Geometry & Topology

Volume 29 (2025)

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Geometry & Topology **29**:5 (2025) 2251–2268
DOI: 10.2140/gt.2025.29.2251
Published: 14 August 2025

Quantitative Thomas–Yau uniqueness

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Under Floer-theoretic conditions, we obtain quantitative estimates on the closeness (Hausdorff distance, flat norm and F-metric) between two Lagrangians, depending on the smallness of Lagrangian angles. Some applications include a strong–weak uniqueness theorem for special Lagrangians, and a characterization of varifold convergence to special Lagrangians in terms of Lagrangian angles.

53C38, 53D12, 57R58

1 Background and introduction

1.1 Special Lagrangians

Inside an n-dimensional compact Kähler manifold (X,ω) with a nowhere-vanishing holomorphic volume form Ω such that $\omega^n/n! = e^{2\rho}(i^{n^2}/2^n)\Omega\wedge\overline{\Omega}$, an n-dimensional compact oriented submanifold L is called *special Lagrangian* if

$$\omega|_L = 0, \quad \operatorname{Im}\Omega|_L = 0. \tag{1}$$

These sit at the crossroad of minimal surface theory and symplectic geometry:

- If the metric is Calabi–Yau (namely $\rho=0$), then special Lagrangians are absolute volume minimizers inside their homology classes.
- Lagrangian submanifolds equipped with (unobstructed) brane structures define objects inside the Fukaya category.

Recall that the *Lagrangian angle* $\theta\colon L\to\mathbb{R}$ is defined by $\Omega|_L = e^{-\rho}e^{i\theta}\,\mathrm{dvol}_L$, so special Lagrangians amount to the condition $\theta=0$. More generally, a Lagrangian is called *quantitative almost calibrated* if $|\theta|\le \frac{1}{2}\pi-\epsilon_0$ for some fixed small $\epsilon_0>0$. Such Lagrangians are important in the Thomas–Yau program; see Li [9] and Thomas and Yau [12]. One of its immediate consequences is the a priori homological mass bound

$$\mathrm{Vol}(L) = \int_L \mathrm{dvol} \le \frac{1}{\sin\epsilon_0}\int_L \operatorname{Re}\Omega.$$

1.2 Thomas–Yau uniqueness

As part of a wider program to relate the existence and uniqueness questions of special Lagrangian branes to the Fukaya category and stability condition, Thomas and Yau [12] proved a remarkable uniqueness theorem, which from the modern perspective reads:

Theorem 1.1 (Imagi [6], Imagi, Joyce and Oliveira dos Santos [7] and Thomas and Yau [12]) *Let L and L' be two compact embedded special Lagrangians with unobstructed brane structures, defining isomorphic objects in the derived Fukaya category. Then their supports coincide.*

The *Floer degree* of a transverse intersection point $p \in \mathrm{CF}^*(L_1, L_2)$ between two Lagrangians L_1 and L_2 is

$$\mu_{L_1,L_2}(p) = \frac{1}{\pi}\left(\sum_1^n \alpha_i - \theta_{L_2}(p) + \theta_{L_1}(p)\right) \in \mathbb{Z}, \tag{2}$$

where inside $T_pX \simeq \mathbb{C}^n$ we can put the tangent spaces into the standard form

$$T_pL_1 = \mathbb{R}^n, \qquad T_pL_2 = (e^{i\alpha_1}, \dots, e^{i\alpha_n})\mathbb{R}^n \quad \text{for } 0 < \alpha_i < \pi.$$

The proof idea of Theorem 1.1 can be outlined as follows. Assume $L \neq L'$.

- By making C^∞-small Hamiltonian perturbations of L and L', we can replace L and L' by isomorphic objects L_1 and L_2 *intersecting transversely*. Moreover, either by a Morse-theoretic argument [12], or using real analyticity via a Łojasiewicz inequality [7], one can remove the degree-zero intersection points, to ensure the Floer cochain group $\mathrm{CF}^0(L_1, L_2)$ is 0.
- Consequently, the Floer cohomology $\mathrm{HF}^0(L, L) \simeq \mathrm{HF}^0(L_1, L_2)$ is 0. In particular the unit of the Floer cohomology ring vanishes, which is impossible, because it violates Poincaré duality (alternatively, because the image of the unit under the open–closed map is the homology class $[L] \in H_n(X, \Lambda_{\mathrm{nov}})$, which cannot be zero).

We take note of a few conceptual features:

- The proof of Thomas–Yau uniqueness is analogous to a *strong maximum principle* argument. The role of symplectic topology is to force the existence of a degree-zero intersection point, which is analogous to the existence of a maximum, and a local calculation concerning the intersection point results in a contradiction.
- The metric is not necessarily Calabi–Yau. The compactness of X can be replaced by any other standard settings where Floer theory makes sense.
- The known proofs rely essentially on smoothness assumptions of the Lagrangians.

Remark 1.2 (for the analysts) Setting up Floer cohomology (or more generally, the Fukaya category) requires a large baggage train, such as brane structure (spin structure, local system and bounding cochains), complicated perturbation schemes and algebraic machinery. For background see Joyce [8, Section 2.5] for a lightning-quick overview, and Auroux [5] for a gentle introduction on the Fukaya category.

The Thomas–Yau argument (as well as the rest of this paper) is *not sensitive to the details of the Fukaya category*. Knowledge on the setup is not essential; the only facts from Floer theory that one really needs for this paper are:

- Floer cohomology groups are invariant under Hamiltonian deformations of the Lagrangians.

- Nonzero Floer cohomology implies the existence of Lagrangian intersection points in the corresponding degree, provided the Lagrangian intersections are all transverse.

- If two Lagrangian branes are in the same derived Fukaya category class, then they live in the same homology class. (This follows from the property of the open–closed map.)

1.3 Quantitative Thomas–Yau uniqueness

A common theme in geometric analysis is *rigidity theorems*, eg a strong maximum principle naturally suggests a Harnack inequality. Analogously, we ask:

Question 1 Suppose L and L' have *small* Lagrangian angles (eg $\|\theta\|_{C^0} \ll 1$ or $\|\theta\|_{L^1} \ll 1$). Then do they have to be *uniformly* close to each other (eg in the Hausdorff distance for subsets, in the flat norm distance for integral currents or the F-metric for varifolds)?

Remark 1.3 A paper of Abouzaid and Imagi [1] studies symplectic topological consequences of special Lagrangians lying inside a small C^0*-neighbourhood* of a given special Lagrangian, under additional hypotheses on the fundamental group and its representations.

Remark 1.4 New ideas are needed for this question: In a naive strategy, one takes a sequence of L_i and L_i', with Lagrangian angles converging to zero, and attempts to use compactness theorems in geometric measure theory to extract subsequential limits L_∞ and L'_∞, which should be special Lagrangian integral currents. To deduce that L_i is close to L_i', one would like to prove $L_\infty = L'_\infty$. This would require a *singular Lagrangian* version of the Thomas–Yau uniqueness theorem, which is yet unknown. (The interested reader may see Li [9, section 5.7] for some heuristic ideas.)

Our main technical result, Theorem 2.1, gives one answer to the question above. Morally, it provides Floer-theoretic conditions such that if a quantitative almost-calibrated Lagrangian has small enough L^1-norm on its Lagrangian angle, then it is close to a given special Lagrangian, with respect to several distance notions from geometric measure theory (Hausdorff distance, flat norm and F-metric), with quantitative estimates depending on explicit powers of $\|\theta\|_{L^1}$.

