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Alexander Berglund
Tomáš Zeman

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Algebraic models for classifying spaces of fibrations

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We prove new structural results for the rational homotopy type of the classifying space $B\mathrm{aut}(X)$ of fibrations with fiber a simply connected finite CW-complex X.

We first study nilpotent covers of $B\mathrm{aut}(X)$ and show that their rational cohomology groups are algebraic representations of the associated transformation groups. For the universal cover, this yields an extension of the Sullivan–Wilkerson theorem to higher homotopy and cohomology groups. For the cover corresponding to the kernel of the homology representation, this proves algebraicity of the cohomology of the homotopy Torelli space.

For the cover that classifies what we call *normal unipotent fibrations*, we then prove the stronger result that there exists a nilpotent dg Lie algebra $\mathfrak{g}(X)$ in algebraic representations that models its equivariant rational homotopy type. This leads to an algebraic model for the space $B\mathrm{aut}(X)$ and to a description of its rational cohomology ring as the cohomology of a certain arithmetic group $\Gamma(X)$ with coefficients in the Chevalley–Eilenberg cohomology of $\mathfrak{g}(X)$. This has strong structural consequences for the cohomology ring and, in certain cases, allows it to be completely determined using invariant theory and calculations with modular forms. We illustrate these points with concrete examples.

As another application, we significantly improve on certain results on self-homotopy equivalences of highly connected even-dimensional manifolds due to Berglund and Madsen, and we prove parallel new results in odd dimensions.

55P62, 55R40; 11F75, 55R15

1 Introduction

In this paper we prove new structural results about the classifying space $B\,\mathrm{aut}(X)$ of the topological monoid of self-homotopy equivalences of a simply connected finite CW-complex X from the point of view of rational homotopy theory.

The fundamental group of $B\,\mathrm{aut}(X)$ may be identified with the group $\mathscr{E}(X)$ of homotopy classes of self-homotopy equivalences of X. By a deep result due to Sullivan [66] and Wilkerson [71], the group $\mathscr{E}(X)$ maps onto an arithmetic group with finite kernel. Our first result extends this in two directions: to more general deck transformation groups and to higher homotopy and (co)homology groups.

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Theorem 1.1 *Let $\mathscr{E}_G(X)$ be the deck transformation group of the cover $B\operatorname{aut}_G(X)$ of $B\operatorname{aut}(X)$ associated to a subgroup G of $\mathscr{E}(X)$. If G acts nilpotently on $H_*(X;\mathbb{Q})$, then $\mathscr{E}_G(X)$ maps onto an arithmetic group with finite kernel. Moreover, the representations of $\mathscr{E}_G(X)$ in the rational homology and simple homotopy groups*[1] *of $B\operatorname{aut}_G(X)$ are algebraic representations of the ambient algebraic group.*

This appears as Theorem 3.22 in the text. For G the trivial group, this is precisely the result of Sullivan and Wilkerson, enhanced by the statement that the representations of $\mathscr{E}(X)$ in the higher homotopy and homology groups of the universal cover of $B\operatorname{aut}(X)$ are algebraic; see Section 3.6.1. For G the kernel of the action of $\mathscr{E}(X)$ on $H_*(X;\mathbb{Z})$, this yields a homotopical counterpart of the algebraicity result of Kupers and Randal-Williams [49] for the cohomology of Torelli groups of the manifolds W_g — for *arbitrary* simply connected finite complexes X; see Section 3.6.2.

Our main result extends Theorem 1.1 further to a space-level statement, for a particular choice of cover, and we use this to construct an algebraic model for $B\operatorname{aut}(X)$.

The problem of finding an algebraic model for $B\operatorname{aut}(X)$ has been raised in Gómez-Tato, Halperin and Tanré [39, Section 7.3] and Lazarev [50, Section 7.2]. This problem is highly nontrivial, because the space $B\operatorname{aut}(X)$ is not nilpotent in general and therefore not amenable to the classical methods of rational homotopy theory. Algebraic models are known for the universal cover of $B\operatorname{aut}(X)$ (see Sullivan [66] and Tanré [67]) and for certain nilpotent covers (see Félix, Fuentes and Murillo [33]), but there are no general results that incorporate the action of the deck transformation groups into such models, let alone integrate such information to structural results about $B\operatorname{aut}(X)$. In this paper, we offer a solution. Instead of trying to incorporate the action of $\mathscr{E}(X)$ into models for the universal cover, which is the approach suggested in [39; 50], the key to our solution is pass to a cover with better properties.

Theorem 1.2 *There is a normal cover $B\operatorname{aut}_u(X)$ of $B\operatorname{aut}(X)$, with deck transformation group $\Gamma(X)$, such that*

1. *$\Gamma(X)$ is an arithmetic subgroup of a reductive algebraic group $R(X)$,*
2. *$B\operatorname{aut}_u(X)$ is $\Gamma(X)$-equivariantly rationally equivalent to the nerve $\langle\mathfrak{g}(X)\rangle$ of a nilpotent dg Lie algebra $\mathfrak{g}(X)$ of algebraic representations of $R(X)$.*

The proof is given in Section 3.7. Here, we say that two connected spaces are *rationally equivalent* if they can be connected by a zig-zag of maps that induce rational isomorphisms[2] on all homotopy and homology groups. The *nerve* is the functor from dg Lie algebras to topological spaces that effects the

[1] By the *simple* homotopy groups of a connected space, we mean the homotopy groups modulo the action of the fundamental group; see Definition 3.20.

[2] We call a group homomorphism a rational isomorphism if its kernel is torsion and if for each element x of the target, the k^{th} power x^k belongs to the image for some $k \neq 0$; see Definition 2.1. For abelian groups, this is equivalent to inducing an isomorphism after tensoring with $\mathbb{Q}$.

equivalence between the homotopy category of nilpotent dg Lie algebras and the homotopy category of rational nilpotent spaces; see Section 2.4. The cover $B\operatorname{aut}_u(X)$ can be characterized as the classifying space for what we call *normal unipotent fibrations*; see Section 4.2.

As a first application, Theorem 1.2 yields the following commutative dg algebra model for $B\operatorname{aut}(X)$.

Corollary 1.3 *The space $B\operatorname{aut}(X)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{g}(X)\rangle_{h\Gamma(X)}$ and there is a zig-zag of quasi-isomorphisms of commutative dg algebras*

$$\Omega^*(B\operatorname{aut}(X)) \sim \Omega^*(\Gamma(X), C^*_{\mathrm{CE}}(\mathfrak{g}(X))).$$

This is proved as Theorem 3.39 in the text. Here $\Omega^*(B)$ stands for Sullivan's polynomial differential forms on B. It is a commutative dg algebra model for the singular cochains $C^*(B;\mathbb{Q})$. The right-hand side is a commutative dg algebra model for the cochains on the group $\Gamma(X)$ with coefficients in the Chevalley–Eilenberg cochains of $\mathfrak{g}(X)$; see Section 3.8.

Theorem 1.2 can be interpreted as a "space-level" enhancement of Theorem 1.1 in the following sense: not only are the rational cohomology groups of $B\operatorname{aut}_u(X)$ algebraic representations of the ambient algebraic group $R(X)$, but the entire *rational homotopy type* of $B\operatorname{aut}_u(X)$, represented by the dg Lie algebra $\mathfrak{g}(X)$, is an algebraic representation of $R(X)$. This subsumes algebraicity of cohomology, because there is an isomorphism of $\Gamma(X)$-modules

$$H^*(B\operatorname{aut}_u(X);\mathbb{Q}) \cong H^*_{\mathrm{CE}}(\mathfrak{g}(X)),$$

and the right-hand side is obviously an algebraic representation of $R(X)$ if $\mathfrak{g}(X)$ is. However, space-level algebraicity has much stronger consequences than algebraicity of cohomology. Together with semisimplicity of algebraic representations of reductive groups, it implies the following result, which reduces the computation of the rational cohomology ring of $B\operatorname{aut}(X)$ to the computation of Chevalley–Eilenberg cohomology and cohomology of arithmetic groups with coefficients in algebraic representations.

Corollary 1.4 *There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}(X);\mathbb{Q}) \cong H^*(\Gamma(X), H^*_{\mathrm{CE}}(\mathfrak{g}(X))). \tag{1}$$ □

The proof is given in Section 3.9. This result implies, but is stronger than, collapse of the rational Serre spectral sequence of the homotopy fiber sequence

$$B\operatorname{aut}_u(X) \to B\operatorname{aut}(X) \to B\Gamma(X). \tag{2}$$

Indeed, collapse of the spectral sequence only implies an isomorphism of algebras after passing to the associated graded algebra of some filtration, whereas Corollary 1.4 says that $H^*(B\operatorname{aut}(X);\mathbb{Q})$ is isomorphic to the E_2-page of this spectral sequence as a graded algebra. In particular, this algebra admits a bigrading by

$$H^{p,q} = H^p(\Gamma(X), H^q_{\mathrm{CE}}(\mathfrak{g}(X))). \tag{3}$$

Combined with the finiteness of the virtual cohomological dimension of arithmetic groups (see Borel and Serre [21]), this implies the following result, which reduces the computation of the ring $H^*(B\operatorname{aut}(X);\mathbb{Q})$ modulo nilpotent elements to invariant theory.

