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A tropical computation of refined toric invariants

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In 2015, G Mikhalkin introduced a refined count for real rational curves in toric surfaces. The counted curves have to pass through some real and complex points located on the toric boundary of the surface, and the count is refined according to the value of a so-called quantum index. This count happens only to depend on the number of complex points on each toric divisor, leading to an invariant. First, we give a way to compute the quantum index of any oriented real rational curve, getting rid of the previously needed "purely imaginary" assumption on the complex points. Then we use the tropical geometry approach to relate these classical refined invariants to tropical refined invariants, defined using Block–Göttsche multiplicity. This generalizes the result of Mikhalkin relating both invariants in the case where all the points are real, and the result of the author where complex points are located on a single toric divisor.

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1 Introduction

1.1 Enumeration of rational curves in toric surfaces

Consider rational curves in $(\mathbb{C}^*)^2$. Such a curve corresponds to the choice of two rational functions F and G on the projective line $\mathbb{C}P^1$. We thus get a parametrized curve $\varphi: t \mapsto (F(t), G(t)) \in (\mathbb{C}^*)^2$ defined outside the poles and zeros of F and G. The collection of orders of vanishing of F and G at each of these points is called the degree Δ of the curve. It is then possible to define a toric surface $\mathbb{C}\Delta$, which is a compactification of $(\mathbb{C}^*)^2$, so that φ extends at the zeros and poles of F and G, but with values in $\mathbb{C}\Delta$ instead of $(\mathbb{C}^*)^2$. For instance, if $(a_i, b_i, c_i)_{1\leqslant i\leqslant d}$ are $3d$ distinct scalars, we can consider parametrized curves of the form

$$t \mapsto \left(\prod \frac{t-a_i}{t-c_i}, \prod \frac{t-b_i}{t-c_i}\right),$$

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which is a rational curve of degree d. The associated toric surface is the standard projective plane $\mathbb{C}P^2$, and a_i, b_i, c_i are sent to the coordinate axes of $\mathbb{C}P^2$. However, this representation tends to use the canonical basis of $(\mathbb{C}^*)^2$. We prefer to adopt a viewpoint independent of this choice. To do this, let N be a two-dimensional lattice and M its dual lattice. They are respectively called the lattices of cocharacters and characters. The version independent of coordinates of $(\mathbb{C}^*)^2$ is $N_{\mathbb{C}^*} = N \otimes \mathbb{C}^*$. As the group law on $\mathbb{C}^*$ is multiplicative, we prefer to denote the tensor product $n \otimes z$ by z^n. To each element $n \in N$ is then associated a function $\mathbb{C}^* \to N_{\mathbb{C}^*}$, $z \mapsto z^n = n \otimes z$, called a cocharacter, while to any element m of M is associated a character, also called its monomial, $\chi^m \colon N_{\mathbb{C}^*} \to \mathbb{C}^*$, $z^n \mapsto z^{\langle m,n\rangle}$. In the case of $(\mathbb{C}^*)^2$, these monomials are the classical monomials in the two coordinates z and w. Using this point of view, a rational curve can be parametrized by

$$\varphi \colon \mathbb{C}P^1 \to N \otimes \mathbb{C}^*, \quad t \mapsto \chi \prod_{i=1}^{m} (t-\alpha_i)^{n_i},$$

where $\chi \in N_{\mathbb{C}^*}$ and $\alpha_i \in \mathbb{C}$ are some scalars and $n_i \in N$ are some cocharacters. the multiset of lattice vectors $\Delta = (n_i) \subset N$ is called the degree of the curve. To Δ is associated a polygon P_Δ in M, unique up to translation, characterized as the unique polygon such that the outer normal vectors to the sides of P_Δ are the vectors of Δ, counted with a multiplicity equal to their lattice length. For instance, the polygon associated to $\Delta_d = \{(-1,0)^d, (0,-1)^d, (1,1)^d\}$ is the standard triangle of side length d. To P_Δ is associated a toric surface $\mathbb{C}\Delta \supset N_{\mathbb{C}^*}$. The toric divisors are in bijection with the directions of the vectors in Δ. Thus, to each cocharacter of Δ is associated a toric divisor.

The complex conjugation $z^n \mapsto \bar{z}^n$ extends to $\mathbb{C}\Delta$ and makes it into a real surface. We say that a rational curve is real if the parametrization can also be chosen invariant by conjugation. In the above notation, this means that $\chi \in N_{\mathbb{R}^*} = N \otimes \mathbb{R}^*$, and $\alpha_i \in \mathbb{R}$, or α_i and $\overline{\alpha}_i$ both appear with the same exponent cocharacter $n_i \in N$.

Unless stated otherwise, the vectors in the multiset Δ will be chosen to be primitive, ie of lattice length 1. Some lattice length would correspond to a tangency between the curve and the toric divisor where they meet. The dimension of the space of rational curves of degree Δ is $m-1$: there are m points α_i to choose, and χ, which varies in a two-dimensional space, but the space of reparametrization of $\mathbb{C}P^1$ has dimension 3. Thus, it seems natural to count the number of curves that satisfy $m-1$ chosen constraints. The easiest constraint that comes to mind is to pass through a given point. Let $\mathcal{P}$ be a set of generic points in $\mathbb{C}\Delta$, called a configuration. Let $\mathcal{S}^{\mathbb{C}}(\mathcal{P})$ be the set of complex rational curves that pass through $\mathcal{P}$. Then the cardinal N_Δ of this set does not depend on the choice of $\mathcal{P}$ as long as it is generic. This number is equal to the degree of some Severi variety, which explains their invariance.

Over the real numbers, the situation is more complicated: the number of real curves passing through a real configuration may depend on the choice of the configuration. By real configuration, we mean a set of cardinal $m-1$ consisting of real points and pairs of complex conjugate points. Indeed, if a real curve pass through some point, as it is invariant by conjugation, it also passes through the conjugate point. Let $\mathcal{S}^{\mathbb{R}}(\mathcal{P})$ be the number of real curves passing through $\mathcal{P}$. The cardinal of $\mathcal{S}^{\mathbb{R}}(\mathcal{P})$ depends on $\mathcal{P}$. However, J-Y Welschinger [21] proved that, for toric del Pezzo surfaces, if the curves are counted with a suitable sign, their number only depends on the number s of pairs of complex conjugate points in the configuration $\mathcal{P}$. This invariant is denoted by $W_{\Delta,s}$.

The properties and the computation of both invariants N_Δ and $W_{\Delta,s}$ have been at the center of many works. One particular approach to their computation unveiled a tremendously rich field of studies, and could to some extent be considered as the foundation of tropical geometry: the correspondence theorem of G Mikhalkin [11]. He proved a theorem relating the computation of N_Δ and $W_{\Delta,0}$ ($m-1$ real points and no complex conjugate points) to a count of tropical curves along with an algorithm to effectively compute these tropical numbers. Although the values of N_Δ were already known, it was the first computation of $W_{\Delta,0}$. The computation of the values of $W_{\Delta,s}$ for any value of s was dealt with by E Shustin [18]. Since, the correspondence has first been generalized to any dimension by T Nishinou and B Siebert [15] using logarithmic stable maps. Other approaches to the correspondence theorem have led to different proofs of it. The reader can refer to Mikhalkin [12], Shustin [17] or I Tyomkin [19].

The computation of N_Δ and $W_{\Delta,0}$ through the tropical geometry approach counts tropical curves using two distinct multiplicities. This multiplicity is a product over the vertices of the tropical curves. Following this computation, F Block and L Göttsche [2] proposed to refine these multiplicities into a polynomial one, also a product over the vertices. This new refined multiplicity specializes to both previous multiplicities when evaluated at ± 1. Moreover, it was proved by I Itenberg and Mikhalkin [10] to also lead to an invariant in the tropical world, but one whose analog in the classical world remains mysterious. Moreover, several other enumerative problems can be solved using the tropical geometry approach, and the tropical side can be *refined*, in the sense that multiplicities can be changed into polynomial ones and we still get an invariant count of solution, making refined invariants appear in several other situations, naturally or less naturally; see for example F Schroeter and Shustin [16], L Blechman and Shustin [1] or P Bousseau [5]. Conjecturally, the refined invariant in our situation should coincide with the refinement of the Euler characteristic of Severi degrees by the Hirzebruch genera, as proposed by Göttsche and V Shende [9]. Some advances in that direction have been made by J Nicaise, S Payne and Schroeter [14].

This conjectural meaning does not prevent these invariants from bearing other interpretations, although it would probably emphasize some deep connection between them. One of these other meanings is given by Mikhalkin [13]. He provides a way of refining the count of real oriented curves passing through some configuration of points on the boundary of a toric surface, and proved that the computation in some particular case can be dealt with tropically using the refined multiplicities. Details are explained below.

1.2 Quantum indices of real curves

Mikhalkin [13] introduced a quantum index for oriented real curves. However, the definition in [13] is restricted to real oriented curves which have real or purely imaginary intersection points with the toric boundary. Let $\varphi\colon \mathbb{C}P^1 \to \mathbb{C}\Delta$ be an oriented real rational curve of degree Δ, which means that the curve is endowed with one of the two possible orientations of its real part $\mathbb{R}P^1$, and $\mathrm{Sq}\colon \mathbb{C}\Delta \to \mathbb{C}\Delta$ be the extension of the square map $z^n \mapsto z^{2n}$ to $\mathbb{C}\Delta$. We say that φ has real or purely imaginary intersection points with the toric boundary if all the intersection points of $\mathrm{Sq}\circ\varphi$ with the toric boundary are real. This means that the coordinates of the intersection points, which are the evaluations of some monomials χ^m, are real or purely imaginary. The definition can be extended to any real oriented curve provided that we allow a slight modification.

In the case of real or purely imaginary intersection points, the quantum index is defined as the log-area of the curve, which happens to be a half-multiple of π^2. Let $\varphi\colon \mathbb{C}P^1 \dashrightarrow N_{\mathbb{C}^*}$ be an oriented real rational curve. The dense torus orbit $N_{\mathbb{C}^*}$ is endowed with the logarithmic map and the argument map

$$\begin{aligned} \mathrm{Log}\colon N_{\mathbb{C}^*} &\to N\otimes\mathbb{R}, & z^n &\mapsto \log|z|n := n\otimes\log|z|,\\ \arg\colon N_{\mathbb{C}^*} &\to N\otimes(\mathbb{R}/2\pi\mathbb{Z}), & z^n &\mapsto \arg(z)n := n\otimes\arg(z). \end{aligned}$$

The images of a curve under these maps are respectively called the *amoeba* and the *coamoeba*. Let ω be a generator of $\Lambda^2 M$, which is thus a nondegenerate 2-form on N. It extends to respective volume forms $\omega_{|\bullet|}$ and ω_θ on the vector space $N_\mathbb{R}$ and the argument torus $N\otimes(\mathbb{R}/2\pi\mathbb{Z})$. We can look at the (signed) area of the amoeba and coamoeba for these area forms. Let $\mathbb{H} = \{\mathfrak{Im}\, t > 0\} \subset \mathbb{C}P^1$ be the Poincaré half-plane, which is one of the two connected components of $\mathbb{C}P^1 \setminus \mathbb{R}P^1$, determined by the fact that the orientation of $\mathbb{R}P^1$ as its boundary coincides with the chosen orientation of the curve. We define

$$\mathscr{A}_{\mathrm{Log}}(\mathbb{H},\varphi) = \int_{\mathbb{H}} \varphi^*\,\mathrm{Log}^*\,\omega_{|\bullet|}, \quad \mathscr{A}_{\arg}(\mathbb{H},\varphi) = \int_{\mathbb{H}} \varphi^*\arg^*\omega_\theta.$$

The reason to integrate only over $\mathbb{H}$ and not $\mathbb{C}P^1$ is that the integral over the complex conjugate connected component $\overline{\mathbb{H}}$ is the opposite, and the whole area is zero. Moreover, these two areas are equal:

$$\mathscr{A}_{\mathrm{Log}}(\mathbb{H},\varphi) = \mathscr{A}_{\arg}(\mathbb{H},\varphi).$$

Their common value is denoted by $\mathscr{A}(\mathbb{H},\varphi)$.

If the intersection points with the toric boundary are real or purely imaginary, their common area is a half-integer multiple of π^2. Otherwise, we can shift the log-area to get a half-integer multiple of π^2 and we have the following theorem, which is a variation of Mikhalkin's theorem [13].

Proposition 3.1 (Mikhalkin) *Let $\varphi\colon \mathbb{C}P^1 \to \mathbb{C}\Delta$ be a oriented real rational curve. Let $\varepsilon_j\theta_j$, with $\varepsilon_j = \pm 1$ and $0 < \theta_j < \pi$, be the arguments of the complex intersection points of $\mathbb{H}$ with the toric boundary. Then*

$$\mathscr{A}(\mathbb{H},\varphi) - \pi\sum \varepsilon_j(2\theta_j - \pi) = k(\mathbb{H},\varphi)\pi^2 \in \tfrac{1}{2}\pi^2\mathbb{Z}.$$

The half-integer $k(\mathbb{H},\varphi)$ is called the quantum index of the oriented curve.

Remark 1.1 If the complex intersection points are purely imaginary, we recover the fact from [13] that the log-area is a half-integer multiple of π^2 since the correction term is zero. ◁

Due to the additive nature of the area, the quantum index is also additive. This can be used to compute the quantum index of a family of curves close to the tropical limit.

Furthermore, the behavior of the log-area with respect to monomial maps allows one to compute the quantum index of any rational curve with at least one real intersection point with the toric boundary, using only the computation of rational curves of degree less than 2, namely a line, a parabola, and an ellipse tangent to one of the coordinate axes not intersecting the other ones. This allows us to provide a formula to compute the log-area of any rational curve provided there is at least one real intersection point with the toric boundary. See Theorem 3.12.

1.3 Refined enumerative geometry

Now let $\mathcal{P}$ be a real configuration, ie consisting of real points and pairs of complex conjugate points, of m points on the toric boundary such that any toric divisor contains a number of points equal to the integral length of the corresponding side in the polygon P_Δ. Assume points are chosen different from the intersection points between toric divisors. Label the sides of P_Δ by $1, \dots, p$. Let s_i be the number of pairs of complex conjugate points on the toric divisor corresponding to the i^{th} side. The Viète formula ensures that there exists a curve of degree Δ passing through $\mathcal{P}$ if and only if the points of $\mathcal{P}$ satisfy the Menelaus condition, which we therefore assume: the product of the coordinates of the points is $(-1)^m$. Concretely, this means that the position of the last point is forced by the choice of the other points.

Let $\mathcal{S}(\mathcal{P})$ be the set of oriented real rational curves of degree Δ such that, for each point $p \in \mathcal{P}$, the curve passes through p or $-p$. Notice that, if a real curve passes through a complex point, it also passes through the complex conjugate point. This set of curves could also be defined in the following way: recall the square map $\mathrm{Sq}\colon \mathbb{C}\Delta \to \mathbb{C}\Delta$; then $\mathcal{S}(\mathcal{P})$ is the set of oriented real rational curves C such that $\mathrm{Sq}(C)$ passes through $\mathrm{Sq}(\mathcal{P})$. The curve $\mathrm{Sq}(C)$ is tangent to the toric divisors at each of its intersection points.

For an oriented curve $(S, \varphi) \in \mathcal{S}(\mathcal{P})$, with $\varphi\colon \mathbb{C}P^1 \to \mathbb{C}C$ a parametrization of the curve, and the orientation given by $S \subset \mathbb{C}P^1 \setminus \mathbb{R}P^1$ one of the connected components, we consider the composition of the parametrization φ with the logarithmic map, restricted to the real locus,

$$\mathrm{Log}\,|\varphi|\colon \mathbb{R}P^1 \to N \otimes \mathbb{R} = N_{\mathbb{R}}.$$

Its image is $\mathrm{Log}(\varphi(\mathbb{R}P^1)) \subset N_{\mathbb{R}}$. We now consider the logarithmic Gauss map that associates to each point of $\mathbb{R}P^1$ the tangent direction to $\mathrm{Log}(\varphi(\mathbb{R}P^1))$, which is a point in $\mathbb{P}^1(N_{\mathbb{R}}) \simeq \mathbb{R}P^1$, oriented by ω. Since both copies of $\mathbb{R}P^1$ are oriented, one can consider the degree $\mathrm{Rot}_{\mathrm{Log}}(S, \varphi)$ of the logarithmic Gauss map, ie when the domain $\mathbb{R}P^1$ is oriented by the choice of the complex orientation of the curve, and the target $\mathbb{P}^1(N_{\mathbb{R}}) \simeq \mathbb{R}P^1$ oriented by ω. Let $\sigma(S, \varphi) = (-1)^{(m - \mathrm{Rot}_{\mathrm{Log}}(S,\varphi))/2}$, which is equal to ± 1. We then set

$$R_{\Delta,k}(\mathcal{P}) = \sum_{(S,\varphi) \in \mathcal{S}_k(\mathcal{P})} \sigma(S, \varphi) \in \mathbb{Z},$$

where $\mathcal{S}_k(\mathcal{P})$ denotes the subset of $\mathcal{S}(\mathcal{P})$ of oriented real rational curves having quantum index k. Finally, we define

$$R_\Delta(\mathcal{P}) = \frac{1}{4}\sum_k R_{\Delta,k}(\mathcal{P})q^k \in \mathbb{Z}[q^{\pm 1/2}].$$

Although it is not stated in these terms, the following invariance statement is proven by Mikhalkin [13]. More precisely, using the definition of quantum index given in Proposition 3.1, the exact same proof applies.

Theorem 1.2 [13] *As long as the configuration $\mathcal{P}$ is generic, the Laurent polynomial $R_\Delta(\mathcal{P})$ only depends on $s = (s_1, \dots, s_p)$.*

The above Laurent polynomial only depending on s is denoted by $R_{\Delta,s}$.

Remark 1.3 It is important for the result and its proof to consider curves passing not only through $\mathcal{P}$ but also through the symmetric points, otherwise the invariance might fail. One other way of formulating the enumerative problem is to count curves after the square map, ie curves having tangencies with the toric divisors. As this yields more complicated computations, we choose to stick with the first formulation and pairs of opposite points. ◁

In the case of a totally real configuration of points, ie $s_i = 0$, Mikhalkin, using the correspondence theorem, proved that the invariant $R_{\Delta,(0)}$ coincides, up to a normalization, with the tropical refined invariant $N_\Delta^{\partial,\mathrm{trop}}$. This invariant is obtained as follows. (For a more complete description, see Section 4.) We consider tropical curves of degree $\Delta \subset N$. For each vector n_j in Δ except one, one chooses an affine line directed by n_j, and such that the configuration of lines is generic. Then we count rational tropical curves of degree Δ whose ends are contained in the chosen lines, using the Block–Göttsche multiplicities from [2]. The remaining end, corresponding to the last element in Δ, is always contained in a line determined by the position of the other ones. This is known as the *tropical Menelaus theorem* [13, Proposition 39]. The result does not depend on the chosen configuration, and is denoted by $N_\Delta^{\partial,\mathrm{trop}}$.

Theorem 1.4 (Mikhalkin [13]) $R_{\Delta,(0)} = (q^{1/2} - q^{-1/2})^{m-2} N_\Delta^{\partial,\mathrm{trop}}$.

Remark 1.5 The exponent $m-2$ corresponds to the number of vertices of a tropical curve involved in the enumerative problem defining $N_\Delta^{\partial\mathrm{trop}}$. Thus, the normalization $(q^{1/2} - q^{-1/2})^{m-2}$ amounts to clear the denominators of the Block–Göttsche multiplicities from [2]. We recall their definition in Section 4. ◁

The results of Mikhalkin reduce the computation of the invariants $R_{\Delta,(0)}$ to a tropical computation of $N_\Delta^{\partial,\mathrm{trop}}$. In this paper we prove that the computation of all the $R_{\Delta,s}$ can also be carried out using tropical geometry.

More precisely, we have the following theorem. Let $\Delta = (n_j) \subset N$ still be a multiset of primitive vectors whose total sum is zero, allowing us to consider curves of degree Δ inside the toric surface $\mathbb{C}\Delta$, and choose decompositions $r_j + 2s_j = \Delta(n_j)$, where $\Delta(n_j)$ denotes the number of times the primitive vector n_j appears in Δ. We denote by $\Delta(s)$ the family $(\Delta \setminus \{n_j^{2s_j}\}_j) \cup \{(2n_j)^{s_j}\}_j$, where $2s_j$ copies of n_j are replaced by s_j copies of $2n_j$. Assume that, for some i, we have $r_i > 0$, meaning there is at least one real intersection point with the toric boundary. The degree $\Delta(s)$, not consisting only of primitive vectors anymore, still allows us to consider tropical curves of degree $\Delta(s)$, and the associated refined invariant $N^{\partial,\text{trop}}_{\Delta(s)}$.

Theorem 4.8 *For Δ, s chosen as above, the refined invariants satisfy*

$$R_{\Delta,s} = 2^{|s|} \frac{(q^{1/2} - q^{-1/2})^{m-2-|s|}}{(q - q^{-1})^{|s|}} N^{\partial,\text{trop}}_{\Delta(s)},$$

where $|s| = \sum_1^p s_i$ is the total number of complex pairs.

This paper is a sequel to [3], in which we computed the invariants $R_{\Delta,s}$ when s is of the form $(s_1, 0, \dots, 0)$, meaning that all complex points lie on the same toric divisor, and complex points were purely imaginary. Briefly, generic configurations $\mathcal{P}$ for which the refined count gives the value of the invariant need to be regular values of the function that maps a curve to the coordinates of its intersection point with the toric boundary. Moreover, one previously needed that intersection points with the toric boundary were purely imaginary to define the quantum index as the logarithmic area. For those specific values of s, one can show the existence of regular values for which complex points are purely imaginary. The purely imaginary assumption also prevents an effective computation using tropical geometry. The new general definition of the quantum index in the case of curves with non-purely imaginary intersection points allows one to compute the invariant in the general case.

The paper is organized as follows. In Section 2, we recall the basics about tropical curves, real tropical curves and tropicalization. We also describe the possible real structures one can put on a rational tropical curve. Then Section 3 is devoted to the definition of quantum indices in the general setting, ie complex intersection points with the boundary may not be purely imaginary. We also compute the quantum index of some auxiliary curves, leading to a general formula in Theorem 3.12. In Section 4, we define properly the classical and tropical enumerative problems that yield to refined invariants in both settings, allowing us to state our main result. The rest of the paper is devoted to the proof of this same result. In Section 5, we prove a real version of the correspondence theorem from Tyomkin [20]. In Section 6, we use this theorem and make the refined count of solutions close to a tropical curve to relate the multiplicities in both enumerative problems and finally prove Theorem 4.8.

Acknowledgments I am grateful to Ilia Itenberg for numerous discussions leading to this result, and to Grigory Mikhalkin for several explanations about the quantum indices.

2 Tropical curves and real tropical curves

We assume the reader is familiar with the following concepts:

- plane tropical curves, which are corner loci of tropical polynomials;

- abstract tropical curves (resp. real tropical curves), which are metric graphs (resp. with an involution) with unbounded ends;
- (real) parametrized tropical curves, which are balanced affine integer maps from a (real) tropical curve to $\mathbb{R}^2$ (respecting the involution), along with their degree Δ;
- the tropicalization process;
- the evaluation map on the tropical moduli space $\mathcal{M}_0(\Delta, N_{\mathbb{R}})$ and the refined Block–Göttsche multiplicity

$$m_\Gamma^q = \prod_V [m_V^{\mathbb{C}}], \tag{1}$$

where $[a] = (q^{a/2} - q^{-a/2})/(q^{1/2} - q^{-1/2})$ denotes the q-analog.

For the unfamiliar reader as well as people looking for a reminder, we refer to [3, Section 2] for all necessary concepts.

We recall the following propositions, which describe the possible parametrizations of a simple plane tropical curve by a (real) tropical curve. These can also be found in [3, Section 2].

