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Aaron Landesman
Daniel Litt
Will Sawin

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Let $\Sigma_{g'} \to \Sigma_g$ be a cover of an orientable surface of genus g by an orientable surface of genus g', branched at n points, with Galois group H. Such a cover induces a virtual action of the mapping class group $\mathrm{Mod}_{g,n+1}$ of a genus g surface with $n+1$ marked points on $H^1(\Sigma_{g'}, \mathbb{C})$. When g is large in terms of the group H, we calculate precisely the connected monodromy group of this action. The methods are Hodge-theoretic and rely on a "generic Torelli theorem with coefficients".

14H30; 14D07, 14H10, 14H40, 14H60, 57K20

1 Introduction

A classical result in geometric topology states that the action of the mapping class group Mod_g of a surface Σ_g of genus g on its first cohomology, $H^1(\Sigma_g, \mathbb{Z})$, is via the full group of automorphisms preserving the cup product, namely $\mathrm{Sp}_{2g}(\mathbb{Z})$. Let the hyperelliptic mapping class group denote the subgroup of mapping classes commuting with an involution having genus-zero quotient. If one restricts to the hyperelliptic mapping class group the image is still of finite index in $\mathrm{Sp}_{2g}(\mathbb{Z})$; see [A'Campo 1979].

Finally, one may consider the monodromy representation on the cohomology of Prym varieties arising from connected étale double covers of genus g curves. It follows from [Looijenga 1997] that the image of this representation is finite index in $\mathrm{Sp}_{2g-2}(\mathbb{Z})$.

In order to generalize the above cases, fix an *arbitrary* finite group H and consider a maximal family of (possibly branched) Galois H-covers of curves of genus g. What is the monodromy representation on the first cohomology of these covering curves? The three cases considered above correspond to the very special cases $H = \{\mathrm{id}\}$, the case $g = 0$ and $H = \mathbb{Z}/2\mathbb{Z}$, and the case where $g \geq 2$, $H = \mathbb{Z}/2\mathbb{Z}$, and the covers are unramified. Our main result asserts that once the genus g of the base curve is sufficiently large, the connected monodromy group of this family of H-covers is as large as possible. Namely, just as the action of the mapping class group on $H^1(\Sigma_g, \mathbb{Z})$ cannot be via all of $\mathrm{GL}_{2g}(\mathbb{Z})$ because it must preserve the cup product, a symplectic form, the monodromy of families of H-covers cannot be the full general linear group, but must preserve the symplectic form and respect the H-action. Moreover, as local systems of geometric origin are semisimple, it must be semisimple. These considerations show that the identity component of the Zariski closure of the image of the monodromy representation is contained in the derived subgroup of the centralizer of H in the symplectic group. We will show that it is in fact equal to this group, once the genus of the base curve is sufficiently large.

1.1 Statement of results We next set up notation to state our main results more precisely. Let $\Sigma_{g,n}$ be an orientable surface of genus g, with n punctures, and let $\mathrm{Mod}_{g,n}$ be the pure mapping class group of $\Sigma_{g,n}$. That is, $\mathrm{Mod}_{g,n} = \pi_0(\mathrm{Homeo}^+(\Sigma_{g,n}))$, where $\mathrm{Homeo}^+(\Sigma_{g,n})$ is the space of orientation-preserving homeomorphisms of $\Sigma_g = \Sigma_{g,0}$ fixing each of the n punctures. The goal of this paper is to study certain natural representations of these mapping class groups, arising from finite unramified covers of $\Sigma_{g,n}$. We call these representations *higher Prym representations*, following the terminology of Putman and Wieland [2013].

Let H be a finite group and let $\Sigma_{g',n'} \to \Sigma_{g,n}$ be a finite unramified Galois H-cover. We will next construct a homomorphism from a finite-index subgroup of $\mathrm{Mod}_{g,n+1}$ to the centralizer of H in $\mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))$. Fix a point x of $\Sigma_{g,n}$. All finite unramified H-torsors $\Sigma_{g',n'} \to \Sigma_{g,n}$ arise from surjection $\varphi\colon \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$, with isomorphism of torsors corresponding to conjugation (by H) of surjections. For any $x' \in \Sigma_{g',n'}$ mapping to x, we can identify the kernel of φ with $K := \pi_1(\Sigma_{g',n'}, x')$. The mapping class group $\mathrm{Mod}_{g,n+1}$ acts (up to isotopy) on $(\Sigma_{g,n}, x)$, where we view x as an $(n+1)^{\mathrm{st}}$ marked point of Σ_g. Thus, $\mathrm{Mod}_{g,n+1}$ acts on $\pi_1(\Sigma_{g,n}, x)$, and hence on the finite set of homomorphisms $\pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$. The stabilizer $\mathrm{Mod}_\varphi \subset \mathrm{Mod}_{g,n+1}$ of φ acts on the kernel K of φ. The induced action on $K^{\mathrm{ab}} = H_1(\Sigma_{g',n'}, \mathbb{Z})$ preserves the kernel of the natural morphism

$$H_1(\Sigma_{g',n'}, \mathbb{Z}) \to H_1(\Sigma_{g'}, \mathbb{Z}),$$

and thus φ gives rise to a virtual action of $\mathrm{Mod}_{g,n+1}$ on $H_1(\Sigma_{g'}, \mathbb{Z})$, ie an action of Mod_φ on $H_1(\Sigma_{g'}, \mathbb{Z})$.

This action manifestly commutes with the action of H and thus defines a homomorphism

$$R_\varphi\colon \mathrm{Mod}_\varphi \to \mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H, \tag{1-1}$$

where Mod_φ is the stabilizer of φ as defined above, and $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$ denotes the centralizer of the action of H in $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))$. We may equivalently think of this as a virtual homomorphism from $\mathrm{Mod}_{g,n+1}$ to $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$, where a virtual homomorphism is a homomorphism from a finite-index subgroup.

We are interested in the *connected monodromy group* of this virtual action of $\mathrm{Mod}_{g,n+1}$, or, in other words, the identity component of the Zariski closure of the image of R_φ inside $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$. The slogan for this work is:

1.2 Slogan Monodromy groups should be as big as possible.

Based on this, we expect that, outside of a few exceptional cases, this Zariski closure should contain the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$. (After passing to a further finite-index subgroup, we may assume the monodromy group is connected, and connected components of monodromy groups of local systems of geometric origin are semisimple, so the commutator subgroup is the largest possible.) Our main result is that this expectation holds under a lower bound on the genus:

1.3 Theorem *Let H be a finite group and let $\Sigma_{g',n'} \to \Sigma_{g,n}$ be an H-cover associated to a surjection $\varphi\colon \pi_1(\Sigma_{g,n},x) \twoheadrightarrow H$, where x is a basepoint of $\Sigma_{g,n}$. Let $\overline{r}$ be the maximal dimension of an irreducible representation of H. Suppose that either*

(1) *$n=0$ and $g \ge 2\overline{r}+2$, or*

(2) *n is arbitrary and $g > \max(2\overline{r}+1, \overline{r}^2)$.*

Let Mod_φ be the stabilizer of φ inside $\mathrm{Mod}_{g,n+1}$. Then, using notation as in (1-1), the identity component of the Zariski closure of the image of

$$R_\varphi\colon \mathrm{Mod}_\varphi \to \mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$$

is the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$.

We prove Theorem 1.3 in Section 7.15. We refer to representations as in Theorem 1.3 as *higher Prym representations*.

1.4 Remark The constant $\overline{r}$ reflects the group-theoretic properties of H. For example, $\overline{r} \le \sqrt{\#H}$, and $\overline{r}$ divides the index of any normal abelian subgroup of H. The case that the covering group H is abelian was considered by Looijenga [1997]. In this case, the representation was called a Prym representation. When H is abelian, $\overline{r}=1$, and so it suffices to take $g \ge 4$ in the statement of Theorem 1.3. If H is dihedral, then $\overline{r}=2$.

1.5 Remark One may reformulate Theorem 1.3 as follows: the Zariski-closure of the virtual image of $\mathrm{Mod}_{g,n+1}$ under R_φ in $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$ is the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$. Here the *Zariski-closure of the virtual image* is the intersection of the Zariski-closures of the images of all finite-index subgroups on which R_φ is defined.

1.6 Remark The Putman–Wieland conjecture [2013] is heavily influenced by Slogan 1.2. It predicts that if $g > 2$, the virtual action of $\mathrm{Mod}_{g,n+1}$ on $H^1(\Sigma_{g'},\mathbb{C})$ has no nonzero finite orbits. (In [Putman and Wieland 2013], this conjecture was made for all $g \geq 2$. However, Marković [2022, Theorem 1.3] gave a counterexample when $g = 2$.) The virtual action of $\mathrm{Mod}_{g,n+1}$ on $H^1(\Sigma_{g'},\mathbb{C})$ has no nonzero finite orbits if and only if the Zariski closure of $\mathrm{Mod}_{g,n+1}$ in $\mathrm{Aut}(H^1(\Sigma_{g'},\mathbb{C}))$ has no nonzero finite orbits. Also the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$ has no nonzero finite orbits on $H^1(\Sigma_{g'},\mathbb{C})$, using (7-1). Therefore, the Putman–Wieland conjecture for a given covering $\Sigma_{g',n'} \to \Sigma_{g,n}$ is implied by the statement that the Zariski closure of $\mathrm{Mod}_{g,n+1}$ in $\mathrm{Aut}(H^1(\Sigma_{g'},\mathbb{C}))$ is the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$.

In particular, since the analogue of the Putman–Wieland conjecture does not hold for $g = 2$ due to Marković's counterexample [2022, Theorem 1.3], as mentioned above, it follows that no analogue of Theorem 1.3 can hold for $g = 2$. That is, some lower bound on the genus is necessary. We prove Theorem 1.3 when the genus g is bounded below by a function depending on the maximal dimension of any irreducible H-representation. It remains possible, however, that the conclusion of Theorem 1.3 holds whenever $g > 2$, independent of H. As explained in the previous paragraph, if the conclusion of Theorem 1.3 were to hold whenever $g > 2$, one would obtain the Putman–Wieland conjecture as a consequence.

It is also unsurprising that the proof of Theorem 1.3 builds on the techniques used by the first two authors to prove the Putman–Wieland conjecture for g sufficiently large in terms of H; see [Landesman and Litt 2024a, Theorem 7.2.1].

1.7 Remark It is reasonable to hope that an even stronger statement, describing the exact image of a finite-index subgroup of $\mathrm{Mod}_{g,n+1}$ and not its Zariski closure, might be true, but a proof of this would require additional ideas. In particular, it is natural to ask if the image is an arithmetic subgroup of its Zariski closure.

In proving Theorem 1.3, it is natural to decompose $H^1(\Sigma_{g'},\mathbb{C})$ into isotypic components corresponding to different irreducible representations of H. We introduce some notation to describe these:

Let G be a group and $\rho\colon G \to \mathrm{GL}_r(\mathbb{C})$ an irreducible representation of G. If ρ is self-dual, then by Schur's lemma, $(\rho \otimes \rho)^G$ is one-dimensional. As $\rho \otimes \rho = \mathrm{Sym}^2(\rho) \oplus \wedge^2\rho$, exactly one of $(\mathrm{Sym}^2(\rho))^G$, $(\wedge^2\rho)^G$ is nonzero.

1.8 Definition Let ρ be an irreducible self-dual finite-dimensional representation of a group G. If $(\mathrm{Sym}^2(\rho))^G \neq 0$, we say ρ is *orthogonally self-dual*. If $(\wedge^2\rho)^G \neq 0$ we say ρ is *symplectically self-dual*.

If $\mathbb{V}^\rho$ is a unitary local system on $\Sigma_{g,n}$ corresponding to a representation ρ of $\pi_1(\Sigma_{g,n}, x)$, we define the *weight-one* piece $W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho) \subset H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$ to be

$$W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho) := H^1(\Sigma_g, j_*\mathbb{V}^\rho),$$

where $j : \Sigma_{g,n} \hookrightarrow \Sigma_g$ is the natural inclusion. This agrees with the usual notion of weights in algebraic geometry. The groups $W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$ and $W_1H^1(\Sigma_{g,n}, \mathbb{V}^{\rho^\vee})$ are naturally dual. In particular, if ρ is self-dual, $W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$ carries a natural perfect pairing with itself. By the graded-commutativity of the cup product, $W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$ is symplectically self-dual if ρ is orthogonally self-dual, and orthogonally self-dual if ρ is symplectically self-dual.

If ρ factors through a surjection $\varphi\colon \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$ with H a finite group, there is a natural isomorphism $\rho^\sigma \xrightarrow{\sim} \rho$ for each $\sigma \in \mathrm{Mod}_\varphi$, the stabilizer of φ in $\mathrm{Mod}_{g,n+1}$. Hence Mod_φ acts naturally on $W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$, via a homomorphism we name $R_{\varphi,\rho}$. A large part of the proof of Theorem 1.3 consists of checking:

1.9 Theorem *Let H be a finite group and let $\varphi\colon \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$ be a surjective homomorphism, where x is a basepoint of $\Sigma_{g,n}$. Suppose $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ is an irreducible representation of dimension r, with r satisfying either*

(1) *$n = 0$ and $g \geq 2r + 2$, or*

(2) *n is arbitrary and $g > \max(2r + 1, r^2)$.*

Let $\mathbb{V}^\rho$ denote the local system on $\Sigma_{g,n}$ associated to $\varphi \circ \rho$. Let Mod_φ be the stabilizer of φ inside $\mathrm{Mod}_{g,n+1}$ and

$$R_{\varphi,\rho}\colon \mathrm{Mod}_\varphi \to \mathrm{GL}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$$

the natural homomorphism.

Then the image of $R_{\varphi,\rho}$ is Zariski-dense in

(1) *$\mathrm{SO}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is symplectically self-dual,*

(2) *$\mathrm{Sp}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is orthogonally self-dual, and*

(3) *the product of $\mathrm{SL}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ with a finite subgroup of the center of $\mathrm{GL}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is not self-dual.*

We prove Theorem 1.9 in Section 7.15.

One may draw a number of concrete corollaries of Theorems 1.3 and 1.9.

1.10 Corollary *Let H be a finite group and X a very general H-curve. Let $\overline{r}$ be the maximal dimension of an irreducible representation of H. Suppose either*

(1) *H acts freely on X and the genus of X/H is at least $2\overline{r}+2$, or*

(2) *the genus of X/H is greater than $\max(2\overline{r}+1, \overline{r}^2)$.*

Then the Mumford–Tate group of $H^1(X,\mathbb{Q})$ contains the commutator subgroup of $\mathrm{Sp}(H^1(X,\mathbb{Q}))^H$ and is contained in $\mathrm{GSp}(H^1(X,\mathbb{Q}))^H$.

1.11 Corollary *Let X be as in Corollary 1.10. Then the endomorphism algebra of $\mathrm{Jac}(X)$ is $\mathbb{Q}[H]$.*

These corollaries are proven in Section 7.17. We expect Corollary 1.11 will likely have applications in equivariant birational geometry, generalizing the applications in [Hassett and Tschinkel 2022, Section 7] of results of [Grunewald et al. 2015].

We conclude the paper with a result on the monodromy of certain special Kodaira fibrations, that is, surfaces with a smooth projective map to a smooth curve. We refer to these as *Kodaira–Parshin fibrations* (see Definition 9.1); loosely speaking, these Kodaira–Parshin fibrations parametrize families of covers of a fixed curve, with a moving branch point. See Theorem 9.8 for a precise algebraic statement. These results have a purely topological interpretation — namely, they say that for the representations considered in Theorem 1.3, their restrictions to certain "point-pushing" subgroups have large image; see Corollary 9.9 for a precise statement.

1.12 The large n regime Another regime in which we expect big monodromy is the case that we fix g and let n grow large. Specifically, we conjecture the analogue of Theorem 1.3 in this context.

1.13 Conjecture Let H be a finite group and let $\Sigma_{g',n'} \to \Sigma_{g,n}$ be the H-cover associated to a homomorphism $\varphi\colon \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$, where x is a basepoint of $\Sigma_{g,n}$. Let $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible H-representation and $\mathbb{V}^\rho$ the local system associated to $\rho \circ \varphi$. We conjecture there is a function $c(g, \dim\rho)$ with the following property. Suppose there are $\Delta > c(g, \dim\rho)$ points of $\Sigma_g - \Sigma_{g,n}$ such that a small loop around each of these points is sent to a nonidentity matrix under the composition $\pi_1(\Sigma_{g,n}) \xrightarrow{\varphi} H \xrightarrow{\rho} \mathrm{GL}_r(\mathbb{C})$. Then, the image of the stabilizer Mod_φ of φ in $\mathrm{Mod}_{g,n+1}$ inside $\mathrm{GL}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ is Zariski-dense in

(1) $\mathrm{SO}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is symplectically self-dual,

(2) $\mathrm{Sp}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is orthogonally self-dual, and

(3) the product of $\mathrm{SL}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ with a finite subgroup of the center of $\mathrm{GL}(W_1H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is not self-dual.

1.14 Remark (motivation from arithmetic statistics) A number of works in arithmetic statistics over function fields have proven results in a large q limit setting by computing relevant monodromy groups with finite coefficients associated to spaces of H-covers. See [Achter 2008; Ellenberg et al. 2016; Feng et al. 2023; Park and Wang 2024; Ellenberg and Landesman 2023] for a few examples of this; the last three references are connected to H-covers for a particular group H via [Ellenberg and Landesman 2023, Proposition 6.4.5].

Verifying Conjecture 1.13 would give some evidence for analogous conjectures in number theory, as it would suggest a similar big monodromy result should be true with finite coefficients.

As evidence for Conjecture 1.13, we prove the following implication of the conjecture. If the monodromy is Zariski dense in the subgroups listed in Conjecture 1.13, then it is not contained in any nontrivial parabolic, and so in particular it does not fix any vectors.

1.15 Theorem *Let H be a finite group, $\Sigma_{g',n'} \to \Sigma_{g,n}$ an H-cover, and $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ an irreducible H-representation. Suppose there are*

$$\Delta > \frac{3r^2}{\sqrt{g+1}} + 8r$$

points of $\Sigma_g - \Sigma_{g,n}$ such that a small loop around each of these points is sent to a nonidentity matrix under the composition $\pi_1(\Sigma_{g,n}) \xrightarrow{\varphi} H \xrightarrow{\rho} \mathrm{GL}_r(\mathbb{C})$. Then, setting $\mathrm{Mod}_\varphi \subset \mathrm{Mod}_{g,n+1}$ to be the stabilizer of φ, there are no nonzero vectors with finite orbit under the image of Mod_φ in $\mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$, for $\mathbb{V}^\rho$ the local system associated to $\rho \circ \varphi$.

We prove Theorem 1.15 in Section 8.5.

1.16 Remark Theorem 1.15 verifies new cases of the Putman–Wieland conjecture [2013, Conjecture 1.2]. See also [Landesman and Litt 2024a, Section 1.8.4] for a summary of other known cases of the Putman–Wieland conjecture. Landesman and Litt [2024a, Theorem 1.4.2] give a variant of Theorem 1.15 when g is large relative to r, in comparison to Theorem 1.15, where we think of Δ as being large relative to g and r.

1.17 Remark We have opted to write Theorem 1.15 with a bound on Δ depending only on the rank r of our given representation ρ and on the genus g of $\Sigma_{g,n}$. We expect, however, that our methods could give stronger bounds in terms of the eigenvalues of the local monodromy of $\rho \circ \varphi$.

1.18 Previous work There is a great deal of past work related to big monodromy for higher Prym representations. In the case that H is abelian, Theorem 1.3 follows from [Looijenga 1997, Corollary 2.6]. Higher Prym representations corresponding to certain nonabelian covering groups H were considered in [Grunewald et al. 2015], and shown to give a rich class of representations of mapping class groups under certain conditions, namely when the cover is "ϕ-redundant" [Grunewald et al. 2015, Theorem 1.2 and 1.6].

A variant for free groups was previously considered by Grunewald and Lubotzky [2009]. See also the recent paper by Looijenga [2025] for a criterion for big monodromy along somewhat different lines. In the four papers above, the representations above were in fact shown to have *arithmetic image*, meaning that they have finite index in the integral points of their Zariski closure. Our techniques seem not to be able to establish anything towards arithmeticity. There are known examples of families of cyclic branched covers of genus-zero curves with nonarithmetic monodromy [Deligne and Mostow 1986].

