We denote the set described above by $\mathcal{P}_p$. Each polygon is characterized by a p-tuple $(\theta^1, \dots, \theta^p) \in (\mathbb{R}/2\pi\mathbb{Z})^p$, where θ^i is the angle of vertex i in the Delaunay circle and $\theta^i \neq \theta^j$ for $i \neq j$.

Definition 3.2 Let p be fixed. Consider a sequence of polygons $(P_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_p$. We say that this sequence is *Delaunay-convergent* if the following conditions hold:

- the cyclic ordering of the vertices in the circle is constant,
- each vertex $(\theta_n^i)_{n\in\mathbb{N}}$ converges in the circle.

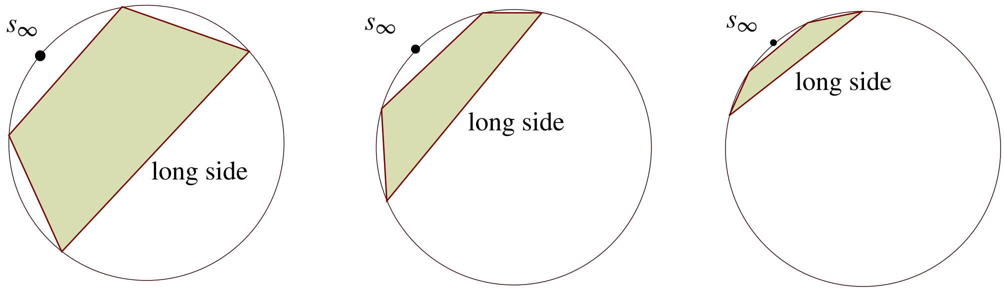

Figure 5: The three first polygons of a degenerating sequence of type 1 whose vertices all converge toward s_∞.

Besides, a Delaunay-convergent polygon is

- a *polygon of type* 1 if all the vertices of $(P_n)_{n\in\mathbb{N}}$ converge towards a given point s_∞ of the circle containing all vertices of P_n (see Figure 5),
- a *polygon of type* 2 if the set of vertices of $(P_n)_{n\in\mathbb{N}}$ converges towards a set of exactly two points s^1_∞ and s^2_∞ of the Delaunay circle (see Figure 6),
- a *polygon of type* 3 if the set of vertices of $(P_n)_{n\in\mathbb{N}}$ converges towards a set of at least three vertices.

In the second case, the slope of the limit edge in $\mathbb{RP}^1$, relating the two remaining vertices, is called the *limit slope*.

By compactness, one can from any sequence of polygons $(P_n)_{n\in\mathbb{N}}$ extract a Delaunay-convergent subsequence.

We now introduce the following terminology which will be useful when proving our main theorem:

Definition 3.3 If $(P_n)_{n\in\mathbb{N}}$ is of type 1, the longest side of the polygon, corresponding to the closest one from the center of the circle in which it is inscribed, is called the *long side*, while the other sides will be called *short*. The *direction* of the polygon in $\mathbb{RP}^1$ is the direction given by its long side

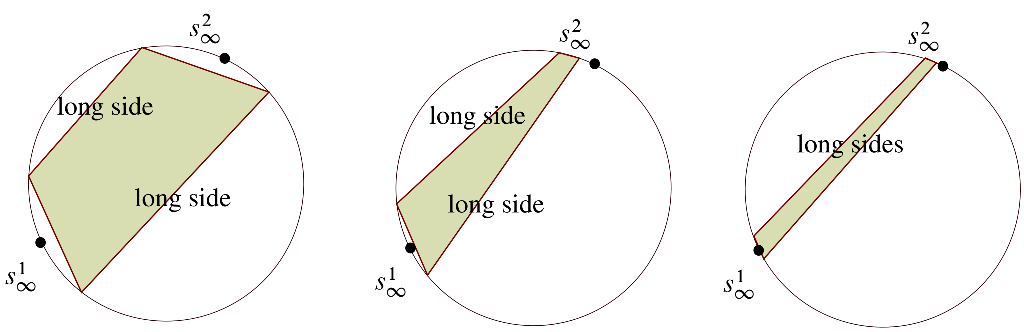

Figure 6: The three first polygons of a degenerating sequence of type 2 whose vertices all converge toward either s^1_∞ or s^2_∞.

If $(P_n)_{n\in\mathbb{N}}$ is of type 2 or 3, the longest sides whose vertices converge to different limit point will be called the *long sides* while the others sides are called *short*. In the type-2 case, the directions tangent to the circle at the two remaining vertices are called the *short side limit slopes* as the short sides are asymptotic to this direction.

Be aware that the terminology is about sequences of polygons, and more precisely about their asymptotic behavior: one can change finitely many polygons of the sequence without changing its long or short sides.

By a harmless abuse of notation, we will refer to a sequence of polygons $(P_n)_{n\in\mathbb{N}}$ in a Delaunay-convergent sequence as a *polygon*. We will use the terms of degenerating polygons. The terms long sides and short sides for these polygons will refer similarly to sequences of edges.

3.3 Delaunay-convergent sequences of dilation surfaces

In this subsection we consider sequences of dilation surfaces of fixed topological type (the underlying marked topological surfaces are isomorphic).

Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of dilation surfaces of same topological type. Up to extracting a subsequence, we can assume that their Delaunay polygonations are all combinatorially equivalent. Precisely, this means that for any $n\in\mathbb{N}$ their Delaunay polygonations have the same pattern. We label for each n the set I of polygons $(P_{i,n})_{i\in I}$ in such a way that

- the sequence $(P_{i,n})_{n\in\mathbb{N}}$ has always the same numbers of sides,
- one can mark the sides of the polygons so that the gluing pattern of the sides of the marked polygons $(P_{i,n})_{n\in\mathbb{N}}$ is constant with respect to the marking.

In that case we say that sequence of surfaces $(\Sigma_n)_{n\in\mathbb{N}}$ has *constant Delaunay pattern*.

Definition 3.4 A sequence $(\Sigma_n)_{n\in\mathbb{N}}$ is said to be *Delaunay-convergent* if

(1) the sequence $(\Sigma_n)_{n\in\mathbb{N}}$ has constant Delaunay pattern,

(2) every polygon $(P_{i,n})_{n\in\mathbb{N}}$ is Delaunay convergent (see Definition 3.2).

We refer to the edges of these polygons as the *Delaunay edges* of the pattern.

For a given sequence of dilation surfaces of fixed topological type there are finitely many Delaunay patterns, so we can always extract a Delaunay-convergent subsequence.

3.4 Maximal domains of type 1

Properties of Delaunay polygonations induce constraints on the Delaunay patterns involving polygons of type 1.

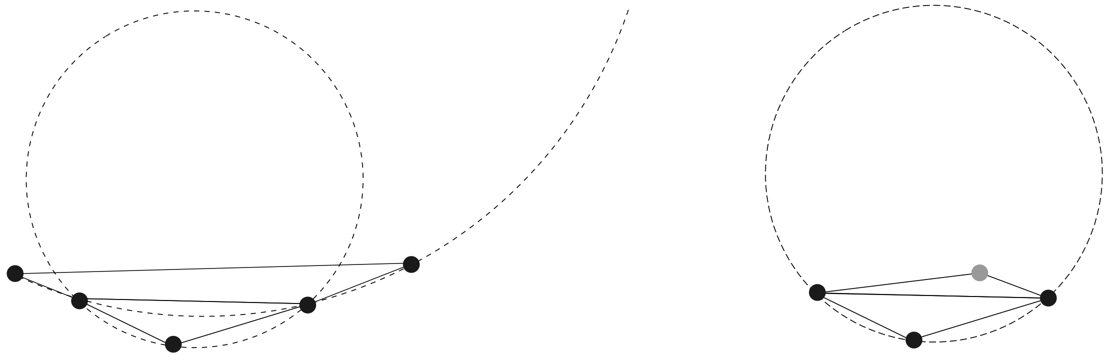

Figure 7: Left: A configuration of two polygons of type 1. None of the singularities lie inside the two Delaunay disks. As the polygons shrink, the Delaunay disks tend to cover half a plane. Right: A forbidden configuration. The gray dot lies inside a large Delaunay disk.

Proposition 3.5 *In a Delaunay-convergent sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of polygonable dilation surfaces, the long side $(L_n)_{n\in\mathbb{N}}$ of a polygon $(P_n)_{n\in\mathbb{N}}$ of type 1 can only be incident to a short side of a polygon (of any type).*

Proof We assume that in addition to being a side of $(P_n)_{n\in\mathbb{N}}$, the edge $(L_n)_{n\in\mathbb{N}}$ is a long side of a polygon $(Q_n)_{n\in\mathbb{N}}$. In these cases, vertices of $(P_n)_{n\in\mathbb{N}}$ distinct from the ends of $(L_n)_{n\in\mathbb{N}}$ are included for n large enough in the interior of the Delaunay disk of polygons $(Q_n)_{n\in\mathbb{N}}$; see Figure 7. This is a contradiction. □

Following Proposition 3.5, the long side of a polygon of type 1 is always incident to the short side of another polygon. We say that two polygons of type 1 belong to the same *domain of type* 1 if the long side of the first is incident to a short side of the second. The equivalence relation generated by these relations defines classes. This way, each polygon of type 1 belongs to a unique *maximal domain of type* 1.