Proposition 2.1 [11] *Let C be a rational plane tropical curve, and let u_e be a directing primitive lattice vector for each end e, oriented toward infinity. Let w_e be the weight of e. Then C is the image of a unique rational parametrized tropical curve of degree $\Delta = \{w_e u_e\}_e$.*

Proposition 2.2 [3, Proposition 2.13] *Let C be an irreducible rational plane tropical curve of degree $P_\Delta \subset M$ having ends of weight 1 or 2. Let $\Delta \subset N$ be the degree associated to P_Δ consisting only of primitive lattice vectors. let $h_0\colon \Gamma_0 \to N_{\mathbb{R}}$ be the unique rational parametrization of C given by Proposition 2.1. Every real rational parametrized curve of degree Δ having the image C is obtained by cutting the graph outside a connected subgraph that contains the odd edges.*

Finally, we recall the definition of *moment of an edge*, which can also be found in [3, Section 2]. Let $h\colon \Gamma \to N_{\mathbb{R}}$ be a parametrized tropical curve. Let ω be a generator of $\Lambda^2 M$, ie a nondegenerate 2-form on N. It extends to an area form on $N_{\mathbb{R}} \simeq \mathbb{R}^2$. Let $e \in \Gamma_\infty^1$ be an end oriented toward infinity, directed by n_e. We define two moments: the *Menelaus moment* and the *primitive moment*.

Definition 2.3 Let $h\colon \Gamma \to N_{\mathbb{R}}$ be a parametrized tropical curve.

- The *primitive moment* is $\omega(\iota_{n_j/l(n_j)}, p)$, where p is any point on the end.
- The *Menelaus moment*, or briefly the *moment*, is $\omega(n_j, p)$, where p is any point on the end.

We similarly define the moment of a bounded edge if we specify its orientation. The moment of a bounded edge is reversed when its orientation is reversed. The Menelaus moments satisfy the tropical Menelaus theorem.

Proposition 2.4 (tropical Menelaus theorem [13, Proposition 39]) *For a parametrized tropical curve of degree Δ, the sum of Menelaus moments is* 0:

$$\sum_{n_e \in \Delta} \mu_e = 0.$$

3 Quantum indices of real curves

3.1 The quantum index of a type I real curve

Let $\varphi\colon \mathbb{C}C \to \mathbb{C}\Delta$ be a real parametrized curve of type I, which means that $\mathbb{C}C \setminus \mathbb{R}C$ is disconnected, and let S be one of its two connected components, which induces an orientation of $\mathbb{R}C$ as its boundary, called complex orientation. We can assume the curve to be rational but this is not needed for the definition of the quantum index.

Let $\alpha_1, \dots, \alpha_r \in \mathbb{R}C$ and $\beta_1, \dots, \beta_s \in S \subset \mathbb{C}C \setminus \mathbb{R}C$ be the parameters of the intersection points between the curve and the toric boundary of $\mathbb{C}\Delta$. Let $n_j : m \mapsto \mathrm{val}_{\alpha_j}(\varphi^* \chi^m)$ and $n'_j : m \mapsto \mathrm{val}_{\beta_j}(\varphi^* \chi^m)$ be the associated weight vectors. In the rational case, the parametrization would then be

$$t \mapsto \chi \prod_{i=1}^{r} (t - \alpha_i)^{n_i} \prod_{j=1}^{s} (t - \beta_j)^{n'_j} (t - \overline{\beta}_j)^{n'_j}.$$

For each intersection points with the boundary, lying on a toric divisor associated to the primitive direction $n \in N$, its coordinate is measured using the monomial $\iota_n \omega \in M$. This coordinate is also called the moment of the point.

We have logarithmic and argument maps, defined on $N_{\mathbb{C}^*}$,

$$\begin{aligned} \mathrm{Log}\colon N_{\mathbb{C}^*} &\to N_{\mathbb{R}}, & z^n &\mapsto n \otimes \log |z|, \\ \arg\colon N_{\mathbb{C}^*} &\to N \otimes (\mathbb{R}/2\pi\mathbb{Z}) = N_{2\pi}, & z^n &\mapsto n \otimes \arg(z). \end{aligned}$$

The argument is here taken mod 2π but it can also be taken mod π. We then define $N_\pi = N \otimes (\mathbb{R}/\pi\mathbb{Z})$. In either case, the codomain is endowed with a volume form induced by the volume form ω on N. If coordinates on $N_{\mathbb{C}^*}$ are $(z_1, z_2) = (e^{x_1 + i\theta_1}, e^{x_2 + i\theta_2})$, where $x_i \in \mathbb{R}$ and $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$, then the forms are

$$\omega_{|\bullet|} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \quad \text{and} \quad \omega_\theta = \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2.$$

For the form ω_θ, the volume of $N_{2\pi}$ is $4\pi^2$, and the volume of N_π is π^2. Due to the vanishing of the meromorphic form

$$\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 + i(\cdots)$$

on S, the pullbacks of $\omega_{|\bullet|}$ and ω_θ to S coincide, and therefore

$$\mathscr{A}(S, \varphi) = \int_S \varphi^* \mathrm{Log}^* \omega_{|\bullet|} = \int_S \varphi^* \arg^* \omega_\theta$$

is well defined and called the logarithmic area of (S, φ). See [13, Lemma 28].

Recall that the coordinate of each complex intersection point is given by $\varphi^*\chi^{\iota_{n'_j}\omega}|_{\beta_j}$. Let $\varepsilon_j\theta_j = \arg\varphi^*\chi^{\iota_{n'_j}\omega}|_{\beta_j}$, with $\varepsilon_j = \pm 1$ and $\theta_j \in]0;\pi[$, denote their arguments. Due to the vanishing of some coordinate, the argument map is not defined at the parameters α_j or β_j. Still, the map can be extended to the oriented blow-up of S at α_j and β_j.

- Under the coamoeba map to $N_{2\pi}$, each circle of the blow-up corresponding to β_j is sent to a geodesic in the direction n'_j, and the value of $\iota_{n'_j}\omega$ at any point of the geodesic is $\varepsilon_j\theta_j$. This geodesic is traveled a number of times equal to the integral length of n'_j.
- The half-circles corresponding to real intersection points are sent to geodesics in the direction associated to the corresponding divisor, but traveled a number of times equal to the integral length of n_j. In particular, if the integral length is odd, there is a half-way travel.
- If the arguments are instead taken in N_π, the real intersection points correspond to full geodesics, and complex intersection points to geodesics traveled twice.

Moreover, following the results of [7], the complement of the geodesics is endowed with an integer function corresponding to the signed number of preimages by $(\arg\circ\varphi)|_S$. This function is a 2-cochain on the argument torus N_π (or $N_{2\pi}$). This last property allows us to determine the function up to a shift. The knowledge of this function is enough to determine the total signed area. For the coamoeba of the whole curve $\mathbb{C}C$, as the total area is known to be zero, the shift can be determined. For a half-curve S, determining the shift can be more complicated.

As noticed by Mikhalkin [13], when the intersection points are purely imaginary, ie $\theta_j = \frac{\pi}{2}$, the logarithmic area $\mathscr{A}(S)$ is a half-integer multiple of π^2 for the following reason. Consider the quotient map

$$N_\pi \twoheadrightarrow N_\pi/\{\pm\mathrm{id}\}$$

from the argument torus mod π to its quotient by the antipodal map. Since each map $2\arg\circ\varphi|_S : S \to N_\pi$ defines a chain in the argument torus N_π, we can push-forward to the quotient and get a chain in the quotient $N_\pi/\{\pm\mathrm{id}\}$, which is homeomorphic to a sphere. However, due to the assumption on the arguments, this chain has no boundary anymore. The boundary previously consisted of the geodesics in N_π, but, as they each pass through a fixed point of the antipodal map, they cancel themselves in passing to the quotient. The volume form ω_θ also passes to the quotient, and therefore the total area is an integer multiple of the area of $N_\pi/\{\pm\mathrm{id}\}$, which is $\frac{1}{2}\pi^2$.

In the general case where the θ_j are not necessarily $\frac{\pi}{2}$, we have the following result.

Proposition 3.1 *There exists a half-integer, called the quantum index, such that*

$$k\pi^2 = \mathscr{A}(S) - \pi\sum_j \varepsilon_j(2\theta_j - \pi).$$

Proof We can shift the geodesics in the following way: for each point β_j with corresponding geodesic B_j in $N_{2\pi}$, add the chain whose oriented boundary consists of $\overline{B}_j$ (which is B_j with the opposite orientation) and $\widehat{B}_j$, which is the translate of B_j whose value under $\iota_{n'_j}\omega$ is now $\varepsilon_j\frac{\pi}{2}$. We get a new chain $\widehat{S}$ whose area satisfies

$$\langle\omega_\theta,\widehat{S}\rangle = \langle\omega_\theta,S\rangle + \pi\sum_j \varepsilon_j(2\theta_j-\pi).$$

The chain $\widehat{S}$ is now subject to passing to the quotient by the antipodal map, after reducing mod π, and therefore $\langle\omega_\theta,\widehat{S}\rangle \in \frac{1}{2}\pi^2\mathbb{Z}$. □

We finish by recalling the result from [3] stating that the log-area is well behaved under the monomial maps, which are covering maps from the complex torus to itself.

Lemma 3.2 [3] *Let $\varphi\colon \mathbb{C}C \dashrightarrow N_{\mathbb{C}^*}$ be a type I real curve with a choice of a connected component $S\subset \mathbb{C}C\setminus\mathbb{R}C$, inducing a complex orientation, and let $\alpha\colon N_{\mathbb{C}^*}\to N'_{\mathbb{C}^*}$ be a monomial map, associated to a morphism $A\colon N\to N'$. We consider the composition*

$$\psi\colon \mathbb{C}C \xrightarrow{\varphi} N_{\mathbb{C}^*} \xrightarrow{\alpha} N'_{\mathbb{C}^*}.$$

Let ω and ω' be the volume forms on N and N', respectively, such that $A^*\omega' = \det A\omega$. Then

$$\mathscr{A}(S,\psi) = \det(A)\mathscr{A}(S,\varphi).$$

Remark 3.3 The proposition deals with the computation of log-area in the general case of a real curve. This log-area needs to be shifted in order to get the quantum index. ◁

3.2 The quantum index near the tropical limit

Mikhalkin [13] proved the following result, which computes the log-areas of curves in a family near the tropical limit.

Proposition 3.4 [13] *Let $C^{(t)} = (f_t\colon \mathbb{C}P^1\to N_{\mathbb{C}^*})$ be a family of type I real parametrized rational curves, enhanced with a family of connected components of the complex locus $S^{(t)}$, inducing complex orientations of the curves. We assume that the family tropicalizes to a parametrized tropical curve $h\colon\Gamma\to N_{\mathbb{R}}$ such that components $S^{(t)}$ specialize to components S_w of C_w for every vertex $w\in\mathrm{Fix}(\sigma)$, thus inducing complex orientations of the curves C_w. Then, for t large enough,*

$$\mathscr{A}(S^{(t)},f_t) = \sum_w \mathscr{A}(S_w,f_w),$$

where the sum is indexed over the fixed vertices of Γ.

Remark 3.5 In particular — and this happens in the proof of the correspondence theorem — for one to know the log-area of curves near the tropical limit, one only needs to know the log-areas of the curves associated to the vertices of the tropical curve, and the way they are glued together along the edges. This means that the quantum index may be computed in the patchworking construction. ◁

3.3 Specific computations

In this section, we compute the log-areas of some auxiliary rational curves. This includes complex conjugate curves, and some curves of degree at most 2. Using Lemma 3.2, this enables the computation of the log-area of oriented curves having up to five intersection points with the toric boundary. Some of these computations were already treated in [3] but we include them here for sake of completeness.

3.3.1 Log-area of a complex curve We begin by proving that the log-area of a complex curve is zero. This justifies the fact that the quantum index near the tropical limit is obtained as a sum over the fixed vertices, since pairs of exchanged vertices do not contribute. The following statement is not specific to rational curves or real curves.

Lemma 3.6 *Let $\varphi\colon \mathbb{C}C \dashrightarrow N_{\mathbb{C}^*}$ be a complex parametrized curve, with $\mathbb{C}C$ a smooth Riemann surface. Then*

$$\mathscr{A}(\mathbb{C}C,\varphi)=0.$$

Proof Let $\mathbb{C}C^o$ be the open set of $\mathbb{C}C$ where φ is defined. We consider the map $\operatorname{Log}\circ\varphi\colon \mathbb{C}C^o\to N_{\mathbb{R}}$. This is a proper map between smooth oriented manifolds; therefore, it has a well-defined degree, which corresponds both to the number of preimages counted with signs over a generic point, and to the map $\mathbb{R}=H^2_c(N_{\mathbb{R}})\xrightarrow{(\operatorname{Log}\circ\varphi)^*}H^2_c(\mathbb{C}C^o)=\mathbb{R}$ between compactly supported cohomology groups. Since the map is not surjective, its degree is zero. Hence, if $\widetilde{\omega}$ is a compactly supported 2-form on $N_{\mathbb{R}}$, then $\int_{\mathbb{C}C^o}(\operatorname{Log}\circ\varphi)^*\widetilde{\omega}=0$. Thus, by writing ω as an (infinite) sum of compactly supported 2-forms using partitions of unity, we get the result. □

3.3.2 Log-area of a line We recall the computation of the quantum index of a line. This was dealt with in [13]. A line has degree $\Delta^{\text{line}}=\{(-1,0),(0,-1),(1,1)\}$.

Lemma 3.7 *The log-area of the parametrized line $t\mapsto(t,1-t)$ is $\frac{1}{2}\pi^2$.*

Proof This computation can be done by hand using the logarithmic map, or using the argument map. The coamoeba is depicted in Figure 1; a half-curve corresponds to one of the two triangles. Therefore, the signed area corresponds to the area of one of the triangles. □

3.3.3 Log-area of a parabola We now consider rational curves of degree

$$\Delta^{\text{par}}=\{(-1,1),(1,1),(0,-1)^2\}.$$

We assume that the two intersection points with the toric divisor associated to $(0,-1)$ are complex conjugate. Choose a coordinate on the curve such that these complex points are $\pm i$ and the intersection point with the toric divisor associated to $(1,1)$ is ∞. There are two such coordinates which differ by their orientation. Thus, the orientation fixes the coordinate uniquely. Up to a multiplicative translation, the oriented curve has a parametrization

$$\varphi\colon t\mapsto\Big(t-c,\frac{t^2+1}{t-c}\Big),\quad t\in\mathbb{C}P^1,$$

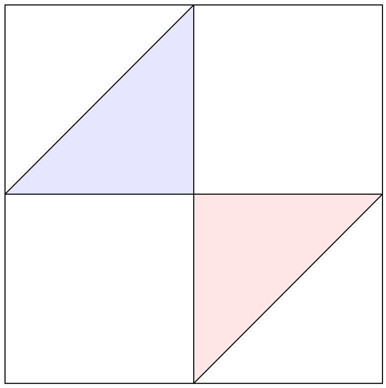

Figure 1: Coamoeba of a line.

where $c \in \mathbb{R}$ is the coordinate of the last intersection point with the toric boundary. The reversing of the orientation leads to the change of c by $-c$.

Lemma 3.8 *The log-area of the parabola* $\varphi\colon t \mapsto (t-c, (t^2+1)/(t-c))$ *is* $2\pi \arctan c$.

Proof The coamoeba along with its order map is depicted in Figure 2. The order map has value 1 on the blue triangles and -1 on the red ones. The abscissa of the two vertical geodesics are both opposite arguments of the complex intersection points with the boundary. The one with parameter i is

$$\arg(i-c) = \operatorname{arccot}(-c) \in]0;\pi[.$$

This is the coamoeba of the whole curve. However, as z is a coordinate on the curve, we can restrict to the coamoeba of the half-curve parametrized by $\mathbb{H}$ if we restrict to the triangles in the right half of the square. Therefore, the log-area is equal to

$$\mathscr{A}(\mathbb{H},\varphi) = \operatorname{arccot}(-c)^2 - (\pi - \operatorname{arccot}(-c))^2 = 2\pi \operatorname{arccot}(-c) - \pi^2 = 2\pi \arctan c. \qquad \square$$

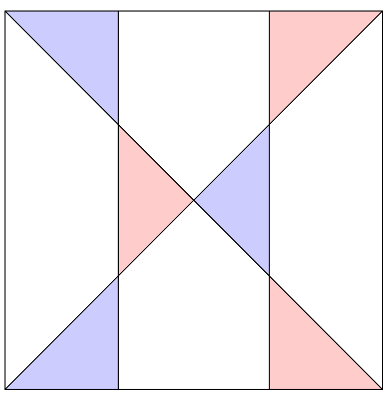

Figure 2: Coamoeba of a parabola with order map: -1 for red (triangles with left vertical side), $+1$ for blue (triangles with right vertical side).

Remark 3.9 Such curves have an equation of the form $w = az + b + c/z$, which becomes

$$\widetilde{w} = zw = az^2 + bz + c$$

after a change of toric coordinates. This is why we call them parabolas. ◁

3.3.4 Log-area of a tangent ellipse We now consider an ellipse tangent to one of the coordinate axes. An ellipse means a curve of degree 2 in $\mathbb{C}P^2$, ie a conic. We assume that its intersection points with the coordinate axes $\{w = 0\}$ and $\{z = 0\}$ are complex, and that it has a unique intersection point with the infinite axis. In the standard nomenclature, a conic tangent to the infinite axis is called a parabola, but we call it an tangent ellipse to distinguish it from the previous parabola case studied.

As in the parabola base, we choose a suitable coordinate on the oriented curve so that the infinite point has coordinate ∞. The other condition to fix this coordinate is that both parameters corresponding to complex intersection points have modulus 1. This is possible by taking as 0 the center of the circle containing all four points, with a suitable scaling. Let $\theta, \phi \in]0; \pi[$ be the arguments of these parameters. The parametrization is, up to a multiplicative translation,

$$\psi : t \mapsto (t^2 - 2t\cos\phi + 1, t^2 - 2t\cos\theta + 1), \quad t \in \mathbb{C}P^1.$$

Moreover, up to a permutation of the coordinate axes, one can assume that $\theta < \phi$, and, up to a change of the orientation of the curve, which can be done by changing t to $-t$ in the parametrization, we can assume that $\theta + \phi < \pi$. Indeed, the change of t to $-t$ is the curve where θ and ϕ have been changed to $\pi - \theta$ and $\pi - \phi$. If $\theta + \phi > \pi$, then $(\pi - \theta) + (\pi - \phi) < \pi$. This also reverses their order.

Lemma 3.10 *The log-area of the oriented curve*

$$\psi : t \mapsto (t^2 - 2t\cos\phi + 1, t^2 - 2t\cos\theta + 1), \quad t \in \mathbb{C}P^1,$$

is

$$\mathscr{A}(\mathbb{H}, \psi) = 2\pi(\theta - \phi) = 4\pi \arctan\left(\frac{r' - r}{s' + s}\right),$$

where $r + is$ and $r' + is'$ are the coordinates of $e^{i\theta}$ and $e^{i\phi}$ in any real coordinatization of the curve keeping the last intersection point with coordinate ∞.

Proof We use the coamoeba of the half-curve to determine the logarithmic area. The coamoeba is depicted in Figure 3. The fact that it is a chain allows us to determine the degree map up to a shift by some $k \in \mathbb{Z}$. The degree is then as depicted in Figure 3, left, up to k. We then have to show that $k = 0$. To do that, let us first compute the positions of the geodesics:

- The geodesic with slope 1 is the geodesic corresponding to the tangent point with the infinite axis, and passes through $(0, 0)$ since the curve is located in the quadrant $\mathbb{R}^2_{>0}$.

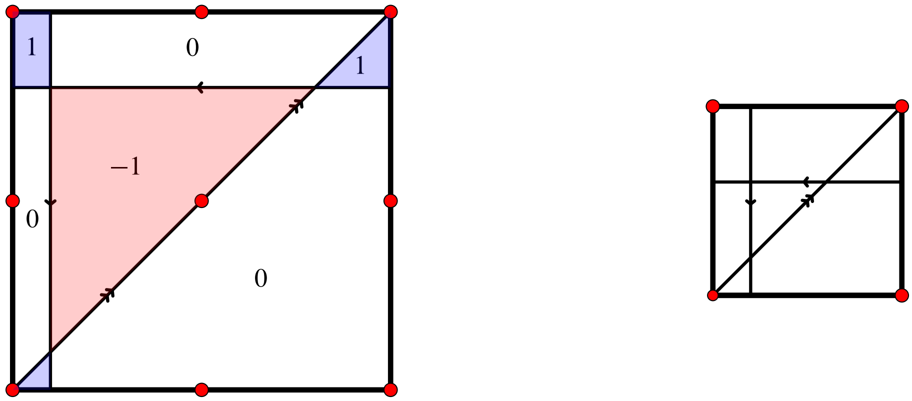

Figure 3: Coamoeba in $N_{2\pi}$ (left) and N_π (right) of a half-ellipse tangent to the infinite axis.

- The abscissa of the vertical geodesic is the argument of the intersection point with the axis $\{w=0\}$:
$$e^{2i\theta} - 2e^{i\theta}\cos\phi + 1 = 2e^{i\theta}(\cos\theta - \cos\phi).$$
As $\cos\theta - \cos\phi > 0$, the argument is θ.
- The height of the horizontal geodesic is the argument of the intersection point with the coordinate axis $\{z=0\}$, which has coordinate
$$e^{2i\phi} - 2e^{i\phi}\cos\theta + 1 = 2e^{i\phi}(\cos\phi - \cos\theta).$$
This time, the argument is $\phi+\pi$.

For each generic point in $N_{2\pi}$, the absolute value of the degree is a lower bound on the number of preimages by the coamoeba map. For a point in N_π, we can also use the degree, but we obtain a finer lower bound by adding the absolute values of the degrees for each of its four preimages by the covering map $N_{2\pi} \to N_\pi$. We now provide an upper bound on the number of preimages in N_π to get a bound on k. Let (α,β) be a generic point in N_π. The preimages of (α,β) by the coamoeba map are the points of the half-curve lying in $e^{i\alpha}\mathbb{R}^* \times e^{i\beta}\mathbb{R}^*$. This set is the fixed-point set of the twisted conjugation

$$\operatorname{conj}_{\alpha,\beta}(z,w) = (e^{2i\alpha}\bar{z}, e^{2i\beta}\overline{w});$$

therefore, they are a subset of the fixed points of the whole curve C by this conjugation. The fixed points coincide with the intersection $C \cap \operatorname{conj}_{\alpha,\beta}(C)$. These are two conics; thus, they have at most four intersection points. Therefore, the number of preimages is at most 4.

We get inequalities, by looking at each of the three regions of the complement of the geodesics in N_π,

$$\begin{cases} 3|k-1| + |k| \leqslant 4 & \text{for the central triangle,} \\ 3|k| + |k-1| \leqslant 4 & \text{for the hexagon,} \\ 3|k| + |k+1| \leqslant 4 & \text{for the split triangle.} \end{cases}$$

The two first inequalities imply $k=0$ or 1. The last inequality forces $k=0$. In fact, the degree in N_π is bounded by 4, and any change in the degree in $N_{2\pi}$ changes the degree in N_π by 4, which is why there are only a few allowed values.

To conclude, we now only have to compute the signed area, which is equal to the difference between the areas of the triangles. The red triangle has side length $\pi+\phi-\theta$ and sign -1, while the blue one has side length $\theta+\pi-\phi$, and sign $+1$, so

$$\mathscr{A} = -\tfrac{1}{2}(\pi+\phi-\theta)^2 + \tfrac{1}{2}(\pi+\theta-\phi)^2 = 2\pi(\theta-\phi).$$

Without the assumptions $\theta<\phi$ and $\theta+\phi<\pi$, the result is still valid:

- If $\theta>\phi$ and $\phi+\theta<\pi$, the change of the axis reduces to the first case by changing the role of θ and ϕ, but also changes the sign of the volume form, so the formula gives $-2\pi(\phi-\theta)=2\pi(\theta-\phi)$, and the same formula holds.
- If $\theta>\phi$ and $\theta+\phi>\pi$, replacing (θ,ϕ) by $(\pi-\theta,\pi-\phi)$ reduces to the first case, but the orientation of the curve has to be changed. Therefore, the log-area becomes $-2\pi(\pi-\theta-(\pi-\phi))=2\pi(\theta-\phi)$.
- The last case is a combination of both previous cases.