Corollary 1.5 *The ring homomorphism*

$$H^*(B\operatorname{aut}(X);\mathbb{Q}) \to H^*(B\operatorname{aut}_u(X);\mathbb{Q})^{\Gamma(X)} \tag{4}$$

is split surjective and its kernel I is a nilpotent ideal such that $I^n = 0$ for all $n > \operatorname{vcd}(\Gamma(X))$. In particular, (4) induces an isomorphism modulo nilradicals. In other words, the algebraic variety of the graded commutative ring $H^*(B\operatorname{aut}(X);\mathbb{Q})$ is isomorphic to that of the invariant ring $H^*_{\mathrm{CE}}(\mathfrak{g}(X))^{\Gamma(X)}$.

Proof Under the isomorphism (1), the homomorphism (4) corresponds to the projection $H^{*,*} \to H^{0,*}$, which is clearly split. The kernel is the ideal $H^{\geq 1,*}$, the n^{th} power of which is contained in $H^{\geq n,*}$, which vanishes for $n > \operatorname{vcd}(\Gamma(X))$. □

Let us point out another nontrivial facet of Corollary 1.4.

Corollary 1.6 *The ring homomorphism*

$$H^*(\Gamma(X),\mathbb{Q}) \to H^*(B\operatorname{aut}(X);\mathbb{Q})$$

is split injective.

Proof The homomorphism in question corresponds to the inclusion of the subring $H^{*,0}$ into $H^{*,*}$, which is clearly split. □

This means in particular that all cohomology classes of the arithmetic group $\Gamma(X)$ are faithfully represented as characteristic classes of fibrations with fiber X. This is especially striking in view of the fact that many arithmetic groups can be realized as $\Gamma(X)$ for some X, cf [66, Theorem 10.3(iv)].

We have chosen to state our main result as an existence theorem in order to highlight the strong consequences of the mere existence of a dg Lie model of algebraic representations, but the ingredients $R(X)$, $\Gamma(X)$ and $\mathfrak{g}(X)$ can be given concrete descriptions; see Section 4. In Section 5 we use these concrete descriptions to make several explicit computations. Let us discuss one of these computations here. For the n-fold product of a d-dimensional sphere, $S^{d\times n} = S^d \times \cdots \times S^d$, Corollary 1.4 assumes the following form.

Theorem 1.7 *For d odd, there is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}(S^{d\times n});\mathbb{Q}) \cong H^*(\Gamma(S^{d\times n}),\operatorname{Sym}^\bullet(V_n[d+1])), \tag{5}$$

where $V_n[d+1]$ denotes the standard representation of $\mathsf{GL}_n(\mathbb{Q})$ put in degree $d+1$,

$$\Gamma(S^{d\times n}) = \begin{cases} \mathsf{GL}_n(\mathbb{Z}) & \text{if } d = 1,3,7, \\ \mathsf{GL}_n^{\Sigma}(\mathbb{Z}) & \text{if } d \neq 1,3,7, \end{cases}$$

and $\mathsf{GL}_n^{\Sigma}(\mathbb{Z}) \leqslant \mathsf{GL}_n(\mathbb{Z})$ is the congruence subgroup of matrices with exactly one odd entry in each row.

For $n = 2$ the right-hand side of (5) can be computed in terms of modular forms via the Eichler–Shimura isomorphism. We also carry out computations for $n = 3$, but for larger n the cohomology of $\mathsf{GL}_n(\mathbb{Z})$, or its congruence subgroups, with coefficients in algebraic representations is not fully known. See Section 5.1 for more details and a further discussion.

The above example illustrates that even in cases where $\Gamma(X)$ and $H^*_{\mathrm{CE}}(\mathfrak{g}(X))$ can be described explicitly, a complete computation of the right-hand side of (1) is in general out of reach, due to the difficulty of calculating the cohomology of arithmetic groups. This suggests a different perspective on the isomorphism (1). Rather than interpreting it as a computation of $H^*(B\operatorname{aut}(X);\mathbb{Q})$, it tells us that cohomology classes of arithmetic groups in the right-hand side, say classes constructed using automorphic forms, can in principle be represented as characteristic classes of fibrations. Connections between characteristic classes and automorphic forms have been observed before in special cases; see eg Furusawa, Tezuka and Yagita [36]. Our results show that this is not an isolated phenomenon. This suggests that there is a deep connection between characteristic classes of fibrations and cohomology of arithmetic groups.

Another important application of our results is that they lead to significant simplifications and improvements of certain key results of Berglund and Madsen [16] about the cohomology of $B\operatorname{aut}(W_g)$, and related spaces, for the manifold

$$W_g = \#^g\, S^d \times S^d.$$

In fact, this was the original motivation for this work. This is discussed in Section 5.2. Our methods also yield parallel results for highly connected odd-dimensional manifolds that were unattainable by the methods of [16]; see Section 5.3. This is used in the work of Stoll [65].

In the final section, Section 5.4, we study an example that, among other things, illuminates the advantage of working over $\Gamma(X)$ rather than $\mathscr{E}(X)$.

1.1 Some comments on related work

The first paragraph on page 314 in Sullivan's [66] contains, without proof, the idea of modeling $B\operatorname{aut}(|\Lambda|)$, where $|\Lambda|$ is the realization of a minimal Sullivan algebra Λ, by taking the nerve of the maximal nilpotent ideal of $\operatorname{Der}\Lambda$ modulo the action of the reductive part of $\operatorname{Aut}\Lambda$. This idea seems to have been largely overlooked in the subsequent rational homotopy theory literature; we are not aware of any source where this idea and its consequences have been properly developed (and in fact we only became aware of this paragraph in the final stages of writing this paper). Theorem 4.14 could be viewed as giving a precise formulation and proof. The key points of the present paper — the treatment of $B\operatorname{aut}(X)$ for nonrational X, the algebraicity of the cohomology of nilpotent covers of $B\operatorname{aut}(X)$, the existence of algebraic Lie models and its strong consequences for the structure of the cohomology ring of $B\operatorname{aut}(X)$ — are to the best of our knowledge new. Our results could be regarded as a strong vindication of Sullivan's idea.

We were inspired by Oprea [57] (via Burghelea [26]) for the idea of passing to the maximal reductive quotient of $\mathscr{E}(X_{\mathbb{Q}})$ to rectify homotopy actions on the algebraic models. The idea of studying the fiberwise rationalization of $B\operatorname{aut}(X) \to B\Gamma(X)$ as we do here is similar in spirit to studying relative Malcev completions of mapping class groups as done by Hain [41]. The algebraicity result for the cohomology of the Torelli group of W_g of Kupers and Randal-Williams [49] inspired us to study similar questions for $B\operatorname{aut}(X)$.

Lazarev [50, Theorem 5.1] constructs Lie models for the universal cover of $B\operatorname{aut}(X_{\mathbb{Q}})$ and shows that the action of the Lie algebra of $\mathscr{E}(X_{\mathbb{Q}})$ on the higher homotopy groups of $B\operatorname{aut}(X_{\mathbb{Q}})$ can be computed in terms of Chevalley–Eilenberg and Harrison cohomology. However, he does not address algebraicity of the representations and he does not construct group actions on the Lie models; in fact he raises this as a problem [50, Section 7.2].

Félix, Fuentes and Murillo [33] construct a Lie model for the space $B\operatorname{aut}_G(X)$ when $G \leqslant \mathscr{E}(X)$ is a subgroup that acts nilpotently on $H_*(X;\mathbb{Z})$. We recover this Lie model; see Corollary 3.15 and Remark 3.17. A crucial advantage of our approach is that it lets us incorporate the action of the deck transformation group. This aspect is not addressed in [33]. This is what allows us to construct an algebraic model for the full space $B\operatorname{aut}(X)$ and not only for nilpotent covers of it.

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2 Background and preliminaries

2.1 Localization of nilpotent groups and spaces

We begin by introducing some terminology and recalling some facts about localizations of nilpotent groups and spaces, mainly following [44].

Definition 2.1 We call a group G uniquely divisible, or $\mathbb{Q}$-local, if the equation $x^k = a$ has a unique solution $x \in G$ for every nonzero integer k and every $a \in G$. For a group homomorphism $f\colon G \to H$, we use the following terminology:

1. (1) f is $\mathbb{Q}$-injective if every element of $\ker(f)$ has finite order.
2. (2) f is $\mathbb{Q}$-surjective if for all $x \in H$, we have $x^k \in \operatorname{im}(f)$ for some $k \neq 0$.
3. (3) f is a $\mathbb{Q}$-isomorphism if it is both $\mathbb{Q}$-injective and $\mathbb{Q}$-surjective.

Every nilpotent group G admits a $\mathbb{Q}$-localization $r\colon G \to G_{\mathbb{Q}}$, characterized up to isomorphism by the properties that $G_{\mathbb{Q}}$ is a $\mathbb{Q}$-local nilpotent group and that r is a $\mathbb{Q}$-isomorphism; see [44, page 7]. A homomorphism $G \to H$ between nilpotent groups is a $\mathbb{Q}$-isomorphism if and only if the induced homomorphism $G_{\mathbb{Q}} \to H_{\mathbb{Q}}$ is an isomorphism. For abelian groups G, one has $G_{\mathbb{Q}} \cong G \otimes \mathbb{Q}$.