Besides, in a maximal domain of type 1, any internal edge is a long side of a polygon while being a short side of another (it may happen that the polygons coincide). Thus the incidence graph of a maximal domain is actually an oriented graph with a unique oriented edge leaving each vertex (since each polygon of type 1 has a unique long side). It follows from that there are two types of maximal domains of type 1:

- *noncyclic domains*, where the incidence graph is a rooted tree (the edges being oriented towards the root),
- *cyclic domains*, where the oriented incidence graph contains a unique (oriented) cycle.

Since a maximal domain of type 1 is connected, there is no other type of graphs of incidence.

3.5 Maximal domains of type 2

Polygons of type 2 have two long sides. Two polygons of type 2 glued along an edge that is a long side for each of them belong to a same domain of type 2. This way, each polygon of type 2 belongs to a

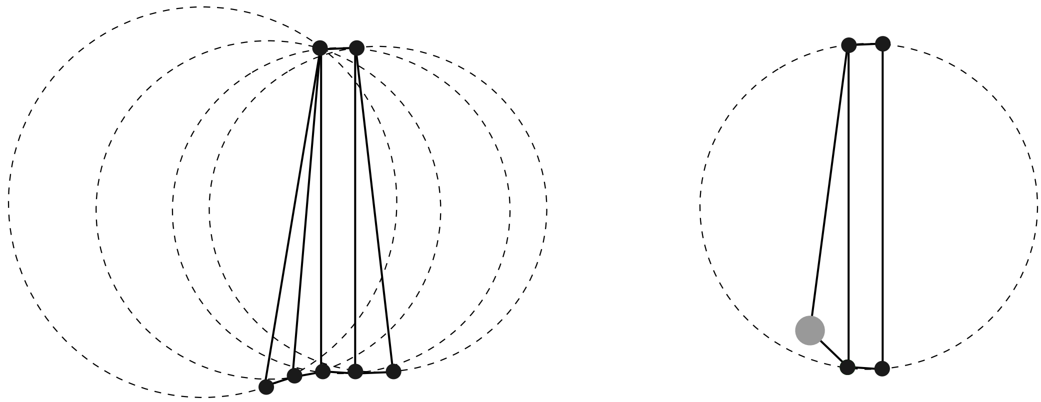

Figure 8: Left: An admissible configuration of four degenerating polygons of type 2. None of the singularities lie inside the Delaunay disks. As the polygons shrink, the Delaunay disks tend to cover a half-plane. Right: A forbidden configuration. The gray dot lies inside a large Delaunay disk.

unique *maximal domain of type* 2. These domains can be *cyclic* or not. In the noncyclic case we call the two long edges that are not glued with another polygon of type 2 the *extremal edges*.

Remark 3.6 The case of a long side of a polygon of type 2 glued along a short side of another polygon of type 2 can happen. We can obtain such a configuration in a variant of the degeneration presented in Figure 9. If the modulus of the connecting flat cylinder decreases to zero (instead of going to infinity), the cylinder is a maximal domain of type 2 and its upper extremal edge is glued on the short side of a polygon of type 2.

An observation that will be needed in the proof of Theorem 1.1 is that short sides of maximal domains of type 2 form "concavely shaped" curves, as shown in Figure 8. It proceeds from the following statement:

Proposition 3.7 *We consider a Delaunay-convergent sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of polygonable dilation surfaces. In the polygon formed by two degenerating polygons of type 2 or 3 glued along a common long side, the magnitude of the limit inner angle between two consecutive short sides is at least π.*

Proof Two consecutive short sides belonging to the same polygon of type 2 or 3 have the same limit slope because their endpoints converge to the same limit point in the Delaunay circle (see Definition 3.3). Therefore, the limit inner angle between them is equal to π.

Now we consider the case of two consecutive short sides $[A_n, B_n]_{n\in\mathbb{N}}$ and $[B_n, C_n]_{n\in\mathbb{N}}$ belonging respectively to two distinct incident degenerating polygons $(P_n^1)_{n\in\mathbb{N}}$ and $(P_n^2)_{n\in\mathbb{N}}$ of type 2 or 3. These two sides have well-defined limit slopes (corresponding to the slope of the tangent line at their limit point in their Delaunay circle). We will assume for contradiction that the limit inner angle θ between these sides at B_n is strictly smaller than π.

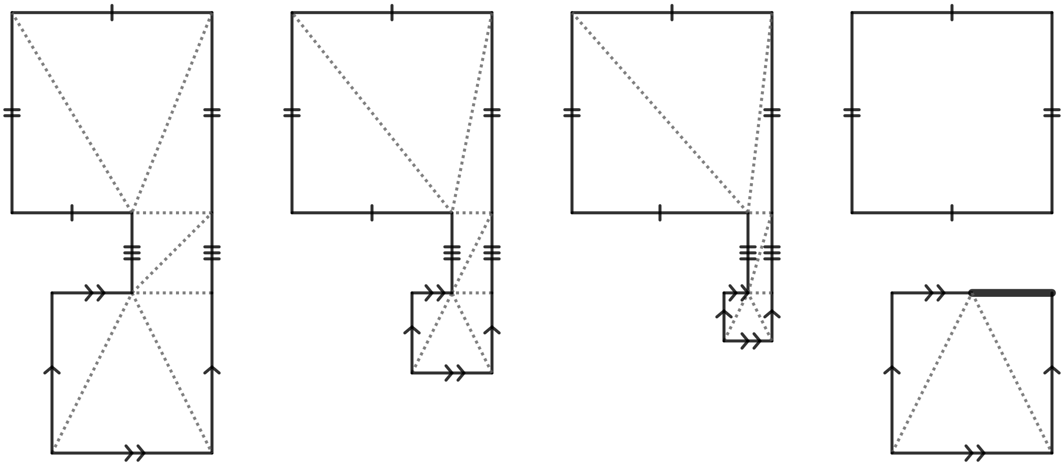

Figure 9: The three first drawings represent some terms of a degenerating dilation surface of genus two with one singularity whose "connecting" flat cylinder degenerates as its modulus goes to infinity. At the level of the Delaunay polygonation, the two parts left after removing the connecting cylinder converge to: (i) a flat torus which is the "Delaunay limit" of the upper part, (ii) a torus with one horizontal boundary component (in bold).

It follows that there is $N > 0$ such that for any $n \geq N$, the segment $[B_n C_n]$ intersects the Delaunay disk $\mathcal{D}_n^1$ of P_n^1. Since by hypothesis C_n cannot belong to P_n^1, the segment $[B_n C_n]$ intersects the boundary of P_n^1 in some point C'_n. The triangle $A_n B_n C'_n$ is inscribed in the Delaunay circle that bounds $\mathcal{D}_n^1$.

Since $P_n^1 \cup P_n^2$ is contractible, it can be endowed with a flat metric in such a way that $[A_n, B_n]_{n\in\mathbb{N}}$ and $[B_n, C_n]_{n\in\mathbb{N}}$ have meaningful lengths. The latter metric is normalized by fixing the radius of the Delaunay disk $\mathcal{D}_n^1$ to 1. As $n \to +\infty$, the length of $[A_n, B_n]$ shrinks to zero. Since the inner angle at B_n converges to $\theta < \pi$, the length of $[B_n C'_n]$ converges to some nonzero limit. It follows that the length of $[B_n C_n]$ cannot decrease to zero as n tends to infinity. In $(P_n^2)_{n\in\mathbb{N}}$, the length of $[B_n C_n]$ does not become negligible in comparison with the length of the common edge between $(P_n^1)_{n\in\mathbb{N}}$ and $(P_n^2)_{n\in\mathbb{N}}$. In other words, $[B_n C_n]$ is not a short side of $(P_n^2)_{n\in\mathbb{N}}$, and we get a contradiction. □

3.6 Delaunay limits

For any Delaunay-convergent sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of closed dilation surfaces, we can define a *Delaunay limit* Σ_∞ formed by the polygons that do not completely degenerate; see Figure 9 for an example.

The limit surface Σ_∞ will be a polygonable dilation surface. However, we should be careful. The limit surface can have several connected components. It can also have a boundary, and it can even be empty.

Definition 3.8 Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a Delaunay-convergent sequence of closed dilation surfaces. We define the *Delaunay limit* Σ_∞ in the following way:

- Σ_∞ is the union of limits of polygons of type 3 in $(\Sigma_n)_{n\in\mathbb{N}}$,

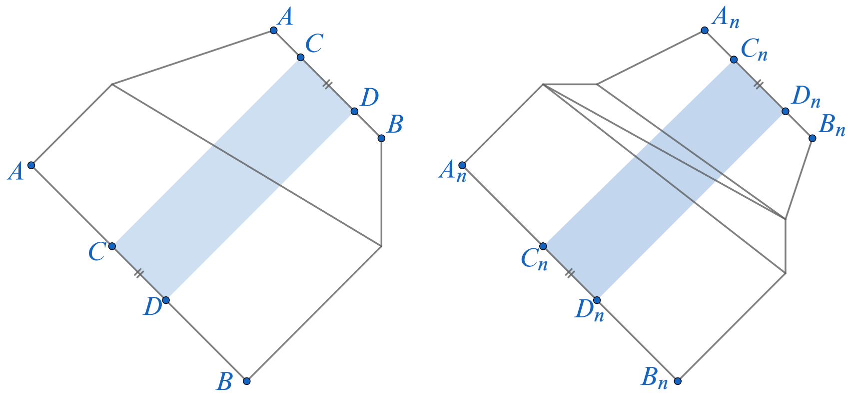

Figure 10: The bands here are shaded. Left: the neighborhood of some periodic orbit. Right: the corresponding band, defined by using the affine identification $\mathcal{A}_n$ of the edges.