Further arithmeticity results associated to Prym representations, primarily in the case that H is abelian, were given by McMullen [2013], Venkataramana [2014a], and Venkataramana [2014b]. Salter and Tshishiku [2020] prove arithmeticity of monodromy groups of certain Kodaira fibrations, corresponding to H-covers with H a finite Heisenberg group. In a more arithmetic direction, big monodromy associated to certain Prym representations played a crucial role in the recent proof of Faltings' theorem given by Lawrence and Venkatesh [2020, Theorem 8.1].

In the setting of covers of projective lines, ie in a $g = 0$ analogue of the setting of Theorem 1.3, a big mod ℓ monodromy result was proven by Jain [2016, Theorem 5.4.2] when H has trivial Schur multiplier for $\ell \nmid 2|H|$. Big mod ℓ monodromy can often be used to prove that the ℓ-adic closure of the image of the mapping class group is large, which implies big Zariski closure. A key tool is a result of Conway and Parker (see eg [Fried and Völklein 1991, Appendix], as well as [Ellenberg et al. 2016, Proposition 3.4] and [Wood 2021]) which gives a stabilization of the braid group action on finite quotients of $\pi_1(\Sigma_{g,n})$. An analogue of this tool in the higher-genus case is work of Dunfield and Thurston [2006, Proposition 6.16], which shows a stabilization in g of the mapping class group action on finite quotients of $\pi_1(\Sigma_g)$ as g grows. Samperton [2020] gives a partial analogue of these results for the mapping class group action on $\pi_1(\Sigma_{g,n})$ with $n > 0$.

It may be possible to use [Dunfield and Thurston 2006, Proposition 6.16] to prove a higher-genus analogue of [Jain 2016, Theorem 5.4.2]. Such a result would have the advantage over Theorem 1.3 that it would give more precise information on the ℓ-adic image of the mapping class group, but the disadvantage that the required lower bound on the genus would be ineffective. Partial results in this direction were obtained in recent work of Sawin and Wood [2024] in the course of studying Heegaard splittings of 3-manifolds. This work uses [Dunfield and Thurston 2006, Proposition 6.16] to compute the intersections of maximal isotropic subspaces with their translates by random elements drawn from the monodromy group of Prym representations, though falls short of computing the actual monodromy group.

Finally, recent work of Landesman and Litt [2024a; 2023a; 2023b] shows that the monodromy group for higher Prym representations is not too small, in the sense that it has no nonzero finite orbit vectors. Here, we build on the methods developed in those papers to prove the monodromy group is as big as possible.

We next describe the primary new ideas of this work that do not appear in [Landesman and Litt 2024a; 2023a; 2023b].

1.19 Innovations of the proof To explain the main new ideas going into our paper, we begin with a sketch of the proof of Theorem 1.9.

Setup for the proof Consider a family of n-pointed curves $\pi\colon \mathscr{C} \to \mathscr{M}$, with associated family of punctured curves $\pi^\circ\colon \mathscr{C}^\circ \to \mathscr{M}$, such that the induced map $\mathscr{M} \to \mathscr{M}_{g,n}$ is dominant étale. Let $\mathbb{V}$ be a complex local system on $\mathscr{C}^\circ$ with finite monodromy, whose restriction to a fiber of π° has monodromy given by ρ. It suffices to compute the connected monodromy group of $W_1 R^1\pi^\circ_*\mathbb{V}$, as the monodromy representation on this local system factors through the representation we are interested in. The main idea of the proof is to analyze the derivative of the period map associated to $W_1 R^1\pi^\circ_*\mathbb{V}$.

A novel technique: functorial reconstruction Using techniques building on those developed by Landesman and Litt [2024b; 2024a] we show in Theorem 6.2 and Proposition 6.4 that (given our assumptions on g and $\dim \mathbb{V}$) the monodromy representation ρ can be *functorially reconstructed* from the derivative of the period map associated to $W_1 R^1\pi^\circ_*\mathbb{V}$ at a generic point of $\mathscr{M}$ along so-called Schiffer variations. We think of this reconstruction as a new kind of "generic Torelli theorem with coefficients" — this is our main technical tool.

This enables a novel strategy to obtain information about the local system $W_1 R^1\pi^\circ_*\mathbb{V}$. We first assume for the sake of contradiction that the monodromy group of $W_1 R^1\pi^\circ_*\mathbb{V}$ has some undesirable property. Using the general theory of variations of Hodge structures, we describe the consequences of this property for the variation of Hodge structures, and in particular for the derivative of its period map. We then examine the consequences those properties have on the local system obtained by applying our functorial reconstruction algorithm, and finally show that these contradict known properties of $\mathbb{V}$.

Proving Theorem 1.9 by repeatedly applying functorial reconstruction For example, to show $W_1 R^1\pi^\circ_*\mathbb{V}$ is irreducible as a representation of the monodromy group, we assume for the sake of contradiction that it is reducible. It would be convenient if this implied that $W_1 R^1\pi^\circ_*\mathbb{V}$ is reducible as a representation of Hodge structures, but this is not the case — it could instead be, for example, the tensor product of a fixed irreducible Hodge structure with an irreducible variation of Hodge structure. However, in this case the derivative of the period map is still reducible in a suitable sense, and in fact one can check that this holds for any variation of Hodge structures whose underlying local system is reducible. Applying the reconstruction algorithm to a reducible derivative of the period map, we obtain a reducible local system, contradicting the irreducibility of $\mathbb{V}$.

A slight enhancement of this argument, namely Theorem 6.7, shows that the connected monodromy group of $W_1 R^1\pi^\circ_*\mathbb{V}$ is in fact a *simple* group, acting irreducibly. By the classification of simple factors of Mumford–Tate groups of Abelian varieties, this group necessarily acts through a minuscule representation; see [Zarkhin 1984, Theorem 0.5.1(b)].

We rule out nonstandard representations in Section 7 via the same strategy. We show that, for these representations, the rank of the derivative of the period map along a Schiffer variation is necessarily large.

Applying the reconstruction algorithm gives a local system of large rank — in fact, larger than the rank of $\mathbb{V}$, leading to a contradiction.

The last case remaining to rule out is when the monodromy group of $W_1 R^1 \pi^\circ_* \mathbb{V}$ is a classical group but $\mathbb{V}$ is not self-dual. In this case we show that $W_1 R^1 \pi^\circ_* \mathbb{V}$ is self-dual as a variation of Hodge structures, and deduce from the reconstruction algorithm that $\mathbb{V}$ is self-dual, obtaining a contradiction. This concludes our sketch of the idea of the proof of Theorem 1.9.

Proving Theorem 1.3 To deduce Theorem 1.3, one may explicitly describe the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))^H$ as a product of the groups appearing in Theorem 1.9 — that theorem implies that the connected monodromy group of $H^1(\Sigma_{g'}, \mathbb{C})$ surjects onto each of the simple factors of $\mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))^H$. It then suffices by the Goursat–Kolchin–Ribet criterion of Katz [1990, Proposition 1.8.2] to show that for ρ_1, ρ_2 irreducible H-representations, with associated local systems $\mathbb{U}_1, \mathbb{U}_2$ on $\mathscr{C}^\circ$, an isomorphism $W_1 R^1 \pi^\circ_* \mathbb{U}_1 \simeq W_1 R^1 \pi^\circ_* \mathbb{U}_2$ necessarily comes from an isomorphism $\rho_1 \simeq \rho_2$. We deduce this from our functorial reconstruction results.

A new technical ingredient: global generation Although the heart of our functorial reconstruction results rest on understanding the derivative of a certain period map, following similar techniques introduced by Landesman and Litt [2023a; 2024a], there are a number of substantial innovations. In particular, the key to the proof of Theorem 6.2 is Proposition 4.9, which analyzes the global generation properties of flat vector bundles under isomonodromic deformation. In past work, the first two authors were only able to prove that the relevant vector bundles were *generically* globally generated, as opposed to actually being globally generated. By analyzing the obstruction for generically globally generated bundles to be globally generated, we are able to push the methods developed previously farther.

The proof of our main results underlies a new connection between *global generation* of certain vector bundles on curves and questions about big monodromy; see Section 10 for further details. As far as we are aware this connection is totally novel.

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2 Notation and preliminaries on moduli

2.1 Notation Throughout this paper, we work over the complex numbers. Suppose we are given a smooth proper family of curves $\pi : \mathscr{C} \to \mathscr{M}$ with geometrically connected fibers and n sections $s_1, \dots, s_n : \mathscr{M} \to \mathscr{C}$ with disjoint images $\mathscr{D}_1, \dots, \mathscr{D}_n \subset \mathscr{C}$. Let $\mathscr{C}^\circ = \mathscr{C} - \{\mathscr{D}_1, \dots, \mathscr{D}_n\}$. If the induced map $\mathscr{M} \to \mathscr{M}_{g,n}$ is dominant étale, we say π is a *versal family of n-pointed curves of genus g*, and refer to $\pi^\circ := \pi|_{\mathscr{C}^\circ}$ as the associated *versal family of n-punctured curves*.

Suppose, moreover, we have a diagram

(2-1)
$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{f} & \mathscr{C} \\ & \underset{\pi'}{\searrow} & \pi \downarrow \uparrow s_1,\ldots,s_n \\ & & \mathscr{M} \end{array}$$

where π is a versal family of n-pointed curves of genus g, and π' is a smooth proper curve with geometrically connected fibers of genus g'. Suppose f is finite, Galois and unramified away from $\bigcup_{i=1}^n \mathscr{D}_i$. Let $\mathscr{X}^\circ = f^{-1}(\mathscr{C}^\circ)$, let $\pi^\circ := \pi|_{\mathscr{C}^\circ}$, and let $f^\circ = f|_{\mathscr{X}^\circ}$. For $m \in \mathscr{M}$ a point, we set $C = \mathscr{C}_m$, $C^\circ = \mathscr{C}_m^\circ$ and $D = C - C^\circ$; let $c \in C^\circ$ be a point. As f° is finite Galois, it induces a surjection $\pi_1(\mathscr{C}^\circ, c) \twoheadrightarrow H$ for some finite group H. As π' has geometrically connected fibers and f is Galois, the composition

$$\psi : \pi_1(C^\circ, c) \to \pi_1(\mathscr{C}^\circ, c) \to H$$

is surjective.

Let $x \in \Sigma_{g,n}$ be a basepoint and $\varphi : \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$ a surjection. A *versal family of φ-covers* is the data of a diagram as in (2-1) above, together with

(1) a point $c \in \mathscr{C}^\circ$, $m = \pi^\circ(c)$, $C^\circ = \mathscr{C}_m^\circ$, and an identification $i : (\Sigma_{g,n}, x) \simeq (C^\circ, c)$, such that

(2) under this identification the map ψ above identifies with φ.

If $\rho : H \to \mathrm{GL}_r(\mathbb{C})$ is a representation of a finite group H, and $\varphi : \pi_1(\Sigma_{g,n}, x) \to H$ is a map, we often use $\mathbb{V}^\rho$ to denote the local system on $\Sigma_{g,n}$ associated to $\rho \circ \varphi$.

2.2 Remark Versal families of φ-covers exist by [Wewers 1998, Theorem 4]. One may also construct such a versal family by taking an open substack of the stack $\mathscr{B}_{g,n}(H)$ as constructed in [Abramovich et al. 2003, Section 2.2] (where the group H here is called G there). The above constructions give versal families of Deligne–Mumford stacks. Hence, if one wishes, one may pass to a dominant étale cover of the $\mathscr{M}$ thus constructed, making all the objects in question schemes.

2.3 Preliminaries on moduli In this section we explain how to interpret the main theorems of this paper as being about the monodromy of (summands of) $R^1\pi'_*\mathbb{C}$, for π' as in (2-1), arising from a versal family of H-covers.

Recall that, given a cover $\Sigma_{g',n'} \to \Sigma_{g,n}$, we are studying the action of finite-index subgroups of $\mathrm{Mod}_{g,n+1}$ on the abelianization of $\pi_1(\Sigma_{g',n'}, x')$, a finite-index subgroup of $\pi_1(\Sigma_{g,n}, x)$. One may interpret the action of $\mathrm{Mod}_{g,n+1}$ on $\pi_1(\Sigma_{g,n}, x)$ as follows. Let $\mathscr{M}_{g,n}$ be the Deligne–Mumford moduli stack of genus g curves with n marked points, $\mathscr{C}_{g,n}/\mathscr{M}_{g,n}$ be the universal family, and $\mathscr{C}^\circ_{g,n}$ the associated family of n-punctured curves. Note that $\mathscr{C}^\circ_{g,n}$ is canonically isomorphic to $\mathscr{M}_{g,n+1}$.

Now consider the map

$$p_1 \colon \mathscr{C}^\circ_{g,n} \times_{\mathscr{M}_{g,n}} \mathscr{C}^\circ_{g,n} \to \mathscr{C}^\circ_{g,n}$$

given by projection onto the first factor. This map has a canonical section given by the diagonal Δ. Thus there is a short exact sequence of fundamental groups

$$1 \to \pi_1(\Sigma_{g,n}) \to \pi_1(\mathscr{C}^\circ_{g,n} \times_{\mathscr{M}_{g,n}} \mathscr{C}^\circ_{g,n}) \xrightarrow{p_{1*}} \pi_1(\mathscr{C}^\circ_{g,n}) \to 1,$$

split by Δ_*, inducing a natural action of $\pi_1(\mathscr{C}^\circ_{g,n})$ on $\pi_1(\Sigma_{g,n})$. It follows from eg [Farb and Margalit 2012, Section 10.6.3 and page 353] that there is a natural identification of $\pi_1(\mathscr{C}^\circ_{g,n}) = \pi_1(\mathscr{M}_{g,n+1})$ with $\mathrm{Mod}_{g,n+1}$, under which this action identifies with the one we are studying.

Fix a surjection $\varphi \colon \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$ and a versal family of φ-covers with notation as in (2-1). In analogy with the fiber product construction above, we may consider the map

$$q_1 \colon \mathscr{C}^\circ \times_{\mathscr{M}} \mathscr{X} \to \mathscr{C}^\circ$$

given by projection onto the first coordinate. We claim that the monodromy of $R^1 q_{1*}\mathbb{C}$ factors through the representation R_φ studied in Theorem 1.3, and indeed factors through a finite-index subgroup of Mod_φ.

We first construct a map from $\pi_1(\mathscr{C}^\circ)$ to Mod_φ. The family of $(n+1)$-pointed curves

$$\mathscr{C}^\circ \times_{\mathscr{M}} \mathscr{C} \to \mathscr{C}^\circ$$

(with sections given by the pullbacks of $s_1, \dots, s_n$ and the tautological section induced by the diagonal) induces a map $\mathscr{C}^\circ \to \mathscr{M}_{g,n+1}$. By the discussion above and [Landesman and Litt 2024a, Lemma 2.1.4], the image of the induced map on fundamental groups $\pi_1(\mathscr{C}^\circ) \to \pi_1(\mathscr{M}_{g,n+1})$ has finite index in $\pi_1(\mathscr{M}_{g,n+1}) = \mathrm{Mod}_{g,n+1}$. By the definition of $\mathscr{M}$, the image of the induced map is contained in Mod_φ, as defined as in Theorem 1.3, and hence the image has finite index in Mod_φ.

Unwinding this discussion, the monodromy representation on $R^1 q_{1*}\mathbb{C}$ factors through the representation R_φ, with image a finite-index subgroup of the image of R_φ. Thus to understand the image of R_φ, it suffices to understand the image of this monodromy representation associated to $R^1 q_{1*}\mathbb{C}$.

Moreover q_1 is itself the pullback of π' along π°, which induces a surjection on fundamental groups. Thus the image of the monodromy representation associated to $R^1 q_{1*}\mathbb{C}$ is the same as that of the monodromy representation associated to $R^1 \pi'_*\mathbb{C}$.

Now let $\rho \colon H \to \mathrm{GL}_r(\mathbb{C})$ be a representation and $\mathbb{V}^\rho$ the associated local system on $\mathscr{C}^\circ$. A completely analogous argument shows that the image of the monodromy representation associated to $W_1 R^1 \pi^\circ_* \mathbb{V}^\rho$ is the same up to finite index as the representation $R_{\varphi,\rho}$ considered in Theorem 1.9.

3 Review of parabolic bundles and period maps

Let C be a smooth projective curve over $\mathbb{C}$, and $D = x_1 + \cdots + x_n$ a reduced effective divisor on C.

3.1 Definition A *parabolic vector bundle* $E_\star$ on (C, D) is a vector bundle E on C, a decreasing filtration $E_{x_j} = E_j^1 \supsetneq E_j^2 \supsetneq \cdots \supsetneq E_j^{n_j+1} = 0$ for each $1 \le j \le n$, and an increasing sequence of real numbers $0 \le \alpha_j^1 < \alpha_j^2 < \cdots < \alpha_j^{n_j} < 1$ for each $1 \le j \le n$, referred to as *weights*. Here, E_{x_j} refers to the fiber of E at x_j, and hence the filtration is merely a filtration of vector spaces on the fiber of E at x_j, not a filtration of the vector bundle E. We use $E_\star = (E, \{E_j^i\}, \{\alpha_j^i\})$ to denote the data of a parabolic bundle. Given a parabolic bundle $E_\star$, we will often write E_0 for the underlying vector bundle E.

3.2 Parabolic bundles admit a notion of *parabolic stability*, analogous to the usual notion of stability for vector bundles, which we next recall. First, the *parabolic degree* of a parabolic bundle $E_\star$ is

$$\deg(E_\star) := \deg(E) + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \alpha_j^i \dim(E_j^i / E_j^{i+1}).$$

Then, the *parabolic slope* is defined by $\mu_\star(E_\star) := \deg(E_\star)/\operatorname{rk}(E_\star)$. Any subbundle $F \subset E$ has an induced parabolic structure $F_\star \subset E_\star$ defined as follows: we take the filtration over x_j on F is obtained from the filtration

$$F_{x_j} = E_j^1 \cap F_{x_j} \supset E_j^2 \cap F_{x_j} \supset \cdots \supset E_j^{n_j+1} \cap F_{x_j} = 0$$

by removing redundancies. For the weight associated to $F_j^i \subset F_{x_j}$ one takes

$$\max_{k, 1 \le k \le n_j} \{\alpha_j^k : F_j^i = E_j^k \cap F_{x_j}\}.$$

A parabolic bundle $E_\star$ is *parabolically semistable* if for every nonzero subbundle $F \subset E$ with induced parabolic structure $F_\star$, we have $\mu_\star(F_\star) \le \mu_\star(E_\star)$. Similarly, a parabolic bundle $E_\star$ is *parabolically stable* if for every nonzero subbundle $F \subset E$ with induced parabolic structure $F_\star$, we have $\mu_\star(F_\star) < \mu_\star(E_\star)$.

We next introduce the notation $E_\star^\rho$ for the parabolic bundle corresponding to a unitary representation ρ.

3.3 Notation Let C be a curve and $D \subset C$ a divisor. Under the Mehta–Seshadri correspondence from [Mehta and Seshadri 1980], there is a bijection between irreducible unitary representations of $\pi_1(C - D)$ and parabolic degree 0 stable parabolic vector bundles on C, with parabolic structure along D. Given an irreducible unitary representation

$$\rho \colon \pi_1(C - D) \to U(r)$$

of dimension r, we use $E_\star^\rho$ to denote the parabolic bundle associated to ρ. The underlying vector bundle E_0^ρ of $E_\star^\rho$ is the Deligne canonical extension of the flat bundle on $C - D$ associated to ρ, ie E_0^ρ carries a flat connection

$$\nabla \colon E_0^\rho \to E_0^\rho \otimes \Omega_C^1(\log D)$$

with monodromy ρ and whose residues have eigenvalues with real parts in $[0, 1)$. See [Landesman and Litt 2024b, Definition 3.3.1] for details of how to associate a parabolic structure to a connection.

Such unitary representations will often arise as follows: we will start with a surjection $\varphi\colon \pi_1(C-D) \twoheadrightarrow H$ for some finite group H. Then for each irreducible representation ρ of H, the representation $\rho\circ\varphi$ is unitary, and we will abuse notation to denote it by ρ as well.

Let $D^{\rho}_{\mathrm{nontriv}} \subset D$ denote the subset of points $p \in D$ such that the local inertia at p under ρ is not the identity. Set Δ to be the number of points in $D^{\rho}_{\mathrm{nontriv}}$.