We have proven that the formula is valid for any (θ,ϕ). Last, we relate the values of θ and ϕ to the arguments of the parameters under any coordinate of the curve sending the tangent point to ∞. To see that the given formula is true, we see that the quantity $(r'-r)/(s+s')$ is unchanged by a real translation and a positive scaling. If we take the coordinate where $r+is=e^{i\theta}$ and $r'+is'=e^{i\phi}$, we have

$$\frac{r'-r}{s'+s} = \frac{\cos\phi-\cos\theta}{\sin\phi+\sin\theta} = \frac{2\sin\bigl(\frac{1}{2}(\theta+\phi)\bigr)\sin\bigl(\frac{1}{2}(\theta-\phi)\bigr)}{2\cos\bigl(\frac{1}{2}(\theta-\phi)\bigr)\sin\bigl(\frac{1}{2}(\theta+\phi)\bigr)} = \tan\bigl(\tfrac{1}{2}(\theta-\phi)\bigr).$$

Hence, the desired formula holds. We have proven that $\mathscr{A}=2\pi(\theta-\phi)=4\pi\arctan((r'-r)/(s+s'))$. □

Remark 3.11 The computation of the logarithmic area is also technically doable using the logarithm instead of the argument map. We have to integrate, over the Poincaré half-plane $\mathbb{R}\times\,]0;\infty[$, the Jacobian of the logarithmic map. The Jacobian is a rational function in the two coordinates of $\mathbb{H}$ whose denominator is already factorized. The integral over $\mathbb{R}$ can be dealt with using the residue formula, and the remaining integral over $]0;\infty[$ dealt with using standard methods. However, the computation passes through several decompositions and can be considered painful. ◁

3.4 Log-area of a real rational curve

Using Lemma 3.2 and the previous computations, it is possible to compute the logarithmic area of any real oriented rational curve which has

- three real punctures,
- two real punctures and two complex ones,
- one real and two pairs of complex ones.

This is already enough to compute the logarithmic area of a family of curves close to the tropical limit provided that the tropical limit curve is generic. We now show that this also allows us to compute the logarithmic area of any oriented real rational curve provided that there is at least one real boundary point and we are given a parametrization of the oriented rational.

Let us be given a parametrized oriented real rational curve, with $r \geqslant 1$,

$$\varphi\colon \mathbb{C}P^1 \to N_{\mathbb{C}^*}, \quad t \mapsto \chi \prod_1^r (t-\alpha_i)^{n_i} \prod_1^s (t^2 - 2t\,\mathfrak{Re}\,\beta_j + |\beta_j|^2)^{n'_j}.$$

Theorem 3.12 *If $r \geqslant 1$, assuming that $\alpha_r = \infty$, the logarithmic area is given by*

$$\mathscr{A}(\mathbb{H},\varphi) = \sum_{i<i'} \omega(n_i,n_{i'})\cdot \tfrac{1}{2}\pi^2 \operatorname{sgn}(\alpha_{i'}-\alpha_i) + \sum_{j<j'} \omega(n'_j,n'_{j'})\cdot 2\pi \arctan\left(\frac{\alpha_i - \mathfrak{Re}\,\beta_j}{\mathfrak{Im}\,\beta_j}\right) + \sum_{i,j} \omega(n_i,n'_j)\cdot 4\pi \arctan\left(\frac{\mathfrak{Re}\,\beta_{j'} - \mathfrak{Re}\,\beta_j}{\mathfrak{Im}\,\beta_{j'} + \mathfrak{Im}\,\beta_j}\right).$$

Proof We use a higher-dimensional analog of Lemma 3.2. If we denote by $(e_1,\dots,e_{r-1},f_1,\dots,f_s)$ the canonical base of $\mathbb{Z}^{r+s-1}$, there is unique linear map $A\colon \mathbb{Z}^{r+s-1} \to N$ such that

$$A(e_i) = n_i, \quad A(f_j) = n'_j.$$

If we denote by $\alpha\colon (\mathbb{C}^*)^{r+s-1} \to N_{\mathbb{C}^*}$ the associated monomial map, the parametrized curve factors as $\varphi = \alpha \circ \psi$, where ψ is

$$\psi\colon \mathbb{C}P^1 \to (\mathbb{C}^*)^{r+s-1}, \quad t \mapsto \chi' \prod_1^r (t-\alpha_i)^{e_i} \prod_1^s (t^2 - 2t\,\mathfrak{Re}\,\beta_j + |\beta_j|^2)^{f_j},$$

where χ' is chosen so that $\alpha(\chi') = \chi$. Using the fact that $\operatorname{Log}\circ\alpha = A\circ\operatorname{Log}$, we relate $\mathscr{A}(\mathbb{H},\varphi)$ to the log-area of the pullback of the volume form ω by A:

$$\mathscr{A}(\mathbb{H},\varphi) = \int_{\mathbb{H}} \psi^* \operatorname{Log}^* A^*\omega.$$

The 2-form $A^*\omega \in (\Lambda^2\mathbb{Z}^{r+s-1})^*$ decomposes as

$$A^*\omega = \sum_{i<i'} \omega(n_i,n_{i'}) e_i^* \wedge e_{i'}^* + \sum_{j<j'} \omega(n'_j,n'_{j'}) f_j^* \wedge f_{j'}^* + \sum_{i,j} \omega(n_i,n'_j) e_i^* \wedge f_j^*.$$

Therefore, we only need to compute the logarithmic area of $(\mathbb{H},\psi)$ for the canonical forms $e_i^*\wedge e_{i'}^*$, $f_j^*\wedge f_{j'}^*$ and $e_i^*\wedge f_j^*$. However, we notice that these forms are the pullbacks of the canonical volume form on $\mathbb{Z}^2$ by the projections

$$p_{ii'}\colon x \mapsto (x_i,x_{i'}), \quad p_{jj'}\colon x \mapsto (x'_j,x'_{j'}), \quad p_{ij}\colon x \mapsto (x_i,x'_j), \quad x \in \mathbb{Z}^{r+s-1}.$$

Using the same trick, the computations of the log-area for these 2-forms are equal to the log-areas of some plane curves which have previously been studied. Therefore, we are left with computing the log-area of a plane curve whose degree is one of the degrees for which the log-area has been computed; it is either a line, a parabola or a tangent ellipse. In each case, we perform a change of variable in order to get back to the canonical choice of coordinate used in the computation of that specific case. In order to simplify the formula, we have assumed that $\alpha_r = \infty$. Therefore, the log-area is, finally,

$$\mathscr{A}(\mathbb{H},\varphi) = \sum_{i<i'} \omega(n_i, n_{i'}) \cdot \tfrac{1}{2}\pi^2 \varepsilon_{ii'} + \sum_{j<j'} \omega(n'_j, n'_{j'}) \cdot 2\pi \arctan\left(\frac{\alpha_i - \mathfrak{Re}\,\beta_j}{\mathfrak{Im}\,\beta_j}\right) + \sum_{i,j} \omega(n_i, n'_j) \cdot 4\pi \arctan\left(\frac{\mathfrak{Re}\,\beta_{j'} - \mathfrak{Re}\,\beta_j}{\mathfrak{Im}\,\beta_{j'} + \mathfrak{Im}\,\beta_j}\right),$$

where $\varepsilon_{ii'}$ is ± 1 according to whether α_i, $\alpha_{i'}$ and α_r are in the cyclic order induced by the orientation of the curve, ie $\alpha_i < \alpha_{i'}$. □

Remark 3.13 If all the intersection points are real, we recover Mikhalkin's result [13] for toric type I curves, stating that the quantum index, here equal to the log-area up to a π^2 factor, only depends on the cyclic order in which the oriented curve meets the divisors, here embodied in the collection of signs $\operatorname{sgn}(\alpha_{i'} - \alpha_i)$. ◁

4 Tropical enumerative problem and refined curve counting

Let $\Delta \subset N$ be a family of m primitive lattice vectors, with total sum 0. As described in the introduction, there is an associated lattice polygon P_Δ having m lattice points on its boundary. The toric surface obtained from Δ is denoted by $\mathbb{C}\Delta$. Let $E_1, \dots, E_p$ denote the sides of the polygon P_Δ and let $n_1, \dots, n_p \in N$ be their normal primitive vectors. Let $s = (s_1, \dots, s_p)$, where $s_i \leqslant \frac{1}{2} l(E_i)$ for each i, be a tuple of positive integers. Let $r_i = l(E_i) - 2s_i$, so $\sum_1^p r_i + 2s_i = m$. Let

$$\Delta(s) = \{n_1^{r_1}, (2n_1)^{s_1}, \dots, n_p^{r_p}, (2n_p)^{s_p}\}.$$

The size of s is

$$|s| = \sum_1^p s_i.$$

4.1 Tropical problem

The tropical curves of degree $\Delta(s)$ have $m - |s|$ ends, and therefore the moduli space $\mathcal{M}_0(\Delta(s), N_\mathbb{R})$ of parametrized rational tropical curves of degree $\Delta(s)$ in $N_\mathbb{R} \simeq \mathbb{R}^2$ has dimension $m - |s| - 1$. We have the evaluation map

$$\mathrm{ev}\colon \mathcal{M}_0(\Delta(s), N_\mathbb{R}) \to \mathbb{R}^{m-|s|-1},$$

which associates to a parametrized tropical curve the moments of every end but the first. Recall that the moment of the first end is equal to minus the sum of the other moments because of the tropical Menelaus theorem. Let $\mu \in \mathbb{R}^{m-|s|-1}$ be a generic family of moments. We look for parametrized rational tropical curves Γ of degree $\Delta(s)$ such that $\mathrm{ev}(\Gamma) = \mu$.

Due to genericity, as noticed in [4], every parametrized rational tropical curve Γ such that $\mathrm{ev}(\Gamma)=\mu$ is a simple nodal tropical curve and thus has a well-defined refined multiplicity. We then set

$$N_{\Delta(s)}^{\partial,\mathrm{trop}}(\mu)=\sum_{\mathrm{ev}(\Gamma)=\mu} m_\Gamma^q \in \mathbb{Z}[q^{\pm 1/2}].$$

Theorem 4.1 [4] *The value of $N_{\Delta(s)}^{\partial,\mathrm{trop}}(\mu)$ is independent of μ provided that it is generic.*

Remark 4.2 The ends of weight 2 are destined to be split, and thus correspond to transverse intersection points between the curve and the toric boundary. Higher-order ends would correspond to tangencies with the boundary. ◁

Remark 4.3 Theorem 4.1 can be seen as a particular case of the theorem about refined broccoli invariants proven by Göttsche and Schroeter [8]. However, that theorem has more general assumptions, while the context of Theorem 4.1 is more specific. That is why the proof provided in [4] is easier. ◁

4.2 Classical problem

Keeping previous notation, let $\mathcal{P}$ be a configuration of m points on the toric boundary $\partial\mathbb{C}\Delta$ such that:

- Each toric divisor associated to a side E_i of P_Δ contains exactly r_i real points chosen in the boundary of the positive orthant $N_{\mathbb{R}_{>0}}$ and s_i pairs of complex conjugate points.
- The configuration satisfies the Menelaus condition.

Let $\mathcal{S}(\mathcal{P})$ be the set of oriented real rational curves of degree $2\Delta(s)$ such that, for every $p\in\mathcal{P}$, the curve that passes through p is tangent to the toric divisor. As the curves are oriented, each real curve is counted twice: once with each of its orientations. Notice that, if a curve has a tangency point at one of the points of a pair of complex conjugate points, it is also tangent to the complex divisor at the complex conjugate point. We denote by $\mathcal{S}_k(\mathcal{P})$ the subset of $\mathcal{S}(\mathcal{P})$ formed by oriented curves with quantum index k.

Remark 4.4 A rational curve that is tangent to the toric divisors at each intersection point can be written as the image of a rational curve by the square map, which is the extension of $N_{\mathbb{C}^*}\to N_{\mathbb{C}^*}$, $z^n\mapsto z^{2n}$, to $\mathbb{C}\Delta$. Given in coordinates, it squares each coordinate. ◁

We stress that the tropical curves we look for are of degree

$$\Delta(s)=\{n_1,\dots,n_r,2n'_1,\dots,2n'_s\},$$

while, due to the tangency conditions, the complex curves are of degree

$$2\Delta(s)=\{2n_1,\dots,2n_r,(2n'_1)^2,\dots,(2n'_s)^2\},$$

meaning they have a parametrization of the form

$$\varphi\colon t\mapsto \chi\prod_1^r (t-\alpha_i)^{2n_i}\prod_1^s (t-\beta_j)^{2n'_j}(t-\overline{\beta}_j)^{2n'_j}.$$

Let $\varphi\colon \mathbb{C}P^1 \to \mathbb{C}\Delta$ be an oriented real parametrized rational curve, denoted by (S,φ). The logarithmic Gauss map sends a point $p \in \mathbb{R}P^1$ to the tangent direction to $\operatorname{Log}\varphi(\mathbb{R}P^1)$ inside $N_{\mathbb{R}}$. We get a map

$$\gamma\colon \mathbb{R}P^1 \to \mathbb{P}^1(N_{\mathbb{R}}).$$

The first space $\mathbb{R}P^1$ is oriented since the curve is oriented, while $\mathbb{P}^1(N_{\mathbb{R}})$ is oriented by ω. The degree of this map is denoted by $\operatorname{Rot}_{\operatorname{Log}}(S,\varphi) \in \mathbb{Z}$. If the curve has transverse intersections with the divisors, it has the same parity as the number of boundary points m. We then set

$$\sigma(S,\varphi) = (-1)^{(m-\operatorname{Rot}_{\operatorname{Log}}(S,\varphi))/2} \in \{\pm 1\}.$$

Now let

$$R_{\Delta,k}(\mathcal{P}) = \sum_{(S,\varphi)\in\mathcal{S}_k(\mathcal{P})} \sigma(S,\varphi),$$

and

$$R_{\Delta}(\mathcal{P}) = \sum_k R_{\Delta,k}(\mathcal{P}) q^{k/4} \in \mathbb{Z}[q^{\pm 1/2}].$$

Remark 4.5 The shift by m to the logarithmic rotation number is only to keep track of its residue mod 2, while we are interested in its residue mod 4, so one could also choose another convention, for instance the logarithmic rotation number of a maximal curve. ◁

Theorem 4.6 (Mikhalkin [13]) *As long as $r = \sum_1^p r_i \geqslant 1$, the value of $R_\Delta(\mathcal{P})$ is independent of the configuration $\mathcal{P}$ as long as it is generic. It only depends on Δ and s.*

The obtained polynomial, independent of $\mathcal{P}$, is denoted by $R_{\Delta,s}$.

Remark 4.7 Although this theorem is not stated in these terms in [13], because the only case considered is the one of purely imaginary points, where the quantum index coincides with the log-area, the proof does not use this specific assumption, and thus applies also in this setting. ◁

4.3 Statement of the result

The refined tropical count and the refined classical count find their relation through the following theorem, proven in Section 6. In what follows, we use the positive and negative q-analogs: for $a \in \mathbb{R}$, let

$$[a]_+ = q^a + q^{-a} \quad \text{and} \quad [a] = q^a - q^{-a}.$$

Theorem 4.8

$$R_{\Delta,s} = 2^{|s|} \frac{\left[\frac{1}{2}\right]^{m-2-|s|}}{[1]^{|s|}} N_{\Delta(s)}^{\partial,\mathrm{trop}} = 2^{|s|} \frac{\left[\frac{1}{2}\right]^{m-2-2|s|}}{\left[\frac{1}{2}\right]_+^{|s|}} N_{\Delta(s)}^{\partial,\mathrm{trop}}.$$

Göttsche and Schroeter [8] proposed a refined way to count so-called *refined broccoli curves* having fixed ends and passing through a fixed configuration of "real and complex" points. In the case where there are only marked ends and no marked points, this count coincides with the count of plane tropical curves passing through the configuration with usual Block–Göttsche multiplicities from [9] up to a multiplication by a constant term depending on the degree and easily computed. More precisely, provided there are no marked points, the refined broccoli multiplicity is just the refined multiplicity from [9] enhanced by a product over the ends of weight higher than 2, coinciding with this aforementioned constant term. In our case, since the only multiple edges are marked and of weight 2 (only one real end is unmarked and of weight 1), this factor is $(q+q^{-1})/(q^{1/2}+q^{-1/2})$ for each of the $|s|$ ends. If we denote by $\mathrm{BG}_{\Delta(s)}(q)$ the refined invariant obtained in [8], then

$$R_{\Delta,s}(q) = \frac{\left[\frac{1}{2}\right]^{m-2-2|s|}}{[1]_+^{|s|}} \mathrm{BG}_{\Delta(s)}(q).$$

5 Realization and correspondence theorem in the real case

The proof of Theorem 4.8 uses a correspondence theorem between classical real curves and tropical curves. The proof of this theorem follows the same steps as in [20] and presents similar calculations. Its proof proceeds as follows: find a suitable set of coordinates on the space of family of curves tropicalizing to a given tropical curve, and then prove that the Jacobian of the evaluation map is invertible, so that Hensel's lemma can be applied. The first subsections are dedicated to notation that is specific to the correspondence theorem. The last subsection proves that the correspondence theorem can be applied in our setting, showing that the Jacobian is indeed invertible.

5.1 Notation

Let (Γ,σ) be a real rational abstract tropical curve with set of ends I of size m. The quotient map is denoted by $\pi:\Gamma\to\Gamma/\sigma$. The action of σ on I induces the decomposition

$$I=\{x_1,\dots,x_r,z_1^{\pm},\dots,z_s^{\pm}\},$$

where $\sigma(x_i)=x_i$ and $\sigma(z_i^{\pm})=z_i^{\mp}$. Assume that $r\geqslant 1$, and orient the edges of both Γ and Γ/σ away from x_r. This makes them rooted trees, thus inducing a partial order $\prec$ on the curve Γ. If $\gamma\in\Gamma^1$ is a bounded edge of γ (same for Γ/σ), let $\mathfrak{t}(\gamma)$ and $\mathfrak{h}(\gamma)$ be the tail and the head of γ. Notice that $\mathfrak{h}(\gamma)\notin\mathrm{Fix}(\sigma)$ if and only if $\gamma\notin\mathrm{Fix}(\sigma)$. We also endow the set I/σ with a total order.

- If $w\in\Gamma^0$ is a vertex of Γ, let I_w^∞ be the set of ends of Γ which are greater than w for the order $\prec$. We use similar notation $(I/\sigma)_w^\infty$ for $w\in(\Gamma/\sigma)^0$. Notice that, if $w\in\mathrm{Fix}(\sigma)$, then I_w^∞ is stable by σ, and, if $w\notin\mathrm{Fix}(\sigma)$, then at most one element of each pair $\{z_j^{\pm}\}$ belongs to I_w^∞.
- If $\gamma^\sigma\in(\Gamma/\sigma)^1$, let $\iota(\gamma^\sigma)=\min(I/\sigma)_{\mathfrak{h}(\gamma^\sigma)}^\infty$.
- The order on I/σ along with this map ι induces an order on the edges of Γ/σ having the same tail. We thus can speak about the smallest and biggest edge leaving a vertex $\pi(w)$ of Γ/σ.
- If $w\in\mathrm{Fix}(\sigma)$, then we have three cases for the lift of an edge γ^σ such that $\mathfrak{t}(\gamma^\sigma)=\pi(w)$:

(RR) The edge $\gamma^\sigma=\{\gamma\}$ lifts to a fixed edge γ of Γ, and $\iota(\gamma^\sigma)=\{x_j\}$ is a real marking. We then set $\iota(\gamma)=x_j$.

(RC) The edge $\gamma^\sigma = \{\gamma\}$ lifts to a fixed edge γ of Γ but $\iota(\gamma^\sigma) = \{z_j^\pm\}$ is a complex marking. It means that the curve Γ splits at some point on the path from w to $\iota(\gamma^\sigma)$, but not right away in γ.

(CC) The edge $\gamma^\sigma = \{\gamma^+, \gamma^-\}$ lifts to a pair of exchanged edges in Γ. Then $\iota(\gamma^\sigma) = \{z_j^\pm\}$ and the tail is smaller than z_j^+.

- If $w \notin \mathrm{Fix}(\sigma)$, for every edge γ such that $\mathfrak{t}(\gamma) = w$ we have a unique complex end $\iota(\gamma) \in \iota(\gamma^\sigma) = \{z_j^\pm\}$ accessible by w.

We denote by $v_\mathbb{R}$ (resp. $v_\mathbb{C}$) the number of fixed vertices (resp. pairs of exchanged vertices), and by $e_\mathbb{R}$ (resp. $e_\mathbb{C}$) the number of fixed bounded edges (resp. pairs of exchanged bounded edges).

5.2 Space of rational curves with given tropicalization

Let Γ be an abstract tropical curve with m ends. Let $(C^{(t)}, x_1, \dots, x_r, z_1^\pm, \dots, z_s^\pm)$ be a stable family of real smooth rational curve with a real configuration of marked points, tropicalizing on Γ, the dual graph of $C^{(0)}$.

Remark 5.1 By taking a coordinate, the marked curve $(C^{(t)}, \boldsymbol{x}, \boldsymbol{z}^\pm)$ can be seen as $\mathbb{P}^1(\mathbb{C}((t))) \simeq \mathbb{C}((t)) \cup \{\infty\}$, the projective line over the field of Laurent series, along with $r + 2s$ Laurent series which are the marked points, taken up to a change of coordinate in $\mathrm{PGL}_2(\mathbb{R}((t)))$. The first r Laurent series are in $\mathbb{R}((t))$, and the last s are taken in $\mathbb{C}((t)) \setminus \mathbb{R}((t))$ along with their conjugates. ◁

We associate to each vertex $w \in \Gamma^0$ a coordinate y_w on $C^{(t)}$ such that $y_{\sigma(w)} = \overline{y_w \circ \sigma}$, and y_w specializes to a coordinate on the irreducible component of $C^{(0)}$ associated to w.

- If $w \in \mathrm{Fix}(\sigma)$, let γ_a^σ and γ_b^σ be the smallest and biggest edges emanating from $\pi(w)$ in Γ/σ. We treat several cases according to the type (RR), (RC), (CC) of each edge:

(RR/RR) Edges γ_a^σ and γ_b^σ lift to edges γ_a and γ_b of Γ, which have well-defined real markings $x_a = \iota(\gamma_a)$ and $x_b = \iota(\gamma_b)$. Then we take y_w such that $y_w(x_r) = \infty$, $y_w(x_a) = 0$ and $y_w(x_b) = 1$. Moreover, $\overline{y_w \circ \sigma} = y_w$.

(RR/RC) If γ_a^σ and γ_b^σ lift up to edges $\gamma_a, \gamma_b \in \mathrm{Fix}(\sigma)$ with $\iota(\gamma_a) = x_a$ and $\iota(\gamma_b) = z_b^\pm$, then $\mathfrak{Re}\, z_b^\pm$ is a well-defined real Laurent series whose specialization on the component associated to w is different from the one of x_a. Then we can take y_w such that $y_w(x_r) = \infty$, $y_w(x_a) = 0$ and $y_w(\mathfrak{Re}\, z_b^\pm) = 1$.

(RC/RR) We do the same with $\mathfrak{Re}\, z_a^\pm$ and x_b.

(RC/RC) If γ_a^σ and γ_b^σ are both of type (RC), we do the same with $\mathfrak{Re}\, z_a^\pm$ and $\mathfrak{Re}\, z_b^\pm$.

(CC/−) If γ_a^σ is of type (CC), then γ_a^σ lifts to a pair of exchanged edges $\{\gamma_a^\pm\}$ both emanating from w. They both have a well-defined $\iota(\gamma_a^\pm) = z_a^\pm$. Then we take y_w such that $y_w(x_r) = \infty$, $y_w(z_a^+) = i$ and $y_w(z_a^-) = -i$, which is also a real coordinate.

(−/CC) If γ_a^σ is of type (RR) or (RC) and γ_b^σ is of type (CC), we do the same with $z_b^\pm$.