We now discuss qualitative consequences of Theorem 2.1, to be proved in Section 3. Theorem 2.1 can be interpreted as the quantitative version of a *strong–weak uniqueness* theorem.

Definition 1.5 Let L_i be a sequence of smooth embedded unobstructed Lagrangian branes which lie in a fixed nonzero derived Fukaya category class (so in particular the same rational homology class), and are all quantitatively almost calibrated. We say a Lagrangian C^∞-submanifold (resp. Lagrangian integral current) L_∞ is a C^∞*-limit* (resp. *varifold limit*) if the L_i converge to L_∞ in the C^∞-topology (resp. in both the weak topology of varifolds and the flat norm topology of integral currents). Notice there is no requirement that L_∞ is equipped with any brane structure.

Theorem 1.6 (strong–weak uniqueness) *Fix a derived Fukaya category class, and let L_∞ be a C^∞-limit and L'_∞ a varifold limit. Assume both are special Lagrangian $\theta_{L_\infty} = \theta_{L'_\infty} = 0$. Then $L_\infty = L'_\infty$ as integral currents.*

Provided a derived Fukaya category class admits a special Lagrangian C^∞-limit, we have a satisfactory characterization of varifold topology convergence in terms of the Lagrangian angle function.

Theorem 1.7 *Fix a derived Fukaya category class, and assume there is a sequence of smooth embedded representatives L_i, converging in C^∞ topology to a special Lagrangian L_∞. Let L'_i be a sequence of quantitatively almost-calibrated Lagrangian branes in the same derived Fukaya category class. Then L'_i converges in the varifold topology to L_∞ if and only if $\|\theta_{L'_i}\|_{L^1} \to 0$.*

Remark 1.8 Since in our setting $\|\theta\|_{L^\infty}$ and the mass of the Lagrangian both have uniform bounds, by interpolation $\|\theta\|_{L^1} \to 0$ is equivalent to $\|\theta\|_{L^p} \to 0$ for any fixed $p > 0$.

2 Proof of quantitative Thomas–Yau uniqueness

2.1 Statement of quantitative Thomas–Yau uniqueness

Our main technical result is a quantitative version of the Thomas–Yau uniqueness theorem.

Theorem 2.1 *Let L and L' be two compact smoothly embedded Lagrangians with unobstructed brane structures, in the same fixed homology classes in $H_n(X,\mathbb{Q})$, such that $\mathrm{HF}^0(L,L') \neq 0$ or $\mathrm{HF}^0(L',L) \neq 0$ holds. Assume the quantitative almost-calibrated condition $\|\theta_{L'}\|_{C^0} \le \frac{1}{2}\pi - \epsilon_0$ for some fixed $\epsilon_0 > 0$, and moreover that there is some ϵ small enough that the Lagrangian angles satisfy*

- $\|\theta_L\|_{C^0} \le \epsilon \ll 1$,
- $\mathrm{Vol}(\{\theta_{L'} > \epsilon\}) \ll \epsilon^n$.

Then the Hausdorff distance between L and L' is uniformly small:

$$\begin{cases} \sup_{p\in L} \mathrm{dist}(p, L') \le C\epsilon^{1/(2n)}, \\ \sup_{p\in L'} \mathrm{dist}(p, L) \le C\epsilon^{1/(4n^2)}. \end{cases} \tag{3}$$

The flat norm distance between L and L' is bounded uniformly by $C\epsilon^{1/(4n)+1/(4n^2)}$. The F-metric between L and L' is bounded by $C\epsilon^{1/(8n)}$. The constants depend on ϵ_0, X and L, but not on the regularity bounds on L', nor on the small ϵ.

A few explanations may clarify the significance of the assumptions:

- If L and L' are isomorphic nonzero objects in the derived Fukaya category, then $\mathrm{HF}^0(L,L') \neq 0$, $\mathrm{HF}^0(L',L) \neq 0$ and $[L] = [L'] \in H_n(X,\mathbb{Q})$.

- The weak L^1-type bound $\mathrm{Vol}(\{\theta_{L'} > \epsilon\}) \ll \epsilon^n$ is implied by $\|\theta_{L'}\|_{L^1(L')} \ll \epsilon^{n+1}$. This is much weaker than assuming C^0-smallness of $\theta_{L'}$. This weakening is needed for Theorem 1.7, because if a sequence L_i' of quantitatively almost-calibrated Lagrangians converge to a special Lagrangian in the weak topology of varifolds (ie as Radon measures on the Grassmannian bundle), then $\|\theta_{L_i'}\|_{L^1} \to 0$ as $i \to \infty$, but $\|\theta_{L_i'}\|_{C^0}$ does not necessarily converge to zero.

- As we will indicate in a series of remarks, the embedded assumption on the Lagrangians in Theorem 2.1 can be relaxed to immersed Lagrangians with *connected domains* and transverse self-intersection points, with rather minor changes. The connected domain assumption is quite crucial to our arguments here (see Remark 2.3), and also appears in the literature on Thomas–Yau uniqueness for immersed Lagrangians [6]. On the other hand, the hypothesis of Theorem 2.1 is a little weaker than assuming L and L' lie in the same derived Fukaya category class, and does not make explicit use of holomorphic curves.

2.2 The small $\|\theta\|_{C^0}$ case

In the special case where $\|\theta\|_{C^0} \ll 1$ is small, we show that the set of points on L lying near L' is sufficiently dense. This argument contains the Floer-theoretic ingredient in our strategy. We do not yet use $[L] = [L'] \in H_n(X)$.

Lemma 2.2 *Assume $\|\theta\|_{C^0} \le \epsilon \ll 1$ for both L and L'. For all $p \in L$, there is some $p' \in L \cap B(C_1\epsilon^{1/2n})$ such that* $\mathrm{dist}(p', L') \le C\epsilon^{(n+1)/(2n)}$. *In particular* $\mathrm{dist}(p, L') \le C\epsilon^{1/2n}$.

Proof We perform a Hamiltonian perturbation of the L within the Weinstein tubular neighbourhood $T^*L \simeq NL$ of the Lagrangian L. For a smooth Hamiltonian function $h\colon X \to \mathbb{R}$, we get a Hamiltonian vector field X_h by $\iota_{X_h}\omega = -dh$ (equivalently $X_h = J\nabla h$), and for $\epsilon\|dh\|_{C^1} \ll 1$, the time ϵ flow sends L to $\varphi_\epsilon(L)$, whose Lagrangian angle is denoted by $\theta_{\varphi_\epsilon(L)}$. In our later applications, we can further assume that along L, the vector field X_h is orthogonal to the tangent spaces of L, or equivalently $X_h = J\nabla^L h$ where $\nabla^L h$ is the gradient of h on L. Alternatively, to the same effect, we can also choose the perturbed Lagrangian as the graph of $\epsilon\, dh$ inside the Weinstein neighbourhood T^*L.