- two sides of limit polygons of $(\Sigma_n)_{n\in\mathbb{N}}$ are identified to each other if they are incident in the Delaunay pattern of the sequence,
- the sides of limit polygons are also identified to each other if they are connected by a noncyclic maximal domain of type 2 (see Section 3.5).

As they degenerate to polygons with empty interior, polygons of type 1 and 2 completely disappear in the Delaunay limit Σ_∞ (see Figures 9 and 11).

The simple, but key, feature about this notion of limit is that "carrying a cylinder" is an open property. Namely, if a sequence of dilation surfaces has a Delaunay limit which carries a cylinder, then for any large enough index of the sequence the corresponding surface also carries a cylinder.

Proposition 3.9 *We consider a Delaunay-convergent sequence $(\Sigma_n)_{n\in\mathbb{N}}$. If its Delaunay limit Σ_∞ is nonempty and contains a closed geodesic in some direction $\theta \in \mathbb{RP}^1$, then for any $\epsilon > 0$, there is $N > 0$ such that for any $n \geq N$, Σ_n contains a closed geodesic in a direction of $]\theta-\epsilon, \theta+\epsilon[$.*

Proof We denote by γ a closed geodesic of slope θ in the limit surface Σ_∞. Such a geodesic must cross an edge of the Delaunay polygonation as there is no closed geodesic contained in the interior of a Delaunay polygon. Let us denote by $[A,B]$ an edge crossed by γ and by $[A,B]_\gamma$ the intersection of $[A,B]$ with γ. By definition, closed geodesics do not contain any singularity. It follows that $[A,B]_\gamma$ belongs to the interior of $[A,B]$ (it is not a singularity). Moreover, as $[A,B]_\gamma$ belongs to a periodic leaf of $\mathcal{F}_\theta$, the foliation $\mathcal{F}_\theta$ on Σ_∞ has a well-defined first return map on a neighborhood $[C,D]$ of $[A,B]_\gamma$; see Figure 10.

By definition of the Delaunay limit, the edge $[A,B]$ is the limit edge of a sequence of long sides $([A_n,B_n])_{n\in\mathbb{N}}$ of a polygon of type 3 in the Delaunay polygonation of $(\Sigma_n)_{n\in\mathbb{N}}$. As the polygons converge, the unique (up to translation) complex affine mapping $\mathcal{A}_n$ of the plane that maps $[A_n,B_n]$ to $[A,B]$ converges to the identity as $n\to\infty$. We set $x_n := \mathcal{A}_n^{-1}([A,B]_\gamma)$. Note that $x_n \to [A,B]_\gamma$ as $n\to\infty$.

For n large enough, the leaf of the foliation $\mathcal{F}_\theta$ in Σ_n starting at x_n crosses the edges corresponding to the edges crossed by γ in Σ_∞ (several edges of Σ_n can correspond to the same edge of Σ_∞ if they are long sides of the same noncyclic maximal domain of type 2) and then crosses back $[A_n, B_n]$ at some point.

As the finitely many polygons encountered converge toward nondegenerate polygons or degenerates toward "edges", there is a bound $N > 0$ such that for any $n \geq N$, the first return map T_n of $[A_n, B_n]$ is well defined on a neighborhood $[C_n, D_n] := \mathcal{A}_n^{-1}([C, D])$.

All the oriented leaves of $\mathcal{F}_\theta$ starting from $[C_n, D_n]$, taken up to their first return on $[A_n, B_n]$, give rise to a band $\mathcal{B}_n$ (a parallelogram) whose sides contained in $[A_n, B_n]$ partially coincide. In particular, $\mathcal{B}_n$ contains a closed geodesic. Similarly, we define $\mathcal{B}_\infty$ in Σ_∞.

For an arbitrarily small $\eta > 0$, we can choose a neighborhood $[C, D]$ of $[A, B]_\gamma$ such that the slopes of the two diagonals of $\mathcal{B}_\infty$ are contained in $]\theta - \eta, \theta + \eta[$. Then, provided n is large enough, the slopes of the diagonals of $\mathcal{B}_n$ can be made arbitrarily close to slopes of the diagonals of $\mathcal{B}_\infty$. The slope of a closed geodesic contained in $\mathcal{B}_n$ belongs to an interval whose ends are the slopes of the diagonals of $\mathcal{B}_n$. It follows that for any $\epsilon > 0$, there is $N > 0$ such that for any $n \geq N$, $\mathcal{B}_n$ contains a closed geodesic whose slope is contained in $]\theta - \epsilon, \theta + \epsilon[$. □

We also need to keep track of the polygons involved in the Delaunay limit. To this purpose, we introduce the notion of *core sequence*:

Definition 3.10 For a given Delaunay-convergent sequence $(\Sigma_n)_{n\in\mathbb{N}}$, the *core sequence* $(C\Sigma_n)_{n\in\mathbb{N}}$ is defined for each n as the union of

- polygons of type 3,
- noncyclic maximal domains of type 2 in which at least one extremal edge is incident to a long side of a polygon of type 3.

Remark 3.11 It follows from Definitions 3.8 and 3.10 that each connected component of Σ_∞ corresponds to a unique connected component of $C\Sigma_n$. Boundary saddle connections of Σ_∞ correspond to long boundary edges of the core.

Maximal domains of type 1, maximal domains of type 2 and connected components of the core are the fundamental pieces of the decomposition we will use in the proof of Theorem 1.1.

Definition 3.12 Polygons of $(\Sigma_n)_{n\in\mathbb{N}}$ are grouped into *Delaunay pieces* that are:

- the connected components of the core $(C\Sigma_n)_{n\in\mathbb{N}}$,
- the maximal domains of type 1,
- the maximal domains of type 2 that do not belong to the core.

It follows from Definition 3.10 that every polygon of $(\Sigma_n)_{n\in\mathbb{N}}$ belongs to a unique Delaunay piece.

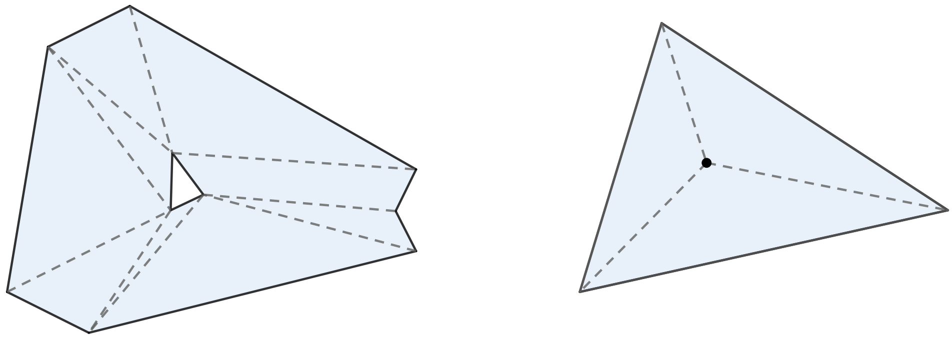

Figure 11: Left: The surface $C\Sigma_n$ of the core sequence with two boundary components. The exterior boundary component contains three long sides, while the interior boundary component is formed by short sides only. Right: The Delaunay limit Σ_∞ with an exterior boundary formed by three sides and a singularity at the center.

The important feature of the decomposition into Delaunay pieces is that their boundary edges cannot be long sides for both of their incident polygons.

Lemma 3.13 *If a Delaunay edge $(L_n)_{n\in\mathbb{N}}$ of $(\Sigma_n)_{n\in\mathbb{N}}$ is a long side for its two incident polygons, then it cannot belong to the boundary of a Delaunay piece.*

Proof It follows from Proposition 3.5 that neither of the incident polygons of $(L_n)_{n\in\mathbb{N}}$ can be polygons of type 1. If both of them are polygons of type 2, then they belong to the same maximal domain and $(L_n)_{n\in\mathbb{N}}$ is not a boundary edge of a Delaunay piece. If at least one of the two polygons incident to $(L_n)_{n\in\mathbb{N}}$ is of type 3, then it follows from Definition 3.10 that $(L_n)_{n\in\mathbb{N}}$ is an interior edge of some connected component of the core $(C\Sigma_n)_{n\in\mathbb{N}}$. □

4 Overview of the proof of Theorem 1.1

As mentioned above, we will argue by induction on the complexity of the closed surface Σ (see Definition 2.4). It is easy to deal with the case of smallest complexity: the flat tori formed by a pair of triangles. The induction assumption is that any closed dilation surface of complexity lower than k carries cylinders in a dense set of directions. We want to show that it is still the case for surfaces of complexity $k+1$.