- If $w \notin \mathrm{Fix}(\sigma)$, then I_w^∞ consists only of complex markings, all edges emanating from w are of type (CC) and we have a well-defined $\iota(\gamma)$ for each of them. Let z_a^ε and z_b^η be the smallest and biggest elements in I_w^∞. We take y_w such that $y_w(x_r) = \infty$, $y_w(z_a^\varepsilon) = 0$ and $y_w(z_b^\eta) = 1$. This choice ensures that $y_{\sigma(w)} = \overline{y_w \circ \sigma}$.

The functions y_w are all coordinates on C sending x_r to ∞; therefore, we can pass from one to another by a real affine function, which we now describe. The proof is straightforward, by checking that both coordinates coincide in three points.

Proposition 5.2 *Let $\gamma \in \Gamma^1$ be a bounded edge.*

- *If $\gamma \notin \mathrm{Fix}(\sigma)$, let z_a^ε and z_b^η be the smallest and biggest elements in $I_{\mathfrak{h}(\gamma)}^\infty$; then*

$$y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(z_a^\varepsilon)}{y_{\mathfrak{t}(\gamma)}(z_b^\eta) - y_{\mathfrak{t}(\gamma)}(z_a^\varepsilon)} \quad \text{and} \quad |\gamma| = \mathrm{val}(y_{\mathfrak{t}(\gamma)}(z_b^\eta) - y_{\mathfrak{t}(\gamma)}(z_a^\varepsilon)).$$

- *If $\gamma \in \mathrm{Fix}(\sigma)$, we make several cases according to the type of $\mathfrak{h}(\gamma)$:*

(RR/RR) $$y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(x_a)}{y_{\mathfrak{t}(\gamma)}(x_b) - y_{\mathfrak{t}(\gamma)}(x_a)} \quad \text{and} \quad |\gamma| = \mathrm{val}(y_{\mathfrak{t}(\gamma)}(x_b) - y_{\mathfrak{t}(\gamma)}(x_a)).$$

(RR/RC) $$y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(x_a)}{\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - y_{\mathfrak{t}(\gamma)}(x_a)} \quad \text{and} \quad |\gamma| = \mathrm{val}(\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - y_{\mathfrak{t}(\gamma)}(x_a)).$$

(RC/RR) $$y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(x_a)}{y_{\mathfrak{t}(\gamma)}(x_b) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)} \quad \text{and} \quad |\gamma| = \mathrm{val}(y_{\mathfrak{t}(\gamma)}(x_b) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)).$$

(RC/RC) $$y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)}{\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)} \quad \text{and} \quad |\gamma| = \mathrm{val}(\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)).$$

(CC/−) $$y_{\mathfrak{h}(\gamma)} = \frac{y_{\mathfrak{t}(\gamma)} - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)}{\mathfrak{Im}\, y_{\mathfrak{t}(\gamma)}(z_a^+)} \quad \text{and} \quad |\gamma| = \mathrm{val}(\mathfrak{Im}\, y_{\mathfrak{t}(\gamma)}(z_a^+)).$$

(−/CC) *Same with a switched by b.*

For every edge we now define $\alpha_\gamma \in \mathbb{C}[\![t]\!]^\times$ and $\beta_\gamma \in \mathbb{C}[\![t]\!]$, which will be the coordinates on the space of real marked curves tropicalizing on Γ. Once again, the definition goes through the distinction of the type of edges emanating from $\mathfrak{h}(\gamma)$. Let $\gamma \in \Gamma_b^1$ be a bounded edge. Then

(RR/RR) $$\alpha_\gamma = t^{-|\gamma|}(y_{\mathfrak{t}(\gamma)}(x_b) - y_{\mathfrak{t}(\gamma)}(x_a)),$$

(RR/RC) $$\alpha_\gamma = t^{-|\gamma|}(\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - y_{\mathfrak{t}(\gamma)}(x_a)),$$

(RC/RR) $$\alpha_\gamma = t^{-|\gamma|}(y_{\mathfrak{t}(\gamma)}(x_b) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)),$$

(RC/RC) $$\alpha_\gamma = t^{-|\gamma|}(\mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_b^\pm) - \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z_a^\pm)),$$

(CC/−) $$\alpha_\gamma = t^{-|\gamma|}\, \mathfrak{Im}\, y_{\mathfrak{t}(\gamma)}(z_a^+),$$

(−/CC) $$\alpha_\gamma = t^{-|\gamma|}\, \mathfrak{Im}\, y_{\mathfrak{t}(\gamma)}(z_b^+).$$

Let $\gamma \in \Gamma^1$ be a not necessarily bounded edge.

- If $\gamma \notin \mathrm{Fix}(\sigma)$ is of type (CC), then $\beta_\gamma = y_{\mathfrak{t}(\gamma)}(z^{\varepsilon}_{\iota(\gamma)})$, where $z^{\varepsilon}_{\iota(\gamma)}$ is the lift of $z_{\iota(\gamma)}$ accessible by γ.
- If $\gamma \in \mathrm{Fix}(\sigma)$ is of type (RR), then $\beta_\gamma = y_{\mathfrak{t}(\gamma)}(x_{\iota(\gamma)})$.
- If $\gamma \in \mathrm{Fix}(\sigma)$ is of type (RC), then $\beta_\gamma = \mathfrak{Re}\, y_{\mathfrak{t}(\gamma)}(z^{\pm}_{\iota(\gamma)})$.

We now can define the function

$$\Psi_\gamma(y) = \beta_\gamma + t^{|\gamma|}\alpha_\gamma y,$$

which allows an easy description of the relations between the y_w.

Proposition 5.3 *The Laurent series α_γ and β_γ satisfy the following properties*:

(i) *If γ is an edge, one has $\alpha_\gamma \in \mathbb{C}[\![t]\!]^\times$. Moreover, if $\gamma \neq \gamma'$ are two different edges with the same tail $\mathfrak{t}(\gamma) = \mathfrak{t}(\gamma')$, then $\beta_\gamma - \beta_{\gamma'} \in \mathbb{C}[\![t]\!]^\times$.*

(ii) *They are real: $\alpha_{\sigma(\gamma)} = \overline{\alpha}_\gamma$ and $\beta_{\sigma(\gamma)} = \overline{\beta}_\gamma$. In particular, $\alpha_\gamma, \beta_\gamma \in \mathbb{R}[\![t]\!]$ if $\gamma \in \mathrm{Fix}(\sigma)$.*

(iii) *For each edge γ, one has $y_{\mathfrak{t}(\gamma)} = \Psi_\gamma(y_{\mathfrak{h}(\gamma)})$. In particular, for any marked point q,*

$$y_{\mathfrak{t}(\gamma)} - y_{\mathfrak{t}(\gamma)}(q) = t^{|\gamma|}\alpha_\gamma(y_{\mathfrak{h}(\gamma)} - y_{\mathfrak{h}(\gamma)}(q)).$$

Moreover, Ψ_γ is real in the sense that $\Psi_{\sigma(\gamma)}(y) = \overline{\Psi_\gamma(\bar{y})}$.

(iv) *Let w, w' be two vertices, and $\gamma_1 \cdots \gamma_d$ the geodesic path from w to w'. Let $\varepsilon_i = \pm 1$ according to whether the orientation of γ_i in Γ and the geodesic path agree or not. Then*

$$y_w = \Psi^{\varepsilon_1}_{\gamma_1} \circ \cdots \circ \Psi^{\varepsilon_d}_{\gamma_d}(y_{w'}),$$

and, in particular, for any marked point q,

$$y_w - y_w(q) = t^{\sum_1^d \varepsilon_i|\gamma_i|}\left(\prod_1^d \alpha^{\varepsilon_i}_{\gamma_i}\right)(y_{w'} - y_{w'}(q)).$$

(v) *Let q be a marked point associated with an end $e \in \Gamma^1_\infty$, w be a vertex, and $\gamma_1, \dots, \gamma_d, e$ be the geodesic path from w to q. Then*

$$y_w(q) = \Psi^{\varepsilon_1}_{\gamma_1} \circ \cdots \circ \Psi^{\varepsilon_d}_{\gamma_d}(\beta_e),$$

and, in particular, for every marked point q associated with the end e, if v_r is the vertex adjacent to the end associated to x_r,

$$y_{v_r}(q) = \beta_{\gamma_1} + t^{|\gamma_1|}\alpha_{\gamma_1}(\cdots(\beta_{\gamma_d} + t^{|\gamma_d|}\alpha_{\gamma_d}\beta_e)).$$

(vi) *For every marked point q_i and vertex $w \in \Gamma^0$, $\mathrm{val}(y_w(q)) \geqslant 0$ if and only if q is accessible by w.*

(vii) *For every edge $\gamma \in \Gamma^1$ and every marked point $q \in I^\infty_{\mathfrak{t}(\gamma)} \setminus I^\infty_{\mathfrak{h}(\gamma)}$, we have $\mathrm{val}(y_{\mathfrak{t}(\gamma)}(q) - \beta_\gamma) = 0$.*

Proof It suffices to check every statement:

- (i) and (ii) follow from the definition of α and β.

- (iii) comes from the definition of Ψ_γ and from the fact that α and β are real.
- (iv) and (v) are just iterations from (iii).
- (vi) comes from (v), and (vii) follows from (i) and (v). □

Remark 5.4 Formally speaking, this proposition is the direct translation of [20, Proposition 4.3], in the setting of curves with a real structure. Although the formulas seem quite repulsive at first glance, the meaning of each object must be clear. The formal series α_γ and β_γ allow one to recover the coordinates of the marked points, following the formula (v) of Proposition 5.3. The formal series α_γ are the *"phase length"* of the edge γ, in contrast to $t^{|\gamma|}$, which could be called *"valuation length"*, while the formal series β_γ are the directions one needs to follow at each w in order to get to the points of $I^\infty_{\mathfrak{h}(\gamma)}$. In other terms, α and β provide the necessary coefficients to find the coordinates of the marked points. One could say that the abstract tropical curve Γ only remembers the valuation information, while the formal series α and β encode the phase information. ◁

Let $v = v_r$ be the vertex adjacent to the end x_r; the tuple

$$(y_v(x_1), \dots, y_v(x_{r-1}), y_v(z_1^+), \dots, y_v(z_s^+)) \in \mathbb{R}((t))^{r-1} \times \mathbb{C}((t))^s$$

provides a system of coordinates on the moduli space of real rational marked curves, which we can restrict on the moduli space of curves tropicalizing on Γ. Notice that the choice of y_v fixes the value of some members of the tuple. The definition of α and β along with a quick induction ensures that they can be written in terms of $(y_v(q))_q$. Conversely, the formula from Proposition 5.3(v) allows us to recover $(y_v(q))_q$ from α and β. Therefore, they also provide a system of coordinates. Moreover, Proposition 5.3 describes the set of possible values of α and β, since the formula from Proposition 5.3(v) gives the values of the points to choose on $\mathbb{P}^1(\mathbb{C}((t)))$, in order to make it into a marked curve with the right tropicalization and the right formal series α and β.

We denote by $\mathscr{A}$ the space $(\mathbb{R}[\![t]\!]^\times)^{e_\mathbb{R}} \times (\mathbb{C}[\![t]\!]^\times)^{e_\mathbb{C}}$ of possible values of α. We denote by $\mathscr{B}$ the space of possible values of β satisfying the conditions of Proposition 5.3.

5.3 Space of morphisms with given tropicalization

Let $h\colon \Gamma \to N_\mathbb{R}$ be a real rational parametrized tropical curve of degree Δ. In this subsection we give an explicit description of morphisms $f\colon (C, \boldsymbol{x}, \boldsymbol{z}^\pm) \to N_{\mathbb{C}((t))^*} = N \otimes \mathbb{C}((t))^*$ of degree Δ tropicalizing to it, and for which $(C, \boldsymbol{x}, \boldsymbol{z}^\pm)$ is a smooth connected marked rational curve.

Let $f\colon C \to \mathrm{Hom}(M, \mathbb{C}((t))^*)$ be a real morphism that tropicalizes to h. By assumption, in the coordinate y_w, the morphism f takes the form

$$f(y_w) = t^{h(w)} \chi_w \prod_{i \in I_w^\infty} (y_w - y_w(q_i))^{n_i} \prod_{i \notin I_w^\infty} \left(\frac{y_w}{y_w(q_i)} - 1 \right)^{n_i} \in N_{\mathbb{C}((t))^*}.$$

Remark 5.5 The choice of normalization in the product ensures that χ_w belongs, in fact, to $N \otimes \mathbb{C}[\![t]\!]^\times$. Both products are indexed by the ends of Γ, and, although the writing does not emphasize this aspect, there are real ends and complex ends. Furthermore, there is a constant term (-1) in the second product, corresponding to the root x_r, for which $y_w(x_r) = \infty$ for any w. ◁

We now relate the expression of f in two different vertices. We assume that they are connected by an edge γ. Let

$$\phi_\gamma = \prod_{i \in I^\infty_{\mathfrak{t}(\gamma)} \setminus I^\infty_{\mathfrak{h}(\gamma)}} (y_{\mathfrak{t}(\gamma)}(q_i) - \beta_\gamma)^{n_i} \prod_{i \notin I^\infty_{\mathfrak{t}(\gamma)}} \left(1 - \frac{\beta_\gamma}{y_{\mathfrak{t}(\gamma)}(q_i)}\right)^{n_i} \in N_{\mathbb{C}((t))^*}.$$

As the coordinate $y_{\mathfrak{t}(\gamma)}(q_i)$ of a marked point q_i can be expressed in terms of the variables α and β, we can view ϕ_γ as a function $\phi_\gamma(\alpha, \beta)$.

Proposition 5.6 [3] *We have the following properties of χ_w and ϕ_γ:*

(i) *The χ_w are real: $\chi_{\sigma(w)} = \overline{\chi}_w$. In particular, if $w \in \mathrm{Fix}(\sigma)$, χ_w takes values in $\mathbb{R}[\![t]\!]^\times$.*

(ii) *The cocharacters ϕ_γ are real: $\phi_{\sigma(\gamma)} = \overline{\phi}_\gamma$. In particular, if $\gamma \in \mathrm{Fix}(\sigma)$ is a fixed edge, the cocharacter ϕ_γ takes values in $\mathbb{R}[\![t]\!]^\times$.*

(iii) *For any edge γ, let n_γ denote the slope of h on γ. One has the "transfer equation"*

$$\phi_\gamma \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}} \cdot \alpha_\gamma^{n_\gamma} = 1 \in N_{\mathbb{C}((t))^*}.$$

Conversely, if we are given a real family of $\chi_w \in N \otimes \mathbb{C}[\![t]\!]^\times$ such that Proposition 5.6 holds for any γ, then the maps defined by the formulas

$$f(y_w) = t^{h(w)} \chi_w \prod_{i \in I^\infty_w} (y_w - y_w(q_i))^{n_i} \prod_{i \notin I^\infty_w} \left(\frac{y_w}{y_w(q_i)} - 1\right)^{n_i}$$

agree and define a real morphism f tropicalizing to $h\colon \Gamma \to N_\mathbb{R}$.

If G is an abelian group, let $N_G = N \otimes G$. Let $\mathscr{X} = N^{v_\mathbb{R}}_{\mathbb{R}[\![t]\!]^\times} \times N^{v_\mathbb{C}}_{\mathbb{C}[\![t]\!]^\times}$ be the space where the tuple χ is chosen. Then the space of morphisms $f\colon (C, \boldsymbol{x}, \boldsymbol{z}^\pm) \to N_{\mathbb{C}((t))^*}$ tropicalizing to $h\colon \Gamma \to N_\mathbb{R}$ is the subset of $\mathscr{X} \times \mathscr{A} \times \mathscr{B}$ given by

$$\phi_\gamma(\alpha, \beta) \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}} \cdot \alpha_\gamma^{n_\gamma} = 1 \in N_{\mathbb{C}((t))^*} \quad \text{for all } \gamma \in \Gamma^1_b.$$

The tuples α and β deal with the tropicalization of the curve, and the tuple χ with the tropicalization of the morphism.

5.4 Evaluation map

We now use the previous description to write down the conditions that a curve whose moments are fixed must satisfy. For each $n_j \in \Delta$, let $m_j = \iota_{n_j}\omega$ be the monomial used to measure the moment of the corresponding end e_j. Notice that we could use any vector proportional to n_j, but, to avoid heavier notation, we keep n_j.

Let q_j be a real or complex marked point, and v_j be the adjacent vertex of the associated end e_j. In the coordinate y_{v_j}, the expression of the moment of the marked point q_j takes the form

$$f^*\chi^{m_j}|_{q_j} = t^{h(v_j)(m_j)}\chi_{v_j}(m_j)\prod_{i\in I^\infty_{v_j}}(\beta_{e_j}-y_{v_j}(q_i))^{\omega(n_j,n_i)}\prod_{i\notin I^\infty_{v_j}}\left(\frac{\beta_{e_j}}{y_{v_j}(q_i)}-1\right)^{\omega(n_j,n_i)}.$$

We then put

$$\varphi_j = \prod_{i\in I^\infty_{v_j}}(\beta_{e_j}-y_{v_j}(q_i))^{\omega(n_j,n_i)}\prod_{i\notin I^\infty_{v_j}}\left(\frac{\beta_{e_j}}{y_{v_j}(q_i)}-1\right)^{\omega(n_j,n_i)} \in \mathbb{C}[\![t]\!]^\times,$$

which, according to Proposition 5.3, is an invertible formal series only depending on α and β. The series is invertible since the only terms of the product having positive valuation are taken with a zero exponent.

Proposition 5.7 *The formal series φ_j are real: $\varphi_{\sigma(j)} = \overline{\varphi}_j$ for every end e_j, and, in particular, if x_j is a real marked point, then $\varphi_j \in \mathbb{R}[\![t]\!]^\times$. Moreover, φ is a function of α and β, ie it does not depend on χ.*

Proof It follows from the fact that all the quantities that intervene in the definition of φ_j are real. The second part is obvious. □

Thus, with this new notation, the evaluation map takes the form

$$f^*\chi^{m_j}|_{q_j} = t^{h(v_j)(m_j)}\chi_{v_j}(m_j)\varphi_j.$$

5.5 Correspondence theorem

Now we look at the map

$$\Theta\colon \mathscr{X}\times\mathscr{A}\times\mathscr{B} \to (N^{e_\mathbb{R}}_{\mathbb{R}[\![t]\!]^\times}\times N^{e_\mathbb{C}}_{\mathbb{C}[\![t]\!]^\times})\times(\mathbb{R}[\![t]\!]^\times)^{r-1}\times(\mathbb{C}[\![t]\!]^\times)^s,$$
$$(\chi,\alpha,\beta)\mapsto\left(\left(\phi_\gamma\cdot\frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_{\mathfrak{h}(\gamma)}}\cdot\alpha_\gamma^{n_\gamma}\right)_{\gamma\in\Gamma^1_b},(\chi_{v_j}(m_j)\varphi_j)_j\right).$$

This map is the same as in [20] but for a curve endowed with a nontrivial real involution. That is why we take every real vertex or real edge with real coefficients, and only one of every pair of complex vertices or complex edges with complex coefficients.

Lemma 5.8 [3] *The dimensions of the source and target spaces are the same.*

Let $\zeta\in(\mathbb{R}((t))^\times)^{r-1}\times(\mathbb{C}((t))^*)^s$ and $(\mu_j) = (\mathrm{val}\zeta_j)$ be families of complex and tropical moments, both assumed to be generic. We denote by $\zeta_j^\Gamma = \zeta_j t^{-\mu_j} \in \mathbb{C}[\![t]\!]^\times$ the complex moments normalized to have a zero valuation. The next proposition is straightforward to check.

Proposition 5.9 *Any family of classical curves of degree $2\Delta(s)$ that are tangent to the toric divisors at any point of $\mathscr{P}$ tropicalizes on a real parametrized tropical curve $h\colon \Gamma \to N_{\mathbb{R}}$ of degree $2\Delta(s)$ satisfying* $\mathrm{ev}(\Gamma) = \mu$, *and specializes to a tuple $(\chi, \alpha, \beta) \in \mathscr{X} \times \mathscr{A} \times \mathscr{B}$ satisfying $\Theta(\chi, \alpha, \beta) = (1, \zeta^{\Gamma})$.*

Remark 5.10 As the constraints have been chosen generically, the plane tropical curve $h(\Gamma)$ is trivalent. Moreover, due to the tangency with the toric divisors, the tropical curve can in fact be written as twice a tropical curve having degree $\Delta(s)$ instead (ie take a tropical curve of degree $\Delta(s)$ and multiply the weight of edge by 2).

According to [11, Proposition 4.3], the tropical curve has a unique parametrization of degree $\Delta(s)$. The space $\mathscr{X} \times \mathscr{A} \times \mathscr{B}$ depends on the choice of the parametrized tropical curve (Γ, h). ◁

Let $h_0\colon \Gamma_0 \to N_{\mathbb{R}}$ be a parametrized tropical curve of degree $\Delta(s)$ with $\mathrm{ev}(\Gamma_0) = \mu$. We denote by $2\Gamma_0$ the curve where all the weights have been doubled. To prove the correspondence, we need to find the classical curves tropicalizing on $2\Gamma_0$ and having tangencies at the points of $\mathscr{P}$. Such a curve corresponds to a point in the moduli space $\mathscr{X} \times \mathscr{A} \times \mathscr{B}$. Finding these curves thus amounts to solving for (χ, α, β) the equation $\Theta(\chi, \alpha, \beta) = (1, \zeta^{\Gamma})$. We say that $(\chi_0, \alpha_0, \beta_0)$ is a first-order solution if $\Theta(\chi_0, \alpha_0, \beta_0) = (1, \zeta^{\Gamma}) \bmod t$.

Let $h_0\colon \Gamma_0 \to N_{\mathbb{R}}$ be a parametrized tropical curve of degree $\Delta(s)$ with $\mathrm{ev}(\Gamma_0) = \mu$. Notice that, as μ is generic, $C_{\mathrm{trop}} = h(\Gamma_0)$ is a nodal curve. The real rational curves with image C_{trop} are described in Proposition 2.2. Assume that (Γ, h) has no flat vertex. We prove in the next section that it is in fact a necessary condition for (Γ, h) to have first-order solutions. We then have the following theorem, which allows us to lift first-order solutions to true solutions, provided that the Jacobian is invertible.

Theorem 5.11 *For each real parametrized tropical curve (Γ, h) of degree Δ, obtained from a parametrized tropical curve of degree $\Delta(s)$ passing through μ, without any flat vertex, and each first-order solution* $\Theta(\chi_0, \alpha_0, \beta_0) = (1, \zeta^{\Gamma})$:

- *The Jacobian of Θ at $(\chi_0, \alpha_0, \beta_0)$ is invertible at first order.*
- *There is a unique lift of $(\chi_0, \alpha_0, \beta_0)$ to a true solution (χ, α, β) in $\mathscr{X} \times \mathscr{A} \times \mathscr{B}$.*

Remark 5.12 Most of the proof consists in showing that the Jacobian is indeed invertible for all the considered first-order solutions. ◁

Proof Let $h\colon \Gamma \to N_{\mathbb{R}}$ be one of the above real parametrized tropical curves, meaning that $h(\Gamma)$ is a plane tropical curve whose parametrization $h_0\colon \Gamma_0 \to N_{\mathbb{R}}$ as a curve of degree $\Delta(s)$ satisfies $\mathrm{ev}(\Gamma_0) = \mu$, and that (Γ, h) has no flat vertex. Let $(\chi_0, \alpha_0, \beta_0)$ be a first-order solution. To lift it to a true solution for $t \neq 0$, we use the multivariable version of Hensel's lemma over the ring of formal series. See for instance [6, Theorem 3.3]. To apply the multivariable version of Hensel's lemma, we just need to prove the invertibility of the Jacobian at $(\chi_0, \alpha_0, \beta_0)$. In particular, the second statement follows from the first, which we now show by induction. For simplicity, and because of the multiplicative nature of the map Θ, we use logarithmic coordinates for every variable except β. It means that we look at $\log\Theta$, depending on the new variables $(\log\chi, \log\alpha, \beta)$. Each time, the logarithm is taken coordinate by coordinate. Hence, if γ is an edge, $\log\alpha_\gamma$ and β_γ are scalars, while, if w is a real vertex, $\log\chi_w \in N_{\mathbb{R}}$.