3.4 Definition Given a parabolic bundle $E_\star = (E, \{E^i_j\}, \{\alpha^i_j\})$, let $J \subset \{1, \dots, n\}$ denote the set of integers $j \in \{1, \dots, n\}$ for which $\alpha^1_j = 0$, and define

$$\widehat{E}_0 := \ker\Bigl(E \to \bigoplus_{j\in J} E_{x_j}/E^2_j\Bigr). \tag{3-1}$$

(This is a special case of more general notation used for coparabolic bundles as in [Landesman and Litt 2024b, 2.2.8] or the equivalent [Boden and Yokogawa 1996, Definition 2.3], but is all we will need for this paper.) In particular, $\widehat{E}_0 \subset E$ is a subsheaf. Note that if $E_\star$ is the parabolic bundle associated to a unitary representation of $\pi_1(C-D)$, the natural logarithmic connection on E_0 descends to a logarithmic connection on $\widehat{E}_0$ (albeit with different residues), by a local calculation.

3.5 Proposition *Let $\mathbb{V}$ be a unitary local system on $C-D$, and let $E_\star$ be the associated parabolic bundle on C. Then there is a natural mixed Hodge structure on $H^1(C-D,\mathbb{V})$, and natural isomorphisms*

$$(F^1\cap W_1)H^1(C-D,\mathbb{V}) = H^0(C, \widehat{E}_0\otimes\omega_C(D)),$$
$$W_1H^1(C-D,\mathbb{V})/(F^1\cap W_1)H^1(C-D,\mathbb{V}) = H^1(C, E_0).$$

Proof Everything except the last line is [Landesman and Litt 2023a, Lemma 3.2]. The last line is just unwinding definitions; see eg [Landesman and Litt 2023a, Section 3] or [Landesman and Litt 2024a, Theorem 4.1.1]. □

Now suppose $\pi\colon \mathscr{C} \to \mathscr{M}$ is a versal family of punctured n-pointed curves, $\pi^\circ\colon \mathscr{C}^\circ \to \mathscr{M}$ is the associated punctured family, and $\mathbb{V}$ a unitary local system on $\mathscr{C}^\circ$. It is well-known (see [Landesman and Litt 2024a, Theorem 4.1.1]) that $R^1\pi^\circ_*\mathbb{V}$ carries an admissible complex variation of Hodge structures, with the fibers of $W_1R^1\pi^\circ_*\mathbb{V}$ as described in Proposition 3.5. Let $m \in \mathscr{M}$ be a point, $C = \pi^{-1}(m)$, $C^\circ = (\pi^\circ)^{-1}(m)$, and $D = C - C^\circ$. Let ρ be the monodromy representation associated to $\mathbb{V}|_{C^\circ}$. The derivative of the period map associated to $W_1R^1\pi^\circ_*\mathbb{V}$ is, by Proposition 3.5, a map

$$dP^\rho_m\colon T_m\mathscr{M} \to \operatorname{Hom}(H^0(C, \widehat{E}^\rho_0\otimes\omega_C(D)), H^1(C, E^\rho_0)).$$

As π° is a versal family, for each $m\in\mathscr{M}$ we have that $T^*_m\mathscr{M} = H^0(\omega_C^{\otimes 2}(D))$, and so by Serre duality we may adjointly obtain a map

$$H^0(C, \widehat{E}^\rho_0\otimes\omega_C(D))\otimes H^0(C, (E^\rho_0)^\vee\otimes\omega_C) \to H^0(C, \omega_C^{\otimes 2}(D)).$$

There is another natural map between these vector spaces, which we denote by B^ρ_m and describe next.

There is a bilinear form

$$\hat{E}_0^\rho \otimes (E_0^\rho)^\vee \to \mathcal{O}_C \tag{3-2}$$

arising from the inclusion $\hat{E}_0^\rho \to E_0^\rho$ and the pairing $E_0^\rho \otimes (E_0^\rho)^\vee \to \mathcal{O}_C$. Twisting this bilinear form by $\omega_C^{\otimes 2}(D)$ yields the map of sheaves

$$\hat{E}_0^\rho \otimes \omega_C(D) \otimes (E_0^\rho)^\vee \otimes \omega_C \to \omega_C^{\otimes 2}(D).$$

Taking global sections yields the map

$$H^0(C, \hat{E}_0^\rho \otimes \omega_C(D) \otimes (E_0^\rho)^\vee \otimes \omega_C) \to H^0(C, \omega_C^{\otimes 2}(D)). \tag{3-3}$$

Finally, we define

$$B_m^\rho \colon H^0(C, \hat{E}_0^\rho \otimes \omega_C(D)) \otimes H^0(C, (E_0^\rho)^\vee \otimes \omega_C) \to H^0(C, \omega_C^{\otimes 2}(D)) \tag{3-4}$$

to be the composition of the multiplication map

$$H^0(C, \hat{E}_0^\rho \otimes \omega_C(D)) \otimes H^0(C, (E_0^\rho)^\vee \otimes \omega_C) \to H^0(C, \hat{E}_0^\rho \otimes \omega_C(D) \otimes (E_0^\rho)^\vee \otimes \omega_C)$$

with (3-3).

By combining Proposition 3.5 above with [Landesman and Litt 2024a, Theorem 5.1.6], we obtain the following description of the derivative of the period map.

3.6 Proposition *Let $\pi \colon \mathscr{C} \to \mathscr{M}$ be a versal family of marked curves of genus g as in Notation 2.1, and let $\pi^\circ \colon \mathscr{C}^\circ \to \mathscr{M}$ be the associated punctured family. Fix $m \in \mathscr{M}$ and set $C = \pi^{-1}(m)$, $C^\circ = (\pi^\circ)^{-1}(m)$ and $D = C - C^\circ$. Let $\mathbb{V}$ be a unitary local system on $\mathscr{C}^\circ$, and let ρ be the monodromy representation of $\mathbb{V}|_{C^\circ}$. Then the derivative of the period map associated to $W_1 R^1 \pi_*^\circ \mathbb{V}$ is identified with the map B_m^ρ defined in (3-4).*

The map dP_m^ρ described above gives rise to a number of other maps by adjointness; we will find it convenient to name some of them. We have denoted the bilinear pairing of (3-4) by B_m^ρ. We will denote by θ_m^ρ the adjoint map

$$\theta_m^\rho \colon H^0(C, \hat{E}_0^\rho \otimes \omega_C(D)) \to H^1(C, E_0^\rho) \otimes H^0(C, \omega_C^{\otimes 2}(D)), \tag{3-5}$$

where we identify $H^1(C, E_0)$ with $H^0(C, (E_0^\rho)^\vee \otimes \omega_C)^\vee$ by Serre duality.

4 Global generation of vector bundles on generic curves

4.1 A generalization of the non-ggg lemma We will need a generalization of the non-ggg (generically globally generated) lemma from [Landesman and Litt 2024b, Proposition 6.3.6]; see also [loc. cit., Proposition 6.3.1]. In order to prove this generalization, we will make use of the following lemma, bounding the rank of the global sections of a subbundle of a vector bundle in terms of various numerical invariants.

4.2 Lemma *Let F be a vector bundle on a smooth proper curve C of genus g, and let $V \subset F$ be a subbundle with $c := \operatorname{rk}(F) - \operatorname{rk}(V)$. Suppose that $0 = N^0 \subset N^1 \subset \cdots \subset N^k = V$ is the Harder Narasimhan filtration of V. Let μ be a rational number. If*

(1) $h^0(C, F) - h^0(C, V) = \delta$,

(2) $0 \le \mu(N^i/N^{i-1}) \le 2g - 2$ *for* $1 \le i \le k$, *and*

(3) $\mu(F) \ge \mu$,

then

$$(2g - 1 - \mu)\operatorname{rk}(F) \ge cg - \delta.$$

Proof Combining [Landesman and Litt 2024b, Lemma 6.2.1] with Riemann–Roch for F gives

$$\tfrac{1}{2}\deg(V) + \operatorname{rk}(V) \ge h^0(C, V) = h^0(C, F) - \delta \ge \deg(F) - (g - 1)\operatorname{rk}(F) - \delta.$$

By assumption (2) we obtain $\mu(V) \le 2g - 2$, ie $\deg(V) \le (2g - 2)\operatorname{rk}(V)$, giving

$$g\operatorname{rk}(V) \ge \deg(F) - (g - 1)\operatorname{rk}(F) - \delta.$$

Then, $\operatorname{rk}(V) = \operatorname{rk}(F) - c$ and $\mu(F) \ge \mu$ gives

$$\begin{aligned} g(\operatorname{rk}(F) - c) = g(\operatorname{rk}(V)) &\ge \deg(F) - (g - 1)\operatorname{rk}(F) - \delta \\ &\ge \mu(F)\operatorname{rk}(F) - (g - 1)\operatorname{rk}(F) - \delta \\ &\ge \mu\operatorname{rk}(F) - (g - 1)\operatorname{rk}(F) - \delta. \end{aligned}$$

Simplifying this gives

$$(2g - 1 - \mu)\operatorname{rk}(F) \ge cg - \delta. \tag{4-1}$$ □

4.3 Proposition *Let μ be a rational number. Suppose E is a vector bundle on a smooth proper connected genus g curve and any nonzero quotient bundle $E \twoheadrightarrow Q$ satisfies $\mu(Q) \ge \mu$. Let $U \subset E$ be a proper subbundle with $c := \operatorname{rk}(E) - \operatorname{rk}(U) > 0$ and $\delta := h^0(C, E) - h^0(C, U)$. If $\mu < 2g - 1$ then we have $\operatorname{rk}(E) \ge (cg - \delta)/(2g - 1 - \mu)$, while if $\mu \ge 2g - 1$ then we have $\delta \ge cg + \mu - (2g - 1)$.*

Proof As a first step, we may reduce to the case U is generically globally generated by replacing U with the saturation of the image of $H^0(C, U) \otimes \mathcal{O}_C \to U \to V$. Next, let N^i denote the largest filtered part of the Harder–Narasimhan filtration of U, whose associated graded sequence of vector bundles all have slopes $> 2g - 2$. Because $U \subset E$ was assumed saturated, and $N^i \subset U$ is saturated, the quotient $U/N^i \subset E/N^i$ is also a saturated inclusion of vector bundles.

Now, let $F := E/N^i$ and let $V := U/N^i$. We now wish to apply Lemma 4.2 to the subbundle $V \subset F$, which will tell us

$$(2g - 1 - \mu)\operatorname{rk}(F) \ge cg - \delta. \tag{4-2}$$

If we also assume $2g-1>\mu$, dividing both sides of (4-2) by $2g-1-\mu$ gives our desired inequality

$$\mathrm{rk}(E) \geq \mathrm{rk}(E/N^i) \geq \frac{cg-\delta}{2g-1-\mu}.$$

On the other hand, if $\mu \geq 2g-1$, then rearranging (4-2), and using that $\mathrm{rk}(F) \geq 1$ (since $V \subset F$ is a proper subbundle), gives the claimed inequality

$$\delta \geq cg-(2g-1-\mu)\,\mathrm{rk}(F) = cg+(\mu-(2g-1))\,\mathrm{rk}(F) \geq cg+\mu-(2g-1).$$

To conclude, it remains to verify conditions (1)–(3) of Lemma 4.2.

For condition (1), we will show

$$\delta = h^0(C,E)-h^0(C,U) = h^0(C,E/N^i)-h^0(C,U/N^i).$$

Indeed, since $H^1(C,N^i)=0$ by [Landesman and Litt 2024b, Lemma 6.3.5], we find $h^0(C,E/N^i) = h^0(C,E)-h^0(C,N^i)$ and $h^0(C,U/N^i) = h^0(C,U)-h^0(C,N^i)$. This implies $h^0(C,E)-h^0(C,U) = h^0(C,E/N^i)-h^0(C,U/N^i)$, and the former is δ by definition.

Condition (3) holds because F is a quotient of E, and so $\mu(F) \geq \mu$ by assumption.

Finally, we verify condition (2). We wish to show each associated graded piece of the Harder–Narasimhan filtration of V has slope between 0 and $2g-2$. The upper bound follows from the construction of V as U/N^i. We conclude by verifying the lower bound. Recall we assumed above that U is generically globally generated. Therefore, V is generically globally generated as it is a quotient of U, and hence any quotient of V is itself generically globally generated, and thus has positive slope. Applying this to the smallest slope associated graded piece of the Harder–Narasimhan filtration verifies (2). □

4.4 Lemma *Suppose $E_\star = (E,\{E^i_j\},\{\alpha^i_j\})$ is a nonzero parabolic bundle on (C,D), where C is a smooth proper connected genus g curve and $D = x_1+\cdots+x_n$ is a reduced effective divisor. Assume $E_\star$ is parabolically semistable of slope $\mu+n$ with $\mu<2g-1$. Suppose $\widehat{E}_0$ has a proper subbundle $U \subset \widehat{E}_0$ with $c := \mathrm{rk}(\widehat{E}_0)-\mathrm{rk}(U)$ and $\delta := h^0(C,\widehat{E}_0)-h^0(C,U)$. Then*

$$\mathrm{rk}(E) = \mathrm{rk}(\widehat{E}_0) \geq \frac{cg-\delta}{2g-1-\mu}.$$

Proof Note that [Landesman and Litt 2024b, Lemma 6.3.4] implies any quotient of vector bundles $\widehat{E}_0 \twoheadrightarrow Q$ satisfies $\mu(Q) \geq \mu$. We may therefore conclude by applying Proposition 4.3 to the vector bundle $\widehat{E}_0$. □

4.5 Lemma *The bundle $E_0^{\rho^\vee}$ is semistable if and only if $\widehat{E}_0^\rho$ is semistable. The bundle $E_0^{\rho^\vee}$ is stable if and only if $\widehat{E}_0^\rho$ is stable.*

Proof Stability of $\widehat{E}_0^\rho$ is equivalent to stability of $\widehat{E}_0^\rho(D)$. In turn, this is equivalent to stability of $(E_0^{\rho^\vee})^\vee$ by [Yokogawa 1995, (3.1)], which shows $\widehat{E}_0^\rho(D) \simeq (E_0^{\rho^\vee})^\vee$. Finally, stability of $(E_0^{\rho^\vee})^\vee$ is equivalent to stability of $E_0^{\rho^\vee}$. The same holds with semistability in place of stability. □

4.6 Global generation and deformation theory In this section, we deduce Proposition 4.9 from Lemma 4.4 by passing to a generic curve. We note that Proposition 4.9 is a result about *global generation* of vector bundles, as opposed to just generic global generation.

Suppose C is a curve, D is a reduced effective divisor in C, and E is a vector bundle on C. Below we denote by $\mathrm{At}_{(C,D)}(E)$ the preimage of $T_C(-D)$ under the natural map $\mathrm{At}_C(E) \to T_C$, where $\mathrm{At}_C(E)$ is the Atiyah bundle of E. For some background on Atiyah bundles relevant to the context of this paper, see [Landesman and Litt 2024b, Section 3]. We begin by recalling a couple of general facts from deformation theory relating to the Atiyah bundle.

4.7 Lemma *Let C be a smooth projective curve over a field k, and D a reduced effective divisor on C. Let E be a vector bundle on C. Then the space of first-order deformations of the triple (C, D, E) is naturally in bijection with the first cohomology of the Atiyah bundle, $H^1(C, \mathrm{At}_{(C,D)}(E))$.*

Proof See [Landesman and Litt 2024b, Proposition 3.5.5], or see [Sernesi 2006, Theorem 3.3.11] for a proof in the case where E is a line bundle and D is empty; the general case is identical. □

4.8 Lemma *With notation as in Lemma 4.7, let $s \in \Gamma(C, E)$ be a global section. Let*

$$\psi : \mathrm{At}_{(C,D)}(E) \to E$$

be the map sending a differential operator ξ to $\xi(s)$. Then given $v \in H^1(C, \mathrm{At}_{(C,D)}(E))$, corresponding to some first-order deformation $(\mathscr{C}, \mathscr{D}, \mathscr{E})$ over $k[\epsilon]/\epsilon^2$, the section s extends to a global section of $\mathscr{E}$ if and only if

$$\psi(v) \in H^1(C, E)$$

vanishes.

In particular, if E carries a flat connection ∇ with logarithmic singularities (equivalently, the natural map $\mathrm{At}_{(C,D)}(E) \to T_C(-D)$ is equipped with a section q^∇), then s extends to a first-order neighborhood in the universal isomonodromic deformation of E if and only if the map

$$\psi \circ q^\nabla : T_C(-D) \to E$$

induces the zero map

$$H^1(C, T_C(-D)) \to H^1(C, E).$$

Proof For the first paragraph, see [Sernesi 2006, Proposition 3.3.14] for the case where E is a line bundle and D is empty; the general case is identical. For the second, fix $v \in H^1(C, T_C(-D))$, corresponding to a first-order deformation of (C, D). The element $q^\nabla(v) \in H^1(\mathrm{At}_{(C,D)}(E))$ corresponds to, by [Landesman and Litt 2024b, Proposition 3.5.7], the first-order isomonodromic deformation of E in the direction v. Thus by the first paragraph, $\psi \circ q^\nabla(v) = 0$; as v was arbitrary, this completes the proof. □

4.9 Proposition *Fix a unitary representation $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$. Let (C, D) be a general n-pointed curve of genus $g \geq 2 + 2r$. Then $\hat{E}_0^{\rho} \otimes \omega_C(D)$ is not only generically globally generated, but even globally generated.*

Proof It suffices to consider the case that ρ is irreducible and nontrivial (as ω_C is globally generated). Letting $\mathbb{V}^{\rho}$ denote the local system on C° associated to ρ, we may assume that $H^1(C, \hat{E}_0^{\rho} \otimes \omega_C(D)) = 0$, as this is dual to $H^0(C, E_0^{\rho^\vee}) = H^0(C^{\circ}, \mathbb{V}^{\rho^\vee})$.

Suppose that the theorem is false, so that for a general n-pointed curve (C, D), $\hat{E}_0^{\rho} \otimes \omega_C(D)$ is not globally generated. Equivalently, there exists $p \in C$ such that $H^1(C, \hat{E}_0^{\rho} \otimes \omega_C(D - p)) \neq 0$. Serre-dually, $H^0(C, E_0^{\rho^\vee}(p))$ is nonzero. As (C, D) is general, we may assume that there exists a section $s \in H^0(C, E_0^{\rho^\vee}(p))$ such that for every first-order deformation $(\widetilde{C}, \widetilde{D})$ of (C, D), there exists a first-order deformation $\tilde{p}$ of p to $\widetilde{C}$ such that s survives to $H^0(\widetilde{C}, \widetilde{E}_0^{\rho^\vee}(\tilde{p}))$, where $\widetilde{E}_0^{\rho^\vee}$ is the isomonodromic deformation of $E_0^{\rho^\vee}$ to $(\widetilde{C}, \widetilde{D})$. If $p \in D$ but $\tilde{p} \not\subset \widetilde{D}$, then we may, by passing to a different general curve (C', D'), assume that $p \notin D$. Thus we can and do assume in what follows that if $p \in D$, then $\tilde{p} \subset \widetilde{D}$ for all first-order deformations $(\widetilde{C}, \widetilde{D})$ of (C, D).

There are two cases, depending on whether or not $p \in D$; we set up notation to handle them simultaneously. If $p \in D$, we set $D' = D$; otherwise, we set $D' = D + p$. The bundle $\mathscr{O}(p)$ has a natural connection on it with regular singularities at p and trivial monodromy, and with residue -1 at p. We give $E_0^{\rho^\vee}(p)$ the tensor product connection, which has regular singularities along D'.

First-order deformations of (C, D) are parametrized by $H^1(C, T_C(-D))$, and first-order deformations of (C, D') are parametrized by $H^1(C, T_C(-D'))$. By Lemma 4.8, applied to (C, D') and the vector bundle $E_0^{\rho^\vee}(p)$, a section $s \in H^0(C, E_0^{\rho^\vee}(p))$ extends to the first-order deformation parametrized by $\eta \in H^1(C, T_C(-D'))$ if and only if the image of η under the map

$$H^1(C, T_C(-D')) \xrightarrow{H^1(\nabla s)} H^1(E_0^{\rho^\vee}(p))$$

is zero, where $\nabla s\colon T_C(-D') \to E_0^{\rho^\vee}(p)$ is the map sending a vector field X to $\nabla_X s$, the derivative of s with respect to the natural connection on $E_0^{\rho^\vee}(p)$ with regular singularities along D', described in the previous paragraph. Note that ∇s is nonzero, since if it were zero, s would be flat, which is impossible as the monodromy representation $\rho^\vee$ is nontrivial.