It follows from Lemma 2.8 that for a dilation surface Σ without boundary, an open set of $\mathbb{RP}^1$ that does not contain any direction of saddle connection contains the direction of a closed geodesic. Therefore, it remains to prove that any direction $\theta \in \mathbb{RP}^1$ that is approached by directions of saddle connections is also approached by directions of closed geodesics. In other words, any open subset of $\mathbb{RP}^1$ containing the direction of a saddle connection should also contain the direction of a closed geodesic. Up to the action of an element of $\mathrm{SL}_2(\mathbb{R})$, we can assume that Σ contains a vertical saddle connection γ and U is an open subset of $\mathbb{RP}^1$ containing the vertical direction.

To rely on the induction assumption, we will use the Teichmüller flow g_t that contracts the vertical direction, and therefore the saddle connection γ, and that expands the horizontal direction. We have seen in Section 3.3 that one can extract a sequence of times $t_n \to +\infty$ such that the sequence $(\Sigma_n)_{n\in\mathbb{N}} := (g_{t_n}\Sigma)_{n\in\mathbb{N}}$ is Delaunay convergent. We are doing induction on closed surfaces, and this is why the case where Σ_∞ has a boundary will need to be handled separately. We first rule out the case in which $(\Sigma_n)_{n\in\mathbb{N}}$ Delaunay-converges toward a closed surface Σ_∞ of the same complexity (see Section 3.6 for a precise definition of Delaunay limits).

Proposition 4.1 *Let Σ be a closed dilation surface that carries a vertical saddle connection and a sequence of times $t_n \to +\infty$ such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges toward a surface Σ_∞. Then one of the following statements holds*:

- *for any $\epsilon > 0$, Σ carries a cylinder whose direction belongs to $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$,*
- *Σ_∞ is of strictly smaller complexity than Σ.*

Proposition 4.1 is proved in Section 5. Note that in the first case there is nothing more to be proven. In the second case, as soon as Σ_∞ is nonempty and contains a component without boundary, we can also conclude. Indeed, the induction assumption guarantees that directions of closed geodesics of Σ_∞ are dense in $\mathbb{RP}^1$. In particular, Σ_∞ contains closed geodesics whose directions are arbitrarily close to the vertical direction.

One can now conclude using Proposition 3.9 that provided n is large enough, Σ_n contains a closed geodesic whose direction is arbitrarily close to the vertical direction. It follows that $\Sigma = g_{t_n}\Sigma_n$ contains a closed geodesic whose direction is even closer to the vertical direction.

It then remains to deal with two cases:

- Σ_∞ is empty,
- every connected component of Σ_∞ has a nonempty boundary.

We will deal with these two cases at once by thoroughly examining how the Delaunay polygonations, and especially the Delaunay pieces, degenerate under the Teichmüller flow.

The key notion here is that of *short and long boundary*. We will say that a Delaunay piece (see Definition 3.12) *has a long boundary side* if one of its boundary sides is the long boundary (in the sense of Definition 3.3) of a Delaunay polygon that belongs to the Delaunay piece.

Recall that a Delaunay piece is either a component of the core, or a maximal domain of type 1 or 2. A boundary side of the core can be short or long, but the short ones disappear in Σ_∞ by construction. In particular, if the limit of a Delaunay piece that belongs to the core has a boundary side, then the Delaunay piece in question must have a long boundary side. A maximal domain of type 1 or 2 must have, by construction, a long boundary side, except in the case where it is cyclic. We summarize the content of this discussion within the following structural lemma:

Lemma 4.2 *Let Σ be a dilation surface and $t_n \to +\infty$ be such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges toward a surface Σ_∞ of strictly smaller complexity than Σ. Then at least one the following conditions holds*:

(1) *one of the Delaunay pieces converges toward a closed nonempty dilation surface of smaller complexity*,

(2) *one of the Delaunay pieces is a cyclic maximal domain of type* 1 *or of type* 2,

(3) *all the Delaunay pieces have at least one long boundary side.*

As mentioned above, the first case is dealt with using the induction assumption. The second case will follow from the next result, which will be proven in Section 6.

Proposition 4.3 *Let Σ be a dilation surface*, *and $t_n \to +\infty$ be such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges and such that at least one of its Delaunay pieces is a cyclic maximal domain of type* 1 *or of type* 2. *Then for any $\epsilon > 0$, Σ carries a cylinder whose direction belongs to $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$.*

One is then left to analyze the last case, in which all the Delaunay pieces have at least one long boundary side. This is the most subtle part of the article.

Proposition 4.4 *Let Σ be a dilation surface*, *and $t_n \to +\infty$ be such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges and such that all the Delaunay pieces in $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ have at least one long boundary side.*

Then, *for any open set $U \subset \mathbb{RP}^1$*, *there is $N > 0$ such that for any $n \geq N$*, *Σ_n contains closed geodesics whose directions belong to U.*

The above proposition shows in particular that Σ carries closed geodesics whose directions are as close as we want to the vertical one as, as usual, nonhorizontal closed geodesics of Σ_n are images of almost vertical ones of Σ under the Teichmüller flow. This proves that any open set of $\mathbb{RP}^1$ containing the vertical direction contains the direction of a closed geodesic of Σ. The three cases of Lemma 4.2 are settled, and Theorem 1.1 is now proven.

The proof of Proposition 4.4 will be given in Section 7.

5 The nondegenerating case (proof of Proposition 4.1)

In this section, we apply the Teichmüller flow to a closed dilation surface Σ containing a vertical saddle connection. For a sequence of times $t_n \to +\infty$ such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges to a limit surface Σ_∞ with the same complexity as Σ, we prove that Σ_∞ (and subsequently Σ) contains closed geodesics whose directions are arbitrarily close to $\frac{1}{2}\pi$.

Remark 5.1 Before entering the proof of the above proposition, let us mention that there is a class of dilation surfaces, called quasi-Hopf surfaces (see [6] for details), having a vertical saddle connection and whose Teichmüller orbit is periodic. These surfaces decompose into disjoint dilation cylinders whose boundary saddle connections are either horizontal or vertical. We cannot extract from the Teichmüller orbit of these surfaces a sequence that Delaunay-converges to a surface of smaller complexity.

We recall that for positive times, the Teichmüller flow expands the horizontal direction and contracts the vertical direction. We first prove the existence of a vertical saddle connection γ_∞ in Σ_∞.

Lemma 5.2 *Let Σ be a closed dilation surface that carries a vertical saddle connection and a sequence of times $t_n \to +\infty$ such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges toward a surface Σ_∞ having the same complexity as Σ. Then the limit surface Σ_∞ carries at least one vertical saddle connection.*

Proof We first prove that for n large enough the vertical saddle connection of $g_{t_n}\Sigma$ belongs to an edge of the Delaunay polygonation. Indeed, in the dilation surface Σ, there is an affine immersion of an elliptic domain D of eccentricity $e < 1$ such that vertical saddle connection γ coincides with the image of the major axis. The ratio of lengths between the (horizontal) semiminor axis and the (vertical) semimajor axis is $\sqrt{1-e^2}$. Teichmüller flow will deform the ellipse. For $T = -\frac{1}{4}\ln(1-e^2) > 0$, the saddle connection $g_T\gamma$ in the surface $g_T\Sigma$ is the vertical diameter of the immersed disk $g_T D$. Consequently, for any $t \geq T$, $g_t\gamma$ is an edge of the Delaunay polygonation of the dilation surface $g_t\Sigma$.

We now prove that the limit surface carries indeed a vertical saddle connection (that belongs to the Delaunay polygonation). By definition of being Delaunay-convergent, all the Delaunay polygons of the sequence $g_{t_n}\Sigma$ converge toward a limit polygon of Σ_∞. A polygon cannot converge toward a polygon with more sides, so the complexity can only decrease. Assuming that the complexity of Σ_∞ and Σ are the same, all the limit polygons keep the same number of sides. In particular, the side corresponding to the vertical connection does not vanish, and the limit surface carries a vertical saddle connection as well. □

On a dilation surface, vertical saddle connections have a top and a bottom endpoint. For any vertical saddle connection γ, we denote by $\mathcal{R}(\gamma)$ the vertical ray satisfying the following properties:

- The starting point M of $\mathcal{R}(\gamma)$ is the top endpoint of the saddle connection γ.
- At M, $\mathcal{R}(\gamma)$ and γ form an angular sector of amplitude π contained in the right half-plane (by convention). In other words, $\mathcal{R}(\gamma)$ is obtained from γ by turning counterclockwise around M by an angle of π.

We will prove that Σ_∞ contains a cylinder with vertical boundary saddle connection by exhibiting a cyclic sequence of vertical saddle connections.