Notice the Lagrangian angle θ is a locally defined smooth function on the Grassmannian bundle of Lagrangian subspaces of the tangent space, and the flow φ_ϵ acting on the Grassmannian bundle is close to the identity, so by Taylor expansion

$$\theta_{\varphi_\epsilon(L)} = \theta_L + \epsilon\mathcal{L}_{X_h}\theta + O(\epsilon^2\|X_h\|_{C^1}^2) = \theta_L + \epsilon\mathcal{L}_{X_h}\theta + O(\epsilon^2\|dh\|_{C^1}^2).$$

The linearized operator $\mathcal{L}(h) = \mathcal{L}_{X_h}\theta$ can be computed in the case of a general weighting function ρ as follows, similar to the computation in Thomas–Yau [12, Lemma 2.3] in the Calabi–Yau metric case. We observe that by the closedness of the holomorphic volume form Ω,

$$\mathcal{L}_{X_h}\Omega = -i\mathcal{L}_{JX_h}\Omega = i(d\iota_{-JX_h}\Omega + \iota_{-JX_h}\, d\Omega) = i\, d\iota_{-JX_h}\Omega.$$

By the definition of the Lagrangian angle $\Omega|_L = e^{-\rho} e^{i\theta}\,\mathrm{dvol}_L$, and the assumption that $-JX_h = \nabla^L h$ is tangent to L, the left side converts to

$$e^{-\rho} e^{i\theta}(-\mathcal{L}_{X_h}\rho + i\,\mathcal{L}_{X_h}\theta)\,\mathrm{dvol}_L + e^{-\rho} e^{i\theta}\mathcal{L}_{X_h}(\mathrm{dvol}_L),$$

while the right side converts to

$$i\,d(e^{-\rho} e^{i\theta}\iota_{\nabla^L h}\,\mathrm{dvol}_L) = i\,d(e^{-\rho} e^{i\theta} *_L dh) = i e^{i\theta} d(e^{-\rho} *_L dh) - e^{-\rho} e^{i\theta} d\theta \wedge *_L dh.$$

Now dividing both sides by $e^{i\theta}$ and then comparing the imaginary parts, we obtain the formula

$$\mathcal{L}(h) e^{-\rho}\,\mathrm{dvol} = d(e^{-\rho} *_L dh),$$

namely that $\mathcal{L}(h)$ is the drift Laplacian

$$\mathcal{L}(h) = e^{\rho}\,\mathrm{div}_L(e^{-\rho}\nabla^L h). \tag{4}$$

In the Calabi–Yau metric case ($\rho = 0$) this reduces to the Laplacian. By construction,

$$|\theta_{\varphi_\epsilon(L)} - \theta_L - \epsilon\mathcal{L}(h)| \le C\epsilon^2\|dh\|^2_{C^1}.$$

Given any point $p \in L$, we prescribe a smooth function $f : L \to \mathbb{R}$ such that $f = 4$ outside $B(p,\eta)$ for some $\eta \ll 1$ to be determined, the weighted integral $\int_L e^{-\rho} f\,\mathrm{dvol}_L$ is 0, and $|f| + |df|\eta \le C\eta^{-n}$ inside $B(p,\eta)$. We solve $\mathcal{L}(h) = f$ with $\int_L h e^{-\rho}\,\mathrm{dvol}_L = 0$, to find the smooth function h with bound $|dh| + \eta|\nabla\,dh| \le C\eta^{1-n}$. Using the smoothness of L, we regard its Weinstein neighbourhood as the cotangent bundle T^*L, which is isomorphic to the normal bundle NL via the metric. We can then extend h to a smooth function on X with the same bounds such that near L the function h is constant on the cotangent fibres. This ensures that the Hamiltonian vector field X_h equals $J\nabla^L h$ along L.

We choose $\eta = C_1\epsilon^{1/2n}$ for some large constant C_1 independent of ϵ. Then the displacement of the time ϵ-flow is bounded by the C^0 norm of the vector field ϵX_h, which is of order $C\epsilon|dh| \le C\epsilon\eta^{1-n} \le C\epsilon^{(n+1)/(2n)} \ll 1$, so $\varphi_\epsilon(L)$ is well defined inside the Weinstein neighbourhood. Furthermore,

$$|\theta_{\varphi_\epsilon(L)} - \theta_L - \epsilon\mathcal{L}(h)| \le C\epsilon^2\|dh\|^2_{C^1} \le C(\epsilon\eta^{-n})^2 \ll \epsilon,$$

whence on $\varphi_\epsilon(L \setminus B(p,\eta))$,

$$\theta_{\varphi_\epsilon(L)} \ge \theta_L + \epsilon\mathcal{L}(h) - \epsilon \ge \epsilon\mathcal{L}(h) - 2\epsilon \ge 4\epsilon - 2\epsilon \ge 2\epsilon.$$

Without loss $\varphi_\epsilon(L)$ is transverse to L' by genericity. By formula (2), any Lagrangian intersection $q \in \mathrm{CF}^*(\varphi_\epsilon(L), L')$ has Floer degree

$$\mu(q) \ge \frac{1}{\pi}(\theta_{\varphi_\epsilon(L)} - \theta_{L'}) \ge \frac{1}{\pi}\epsilon,$$

and hence degree-zero intersections cannot occur on $\varphi_\epsilon(L \setminus B(p,\eta))$. By the Hamiltonian invariance of Floer cohomology

$$\mathrm{HF}^0(\varphi_\epsilon(L), L') = \mathrm{HF}^0(L, L') \ne 0.$$

This forces there to be an intersection $q \in \varphi_\epsilon(L \cap B(p,\eta)) \cap L'$. Thus there is $p' \in L \cap B(p,\eta)$ whose distance to q is less than $C\epsilon|dh| \le C\epsilon^{(n+1)/(2n)}$. □

Remark 2.3 If we replace the embedded Lagrangians by immersed Lagrangians with connected domain and transverse self intersections, we can still use the Floer cohomology of Akaho and Joyce [2]. The main caveat is that in order for the Hamiltonian function $h: L \to \mathbb{R}$ to extend to X, we need h to take the same value at the finitely many self-intersection points. These finitely many linear constraints are easy to meet, by relaxing $f = 4$ outside $B(p,\eta)$ to the more flexible condition $f \geq 4$ outside $B(p,\eta)$.

The connected domain hypothesis on L is important for solving the equation $\mathcal{L}(h) = f$. If we drop this hypothesis, then take for instance L the disjoint union of two special Lagrangians and L' to be one of the components, and the lemma would be false.

Remark 2.4 Lemma 2.2 easily gives a *new proof* to the original Thomas–Yau uniqueness theorem (Theorem 1.1), not relying on Morse theory or real analyticity. Indeed, by taking the $\epsilon \to 0$ limit, we deduce that each point on L has zero distance to L', and hence $L \subset L'$. Since the roles of L and L' are symmetric in Theorem 1.1, we recover $L = L'$. The rest of this paper treats the additional difficulties caused by giving up quantitative regularity bounds on L', and by relaxing the C^0-smallness of $\theta_{L'}$ to merely smallness on most of the measure.

2.3 Generic perturbation technique

The technique of controlling the number of intersection points by utilizing a sufficiently rich family of perturbations originated from Arnold [4], who used it to study the dynamical growth of intersection points. Proposition 2.5 is a detailed exposition to clarify the dependence of constants. Seidel [11, Lecture 5] contains an application to bound the rank of Lagrangian Floer cohomology.