To compute the Jacobian relative to coordinates $(\log\chi, \log\alpha, \beta)$ at $t=0$, we can first put $t=0$. Thus, for every bounded edge $\gamma\in\Gamma_b^1$,

$$\phi_\gamma|_{t=0} = \prod_{\substack{\gamma'\neq\gamma \\ \mathfrak{t}(\gamma')=\mathfrak{t}(\gamma)}} (\beta_{\gamma'}|_{t=0}-\beta_\gamma|_{t=0})^{n_{\gamma'}} \in N_{\mathbb{R}^*},$$

and, for every end e_j,

$$\varphi_j|_{t=0} = \pm \prod_{\substack{\gamma'\neq e_j \\ \mathfrak{t}(\gamma')=v_j}} (\beta_{e_j}|_{t=0}-\beta_{\gamma'}|_{t=0})^{\omega(n_j,n_{\gamma'})} \in \mathbb{C}^*.$$

Notice that, at first order, ϕ_γ depends only on β and not α. For convenience of notation, all the following computations are taken at the first order, and we drop "$|_{t=0}$" out of the notation.

For a bounded edge $\gamma\in\Gamma_b^1$, let $N_\gamma = \phi_\gamma\cdot(\chi_{\mathfrak{t}(\gamma)}/\chi_{\mathfrak{h}(\gamma)})\cdot\alpha_\gamma^{n_\gamma} \in N_{\mathbb{K}[\![t]\!]^\times}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ according to whether or not $\gamma\in\mathrm{Fix}(\sigma)$. Similarly, let $X_j = \chi_{v_j}(m_j)\varphi_j \in \mathbb{R}[\![t]\!]^\times$ for a real end x_j, and $Z_j = \chi_{v_j}(m_j)\varphi_j \in \mathbb{C}[\![t]\!]^\times$ for a complex end $z_j^\pm$. The variables N_γ, X_j and Z_j index the lines of the Jacobian matrix $\partial\log\Theta/\partial(\log\chi,\log\alpha,\beta)$. Thus,

$$\log N_\gamma = \sum_{\substack{\gamma'\neq\gamma \\ \mathfrak{t}(\gamma')=\mathfrak{t}(\gamma)}} n_{\gamma'}\log(\beta_{\gamma'}-\beta_\gamma) + \log\chi_{\mathfrak{t}(\gamma)} - \log\chi_{\mathfrak{h}(\gamma)} + n_\gamma\log\alpha_\gamma \in N_{\mathbb{R}},$$

$$\log X_j = \sum_{\substack{\gamma'\neq e_j \\ \mathfrak{t}(\gamma')=v_j}} \omega(n_j,n_{\gamma'})\log(\beta_{e_j}-\beta_{\gamma'}) + \log\chi_{v_j}(m_j) \in \mathbb{R},$$

$$\log Z_j = \sum_{\substack{\gamma'\neq e_j \\ \mathfrak{t}(\gamma')=v_j}} \omega(n_j,n_{\gamma'})\log(\beta_{e_j}-\beta_{\gamma'}) + \log\chi_{v_j}(m_j) \in \mathbb{C}.$$

Notice that the vertices are either trivalent, quadrivalent including a pair of exchanged edges, or pentavalent including two pairs of exchanged edges. This means that the sum over γ' is one constant term if $\mathfrak{t}(\gamma)$ is trivalent, contains one or two terms depending on a real parameter β if $\mathfrak{t}(\gamma)$ is quadrivalent, and two or three terms depending on a complex parameter β if it is pentavalent, as depicted below. The following lemmas are used to prove by induction that the Jacobian is invertible.

Lemma 5.13 *Let V be a real trivalent vertex adjacent to two real ends.*

- *If V is the only vertex, the Jacobian is invertible,*
- *If not, the invertibility of the Jacobian reduces to the invertibility of the Jacobian for the curve where V is removed and the unique bounded edge adjacent to V is replaced with an end of the same direction.*

Proof Let γ be the edge with $\mathfrak{h}(\gamma)=V$. We assume that γ is a bounded edge. The matrix has the form

$$\operatorname{Jac}\Theta = \begin{array}{c|ccc|cc|c|}
\multicolumn{4}{c}{} & \multicolumn{2}{c}{\chi_V} & \multicolumn{1}{c}{\alpha_\gamma} \\ \cline{5-7}
\multicolumn{1}{c}{} & & & & 0 & 0 & 0 \\
\multicolumn{1}{c}{} & & * & & \vdots & \vdots & \vdots \\
\multicolumn{1}{c}{} & & & & 0 & 0 & 0 \\ \cline{2-7}
N_\gamma & * & \cdots & * & -1 & 0 & n_\gamma \\
 & * & \cdots & * & 0 & -1 & \\ \cline{2-7}
X_0 & 0 & \cdots & 0 & \multicolumn{2}{c|}{m_0} & 0 \\ \cline{2-7}
X_1 & 0 & \cdots & 0 & \multicolumn{2}{c|}{m_1} & 0 \\ \cline{2-7}
\end{array},$$

where $n_\gamma = n_0 + n_1$. Indeed, at first order we have

$$\begin{aligned}
\log N_\gamma &= \sum_{\substack{\gamma'\neq\gamma \\ \mathfrak{t}(\gamma')=\mathfrak{t}(\gamma)}} n_{\gamma'} \log(\beta_{\gamma'}-\beta_\gamma) + \log\chi_{\mathfrak{t}(\gamma)} - \log\chi_V + n_\gamma \log\alpha_\gamma \in N_{\mathbb{R}}, \\
\log X_0 &= \log\chi_V(m_0), \\
\log X_1 &= \log\chi_V(m_1).
\end{aligned}$$

Therefore, we have the partial derivatives

$$\frac{\partial \log N_\gamma}{\partial \log\chi_V} = -I_2, \quad \frac{\partial \log N_\gamma}{\partial \log\alpha_\gamma} = n_\gamma \qquad \text{and} \qquad \frac{\partial \log X_0}{\partial \log\chi_V} = m_0, \quad \frac{\partial \log X_1}{\partial \log\chi_V} = m_1.$$

By expanding with respect to the last two rows, since (m_0, m_1) are free, we are left with the determinant

$$\begin{array}{c|ccc|c|}
\multicolumn{4}{c}{} & \multicolumn{1}{c}{\alpha_\gamma} \\ \cline{5-5}
\multicolumn{1}{c}{} & & & & 0 \\
\multicolumn{1}{c}{} & & * & & \vdots \\
\multicolumn{1}{c}{} & & & & 0 \\ \cline{2-5}
N_\gamma & * & \cdots & * & n_\gamma \\
 & * & \cdots & * & \\ \cline{2-5}
\end{array}.$$

If γ was an end, we would be left with the empty matrix and we would have proven invertibility. Otherwise, the last two rows correspond to a copy of $N_{\mathbb{R}}$, and are thus given by two elements of $M_{\mathbb{R}}$, the dual of $N_{\mathbb{R}}$. Up to a change of basis, one can assume that one of these elements of $M_{\mathbb{R}}$ is $\omega(n_\gamma, -)$, which takes value 0 on n_γ. Thus, by expanding with respect to the column, noticing that $\omega(n_\gamma, N_\gamma)$ is precisely the line corresponding to the end in the curve where V is removed and γ replaced by an end, we are reduced to the invertibility of the corresponding matrix. □

Lemma 5.14 *Let V be a complex trivalent vertex adjacent to two complex ends. The invertibility of the Jacobian reduces to the invertibility of the Jacobian for the curve where V and $\sigma(V)$ are removed and the unique bounded edges adjacent to V and $\sigma(V)$ are replaced with ends of the same direction.*

Proof Let γ be the bounded edge with $\mathfrak{h}(\gamma)=V$. The matrix has the form

$$
\operatorname{Jac}\Theta=
\begin{array}{c|cccc|cc|cc|cc|}
\multicolumn{5}{c}{} & \multicolumn{2}{c}{|\chi_V|} & \multicolumn{2}{c}{\arg\chi_V} & \multicolumn{2}{c}{\alpha_\gamma} \\
\cline{6-11}
\multicolumn{5}{c|}{*} & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{0} & 0 & 0 \\
\multicolumn{5}{c|}{} & \multicolumn{2}{c}{\vdots} & \multicolumn{2}{c|}{\vdots} & \vdots & \vdots \\
\multicolumn{5}{c|}{} & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{0} & 0 & 0 \\
\hline
 & * & \cdots & \cdots & * & -1 & 0 & 0 & 0 & & \\
|N_\gamma| & * & \cdots & \cdots & * & 0 & -1 & 0 & 0 & n_\gamma & 0 \\
 & * & \cdots & \cdots & * & 0 & 0 & -1 & 0 & & \\
\arg N_\gamma & * & \cdots & \cdots & * & 0 & 0 & 0 & -1 & 0 & n_\gamma \\
\hline
Z_0 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c}{m_0} & \multicolumn{2}{c|}{0} & 0 & 0 \\
 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{m_0} & 0 & 0 \\
\hline
Z_1 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c}{m_1} & \multicolumn{2}{c|}{0} & 0 & 0 \\
 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c}{0} & \multicolumn{2}{c|}{m_1} & 0 & 0 \\
\hline
\end{array}.
$$

Indeed, the coordinates are now complex: α_γ is a complex scalar, χ_V has complex values, $N_\gamma\in N_{\mathbb{C}^*}$ and $Z_0,Z_1\in\mathbb{C}^*$. To get real coordinates, we split them into real and imaginary parts. But, as we take logarithmic coordinates, the real/imaginary decomposition emphasizes the modulus/argument decomposition of the coordinates. One has

$$\log\alpha_\gamma=\log|\alpha_\gamma|+i\arg\alpha_\gamma,\quad \log\chi_V=\log|\chi_V|+i\arg\chi_V.$$

Therefore,

$$
\begin{aligned}
\log N_\gamma &= \sum_{\substack{\gamma'\neq\gamma\\ \mathfrak{t}(\gamma')=\mathfrak{t}(\gamma)}} n_{\gamma'}\log(\beta_{\gamma'}-\beta_\gamma)+\log\chi_{\mathfrak{t}(\gamma)}-\log|\chi_V|-i\arg\chi_V+n_\gamma\log|\alpha_\gamma|+in_\gamma\arg\alpha_\gamma,\\
\log Z_j &= \log|\chi_V|(m_j)+i\arg\chi_V(m_j)+c.
\end{aligned}
$$

Hence, we get the announced Jacobian matrix. Then the computation is entirely similar, except it is done twice: expand with respect to the last four rows, do a change of basis in the new last four rows and expand with respect to the last two columns, thus reducing to the Jacobian of the announced curve. □

Lemma 5.15 *Let V be a real quadrivalent vertex adjacent to a real end and two exchanged complex ends.*

- *If V is the only vertex, the Jacobian is invertible,*
- *If not, the invertibility of the Jacobian reduces to the invertibility of the Jacobian for the curve where V is removed and the unique bounded edge adjacent to V is replaced with an unbounded edge of the same direction.*

Proof If the vertex V is adjacent to two complex ends (directed by n_0), and to a real end (directed by n_1), let β_1 be the β coordinate associated to the real end. Let γ be the edge with $\mathfrak{h}(\gamma)=V$, thus directed by $2n_0+n_1$. The determinant takes the form

$$
\operatorname{Jac}\Theta=\begin{array}{c}\\ \\ \\ N_\gamma\\ \\ Z_0\\ \\ X_1\end{array}
\begin{array}{c|cc|c|c|}
\multicolumn{1}{c}{} & \multicolumn{2}{c}{\chi_V} & \multicolumn{1}{c}{\alpha_\gamma} & \multicolumn{1}{c}{\beta_1}\\
\cline{2-5}
 & 0 & 0 & 0 & 0\\
* & \vdots & \vdots & \vdots & \vdots\\
 & 0 & 0 & 0 & 0\\
\hline
\multicolumn{1}{|c|}{* \ \cdots \ *} & -1 & 0 & & 0\\
\multicolumn{1}{|c|}{* \ \cdots \ *} & 0 & -1 & n_\gamma & 0\\
\hline
\multicolumn{1}{|c|}{0 \ \cdots \ 0} & \multicolumn{2}{c|}{m_0} & 0 & \beta_1\\
\multicolumn{1}{|c|}{0 \ \cdots \ 0} & \multicolumn{2}{c|}{0} & 0 & 1\\
\hline
\multicolumn{1}{|c|}{0 \ \cdots \ 0} & \multicolumn{2}{c|}{m_1} & 0 & -2\beta_1\\
\hline
\end{array}\,.
$$

Indeed, we have

$$
\begin{aligned}
\log N_\gamma &= \cdots+\log\chi_{\mathfrak{t}(\gamma)}-\log\chi_V+n_\gamma\log\alpha_\gamma,\\
\log Z_0 &= \omega(n_0,n_1)\log(i-\beta_1)+\log\chi_V(m_0),\\
\log X_1 &= \omega(n_1,n_0)\log(\beta_1-i)+\omega(n_1,n_0)\log(\beta_1+i)+\log\chi_V(m_1)\\
&= \omega(n_1,n_0)\log(\beta_1^2+1)+\log\chi_V(m_1).
\end{aligned}
$$

Hence, we recover the partial derivatives $\partial\log N_\gamma/\partial\log\chi_V$, $\partial\log N_\gamma/\partial\log\alpha_\gamma$, $\partial\log Z_0/\partial\log\chi_V$ and $\partial\log X_1/\partial\log\chi_V$, but we also get the new partial derivatives

$$
\frac{\partial\log X_1}{\partial\beta_1}=\omega(n_1,n_0)\frac{2\beta_1}{\beta_1^2+1}\quad\text{and}\quad\frac{\partial\log Z_0}{\partial\beta_1}=\omega(n_0,n_1)\frac{-1}{i-\beta_1}=\frac{\omega(n_0,n_1)}{\beta_1^2+1}(\beta_1+i).
$$

Notice that we dropped out the nonzero constant factor $\omega(n_0,n_1)/(\beta_1^2+1)$ in the last column. Then one can expand with respect to the penultimate row, and then the second resulting last rows. We get the empty determinant if γ is an end, thus proving invertibility. Otherwise, as before, we make a change of basis in the rows associated to N_γ and expand with respect to the last column, thus reducing to the associated determinant. □

Lemma 5.16 *Let V be a real pentavalent vertex adjacent to two real ends.*

- *If V is the only vertex, the Jacobian is invertible,*
- *If not, the invertibility of the Jacobian reduces to the invertibility of the Jacobian for the curve where V is removed and the unique bounded edge adjacent to V is replaced with an unbounded edge of the same direction.*

Proof Let the adjacent ends be labeled 0 and 1, and let γ be the unique edge with $\mathfrak{h}(\gamma)=V$. In the coordinate y_V, the ends associated to the label 0 are $\pm i$, and the ends associated to 1 are β and $\overline{\beta}$. The evaluation matrix takes the form

$$
\operatorname{Jac}\Theta = \begin{array}{c|cccc|cc|c|cc|}
 & & & & & \multicolumn{2}{c|}{\chi_V} & \alpha & \multicolumn{2}{c|}{\beta} \\
\cline{6-10}
 & & & * & & \multicolumn{2}{c|}{\begin{matrix}0\\ \vdots \\ 0\end{matrix}} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \multicolumn{2}{c|}{\begin{matrix}0\\ \vdots \\ 0\end{matrix}} \\
\hline
N_\gamma & * & \cdots & \cdots & * & -1 & 0 & n_\gamma & 0 & 0 \\
 & * & \cdots & \cdots & * & 0 & -1 & & 0 & 0 \\
\hline
Z_0 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c|}{m_0} & 0 & \multicolumn{2}{c|}{\partial\log Z_0/\partial\beta} \\
 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c|}{0} & 0 & & \\
\hline
Z_1 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c|}{m_1} & 0 & \multicolumn{2}{c|}{\partial\log Z_1/\partial\beta} \\
 & 0 & \cdots & \cdots & 0 & \multicolumn{2}{c|}{0} & 0 & & \\
\hline
\end{array},
$$

where $\partial\log Z_0/\partial\beta$ and $\partial\log Z_1/\partial\beta$ are some matrices in $\mathcal{M}_2(\mathbb{R})$. Indeed, we still have the previously computed partial derivative, but we have the new partial derivatives depending on β to compute. First, since

$$\log Z_1 = \omega(n_1,n_0)\log(\beta-i) + \omega(n_1,n_0)\log(\beta+i) + \cdots = \omega(n_1,n_0)\log(\beta^2+1) + \cdots,$$

which is a function of the complex variable β, we have

$$\frac{\partial\log Z_1}{\partial\beta} = \omega(n_1,n_0)\frac{2\beta}{\beta^2+1} \in \mathbb{C} \subset \mathcal{M}_2(\mathbb{R}),$$

where the scalar complex represents the multiplication from $\mathbb{C}$ to $\mathbb{C}$:

$$\mathbb{C} \to \mathcal{M}_2(\mathbb{R}), \quad r + is \mapsto \begin{pmatrix} r & -s \\ s & r \end{pmatrix}.$$

Next,

$$\log Z_0 = \omega(n_0,n_1)\log(i-\beta) + \omega(n_0,n_1)\log(i-\overline{\beta}) + \cdots,$$

which depends on β but also $\overline{\beta}$. Same as before, the derivative of the first term is $\omega(n_0,n_1)/(\beta-i)$, while the second, being precomposed by the conjugation, whose matrix is $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, has derivative

$$\omega(n_0,n_1)\frac{1}{\overline{\beta}-i} \circ J \in \mathcal{M}_2(\mathbb{R}),$$

where $\circ$ is the product of matrices. The total partial derivative is

$$\frac{\partial\log Z_0}{\partial\beta} = \omega(n_0,n_1)\left(\frac{1}{\beta-i} + \frac{1}{\overline{\beta}-i} \circ J\right) \in \mathcal{M}_2(\mathbb{R}).$$

We now do our usual expanding in the determinant. Provided that the determinant consisting of the second rows of $\partial\log Z_0/\partial\beta$ and $\partial\log Z_1/\partial\beta$ is nonzero, we can expand with respect to the last and second-to-last rows. Then, as in all previous cases, we can expand with respect to the last two rows since (m_0,m_1) is

Figure 4: Vertices adjacent to two real ends (left) or a real end and two complex ends (right).

free, and either get the empty determinant, or do the change of basis in the rows corresponding to N_γ and reduce to a smaller determinant. Thus, it suffices to prove the invertibility of the 2×2 determinant consisting of the second row of both 2×2 matrices. Identifying the second row with $\mathbb{C}$ by the map

$$\mathbb{C} \to \mathcal{M}_{1,2}(\mathbb{R}), \quad r + is \mapsto (s \ \ r),$$

the second row of $\partial \log Z_1/\partial\beta$ is $2\beta/(\beta^2+1)$, while the second row of $\partial \log Z_0/\partial\beta$ is $1/(\beta-i)-1/(\beta+i)$. Indeed,

$$\frac{1}{\overline{\beta}-i} \circ J = J \circ \frac{1}{\beta+i},$$

and multiplication by J on the left just changes the sign of the second row. Thus, we get $2i/(\beta^2+1)$. Therefore, up to the multiplication by $2/(\beta^2+1)$, we have to show that the family (β, i) is free over $\mathbb{R}$, which is the case if β is not purely imaginary. If β was purely imaginary, the moments of the complex ends would be real, and this is excluded by genericity. See Section 6 for more details on why β cannot be purely imaginary. □

The previous lemmas prove that the Jacobian is always invertible. Therefore, this finishes the proof of the correspondence theorem. □

6 Proof of Theorem 4.8

Now that we have a correspondence theorem that allows us to lift every first-order solution (provided the coordinates β are not purely imaginary), we only need to count the first-order solutions for any rational tropical curve solution to the tropical enumerative problem. To do this, we first solve the enumerative problems associated to the vertices of a real rational tropical curve. Then we use these local resolutions to solve inductively the system $\Theta = (1, \zeta^\Gamma)$ at first order. There are three local enumerative problems: one for trivalent vertices, one for quadrivalent vertices, and one for pentavalent vertices. Each time, we do the refined count of real oriented curves having fixed moments, by doing the same count but over the curves with fixed primitive moments, which explains why the computation is rather long.

6.1 Local enumerative problems

6.1.1 Line problem: trivalent vertex We consider curves of degree

$$\Delta(n_1, n_2, n_3) = \{n_1, n_2, n_3\},$$

where $n_1 + n_2 + n_3 = 0$, to which are assigned triangles of lattice area $m_\Delta = |\det(n_1, n_2)|$. A curve of this degree is the image of a line under a monomial map followed by a multiplicative translation. It has a parametrization of the form

$$\varphi : t \mapsto \chi \cdot t^{n_1}(t-1)^{n_2}.$$

Let $\mathbb{C}E_j$ be the toric divisor associated to the vector n_j. Let l_j be the integral length of n_j. We here recall the resolution of two classic enumerative problems:

Problem 6.1 Take one point on $\mathbb{C}E_1$ and one on $\mathbb{C}E_2$.

- How many complex curves of degree $\Delta(n_1, n_2, n_3)$ pass through the two fixed points on the toric divisors $\mathbb{C}E_1$ and $\mathbb{C}E_3$?
- What if the points are real and the curve is asked to be real?

The first proposition is proved in [11].

Proposition 6.2 (Mikhalkin [11]) *Let $p_1 \in \mathbb{C}E_1$ and $p_2 \in \mathbb{C}E_2$ be two complex points. There are $m_\Delta / l_1 l_2$ curves of degree $\Delta(n_1, n_2, n_3)$ passing through p_1 and p_2.*

We now turn our focus to a slight variation on the same problem: we look for curves having a fixed moment. The difference is that the moment is not measured with the primitive vector n_j / l_j but with the nonprimitive vector n_j. Geometrically, this means that we count curves passing by several pairs of points at the same time, which differ by the action of an $l_j{}^{\text{th}}$ root of unity.

Proposition 6.3 *Let $\mu_1, \mu_2 \in \mathbb{C}^*$. There are m_Δ curves of degree $\Delta(n_1, n_2, n_3)$ having moments μ_1 and μ_2. Moreover, the curves are equally distributed among the possible intersection points with $\mathbb{C}E_3$.*

Proof As the curve is uniquely determined by the cocharacter χ, the problem amounts to solving

$$\chi(m_1) = \mu_1, \quad \chi(m_2) = \mu_2,$$

where $\chi(m)$ denotes the evaluation of $\chi \in N_{\mathbb{C}^*}$ at a monomial $m \in M$. Taking the logarithm solves uniquely for the modulus. Taking the argument leads to a system for $\arg \chi \in \operatorname{Hom}(M, \mathbb{R}/2\pi\mathbb{Z})$, knowing it on the lattice $\mathbb{Z}m_1 + \mathbb{Z}m_2$, which is a sublattice of index m_Δ. This leads to $m_\Delta = |\det(n_1, n_2)|$ solutions: Let (e_1, e_2) be a basis of M such that $(d_1 e_1, d_2 e_2)$ is a basis of $m_1\mathbb{Z} + m_2\mathbb{Z}$, with $d_1 d_2 = m_\Delta$. We have d_1 possible values for $\chi(e_1)$ and d_2 for $\chi(e_2)$.

The value of $m_3 = -(m_1 + m_2)$ must satisfy $\chi(m_3) = 1/\mu_1\mu_2$. By choosing $e_1 = m_3/l_3$, it thus can take l_3 values. Thus there are each time m_Δ / l_3 for each value of $\chi(m_3/l_3)$. □

Finally, when the points are real and the curve is real, we have the following proposition.