Thus it suffices to show that

$$\ker(H^1(C, T_C(-D')) \xrightarrow{H^1(\nabla s)} H^1(E_0^{\rho^\vee}(p)))$$

does not surject onto $H^1(C, T_C(-D))$ under the natural map induced by the inclusion $T_C(-D') \to T_C(-D)$.

Since $\deg D' \geq \deg D - 1$, it is enough to show that the rank of the map induced by ∇s on H^1 is at least 2, or equivalently that the Serre dual map

$$\hat{E}_0^{\rho} \otimes \omega_C(D - p) \to \omega_C^{\otimes 2}(D')$$

induces a map of rank at least 2 on H^0. Setting U to be the kernel of this map, $\mu := 2g-3$, and δ to be the rank of the map

$$H^0(\hat{E}_0^\rho \otimes \omega_C(D-p)) \to H^0(\omega_C^{\otimes 2}(D')),$$

we have by Lemma 4.4 that

$$r \geq \tfrac{1}{2}(g-\delta),$$

and hence that $\delta \geq g-2r$. Hence $\delta \geq 2$ as long as $g-2r \geq 2$, which holds by assumption. □

5 Preliminaries on variations of Hodge structure

In this section, we freely use the terminology of Tannakian categories. For background, we suggest the reader consult [Deligne et al. 1982, II].

Let Y be a smooth variety over $\mathbb{C}$ and x a point of Y.

5.1 Definition (integral K-VHS) Let K be a number field. A K-variation of Hodge structures on Y is a K-local system $\mathbb{V}$ on Y equipped with a decreasing filtration $F^\bullet$ on $W_{K/\mathbb{Q}}\mathbb{V} \otimes \mathscr{O}_Y$, where $W_{K/\mathbb{Q}}$ is the Weil restriction, turning $W_{K/\mathbb{Q}}\mathbb{V}$ into a polarizable $\mathbb{Q}$-variation of Hodge structure so that the natural action of K is via morphisms of variations of Hodge structure. We say that a K-variation $\mathbb{V}$ is *integral* if there exists a locally constant sheaf of locally free $\mathscr{O}_K$-modules $\mathbb{W}$ on Y and an isomorphism $\mathbb{W} \otimes_{\mathscr{O}_K} K \simeq \mathbb{V}$.

We denote by $\mathrm{VHS}(Y, K)$ the neutral Tannakian category of semisimple K-variations of mixed Hodge structure (ie direct sums of irreducible pure K-variations of Hodge structure), and by $\mathrm{VHS}(Y, \mathscr{O}_K)$ the full (neutral Tannakian) subcategory consisting of direct sums of integral pure K-variations of Hodge structure. Note that $\mathrm{VHS}(Y, \mathscr{O}_K)$ is a K-linear category, not an $\mathscr{O}_K$-linear category. We equip this category with the fiber functor sending a local system to its fiber at x.

If $\mathbb{V}$ is a $\mathbb{Q}$-VHS, the *generic Mumford–Tate group* of $\mathbb{V}$ is the identity component of the Tannakian group associated to the full subcategory of $\mathrm{VHS}(Y, \mathbb{Q})$ generated by $\mathbb{V}$ (see eg [Moonen 2017, penultimate paragraph of Section 4.1] for more discussion and a comparison to other definitions of the generic Mumford–Tate group; in particular, if x is very general, there is a canonical isomorphism between the generic Mumford–Tate group and the Mumford–Tate group of $\mathbb{V}_x$).

For any local system $\mathbb{V}$ on a variety Y, with associated monodromy representation $\rho\colon \pi_1(Y, y) \to \mathrm{GL}(\mathbb{V}_y)$, the *algebraic monodromy group* of $\mathbb{V}$ is the identity component of the Zariski closure of the image of ρ.

5.2 Lemma *Let $\mathbb{V}$ be an integral K-variation of Hodge structure on a smooth variety Y. Then the algebraic monodromy group of $\mathbb{V}$ is a normal subgroup of the derived subgroup of the generic Mumford–Tate group of the Weil restriction $W_{K/\mathbb{Q}}\mathbb{V}$.*

Proof This is immediate from [André 1992, Theorem 1 on page 10]. □

If $\mathbb{V}$ is a K-variation of Hodge structure on Y and $\iota\colon K \hookrightarrow \mathbb{C}$ is an embedding, then $\mathbb{V} \otimes_{K,\iota} \mathbb{C}$ naturally obtains the structure of a $\mathbb{C}$-variation of Hodge structure.

5.3 Definition (infinitesimal VHS) A *weak infinitesimal variation of Hodge structure on Y at x*, or weak IVHS, is a finite-dimensional $\mathbb{Z}$-graded complex vector space $V^\bullet$ equipped with an action by T_xY by commuting linear operators of degree -1, ie with a map $\delta^i\colon T_xY \to \mathrm{Hom}(V^i, V^{i-1})$, such that $\delta^{i-1}(w_1) \circ \delta^i(w_2)(v) = \delta^{i-1}(w_2) \circ \delta^i(w_1)(v)$ for all $w_1, w_2 \in T_xY$, $v \in V^i$. A morphism of weak IVHS is a graded map of vector spaces commuting with the action of T_xY. We denote the category of weak IVHS on Y at x by $\mathrm{IVHS}(Y, x)$.

5.4 Remark We call the above *weak* IVHS because the conditions we impose are weaker than the standard conditions on infinitesimal variations of Hodge structure; see for example [Carlson et al. 1983].

5.5 Proposition *The category* $\mathrm{IVHS}(Y, x)$ *is a neutral Tannakian category, and the forgetful functor* $\mathrm{IVHS}(Y, x) \to \mathrm{Vect}_{\mathbb{C}}$ *sending $V^\bullet$ to its underlying vector space is a fiber functor. The corresponding Tannakian group is $T_xY \rtimes \mathbb{G}_m$, where $\mathbb{G}_m$ acts on T_xY by inverse scaling.*

Proof It suffices to prove that $\mathrm{IVHS}(Y, x)$ is equivalent to the category of representations of $T_xY \rtimes \mathbb{G}_m$, with the equivalence respecting the tensor product and forgetful functor to vector spaces, as the category of representations of any pro-algebraic group is a neutral Tannakian category with fiber functor the forgetful functor and Tannakian group the initial pro-algebraic group [Deligne et al. 1982, II, Example 1.25].

Given a weak infinitesimal variation of Hodge structures, we obtain an action of the algebraic group T_xY on $V^\bullet$ by exponentiating δ, ie for $v \in V^i$ and $w \in T_xY$ we have

$$w \cdot v = v + \delta^i(w)(v) + \frac{\delta^{i-1}(w) \circ \delta^i(w)(v)}{2!} + \frac{\delta^{i-2}(w) \circ \delta^{i-1}(w) \circ \delta^i(w)(v)}{3!} + \cdots.$$

(The power series converges since $V^{i-k} = 0$ for all k sufficiently large.) The relation

$$\delta^{i-1}(w_1) \circ \delta^i(w_2)(v) = \delta^{i-1}(w_2) \circ \delta^i(w_1)(v)$$

implies that $w_1 \cdot (w_2 \cdot v) = (w_1 + w_2) \cdot v$ and thus this is an action of the additive group T_xY on $V^\bullet$.

We also obtain an action of $\mathbb{G}_m$ on $V^\bullet$ by, for $v \in V^i$ and $\lambda \in \mathbb{G}_m$, taking $\lambda \cdot v = \lambda^i v$. Then for $\lambda \in \mathbb{G}_m$, $w \in T_xY$ and $v \in V^\bullet$ it is not hard to check that we have $\lambda \cdot (w \cdot (\lambda^{-1} \cdot v)) = (\lambda^{-1} w) \cdot v$. This relation implies that the two actions combine to give an action of $T_xY \rtimes \mathbb{G}_m$ on $V^\bullet$.

Conversely, given a vector space V with an action of $\mathbb{G}_m$, we obtain a grading by taking V^i to be the subspace on which $\lambda \in \mathbb{G}_m$ acts with eigenvalue λ^i for all λ, and the derivative at the identity of the action of the additive group T_xY defines an action of T_xY by commuting linear operators.

These two constructions are inverse, as can be checked separately on the T_xY and $\mathbb{G}_m$ parts, the first part being the standard equivalence between representations of a unipotent algebraic group and nilpotent representations of its Lie algebra, and the second being the standard equivalence between graded vector spaces and representations of $\mathbb{G}_m$.

They are compatible with tensor product (for the natural notion of tensor product on weak infinitesimal variations of Hodge structures) and, trivially, with the forgetful functor to the underlying vector space. □

Given a K-variation of Hodge structure $\mathbb{V}$ on Y and an embedding $\iota\colon K \hookrightarrow \mathbb{C}$, we can form the graded vector space $\bigoplus_{i\in\mathbb{Z}}(F^i\mathbb{V}_x \otimes_{K,\iota} \mathbb{C})/(F^{i+1}\mathbb{V}_x \otimes_{K,\iota} \mathbb{C})$, which admits an action of the tangent space T_xY by commuting linear maps of degree -1 (by Griffiths transversality). This formation is functorial, giving rise to a functor GH_x, where GH stands for "graded Hodge",

$$\mathrm{GH}_x\colon \mathrm{VHS}(Y,K) \to \mathrm{IVHS}(Y,x) \tag{5-1}$$

(which depends on ι, but we suppress this dependence from the notation).

Composing this functor with the Weil restriction functor $W_{K/\mathbb{Q}}$ induces a homomorphism from $T_xY \rtimes \mathbb{G}_m$ to the generic Mumford–Tate group of $W_{K/\mathbb{Q}}\mathbb{V}$, since $T_xY \rtimes \mathbb{G}_m$ is connected and hence any homomorphism from $T_xY \rtimes \mathbb{G}_m$ to the Tannakian group lands in its identity component.

5.6 Lemma *Let K be a number field, Y a smooth complex variety, x a point of Y, and $\mathbb{V}$ a pure integral K-VHS on Y. Let M be the monodromy group of $\mathbb{V}$, ie the Zariski closure of the monodromy representation $\pi_1(Y,x) \to \mathrm{GL}(\mathbb{V}_x)$, and let $\mathfrak{m}$ be the Lie algebra of the identity component of M.*

Suppose $\mathbb{V}$ has at most $k+1$ nonvanishing Hodge numbers. Then either $\mathrm{GH}_x(\mathbb{V})$ splits as a direct sum in $\mathrm{IVHS}(Y,x)$ or $\mathfrak{m}$ acts irreducibly on $\mathbb{V}$ and has at most k simple factors.

In particular, if $\mathbb{V}$ has at most 2 nonvanishing Hodge numbers, then either $\mathrm{GH}_x(\mathbb{V})$ splits, or the identity component of M is a simple algebraic group acting irreducibly on $\mathbb{V}$.

Proof Since monodromy groups are preserved when a local system is extended to a larger coefficient field, we may assume K is Galois over $\mathbb{Q}$.

Let G be the generic Mumford–Tate group of $W_{K/\mathbb{Q}}\mathbb{V}$ and $\mathfrak{g}$ the Lie algebra of G. The action of $T_xY \rtimes \mathbb{G}_m$ on $\mathrm{gr}^{F^\bullet}(W_{K/\mathbb{Q}}\mathbb{V}_x \otimes \mathbb{C})$ is given by a homomorphism $T_xY \rtimes \mathbb{G}_m \to G_{\mathbb{C}}$.

Because the category $\mathrm{VHS}(Y,\mathscr{O}_K)$ is semisimple, G is reductive. Thus if $\mathbb{V}$ is reducible as a representation of G_K (that is, after passing to a finite étale cover, as a K-VHS), it splits as a direct sum of two representations of G, hence a direct sum of two representations of $T_xY \rtimes \mathbb{G}_m$, so $\mathrm{GH}_x(\mathbb{V})$ splits as a sum of two graded vector spaces with actions of T_xY. Hence we may assume $\mathbb{V}$ is irreducible as a representation of G_K, and thus irreducible as a representation of $\mathfrak{g}$.

Because G is reductive, $\mathfrak{g}$ splits as a product of n simple Lie algebras $\mathfrak{g}_1,\dots,\mathfrak{g}_n$ times a trivial Lie algebra, and because $\mathbb{V}$ is irreducible as a representation of $\mathfrak{g}$, it is a tensor product of n nontrivial irreducible representations $V_1,\dots,V_n$ of the n simple Lie algebras with a one-dimensional representation of the trivial Lie algebra. Each V_i is a representation of the Lie algebra of $T_xY \rtimes \mathbb{G}_m$, and thus admits a $\mathbb{C}$-grading and an action of T_xY by commuting linear maps of degree -1.

If T_xY acts trivially on some V_i, then V_i splits as a direct sum since every graded vector space of dimension > 1 splits as a direct sum of graded vector spaces (and one-dimensional representations of simple Lie algebras are trivial), so $\mathbb{V}$ splits as a direct sum of graded vector spaces with actions of T_xY. So we may assume T_xY acts nontrivially on each V_i. It follows that each V_i has vectors of at least two different grades, so the tensor product $\mathbb{V}$ of the V_i has vectors of at least $n+1$ different grades, and thus $n \le k$.

As M is a normal subgroup of G, the Lie algebra $\mathfrak{m}$ of the identity component of M is an ideal of G. Therefore, $\mathfrak{m}$ is a sum of some of the $\mathfrak{g}_i$, possibly with a trivial algebra. Since monodromy groups of $\mathbb{Q}$-VHS's are simple, $\mathfrak{m}$ is a sum of some of the $\mathfrak{g}_i$. If some $\mathfrak{g}_i$ does not appear in this sum, then its adjoint representation corresponds to a $\mathbb{Q}$-VHS (possibly on a cover of X) with trivial monodromy, hence a constant Hodge structure, so the derivative of its period map vanishes, and thus T_xY acts trivially on this adjoint representation. But this implies that the image of T_xY in $\mathfrak{g}_i$ is zero, and thus T_xY acts trivially on V_i, contradicting our assumption that T_xY acts nontrivially on each V_i.

Thus $\mathfrak{m}$ is the sum of all the $\mathfrak{g}_i$ and thus acts irreducibly on $\mathbb{V}$ and, in addition, has $n \le k$ simple factors. □

5.7 Lie algebras and weights We discuss some generalities on representations of the Lie algebra of the generic Mumford–Tate group.

For $\mathbb{V}$ a $\mathbb{Q}$-VHS on a variety X and x a point of X, Proposition 5.5 gives a homomorphism $T_xY \rtimes \mathbb{G}_m \to G$, where G is the generic Mumford–Tate group of $\mathbb{V}$. Thus we obtain a Lie algebra homomorphism $T_xY \rtimes \mathbb{C} \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, $\mathbb{C}$ is the Lie algebra of $\mathbb{G}_m$, and $\mathbb{C}$ acts on T_xY by $[1, v] = -v$ for $v \in T_xY$, where the minus sign appears because $\mathbb{G}_m$ acts on T_xY by inverse scaling, by Proposition 5.5.

For any representation of $\mathfrak{g}$, we refer to the eigenvalues of $\mathbb{C} \subseteq T_xY \rtimes \mathbb{C}$ on that representation as the weights and the generalized eigenspace of a given eigenvalue as the weight space. For the representation arising from the action of G on $\mathbb{V}_x$, these weights agree with the Hodge weights of $\mathbb{V}$, and in particular are integers, but for arbitrary representations they will be complex numbers.

For any representation that factors through a finite covering of G, the action of $\mathbb{C}$ factors through a finite covering of $\mathbb{G}_m$. This implies that the weights will be rational numbers, and the action of $\mathbb{C}$ is semisimple so the generalized eigenspaces will be eigenspaces.

The identity $[1, v] = -v$ for $v \in T_xY$ implies that the elements of T_xY send elements of weight w to elements of weight $w - 1$.

5.8 Example Let $\mathbb{V}$ be an integral K-variation of Hodge structure on X, with Mumford–Tate (isogenous to) GL_ν acting through the $\wedge^k \colon \mathrm{GL}_\nu \to \mathrm{GL}_{\binom{\nu}{k}}$, the k^{th} wedge power of the standard representation, where $1 < k < \nu - 1$. Let $\mathrm{std}\colon \mathfrak{gl}_\nu \to \mathfrak{gl}_\nu$ be the standard representation. As in the discussion above, we have a Lie algebra homomorphism $\iota\colon T_xY \rtimes \mathbb{C} \to \mathfrak{g} = \mathfrak{gl}_\nu$, so that the weights of $\wedge^k \circ \iota(1)$ are the Hodge weights of $\mathbb{V}$.

We may in this case consider $\mathrm{std} \circ \iota(1)$, which acts on $\mathbb{C}^\nu$ with weights in $\frac{1}{k}\mathbb{Z}$. By definition, $\wedge^k \circ \iota(1) = \bigwedge^k(\mathrm{std} \circ \iota(1))$. Thus, letting $a_1, \dots, a_\nu$ be the weights of $\mathrm{std} \circ \iota(1)$, we have that the weights of $\wedge^k \circ \iota(1)$ are precisely the sums $a_{i_1} + \cdots + a_{i_k}$, where $1 \le i_1, \dots, i_k \le \nu$ are distinct integers.

The following case will be used in the proof Lemma 7.9. Suppose every weight of $\wedge^k \circ \iota(1)$ is either 0 or 1. Then the weights of $\mathrm{std} \circ \iota(1)$ must be either

$$\left\{\frac{1}{k}, \dots, \frac{1}{k}, -\frac{k-1}{k}\right\} \quad \text{or} \quad \{0, \dots, 0, 1\},$$

by an elementary combinatorial analysis.

6 Generic Torelli theorems for unitary local systems

Let $\pi\colon \mathscr{C} \to \mathscr{M}$ be a versal family of n-punctured curves of genus g, and let $\pi^\circ\colon \mathscr{C}^\circ \to \mathscr{M}$ be the associated punctured versal family, defined as in Notation 2.1. Let $m \in \mathscr{M}$ be a general point and $C = \mathscr{C}_m$, $C^\circ = \mathscr{C}^\circ_m$ and $D = C - C^\circ$. Let $\mathbb{U}$ be a unitary local system on $\mathscr{C}^\circ$. The goal of this section is to show that, in many cases, $\mathbb{U}|_{C^\circ}$ can be functorially recovered from the local system $W_1 R^1\pi^\circ_*\mathbb{U}$. In fact, it will be recoverable from the associated weak IVHS at a general point of m. We view this as a generic Torelli theorem for curves with a unitary local system, and indeed the proof is closely related to classical proofs of the generic Torelli theorem, as in [Harris 1985].

6.1 Reconstructing unitary bundles In this section we reconstruct from $W_1 R^1\pi^\circ_*\mathbb{U}$ the vector bundle $\widehat{E}_0$ corresponding to the parabolic bundle on (C, D) associated to $\mathbb{U}|_{C^\circ}$. In fact we will reconstruct $\widehat{E}_0$ from $\mathrm{GH}_m(W_1 R^1\pi^\circ_*\mathbb{U})$, with GH as defined in (5-1). In the case $D = \varnothing$, this in fact recovers $\mathbb{U}|_C$ by the Narasimhan–Seshadri correspondence [Narasimhan and Seshadri 1965]. When D is nonempty, some additional work will be required to recover $\mathbb{U}|_C$.

6.2 Theorem *With notation as above, let $\mathbb{U}$ be a unitary local system on C°, with monodromy representation ρ and associated parabolic bundle $E_\star$ on (C, D). Let $r = \mathrm{rk}(\mathbb{U})$, and assume $g \ge 2 + 2r$. Let ψ^ρ_m be the map*

$$H^0(C, \widehat{E}_0 \otimes \omega_C(D)) \otimes \mathscr{O}_C \xrightarrow{\theta^\rho_m \otimes \mathrm{id}} H^1(C, E_0) \otimes H^0(C, \omega_C^{\otimes 2}(D)) \otimes \mathscr{O}_C \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} H^1(C, E_0) \otimes \omega_C^{\otimes 2}(D)$$

obtained as the composition of the map induced by θ^ρ_m, defined in (3-5), with the map induced by the evaluation map $\mathrm{ev}\colon H^0(C, \omega_C^{\otimes 2}(D)) \otimes \mathscr{O}_C \to \omega_C^{\otimes 2}(D)$. Then ψ^ρ_m factors through the evaluation map

$$\mathrm{ev}\colon H^0(C, \widehat{E}_0 \otimes \omega_C(D)) \otimes \mathscr{O}_C \to \widehat{E}_0 \otimes \omega_C(D),$$

and induces an isomorphism

$$\widehat{E}_0 \otimes \omega_C(D) \xrightarrow{\sim} \mathrm{im}(\psi^\rho_m).$$

Proof It suffices to show that

$$\ker(\psi_m^\rho) = \ker(\mathrm{ev}\colon H^0(C, \widehat{E}_0 \otimes \omega_C(D)) \otimes \mathscr{O}_C \to \widehat{E}_0 \otimes \omega_C(D)), \tag{6-1}$$

as $\widehat{E}_0 \otimes \omega_C(D)$ is globally generated by Proposition 4.9, using that m is a general point of $\mathscr{M}$.