Let M be an m-dimensional compact manifold, N and N' be two compact submanifolds of complementary dimension and $U \subset N'$ be an open subset. Let V be a p-dimensional space of vector fields on M, which is surjective to the normal space TM/TN at every point of N. Fix a Euclidean metric on V, and consider the p-dimensional family of diffeomorphisms ϕ_t of M obtained by exponentiating the vector fields in $P = B(0,\epsilon) \subset V$ for some small ϵ. This induces a p-parameter deformation family $\mathcal{N} \to P$ for N whose fibres are $\phi_t(N)$ for $t \in P$. There is an evaluation map $\pi: \mathcal{N} \to M$, which by construction is surjective on tangent spaces.

Proposition 2.5 *There exists some $t \in P$ such that $\phi_t(N)$ intersects N' transversely, and the number of intersection points is*

$$|\phi_t(N) \cap U| \leq C\epsilon^{-n}\,\mathrm{Vol}(U),$$

where the constant C depends only on M, V and the C^1-regularity of N, but not on the small ϵ and the regularity bounds of N'.

Proof The transversality holds for a.e. $t \in P$ by Sard and Smale, so that $|\phi_t(N) \cap U|$ is a well defined integer for a.e. $t \in P$. By the coarea formula,

$$\int_P |\phi_t(N) \cap U|\, dt \leq \mathrm{Vol}(\mathcal{N} \cap \pi^{-1}(U)),$$

where the volume is computed with respect to the restriction of the product metric on $M \times P$. Now $\pi : \mathcal{N} \cap \pi^{-1}(U) \to U$ is surjective on tangent spaces, with Jacobian factor bounded below by C^{-1}, so

$$\mathrm{Vol}(\mathcal{N} \cap \pi^{-1}(U)) \leq C \int_U \mathrm{Vol}(\mathcal{N} \cap \pi^{-1}(y))\, dy.$$

For each $y \in U$, the contributions to $\mathcal{N} \cap \pi^{-1}(y)$ come from the deformations of the small local region $N \cap B(y, C\epsilon)$. The assumption that V is surjective to TM/TN at every point of N, together with the C^1-regularity of N, allows us to find a codimension-n subspace $V_y \subset V$ such that the projection

$$\mathcal{N} \cap \pi^{-1}(y) \to P = B(0, \epsilon) \subset V \to V_y$$

exhibits $\mathcal{N} \cap \pi^{-1}(y)$ as part of a C^1-graph over the ϵ-ball inside V_y, whose volume is bounded by $C\epsilon^{p-n}$. We deduce

$$\mathrm{Vol}(\mathcal{N} \cap \pi^{-1}(y)) \leq C\epsilon^{p-n}.$$

Combining the above,

$$\int_P |\phi_t(N) \cap U|\, dt \leq C\epsilon^{p-n}\, \mathrm{Vol}(U),$$

so we can select some $t \in P$ with

$$|\phi_t(N) \cap U| \leq C\, \mathrm{Vol}(U)\epsilon^{p-n}(\mathrm{Vol}(P))^{-1} \leq C\, \mathrm{Vol}(U)\epsilon^{-n}. \qquad \square$$

In our application, the idea is that under sufficiently generic Hamiltonian deformation, subsets with very small measure do not contribute to Lagrangian intersection points. This allows us to relax Lemma 2.2 to a weak L^1-assumption on the Lagrangian angle of L'.

Proposition 2.6 *Assume $\|\theta_L\|_{C^0} \leq \epsilon$, while $\mathrm{Vol}(\{|\theta_{L'} > \epsilon\}) \ll \epsilon^{-n}$. Then there are constants C_1 and C independent of ϵ such that for all $p \in L$, there is some $p' \in L \cap B(C_1\epsilon^{1/2n})$ satisfying $\mathrm{dist}(p', L') \leq C\epsilon^{(n+1)/(2n)}$.*

Proof We can find a large-dimensional vector space V of Hamiltonian vector fields which is surjective to the normal space TX/TL at every point of L. We can pick the Euclidean metric on V so that for any v in the unit ball of V, the induced vector field acting on the Grassmannian bundle over X has C^0-norm $\ll 1$. In particular, the Hamiltonian diffeomorphisms ϕ_t obtained via exponentiating vector fields in $P = B(0, \epsilon) \subset V$ only change the Lagrangian angle function by an amount $\ll \epsilon$.

We now reexamine the argument of Lemma 2.2. For any $p \in L$, we can construct the C^1-small Hamiltonian deformation $\varphi_\epsilon(L)$ of L such that on $\varphi_\epsilon(L)(L \setminus B(p, \eta))$, we have $\theta_{\varphi_\epsilon(L)} \geq 2\epsilon$. Thus for any $t \in P$, the Hamiltonian deformation $\phi_t\varphi_\epsilon(L)$ satisfies away from $B(p, \eta)$

$$\theta_{\phi_t\varphi_\epsilon(L)} \geq \theta_{\varphi_\epsilon(L)} - \tfrac{1}{2}\epsilon \geq \tfrac{3}{2}\epsilon.$$

Let $U = \{\theta_{L'} > \epsilon\} \subset L'$. Then according to Proposition 2.5, there is some $t \in P$ such that $\phi_t\varphi_\epsilon(L)$ is transverse to L', and the number of intersection points is

$$|\phi_t\varphi_\epsilon(L) \cap U| \leq C\epsilon^{-n}\, \mathrm{Vol}(U) \ll 1.$$

Thus $\phi_t\varphi_\epsilon(L)\cap U$ is in fact *empty*. This forces $\theta_{L'}\le\epsilon$ at the Lagrangian intersections $\phi_t\varphi_\epsilon(L)\cap L'$, and we conclude as in Lemma 2.2. □

Consequently, we get one half of the Hausdorff distance estimate:

Corollary 2.7 *Under the same conditions*,

$$\sup_{p\in L}\operatorname{dist}(p,L')\le C\epsilon^{1/2n}. \tag{5}$$

Remark 2.8 We have so far not used the condition $[L]=[L']\in H_n(X,\mathbb{Q})$. Without this condition, the constant C in Corollary 2.7 really requires some regularity bound on L. For instance, we consider a surgery exact triangle $L_1\to L_2\to L_3\to L_1[1]$, where L_2 is the Lagrangian connected sum of L_1 and L_3 with very small neck, and all Lagrangians angles can be made arbitrarily small in C^0. The mere assumption that $\mathrm{HF}^0(L_1,L_2)\ne 0$ cannot imply that $\sup_{p\in L_2}\operatorname{dist}(p,L_1)$ is small. The proof breaks down because the C^1-regularity bound of L_2 is highly degenerate in the neck region.

2.4 Monotonicity inequality

In minimal surface theory, the famous monotonicity formula says that for an n-dimensional submanifold N inside a smooth compact ambient manifold, if the mean curvature $\|\vec{H}\|_{L^\infty}$ is less than $+\infty$, then inside coordinate balls the volume ratio

$$e^{Cr}r^{-n}\operatorname{Vol}(B(p,r))$$

increases with the radius r. One useful consequence is that the volume of $N\cap B(p,r)$ has a lower bound for $p\in N$. We *only impose assumptions on the Lagrangian angle* θ, *but not on* $|\vec{H}|$ *directly*. We shall see that the *volume lower bound* is still satisfied. Our style of argument is inspired by Neves [10, Section 3.3]; see also [9, Section 5.2].

Recall a special case of the optimal isoperimetric inequality of Almgren. Denote by ω_n the volume of the n-dimensional unit ball in $\mathbb{R}^n$.

