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Tian's stabilization problem for toric Fanos

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In 1988, Tian posed the stabilization problem for equivariant global log canonical thresholds. We solve it in the case of toric Fano manifolds. This is the first general result on Tian's problem. A key new estimate involves expressing complex singularity exponents associated to orbits of a group action in terms of support and gauge functions from convex geometry. These techniques also yield a resolution of another conjecture of Tian from 2012 on more general thresholds associated to Grassmannians of plurianticanonical series.

14C20, 14J45, 14J50, 14M25; 14L30, 32Q20, 32U05, 52A40

In memory of Eugenio Calabi

1 Introduction

This article is the first in a series in which we study asymptotics of invariants related to existence of canonical metrics on Kähler manifolds.

Here we focus on Tian's $\alpha_{k,G}$- and $\alpha_{k,m,G}$-invariants that were defined in 1988 and 1991, and to a large extent are still rather mysterious. The presence of the compact symmetry group G is a major source of difficulty and new ideas are needed here as these invariants have not been previously systematically studied or computed. In particular, we provide a formula for such invariants, valid for all toric Fano manifolds, leading to a resolution of Tian's stabilization conjecture in this setting, which is also the first general result on Tian's conjecture.

A sequel to this article deals with the δ_k-invariants of Fujita and Odaka (also called k^{th} stability thresholds) on toric Fano manifolds. It turns out that for these invariants there is a quantitative dichotomy regarding stabilization, and when stabilization fails we derive their complete asymptotic expansion.

1.1 Tian's stabilization problem

Let (X, L, ω) be a polarized Kähler manifold of dimension n with L a very ample line bundle over X, and ω a Kähler form representing $c_1(L)$. The space

(1) $$\mathcal{H}_L := \{\varphi : \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\} \subset C^\infty(X)$$

of Kähler potentials of metrics cohomologous to ω was introduced by Calabi in a short visionary talk in the joint AMS–MAA annual meetings held at Johns Hopkins University in December 1953 [7]. In a groundbreaking article some 35 years later, Tian proved (motivated by a question of Yau [41, page 139]) that $\mathcal{H}_L$ is approximated (or "quantized") in the C^2 sense by the finite-dimensional spaces $\mathcal{H}_k$ consisting of pullbacks of Fubini–Study metrics on $\mathbb{P}(H^0(X, L^k)^*)$ under all possible Kodaira embeddings induced by $H^0(X, L^k)$ [36]. A decade later this was improved to a complete asymptotic expansion (see Catlin [10] and Zelditch [42]) and so the $\mathcal{H}_k$ can be considered as the Taylor (or Fourier, depending on the point of view) expansion of $\mathcal{H}_L$. The theme that holomorphic and other invariants associated to X and $\mathcal{H}$ may be quantized using the spaces $\mathcal{H}_k$ has dominated Kähler geometry for the last 35 years.

In the 1980s, Futaki's invariant was a new obstruction for the existence of Kähler–Einstein metrics, but there were no invariants that guaranteed existence. At best, there were constructions that utilized symmetry to reduce the Kähler–Einstein equation to a simpler equation that could be solved and lead to specific examples (another theme pioneered by Calabi, which makes a surprise appearance to close this paper; see Example 8.12). Given a maximal compact subgroup G of the automorphism group $\operatorname{Aut} X$, Tian introduced the invariant

(2) $$\alpha_G := \sup\left\{c > 0 : \sup_{\varphi \in \mathcal{H}^G} \int_X e^{-c(\varphi - \sup\varphi)} \omega^n < \infty\right\}$$

(where $\mathcal{H}^G$ denotes the G-invariant elements of $\mathcal{H}_L$), and its quantized version (Definition 4.1)

$$\alpha_{k,G},$$

computed over the G-invariant elements of $\mathcal{H}_k$, and obtained the sufficient condition

$$\alpha_G > \frac{n}{n+1} \tag{3}$$

for the existence of a Kähler–Einstein metric when $L = -K_X$ (which we will henceforth assume unless otherwise stated); see Tian [34, Theorem 4.1]. Initially, the main interest in the invariants α_G was as the first systematic tool for constructing Kähler–Einstein metrics on Fano manifolds, but later it was also conjectured by I Cheltsov and established by Demailly that α_G actually coincides with the G-equivariant global log canonical threshold from algebraic geometry; see Cheltsov and Shramov [13]. Since $\mathcal{H}_k \subset \mathcal{H}_L$, it follows that $\alpha_G \leq \inf_k \alpha_{k,G}$ [36, page 128], yet this does not help obtain (3). Instead, Tian posed the following difficult question that would reduce the computation of the invariant α_G from the infinite-dimensional space $\mathcal{H}_L$ to a finite-dimensional one $\mathcal{H}_k$, and establish a highly nontrivial relation between the different $\mathcal{H}_k$; see Tian [35; 36, Question 1]:

Problem 1.1 *Let X be Fano, $L = -K_X$, and let G be a maximal compact subgroup $G \subset \operatorname{Aut} X$. Is $\alpha_{k,G} = \alpha_G$ for all sufficiently large $k \in \mathbb{N}$?*

It is interesting to note that a slight variation of Tian's α- and α_k-invariants turned out to lead about three decades later [43; 30] to the very closely related δ- and δ_k-invariants of Fujita and Odaka [19] that are in turn a slight (and ingenious) variation on global log canonical thresholds, and turn out to essentially characterize the existence of Kähler–Einstein metrics. We return to these invariants in a sequel [25].

1.2 A Demailly type identity in the presence of symmetry

Another (easier) problem motivating this article concerns the by-now classical relation between Tian's (holomorphic) invariants and the (algebraic) global log canonical thresholds. The relationship was first conjectured by Cheltsov (see Cheltsov and Shramov [13]) and and proved by Demailly and Shi; see Shi [31]. However, so far, this relationship has only been shown for the α- and α_k-invariants, or for the α_G-invariant (see [13, Theorem A.3], [31, Proposition 2.1], and [13, (A.1)], respectively), and not for the more subtle invariants $\alpha_{k,G}$. Inspired by Demailly, we introduce (Definition 4.4) the k^{th} G-equivariant global log canonical threshold

$$\operatorname{glct}_{k,G}$$

as an algebraic counterpart of Tian's $\alpha_{k,G}$ (Definition 4.1). A natural question is:

Problem 1.2 *Let X be Fano and $L = -K_X$, and let G be a compact subgroup $G \subset \operatorname{Aut} X$. Is $\operatorname{glct}_{k,G} = \alpha_{k,G}$?*

Note that the proof of Demailly in the equivariant case [13, (A.2)] works for the limit as $k \to \infty$ but not on a fixed level k.

1.3 Results

We resolve both Problems 1.1 and 1.2 in the toric setting. It is perhaps not well known, but Calabi was interested in toric geometry and computed certain geodesics in $\mathcal{H}_L$ in the toric setting, although he never published the result [9].

As standard, we allow the slightly more flexible situation of any (and not just a maximal) compact subgroup of the normalizer

$$N((\mathbb{C}^*)^n)$$

of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$. We refer the reader to Section 2, where these and other toric notation is set up carefully. Denote by

$$\operatorname{Aut} P \subseteq \mathrm{GL}(M) \cong \mathrm{GL}(n, \mathbb{Z}) \tag{4}$$

the subgroup of the automorphism group of the lattice M that leaves the polytope P (27) invariant. It is necessarily a finite group (see Section 2). In fact, $\operatorname{Aut} P$ is the quotient of the normalizer $N((\mathbb{C}^*)^n)$ of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$ by $(\mathbb{C}^*)^n$, so that $N((\mathbb{C}^*)^n)$ consists of finitely many components, each isomorphic to a complex torus [3, Proposition 3.1]. For $H \subset \operatorname{Aut} P$, let

$$G(H) := H \ltimes (S^1)^n \subset N((\mathbb{C}^*)^n) \subset \operatorname{Aut} X \tag{5}$$

denote the compact group generated by H and $(S^1)^n$ (the latter is the maximal compact subgroup of the complex torus $(\mathbb{C}^*)^n$).

Our first result resolves Problem 1.2 in this generality:

Proposition 1.3 *Let X be toric Fano and $L = -K_X$. Let $P \subset M_{\mathbb{R}}$* (see (18) and (27)) *be the polytope associated to $(X, -K_X)$, let $H \subset \operatorname{Aut} P$, and let $G(H)$ be as in* (5)*. Then $\operatorname{glct}_{k,G(H)} = \alpha_{k,G(H)}$.*

The proof of Proposition 1.3 relies on the structure of the ring of holomorphic sections on a toric line bundle established in Section 4.3. For general nontoric cases, such a proposition will hold once a similar result on the space of sections can be established.

Using this result, and several new estimates, we can resolve Tian's Problem 1.1 in the toric setting in a surprisingly strong sense, showing that equality holds for *all* $k \in \mathbb{N}$. We also allow for all groups $G(H)$ (and not just the maximal toric one $G(\operatorname{Aut} P)$).

To state the precise result we introduce some more notation. For $H \subset \operatorname{Aut} P$, denote by

$$P^H := \{y \in P : h.y = y \text{ for all } h \in H\} \subset P \subset M_{\mathbb{R}} \tag{6}$$

the fixed-point set of H in P, and let

$$\pi_H := \frac{1}{|H|} \sum_{\eta \in H} \eta \in \operatorname{End}(M_{\mathbb{Q}}) \tag{7}$$

be the map that takes a point in $M_{\mathbb{R}}$ to the average of its H-orbit. Note that π_H is a projection map (see Section 6.3 for details).

Theorem 1.4 *Let X be toric Fano associated to a fan Δ whose rays are generated by primitive elements v_i in the lattice N dual to M. Let $P \subset M_{\mathbb{R}}$ (see (18) and (27)) be the polytope associated to $(X, -K_X)$, let $H \subset \operatorname{Aut} P$, and let $G(H)$ be as in (5). Then for any $k \in \mathbb{N}$,*

$$\alpha_{k,G(H)} = \sup\Big\{c \in (0,1) : -\frac{c}{1-c} P^H \subset P\Big\} = \min_{u \in \operatorname{Ver} P^H} \frac{1}{\max_i \langle u, v_i\rangle + 1} = \min_{u \in \pi_H(\operatorname{Ver} P)} \frac{1}{\max_i \langle u, v_i\rangle + 1}, \tag{8}$$

where P^H and π_H are defined in (6)–(7) and $\operatorname{Ver}(\cdot)$ denotes the vertex set of a polytope. In particular, $\alpha_{k,G(H)}$ is independent of $k \in \mathbb{N}$ and is equal to $\alpha_{G(H)}$.

There are a few new ingredients in the proof of Theorem 1.4. The first is a useful formula for the spaces $\mathcal{H}_k^{G(H)}$ in terms of the H-orbits of the finite group action (Lemma 4.5). This together with a trick that amounts to estimating the singularities associated to a basis of sections in terms of the finite group action orbit of a section yields a useful formula for $\alpha_{k,G(H)}$ (Proposition 4.6), as well as the equality $\alpha_{k,G(H)} = \operatorname{glct}_{k,G(H)}$, ie a solution to Problem 1.2 (Proposition 1.3). These then yield a corresponding useful formula for $\alpha_{G(H)}$ (Corollary 4.7). One may prove, using the results of Sections 3–4, that $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)}$ for any fixed k and all $\ell \in \mathbb{N}$ (Proposition 5.1). In Proposition 5.3 it is shown that there is a special k_0 for which $\alpha_{k_0\ell,G(H)} = \alpha_{G(H)}$ for all $\ell \in \mathbb{N}$, as well as observed that this does not seem to imply Tian's conjecture (Remark 5.6). Finally, key new estimates occur in Section 6. First, we show that rather general complex singularity exponents associated to collections of toric monomials are independent of k (Proposition 6.2). Our proof uses a new observation about the relation between the support functions of collections of lattice points associated to the toric monomials and complex singularity exponents. We then apply this to our $G(H)$-invariant setting, using the aforementioned expression of $\mathcal{H}_k^{G(H)}$ and a reduction lemma to the H-invariant subspace (Lemma 6.10), to conclude the proof of Theorem 1.4.

It is perhaps of some interest to include here a rather immediate application of this circle of ideas to a slightly more technical set of invariants, also introduced by Tian, which we call Tian's Grassmannian α-invariants. These invariants are defined a little differently, algebraically, and are denoted by $\alpha_{k,m}$ or $\alpha_{k,m,G}$ (Definition 7.1). At least in the nonequivariant setting (as well as in the torus-equivariant setting, see Remark 7.2) these can be considered as generalizations of the α_k as $\alpha_k = \alpha_{k,1}$; see Cheltsov and Shramov [13] and Shi [31]. The invariants $\alpha_{k,2}$ were used by Tian implicitly in his proof of Calabi's conjecture for del Pezzo surfaces [36, Appendix A] (cf [37, Theorem 6.1]) to overcome the most difficult case (of a cubic surface with an Eckardt point) where equality holds in (3), and this was improved by Shi to $\alpha_{k,2} > 2/3 = \alpha_{k,1}$ in that case; see Cheltsov [11] and Shi [31, Theorem 1.3].

Tian also posed a stabilization conjecture for these invariants in 2012 [38, Conjecture 5.3]:

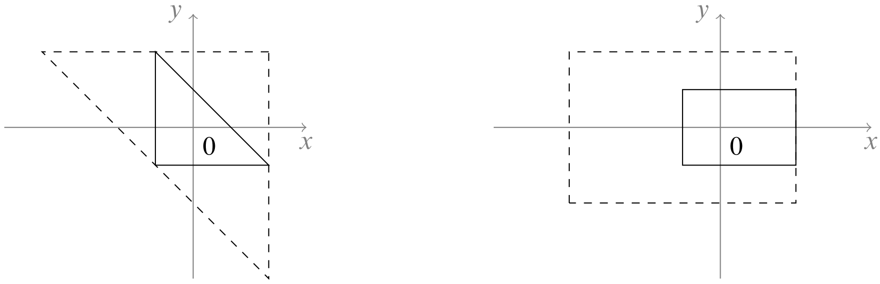

Figure 1: The polytope P (solid line) and the level set $\{\|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\}$ (dashed line). For $P = \mathrm{co}\{(-1,-1),(2,-1),(-1,2)\}$, the maximum is only attained at the vertices of P. In particular, $(*_P)$ holds. For $P = [-1,2]\times[-1,1]$, the maximum is attained on the line segment $\{2\}\times[-1,1]$. In particular, $(*_P)$ does not hold.

Conjecture 1.5 *Let X be Fano. Fix $m \in \mathbb{N}$. For sufficiently large k, $\alpha_{k,m,G}$ is constant.*

We completely resolve Conjecture 1.5 in the toric setting. Theorem 1.4 resolved Problem 1.1 in the affirmative (corresponding to the case $m = 1$ of Conjecture 1.5). For $m \geq 2$, Conjecture 1.5 turn out to be only partially true as determined by a novel convex geometric obstruction we introduce:

$$(*_P) \qquad \|\cdot\|_{-P}|_{P\setminus \mathrm{Ver}\, P} < \max_P \|\cdot\|_{-P}.$$

Note that this condition depends only on P (and not on m or k). The condition $(*_P)$ means that the function $\|\cdot\|_{-P}$ on P achieves its maximum only at the vertices of P, ie

$$(*_P) \qquad \mathrm{argmax}_P \|\cdot\|_{-P} \subset \mathrm{Ver}\, P.$$

When $(*_P)$ fails, the maximum is achieved also at some point that is not a vertex of P.

Theorem 1.6 *Let X be toric Fano with associated polytope P (28) and let $\alpha := \alpha_{(S^1)^n}$ be as in (2) with $L = -K_X$. Conjecture 1.5 holds if and only if $(*_P)$ fails. More precisely, if $(*_P)$ holds,*

$$(9) \qquad \alpha_{k,m,(S^1)^n} > \alpha \quad \text{for } k \in \mathbb{N} \text{ and } m \in \mathbb{N}\setminus\{1\},$$

otherwise

$$(10) \qquad \alpha_{k,m,(S^1)^n} = \alpha \quad \text{for } m \in \mathbb{N} \text{ and for sufficiently large } k \in \mathbb{N}.$$

Theorem 1.6 is proven in Section 7.1, where we also explain the intuition behind it (see also Examples 8.6 and 8.9). For now, let us elucidate the condition $(*_P)$ a bit. The level set $\{\|\cdot\|_{-P} = \lambda\}$ is the dilation $\lambda\partial(-P)$, and $\{\|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\}$ is the largest dilation that intersects P (by Lemma 7.4). Thus condition $(*_P)$ states that P intersects $\max_P \|\cdot\|_{-P}\,\partial(-P)$ only at vertices. When $(*_P)$ fails, convexity arguments show the intersection will contain a positive-dimensional face of P. See Figure 1 for two examples.

Relation to earlier works Theorem 1.4 strengthens and clarifies work of Song [33] and Li and Zhu [28]. Song obtained a formula for α_G but not for $\alpha_{k,G}$. In particular, there seems to be a gap in the proof of [33, Theorem 1.2] that claims that $\alpha_G = \alpha_{k,G}$ for all sufficiently large k. This claim relies on proving that for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$, $\alpha_G = \alpha_{k_0\ell,G}$, and then invoking that $\alpha_{k,G}$ is eventually monotone in k and hence must be independent of k for sufficiently large k. Unfortunately, the proof of monotonicity is omitted from [33, page 1257, line 7], and it appears to be difficult to reproduce. It seems that such monotonicity is not currently known (cf Remark 5.6). Indeed, there is no obvious relationship between the various $\mathcal{H}_k^G$ coming from different Kodaira embeddings. As noted above, Song showed (for $H = \mathrm{Aut}\, P$) that $\alpha_{G(H)} = \alpha_{k_0\ell,G(H)}$ for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$. Li and Zhu showed the same identity (essentially for $H = \{\mathrm{id}\}$) for a group compactification of a reductive complex Lie group and also claimed, similarly to Song, that this implies eventual constancy in k in that setting [28, Theorem 1.3, page 233]. Unfortunately, they also do not provide a proof of the needed eventual monotonicity or constancy. In the nonequivariant setting, and for rather general Fano varieties for which $\alpha \le 1$, Birkar showed the deep result that $\alpha = \alpha_{k_0\ell}$ for some $k_0 \in \mathbb{N}$ and all $\ell \in \mathbb{N}$ [5, Theorem 1.7]. However, this result also does not imply Tian stabilization due to the aforementioned unknown monotonicity. Thus Theorem 1.4 seems to be the first general result on Tian's stabilization Problem 1.1.

