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**Joyce structures on spaces of quadratic differentials**

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Consider the space parametrising complex projective curves of genus  $g$  equipped with a quadratic differential with simple zeroes. We use the geometry of isomonodromic deformations to construct a complex hyperkähler structure on the total space of its tangent bundle. This provides nontrivial examples of the Joyce structures introduced by the author in relation to Donaldson–Thomas theory.

14H60, 14H70

## 1 Introduction

Since Hitchin’s classic papers [1987a; 1987b], moduli spaces of Higgs bundles on algebraic curves have appeared in many areas of pure mathematics and mathematical physics. Consider for definiteness the space  $\mathcal{M}_C(E, \Phi)$  parametrising  $\mathrm{SL}_2(\mathbb{C})$  Higgs bundles  $(E, \Phi)$  on a smooth complex projective curve  $C$  of genus  $g > 1$ . Two of its most important features are

- (i) a hyperkähler metric, defined using the non-Abelian Hodge correspondence, and
- (ii) a proper map  $\mathcal{M}_C(E, \Phi) \rightarrow H^0(C, \omega_C^{\otimes 2})$ , whose target parametrises quadratic differentials on  $C$ , and whose general fibres are abelian varieties, homeomorphic to  $(S^1)^{6g-6}$ .

In this paper we study a moduli space  $\mathcal{M}(C, E, \nabla, \Phi)$  which in some respects resembles a complexification of  $\mathcal{M}_C(E, \Phi)$ . For a given integer  $g > 1$  it parametrises the data of a curve  $C$  of genus  $g$ , together with an  $\mathrm{SL}_2(\mathbb{C})$  bundle  $E$  on  $C$ , equipped with both a flat connection  $\nabla$  and a Higgs field  $\Phi$ . We construct

- (i) a meromorphic complex hyperkähler metric, defined using isomonodromic flows, and
- (ii) a map  $\mathcal{M}(C, E, \nabla, \Phi) \rightarrow \mathcal{M}(C, Q)$  whose fibres are algebraic tori  $(\mathbb{C}^*)^{6g-6}$ . The target  $\mathcal{M}(C, Q)$  of this map is the “generic Hitchin base” parametrising curves  $C$  of genus  $g$  equipped with a quadratic differential  $Q \in H^0(C, \omega_C^{\otimes 2})$  with simple zeroes.

It is important to note that the complex hyperkähler metric we construct on  $\mathcal{M}(C, E, \nabla, \Phi)$  is a much simpler object than the hyperkähler metric on  $\mathcal{M}_C(E, \Phi)$ . In particular it is algebraic, in contrast to the Hitchin metric, which is highly transcendental. A striking demonstration of this difference appears on generalising to the setting where the Higgs fields and connections have poles of fixed orders [Zikidis  $\geq$  2025]. One can then take the curve  $C$  to have genus  $g = 0$ , and in simple examples the resulting complex hyperkähler structures can be written explicitly in terms of rational functions [Bridgeland and Masoero 2023]. No such explicit formulae are expected for the Hitchin metric.

We expect our construction to generalise to gauge groups other than  $\mathrm{SL}_2(\mathbb{C})$  but this will require new ideas. A key point in our construction is that the space of quadratic differentials  $H^0(C, \omega_C^{\otimes 2})$  is the cotangent fibre to the moduli space of curves. To generalise to the gauge group  $G = \mathrm{SL}_m(\mathbb{C})$ , for example, would require a space of “higher complex structures” whose cotangent fibre is the Hitchin base  $\bigoplus_{k=2}^m H^0(C, \omega^{\otimes k})$ . A natural candidate is provided by the work of Fock and Thomas [2021]. The expectation is then that the total space of the cotangent bundle of this space should carry a meromorphic Joyce structure generalising the one constructed here.

### 1.1 DT invariants and [GMN]

In a celebrated development, Gaiotto, Moore and Neitzke [Gaiotto et al. 2010; 2013] uncovered a deep relation between the hyperkähler geometry of the moduli spaces  $\mathcal{M}_C(E, \Phi)$  and the BPS invariants of a class of four-dimensional  $N = 2$  supersymmetric gauge theories known as theories of class  $S[A_1]$ . More precisely, they introduced a class of nonlinear Riemann–Hilbert (RH) problems defined by the BPS invariants, and showed that their solutions describe twistor lines in the twistor space of  $\mathcal{M}_C(E, \Phi)$ . In mathematical terms these BPS invariants can be understood as the Donaldson–Thomas (DT) invariants of a certain three-dimensional Calabi–Yau ( $\mathrm{CY}_3$ ) triangulated category [Bridgeland and Smith 2015].

Our interest in the moduli spaces  $\mathcal{M}(C, E, \nabla, \Phi)$  stems from a general programme [Bridgeland 2019; 2021] which attempts to encode the DT invariants of a  $\mathrm{CY}_3$  triangulated category in a geometric structure on its space of stability conditions. This procedure is currently highly conjectural, and involves a class of RH problems closely related to those considered by Gaiotto, Moore and Neitzke, and obtained from them by a procedure known in physics as the conformal limit [Gaiotto 2014]. In this limit it appears that the geometry of the Hitchin space considered in [Gaiotto et al. 2013] should be replaced by the simpler geometry of the space  $\mathcal{M}(C, E, \nabla, \Phi)$  considered here.

The geometric structure on spaces of stability conditions envisaged in [Bridgeland 2021] is a kind of nonlinear Frobenius structure, and was christened a Joyce structure in honour of the paper [Joyce 2007], where the main ingredients were first identified. In later work [Bridgeland and Strachan 2021] it was shown that a Joyce structure on a complex manifold induces a complex hyperkähler structure on the total space of its tangent bundle. The main result of this paper is a construction of a meromorphic Joyce structure on the space  $\mathcal{M}(C, Q)$ .

The space  $\mathcal{M}(C, Q)$  can be identified with a space of stability conditions on a  $\mathrm{CY}_3$  category by the work of Haiden [2024]. We leave for future research the problem of using the Joyce structure constructed here to solve the RH problems of [Bridgeland 2019] defined by the DT invariants of Haiden’s category. This would probably be more easily accomplished in the setting of meromorphic quadratic differentials [Bridgeland and Smith 2015; Zikidis  $\geq$  2025], using the Fock–Goncharov cluster structure on the wild character variety [Fock and Goncharov 2006] and results from exact WKB analysis as in [Gaiotto et al. 2013]. One particular example was treated in detail in this way by Bridgeland and Masoero [2023], and other partial results have been obtained by Allegretti [2019; 2021].

## 1.2 Summary of the construction

We fix a genus  $g > 1$  throughout the paper. The space  $\mathcal{M}(C, Q)$  as defined above is a smooth Deligne–Mumford stack or, if we work in the analytic category, a complex orbifold. Since this may be uncomfortable for some readers, we shall also fix an integer  $\ell > 0$ , and insist that all curves  $C$  are equipped with a level  $\ell$  structure. This extra data plays no essential role in our constructions so we will omit it from the notation. We always assume  $\ell > 2$ , since this has the pleasant consequence that all moduli spaces appearing are smooth quasiprojective varieties. But the reader happy with stacks can eliminate the level structures by taking  $\ell = 1$  and working instead with smooth Deligne–Mumford stacks.

Let us then introduce the smooth quasiprojective variety  $M = \mathcal{M}(C, Q)$  parametrising pairs  $(C, Q)$  consisting of a smooth projective complex curve  $C$ , equipped as always with a level  $\ell$  structure, and a quadratic differential  $Q \in H^0(C, \omega_C^{\otimes 2})$  with simple zeroes. Associated to a point  $(C, Q) \in M$  is a smooth spectral curve  $\Sigma$  cut out in the cotangent bundle  $T_C^*$  by the equation  $y^2 = Q$ . The projection  $p: \Sigma \rightarrow C$  is a branched double cover with a covering involution  $\sigma$ . The tangent space to  $M$  at the point  $(C, Q)$  can then be identified with the anti-invariant cohomology group  $H^1(\Sigma, \mathbb{C})^-$ . The dual of the integral anti-invariant homology defines an integral affine structure  $T_M^{\mathbb{Z}} \subset T_M$  and we consider the quotient  $X^{\#} = T_M / T_M^{\mathbb{Z}}$ . The fibre of the induced projection  $\pi: X^{\#} \rightarrow M$  over the point  $(C, Q)$  is the quotient of the group  $H^1(\Sigma, \mathbb{C}^*)^-$  by the finite subgroup  $p^*(H^1(C, \{\pm 1\}))$  and is isomorphic to  $(\mathbb{C}^*)^{6g-6}$ .

A Joyce structure on  $M$  is essentially the data of a pencil of flat symplectic nonlinear connections  $h_{\epsilon}$  on the bundle  $\pi: X^{\#} \rightarrow M$  parametrised by  $\epsilon \in \mathbb{C}^*$ . The associated complex hyperkähler structure on  $X^{\#}$  is then defined by taking the eigenspaces of the operators  $I, J, K$  to be the horizontal subbundles of  $T_{X^{\#}}$  defined by certain elements of this pencil. We construct the nonlinear connections  $h_{\epsilon}$  as follows. We can realise elements of  $H^1(\Sigma, \mathbb{C}^*)^-$  as the holonomy of anti-invariant line bundles with connection  $(L, \partial)$  on  $\Sigma$ . The usual spectral correspondence associates to  $L$  a Higgs bundle  $(E, \Phi)$  on the curve  $C$ . Using an extension of this correspondence to connections, valid under a genericity assumption on  $L$ , we can use  $\partial$  to induce a connection  $\nabla$  on  $E$ . The required family of nonlinear connections  $h_{\epsilon}$  is then given by the isomonodromy flows for the connections  $\nabla - \epsilon^{-1}\Phi$ .

The complex hyperkähler structure we construct on  $M$  has poles; these arise from two interesting issues. Firstly, the extension of the spectral correspondence to connections requires a genericity assumption on the line bundle  $L$ . This relates to the theta divisor in the generalised Prym variety of the double cover  $p: \Sigma \rightarrow C$ . Secondly, given a fixed curve  $C$  equipped with a Higgs bundle  $(E, \Phi)$ , we need to lift deformations of the quadratic differential  $Q = \text{tr}(\Phi^2)$  to deformations of the Higgs field  $\Phi$ . This relates to the wobbly locus in the space of Higgs bundles; see [Donagi and Pantev 2009].

The extended spectral correspondence in our construction can be viewed as an abelianization procedure for flat connections in the presence of a quadratic differential. In the case of meromorphic quadratic differentials, this de Rham abelianization can be compared with the Betti abelianization of [Hollands and Neitzke 2016; Nikolaev 2021], which depends on the choice of a spectral network on  $C$ . Their

relationship is highly nontrivial, and in fact, if we take the spectral network to be the WKB triangulation of the quadratic differential, one can view the solutions to the RH problems discussed above as intertwining these two abelianisation procedures.

**Plan of the paper** The aim of the paper is to construct a meromorphic Joyce structure on the space  $M = \mathcal{M}(C, Q)$ . A Joyce structure on a complex manifold  $M$  is a combination of two ingredients: a period structure and a pencil of nonlinear connections on the tangent bundle. The definitions of all these terms can be found in Section 2.

The required period structure on  $M$  is well-known and is described in Section 3. In Section 4 we recall the standard correspondence between Higgs bundles on  $C$  and line bundles on the spectral curve  $\Sigma$ , and explain how it can be extended to bundles with connection. Section 5 introduces the essential diagrams of moduli spaces which will be used to construct the pencil of nonlinear connections. We also prove two crucial generic finiteness results.

In Section 6 we recall the Atiyah–Bott symplectic form on the moduli space of flat connections and prove that our extended spectral correspondence preserves it. The meromorphic Joyce structure on  $M$  is finally constructed in Section 7 using isomonodromic flows. In Section 8, we describe an interesting compatibility relation between this Joyce structure and the Lagrangian submanifolds in  $M$  obtained by fixing the curve  $C$ . We include in the appendix a summary of the scheme-theoretic definitions and constructions of the various moduli spaces used in the main text.

**Conventions and notation** We use rather unconventional conventions for labelling moduli spaces. In general a symbol  $\mathcal{M}_A(B)$  denotes the moduli space of objects of type  $B$  on a fixed object  $A$ . So for example  $\mathcal{M}_C(E, \Phi)$  denotes the moduli space of  $\mathrm{SL}_2(\mathbb{C})$  Higgs bundles  $(E, \Phi)$  on a fixed curve  $C$ , whereas  $\mathcal{M}(C, E, \Phi)$  denotes the moduli space where  $C$  is also allowed to vary. We can only apologise for the initially nonsensical appearance of statements such as “Take a point  $(C, Q) \in \mathcal{M}(C, Q)$ ”, and hope that this proves less of an inconvenience than having to constantly consult a dictionary of the large number of moduli spaces that appear.

The paper contains many connections, both linear and nonlinear. Linear connections on a vector bundle  $E$  are specified by their covariant derivative  $E \rightarrow E \otimes \Omega^1$  and are usually denoted by the symbols  $\nabla$  or  $\partial$ . Nonlinear connections on a map  $\pi: X \rightarrow M$  are specified by a bundle map  $\pi^*(T_M) \rightarrow T_X$  and are denoted by small latin letters  $h, j$ , etc. Throughout the paper we encounter families of connections parametrised by  $\epsilon \in \mathbb{C}^*$ , which we refer to as pencils. Often the inverse  $\zeta = \epsilon^{-1}$  would seem to be a more natural parameter, but we will nonetheless use  $\epsilon$  since in the relations with mathematical physics this is the most natural variable, relating variously to the string coupling, Planck’s constant, etc.

We work with both complex manifolds and algebraic varieties. Except in the appendix, all algebraic varieties appearing are smooth and quasiprojective over  $\mathbb{C}$ . We view them as a subcategory of the category of complex manifolds. The derivative of a map of complex manifolds  $f: X \rightarrow Y$  is denoted  $f_*: T_X \rightarrow f^*(T_Y)$ . The map  $f$  is called étale if  $f_*$  is an isomorphism.

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## 2 Joyce structures

This section introduces the notion of a Joyce structure on a complex manifold  $M$ . The definition arose from a line of work in Donaldson–Thomas theory [Bridgeland 2021] which originated with a paper of Joyce [2007]. We first define the notion of a pre-Joyce structure, which consists of a pencil of flat symplectic nonlinear connections on the tangent bundle  $\pi: X = T_M \rightarrow M$ . Following [Bridgeland and Strachan 2021], we show that a pre-Joyce structure on  $M$  induces a complex hyperkähler structure on  $X$ . This construction is well known in the twistor-theory literature, see for example [Dunajski and Mason 2000], and goes back to the work of Plebański [1975]. A Joyce structure is then defined to be a pre-Joyce structure with certain extra symmetries. The description of these symmetries involves a strengthening of the notion of an integral affine structure which we call a period structure.

### 2.1 Nonlinear connections

We begin by briefly summarising some basic facts about nonlinear connections in the sense of Ehresmann. We work with complex manifolds and holomorphic maps, but everything in this section holds also in the smooth setting.

Let  $\pi: X \rightarrow M$  be a holomorphic submersion of complex manifolds. Denote the fibres by  $X_m = \pi^{-1}(m)$ . The derivative of  $\pi$  gives rise to a short exact sequence of vector bundles

$$(1) \quad 0 \rightarrow T_{X/M} \xrightarrow{i} T_X \xrightarrow{\pi_*} \pi^*(T_M) \rightarrow 0.$$

**Definition 2.1** A nonlinear connection on the map  $\pi$  is a bundle map  $h: \pi^*(T_M) \rightarrow T_X$  satisfying  $\pi_* \circ h = 1$ .

Writing  $H = \text{im}(h)$  and  $V = T_{X/M}$ , the tangent bundle decomposes as a direct sum  $T_X = H \oplus V$ . We call tangent vectors and vector fields horizontal or vertical if they lie in  $H$  or  $V$ , respectively. Note that a vector field  $u \in H^0(M, T_M)$  can be lifted to a horizontal vector field  $h(u) \in H^0(X, T_X)$  by composing the pullback  $\pi^*(u) \in H^0(X, \pi^*(T_M))$  with the map  $h$ .