Proposition 2.9 [3, Theorem 10] *Let T be an $(n-1)$-dimensional integral current inside $\mathbb{R}^N$ with $\partial T=0$. Then there is an integral n-current Q inside $\mathbb{R}^N$ with $\partial Q=T$ and*

$$\operatorname{Mass}(Q)\le n^{-n/(n-1)}\omega_n^{-1/(n-1)}\operatorname{Mass}(T)^{n/(n-1)}.$$

Remark 2.10 The inequality is saturated by the n-dimensional unit ball; this sharpness of constant will be important for getting the sharp coefficient in Proposition 2.11(ii), which is essential for proving that the points on L' close to L occupy almost the full measure in L' as in Lemma 2.14. Moreover, if T is contained in some ball $B(R)$, then Q can also be taken inside $B(R)$, because there is a retraction of $\mathbb{R}^N$ to $B(R)$ with Lipschitz constant 1.

We now work inside the Kähler manifold X. Around any fixed $p \in X$, we can take local holomorphic coordinates $z_1, \dots, z_n$ such that for small $|z|$,

$$\omega = \frac{\sqrt{-1}}{2} \sum dz_i \wedge d\bar{z}_i + O(|z|), \quad e^{\rho(p)}\Omega = dz_1 \wedge \cdots \wedge dz_n + O(|z|), \quad g = \sum |dz_i|^2 + O(|z|).$$

Beware that the smooth function ρ measures the failure of the metric to be Calabi–Yau.

Proposition 2.11 *Let L' be a smooth Lagrangian with $|\theta| \le \frac{1}{2}\pi - \epsilon_0$, and assume there exists $p' \in L'$ with $|p'| \le \lambda \ll 1$. We shall use $C(\epsilon_0)$ to denote constants depending only on ϵ_0 and the coordinate chart.*

- (i) *For the Euclidean balls $B_{\mathrm{Eucl}}(r)$ with radius $r > \lambda$ contained in the chart, we have the volume lower bound $\mathrm{Vol}(L' \cap B_{\mathrm{Eucl}}(r)) \ge C(\epsilon_0)(r-\lambda)^n$.*
- (ii) *Assume furthermore that $\mathrm{Vol}(\{\theta_{L'} > \epsilon\} \cap L') \le \lambda^n$ where $\lambda \ge \epsilon^2$. Then for the Euclidean balls $B_{\mathrm{Eucl}}(r)$ with radius $r > C(\epsilon_0)\lambda$ contained in the chart, we have the sharper volume lower bound*

$$\mathrm{Vol}(L' \cap B_{\mathrm{Eucl}}(r)) \ge \omega_n (r - C(\epsilon_0)\lambda)^n (1 - C(\epsilon_0) r). \tag{6}$$

Proof For a.e. $0 < r$ smaller than the radius of the coordinate ball (of order $O(1)$), the level set $L' \cap \partial B_{\mathrm{Eucl}}(r)$ is smooth, and the isoperimetric inequality allows us to find $Q \subset B_{\mathrm{Eucl}}(r)$ with $\partial Q = L' \cap \partial B_{\mathrm{Eucl}}(r)$ and mass bound

$$\mathrm{Mass}(Q) \le n^{-n/(n-1)} \omega_n^{-1/(n-1)} \mathcal{H}^{n-1}(L' \cap \partial B_{\mathrm{Eucl}}(r))^{n/(n-1)}.$$

Since $\partial(L' \cap B_{\mathrm{Eucl}}(r)) = \partial Q$, the form $\mathrm{Re}\,\Omega$ is closed and $\mathrm{Re}\,\Omega|_Q \le (1 + O(r)) e^{-\rho(p)}\,\mathrm{dvol}_Q$,

$$\int_{L' \cap B_{\mathrm{Eucl}}(r)} \mathrm{Re}\,\Omega = \int_Q \mathrm{Re}\,\Omega \le (1 + O(r)) e^{-\rho(p)}\,\mathrm{Mass}(Q).$$

Under the quantitative almost-calibrated condition $|\theta| \le \frac{1}{2}\pi - \epsilon_0$,

$$\mathrm{Vol}(L' \cap B_{\mathrm{Eucl}}(r)) = \int_{L' \cap B_{\mathrm{Eucl}}(r)} \mathrm{dvol} \le \frac{1}{\sin \epsilon_0} \int_{L' \cap B_{\mathrm{Eucl}}(r)} e^{\rho}\,\mathrm{Re}\,\Omega.$$

Combining the above,

$$\mathrm{Vol}(L' \cap B_{\mathrm{Eucl}}(r)) \le C(\epsilon_0) \mathcal{H}^{n-1}(L' \cap \partial B_{\mathrm{Eucl}}(r))^{n/(n-1)}.$$

The function $f(r) = \mathrm{Vol}(L' \cap B_{\mathrm{Eucl}}(r))$ is increasing in r. Using the coarea formula, we rewrite the differential inequality as

$$f(r) \le C(\epsilon_0) f'(r)^{n/(n-1)}.$$

For $r > \lambda$, we have $f(r) > 0$, and the increasing function f satisfies

$$(f^{1/n})' \ge C(\epsilon_0),$$

whence $f \ge C(\epsilon_0)(r-\lambda)^n$.

Next we assume $\mathrm{Vol}(\{\theta_{L'} > \epsilon\} \cap L') \le \lambda^n$ and deduce the sharper volume lower bound. For $O(1) > r > C\lambda$ with C depending only on the volume lower bound constant, we have $f(r) > \lambda^n$. Notice that on $L' \cap \{\theta_{L'} \le \epsilon\} \cap B_{\mathrm{Eucl}}(r)$,

$$\int_{L'\cap B_{\mathrm{Eucl}}(r)\cap\{\theta_{L'}\le\epsilon\}} \mathrm{dvol} \le \frac{1}{\cos\epsilon}\int_{L'\cap B_{\mathrm{Eucl}}(r)\cap\{\theta_{L'}\le\epsilon\}} e^\rho \,\mathrm{Re}\,\Omega \le \frac{1}{\cos\epsilon}\int_{L'\cap B_{\mathrm{Eucl}}(r)} e^\rho \,\mathrm{Re}\,\Omega.$$

Hence

$$\begin{aligned} f(r) &\le \lambda^n + \frac{1}{\cos\epsilon}\int_{L'\cap B_{\mathrm{Eucl}}(r)} e^\rho \,\mathrm{Re}\,\Omega \le \lambda^n + \frac{1}{\cos\epsilon}(1+O(r))\,\mathrm{Mass}(Q) \\ &\le \lambda^n + \frac{1}{\cos\epsilon}(1+O(r))n^{-n/(n-1)}\omega_n^{-1/(n-1)} f'(r)^{n/(n-1)}. \end{aligned}$$

Now $1/\cos\epsilon = 1 + O(\epsilon^2) = 1 + O(\lambda) = 1 + O(r)$, so the $1/\cos\epsilon$ factor can be absorbed into the $1 + O(r)$ factor by changing the constant. Rearranging the terms,

$$f'(r) \ge n\omega_n^{1/n}(1-O(r))(f-\lambda^n)^{(n-1)/n}.$$

We deduce

$$\frac{d}{dr}(f-\lambda^n)^{1/n} \ge \omega_n^{1/n}(1-O(r)),$$

and after integration

$$f \ge \omega_n(r-C\lambda)^n(1-O(r)),$$

as required. □

Remark 2.12 In the context of special Lagrangians in $\mathbb{C}^n$, a standard way to deduce the monotonicity formula is to compare the volume of $L' \cap B(r)$ with the cone over $L' \cap \partial B(r)$. If we follow this strategy for smooth Lagrangians inside $\mathbb{C}^n$ with $|\theta| \le \epsilon$, then we would deduce

$$\mathrm{Vol}(L'\cap B(r)) \le \frac{1}{\cos\epsilon}\int_{L'\cap B(r)} \mathrm{Re}\,\Omega \le \frac{r}{n\cos\epsilon}\mathcal{H}^{n-1}(L'\cap B(r)),$$

whence $d/dr \log \mathrm{Vol}(L' \cap B(r)) \ge n\cos\epsilon/r$. Unfortunately, this would only prove the monotonicity of $\mathrm{Vol}(L' \cap B(r))r^{-n\cos\epsilon}$, which is *not sufficient* to deduce the volume lower bound above.