Recall that a connected topological space X is called nilpotent if the group $\pi_1(X)$ is nilpotent and if its action on $\pi_n(X)$ is nilpotent for every $n \geq 2$, in the sense that there is a filtration of $\pi_n(X)$ by $\pi_1(X)$-submodules such that the action on the filtration quotients is trivial.

A nilpotent space X is called $\mathbb{Q}$-local if the group $\pi_n(X)$ is $\mathbb{Q}$-local for each $n \geq 1$. Every nilpotent space X admits a $\mathbb{Q}$-localization, or rationalization, $r\colon X \to X_{\mathbb{Q}}$, characterized up to homotopy by the properties that $X_{\mathbb{Q}}$ is a $\mathbb{Q}$-local nilpotent space and that $\pi_n(r)\colon \pi_n(X) \to \pi_n(X_{\mathbb{Q}})$ is a $\mathbb{Q}$-localization for every n.

Definition 2.2 We call a map $f\colon X \to Y$ between connected topological spaces a *rational homotopy equivalence* if $\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$ is a $\mathbb{Q}$-isomorphism for all n, and a *rational homology equivalence* if $H_n(f)\colon H_n(X;\mathbb{Q}) \to H_n(Y;\mathbb{Q})$ is an isomorphism for all n.

We say that f is a *rational equivalence* if it is both a rational homotopy equivalence and a rational homology equivalence.

It is well-known that a map between nilpotent spaces is a rational homotopy equivalence if and only if it is a rational homology equivalence. We will need an extension of this fact to virtually nilpotent spaces. Recall from [30; 31] that a connected space X is called *virtually nilpotent* if $\pi_1(X)$ has a nilpotent subgroup of finite index and for each $n \geq 2$, there is a finite-index subgroup of $\pi_1(X)$ that acts nilpotently on $\pi_n(X)$. Equivalently, X is virtually nilpotent if each Postnikov section $P_n X$ of X admits a finite cover $E \to P_n X$ such that E is nilpotent.

Lemma 2.3 *Let $f\colon X \to Y$ be a map from a virtually nilpotent space X to a nilpotent space Y. If f is a rational homotopy equivalence, then f is a rational homology equivalence.*

Proof It suffices to show that f induces a rational homology equivalence on each Postnikov section, so we may without loss of generality assume that X admits a finite cover $p\colon E \to X$ such that E is nilpotent. Clearly, p is a rational homotopy equivalence. The composite $fp\colon E \to Y$ is then a rational homotopy equivalence between nilpotent spaces, so it is a rational homology equivalence. Since p is a finite cover, a transfer argument shows that $H_*(p;\mathbb{Q})\colon H_*(E;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is surjective. We just saw that $H_*(f;\mathbb{Q}) \circ H_*(p;\mathbb{Q}) = H_*(fp;\mathbb{Q})$ is an isomorphism, so $H_*(p;\mathbb{Q})$ must be injective as well. It follows that $H_*(f;\mathbb{Q})$ is an isomorphism. □

Remark 2.4 The converse is true if X is nilpotent, but false in general: the map $\mathbb{R}\mathrm{P}^2 \to *$ is a rational homology equivalence from a virtually nilpotent space to a nilpotent space, but the induced map on π_2 is not a $\mathbb{Q}$-isomorphism.

Also, one cannot relax nilpotence of Y to virtual nilpotence. The universal cover $S^2 \to \mathbb{R}\mathrm{P}^2$ provides an example of a map from a nilpotent space to a virtually nilpotent space which is a rational homotopy equivalence but not a rational homology equivalence.

2.2 Affine algebraic groups and arithmetic groups

In this section we will collect the basic facts about affine algebraic groups over $\mathbb{Q}$ and arithmetic groups that we will need, following mainly [45; 55; 62]. All algebras, vector spaces, undecorated tensor products, affine schemes, etc, should be taken to be over $\mathbb{Q}$ unless explicitly specified otherwise.

Recall that an affine algebraic group is a group object in the category of affine schemes. A linear algebraic group is an algebraic subgroup of GL_n for some n. It is a well-known fact that every affine algebraic group admits a faithful algebraic representation [55, Corollary 4.10], so the notions of affine algebraic group and linear algebraic group essentially coincide.

2.2.1 Unipotent and reductive groups Recall that a representation V is called unipotent (or nilpotent) if there is a sequence of subrepresentations

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V$$

such that V_i/V_{i-1} is a trivial representation for every i.

An affine algebraic group U is called unipotent if every algebraic representation V of U is unipotent. Equivalently, U is unipotent if and only if it is an algebraic subgroup of the group $\mathbb{U}_n$ of $n \times n$ upper-triangular matrices with ones along the diagonal for some n; see [55, Theorem 14.5]. The group of $\mathbb{Q}$-points $U(\mathbb{Q})$ of a unipotent group U is, in particular, nilpotent and uniquely divisible.

Every affine algebraic group G admits a largest normal unipotent subgroup, called the unipotent radical G_u of G; see [45, Theorem 10.5]. An affine algebraic group G is called reductive if G_u is trivial. The representation theory of reductive groups in characteristic zero is particularly well-behaved:

Theorem 2.5 [55, Corollary 22.43], [45, page 78] *Every finite-dimensional algebraic representation of a reductive group is semisimple.* □

In other words, the category $\mathrm{Rep}_{\mathbb{Q}}(G)$ of finite-dimensional algebraic representations of G is a semisimple abelian category whenever G is reductive.

Another feature of affine algebraic groups in characteristic zero is the existence of Levi decompositions. There is an extension

$$1 \to G_u \to G \to G/G_u \to 1 \tag{6}$$

of the maximal reductive quotient of G by the unipotent radical.

Theorem 2.6 [45, Theorem 14.2] *The extension (6) is (noncanonically) split and moreover, any two splittings are conjugate in the action of G_u.* □

The following lemma gives a concrete description of the unipotent radical and the maximal reductive quotient. It is presumably well-known, but we include a proof as we have not found the precise statement we give here in the literature.

Recall that a composition series of a representation V is a filtration

$$0 = V_0 \subseteq \cdots \subseteq V_n = V$$

by subrepresentations such that each V_i/V_{i-1} is a simple representation. The "semisimplification" of V is the associated graded representation

$$V^{\mathrm{ss}} = \bigoplus_{i=1}^{n} V_i/V_{i-1}.$$

It is semisimple by construction and its isomorphism type is independent of the choice of composition series by the Jordan–Hölder theorem. Note that V is isomorphic to V^{ss} if and only if V is semisimple.

Lemma 2.7 *Let G be an affine algebraic group defined over $\mathbb{Q}$. If V is a representation of G with unipotent kernel, then the unipotent radical G_u and the maximal reductive quotient G/G_u may be identified with the kernel and the image, respectively, of the homomorphism $G \to \mathsf{GL}(V^{\mathrm{ss}})$.*

Proof Let K and N denote the kernel of the action of G on V and V^{ss}, respectively. Clearly, both K and N are normal in G and $K \leqslant N$. Furthermore, K is unipotent by hypothesis, and N/K is unipotent because V is a faithful unipotent representation of it. Since unipotent groups are closed under extensions (see [55, Corollary 14.7]), it follows that N is unipotent.

To show that N is the unipotent radical, we need to show that every normal unipotent subgroup U of G acts trivially on V^{ss}. For this, it suffices to show that U acts trivially on every simple G-representation W. Since U is unipotent, there is a nonzero $w \in W$ that is fixed by U; see [55, Proposition 14.1]. Since U is normal in G, it also fixes gw for every $g \in G$. Indeed, for every $u \in U$ we have $g^{-1}ug \in U$, whence $(g^{-1}ug)w = w$ so that $ugw = gw$. Since W is simple, w generates W as a G-module, so U acts trivially on W. □

This has the following consequence:

Lemma 2.8 *Let G be an affine algebraic group defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$. If V is a representation of G with unipotent kernel, then the Lie algebra of the unipotent radical, $\mathrm{Lie}\, G_u$, may be identified with the maximal ideal*

$$\mathrm{nil}_V\, \mathfrak{g} \subseteq \mathfrak{g},$$

consisting of elements which act nilpotently on V.

Proof By [55, Section 10.14], the functor $G \mapsto \mathrm{Lie}\, G$ from affine algebraic groups to Lie algebras commutes with pullbacks, so in particular it preserves kernels of morphisms. Thus by the preceding lemma,

$$\mathrm{Lie}\, G_u = \ker\left(\mathfrak{g} \longrightarrow \mathfrak{gl}(V^{\mathrm{ss}})\right). \tag{7}$$

Thus we have that $\mathrm{Lie}\, G_u \subseteq \mathrm{nil}_V\, \mathfrak{g}$.