Consider a smooth path  $\gamma: [0, 1] \rightarrow M$ . Given a point  $x \in X_{\gamma(0)}$  we can look for a lifted path  $\alpha: [0, \delta] \rightarrow X$  satisfying  $\alpha_*(d/dt) = h(\gamma_*(d/dt))$  and  $\alpha(0) = x$ . Such a lift will exist for small enough  $\delta > 0$ . For  $t \in [0, \delta]$  we call  $\alpha(t) \in X_{\gamma(t)}$  the time  $t$  parallel transport of the point  $x$  along the path  $\gamma$ . Given a point  $x_0 \in X_{\gamma(0)}$  we can find a  $\delta > 0$  and open subsets  $U_t \subset X_{\gamma(t)}$  with  $x_0 \in U_0$ , such that time  $t$  parallel transport along  $\gamma$  defines an isomorphism  $\text{PT}_\gamma(t): U_0 \rightarrow U_t$  for each  $t \in [0, \delta]$ .

Given complex manifolds  $M$  and  $N$  there is a connection on the projection map  $\pi_M: M \times N \rightarrow M$  induced by the canonical splitting  $T_{M \times N} = \pi_M^*(T_M) \oplus \pi_N^*(T_N)$ . A connection  $h$  on  $\pi: X \rightarrow M$  is called flat if it is locally isomorphic to a connection of this form. More precisely:

**Definition 2.2** *The connection  $h$  is flat if the following equivalent conditions hold:*

- (i) *For every  $x \in X$  there are local coordinates  $(x_1, \dots, x_n)$  on  $X$  at  $x$ , and  $(y_1, \dots, y_d)$  on  $M$  at  $\pi(x)$ , such that  $x_i = \pi^*(y_i)$  and  $h(\partial/\partial y_i) = \partial/\partial x_i$  for  $1 \leq i \leq d$ .*
- (ii) *The subbundle  $H = \text{im}(h) \subset T_X$  is involutive:  $[H, H] \subset H$ .*

Suppose given a relative symplectic form  $\Omega_\pi \in H^0(X, \wedge^2 T_{X/M}^*)$  on the map  $\pi$ . It restricts to a symplectic form  $\Omega_m \in H^0(X_m, \wedge^2 T_{X_m}^*)$  on each fibre  $X_m$ . Note that since  $T_X/\text{im}(h) = T_{X/M}$ , the relative form  $\Omega_\pi$  can be lifted uniquely to a form  $\Omega \in H^0(X, \wedge^2 T_X^*)$  satisfying  $\ker(\Omega) = \text{im}(h)$ . We say that the connection  $h$  preserves  $\Omega_\pi$  if, for any path  $\gamma: [0, 1] \rightarrow M$ , the partially defined parallel transport maps  $\text{PT}_\gamma(t): X_{\gamma(0)} \rightarrow X_{\gamma(t)}$  take  $\Omega_{\gamma(0)}$  to  $\Omega_{\gamma(t)}$ .

**Lemma 2.3** (i) *The connection  $h$  preserves  $\Omega_\pi$  precisely if  $i_{v_1}i_{v_2}(d\Omega) = 0$  for any two vertical vector fields  $v_1, v_2 \in H^0(X, T_{X/M})$ .*  
(ii) *If the connection  $h$  is flat, then it preserves  $\Omega_\pi$  precisely if  $d\Omega = 0$ .*

**Proof** The first statement is [Gotay et al. 1983, Theorem 4]. For the second, take three vector fields  $u_1, u_2, u_3$  on  $X$  and consider the expression defining  $d\Omega(u_1, u_2, u_3)$ . We can assume that each  $u_i$  is either horizontal or vertical. Since  $i_h(\Omega) = 0$  for any horizontal vector field  $h$ , and horizontal vector fields are closed under the Lie bracket, we have  $d\Omega(u_1, u_2, u_3) = 0$  as soon as two of the  $u_i$  are horizontal. The claim then follows from (i).  $\square$

Suppose that a discrete group  $G$  acts freely and properly on  $X$  preserving the map  $\pi$ . Then  $Y = X/G$  is a complex manifold and the quotient map  $q: X \rightarrow Y$  is étale. There is an induced submersion  $\eta: Y \rightarrow M$  and a factorisation  $\pi = \eta \circ q$ . A connection  $h: \pi^*(T_M) \rightarrow T_X$  will be called  $G$ -invariant if  $g_* \circ h = h$  for all  $g \in G$ . There is then an induced connection  $j: \eta^*(T_M) \rightarrow T_Y$  on  $\eta$ , uniquely defined by the condition that  $q_* \circ h = q^*(j)$ . We say that the connection  $h$  descends along the quotient map  $q$ .

## 2.2 Pre-Joyce structures

Let  $M$  be a complex manifold and let  $\pi: X = T_M \rightarrow M$  be the total space of the tangent bundle of  $M$ . There is a canonical isomorphism  $\nu: \pi^*(T_M) \rightarrow T_{X/M}$  obtained by composing the chain of identifications

$$(2) \quad \pi^*(T_M)_x = T_{M, \pi(x)} = T_{T_{M, \pi(x)}, x} = T_{X_{\pi(x)}, x} = T_{X/M, x},$$

and we set  $v = i \circ \nu$ . A connection  $h: \pi^*(T_M) \rightarrow T_X$  on  $\pi$  then defines a family of such connections  $h_\epsilon = h + \epsilon^{-1}v$  parametrised by  $\epsilon \in \mathbb{C}^*$ . We call such a family a  $\nu$ -pencil of connections.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_{X/M} & \xrightarrow{i} & T_X & \xrightarrow{\pi^*} & \pi^*(T_M) \longrightarrow 0 \\
 & & & & & \curvearrowright^{h_\epsilon} & \\
 & & & & & \curvearrowleft_v & 
 \end{array}$$

Suppose that  $M$  is equipped with a holomorphic symplectic form  $\omega \in H^0(M, \wedge^2 T_M^*)$ . Via the isomorphism  $\nu$  we obtain a relative symplectic form  $\Omega_\pi \in H^0(X, \wedge^2 T_{X/M}^*)$ . We say that a connection on  $\pi$  is symplectic if it preserves  $\Omega_\pi$ .

**Definition 2.4** A pre-Joyce structure on a complex manifold  $M$  consists of

- (i) a holomorphic symplectic form  $\omega$  on  $M$ , and
- (ii) a nonlinear connection  $h$  on the tangent bundle  $\pi: X = T_M \rightarrow M$ ,

such that for each  $\epsilon \in \mathbb{C}^*$ , the connection  $h_\epsilon = h + \epsilon^{-1}v$  is flat and symplectic.

To clarify this definition we now describe it in local coordinates, although the resulting expressions will play no role in what follows. Given a local coordinate system  $(z_1, \dots, z_n)$  on  $M$  there are associated linear coordinates  $(\theta_1, \dots, \theta_n)$  on the tangent spaces  $T_{M,m}$  obtained by writing a tangent vector in the form  $\sum_i \theta_i \cdot (\partial/\partial z_i)$ . We thus get induced local coordinates  $(z_i, \theta_j)$  on the space  $X = T_M$ . In these coordinates,  $\nu(\partial/\partial z_i) = \partial/\partial \theta_i$ .

We always assume that the coordinates  $z_i$  are Darboux, in the sense that

$$\omega = \frac{1}{2} \sum_{p,q} \omega_{pq} \cdot dz_p \wedge dz_q,$$

with  $\omega_{pq}$  a constant skew-symmetric matrix. We denote by  $\eta_{pq}$  the inverse matrix.

The fact that the connection  $h$  is flat and symplectic ensures that we can write it in Hamiltonian form

$$(3) \quad h\left(\frac{\partial}{\partial z_i}\right) = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta_{pq} \cdot \frac{\partial W_i}{\partial \theta_p} \cdot \frac{\partial}{\partial \theta_q},$$

for functions  $W_i: X \rightarrow \mathbb{C}$ . The connection  $h_\epsilon$  is then flat for all  $\epsilon \in \mathbb{C}^*$  if we can take  $W_i = \partial W / \partial \theta_i$  for a single function  $W: X \rightarrow \mathbb{C}$ , which moreover satisfies

$$(4) \quad \frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q}.$$

The function  $W$  is called the Plebański function, and the partial differential equations (4) are known as Plebański’s second heavenly equations [Dunajski and Mason 2000].

### 2.3 Complex hyperkähler structures

By a complex hyperkähler structure on a complex manifold  $X$  we mean the data of a holomorphic metric  $g: T_X \otimes T_X \rightarrow \mathcal{O}_X$ , together with endomorphisms  $I, J, K \in \text{End}_X(T_X)$  satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1,$$

which preserve  $g$ , and which are parallel for the holomorphic Levi-Civita connection  $\nabla$ :

$$(5) \quad g(R(u_1), R(u_2)) = g(u, v), \quad \nabla(R = 0, \quad R \in \{I, J, K\}.$$

Such structures have appeared before in the literature, often under different names.

Let  $M$  be a complex manifold with a holomorphic symplectic form  $\omega$ . A nonlinear connection  $h$  on the tangent bundle  $\pi: X = T_M \rightarrow M$  gives a decomposition

$$(6) \quad T_X = \text{im}(v) \oplus \text{im}(h) \cong \pi^*(T_M) \otimes_{\mathbb{C}} \mathbb{C}^2.$$

We can define a metric  $g: T_X \otimes T_X \rightarrow \mathcal{O}_X$  by taking the tensor product of  $\pi^*(\omega)$  with the standard symplectic form on  $\mathbb{C}^2$ , and an action of the quaternions on  $T_X$  by identifying the complexification of the quaternions  $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$  with the algebra  $\text{End}_{\mathbb{C}}(\mathbb{C}^2)$ . With appropriate conventions this leads to the formulae

$$(7) \quad \begin{aligned} I \circ h &= i \cdot h, & J \circ h &= -v, & K \circ h &= i \cdot v, \\ I \circ v &= -i \cdot v, & J \circ v &= h, & K \circ v &= i \cdot h, \end{aligned}$$

which should be interpreted as equalities of maps  $\pi^*(T_M) \rightarrow T_X$ , and

$$(8) \quad g(h(u_1), v(u_2)) = \frac{1}{2}\omega(u_1, u_2), \quad g(h(u_1), h(u_2)) = 0 = g(v(u_1), v(u_2)).$$

It is easily checked that  $g$  is preserved by the endomorphisms  $I, J, K$ .

The following result implies in particular that a pre-Joyce structure on a complex manifold  $M$  induces a complex hyperkähler structure on the total space  $X = T_M$ .

**Theorem 2.5** *The endomorphisms  $I, J, K$  are parallel for the Levi-Civita connection  $\nabla$  associated to  $g$  precisely if the connection  $h_\epsilon$  is flat and symplectic for all  $\epsilon \in \mathbb{C}^*$ .*

**Proof** We begin with a general remark. Let  $g: T_X \times T_X \rightarrow \mathcal{O}_X$  be a metric on a complex manifold  $X$  with associated Levi-Civita connection  $\nabla$ . Let  $R \in \text{End}_X(T_X)$  be an endomorphism which is compatible with  $g$  and satisfies  $R^2 = -1$ . We can then define a 2-form  $\Omega$  on  $X$  by setting  $\Omega_R(u_1, u_2) = g(R(u_1), u_2)$ . Let  $H \subset T_X$  denote the  $+i$  eigenbundle of  $R$ . Then standard proofs from Kähler geometry apply unchanged in this holomorphic context to give implications

$$(9) \quad \nabla(R) = 0 \implies [H, H] \subset H, \quad \nabla(R) = 0 \iff d\Omega_R = 0.$$

Return now to the setting above. For  $\epsilon \in \mathbb{C}^*$  we introduce the endomorphism

$$(10) \quad J_\epsilon = I - i\epsilon^{-1}(J + iK).$$

A simple calculation using the definitions (7) shows that  $J_\epsilon^2 = -1$ , and that the  $+i$  eigenbundle of  $J_\epsilon$  coincides with  $H_\epsilon = \text{im}(h_\epsilon)$ .

As in Section 2.2, the symplectic form  $\omega$  on  $M$  induces a relative symplectic form  $\Omega_\pi$  on the projection  $\pi: X \rightarrow M$ . Moreover, as explained before Lemma 2.3, there is then a unique 2-form  $\Omega_\epsilon$  on  $X$  satisfying the conditions

$$(11) \quad \ker(\Omega_\epsilon) = H_\epsilon \quad \text{and} \quad \Omega_\epsilon(v(u_1), v(u_2)) = \omega(u_1, u_2),$$

where  $u_1, u_2$  are arbitrary vector fields on  $M$ . Another calculation using (7) and (8) shows that this form is given explicitly by the formula

$$(12) \quad \Omega_\epsilon = \epsilon^{-2} \cdot \Omega_+ + 2i\epsilon^{-1} \cdot \Omega_I + \Omega_-, \quad \text{where } \Omega_\pm = \Omega_{J \pm iK}.$$

We can now prove the theorem. Suppose first that  $I, J, K$  are parallel. Then  $J_\epsilon$  is parallel for all  $\epsilon \in \mathbb{C}^*$ , and applying (9) with  $R = J_\epsilon$  we find that  $[H_\epsilon, H_\epsilon] \subset H_\epsilon$  and hence that  $h_\epsilon$  is flat. Since  $\Omega_\epsilon$  is also parallel and hence closed, applying Lemma 2.3 shows that  $h_\epsilon$  is symplectic. Conversely suppose that for all  $\epsilon \in \mathbb{C}^*$  the connection  $h_\epsilon$  is flat and symplectic. Then by Lemma 2.3 again,  $d\Omega_\epsilon = 0$  for all  $\epsilon \in \mathbb{C}^*$ , and this easily implies that  $d\Omega_R = 0$  for  $R \in \{I, J, K\}$ . By (9) we conclude that  $I, J, K$  are parallel.  $\square$

### 2.4 Period structures

Let  $M$  be a complex manifold and  $\mathcal{H}$  a holomorphic vector bundle on  $M$ . By a lattice in  $\mathcal{H}$  we mean a locally constant subsheaf of abelian groups  $\mathcal{H}^{\mathbb{Z}} \subset \mathcal{H}$  such that the multiplication map  $\mathcal{H}^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_M \rightarrow \mathcal{H}$  is an isomorphism. There is an induced flat linear connection  $\nabla$  on  $\mathcal{H}$  whose flat sections are  $\mathbb{C}$ -linear combinations of the sections of  $\mathcal{H}^{\mathbb{Z}}$ .

**Definition 2.6** *A period structure on a complex manifold  $M$  consists of*

- (P1) *a lattice  $T_M^{\mathbb{Z}} \subset T_M$ , whose associated flat connection we denote by  $\nabla$ , and*
- (P2) *a vector field  $Z \in \Gamma(M, T_M)$  satisfying  $\nabla(Z) = \text{id}$ .*

Let  $(T_M^{\mathbb{Z}}, \nabla, Z)$  be a period structure on a complex manifold  $M$  and take a basepoint  $p \in M$ . A basis of the free abelian group  $T_{M,p}^{\mathbb{Z}}$  extends uniquely to a basis of  $\nabla$ -flat sections  $\phi_1, \dots, \phi_n$  of  $T_M$  over a contractible open neighbourhood  $p \in U \subset M$ . Writing the vector field  $Z$  in the form  $Z = \sum_i z_i \cdot \phi_i$  then defines holomorphic functions  $z_i: U \rightarrow \mathbb{C}$ , and condition (P2) implies that  $(z_1, \dots, z_n)$  is a local coordinate system on  $M$ , and that  $\phi_i = \partial/\partial z_i$ . Note in particular that the connection  $\nabla$  is necessarily torsion-free.

Recall that an integral affine structure on a complex manifold  $M$  consists of a lattice  $T_M^{\mathbb{Z}} \subset T_M$  whose associated flat connection  $\nabla$  is torsion-free [Kontsevich and Soibelman 2006]. A local coordinate system  $(z_1, \dots, z_n)$  is then called integral affine if the tangent vectors  $\partial/\partial z_i$  lie in the lattice  $T_M^{\mathbb{Z}}$ . Such coordinate systems are uniquely defined up to affine transformations of the form  $z_i \mapsto \sum_j a_{ij} z_j + v_i$  with  $(a_{ij}) \in \text{GL}_n(\mathbb{Z})$  and  $(v_i) \in \mathbb{C}^n$ .