Proposition 6.4 *Let $p_1 \in \mathbb{R}E_1$ and $p_2 \in \mathbb{R}E_2$ be two points with respective coordinates μ_1 and μ_2. There are four real curves passing through $\pm p_1$ an $\pm p_2$. This leads to eight oriented real curves, and their refined count is*

$$4(q^{m_\Delta/2} - q^{-m_\Delta/2}) = 4\left[\tfrac{1}{2}m_\Delta\right].$$

Proof We solve the system

$$\chi\left(\frac{m_1}{l_1}\right) = \pm\mu_1, \quad \chi\left(\frac{m_2}{l_2}\right) = \pm\mu_2$$

for $\chi \in N_{\mathbb{R}^*} \simeq N_{\mathbb{R}} \times N_{\{\pm 1\}}$. The isomorphism is given by the logarithm of the absolute value and the sign. We solve uniquely for the absolute value. Concerning the choice of the sign, any choice of sign for χ yields a solution since we do not care about the sign in front of μ_1 and μ_2. Therefore, there are four solutions, leading to eight oriented real curves when accounting for both orientations of each of them. Their logarithmic rotation number and logarithmic area are computed with Lemma 3.2. □

6.1.2 Parabola problem: quadrivalent vertex

Parametrization and sign We consider a real oriented curve of degree

$$\Delta^{\mathrm{par}}(u_1,u_2,u_3) = \{(0, u_3-u_2)^2, (u_1, 2u_2), (-u_1, -2u_3)\},$$

which is associated to a triangle $P^{\mathrm{par}}_\Delta(u_1,u_2,u_3)$ of lattice area m_Δ depicted in Figure 5. The sides of the triangle are denoted by E_1, E_2 and E_3. The associated toric divisors in the associated toric surface $\mathbb{C}\Delta(u_1,u_2,u_3)$ are denoted by $\mathbb{C}E_j$. We denote by l_j the integral length of the corresponding vectors in $\Delta^{\mathrm{par}}(u_1,u_2,u_3)$. Thus, $l_1 = u_3-u_2$ and $l_3 = \gcd(u_1, 2u_2)$. A curve having such a degree has a parametrization of the form

$$t \mapsto (a(t-c)^{u_1}, b(t-c)^{2u_2}(t^2+1)^{u_3-u_2}),$$

where $a, b \in \mathbb{R}^*$ and $c \in \mathbb{R}$ are real parameters. The coordinate of parametrization is chosen so that the intersection points with $\mathbb{C}E_1$ have coordinate $\pm i$ and the intersection point with $\mathbb{C}E_2$ has coordinate ∞. There are two such coordinates. The choice of i determines an orientation of the curve and fixes the coordinate. Such a curve is the image of a curve of degree $\Delta^{\mathrm{par}} = \{(1,1), (-1,1), (0,-1)^2\}$, having parametrization

$$t \mapsto \left(t-c, \frac{t^2+1}{t-c}\right),$$

by a monomial map followed by a multiplicative translation. The matrices of the monomial map between the lattices of cocharacters and characters are, respectively,

$$A = \begin{pmatrix} u_1 & 0 \\ u_2+u_3 & u_3-u_2 \end{pmatrix} : N_0 \to N \quad \text{and} \quad A^T = \begin{pmatrix} u_1 & u_2+u_3 \\ 0 & u_3-u_2 \end{pmatrix} : M \to M_0.$$

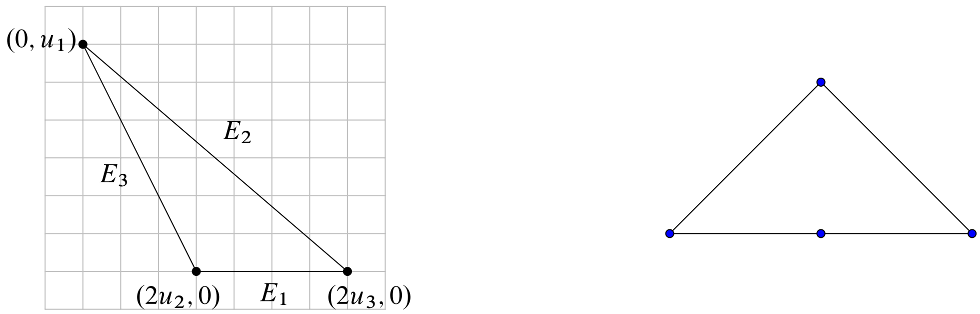

Figure 5: The polygon $P_\Delta^{\mathrm{par}}(u_1,u_2,u_3)$ and the parabola triangle used for its parametrization.

Proposition 6.5 *The curves having parametrization*

$$t \mapsto (a(t-c)^{u_1}, b(t-c)^{2u_2}(t^2+1)^{u_3-u_2})$$

have logarithmic rotation index 0 *and log-area*

$$\pi \cdot \tfrac{1}{2} m_\Delta (2\operatorname{arccot}(-c) - \pi).$$

Proof The log-area is computed using Lemma 3.2 and the computation of the log-area of a parabola. As the monomial map acts as the linear map A on the logarithmic side, the logarithmic rotation number of the image curve is equal to the one of the curve in the domain multiplied by the sign of the determinant of the monomial map. But, as a parabola has logarithmic rotation index 0, it stays 0. □

Enumerative problems We consider several similar enumerative problems dealing with parabolas. First, we look for real oriented curves of degree $\Delta^{\mathrm{par}}(u_1,u_2,u_3)$ having fixed intersection points with the toric boundary of $\mathbb{C}\Delta^{\mathrm{par}}(u_1,u_2,u_3)$. It means their primitive moments are fixed (see Definition 2.3). Let $\mu_2 \in \mathbb{R}^*$ be the coordinate of a point on $\mathbb{C}E_3$, and $\mu_1 e^{\pm i\pi\theta} \in \mathbb{C}^*$, with $\mu_1 > 0$ and $0 < \theta < 1$, be the coordinate of a pair of complex conjugate points on $\mathbb{C}E_1$. The coordinate on the toric divisor $\mathbb{C}E_1$ is the monomial z, and the coordinate on $\mathbb{C}E_3$ is the monomial $z^{2u_2/l_3}w^{-u_1/l_3}$. These are in fact primitive lattice vectors directing the side of the triangle $P_\Delta^{\mathrm{par}}(u_1,u_2,u_3)$ from Figure 5.

Problem 6.6 How many oriented real rational curves of degree $\Delta^{\mathrm{par}}(u_1,u_2,u_3)$ pass through the pair of complex conjugate points on $\mathbb{C}E_1$ and one of the opposite real points on $\mathbb{C}E_3$? Their count is refined by their log-area and counted with sign.

Using a parametrization for the oriented real rational curve, the enumerative problem amounts to solving the system

$$a(i-c)^{u_1} = \mu_1 e^{\pm i\pi\theta}, \quad a^{2u_2/l_3} b^{-u_1/l_3} (c^2+1)^{-u_1(u_3-u_2)/l_3} = \pm\mu_2,$$

where the $\pm$ in front of θ and μ_2 can be arbitrary. We get the following proposition.

Proposition 6.7 *For $0<\theta<1$, the refined count of real oriented curves solution to the enumerative Problem 6.6 is*

$$2(q^{l_1(2\theta-1)}+q^{-l_1(2\theta-1)})\frac{q^{m_\Delta/2}-q^{-m_\Delta/2}}{q^{l_1}-q^{-l_1}}=2[l_1(2\theta-1)]_+\frac{\left[\frac{1}{2}m_\Delta\right]}{[l_1]}.$$

Proof We solve successively by taking $e^{i\pi\theta}$ and $e^{-i\pi\theta}$ in the first equation.

The argument mod π of the first equation gives us

$$u_1\arg(i-c)\equiv\pi\theta \mod \pi.$$

This is equivalent to

$$\operatorname{arccot}(-c)\equiv\frac{\theta+k}{u_1}\pi \mod \pi \quad \text{with } 0\leqslant k<u_1.$$

As arccot has image $]0;\pi[$, for each k we get a unique solution $c_k=-\cot((\theta+k)\pi/u_1)$ provided that $0<\theta+k<u_1$. The first equation then uniquely solves for $a=\mu_1 e^{i\pi\theta}/(i-c_k)^{u_1}\in\mathbb{R}^*$. Finally, the second equation solves for b up to sign. In total, we get $2u_1$ oriented curves for the solution with $e^{i\pi\theta}$.

The log-area of each oriented curve is given by $\frac{1}{2}m_\Delta(2\pi\operatorname{arccot}(-c)-\pi^2)$. As $m_\Delta=2l_1u_1$, this is equal to

$$l_1(2(\theta+k)-u_1)\pi^2 \quad \text{for } 0\leqslant k<u_1.$$

The count of real oriented curves passing through the fixed points and refined by the value of the log-area is thus

$$2\sum_{k=0}^{h_1-1}q^{l_1(2\theta+2k-u_1)}=2q^{l_1(2\theta-1)}\frac{q^{m_\Delta/2}-q^{-m_\Delta/2}}{q^{l_1}-q^{-l_1}}=2q^{l_1(2\theta-1)}\frac{\left[\frac{1}{2}m_\Delta\right]}{[l_1]}.$$

For the system with $e^{-i\pi\theta}$, the solution is identical, by replacing θ with $-\theta$, except for a change in the possible values for k, since we must still have $0<-\theta+k<u_1$. As a matter of fact, this is equivalent to replacing θ by $1-\theta$ in the previous formula. Adding both contributions, and considering both signs of μ_1, we get the desired refined count. □

We can now slightly modify the problem by also allowing the curves to pass through the opposite pair of complex points.

Problem 6.8 How many real rational curves of degree $\Delta^{\mathrm{par}}(u_1,u_2,u_3)$ pass through one of the opposite pairs of complex points on $\mathbb{C}E_1$, and one of the opposite real points on $\mathbb{C}E_2$?

Corollary 6.9 *For $0<\theta<1$, the refined count of real oriented curves passing through the symmetric configuration is equal to*

$$4[l_1(2\theta-1)]_+\frac{\left[\frac{1}{2}m_\Delta\right]}{[l_1]}.$$

Proof Just add the counts for the values θ and $1-\theta$, which happen to be equal. □

Finally, we solve the final problem, where we impose the moment of the curve rather than its primitive moment, which would mean imposing the position of its intersection point.

Problem 6.10 Let $\mu_1 = e^{\pm i\pi\theta} \in \mathbb{C}^*$ and $\mu_2 \in \mathbb{R}^*$. What is the refined count of real oriented curves of degree $\Delta^{\mathrm{par}}(u_1, u_2, u_3)$ having moments $\pm\mu_1$ on $\mathbb{C}E_1$ and $\pm\mu_2$ on $\mathbb{C}E_2$?

The solution is a direct corollary of the solution of the previous problem by taking the $l_1{}^{\text{th}}$ root of the first equation.

Corollary 6.11 *The refined number of oriented real rational curves of degree $\Delta^{\mathrm{par}}(u_1, u_2, u_3)$ with moments $\pm e^{\pm i\pi\theta} \in \mathbb{C}^*$ on $\mathbb{C}E_1$ and $\pm\mu_2 \in \mathbb{R}^*$ on $\mathbb{C}E_2$ is*

$$4[2\theta - 1]_+ \frac{\left[\frac{1}{2}m_\Delta\right]}{[1]}.$$

Proof Let $l = l_1$. The moment on $\mathbb{C}E_1$ is known. Therefore, the primitive moment is its l^{th} root. So it can take the values

$$\pm\frac{\theta+k}{l}\pi, \quad \pm\frac{1-\theta+k}{l}\pi \quad \text{for } 0 \leqslant k < l.$$

We have $4l$ of these values: l roots for each of the four arguments $\pm\theta\pi$ and $(1 \pm \theta)\pi$. Half of them belong to $]0;\pi[$ and they are obtained by taking $+$ signs in front of the fractions. These values can be grouped into packs of 4: if $\varphi\pi$ is a possible primitive moment, so are $-\varphi\pi$, $(1-\varphi)\pi$ and $(1+\varphi)\pi$. In the choices of parametrization of the values by k, this association is

$$1 - \frac{\theta+k}{l} = \frac{1-\theta+l-1-k}{l};$$

thus, $\pm(\theta+k)/l$ corresponds to $\pm(1-\theta+k')/l$, where $k+k' = l-1$. So we can take $(\theta+k)/l$ to be a positive representative of each group of four. The refined count is known to be

$$4\left[l\left(2\frac{\theta+k}{l} - 1\right)\right]_+ \frac{\left[\frac{1}{2}m_\Delta\right]}{[l]}.$$

To conclude, we just need to add these contributions and notice that

$$\begin{aligned}\sum_0^{l-1}[2\theta + 2k - l]_+ &= q^{2\theta-l}\sum_0^{l-1} q^{2k} + q^{l-2\theta}\sum_0^{l-1} q^{-2k} \\ &= q^{2\theta-l}\frac{q^{2l}-1}{q^2-1} + q^{l-2\theta}\frac{1-q^{-2l}}{1-q^{-2}} \\ &= (q^{2\theta-1} + q^{1-2\theta})\frac{q^l - q^{-l}}{q - q^{-1}} \\ &= [2\theta - 1]_+ \frac{[l]}{[1]}.\end{aligned}$$

□

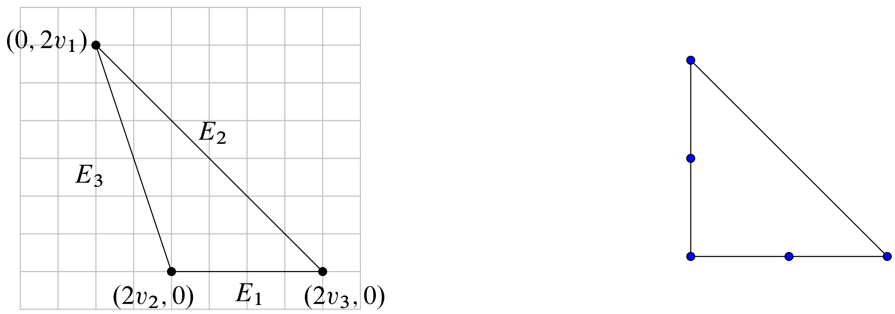

Figure 6: The polygon $P_\Delta^{\text{con}}(v_1, v_2, v_3)$ and the conic triangle used for its parametrization.

Remark 6.12 If the vectors are primitive, ie $l_1 = 1$, by making $\theta \to \frac{1}{2}$ and dividing by 4, we get the value of the refined invariant announced in Theorem 4.8:

$$R_{\Delta^{\text{par}}(u_1,u_2,u_3)} = 2\frac{q^{m_\Delta/2} - q^{-m_\Delta/2}}{q - q^{-1}} = 2\frac{\left[\frac{1}{2}\right]}{[1]} N^{\partial,\text{trop}}_{\Delta^{\text{par}}(u_1,u_2,u_3)}. \qquad \triangleleft$$

6.1.3 Tangent ellipse problem: pentavalent vertex

Parametrization and sign We consider real oriented curves of degree

$$\Delta^{\text{con}}(v_1, v_2, v_3) = \{(0, v_3 - v_2)^2, (v_1, v_2)^2, (-2v_1, -2v_3)\},$$

which is dual to the triangle $P_\Delta^{\text{con}}(v_1, v_2, v_3)$ depicted in Figure 6. We label the sides E_1, E_2 and E_3, and name the toric divisors in the toric surface accordingly. A curve having such a degree has a parametrization of the form

$$t \mapsto \big(a(t^2 - 2t\cos(x_3\pi) + 1)^{v_1}, b(t^2 - 2t\cos(x_3\pi) + 1)^{v_2}(t^2 - 2t\cos(x_1\pi) + 1)^{v_3 - v_2}\big),$$

where $a, b \in \mathbb{R}^*$ are real parameters and $0 < x_1, x_3 < 1$ are the normalized arguments of the parameters for the complex intersection points. The coordinate is chosen, as in Section 3.3.4, so that the intersection point with $\mathbb{C}E_2$ has coordinate ∞, and the intersection points with $\mathbb{C}E_1$ have coordinates of modulus 1. The choice of the orientation fixes one among the two coordinates satisfying this assumption. In this setting, $e^{i\pi x_1}$ corresponds to the coordinate of the intersection points with $\mathbb{C}E_1$, and similarly for $\mathbb{C}E_3$. The normalized arguments x_1 and x_3 cannot be equal, otherwise the curve is a double cover of a curve of smaller degree $\frac{1}{2}\Delta^{\text{con}}(v_1, v_2, v_3)$.

A curve of degree $\Delta^{\text{con}}(v_1, v_2, v_3)$ is the image of a conic in the projective plane parametrized by

$$t \mapsto (t^2 - 2t\cos(x_3\pi) + 1, t^2 - 2t\cos(x_1\pi) + 1),$$

under a monomial map followed by a translation. The monomial map between the lattices of cocharacters and characters have respective matrices

$$A = \begin{pmatrix} v_1 & 0 \\ v_2 & v_3 - v_2 \end{pmatrix} : N_0 \to N \quad \text{and} \quad A^T = \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 - v_2 \end{pmatrix} : M \to M_0.$$

The monomial map is a ramified cover of degree $v_1(v_3-v_2)=\frac{1}{4}m_\Delta$, which is ramified over the toric boundary. The degree of the covering map restricted to a $\mathbb{C}E_j$ divisor is equal to the total degree divided by the integral length of the corresponding size in the associated triangle. Let l_j be the integral length of the vectors in $\Delta^{\mathrm{con}}(v_1,v_2,v_3)$ associated to the divisor, and h_j its relative height, so that $h_j l_j = v_1(v_3-v_2)$. In our case, $l_3=\gcd(v_1,v_2)$ and $l_1=v_3-v_2$.

The coordinate on $\mathbb{C}E_j$ is one of the two primitive monomials directing the side E_j. For instance, the coordinate on $\mathbb{C}E_3$ is $z^{v_2/l_3}w^{-v_1/l_3}$. Its pullback by the monomial map is $w^{-v_1(v_3-v_2)/l_3}=w^{-h_3}$. The degree of the cover restricted to the toric divisor $\mathbb{C}E_j$ is h_j.

Just as in the parabola case, the logarithmic rotation number of a curve and its image by a monomial map are equal up to sign. Then the following proposition computes the logarithmic rotation number of curves of degree $\Delta^{\mathrm{con}}(v_1,v_2,v_3)$.

Lemma 6.13 *The logarithmic rotation number of*

$$t\mapsto (t^2-2t\cos(x_3\pi)+1,\, t^2-2t\cos(x_1\pi)+1),$$

is ± 1, *equal to the sign of* x_3-x_1.

Proof The logarithmic map thus takes the form

$$t\mapsto \bigl(\log(t^2-2t\cos(x_3\pi)+1),\log(t^2-2t\cos(x_1\pi)+1)\bigr).$$

We can now express the logarithmic Gauss map as

$$\gamma\colon t\mapsto \left[\frac{t-\cos(x_3\pi)}{t^2-2t\cos(x_3\pi)+1} : \frac{t-\cos(x_1\pi)}{t^2-2t\cos(x_1\pi)+1}\right]\in\mathbb{R}P^1.$$

To compute its degree, we compute the signed number of preimages over a fixed direction in $\mathbb{R}P^1$. We choose the direction to be $[0:1]$. The unique solution is $t=\cos(x_3\pi)$. We need to know if the signed contribution is $+1$ or -1. To do this, we have to notice the orientation of $\mathbb{R}P^1$ given by the orientation of $\mathbb{R}^2$ is also given by the canonical orientation of $\mathbb{R}$ via the coordinate $-u/v$, where $[u:v]\in\mathbb{R}P^1$. Therefore, the sign is given by the sign of the second coordinate

$$\frac{\cos(x_3\pi)-\cos(x_1\pi)}{\cos^2(x_3\pi)-2\cos(x_3\pi)\cos(x_1\pi)+1}.$$

Finally, the logarithmic rotation number is $+1$ if $\cos(x_3\pi)<\cos(x_1\pi)$ and -1 otherwise. □

Enumerative problems We now consider the same enumerative problems as Problems 6.6, 6.8 and 6.10 but for the degree $\Delta^{\mathrm{con}}(v_1,v_2,v_3)$. First, we look for real oriented curves of degree $\Delta^{\mathrm{con}}(v_1,v_2,v_3)$ having fixed intersection with the toric boundary of $\mathbb{C}\Delta^{\mathrm{con}}(v_1,v_2,v_3)$. Up to the real torus action, we can assume that the coordinates of the chosen intersection points are of modulus 1, given by $e^{\pm i\theta_1\pi}$ on $\mathbb{C}E_1$ and $e^{\pm i\theta_3\pi}$ on $\mathbb{C}E_3$. We choose $0<\theta_j<1$.

Problem 6.14 How many oriented real rational curves have degree $\Delta^{\mathrm{con}}(v_1,v_2,v_3)$ and pass through the fixed pairs of complex conjugate points on $\mathbb{C}E_1$ and $\mathbb{C}E_3$?

Using the given parametrization of such a curve, the problem amounts to the system

$$\begin{cases} a(\cos(x_1\pi)-\cos(x_3\pi))^{h_1}e^{i\pi h_1 x_1}=e^{\pm i\pi\theta_1}, \\ a^{-m_2/l_3}b^{m_1/l_3}(\cos(x_3\pi)-\cos(x_1\pi))^{h_3}e^{i\pi h_3 x_3}=e^{\pm i\pi\theta_3}. \end{cases}$$

The two equations are obtained by evaluating the primitive monomials z and $z^{-v_2/l_3}w^{v_1/l_3}$, respectively, at $e^{i\pi x_1}$ and $e^{i\pi x_3}$ to obtain the coordinates of the intersection points with the toric boundary. Notice that $h_1=v_1$ and $h_3=v_1(v_3-v_2)/l_3$ are the degrees of the monomial ramified cover restricted to the toric divisors.

The choice of the signs in front of the θ_j corresponds to the image of $e^{i\pi x_1}$ and $e^{i\pi x_3}$ being either point in each of both pairs of complex conjugate points. In fact, counting real oriented curves is equivalent to counting real holomorphic disks with boundary in the real part of the complex manifold. The disk only passes through one point of each pair of complex conjugate points.

Proposition 6.15 *For $0<\theta_1,\theta_3<1$, the refined contribution is*:

- *If v_1/l_3 is odd*,

$$\sum_{\substack{\widehat{\theta}_j=\theta_j,1-\theta_j \\ 0\leqslant k_1<h_1 \\ 0\leqslant k_3<h_3}} \sigma_{k_1,k_3}q^{2l_3(\widehat{\theta}_3+k_3)-2l_1(\widehat{\theta}_1+k_1)},$$

  where the sum has $4h_1h_3$ terms accounting for the values $\widehat{\theta}_j$, taken equal to θ_j and $1-\theta_j$ for $j=1$ and 3. The sign σ_{k_1,k_3} is the sign of the exponent.

- *If v_1/l_3 is even*,

$$2\sum_{\substack{0\leqslant k_1<h_1 \\ 0\leqslant k_3<h_3 \\ k_1+k_3\equiv\epsilon(\widehat{\theta}_1,\widehat{\theta}_3) \bmod 2}} \sigma_{k_1,k_3}q^{2l_3(\widehat{\theta}_3+k_3)-2l_1(\widehat{\theta}_1+k_1)},$$

  where the sum has $2h_1h_3$ terms accounting for the values $\widehat{\theta}_j$ and the congruence condition for k_1,k_3:

$$\epsilon(\theta_1,\theta_3)=\epsilon(1-\theta_1,1-\theta_3)=0,\quad \epsilon(\theta_1,1-\theta_3)=\epsilon(1-\theta_1,\theta_3)=1.$$

  The sign σ_{k_1,k_3} is still the sign of the exponent.

Remark 6.16 The computation of these sums depends in a complicated way on the value of (l_1,l_3). It can be computed in the case $l_1=l_3=1$ and this is in fact the only case that we need. This is done later in this subsection. ◁

Proof Let us carry out the resolution as proposed before the proposition:

- First, take the image by the logarithm of modulus to get the system

$$\begin{cases} \log|a|+h_1\log|\cos(x_3\pi)-\cos(x_1\pi)|=0, \\ -(m_2/l_3)\log|a|+(v_1/l_3)\log|b|+h_3\log|\cos(x_1\pi)-\cos(x_1\pi)|=0. \end{cases}$$

This allows us to express the modulus of a and b in terms of $|\cos(x_3\pi)-\cos(x_1\pi)|$, which is known if x_1 and x_3 are known.