We first observe that $H^1(C, E_0)$ is, by Serre duality, dual to $H^0(C, E_0^\vee \otimes \omega_C)$. As m is general, $E_0^\vee \otimes \omega_C$ is globally generated by Proposition 4.9 since, setting $F_\star$ to be the parabolic dual to $E_\star$, we have $E_0^\vee \otimes \omega_C = \widehat{F}_0 \otimes \omega_C(D)$.

Let s be a local section to $H^0(C, \widehat{E}_0 \otimes \omega_C(D)) \otimes \mathscr{O}_C$. Now s is in the kernel of ψ_m^ρ if and only if, for all $t \in H^1(C, E_0)^\vee = H^0(C, E_0^\vee \otimes \omega_C)$, we have that $t(\psi_m^\rho(s)) = 0$ as a local section to $\omega_C^{\otimes 2}(D)$. Note that $t(\psi_m^\rho(s)) = B_m^\rho(s,t)$, for B_m^ρ as defined in (3-4). Now, $B_m^\rho(s,t)$ vanishes for all t if and only if $\mathrm{ev}(s) = 0$, as $E_0^\vee \otimes \omega_C$ is (generically) globally generated and B_m^ρ induces a perfect pairing between the generic fibers of $\widehat{E}_0 \otimes \omega_C(D)$ and $E_0^\vee \otimes \omega_C$. This implies (6-1). □

6.3 Functoriality The upshot of Theorem 6.2 is that $\widehat{E}_0 \otimes \omega_C(D)$ (and hence $\widehat{E}_0$ itself) may be constructed from $\mathrm{GH}_m(W_1 R^1\pi_*\mathbb{U})$ for generic m. This construction is functorial: given $\mathbb{U}$ and $\mathbb{U}'$ unitary local systems on $\mathscr{C}^\circ$, and given a map $\mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}) \to \mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}')$, one obtains a map $\widehat{E}_0 \to \widehat{E}'_0$, where $E'_\star$ is the parabolic bundle on C corresponding to $\mathbb{U}'|_{C^\circ}$. The goal of this section is to show that in many cases this map is necessarily flat for the natural unitary connections on $\widehat{E}_0, \widehat{E}'_0$. This is automatic from the Narasimhan–Seshadri correspondence if $D = \varnothing$, but not in general.

6.4 Proposition *With notation as in Notation 2.1, let $m \in \mathscr{M}$ be general, set $C = \mathscr{C}_m$, $C^\circ = \mathscr{C}_m^\circ$ and $D = C - C^\circ$. Let $\mathbb{U}$ and $\mathbb{U}'$ be unitary local systems on $\mathscr{C}^\circ$ such that $\mathbb{U}|_{C^\circ}$ and $\mathbb{U}'|_{C^\circ}$ are irreducible, of dimensions r and r', respectively. Assume that*

$$\mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}) \quad \text{and} \quad \mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}')$$

are isomorphic to one another. Suppose that $g \geq \max(2+2r, 2+2r')$ and either

(1) $D = \varnothing$, *or*

(2) $g > rr'$.

Then the natural map

$$\mathrm{Hom}_{\mathrm{LocSys}}(\mathbb{U}|_{C^\circ}, \mathbb{U}'|_{C^\circ}) \to \mathrm{Hom}_{\mathrm{IVHS}(\mathscr{M},m)}(\mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}), \mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}')) \tag{6-2}$$

is a bijection. In particular, $\mathbb{U}|_{C^\circ}$ is isomorphic to $\mathbb{U}'|_{C^\circ}$.

Proof Let ρ and ρ' be the monodromy representations of $\mathbb{U}$ and $\mathbb{U}'$, and let $E_\star^\rho, E_\star^{\rho'}$ be the parabolic bundles associated to these local systems. By Theorem 6.2, there exists an isomorphism $\widehat{E}_0^\rho \simeq \widehat{E}_0^{\rho'}$, and hence $r = r'$.

We now prove (6-2) is injective. Indeed, a map $\mathbb{U}|_{C^\circ} \to \mathbb{U}'|_{C^\circ}$ is determined by the induced map $\widehat{E}_0^{\rho} \to \widehat{E}_0^{\rho'}$, which is in turn determined by the induced map

$$H^0(C, \widehat{E}_0^{\rho} \otimes \omega_C(D)) \to H^0(C, \widehat{E}_0^{\rho'} \otimes \omega_C(D)),$$

as the vector bundles in question are globally generated by Proposition 4.9.

To prove the surjectivity of (6-2), consider a map $\psi\colon \mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}) \to \mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U}')$. By Theorem 6.2, it induces a map $\widetilde{\psi}\colon \widehat{E}_0^{\rho} \to \widehat{E}_0^{\rho'}$. We must show this map is flat. Equivalently, we wish to show that the map

$$\nabla\widetilde{\psi}\colon T_C(-D) \to \mathrm{Hom}(\widehat{E}_0^{\rho}, \widehat{E}_0^{\rho'})$$

sending a vector field X to $\nabla_X\widetilde{\psi}$ is identically zero. Here ∇_X is differentiation along X, using the connection ∇ on $\mathrm{Hom}(\widehat{E}_0^{\rho}, \widehat{E}_0^{\rho'})$. When $D = \varnothing$ this is immediate by the functoriality of the Narasimhan–Seshadri correspondence, so we need only consider the case $D \neq \varnothing$.

By Lemma 4.8, we know that $\nabla\widetilde{\psi}$ induces the zero map

$$H^1(C, T_C(-D)) \to H^1(\mathrm{Hom}(\widehat{E}_0^{\rho}, \widehat{E}_0^{\rho'})),$$

as m is general and hence ψ extends to a first-order neighborhood of m. Serre-dually, the map

$$\mathrm{Hom}(\widehat{E}_0^{\rho'}, \widehat{E}_0^{\rho}) \otimes \omega_C \to \omega_C^{\otimes 2}(D),$$

obtained by dualizing $\nabla\widetilde{\psi}$ and tensoring with ω_C, induces zero on H^0. Thus if $\nabla\widetilde{\psi}$ is nonzero, the bundle $\mathrm{Hom}(\widehat{E}_0^{\rho'}, \widehat{E}_0^{\rho}) \otimes \omega_C$ is not generically globally generated.

Now as $g > rr' = r^2$, $E_0^{(\rho')^\vee}$ and $E_0^{\rho^\vee}$ are semistable, by [Landesman and Litt 2024b, Corollary 6.1.2]; hence by Lemma 4.5, the same is true for $\widehat{E}_0^{\rho'}$ and $\widehat{E}_0^{\rho}$. As these bundles are isomorphic by the first paragraph above, we have that $\mathrm{Hom}(\widehat{E}_0^{\rho'}, \widehat{E}_0^{\rho}) \otimes \omega_C$ is semistable of slope $2g-2$ and rank $rr' = r^2$. As $g > rr'$ it is thus generically globally generated by [loc. cit., Proposition 6.3.1(b)], whence the proof is complete. □

The following will not be used in what follows, but we felt it might be of independent interest.

6.5 Construction With notation as in Notation 2.1, let $\mathbb{U}$ be a unitary local system on $\mathscr{C}^\circ$ of rank r. Let $g > r^2$. Let $W_1 R^1\pi_*^\circ\mathbb{U}$ be the above defined $\mathbb{C}$-VHS and let $m \in \mathscr{M}$ be general, with $C^\circ = \mathscr{C}_m^\circ$ and $C = \mathscr{C}_m$. We next sketch how to directly reconstruct the connection on $\widehat{E}_0$, where $E_\star$ is the parabolic bundle associated to $\mathbb{U}|_{C^\circ}$.

We may recover the connection from the restriction of $W_1 R^1\pi_*^\circ\mathbb{U}$ to a small neighborhood of m, though we do not know how to do so from $\mathrm{GH}_m(W_1 R^1\pi_*^\circ\mathbb{U})$.

Recall that the data of a logarithmic connection on a bundle F is the same as an $\mathscr{O}_C$-linear splitting s of the natural map

$$\mathrm{At}_{(C,D)}(F) \to T_C(-D)$$

(see eg [loc. cit., Proposition 3.1.6]). The induced map

$$H^1(C, T_C(-D)) \to H^1(C, \mathrm{At}_{(C,D)}(F))$$

may be interpreted as the map sending a first-order deformation $(\widetilde{C}, \widetilde{D})$ of (C, D) to the triple $(\widetilde{C}, \widetilde{D}, \widetilde{E})$, where $\widetilde{E}$ is the isomonodromic deformation of E; see [Landesman and Litt 2024b, Proposition 3.5.7]. This latter map may be recovered (taking $F = \widehat{E}_0$) from $W_1 R^1\pi_*^\circ \mathbb{U}$ if $g \geq r^2$, by performing the construction of Theorem 6.2 in families. Serre-dually, we may recover the equivalent data of the Serre-dual map

$$H^0(C, (\mathrm{At}_{(C,D)}(\widehat{E}_0))^\vee \otimes \omega_C) \to H^0(C, \omega_C^{\otimes 2}(D)).$$

Now $\omega_C^{\otimes 2}(D)$ is globally generated, and $(\mathrm{At}_{(C,D)}(\widehat{E}_0))^\vee \otimes \omega_C$ is generically globally generated by similar arguments to those in the proof of Proposition 6.4.

Since a map of generically globally generated vector bundles is determined by the induced map on global sections, the argument is complete.

6.6 Simplicity of the monodromy representation We now prove that the monodromy representation is simple.

6.7 Theorem *With notation as in Notation 2.1, let $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible H-representation, and let $\mathbb{V}$ be the corresponding local system on $\mathscr{C}^\circ$. Suppose that either $n = 0$ and $g > 2r + 1$, or n is arbitrary and $g > \max(2r + 1, r^2)$. Then the variation of Hodge structure $W_1 R^1\pi_*^\circ \mathbb{V}$ has simple monodromy group, acting irreducibly.*

Proof Note first that $\mathbb{V}$ is an integral variation of Hodge structure because every representation of a finite group is defined over the ring of integers over some number field and is polarizable because representations of finite groups are unitary. Thus $W_1 R^1\pi_*^\circ \mathbb{V}$ is an integral variation of Hodge structures because these are stable under derived pushforward and passing to subspaces in the weight filtration. (See eg [Landesman and Litt 2024a, Theorem 4.1.1] for a discussion without the integrality condition.)

Now suppose that the conclusion of the theorem is false, ie that either the monodromy group of $W_1 R^1\pi_*^\circ \mathbb{V}$ is not simple, or it does not act irreducibly. Then by Lemma 5.6, $\mathrm{GH}_m(W_1 R^1\pi_*^\circ \mathbb{V})$ splits as a direct sum for general $m \in \mathscr{M}$. Thus we have

$$\dim \mathrm{Hom}_{\mathrm{IVHS}(\mathscr{M},m)}(\mathrm{GH}_m(W_1 R^1\pi_*^\circ \mathbb{V}), \mathrm{GH}_m(W_1 R^1\pi_*^\circ \mathbb{V})) \geq 2.$$

But taking $\mathbb{U} = \mathbb{U}' = \mathbb{V}$ in Proposition 6.4, we have

$$\dim \mathrm{Hom}_H(\rho, \rho) \geq 2.$$

But this contradicts the irreducibility of ρ, by Schur's lemma. □

6.8 Remark We could have argued using Theorem 6.2, instead of Proposition 6.4, as follows. The splitting of $\mathrm{GH}_m(W_1 R^1\pi^\circ_*\mathbb{V})$ implies, by the construction of Theorem 6.2, that $\hat{E}_0^\rho$ itself splits as a direct sum. But forthcoming work of Ramirez-Cote, following ideas of Landesman and Litt [2024b], shows that $\hat{E}_0^\rho$ is stable for irreducible ρ when $g \geq r^2$, contradicting this splitting.

6.9 Corollary *With notation as in Notation 2.1, let $\rho_i\colon H \to \mathrm{GL}_{r_i}(\mathbb{C})$ for $i = 1, 2$ be irreducible representations, and let $\mathbb{V}_1$ and $\mathbb{V}_2$ be the corresponding local systems on $\mathscr{C}^\circ$. Suppose that $g \geq \max(2+2r_1, 2+2r_2)$ and either $D = \varnothing$ or $g > r_1 r_2$. If the local systems $W_1 R^1\pi^\circ_*\mathbb{V}_1$ and $W_1 R^1\pi^\circ_*\mathbb{V}_2$ are isomorphic, then ρ_1 and ρ_2 are conjugate.*

Proof By Theorem 6.7, the local systems $W_1 R^1\pi^\circ_*\mathbb{V}_1, W_1 R^1\pi^\circ_*\mathbb{V}_2$ are irreducible. Hence, by the theorem of the fixed part applied to the tensor product $(W_1 R^1\pi^\circ_*\mathbb{V}_1)^\vee \otimes W_1 R^1\pi^\circ_*\mathbb{V}_2$, any isomorphism between them is necessarily an isomorphism of $\mathbb{C}$-VHS (up to a shift of the weight and Hodge filtrations, but these shifts must vanish by consideration of the weight and Hodge numbers of both sides), and in particular induces an isomorphism $\mathrm{GH}_m(W_1 R^1\pi^\circ_*\mathbb{V}_1) \simeq \mathrm{GH}_m(W_1 R^1\pi^\circ_*\mathbb{V}_2)$ for any $m \in \mathbb{M}$. Now the result follows by Proposition 6.4. □

7 Proofs of Theorems 1.3 and 1.9: big monodromy for g large

In this section, we prove our main results, Theorems 1.3 and 1.9. We next describe the possibilities for the connected monodromy groups of the variations of Hodge structure appearing in Theorem 6.7 by applying a result of Deligne (implicit in his classification of Shimura varieties of Abelian type [Deligne 1979, 1.3.6], and explicit in [Zarkhin 1984, Theorem 0.5.1(b)]), and proceed to rule out many of these possibilities through several lemmas. We conclude the proofs in Section 7.15.

7.1 Describing the possibilities for the monodromy representation With notation as in Theorem 6.7, the Hodge filtration of $W_1 R^1\pi^\circ_*\mathbb{V}$ has only two parts, and the monodromy group of $W_1 R^1\pi^\circ_*\mathbb{V}$ is simple by Theorem 6.7. As it is a normal subgroup of the generic Mumford–Tate group of $W_1 R^1\pi^\circ_*\mathbb{V}$, by [André 1992, Theorem 1 on page 10], it follows that it must be a simple factor of this group. It follows from [Zarkhin 1984, Theorem 0.5.1(b)] (which Zarkhin attributes to Deligne) that the monodromy group is isogenous to $\mathrm{SL}_\nu, \mathrm{Sp}_{2\nu}, \mathrm{SO}_\nu$ for some ν, acting in one of the following ways:

(1) The standard representation.

(2) The trivial representation.

(3) The spin or half-spin representation of the spin cover of the group SO_ν.

(4) A wedge power of the standard representation of the group SL_ν.

We first rule out the trivial representation.

7.2 Lemma *With notation as in Notation 2.1, let $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible H-representation, and let $\mathbb{V}$ be the corresponding local system on $\mathscr{C}^\circ$. If $g > 2$, the connected monodromy group of $W_1 R^1\pi^\circ_*\mathbb{V}$ is nontrivial.*

Proof We are free to pass to finite étale covers of the base $\mathscr{M}$ in our setup. Therefore, we may assume the Zariski closure of monodromy is already connected. We also know the monodromy representation is irreducible by Theorem 6.7. Further,

$$\dim W_1 R^1\pi^\circ_*\mathbb{V}_m \geq (2g-2)\,\mathrm{rk}(\mathbb{V}) > 1,$$

since we are assuming $g \geq 2$, so we conclude that the monodromy group is not acting via the trivial representation. □

7.3 Rank estimates

7.4 Lemma *With notation as in Notation 2.1, let $\mathbb{U}$ be a unitary local system on $\mathscr{C}^\circ$ and $m \in \mathscr{M}$ a general point. Let ρ be the monodromy representation of $\mathbb{U}|_{C^\circ}$ and let $E_\star$ be the associated parabolic bundle on (C, D). Let $r = \mathrm{rk}(\mathbb{U})$, and assume $g \geq 2 + 2r$. For p in C, consider the one-dimensional space of $H^0(C, \omega_C^{\otimes 2}(D))^\vee = T_m\mathscr{M}$ spanned by the functional sending a global section to its value at p, and then composing with a linear map $\omega_C^{\otimes 2}(D)|_p \to \mathbb{C}$.*

For p a general point of C and α_p an element of the associated one-dimensional subspace of $T_m\mathscr{M}$, the rank of $dP^\rho_m(\alpha_p) \in \mathrm{Hom}(H^0(C, \hat{E}_0 \otimes \omega_C(D)), H^1(C, E_0))$ is equal to r.

Here dP^ρ_m is defined as in Section 3, immediately above Proposition 3.6.

Proof Theorem 6.2 shows that $\hat{E}_0 \otimes \omega_C(D)$, a sheaf of rank r, is the image of the map ψ^ρ_m, defined as in the statement of that theorem. For any map $f\colon V \to W$ of vector bundles, the rank of the image of f is equal to the rank of the fiber $f_q\colon V_q \to W_q$ of f at a general point q. So it suffices to check that the rank of the fiber of ψ^ρ_m at a general point p in C is equal to the rank of $dP_m(\alpha_p)$.

The fiber of the evaluation map $\mathrm{ev}\colon H^0(C, \omega_C^{\otimes 2}(D)) \otimes \mathscr{O}_C \to \omega_C^{\otimes 2}(D)$ at p is, up to scalars, the linear form $\alpha_p \in (H^0(C, \omega_C^{\otimes 2}(D)) \otimes \mathscr{O}_C)^\vee$.

Since ψ^ρ_m is the composition of θ^ρ_m with $\mathrm{id} \otimes \mathrm{ev}$, the fiber of ψ^ρ_m is the composition of θ^ρ_m with $\mathrm{id} \otimes \alpha_p$. Previously, in (3-5), we defined θ^ρ_m as the adjoint of

$$dP^\rho_m\colon H^0(C, \omega_C^{\otimes 2}(D))^\vee \to \mathrm{Hom}(H^0(C, \hat{E}_0 \otimes \omega_C(D)), H^1(C, E_0)),$$

namely by

$$\theta^\rho_m\colon H^0(C, \hat{E}^\rho_0 \otimes \omega_C(D)) \to H^1(C, E_0) \otimes H^0(C, \omega_C^{\otimes 2}(D)).$$

The defining property of this adjoint is that composing with $\mathrm{id} \otimes \alpha$ for a linear form $\alpha \in H^0(C, \omega_C^{\otimes 2}(D))^\vee$ gives a map $H^0(C, \hat{E}^\rho_0 \otimes \omega_C(D)) \to H^1(C, E_0)$ defined by $dP^\rho_m(\alpha)$, so the composition of θ^ρ_m with $\mathrm{id} \otimes \alpha_p$ is $dP^\rho_m(\alpha_p)$, as desired. □

7.5 Remark The subspaces of $H^0(C, \omega_C^{\otimes 2}(D))^\vee = H^1(C, T_C(-D))$ spanned by the α_p of Lemma 7.4 are known as *Schiffer variations*, see eg [Collino and Pirola 1995, 1.2.4]. There is some history of using (variants of) Schiffer variations for Torelli-style results, eg in [Voisin 2022].