Given a period structure on a complex manifold  $M$  we obtain an integral affine structure by forgetting the vector field  $Z$ . A system of integral affine coordinates  $(z_1, \dots, z_n)$  will be called integral linear if  $Z = \sum_i z_i \cdot (\partial/\partial z_i)$ . Such coordinate systems are uniquely defined up to linear transformations of the form  $z_i \mapsto \sum_j a_{ij} z_j$  with  $(a_{ij}) \in \text{GL}_n(\mathbb{Z})$ . Thus a period structure can be thought of as an integral *linear* structure.

Using the connection  $\nabla$ , we can lift the vector field  $Z \in \Gamma(M, T_M)$  to a horizontal vector field  $E \in \Gamma(X, T_X)$ . Let us consider the case when there is a  $\mathbb{C}^*$  action on the manifold  $M$  whose generating vector field is  $Z$ . There is a  $\mathbb{C}^*$  action on  $X = T_M$  obtained by combining the induced action of  $\mathbb{C}^*$  on  $X = T_M$  with the rescaling action on the fibres of  $\pi: X = T_M \rightarrow M$  of weight  $-1$ . If  $m_t: M \rightarrow M$  denotes the action of  $t \in \mathbb{C}^*$  on  $M$ , this is the action for which  $t \in \mathbb{C}^*$  sends  $v \in T_{M,m}$  to  $(m_t)_*(t^{-1}v) \in T_{M,m_t(v)}$ .

**Lemma 2.7** *The generating vector field for this  $\mathbb{C}^*$  action on  $X$  is the horizontal lift  $E$ .*

**Proof** If we take a system of integral linear coordinates  $(z_1, \dots, z_n)$  on  $M$  then by definition  $Z = \sum_i z_i \cdot (\partial/\partial z_i)$ . Taking associated coordinates  $(z_i, \theta_j)$  on  $X = T_M$  as before the  $\nabla$ -horizontal lift of  $Z$  is the vector field  $E = \sum_i z_i \cdot (\partial/\partial z_i)$  on  $X$ . The  $\mathbb{C}^*$  action on  $X$  induced by that on  $M$  is given by  $(z_i, \theta_j) \mapsto (tz_i, t\theta_j)$ . Composing with the contraction in the fibres we obtain the action  $(z_i, \theta_j) \mapsto (tz_i, \theta_j)$ , whose generating vector field is  $E$ . □

**Definition 2.8** *A period structure with skew form on a complex manifold  $M$  consists of a period structure  $(T_M^{\mathbb{Z}}, \nabla, Z)$ , together with a skew-symmetric form*

$$(13) \quad \eta: T_M^* \times T_M^* \rightarrow \mathcal{O}_M,$$

such that  $\eta/2\pi i$  takes integral values on the lattices  $(T_M^{\mathbb{Z}})^* \subset T_M^*$ .

The pairing  $\eta$  is necessarily parallel for the flat connection  $\nabla$ , and it follows that it defines a holomorphic Poisson structure on  $M$ . We will be particularly interested in the case when the kernel of  $\eta$  is zero. Viewing  $\eta$  as a linear map  $T_M^* \rightarrow T_M$ , its inverse defines a complex symplectic form  $\omega \in H^0(M, \wedge^2 T_M^*)$ .

### 2.5 Joyce structures

Let  $M$  be a complex manifold with a period structure  $(T_M^{\mathbb{Z}}, \nabla, Z)$ . The rescaled lattice  $(2\pi i)T_M^{\mathbb{Z}} \subset T_M$  acts on  $X = T_M$  by translations in the fibres. We introduce the quotient

$$(14) \quad X^\# = T_M^\# = T_M / (2\pi i) T_M^{\mathbb{Z}}.$$

We also consider the involution  $\iota: X \rightarrow X$  which acts by  $-1$  on the fibres of  $\pi: X \rightarrow M$ .

**Definition 2.9** Let  $M$  be a complex manifold, and let  $\pi : X = T_M \rightarrow M$  denote the total space of the holomorphic tangent bundle. A Joyce structure on  $M$  consists of

- (a) a period structure with skew form  $(T_M^{\mathbb{Z}}, Z, \nabla, \eta)$  on  $M$ , and
- (b) a pre-Joyce structure  $(\omega, h)$  on  $M$ ,

satisfying the following conditions:

- (J1) The symplectic form  $\omega$  is the inverse of the skew form  $\eta$ .
  - (J2) The connection  $h$  is invariant under the action of the lattice  $(2\pi i) T_M^{\mathbb{Z}} \subset T_M$ .
  - (J3) If  $E$  is the  $\nabla$ -horizontal lift of the vector field  $Z$ , then for any vector field  $v$  on  $M$
- (15) 
$$h([Z, v]) = [E, h(v)].$$
- (J4) The connection  $h$  is invariant under the action of the involution  $\iota : X \rightarrow X$ .

Note that once the period structure with skew form  $(T_M^{\mathbb{Z}}, Z, \nabla, \eta)$  on  $M$  is fixed, the Joyce structure involves only one further piece of data, namely the nonlinear connection  $h$ . For the Joyce structures appearing in this paper the period structure is elementary and well-known, so all our work will go into defining the nonlinear connection  $h$ .

Let us express the conditions of Definition 2.9 in terms of a local coordinates as in Section 2.2. If we take a system of integral linear coordinates  $(z_1, \dots, z_n)$  on  $M$  then by definition  $Z = \sum_i z_i \cdot (\partial/\partial z_i)$ . Taking associated coordinates  $(z_i, \theta_j)$  on  $X = T_M$  as before, the  $\nabla$ -horizontal lift of  $Z$  is the vector field  $E = \sum_i z_i \cdot (\partial/\partial z_i)$  on  $X$ . The symmetries (J2)-(J4) then translate into the following conditions on the Plebański function:

(16) 
$$\frac{\partial^2 W}{\partial \theta_p \partial \theta_q}(z_i, \theta_j + 2\pi i k_j) = \frac{\partial^2 W}{\partial \theta_p \partial \theta_q}(z_i, \theta_j),$$

(17) 
$$W(\lambda z_i, \theta_j) = \lambda^{-1} W(z_i, \theta_j),$$

(18) 
$$W(z_i, -\theta_j) = -W(z_i, \theta_j),$$

where  $(k_1, \dots, k_n) \in \mathbb{Z}^n$  and  $\lambda \in \mathbb{C}^*$ .

The construction we describe in this paper produces what we shall call a meromorphic Joyce structure. This means that the connection  $h$  has poles on certain subsets of  $X$ . More precisely, there is an effective divisor  $D \subset X$ , and  $h$  is defined by a bundle map  $h : \pi^*(T_M) \rightarrow T_X(D)$  satisfying

(19) 
$$(\pi_* \otimes \mathcal{O}_X(D)) \circ h = 1_{\pi^*(T_M)} \otimes s_D,$$

where  $s_D : \mathcal{O}_X \rightarrow \mathcal{O}_X(D)$  is the canonical inclusion. This means that when expressed in terms of local coordinates as above, the Plebański function  $W(z_i, \theta_j)$  is a meromorphic function.

### 3 The period structure on the space of quadratic differentials

For the rest of the paper we fix an integer  $g > 1$  and a level  $\ell > 2$ . As discussed in Section 1.2 we insist that all curves  $C$  are equipped with a level  $\ell$  structure, although we suppress this from the notation. In this section we introduce the space  $M = \mathcal{M}(C, Q)$ , which will form the base of our Joyce structure. It parametrises pairs  $(C, Q)$  consisting of a smooth projective curve  $C$  of genus  $g$  equipped with a level  $\ell$  structure, and a quadratic differential  $Q \in H^0(C, \omega_C^{\otimes 2})$  with simple zeroes. Any such pair  $(C, Q)$  determines a branched double cover  $p: \Sigma \rightarrow C$ , which we call the spectral curve. We construct a period structure with skew form on  $M$ , and give a moduli-theoretic description of the fibres of the map (14) in terms of line bundles with connection on  $\Sigma$ .

#### 3.1 Moduli space of quadratic differentials

Let us begin by recalling the definition of a level structure. Given a smooth complex projective curve  $C$  of genus  $g$ , the homology group  $H_1(C, \mathbb{Z}/\ell)$  is a free  $\mathbb{Z}/\ell$ -module of rank  $2g$ . The intersection form defines a nondegenerate skew-symmetric form

$$(20) \quad \langle -, - \rangle: H_1(C, \mathbb{Z}/\ell) \times H_1(C, \mathbb{Z}/\ell) \rightarrow \mathbb{Z}/\ell.$$

A level  $\ell$  structure on  $C$  is a basis  $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$  for  $H_1(C, \mathbb{Z}/\ell)$  which is symplectic, in the sense that  $\langle \alpha_i, \alpha_j \rangle = 0 = \langle \beta_i, \beta_j \rangle$  and  $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$ .

Let  $\mathcal{M}(C)$  denote the moduli space of smooth complex projective curves of genus  $g$  equipped with a level  $\ell$  structure. As we recall in the appendix, given our assumptions  $g > 1$  and  $\ell > 2$  this is a smooth quasiprojective complex variety of dimension  $3g - 3$ .

The tangent space to  $\mathcal{M}(C)$  at a curve  $C$  is the cohomology group  $H^1(C, T_C)$ , and Serre duality gives  $H^0(C, \omega_C^{\otimes 2}) = H^1(C, T_C)^*$ , so the cotangent bundle  $T_{\mathcal{M}(C)}^*$  parametrises pairs  $(C, Q)$  consisting of a smooth complex projective curve  $C$  equipped with a level  $\ell$  structure, together with an element  $Q \in H^0(C, \omega_C^{\otimes 2})$ . We define  $\mathcal{M}(C, Q) \subset T_{\mathcal{M}(C)}^*$  to be the open subset of pairs  $(C, Q)$  for which  $Q$  has simple zeroes. Then  $M = \mathcal{M}(C, Q)$  is a smooth quasiprojective complex variety of dimension  $6g - 6$ .

#### 3.2 Spectral curve

Let  $C$  be a smooth complex projective curve of genus  $g$ , and  $Q \in H^0(C, \omega_C^{\otimes 2})$  a quadratic differential with simple zeroes. The spectral curve  $\Sigma$  associated to the pair  $(C, Q)$  is the smooth projective curve cut out inside the total space of the cotangent bundle  $T_C^*$  by the equation  $y^2 = Q$ . The projection  $(x, y) \mapsto x$  defines a branched double cover  $p: \Sigma \rightarrow C$ , and there is a covering involution  $\sigma: \Sigma \rightarrow \Sigma$  defined by  $(x, y) \mapsto (x, -y)$ . The assumption that  $Q$  has simple zeroes ensures that  $\Sigma$  is smooth, and the fact that  $Q$  has at least one zero implies that  $\Sigma$  is connected.

Denote by  $R \subset \Sigma$  the branch divisor of the map  $p: \Sigma \rightarrow C$ . It has degree  $4g - 4$ , so by the Riemann–Hurwitz formula the spectral curve  $\Sigma$  has genus  $4g - 3$ . The dual of the derivative of  $p$  defines a canonical section  $s: p^*(\omega_C) \rightarrow \omega_\Sigma$  fitting into a short exact sequence

$$(21) \quad 0 \rightarrow p^*(\omega_C) \xrightarrow{s} \omega_\Sigma \rightarrow \mathcal{O}_R \rightarrow 0.$$

On the other hand the square-root of  $p^*(Q)$  defines a section of  $p^*(\omega_C)$  with simple zeroes on  $R$ , and hence a short exact sequence

$$(22) \quad 0 \rightarrow \mathcal{O}_\Sigma \xrightarrow{\phi} p^*(\omega_C) \rightarrow \mathcal{O}_R \rightarrow 0.$$

We define the invariant and anti-invariant homology groups

$$H_1(\Sigma, \mathbb{Z})^\pm = \{\gamma \in H_1(\Sigma, \mathbb{Z}) : \sigma^*(\gamma) = \pm\gamma\},$$

and similarly for the cohomology groups  $H^1(\Sigma, \mathbb{Z})^\pm, H^1(\Sigma, \mathbb{C})^\pm$ , etc. There is a short exact sequence of free abelian groups

$$(23) \quad 0 \rightarrow H_1(\Sigma, \mathbb{Z})^- \rightarrow H_1(\Sigma, \mathbb{Z}) \xrightarrow{p_*} H_1(C, \mathbb{Z}) \rightarrow 0,$$

the map  $p_*$  being surjective because  $p$  is ramified.

Taking maps of (23) into  $\mathbb{Z}$  shows that the image of  $p^*: H^1(C, \mathbb{Z}) \rightarrow H^1(\Sigma, \mathbb{Z})$  coincides with the subgroup  $H^1(\Sigma, \mathbb{Z})^+$ . The anti-invariant homology group  $H_1(\Sigma, \mathbb{Z})^-$  is therefore free of rank  $6g - 6$ . We also consider the extended group

$$\tilde{H}^1(\Sigma, \mathbb{Z})^- := \text{Hom}_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z})^-, \mathbb{Z}) = H^1(\Sigma, \mathbb{Z})/H^1(C, \mathbb{Z}).$$

### 3.3 Period structure

Introduce the vector bundle  $\mathcal{H} \rightarrow M$  whose fibre over a point  $(C, Q)$  is the vector space  $H^1(\Sigma, \mathbb{C})^-$ . It contains a lattice  $\mathcal{H}^{\mathbb{Z}} \subset \mathcal{H}$  whose fibres are the groups  $\tilde{H}^1(\Sigma, \mathbb{Z})^-$ . The associated flat connection  $\nabla^{\text{GM}}$  on  $\mathcal{H}$  is the Gauss–Manin connection. The dual bundle  $\mathcal{H}^*$  has fibres  $H_1(\Sigma, \mathbb{C})^-$  and contains the dual lattice  $(\mathcal{H}^{\mathbb{Z}})^*$  with fibres  $H_1(\Sigma, \mathbb{Z})^-$ . The intersection form defines a parallel skew-symmetric form on  $\mathcal{H}^*$ , which takes integral values on  $(\mathcal{H}^{\mathbb{Z}})^*$ .

For each point  $(C, Q) \in M$ , the tautological 1-form  $y dx$  on  $T^*C$  restricts to a 1-form  $\lambda \in H^0(\Sigma, \omega_\Sigma)$  satisfying  $\lambda^{\otimes 2} = p^*(Q)$  and  $\sigma^*(\lambda) = -\lambda$ . This should not be confused with the section  $\phi$  appearing in (22): there is a relation  $\lambda = s \circ \phi$ . The de Rham cohomology class associated to  $\lambda$  is an element of  $H^1(\Sigma, \mathbb{C})^-$ , and the resulting map

$$(24) \quad \delta: M \rightarrow \mathcal{H}, \quad (C, Q) \mapsto [\lambda] \in H^1(\Sigma, \mathbb{C})^-,$$

is a holomorphic section of the bundle  $\mathcal{H}$ .

**Theorem 3.1** [Veech 1990] *The covariant derivative of  $\delta$  with respect to the Gauss–Manin connection defines an isomorphism  $\nabla^{\text{GM}}(\delta): T_M \rightarrow \mathcal{H}$ .*

Taking a basis  $(\gamma_1, \dots, \gamma_n) \subset H_1(\Sigma, \mathbb{Z})^-$  at some point  $(C, Q) \in M$  and extending to nearby points using the Gauss–Manin connection gives locally defined functions on  $M$

$$(25) \quad z_i = Z(\gamma_i) = ([\lambda], \gamma_i) = \int_{\gamma_i} \sqrt{Q} \quad \text{for } 1 \leq i \leq n.$$

Theorem 3.1 is then the statement that these functions are local coordinates on  $M$ . Note that the associated linear coordinates  $\theta_i = (dz_i, -)$  on the fibres of the bundle  $T_M \rightarrow M$  considered in Section 2.2 correspond, under the isomorphism of Theorem 3.1, to the functions on the fibres of the bundle  $\mathcal{H} \rightarrow M$  given by pairing with the classes  $\gamma_i$ .