Lemma 5.3 *For any vertical saddle connection γ in the limit surface Σ_∞, $\mathcal{R}(\gamma)$ is a vertical saddle connection too.*

Proof We argue by contradiction, assuming that Σ_∞ contains a vertical saddle connection γ such that $\mathcal{R}(\gamma)$ is not a saddle connection. We denote by s_{top} the top singularity of γ and choose a continuous

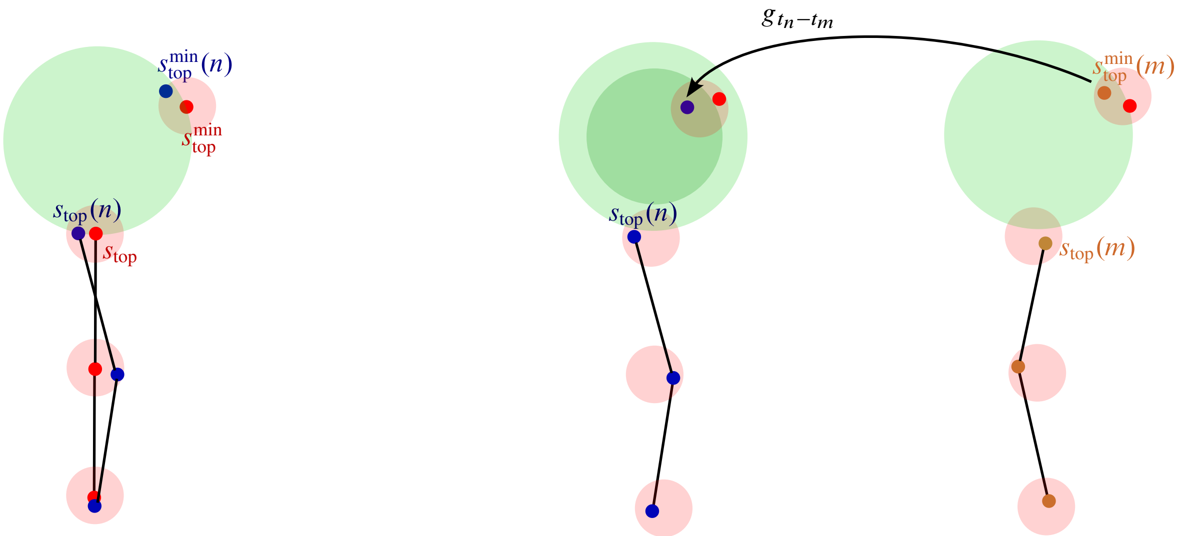

Figure 12: Left: In green, the maximal disk defining the singularity $s_{\mathrm{top}}^{\min}$. The red dots correspond to the singularities encountered on an $\mathcal{R}$-orbit in Σ_∞. The blue dots correspond to the associated singularities in Σ_n. Right: The left part corresponds to the surface Σ_n as the right one to Σ_m. The shadow of the closest singularity on Σ_m is mapped within a 2ϵ shorter disk of Σ_n.

parametrization $\Gamma(t)$ of the ray $\mathcal{R}(\gamma)$ with $\Gamma(0) = s_{\mathrm{top}}$; see Figure 12. For t small enough, we define $\mathcal{D}_t$ to be the unique immersed disk whose center is $\Gamma(t)$ and whose radius is given by the segment $[s_{\mathrm{top}}, \Gamma(t)]$. Note that for any $t > t'$ we have $\mathcal{D}_{t'} \subset \mathcal{D}_t$. We then define the "closest" singularity $s_{\mathrm{top}}^{\min}$ to γ_∞ as the first singularity encountered when considering the increasing sequence of disks $(\mathcal{D}_t)_{t\geq 0}$. Note that such a sequence must actually encounter a singularity because of Theorem 2.11.

By hypothesis, $s_{\mathrm{top}}^{\min}$ does not belong to $\mathcal{R}(\gamma)$. We will reach our contradiction by showing that there is a closer singularity to s_{top} than $s_{\mathrm{top}}^{\min}$. In order to do so, we rely on the assumption that Σ_∞ is the Delaunay limit of $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$. All polygons of the Delaunay triangulation converge toward their limit polygon, and all the quantities indexed by n must converge toward their ∞-indexed corresponding quantity. For $N > 0$ large enough, any surface $g_{t_n}\Sigma$ contains a well-defined saddle connection γ_n corresponding to γ_∞ in Σ_∞. Analogously we denote by $s_{\mathrm{top}}(n)$ and $s_{\mathrm{top}}^{\min}(n)$ the top singularity of such a sequence and the singularity in $g_{t_n}\Sigma$ corresponding to $s_{\mathrm{top}}^{\min}$; see Figure 12.

Then let $m > n$ such that $t_m - t_n > 0$. By construction $g_{t_n-t_m}\Sigma_m = \Sigma_n$. Note that $t_n - t_m < 0$ so that the Teichmüller flow is now expanding (by a definite amount independent of n, m and ϵ) in the vertical direction and contracting the horizontal one. Note also that the image of $s_{\mathrm{top}}^{\min}(m)$ under $g_{t_n-t_m}$ must be a singularity of Σ_n. Since ϵ is arbitrary, one can take it as small as for the singularity $g_{t_n-t_m}(s_{\mathrm{top}}^{\min}(m))$ to be inside the disk centered at the same point as the maximal disk defining $s_{\mathrm{top}}^{\min}$ but of radius 2ϵ shorter. This contradicts our initial assumption: as $g_{t_n-t_m}(s_{\mathrm{top}}^{\min}(m))$ must be ϵ close to a singularity of Σ_∞, this new singularity would be inside the maximal disk defining $s_{\mathrm{top}}^{\min}$; see Figure 12. □

Proof of Proposition 4.1 Assuming that Σ_∞ and Σ have the same complexity, Lemma 5.2 proves that Σ_∞ contains a vertical saddle connection γ_∞. Using repeatedly Lemma 5.3, we get that γ_∞ belongs to a periodic sequence of vertical saddle connections. In other words, two consecutive saddle connections in this sequence differ by an angle of π. The curve formed by these saddle connections becomes simple if moved slightly toward the right. Such a simple curve must be the boundary of a cylinder. Consequently, the vertical saddle connection γ_∞ belongs to the boundary of some cylinder of Σ_∞. Proposition 3.9 guarantees that for any $\epsilon > 0$ there is $N > 0$ such that for any $n \geq N$, $g_{t_n}\Sigma$ contains a closed geodesic whose direction belongs to $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$. For positive times, the Teichmüller flow g_t expands the horizontal direction and shrinks the vertical direction. Thus, a fortiori, the same holds for Σ. □

6 Cyclic maximal domains of type 1 and 2 (proof of Proposition 4.3)

We will show that the existence of a cyclic maximal domain of type 1 or 2 (see Sections 3.4 and 3.5) in the Delaunay limit Σ_∞ of a Delaunay-convergent subsequence $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ of positive Teichmüller orbit of a dilation surface Σ implies the existence of closed geodesic in Σ whose direction is arbitrarily close to the vertical direction.

We will prove Proposition 4.3 by contradiction. We first give estimates on the moduli and directions of cylinders in surfaces of the positive Teichmüller orbit of a dilation surface without closed geodesics in a neighborhood of the vertical direction.

Lemma 6.1 *For $\delta > 0$, we consider a closed dilation surface Σ that does not contain any closed geodesic whose direction belongs to the interval $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$.*

Then there is a positive constant $C_\delta > 0$ such that for any $t \geq 0$ the modulus of any cylinder of the surface $g_t\Sigma$ is bounded above by C_δ.

Besides, for any $\epsilon > 0$, there is a time T_ϵ such that for any $t \geq T_\epsilon$ directions of closed geodesics of $g_t\Sigma$ are contained in $[\pi - \epsilon, \pi]$.

Proof The second claim follows immediately from the action of the Teichmüller flow on the interval $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$ in $\mathbb{RP}^1$.

Recall that Corollary 2.6 asserts that every cylinder of Σ is of modulus at most M for some $M > 0$. We will prove that the moduli of cylinders of surfaces $(g_t\Sigma)_{t\in\mathbb{R}^+}$ satisfy a global upper bound $C_\delta = M/\sin^2(\delta)$. Note that cylinders of $g_t\Sigma$ correspond to cylinders of Σ.

We first consider a flat cylinder C of Σ. Normalizing its area to 1, its modulus is equal to h^{-2} where h is the (normalized) length of its closed geodesics. These geodesics have a direction θ which is δ far away from $\frac{1}{2}\pi$ by assumption. Now we consider the images of C under the action of the Teichmüller flow. The normalized area remains identical while the lengths h_t of closed geodesics of g_tC satisfy

$$\frac{h_t}{h} = \sqrt{e^{2t}\cos^2(\theta) + e^{-2t}\sin^2(\theta)}.$$

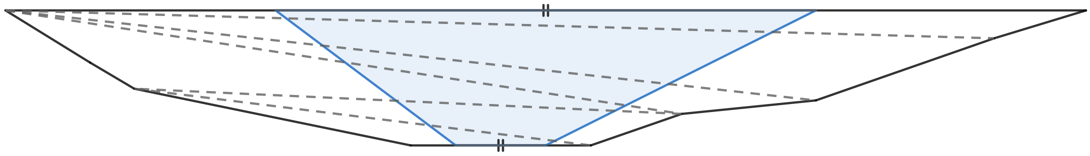

Figure 13: A maximal domain of type 1 containing a dilation cylinder.