- To find x_1, x_3 and the signs of a and b, we take the argument mod π of both equations

$$h_1x_1\pi \equiv \pm\theta_1\pi \mod \pi, \qquad h_3x_3\pi \equiv \pm\theta_3\pi \mod \pi.$$

Therefore,

$$x_1 = \frac{\widehat{\theta}_1 + k_1}{h_1}, \quad 0 \leqslant k_1 < h_1, \qquad x_3 = \frac{\widehat{\theta}_3 + k_1}{h_3}, \quad 0 \leqslant k_3 < h_3,$$

where $\widehat{\theta}_j$ is taken equal to θ_j or $1-\theta_j$. This convention replaces the $\pm\theta$ to get the same interval for k_1 in both cases. This solves for x_1 and x_3. We get $4h_1h_3$ solutions. Then, going back to the principal system, we find the sign of a in the first equation and the sign of b in the second equation according to the parity of v_1/l_3. Solving for b gives one solution if v_1/l_3 is odd, or zero or two solutions according to sign if it is even. In each case, we get $4h_1h_3$ solutions, for which we need to compute the contribution.

– Assume v_1/l_3 is odd. Then there is a unique solution for the signs of a and b for each value of (k_1,k_3). Moreover, the log-area is equal to

$$\mathscr{A} = \det A \times 2\pi(x_3\pi - x_1\pi) = 2h_jl_j\left(\frac{\widehat{\theta}_3+k_3}{h_3} - \frac{\widehat{\theta}_1+k_1}{h_1}\right)\pi^2 = 2l_3(\widehat{\theta}_3+k_3)\pi^2 - 2l_1(\widehat{\theta}_1+k_1)\pi^2,$$

and the logarithmic rotation number is equal to its sign. Therefore, the total refined count is equal to

$$\sum_{\substack{0\leqslant k_1<h_1\\ 0\leqslant k_3<h_3}} \sigma_{k_1,k_3} q^{2l_3(\widehat{\theta}_3+k_3)-2l_1(\widehat{\theta}_1+k_1)},$$

where the sum has $4h_1h_3$ terms accounting for the values $\widehat{\theta}_j$, and the sign σ_{k_1,k_3} is the sign of the exponent.

– Now, assume that v_1/l_3 is even. Then $h_1 = v_1$ is also even, as is $h_3 = v_1(v_3-v_2)/l_3$. However, v_2/l_3 is odd, as $l_3 = \gcd(v_1,v_2)$. Taking care of the parity of these integers and taking $\arg(a) \equiv \varepsilon\pi \mod 2\pi$, the argument mod 2π of the principal system gives us

$$\varepsilon + \widehat{\theta}_1 + k_1 \equiv \pm\theta_1 \mod 2, \qquad \varepsilon + \widehat{\theta}_3 + k_3 \equiv \pm\theta_3 \mod 2.$$

Both equations give the value of ε, and these have to coincide. This is the case if and only if

$$k_1 + k_3 \equiv 0 \bmod 2 \quad \text{for the pairs } (\theta_1,\theta_3) \text{ and } (-\theta_1,-\theta_3),$$
$$k_1 + k_3 \equiv 1 \bmod 2 \quad \text{for the pairs } (-\theta_1,\theta_3) \text{ and } (\theta_1,-\theta_3).$$

The total signed contribution is then

$$2 \sum_{\substack{0\leqslant k_1<h_1\\ 0\leqslant k_3<h_3\\ k_1+k_3\equiv\epsilon(\widehat{\theta}_1,\widehat{\theta}_3) \bmod 2}} \sigma_{k_1,k_3} q^{2l_3(\widehat{\theta}_3+k_3)-2l_1(\widehat{\theta}_1+k_1)},$$

where the sum has $2h_1h_3$ terms accounting for the values $\widehat{\theta}$ and the congruence condition for k_1 and k_3. Once again, the sign σ_{k_1,k_3} is the sign of the exponent. □

Now we get to the first variant of the enumerative problem and allow the complex points to take the opposite value.

Problem 6.17 How many oriented real rational curves of degree $\Delta^{\mathrm{con}}(v_1, v_2, v_3)$ have primitive moments $\pm e^{i\pi\theta_1}$ on $\mathbb{C}E_1$ and $\pm e^{i\pi\theta_3}$ on $\mathbb{C}E_3$?

As in the parabola case, the solution consists in adding the contribution for the different possibilities.

Corollary 6.18 *The refined number of oriented real rational curves of degree $\Delta^{\mathrm{con}}(v_1, v_2, v_3)$ having primitive moments $\pm e^{i\pi\theta_1}$ on $\mathbb{C}E_1$ and $\pm e^{i\pi\theta_3}$ on $\mathbb{C}E_3$ is*

$$4 \sum_{\substack{0\leq k_1<h_1 \\ 0\leq k_3<h_3}} \sigma_{k_1,k_3} q^{2l_3(\widehat{\theta}_3+k_3)-2l_1(\widehat{\theta}_1+k_1)}.$$

Proof One just needs to consider the pairs $(1-\theta_1, \theta_3)$, $(\theta_1, 1-\theta_3)$ and $(1-\theta_1, 1-\theta_3)$ in place of (θ_1, θ_3) in Proposition 6.15.

- If v_1/l_3 is odd, the contribution is the same for each of the pairs. Thus, we get the announced contribution.

- If v_1/l_3 is even, let us write carefully the contributions. The contribution for (θ_1, θ_3) given by Proposition 6.15 is

$$2 \sum_{\substack{0\leq k_1<h_1 \\ 0\leq k_3<h_3 \\ k_1+k_3\equiv 0 \bmod 2}} \sigma q^{2l_3(\theta_3+k_3)-2l_1(\theta_1+k_1)} + \sigma q^{2l_3(1-\theta_3+k_3)-2l_1(1-\theta_1+k_1)} \\ + 2 \sum_{\substack{0\leq k_1<h_1 \\ 0\leq k_3<h_3 \\ k_1+k_3\equiv 1 \bmod 2}} \sigma q^{2l_3(1-\theta_3+k_3)-2l_1(\theta_1+k_1)} + \sigma q^{2l_3(\theta_3+k_3)-2l_1(1-\theta_1+k_1)}.$$

The contribution is the same when replacing (θ_1, θ_3) by $(1-\theta_1, 1-\theta_3)$. However, when replacing by $(\theta_1, 1-\theta_3)$ and $(1-\theta_1, \theta_3)$, the congruences on k_1 and k_3 are exchanged. Then, adding all the contributions, we get

$$4 \sum_{\substack{0\leq k_1<h_1 \\ 0\leq k_3<h_3}} \sigma q^{2l_3(\theta_3+k_3)-2l_1(\theta_1+k_1)} + \sigma q^{2l_3(1-\theta_3+k_3)-2l_1(1-\theta_1+k_1)} \\ + 4 \sum_{\substack{0\leq k_1<h_1 \\ 0\leq k_3<h_3}} \sigma q^{2l_3(1-\theta_3+k_3)-2l_1(\theta_1+k_1)} + \sigma q^{2l_3(\theta_3+k_3)-2l_1(1-\theta_1+k_1)},$$

which is the announced result. □

Finally, we can turn our focus to the last problem, which is to make the refined count of curves have fixed moments rather than fixed primitive moments.

Problem 6.19 Let $e^{i\pi\theta_1}, e^{i\pi\theta_3} \in \mathbb{C}^*$. How many oriented real rational curves of degree $\Delta^{\mathrm{con}}(v_1, v_2, v_3)$ have moments $\pm e^{\pm i\pi\theta_1}$ on $\mathbb{C}E_1$ and $\pm e^{\pm i\pi\theta_1}$ on $\mathbb{C}E_3$?

The proof goes by the same trick as in the parabola case for solving Problem 6.10: we extract l_1 and l_3 roots to find the possible values of the respective primitive moments.

Corollary 6.20 *Let $e^{i\pi\theta_1}, e^{i\pi\theta_3} \in \mathbb{C}^*$. The refined number of oriented real rational curves of degree $\Delta^{\mathrm{con}}(v_1, v_2, v_3)$ having moments $\pm e^{\pm i\pi\theta_1}$ on $\mathbb{C}E_1$ and $\pm e^{\pm i\pi\theta_1}$ on $\mathbb{C}E_3$ is*

$$[2\theta_3-1]_+[2\theta_1-1]_+\frac{\left[\frac{1}{2}m_\Delta\right]}{[1][1]}-\frac{m_\Delta}{2[1]}\begin{cases}[2\theta_3-1]_+[2\theta_1]_+ & \text{if } \theta_1<\theta_3,\ \theta_1+\theta_3<1,\\ [2\theta_3-1]_+[2(1-\theta_1)]_+ & \text{if } \theta_1>\theta_3,\ \theta_1+\theta_3>1,\\ [2\theta_1-1]_+[2\theta_3]_+ & \text{if } \theta_1>\theta_3,\ \theta_1+\theta_3<1,\\ [2\theta_1-1]_+[2(1-\theta_3)]_+ & \text{if } \theta_1<\theta_3,\ \theta_1+\theta_3>1.\end{cases}$$

Proof As for Problem 6.10, the primitive moment on $\mathbb{C}E_1$ can be

$$\pm\frac{\theta_1+p_1}{l_1}\pi,\quad \pm\frac{1-\theta_1+p_1}{l_1}\pi\quad \text{for } 0\leqslant p_1<l_1,$$

and the same with 1 replaced by 3 for $\mathbb{C}E_3$. They are grouped by four: pairs of opposite pairs of complex conjugate points. The representatives of the arguments lying in $]0;\pi[$ can be taken to be $(\theta_j+p_j)/l_j$ for $j=1$ or 3. The other possible choices would be to replace θ_j by $1-\theta_j$ for some of them. Thus, we only have to add the contributions for these arguments, and, using Proposition 6.15, we get

$$\sum_{\substack{0\leqslant p_1<l_1\\ 0\leqslant p_3<l_3}} 4 \sum_{\substack{0\leqslant k_1<h_1\\ 0\leqslant k_3<h_3}} \sigma\Big(q^{2l_3((\theta_3+p_3)/l_3+k_3)-2l_1((\theta_1+p_1)/l_1+k_1)} + q^{2l_3(1-(\theta_3+p_3)/l_3+k_3)-2l_1(1-(\theta_1+p_1)/l_1+k_1)}\Big).$$

We now compute explicitly. First, notice that the exponent has value

$$2l_3\left(\frac{\theta_3+p_3}{l_3}+k_3\right)-2l_1\left(\frac{\theta_1+p_1}{l_1}+k_1\right)=2(\theta_3+p_3+k_3l_3)-2(\theta_1+p_1+k_1l_1).$$

If there is a $1-(\theta_j+p_j)/l_j$, we can make the change of index $p_j'=l_j-1-p_j$, so that the sum goes over the same values but we have a $1-\theta_j$ instead. Moreover, the index $p_j+k_jl_j$ goes through the integer interval $[\![0;l_jh_j-1]\!]$, and $h_jl_j=\frac{1}{4}m_\Delta$ is constant. Thus, the double sum over a rectangle each time is in fact a unique sum over a square of side length $\frac{1}{4}m_\Delta$. The sum is then

$$\sum_{0\leqslant k_1,k_3<m_\Delta/4}\sigma q^{2(\widehat{\theta}_3+k_3)-2(\widehat{\theta}_1+k_1)} = \sum_{0\leqslant k_1,k_3<m_\Delta/4}(\sigma q^{2\theta_3-2\theta_1}+\sigma q^{2-2\theta_3-2\theta_1}+\sigma q^{2\theta_3+2\theta_1-2}+\sigma q^{2\theta_1-2\theta_3})q^{2k_3-2k_1}.$$

This sum is computed in the following lemma, leading to the announced result.

Lemma 6.21 *For* $0 < \theta, \varphi < 1$,

$$S(\theta,\varphi) = \sum_{0\leqslant i,j<n} (\sigma q^{2(\varphi-\theta)} + \sigma q^{-(2\varphi-2\theta)} + \sigma q^{2(1-\varphi-\theta)} + \sigma q^{-2(1-\varphi-\theta)}) q^{2j-2i}$$

$$= [2\varphi-1]_+[2\theta-1]_+\frac{[2n]}{[1][1]} - \frac{2n}{[1]}\begin{cases}[2\varphi-1]_+[2\theta]_+ & \text{if } \theta<\varphi,\ \theta+\varphi<1,\\ [2\varphi-1]_+[2(1-\theta)]_+ & \text{if } \theta>\varphi,\ \theta+\varphi>1,\\ [2\varphi]_+[2\theta-1]_+ & \text{if } \theta>\varphi,\ \theta+\varphi<1,\\ [2(1-\varphi)]_+[2\theta-1]_+ & \text{if } \theta<\varphi,\ \theta+\varphi>1.\end{cases}$$

Proof We split the sum in three parts:

- If $j > i$, the exponent is always positive due to the assumption on θ and φ, and therefore $\sigma = +1$. Then

$$q^{2(\varphi-\theta)} + q^{-(2\varphi-2\theta)} + q^{2(1-\varphi-\theta)} + q^{-2(1-\varphi-\theta)} = [2\varphi-2\theta]_+ + [2-2\varphi-2\theta]_+ = [2\varphi-1]_+[2\theta-1]_+$$

and we compute

$$\begin{aligned}\sum_{j=1}^{n-1}\sum_{i=0}^{j-1} q^{2(j-i)} &= \sum_{j=1}^{n-1}\sum_{i=0}^{j-1} q^{2(j-i)}\\ &= \sum_{j=1}^{n-1} q^{2j}\frac{1-q^{-2j}}{1-q^{-2}}\\ &= \frac{q}{[1]}q^2\frac{q^{2(n-1)}-1}{q^2-1} - (n-1)\frac{q}{[1]}\\ &= \frac{q^{2n}-q^2}{[1]^2} - (n-1)\frac{q}{[1]}.\end{aligned}$$

- When we add the sum for $i > j$, the value is the same if we replace q by q^{-1}, since the sign exponent $2(j-i)$ changes its sign. The sign σ this time is -1. Hence,

$$\begin{aligned}\frac{q^{2n}-q^2}{[1]^2} - \frac{q^{-2n}-q^{-2}}{[1]^2} - (n-1)\frac{q}{[1]} - (n-1)\frac{q^{-1}}{[1]} &= \frac{[2n]}{[1]^2} - \frac{[2]}{[1]^2} - (n-1)\frac{[1][1]_+}{[1]^2}\\ &= \frac{[2n]}{[1][1]} - n\frac{[1]_+}{[1]}.\end{aligned}$$

- For the central sum, since $i = j$, we have the following factorizations:
  - If $\theta < \varphi$ and $\theta+\varphi < 1$,

$$n(q^{2(\varphi-\theta)} - q^{-(2\varphi-2\theta)} + q^{2(1-\varphi-\theta)} - q^{-2(1-\varphi-\theta)}) = -n[2\varphi-1]_+[2\theta-1].$$

  - If $\theta > \varphi$ and $\theta+\varphi > 1$,

$$n(-q^{2(\varphi-\theta)} + q^{-(2\varphi-2\theta)} - q^{2(1-\varphi-\theta)} + q^{-2(1-\varphi-\theta)}) = n[2\varphi-1]_+[2\theta-1].$$

- If $\theta > \varphi$ and $\theta + \varphi < 1$,
$$n(-q^{2(\varphi-\theta)} + q^{-(2\varphi-2\theta)} + q^{2(1-\varphi-\theta)} - q^{-2(1-\varphi-\theta)}) = -n[2\theta-1]_+[2\varphi-1].$$
- If $\theta < \varphi$ and $\theta + \varphi > 1$,
$$n(q^{2(\varphi-\theta)} - q^{-(2\varphi-2\theta)} - q^{2(1-\varphi-\theta)} + q^{-2(1-\varphi-\theta)}) = n[2\theta-1]_+[2\varphi-1].$$

We can pass from one factorization to another using the symmetries $(\theta, \varphi) \mapsto (\varphi, \theta)$, $\theta \mapsto 1-\theta, \dots$. This corresponds to the action of the dihedral group over the regions delimited by the inequalities. The total contribution is then
$$[2\varphi-1]_+[2\theta-1]_+\frac{[2n]}{[1][1]} - \frac{n}{[1]}([1]_+[2\varphi-1]_+[2\theta-1]_+ - [1]G(\theta,\varphi)),$$
where $G(\theta, \varphi)$ is the above factor without the n. If $\theta < \varphi$ and $\theta + \varphi < 1$, the second term can be factorized, leading to
$$[2\varphi-1]_+[2\theta-1]_+\frac{[2n]}{[1][1]} - \frac{2n}{[1]}[2\varphi-1]_+[2\theta]_+.$$
The other cases are obtained by symmetry or treated similarly. □

6.2 Global enumerative problem

We now can prove Theorem 4.8, relating $R_{\Delta,s}$ and $N^{\partial,\mathrm{trop}}_{\Delta(s)}$. This is a consequence of Proposition 6.23, which proves that the multiplicities of rational tropical curves of degree $\Delta(s)$ given through the correspondence are proportional to the refined Block–Göttsche (equation (1)) used to compute $N^{\partial,\mathrm{trop}}_{\Delta(s)}$.

Let $\mathcal{P}_t$ be a symmetric configuration of points depending on a parameter t, chosen as in Section 4.2. For a pair of complex conjugate points $p, \bar{p} \in \mathcal{P}_t$, let $\pm\theta_p\pi$ be their arguments, with $0 < \theta_p < 1$. We keep similar notation for the rest. Given a parametrized tropical curve $h_0\colon \Gamma_0 \to N_\mathbb{R}$ of degree $\Delta(s)$ with $\mathrm{ev}(\Gamma_0) = \mu$, the task of computing its multiplicity amounts to two things. The first is to find the parametrized real tropical curves of degree Δ having the same image and admitting first-order solutions. The second task consists in finding the first-order solutions to $\Theta(\chi, \alpha, \beta) = (1, \zeta^\Gamma)$.

The first problem is taken care of with the following lemma.

Lemma 6.22 *The parametrized real tropical curves with a trivalent flat vertex cannot be the tropicalization of a family of parametrized real rational curves passing through the symmetric configuration $\mathcal{P}_t$.*

Proof Let $h_0\colon \Gamma_0 \to N_\mathbb{R}$ be a parametrized rational tropical curve of degree $\Delta(s)$ such that $\mathrm{ev}(\Gamma_0) = \mu$. Let $C_{\mathrm{trop}} = h_0(\Gamma_0)$ be its image, which is a plane tropical curve. The different possible real structures are described in Proposition 2.2. Let (Γ, h) be one of them.

Assume that there exists a trivalent flat vertex w in (Γ, h), with two outgoing edges leading to branches exchanged by the involution. Let $f_t\colon \mathbb{C}P^1 \dashrightarrow N \otimes \mathbb{C}((t))^*$ be a parametrized real rational curve tropicalizing to (Γ, h).

Let V be a vertex and denote by n_i the outgoing slope of its adjacent edges. Also let $m_i = \iota_{n_i}\omega$. Due to the balancing condition, one has $\prod_i \chi_V(m_i) = 1$. Thus, for every bounded edge γ,

$$\chi_{\mathfrak{h}(\gamma)}(m_\gamma) = \prod_{\mathfrak{t}(\gamma')=\mathfrak{h}(\gamma)} \chi_{\mathfrak{h}(\gamma)}(m_{\gamma'}).$$

Moreover, the vertices in each branch are trivalent; therefore, at first order, $\phi_\gamma = (\pm 1)^{n_{\gamma'}}$, except for an edge γ such that $\mathfrak{t}(\gamma) = V$, where $\phi_\gamma = (2i)^{n_\gamma}$. In any case, the transfer relation $\phi_\gamma \cdot (\chi_{\mathfrak{t}(\gamma)}/\chi_{\mathfrak{h}(\gamma)}) \cdot \alpha_\gamma^{n_\gamma} = 1$ evaluated at m_γ and at first order gives us

$$\chi_{\mathfrak{t}(\gamma)}(m_\gamma) = \pm \chi_{\mathfrak{h}(\gamma)}(m_\gamma).$$

Multiplying these relations for every vertex in the branch, for γ one of the complex edges adjacent to V, we get

$$\chi_V(m_\gamma) = \pm \prod_e \chi_{v_e}(m_e),$$

where the product is indexed by the ends belonging to the branch, and v_e denotes the adjacent vertex to a branch e, and m_e the character used to computes its moment. Due to the condition of passing through $\mathcal{P}_t$, the argument of $\chi_{v_e}(m_e)$ is $\pm\theta_e$ or $\pi \pm \theta_e$, ie the argument of the point through which the curve passes. Therefore, the argument of $\chi_V(m_\gamma)$ is a sum of $\pm\theta_e$. In particular, for generic points, it is not real, which should be the case if V is fixed: $\chi_V(m_\gamma)$ would be real. Hence, the curve cannot have any trivalent flat vertex if the configuration is generic. □

We now count the first-order solutions, which lead to true solutions thanks to the correspondence theorem. It happens that knowledge of the first order is sufficient to recover the sign and the logarithmic area. The refined count of these first-order solutions thus gives the right multiplicity used to count tropical curves.

Proposition 6.23 *Let $\mathcal{P}_t$ be a symmetric real configuration of points as previously chosen, tropicalizing on a family of moments μ. Let $h_0\colon \Gamma_0 \to N_\mathbb{R}$ be a parametrized tropical curve of degree $\Delta(s)$ having moments μ. The refined count according to the log-area and sign σ is given by*

$$m'_{\Gamma_0} = 4\frac{2^{|s|}}{[1]^{|s|}} \prod_V \left[\tfrac{1}{2} m_V\right],$$

where the product is over the trivalent vertices of Γ_0.

Before proving this proposition, we can now prove Theorem 4.8.

Proof of Theorem 4.8 This is a consequence of the correspondence theorem, which states that each of the first-order solutions lifts to a unique solution, given by Proposition 6.23: we obtain $R_{\Delta,s}$ by counting curves with multiplicities $\frac{1}{4}m'_{\Gamma_0}$, while $N^{\partial,\mathrm{trop}}_{\Delta(s)}$ is obtained by counting them with multiplicity $m^q_{\Gamma_0}$. Finally,

$$\tfrac{1}{4} m'_{\Gamma_0} = \frac{2^{|s|}\left[\frac{1}{2}\right]^{|\Delta|-2-|s|}}{[1]^{|s|}} m^q_{\Gamma_0}.$$ □

We now prove Proposition 6.23.

Proof of Proposition 6.23 The proof proceeds by induction on the number of vertices of the curve Γ. Exceptionally, to suit the induction, a vector n_j of N directing an end e_j may not be primitive. In that case, we denote by $m_j = \omega(n_j/l(n_j), -)$ the dual vector, but still of lattice length 1.

Let (Γ_0, h_0) be a parametrized tropical curve of degree $\Delta(s)$ with $\mathrm{ev}(\Gamma_0, h_0) = \mu$. The real tropical curves that may have first-order solutions are real tropical curves (Γ, h, σ) with the same image and without any flat vertex. The proof proceeds as follows: we compute the refined number of first-order solutions by evicting one by one all the vertices for any real tropical curve (Γ, h). Then we add all the refined counts for every real tropical curve (Γ, h) such that $h(\Gamma) = h_0(\Gamma_0)$.

Let (Γ, h) be a real tropical curve with $h(\Gamma) = h_0(\Gamma_0)$ and without flat vertex. For a vertex, we have two possibilities: either it is a complex vertex, or it is a real vertex. In the second case, we assume that it is an extremal vertex of the subgraph $\mathrm{Fix}(\sigma)$, meaning that every vertex $W \succ V$ is a complex vertex. Let V be such a vertex. We have three possibilities: V is trivalent and adjacent to two real ends, V is quadrivalent and adjacent to one real end, or V is pentavalent. The next four points do the following:

(i) evict a trivalent real vertex,

(ii) evict a complex branch,

(iii) evict a quadrivalent vertex,

(iv) evict a pentavalent vertex.