We can use the isomorphism of Theorem 3.1 to transfer the data  $(\mathcal{H}^{\mathbb{Z}}, \nabla^{\text{GM}}, \delta)$  from the bundle  $\mathcal{H}$  to the tangent bundle  $T_M$ . This defines a period structure  $(T_M^{\mathbb{Z}}, \nabla, Z)$  on  $M$  for which the periods (25) are integral linear coordinates. The required identity  $\nabla(Z) = \text{id}$  holds by definition. The intersection form on  $\mathcal{H}^*$  induces a skew-symmetric form  $T_M^* \times T_M^* \rightarrow \mathcal{O}_M$  which takes integral values on the lattice  $(T_M^{\mathbb{Z}})^* \subset T_M^*$ . We obtain a period structure with skew form by taking  $\eta$  to be this form multiplied by  $2\pi i$ . Since the intersection form is nondegenerate, the inverse to  $\eta$  defines a symplectic form  $\omega \in H^0(M, \wedge^2 T_M^*)$ .

### 3.4 Prym variety and abelian connections

We denote by  $J(C)$  and  $J(\Sigma)$  the Jacobians of the curves  $C$  and  $\Sigma$ . Set

$$J(\Sigma)^- = \{M \in J(\Sigma) : M \otimes \sigma^*(M) \cong \mathcal{O}_\Sigma\}, \quad J^2(C) = \{P \in J(C) : P^{\otimes 2} \cong \mathcal{O}_C\}.$$

The pullback map  $p^*: J(C) \rightarrow J(\Sigma)$  is injective [Mumford 1974, Section 3], and we identify  $J(C)$  with its image. The Prym variety is defined by either of the two quotients

$$(26) \quad P(\Sigma) = J(\Sigma)/J(C) = J(\Sigma)^-/J^2(C).$$

To see that the two quotients in (26) are indeed the same, consider the map  $j: J(\Sigma)^- \rightarrow J(\Sigma)/J(C)$  induced by the inclusion  $J(\Sigma)^- \subset J(\Sigma)$ . Then  $j$  is surjective because for any  $M \in J(\Sigma)$  we can write  $M = N^{\otimes 2}$ , and then

$$M = (N \otimes \sigma^*(N^*)) \otimes (N \otimes \sigma^*(N)).$$

The first factor clearly lies in  $J(\Sigma)^-$ , and it is proved in [Mumford 1974, Section 3] that the second lies in  $J(C)$ . The kernel of  $j$  is the intersection  $J(C) \cap J(\Sigma)^- \subset J(\Sigma)$ , and since any element  $M \in J(C)$  satisfies  $\sigma^*(M) = M$ , this coincides with  $J^2(C)$ .

We also consider the spaces  $J^\#(C)$  and  $J^\#(\Sigma)$  of line bundles with connection. We can again identify  $J^\#(C)$  with the image of the pullback map  $p^*: J^\#(C) \rightarrow J^\#(\Sigma)$ . We set

$$J^\#(\Sigma)^- = \{(M, \partial) \in J^\#(\Sigma) : (M, \partial) \otimes \sigma^*(M, \partial) \cong (\mathcal{O}_\Sigma, d)\}.$$

Note that if  $(N, \partial_N)$  is a line bundle with connection on  $C$ , and  $P^{\otimes 2} \cong N$ , then there is a unique connection  $\partial_P$  such that  $(P, \partial_P)^{\otimes 2} \cong (N, \partial_N)$ . In particular, each  $P \in J^2(C)$  has a unique connection  $\partial_P$  satisfying  $(P, \partial_P)^{\otimes 2} = (\mathcal{O}_C, d)$ , and we can therefore identify  $J^2(C)$  with a subgroup of  $J^\#(\Sigma)$ . We define

$$(27) \quad P^\#(\Sigma) = J^\#(\Sigma)/J^\#(C) = J^\#(\Sigma)^-/J^2(C),$$

with a similar argument as before showing that these two quotients are equal.

The exact sequence (23) shows that

$$\tilde{H}^1(\Sigma, \mathbb{C}^*)^- := \text{Hom}_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z})^-, \mathbb{C}^*) \cong H^1(\Sigma, \mathbb{C}^*)/H^1(C, \mathbb{C}^*).$$

The Riemann–Hilbert isomorphism  $J^\#(\Sigma) \cong H^1(\Sigma, \mathbb{C}^*)$  then induces an isomorphism

$$(28) \quad P^\#(\Sigma) \cong \tilde{H}^1(\Sigma, \mathbb{C}^*)^-.$$

### 3.5 Anti-invariant branched connections

Let  $F$  be a vector bundle on the spectral curve  $\Sigma$ . By a branched connection on  $F$  we mean a meromorphic connection  $\partial: F \rightarrow F \otimes \omega_\Sigma(R)$  with simple poles on the branch divisor  $R \subset \Sigma$ . The line bundle  $\mathcal{O}_\Sigma(R)$  has a canonical branched connection  $d: \mathcal{O}_\Sigma(R) \rightarrow \mathcal{O}_\Sigma(R) \otimes \omega_\Sigma(R)$  induced by the de Rham differential applied to functions on  $\Sigma$  with poles on  $R$ . This connection has a simple pole with residue  $-1$  at each point of  $R$ .

The short exact sequence (22) induces an isomorphism  $p^*(\omega_C) \cong \mathcal{O}_\Sigma(R)$ . We therefore obtain a canonical branched connection  $d_\phi$  on  $p^*(\omega_C)$ . It is uniquely defined by the condition that  $\phi$  is a flat map of bundles with meromorphic connection  $(\mathcal{O}_\Sigma, d) \rightarrow (p^*(\omega_C), d_\phi)$ . We say that a line bundle with branched connection  $(L, \partial)$  on  $\Sigma$  is anti-invariant if

$$(29) \quad (L, \partial_L) \otimes \sigma^*(L, \partial_L) \cong (p^*(\omega_C), d_\phi).$$

It follows that  $\partial$  has a simple pole with residue  $-\frac{1}{2}$  at each point of  $R$ .

Let  $J_{\text{br}}^\#(\Sigma)$  denote the space of line bundles  $L$  on  $\Sigma$  equipped with anti-invariant branched connections  $\partial$ . The group  $J^2(C) \subset J^\#(\Sigma)$  acts on this space by tensor product, and in analogy with (27) we define

$$P_{\text{br}}^\#(\Sigma) = J_{\text{br}}^\#(\Sigma)/J^2(C).$$

**Lemma 3.2** *There is a canonical isomorphism  $P^\#(\Sigma) \cong P_{\text{br}}^\#(\Sigma)$ .*

**Proof** Tensor product gives  $J_{\text{br}}^\#(\Sigma)$  the structure of a torsor over  $J^\#(\Sigma)^-$ , so choosing a point  $(L_0, \partial_0) \in J_{\text{br}}^\#(\Sigma)$  gives a noncanonical identification of the two spaces

$$(30) \quad (L, \partial) \in J^\#(\Sigma) \mapsto (L, \partial) \otimes (L_0, \partial_0) \in J_{\text{br}}^\#(\Sigma)^-,$$

which descends to the quotients by  $J^2(C)$ . To obtain a canonical bijection, take a spin structure  $\omega_C^{1/2}$  on  $C$  and let  $\partial_0$  be the unique branched connection on  $L_0 = p^*(\omega_C^{1/2})$  satisfying  $(L_0, \partial_0)^{\otimes 2} = (p^*(\omega_C), d_\phi)$ . Since  $\omega_C^{1/2}$  is uniquely defined up to the action of  $J^2(C)$ , the resulting isomorphism  $P^\#(\Sigma) \cong P_{\text{br}}^\#(\Sigma)$  is canonically defined.  $\square$

## 4 The spectral correspondence

Let us fix a smooth projective curve  $C$  of genus  $g$  and a quadratic differential  $Q \in H^0(C, \omega_C^{\otimes 2})$  with simple zeroes. Let  $p: \Sigma \rightarrow C$  be the associated spectral curve with its covering involution  $\sigma$ . There is a very well-known correspondence relating  $\text{SL}_2(\mathbb{C})$  Higgs bundles on  $C$  to anti-invariant line bundles on  $\Sigma$ . In this section we show how to extend this construction to include connections. The existence of this extension seems to be little known, although it is discussed by Donagi and Pantev [2009, Section 3.2] and also appears in a paper of Arinkin [2005].

### 4.1 Definition

The  $\text{SL}_2(\mathbb{C})$  spectral correspondence [Hitchin 1987a, Section 8; Beauville et al. 1989, Section 3] defines a bijection between

- (i) rank 2 vector bundles  $E$  on  $C$ , with  $\det(E) \cong \mathcal{O}_C$ , equipped with a Higgs field  $\Phi: E \rightarrow E \otimes \omega_C$  with  $\text{tr}(\Phi) = 0$  and  $\frac{1}{2} \text{tr}(\Phi^2) = Q$ , and
- (ii) line bundles  $L$  on  $\Sigma$  satisfying  $L \otimes \sigma^*(L) \cong p^*(\omega_C)$ .

The line bundle  $L$  is obtained from the eigendecomposition of the pullback of the Higgs field  $p^*(\Phi)$ . In the reverse direction, the line bundle  $L$  is sent to the bundle  $E = p_*(L)$  equipped with the Higgs field  $\Phi$  which is the pushforward of the map  $1 \otimes \phi: L \rightarrow L \otimes p^*(\omega_C)$ .

We shall need an extension of this correspondence involving connections on the bundles  $L$  and  $E = p_*(L)$ . More precisely, the extension relates

- (i) connections  $\nabla$  on  $E$  inducing the trivial connection on  $\det(E) \cong \mathcal{O}_C$ ,
- (ii) anti-invariant branched connections  $\partial$  on the line bundle  $L$ .

We do not claim that this correspondence is a bijection in general, but it does define a birational map of the relevant moduli spaces: see Theorem 5.1 below.

The extended correspondence is defined as follows. Take a bundle  $E = p_*(L)$  on  $C$  and a connection  $\nabla$  on  $E$  as in (i). The natural transformation  $p^* p_*(L) \rightarrow L$  gives rise to a short exact sequence

$$(31) \quad 0 \rightarrow p^*(E) \xrightarrow{f} L \oplus \sigma^*(L) \rightarrow \mathcal{O}_R \rightarrow 0.$$

Since the map  $f$  is an isomorphism away from the divisor  $R \subset \Sigma$ , there is a unique meromorphic connection  $\tilde{\nabla}$  on  $L \oplus \sigma^*(L)$  with poles along  $R$  such that the map  $f$  becomes a map of bundles with meromorphic connection  $p^*(E, \nabla) \rightarrow (L \oplus \sigma^*(L), \tilde{\nabla})$ . The local calculation in the next subsection shows that the poles of  $\tilde{\nabla}$  at the points of  $R$  are simple. Taking the component of  $\tilde{\nabla}$  along  $L$  then gives the required branched connection  $\partial$ .

To prove that  $(L, \partial)$  satisfies (29), take determinants of (31) to get a map  $\det(f): \mathcal{O}_\Sigma \rightarrow L \otimes \sigma^*(L)$ . The support of the cokernel is precisely  $R$  and it follows for degree reasons that  $\text{coker}(\det(f)) = \mathcal{O}_R$ . The sequence (22) then shows that we can identify  $L \otimes \sigma^*(L)$  with  $p^*(\omega_C)$  in such a way that the map  $\det(f)$  coincides with the map  $\phi$ . But then  $\det(\tilde{\nabla}) = \partial \otimes 1 + 1 \otimes \sigma^*(\partial)$  is related to the trivial connection  $\det(p^*(\nabla))$  by the meromorphic gauge change  $\phi$ , and hence coincides with  $d\phi$ .

### 4.2 Local computation at branchpoint

Consider a  $\sigma$ -invariant neighbourhood  $U \subset \Sigma$  of a branchpoint  $p \in R$ . Choose a local coordinate  $w: U \rightarrow \mathbb{C}$  satisfying  $\sigma^*(w) = -w$ , and hence  $w(p) = 0$ . Choose also a local nonvanishing section  $s \in H^0(U, L)$ . In terms of the basis of sections  $(s, \sigma^*(s))$  we can write the induced connection  $\tilde{\nabla}$  on  $L \oplus \sigma^*(L)$  in the form

$$\tilde{\nabla} = d + \begin{pmatrix} \alpha(w) & \beta(w) \\ \gamma(w) & \delta(w) \end{pmatrix} dw,$$

where  $\alpha, \beta, \gamma, \delta: U \rightarrow \mathbb{C}$  are meromorphic functions defined in a neighbourhood of  $0 \in \mathbb{C}$ , and regular away from 0. The invariance of the pullback connection on  $p^*(E)$  implies that  $\gamma(w) = -\beta(-w)$  and  $\delta(w) = -\alpha(-w)$ .

The sequence (31) shows that a section of  $p^*(E)$  over  $U$  is determined by sections of  $L$  and  $\sigma^*(L)$  which agree at the branchpoint  $p$ . Since derivatives of regular sections of  $p^*(E)$  are also regular sections of  $p^*(E)$  we can write

$$(32) \quad \tilde{\nabla}_{\frac{\partial}{\partial w}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha(w) + \beta(w) \\ -\alpha(-w) - \beta(-w) \end{pmatrix} = \begin{pmatrix} c_+ \\ c_+ \end{pmatrix} + O(w),$$

$$(33) \quad \tilde{\nabla}_{\frac{\partial}{\partial w}} \begin{pmatrix} w \\ -w \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + w \begin{pmatrix} \alpha(w) - \beta(w) \\ \alpha(-w) - \beta(-w) \end{pmatrix} = \begin{pmatrix} c_- \\ c_- \end{pmatrix} + O(w),$$

with  $c_{\pm} \in \mathbb{C}$ . It follows that  $\alpha(w)$  and  $\beta(w)$  have at worst simple poles at  $w = 0$  and we can therefore write

$$\alpha(w) = -\frac{a}{2w} + c + O(w), \quad \beta(w) = \frac{b}{2w} - d + O(w),$$

with  $a, b, c, d \in \mathbb{C}$ . Equation (32) then implies that  $b = a$  and  $d = c$ , and (33) implies that  $a + b = 2$ .

Thus

$$\alpha(w) = -\frac{1}{2w} + c + O(w), \quad \beta(w) = \frac{1}{2w} - c + O(w),$$

for some element  $c \in \mathbb{C}$ . In particular,  $\tilde{\nabla}$  has simple poles on  $R$ . The induced branched connection on  $L$  is given by  $\partial = d + \alpha(w) dw$ .

### 4.3 Explicit description

It will be useful later to have a more detailed description of the extended spectral correspondence described in Section 4.1.

The functor  $p_* : \text{Coh}(\Sigma) \rightarrow \text{Coh}(C)$  has a left adjoint  $p^*$  and a right adjoint  $p^!$ . They are related by  $p^!(-) = p^*(-) \otimes \omega_{\Sigma/C}$ , where  $\omega_{\Sigma/C} = \omega_{\Sigma} \otimes p^*(\omega_C^\vee)$  is the relative dualising sheaf. The short exact sequence (21) induces an identification of  $\omega_{\Sigma/C}$  with  $\mathcal{O}_{\Sigma}(R)$ . We can then identify  $p^!$  with the functor  $p^*(-) \otimes \mathcal{O}_{\Sigma}(R)$ . There are natural transformations

$$(34) \quad \eta : p^* \circ p_* \rightarrow 1, \quad \chi : 1 \rightarrow p^! \circ p_*.$$

Let  $L$  be a line bundle on  $\Sigma$ . Setting  $f = (\eta_L, \eta_{\sigma^*(L)})$  and  $g = (\chi_L, \chi_{\sigma^*(L)})$  gives maps

$$(35) \quad p^*(E) \xrightarrow{f} L \oplus \sigma^*(L) \xrightarrow{g} p^*(E) \otimes \mathcal{O}_{\Sigma}(R).$$

We claim that they are mutually inverse away from the ramification divisor:

**Lemma 4.1** *Let  $s_R : \mathcal{O}_{\Sigma} \rightarrow \mathcal{O}_{\Sigma}(R)$  denote the canonical inclusion. Then*

$$(36) \quad g \circ f = 1_{p^*(E)} \otimes s_R, \quad f \otimes \omega_{\Sigma/C} \circ g = 1_{L \oplus \sigma^*(L)} \otimes s_R.$$

**Proof** As in the previous section we can view local sections of  $p^*(E)$  as consisting of pairs of local sections  $u$  of  $L$  and  $v$  of  $\sigma^*(L)$  whose restrictions to the branch divisor  $R$  coincide. Then  $\eta_L : p^*(E) \rightarrow L$  sends such a pair to the local section  $u$ , and  $\chi_L(-R) : L(-R) \rightarrow p^*(E)$  sends a local section  $u$  of  $L$  which vanishes on  $R$  to the pair  $(u, 0)$ . The result follows.  $\square$

The extended correspondence is defined by viewing the maps  $f$  and  $g$  as meromorphic gauge transformations, and using them to transfer the connection  $p^*(\nabla)$  from  $p^*(E)$  to  $L \oplus \sigma^*(L)$ . We then take the first component to obtain a meromorphic connection  $\partial$  on  $L$  with poles along  $R$ . The connection  $\partial$  is therefore given by the composite map

$$(37) \quad L \xrightarrow{\chi_L} p^*(E) \otimes \mathcal{O}_{\Sigma}(R) \xrightarrow{p^*(\nabla) \otimes 1 + 1 \otimes d} p^*(E) \otimes \mathcal{O}_{\Sigma}(R) \otimes \omega_{\Sigma}(R) \xrightarrow{\eta_L \otimes \omega_{\Sigma}(2R)} L \otimes \omega_{\Sigma}(2R).$$

Here  $d$  denotes the canonical branched connection on  $\mathcal{O}_{\Sigma}(R)$  induced by the de Rham differential, as in Section 3.5. Note that although the resulting connection  $\partial$  a priori has double poles along  $R$ , the local calculation in Section 4.2 shows that these poles are in fact of order one.