Similarly, Theorem 1.6 seems to be the first general result on Conjecture 1.5. Indeed, Li and Zhu showed the same type of result Song obtained in the $m = 1$ setting, ie that $\alpha_{k_0\ell,m,(S^1)^n} = \alpha_{(S^1)^n}$ under a condition depending on k_0 and m, and hence different from our $(*_P)$ (with the minor caveat that their statement as written [28, Theorem 1.4] is incorrect, though can be easily fixed by replacing "facet" by "face", see Section 7.1). However, again, due to the lack of monotonicity, they do not obtain a resolution of Conjecture 1.5 though they do obtain the first counterexamples to it when $m \ge 2$.

Combining Theorem 1.4 and Demailly's theorem [13, (A.1)] also recovers Song's formula for the $\alpha_{G(\mathrm{Aut}\, P)}$ (that itself generalized Batyrev and Selivanova's formula that $\alpha_{G(\mathrm{Aut}\, P)} = 1$ whenever $P^{\mathrm{Aut}\, P} = \{0\}$ (recall (6)) [3, Theorem 1.1, page 233]). Cheltsov and Shramov claimed a more general formula for α_H (ie without the real torus symmetry included in $G(H)$, recall (5)) however (as kindly pointed out to us by Cheltsov) there is an error in the proof of [13, Lemma 5.1] as the toric degeneration used there need not respect the H-invariance. Further generalizations of Song's formula for α_G to general polarizations and group compactifications are due to Delcroix [14; 15] and Li, Shi, and Yao [27], and our methods should generalize to those settings as well as to the setting of log toric Fano pairs and edge singularities [12, Sections 6–7]. Although this is not the topic of our article, for completeness we mention in passing that the stabilization of the $\alpha_k(L)$-invariants for arbitrary toric varieties (ie not necessarily Fano) but for the *nonequivariant* setting (ie $H = \{\mathrm{id}\}$) is true, ie $\alpha_{k,(S^1)^n}(L) = \alpha(L)$ for all $k \in \mathbb{N}$, yet is much-simpler; the crux of our work is to address the equivariant setting that presents new challenges.

Theorem 1.7 *Let X be a toric variety and L ample, with associated lattice polytope $P=\bigcap_{i=1}^{d}\{y\in M_{\mathbb{R}}:\langle y,v_i\rangle\geq -b_i\}$ with $v_i\in N$ and $b_i\in\mathbb{Z}$. For $k\in\mathbb{N}$,*

$$\alpha_{k,(S^1)^n}(L)=\min_{u\in P\cap M/k}\frac{1}{\max_i\langle u,v_i\rangle+b_i}=\min_{u\in\mathrm{Ver}\,P}\frac{1}{\max_i\langle u,v_i\rangle+b_i}=\alpha(L).$$

The same proof for Theorem 1.4 works in this case, with the additional assumption that $H=\{\mathrm{id}\}$, except that the coefficient 1 is replaced by b_i. Proposition 4.6, for instance, gives the left-hand side. Blum and Jonsson's formula [6, (7.2)] gives the right-hand side. And the equality again relies on $\mathrm{Ver}\,P\subset M$ and the fact that a convex function on a convex polytope attains its maximum on a vertex.

Finally, it is also worth mentioning that Tian also posed more general conjectures [38, Conjecture 5.4] for general polarizations (ie L not being $-K_X$) for which there are already some counterexamples [1].

Organization Section 2 sets up the necessary notation concerning toric varieties and convex analysis. Section 3 constructs natural equivariant reference Hermitian metrics and volume forms. Proposition 1.3 is proved in Section 4.4. Section 5 explains a trick that allows us to deal with divisible $k\in\mathbb{N}$, but also highlights the difficulties in dealing with general k. Theorem 1.4 is proved in Section 6. Theorem 1.6 is proved in Section 7. We conclude with examples in Section 8.

Acknowledgments Eugenio Calabi has had a profound influence on Rubinstein, who first met him as a graduate student in 2007. This article is dedicated to his memory and to the celebration of his mathematical culture, curiosity, and generosity. Calabi's contributions pop up in several places in this article, starting from the space of Kähler metrics, the space of toric metrics, and the Calabi–Hirzebruch manifold at the very end.

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2 Toric and convex analysis setup

2.1 Notions from convexity

Consider an n-dimensional real vector space $V\cong\mathbb{R}^n$ and let $V^*\cong\mathbb{R}^n$ denote its dual with the pairing denoted by $\langle\cdot,\cdot\rangle$. Given a set $A\subset V$, denote by

$$A^\circ=\{y\in V^*:\langle x,y\rangle\leq 1\text{ for all }x\in A\}$$

the polar of A [29, page 125], and by

(11) $$\mathrm{co}\,A$$

the convex hull of A [29, page 12]. For a finite set [29, Theorem 2.3],

$$\operatorname{co}\{p_1,\dots,p_\ell\} = \left\{ \sum_{i=1}^{\ell} \lambda_i p_i : \sum_{i=1}^{\ell} \lambda_i = 1, \lambda \in [0,1]^\ell \right\}. \tag{12}$$

Also, set

$$-K := \{-x : x \in K\}.$$

Note $(-K)^\circ = -K^\circ$. Also, $K^\circ = (\operatorname{co} K)^\circ \subset V^*$ whenever $K \subset V$. The polar can also be described via the support function $h_K\colon V^* \to \mathbb{R}$,

$$h_K(y) := \sup_{x\in K} \langle x, y\rangle \quad \text{for } y \in V^* \cong \mathbb{R}^n, \tag{13}$$

by $K^\circ = \{h_K \le 1\}$.

A dual notion to the support function is the near-norm function

$$\|x\|_K := \inf\{t \ge 0 : x \in tK\} \tag{14}$$

associated to any compact convex set K with $0 \in \operatorname{int} K$. Note that [29, Corollary 14.5]

$$\|\cdot\|_K = h_{K^\circ}. \tag{15}$$

Also note that $\|\cdot\|_K$ is a norm when K is centrally symmetric (ie $K = -K$), otherwise it is only a near-norm in the sense that it satisfies all the properties of a norm but is only $\mathbb{R}_+$-homogeneous: $\|\lambda x\|_K = \|x\|_K$ for $\lambda \in \mathbb{R}_+$ (and not fully $\mathbb{R}$-homogeneous). The infimum in (14) is achieved: for a minimizing sequence $\{t_i\}$, $x/t_i \in K$ for any i, so

$$x \in \|x\|_K K \tag{16}$$

since K is closed.

Lemma 2.1 *Let V be an $\mathbb{R}$-vector space. Consider the polytope*

$$A = \bigcap_{j=1}^{d} \{x \in V : \langle x, v_j\rangle \le 1\},$$

where $v_j \in V^*$. Then

$$\|x\|_A = \max_{1\le j\le d} \langle x, v_j\rangle.$$

Proof Notice that $x \in tA$ if and only if for any $1 \le j \le d$, $\langle x, v_j\rangle \le t$. Thus

$$\|x\|_A = \inf\Bigl\{t \ge 0 : \max_{1\le j\le d} \langle x, v_j\rangle \le t\Bigr\} = \max_{1\le j\le d} \langle x, v_j\rangle.$$

Alternatively, observe $A = \{v_1,\dots,v_d\}^\circ$ and $A^\circ = (\{v_1,\dots,v_d\}^\circ)^\circ = \operatorname{co}\{v_1,\dots,v_d\}$ [29, Theorem 14.5] and then use (57) and (15). □

2.2 Toric algebra

Consider a lattice of rank n and its dual lattice,

$$N \quad \text{and} \quad M := N^* := \mathrm{Hom}(N, \mathbb{Z}). \tag{17}$$

Both N and M are isomorphic to $\mathbb{Z}^n$, but we do not specify the isomorphism (see eg the proof of Lemma 2.3 for this point). The notation is useful as it serves to distinguish between objects living in one lattice and its dual (although of course in computations we simply work on $\mathbb{Z}^n$; see Section 8). Denote the corresponding $\mathbb{R}$-vector space and its dual (both isomorphic to $\mathbb{R}^n$) by

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}}^*. \tag{18}$$

A rational convex polyhedral cone in $N_{\mathbb{R}}$ takes the form

$$\sigma = \sigma(v_1, \ldots, v_d) := \left\{ \sum_{i=1}^{d} a_i v_i : a_i \geq 0,\, v_i \in N \right\}. \tag{19}$$

The rays $\mathbb{R}_+ v_i$ for $i \in \{1, \ldots, d\}$ are called the generators of the cone [20, page 9]. They are (1-dimensional) cones themselves, of course. Our convention will be that the

$$v_i \text{ for } i \in \{1, \ldots, d\} \text{ are primitive elements of the lattice } N, \tag{20}$$

which means there is no $m \in \mathbb{N} \setminus \{1\}$ such that $v_i/m \in N$. A cone is called strongly convex if $\sigma \cap -\sigma = \{0\}$ [20, page 14]. A face of σ is any intersection of σ with a supporting hyperplane.

Definition 2.2 A *fan* $\Delta = \{\sigma_i\}_{i=1}^{\delta}$ in N is a finite set of rational strongly convex polyhedral cones σ_i in $N_{\mathbb{R}}$ such that

(i) each face of a cone in Δ is also (a cone) in Δ,

(ii) the intersection of two cones in Δ is a face of each.

Such a fan gives rise to a toric variety $X(\Delta)$: each cone σ_i in Δ gives rise to an affine toric variety [20, Section 1.3] that serves as (a Zariski open) chart in $X(\Delta)$ with the transition between the charts constructed by (i) and (ii) above [20, page 21]. For instance, the zero cone corresponds to the open dense orbit $(\mathbb{C}^*)^n$ [32, page 64], and more generally there is a bijection between the cones $\{\sigma_i\}_{i=1}^{\delta}$ and the orbits of the complex torus $(\mathbb{C}^*)^n$ in $X(\Delta)$ [32, Proposition 5.6.2], with the nonzero cones corresponding precisely to all the toric subvarieties of $X(\Delta)$ of positive codimension.

When $X(\Delta)$ is a smooth toric Fano variety (as we always assume), the fan Δ must arise from an integral polytope as follows (but in general, ie for singular toric varieties, this need not be the case [20, page 25]). Let Δ be a fan such that $X(\Delta)$ is smooth Fano and let $\sigma_1, \ldots, \sigma_d$ be its 1-dimensional cones (ie rays) generated by primitive generators

$$\Delta_1 := \{v_1, \ldots, v_d\} \subset N, \tag{21}$$

so $\sigma_i = \mathbb{R}_+ v_i$, and set (recall (11))

$$Q := \operatorname{co}\Delta_1 = \operatorname{co}\{v_1, \dots, v_d\} = \operatorname{co}\Delta_1 \subset N_{\mathbb{R}}. \tag{22}$$

Then Δ is equal to the collection of cones over each face of Q plus the zero cone [20, page 25]. In other words if $F \subset Q$ is a face, then

$$\sigma_F := \{rx \in N_{\mathbb{R}} : r \geq 0,\, x \in F\} \tag{23}$$

is the union of all rays through F and the origin, and

$$\Delta = \{\sigma_F\}_{F \subset Q} \cup \{0\},$$

where 0 denotes the zero cone. Denote by

$$\operatorname{Ver} A \tag{24}$$

the vertices of a polytope A. Note that $\operatorname{Ver} F \subset \operatorname{Ver} Q = \Delta_1$, and by (19)–(20),

$$\sigma_F = \sigma(\operatorname{Ver} F). \tag{25}$$

Smoothness of X means that the generators of σ_F form a $\mathbb{Z}$-basis for N [20, page 29]. By (25) this means

$$\text{the vertices of } F \text{ form a } \mathbb{Z}\text{-basis for } N \text{ (for any facet } F \subset Q). \tag{26}$$

Thus each facet F of Q is an $(n-1)$-simplex whose vertices form a $\mathbb{Z}$-basis of N. In Lemma 2.3 we show this means the vertices of the polar polytope belong to the dual lattice M.

When $L = -K_X$, there is an $\operatorname{Aut} X$ action on $H^0(X, -kK_X)$ for every $k \in \mathbb{N}$. To get an induced linear action on $M_{\mathbb{Q}}$ we must restrict to the normalizer $N((\mathbb{C}^*)^n)$ of the complex torus $(\mathbb{C}^*)^n$ in $\operatorname{Aut} X$. The representation of $(\mathbb{C}^*)^n$ on $H^0(X, -kK_X)$ splits into 1-dimensional spaces, whose generators are called the monomial basis. There is a one-to-one correspondence between the monomial basis of $H^0(X, -kK_X)$ and points in $kP \cap M$, and the quotient $N((\mathbb{C}^*)^n)/(\mathbb{C}^*)^n$ is a linear group that can be identified with $\operatorname{Aut} P \subset \operatorname{GL}(M) \cong \operatorname{GL}(n, \mathbb{Z})$; see (4). Since P is defined as the convex hull of vertices in M, it follows that $\operatorname{Aut} P$ is finite. Alternatively, this can be seen by observing that $N_{\mathbb{R}}$ is canonically isomorphic to the quotient of $(\mathbb{C}^*)^n$ by its maximal compact subgroup $(S^1)^n$ [3, page 229] and the induced action on $M_{\mathbb{R}}$ is then defined by transposing via the pairing. Conversely, all compact subgroups of $N((\mathbb{C}^*)^n)$ that contain $(S^1)^n$ are generated by $(S^1)^n$ and a finite subgroup H of $\operatorname{Aut} P$ [3, Proposition 3.1], and we denote such a group by $G(H) \subset \operatorname{Aut} X$ as in (5). We describe in the proof of Lemma 3.1 concretely how the action of $\operatorname{Aut} P$ is expressed in coordinates. For a finite group H or finite set $\mathcal{A}$ we denote by

$$|H|, \text{ respectively, } |\mathcal{A}|,$$

its order or cardinality.

Oftentimes we will work with

$$P := -Q^\circ = \{-v_1, \dots, -v_d\}^\circ = -\{v_1, \dots, v_d\}^\circ \subset M_{\mathbb{R}}, \tag{27}$$

as it has the nice geometric property of faces of (real) dimension k corresponding to toric subvarieties of dimension k, and since the metric properties (eg volume) of P correspond to those of X. (Moreover, P can also be realized as the Delzant (moment) polytope associated to any $(S^1)^n$-invariant Kähler metric representing the anticanonical class.) In particular,

$$P = \bigcap_{i=1}^{d} \{y \in M_{\mathbb{R}} : \langle y, -v_i \rangle \le 1\} = \{h_{-\Delta_1} \le 1\} = \bigl\{y \in M_{\mathbb{R}} : \max_j \langle -v_j, y \rangle \le 1\bigr\}, \tag{28}$$

and irreducible toric divisors correspond to facets of P

$$\{y \in P : \langle y, -v_i \rangle = 1\}. \tag{29}$$

Note that P contains the origin in its interior. Also note that (27) is the standard convention since then lattice points of P correspond to monomials via (34). Batyrev and Selivanova use $-P$ instead.

Lemma 2.3 *Let $P \subset M_{\mathbb{R}}$ be the polytope associated to a smooth toric variety X. Then P is an integral lattice polytope*, *ie* $\operatorname{Ver} P \subset M$.

Proof By duality, if $u \in M_{\mathbb{R}}$ is a vertex of P then

$$F = \{v \in Q : \langle u, -v \rangle = 1\} \subset N_{\mathbb{R}} \tag{30}$$

is a facet of $Q = -P^\circ \subset N_{\mathbb{R}}$. Since X is smooth, the vertices of F form a $\mathbb{Z}$-basis for N by (26). Choose coordinates on N associated to this $\mathbb{Z}$-basis, ie the vertices of F are the standard basis vectors $e_1, \dots, e_n$. Thus the facet $F = \operatorname{co}\{e_1, \dots, e_n\}$ is the standard $(n-1)$-simplex in $\mathbb{R}^n$ cut out by the equation $\langle u, -v \rangle = 1$ where $u = (-1, \dots, -1) \in M$, as desired.

Alternatively, if one does not wish to choose coordinates but rather work invariantly, let

$$\operatorname{Ver} F := \{f_1, \dots, f_n\} \subset N,$$

and note $\operatorname{span}_{\mathbb{Z}} \operatorname{Ver} F = N$. Thus any $v \in N$ can be written uniquely as $v = \sum_{i=1}^n a_i f_i$ (with $a_i \in \mathbb{Z}$), and so $u \in M_{\mathbb{R}} = \operatorname{Hom}(N, \mathbb{R})$ can be identified (recall (17)) with the map

$$N \ni v \mapsto -\sum_{i=1}^{n} a_i.$$

As $\sum_{i=1}^n a_i \in \mathbb{Z}$, this map actually belongs to $\operatorname{Hom}(N, \mathbb{Z}) = M$. □

Another useful fact is a sort of maximum principle for convex polytopes, saying essentially that a convex function on a polytope achieves its maximum at some vertex (regardless of continuity).

Lemma 2.4 *Let A be a convex polytope and $f : A \to \mathbb{R} \cup \{\infty\}$ a convex function. Then*:

(i) $\sup_A f = \sup_{\mathrm{Ver}\, A} f$.

(ii) *If f is bounded on* Ver A *it is bounded on A and its maximum is achieved in a vertex.*

(iii) *If f attains its finite maximum on* int A *it is constant.*

(iv) *If f attains its finite maximum on the relative interior of a face $F \subset A$ it is constant on F.*

Proof Write $\mathrm{Ver}\, A = \{p_1, \dots, p_\ell\}$ and assume $f(p_1) \le \dots \le f(p_\ell)$. By convexity, $A = \mathrm{co}\,\mathrm{Ver}\, A$, and (12) implies that any $x \in A$ can be expressed as

(31) $$x = \sum_{i=1}^{\ell} \lambda_i p_i \quad \text{where} \quad \sum_{i=1}^{\ell} \lambda_i = 1 \text{ and } \lambda \in [0,1]^\ell.$$

Then $f(x) \le \sum_{i=1}^{\ell} \lambda_i f(p_i) \le \sum_{i=1}^{\ell} \lambda_i f(p_\ell) = f(p_\ell)$, proving (i) and (ii).