(i) First, assume that V is adjacent to two ends indexed by 0 and 1. Let γ be the unique edge with $\mathfrak{h}(\gamma) = V$. If V is the only vertex of the tropical curve, the refined count has already been proven to be $4\left[\frac{1}{2}m_V\right]$. Otherwise, γ is bounded and the coordinates associated to V and γ are α_γ and χ_V. Then we have to solve, for $\chi_V : M \to \mathbb{R}^*$, the system

$$\chi_V(m_0) = \pm\zeta_0^\Gamma \in \mathbb{R}^*, \quad \chi_V(m_1) = \pm\zeta_1^\Gamma \in \mathbb{R}^*.$$

Recall that the vectors n_0 and n_1 might not be primitive, but m_0 and m_1 are. According to Proposition 6.4, this system leads to four solutions: if (e_1^*, e_2^*) is a basis of M, the absolute value of $\chi_V(e_1^*)$ and $\chi_V(e_2^*)$ is uniquely determined, while the sign may be chosen arbitrarily. Notice that some choices of signs for the right-hand side of the system may provide several solutions, while others provide none. Let m_γ be the primitive vector dual to n_γ. Let $\widetilde{m}_\gamma$ be such that $(m_\gamma, \widetilde{m}_\gamma)$ is a basis of M. These four solutions separate themselves into two groups of two, according to the sign of $\chi_V(m_\gamma)$. We have the transfer equation

$$\phi_\gamma \cdot \frac{\chi_{\mathfrak{t}(\gamma)}}{\chi_V} \cdot \alpha_\gamma^{n_\gamma} = 1 \in N_{\mathbb{R}^*}.$$

We evaluate at m_γ, leading to

$$\phi_\gamma(m_\gamma)\chi_{\mathfrak{t}(\gamma)}(m_\gamma) = \chi_V(m_\gamma) \in \mathbb{R}^*.$$

Recall that, according to the choice of signs of $\pm\zeta_j$, the sign of $\chi_V(m_\gamma)$ may change. So we might as well add $\pm$, leading to

$$\phi_\gamma(m_\gamma)\chi_{t(\gamma)}(m_\gamma) = \pm|\chi_V(m_\gamma)| \in \mathbb{R}^*.$$

Let us replace the bounded edge γ by an end with direction n_γ, leading to a new parametrized tropical curve (Γ', h'). The above equation is the equation associated to this new end in Γ', in the corresponding system $\Theta(\chi, \alpha, \beta) = (1, \zeta^\Gamma)$. Thus, we can proceed by induction. Let $4R$ denote the refined signed count of oriented curves lifting Γ'. These $4R$ oriented curves separate themselves into four groups of R oriented curves according to the signs the function $\phi_\gamma \cdot \chi_{t(\gamma)}$ takes on the basis $(m_\gamma, \widetilde{m}_\gamma)$, using the action of the deck transformation group $\{\pm 1\}^2$. On the other side, we get only the two solutions among those for which $\phi_\gamma(m_\gamma)\chi_{t(\gamma)}(m_\gamma) = \chi_V(m_\gamma)$. Last, we need to solve for α_γ. By evaluating at $\widetilde{m}_\gamma$, we get

$$\alpha_\gamma^{\langle n_\gamma, \widetilde{m}_\gamma\rangle} = \frac{\chi_V(\widetilde{m}_\gamma)}{\phi_\gamma(\widetilde{m}_\gamma) \cdot \chi_{t(\gamma)}(\widetilde{m}_\gamma)}.$$

Notice that, through the presence of ϕ_γ and $\chi_{t(\gamma)}$, the right-hand term depends on which of the $4R$ solutions is chosen. The solution as well as the number of solutions depends on the sign of the right-hand side.

- If n_γ has odd integer length, then we solve uniquely for α_γ for each possible sign of $\chi_V(\widetilde{m}_\gamma)$. The sign of $\chi_V(m_\gamma)$ is already determined since

  $$\phi_\gamma(m_\gamma)\chi_{t(\gamma)}(m_\gamma) = \chi_V(m_\gamma).$$

  Thus, each of the oriented curves in each of the groups have two possible solutions for χ_V. This corresponds to the gluing of two possible curves, one increasing the logarithmic rotation number by one, the other decreasing it by one. In each case, the orientation of the curve propagates, and the signed count becomes

  $$4 \times (q^{m_V/2} - q^{-m_V/2})R.$$

- If n_γ has even integer length, then the sign of $\chi_V(m_\gamma)$ is still determined, and the sign of $\chi_V(\widetilde{m}_\gamma)$ is forced in order to have at least one solution for α_γ. In that case, we have only one of the four possible curves that we can glue, but there are two possible values for α_γ, which means two gluings. One of these choices decreases the logarithmic rotation number by one, while the other increases it by one. The signed count then becomes again

  $$4 \times (q^{m_V/2} - q^{-m_V/2})R.$$

(ii) Before considering quadrivalent and pentavalent vertices, we emphasize the repartition of the complex points inside a complex branch. Consider a complex branch $\mathcal{B}$ of the real tropical curve Γ. This means that there exists some quadrivalent or pentavalent vertex V, and $\mathcal{B}$ is the subgraph of Γ accessible via one of the complex edges adjacent to V. Let γ_0 be this edge, called the root of the branch. Notice that, for a branch $\mathcal{B}$, we have the conjugated branch $\sigma(\mathcal{B})$.

For any edge γ, recall we distinguish between its moment, which is $\chi_{\mathfrak{t}(\gamma)}(m_\gamma)$, and its primitive moment, which is $\chi_{\mathfrak{t}(\gamma)}(m_\gamma/l(m_\gamma))$. The goal of this point is to express the moment of the root of the branch in terms of the moments of the ends, and then count the possible lifts.

Let us index by $[\![1;p]\!]$ the (complex) ends belonging to $\mathcal{B}$. Each end indexed by j is associated to a pair of complex conjugate points having moment $\pm e^{\pm i\pi\theta_j}$. To distinguish between $\mathcal{B}$ and $\sigma(\mathcal{B})$, we assume that the point belonging to $\mathcal{B}$ associated to the first end has moment $e^{i\pi\theta_1}$ or $-e^{-i\pi\theta_1}$. Then there are 2^{p-1} possible repartitions for the other points according to whether it is the point with argument $\theta_j\pi$ or $-\theta_j\pi$ that belongs to $\mathcal{B}$.

For any repartition of the arguments, we have:

- Due to the balancing condition, at any vertex W with ingoing edge γ and outgoing edges γ_1 and γ_2, one has
$$\chi_V(m_\gamma)=\chi_V(m_1)\chi_V(m_2).$$
- For any edge γ, due to the transfer equation evaluated at $m_\gamma/l(m_\gamma)$, one has
$$\chi_{\mathfrak{h}(\gamma)}\left(\frac{m_\gamma}{l(m_\gamma)}\right)=\phi_\gamma\left(\frac{m_\gamma}{l(m_\gamma)}\right)\chi_{\mathfrak{t}(\gamma)}\left(\frac{m_\gamma}{l(m_\gamma)}\right)=\pm\chi_{\mathfrak{t}(\gamma)}\left(\frac{m_\gamma}{l(m_\gamma)}\right),$$
and thus also
$$\chi_{\mathfrak{h}(\gamma)}(m_\gamma)=\phi_\gamma(m_\gamma)\chi_{\mathfrak{t}(\gamma)}(m_\gamma)=\pm\chi_{\mathfrak{t}(\gamma)}(m_\gamma).$$

Therefore, if the moments of the ends are fixed, the moment of every bounded edge is also fixed, and is equal up to sign to the product of the moments of the ends.

We can now compute the number of first-order solutions for every possible primitive moment of the root γ_0. Fix one of these possible values. Let $V_0=\mathfrak{h}(\gamma_0)$. Proposition 6.3 ensures that there are $m_{V_0}/(4l(m_{\gamma_0}))$ choices for χ_{V_0}. The 4 is to account for the following fact: the multiplicity involved in Proposition 6.3 is the multiplicity in the parametrized curve where V_0 is the vertex in a pair of complex conjugate vertices. This multiplicity is equal to the multiplicity of V_0 in the curve Γ_0 divided by 4, as each adjacent slope is only half the slope in Γ_0. This choice now fixes the primitive moment of the edges outgoing from V_0 and one can proceed by induction. We get a number of first-order solutions equal to
$$\prod_{W\in\mathcal{B}-\{V\}}\frac{m_W}{4l(m_{\gamma_W})},$$
where γ_W denotes the unique edge such that $\mathfrak{h}(\gamma)=W$. This number of solutions has to be multiplied by the number of possible first-order solutions of α_γ. We solve for α_γ by evaluating the transfer equation at some vector completing $m_\gamma/l(m_\gamma)$ into a basis of M. We thus get $l(m_\gamma)$ solutions. In total, we get $\prod_W \frac{1}{4}m_W$ first-order solutions for each possible value of the primitive moment of γ_0.

To finish this part of the computation, we give the set Θ of possible values of the moment of the root edge γ_0. According to the above, it is, up to sign, the product of the moments of the ends in $\mathcal{B}$. The possible values are

$$\pm e^{i\pi(\pm\theta_1\pm\theta_2\pm\cdots\pm\theta_p)}.$$

There are 2^{p+1} such values, ie 2^p pairs of complex conjugate values, themselves consisting of 2^{p-1} pairs of opposite complex conjugate pairs. On the other side, as the moment of each end can take four values, and the complex conjugate branch takes the complex conjugate moments, there are $\frac{1}{2}\cdot 4^p=2^{2p-1}$ repartitions of the moments. They are equally spread between the 2^p pairs of complex conjugate moments. Thus, each pair of complex conjugate moments is reached by $2^{2p-1}/2^p=2^{p-1}$ repartitions. Another set of 2^{p-1} repartitions is assigned to the opposite pair of complex conjugate moments.

Finally, the multiplicity to account for the first-order solutions corresponding to coordinates in the branch is

$$2^{p-1}\prod_W \tfrac{1}{4}m_W=\prod_W \tfrac{1}{2}m_W,$$

since there are $p-1$ vertices, and we are left to solve the problem where the branch is replaced by a pair of complex conjugate ends, and the condition imposed is a fixed moment.

(iii) Now, assume that V is a quadrivalent vertex adjacent to a real end and two exchanged complex edges $\gamma^{\mathbb{C}}$ and $\sigma(\gamma^{\mathbb{C}})$ which might be unbounded or not. We denote by $\mathcal{B}$ the branch accessible by $\gamma^{\mathbb{C}}$. Let n_0 be the common slope of h on the complex edges, and n_1 be the slope of h on the real end. Let $V_0=\mathfrak{h}(\gamma^{\mathbb{C}})$ be the first complex vertex in the branch if there is one. Let β and χ_V be the coordinates associated to V.

Using the previous point, to find χ_V and β, we have to solve the moment problem for a parabola, equivalent to the system

$$\chi_V(m_0)(\beta-i)^{\langle m_0,n_1\rangle}=\pm\chi_{V_0}(m_0),\quad \chi_V m_1(\beta^2+1)^{\langle m_1,n_0\rangle}=\pm\zeta_1^\Gamma.$$

The first equation is the first order of the transfer equation of $\gamma^{\mathbb{C}}$ evaluated at vector $m_0=\iota_{n_0}\omega$, while the second equation is the moment equation for the real end. If the complex edges adjacent to V are ends, the first equation is replaced with the moment equation $\chi_V(m_0)(\beta-i)^{\langle m_0,n_1\rangle}=\pm\zeta_0^\Gamma\in\mathbb{C}^*$. This system is the parabola problem for a curve of degree given by the outgoing edges of V, and with a pair of complex points having moments $\pm e^{i\pi\theta}$, where θ takes the values provided by the previous point. For each possible value, the refined count was proven to be

$$4[2\theta-1]_+\frac{\left[\frac{1}{2}m_V\right]}{[1]}.$$

This contribution has to be multiplied by $\prod_W \frac{1}{2}m_W$ and added over the 2^{p-1} values that may be taken by θ.

(iv) If the vertex V is pentavalent, adjacent to two pairs of exchanged complex edges behind which we find four branches, similarly, the number of solutions in the branches has already been taken care of: the refined count for the moment problem associated to V and each possible value of the moments of the adjacent complex edges is multiplied by $\prod_W \frac{1}{2}m_W$, where W goes through the vertices in both branches. Let $0 < \theta, \varphi < 1$ be such that the moments of the complex edges may take the values $\pm e^{\pm i\pi\theta}$ and $\pm e^{\pm i\pi\varphi}$, where φ and θ may take several values, described by the second point of this proof. Then the refined count is given by Corollary 6.20. The total contribution is computed later on.

All the cases of multiplicities have been dealt with for a specific real tropical curve. The last thing to do is to add the various contributions of all the real tropical curves of degree Δ that can be obtained from a specific curve of degree $\Delta(s)$. These multiple choices are described by Proposition 2.2.

Let γ be an edge of Γ_0, and let $\mathcal{B} = \mathcal{B}(\gamma)$ be the branch of Γ_0 consisting of points accessible via γ in the oriented structure of the tropical curve. We distinguish two cases according to whether the lift of γ in Γ is a pair of complex edges or a fixed edge.

- If the lift consists of a pair of complex edges, let Θ_γ be the set of values of normalized arguments (in $]0;1[$) and up to sign that the moment of the edge may take when we vary the moments of the unbounded edges, so that we have exactly one representative by a 4-tuple. In the second point, we proved that, if the branch $\mathcal{B}$ has p ends, $|\Theta_\gamma| = 2^{p-1}$.
- Otherwise, the lift is a fixed edge.

Now, for an edge γ in a component of Γ_{even}, let R_γ be the number of first-order solutions in the branch $\mathcal{B}(\gamma)$ if the edge γ is lifted to a fixed edge of γ, and let C_γ be the number of first-order solutions in the branch if γ is lifted to a pair of complex conjugate edges.

Let γ be an edge, and let γ_1 and γ_2 be the two edges emanating from $\mathfrak{h}(\gamma)$. We drop the index 4 quantities concerning γ and index 1 and 2 quantities concerning γ_1 or γ_2. We have already proven the following statements:

- The complex contribution being a product over the vertices in the branch, they satisfy
$$C = C_1 C_2 \cdot \tfrac{1}{2} m_{\mathfrak{h}(\gamma)}.$$
- If γ was an end having moment $\pm e^{\pm i\pi\theta}$, we would have $\Theta = \{\theta\}$.
- The possible moments satisfy
$$\Theta = (\Theta_1 + \Theta_2) \cup (1 + \Theta_1 + \Theta_2) \mod 1.$$
This relation is compatible with $2 \cdot 2^{p_1-1} \cdot 2^{p_2-1} = 2^{p_1+p_2-1}$.

Finally, we have the following recursive relation, which emphasizes the following disjunction: if a bounded edge γ is lifted to a fixed edge, the lift of $\mathfrak{h}(\gamma)$ in Γ is either trivalent real vertex, a pentavalent vertex, or a quadrivalent vertex, and the two adjacent exchanged edges may be lifts of either γ_1 or γ_2. Therefore,

$$R = \left[\tfrac{1}{2} m_{\mathfrak{h}(\gamma)}\right] R_1 R_2 + \sum_{(\theta,\varphi)\in\Theta_1\times\Theta_2} S(\theta,\varphi) C_1 C_2 + \sum_{\theta_1\in\Theta_1} [2\theta_1 - 1]_+ \frac{\left[\frac{1}{2} m_{\mathfrak{h}(\gamma)}\right]}{[1]} C_1 R_2 + \sum_{\theta_2\in\Theta_2} [2\theta_2 - 1]_+ \frac{\left[\frac{1}{2} m_{\mathfrak{h}(\gamma)}\right]}{[1]} R_1 C_2.$$

Here:

- The first term corresponds to $\mathfrak{h}(\gamma)$ being lifted to a trivalent vertex, and the $\left[\frac{1}{2} m_{\mathfrak{h}(\gamma)}\right]$ accounts for the refined number of first-order solutions associated to $\chi_{\mathfrak{h}(\gamma)}$.
- The first sum corresponds to $\mathfrak{h}(\gamma)$ being lifted to a pentavalent vertex. Recall that the refined number of local lifts is $S(\theta,\varphi)$, and is computed in Corollary 6.20.
- The second sum corresponds to $\mathfrak{h}(\gamma)$ being lifted to a quadrivalent vertex whose adjacent complex edges are lifts of γ_1.
- Same with γ_2.

The final step is to compute

$$\left[\tfrac{1}{2} m_{\mathfrak{t}(\gamma)}\right] R + \frac{\left[\frac{1}{2} m_{\mathfrak{t}(\gamma)}\right]}{[1]} C \sum_{\theta\in\Theta} [2\theta - 1]_+$$

for the root edge of a component of Γ_{even}, and a surprising simplification occurs, leading to a mere product over the vertices. Using the recursive formulas, we prove the following lemma.

Lemma 6.24 *If γ_1 and γ_2 are the edges leaving V and γ is such that $V = \mathfrak{h}(\gamma)$, then*

$$R + \frac{C}{[1]} \sum_{\theta\in\Theta} [2\theta - 1]_+ = \left[\tfrac{1}{2} m_V\right] \left(R_1 + \frac{C_1}{[1]} \sum_{\theta_1\in\Theta_1} [2\theta_1 - 1]_+ \right) \left(R_2 + \frac{C_2}{[1]} \sum_{\theta_2\in\Theta_2} [2\theta_2 - 1]_+ \right).$$

Proof We use the recursive relation to get

$$R + \frac{C}{[1]} \sum_{\theta\in\Theta} [2\theta - 1]_+ = \left[\tfrac{1}{2} m_V\right] \left[R_1 R_2 + R_1 \frac{C_2}{[1]} \sum_{\Theta_2} [2\theta_2 - 1]_+ + R_2 \frac{C_1}{[1]} \sum_{\Theta_1} [2\theta_1 - 1]_+ \right] + \tfrac{1}{2} m_V \frac{C_1 C_2}{[1]} \sum_{\Theta} [2\theta - 1]_+ + C_1 C_2 \sum_{\Theta_1\times\Theta_2} S(\theta_1,\theta_2).$$

We now relate the sum over Θ and the sum over $\Theta_1\times\Theta_2$.

- The value of $S(\theta_1,\theta_2)$ given by Corollary 6.20 depends on whether $\theta_1<\theta_2$ and $\theta_1+\theta_2<1$. It is a sum of two terms. The first does not depend on this condition while the value of the second does. Thus,

$$\sum_{\Theta_1\times\Theta_2} S(\theta_1,\theta_2)=\frac{\left[\frac{1}{2}m_V\right]}{[1][1]}\sum_{\Theta_1\times\Theta_2}[2\theta_1-1]_+[2\theta_2-1]_+-\frac{m_V}{2[1]}\sum_{\substack{\theta_1<\theta_2\\ \theta_1+\theta_2<1}}[2\theta_1]_+[2\theta_2-1]_+ \\ -\frac{m_V}{2[1]}\sum_{\substack{\theta_1>\theta_2\\ \theta_1+\theta_2>1}}[2(1-\theta_1)]_+[2\theta_2-1]_+-\frac{m_V}{2[1]}\sum_{\substack{\theta_1>\theta_2\\ \theta_1+\theta_2<1}}[2\theta_1-1]_+[2\theta_2]_+ \\ -\frac{m_V}{2[1]}\sum_{\substack{\theta_1<\theta_2\\ \theta_1+\theta_2>1}}[2\theta_1-1]_+[2(1-\theta_2)]_+.$$

- For $\sum_\Theta[2\theta-1]_+$, we remember that $\Theta=(\Theta_1+\Theta_2)\cup(1+\Theta_1+\Theta_2)$, where the representatives are taken in $]0;1[$, mod 1 and up to sign. Since each θ stands for a quadruple $\pm e^{\pm i\pi\theta}$, the sign corresponds to the conjugation and the mod 1 to taking opposite moments. Thus, if $\theta_1,\theta_2\in\Theta_1\times\Theta_2$, with $0<\theta_1,\theta_2<1$, the choice of representative for both sets $\Theta_1+\Theta_2$ and $1+\Theta_1+\Theta_2$ also depends on the inequalities $\theta_1<\theta_2$ and $\theta_1+\theta_2<1$:

  - If $\theta_1<\theta_2$ and $\theta_1+\theta_2<1$, representatives in $]0;1[$ can be taken to be $\theta_2-\theta_1$ and $\theta_1+\theta_2$, and we have
  $$[2(\theta_2-\theta_1)-1]_+[2(\theta_1+\theta_2)-1]_+=[2\theta_1]_+[2\theta_2-1]_+.$$
  - If $\theta_1>\theta_2$ and $\theta_1+\theta_2>1$, representatives in $]0;1[$ can be taken to be $\theta_1-\theta_2$ and $\theta_1+\theta_2-1$, and we have
  $$[2(\theta_1-\theta_2)-1]_+[2(\theta_1+\theta_2-1)-1]_+=[2(1-\theta_1)]_+[2\theta_2-1]_+.$$
  - If $\theta_1>\theta_2$ and $\theta_1+\theta_2<1$, representatives in $]0;1[$ can be taken to be $\theta_1-\theta_2$ and $\theta_1+\theta_2$, and we have
  $$[2(\theta_1-\theta_2)-1]_+[2(\theta_1+\theta_2)-1]_+=[2\theta_1-1]_+[2\theta_2]_+.$$
  - If $\theta_1<\theta_2$ and $\theta_1+\theta_2>1$, representatives in $]0;1[$ can be taken to be $\theta_2-\theta_1$ and $\theta_1+\theta_2-1$, and we have
  $$[2(\theta_2-\theta_1)-1]_+[2(\theta_1+\theta_2-1)-1]_+=[2\theta_1-1]_+[2(1-\theta_2)]_+.$$

Thus, the sums where the disjunction occurs cancel themselves. We are left with

$$\begin{aligned}
&R+\frac{C}{[1]}\sum_{\theta\in\Theta}[2\theta-1]_+\\
&=\big[\tfrac{1}{2}m_V\big]\bigg[R_1R_2+R_1\frac{C_2}{[1]}\sum_{\Theta_2}[2\theta_2-1]_++R_2\frac{C_1}{[1]}\sum_{\Theta_1}[2\theta_1-1]_++\frac{C_1C_2}{[1][1]}\sum_{\Theta_1\times\Theta_2}[2\theta_1-1]_+[2\theta_2-1]_+\bigg]\\
&=\big[\tfrac{1}{2}m_V\big]\bigg[R_1R_2+R_1\frac{C_2}{[1]}\sum_{\Theta_2}[2\theta_2-1]_++R_2\frac{C_1}{[1]}\sum_{\Theta_1}[2\theta_1-1]_++\frac{C_1C_2}{[1][1]}\Big(\sum_{\Theta_1}[2\theta_1-1]_+\Big)\Big(\sum_{\Theta_2}[2\theta_2-1]_+\Big)\bigg]\\
&=\big[\tfrac{1}{2}m_V\big]\Big(R_1+\frac{C_1}{[1]}\sum_{\Theta_1}[2\theta_1-1]_+\Big)\Big(R_2+\frac{C_2}{[1]}\sum_{\Theta_2}[2\theta_2-1]_+\Big). \qquad\square
\end{aligned}$$

With this easier recursive relation, using that for an end,

$$R+\frac{C}{[1]}\sum_{\Theta}[2\theta-1]_+=0+\frac{1}{[1]}[2\theta-1]_+=\frac{[2\theta-1]_+}{[1]},$$

we immediately get that the contribution for a branch $\mathcal{B}$ with p ends is

$$\frac{1}{[1]^p}\prod_{i=1}^{p}[2\theta_i-1]_+\prod_{W\in\mathcal{B}}\big[\tfrac{1}{2}m_W\big].$$

To conclude, finishing the recursion over the real part of the curve using (i), we get a refined number of first-order solutions equal to

$$4\prod_i\frac{[2\theta_i-1]_+}{[1]}\prod_V\big[\tfrac{1}{2}m_V\big].$$

Last, to get quantum indices instead of log-areas, it suffices to make each θ_i go to 1, finally leading to the announced result

$$4\frac{2^{|s|}}{[1]^{|s|}}\prod_V\big[\tfrac{1}{2}m_V\big],$$

since $[0]_+=2$. $\square$

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