## 5 Two diagrams of moduli spaces

In this section we introduce a diagram of moduli spaces which will play a key role in our construction of the Joyce structure on the space  $\mathcal{M}(C, Q)$ . These moduli spaces parametrise smooth projective curves  $C$  of genus  $g$  equipped with various extra structures involving vector bundles, Higgs fields and connections.

We defer discussion of the existence and basic properties of the moduli spaces to the appendix. The main result of this section focuses on the two most interesting maps in our diagram:  $\alpha$  and  $\beta_\epsilon$ . We show that the first is birational and the second is generically étale.

### 5.1 The diagrams

We use the following notation:

- $C$  is a complex projective curve of genus  $g$ , equipped with a level  $\ell$  structure.
- $Q \in H^0(C, \omega_C^{\otimes 2})$  is a quadratic differential on  $C$  with simple zeroes.
- $p: \Sigma \rightarrow C$  is the spectral curve defined by the quadratic differential  $Q$ .
- $E$  is a stable rank 2 vector bundle on  $C$  with trivial determinant.
- $\Phi$  is a trace-free Higgs field on  $E$  such that  $\frac{1}{2} \text{tr}(\Phi^2)$  has simple zeroes.
- $\nabla$  is a connection on  $E$  inducing the trivial connection on  $\det(E)$ .
- $L$  is a line bundle on  $\Sigma$  such that  $L \otimes \sigma^*(L) \cong p^*(\omega_C)$  and  $p_*(L)$  is stable.
- $\partial$  is an anti-invariant branched connection on  $L$ .

Let us fix a parameter  $\epsilon \in \mathbb{C}^*$  and contemplate the diagram

$$\begin{array}{ccccc}
 & & \mathcal{M}(C, E, \nabla, \Phi) & & \\
 & \swarrow \alpha & & \searrow \beta_\epsilon & \\
 (38) \quad \mathcal{M}(C, Q, L, \partial) & & & & \mathcal{M}(C, Q, E, \nabla) \xrightarrow{\rho'} \mathcal{M}(C, E, \nabla) \\
 \pi_3 \downarrow & & & & \pi_2 \downarrow \qquad \qquad \qquad \pi_1 \downarrow \\
 \mathcal{M}(C, Q) & \xleftarrow{=} & \mathcal{M}(C, Q) & \xrightarrow{\rho} & \mathcal{M}(C)
 \end{array}$$

Each moduli space parametrises the indicated objects, and the maps  $\rho, \rho'$  and  $\pi_i$  are the obvious projections. Note that  $\mathcal{M}(C)$  and  $\mathcal{M}(C, Q) = M$  are the moduli spaces appearing in Section 3.1. The map  $\alpha$  is the extended spectral correspondence discussed in the previous subsection, and  $\beta(\epsilon)$  is defined by the rule

$$\beta_\epsilon(C, E, \nabla, \Phi) = (C, \frac{1}{2} \text{tr}(\Phi^2), E, \nabla - \epsilon^{-1} \Phi).$$

We refer the reader to the appendix for further details on the moduli spaces appearing in (38). Given our standing assumptions  $g > 1$  and  $\ell > 2$ , all the spaces appearing are smooth quasiprojective complex varieties which co-represent the relevant moduli functors.

Consider the bundle  $\mathcal{H}^\# = \mathcal{H}/(2\pi i)\mathcal{H}^{\mathbb{Z}}$  over  $M$  with fibres  $\tilde{H}^1(\Sigma, \mathbb{C}^*)^-$ . Under the isomorphism of Theorem 3.1 it corresponds to the quotient (14). We now consider a second diagram of spaces, which can

be attached to the left-hand side of (38):

$$(39) \quad \begin{array}{ccccc} T_M^\# & \xrightarrow{\nabla^{\text{GM}}(\delta)} & \mathcal{H}^\# & \xleftarrow{\tau} & \mathcal{M}(C, Q, L, \partial) \\ \pi_5 \downarrow & & \pi_4 \downarrow & & \pi_3 \downarrow \\ M & \xleftarrow{=} & M & \xleftarrow{=} & \mathcal{M}(C, Q) \end{array}$$

Note that the fibre of the map  $\pi_3$  over a pair  $(C, Q)$  is the open subset of the space  $J_{\text{br}}^\#(\Sigma)$  defined by the condition that  $p_*(L)$  is stable. The map  $\tau$  is then given on fibres by the composite of the quotient map  $J_{\text{br}}^\#(\Sigma) \rightarrow P_{\text{br}}^\#(\Sigma)$  with the isomorphism of Lemma 3.2 and the Riemann–Hilbert isomorphism (28). It is an étale map of complex manifolds.

### 5.2 Generic finiteness results

The following result contains the two main nontrivial facts we will need for the construction of our Joyce structure.

**Theorem 5.1** *Fix a genus  $g > 1$  and a level  $\ell > 2$ . Then*

- (i) *the map  $\alpha$  is birational, and*
- (ii) *the map  $\beta_\epsilon$  is generically étale.*

**Proof** Note first that the sources and targets of the maps  $\alpha$  and  $\beta_\epsilon$  are smooth quasiprojective varieties of the same dimension. By generic smoothness and the dimension theorem, for (i) it will be enough to prove that the general fibre of  $\alpha$  is a single point, and for (ii) that the general fibre of  $\beta_\epsilon$  is finite, or equivalently, that  $\beta_\epsilon$  is dominant.

For (i), suppose we have two connections on  $E$  giving rise to the same connection on  $L$ . In terms of the local computation of Section 4.2 this means that we have two possible  $\beta_i$  with the same  $\alpha$ , and in particular, the same  $c \in \mathbb{C}$ . Then the difference  $\beta_2(w) - \beta_1(w)$  is regular and vanishes at the branchpoint  $w = 0$ . Globally the difference  $(\beta_2(w) - \beta_1(w)) dw$  corresponds to a section  $\sigma^*(L) \rightarrow L \otimes \omega_\Sigma$ . But since  $\Sigma$  has genus  $4g - 3$ , and  $R$  has degree  $4g - 4$

$$(40) \quad \chi(\sigma^*(L), L \otimes \omega_\Sigma(-R)) = \chi(\omega_\Sigma(-R)) = (4g - 3) - 1 - (4g - 4) = 0,$$

and it follows that for generic  $L$  any such section is zero.

For (ii), note first that by the relation  $\nabla_\epsilon = \nabla - \epsilon^{-1}\Phi$ , the source of  $\beta_\epsilon$  may be equivalently viewed as parametrising  $C, E, \Phi$  and  $\nabla_\epsilon$ , and therefore  $\beta_\epsilon$  is a base-change of the map

$$(41) \quad \gamma: \mathcal{M}(C, E, \Phi) \rightarrow \mathcal{M}(C, E, Q)$$

defined by  $Q = \frac{1}{2} \text{tr}(\Phi^2)$ . That this map is dominant follows from the proof of [Beauville et al. 1989, Theorem 1]. Namely, we first extend  $\gamma$  by dropping the condition that  $Q$  has simple zeroes, and then show that the resulting map is dominant. But by the results of [Laumon 1988], for each curve  $C$  there exists a bundle  $E$  for which the fibre of  $\gamma$  over the point  $(C, E, 0)$  is a single point. □

Note that the proof of Theorem 5.1 actually gives more: the open locus over which the map  $\alpha$  is birational intersects each fibre  $\pi_3^{-1}(m)$ , and similarly, the open locus over which the map  $\beta_\epsilon$  is étale intersects each fibre  $\pi_2^{-1}(m)$ . We record this as:

**Corollary 5.2** *Let  $C$  be a smooth projective curve of genus  $g > 1$ , and  $Q \in H^0(C, \omega_C^{\otimes 2})$  a quadratic differential with simple zeroes. Let  $p: \Sigma \rightarrow C$  be the associated spectral curve. Then for each  $\epsilon \in \mathbb{C}^*$ , the composite  $\beta_\epsilon \circ \alpha^{-1}$  defines a dominant rational map*

$$(42) \quad \gamma_\epsilon: \mathcal{M}_\Sigma(L, \partial) \dashrightarrow \mathcal{M}(E, \nabla)$$

*between the moduli space of anti-invariant branched connections on  $\Sigma$ , and the moduli space of rank 2 connections on  $C$  with trivial determinant.* □

## 6 Symplectic forms and their preservation

In this section we introduce the relevant symplectic forms on the moduli spaces and show that our maps preserve them.

### 6.1 The Atiyah–Bott symplectic form

Let  $E$  be a bundle on  $C$  and  $\nabla: E \rightarrow E \otimes \omega_C$  a connection. There is an associated de Rham complex

$$(43) \quad 0 \rightarrow \mathcal{E}nd_{\theta_C}(E) \xrightarrow{\nabla} \mathcal{E}nd_{\theta_C}(E) \otimes \omega_C \rightarrow 0.$$

We denote by  $\mathbb{H}_{\text{dR}}^i(C, \nabla)$  the  $i^{\text{th}}$  hypercohomology of this complex. The long exact sequence in hypercohomology gives relations

$$(44) \quad \mathbb{H}_{\text{dR}}^0(C, \nabla) = H^0(C, \mathcal{E}nd_{\theta_C}(E)), \quad \mathbb{H}_{\text{dR}}^2(C, \nabla) = H^1(C, \mathcal{E}nd_{\theta_C}(E) \otimes \omega_C),$$

and a short exact sequence

$$(45) \quad 0 \rightarrow H^0(C, \mathcal{E}nd_{\theta_C}(E) \otimes \omega_C) \rightarrow \mathbb{H}_{\text{dR}}^1(C, \nabla) \rightarrow H^1(C, \mathcal{E}nd_{\theta_C}(E)) \rightarrow 0.$$

Consider the moduli space  $\text{Flat}_C(r)$  of rank  $r$  stable bundles on  $C$  equipped with a flat connection. It is a smooth, quasiprojective scheme whose tangent space at a point  $(E, \nabla)$  is the group  $\mathbb{H}_{\text{dR}}^1(C, \nabla)$ . There is a canonical isomorphism  $\int_C: H^1(C, \omega_C) \rightarrow \mathbb{C}$ . The Atiyah–Bott form is the composite of the wedge product

$$(46) \quad \mathbb{H}_{\text{dR}}^1(C, \nabla) \times \mathbb{H}_{\text{dR}}^1(C, \nabla) \xrightarrow{\wedge} \mathbb{H}_{\text{dR}}^2(C, \nabla)$$

with the canonical maps

$$(47) \quad \mathbb{H}_{\text{dR}}^2(C, \nabla) \xrightarrow{\cong} H^1(C, \mathcal{E}nd_{\theta_C}(E) \otimes \omega_C) \xrightarrow{\text{tr}} H^1(C, \omega_C) \xrightarrow{\int_C} \mathbb{C}.$$

There is a forgetful map

$$(48) \quad \pi : \text{Flat}_C(r) \rightarrow \text{Bun}_C(r)$$

to the moduli space of rank  $r$ , degree 0 stable bundles on  $C$ . The tangent space at a point of  $\text{Bun}_C(r)$  is the cohomology group  $\text{Ext}_C^1(E, E)$ . The map  $\pi$  is an affine bundle for the vector bundle over  $\text{Bun}_C(r)$  with fibres  $\text{Hom}_C(E, E \otimes \omega_C)$ , which can be identified with the cotangent bundle of  $\text{Bun}_C(r)$  using the Serre duality pairing

$$(49) \quad \text{Hom}_C(E, E \otimes \omega_C) \times \text{Ext}_C^1(E, E) \rightarrow \text{Ext}^1(E, E \otimes \omega_C) \xrightarrow{\text{tr}} H^1(C, \omega_C) \xrightarrow{\int_C} \mathbb{C}.$$

The short exact sequence (1) defined by the derivative of the map  $\pi$  can be identified with (45), and the following result is then immediate.

**Lemma 6.1** *The Atiyah–Bott form on  $\text{Flat}_C(r)$  is uniquely characterised by the following two properties:*

- (i) *The fibres of the map (48) are Lagrangian.*
- (ii) *At any point  $(E, \nabla)$  the induced pairing between the vertical tangent space for the map (48) and the cotangent space  $T_E^* \text{Bun}_C(r)$  is the Serre duality pairing (49).*

Note that in the case of bundles of rank  $r = 1$  the de Rham complex for  $(E, \nabla)$  becomes the usual de Rham complex of  $C$  and hence  $\mathbb{H}_{\text{dR}}^1(C, \nabla) = H^1(C, \mathbb{C})$ . The Atiyah–Bott pairing can then be identified with the intersection form.

## 6.2 Properties of traces

Let  $X$  be a variety, and  $E$  a vector bundle on  $X$ . Taking the trace of endomorphisms defines a map  $\text{tr} : \text{End}_{\mathcal{O}_X}(E) \rightarrow \mathcal{O}_X$ . Tensoring by a line bundle  $L$  and taking cohomology gives linear maps

$$(50) \quad \text{tr}_E : \text{Ext}_X^p(E, E \otimes L) \rightarrow H^p(X, L).$$

In the proof of Theorem 6.2 we shall need the following properties of these maps, whose proofs we leave to the reader:

- (T1) Given bundles  $E, F$  and a line bundle  $L$ , and elements  $g \in \text{Ext}_X^p(E, F)$  and  $h \in \text{Ext}_X^q(F, E \otimes L)$ , there is an identity

$$(51) \quad \text{tr}_E(h \circ g) = \text{tr}_F((g \otimes L) \circ h) \in H^{p+q}(X, L).$$

- (T2) Given a bundle  $E$  and line bundles  $L, M$  and elements  $f \in \text{Ext}_X^p(E, E \otimes L)$  and  $g \in \text{Ext}_X^q(L, M)$ , there is an identity

$$(52) \quad g \circ \text{tr}_E(f) = \text{tr}_E((1_E \otimes g) \circ f).$$

(T3) Suppose given two bundles  $E_1, E_2$  and a line bundle  $L$ , and an element  $f \in \text{Ext}_X^p(G, G \otimes L)$ , where  $G = E_1 \oplus E_2$ . Let  $s_i: E_i \rightarrow G$  be the canonical inclusions, and  $\pi_i: G \rightarrow E_i$  the canonical inclusions. Then

$$(53) \quad \text{tr}_G(f) = \text{tr}_{E_1}(\pi_1 \circ f \circ s_1) + \text{tr}_{E_2}(\pi_2 \circ f \circ s_2).$$

(T4) Given a map of varieties  $f: X \rightarrow Y$ , a bundle  $E$  on  $Y$ , a line bundle  $L$  on  $Y$  and an element  $s \in \text{Ext}^p(E, E \otimes L)$ , there is an identity

$$(54) \quad \text{tr}_{f^*(E)}(f^*(s)) = f^*(\text{tr}_E(s)) \in H^p(X, f^*(L)).$$

### 6.3 Preservation of symplectic forms

Each of the maps  $\pi_1, \dots, \pi_5$  appearing in the diagram (38) and (39) carries a natural relative symplectic form, and we claim that the horizontal maps in these diagrams preserve these forms. For the most part these statements are rather obvious, but Theorem 6.2 below is quite nontrivial.