7.6 Corollary *With notation as above, let $\mathbb{U}$ be a unitary local system on C°, with monodromy representation ρ and associated parabolic bundle $E_\star$ on (C, D). Let $r = \mathrm{rk}(\mathbb{U})$, and assume $g \geq 2 + 2r$. We have*

$$r \geq \inf\{\mathrm{rank}(dP_m^\rho(\alpha)) \mid \alpha \in T_m\mathscr{M}, dP_m^\rho(\alpha) \neq 0\}.$$

Proof This follows from Lemma 7.4 since any member of a set is at least its minimal value, and r is a member of this set because $r = \mathrm{rank}(dP_m^\rho(\alpha_p))$ with $dP_m^\rho(\alpha_p) \neq 0$, as $r > 0$. □

7.7 Lemma *For ρ a unitary representation of $\pi_1(C - D)$ of rank r, we have*

$$\dim H^1(C, E_0^\rho) \geq (g-1)r \quad \text{and} \quad \dim H^0(C, \hat{E}_0^\rho \otimes \omega_C(D)) \geq (g-1)r.$$

Proof Since $E_\star^\rho$ has parabolic degree 0, the degree of E_0^ρ is ≤ 0, and so $H^1(C, E_0^\rho) \geq (g-1)r$ by Riemann–Roch. By Serre duality, [Landesman and Litt 2024b, Proposition 2.6.6],

$$\dim H^0(C, \hat{E}_0^\rho \otimes \omega_C(D)) = \dim H^1(C, \hat{E}_0^{\rho^\vee}).$$

The latter is $\geq (g-1)r$ by the first part. □

Combining Corollary 7.6 and Lemma 7.7, we obtain an inequality expressed only in terms of the weak IVHS of $W_1 R^1 \pi_*^\circ \mathbb{V}$, ie

$$\dim H^1(C, E_0^\rho) \geq (g-1) \inf\{\mathrm{rank}(dP_m^\rho(\alpha)) \mid \alpha \in T_m\mathscr{M}, dP_m^\rho(\alpha) \neq 0\},$$
$$\dim H^0(C, \hat{E}_0^\rho \otimes \omega_C(D)) \geq (g-1) \inf\{\mathrm{rank}(dP_m^\rho(\alpha)) \mid \alpha \in T_m\mathscr{M}, dP_m^\rho(\alpha) \neq 0\}.$$

7.8 Ruling out nonstandard minuscule representations In this section we show that if the monodromy group of $W_1 R^1 \pi_*^\circ \mathbb{V}$ is SL_ν or SO_ν, it must act via the standard representation. We will use the notation discussed in Section 5.7.

7.9 Lemma *With notation as in Notation 2.1, let $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible H-representation, and let $\mathbb{V}$ be the corresponding local system on $\mathscr{C}^\circ$. Suppose $g \geq 2r + 2$. If the connected monodromy group of $W_1 R^1 \pi_*^\circ \mathbb{V}$ is isogenous to SL_ν acting via $\wedge^k$ of the standard representation, then either $k = 1$ or $k = \nu - 1$.*

Proof The identity component of the normalizer of the monodromy group SL_ν is $\mathrm{GL}_\nu/\mu_{\gcd(k,\nu)}$, with $\mu_{\gcd(k,\nu)}$ embedded as scalar matrices, so its Lie algebra is $\mathfrak{gl}_\nu$; in particular, we are provided with a natural Lie algebra homomorphism $T_x Y \rtimes \mathbb{C} \to \mathfrak{gl}_\nu$ as in Section 5.7. Let $(a_1, \dots, a_\nu)$ be the weights of the standard representation of $\mathfrak{gl}_\nu$.

We first determine the possible values for the a_i; see also Example 5.8. Since the k^{th} wedge power of the standard representation has only weights 0 and 1, as the Hodge structure $W_1 R^1\pi^\circ_*\mathbb{V}$ has only two parts, we must have that $\sum_{i\in I} a_i \in \{0,1\}$ for any subset $I \subset \{1,\dots,\nu\}$ of size k. By Lemma 7.2, $k \neq 0, \nu$. Thus Lemma 7.10 below shows that the only possibilities for $(a_1,\dots,a_\nu)$, up to reordering, are

$$a_1 = 1, \quad a_2 = \cdots = a_\nu = 0 \qquad \text{or} \qquad a_1 = \frac{1-k}{k}, \quad a_2 = \cdots = a_\nu = \frac{1}{k}.$$

We now consider how elements of $T_m\mathscr{M}$ act on the standard representation of $\mathfrak{gl}_\nu$. In the first case, any element of $T_m\mathscr{M}$ must send the weight 1 space to the weight 0 space and the weight 0 space to zero. Since the weight 1 space is one-dimensional, any element of $T_m\mathscr{M}$ must be zero or nilpotent of rank 1. In the second case, the element sends the weight $1/k$ space to the weight $(1-k)/k$ space, and the weight $(1-k)/k$ space to zero, and again must be zero or nilpotent of rank 1.

The action of a nilpotent element of rank one in $\mathfrak{gl}_\nu$ on the representation $\wedge^k$ has rank $\binom{\nu-2}{k-1}$ by Lemma 7.11 below. So the minimum nonzero rank of an element of $T_m\mathscr{M}$ acting on $\mathbb{V}$ is $\binom{\nu-2}{k-1}$. It follows from Corollary 7.6 that $r \geq \binom{\nu-2}{k-1}$.

On the other hand, $\dim H^1(C, E_0^\rho)$ and $\dim H^0(C, \hat{E}_0^\rho) \otimes \omega_C(D)$ are either respectively equal to $\binom{\nu-1}{k-1}$ and $\binom{\nu-1}{k}$, or respectively equal to $\binom{\nu-1}{k}$ and $\binom{\nu-1}{k-1}$. By Lemma 7.7, these are both at least $(g-1)r$, which gives

$$\binom{\nu-1}{k-1} \geq (g-1)r \geq (g-1)\binom{\nu-2}{k-1} \quad \text{and} \quad \binom{\nu-1}{k} \geq (g-1)r \geq (g-1)\binom{\nu-2}{k-1}.$$

Dividing both sides by $\binom{\nu-2}{k-1}$ we obtain

$$\frac{\nu-1}{\nu-k} \geq (g-1) \quad \text{and} \quad \frac{\nu-1}{k} \geq (g-1),$$

so

$$1 = \frac{k}{\nu} + \frac{\nu-k}{\nu} < \frac{k}{\nu-1} + \frac{\nu-k}{\nu-1} \leq \frac{1}{g-1} + \frac{1}{g-1},$$

which implies $g-1 < 2$, contradicting the assumption that $g \geq 2r+2 \geq 4$. □

7.10 Lemma *Let ν and k be natural numbers with $1 < k < \nu$. Let $a_1,\dots,a_\nu$ be a tuple of complex numbers such that each sum of exactly k of the $a_1,\dots,a_\nu$ is either 0 or 1, with both possibilities occurring. Then, up to reordering, we have either*

$$a_1 = 1, \quad a_2 = \cdots = a_\nu = 0 \qquad \text{or} \qquad a_1 = \frac{1-k}{k}, \quad a_2 = \cdots = a_\nu = \frac{1}{k}.$$

Proof We first observe that there can be at most two distinct values appearing in $\{a_1,\dots,a_\nu\}$, as otherwise one could find size k subsets summing to three different values. Further, if a_1 is distinct from a_2, then, after possibly switching a_1 and a_2, all other values of a_i must agree with a_2, using that $1 < k < \nu-1$, or else we could again obtain three distinct sums from subsets of size k. Hence, we must have $a_2 = \cdots = a_\nu$, and then a subset of size k drawn from $\{a_2,\dots,a_\nu\}$ must sum to 0 or 1, yielding the two possibilities claimed above. □

7.11 Lemma *Let $\nu \geq 2$ and k be natural numbers and Let $N \in \mathfrak{gl}_\nu$ be a nilpotent element of rank 1. Then the action of N on the representation $\wedge^k$ of $\mathfrak{gl}_\nu$ is by a matrix of rank $\binom{\nu-2}{k-1}$.*

Proof We may choose a basis $e_1, \dots, e_\nu$ where the element sends e_1 to e_2 and each other basis vector to 0, and then the image is generated by wedges of e_2 with $k-1$ of the remaining $\nu-2$ basis vectors $e_3, \dots, e_\nu$. □

7.12 Lemma *With notation as in Notation 2.1, let $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible H-representation, and let $\mathbb{V}$ be the corresponding local system on $\mathscr{C}^\circ$. Suppose $g \geq 2r+2$. The connected monodromy group of $W_1 R^1\pi_*^\circ\mathbb{V}$ cannot be isogenous to Spin_ν acting via a spin or half-spin representation unless the monodromy group is also a classical group acting via its standard representation.*

Note that (as discussed in the first paragraph of the proof of Lemma 7.12) the group Spin_ν acting via the spin representation is isogenous to a classical group via an isogeny sending the spin representation to the standard representation, if and only if $\nu = 3, 4, 5, 6, 8$.

Proof We first observe that for $\nu = 3, 4, 5, 6$, the spin group is respectively SL_2, $\mathrm{SL}_2 \times \mathrm{SL}_2$, Sp_4, SL_4, and the spin representation in each case is the standard representation (or, in $\nu = 2$, the standard representation of one of the two factors). For $\nu = 8$, the spin group is not a classical group but each half-spin representation factors through the standard representation of a different quotient of Spin_8 isomorphic to SO_8. (The existence of these three different SO_8 quotients is known as triality). So we may assume that $\nu > 8$ or $\nu = 7$.

Now, the identity component of the normalizer of the monodromy group is GSpin_ν, with Lie algebra $\mathfrak{go}_\nu$. The weights of the standard representation of $\mathfrak{go}_\nu$ come in opposite pairs $b+a_1, b-a_1, \dots, b+a_k, b-a_k$ if $\nu = 2k$ is even, while if $\nu = 2k+1$ is odd, the weights have the form $b+a_1, b-a_1, \dots, b+a_k, b-a_k, b$. In either case we may assume all the values a_i are nonnegative.

First, we recall properties of the spin representation, as defined by Chevalley [1954, pages 119–122]. In the case that $\nu = 2k$ is even, there are two irreducible half-spin representation of dimension 2^{k-1}, and if $\nu = 2k+1$ is odd, then there is an irreducible spin representation of dimension 2^k.

The weights of the spin representation are then obtained as the 2^k sums $\frac{1}{2}(\pm a_1 \pm a_2 \pm \cdots \pm a_k) + \frac{1}{2}kb$. One half-spin representation has weights which are sums as above with an odd number of minus signs, and the other consists of weights which are sums as above with an even number of minus signs. Applying Lemma 7.13, in case (1) if $\nu = 2k+1$ is odd (where we have $\nu \geq 7$ so that $k \geq 3$) and in case (2) if $\nu = 2k$ is even (where we have $\nu > 8$ so $k > 4$), we can assume $a_1 = 1$ and $a_2, \dots, a_k = 0$.

It follows that the action of a generator of the Lie algebra $\mathbb{C}$ on the standard representation of $\mathfrak{so}_\nu$ has a one-dimensional $(1+b)$-eigenspace, a one-dimensional $(b-1)$-eigenspace, and a $(\nu-2)$-dimensional b-eigenspace. Applying Lemma 7.14, where w is obtained by subtracting the scalar b from the generator 1 of the Lie algebra $\mathbb{C}$ and the commutator relation comes from Section 5.7, we see that elements of $T_m\mathcal{M}$ acting on $\mathbb{V}$ have rank either 0, 2^{k-3} or 2^{k-2}. In particular, the minimum nonzero rank is $\geq 2^{k-3}$, and hence $r \geq 2^{k-1}$. But the weight-0 and weight-1 spaces of the spin representation both have dimension 2^{k-2}, so that $\dim H^1(C, E_0^\rho)$ and $\dim H^0(C, \hat{E}_0^\rho) \otimes \omega_C(D))$ are each equal to 2^{k-2} and hence $\leq 2r$. Since $g \geq 2r+2 > 3$, this contradicts Lemma 7.7. □

7.13 Lemma *Let k be a natural number. Let $a_1, \dots, a_k$ and b be rational numbers. Suppose that either*:

1. $k \geq 1$ *and all sums of the form* $\frac{1}{2}(\pm a_1 \pm a_2 \pm \cdots \pm a_k) + \frac{1}{2}kb$ *are equal to* 0 *or* 1, *with both values attained.*
2. $k > 4$ *and either*
   - (a) *all sums of the form* $\frac{1}{2}(\pm a_1 \pm a_2 \pm \cdots \pm a_k) + \frac{1}{2}kb$ *with an even number of minus signs are equal to* 0 *or* 1, *with both values attained*, *or*
   - (b) *all sums of the form* $\frac{1}{2}(\pm a_1 \pm a_2 \pm \cdots \pm a_k) + \frac{1}{2}kb$ *with an odd minus sign are equal to* 0 *or* 1, *with both values attained.*

Then, *up to reordering and swapping signs*, *we have* $a_1 = 1$, $a_2, \dots, a_k = 0$, *and* $b = 1/k$.

Proof By swapping signs, we may assume all the a_i are nonnegative.

In case (1), if two of the a_is are nonzero, then without loss of generality we may assume a_1 and a_2 are both nonzero, and we obtain three distinct sums

$$\frac{1}{2}\Big(-a_1 - a_2 + \sum_{i=3}^{k} a_k + kb\Big), \quad \frac{1}{2}\Big(-a_1 + a_2 + \sum_{i=3}^{k} a_k + kb\Big), \quad \frac{1}{2}\Big(a_1 + a_2 + \sum_{i=3}^{k} a_k + kb\Big).$$

So all but one of the a_i are 0, and up to reordering only a_1 may be nonzero, so the only possible weights are $\frac{1}{2}a_1 + \frac{1}{2}kb$ and $-\frac{1}{2}a_1 + \frac{1}{2}kb$. Since the difference between these must be 1, we must have $a_1 = 1$, which implies $b = 1/k$.

In case (2), if two of the a_i are nonzero, without loss of generality a_1 and a_2, the sums with an even number of minus signs will include the three distinct values

$$\frac{1}{2}\Big(a_1 + a_2 + a_3 + a_4 + \sum_{i=5}^{k} a_k + kb\Big),$$

$$\frac{1}{2}\Big(-a_1 + a_2 - a_3 + a_4 + \sum_{i=5}^{k} a_k + kb\Big),$$

$$\frac{1}{2}\Big(-a_1 - a_2 - a_3 - a_4 + \sum_{i=5}^{k} a_k + kb\Big).$$

Similarly, the sums with an odd number of minus signs will include the three distinct values

$$\frac{1}{2}\Bigl(a_1+a_2+a_3+a_4-a_5+\sum_{i=6}^{k} a_k+kb\Bigr),$$

$$\frac{1}{2}\Bigl(-a_1+a_2-a_3+a_4-a_5+\sum_{i=6}^{k} a_k+kb\Bigr),$$

$$\frac{1}{2}\Bigl(-a_1-a_2-a_3-a_4-a_5+\sum_{i=6}^{k} a_k+kb\Bigr).$$

So only one of the a_i may be nonzero. We conclude the argument the same way as in case (1). □

7.14 Lemma *Let $\nu \ge 7$ be a natural number. Let $w \in \mathfrak{so}_\nu$ be an element with a one-dimensional 1-eigenspace, a one-dimensional (-1)-eigenspace, and a $(\nu-2)$-dimensional 0-eigenspace. Let $N \in \mathfrak{so}_\nu$ be an element such that $wN - Nw = -N$. Then if $\nu = 2k-1$ is odd, the action of N on the 2^{k-1}-dimensional spin representation of $\mathfrak{so}_\nu$ has rank either 2^{k-2}, 2^{k-3} or 0. If $\nu = 2k$ is even, the action of N on either of the 2^{k-1}-dimensional half-spin representations of $\mathfrak{so}_\nu$ has rank either 2^{k-2}, 2^{k-3} or 0.*

Proof The action of N sends the one-dimensional weight 1-eigenspace to the $(\nu-2)$-dimensional 0-eigenspace, and sends the 0-eigenspace to the (-1)-eigenspace. Thus N is determined by its action on a generator of the 1-eigenspace space as the condition that it is equal to minus its transpose will then determine its action on the 0-eigenspace. Fix a two-dimensional subspace W of the 0-eigenspace where the quadratic form is nondegenerate. Then W contains nontrivial elements where the quadratic form attains any fixed value, including zero, and thus W intersects each orbit of $\mathrm{SO}_{\nu-2}$ on the 0-eigenspace.

Hence any element of the 0-eigenspace is $\mathrm{SO}_{\nu-2}$-conjugate to an element of W, and thus N is conjugate to an element that sends the generator of the 1-eigenspace to an element of W. It follows from $N=-N^T$ that N sends every element of the 0-eigenspace perpendicular to W to the zero element of the (-1)-eigenspace, ie N sends the orthogonal complement of W in the 0-eigenspace to 0. Elements of $\mathfrak{so}_\nu$ sending the orthogonal complement of W in the 0-eigenspace to 0 form a Lie algebra, isomorphic to $\mathfrak{so}_4$, so we conclude that N is conjugate to a nilpotent element of $\mathfrak{so}_4$.

The restriction to $\mathfrak{so}_4$ of the spin representation of $\mathfrak{so}_\nu$ for $\nu = 2k-1$ is odd is isomorphic to the sum of 2^{k-3} copies of each half-spin representation of $\mathfrak{so}_4$, and the same is true for the restriction to $\mathfrak{so}_4$ of the half-spin representation of $\mathfrak{so}_\nu$ for $\nu = 2k$. This may be checked by comparing weights, since a representation of a simple Lie group is uniquely determined by its weight multiplicities. Since $\mathfrak{so}_4$ is isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, with each spin representation the standard representation of one factor, every nilpotent element is a pair of two nilpotent elements in $\mathfrak{sl}_2$, each of which may be zero. The rank of the element is 0 if both elements of the pair vanish, 2^{k-3} if one element is nonvanishing, or 2^{k-2} if both are nonvanishing. □

7.15 Proof of main theorems We next give the proof of Theorem 1.9. This theorem says that, under suitable hypotheses on n, g and r, the monodromy map $R_{\varphi,\rho}\colon \mathrm{Mod}_\varphi \to \mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ has image with Zariski closure $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ when ρ is symplectically self-dual, $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ when ρ is orthogonally self-dual, and the product of $\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ with a finite central subgroup when ρ is not self-dual.

Proof of Theorem 1.9 As in Notation 2.1, let $\pi\colon \mathscr{C} \to \mathscr{M}$ be a versal family of n-pointed curves of genus g, with associated punctured versal family $\pi^\circ\colon \mathscr{C}^\circ \to \mathscr{M}$, and suppose $f\colon \mathscr{X} \to \mathscr{C}$ gives a versal family of H-covers. Let $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible representation of H, let $\mathbb{V}^\rho$ denote the associated local system on $\Sigma_{g,n}$ and let $\mathbb{U}^\rho$ be the associated local system on $\mathscr{C}^\circ$; we wish to analyze the connected monodromy group of $R^1\pi^\circ_*\mathbb{U}^\rho$. By the discussion of Section 2.3, this suffices.

By Theorem 6.7, the connected monodromy group associated to $W_1 R^1\pi^\circ_*\mathbb{U}^\rho$ is a simple group, acting irreducibly. As described in Section 7.1, there are several possibilities for the monodromy representation. Recall we are assuming $g \ge 2r+2$, so the bounds on g in Lemmas 7.2, 7.9 and 7.12 are satisfied. By removing the possibilities excluded via Lemmas 7.2, 7.9 and 7.12, we see that the monodromy representation must act via the standard representation of SL_ν, SO_ν or Sp_ν, ie must be $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$, $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ or $\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$.

We next show that the identity component of the monodromy group is

(1) $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is symplectically self-dual,
(2) $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is orthogonally self-dual, and
(3) $\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ if ρ is not self-dual.

We first handle the case that ρ is self-dual.

If ρ is symplectically self-dual, the antisymmetric pairing on ρ induces a symmetric pairing on the weight-one piece $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$, which implies that the connected monodromy group must be contained in $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$. This implies that the monodromy cannot be $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ or $\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ so must be $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$. If ρ is orthogonally self-dual, the monodromy must be contained in $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$, and we similarly obtain that it must be equal to $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$.

To conclude the calculation of the identity component, it remains to show that the identity component of the Zariski closure of the image of monodromy is $\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ when ρ is not self dual. As explained above, there are only three possibilities, and hence it remains to show that the monodromy cannot be $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ or $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$. If the monodromy has this form, then there is an isomorphism of local systems between $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$ and $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho)^\vee$, which by Poincaré duality is $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho^\vee})$. It follows from Corollary 6.9 that ρ and $\rho^\vee$ are conjugate, contradicting the assumption that ρ is not self-dual.