To see (iii), again express any $x \in A$ using (31). Suppose that $x_{\mathrm{int}} \in \mathrm{int}\, A$ achieves the (finite) maximum of f. Note that $x_{\mathrm{int}} \in \mathrm{int}\, A$ means one has the representation (31) for x_{int} for some $\lambda_{\mathrm{int}} \in (0,1)^\ell$. Choose $\delta = \delta(x) \in (0,1)$ so that $\lambda_{\mathrm{int}} - \delta\lambda \in \mathbb{R}_+^\ell$. Define

$$\lambda' := \frac{1}{1-\delta}(\lambda_{\mathrm{int}} - \delta\lambda) \in \mathbb{R}_+^\ell.$$

Note that λ' satisfies $\sum_{i=1}^{\ell} \lambda'_i = 1$, so letting $x' = \sum_{i=1}^{\ell} \lambda'_i p_i$ we have $x' \in A$ by (31). Also, $\delta x + (1-\delta)x' = x_{\mathrm{int}}$. Thus $\max_A f = f(x_{\mathrm{int}}) \le \delta f(x) + (1-\delta) f(x') \le \max_A f$, forcing equality, ie $f(x) = \max_A f$ (since $\delta > 0$ and $\max_A f < \infty$), proving (iii). The proof of (iv) is identical by working on the polytope F. □

Finally, we recall Ehrhart's theorem on the polynomiality of the number of lattice points in dilations of lattice polytopes [16; 17; 23, Theorem 19.1]. Set

(32) $$E_P(k) := |kP \cap M| \quad \text{for } k \in \mathbb{N}.$$

Proposition 2.5 *Let M be a lattice and $P \subset M_{\mathbb{R}}$ be a lattice polytope of dimension n. Then*

$$E_P(k) = \sum_{i=0}^{n} a_i k^i \quad \text{for any } k \in \mathbb{N},$$

with $a_n = \mathrm{Vol}(P)$, and

(33) $$k \le k' \implies E_P(k) \le E_P(k').$$

3 A natural equivariant Hermitian metric and volume form

In light of Lemma 4.2 below it makes sense to choose a convenient pair (μ, h) of a volume form and a Hermitian metric. In fact, we are free to choose such a pair for each k. The special feature of working with $L = -K_X$ is that in fact a volume form essentially doubles as a Hermitian metric, which is sometimes a bit confusing to keep track of in terms of notation, but is quite convenient for computations. This section serves to explain this choice $(\mu_k = h_k^{1/k}, h_k)$; see (40) and (42), originally due to Song [33, Lemma 4.3]. We emphasize that the Hermitian metric must additionally be chosen G-invariant in Definition 4.1, and this is confirmed for h_k in Lemma 3.1.

Let X be a toric Fano manifold and P its associated polytope. There is a natural basis of the space of holomorphic sections $H^0(X, -kK_X)$ defined by the monomials z^{ku} where $u \in P \cap (1/k)M$. That is, there exists an invariant frame e over the open orbit such that

$$s_{k,u}(z) = z^{ku} e. \tag{34}$$

What does e actually look like? This is most naturally expressed in terms of the *monomial basis*. When $k = 1$ and $u \in P \cap M$ [20, Section 4.3],

$$s_{1,u} = z^u \prod_{i=1}^{n} z_i \partial_{z_1} \wedge \cdots \wedge \partial_{z_n}.$$

In general, for any $k \in \mathbb{N}$ and $u \in P \cap (1/k)M$,

$$s_{k,u} = z^{ku} \left(\prod_{i=1}^{n} z_i \right)^k (\partial_{z_1} \wedge \cdots \wedge \partial_{z_n})^{\otimes k}. \tag{35}$$

In other words,

$$e = \left(\prod_{i=1}^{n} z_i \right)^k (\partial_{z_1} \wedge \cdots \wedge \partial_{z_n})^{\otimes k}. \tag{36}$$

Next, let us construct a canonical Hermitian metric h_k on $-kK_X$. Since $-kK_X$ is very ample [20, page 70], it is natural to pull back the Fubini–Study Hermitian metric via the Kodaira embedding. It turns out that choosing the Kodaira embedding given by the monomial basis

$$\iota_k : X \ni z \mapsto [s_{k,u}(z)/e(z)]_{u \in P \cap (1/k)M} = [z^{ku}]_{u \in P \cap (1/k)M} \in \mathbb{P}^{E_P(k)-1} \tag{37}$$

will yield the desired h_k; importantly, the resulting h_k will be *torus-invariant*, smooth, and essentially transform computations on X to P. To wit, the Fubini–Study metric on $\mathcal{O}(1) \to \mathbb{P}^{E_P(k)-1}$ is (where $E_P(k)$ is the number of lattice points in kP)

$$h_{\mathrm{FS}}(Z_i, Z_j) := \frac{Z_i \bar{Z}_j}{\sum_\ell |Z_\ell|^2},$$

and we define

$$h_k := \iota_k^* h_{\mathrm{FS}}. \tag{38}$$

Note that each homogeneous coordinate $Z_i \in H^0(\mathbb{P}^{E_P(k)-1}, \mathcal{O}(1))$ pulls back via ι_k to one of the monomial sections $s_{k,u}$ (which one depends on the ordering for the elements of $P \cap (1/k)M$ chosen in (37)). Thus to express h_k it suffices to compute it on the monomial basis of $H^0(X, -kK_X)$:

$$h_k(s_{k,u_1}, s_{k,u_2})(z) = h_{\mathrm{FS}}(s_{k,u_1}, s_{k,u_2})(\iota_k(z)) = \frac{z^{ku_1}\overline{z^{ku_2}}}{\sum_{u \in P\cap(1/k)M} |z^{ku}|^2}. \tag{39}$$

Comparing (35) and (39) means that h_k can be written as

$$h_k = \frac{(dz^1 \wedge \overline{dz^1} \wedge \cdots \wedge dz^n \wedge \overline{dz^n})^{\otimes k}}{\left(\prod_{i=1}^n |z_i|^2\right)^k \sum_{u \in P\cap(1/k)M} |z^{ku}|^2}. \tag{40}$$

In conclusion, $h_k^{1/k}$ is a smooth metric on $-K_X$ (h_k being obtained as a pullback of a smooth metric under the Kodaira *embedding*), and hence it is a smooth volume form on X. To express this volume form, on the open orbit $(\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n$ consider the holomorphic coordinates

$$w_i := \tfrac{1}{2}x_i + \sqrt{-1}\theta_i = \log z_i \in \mathbb{C}^n. \tag{41}$$

In these coordinates this volume form, on the open orbit, is

$$\mu_k := h_k^{1/k} = \frac{dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_n}{\left(\sum_{u \in P\cap M/k} e^{\langle ku, x\rangle}\right)^{1/k}}. \tag{42}$$

Lemma 3.1 *Let $G(H) \subseteq \mathrm{Aut}\, X$ (5) be a subgroup generated by $(S^1)^n$ and a subgroup H of $\mathrm{Aut}\, P$. Then h_k (40) is $G(H)$-invariant.*

Proof From (42) it is evident that h_k is independent of $(\theta_1, \dots, \theta_n)$, ie it is $(S^1)^n$-invariant. An automorphism $\sigma \in \mathrm{Aut}\, P \subseteq \mathrm{GL}(M)$ can be represented (via choosing a basis for the lattice M) by a matrix in $\mathrm{GL}(n, \mathbb{Z}) \cong \mathrm{GL}(M)$. Since σ preserves the polytope P, $\det\sigma \in \{\pm 1\}$ (σ could be orientation reversing, eg in the case of a reflection). The induced action of σ on the dual space $N_{\mathbb{R}}$ is naturally represented (via the pairing between M and N) by the transpose matrix, which we denote by σ^T, and this action actually comes from the $\mathbb{C}$-linear action of σ^T on $N_{\mathbb{C}} \cong \mathbb{C}^n$ (41). Thus

$$\sigma.(dx_1 \wedge \cdots \wedge dx_n) = d(x_1 \circ \sigma) \wedge \cdots \wedge d(x_n \circ \sigma) = \det(\sigma^T) dx_1 \wedge \cdots \wedge dx_n$$

(here $\sigma.$ denotes the action of σ on forms, ie by pullback), and $\sigma.(d\theta_1 \wedge \cdots \wedge d\theta_n) = \det(\sigma^T) d\theta_1 \wedge \cdots \wedge d\theta_n$. Since $(\det\sigma^T)^2 = 1$ it remains to consider the denominator of (42),

$$\begin{aligned}\sigma. \sum_{u \in P\cap M/k} e^{k\langle u, x\rangle} &= \sum_{u \in P\cap M/k} e^{k\langle u, \sigma^T.x\rangle} = \sum_{u \in P\cap M/k} e^{k\langle \sigma.u, x\rangle} = \sum_{u \in \sigma(P\cap M/k)} e^{k\langle u, x\rangle} \\ &= \sum_{u \in P\cap M/k} e^{k\langle u, x\rangle},\end{aligned}$$

since σ preserves both P and M/k. In particular, h_k is invariant under σ, concluding the proof. □

Remark 3.2 An alternative, more invariant, proof of Lemma 3.1 is as follows. By (38), $\sigma.h_k = (\iota_k \circ \sigma)^* h_{\mathrm{FS}}$. Now $\iota_k \circ \sigma$ induces the exact same Kodaira embedding if $\sigma \in (S^1)^n < G(H)$. So it suffices to consider $\sigma \in H$; then, since H preserves $P \cap M/k$, one obtains the same Kodaira embedding up to permutation of the coordinates in $\mathbb{P}^{E_P(k)-1}$. Either way, one obtains the same pullback of the Fubini–Study metric as can be from the definition of the Fubini–Study metric or directly from (40), ie $\sigma.h_k = h_k$.

4 An algebraic $\alpha_{k,G}$-invariant and a Demailly type result

4.1 Analytic definition

Analogous to the classical α_G-invariant, Tian [36, page 128] defined the $\alpha_{k,G}$-invariant. For this, one restricts to a G-invariant subset of $\mathcal{H}_k$. To write the subset explicitly in terms of global Kähler potentials it is necessary to choose a continuous G-invariant Hermitian metric h on $-kK_X$:

(43) $$\mathcal{H}_k^G(h) := \Big\{ \varphi = \frac{1}{k} \log \sum_i |s_i|_h^2 : \varphi \text{ is } G\text{-invariant and } \{s_i\} \text{ is a basis of } H^0(X, -kK_X) \Big\} \subset C^\infty(X).$$

Definition 4.1 Let $G \subseteq \mathrm{Aut}\, X$ be a compact subgroup. Let h be a fixed continuous G-invariant Hermitian metric on $-kK_X$, and μ a fixed continuous volume form on X. Then

$$\alpha_{k,G}(h,\mu) := \sup\Big\{ c > 0 : \sup_{\varphi \in \mathcal{H}_k^G} \int_X e^{-c(\varphi - \sup \varphi)}\, d\mu < \infty \Big\}.$$

Lemma 4.2 *Definition 4.1 does not depend on the choice of h or μ.*

For this reason we will simply denote the invariants by $\alpha_{k,G}$ from now on.

Proof Since X is compact, any two continuous volume forms are uniformly bounded and hence define the same L^1 spaces. Next, given two continuous G-invariant Hermitian metrics h and $\tilde{h}$ on $-kK_X$ there is an isomorphism from $\mathcal{H}_k^G(h)$ to $\mathcal{H}_k^G(\tilde{h})$ given by $\varphi \mapsto \varphi + (1/k)\log(\tilde{h}/h)$. Observe that $(1/k)\log(\tilde{h}/h)$ is (again by compactness of X) a uniformly bounded function on X. Hence, for a fixed $c > 0$,

$$\sup_{\varphi \in \mathcal{H}_k^G(h)} \int_X e^{-c(\varphi - \sup\varphi)}\, d\mu < \infty \iff \sup_{\varphi \in \mathcal{H}_k^G(\tilde{h})} \int_X e^{-c(\varphi - \sup\varphi)}\, d\mu < \infty,$$

as desired. □

4.2 Algebraic definition

Consider a (complex) nonzero vector subspace V of $H^0(X, kL)$. Associated to it is the (not necessarily complete) linear system $|V| := \mathbb{P}V \subset |kL| := \mathbb{P}H^0(X, kL)$ [22, page 137].

Lemma 4.3 *Let V be vector subspace of $H^0(X,kL)$ of dimension $p>0$. For any basis $\nu_1,\dots,\nu_p\in H^0(X,kL)$ of V, the number*

$$\sup\left\{c>0:\left(\sum_{j=1}^{p}|\nu_j(z)|^2\right)^{-c} \text{ is locally integrable on } X\right\}$$

is the same.

Proof Let $\{\nu_1^{(\ell)},\dots,\nu_p^{(i)}\}$ for $\ell\in\{1,2\}$, be two bases for $V\in H^0(X,kL)$. Let $A\in \mathrm{GL}(p,\mathbb{C})$ be the change-of-basis matrix, ie $\nu_j^{(2)}=A_j^i\nu_i^{(1)}$. Observe that A^HA is a positive Hermitian matrix-valued on X, and denote its eigenvalues by $0<\lambda_1\le\dots\le\lambda_p$. Define $\nu^{(\ell)}(z):=(\nu_1^{(\ell)}(z),\dots,\nu_p^{(\ell)}(z))\in\mathbb{C}^p$ and $|\nu^{(\ell)}(z)|^2=\sum_{i=1}^p|\nu_i^{(\ell)}(z)|^2$. Then

$$\frac{|\nu^{(2)}(z)|^2}{|\nu^{(1)}(z)|^2}=\frac{|A\nu^{(1)}(z)|^2}{|\nu^{(1)}(z)|^2}\in[\lambda_1,\lambda_p].$$

So $|\nu^{(2)}|^2$ is locally integrable if and only if $|\nu^{(1)}|^2$ is. □

Thus define the log canonical threshold of the linear system $|V|$ by

(44) $$\mathrm{lct}|V|:=\sup\left\{c>0:\left(\sum_j|\nu_j(z)|^2\right)^{-c} \text{ is locally integrable on } X\right\}.$$

When $L=-K_X$ there is an $\mathrm{Aut}\,X$ action on $H^0(X,-kK_X)$ for every $k\in\mathbb{N}$. In [13, Theorem A.3, (A.1)] Demailly noted that then α_G-invariants (for compact subgroup $G\subset\mathrm{Aut}\,X$) can be algebraically computed as

(45) $$\alpha_G=\inf_{k\in\mathbb{N}}k\inf_{\substack{|V|\subset|-kK_X|\\ V^G=V\neq 0}}\mathrm{lct}|V|,$$

where

$$V^G:=\{v\in V: g.v\in V \text{ for all } g\in G\}.$$

Note that in (45) $V\neq 0$ ranges over all G-invariant vector subspaces of $H^0(X,-kK_X)$ (ie of any positive dimension). For example, $\mathrm{lct}|-kK_X|=\infty$. From the definition, if $V_1\subset V_2$ are two such subspaces it suffices to compute $\mathrm{lct}|V_1|$ since

(46) $$\mathrm{lct}|V_1|\le\mathrm{lct}|V_2|.$$

It is thus natural to define the algebraic counterpart of the $\alpha_{k,G}$-invariant as follows:

Definition 4.4 Let X be a Fano manifold and $G\subset\mathrm{Aut}\,X$ be a compact subgroup of the automorphism group. Define

(47) $$\mathrm{glct}_{k,G}:=k\inf_{\substack{|V|\subset|-kK_X|\\ V^G=V}}\mathrm{lct}|V|.$$

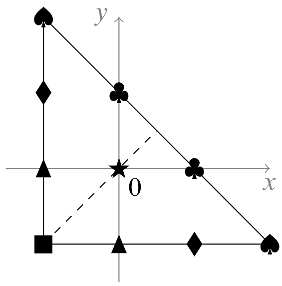


Figure 2: The six orbits $O_1^{(1)}, \dots, O_6^{(1)}$ of the action of the group generated by the reflection about $y = x$ on the polytope corresponding to $\mathbb{P}^2$ with $k = 1$.

4.3 Characterization of the equivariant Bergman spaces

First, we show that in the toric setting, the space $\mathcal{H}_k^{G(H)}$ consists of Kähler potentials induced by Kodaira embeddings of multiples of monomials sections, with the norming constants constant along orbits of H.

Denote by

$$O_1^{(k)}, \dots, O_N^{(k)}, \tag{48}$$

the orbits of H in $k^{-1}M \cap P$ (see Figure 2 for an example).

Lemma 4.5 *Let $\{s_{k,u}\}_{u \in k^{-1}M \cap P}$ be the monomial basis (34) of $H^0(X, -kL)$. Let $G(H) \subseteq \operatorname{Aut} X$ (5) be a subgroup generated by $(S^1)^n$ and a subgroup H of $\operatorname{Aut} P$. Then* (recall (43) and (48))

$$\mathcal{H}_k^{G(H)} := \mathcal{H}_k^{G(H)}(h_k) = \left\{ k^{-1} \log \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|_{h_k}^2 : \lambda_i > 0 \right\}.$$

Proof Recall the natural basis $\{s_{k,u}\}_{u \in P \cap M/k}$ defined by the lattice monomials (34). Given any other basis $\{s_i\}$ for $H^0(X, -kK_X)$, let $A \in \mathrm{GL}(E_P(k), \mathbb{C})$ be the change-of-basis matrix, so

$$s_i = A_i^u s_{k,u} \quad \text{for } i \in \{1, \dots, E_P(k)\}$$

(we use the Einstein summation convention).