Let us consider each vertical map in turn. For the map  $\pi_3$  we take the Atiyah–Bott form for rank 1 bundles with connection, restricted to the subset of anti-invariant connections. The tangent space to the fibres is  $H^1(\Sigma, \mathbb{C})^-$  and the symplectic form is just the usual intersection form. We then take the same form on  $\pi_4$  and by definition of the symplectic form  $\omega$  on  $M$  this induces the required relative symplectic form on the map  $\pi_5$ .

The map  $\pi_1$  is equipped with a relative form whose restriction to each fibre is the Atiyah–Bott form discussed above. Note that since we are in the  $\text{SL}_2(\mathbb{C})$  setting we should impose trace-free conditions appropriately. Since the right-hand square in (38) is Cartesian this induces a relative symplectic form on  $\pi_2$ . In terms of the map  $\gamma_\epsilon = \beta_\epsilon \circ \alpha^{-1}$  of Corollary 5.2 what is then left to prove is

**Theorem 6.2** *The pullback of the Atiyah–Bott form on  $\mathcal{M}_C(E, \nabla)$  by the rational map  $\gamma_\epsilon$  is twice the Atiyah–Bott form on  $\mathcal{M}_\Sigma(L, \partial)$ .*

**Proof** We freely use notation from Section 4.3. At the level of bundles  $\gamma_\epsilon$  is defined by  $E = p_*(L)$ . We have a map of short exact sequences in which all vertical arrows are isomorphisms

$$\begin{array}{ccccc} \text{Hom}_\Sigma(L, L \otimes \omega_\Sigma)_0 & \longrightarrow & T_{(L, \partial)}\mathcal{M}_\Sigma(L, \partial) & \longrightarrow & \text{Ext}_\Sigma^1(L, L)_0 \\ \gamma_{\epsilon,*} \uparrow & & \gamma_{\epsilon,*} \uparrow & & \downarrow p_* \\ \text{Hom}_C(E, E \otimes \omega_C)_0 & \longrightarrow & T_{(E, \nabla)}\mathcal{M}_C(E, \nabla) & \longrightarrow & \text{Ext}_C^1(E, E)_0 \end{array}$$

Take  $L$  an anti-invariant line bundle on  $\Sigma$  and set  $E = p_*(L)$ . Take elements

$$(55) \quad v \in \text{Hom}_C(E, E \otimes \omega_C), \quad w \in \text{Hom}_\Sigma^1(L, L).$$

We assume that  $\text{tr}(v) = 0$  and  $w$  is anti-invariant, meaning that  $w \otimes 1_{\sigma^*(L)} + 1_L \otimes \sigma^*(w) = 0$ . What we must show is that

$$(56) \quad \int_C \text{tr}_E(v \circ p_*(w)) = \int_\Sigma \text{tr}_L(\alpha_*(v) \circ w).$$

By construction,  $\alpha_*(v)$  is given by the composite

$$(57) \quad L \xrightarrow{\chi_L} p^*(E) \otimes \omega_{\Sigma/C} \xrightarrow{p^*(v) \otimes \omega_{\Sigma/C}} p^*(E) \otimes \omega_\Sigma \xrightarrow{\eta_L \otimes \omega_\Sigma} L \otimes \omega_\Sigma.$$

Thus we have

$$(58) \quad \int_\Sigma \text{tr}_L(\alpha_*(v) \circ w) = \int_\Sigma \text{tr}_L(\eta_L \otimes \omega_{\Sigma/C} \circ p^!(v) \circ \chi_L \circ w).$$

On the other hand, using the relation  $\int_C \psi = \int_\Sigma (s \circ p^*(\psi))$ , valid for all elements  $\psi \in H^1(C, \omega_C)$ , we can write

$$(59) \quad \int_C \text{tr}_E(v \circ p_*(w)) = \int_\Sigma s \circ p^* \text{tr}_E(v \circ p_*(w))$$

$$(60) \quad \stackrel{(T4)}{=} \int_\Sigma s \circ \text{tr}_{p^*(E)}(p^*(v) \circ p^* p_*(w))$$

$$(61) \quad \stackrel{(T2)}{=} \int_\Sigma \text{tr}_{p^*(E)}((1_{p^*(E)} \otimes s) \circ p^*(v) \circ p^* p_*(w))$$

$$(62) \quad = \int_\Sigma \text{tr}_{p^*(E)}(g \circ f \circ p^*(v) \circ p^* p_*(w))$$

$$(63) \quad \stackrel{(T1)}{=} \int_\Sigma \text{tr}_{L \oplus \sigma^*(L)}(f \otimes \omega_{\Sigma/C} \circ p^!(v) \circ p^! p_*(w) \circ g)$$

$$(64) \quad \stackrel{(T3)}{=} \int_\Sigma \text{tr}_L(\eta_L \otimes \omega_{\Sigma/C} \circ p^!(v) \circ \chi_L \circ w) + \int_\Sigma \text{tr}_{\sigma^*(L)}(\eta_{\sigma^*(L)} \otimes \omega_{\Sigma/C} \circ p^!(v) \circ \chi_{\sigma^*(L)} \circ \sigma^*(w)),$$

where we used the naturality of  $\chi$  as well as (T3) in the final step. The same argument together with the assumption  $\text{tr}_E(v) = 0$  shows that

$$(65) \quad \int_\Sigma \text{tr}_L(\eta_L \otimes \omega_{\Sigma/C} \circ p^!(v) \circ \chi_L) + \int_\Sigma \text{tr}_{\sigma^*(L)}(\eta_{\sigma^*(L)} \otimes \omega_{\Sigma/C} \circ p^!(v) \circ \chi_{\sigma^*(L)}) = 0.$$

For line bundles, (T2) shows that the trace of the composite is the composite of the traces. Using the assumption that  $w$  is anti-invariant we see that the two terms in (62) are equal, which proves the claim.  $\square$

## 7 Construction of the Joyce structure

In this section we finally construct the required meromorphic Joyce structure on the complex manifold  $M = \mathcal{M}(C, Q)$ . The key ingredient is the isomonodromy connection on the map  $\pi_1$ , which is both flat and symplectic. We pull this back across the diagrams (38) and (39) using the generic finiteness results of Theorem 5.1. This then gives the required nonlinear connections  $h_\epsilon$  on the tangent bundle of  $M$ .

### 7.1 Nonlinear connections

Let  $\pi : X \rightarrow Y$  be a smooth map of varieties. We can define the notion of a nonlinear connection on  $\pi$  exactly as before. Namely, the derivative of  $\pi$  gives a short exact sequence

$$0 \rightarrow T_{X/M} \xrightarrow{i} T_X \xrightarrow{\pi_*} \pi^*(T_Y) \rightarrow 0,$$

and we define a nonlinear connection on  $\pi$  to be a map of bundles  $h : \pi^*(T_Y) \rightarrow T_X$  satisfying  $\pi_* \circ h = \text{id}$ . Let  $D \subset X$  be an effective Cartier divisor with corresponding section  $s_D : \mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ . We define a meromorphic connection on  $\pi$  with poles along  $D$  to be a map of bundles  $h : \pi^*(T_Y) \rightarrow T_X(D)$  satisfying  $(\pi_* \otimes \mathcal{O}_X(D)) \circ h = 1_{\pi^*(T_Y)} \otimes s_D$ .

We shall need the following simple facts, whose proofs we leave to the reader:

- (C1) Suppose given smooth maps  $\pi : X \rightarrow Y$  and  $\eta : Y \rightarrow Z$  and connections  $h : \pi^*(T_Y) \rightarrow T_X$  on  $\pi$  and  $j : \eta^*(T_Z) \rightarrow T_Y$  on  $\eta$ . Then the composite  $h \circ \pi^*(j) : \pi^*\eta^*(T_Z) \rightarrow T_X$  is a connection on the map  $\eta \circ \pi : X \rightarrow Z$ .
- (C2) If  $\pi : X \rightarrow Y$  is étale then  $\pi_*^{-1} : \pi^*(T_Y) \rightarrow T_X$  is the unique connection on  $\pi$ .
- (C3) Given a Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g} & X \\ \eta \downarrow & & \downarrow \pi \\ Z & \xrightarrow{f} & Y \end{array}$$

with  $\pi$  smooth, and a connection  $h : \pi^*(T_Y) \rightarrow T_X$  on  $\pi$ , there exists a unique connection  $j : \eta^*(T_Z) \rightarrow T_W$  on  $\eta$  such that

$$(64) \quad g_* \circ j = g^*(h) \circ \eta^*(f_*) : \eta^*(T_Z) \rightarrow g^*(T_X).$$

- (C4) Suppose given a smooth map  $\pi : X \rightarrow Y$  and an open subset  $U \subset X$ . Denote the inclusion map by  $i : U \rightarrow X$ , and set  $\pi_0 = \pi \circ i$ . Then any connection  $h_0 : \pi_0^*(T_Y) \rightarrow T_U$  on  $\pi_0$  extends to a meromorphic connection on  $\pi$ . More precisely, there is an effective divisor  $D \subset X$  and a meromorphic connection  $h : \pi^*(T_Y) \rightarrow T_X(D)$  on  $\pi$  with poles along  $D$  such that  $i^*(h) = \text{id}_{T_U} \otimes i^*(s_D) \circ h_0$ .

### 7.2 Isomonodromy connection

Return to the diagram (38), and recall that we are imposing the condition that the bundles  $E$  are stable. In particular, the map

$$(65) \quad \pi_1 : \mathcal{M}(C, E, \nabla) \rightarrow \mathcal{M}(C)$$

is smooth. There is a flat nonlinear connection on this map known as the isomonodromy connection and constructed as follows. There is a map

$$(66) \quad \pi_0 : \mathcal{M}(C, \mathcal{E}) \rightarrow \mathcal{M}(C)$$

whose fibre over a curve  $C$  is the space of  $\mathrm{SL}_2(\mathbb{C})$  local systems on  $C$ . The relative Riemann–Hilbert correspondence [Deligne 1970, Theorem 2.23] gives an open embedding  $\mu: \mathcal{M}(C, E, \nabla) \rightarrow \mathcal{M}(C, \mathcal{E})$  sending a bundle with connection  $(E, \nabla)$  to its monodromy local system. The fact that the universal family of curves over  $\mathcal{M}(C)$  is locally trivial as a family of smooth surfaces defines a Gauss–Manin connection on  $\pi_0$ , which by pullback along  $\mu$  then induces the isomonodromy connection on  $\pi_1$ .

We shall need two properties of the isomonodromy connection. Firstly, despite the fact that the monodromy map  $\mu$  is highly transcendental, the isomonodromy connection is nonetheless an algebraic object. The basic reason is that the condition for a relative connection  $(\mathcal{E}, \nabla)$  on a family of curves  $f: \mathcal{C} \rightarrow T$  to be isomonodromic can be rephrased as the lifting of the relative flat connection on the bundle  $\mathcal{E}$  over  $\mathcal{C}$  to an actual flat connection, and the isomonodromy connection then arises from a zero curvature condition. A more abstract approach using crystals was explained by Simpson [1994b, Section 8].

The second property of the isomonodromy connection we need is that it is symplectic. Goldman [1984] proved that the Riemann–Hilbert map  $\mu$  takes the Atiyah–Bott symplectic form on  $\mathcal{M}_C(E, \nabla)$  to a natural symplectic form on the character variety defined using group cohomology. The only important point for us is that this second symplectic form is defined topologically, and is therefore independent of the complex structure on  $C$ . It follows that the parallel transport maps for the isomonodromy connection preserve the Atiyah–Bott symplectic form on the fibres of  $\pi_1$ .

The right-hand square in (38) is Cartesian so we can pull back the isomonodromy connection using (C3) to obtain a connection on  $\pi_2$ . Using (C1), (C2) and Theorem 5.1 we obtain a connection on an open subset of  $\pi_3$ . Note that the bundle of groups  $J^2(C)$  over  $\mathcal{M}(C)$  acts by tensor product on the upper row of the diagram (38). Tensoring  $(E, \nabla)$  by an element  $(P, \partial_P) \in J^2(C)$  multiplies the monodromy by a homomorphism  $H_1(C, \mathbb{Z}) \rightarrow \{\pm 1\}$ . This action clearly preserves the isomonodromy connection, and the connection on  $\pi_3$  therefore descends along the map  $\tau$  appearing in (39). Using (C4) we can extend it to a meromorphic connection. Continuing across this diagram, we finally obtain a meromorphic connection  $h_\epsilon$  on a dense open subset of  $\pi_5: T_M^\# \rightarrow M$ .

### 7.3 Joyce structure

In the last section we showed how to construct, for each  $\epsilon \in \mathbb{C}^*$ , a nonlinear connection  $h_\epsilon$  on a dense open subset of  $\pi_5$ . This connection is flat and symplectic because it is a pullback of the isomonodromy connection. To produce a pre-Joyce structure it remains to prove that as  $\epsilon \in \mathbb{C}^*$  varies these connections form a  $v$ -pencil, ie that  $h_\epsilon = h + \epsilon^{-1}v$ , where  $h = h_\infty$ .

Consider the automorphism

$$r_\epsilon: \mathcal{M}(C, E, \nabla, \Phi) \rightarrow \mathcal{M}(C, E, \nabla, \Phi), \quad (C, E, \nabla, \Phi) \mapsto (C, E, \nabla + \epsilon^{-1}\Phi, \Phi).$$

Then  $\beta_\epsilon = \beta_\infty \circ r_{-\epsilon}$ , and it follows that  $h_\epsilon = (r_\epsilon)_* \circ h$  as maps of bundles  $\pi^*(T_M) \rightarrow T_X$ . It will be enough to show that  $(r_\epsilon)_* = \mathrm{id} + \epsilon^{-1}(v \circ \pi_*)$ .

Recall the tautological differential  $\lambda \in H^0(\Sigma, \omega_\Sigma)$  of Section 3.1 whose periods (25) around a basis of cycles  $(\gamma_1, \dots, \gamma_n) \subset H_1(\Sigma, \mathbb{Z})^-$  define local flat integral coordinates  $z_i$  on  $M$ . After transferring along the birational map  $\alpha$ , the map  $r_\epsilon$  becomes

$$r_\epsilon: \mathcal{M}(C, Q, L, \partial) \rightarrow \mathcal{M}(C, Q, L, \partial), \quad (C, Q, L, \partial) \mapsto (C, Q, L, \partial + \epsilon^{-1}\lambda).$$

Thus  $r_\epsilon$  is the operation of tensoring  $(L, \partial_L) \in J_{\text{br}}^\#(\Sigma)$  with  $(\mathcal{O}_\Sigma, d + \epsilon^{-1}\lambda) \in J^\#(\Sigma)^-$ . Note that the monodromy of this second connection around a cycle  $\gamma_i$  is just  $\exp(\epsilon^{-1}z_i)$ . We now transfer the automorphism  $r_\epsilon$  across the diagram (39). The map  $\sigma$  takes a point of the space  $\mathcal{M}(C, Q, L, \partial)$  to the monodromy of the product (30). Thus taking fibre coordinates  $\theta_i$  as in Section 2.2, we finally arrive at the automorphism of  $X = T_M$  given in local coordinates by  $\theta_i \mapsto \theta_i + \epsilon^{-1}z_i$ , and the claim follows.

The final step is to check the compatibility between the period structure and the pre-Joyce structure. Conditions (J1) and (J2) of Definition 2.9 hold by construction. It remains to consider (J3) and (J4).

For (J3) we consider the  $\mathbb{C}^*$  action on  $M = \mathcal{M}(C, Q)$  for which  $t \in \mathbb{C}^*$  acts by  $t \cdot (C, Q) = (C, t^2 \cdot Q)$ . Combining the induced action on  $X = T_M$  with the rescaling action on the fibres as in the paragraph before Lemma 2.7 gives a  $\mathbb{C}^*$  action on  $X = T_M$ . It is not hard to see that this descends to  $X^\#$ , and that when transferred across the diagrams (38) and (39) it becomes the  $\mathbb{C}^*$  action on  $\mathcal{M}(C, E, \nabla, \Phi)$  which sends  $\Phi \mapsto t\Phi$  and leaves  $(C, E, \nabla)$  fixed. We denote by  $m_t: M \rightarrow M$  and  $n_t: X \rightarrow X$  the resulting actions of  $t \in \mathbb{C}^*$ . Note that the involution  $\iota: X \rightarrow X$  coincides with  $n_{-1}$ .