Having described the connected monodromy group, we now describe the monodromy group itself. If ρ is orthogonally self-dual, Poincaré duality gives a symplectic form on $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$, so the monodromy group is contained in $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ and contains $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$, and thus must equal $\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$.

Similarly, if ρ is symplectically self-dual, Poincaré duality gives a symmetric form on $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$, so the monodromy group is contained in $O(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ and contains $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$, and thus must equal either $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ or $O(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$. However, we now check that the monodromy group is not $O(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$: The representation ρ, being symplectic and unitary, necessarily has the structure as a representation over the quaternions $\mathbb{H}$, ie has an $\mathbb{R}$-linear action of the quaternions compatible with the action of H. Hence the complex local system $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$ has an $\mathbb{R}$-linear action of the quaternions compatible with the mapping class group action. Thus, for each element σ in the mapping class group, the eigenspace with eigenvalue 1 of σ, acting on $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho)$, has an action of the quaternions and hence is an even-dimensional complex vector space. However, any elements of $O_N - \mathrm{SO}_N$ for even N have an odd-dimensional 1-eigenspace, so the image of the mapping class group is contained in $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ and thus the monodromy group must be $\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$.

Finally, if ρ is not self-dual, the monodromy group is a subgroup of $\mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ whose identity component is $\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ and hence must be the product of $\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$ with a finite subgroup of the center of $\mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^\rho))$. □

For the proof of Theorem 1.3, we will also need the following explicit description of the symplectic centralizer.

7.16 Lemma *Let $\rho_1, \dots, \rho_\alpha$ denote the orthogonally self-dual complex irreducible representations of H, let $\rho_{\alpha+1}, \dots, \rho_{\alpha+\beta}$ denote the symplectically self-dual irreducible complex representations of H, and let $(\rho_{\alpha+\beta+1}, \rho_{\alpha+\beta+\gamma+1}), \dots, (\rho_{\alpha+\beta+\gamma}, \rho_{\alpha+\beta+2\gamma})$ denote the dual pairs of complex irreducible representations of H. We use $\mathbb{V}^\rho$ to denote the local system on $\Sigma_{g,n}$ corresponding to ρ. Then the isomorphism*

$$H^1(\Sigma_{g'}, \mathbb{C}) \xrightarrow{\sim} \prod_{i=1}^{\alpha+\beta+2\gamma} \rho_i^\vee \otimes W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i})$$

induces an isomorphism

$$\mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))^H \xrightarrow{\sim} \prod_{i=1}^{\alpha} \mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i})) \times \prod_{i=\alpha+1}^{\alpha+\beta} O(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i})) \times \prod_{i=\alpha+\beta+1}^{\alpha+\beta+\gamma} \mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i})). \tag{7-1}$$

Hence the commutator of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$ *is*

$$\prod_{i=1}^{\alpha}\mathrm{Sp}(W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i}))\times\prod_{i=\alpha+1}^{\alpha+\beta}\mathrm{SO}(W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i}))\times\prod_{i=\alpha+\beta+1}^{\alpha+\beta+\gamma}\mathrm{SL}(W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i})). \tag{7-2}$$

The above lemma follows from the explicit description of the symplectic centralizer given in [Jain 2016, Theorem 3.1.10] (which seems to implicitly work over finite fields, but the same proof works over the complex numbers).

We next prove our main result, Theorem 1.3, which states that under suitable hypotheses on g, n and the maximal dimension $\overline{r}$ of an irreducible representation of H, the identity component of the Zariski closure of the monodromy map $R_\varphi:\mathrm{Mod}_\varphi\to\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$ is the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$.

Proof of Theorem 1.3 Let G be the identity component of the Zariski closure of the image of the mapping class group in (7-1). In other words, G is the connected monodromy group of $\bigoplus_{i=1}^{\alpha+\beta+\gamma}W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i})$. Checking Theorem 1.3, ie that the virtual image of the mapping class group is Zariski dense in the commutator subgroup of this group, is equivalent to checking that G contains (7-2). By the discussion of Section 2.3, we may interpret all the representations in question as monodromy representations associated to local systems on the base $\mathscr{M}$ of a versal family of φ-covers.

To check that G contains (7-2), we apply the Goursat–Kolchin–Ribet criterion of Katz [1990, Proposition 1.8.2]. We let V_i be the representation of G acting on $W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i})$. Let G_i be the image of G in $\mathrm{GL}(V_i)$, which we know from Theorem 1.9 is $\mathrm{Sp}(W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i}))$ for i from 1 to α, $\mathrm{SO}(W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i}))$ for i from $\alpha+1$ to $\alpha+\beta$, and $\mathrm{SL}(W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i}))$ for i from $1+\alpha+\beta$ to $\alpha+\beta+\gamma$. Then [loc. cit., Proposition 1.8.2] guarantees that $G^{0,\mathrm{der}}=\prod_{i=1}^{\alpha+\beta+\gamma}G_i^{0,\mathrm{der}}$, which is the desired (7-2), as long as four conditions are satisfied, which we verify next.

The first condition is that for each i, $G_i^{0,\mathrm{der}}$ operates irreducibly on V_i, and its Lie algebra is simple. Theorem 1.9 guarantees that the action is by the standard representation, which is irreducible except in the case of the two-dimensional standard representation of SO_2, and the Lie algebra is simple except in the case of the four-dimensional standard representation of SO_4. But our assumptions on r and g imply each representation has dimension $\geq 2r_i(g-1)\geq 2r_i(2r_i+1)\geq 6$ where $r_i=\dim\rho_i$, so this condition is always satisfied.

The second condition is that for any $i\neq j$, $(G_i^{0,\mathrm{der}},V_i)$ and $(G_j^{0,\mathrm{der}},V_j)$ are Goursat-adapted in the sense of [Katz 1990, Section 1.8], but this follows by [loc. cit., Example 1.8.1] from the fact that G_i is a classical group and V_i is its standard representation of dimension ≥ 6 with the possible exception that $G_i\cong G_j\cong\mathrm{SO}_8$. However, we know from Theorem 1.9 that G_i is SO_n only if ρ_i is symplectically self-dual, which implies ρ_i has rank $r_i\geq 2$ since all symplectically self-dual representations are even-dimensional and thus $\dim V_i\geq 2r_i(2r_i+1)\geq 20>8$, so G_i cannot be SO_8.

The third and fourth conditions say that for each $i \neq j$ and each character χ of G, neither the representation V_i nor its dual is isomorphic to $V_j \otimes \chi$. Since G is the connected monodromy group of a local system of geometric origin, it is necessarily simple, and so χ is finite-order. Thus by passing to a finite cover of $\mathscr{M}$, the character χ becomes trivial, showing that $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i})$ is isomorphic to $W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_j})$ or its dual over this finite covering. This case is ruled out by Corollary 6.9 unless ρ_i is conjugate to ρ_j or its dual, which is impossible as $i \neq j$ and $i, j \leq \alpha+\beta+\gamma$ so they cannot be part of a dual pair.

Since all the conditions of the Goursat–Kolchin–Ribet criterion of Katz [1990, Proposition 1.8.2] are satisfied, $G^{0,\mathrm{der}} = \prod_{i=1}^{\alpha+\beta+\gamma} G_i^{0,\mathrm{der}}$ and thus $\prod_{i=1}^{\alpha+\beta+\gamma} G_i^{0,\mathrm{der}}$ is a subgroup of G, as desired. □

7.17 Proofs of corollaries We next prove Corollary 1.10, which states that, for X a very general H-curve, under suitable hypotheses on g and H, the Mumford–Tate group of $H^1(X, \mathbb{Q})$ contains the commutator subgroup of $\mathrm{Sp}(H^1(X, \mathbb{Q}))^H$ and is contained in $\mathrm{GSp}(H^1(X, \mathbb{Q}))^H$.

Proof of Corollary 1.10 This is immediate from André's theorem of the fixed part [André 1992, Theorem 1 on page 10] and Theorem 1.3; André's theorem implies the generic Mumford–Tate group contains the monodromy group, namely the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))^H$. On the other hand, it is contained in the centralizer of H in $\mathrm{GSp}(H^1(\Sigma_{g'}, \mathbb{C}))$, as it centralizes H and preserves the symplectic pairing on H^1 up to scaling (as the symplectic pairing corresponds to a Hodge class). □

We next prove Corollary 1.11, which, under the same hypotheses as in the previous corollary, states that the endomorphism algebra of the Jacobian of a very general H-curve X is $\mathbb{Q}[H]$.

Proof of Corollary 1.11 Let G be the Mumford–Tate group of $H^1(X, \mathbb{Q})$. It suffices to show that the natural map $\mathbb{Q}[H] \to \mathrm{End}_{\mathrm{HS}}(H^1(X, \mathbb{Q})) = \mathrm{End}_G(H^1(X, \mathbb{Q}))$ is a bijection, where HS is the category of $\mathbb{Q}$-polarizable variations of Hodge structure. As G contains the commutator subgroup S of $\mathrm{Sp}(H^1(X, \mathbb{Q}))^H$ by Corollary 1.10, it moreover suffices to show that the map

$$\mathbb{Q}[H] \to \mathrm{End}_S(H^1(X, \mathbb{Q}))$$

is a bijection; we may do so after tensoring with $\mathbb{C}$, whence $\mathbb{C}[H] = \prod_{\rho_i} \mathrm{End}(\rho_i)$, where the ρ_i run over the irreducible complex representations of H. Similarly, for $\mathbb{V}^{\rho_i}$ the local system on $X/H - D$ associated to ρ_i,

$$H^1(X, \mathbb{C}) = \bigoplus \rho_i \otimes W_1 H^1(X/H - D, \mathbb{V}^{\rho_i}),$$

where $D \subset X/H$ is the branch locus of the natural map $X \to X/H$. By the description of the symplectic centralizer, Lemma 7.16, the $W_1 H^1(X/H - D, \rho_i)$ are simple and pairwise nonisomorphic as S-representations. Hence

$$\mathrm{End}_S(H^1(X, \mathbb{C})) = \prod_{\rho_i} \mathrm{End}_{\mathbb{C}}(\rho_i)$$

has dimension $|H|$. As the map we are studying is evidently injective, it is necessarily surjective as well by a dimension count. □

8 Proof of Theorem 1.15: big monodromy for n large

We now prove Theorem 1.15. We will require the following slight strengthening of [Landesman and Litt 2024b, Theorem 1.3.4], which was not quite stated optimally. See [Landesman and Litt 2024b, Definitions 1.2.1 and 1.2.3] for the definitions of *hyperbolic curve* and *analytically very general*.

8.1 Theorem *Let (C, D) be hyperbolic of genus g and let (E, ∇) be a flat vector bundle on C with regular singularities along D, and irreducible monodromy. Suppose that (E', ∇') is an isomonodromic deformation of (E, ∇) to an analytically general nearby curve, with Harder–Narasimhan filtration*

$$0 = (F')^0 \subset (F')^1 \subset \cdots \subset (F')^m = E'.$$

For $1 \le i \le m$, let μ_i denote the slope of $\mathrm{gr}^i_{\mathrm{HN}} E' := (F')^i/(F')^{i-1}$. Suppose that E' is not semistable. Then for every $0 < i < m$, there exists $j < i < k$ with

$$\mathrm{rk}(\mathrm{gr}^{j+1}_{\mathrm{HN}} E') \cdot \mathrm{rk}(\mathrm{gr}^{k}_{\mathrm{HN}} E') \ge g+1 \quad \text{and} \quad 0 < \mu_{j+1} - \mu_k \le 1.$$

Proof In fact this is precisely the output of the proof of [Landesman and Litt 2024b, Theorem 1.3.4]. □

8.2 Corollary *With notation as in Theorem 8.1, let $\mu_{\mathrm{diff}} = \mu_1 - \mu_m$. Then*

$$\mu_{\mathrm{diff}} \le \frac{3\,\mathrm{rk}(E)}{2\sqrt{g+1}} + 3.$$

Proof Set s to be the greatest integer which is strictly less than $\mu_{\mathrm{diff}}/3$. We first show that

$$\mathrm{rk}(E) \ge 2\sqrt{g+1} \cdot s.$$

Fix disjoint closed intervals $I_1, \dots, I_s$ in $[\mu_m, \mu_1]$, each of length 3. For each interval, $I_t = [a_t, a_t+3]$, there exists i_t with $\mu_t \in [a_t+1, a_t+2]$, by Theorem 8.1. By the same theorem, there exists $j_t < i_t < k_t$ with

(8-1) $$\mathrm{rk}(\mathrm{gr}^{j_t+1}_{\mathrm{HN}} E') \cdot \mathrm{rk}(\mathrm{gr}^{k_t}_{\mathrm{HN}} E') \ge g+1$$

and

$$0 < \mu_{j_t+1} - \mu_{k_t} \le 1.$$

In particular, $\mu_{j_t+1}, \mu_{k_t} \in I_t$, and hence all the integers j_t+1, k_t are distinct.

Now we have

$$\mathrm{rk}(E) = \mathrm{rk}(E') \ge \sum_{t=1}^{s} \bigl(\mathrm{rk}(\mathrm{gr}^{j_t+1}_{\mathrm{HN}} E') + \mathrm{rk}(\mathrm{gr}^{k_t}_{\mathrm{HN}} E')\bigr),$$

which is bounded below by $(2\sqrt{g+1}) \cdot s$ by the AM–GM inequality and (8-1). So $s \le \mathrm{rk}(E)/(2\sqrt{g+1})$, which implies

$$\mu_{\mathrm{diff}} \le 3s + 3 \le \frac{3\,\mathrm{rk}(E)}{2\sqrt{g+1}} + 3.$$ □

8.3 Lemma *With notation as in Notation 3.3, one of E_0^{ρ} and $E_0^{\rho^\vee}$ has slope less than or equal to $-\Delta/2r$.*

Proof It suffices to show that $E_0^{\rho} \oplus E_0^{\rho^\vee} = E_0^{(\rho\oplus\rho^\vee)}$ has degree less than or equal to $-\Delta$. But if λ is an eigenvalue of local monodromy of ρ at a point x of $D^{\rho}_{\mathrm{nontriv}}$, then λ^{-1} is an eigenvalue of local monodromy of $\rho^\vee$ at x. Hence if $\alpha \neq 0$ is a parabolic weight of $E_\star^{\rho}$ at x, then $1-\alpha$ is a parabolic weight of $E_\star^{\rho^\vee}$ at x. In particular, the sum of the parabolic weights of $E_\star^{(\rho\oplus\rho^\vee)}$ at x is at least 1 for each $x \in D^{\rho}_{\mathrm{nontriv}}$.

Hence,

$$\deg E_0^{(\rho\oplus\rho^\vee)} = -\sum_{x_j \in D^{\rho}_{\mathrm{nontriv}}} \sum_{i=1}^{n_j} \alpha_j^i \dim(E_j^i/E_j^{i+1}) \le -\sum_{x\in D^{\rho}_{\mathrm{nontriv}}} 1 \le -\Delta. \qquad \square$$

8.4 Lemma *With notation as in Notation 3.3, suppose (C, D) is a general n-pointed curve. If*

$$\Delta > \frac{3r^2}{\sqrt{g+1}} + 8r, \tag{8-2}$$

then at least one of $(E_0^{\rho})^\vee \otimes \omega_C$ and $(E_0^{\rho^\vee})^\vee \otimes \omega_C$ is globally generated.

Proof Without loss of generality we may (by replacing ρ with $\rho^\vee$ if necessary) assume

$$\mu(E_0^{\rho}) \le -\frac{\Delta}{2r},$$

by Lemma 8.3. We will show that in this case $(E_0^{\rho})^\vee \otimes \omega_C$ is globally generated.

Let $\mu_1 > \cdots > \mu_m$ be the set of slopes of the graded pieces of the Harder–Narasimhan filtration of E_0^{ρ}, as in Theorem 8.1, and let $\mu_{\mathrm{diff}} = \mu_1 - \mu_m$. By Corollary 8.2, we have

$$\mu_{\mathrm{diff}} \le \frac{3r}{2\sqrt{g+1}} + 3.$$

Hence

$$\mu_1 \le -\frac{\Delta}{2r} + \mu_{\mathrm{diff}} \le -\frac{\Delta}{2r} + \frac{3r}{2\sqrt{g+1}} + 3 < -1$$

by rearranging the assumption (8-2). Thus the Harder–Narasimhan slopes of $(E_0^{\rho})^\vee \otimes \omega_C$ are all greater than $2g-1$.

We claim any vector bundle V such that each graded piece of its Harder–Narasimhan filtration has slope more than $2g-1$ is globally generated. This is well known, but we explain it for completeness. A vector bundle V is globally generated at p if $H^1(C, V(-p)) = 0$, as then $H^0(C, V) \to H^0(C, V|_p)$ is surjective. By [Landesman and Litt 2024b, Lemma 6.3.5], if W is a vector bundle such that each graded piece of its Harder–Narasimhan filtration has slope more than $2g-2$, $H^1(C, W) = 0$. Therefore, $H^1(C, V(-p)) = 0$ for any point p, so V is globally generated. This shows $(E_0^{\rho})^\vee \otimes \omega_C$ is globally generated. $\square$

8.5 Proof of Theorem 1.15 We are aiming to prove that, once Δ is sufficiently large (larger than $(3r^2/\sqrt{g+1})+8r$), there are no nonzero vectors with finite orbit in $W_1H^1(\Sigma_{g,n},\mathbb{V}^\rho)$ under the image of Mod_φ; here, Mod_φ is the stabilizer of $\varphi\colon \pi_1(\Sigma_{g,n})\to H$ in $\mathrm{Mod}_{g,n+1}$.

As the virtual representations of $\mathrm{Mod}_{g,n+1}$ on

$$W_1H^1(\Sigma_{g,n},\mathbb{V}^\rho),\, W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho^\vee})$$

are semisimple and dual to one another, it suffices to prove this for one of ρ and $\rho^\vee$, so we may without loss of generality assume by Lemma 8.4 that $(E_0^\rho)^\vee\otimes\omega_C$ is globally generated for (C,D) a general n-pointed curve. A similar argument to that given in [Landesman and Litt 2023a, Proposition 3.4] shows that the virtual action of $\mathrm{Mod}_{g,n+1}$ on $\mathrm{GL}(W_1H^1(\Sigma_{g,n},\mathbb{V}^\rho))$ has no nonzero finite-orbit vectors.

To make this proof slightly more self-contained, we recall briefly the idea of the proof of [Landesman and Litt 2023a, Proposition 3.4]. Namely, the derivative of the period map as described in Proposition 3.6 associated to the Hodge filtration of $W_1H^1(\Sigma_{g,n},\mathbb{V}^\rho)$ can be identified with a map

$$H^0(C,\hat{E}_0^\rho\otimes\omega_C(D))\otimes H^0(C,(E_0^\rho)^\vee\otimes\omega_C)\to H^0(C,\omega_C^{\otimes 2}(D)).$$

If there is a vector with finite orbit in $W_1H^1(\Sigma_{g,n},\mathbb{V}^\rho)$ then the adjoint map

$$H^0(C,\hat{E}_0^\rho\otimes\omega_C(D))\to \mathrm{Hom}\big(H^0(C,(E_0^\rho)^\vee\otimes\omega_C),H^0(C,\omega_C^{\otimes 2}(D))\big)$$

has a nonzero kernel. A vector in the kernel yields a nonzero map

$$\phi\colon (E_0^\rho)^\vee\otimes\omega_C\to\omega_C^{\otimes 2}(D)$$

inducing the 0 map on global sections. Then, $\ker\phi\subset(E_0^\rho)^\vee\otimes\omega_C$ would be a proper subbundle inducing an isomorphism on global sections, implying $(E_0^\rho)^\vee\otimes\omega_C$ is not generically globally generated, hence not globally generated, a contradiction. □

9 Big monodromy for Kodaira fibrations

In this section we analyze the connected monodromy groups of certain smooth proper families

$$\pi: S\to Z$$

over $\mathbb{C}$, referred to as *Kodaira–Parshin fibrations*, where Z is a smooth curve and π is smooth and proper of relative dimension one, with connected fibers. Loosely speaking, these families will parametrize covers of a fixed curve C, branched at a moving point. We next give a few definitions to fix notation for Kodaira–Parshin fibrations.