For the reverse inclusion, it suffices to show that $\mathrm{nil}_V\,\mathfrak{g}$ is the Lie algebra of a normal unipotent subgroup H of G, as then $H \leqslant G_u$ and $\mathrm{nil}_V\,\mathfrak{g} \subseteq \mathrm{Lie}\,G_u$. Let K be the kernel of the action of G on V, and let $\mathfrak{h}'$ be the image of $\mathrm{nil}_V\,\mathfrak{g}$ under the action morphism $\mathfrak{g} \to \mathfrak{gl}(V)$. Then $\mathfrak{h}'$ consists of nilpotent endomorphisms of V, so by Engel's theorem, $\mathfrak{h}'$ is a nilpotent subalgebra of $\mathrm{Lie}(G/K) \subseteq \mathfrak{gl}(V)$. Thus by [55, Theorem 14.37] there is a unipotent subgroup $H' \leqslant G/K \leqslant \mathsf{GL}(V)$ such that $\mathfrak{h}' = \mathrm{Lie}\,H'$. Let H be the preimage of H' in G. Then H is an extension of the unipotent group H' by the unipotent group K, hence unipotent, and $\mathrm{Lie}\,H$ is therefore nilpotent. Moreover, since the functor Lie commutes with pullbacks, we get that $\mathrm{nil}_V\,\mathfrak{g} = \mathrm{Lie}\,H$. Finally, since $\mathrm{nil}_V\,\mathfrak{g}$ is an ideal of $\mathfrak{g}$, the subgroup H of G is normal. □

The quotient G/H of an affine algebraic group G by a normal algebraic subgroup H always exists [55, Theorem 5.14], but the rational points of the quotient $(G/H)(\mathbb{Q})$ need not agree with $G(\mathbb{Q})/H(\mathbb{Q})$ in general. They do agree, however, if H is unipotent.

Lemma 2.9 *Let G be an affine algebraic group. If H is a normal unipotent algebraic subgroup of G, then the natural homomorphism*

$$G(\mathbb{Q})/H(\mathbb{Q}) \to (G/H)(\mathbb{Q})$$

is an isomorphism.

Proof This is a consequence of the vanishing of Galois cohomology,

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), H),$$

for unipotent groups H; see [71, Theorem 9.5]. □

Similarly, the normalizer $N_G(H)$ of an algebraic subgroup H of an affine algebraic group G always exists and its rational points, $N_G(H)(\mathbb{Q})$, are contained in $N_{G(\mathbb{Q})}(H(\mathbb{Q}))$, the normalizer of $H(\mathbb{Q})$ in $G(\mathbb{Q})$ in the ordinary sense [55, Proposition 1.83]. In general, equality need not hold, but it does if H is unipotent.

Lemma 2.10 *If H is a unipotent algebraic subgroup of an affine algebraic group G, then $N_G(H)(\mathbb{Q}) = N_{G(\mathbb{Q})}(H(\mathbb{Q}))$.*

Proof This follows from [55, Proposition 1.84] upon noting that every affine algebraic group over $\mathbb{Q}$ is smooth [55, Theorem 3.23] and that $H(\mathbb{Q})$ is dense in H if H is unipotent (eg by [55, Theorem 17.93]). □

2.2.2 Arithmetic subgroups Given a linear algebraic group $G \leqslant \mathsf{GL}_n$, we write

$$G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathsf{GL}_n(\mathbb{Z})$$

for the integer matrices inside the rational points of G. By definition, an arithmetic subgroup Γ of G is a subgroup of $G(\mathbb{Q})$ which is commensurable with $G(\mathbb{Z})$, ie the intersection $\Gamma \cap G(\mathbb{Z})$ has finite index in both $G(\mathbb{Z})$ and Γ.

Arithmeticity can be thought of as a strong finiteness property. In particular, arithmetic groups are finitely presented [62].

Lemma 2.11 [62, Section 1.1] *Let G be an affine algebraic group over $\mathbb{Q}$ and let $H \leqslant G$ be an algebraic subgroup. If Γ is arithmetic in $G(\mathbb{Q})$, then $\Gamma \cap H(\mathbb{Q})$ is arithmetic in $H(\mathbb{Q})$.*

Lemma 2.12 [18, Theorem 6] *Let $\varphi\colon G \to G'$ be a surjective morphism of affine algebraic groups defined over $\mathbb{Q}$. If Γ is an arithmetic subgroup of $G(\mathbb{Q})$, then $\varphi(\Gamma)$ is an arithmetic subgroup of $G'(\mathbb{Q})$.*

We record the following, presumably well-known, characterization of arithmetic subgroups of unipotent groups for later use. We include a proof for completeness.

Lemma 2.13 *Let U be a unipotent algebraic group defined over $\mathbb{Q}$ and let Γ be a subgroup of $U(\mathbb{Q})$.*

(1) *Γ is Zariski dense in U if and only if the inclusion of Γ into $U(\mathbb{Q})$ is $\mathbb{Q}$-surjective.*
(2) *Γ is an arithmetic subgroup of $U(\mathbb{Q})$ if and only if it is finitely generated and Zariski dense.*

Proof We may assume that U is a subgroup of $\mathbb{U}_n$ for some n. Following [61, page 104], consider the Lie subalgebra $\mathscr{L}(\Gamma)$ of the Lie algebra $\mathfrak{u}$ of U given by the linear span of the image of Γ under the bijection $\log\colon U(\mathbb{Q}) \to \mathfrak{u}$. The associated unipotent algebraic subgroup $\exp\mathscr{L}(\Gamma) \leqslant U$ contains Γ, so it must be equal to U if Γ is Zariski dense. The inclusion $\Gamma \to U(\mathbb{Q})$ is then $\mathbb{Q}$-surjective by [61, Theorem 2(iv), page 104]. Conversely, if the inclusion of Γ into $U(\mathbb{Q})$ is $\mathbb{Q}$-surjective, then so is the inclusion of the Zariski closure $\bar{\Gamma}$. This is then a $\mathbb{Q}$-isomorphism between nilpotent uniquely divisible groups, so it must be an equality. This proves the first statement.

It follows from [61, Exercise 13, page 123] that every finitely generated dense subgroup of $U(\mathbb{Q})$ is arithmetic. Conversely, if Γ is arithmetic, then it is finitely generated, and we will now argue that the inclusion $\Gamma \to U(\mathbb{Q})$ is $\mathbb{Q}$-surjective. Inclusions of finite-index subgroups are clearly $\mathbb{Q}$-isomorphisms, so it suffices to show that the inclusion of $U(\mathbb{Z})$ into $U(\mathbb{Q})$ is $\mathbb{Q}$-surjective, ie that for every $A \in U(\mathbb{Q})$, there is a positive integer k such that $A^k \in U(\mathbb{Z})$. Since A is a unipotent $n \times n$ matrix, the matrix $N = A - I$ satisfies $N^n = 0$. Pick a positive integer d such that dN^i has integer entries for all $i \geq 1$, and let $k = 1!2!\cdots n!d$. Then the matrix

$$A^k = \sum_{i=0}^{n-1} \binom{k}{i} N^i$$

has integer entries, because each coefficient $\binom{k}{i}$ is divisible by d. □

Since $U(\mathbb{Q})$ is nilpotent and $\mathbb{Q}$-local when U is unipotent, the group $U(\mathbb{Q})$ will be a $\mathbb{Q}$-localization of the nilpotent group $\Gamma \leqslant U(\mathbb{Q})$ whenever Γ is dense in U.

Remark 2.14 One cannot relax unipotence to nilpotence in Lemma 2.13. The multiplicative group is abelian and in particular nilpotent, but the inclusion of $\mathbb{Z}^\times$ into $\mathbb{Q}^\times$ is not $\mathbb{Q}$-surjective.

2.3 Nilpotent radicals of dg Lie algebras

In this section we discuss the notion of nilradical in the setting of differential graded Lie algebras. Let $(\mathfrak{g}, \delta)$ be a dg Lie algebra, possibly unbounded as a chain complex. We write $\Gamma^k \mathfrak{g}$ for the lower central series of $\mathfrak{g}$, so $\Gamma^1 \mathfrak{g} = \mathfrak{g}$ and $\Gamma^{k+1} \mathfrak{g} = [\Gamma^k \mathfrak{g}, \mathfrak{g}]$. Given an integer k, we write $\mathfrak{g}\langle k \rangle$ for the truncation of $\mathfrak{g}$ given by

$$\mathfrak{g}\langle k \rangle_n = \begin{cases} 0 & \text{if } n < k, \\ \ker(\delta\colon \mathfrak{g}_n \to \mathfrak{g}_{n-1}) & \text{if } n = k, \\ \mathfrak{g}_n & \text{if } n > k. \end{cases}$$

We call $\mathfrak{g}$ *connected* if $\mathfrak{g} = \mathfrak{g}\langle 0 \rangle$ and we call $\mathfrak{g}$ *simply connected* if $\mathfrak{g} = \mathfrak{g}\langle 1 \rangle$. We say that $\mathfrak{g}$ is of *finite type* if $\mathfrak{g}_i$ is finite-dimensional for every i.

Definition 2.15 A connected dg Lie algebra $\mathfrak{g}$ is *nilpotent* if the following equivalent conditions are satisfied:

(i) for every n, we have $(\Gamma^k \mathfrak{g})_n = 0$ for $k \gg n$;

(ii) $\mathfrak{g}_0$ is a nilpotent Lie algebra which acts nilpotently on $\mathfrak{g}_n$ for every $n > 0$.

Definition 2.16 The *nilradical* of $\mathfrak{g}$, denoted by $\operatorname{nil} \mathfrak{g}$, is the maximal nilpotent ideal of $\mathfrak{g}\langle 0 \rangle$, provided such an ideal exists.

Remark 2.17 Ordinary Lie algebras may be identified with dg Lie algebras concentrated in degree 0. For these, the above definitions specialize to the usual definitions of nilpotence and nilradicals. In particular, the nilradical does not necessarily exist unless certain finiteness conditions are imposed. If the nilradical exists, however, then it is unique: if I, J are nilpotent ideals of $\mathfrak{g}$, then so is $I + J$. Thus if I is maximal, then $J \subseteq I + J = I$.