After Lemma 2.7, to check (J3) it will be enough to show that for any vector field  $v$  on  $M$  we have  $(n_t)_*(h(v)) = h((m_t)_*(v))$ . Taking  $t = -1$  this will also imply (J4). To prove this identity it is enough to show that  $(n_t)_*(h(v))$  is a horizontal vector field for  $h$ . But this is clear by construction of  $h$  since in the diagram (38),  $\rho' \circ n_t = \rho'$ .

### 7.4 Restriction to the zero-section

It was explained in [Bridgeland and Strachan 2021, Section 3.2] that the involution property (iii) above implies that, when restricted to the zero-section  $M \subset X = T_M$ , the holomorphic Levi-Civita connection of the complex hyperkähler structure on  $X$  induces a flat, torsion-free connection  $\nabla^J$  on the tangent bundle of  $M$ . This connection was referred to in [Bridgeland 2021, Section 7] as the linear Joyce connection, and is given in coordinates by

$$(67) \quad \nabla_{\frac{\partial}{\partial z_i}}^J \left( \frac{\partial}{\partial z_j} \right) = \sum_{p,q} \eta_{qp} \cdot \frac{\partial^3 W}{\partial \theta_i \partial \theta_j \partial \theta_p} \Big|_{\theta=0} \cdot \frac{\partial}{\partial z_q}.$$

Note, however, that locating the poles of the Joyce structure constructed above is a subtle problem, and in particular it is not clear whether the structure is regular along the zero-section  $M \subset X = T_M$ . This submanifold  $M \subset X$  is the fixed locus of the involution  $\iota$ , and when transferred across the diagram (39) corresponds to the multisection of  $\pi_3$  consisting of points satisfying  $(L, \partial_L)^{\otimes 2} = (p^*(\omega_C), \partial_{\text{can}})$ . Understanding the properties of the Joyce structure at these points is difficult however, because they lie in

the exceptional locus of the birational map  $\alpha$ . Note that in the one example that has been computed in detail [Bridgeland and Masoero 2023], the connection  $\nabla^J$  is indeed well-defined, and turns out to be quite natural.

## 8 Good Lagrangian submanifolds

The spaces of complex and Kähler parameters on a compact Calabi–Yau threefold are expected to appear as complex Lagrangian submanifolds in the stability space of the associated  $CY_3$  triangulated category. It has been a longstanding question to try to abstractly characterise these submanifolds in stability space; see eg [Bridgeland 2009, Section 7]. In this section we give a general definition of a good Lagrangian submanifold  $B \subset M$  in the base of a Joyce structure. We then prove that for the Joyce structures constructed in Section 7, the submanifolds in  $M = \mathcal{M}(C, Q)$  obtained by fixing the curve  $C$  and varying the quadratic differential  $Q$  are good Lagrangians in this sense.

### 8.1 General definition

Consider a Joyce structure on a complex manifold  $M$  and a complex Lagrangian submanifold  $B \subset M$ . Consider the normal bundle  $\pi : N_B \rightarrow B$  fitting into the sequence

$$(68) \quad 0 \rightarrow T_B \xrightarrow{i} T_M|_B \xrightarrow{k} N_B \rightarrow 0.$$

Recall the pencil of connections  $h_\epsilon = h + \epsilon^{-1}v$  on the bundle  $\pi : X = T_M \rightarrow M$ . For any complex submanifold  $B \subset M$ , and any  $\epsilon^{-1} \in \mathbb{C}$ , the connection  $h_\epsilon$  restricts to a connection  $h_\epsilon|_B$  on the bundle  $X_B = T_M|_B \rightarrow B$ .

**Definition 8.1** *A complex Lagrangian submanifold  $B \subset M$  will be called good if the restricted connection  $h_\epsilon|_B$  descends via the map  $k : X_B \rightarrow N_B$  to a connection  $n$  on the normal bundle  $\pi : N_B \rightarrow B$ .*

To explain this condition in more detail, take  $x \in X_B$  with  $\pi(x) = b \in B$ . The bundle map  $k$  defines a map of complex manifolds  $k : X_B \rightarrow N_B$ , and we set  $y = k(x) \in N_B$ .

$$(69) \quad \begin{array}{ccccc} X & \hookrightarrow & X_B & \xrightarrow{k} & N_B \\ \pi \downarrow & & \downarrow & & \downarrow \pi \\ M & \hookrightarrow & B & \xrightarrow{=} & B \end{array}$$

Given a vector  $w \in T_b B \subset T_b M$  the connection  $h_\epsilon$  defines a lift  $h_\epsilon(w) \in T_x X_B \subset T_x X$ , and we define  $n(w) = k_*(h_\epsilon(w)) \in T_y N_B$ . Note that  $n(w)$  is independent of  $\epsilon$ , since  $k_*(v(w)) = 0$ . The condition of Definition 8.1 is that  $n(w)$  depends only on  $y \in N_B$ , not on the element  $x \in X_B$  satisfying  $k(x) = y$ . When this condition holds, the map  $n$  defines a connection on  $\pi : N_B \rightarrow B$ .

**Lemma 8.2** *If  $B \subset M$  is a good Lagrangian then the induced connection  $n$  on the normal bundle  $\pi : N_B \rightarrow B$  is flat.*

**Proof** Recall that if  $f : M \rightarrow N$  is a map of complex manifolds, and  $u, w$  are vector fields on  $M, N$ , respectively, then  $u, w$  are said to be  $f$ -related if  $f_*(u_m) = w_{f(m)}$  for all  $m \in M$ . Given vector fields  $u_1, u_2$  on  $M$  which are  $f$ -related to vector fields  $w_1, w_2$  on  $N$ , it is easily checked that  $[u_1, u_2]$  is  $f$ -related to  $[w_1, w_2]$ . We will apply this to the map  $k : X_B \rightarrow N_B$ .

Given a vector field  $u$  on  $B$ , we can extend it to a vector field on  $M$  which we also denote by  $u$ . We can then use the connection  $h_\epsilon$  to lift it to the vector field  $h_\epsilon(u)$  on  $X$ . The restriction of this vector field to  $X_B$  is a vector field on  $X_B$ , and is independent of the chosen extension. The good Lagrangian condition states that this vector field on  $X_B$  is  $k$ -related to a vector field on  $N_B$ , which by definition is  $n(u)$ .

The connection  $h_\epsilon$  being flat is the condition that for any vector fields  $u_1, u_2$  on  $M$  we have  $h_\epsilon([u_1, u_2]) = [h_\epsilon(u_1), h_\epsilon(u_2)]$ . But then it follows that  $h_\epsilon([u_1, u_2])$  is  $k$ -related to  $[n(u_1), n(u_2)]$ , which by definition of  $n$  implies that  $n([u_1, u_2]) = [n(u_1), n(u_2)]$  and hence that the connection  $n$  is flat.  $\square$

To express the good Lagrangian condition more concretely, take local Darboux coordinates  $(z_1, \dots, z_{2d})$  on  $M$  and assume that  $B \subset M$  is given by the equations  $z_{d+1} = \dots = z_{2d} = 0$ , and that  $\omega_{pq} = \pm 1$  if  $q-p = \pm d$  and is otherwise zero. Lifting the vector fields  $\partial/\partial z_i$  with  $1 \leq i \leq d$  from  $M$  to  $X$  as in (3) gives

$$(70) \quad v_i = \frac{\partial}{\partial \theta_i}, \quad h_i = \frac{\partial}{\partial z_i} + \sum_{j=1}^d \left( \frac{\partial^2 W}{\partial \theta_i \partial \theta_{j+d}} \cdot \frac{\partial}{\partial \theta_j} - \frac{\partial^2 W}{\partial \theta_i \partial \theta_j} \cdot \frac{\partial}{\partial \theta_{j+d}} \right).$$

Applying the projection  $k : T_M|_B \rightarrow N_B$  amounts to setting  $\partial/\partial \theta_i = 0$  for  $1 \leq i \leq d$ . The condition of Definition 8.1 is then that for  $1 \leq i \leq d$  the result of this projection should be independent of the coordinates  $\theta_i$  for  $1 \leq i \leq d$ . This is equivalent to

$$(71) \quad \frac{\partial^3 W}{\partial \theta_i \partial \theta_j \partial \theta_k} = 0 \quad \text{for } 1 \leq i, j, k \leq d$$

along the locus  $z_{d+1} = \dots = z_{2d} = 0$ .

There is a canonical real structure on the tangent bundle  $T_M$  whose fixed locus  $T_M^{\mathbb{R}} \subset T_M$  is the real span of the integral affine structure  $T_M^{\mathbb{Z}} \subset T_M$ . We call a complex Lagrangian  $B \subset M$  nondegenerate if  $T_B \cap T_M^{\mathbb{R}}|_B = (0) \subset T_M|_B$ . When this holds, the restriction of the map  $k$  to the lattice  $T_M^{\mathbb{Z}}|_B \subset T_M|_B$  is injective, and we denote its image by  $N_B^{\mathbb{Z}} \subset N_B$ . When the Lagrangian  $B$  is both good and nondegenerate, the connection  $n$  of Lemma 8.2 descends to a connection on the projection  $\pi : N_B/N_B^{\mathbb{Z}} \rightarrow B$  whose fibres are compact tori  $\mathbb{C}^d/\mathbb{Z}^{2d} \cong (S^1)^{2d}$ .

### 8.2 Class $S[A_1]$ examples

Consider the Joyce structure on the space  $M = \mathcal{M}(C, Q)$  constructed in this paper. Let us fix a curve  $C \in \mathcal{M}(C)$  and consider the Lagrangian submanifold  $B = \mathcal{M}_C(Q) \subset M$  which is the corresponding fibre of the projection  $\rho : \mathcal{M}(C, Q) \rightarrow \mathcal{M}(C)$ . Thus  $B \subset H^0(C, \omega_C^{\otimes 2})$  parametrises quadratic differentials on  $C$  with simple zeroes.

Note that if  $Q' \in H^0(C, \omega_C^{\otimes 2})$  is a quadratic differential on  $C$ , then  $p^*(Q')$  vanishes to order two along the branch divisor  $R \subset \Sigma$ . It follows that the tangent space to  $B$  at a point  $(C, Q)$  can be identified with  $H^0(\Sigma, \omega_\Sigma)^-$  via the map

$$(72) \quad T_b B = H^0(C, \omega_C^{\otimes 2}) \rightarrow H^0(\Sigma, \omega_\Sigma)^-, \quad Q' \mapsto p^*(Q')/2\lambda,$$

where  $\lambda$  is the tautological 1-form on  $\Sigma$  appearing in (24). Under the isomorphism of Theorem 3.1 the sequence (68) then corresponds to the Hodge filtration

$$(73) \quad 0 \rightarrow H^0(\Sigma, \omega_\Sigma)^- \xrightarrow{i} H^1(\Sigma, \mathbb{C})^- \xrightarrow{k} H^1(\Sigma, \theta_\Sigma)^- \rightarrow 0.$$

It follows from the isomorphism (28) that the fibres

$$(74) \quad H^1(\Sigma, \theta_\Sigma)^- / \tilde{H}^1(\Sigma, \mathbb{Z}) = P^\#(\Sigma) / H^0(\Sigma, \omega_\Sigma)^-$$

of the map  $\pi : N_B / N_B^{\mathbb{Z}} \rightarrow B$  are the Prym varieties  $P(\Sigma)$  appearing in Section 3.4.

**Lemma 8.3** *For each curve  $C$  the submanifold  $B = \mathcal{M}_C(Q) \subset M = \mathcal{M}(C, Q)$  is a good Lagrangian. The horizontal leaves of the induced meromorphic flat connection on  $\pi : N_B / N_B^{\mathbb{Z}} \rightarrow B$  are defined by the condition that  $E = p_*(L)$  is constant.*

**Proof** We use the notation  $\mathcal{M}_C(Q, L, \partial)$  to denote the space parametrising data  $(Q, L, \partial)$  on the fixed curve  $C$ , and similarly for  $\mathcal{M}_C(Q, E, \nabla_\epsilon)$ , etc. Let  $w$  be a vector field on  $B \subset M$  and let  $u = h_\epsilon(w)$  be the lift to a vector field on  $X_B \subset X$ . Transferring across the diagram (39) we can consider  $u$  to be a vector field on  $\mathcal{M}_C(Q, L, \partial)$ , and we must show that it descends to the space  $\mathcal{M}_C(Q, L)$ . That is, the flow of the line bundle  $L$  under  $u$  should be independent of the connection  $\partial$ . Passing through the diagram (38) we can view  $u$  as a vector field on  $\mathcal{M}_C(Q, E, \nabla_\epsilon)$ , and we must show that it descends to the space  $\mathcal{M}_C(Q, E)$ .

By definition, the connection  $h_\epsilon$  on the projection  $\pi_2$  is pulled back from the isomonodromy connection on  $\pi_1$ . Since  $\rho_*(w) = 0$ , it follows that  $\rho'_*(u) = 0$ . That is,  $u$  is obtained by keeping the pair  $(E, \nabla_\epsilon)$  on  $C$  fixed as  $Q$  varies with  $w$ . It is then clear that  $u$  descends to  $\mathcal{M}_C(Q, E)$ , and the result follows.  $\square$

Consider the diagram of moduli spaces

$$(75) \quad \begin{array}{ccccc} N_B / N_B^{\mathbb{Z}} & \xleftarrow{\tau} & \mathcal{M}_C(Q, L) & \xrightarrow{\kappa} & \mathcal{M}_C(E, \Phi) \\ \pi \downarrow & & \downarrow & & \downarrow \\ B & \xleftarrow{=} & \mathcal{M}_C(Q) & \xleftarrow{=} & \mathcal{M}_C(Q) \end{array}$$

Here  $\kappa$  is the isomorphism defined by the usual spectral construction sending a line bundle  $L$  on  $\Sigma$  to the Higgs bundle  $(E, \Phi)$  on  $C$ , and  $\tau$  is induced by the corresponding map from (39). The forgetful map  $\mathcal{M}_C(E, \Phi)_0 \rightarrow \mathcal{M}_C(E)$  can be identified with the cotangent bundle of  $\mathcal{M}_C(E)$ , and according to Lemma 8.3, when transferred across the diagram (75), the fibres of this map become the horizontal leaves of the connection  $n$ .

## Appendix Definition of the moduli spaces

In this section we give detailed constructions of the moduli spaces appearing in the diagram (38). All schemes are over  $\text{Spec}(\mathbb{C})$ . We fix a genus  $g > 1$  and a level  $\ell > 2$  throughout. We use the terminology bundle for locally free sheaf of finite rank, and line bundle for invertible sheaf.

### A.1 Curves with level structure

A family of genus  $g$  curves is a smooth proper map of schemes  $f: \mathcal{C} \rightarrow S$  of relative dimension 1 whose geometric fibres are connected and of genus  $g$ . Given a map of schemes  $s: S' \rightarrow S$  we can pull back the family by forming the Cartesian diagram

$$(76) \quad \begin{array}{ccc} \mathcal{C}' & \xrightarrow{t} & \mathcal{C} \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{s} & S \end{array}$$

Given a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  as above, there is a locally constant sheaf of free  $\mathbb{Z}/\ell$ -modules  $\mathcal{V}_f = \mathbb{R}^1 f_*(\mathbb{Z}/\ell)$  on  $S$ . This construction commutes with base-change: given a diagram (76) there is a canonical isomorphism  $\mathcal{V}_{f'} \cong s^*(\mathcal{V}_f)$ . In particular, the pullback of  $\mathcal{V}_f$  to a  $\mathbb{C}$ -valued point of  $S$  is the cohomology group  $H^1(C, \mathbb{Z}/\ell) \cong (\mathbb{Z}/\ell)^{\oplus 2g}$  of the corresponding genus  $g$  curve  $C$ . The intersection form on these cohomology groups defines a skew-symmetric  $\mathbb{Z}/\ell$ -bilinear form

$$(77) \quad \varpi_f: \mathcal{V}_f \times \mathcal{V}_f \rightarrow \mathbb{Z}/\ell.$$

Let  $V$  be the free  $\mathbb{Z}/\ell$ -module on the symbols  $a_1, b_1, \dots, a_g, b_g$  equipped with the standard skew-symmetric form defined by

$$(78) \quad \varpi(a_i, a_j) = 0 = \varpi(b_i, b_j) \quad \text{and} \quad \varpi(a_i, b_j) = \delta_{ij} \quad \text{for } 1 \leq i, j \leq g.$$

A level  $\ell$  structure on the family  $f$  is defined to be an isomorphism of  $\mathbb{Z}/\ell$ -modules  $\theta: V \rightarrow H^0(S, \mathcal{V}_f)$  relating the forms  $\varpi$  and  $\varpi_f$ . Given a diagram (76), a level  $\ell$  structure on the family  $f$  defines a pulled-back level structure on the family  $f'$  in the obvious way.