Let $\varphi \in \mathcal{H}_k^{G(H)}$ and let $\{s_i\}$ be the associated basis (43). Expanding φ in terms of the monomials,

$$e^{k\varphi} = \sum_i |s_i|_{h_k}^2 = \sum_{u,u' \in P \cap M/k} \sum_{i,j=1}^{E_P(k)} A_i^u \overline{A_j^{u'}} \langle s_{k,u}, s_{k,u'} \rangle_{h_k} =: |s_{k,0}|_{h_k}^2 \sum_{u,u' \in P \cap M/k} c_{u,u'} z^{ku} \bar{z}^{ku'},$$

for some coefficients $\{c_{u,u'}\}_{u,u' \in P \cap M/k}$. We claim that $\{z^u \bar{z}^{u'}\}_{u,u' \in M}$ are linearly independent. To see that, suppose

$$f(z) = \sum_{u,u' \in M} c_{u,u'} z^u \bar{z}^{u'} = 0$$

with all but finitely many coefficients being 0. We may assume for any $c_{u,u'} \neq 0$ that u and u' lie in the positive orthant, by multiplying by $\prod_i |z_i|^2$ to some sufficiently large power. Then

$$c_{u,u'} = \frac{1}{u!u'!} \frac{\partial^{|u|+|u'|}}{\partial^u z \partial^{u'} \bar{z}}\bigg|_{z=0} f(z) = 0,$$

proving the claim. Now the subgroup $(S^1)^n < G(H)$ acts on the open orbit $(\mathbb{C}^*)^n$ by

$$(\beta_1, \dots, \beta_n).(z_1, \dots, z_n) = (e^{\sqrt{-1}\beta_1} z_1, \dots, e^{\sqrt{-1}\beta_n} z_n). \tag{49}$$

So by $(S^1)^n$-invariance and Lemma 3.1,

$$\beta.e^{k\varphi} = |s_{k,0}|^2_{h_k} \sum_{u,u' \in P \cap M/k} c_{u,u'} e^{\sqrt{-1}\langle \beta, k(u-u')\rangle} z^{ku} \bar{z}^{ku'} = |s_{k,0}|^2_{h_k} \sum_{u,u' \in P \cap M/k} c_{u,u'} z^{ku} \bar{z}^{ku'} = e^{k\varphi},$$

for any $\beta \in (S^1)^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. By comparing the coefficients and using the claim we just demonstrated, it follows that for $e^{k\varphi}$ to be $(S^1)^n$-invariant we must have $c_{u,u'} = 0$ whenever $u \neq u'$ (the converse is also true, of course).

Moreover, we also require $e^{k\varphi}$ to be invariant under the action of H, ie $c_{u,u} = c_{\sigma u, \sigma u}$ for any $\sigma \in H$. In conclusion, let $O_1^{(k)}, \dots, O_N^{(k)}$ be the orbits of the action of H on $P \cap M/k$. Then

$$e^{k\varphi} = \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k},$$

for some coefficients $\{\lambda_i\}_{i=1}^N$. Thus (43) simplifies to

$$\mathcal{H}_k^{G(H)}(h) = \left\{ \varphi = \frac{1}{k} \log \sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} : \lambda_i > 0 \right\},$$

as claimed. □

4.4 Replacing a basis of sections by an orbit of a section

The next result generalizes [31, Proposition 2.1] to the equivariant setting using Lemma 4.5.

Proposition 4.6 *For a subgroup $G(H) \subseteq \mathrm{Aut}\, X$ (5) generated by $(S^1)^n$ and a subgroup H of $\mathrm{Aut}\, P$* (recall (42) and Definition 4.1),

$$\alpha_{k,G(H)} = \sup\left\{ c > 0 : \int_X \left(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \right)^{-\frac{c}{k}} d\mu_k < \infty \text{ for all } u \in P \cap M/k \right\}. \tag{50}$$

Proof Step 1 (estimate a basis-type element using the worst orbit present in its expansion) Comparing (38) and (39) gives a useful expression for h_k (which is not strictly needed for the computation below, but makes it slightly easier to follow):

$$h_k = \Big(\sum_{u \in P \cap M/k} |s_{k,u}|^2 \Big)^{-1}. \tag{51}$$

Now, assuming $\lambda_N = \max_{i=1,\dots,N} \lambda_i$, following the notation of Lemma 4.5,

$$\begin{aligned} \sup \varphi \le \sup \Big(\frac{1}{k} \log \sum_{i=1}^{N} \lambda_N \sum_{u \in O_i^{(k)}} |s_{k,u}|_{h_k}^2 \Big) &= \sup \Big(\frac{1}{k} \log \lambda_N \sum_{i=1}^{N} \sum_{u \in O_i^{(k)}} |s_{k,u}|_{h_k}^2 \Big) \\ &= \sup \Big(\frac{1}{k} \log \lambda_N \sum_{u \in P \cap M/k} |s_{k,u}|_{h_k}^2 \Big) = \frac{1}{k} \log \lambda_N, \end{aligned}$$

since by (51),

$$|s|_{h_k}^2 = \frac{|s|^2}{\sum_{u \in P \cap M/k} |s_{k,u}|^2} \quad \text{for any } s \in H^0(X, -kK_X).$$

If $c > 0$ is such that for each $i \in \{1, \dots, N\}$

$$\int_X \Big(\sum_{u \in O_i^{(k)}} |s_{k,u}|_{h_k}^2 \Big)^{-\frac{c}{k}} d\mu_k \le C_c,$$

then for $\varphi \in \mathcal{H}_k^{G(H)}$ (using Lemma 4.5),

$$\begin{aligned} \int_X e^{-c(\varphi - \sup \varphi)} d\mu_k &= e^{c \sup \varphi} \int_X \Big(\sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|_{h_k}^2 \Big)^{-\frac{c}{k}} d\mu_k \\ &\le e^{(c/k) \log \lambda_N} \int_X \Big(\lambda_N \sum_{u \in O_N^{(k)}} |s_{k,u}|_{h_k}^2 \Big)^{-\frac{c}{k}} d\mu_k \\ &= \lambda_N^{c/k} \lambda_N^{-c/k} \int_X \Big(\sum_{u \in O_N^{(k)}} |s_{k,u}|_{h_k}^2 \Big)^{-\frac{c}{k}} d\mu_k \le C_c. \end{aligned}$$

Step 2 (estimate an orbit-type element using an approximation by degenerating basis-type elements) Conversely, assume $c > 0$ is such that for any $\varphi \in \mathcal{H}_k^{G(H)}$,

$$\int_X e^{-c(\varphi - \sup \varphi)} d\mu_k \le C_c.$$

Since for any $\ell \in \{1, \dots, N\}$

$$\Big(\sum_{u \in O_\ell^{(k)}} |s_{k,u}|_{h_k}^2 \Big)^{-\frac{c}{k}} = \lim_{\lambda_i \to 0, i \ne \ell} \Big(\sum_{i=1}^{N} \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|_{h_k}^2 \Big)^{-\frac{c}{k}},$$

where $\lambda_\ell = 1$, by Fatou's lemma [18, Lemma 2.18],

$$\int_X \Bigg(\sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h^k} \Bigg)^{-\frac{c}{k}} d\mu_k \le \liminf_{\lambda_i \to 0, i \neq \ell} e^{-c \sup \varphi_\lambda} e^{c \sup \varphi_\lambda} \int_X \Bigg(\sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \Bigg)^{-\frac{c}{k}} d\mu_k$$
$$= \Bigg(\sup \sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h_k} \Bigg)^{-\frac{c}{k}} \liminf_{\lambda_i \to 0, i \neq \ell} \int_X e^{-c(\varphi_\lambda - \sup \varphi_\lambda)} d\mu_k$$
$$\le \Bigg(\sup \sum_{u \in O_\ell^{(k)}} |s_{k,u}|^2_{h_k} \Bigg)^{-\frac{c}{k}} C_c,$$

where

$$\varphi_\lambda := \frac{1}{k} \log \sum_{i=1}^N \lambda_i \sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k}.$$

Thus we have shown that

$$\alpha_{k,G(H)} = \sup \Bigg\{ c > 0 : \int_X \Bigg(\sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \Bigg)^{-\frac{c}{k}} d\mu_k < \infty \text{ for all } i \in \{1, \dots, N\} \Bigg\}$$
$$= \sup \Bigg\{ c > 0 : \int_X \Bigg(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k} \Bigg)^{-\frac{c}{k}} d\mu_k < \infty \text{ for all } u \in P \cap M/k \Bigg\}, \tag{52}$$

proving (50). □

From (52):

Corollary 4.7 *Fix $k \in \mathbb{N}$ and let $O_1^{(k)}, \dots, O_N^{(k)}$ be the orbits of the action of H on $k^{-1}M \cap P$. Then*

$$\alpha_{k,G(H)} = \min_{1 \le i \le N} \sup \Bigg\{ c > 0 : \int_X \Bigg(\sum_{u \in O_i^{(k)}} |s_{k,u}|^2_{h_k} \Bigg)^{-\frac{c}{k}} d\mu_k < \infty \Bigg\}.$$

We can now answer affirmatively Problem 1.2 in our setting.

Proof of Proposition 1.3 Let $|V_{O_i^{(k)}}|$ be the linear system generated by $\{(s_{k,u}) : u \in O_i^{(k)}\}$ (recall (48)). By (44) and Corollary 4.7,

$$\alpha_{k,G(H)} = k \min_{i \in \{1,\dots,N\}} \operatorname{lct} |V_{O_i^{(k)}}|.$$

It remains to show

$$\operatorname{glct}_{k,G(H)} = k \min_{i \in \{1,\dots,N\}} \operatorname{lct} |V_{O_i^{(k)}}|.$$

By (46), it suffices in (47) to restrict to irreducible $G(H)$-invariant linear systems; see [27, page 146]. Now, any $(S^1)^n$-invariant linear system $|V|$ is spanned by monomials (see Lemma 4.8 below). By irreducibility, this means $|V|$ is spanned by the monomials coming from some H-orbit O_i, ie $|V| = |V_{O_i^{(k)}}|$. □

Lemma 4.8 *Let $V \subseteq H^0(X, -kK_X)$ be a complex vector subspace invariant under $(S^1)^n$. Then there exists a unique subset $\mathcal{F} \subset P \cap M/k$ such that*

$$V = \operatorname{span}\{s_{k,u}\}_{u\in\mathcal{F}}.$$

Proof It suffices to prove that any section in V is generated by monomials in V. That is, given any section

$$s = \sum_{u\in P\cap M/k} a_u s_{k,u} \in V,$$

if $a_u \neq 0$, then $s_{k,u} \in V$.

By $(S^1)^n$-invariance, for any $\beta \in (S^1)^n$ and $s \in V$, also $\beta.s \in V$ (recall (49)), that is

$$\beta.s = \sum_{u\in P\cap M/k} a_u \beta.s_{k,u} = \sum_{u\in P\cap M/k} a_u e^{\sqrt{-1}\langle\beta,ku\rangle} s_{k,u} \in V.$$

Thus for any $u_0 \in P \cap M/k$,

$$\begin{aligned}\int_{(S^1)^n} e^{-\sqrt{-1}\langle\beta,ku_0\rangle}\beta.s\,d\beta &= \sum_{u\in P\cap M/k} a_u s_{k,u}\int_{(S^1)^n} e^{\sqrt{-1}\langle\beta,ku-ku_0\rangle}\,d\beta \\ &= \sum_{u\in P\cap M/k} a_u s_{k,u}(2\pi)^n\delta_{u-u_0} = (2\pi)^n a_{u_0}s_{k,u_0} \in V.\end{aligned}$$

If $a_{u_0} \neq 0$, then $s_{k,u_0} \in V$. □

Corollary 4.9

$$\alpha_{G(H)} = \inf_k \alpha_{k,G(H)} = \sup\left\{c>0 : \int_X \left(\sum_{\sigma\in H}|s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu_k < \infty \text{ for all } u\in P\cap M/k,\ k\in\mathbb{N}\right\}. \tag{53}$$

Proof By Demailly's theorem (45) [13, (A.1)], (47), and Proposition 1.3,

$$\alpha_{G(H)} = \inf_{k\in\mathbb{N}} \operatorname{glct}_{k,G(H)} = \inf_{k\in\mathbb{N}} \alpha_{k,G(H)}.$$

By (50),

$$\begin{aligned}\alpha_{G(H)} &= \inf_{k\in\mathbb{N}} \sup\left\{c>0 : \int_X \left(\sum_{\sigma\in H}|s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu_k < \infty \text{ for all } u\in P\cap M/k\right\} \\ &= \sup\left\{c>0 : \int_X \left(\sum_{\sigma\in H}|s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu_k < \infty \text{ for all } u\in P\cap M/k, k\in\mathbb{N}\right\}.\end{aligned}$$ □

Remark 4.10 Here it would not have been enough to invoke Tian's theorem [36, Proposition 6.1], and it was necessary to invoke Demailly's theorem [13, (A.1)]. The reason is that the $\alpha_{k,G}$-invariants are defined by requiring certain integrals to be merely finite (ie bounded), but with a constant possibly depending on k. Tian's theorem says that $\mathcal{H}^G$ is approximated by $\mathcal{H}^G_k$, but in approximating $\varphi \in \mathcal{H}^G$ by a sequence $\varphi_k \in \mathcal{H}^G_k$ it might happen that the integrals $\int_X e^{-c(\varphi_k - \sup\varphi_k)}\omega^n$ blow up as k tends to infinity. Demailly precludes that from happening by using the Demailly–Kollár lower semicontinuity of complex singularity exponents.

5 The case of divisible k

We ultimately improve on the results of this section, but we include them since they serve to emphasize the difficulties that still need to be dealt with (see Remark 5.6).

Proposition 5.1 *Let $G(H) \subseteq \mathrm{Aut}\, X$ (5) be a subgroup generated by $(S^1)^n$ and a subgroup H of $\mathrm{Aut}\, P$. For $k, \ell \in \mathbb{N}$ we have $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)}$.*

Proof Since $h_{k\ell}$ and h_k^ℓ (39) are both smooth metrics on $-k\ell K_X$, by compactness of X they are equivalent. We also use the following lemma:

Lemma 5.2 *Let $a_1, \dots, a_N \geq 0$. Then for $\ell \in \mathbb{N}$,*

$$\sum_{i=1}^{N} a_i^\ell \leq \left(\sum_{i=1}^{N} a_i\right)^\ell \leq N^{\ell-1} \sum_{i=1}^{N} a_i^\ell.$$

Proof The first inequality follows by expanding $(\sum_{i=1}^N a_i)^\ell$. For the second inequality, consider the function $f(x) := x^\ell$ on $[0, +\infty)$. Since f is convex, by Jensen's inequality

$$f\left(\frac{1}{N}\sum_{i=1}^{N} a_i\right) \leq \frac{1}{N}\sum_{i=1}^{N} f(a_i),$$

ie $\left(\sum_{i=1}^N a_i\right)^\ell \leq N^{\ell-1} \sum_{i=1}^N a_i^\ell$. □

By Proposition 4.6, and since $M/k\ell \subset M/k$,

$$\begin{aligned}
\alpha_{k\ell,G(H)} &= \sup\left\{c > 0 : \int_X \left(\sum_{\sigma \in H} |s_{k\ell,\sigma u}|^2_{h_{k\ell}}\right)^{-\frac{c}{k\ell}} d\mu_k < \infty \text{ for all } u \in P \cap M/k\ell\right\} \\
&\leq \sup\left\{c > 0 : \int_X \left(\sum_{\sigma \in H} |s_{k\ell,\sigma u}|^2_{h_{k\ell}}\right)^{-\frac{c}{k\ell}} d\mu_k < \infty \text{ for all } u \in P \cap M/k\right\} \\
&= \sup\left\{c > 0 : \int_X \left(\sum_{\sigma \in H} |s^{\otimes \ell}_{k,\sigma u}|^2_{h_k^\ell}\right)^{-\frac{c}{k\ell}} d\mu_k < \infty \text{ for all } u \in P \cap M/k\right\} \\
&= \sup\left\{c > 0 : \int_X \left(\sum_{\sigma \in H} |s_{k,\sigma u}|^{2\ell}_{h_k}\right)^{-\frac{c}{k\ell}} d\mu_k < \infty \text{ for all } u \in P \cap M/k\right\} \\
&= \sup\left\{c > 0 : \int_X \left(\sum_{\sigma \in H} |s_{k,\sigma u}|^2_{h_k}\right)^{-\frac{c}{k}} d\mu_k < \infty \text{ for all } u \in P \cap M/k\right\} = \alpha_{k,G(H)},
\end{aligned}$$

where Lemma 5.2 was invoked in the penultimate equality. □

Proposition 5.3 *Let $G(H) \subseteq \mathrm{Aut}\, X$ (5) be a subgroup generated by $(S^1)^n$ and a subgroup H of $\mathrm{Aut}\, P$. There exists $k_0 \in \mathbb{N}$ such that $\alpha_{K,G(H)} = \alpha_{G(H)}$ for all K divisible by k_0.*

Remark 5.4 The proof will show that k_0 is determined by the fan as follows: let $u_0 \in P^H$ attain $\sup_{u\in P^H} \max_i \langle u, v_i\rangle$. Since P^H is a convex polytope, u_0 will be a vertex of P^H, ie cut out by P^H and the supporting hyperplanes of P containing u_0. The equations defining P^H are determined by $H \subset \mathrm{GL}(M)$, and hence are linear equations with integer coefficients. So are the equations cutting out ∂P. It follows that u_0 is a rational point, ie $u_0 \in M/k_0$ for some k_0. Let k_0 be the smallest such positive integer. The fact that u_0 is a rational point is originally due to Song [33, page 1257], who proved that $\alpha_{k_0,G(\mathrm{Aut}\,P)} = \alpha_{G(\mathrm{Aut}\,P)}$.