It follows that
$$\frac{1}{h_t} \leq \frac{1}{\sin^2(\delta)h},$$
which is desired result.

It then remains to deal with the case where the cylinder C is a dilation cylinder of Σ whose directions of closed geodesics is the interval $]\theta_1, \theta_2[\subset \big]-\frac{1}{2}\pi + \delta, \frac{1}{2}\pi - \delta\big[$. We denote by $\lambda > 1$ the dilation multiplier of C. Recall that the modulus M of such a cylinder is given by the relation
$$M = \frac{|\theta_2 - \theta_1|}{\ln(\lambda)}.$$
The action of the Teichmüller flow preserves the dilation multiplier. On the other hand, g_t transforms any slope θ into
$$\arctan(e^{-2t}\tan(\theta)).$$
Using that for any $x, y < 0$, we have $|\arctan(y) - \arctan(x)| \leq |y - x|$, we get that the size $|\theta_1(t) - \theta_2(t)|$ of the interval $g_t(]\theta_1, \theta_2[)$ satisfies
$$|\theta_1(t) - \theta_2(t)| \leq e^{-2t}|\tan(\theta_1) - \tan(\theta_2)| \leq |\theta_1 - \theta_2| \sup_{\theta \in]-\pi/2+\delta, \pi/2-\delta[} |\tan'(\theta)|.$$
Actually, $\sup_{\theta \in]-\pi/2+\delta, \pi/2-\delta[} |\tan'(\theta)| = 1/\sin^2(\delta)$, so we have
$$|\theta_1(t) - \theta_2(t)| \leq \frac{|\theta_1 - \theta_2|}{\sin^2(\delta)}.$$
Thus the modulus of the image cylinder is bounded above by $M/\sin^2(\delta)$. □

We split the proof of Proposition 4.3 in two statements, corresponding to the maximal domains of type 1 and type 2.

6.1 Cyclic maximal domains of type 1

Polygons of type 1 assemble into maximal domains of type 1 (see Section 3.4). Following Proposition 3.5, the (unique) long side of a degenerating polygon of type 1 must be glued to the short side of any other polygon. A cyclic maximal domain is formed by polygons of type 1 glued long side on short side, as in Figure 13.

We prove that cyclic maximal domains of type 1 contain dilation cylinders whose angular amplitude is bounded below.

Proposition 6.2 *Let Σ be a dilation surface, and $t_n \to +\infty$ be such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges and such that at least one of its Delaunay pieces is a cyclic maximal domain of type* 1. *Then for any $\epsilon > 0$, Σ carries a cylinder whose direction belongs to $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$.*

Proof Assuming for contradiction that for some $\epsilon > 0$, the interval $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$ does not contain any direction of a closed geodesic of Σ, we use Lemma 6.1 to prove that the maximal angular amplitude of dilation cylinders of $\Sigma_n = g_{t_n}\Sigma$ becomes arbitrarily small as n tends to infinity. We will obtain a contradiction by proving that the cyclic maximal domain of type 2 $(X_n)_{n\in\mathbb{N}}$ in $(\Sigma_n)_{n\geq\mathbb{N}}$ contains a dilation cylinder whose angular amplitude is bounded below provided n is large enough.

The incidence graph of $(X_n)_{n\in\mathbb{N}}$ is connected and contains a unique (oriented) cycle C (see Section 3.4 for details). For any $n \in \mathbb{N}$, X_n is a topological cylinder. We consider an edge $(L_n)_{n\in\mathbb{N}}$ between two polygons of the cycle C.

Cutting along the edge $(L_n)_{n\in\mathbb{N}}$ in $(X_n)_{n\in\mathbb{N}}$, we obtain a sequence of simply connected flat surfaces $(P_n)_{n\in\mathbb{N}}$ with a unique boundary component. It is formed by the gluing of polygons of type 1 according to an incidence graph which is a tree.

By definition of a polygon of type 1 (see Definition 3.2), all the sides of $(P_n)_{n\in\mathbb{N}}$ have the same limit direction in $\mathbb{RP}^1$. We normalize $(P_n)_{n\in\mathbb{N}}$ in such a way that all the sides tend to be horizontal and every surface P_n has unit area. In particular, provided n is large enough, P_n is a planar polygon in the classical sense.

Since every polygon of type 1 has a unique long side (see Definition 3.3), provided that n is large enough, P_n has a unique upper side S_n (or a unique lower side, depending on the normalization) and several lower sides $T_n^1, \dots, T_n^{p-1}$ (where p is the number of sides of P_n for any n).

The edge $(L_n)_{n\in\mathbb{N}}$ corresponds to the identification of the unique upper side $(S_n)_{n\in\mathbb{N}}$ with some lower side $(T_n^{i_0})_{n\in\mathbb{N}}$. Since S_n and $T_n^{i_0}$ have the same slope, they cannot be adjacent in the boundary of P_n. Thus the corner angles and the ends of $T_n^{i_0}$ tend to π as n tends to infinity. Provided that n is large enough, rays starting from the ends of the side $T_n^{i_0}$ in directions $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ intersect S_n and there exists a trapezoid M_n in P_n formed by $T_n^{i_0}$ (the lower side of M_n), a side of slope $\frac{1}{4}\pi$, a portion of S_n (the upper side of M_n) and a side of slope $\frac{3}{4}\pi$.

Since any point of the upper side of M_n is identified with a point of $T_n^{i_0}$, the lift of the trapezoid M_n in X_n contains a family of closed geodesics whose slopes sweep an interval of length at least $\frac{1}{2}\pi$ in $\mathbb{RP}^1$ (see Figure 13). Thus, for any large enough n, the surface Σ_n contains a dilation cylinder of angle at least $\frac{1}{2}\pi$. This is the desired contradiction. □

6.2 Cyclic maximal domains of type 2

In Section 3.5, we defined maximal domains of type 2 as collections of polygons of type 2 glued along their long boundary sides. Such a maximal domain is cyclic if the polygons are glued according to a cyclic graph, as in Figure 14.

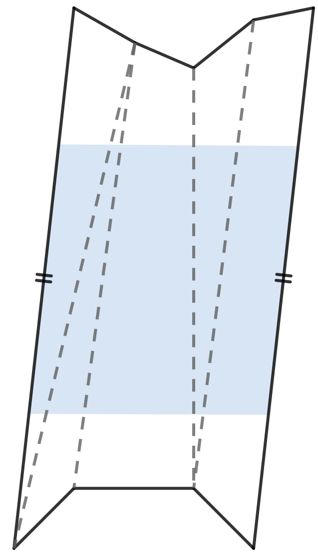

Figure 14: A maximal domain of type 2 containing a cylinder of large modulus (shaded).

We prove that cyclic maximal domains of type 2 contain cylinders of arbitrarily large modulus.

Proposition 6.3 *Let Σ be a dilation surface, and $t_n \to +\infty$ be such that $(g_{t_n}\Sigma)_{n\in\mathbb{N}}$ Delaunay-converges and such that at least one of its Delaunay pieces is a cyclic maximal domain of type* 2*. Then for any $\epsilon > 0$, Σ carries a cylinder whose direction belongs to $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$.*

Proof We proceed similarly as when proving Proposition 6.2. Assuming for contradiction that for some $\epsilon > 0$, the interval $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi + \epsilon\right[$ does not contain any direction of a closed geodesic of Σ, we use Lemma 6.1 to obtain an upper bound $M > 0$ on the modulus of any cylinder in any dilation surface $\Sigma_n = g_{t_n}\Sigma$. We will prove that the cyclic maximal domain of type 2 $(X_n)_{n\in\mathbb{N}}$ in $(\Sigma_n)_{n\in\mathbb{N}}$ contains cylinders of arbitrarily large modulus as n tends to infinity.

In particular, for any $n \in \mathbb{N}$, X_n is a topological cylinder. We cut along some edge $(L_n)_{n\in\mathbb{N}}$ and obtain a sequence of polygons $(P_n)_{n\in\mathbb{N}}$. We normalize each polygon P_n in such a way that the two sides corresponding to edge L_n are vertical and P_n has unit area. These two vertical sides will be referred to as S_n (for the left side) and T_n (for the right side).

In a polygon of type 2 (see Definition 3.2), the ratio between the length $|S_n|$ of S_n and the length $|T_n|$ of T_n converges to 1, and the length of any other side of P_n becomes negligible in comparison with $|S_n|$ and $|T_n|$ (see Figure 14). Since P_n has unit area for any $n \in \mathbb{N}$, $|S_n|$ and $|T_n|$ tend to infinity while the distance between S_n and T_n tends to zero as $n \to +\infty$.

It follows that for any $\epsilon > 0$, there is $N > 0$ such that for any $n \geq N$, the polygon P_n contains a rectangle R_n satisfying the following conditions:

- sides of R_n are either vertical or horizontal,

- the vertical left and right sides are portions of S_n and T_n,
- the length of the vertical sides of R_n is at least $(1-\epsilon)|S_n|$,
- the length of the vertical sides of R_n tend to infinity as $n \to +\infty$,
- the length of the horizontal sides of R_n tend to zero as $n \to +\infty$.