Two families of curves  $f_i: \mathcal{C}_i \rightarrow S$  with level  $\ell$  structure  $\theta_i$  are isomorphic if there is an isomorphism  $g: \mathcal{C}_1 \rightarrow \mathcal{C}_2$  satisfying  $f_2 \circ g = f_1$  and preserving the level structures in the obvious way. There is a functor  $M(g, \ell): (\text{Sch}/\mathbb{C})^{\text{op}} \rightarrow \text{Sets}$  by sending a scheme  $S$  to the set of isomorphism classes of families of genus  $g$  curves over  $S$  equipped with level  $\ell$  structure.

**Theorem A.1** *When  $g > 1$  and  $\ell > 2$ , the functor  $M(g, \ell)$  is represented by a smooth quasiprojective scheme  $\mathcal{M}(g, \ell)$ .*

**Proof** This appears to be standard, although it is hard to find a complete proof in the literature. Grothendieck [1962, Section 2] shows that  $M(g, \ell)$  is representable by an algebraic space for  $\ell \gg 0$ , and

attributes to Serre the statement that  $\ell > 2$  is sufficient. Mumford, Fogarty and Kirwan [Mumford et al. 1994, Theorem 7.9] prove the analogous result on moduli spaces of abelian varieties, which implies that  $\mathcal{M}(g, \ell)$  is representable by a quasiprojective scheme for  $\ell \gg 0$ , and they again attribute to Serre the statement that  $\ell > 2$  is sufficient.  $\square$

The closed points of  $\mathcal{M}(g, \ell)$  parametrise smooth projective genus  $g$  curves  $C$  equipped with a choice of symplectic basis in the homology group  $H_1(C, \mathbb{Z}/\ell)$ .

## A.2 Quadratic differentials

Given a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  we denote by  $\omega_{\mathcal{C}/S}$  the relative cotangent bundle. Given a Cartesian diagram (76) there is a canonical isomorphism  $\omega_{\mathcal{C}'/S'} \cong g^*(\omega_{\mathcal{C}/S})$ .

If  $C$  is any fibre of  $f$  then Serre duality gives  $H^1(C, \omega_C^{\otimes 2}) = H^0(C, T_C)^* = 0$ . Using cohomology and base-change it follows that  $f_*(\omega_{\mathcal{C}/S}^{\otimes 2})$  is a vector bundle on  $S$ . By a quadratic differential on the family of curves  $f$  we mean a section of this vector bundle. Note that

$$(79) \quad H^0(S, f_*(\omega_{\mathcal{C}/S}^{\otimes 2})) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S}^{\otimes 2}).$$

Applying this construction to the universal family of curves defines a vector bundle  $\mathcal{E}$  on the space  $\mathcal{M}(g, \ell)$ .

Define a functor  $Quad(g, \ell): \text{Sch}/\mathbb{C}^{\text{op}} \rightarrow \text{Sets}$  by sending a scheme  $S$  to the set of isomorphism classes of families of genus  $g$  curves equipped with level structures and quadratic differentials.

**Lemma A.2** *The functor  $Quad(g, \ell)$  is represented by smooth quasiprojective variety  $Quad(g, \ell)$  which is the total space of the vector bundle  $\mathcal{E}$  over  $\mathcal{M}(g, \ell)$ .*

The closed points of  $Quad(g, \ell)$  parametrise smooth projective genus  $g$  curves  $C$  equipped with a choice of symplectic basis in the homology group  $H_1(C, \mathbb{Z}/\ell)$  and a quadratic differential  $Q \in H^0(C, \omega_C^{\otimes 2})$ .

Consider a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  and a section  $Q \in H^0(\mathcal{C}, \omega_{\mathcal{C}/S}^{\otimes 2})$ . The relative critical locus of  $Q$  is a closed subscheme of  $\mathcal{C}$ , and since  $f$  is proper, its image is a closed subscheme in  $S$ . This construction commutes with base-change. Applying it to the universal family defines a closed subscheme of  $Quad(g, \ell)$ . We define  $Quad_0(g, \ell) \subset Quad(g, \ell)$  to be the complementary open subscheme. By definition, a closed point of  $Quad(g, \ell)$  lies in this open subscheme precisely if the corresponding quadratic differential  $Q \in H^0(C, \omega_C^{\otimes 2})$  has simple zeroes.

## A.3 Bundles, Higgs fields and flat connections

Let  $f: \mathcal{C} \rightarrow S$  be a family of genus  $g$  curves over a scheme  $S$ . We now give the definitions of the relative moduli spaces of bundles, Higgs bundles and flat connections we will need.

A family of rank  $r$  bundles on  $f$  is simply a rank  $r$  bundle  $E$  on  $\mathcal{C}$ . Two such families  $E_i$  are equivalent if there is a line bundle  $L$  on  $S$  and an isomorphism  $\theta: E_1 \rightarrow E_2 \otimes f^*(L)$ . A family  $E$  is said to have trivial determinant if the associated family of rank 1 bundles  $\wedge^r(E)$  is equivalent to the trivial family  $\mathcal{O}_{\mathcal{C}}$ .

A family of rank  $r$  Higgs bundles is a bundle  $E$  on  $\mathcal{C}$  equipped with a relative Higgs field  $\Phi: E \rightarrow E \otimes \omega_{\mathcal{C}/S}$ . Two such families  $(E_i, \Phi_i)$  are equivalent if there is an isomorphism  $\theta: E_1 \rightarrow E_2 \otimes f^*(L)$  which intertwines  $\Phi_1$  and  $\Phi_2 \otimes 1_{f^*(L)}$ . A family  $(E, \Phi)$  has trivial determinant if the associated family of rank 1 Higgs bundles  $(\wedge^r(E), \wedge^r(\Phi))$  is equivalent to the trivial family  $(\mathcal{O}_{\mathcal{C}}, 0)$ . (This is the usual condition that the Higgs bundle has zero trace.)

A family of rank  $r$  flat connections on  $f$  is a bundle  $E$  on  $\mathcal{C}$  equipped with a relative connection  $\nabla: E \rightarrow E \otimes \omega_{\mathcal{C}/S}$ . Two such families  $(E_i, \nabla_i)$  are equivalent if there is an isomorphism  $\theta: E_1 \rightarrow E_2 \otimes f^*(L)$  which is flat for the induced relative connection on  $\mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(E_1, E_2 \otimes f^*(L))$ . A family  $(E, \nabla)$  has trivial determinant if the corresponding family of rank 1 flat connections  $(\wedge^r(E), \wedge^r(\nabla))$  is equivalent to the trivial family  $(\mathcal{O}_{\mathcal{C}}, d)$ .

In all cases we say that a family on  $\mathcal{C}$  is stable if the restriction of the bundle  $E$  to each geometric fibre of  $f$  is stable. Note that for Higgs bundles and flat connections this is strictly stronger than the usual notion of stability. We use the stronger notion to ensure that the forgetful maps appearing in the diagram (38) are well-defined. Since stability is an open condition it corresponds to passing to open subsets of the usual moduli spaces.

Fix again a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  and assume  $S$  to be of finite type over  $\mathbb{C}$ . There is a functor

$$(80) \quad \text{Bun}(\mathcal{C}/S, r): (\text{Sch}/S)^{\text{op}} \rightarrow \text{Sets}$$

which sends a map  $m: T \rightarrow S$  to the set of equivalence classes of stable families of rank  $r$  bundles with trivial determinant on the pulled back family  $f_T: \mathcal{C} \times_S T \rightarrow T$ . We define moduli functors  $\text{Higgs}(\mathcal{C}/S, r)$  and  $\text{Flat}(\mathcal{C}/S, r)$  in the same way.

**Theorem A.3** *The functors  $\text{Bun}(\mathcal{C}/S, r)$ ,  $\text{Higgs}(\mathcal{C}/S, r)$  and  $\text{Flat}(\mathcal{C}/S, r)$  are co-representable by schemes  $\text{Bun}(\mathcal{C}/S, r)$ ,  $\text{Higgs}(\mathcal{C}/S, r)$  and  $\text{Flat}(\mathcal{C}/S, r)$  respectively. Each of these schemes is smooth and quasiprojective over  $S$ . The obvious forgetful maps  $\text{Higgs}(\mathcal{C}/S, r) \rightarrow \text{Bun}(\mathcal{C}/S, r)$  and  $\text{Flat}(\mathcal{C}/S, r) \rightarrow \text{Bun}(\mathcal{C}/S, r)$  are smooth.*

**Proof** The co-representability follows from the results of Simpson [1994a]. The other statements are easy and well-known. □

We apply these results to the universal family of curves over the moduli space  $\mathcal{M}(g, \ell)$ . We denote the resulting moduli spaces as  $\text{Bun}(g, \ell, r)$ ,  $\text{Higgs}(g, \ell, r)$  and  $\text{Flat}(g, \ell, r)$ . They are smooth quasiprojective schemes. Note for example that  $\text{Bun}(g, \ell, r)$  co-represents the functor which sends a scheme  $S$  to the set

of equivalence classes of pairs  $(f, E)$  consisting of a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  and a stable family of rank  $r$  bundles with trivial determinant  $E$  over  $f$ . Similar remarks apply to  $\text{Higgs}(g, \ell, r)$  and  $\text{Flat}(g, \ell, r)$ .

Given a family of Higgs fields  $\Phi: E \rightarrow E \otimes \omega_{\mathcal{C}/S}$  on a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  we can define a quadratic differential  $Q \in H^0(\mathcal{C}, \omega_{\mathcal{C}/S}^{\otimes 2})$  by setting  $Q = \frac{1}{2} \text{tr}(\Phi^2)$ . This defines a map

$$(81) \quad \text{Higgs}(g, \ell, r) \rightarrow \text{Quad}(g, \ell),$$

and we define  $\text{Higgs}_0(g, \ell, r)$  to be the inverse image of the open subset  $\text{Quad}_0(g, \ell)$  of quadratic differentials with simple zeroes.

#### A.4 Anti-invariant branched connections

Consider a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  equipped with a quadratic differential  $Q \in H^0(S, \omega_{\mathcal{C}/S}^{\otimes 2})$  with no relative critical points. We can form a double cover  $p: \Sigma \rightarrow \mathcal{C}$  by writing the equation  $y^2 = Q$  inside the total space of the bundle  $\omega_{\mathcal{C}/S}$ . This construction commutes with base-change in the obvious way. There is a covering involution  $\sigma: \Sigma \rightarrow \Sigma$  and a branch divisor  $R \subset \Sigma$  which is flat over  $S$ . The composite  $g = f \circ p: \Sigma \rightarrow S$  is a family of smooth genus  $4g - 3$  curves.

A family of branched connections on  $g: \Sigma \rightarrow S$  is a line bundle  $L$  on  $\Sigma$  equipped with a relative meromorphic connection  $\partial: L \rightarrow L \otimes \omega_{\Sigma/S}(R)$ . Two such families  $(L_i, \partial_i)$  are equivalent if there is a line bundle  $N$  on  $S$  and an isomorphism  $\theta: L_1 \rightarrow L_2 \otimes g^*(N)$  which is flat for the induced relative meromorphic connection on  $\mathcal{H}om_{\mathcal{O}_\Sigma}(L_1, L_2 \otimes g^*(N))$ . A family of branched connections  $(L, \partial)$  is anti-invariant if the branched connection  $(L, \partial) \otimes \sigma^*(L, \partial)$  is equivalent to the family of branched connections  $(p^*(\omega_{\mathcal{C}}), \partial_{\text{can}})$ .

There is a functor

$$(82) \quad \text{Flat}^{\text{br}}(\Sigma/S): (\text{Sch}/S)^{\text{op}} \rightarrow \text{Sets}$$

which sends a map  $m: T \rightarrow S$  to the set of equivalence classes of anti-invariant families of branched connections on the pulled back family  $g_T: \Sigma \times_S T \rightarrow T$ .

**Theorem A.4** *The functor  $\text{Flat}^{\text{br}}(\Sigma/S)$  is representable by a scheme  $\text{Flat}^{\text{br}}(\Sigma/S)$  which is smooth and quasiprojective over  $S$ .*

**Proof** For moduli spaces of flat connections with logarithmic singularities we can refer to Nitsure [1993], but since we are dealing with rank 1 connections this is really over-kill. For a more elementary approach we can pass to an étale cover of the functor  $\text{Flat}^{\text{br}}(\Sigma/S)$  by adding the data of a square-root of the line bundle  $\omega_{\mathcal{C}/S}$ . Then, as in the proof of Lemma 3.2, we can replace the branched connections  $(L, \partial_L)$  with regular connections  $(M, \partial_M)$ .  $\square$

We can apply the above construction to the universal family over  $S = \text{Quad}_0(g, \ell)$ . We denote the resulting moduli space by  $\text{Flat}^{\text{br}}(g, \ell)$ . It is a smooth quasiprojective variety. It represents the functor which sends a scheme  $S$  to the set of equivalence classes of quadruples  $(f, Q, L, \vartheta)$  consisting of a family of genus  $g$  curves  $f: \mathcal{C} \rightarrow S$  equipped with a quadratic differential  $Q \in H^0(\mathcal{C}, \omega_{\mathcal{C}/S}^{\otimes 2})$  with simple zeroes, and a family of anti-invariant branched connections  $(L, \vartheta)$  on the associated family of spectral curves  $g: \Sigma \rightarrow S$ .

### A.5 Moduli spaces and maps

We can now define the moduli spaces and maps appearing in (38). Firstly we set  $\mathcal{M}(C) = \mathcal{M}(g, \ell)$  and  $\mathcal{M}(C, Q) = \text{Quad}_0(g, \ell)$ . Then we take rank  $r = 2$  and set  $\mathcal{M}(C, E) = \text{Bun}(g, \ell, 2)$  and

$$(83) \quad \mathcal{M}(C, E, \nabla) = \text{Flat}(g, \ell, 2), \quad \mathcal{M}(C, E, \Phi) = \text{Higgs}_0(g, \ell, 2).$$

We define  $\mathcal{M}(C, E, \nabla, \Phi)$  to be the fibre product of the obvious forgetful maps

$$\begin{array}{ccc} \mathcal{M}(C, E, \nabla, \Phi) & \longrightarrow & \mathcal{M}(C, E, \Phi) \\ \downarrow & & \downarrow \\ \mathcal{M}(C, E, \nabla) & \longrightarrow & \mathcal{M}(C, E) \end{array}$$

Similarly we define  $\mathcal{M}(C, Q, E, \nabla)$  as the fibre product

$$\begin{array}{ccc} \mathcal{M}(C, Q, E, \nabla) & \longrightarrow & \mathcal{M}(C, Q) \\ \downarrow & & \downarrow \\ \mathcal{M}(C, E, \nabla) & \longrightarrow & \mathcal{M}(C) \end{array}$$

Finally we take  $\mathcal{M}(C, Q, L, \vartheta)$  to be the open subvariety of  $\text{Flat}^{\text{br}}(g, \ell)$  defined by the condition that  $E = p_*(L)$  is stable. Each of these spaces are smooth quasiprojective varieties.

The maps in the diagram (38) are just the obvious forgetful maps, with the exception of  $\alpha$  and  $\beta_\epsilon$ . To define  $\alpha$  we follow the same procedure in the text in the relative setting. To define  $\beta_\epsilon$  note that  $\beta_\epsilon = \beta_\infty \circ r_\epsilon$ , where  $r_\epsilon$  is the automorphism of the space  $\mathcal{M}(C, E, \nabla, \Phi)$  defined by  $\nabla \mapsto \nabla + \epsilon^{-1}\Phi$ . The map  $\beta_\infty$  is given by the rule  $(C, E, \nabla, \Phi) \mapsto (C, Q, E, \nabla)$  and is induced using the above fibre-product diagrams from the map (81).

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
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