Proof We can further simplify (53). Recall (7). By the geometric–arithmetic mean inequality,

$$\Big(\frac{1}{|H|}\sum_{\sigma\in H}|s_{k,\sigma u}|^2_{h_k}\Big)^{-\frac{\alpha}{k}} \le \prod_{\sigma\in H}|s_{k,\sigma u}|_{h_k}^{-2\alpha/(|H|k)} = |s_{|H|k,\pi_H(u)}|_{h_{|H|k}}^{-2\alpha/(|H|k)} = \Big(\frac{1}{|H|}\sum_{\sigma\in H}|s_{|H|k,\sigma\pi_H(u)}|^2_{h_{|H|k}}\Big)^{-\alpha/(|H|k)},$$

with the last equality since $\pi_H(u)$ is fixed by H (recall (6)–(7)) so each term in the sum is identical. Therefore the supremum in (53) is unchanged if restricted to those u fixed by H. Thus, using Lemma 5.5 (proven below), (53) simplifies to (recall (6))

$$\begin{aligned}\alpha_{G(H)} &= \sup\Big\{c>0 : \int_X |s_{k,u}|_{h_k}^{-2c/k}\,d\mu_k < \infty \text{ for all } u\in P^H\cap \frac{1}{k}M,\ k\in\mathbb{N}\Big\}\\ &= \inf\Big\{k\,\mathrm{lct}(s_{k,u}) : u\in P^H\cap\frac{1}{k}M,\ k\in\mathbb{N}\Big\} = \inf_{k\in\mathbb{N}}\ \inf_{u\in P^H\cap(1/k)M}\frac{1}{\max_i\langle u,v_i\rangle+1}\\ &= \inf_{u\in P^H}\frac{1}{\max_i\langle u,v_i\rangle+1} = \frac{1}{\max_i\langle u_0,v_i\rangle+1},\end{aligned}$$

where we used the notation of Remark 5.4. By that same remark, $u_0 \in P^H \cap (1/K)M$ whenever $K\in\mathbb{N}$ is divisible by k_0. In particular, by (50), and using Lemma 5.5 again,

$$\alpha_{K,G(H)} \le \sup\Big\{c>0 : \int_X |s_{K,u_0}|_{h_K}^{-2c/K}\,d\mu_k<\infty\Big\} = K\,\mathrm{lct}(s_{K,u_0}) = \min_i \frac{1}{\langle u_0,v_i\rangle+1} = \alpha_{G(H)}.$$

Since by definition $\alpha_{G(H)} \le \alpha_{K,G(H)}$, equality is achieved and $\alpha_{K,G(H)} = \alpha_{G(H)}$ for all K divisible by k_0. □

Lemma 5.5 *For $u \in P\cap k^{-1}M$, $\mathrm{lct}(s_{k,u}) = (1/k)(1/(1+\max_i\langle u,v_i\rangle))$ (recall (21) and (34)).*

Proof This is well known; see eg [6, Corollary 7.4; 27, Theorem 4.1]. It is also a consequence of Proposition 6.8 proven below (put $\mathcal{F} = \{u\}$). □

Remark 5.6 According to Proposition 5.3, the sequence $\{\alpha_{k,G(H)}\}_{k\in\mathbb{N}}$ is constant (equal to $\alpha_{G(H)}$) along the subsequence $k_0, 2k_0, \ldots$. Proposition 5.3 does not yield information for all $k\in\mathbb{N}$ unless $k_0 = 1$, and examples show (see Section 8) that oftentimes $k_0 > 1$. Moreover, even though the sequence also satisfies $\alpha_{k,G(H)} \geq \alpha_{k\ell,G(H)} \geq \alpha_{G(H)}$ for any $k,\ell \in \mathbb{N}$ (Proposition 5.1), these facts combined are still not enough to conclude Tian's conjecture without further work. For instance, the sequence

$$a_k := \begin{cases} \alpha_{G(H)} & \text{if } k \text{ is even,} \\ \alpha_{G(H)} + k^{-1} & \text{if } k \text{ is odd,} \end{cases}$$

satisfies these requirements for $k_0 = 2$ (and also satisfy $\lim_k a_k = \alpha_{G(H)}$). Perhaps a more natural sequence that satisfies all the requirements and even for the k_0 of Remark 5.4 is

$$a_k := \inf\Big\{k \operatorname{lct}(s_{k,u}) : u \in P^H \cap \frac{1}{k}M\Big\},$$

with the proof of Proposition 5.3 showing that $a_{k_0\ell} = a_{k_0} = \alpha_{G(H)}$ for $\ell \in \mathbb{N}$ but possibly $a_k > \alpha_{G(H)}$ for k not divisible by k_0. Thus we are led to develop more refined estimates that are the topic of the next section.

6 Estimating singularities associated to orbits

6.1 Real singularity exponents and support functions

In this subsection we develop a key new technical estimate that expresses real singularity exponents associated to collections of toric monomials in terms of support functions.

Definition 6.1 For nonempty finite sets $\mathcal{F},\mathcal{U} \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}$,

$$c_k(\mathcal{F},\mathcal{U}) := \sup\bigg\{c \in (0,1) : \int_{\mathbb{R}^n} \frac{\big(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\big)^{-c/k}}{\big(\sum_{u\in\mathcal{U}} e^{\langle ku,x\rangle}\big)^{(1-c)/k}}\,dx < \infty\bigg\}.$$

Proposition 6.2 *For $\mathcal{F},\mathcal{U} \subseteq \mathbb{R}^n$ nonempty finite sets with $0 \in \operatorname{int}\operatorname{co}\mathcal{U}$ (recall (11)),*

$$c_k(\mathcal{F},\mathcal{U}) = \sup\{c \in (0,1) : 0 \in (1-c)\operatorname{co}\mathcal{U} + c\operatorname{co}\mathcal{F}\} = \sup\Big\{c \in (0,1) : \Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big) \cap \operatorname{co}\mathcal{U} \neq \varnothing\Big\} > 0.$$

In particular, $c_k(\mathcal{F},\mathcal{U})$ is independent of k.

Proof Using polar coordinates $x = r\nu$ and $dx = r^{n-1}\,dr \wedge dS(\nu)$ with $r \in \mathbb{R}_+$ and $\nu \in S^{n-1}(1) := \{x \in \mathbb{R}^n : |x| = 1\}$,

$$\int_{\mathbb{R}^n} \frac{\big(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\big)^{-c/k}}{\big(\sum_{u\in\mathcal{U}} e^{\langle ku,x\rangle}\big)^{(1-c)/k}}\,dx = \int_{S^{n-1}(1)} f_c(\nu)\,dS(\nu), \tag{54}$$

where $f_c \colon S^{n-1}(1) \to \mathbb{R}_+$ is defined by

$$f_c(\nu) := \int_0^\infty \frac{\big(\sum_{u\in\mathcal{F}} e^{kr\langle u,\nu\rangle}\big)^{-c/k}}{\big(\sum_{u\in\mathcal{U}} e^{kr\langle u,\nu\rangle}\big)^{(1-c)/k}} r^{n-1}\,dr.$$

Notice that for any finite set $\mathcal{A} \subset \mathbb{R}^n$ and $r \in \mathbb{R}_+$ (recall (13)),

$$e^{krh_{\mathcal{A}}(\nu)} \leq \sum_{u\in\mathcal{A}} e^{kr\langle u,\nu\rangle} \leq |\mathcal{A}| e^{krh_{\mathcal{A}}(\nu)}.$$

Hence for $c \in (0,1)$,

(55) $$\frac{|\mathcal{F}|^{-c/k}}{|\mathcal{U}|^{(1-c)/k}} e^{-krg_c(\nu)} \le \frac{\left(\sum_{u\in\mathcal{F}} e^{kr\langle u,\nu\rangle}\right)^{-c/k}}{\left(\sum_{u\in\mathcal{U}} e^{kr\langle u,\nu\rangle}\right)^{(1-c)/k}} \le e^{-krg_c(\nu)},$$

where

(56) $$g_c(x) := c\max_{u\in\mathcal{F}}\langle u,x\rangle + (1-c)\max_{u\in\mathcal{U}}\langle u,x\rangle = h_{c\mathcal{F}+(1-c)\mathcal{U}}(x) = c\max_{u\in\operatorname{co}\mathcal{F}}\langle u,x\rangle + (1-c)\max_{u\in\operatorname{co}\mathcal{U}}\langle u,x\rangle$$
$$= ch_{\operatorname{co}\mathcal{F}}(x) + (1-c)h_{\operatorname{co}\mathcal{U}}(x) = h_{c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U}}(x),$$

and where $A+B := \{x+y : x\in A,\ y\in B\}$ is the Minkowski sum.

We need the following property of support functions:

Claim 6.3 *Let $\mathcal{A}\subset\mathbb{R}^n$ be a nonempty finite set. Then $h_{\mathcal{A}}|_{S^{n-1}(1)} > 0$ if and only if $0\in\operatorname{int}\operatorname{co}\mathcal{A}$.*

Proof Note first that

(57) $$h_{\mathcal{A}} = h_{\operatorname{co}\mathcal{A}}:$$

$h_{\mathcal{A}} \le h_{\operatorname{co}\mathcal{A}}$ since $\mathcal{A}\subset\operatorname{co}\mathcal{A}$, while if $h_{\operatorname{co}\mathcal{A}}(y) = \langle a(y), y\rangle$ for some $a(y)\in\operatorname{co}\mathcal{A}$, and writing $a(y) = \sum_{i=1}^{|\mathcal{A}|}\lambda_i a_i$ with $\sum_{i=1}^{|\mathcal{A}|}\lambda_i = 1$ and $\lambda_i\ge 0$ where $\mathcal{A} = \{a_i\}$, then

$$h_{\operatorname{co}\mathcal{A}}(y) = \sum_{i=1}^{|\mathcal{A}|}\lambda_i\langle a_i, y\rangle \le \sum_{i=1}^{|\mathcal{A}|}\lambda_i h_{\mathcal{A}}(y) = h_{\mathcal{A}}(y).$$

Next, if $0\in\operatorname{int}\operatorname{co}\mathcal{A}$ then $\epsilon B_2^n\subset\operatorname{co}\mathcal{A}$ where $B_2^n := \{x\in\mathbb{R}^n : |x|\le 1\}$ and $h_{\operatorname{co}\mathcal{A}} \ge h_{\epsilon B_2^n} = \epsilon h_{B_2^n} = \epsilon$ when restricted to $S^{n-1}(1)$.

Conversely, if $0\notin\operatorname{int}\operatorname{co}\mathcal{A}$ then by convexity of $\operatorname{co}\mathcal{A}$ there exists a hyperplane H passing through a boundary point of $\operatorname{co}\mathcal{A}$ so that $\operatorname{co}\mathcal{A}$ lies on one side of it and 0 lies on the other side of it. If $\nu\in S^{n-1}(1)$ is normal to H and points toward the side not containing $\operatorname{co}\mathcal{A}$ then $h_{\operatorname{co}\mathcal{A}}(\nu)\le 0$. □

Claim 6.4 *Let $\mathcal{A}\subset\mathbb{R}^n$ be a nonempty finite set. Then $h_{\mathcal{A}}|_{S^{n-1}(1)}$ is somewhere negative if and only if $0\in\operatorname{int}(\mathbb{R}^n\setminus\operatorname{co}\mathcal{A}) = \mathbb{R}^n\setminus\operatorname{co}\mathcal{A}$.*

Proof Suppose $0\notin\operatorname{int}(\mathbb{R}^n\setminus\operatorname{co}\mathcal{A}) = \mathbb{R}^n\setminus\operatorname{co}\mathcal{A}$. Equivalently $0\in\operatorname{co}\mathcal{A}$. Then $h_{\mathcal{A}}\ge h_{\{0\}} = 0$, so $h_{\mathcal{A}}$ is nowhere negative.

Conversely, if $0\notin\operatorname{co}\mathcal{A}$ then by convexity of $\operatorname{co}\mathcal{A}$ there exists a hyperplane H passing through the origin so that $\operatorname{co}\mathcal{A}$ lies *strictly* on one side. Let $\nu\in S^{n-1}(1)$ be a unit normal to H that points toward the side not containing $\operatorname{co}\mathcal{A}$. By the strictness mentioned above and compactness of $\operatorname{co}\mathcal{A}$, for some small $\epsilon>0$, $\operatorname{co}\mathcal{A}+\epsilon\nu$ still lies on the same side of H as $\operatorname{co}\mathcal{A}$. Hence, as in the proof of Claim 6.3, $h_{\operatorname{co}\mathcal{A}+\epsilon\nu}(\nu)\le 0$. In other words,

$$h_{\operatorname{co}\mathcal{A}}(\nu) = h_{\operatorname{co}\mathcal{A}+\epsilon\nu}(\nu) - \epsilon|\nu|^2 \le -\epsilon,$$

as claimed. □

Corollary 6.5 *For $\mathcal{F},\mathcal{U}\subseteq\mathbb{R}^n$ nonempty finite sets with $0\in\operatorname{int}\operatorname{co}\mathcal{U}$ (recall (11)),*

$$\sup\Big\{c\in(0,1):\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\cap\operatorname{co}\mathcal{U}\neq\varnothing\Big\}=\begin{cases}1 & \text{if } 0\in\operatorname{co}\mathcal{F},\\ [1-\min_{S^{n-1}(1)}(h_{\mathcal{F}}/h_{\mathcal{U}})]^{-1} & \text{if } 0\notin\operatorname{co}\mathcal{F}.\end{cases}\tag{58}$$

Proof By assumption $0\in\operatorname{int}\operatorname{co}\mathcal{U}$, so by Claim 6.3, $h_{\mathcal{U}}|_{S^{n-1}(1)}>0$. By (56),

$$g_c=h_{\operatorname{co}\mathcal{U}}\Big(1-c\Big(1-\frac{h_{\operatorname{co}\mathcal{F}}}{h_{\operatorname{co}\mathcal{U}}}\Big)\Big).\tag{59}$$

Case 1 ($0\in\operatorname{co}\mathcal{F}$) If $0\in\operatorname{co}\mathcal{F}$, ie $h_{\operatorname{co}\mathcal{F}}\geq 0$, then $g_c\geq h_{\operatorname{co}\mathcal{U}}(1-c)>0$ for any $c\in(0,1)$. By (59) and Claim 6.3, $0\in\operatorname{int}(c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U})\subseteq c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U}$ and so (58) holds in this case.

Case 2 ($0\notin\operatorname{co}\mathcal{F}$) By assumption and Claim 6.4, $h_{\operatorname{co}\mathcal{F}}<0$ somewhere; by continuity/compactness let ν_0 be a minimizer of the function

$$\frac{h_{\operatorname{co}\mathcal{F}}}{h_{\operatorname{co}\mathcal{U}}}$$

on $S^{n-1}(1)$. In particular,

$$\frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)}=\min_{S^{n-1}(1)}\frac{h_{\mathcal{F}}}{h_{\mathcal{U}}}<0.$$

Set

$$c(\mathcal{F},\mathcal{U}):=\Big[1-\frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)}\Big]^{-1}=\Big[1-\min_{S^{n-1}(1)}\frac{h_{\mathcal{F}}}{h_{\mathcal{U}}}\Big]^{-1}.$$

First, if $c\in(0,c(\mathcal{F},\mathcal{U}))$, for all $\nu\in S^{n-1}(1)$,

$$g_c(\nu)=h_{\operatorname{co}\mathcal{U}}(\nu)\Big(1-c\Big(1-\frac{h_{\operatorname{co}\mathcal{F}}(\nu)}{h_{\operatorname{co}\mathcal{U}}(\nu)}\Big)\Big)\geq h_{\operatorname{co}\mathcal{U}}(\nu)\Big(1-c\Big(1-\frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)}\Big)\Big)=h_{\operatorname{co}\mathcal{U}}(\nu)\Big(1-\frac{c}{c(\mathcal{F},\mathcal{U})}\Big)>0.$$

Thus, by (56) and Claim 6.3, $0\in\operatorname{int}(c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U})\subseteq c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U}$; in particular,

$$\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\cap\operatorname{co}\mathcal{U}\neq\varnothing.$$

So

$$\sup\Big\{c\in(0,1):\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\cap\operatorname{co}\mathcal{U}\neq\varnothing\Big\}\geq c(\mathcal{F},\mathcal{U}).$$

Second, if $c\in(c(\mathcal{F},\mathcal{U}),1)$,

$$g_c(\nu_0)=h_{\operatorname{co}\mathcal{U}}(\nu_0)\Big(1-c\Big(1-\frac{h_{\operatorname{co}\mathcal{F}}(\nu_0)}{h_{\operatorname{co}\mathcal{U}}(\nu_0)}\Big)\Big)=h_{\operatorname{co}\mathcal{U}}(\nu_0)\Big(1-\frac{c}{c(\mathcal{F},\mathcal{U})}\Big)<0.$$

Thus, by Claim 6.4, $0\in\mathbb{R}^n\setminus(c\operatorname{co}\mathcal{F}+(1-c)\operatorname{co}\mathcal{U})$, equivalently,

$$\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\cap\operatorname{co}\mathcal{U}=\varnothing,$$

and

$$\sup\Big\{c\in(0,1):\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\cap\operatorname{co}\mathcal{U}\neq\varnothing\Big\}\leq c(\mathcal{F},\mathcal{U}).$$

Thus also in this case (58) holds. □

The next corollary follows from the proof of Corollary 6.5:

Corollary 6.6 *For $\mathcal{F},\mathcal{U}\subseteq\mathbb{R}^n$ nonempty finite sets with $0\in\operatorname{int}\operatorname{co}\mathcal{U}$ (recall (11)), let*

$$c(\mathcal{F},\mathcal{U}):=\sup\Big\{c\in(0,1):\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\cap\operatorname{co}\mathcal{U}\neq\varnothing\Big\}=\sup\{c\in(0,1):0\in(1-c)\operatorname{co}\mathcal{U}+c\operatorname{co}\mathcal{F}\}.$$

Recall (56). If $c<c(\mathcal{F},\mathcal{U})$, then $g_c>0$; if $c>c(\mathcal{F},\mathcal{U})$, then $g_c<0$ somewhere.

We can now complete the proof of Proposition 6.2. Set

$$c(\mathcal{F},\mathcal{U}):=\sup\Big\{c\in(0,1):\Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big)\cap\operatorname{co}\mathcal{U}\neq\varnothing\Big\}.$$

By assumption $0\in\operatorname{int}\operatorname{co}\mathcal{U}$, thus $c(\mathcal{F},\mathcal{U})>0$. We consider two cases.

Case 1 $(c\in(0,c(\mathcal{F},\mathcal{U})))$ By the proof of Corollary 6.5, $g_c|_{S^{n-1}(1)}>0$. By continuity/compactness it attains its minimum $g_c(\hat{\nu})>0$. Therefore

$$f_c(\nu)\le\int_0^\infty e^{-krg_c(\nu)}r^{n-1}\,dr\le\int_0^\infty e^{-krg_c(\hat{\nu})}r^{n-1}\,dr<\infty, \tag{60}$$

so by (54), $c_k(\mathcal{F},\mathcal{U})\ge c$, ie $c_k(\mathcal{F},\mathcal{U})\ge c(\mathcal{F},\mathcal{U})$.

Case 2 $(c\in(c(\mathcal{F},\mathcal{U}),1))$ By the proof of Corollary 6.5, $g_c(\nu_0)<0$ for some open neighborhood $U\subset S^{n-1}(1)$. Thus, by (55), for some constant $C=C(c,k,\mathcal{F},\mathcal{U})>0$,

$$f_c(\nu)\ge C\int_0^\infty e^{-krg_c(\nu)}r^{n-1}\,dr=\infty\quad\text{for }\nu\in U,$$

so $c_k(\mathcal{F},\mathcal{U})\le c$, ie $c_k(\mathcal{F},\mathcal{U})\le c(\mathcal{F},\mathcal{U})$.