Since the sides S_n and T_n are identified, the lift of the rectangle R_n in X_n contains a family of closed geodesics covering most of the rectangle R_n (the complement of a part of arbitrarily small relative area). Therefore, provided that n is large enough, X_n contains a cylinder of arbitrarily large modulus (see Figure 14). □

7 Long sides and short sides (proof of Proposition 4.4)

We will actually prove the following slightly stronger version of Proposition 4.4, which does not involve the Teichmüller flow:

Proposition 4.4 *Let $(\Sigma_n)_{n\in\mathbb{N}}$ be a Delaunay-convergent sequence of dilation surfaces such that all its Delaunay pieces have at least one long boundary side. Then, for any open set $U \subset \mathbb{RP}^1$, there is $N > 0$ such that for any $n \geq N$, Σ_n contains closed geodesics whose directions belong to U.*

The proof is based on Proposition 2.9. This proposition asserts that, in the case of a dilation surface with boundary, either a given open set of directions contains a cylinder, or a set of trajectories having these directions, a pencil to be precise, must leave across a boundary component of the dilation surface. This proposition then shows that we can concentrate on the case where each trajectory of a Delaunay piece (see Definition 3.12) leaves it by hitting the boundary. It can do it by crossing either a long edge or a short one. We will actually rule out the short edge case in Sections 7.1 and 7.2. Indeed, these boundaries are by definition very small compared to the long edges and it will be unlucky to leave the piece through such a short side. The trajectories of the pencil will then have to leave the Delaunay piece through a long side and then enter a new Delaunay piece through a short side (as by construction Delaunay pieces are glued to one another short side to long side; see Lemma 3.13). If the pencil does not enter in a cylinder, one can repeat the argument to get a sequence of Delaunay pieces such that the pencil enters them by short sides and leaves them by long sides. As there are only finitely many boundary components, such a pencil will cross a given edge twice. The first return map on such an edge is a very dilating mapping, as going from short sides to long sides induces a huge contraction. This concludes the argument, as contracting mappings have periodic orbits.

7.1 Trajectories inside maximal domains of type 1 or 2

For a trajectory whose slope is far enough from the limit directions of the (finitely many) Delaunay edges, we have some control on its behavior in Delaunay pieces formed by degenerating polygons.

Definition 7.1 For any $\epsilon > 0$, $\Theta_\epsilon \subset \mathbb{RP}^1$ is the open subset of slopes whose distance to any limit direction of a Delaunay edge of $(\Sigma_n)_{n\in\mathbb{N}}$ is strictly bigger than ϵ.

The following proposition asserts that a for a given direction in Θ_ϵ a trajectory entering a maximal domain of type 1 by a small edge must exit it through a long one, provided that n is large enough.

Proposition 7.2 *Let $(E_n)_{n\in\mathbb{N}}$ be a short boundary edge of a maximal domain of type 1 $(X_n)_{n\in\mathbb{N}}$ that has at least one boundary long edge.*

For any $\epsilon \in \left]0, \frac{1}{2}\pi\right[$, there is a long boundary edge $(M_n)_{n\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ and $N > 0$ such that for any $n \geq N$, any trajectory of X_n whose direction belongs to Θ_ϵ starting from E_n eventually leaves X_n through the interior of M_n.

Proof Since a maximal domain of type 1 is formed by polygons of type 1, Delaunay edges have the same limit slope. Without loss of generality, we will assume that this unique limit slope is horizontal.

We start by discussing the case of a polygon of type 1. Note that for any $\delta > 0$, there is $N_\delta \in \mathbb{N}$ such that for any $n \geq N_\delta$, every (convex) Delaunay polygon P_n of X_n satisfies the following properties:

- the slope of every Delaunay edge belongs to $]-\delta, \delta[\subset \mathbb{RP}^1$,
- the inner angle between two short sides of P_n is at least $\pi - \delta$,
- the inner angle between a short side and a long side of P_n is at most δ.

If a trajectory of P_n starts from a short side and leaves P_n through another short side, then it cuts out P_n into two polygons. Computing the sum of the inner angles in each of them, we deduce that the slope of t belongs to $[-p\delta, p\delta]$, where p is the number of sides of P_n. Thus, by choosing $\delta \geq \epsilon/q$ where q is the number of Delaunay edges of P_n, one makes sure that for $n \geq N_\delta$, a trajectory of polygon P_n whose slope belongs to Θ_ϵ starting from a short side of P_n leaves it through the interior of its unique long side.

The proof of the noncyclic domain of type 1 follows the exact same line. The key remark being that type-1 polygons piled up long side to short side form a polygon that satisfies the three points above (see Figure 13). Therefore, if we set $\delta = \epsilon/m$ where m is the total number of edges that are short sides of at least one polygon of X_n, following the argumentation above, we see that there is N_δ large enough that for $n \geq N_\delta$ the trajectory visits finitely many long boundaries of X_n and exits X_n, as otherwise the domain would be cyclic. □

We now address the case of maximal domains of type 2.

Proposition 7.3 *Let $(E_n)_{n\in\mathbb{N}}$ be a short boundary edge of a maximal domain of type 2 $(X_n)_{n\in\mathbb{N}}$ that has at least one boundary long edge. We also consider a nonempty open interval $\mathcal{I} \subset \Theta_\epsilon$ for some $\epsilon > 0$.*

There is a long boundary edge $(M_n)_{n\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ and $N > 0$ such that for any $n \geq N$, any trajectory of X_n whose direction belongs to $\mathcal{I}$ starting from E_n eventually leaves X_n through the interior of M_n.

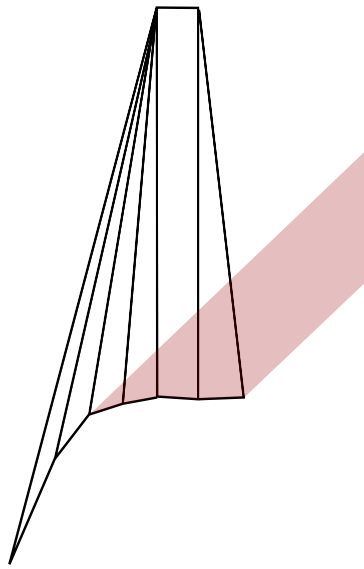

Figure 15: A maximal domain of type 2 with trajectories starting from short sides in a direction far from that of the long sides.

Proof A maximal domain of type 2 with at least one boundary edge actually has two long boundary edges, since its graph of incidence is linear (see Section 3.5). Without loss of generality, we assume that the limit slope of the long Delaunay edges of $(X_n)_{n\in\mathbb{N}}$ is vertical. Therefore, we will refer to the boundary edges of X_n (which is a polygon) as the long left side, the long right side, the short upper sides and the short lower sides.

There is $N > 0$ such that for any $n \geq N$, the slope of any straight segment joining an upper vertex and a lower vertex of X_n is contained in $\left]\frac{1}{2}\pi - \epsilon, \frac{1}{2}\pi\right] + \epsilon\left[\right.$. It follows that the slope of a trajectory of X_n joining a short upper side and a short lower side cannot belong to Θ_ϵ for $n \geq N$. It remains to consider the case of a trajectory t joining two upper (or lower) short sides of X_n.

It follows from Proposition 3.7 that for any $\delta > 0$, there is N_δ such that for any $n \geq N_\delta$, the inner angle between two consecutive upper (or lower) short sides of X_n is at least $\pi - \delta$. Any trajectory t cuts out X_n into two polygons. Computing the sum of inner angles in each of them, we deduce that the trajectory t forms an angle of magnitude smaller than $p\delta$ with one of the short sides of X_n (here p is the number of sides of X_n). Since δ can be made arbitrarily small, there exists a bound $N' > 0$ such that for any $n \geq N'$, a trajectory joining two upper (or lower) short sides of X_n cannot belong to Θ_ϵ.

Consequently, for any n satisfying $n \geq \max(N, N')$, any trajectory starting from a short boundary edge E_n of X_n eventually leaves X_n through the interior of one of its two extremal edges (see Figure 15). If we restrict ourselves to trajectories whose slope belongs to a connected open subset U of Θ_ϵ, a continuity argument proves that two trajectories starting from E_n leave X_n through the same extremal edge. □

7.2 Trajectories inside connected components of the core

The case of Delaunay pieces that are connected components of the core $(C\Sigma_n)_{n\in\mathbb{N}}$ is a bit more complicated. In order to find an open set of directions where trajectories starting from the same short boundary edge leave a component of $C\Sigma_n$ through the same long boundary edge, we first prove the analogous result for connected components of the limit surface Σ_∞, which is an easy consequence of Proposition 2.9.