We think it is an interesting question whether $e^{Cr}\,\mathrm{Vol}(L' \cap B(r))r^{-n}$ is monotone under the mere hypothesis that $\|\theta\|_{C^0}$ is small, with no assumption directly on the mean curvature.

2.5 Hausdorff distance bound

We still need to show that all the points of L' must stay close to L. The strategy is to first show that such points occupy almost the full measure of L', and then use a monotonicity inequality argument to extend this to all points.

We record a weighted volume upper bound:

Lemma 2.13 *If* $\mathrm{Vol}(L' \cap \{\theta_{L'} > \epsilon\}) \le \epsilon^n$, *then* $\int_{L'} e^{-\rho}\,\mathrm{dvol}_{L'} \le \int_{L'} \mathrm{Re}\,\Omega/\cos\epsilon + C\epsilon^n$.

Proof On the subset $L' \cap \{\theta_{L'} \le \epsilon\}$, we integrate $e^{-\rho}\,\mathrm{dvol}_{L'} \le (1/\cos\epsilon)\,\mathrm{Re}\,\Omega|_{L'}$. □

Lemma 2.14 *In the setting of Theorem 2.1,*

$$\int_{L'\cap\{\mathrm{dist}(\cdot,L)\le\epsilon^{1/4n}\}} e^{-\rho}\,\mathrm{dvol}_{L'} \ge (1-C\epsilon^{1/4n})\int_L \mathrm{Re}\,\Omega. \tag{7}$$

Proof We can take $\lambda = C\epsilon^{1/2n}$ such that by Corollary 2.7, any $p\in L$ lies within distance λ to L'. Our setting implies $\lambda\ge\epsilon^2$ and $\mathrm{Vol}(\{\theta_{L'}>\epsilon\})\le\lambda^n$. By Proposition 2.11, and the fact that Euclidean balls and Riemannian balls only differ by relative error $O(r)$, we see that for $O(1)\ge r\ge C\lambda$,

$$\mathrm{Vol}_g(L'\cap B_g(p,r)) \ge \omega_n(r-C\lambda)^n(1-Cr)\quad\text{for all }\ p\in L.$$

The constants are uniform for $p\in L$, so after integration

$$\begin{aligned}\int_L e^{-\rho(p)}\,\mathrm{Vol}_g(L'\cap B_g(p,r))\,\mathrm{dvol}_L(p) &\ge \omega_n(r-C\lambda)^n(1-Cr)\int_L e^{-\rho}\,\mathrm{dvol}_L\\ &\ge \omega_n(r-C\lambda)^n(1-Cr)\int_L \mathrm{Re}\,\Omega.\end{aligned}$$

The left side can be computed by Fubini's theorem as

$$\int_{L'} e^{-\rho(q)}\,\mathrm{dvol}_{L'}(q)\int_{L\cap B_g(q,r)} e^{\rho(q)-\rho(p)}\,\mathrm{dvol}_L(p).$$

The function ρ is smooth, so $|\rho(q)-\rho(p)|\le Cr$. The Lagrangian L has fixed C^1-regularity bound, so for small radius $r\ll 1$,

$$\int_{L\cap B_g(q,r)} \mathrm{dvol}_L(p)\le(1+Cr)\omega_n r^n.$$

Combining the above,

$$\begin{aligned}\omega_n(r-C\lambda)^n(1-Cr)\int_L \mathrm{Re}\,\Omega &\le \int_L e^{-\rho(p)}\,\mathrm{Vol}_g(L'\cap B_g(p,r))\,\mathrm{dvol}_L(p)\\ &\le (1+Cr)\omega_n r^n\int_{L'\cap\{\mathrm{dist}(\cdot,L)\le r\}} e^{-\rho(q)}\,\mathrm{dvol}_{L'}(q),\end{aligned}$$

whence for $C\lambda\le r\le O(1)$,

$$\int_{L'\cap\{\mathrm{dist}(\cdot,L)\le r\}} e^{-\rho(q)}\,\mathrm{dvol}_{L'}(q)\ge\Big(1-Cr-C\frac{\lambda}{r}\Big)\int_L \mathrm{Re}\,\Omega. \tag{8}$$

The choice $r=\epsilon^{1/4n}$ yields the claim. □

Remark 2.15 In the above, we assumed L is embedded. If L is merely immersed with transverse self intersections, then the inequality

$$\int_{L\cap B_g(q,r)} \mathrm{dvol}_L(p)\le(1+Cr)\omega_n r^n$$

will break down for q in the r-neighbourhood of the self-intersection points. Essentially the same proof would then give

$$\int_{L'\cap\{\mathrm{dist}(\cdot,L)\le\epsilon^{1/4n}\}} e^{-\rho}\,\mathrm{dvol}_{L'} \ge (1-C\epsilon^{1/4n})\int_L \mathrm{Re}\,\Omega - Cr^n.$$

Since $r^n \ll \epsilon^{1/4n}$, the r^n term can be absorbed.

We can finally prove the remaining half of the *Hausdorff distance bound*:

Proposition 2.16 *In the setting of Theorem 2.1,*

$$\sup_{p\in L'} \mathrm{dist}(p,L) \le C\epsilon^{1/(4n^2)}. \tag{9}$$

Proof The assumption that $[L]=[L']\in H_n(X,\mathbb{Q})$ and the hypothesis that $\mathrm{Vol}(L'\cap\{\theta_{L'}>\epsilon\})\le\epsilon^n$, together with Lemma 2.13, imply

$$\int_{L'} e^{-\rho}\,\mathrm{dvol}_{L'} \le \frac{1}{\cos\epsilon}\int_L \mathrm{Re}\,\Omega + C\epsilon^n \le (1+C\epsilon^2)\int_L \mathrm{Re}\,\Omega.$$

Combined with Lemma 2.14,

$$\int_{L'\cap\{\mathrm{dist}(\cdot,L)\ge\epsilon^{1/4n}\}} e^{-\rho}\,\mathrm{dvol}_{L'} \le C\epsilon^{1/4n}\int_L \mathrm{Re}\,\Omega.$$

Assume $p\in L'$ with $r_p:=\mathrm{dist}(p,L)>\epsilon^{1/4n}$, so that $L'\cap B(p,r_p-\epsilon^{1/4n})$ is in $L'\cap\{\mathrm{dist}(\cdot,L)\ge\epsilon^{1/4n}\}$. From Proposition 2.11,

$$C(r_p-\epsilon^{1/4n})^n \le \mathrm{Vol}(L'\cap B(p,r_p-\epsilon^{1/4n})) \le C\epsilon^{1/4n}\int_L \mathrm{Re}\,\Omega.$$

Consequently $r_p \le C\epsilon^{1/4n^2}$. □

2.6 Flat norm distance bound

Recall that the flat norm distance between L and L' is defined as

$$\inf\{\mathrm{Mass}(T)+\mathrm{Mass}(R) : L-L'=\partial T+R\},$$

where T and R are integral currents of dimension $n+1$ and n, respectively. Intuitively, the Hausdorff distance bound is an L^∞-type bound on the distance function, while the flat norm behaves like an L^1-type bound. In the corollary below, we leverage the previous information to get a slightly better exponent than the obvious bound from $L^\infty\to L^1$, but we expect this is still not sharp.