In conclusion, $c_k(\mathcal{F},\mathcal{U})=c(\mathcal{F},\mathcal{U})$ and Proposition 6.2 is proved. □

6.2 Complex singularity exponents

Now we go back to our setting of estimating complex singularity exponents, where P was a reflexive polytope coming from a fan. For any $k\in\mathbb{N}$ and a nonempty subset $\mathcal{F}\subseteq P\cap M/k$ set

$$c_k(\mathcal{F}):=\sup\Big\{c\in(0,1):\int_X\Big(\sum_{u\in\mathcal{F}}|s_{k,u}|^2_{h_k}\Big)^{-\frac{c}{k}}\,d\mu_k<\infty\Big\}. \tag{61}$$

Remark 6.7 Consider

$$\tilde{c}_k(\mathcal{F}):=\sup\Big\{c\in(0,\infty):\int_X\Big(\sum_{u\in\mathcal{F}}|s_{k,u}|^2_{h_k}\Big)^{-\frac{c}{k}}\,d\mu_k<\infty\Big\}.$$

Then, by definition (44), $\tilde{c}_k(\mathcal{F})=k\operatorname{lct}|\operatorname{span}\{s_{k,u}\}_{u\in\mathcal{F}}|$. However, Proposition 6.2 would not be applicable then. The point is that for the proof of Tian's conjecture we need to compute equivariant *global* log canonical thresholds and there will always be "admissible" invariant linear series for which k times the log canonical threshold is at most 1 (ie the "worst" ones will not have $\tilde{c}_k>1$), ie for which $c_k=\tilde{c}_k$. This is essentially because our H are linear groups so $\{0\}$ is always an H-orbit and $\operatorname{lct}(s_{k,0})=1/k=c_k(\{0\})=\tilde{c}_k(\{0\})$ (by Lemma 5.5).

Proposition 6.8 *For any $k \in \mathbb{N}$ and a nonempty subset $\mathcal{F} \subseteq P \cap M/k$,*

$$c_k(\mathcal{F}) = \sup\Big\{c \in (0,1) : P \cap \Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big) \neq \varnothing\Big\} > 0.$$

In particular, it is independent of k.

Proof By (42),

$$\int_X \Big(\sum_{u\in\mathcal{F}} |s_{k,u}|^2_{h_k}\Big)^{-\frac{c}{k}} d\mu_k = (2\pi)^n \int_{\mathbb{R}^n} \frac{\big(\sum_{u\in\mathcal{F}} e^{\langle ku,x\rangle}\big)^{-c/k}}{\big(\sum_{u\in P\cap M/k} e^{\langle ku,x\rangle}\big)^{(1-c)/k}} dx.$$

By Proposition 6.2 with $\mathcal{U} = P \cap M/k$,

$$c_k(\mathcal{F}) = \sup\Big\{c \in (0,1) : P \cap \Big(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Big) \neq \varnothing\Big\} > 0,$$

since $\operatorname{co}\mathcal{U} = P$ (as the vertices of P belong to M/k for all $k \in \mathbb{N}$). □

Remark 6.9 Although not needed for our analysis, it might be of independent interest to understand whether the supremum in (61) — and in the definition of $\alpha_{k,G(H)}$ (50) — is attained. In other words, is (recall the notation of Corollary 6.6)

$$\int_{S^{n-1}(1)} f_c(\nu)\, dS(\nu) \tag{62}$$

finite for $c = c(\mathcal{F},\mathcal{U})$? By (54)–(55) it is equivalent to consider this question for the integral

$$\int_{S^{n-1}(1)} \int_0^\infty e^{-krg_c(\nu)} r^{n-1}\, dr\, dS(\nu) = \int_{\mathbb{R}^n} e^{-kh_{(1-c)\operatorname{co}\mathcal{U}+c\operatorname{co}\mathcal{F}}(x)}\, dx. \tag{63}$$

A classical formula in convex geometry says that whenever $K \subset \mathbb{R}^n$ is a compact convex set with the origin in its interior, then $n!|K^\circ| = \int_{\mathbb{R}^n} e^{-h_K(y)}\, dy$; for a detailed proof see [4, (4.2)]. In fact, it is possible to show that when $0 \in \partial K$ the formula still holds in the sense that both sides are equal to ∞ (this is related to, but stronger than, the classical fact that $0 \in \operatorname{int} K$ if and only if K° is bounded [29, Corollary 14.5.1]). To summarize, both (62) and (63) are infinite when $c = c(\mathcal{F},\mathcal{U})$. We remark that one can also use this point of view to give an alternative, but equivalent, proof of Corollary 6.6.

6.3 Group orbits and singularities

Consider the orbit-averaging map π_H (7), taking a point to the average of its image under H. In particular, points on the same orbit have the same image. Let $M_{\mathbb{R}}^H$ be the subspace of fixed points of H in $M_{\mathbb{R}}$. Then $\pi_H|_{M_{\mathbb{R}}^H} = \mathrm{id}_{M_{\mathbb{R}}^H}$, and $M_{\mathbb{R}}^H$ is the image of π_H. Note that unless H is trivial, this is a proper subspace of $\mathbb{M}_{\mathbb{R}}$, so π_H is a projection from $M_{\mathbb{R}}$ to the invariant subspace $M_{\mathbb{R}}^H$, justifying the notation.

Recalling (6),

$$P^H := P \cap M_{\mathbb{R}}^H.$$

Since P is convex, $\pi_H(P) \subseteq P$. Hence $\pi_H(P) \subseteq P^H$. On the other hand, $P^H = \pi_H(P^H) \subseteq \pi_H(P)$. This implies

$$P^H = \pi_H(P). \tag{64}$$

Then (recall (24))

$$P^H = \pi_H(P) = \pi_H(\mathrm{co}(\mathrm{Ver}\, P)) = \mathrm{co}(\pi_H(\mathrm{Ver}\, P)).$$

In particular,

$$\mathrm{Ver}\, P^H \subseteq \pi_H(\mathrm{Ver}\, P). \tag{65}$$

Note that equality does not hold in general (for instance, consider $P = [-1,1]^2$ and H the reflection group about a diagonal).

Lemma 6.10 *Fix $u_0 \in M_{\mathbb{R}}^H$, and let $\mathcal{F}$ be a nonempty H-invariant convex subset of some fiber $\pi_H^{-1}(u_0)$ of π_H. Then $\mathcal{F} \cap P \neq \varnothing$ if and only if $u_0 \in P$.*

Proof If $\mathcal{F} \cap P \neq \varnothing$, then $\pi_H(\mathcal{F} \cap P) \neq \varnothing$. By (64), $\pi_H(\mathcal{F} \cap P) \subset \pi_H(P) = P^H \subset P$, and since $\varnothing \neq \mathcal{F} \subset \pi_H^{-1}(u_0)$, we have $\pi_H(\mathcal{F} \cap P) \subset \pi_H(\mathcal{F}) = \{u_0\}$. Thus $\pi_H(\mathcal{F} \cap P) \subset \{u_0\} \cap P$, and for this to be nonempty we must have $u_0 \in P$.

For the converse, since $\mathcal{F}$ is H-invariant and convex, $\pi_H(\mathcal{F}) \subset \mathcal{F}$. Additionally, as before $\varnothing \neq \mathcal{F} \subset \pi_H^{-1}(u_0)$ implies $\pi_H(\mathcal{F}) = \{u_0\}$. Thus $u_0 \in \mathcal{F}$. Therefore if $u_0 \in P$ then actually $u_0 \in \mathcal{F} \cap P$. □

6.4 Proof of Tian's conjecture

We can now prove our main result:

Proof of Theorem 1.4 Fix $k \in \mathbb{N}$ and let $O_1^{(k)}, \dots, O_N^{(k)}$ be the orbits of the action of H on $P \cap M/k$. Recall that π_H maps an orbit to a singleton:

$$\{o_i^{(k)}\} := \pi_H(O_i^{(k)}).$$

Note that $O_i^{(k)} \subset \pi_H^{-1}(o_i^{(k)})$, and hence (as the action of H and hence also π_H is linear) also $\mathrm{co}\, O_i^{(k)} \subset \pi_H^{-1}(o_i^{(k)})$. Moreover, $\mathrm{co}\, O_i^{(k)}$ is convex and H-invariant. By Corollary 4.7, Proposition 6.8, and Lemma 6.10,

$$\begin{aligned}\alpha_{k,G(H)} &= \min_{1\le i\le N} c(O_i^{(k)}) = \min_{1\le i\le N} \sup\Big\{c \in (0,1) : \Big(-\frac{c}{1-c}\,\mathrm{co}\, O_i^{(k)}\Big) \cap P \neq \varnothing\Big\}\\ &= \min_{1\le i\le N} \sup\Big\{c \in (0,1) : -\frac{c}{1-c} o_i^{(k)} \in P\Big\} = \sup\Big\{c \in (0,1) : -\frac{c}{1-c}\{o_1^{(k)}, \dots, o_N^{(k)}\} \subset P\Big\}.\end{aligned}$$

Since

$$\{o_1^{(k)}, \dots, o_N^{(k)}\} = \pi_H\Big(\bigcup_{i=1}^N O_i^{(k)}\Big) = \pi_H(k^{-1}M \cap P),$$

we have

$$\begin{aligned}(66)\quad \alpha_{k,G(H)} &= \sup\Big\{c\in(0,1): -\frac{c}{1-c}\pi_H(k^{-1}M\cap P)\subset P\Big\}\\ &= \sup\Big\{c\in(0,1): -\frac{c}{1-c}\,\mathrm{co}(\pi_H(k^{-1}M\cap P))\subset P\Big\}\\ &= \sup\Big\{c\in(0,1): -\frac{c}{1-c}\pi_H(\mathrm{co}(k^{-1}M\cap P))\subset P\Big\} = \sup\Big\{c\in(0,1): -\frac{c}{1-c}\pi_H(P)\subset P\Big\}\\ &= \sup\Big\{c\in(0,1): -\frac{c}{1-c}P^H\subset P\Big\} = \sup\Big\{c\in(0,1): -\frac{c}{1-c}\,\mathrm{Ver}\,P^H\subset P\Big\}\\ &= \min_{u\in \mathrm{Ver}\,P^H}\sup\Big\{c\in(0,1): -\frac{c}{1-c}u\in P\Big\}.\end{aligned}$$

Also, by (65),

$$\begin{aligned}\alpha_{k,G(H)} &= \sup\Big\{c\in(0,1): -\frac{c}{1-c}\,\mathrm{Ver}\,P^H\subset P\Big\} \geq \sup\Big\{c\in(0,1): -\frac{c}{1-c}\pi_H(\mathrm{Ver}\,P)\subset P\Big\}\\ &\geq \sup\Big\{c\in(0,1): -\frac{c}{1-c}\pi_H(P)\subset P\Big\} = \alpha_{k,G(H)}.\end{aligned}$$

Therefore

$$(67)\quad \alpha_{k,G(H)} = \sup\Big\{c\in(0,1): -\frac{c}{1-c}\pi_H(\mathrm{Ver}\,P)\subset P\Big\} = \min_{u\in\pi_H(\mathrm{Ver}\,P)}\sup\Big\{c\in(0,1): -\frac{c}{1-c}u\in P\Big\}.$$

Recall (28)

$$P = \bigcap_{j=1}^{d}\{x\in M_{\mathbb{R}} : \langle x, -v_j\rangle \leq 1\}.$$

For any point u and $c\in(0,1)$,

$$-\frac{c}{1-c}u\in P \qquad \text{if and only if} \qquad c\leq \min_j \frac{1}{\langle u, v_j\rangle+1} \quad \text{for all } j\in\{1,\dots,d\}.$$

In sum, combining (66) and (67) implies (8). In particular, $\alpha_{k,G(H)}$ is independent of $k\in\mathbb{N}$ and, by Corollary 4.9, is equal to $\alpha_{G(H)}$. □

7 Grassmannian Tian invariants

The following definition is due to Tian [37, (6.1)]:

Definition 7.1 Let X be a Fano manifold and $G\subseteq \mathrm{Aut}\,X$ a compact subgroup. For $k,m\in\mathbb{N}$, define (recall (44))

$$\alpha_{k,m,G} := k \inf_{\substack{|V|\subset|-kK_X|\\ \dim V=m\\ V^G=V}} \mathrm{lct}|V|,$$

with the convention $\inf\varnothing=\infty$.

Equivalently, these invariants are the minimum of the function $V\mapsto k\,\mathrm{lct}|V|$ over the Grassmannian restricted to the subset of G-invariant m-dimensional subspaces of $H^0(X,-kK_X)$. This subset can be, in some situations, even a finite set (see Remark 7.9). By Remark 7.2 below, these invariants generalize the invariants $\alpha_{k,G}$. Note also that Definition 7.1 is an *algebraic* one. Also, $\alpha_{k,E_P(k),G} = \infty$ (recall (32)) so it is *not* equal to $\alpha_{k,G}$ from Definition 4.1.

In the toric setting it is natural to consider Conjecture 1.5 for the invariants $\alpha_{k,m,(S^1)^n}$. For these invariants we show that Conjecture 1.5 does not quite hold, and we give a precise breakdown of when it does in terms of a natural convex geometric criterion. In contrast with previous sections, we do not work with the additional symmetry corresponding to the groups $G(H)$ (5). The reason for that is that even in simple situations $\alpha_{k,m,G(H)}$ will not be well-behaved; see Example 8.6.

Remark 7.2 Putting $H = \{\mathrm{id}\}$ in Proposition 4.6, $\alpha_{k,(S^1)^n} = \alpha_{k,1,(S^1)^n}$. In this sense, $\alpha_{k,m,(S^1)^n}$ is a generalization of the $\alpha_{k,(S^1)^n}$-invariants.

First we show, as a rather immediate consequence of our work, an explicit formula for the invariants of Definition 7.1.

Corollary 7.3 *Let X be toric Fano with associated polytope P (28). For $k, m \in \mathbb{N}$ (recall (61)),*

$$\alpha_{k,m,(S^1)^n} = \min_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}|=m}} c_k(\mathcal{F}).$$

Proof Fix $k, m \in \mathbb{N}$. Let $V \subseteq H^0(X, -kK_X) = \mathrm{span}\{s_u\}_{u \in P \cap M/k}$ be a subspace invariant under $(S^1)^n$. By Lemma 4.8, there exists a unique $\mathcal{F} \subseteq P \cap M/k$ such that

$$V = \mathrm{span}\{s_u\}_{u \in \mathcal{F}}.$$

Note that $|\mathcal{F}| = \dim V$. By (44) and (61),

$$\alpha_{k,m,(S^1)^n} = k \min_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}|=m}} \mathrm{lct}|\mathrm{span}\{s_u\}_{u \in \mathcal{F}}| = \min_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}|=m}} c_k(\mathcal{F}). \qquad \square$$

Lemma 7.4 *Let $K \subset \mathbb{R}^n$ be a compact convex set with $0 \in \mathrm{int}\, K$. For any set $S \subset \mathbb{R}^n$,*

$$\inf_S \|\cdot\|_K = \inf\{\lambda \geq 0 : S \cap \lambda K \neq \varnothing\}.$$

Proof If $S \cap \lambda K \neq \varnothing$ for some $\lambda \geq 0$, pick $x \in S \cap \lambda K$. Since $x \in \lambda K$, by (14), $\|x\|_K \leq \lambda$. Therefore $\inf_{x \in S} \|x\|_K \leq \inf\{\lambda \geq 0 : S \cap \lambda K \neq \varnothing\}$.

On the other hand, for any $x \in S$ we have $S \cap \|x\|_K K \supseteq \{x\} \neq \varnothing$ by (16). Therefore $\inf_{x \in S} \|x\|_K \geq \inf\{\lambda \geq 0 : S \cap \lambda K \neq \varnothing\}$. $\square$

Proposition 6.8 can be reformulated in terms of near-norms:

Corollary 7.5 *For $k \in \mathbb{N}$ and a nonempty set $\mathcal{F} \subseteq P \cap M/k$ (recall (61)),*

$$c_k(\mathcal{F}) = \frac{1}{1 + \min_{\mathrm{co}\, \mathcal{F}} \|\cdot\|_{-P}}.$$

Proof By Proposition 6.8 and Lemma 7.4,

$$\begin{aligned} c_k(\mathcal{F}) &= \sup\Bigl\{c \in (0,1) : P \cap \Bigl(-\frac{c}{1-c}\operatorname{co}\mathcal{F}\Bigr) \neq \varnothing\Bigr\} = \sup\Bigl\{c \in (0,1) : \operatorname{co}\mathcal{F} \cap \frac{1-c}{c}\Bigl(-P\Bigr) \neq \varnothing\Bigr\} \\ &= \sup\Bigl\{\frac{1}{1+a} : a \in (0,\infty), \operatorname{co}\mathcal{F} \cap (-aP) \neq \varnothing\Bigr\} = [1 + \inf\{a \in (0,\infty) : \operatorname{co}\mathcal{F} \cap (-aP) \neq \varnothing\}]^{-1} \\ &= \bigl[1 + \inf_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\bigr]^{-1} = \bigl[1 + \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\bigr]^{-1}, \end{aligned}$$

using compactness of $\operatorname{co}\mathcal{F}$. □

Combining Corollaries 7.3 and 7.5 yields an explicit formula for $\alpha_{k,m,(S^1)^n}$.

Corollary 7.6 *Let X be toric Fano with associated polytope P (28). For $k, m \in \mathbb{N}$,*

$$\alpha_{k,m,(S^1)^n} = \bigl[1 + \max_{\substack{\mathcal{F} \subseteq P \cap M/k \\ |\mathcal{F}|=m}} \min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P}\bigr]^{-1}.$$

Remark 7.7 For $m = 1$, Corollary 7.6 reads

$$\alpha_{k,(S^1)^n} = \frac{1}{1+\max_{u \in P \cap M/k} \|u\|_{-P}}.$$

Being a near-norm, the function $\|\cdot\|_{-P}$ satisfies the triangle inequality and is positively 1-homogeneous; hence it is convex. By Lemma 2.3, the vertex set of P is in the integral lattice M, and hence also contained in M/k for all $k \in \mathbb{N}$. This combined with Lemma 2.4 gives

$$\alpha_{k,(S^1)^n} = \frac{1}{1+\max_{\operatorname{Ver} P} \|\cdot\|_{-P}} = \frac{1}{1+\max_P \|\cdot\|_{-P}}. \tag{68}$$

This is, of course, a special case of Theorem 1.4 (proved in Section 6), derived here in slightly different notation (using the near-norm instead of the support function). Thus

$$\alpha_{k,(S^1)^n} = \alpha_{(S^1)^n} = \alpha, \tag{69}$$

where the first equality follows from (68) and Demailly's result [13, (A.1)], and the last equality follows by comparing Theorem 1.4 with Blum and Jonsson's formula [6, (7.2)].