Lemma 2.18 *Suppose that $Z_0(\mathfrak{g})$ is finite-dimensional. Then* $\operatorname{nil} \mathfrak{g}$ *exists.*

Proof We may assume that $\mathfrak{g}$ is connected. Note that the positively graded truncation $\mathfrak{g}\langle 1 \rangle$ is always a nilpotent ideal of $\mathfrak{g}$, so the poset of all nilpotent ideals contains a maximal element if and only if the poset of nilpotent ideals above $\mathfrak{g}\langle 1 \rangle$ does. But the latter poset injects into the poset of nilpotent (or indeed all) ideals of the quotient dg Lie algebra $\mathfrak{g}' = \mathfrak{g}/\mathfrak{g}\langle 1 \rangle$. Clearly $\mathfrak{g}'_i = 0$ for $i > 1$, the differential $\mathfrak{g}'_1 \to \mathfrak{g}'_0$ is injective, and $\mathfrak{g}'_0 = \mathfrak{g}_0$ is finite-dimensional, so every chain of ideals of $\mathfrak{g}'$ is finite. □

2.4 Geometric realizations of nilpotent dg Lie algebras

To each nilpotent Lie algebra $\mathfrak{g}$ over $\mathbb{Q}$, one can associate a group $\exp(\mathfrak{g})$ with underlying set $\mathfrak{g}$ and with multiplication given by the Baker–Campbell–Hausdorff formula. The association $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ is part of an equivalence of categories between nilpotent Lie algebras over $\mathbb{Q}$ and nilpotent uniquely divisible groups; see [59, Appendix A]. When $\mathfrak{g}$ is finite-dimensional, $\exp(\mathfrak{g})$ can be given the structure of an affine algebraic group, and the association $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ is part of an equivalence between the category of finite-dimensional nilpotent Lie algebras over $\mathbb{Q}$ and the category of unipotent algebraic groups over $\mathbb{Q}$; see [55, Theorem 14.37].

Now let $\mathfrak{g}$ be a nilpotent dg Lie algebra and consider the simplicial group

$$\exp_\bullet \mathfrak{g} = \exp Z_0(\mathfrak{g} \otimes \Omega_\bullet),$$

where $\Omega_\bullet$ denotes the simplicial commutative differential graded algebra over $\mathbb{Q}$ of polynomial differential forms on the standard simplices.

Proposition 2.19 [13, Theorem 6.2] *For every nilpotent dg Lie algebra* $\mathfrak{g}$, *there is a natural isomorphism of groups*,

$$\exp H_0(\mathfrak{g}) \to \pi_0 \exp_\bullet(\mathfrak{g}), \tag{8}$$

and a natural isomorphism of abelian groups

$$H_k(\mathfrak{g}) \to \pi_k \exp_\bullet(\mathfrak{g})$$

for every $k > 0$, *compatible with the actions of the groups in* (8). □

By [13, Theorem 5.2(2)], a natural model for the classifying space $B \exp_\bullet(\mathfrak{g})$ is given by the nerve, or Maurer–Cartan space. This is the simplicial set defined by

$$\mathsf{MC}_\bullet(\mathfrak{g}) = \mathsf{MC}(\mathfrak{g} \otimes \Omega_\bullet),$$

where MC denotes the set of Maurer–Cartan elements in a dg Lie algebra, ie the solutions to the equation

$$\delta(\tau) + \tfrac{1}{2}[\tau, \tau] = 0.$$

Note that Proposition 2.19 implies that $|\mathsf{MC}_\bullet(\mathfrak{g})|$ is a $\mathbb{Q}$-local nilpotent space if $\mathfrak{g}$ is a nilpotent dg Lie algebra.

Definition 2.20 A *Lie model* for a space B is by definition a dg Lie algebra $\mathfrak{g}$ such that B is rationally homology equivalent to $|\mathsf{MC}_\bullet(\mathfrak{g})|$.

If X is a simply connected space, then Quillen's dg Lie algebra $\lambda(X)$ [59] is a Lie model for X in the sense of the above definition (this follows from [13, Theorem 8.1]). Recall (see eg [8]) that every simply connected space X admits a unique, up to noncanonical isomorphism, Lie model of the form $L_X = (\mathbb{L}(V), d)$, where $\mathbb{L}(V)$ denotes the free graded Lie algebra on the graded vector space $V = s^{-1}\widetilde{H}_*(X;\mathbb{Q})$ and d is decomposable in the sense that the induced differential on $L_X/[L_X, L_X]$ is trivial. Note that L_X is finitely generated if and only if $\widetilde{H}_*(X;\mathbb{Q})$ is finite-dimensional. We will refer to L_X as the minimal Quillen model of X.

Recall from eg [34, Section 22] that the Chevalley–Eilenberg complex of a dg Lie algebra $\mathfrak{g}$ is the dg coalgebra

$$C_*(\mathfrak{g}) = (\Lambda s\mathfrak{g}, d = d_0 + d_1),$$

where d_0 and d_1 are the coderivations characterized by

$$d_0(sx) = -s(dx), \quad d_1(sx_1 \wedge sx_2) = (-1)^{|x_1|} s[x_1, x_2].$$

The Chevalley–Eilenberg cochain algebra is the dual dg algebra $C^*(\mathfrak{g}) = C_*(\mathfrak{g})^\vee$. Denote its cohomology algebra by $H^*_{\mathrm{CE}}(\mathfrak{g})$.

Next, recall that the polynomial differential forms on a simplicial set K is the dg algebra $\Omega^*(K) = \mathrm{Hom}_{\mathrm{sSet}}(K, \Omega_\bullet)$. It is a commutative dg algebra model for the cochains on K; see Section 3.8.1 below for a further discussion. Also, recall that the spatial realization of a commutative dg algebra Λ is the simplicial set $\langle \Lambda \rangle = \mathrm{Hom}_{\mathrm{cdga}}(\Lambda, \Omega_\bullet)$.

Proposition 2.21 *Let $\mathfrak{g}$ be a nilpotent dg Lie algebra of finite type. There is a natural quasi-isomorphism of commutative dg algebras*

$$C^*(\mathfrak{g}) \to \Omega^*(\mathsf{MC}_\bullet(\mathfrak{g})).$$

In particular, if $\mathfrak{g}$ is a Lie model for the space B, then $H^*(B; \mathbb{Q}) \cong H^*_{\mathrm{CE}}(\mathfrak{g})$ and $|\mathsf{MC}_\bullet(\mathfrak{g})|$ is a $\mathbb{Q}$-localization of B.

Proof There is a natural isomorphism of simplicial sets

$$\mathsf{MC}_\bullet(\mathfrak{g}) \xrightarrow{\cong} \langle C^*(\mathfrak{g}) \rangle,$$

see eg [11, Corollary 3.6]. The Chevalley–Eilenberg cochains $C^*(\mathfrak{g})$ is a Sullivan algebra and hence cofibrant (see eg [10, Theorem 2.3]), so the adjoint map $C^*(\mathfrak{g}) \to \Omega^*(\mathsf{MC}_\bullet(\mathfrak{g}))$ is a quasi-isomorphism as a consequence of [23, Theorem 9.4]. □

2.5 The dg Lie algebra of curved derivations

The main result of Quillen's theory [59] is that the functor $X \mapsto \lambda(X)$ induces an equivalence between the rational homotopy category of simply connected pointed spaces and the homotopy category of simply connected dg Lie algebras. If one wants to model unpointed spaces, one has to enlarge the set of morphisms of dg Lie algebras. A possible solution is to work with so called curved morphisms of dg Lie algebras; see [52].

Let L be a positively graded dg Lie algebra and let $L_+ = (L * \mathbb{L}(\tau), d^\tau)$ be the dg Lie algebra obtained by freely adjoining a Maurer–Cartan element τ to L and twisting the differential by τ, so $d^\tau(x) = d(x) + [\tau, x]$ for $x \in L$. It is straightforward to check that morphisms from L_+ to a dg Lie algebra L' correspond to curved morphisms from L to L'. This is analogous to the fact that the space of free maps from a pointed space X to another pointed space Y can be recovered as the space of pointed maps from X_+ to Y, where X_+ is the space obtained from X by adding a disjoint basepoint.

The projection $p: L_+ \to L$ that restricts to the identity on L and sends τ to zero is a morphism of dg Lie algebras. Let $\mathrm{Der}^c(L)$ denote the chain complex of p-derivations from L_+ to L. Its elements are maps $\theta: L_+ \to L$ that satisfy

$$\theta[x,y] = [\theta(x), p(y)] + (-1)^{|\theta||x|}[p(x), \theta(y)],$$

for all $x, y \in L_+$.

As is well known, the mapping cone of the chain map $\mathrm{ad}: L \to \mathrm{Der}\, L$, denoted by $\mathrm{Der}\, L /\!\!/ \mathrm{ad}\, L$ or $\mathrm{Der}\, L \ltimes_{\mathrm{ad}} sL$, admits a dg Lie algebra structure; see eg [67] or [11, page 252]. We now make the observation that this mapping cone may be identified with the chain complex of curved derivations.