Lemma 7.4 *For any nonempty open subset $U \subset \mathbb{RP}^1$ and any connected component X_∞ of Σ_∞ with a nonempty boundary, one of the following statements holds*:

- *there exists a closed geodesic in X_∞ whose slope is contained in U*,
- *there is a nonempty open subset $V \subset U$ such that every trajectory starting from a singularity x of X_∞ in a direction of V eventually leaves X_∞ through the interior of a boundary saddle connection.*

Proof Let $S_{x,U}$ be the set of (oriented) trajectories starting from the singularity x with a slope in U. The topology of $S_{x,U}$ is induced by the canonical projection π_x to $\mathbb{RP}^1$. Assuming that no closed geodesic of X_∞ belongs to a direction of U, it has been proved in Proposition 2.9 that trajectories of $S_{x,U}$ leaving X_∞ through the interior of a boundary saddle connection form an open dense subset of $S_{x,U}$. Since there are finitely many such singularities in Σ_∞ and projections π_x have finitely many preimages, there is an open dense subset V of U such that every trajectory starting from such a singularity x in a direction of V leaves its component through the interior of a boundary saddle connection. □

Since short boundary edges of connected components of the core degenerate to singular points in the Delaunay limit, we obtain a result about trajectories in the connected components of the core:

Proposition 7.5 *Let $(X_n)_{n\in\mathbb{N}}$ be a connected component of the core $(C\Sigma_n)_{n\in\mathbb{N}}$ with at least one long boundary edge. Let $(E_n)_{n\in\mathbb{N}}$ be a short boundary edge of $(X_n)_{n\in\mathbb{N}}$. For any nonempty open subset $U \subset \mathbb{RP}^1$, one of the following statements holds*:

- *there is a bound $N > 0$ such that for any $n \geq N$, there exists a closed geodesic in X_n whose slope is contained in U*,
- *there is a nonempty open subset $V \subset U$, a bound $N > 0$ and a long boundary edge $(M_n)_{n\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$ such that for any $n \geq N$, any trajectory of X_n in a direction $\theta \in V$ starting from E_n eventually leaves X_n through the interior of M_n.*

Proof We first decompose the proof into two subcases, depending whether Σ_∞ contains a closed geodesic whose direction belongs to U or not. In the first case, we deduce from Proposition 3.9 that there exists $N > 0$ such that for any $n \geq N$, Σ_n contains a closed geodesic whose direction belongs to U.

In the second case, we fix ϵ small enough that $U \cap \Theta_\epsilon$ is nonempty. Lemma 7.4 then proves the existence of a nonempty open subset V of $U \cap \Theta_\epsilon$ such that any trajectory of the Delaunay limit X_∞ of $(X_n)_{n\in\mathbb{N}}$

whose direction belongs to V and that starts from a singularity leaves X_∞ through the interior of a boundary saddle connection.

Let $(E_n)_{n\in\mathbb{N}}$ be a short boundary edge of $(X_n)_{n\in\mathbb{N}}$. By construction this short boundary edge converges toward a point x of X_∞ in the limit. For any interval I in V, we consider the two-parameter family $P(E_n, I)$ of trajectories starting from the edge E_n and whose directions belong to I. This family of trajectories accumulates on a pencil $P(x, I)$ of X_∞ as n tends to infinity.

The edge $(E_n)_{n\in\mathbb{N}}$ is a short edge of a chain of polygons of type 2 belonging to $(X_n)_{n\in\mathbb{N}}$ (see Figure 11). Using Proposition 3.7 as in the proof of Proposition 6.3, we deduce that provided that n is large enough, trajectories of $P(E_n, I)$ leave each of these polygons of type 2 through one of its long side (the hypothesis that I is disjoint from Θ_ϵ is crucial here). Then these trajectories finally enter a polygon of type 3 of $(X_n)_{n\in\mathbb{N}}$. A continuity argument proves that trajectories of the pencil $P(x, I)$ leave X_∞ through the interior of the same boundary saddle connection M_∞ which is the limit of a long boundary edge $(M_n)_{n\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$. Up to replacing I by a smaller open interval, we can assume that the intersection of trajectories of $P(x, I)$ with M_∞ is disjoint from a neighborhood of the endpoints of M_∞. We deduce that, provided n is large enough, trajectories of $P(E_n, I)$ leave X_n through the interior of M_n. □

We combine the previous results to exhibit a set of directions and a lower bound that hold for every Delaunay piece of $(\Sigma_n)_{n\in\mathbb{N}}$:

Corollary 7.6 *For any nonempty open subset $U \subset \mathbb{RP}^1$, one of the following statements holds*:

- *there is a bound $N > 0$ such that for any $n \geq N$, there exists a closed geodesic in Σ_n whose slope is contained in U*,
- *there is a nonempty open subset $V \subset U$ and a bound $N > 0$ such that for any $n \geq N$ and any short boundary edge E_n in any Delaunay piece X_n of Σ_n having a long boundary edge*, *there is a long boundary edge M_n such that every trajectory of X_n starting from E_n and whose slope belongs to U eventually leaves X_n through the interior of M_n.*

Proof Provided ϵ is small enough, $\Theta_\epsilon \cap U$ is nonempty. For such a small ϵ, we consider an open interval $I \subset \Theta_\epsilon \cap U$. Since there are finitely many Delaunay pieces and Delaunay edges in $(\Sigma_n)_{n\in\mathbb{N}}$, there is a global bound N_0 such that the second statement holds for trajectories whose slope is in the interval I for any short boundary edge in any Delaunay piece X_n that is a maximal domain of type 1 or 2 (see Propositions 7.2 and 7.3).

Then we apply Proposition 7.5 to a boundary short edge $(E_n)_{n\in\mathbb{N}}$ in a connected component $(X_n)_{n\in\mathbb{N}}$ of the core. If X_n contains a closed geodesic, provided n is large enough, then the first statement of our proposition holds. Otherwise, the second statement of Proposition 7.5 provides a nonempty open subset I' of I and a new bound such that the property also holds for this edge. After finitely many steps, we

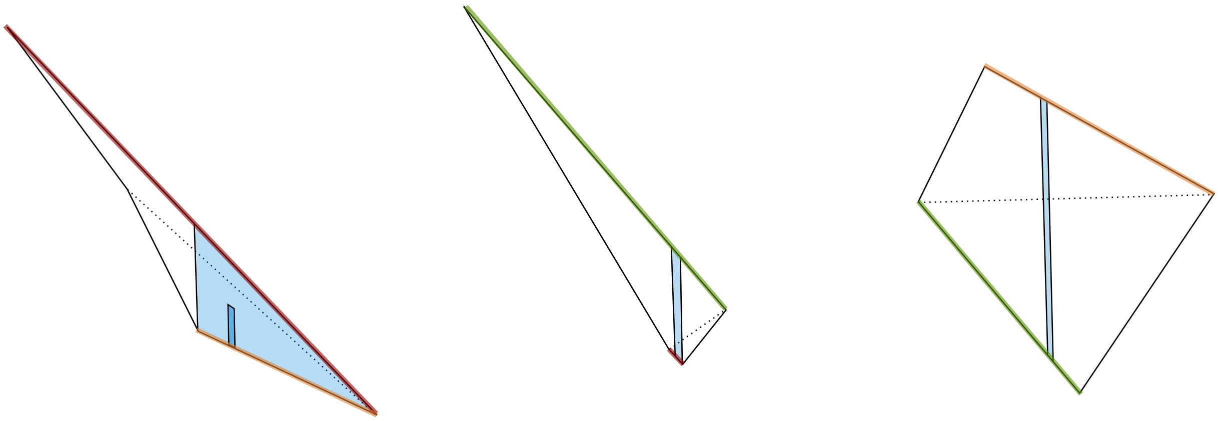

Figure 16: From left to right: a maximal domain D_1 of type 1, a maximal domain D_2 of type 2 and a component C of the core. The orange edge on the left corresponds to E_n. The ribbon of parallel trajectories leaves D_1 to enter D_2 through its short side which goes to C from a long side of D_2. When the ribbon enters back into D_1 it has been contracted by an amount that goes to infinity when $n \to +\infty$.

obtain a nonempty open subset V of $\mathbb{RP}^1$ and a bound $N > 0$ such that the property holds of trajectories whose slope belongs to V for every short boundary edge in every Delaunay piece having a long boundary edge (provided that $n \geq N$). □

Proof of Proposition 4.4 Corollary 7.6 shows that it is enough to prove that any direction d such that for n large enough any trajectory starting from any short edge of any Delaunay piece exits the Delaunay piece through one of its long sides carries a cylinder. As we only have finitely many Delaunay pieces, any such trajectory will have to cross twice some boundary edge $(E_n)_{n\in\mathbb{N}}$ of two Delaunay pieces (one for which it is a short side and one for which it is a long side).

Given a direction d and a short edge $(E_n)_{n\in\mathbb{N}}$, we denote by $P(E_n, d)$ the set of trajectories starting from a point of $(E_n)_{n\in\mathbb{N}}$ of direction d pointing inside the Delaunay piece for which $(E_n)_{n\in\mathbb{N}}$ is the short side.

By definition of E_n, there is a trajectory t of $P(E_n, d)$ which crosses back E_n for n large enough. We claim that all the trajectories of $P(E_n, d)$ cross E_n alongside t. Indeed, as any trajectory of $P(E_n, d)$ only exits a Delaunay piece by its long side and enters one by its short side, the contraction ratio of the first return map on a neighborhood of E_n converges to 0 as $n \to +\infty$. In particular, for n large enough, the image of E_n must be fully contained in E_n, which implies that it has a periodic point. This periodic point of the first return map corresponds to a closed geodesic. □

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