Proposition 7.8 *Let X be toric Fano with associated polytope P (28). For $k \in \mathbb{N}$ and $m' > m$,*

$$\alpha_{k,m',(S^1)^n} \geq \alpha_{k,m,(S^1)^n}. \tag{70}$$

In particular,

$$\alpha_{k,m,(S^1)^n} \geq \alpha_{k,1,(S^1)^n} = \alpha_{k,(S^1)^n} = \alpha.$$

Equality holds if and only if there is $\mathcal{F} \subseteq P \cap M/k$ with $|\mathcal{F}| = m$ satisfying

$$\|\cdot\|_{-P}|_{\operatorname{co}\mathcal{F}} = \max_P \|\cdot\|_{-P}. \tag{71}$$

Remark 7.9 For a general Fano X, it does not follow from Definition 7.1 that $\alpha_{k,m',G} \geq \alpha_{k,m,G}$, ie there may be no monotonicity in m. For instance, there might be no m-dimensional G-invariant subspaces. Thanks to Lemma 4.8 the situation for $G=(S^1)^n$ is particularly simple: there are finitely many m-dimensional $(S^1)^n$-invariant subspaces (in fact, the Grassmannian of m-dimensional subspaces has exactly

$$\binom{E_P(k)}{m}$$

fixed points by the $(S^1)^n$-action), and moreover every $(S^1)^n$-invariant subspace consists of 1-dimensional blocks.

Proof Pick $\mathcal{F}'\subseteq P\cap M/k$ that computes $\alpha_{k,m',(S^1)^n}$ (such a subset exists since $P\cap M/k$ is a finite set). That is, $|\mathcal{F}'|=m'$ and $\alpha_{k,m',(S^1)^n}=[1+\min_{\mathrm{co}\,\mathcal{F}'}\|\cdot\|_{-P}]^{-1}$ (recall Corollary 7.6). For $\mathcal{F}$ a subset of $\mathcal{F}'$ with $|\mathcal{F}|=m$, Corollary 7.6 implies

$$\alpha_{k,m,(S^1)^n}=\frac{1}{1+\max_{\substack{\mathcal{A}\subseteq P\cap M/k\\|\mathcal{A}|=m}}\min_{\mathrm{co}\,\mathcal{A}}\|\cdot\|_{-P}}\le\frac{1}{1+\min_{\mathrm{co}\,\mathcal{F}}\|\cdot\|_{-P}}\le\frac{1}{1+\min_{\mathrm{co}\,\mathcal{F}'}\|\cdot\|_{-P}}=\alpha_{k,m',(S^1)^n},$$

proving (70).

Next, suppose $\alpha_{k,m,(S^1)^n}=\alpha$, ie (recall (68)–(69))

$$\max_{\substack{\mathcal{F}\subseteq P\cap M/k\\|\mathcal{F}|=m}}\min_{\mathrm{co}\,\mathcal{F}}\|\cdot\|_{-P}=\max_P\|\cdot\|_{-P}. \tag{72}$$

For any $\mathcal{F}\subseteq P$, convexity of P (recall (28)) implies $\mathrm{co}\,\mathcal{F}\subseteq P$. Hence

$$\min_{\mathrm{co}\,\mathcal{F}}\|\cdot\|_{-P}\le\max_P\|\cdot\|_{-P}.$$

The equality (72) holds if and only if there is $\mathcal{F}\subseteq P\cap M/k$ such that $|\mathcal{F}|=m$ and

$$\min_{\mathrm{co}\,\mathcal{F}}\|\cdot\|_{-P}=\max_P\|\cdot\|_{-P},$$

which forces $\|\cdot\|_{-P}$ to be the constant $\max_P\|\cdot\|_{-P}$ on $\mathrm{co}\,\mathcal{F}$. □

7.1 Proof of Theorem 1.6 and the intuition behind it

Theorem 1.4 resolved Tian's classical stabilization Problem 1.1 in the affirmative. This corresponds to the case $m=1$ of Conjecture 1.5. Next we show that Conjecture 1.5 is only partially true for $m\ge 2$.

Proof of Theorem 1.6 Let $\mathcal{F}\subset P\cap M/k$ with $|\mathcal{F}|=m$. By convexity of P and as $m\ge 2$, $\mathrm{co}\,\mathcal{F}$ contains an interval in P. In particular, $\mathcal{F}$ contains a point of $P\setminus\mathrm{Ver}\,P$. Thus if $(*_P)$ holds then (71) does not hold (for any such $\mathcal{F}$), and Proposition 7.8 implies (9).

Next, suppose $(*_P)$ fails (and now we relax to $m\ge 1$). Let $\hat{x}\in P\setminus\mathrm{Ver}\,P$ achieve the maximum of $\|\cdot\|_{-P}$ on P. Express $\hat{x}$ as a convex combination of a *subset* of vertices of P, $\{p_1,\dots,p_\ell\}\subset\mathrm{Ver}\,P$, namely

$$\hat{x}=\sum_{i=1}^{\ell}\hat{\lambda}_i p_i\quad\text{where}\quad\sum_{i=1}^{\ell}\hat{\lambda}_i=1\text{ and }\hat{\lambda}\in(0,1)^{\ell}. \tag{73}$$

Note that necessarily

$$\ell > 1. \tag{74}$$

We claim that

$$\|x\|_{-P} = \max_{x\in P}\|\cdot\|_{-P} \quad \text{for any } x \in \mathrm{co}\{p_i\}_{i=1}^{\ell}.$$

Indeed, letting

$$P' := \mathrm{co}\{p_i\}_{i=1}^{\ell} \subset P,$$

the claim follows from $\hat{x} \in \operatorname{int} P' \subset P$ and Lemma 2.4(iii).

Now the lattice polytope $P' := \mathrm{co}\{p_i\}_{i=1}^{\ell}$ has positive dimension by (74). Equivalently, the Ehrhart polynomial of P' has degree at least 1. Hence, by Proposition 2.5, $|P' \cap M/k| \geq m$ for all sufficiently large k. Pick $\mathcal{F} \subseteq P' \cap M/k \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$. Then $\mathrm{co}\,\mathcal{F} \subseteq P'$. By Proposition 7.8, (10) follows. The case $m = 1$ was already treated in Theorem 1.4, but also from this proof we see that any $k \in \mathbb{N}$ works for $m = 1$. □

The proof of Theorem 1.6 and condition $(*_P)$ have a pleasing geometric interpretation. Geometrically, by Proposition 7.8, $\alpha_{k,m,(S^1)^n} = \alpha$ if and only if there is a subset $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and $\mathrm{co}\,\mathcal{F}$ lies entirely in the level set

$$\Big\{\|\cdot\|_{-P} = \max_P \|\cdot\|_{-P}\Big\} \subseteq \partial P. \tag{75}$$

Recall that any convex subset of the boundary of a polytope must lie in a single face. Hence $\mathcal{F}$ must be a subset of a face F of P. We may assume F is the minimal face that contains $\mathcal{F}$, ie $\mathcal{F}$ intersects the relative interior of F. In particular, $\|\cdot\|_{-P}|_F$ attains its maximum in the interior of F and so F has to lie entirely in the level set (75) by Lemma 2.4. Therefore $\alpha_{k,m,(S^1)^n} = \alpha$ if and only if there is a subset $\mathcal{F} \subseteq P \cap M/k$ such that $|\mathcal{F}| = m$ and $\mathcal{F}$ is a subset of a face that lies in the level set (75), or equivalently, there is a face F that lies in the level set (75), and $|F \cap M/k| \geq m$.

Notice that the level set $\{\|\cdot\|_{-P} = \lambda\}$ is the dilation $\lambda\partial(-P)$, and (75) is the largest dilation that intersects P (by Lemma 7.4); see Figure 1. Combining all the above, $(*_P)$ states that

$$P \text{ intersects } \max_P \|\cdot\|_{-P}\,\partial(-P) \text{ only at vertices.}$$

If $(*_P)$ holds, then any such face F consists of a single lattice point. In particular, $|F \cap M/k| = 1$ for any k, and $\alpha_{k,m,(S^1)^n}$ stabilizes if and only if $m = 1$. On the other hand, if $(*_P)$ fails, then there is a face F of positive dimension that lies in the level set (75). Since $|F \cap M/k|$ increases to infinity, for any m, we can find k_0 such that $|F \cap M/k| \geq m$ for $k \geq k_0$, ie $\alpha_{k,m,(S^1)^n}$ stabilizes in k.

7.2 A Demailly result for Grassmannian invariants

The next result was obtained independently by Li and Zhu [28, Proposition 4.1].

Proposition 7.10 *Let X be toric Fano with associated polytope P (28). For $m \in \mathbb{N}$,*

$$\lim_{k\to\infty} \alpha_{k,m,(S^1)^n} = \alpha.$$

Proof By continuity, given $\epsilon > 0$ there exists an open set $U_\epsilon \subset P$ such that

$$\|\cdot\|_{-P}|_{U_\epsilon} > \max_P \|\cdot\|_{-P} - \epsilon. \tag{76}$$

Fix $\delta > 0$ sufficiently small so that there is a closed cube $C_\delta \subset U_\epsilon$ with edge length δ. Now, for $k \geq K$, $|C_\delta \cap M/k| \geq \lfloor K\delta \rfloor^n$. Given $m \in \mathbb{N}$, choose K so $\lfloor K\delta \rfloor^n \geq m$. Then for each $k \geq K$ there exists $\mathcal{F} \subset C_\delta \cap M/k \subset P \cap M/k$ such that $|\mathcal{F}| = m$. Since $\mathcal{F} \subset C_\delta$ and C_δ is convex, $\operatorname{co}\mathcal{F} \subset C_\delta \subset U_\epsilon$. By (76),

$$\min_{\operatorname{co}\mathcal{F}} \|\cdot\|_{-P} \geq \min_{C_\delta} \|\cdot\|_{-P} > \max_P \|\cdot\|_{-P} - \epsilon.$$

So by Corollary 7.6 and (68)–(69),

$$\alpha_{k,m,(S^1)^n} \leq \frac{1}{1+\min_{\operatorname{co}\mathcal{F}}\|\cdot\|_{-P}} < \frac{1}{1+\max_P\|\cdot\|_{-P}-\epsilon} = \frac{\alpha}{1-\epsilon\alpha},$$

for any k such that $\lfloor k\delta \rfloor \geq \sqrt[n]{m}$. Hence $\limsup_{k\to\infty} \alpha_{k,m,(S^1)^n} \leq \alpha/(1-\epsilon\alpha)$. On the other hand, by Proposition 7.8, $\liminf_{k\to\infty} \alpha_{k,m,(S^1)^n} \geq \alpha$. Since $\epsilon > 0$ is arbitrary the proof is complete. □

8 Examples

In Section 8.1 we carefully illustrate Theorems 1.4 and 1.6 in dimension 2 and for all toric del Pezzo surfaces. In Section 8.2 we discuss central symmetry and its characterization in dimension 2 in terms of the α-invariant. In Section 8.3 we draw the connection between $(*_P)$ and work in the convex geometry literature on the asymmetry constant. We illustrate this with computations related to a remarkable toric 3-fold that uniquely minimizes α among all toric (and possibly even nontoric) 3-folds.

8.1 Del Pezzo surfaces

Example 8.1 Let $v_1 = (1,0)$, $v_2 = (0,1)$, and $v_3 = (-1,-1)$. Then $\mathbb{P}^2 = X(\Delta)$ for

$$\Delta = \{\{0\}, \mathbb{R}_+v_1, \mathbb{R}_+v_2, \mathbb{R}_+v_3, \mathbb{R}_+v_1 + \mathbb{R}_+v_2, \mathbb{R}_+v_1 + \mathbb{R}_+v_3, \mathbb{R}_+v_2 + \mathbb{R}_+v_3\}$$

and $P = \{-v_1, -v_2, -v_3\}^\circ = \{y \in M_\mathbb{R} : \langle y, -v_i\rangle \leq 1 \text{ for } i \in \{1,2,3\}\} \subset M_\mathbb{R}$ is depicted in Figure 3. Then $\operatorname{Aut} P \cong D_6 = S_3$, the dihedral/cyclic group of order 6 consisting of the permutations of the 3 vertices, or the 3 homogeneous coordinates $\mathbb{P}^2$, as in Figure 2. For $H = \operatorname{Aut} P$, P^H is the origin, so by Theorem 1.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$. When H is the trivial group $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at any vertex of P. When H is the subgroup $\mathbb{Z}_2$ given by reflection about, say, $y = x$ (ie generated by the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ associated to switching two of the homogeneous coordinates of $\mathbb{P}^2$), P^H is the line segment $\operatorname{co}\{(-1,-1), (\frac{1}{2},\frac{1}{2})\}$ and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at $(-1,-1)$. When H is the subgroup of order 3, $P^H = \{0\}$ and $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$,

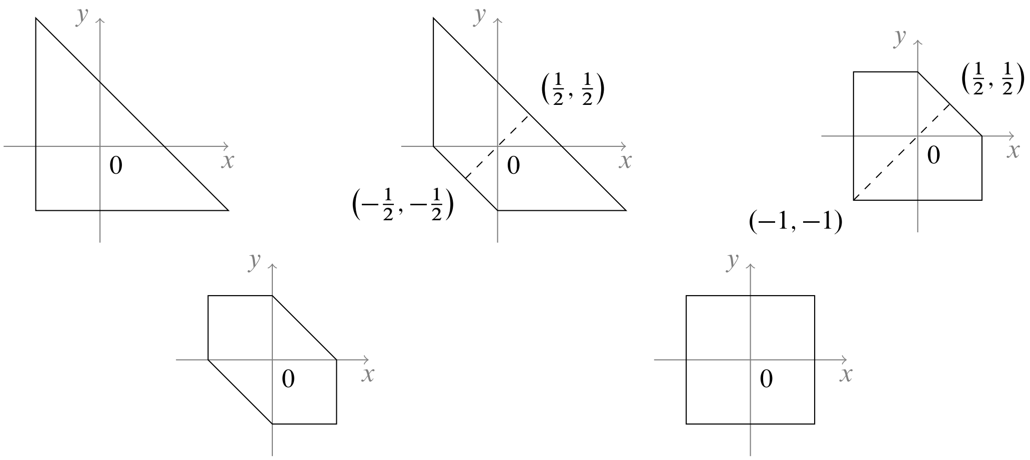


Figure 3: The polytope P for del Pezzo surfaces, namely $\mathbb{P}^2$ blown up at no more than three generically positioned points, and $\mathbb{P}^1 \times \mathbb{P}^1$. For the two K-unstable examples, the automorphism group $H = \operatorname{Aut} P$ is generated by the reflection about $y = x$, and P^H is the intersection of P with this reflection axis. For the other three examples, the automorphism group $H = \operatorname{Aut} P$ is the dihedral group associated to the polygon P, and $P^H = \{0\}$.

Example 8.2 Let P be the polytope for $\mathbb{P}^2$ blown up at one point, ie with v_1, v_2, and v_3 as above and $v_4 = (1,1)$. Then $\operatorname{Aut} P \cong \mathbb{Z}_2$, as shown in Figure 3. There are therefore two choices for H. When $H = \operatorname{Aut} P$, P^H is the line segment $\operatorname{co}\{(-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$. By Theorem 1.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $\pm(\frac{1}{2}, \frac{1}{2})$. When H is the trivial group $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved at either of the vertices $(-1,2)$ or $(2,-1)$.

Example 8.3 Let P be the polytope for $\mathbb{P}^2$ blown up at two points, ie with v_1, v_2, and v_3 as above, $v_4 = (-1,0)$, and $v_5 = (0,-1)$. Again $\operatorname{Aut} P \cong \mathbb{Z}_2$, as shown in Figure 3. For $H = \operatorname{Aut} P$, P^H is the line segment $\operatorname{co}\{(-1,-1), (\frac{1}{2}, \frac{1}{2})\}$. By Theorem 1.4, $\alpha_{k,G} = \alpha_G = \frac{1}{3}$, with minimum achieved at $(-1,-1)$. When H is the trivial group $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{3}$, with minimum achieved still at $(-1,-1)$.

Example 8.4 Let P be the polytope for $\mathbb{P}^2$ blown up at three noncolinear points, ie with $v_1, \dots, v_5$ as in the previous example and $v_6 = (1,1)$. Now $\operatorname{Aut} P \cong D_{12}$, the dihedral group of order 12 consisting of the cyclic permutations of the 6 vertices and the 6 reflections, as in Figure 3. For $H = \operatorname{Aut} P$, P^H is the origin, so by Theorem 1.4, $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$. When H is the trivial group $P^H = P$ and Theorem 1.4 gives $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at any vertex of P. When H is the subgroup $\mathbb{Z}_2$ given by reflection about $y = x$, P^H is the line segment $\operatorname{co}\{(-\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$ and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $\pm(\frac{1}{2}, \frac{1}{2})$. When H is the subgroup $\mathbb{Z}_2$ given by reflection about $y = -x$, $P^H = \operatorname{co}\{(-1,1), (1,-1)\}$, and $\alpha_{k,G(H)} = \alpha_{G(H)} = \frac{1}{2}$, with minimum achieved at $(\pm 1, \mp 1)$. The same holds for the other reflections. When H contains a cyclic permutation, or more than one reflection, P^H is the origin and $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$.

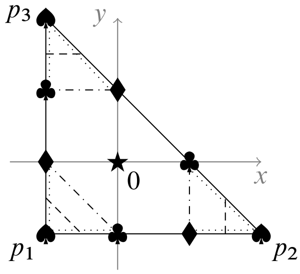


Figure 4: The four orbits $O_1^{(1)}, \dots, O_4^{(1)}$ of the action of the cyclic group of order 3 (generated by cyclic permutation of the homogeneous coordinates of $\mathbb{P}^2$) on the polytope corresponding to $\mathbb{P}^2$ with $k=1$. The dashed-dotted lines depict the sets on which h_{Δ_1} is constant equal to 1 and that compute $\alpha_{1,2,(S^1)^2} = \frac{1}{2}$. These sets are the three components of $-\partial P \cap P$. The dotted lines depict the sets on which h_{Δ_1} is not constant but whose minimum is 1 and that also compute $\alpha_{1,2,(S^1)^2} = \frac{1}{2}$. The dashed lines depict the sets on which h_{Δ_1} is constant equal to $\frac{3}{2}$ and that compute $\alpha_{2,2,(S^1)^2} = \frac{2}{5}$. These sets are the three components of $-\frac{3}{2}\partial P \cap P$.