Corollary 2.17 *In the setting of Theorem 2.1, there is an integral current T with $\partial T=L'-L$ such that $\mathrm{Mass}(T)\le C\epsilon^{1/(4n^2)+1/(4n)}$.*

Proof The Hausdorff bound shows that L' lies within a small C^0-neighbourhood of L. Using the controlled C^1-regularity of L, we can view the C^0-neighbourhood as its normal bundle, and obtain a projection map $\pi: L' \to L$. Since L is homologous to L' by assumption, π has degree 1. We form an $(n+1)$-dimensional integral current T from the union of the line segments joining the points $q \in L'$ to $\pi(q) \in L$. Up to the choice of orientation sign, $\partial T = L' - \pi_*(L') = L' - L$. The lengths of the segments are comparable to $\mathrm{dist}(q, L)$. Thus

$$\mathrm{Mass}(T) \le C \int_{L'} \mathrm{dist}(\cdot, L)\, \mathrm{dvol}_{L'} = C \int_0^{\sup \mathrm{dist}} dr \int_{L' \cap \{\mathrm{dist}(\cdot, L) \ge r\}} \mathrm{dvol}_{L'}.$$

Recall the weak L^1-type estimate (8), which implies

$$\int_{L' \cap \{\mathrm{dist}(\cdot, L) \ge r\}} \mathrm{dvol}_{L'} \le C \begin{cases} 1 & \text{if } r = O(\epsilon^{1/2n}), \\ \epsilon^{1/2n}/r & \text{if } \epsilon^{1/2n} \le r \le \epsilon^{1/4n}, \\ \epsilon^{1/4n} & \text{if } r > \epsilon^{1/4n}. \end{cases}$$

By the Hausdorff distance estimate, $\sup \mathrm{dist}(\cdot, L) \le C\epsilon^{1/4n^2}$. Combining the above gives the mass bound. □

Remark 2.18 If L is immersed rather than embedded, then π would be only Lipschitz near the self-intersection points, but the above argument is unaffected.

2.7 The F-metric bound

We view the C^0 neighbourhood of L as its normal bundle, which has a projection π to its zero section L. Normal geodesic flow allows us to identify the tangent spaces $T_q X$ with $T_{\pi(q)} X$. We now show that L' is graphical over L with small norm, away from a set with small measure. Denote by $\vec{T}(q)$ the unit n-vector $e_1 \wedge \cdots \wedge e_n$ on L', where e_i is an oriented orthonormal frame of $T_q L'$, and by $\vec{T}(\pi(q))$ the analogous unit n-vector from the oriented orthonormal basis of $T_{\pi(q)} L$.

Proposition 2.19 *In the setting of Theorem 2.1, away from a subset $E \subset L$ with $\mathrm{Vol}(E) + \mathrm{Vol}(\pi^{-1}(E)) \le C\epsilon^{1/4n}$, the projection $\pi: L' \setminus \pi^{-1}(E) \to L \setminus E$ is bijective, and*

$$\int_{L' \setminus \pi^{-1}(E)} |\vec{T}(q) - \vec{T}(\pi(q))|^2\, \mathrm{dvol}(q) \le C\epsilon^{1/4n}.$$

Proof The projection $\pi: L' \to L$ is almost metric decreasing:

$$|d\pi|_{T_q L'}| \le 1 + O(\mathrm{dist}(q, \pi(q))).$$

As a general rule, the error coming from comparing ambient Kähler structures at q and $\pi(q)$ is bounded by $O(\mathrm{dist}(q, \pi(q)))$. Let $E_1' = \{\mathrm{dist}(\cdot, L) \ge \epsilon^{1/4n}\} \subset L'$. Then E_1' has measure bounded by $C\epsilon^{1/4n}$ by Lemma 2.13 and (7).

Since $\pi : L' \to L$ has degree 1, it is surjective. Let E be the subset of L with at least two preimages. The volume almost decreasing property of $d\pi$ together with the assumption $|\theta| \le \epsilon$ on L imply

$$2\int_E \operatorname{Re}\Omega = 2\int_E e^{-\rho}\,\mathrm{dvol}(1+O(\epsilon)) \le (1+O(\epsilon^{1/4n^2}))\int_{\pi^{-1}(E)} e^{-\rho}\,\mathrm{dvol}.$$

In particular,

$$\mathrm{Vol}(E)+\mathrm{Vol}(\pi^{-1}(E)) \le C\int_{\pi^{-1}(E)} e^{-\rho}\,\mathrm{dvol} \le C'\Big(\int_{\pi^{-1}(E)} e^{-\rho}\,\mathrm{dvol} - \int_E \operatorname{Re}\Omega\Big). \tag{10}$$

On the other hand, at the points $q \in L' \setminus \pi^{-1}(E) \cup E_1'$, the difference between the tangent planes is bounded:

$$|\vec{T}(q) - \vec{T}(\pi(q))|^2 \le C(1-|d\pi(\vec{T})| + O(\epsilon^{1/4n})).$$

Thus

$$\begin{aligned}\int_{L'\setminus\pi^{-1}(E)} |\vec{T}(q)-\vec{T}(\pi(q))|^2\,\mathrm{dvol}(q) &\le C\epsilon^{1/4n} + C\int_{L'\setminus\pi^{-1}(E)} (1-|d\pi(\vec{T})|)\,\mathrm{dvol}\\ &\le C\epsilon^{1/4n} + C'\int_{L'\setminus\pi^{-1}(E)} (1-|d\pi(\vec{T})|)e^{-\rho}\,\mathrm{dvol}.\end{aligned}$$

Here the contribution from E_1' is absorbed into $C\epsilon^{1/4n}$. From the pointwise inequality on $L'\setminus\pi^{-1}(E)\cup E_1'$,

$$|d\pi(\vec{T})|e^{-\rho} = (1-O(\epsilon^{1/4n}))|d\pi(\vec{T})|\,|\operatorname{Re}\Omega|_L| \ge \operatorname{Re}\Omega(\pi_*\vec{T})(1-O(\epsilon^{1/4n})),$$

the above is bounded by

$$C\epsilon^{1/4n} + C'\Big(\int_{L'\setminus\pi^{-1}(E)} e^{-\rho}\,\mathrm{dvol} - \int_{L\setminus E}\operatorname{Re}\Omega\Big).$$

The coefficient C' can be arranged as the same (large) constant appearing in (10). Adding the two contributions and invoking the weighted volume upper bound in Lemma 2.13,

$$\begin{aligned}\mathrm{Vol}(E)+\mathrm{Vol}(\pi^{-1}(E)) + \int_{L'\setminus\pi^{-1}(E)} |\vec{T}(q)-\vec{T}(\pi(q))|^2\,\mathrm{dvol}(q)\\ \le C\epsilon^{1/4n} + C'\Big(\int_{L'} e^{-\rho}\,\mathrm{dvol} - \int_L \operatorname{Re}\Omega\Big)\\ \le C\epsilon^{1/4n} + C'\Big(\int_{L'} \operatorname{Re}\Omega - \int_L \operatorname{Re}\Omega\Big) = C\epsilon^{1/4n},\end{aligned}$$

as required. □

Remark 2.20 If L is immersed instead of embedded, then we include the $O(\epsilon^{1/4n^2})$ neighbourhood of the self intersections into E, and the rest of the arguments run almost verbatim.