Proposition 2.22 *The map* $\varphi: \mathrm{Der}\, L /\!\!/ \mathrm{ad}\, L \to \mathrm{Der}^c(L)$, *defined by*

$$\varphi(\theta, s\xi) = \theta \circ p + (-1)^{|\xi|}\xi \frac{\partial}{\partial \tau},$$

is an isomorphism of chain complexes, with inverse

$$\varphi^{-1}(\nu) = (\nu|_L, (-1)^{|\nu|+1} s\nu(\tau)).$$

Proof Straightforward calculation. □

In particular, this implies that $\mathrm{Der}^c(L)$ admits a dg Lie algebra structure and that it acts on L by outer derivations in the sense of [11, Section 3.5]. We do not recall the full definition here, but we point out that the outer action of $\mathrm{Der}^c(L)$ on L gives rise to an (ordinary) action by coderivations on the Chevalley–Eilenberg chains $C_*(L) = (\Lambda sL, d)$. This action may be constructed by noting that $\mathrm{s}L_+$ contains $\mathbb{Q}[0] \oplus \mathrm{s}L = \Lambda^{\leq 1}\mathrm{s}L$ as a graded subspace, where the copy of $\mathbb{Q}$ in degree 0 is generated by $\mathrm{s}\tau$. Hence every curved derivation $\phi \in \mathrm{Der}^c(L)$ determines a map $\Lambda^{\leq 1}\mathrm{s}L \to \mathrm{s}L$ by suspension and restriction, and thus it determines a unique coderivation of the cofree coalgebra $\Lambda \mathrm{s}L$. Explicitly,

$$\phi(\mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_n) = (-1)^{|\phi|}\mathrm{s}\phi(\tau) \wedge \mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}x_n + \sum_i \pm \mathrm{s}x_1 \wedge \cdots \wedge \mathrm{s}\phi(x_i) \wedge \cdots \wedge \mathrm{s}x_n. \tag{9}$$

Dually, $\mathrm{Der}^c(L)$ then acts by derivations on the cdga $C^*(L) = C_*(L)^\vee$ of Chevalley–Eilenberg cochains on L.

Remark 2.23 The constructions L_+ and $\mathrm{Der}^c(L)$ essentially agree with the constructions $L\langle\tau\rangle$ and $\mathrm{Der}_\tau(L\langle\tau\rangle)$ considered in [50, pages 44–45].

2.6 Algebraic groups of automorphisms

The following goes back to Sullivan [66, Section 6]. We outline the proof for the reader's convenience.

Theorem 2.24 *Let X be a simply connected finite CW-complex with minimal Quillen model L.*

(i) *The automorphisms of L form an affine algebraic group $\mathscr{A}\mathrm{ut}(L)$ and the automorphisms homotopic to the identity form a unipotent subgroup $\mathscr{A}\mathrm{ut}_h(L)$.*

(ii) *The group $\mathscr{E}(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$-points of the quotient affine algebraic group $\mathscr{A}\mathrm{ut}^h(L) = \mathscr{A}\mathrm{ut}(L)/\mathscr{A}\mathrm{ut}_h(L)$.*

(iii) *The Lie algebra of the algebraic group $\mathscr{A}\mathrm{ut}^h(L)$ is isomorphic to $H_0(\mathrm{Der}^c L)$.*

(iv) *The representation of $\mathscr{E}(X_{\mathbb{Q}})$ in $H_*(X;\mathbb{Q})$ extends to an algebraic representation of $\mathscr{A}\mathrm{ut}^h(L)$ with unipotent kernel.*

Outline of proof Since X is a finite CW-complex, the minimal Quillen model L is finitely generated. We can then define a linear algebraic group $\mathscr{A}\mathrm{ut}\, L$ with functor of points

$$R \mapsto \mathrm{Aut}_{\mathrm{dgl}(R)}(L \otimes R),$$

sending a $\mathbb{Q}$-algebra R to the group of R-linear automorphisms of the dg Lie algebra $L \otimes R$. To see that this indeed is a linear algebraic group, note that a faithful finite-dimensional representation is given by $L_{\le N}$, where N is the maximal degree of a generator.

The subgroup $\mathscr{A}\mathrm{ut}_h(L)$ of automorphisms homotopic to the identity may be identified with the unipotent algebraic group associated to the nilpotent Lie algebra $B_0 \,\mathrm{Der}\, L$ of derivations of L of the form $[d,\theta]$, for some derivation $\theta\colon L \to L$ of degree 1; see [66, Section 6] or [17, Theorem 3.4], or the proof of [33, Lemma 8.1].

Since $X_{\mathbb{Q}}$ is simply connected, there is an isomorphism of groups $\mathscr{E}_*(X_{\mathbb{Q}}) \cong \mathscr{E}(X_{\mathbb{Q}})$. Quillen's equivalence [59] between the category of simply connected pointed spaces, localized at the rational homotopy equivalences, and the category of positively graded dg Lie algebras, localized at the quasi-isomorphisms, coupled with the fact that a quasi-isomorphism between minimal dg Lie algebras is an isomorphism, leads to an isomorphism of groups

$$\mathscr{E}_*(X_{\mathbb{Q}}) \cong \mathrm{Aut}(L)/\mathrm{Aut}_h(L),$$

where $\mathrm{Aut}(L)$ and $\mathrm{Aut}_h(L)$ denote the $\mathbb{Q}$-points of $\mathscr{A}\mathrm{ut}(L)$ and $\mathscr{A}\mathrm{ut}_h(L)$, respectively. It follows from Lemma 2.9 that the right-hand side may be identified with the $\mathbb{Q}$-points of the quotient algebraic group $\mathscr{A}\mathrm{ut}^h(L) = \mathscr{A}\mathrm{ut}(L)/\mathscr{A}\mathrm{ut}_h(L)$.

The Lie algebra of $\mathscr{A}\mathrm{ut}(L)$ may be identified with $Z_0(\mathrm{Der}\, L)$, so the Lie algebra of $\mathscr{A}\mathrm{ut}^h(L)$ may be identified with $Z_0(\mathrm{Der}\, L)/B_0(\mathrm{Der}\, L) = H_0(\mathrm{Der}\, L)$. Since L is positively graded, $H_0(\mathrm{Der}\, L) = H_0(\mathrm{Der}^c\, L)$.

For the last statement, one identifies

$$H_*(X;\mathbb{Q}) = \mathbb{Q} \oplus sL/[L,L],$$

and notes that the right-hand side is an algebraic representation of $\mathscr{A}\mathrm{ut}\, L$ on which $\mathscr{A}\mathrm{ut}_h\, L$ acts trivially. The associated graded $\mathrm{gr}\, L$ of L with respect to the lower central series is isomorphic to the free Lie algebra on $L/[L,L]$ as a representation of $\mathscr{A}\mathrm{ut}\, L$. It follows that the kernel of the representation $L/[L,L]$ acts trivially on $\mathrm{gr}\, L$, which implies that it acts unipotently on L in each degree. □

There is a parallel statement for finite Postnikov stages. The proof is essentially the same so we omit it.

Theorem 2.25 *Let X be a simply connected Postnikov stage of finite type with minimal Sullivan model Λ.*

(i) *The automorphisms of Λ form an affine algebraic group $\mathscr{A}\mathrm{ut}(\Lambda)$ and the automorphisms homotopic to the identity form a unipotent subgroup $\mathscr{A}\mathrm{ut}_h(\Lambda)$.*

(ii) *The group $\mathscr{E}(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$-points of the algebraic group*

$$\mathscr{A}\mathrm{ut}^h \Lambda = \mathscr{A}\mathrm{ut}\,\Lambda / \mathscr{A}\mathrm{ut}_h\,\Lambda.$$

(iii) *The Lie algebra of the algebraic group $\mathscr{A}\mathrm{ut}^h \Lambda$ is isomorphic to $H_0(\mathrm{Der}\,\Lambda)$.*

(iv) *The representation of $\mathscr{E}(X_{\mathbb{Q}})$ in $H_*(X;\mathbb{Q})$ extends to an algebraic representation of $\mathscr{A}\mathrm{ut}^h(\Lambda)$ with unipotent kernel.* □

For a simply connected finite complex X of dimension n, there is an isomorphism $\mathscr{E}(X_{\mathbb{Q}}) \cong \mathscr{E}(P_n X_{\mathbb{Q}})$, where $P_n X$ denotes the n^{th} Postnikov stage of X; see [17, Proposition 3.1(2)]. We therefore have two, a priori different, ways to realize $\mathscr{E}(X_{\mathbb{Q}})$ as the $\mathbb{Q}$-points of an algebraic group, one using Quillen models and one using Sullivan models. Saleh [60] has recently shown that these algebraic group structures agree. In the rest of the paper we will, somewhat imprecisely, refer to $\mathscr{E}(X_{\mathbb{Q}})$ as an algebraic group, with the understanding that we refer to either of the isomorphic algebraic groups above.

The following goes back to Sullivan [66, Theorem 10.3] and Wilkerson [71, Theorem B]. An elaboration of Sullivan's argument can be found in [68].

Theorem 2.26 *The homomorphism $\rho\colon \mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}})$ induced by rationalization has finite kernel and image an arithmetic subgroup.* □

3 Proofs of the main results

Throughout this section, we fix a simply connected finite CW-complex X and we let $\mathrm{aut}(X)$ denote the topological monoid of self-homotopy equivalences of X. The group $\mathscr{E}(X)$ will interchangeably be thought of as the group $\pi_0\,\mathrm{aut}(X)$ of components of the monoid $\mathrm{aut}(X)$ or as the fundamental group $\pi_1 B\,\mathrm{aut}(X)$ of the classifying space.