Example 8.5 Let P be the polytope for $\mathbb{P}^1 \times \mathbb{P}^1$, ie with v_1 and v_2 as above, $v_3 = (-1,0)$, and $v_4 = (0,-1)$, so $P = [-1,1]^2$ and $\operatorname{Aut} P \cong D_8$, the dihedral group of order 8 generated by 4 reflections shown in Figure 3. Similar to previous examples, we obtain $\alpha_{k,G(H)} = \alpha_{G(H)} = 1$ whenever H contains a cyclic permutation or more than one reflection. Otherwise it equals $\frac{1}{2}$.

Example 8.6 Let P be the polytope for $X = \mathbb{P}^2$ from Example 8.1. Recall that $\operatorname{Aut} P = S_3$. Let $H = A_3$, the alternating group of order 3, generated by a cyclic permutation of the three vertices. The group is represented by $\{I, A, A^2\}$ where

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

All orbits of H on $M \cong \mathbb{Z}^2$ have cardinality 3 except the orbit of the origin, which has cardinality 1. Any $G(H)$-invariant subspace of $H^0(X, -kK_X)$ is spanned by a collection of monomials indexed by a union $\bigcup_{i=1}^{\ell} O_i^{(k)}$ of H-orbits on $kP \cap M$. This forces the subspace to have dimension $m = 3\ell$ or $m = 3(\ell-1)+1$. Thus $m \equiv_3 0, 1$ (see Figure 4 for the case $k=1$). As a result, if $m \equiv_3 2$ then $\alpha_{k,m,G(H)} = \infty$. For this reason we only consider in Section 7 the case $H = \{\mathrm{id}\}$.

Example 8.7 Let P be the polytope for $X = \mathbb{P}^2$ from Example 8.1. We compute the obstruction $(*_P)$, ie compute $\operatorname{argmax}_P \|\cdot\|_{-P} = \operatorname{argmax}_P h_Q = \operatorname{argmax}_P h_{\Delta_1}$. Recall $\Delta_1 = \{v_1, v_2, v_3\} = \{(1,0), (0,1), (-1,-1)\}$. Let $y = (y_1, y_2)$ be coordinates on P. Then

$$h_{v_1}(y) = y_1, \quad h_{v_2}(y) = y_2, \quad \text{and} \quad h_{v_3}(y) = -y_1 - y_2,$$

so

$$h_{\Delta_1}(y) = \max\{y_1, y_2, -y_1 - y_2\}.$$

By Lemma 2.4, $\operatorname{Ver} P \cap \operatorname{argmax}_P h_{\Delta_1} \neq \varnothing$. Note,

$$\operatorname{Ver} P = \{p_1, p_2, p_3\} = \{(-1,-1), (2,-1), (-1,2)\}.$$

So

$$\max_P h_Q = \max\{h_Q(p_1), h_Q(p_2), h_Q(p_3)\} = \max\{2,2,2\} = 2,$$

and $\operatorname{Ver} P \subset \operatorname{argmax}_P h_{\Delta_1}$. Suppose that $h_Q(y) = 2$. Then a computation shows that $y \in \operatorname{Ver} P$. Thus $\operatorname{Ver} P = \operatorname{argmax}_P h_{\Delta_1}$. By Theorem 1.6, $\alpha_{k,m,(S^1)^n} > \alpha = \frac{1}{3}$ for all $k \in \mathbb{N}$ and $m \geq 2$.

Let us compute $\alpha_{1,2,(S^1)^n}$. Let $\mathcal{F} = \{f_1, f_2\} \subset P \cap M$. By Corollary 7.6 it suffices to consider "minimal" $\mathcal{F}$ in the sense that for no other $\mathcal{F}'$, $\operatorname{co}\mathcal{F}' \subset \operatorname{co}\mathcal{F}$. So for instance, we do not need to consider $\{p_1, p_2\}$ but instead

$$\{p_1, (0,-1)\}, \{(0,-1),(1,-1)\}, \{(1,-1), p_2\}.$$

In fact, $\min_{\operatorname{co}\{p_1,(0,-1)\}} h_Q = h_Q(0,-1) = 1$ while $\min_{\{p_1,p_2\}} h_Q = h_Q\big(\frac{1}{2},-1\big) = \frac{1}{2}$.

Also, again by Corollary 7.6, we do not need to consider any $\mathcal{F}$ such that $0 \in \operatorname{co}\mathcal{F}$ since on such an interval the support function attains its absolute minimum, ie vanishes.

Next, observe that $h_{\Delta_1} = 1$ on the vertices of the hexagon

$$P_1 := \operatorname{co}\{\pm(1,0), \pm(0,1), \pm(-1,1)\}.$$

By Lemma 2.4,

$$h_{\Delta_1}|_{P_1} \leq h_{\Delta_1}|_{\operatorname{Ver} P_1} = 1.$$

Thus (by Corollary 7.6) we do not need to consider any $\mathcal{F}$ such that $\operatorname{co}\mathcal{F}$ intersects P_1, as such an $\mathcal{F}$ will satisfy $\min_{\operatorname{co}\mathcal{F}} h_Q \leq 1$. Since every $\mathcal{F}$ intersects P_1 it follows that $\alpha_{1,2,(S^1)^2} = \frac{1}{2}$ by Corollary 7.6; see Figure 4.

By the same reasoning we can compute $\alpha_{k,2,(S^1)^2}$ using the hexagons

$$\begin{aligned} P_k &:= \operatorname{co}\Big\{\Big(2-\frac{1}{k},\frac{1}{k}-1\Big),\Big(2-\frac{1}{k},-1\Big),\Big(-1+\frac{1}{k},-1\Big),\Big(-1,-1+\frac{1}{k}\Big),\Big(-1,2-\frac{1}{k}\Big),\Big(-1+\frac{1}{k},2-\frac{1}{k}\Big)\Big\} \\ &= \frac{1}{k}\operatorname{co}\big\{(2k-1,1-k),(2k-1,-k),(-k+1,-k),(-k,-k+1),(-k,2k-1),(-k+1,2k-1)\big\} \\ &\subset P \cap M/k. \end{aligned}$$

By Lemma 2.4,

$$h_{\Delta_1}|_{P_k} \leq h_{\Delta_1}|_{\operatorname{Ver} P_k} = 2 - \frac{1}{k}.$$

Since every $\mathcal{F}$ intersects P_k it follows that

$$\alpha_{k,2,(S^1)^2} = \Big[1 + 2 - \frac{1}{k}\Big]^{-1} = \frac{k}{3k-1}$$

by Corollary 7.6. Note that $\lim_k k/(3k-1) = \frac{1}{3} = \alpha$ (recall Example 8.1 and (69)) in accordance with Proposition 7.10 and that this is a monotone sequence in accordance with Proposition 7.8.

8.2 Central symmetry

The next result should be well known, but we could not find a reference.

Lemma 8.8 *Let X be toric Fano. Let $P \subset M_{\mathbb{R}}$ (see (18) and (27)) be the polytope associated to $(X, -K_X)$. Then $\alpha \le \frac{1}{2}$, and P is centrally symmetric (ie $P = -P$) if and only if $\alpha = \frac{1}{2}$. If P is not centrally symmetric then $\alpha \le \frac{1}{3}$.*

Proof If $P \ne -P$, then there is $u_0 \in P \setminus -P$. By (28), $-P = \{y \in M_{\mathbb{R}} : \max_j \langle v_j, y\rangle \le 1\}$. Thus $\langle u_0, v_{i_0}\rangle > 1$ for some $v_{i_0} \in \Delta_1 \subset N$ (recall (21)). By Theorem 1.4 and (69), or [6, Corollary 7.16],

$$\alpha = \min_{u\in P} \min_i \frac{1}{1+\langle u, v_i\rangle} \le \frac{1}{1+\langle u_0, v_{i_0}\rangle} < \frac{1}{2}.$$

In fact, by convexity one may take $u_0 \in \operatorname{Ver} P \setminus -P$, so $u_0 \in M$ by Lemma 2.3 and hence $\mathbb{Z} \ni \langle u_0, v_{i_0}\rangle \ge 2$, ie $\alpha \le \frac{1}{3}$.

If $P = -P$, by (68) and (69),

$$\alpha = \frac{1}{1+\max_P \|\cdot\|_{-P}} = \frac{1}{1+\max_P \|\cdot\|_P} = \frac{1}{1+1} = \frac{1}{2}$$

by (14) and (16). □

It turns out that any centrally symmetric toric Fano manifold can be written as the product of factors that are either $\mathbb{P}^1$ or the so-called toric del Pezzo variety V^{2k}, namely the unique indecomposable centrally symmetric toric Fano manifold of even dimension $2k$ [39, Theorem 6]. For example, V^2 is $\mathbb{P}^2$ blown up at three noncolinear points, and the only centrally symmetric toric Fano 3-folds are $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times V^2$.

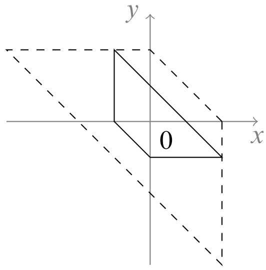
Example 8.9 The purpose of this paragraph is to show that when $n = 2$, condition $(*_P)$ holds for a Delzant polytope P if and only if P is not centrally symmetric, ie $P \ne -P$. Alternatively, if $n = 2$, Conjecture 1.5 holds if and only if $\alpha = \frac{1}{2}$, by Lemma 8.8. However, in dimension $n \ge 3$ this is no longer the case: for instance $\mathbb{P}^2 \times \mathbb{P}^1$ has a noncentrally symmetric polytope P but condition $(*_P)$ fails. For a counterexample that does not admit a centrally symmetric factor, consider the Fano 3-fold 4-11 [2, Table 1.3], namely the blowup of $\mathbb{F}_1 \times \mathbb{P}^1$ along $E \times \{0\}$, where E is the -1-curve on $\mathbb{F}_1$. Condition $(*_P)$ fails along three edges of the polytope.

When $n = 2$, the centrally symmetric Delzant polytopes are associated to $\mathbb{P}^2$ blown up at the three noncolinear points, and $\mathbb{P}^1 \times \mathbb{P}^1$ (Examples 8.4–8.5). Recall the description of $(*_P)$ given in Section 7.1. Since $P = -P$, the intersection of P and $\max_P \|\cdot\|_{-P}\, \partial(-P)$ is precisely all of ∂P, ie $(*_P)$ fails.

Thus it remains to check $\mathbb{P}^2$ blown up at up to two points. We have already showed that $(*_P)$ holds for (the noncentrally symmetric) P coming from $\mathbb{P}^2$ (Example 8.7). It thus remains to check the remaining two cases. The polytope for the 1-point blowup is a subset of the one for $\mathbb{P}^2$ by chopping a corner. Thus the intersection of P and $\max_P \|\cdot\|_{-P}\, \partial(-P) = -2\partial P$ is still just the two vertices $(-1, 2)$ and $(2, -1)$. For the 2-point blowup, the intersection of P and $\max_P \|\cdot\|_{-P}\, \partial(-P) = -2\partial P$ is just the vertex $(-1, -1)$, concluding the proof; see Figure 5.

8.3 The asymmetry constant and a remarkable Fano 3-fold

The asymmetry constant measures how far a body is from being centrally symmetric [21; 24]:



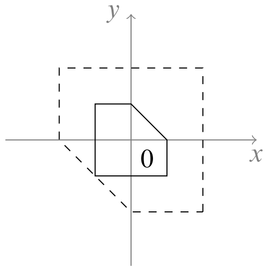


Figure 5: The polytope P for $\mathbb{P}^2$ blown up 1 or 2 points, and the level set $\{\|\cdot\|_{-P} = 2\}$.

Definition 8.10 For a convex polytope $P \ni 0$, define

$$\operatorname{asym}(P) := \min_{a\in P} \max_{P} \|\cdot\|_{-(P-a)} = \min_{a\in P} \min\{\lambda \geq 0 : P \subseteq -\lambda(P-a)\} \geq 1.$$

In other words, $\operatorname{asym}(P)(-P)$ is the smallest dilation of $-P$ that admits a translation containing P.

A corollary of (68)–(69) is

$$\alpha \leq [1+\operatorname{asym}(P)]^{-1}. \tag{77}$$

As we will see in Example 8.11, this bound is often not sharp. In order to see that, let us first describe how to compute $\operatorname{asym}(P)$ in general.

In practice, $\operatorname{asym}(P)$ is the smallest $\lambda \geq 0$ such that

$$-\lambda P + \tau \supseteq P$$

for some $\tau \in \mathbb{R}^n$. In other words, it is the smallest λ among all the pairs $(\tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ satisfying

$$-\lambda P + \tau \supseteq P.$$

Hence it is the solution to the following linear programming problem: minimize

$$f(\tau, \lambda) := \lambda$$

on the region

$$\{(\tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} : -\lambda P + \tau \supseteq P\}.$$

More precisely, write $\operatorname{Ver} P = \{u_i\}_{i=1}^p$ and recall (29) that the d facets of P are given by $\langle \cdot, v_j\rangle = 1$ for $j \in \{1, \dots, d\}$. Then by (28), $\operatorname{asym}(P)$ is the solution to the following linear programming problem: minimize

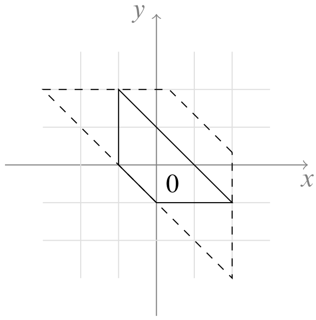
$$f(\tau, \lambda) := \lambda$$

under the constraints that $(\tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ satisfies

$$\langle \tau, v_j\rangle - \lambda \leq \langle u_i, v_j\rangle \quad \text{for all } j \in \{1, \dots, d\} \text{ and } i \in \{1, \dots, p\}. \tag{78}$$

Example 8.11 For the polytopes of Figure 5, one has $\operatorname{asym}(P) = \frac{5}{3}$ for the 1-point blowup and $\operatorname{asym}(P) = \frac{4}{3}$ for the 2-point blowup, confirming the intuition that the latter is "more centrally symmetric" than the former. This also serves to show that in many instances

$$\max_{x\in P} \|\cdot\|_{-P} > \operatorname{asym}(P),$$



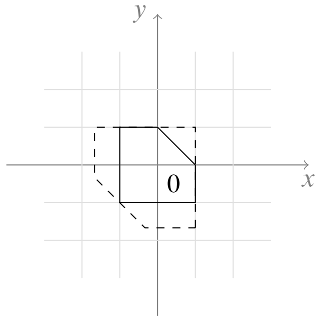


Figure 6: The polytope P for $\mathbb{P}^2$ blown up 1 or 2 points, and the translation of the smallest dilation of $-P$ that contains P. We can read from the figure that $\operatorname{asym}(P) = \frac{5}{3}, \frac{4}{3}$, respectively.

ie (77) is rarely sharp for Delzant polytopes. The only examples for which we know it to be sharp are $\mathbb{P}^n$ and centrally symmetric P.

By John's theorem [26], $\operatorname{asym}(P) \le n$, with equality attained for the simplex associated to $\mathbb{P}^n$ [21]. However, the next example shows that for Delzant polytopes arising from toric Fano manifolds one may have $\max_{x \in P} \|\cdot\|_{-P} > n$, even though this seems rare (at least in small dimensions). This first happens in $n = 3$ and just for a unique Fano 3-fold.

Example 8.12 Consider the Fano 3-fold 2-36 [2, Table 1.3], namely the Calabi–Hirzebruch manifold $\mathbb{F}_{3,2} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2})$ [8]. It is a toric manifold with [40, (4), page 42]

$$\Delta_1 = \{v_1, \dots, v_5\} = \{(0,0,1), (0,0,-1), (1,0,0), (0,1,0), (-1,-1,-2)\}.$$

Let P be the corresponding polytope. We have

$$\operatorname{Ver} P = \{(-1,-1,1), (-1,0,1), (0,-1,1), (-1,-1,-1), (-1,4,-1), (4,-1,-1)\}.$$

To see this, first notice that

$$\bigcap_{i=1}^{4} \{\langle \cdot, v_i \rangle \ge -1\} = [-1,+\infty) \times [-1,+\infty) \times [-1,1],$$

which is an infinite cuboid with vertices $(-1,-1,\pm 1)$. The polytope P is obtained by cutting it with the hyperplane $\langle \cdot, v_5 \rangle = -1$, introducing the other four vertices $(-1,0,1)$, $(0,-1,1)$, $(-1,4,-1)$, and $(4,-1,-1)$, namely the vertices of the isosceles trapezoid which is the facet cut out by the last equation $\langle \cdot, v_5 \rangle = -1$; see Figure 7. Since

$$(-4\partial P) \cap P = \{(-1,-1,-1), (-1,4,-1), (4,-1,-1)\},$$

we have $\alpha = \alpha_{k,1,(S^1)^n} = \frac{1}{5} < \alpha_{k,m,(S^1)^n}$, for $m \ge 2$. Solving the linear programming problem (78) yields $\operatorname{asym}(P) = \frac{14}{5} < 3$.

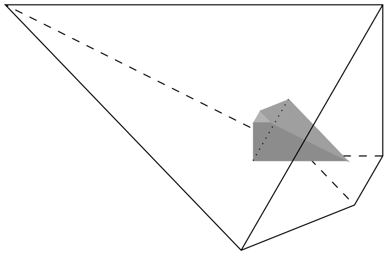


Figure 7: The frustum P (the solid inside) with upper base $\mathrm{co}\{(-1,-1,1),(-1,0,1),(0,-1,1)\}$ and lower base $\mathrm{co}\{(-1,-1,-1),(-1,4,-1),(4,-1,-1)\}$, and the boundary of $-4P$ (edges in solid and dashed lines). Their intersection consists of the three vertices of the lower base of P.

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