9.1 Definition Let Z be a smooth, not necessarily proper, connected curve. A smooth proper morphism $\pi: S\to Z$ of relative dimension one with connected fibers is a *Kodaira–Parshin fibration* if there exists a smooth proper curve C, a reduced effective divisor $D\subset C$, a nonconstant map $f: Z\to C$, and a dominant finite map $q\colon S\to Z\times C$, branched only over the graph of f and $Z\times D$, such that π factors as $\pi = \pi_1 \circ q$, where π_1 is projection onto the first coordinate, as in the diagram

$$\begin{array}{ccccc} S & \xrightarrow{q} & Z \times C & \xrightarrow{\pi_2} & C \\ & \overset{\pi}{\searrow} & \big\downarrow{\scriptstyle \pi_1} & & \\ & & Z & & \end{array} \tag{9-1}$$

9.2 Definition Continuing with notation as in Definition 9.1, fix $z \in Z$, and consider the map $\pi^{-1}(z) \to C$ given as $\pi_2 \circ q|_{\pi^{-1}(z)}$, where $\pi_2 \colon Z \times C \to C$ is projection on to the second coordinate. We refer to the underlying map on topological spaces (in the Euclidean topology) as the *topological type* of the family π.

9.3 Remark The topological type of π, as defined in Definition 9.2, is independent of z up to homeomorphism, by Ehresmann's theorem. That is, every fiber of π is a cover of C of the same topological type. Hence it makes sense to speak of the topological type of π, and not just of π over z.

9.4 Definition Continuing with notation as in Definition 9.2, the topological type of a Kodaira–Parshin fibration is a ramified map of (topological) surfaces $t \colon \Sigma_{g'} \to \Sigma_g$; we refer to the Galois group of (the Galois closure of) this map as the *Galois group* of the Kodaira–Parshin fibration. Note that Σ_g has a *distinguished point*, which corresponds to $f(z)$ under the isomorphism $\Sigma_g \simeq C$. We refer to the conjugacy class of the monodromy about this point in the Galois group H of the cover as the *distinguished class* $[h] \subset H$.

We next recall the classical Kodaira–Parshin trick; see [Parshin 1968, Proposition 7] for the original construction, and also [Landesman and Litt 2024b, Proposition 5.1.1] for a more modern construction in families. See also [Atiyah 1969] and [Kodaira 1967] for closely related constructions.

9.5 Example (the Kodaira–Parshin trick) Let C be a smooth proper curve, and let $p \colon S \to C \times C$ be a dominant finite map branched only over the diagonal Δ. More precisely, S is the normalization of $C \times C - \Delta$ in the function field of a finite étale cover of $C \times C - \Delta$. Let

$$S \xrightarrow{\pi} Z \to C$$

be the Stein factorization of the composition $\pi_1 \circ p$, where $\pi_1 \colon C \times C \to C$ is projection onto the first factor. Then π is a Kodaira–Parshin fibration.

9.6 Remark Define a *Kodaira fibration* to be a surjective smooth proper morphism from a smooth projective surface to a smooth *proper* curve. In the setting of Example 9.5, C itself is proper, so π is in fact a Kodaira fibration. The construction of Example 9.5 is known as the *Kodaira–Parshin trick*, and is used eg by Faltings in his proof of the Mordell conjecture.

The main theorem of this section is a computation of the connected monodromy group of a Kodaira–Parshin fibration of topological type $t \colon \Sigma_{g'} \to \Sigma_g$, when g is large compared to the dimensions of the irreducible representations of H. We give a statement for Kodaira–Parshin fibrations with Galois topological type, but one may easily deduce an analogous statement for arbitrary Kodaira–Parshin fibrations by passing to Galois closures. To state the theorem, we use the following notation.

9.7 Notation Let $\pi: S \to Z$ be a Kodaira–Parshin fibration of topological type $t: \Sigma_{g'} \to \Sigma_g$, with t a Galois cover branched at n points, with Galois group H and distinguished class $[h] \subset H$, as defined in Definition 9.1. Let $\varphi\colon \pi_1(\Sigma_{g,n}, x) \twoheadrightarrow H$ denote the surjection associated to t, for x a chosen basepoint. Let $\rho_1^O, \dots, \rho_\alpha^O$ be the irreducible orthogonally self-dual complex H-representations with $\rho_i^O([h])$ nontrivial, $\rho_1^{\mathrm{Sp}}, \dots, \rho_\beta^{\mathrm{Sp}}$ the irreducible symplectically self-dual H irreducible complex representations with $\rho_i^{\mathrm{Sp}}([h])$ nontrivial, and let $(\rho_1^{\mathrm{SL}}, \rho_{\gamma+1}^{\mathrm{SL}}), \dots, (\rho_\gamma^{\mathrm{SL}}, \rho_{2\gamma}^{\mathrm{SL}})$ be the set of dual pairs of complex irreducible H-representations with $\rho_i^{\mathrm{SL}}([h])$ nontrivial. Let $\overline{s}$ be the maximal dimension of an irreducible representation ρ of H with $\rho([h])$ nontrivial. We use $\mathbb{V}^\rho$ to denote the local system on $\Sigma_{g,n}$ associated to the composition $\rho \circ \varphi$.

9.8 Theorem *With notation as in Notation 9.7, suppose $g > \max(2\overline{s}+1, \overline{s}^2)$. Then the identity component of the Zariski-closure of $\pi_1(Z, z)$ in $\mathrm{GL}(H^1(\pi^{-1}(z), \mathbb{C}))$ is*

$$\prod_{i=1}^{\alpha} \mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i^O})) \times \prod_{i=1}^{\beta} \mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i^{\mathrm{Sp}}})) \times \prod_{i=1}^{\gamma} \mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i^{\mathrm{SL}}})), \tag{9-2}$$

the subgroup of the derived subgroup of the centralizer of H in $\mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))$ corresponding to those H-irreps which are nontrivial on $[h]$.

The following purely topological corollary follows immediately from the definitions, as in Section 2.3 — loosely speaking, it says that in the representations considered in Theorem 1.3, the restriction to the "point-pushing subgroup" still has large image.

9.9 Corollary *With notation as in Notation 9.7, suppose that $n > 0$, and let $h \in \pi_1(\Sigma_{g,n}, x)$ be a loop around a fixed puncture z of $\Sigma_{g,n}$, such that $\varphi(h) = [h]$ is the distinguished class. Suppose that $g > \max(2\overline{s}+1, \overline{s}^2)$. Let P_φ be the stabilizer of φ in the kernel of the map*

$$\mathrm{Mod}_{g,n+1} \to \mathrm{Mod}_{g,n}$$

induced by forgetting z. (Here we view $\Sigma_{g,n}$ as an $(n+1)$-marked surface, with x an additional marked point.) The identity component of the Zariski closure of

$$R_\varphi|_{P_\varphi}\colon P_\varphi \to \mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))^H$$

in $\mathrm{GL}(H^1(\Sigma_{g'}, \mathbb{C}))$ is

$$\prod_{i=1}^{\alpha} \mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i^O})) \times \prod_{i=1}^{\beta} \mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i^{\mathrm{Sp}}})) \times \prod_{i=1}^{\gamma} \mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i^{\mathrm{SL}}})),$$

the subgroup of the derived subgroup of the centralizer of H in $\mathrm{Sp}(H^1(\Sigma_{g'}, \mathbb{C}))$ corresponding to those H-irreps nontrivial on $[h]$. Here R_φ is defined as in Theorem 1.3.

In order to prove Theorem 9.8, we recall some notation from Notation 2.1. Given a branched cover

$$t\colon \Sigma_{g'} \to \Sigma_g$$

of topological surfaces, with Galois group H and n branch points (here consisting of the points of D together with the $f(z)$), there is a versal family of covers over a variety $\mathscr{M}_t$, parametrizing branched covers of Riemann surfaces of topological type t (see eg [Bertin and Romagny 2011, Section 6]), carrying a family of covers $\mathscr{X}_t/\mathscr{C}_t$, as in the diagram

$$\begin{array}{ccc} \mathscr{X}_t & \xrightarrow{f} & \mathscr{C}_t \\ & {\scriptstyle \pi'}\searrow & {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle s_1,\dots,s_n} \\ & & \mathscr{M}_t \end{array}$$

After replacing Z with an étale cover, our given Kodaira–Parshin fibration $S \to Z$ of topological type t is the pullback of π' by a map $\iota\colon Z \to \mathscr{M}_t$. The family of pointed curves $\mathscr{C}_t/\mathscr{M}_t$ induces a dominant map $\mathscr{M}_t \to \mathscr{M}_{g,n}$ with finitely many geometric points in each fiber. By definition, ι dominates a component of the fiber of the composition $\mathscr{M}_t \to \mathscr{M}_{g,n} \to \mathscr{M}_{g,n-1}$, given by forgetting the distinguished point as defined in Definition 9.4.

The following lemma is the main ingredient in the proof, and describes the monodromy of a Kodaira–Parshin family associated to a particular representation.

9.10 Lemma *With notation as above, let $[h] \in H$ be the distinguished class, and let $\rho\colon H \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible H-representation. Suppose $g > \max(2r+1, r^2)$. Then the identity component of the Zariski-closure of the monodromy representation*

$$\pi_1(Z, z) \to \mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho})) \tag{9-3}$$

is trivial if $\rho([h])$ is trivial. If $\rho([h])$ is nontrivial, it is

1. *$\mathrm{SO}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho}))$ if ρ is symplectically self-dual,*
2. *$\mathrm{Sp}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho}))$ if ρ is orthogonally self-dual, or*
3. *$\mathrm{SL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho}))$ if ρ is not self-dual.*

Proof We first observe that if $\rho([h])$ is trivial, the connected component of the monodromy group in question is also trivial. Indeed, in this case, let $K \subset H$ be the kernel of ρ; the monodromy representation in question appears in the cohomology of $S/K \to Z$, which is isotrivial. Indeed, the fibers of this map are covers of C branched at a fixed divisor, and hence do not vary in moduli.

We now suppose $\rho([h])$ is nontrivial. We first claim that it suffices to show that the monodromy representation (9-3) has infinite image. Indeed, by the discussion above the statement of the lemma, (9-3) factors through a representation

$$\pi_1(\mathscr{M}_t) \to \mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho})). \tag{9-4}$$

By Theorem 1.9, the representation (9-4) has image with Zariski-closure SO, Sp or SL, depending on the self-duality properties of ρ. The connected monodromy group of (9-3) is in fact a *normal* subgroup of this group, as the fundamental group of the fiber of the map $\mathscr{M}_t \to \mathscr{M}_{g,n-1}$ is normal in the fundamental group of $\mathscr{M}_t$. Thus if the image in question is infinite, it must be Zariski-dense in the Zariski-closure of the image of (9-4), as this group is simple by Theorem 6.7.

We prove the infinitude of the image of this representation by Hodge-theoretic methods. As the monodromy representation in question is a topological invariant, we may take Z to dominate a component of a general fiber of the map $\mathscr{M}_t \to \mathscr{M}_{g,n-1}$ and hence assume that C is general.

Fix general $z \in Z$, and let $X = \pi^{-1}(z)$, let $D \subset C$ be the branch locus of the cover $X \to C$, and let $p \in C$ be the image of z in C; note that (C, D) is general, and $p \in D$ by the definition of a Kodaira–Parshin fibration. Recall that the theorem of the fixed part implies a VHS with finite monodromy has constant period map. Using this, it suffices to show that the derivative of the period map

$$T_z Z \to \mathrm{Hom}(H^0(C, \hat{E}_0^\rho \otimes \omega_X(D)), H^1(C, E_0^\rho))$$

is nonzero for generic $z \in Z$. There is a natural identification

$$T_z Z = \ker(H^1(C, T_C(-D)) \to H^1(C, T_C(-D+p))),$$

as the map $Z \to C$ is generically étale, so Serre-dually we wish to show that the pairing

$$H^0(C, \hat{E}_0^\rho \otimes \omega_X(D)) \otimes H^0(C, (E_0^\rho)^\vee \otimes \omega_C) \to H^0(C, \omega_C^{\otimes 2}(D-p)) \\ \to \mathrm{coker}\big(H^0(C, \omega_C^{\otimes 2}(D-p)) \to H^0(C, \omega_C^{\otimes 2}(D))\big)$$

is nonzero, where the first map is induced by B_z^ρ, as in Proposition 3.6. Evaluation at p identifies

$$\mathrm{coker}\big(H^0(C, \omega_C^{\otimes 2}(D-p)) \to H^0(C, \omega_C^{\otimes 2}(D))\big)$$

with $\omega_C^{\otimes 2}(D)|_p$. Using that (C, D) is general, so $\hat{E}_0^\rho \otimes \omega_X(D)$ and $(E_0^\rho)^\vee \otimes \omega_C$ are globally generated by Proposition 4.9, it is enough to show that the pairing

$$(\hat{E}_0^\rho \otimes \omega_X(D)) \otimes (E_0^\rho)^\vee \otimes \omega_C \to \omega_X^{\otimes 2}(D)|_p$$

is nonzero. But this pairing is induced by the corresponding pairing of sheaves between $E_0^\rho \otimes \omega_X(D)$ and $(E_0^\rho)^\vee \otimes \omega_C$, which is perfect, via the inclusion $\hat{E}_0^\rho \hookrightarrow E_0^\rho$. So it is nonzero as long as this inclusion is nonzero at p, ie as long as $\rho([h])$ is nontrivial, as desired. □

To complete the proof of Theorem 9.8, we will need to compute the total monodromy group, which is contained in a product of groups indexed by the different irreps of H. To prove our result we will use Goursat's lemma, and the following lemma is a key input to verifying the hypotheses of Goursat's lemma.

9.11 Lemma *Let $\rho_1, \rho_2 \colon H \to \mathrm{GL}_r(\mathbb{C})$ be irreducible H-representations with $\rho_i([h])$ nontrivial for $i = 1, 2$. If $g > r^2$ and the monodromy representations*

$$\pi_1(Z, z) \to \mathrm{GL}(W_1 H^1(\Sigma_{g,n}, \mathbb{V}^{\rho_i})) \tag{9-5}$$

for $i = 1, 2$ are isomorphic to one another, then $\rho_1 \simeq \rho_2$.

Proof We can factor (9-5) as a composition

$$\pi_1(Z,z)\to\pi_1(\mathscr{M}_t)\xrightarrow{\phi_i}\mathrm{GL}(W_1H^1(\Sigma_{g,n},\mathbb{V}^{\rho_i})), \tag{9-6}$$

and by Proposition 6.4, it suffices to show that ϕ_1 is isomorphic to ϕ_2, possibly after passing to a cover of $\mathscr{M}_t$. By [Landesman and Litt 2024a, Lemma 2.2.2] (using that our two representations of $\pi_1(Z,z)$ in question are irreducible by Lemma 9.10), the projectivizations of ϕ_1 and ϕ_2 are isomorphic. Hence, we may assume $\phi_1\simeq\phi_2\otimes\chi$, for χ a character. Observe that χ must be of finite order because its connected monodromy group is simple (as also argued in the penultimate paragraph of the proof of Theorem 1.3). Hence, we may trivialize χ by passing to a cover of $\mathscr{M}_t$, and therefore reduce to the case $\phi_1\simeq\phi_2$, as desired. □

Proof of Theorem 9.8 The proof follows from the Goursat–Kolchin–Ribet criterion of Katz [1990, Proposition 1.8.2], precisely following the argument in the proof of Theorem 1.3; we use Lemma 9.10 in place of Theorem 1.9, and Lemma 9.11 in place of Proposition 6.4. The final statement describing (9-2) follows from Lemma 7.16. □

10 Questions

We conclude with a number of open questions motivated by the preceding results.

10.1 Improving the bounds in our results Theorem 1.3 gives a large monodromy result for the mapping class group action on the homology of an H-cover of a genus g curve, on the assumption that g is large compared to the dimensions of the irreducible representations of H. However, we don't know a counterexample to the statement of Theorem 1.3 as long as $g\geq 3$.

10.2 Question Let H be a finite group and let $\Sigma_{g',n'}\to\Sigma_{g,n}$ be an H-cover. Suppose that $g\geq 3$. Is the identity component of the Zariski closure of the virtual image $\mathrm{Mod}_{g,n+1}$ in $\mathrm{GL}(W_1H^1(\Sigma_{g'}))$ the commutator subgroup of $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{C}))^H$?

It seems natural to conjecture that the answer is yes, strengthening the Putman–Wieland conjecture [2013] as explained in Remark 1.6. Moreover one might expect that as soon as $g\geq 3$, the image is arithmetic:

10.3 Question Does the virtual image of $\mathrm{Mod}_{g,n+1}$ in $\mathrm{Sp}(H^1(\Sigma_{g'},\mathbb{Z}))^H$ have finite index?

While our methods say nothing about Question 10.3, strengthening our results on (generic) global generation would imply a positive answer to Question 10.2 in some cases, as we now explain.

10.4 Question Let $g\geq 3$, and let $\rho\colon\pi_1(\Sigma_{g,n})\to\mathrm{GL}_r(\mathbb{C})$ be a representation with finite image. Fix a very general complex structure (C,D) on a pointed surface $(\Sigma_g,x_1,\dots,x_n)$ and let $E_\star$ be the parabolic bundle corresponding to ρ. Is the vector bundle $\widehat{E}_0\otimes\omega_C(D)$

(a) generically globally generated, or even

(b) globally generated?

10.5 Remark When $n = 0$, a positive answer to Question 10.4(b) for arbitrary r would imply a positive answer to Question 10.2 by the methods of this paper. Similarly, for arbitrary n, a positive answer to Question 10.4(a) would imply a positive answer to the Putman–Wieland conjecture [2013].

10.6 Remark One might naturally pose Question 10.4 for arbitrary unitary representations ρ. We are grateful to Eric Larson and Isabel Vogt for explaining to us that in this case one must necessarily take the meaning of "very general" in the question to *depend* on ρ. That is, every smooth proper curve C of genus $g \geq 2$ admits a stable vector bundle E of degree zero (corresponding, by the Narasimhan–Seshadri correspondence, to some unitary representation) such that $E \otimes \omega_C$ is not generically globally generated.

10.7 Remark A positive answer to Question 10.4 for general unitary representations would provide some evidence for the well-known conjecture that mapping class groups have Kazhdan's property T in genus $g \geq 3$, as it would imply that certain unitary representations of mapping class groups are rigid, following the methods of [Landesman and Litt 2024a]. See [Ivanov 2006, Section 8] for a discussion of this question.

10.8 Arithmetic statistics As mentioned in Remark 1.14, the results of this paper are closely related to a number of results in arithmetic statistics, which concern understanding the monodromy with $\mathbb{Z}/\ell\mathbb{Z}$ coefficients, instead of complex coefficients. As noted in the introduction, Jain [2016] is able to prove a big monodromy result over genus 0 bases with the number n of punctures having monodromy in every conjugacy class sufficiently large. However, his result does not say how large n has to be.

10.9 Question Is it possible to obtain a big monodromy result analogous to Theorem 1.3 which is effective in g, or a result analogous to Conjecture 1.13 which is effective in n, with $\mathbb{Z}/\ell\mathbb{Z}$ coefficients instead of complex coefficients?

If the above were possible, can one use it to deduce some large q limit versions of the Cohen–Lenstra heuristics, as alluded to in Remark 1.14?

10.10 Analogs for free groups Finally, it is natural to pose analogues of the questions here for representations of groups other than the mapping class group $\mathrm{Mod}_{g,n}$. For example, the group $\mathrm{Aut}(F_n)$ acts virtually on finite-index subgroups of the free group F_n, and hence on their abelianizations.

10.11 Question Let $n \geq 3$ and fix a finite-index subgroup $K \subset F_n$. Is the image in $\mathrm{GL}(K^{\mathrm{ab}})$ of the stabilizer of K inside $\mathrm{Aut}(F_n)$ arithmetic? What is its Zariski-closure?

This last question is studied in many interesting special cases by Grunewald and Lubotzky [2009].

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Department of Mathematics FAS, Harvard University
Cambridge, MA, United States

Department of Mathematics, University of Toronto
Toronto, ON, Canada

Department of Mathematics, Princeton University
Princeton, NJ, United States

aaronlandesman@gmail.com, daniel.litt@utoronto.ca, wsawin@math.princeton.edu

Proposed: Benson Farb
Seconded: Dan Abramovich, David Fisher

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