Definition 3.1 For a subgroup $G \leqslant \mathscr{E}(X)$, let $\mathrm{aut}_G(X)$ denote the union of the components of $\mathrm{aut}(X)$ that belong to G, so that there is a pullback square

$$\begin{array}{ccc} \mathrm{aut}_G(X) & \longrightarrow & G \\ \downarrow & & \downarrow \\ \mathrm{aut}(X) & \longrightarrow & \mathscr{E}(X) \end{array}$$

The cover of $B\,\mathrm{aut}(X)$ associated to $G \leqslant \mathscr{E}(X)$ is weakly equivalent to the classifying space $B\,\mathrm{aut}_G(X)$ of the monoid $\mathrm{aut}_G(X)$, defined eg using the geometric bar construction [53, Section 7], and we will tacitly identify these two spaces.

Here is an outline of the proof of the main results:

We fix a subgroup $G \leqslant \mathscr{E}(X)$ that acts nilpotently on the rational homology of X. In Section 3.1 we show that G uniquely determines a unipotent algebraic subgroup $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ such that the homomorphism $\rho\colon \mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}})$ restricts to a $\mathbb{Q}$-isomorphism $G \to U$, and we give several equivalent characterizations of this U. In Section 3.2 we show that the space $B\operatorname{aut}_G(X)$ is virtually nilpotent. In Section 3.3, we start incorporating the action of the deck transformation group $\mathscr{E}_G(X)$ on $B\operatorname{aut}_G(X)$; in particular, we show that $B\operatorname{aut}_G(X)$ is $\mathscr{E}_G(X)$-equivariantly rationally equivalent to $B\operatorname{aut}_U(X_{\mathbb{Q}})$. In Section 3.4 we construct a Lie model for the space $B\operatorname{aut}_U(X_{\mathbb{Q}})$. In general, there is no action of the deck transformation group on this Lie model, but there is an action of a larger group defined in terms of the minimal Quillen model of X. A crucial step in the proof is to relate this algebraically defined action to the action of the deck transformation group. The key is a lemma about conjugation actions on bar constructions, which we prove in Section 3.5. In Section 3.6, we use this to prove Theorem 1.1. In Section 3.7, we observe that if U is the unipotent radical of $\mathscr{E}(X_{\mathbb{Q}})$, then the deck transformation group can be made to act on the Lie model and this leads to the proof of Theorem 1.2.

3.1 Unipotent groups of self-homotopy equivalences

Recall that X is assumed to be a simply connected finite CW-complex. We begin by discussing how the homomorphism induced by rationalization,

$$\rho\colon \mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}}),$$

can be used to characterize subgroups of $\mathscr{E}(X)$ that act nilpotently on $H_*(X;\mathbb{Q})$.

Since X is simply connected, $\mathscr{E}(X)$ is isomorphic to the group $\mathscr{E}_*(X)$ of pointed homotopy classes of pointed self-homotopy equivalences. This group acts on the homotopy and homology groups of X. Moreover, the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ is $\mathscr{E}(X)$-equivariant. The *spherical homology* $SH_n(X)$ is by definition the image of the Hurewicz homomorphism. It is a $\mathscr{E}(X)$-submodule of $H_n(X)$ and a quotient $\mathscr{E}(X)$-module of $\pi_n(X)$.

Proposition 3.2 *Let $G \leqslant \mathscr{E}(X)$ be a subgroup and let R be a subring of $\mathbb{Q}$. The following are equivalent*:

(i) *G acts nilpotently on $H_n(X;R)$ for all n.*

(ii) *G acts nilpotently on $\pi_n(X) \otimes R$ for all n.*

(iii) *G acts nilpotently on $SH_n(X;R)$ for all n.*

For $R = \mathbb{Q}$ these conditions are equivalent to the following:

(iv) *$\rho(G)$ is contained in a unipotent algebraic subgroup of $\mathscr{E}(X_{\mathbb{Q}})$.*

Proof It is well-known that the first two conditions are equivalent (eg the argument in [43, Theorem 2.1] goes through) and they clearly imply the third. For the converse, assume inductively that G acts nilpotently on the R-homotopy groups of the Postnikov section $P_{n-1}(X)$ and use that the Hurewicz homomorphism sits in an exact sequence

$$H_{n+1}(P_{n-1}(X)) \to \pi_n(X) \to H_n(X).$$

This gives rise to an exact sequence

$$H_{n+1}(P_{n-1}(X); R) \to \pi_n(X) \otimes R \to SH_n(X; R) \to 0,$$

where the left and right terms are nilpotent G-modules. It follows from [44, Proposition I.4.3] that $\pi_n(X) \otimes R$ is a nilpotent G-module.

Finally, we prove the equivalence between the first condition and the fourth when $R = \mathbb{Q}$. If G acts nilpotently on $H_*(X;\mathbb{Q})$, then the image of G in $\mathsf{GL}(H_*(X;\mathbb{Q}))$ lies in a unipotent algebraic subgroup U''. The preimage U' of U'' in $\mathscr{E}(X_{\mathbb{Q}})$ contains $\rho(G)$ and it is a unipotent algebraic subgroup since it is an extension of U'' by the kernel of the $\mathscr{E}(X_{\mathbb{Q}})$-representation $H_*(X;\mathbb{Q})$, which is unipotent by Theorem 2.24(iv). Conversely, since $H_*(X;\mathbb{Q})$ is an algebraic representation of $\mathscr{E}(X_{\mathbb{Q}})$, any unipotent algebraic subgroup of the latter acts nilpotently on it. □

Proposition 3.3 *If $G \leqslant \mathscr{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X;\mathbb{Q})$, then there is a unique unipotent algebraic subgroup $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ that satisfies the following equivalent conditions*:

(i) *U is minimal among the unipotent algebraic subgroups that contain $\rho(G)$.*

(ii) *$\rho(G)$ is a Zariski dense subgroup of U.*

(iii) *$\rho(G)$ is an arithmetic subgroup of U.*

(iv) *G is a finite-index subgroup of $\rho^{-1}(U)$.*

(v) *$\rho(G) \leqslant U$ and the induced homomorphism $\rho\colon G \to U$ is a $\mathbb{Q}$-isomorphism.*

Moreover, G is finitely generated in this situation.

Conversely, if $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ is a unipotent algebraic subgroup, then there is a unique commensurability class of subgroups $G \leqslant \mathscr{E}(X)$ such that the above conditions are satisfied.

Proof If $G \leqslant \mathscr{E}(X)$ acts nilpotently on $SH_*(X;\mathbb{Q})$, then $\rho(G)$ is contained in a unipotent algebraic subgroup of $\mathscr{E}(X_{\mathbb{Q}})$ by Proposition 3.2. If we let U be the intersection of all unipotent algebraic subgroups of $\mathscr{E}(X_{\mathbb{Q}})$ that contain $\rho(G)$, then U is clearly the unique minimal unipotent algebraic subgroup that contains $\rho(G)$.

If U is minimal among the unipotent algebraic subgroups that contain $\rho(G)$, then the Zariski closure $\overline{\rho(G)}$ is contained in U since the latter is Zariski closed. But then $\overline{\rho(G)}$ must be unipotent and therefore equal to U by minimality of U.

Now assume that $\rho(G)$ is Zariski dense in U. Clearly, $\rho(G)$ is contained in $U \cap \rho(\mathscr{E}(X))$ and the latter is an arithmetic subgroup of U by Lemma 2.11. Arithmetic subgroups of unipotent groups are nilpotent and finitely generated (Lemma 2.13), and subgroups of finitely generated nilpotent groups are always finitely generated, so it follows that $\rho(G)$ is finitely generated. Since $\rho(G)$ is Zariski dense in U, Lemma 2.13 shows that $\rho(G)$ is an arithmetic subgroup of U. We note in passing that we can use the exact sequence

$$1 \to \ker(\rho) \cap G \to G \to \rho(G) \to 1$$

to deduce that G is finitely generated as well. Indeed, the kernel is finite by Theorem 2.26 and we have just seen that $\rho(G)$ is finitely generated.

Next, if $\rho(G)$ is an arithmetic subgroup of U, then it must have finite index in $\rho(\rho^{-1}(U)) = U \cap \rho(\mathscr{E}(X))$, because the latter is also an arithmetic subgroup of U. As ρ has finite kernel, this implies that G has finite index in $\rho^{-1}(U)$.

The homomorphism $\rho\colon G \to U$ is the composite of the inclusion $G \to \rho^{-1}(U)$ followed by the homomorphism $\rho\colon \rho^{-1}(U) \to U$. If G has finite index in $\rho^{-1}(U)$, then the inclusion is in particular a $\mathbb{Q}$-isomorphism. The homomorphism $\rho\colon \rho^{-1}(U) \to U$ has finite kernel and image an arithmetic subgroup of U, so it is a $\mathbb{Q}$-isomorphism by Lemma 2.13. This shows that $\rho\colon G \to U$ is a $\mathbb{Q}$-isomorphism if G has finite index in $\rho^{-1}(U)$.