Recall the distance between two varifolds L and L' is measured by *F-metric*

$$\sup\Big\{\Big|\int_L f(p, T_pL)\,\mathrm{dvol}_L - \int_{L'} f(q, T_qL')\,\mathrm{dvol}_{L'}\Big|\Big\},$$

where f ranges over all functions on the Grassmannian bundle of n-dimensional tangent subspaces over X, with $|f| \le 1$ and Lipschitz constant at most 1. Given a uniform upper bound on the mass, the F-metric induces the same topology as the weak topology on varifolds (because continuous functions can be approximated by Lipschitz functions).

Corollary 2.21 *In the setting above, the F-metric between L and L' is bounded by $C\epsilon^{1/8n}$.*

Proof Since $|f| \le 1$, the integrals on E, E_1' and $\pi^{-1}(E)$ are bounded by $C\epsilon^{1/4n}$. At $q \in L' \setminus \pi^{-1}(E) \cup E_1'$, by the Lipschitz bound on f,

$$|f(q, T_q L') - f(\pi(q), T_{\pi(q)} L)| \le |q - \pi(q)| + |\vec{T}(q) - \vec{T}(\pi(q))| \le C\epsilon^{1/4n} + |\vec{T}(q) - \vec{T}(\pi(q))|.$$

The discrepancy between the volume forms $\pi^* \operatorname{dvol}_L$ and $\operatorname{dvol}_{L'}$ is bounded by

$$C|\vec{T}(q) - \vec{T}(\pi(q))|^2 \operatorname{dvol}_{L'}.$$

Consequently, the F-metric is bounded by

$$C\epsilon^{1/4n} + \int_{L' \setminus \pi^{-1}(E)} |\vec{T}(q) - \vec{T}(\pi(q))| \operatorname{dvol}_{L'} + C\int_{L' \setminus \pi^{-1}(E)} |\vec{T}(q) - \vec{T}(\pi(q))|^2 \operatorname{dvol}_{L'}.$$

Using Proposition 2.19 and Cauchy–Schwarz, this is bounded by $C\epsilon^{1/8n}$. □

3 Consequences of Theorem 2.1

We now prove the consequences stated in the introduction.

Proof of Theorem 1.6 From the C^∞-limit assumption, $L_i \to L_\infty$ with $\|\theta\|_{C^0} \to 0$. From the varifold limit assumption, $L_i' \to L'$. This implies that $\|\theta\|_{L^1} \to 0$ as follows. Since the Lagrangian angle is a continuous function on the open subset $\{\operatorname{Re}\Omega > 0\}$ in the Grassmannian bundle, the weak topology convergence of L_i' implies

$$\int_{L_i'} |\theta| \operatorname{dvol}_{L_i'} \to \int_{L_\infty'} |\theta| \operatorname{dvol} = 0.$$

All Lagrangians lie in the same homology class, and all the approximants L_i and L_i' lie in the same derived category class. Thus Theorem 2.1 applies to L_i and L_i' for $i \gg 1$ and arbitrarily small ϵ, and the uniform C^∞-regularity on L_i means that all the constants are uniform. The flat norm distance between L_i and L_i' is bounded by $C\epsilon^{1/(4n)+1/(4n^2)}$. Taking the limit as $i \to +\infty$, the flat norm distance between L_∞ and L_∞' is bounded by $C\epsilon^{1/(4n)+1/(4n^2)}$, and letting $\epsilon \to 0$ gives $L_\infty = L_\infty'$. □

Proof of Theorem 1.7 As above, varifold convergence to L^∞ implies $\|\theta_{L_i'}\|_{L^1} \to 0$. Conversely, if $\|\theta_{L_i'}\|_{L^1} \to 0$, then Theorem 2.1 applies to L_i and L_i' for $i \gg 1$ and arbitrarily small ϵ, whence the F-metric between L_i and L_i' is bounded by $C\epsilon^{1/8n}$ for $i \gg 1$ depending on ϵ. In the $i \to +\infty$ limit, the F-metric distance between L_i' and L_∞ tends to zero. Since all Lagrangians have uniform mass bounds, the F-metric induces the same topology as the weak topology on varifolds, whence $L_i' \to L_\infty$ in the varifold topology. □

4 Open questions

We raise some natural questions:

Question 2 With the same assumptions on L and L' as in Theorem 2.1, what would be the *sharp exponent* for the Hausdorff distance bound in terms of ϵ?

This question has many variants. For instance, one can replace $\mathrm{Vol}(\{\theta_{L'} > \epsilon\}) \le \epsilon^n$ with the smallness of $\|\theta_{L'}\|_{L^p(L')}$ (or $\|\theta_{L'}\|_{C^0(L')}$). We expect the exponents of ϵ in Theorem 2.1 to be not optimal. On the other hand, even in the simple setting of Calabi–Yau metrics, assuming L is a fixed smooth special Lagrangian and $\|\theta_{L'}\|_{L^1} < \epsilon$, we do not expect the Hausdorff distance to be uniformly bounded by $O(\epsilon)$. The reason is that the linearization of the special Lagrangian equation is the Poisson equation $\Delta u = f$, and the failure of the Sobolev embedding $W^{2,1} \to C^1$ means we cannot expect u to have C^1-bounds proportional to $\|f\|_{L^1}$. Some nonlinear effects are necessary for a result like Theorem 2.1.

Question 3 Do the uniform constants really depend on the C^∞-regularity bounds of L?

The part of our arguments most sensitive to C^∞-regularity bounds is Lemma 2.2, which involves solving the Poisson equation on L with bounds. If L has some mild degeneration, eg when it is approximately the desingularization of some special Lagrangian with certain isolated conical singularities, then we expect results similar to Theorem 2.1 would hold, with possibly different exponents for ϵ. On the other hand, it is not clear if we can drop all regularity bounds on L beyond the information of the homology class and the Lagrangian angles. A closely related question is whether *strong uniqueness* holds:

Question 4 Consider the variant of Theorem 1.6 where both L_∞ and L'_∞ are varifold limits. Assuming both are special Lagrangian $\theta_{L_\infty} = \theta_{L'_\infty} = 0$, do they coincide as integral currents?

Question 5 If L and L' are immersed Lagrangians defining the same object in the derived Fukaya category, not necessarily with connected domains, then is there still an analogue of the quantitative Thomas–Yau uniqueness theorem? Is there a corresponding strong–weak uniqueness theorem?

Acknowledgement This work was completed while the author was a Clay research fellow based at MIT. He thanks P Seidel for discussions, Y Imagi for comments and the referees for useful suggestions.

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Received: 24 September 2022
Revised: 29 October 2024

Geometry & Topology

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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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Geometry & Topology

Volume 29 Issue 5 (pages 2251–2782) 2025