Finally, suppose ρ restricts to a $\mathbb{Q}$-isomorphism $G \to U$. If $U' \leqslant \mathscr{E}(X_{\mathbb{Q}})$ is a unipotent algebraic subgroup such that $\rho(G) \leqslant U'$, then $G \to U$ factors as $G \to U \cap U'$ followed by the inclusion $U \cap U' \to U$, implying the latter is $\mathbb{Q}$-surjective. But this is then a $\mathbb{Q}$-surjective inclusion of uniquely divisible groups, so it must be an equality. This shows that U is minimal among the unipotent algebraic subgroups that contain $\rho(G)$. By that we have gone full circle, showing the five conditions are equivalent.

Conversely, given U and two subgroups G and G' that satisfy the equivalent conditions, the fourth condition shows that G and G' are finite-index subgroups of $\rho^{-1}(U)$. This implies that $G \cap G'$ has finite index in both G and G'. So the commensurability class determined by U is precisely the set of finite-index subgroups of $\rho^{-1}(U)$. □

3.2 Virtual nilpotence of covers

As before, X is a simply connected finite CW-complex and G is a subgroup of $\mathscr{E}(X)$. The purpose of this section is to show that the space $B\operatorname{aut}_G(X)$ is virtually nilpotent if G acts nilpotently on the rational spherical homology of X. This extends a result of Dror and Zabrodsky [32, Theorem D], which says that $B\operatorname{aut}_G(X)$ is nilpotent if G acts nilpotently on the integral homology of X.

Lemma 3.4 *Let G be a group acting on a finitely generated abelian group A. If G acts nilpotently on $A \otimes \mathbb{Q}$, then there is a finite-index subgroup $K \leqslant G$ that acts nilpotently on A.*

Proof Let K be the kernel of the action of G on the torsion subgroup A_{tor}. Since the group A_{tor} is finite, so is its automorphism group, so the exact sequence

$$1 \to K \to G \to \mathrm{Aut}(A_{\mathrm{tor}})$$

shows that K has finite index in G. If G acts nilpotently on $A \otimes \mathbb{Q}$, then there exists a filtration of G-modules

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = A \otimes \mathbb{Q}$$

such that G acts trivially on V_i/V_{i-1} for each i. Let $W_i \subseteq A$ be the preimage of V_i under the homomorphism $A \to A \otimes \mathbb{Q}$. This yields a filtration of G-modules

$$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = A,$$

and it follows that W_i/W_{i-1} has trivial G-action for $1 \leq i \leq n$. Note that $W_0 = A_{\mathrm{tor}}$, so if G acts nilpotently on A_{tor} it follows that it acts nilpotently on A as well. In general, the G-action on A_{tor} need not be nilpotent, but the action of K on it is trivial by definition. Thus, extending the filtration by $W_{-1} = 0$ yields a filtration witnessing that K acts nilpotently on A. □

Proposition 3.5 *If $G \leqslant \mathscr{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X;\mathbb{Q})$, then the space $B\mathrm{aut}_G(X)$ is virtually nilpotent. It is nilpotent if G acts nilpotently on $SH_*(X;\mathbb{Z})$.*

Proof We will in fact prove the slightly stronger statement that $B\mathrm{aut}_G(X)$ admits a finite cover which is nilpotent. By Lemma 3.4, there is a finite-index subgroup $K \leqslant G$ that acts nilpotently on $SH_*(X;\mathbb{Z})$. By Proposition 3.2, the group K acts nilpotently on $H_*(X;\mathbb{Z})$. It follows from [32, Theorem D] that the space $B\mathrm{aut}_K(X)$ is nilpotent. The space $B\mathrm{aut}_K(X)$ is weakly equivalent to the finite cover of $B\mathrm{aut}_G(X)$ that corresponds to the finite-index subgroup $K \leqslant G$. □

Example 3.6 The space $B\mathrm{aut}_G(X)$ is not virtually nilpotent in general. For example, if we take $X = S^2 \vee S^2$ and $G = \mathscr{E}(X)$, then $G \cong \mathrm{GL}_2(\mathbb{Z})$, which is not virtually nilpotent.

Example 3.7 The space $B\mathrm{aut}_G(X)$ need not be nilpotent even if G acts nilpotently on $SH_*(X;\mathbb{Q})$. To see this, consider the Moore space $X = M(\mathbb{Z}/3\mathbb{Z}, 2)$, ie the homotopy cofiber of a degree-3 self-map of S^2. This space is rationally equivalent to point, so $G = \mathscr{E}(X)$ acts nilpotently on $SH_*(X;\mathbb{Q}) = 0$ for trivial reasons. The group of self-equivalences is not difficult to compute (see eg [63, Theorem 2]); there is an isomorphism

$$\mathscr{E}(X) \cong (\mathbb{Z}/3\mathbb{Z})^\times \ltimes \mathbb{Z}/3\mathbb{Z} \cong \Sigma_3,$$

showing $\pi_1(B\mathrm{aut}(X)) = \mathscr{E}(X)$ is not nilpotent. The criterion in Proposition 3.5 for nilpotency of $B\mathrm{aut}_G(X)$ is not satisfied because the action on $SH_2(X;\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$ is through the projection onto $(\mathbb{Z}/3\mathbb{Z})^\times$ and this action is not nilpotent.

3.3 Equivariant rationalization of covers

In this section, we will show that $B\operatorname{aut}_G(X)$ is rationally equivalent to $B\operatorname{aut}_U(X_{\mathbb{Q}})$ when G and U are as in Proposition 3.3. To keep track of the action of the deck transformation group, we need a model for $B\operatorname{aut}_G(X)$ that is appropriately functorial in $G \leqslant \mathscr{E}(X)$. Recall that the subgroups of $\mathscr{E}(X)$ form the objects of the *orbit category*, where a morphism $G \to H$ is given by a morphism of left $\mathscr{E}(X)$-sets $\mathscr{E}(X)/G \to \mathscr{E}(X)/H$.

Definition 3.8 We define a functor from the orbit category of $\mathscr{E}(X)$ to the category of spaces over $B\operatorname{aut}(X)$ by sending $G \leqslant \mathscr{E}(X)$ to the space

$$B_G\operatorname{aut}(X) = B\big(*, \operatorname{aut}(X), \mathscr{E}(X)/G\big)$$

defined using the geometric bar construction [53, Section 7] of the topological monoid $\operatorname{aut}(X)$ acting on $\mathscr{E}(X)/G$ via the canonical map $\operatorname{aut}(X) \to \mathscr{E}(X)$.

The space $B_G\operatorname{aut}(X)$ can be thought of as a functorial model for the cover of $B\operatorname{aut}(X)$ associated to $G \leqslant \mathscr{E}(X)$. The map

$$B\operatorname{aut}_G(X) = B\big(*, \operatorname{aut}_G(X), *\big) \to B\big(*, \operatorname{aut}(X), \mathscr{E}(X)/G\big) = B_G\operatorname{aut}(X),$$

induced by the inclusion of monoids $\operatorname{aut}_G(X) \to \operatorname{aut}(X)$ and the map of $\operatorname{aut}_G(X)$-spaces $* \to \mathscr{E}(X)/G$ that selects the coset G, is a weak equivalence, so both source and target are models for the classifying space of the monoid $\operatorname{aut}_G(X)$. The advantage of $B_G\operatorname{aut}(X)$ is that it carries an action of the group $\mathscr{E}_G(X)$ of automorphisms of the $\mathscr{E}(X)$-set $\mathscr{E}(X)/G$. We remind the reader that this group, the "deck transformation group", may be identified with $N_{\mathscr{E}(X)}(G)/G$, the normalizer of G in $\mathscr{E}(X)$ modulo G.

In what follows, we fix a subgroup $G \leqslant \mathscr{E}(X)$ that acts nilpotently on $SH_*(X;\mathbb{Q})$ (or equivalently on $H_*(X;\mathbb{Q})$ or $\pi_*(X) \otimes \mathbb{Q}$) and we let $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ be the minimal unipotent algebraic subgroup that contains $\rho(G)$ as in Proposition 3.3.

Lemma 3.9 *The homomorphism $\rho\colon \mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}})$ carries the normalizer of G to the normalizer of U. In particular, there is an induced group homomorphism $\mathscr{E}_G(X) \to \mathscr{E}_U(X_{\mathbb{Q}})$.*

Proof Let $g \in N_{\mathscr{E}(X)}(G)$. We need to show that $\rho(g) \in N_{\mathscr{E}(X_{\mathbb{Q}})}(U)$, ie that ${}^gU = \rho(g)U\rho(g)^{-1}$ agrees with U. By assumption, U is the minimal unipotent algebraic subgroup of $\mathscr{E}(X_{\mathbb{Q}})$ that contains $\rho(G)$. Thus ${}^gU$ is a minimal unipotent algebraic subgroup that contains $\rho(g)\rho(G)\rho(g)^{-1} = \rho(gGg^{-1}) = \rho(G)$, so ${}^gU = U$ by the uniqueness part of Proposition 3.3. □

By Lemma 3.9, $\mathscr{E}_G(X)$ acts on $B_U\operatorname{aut}(X_{\mathbb{Q}})$ via the induced homomorphism $\rho\colon \mathscr{E}_G(X) \to \mathscr{E}_U(X_{\mathbb{Q}})$. We remind the reader that we call a map between connected spaces a *rational equivalence* if it induces a $\mathbb{Q}$-isomorphism on all homotopy groups *and* all homology groups; see Definition 2.2.

Proposition 3.10 *There is a zig-zag of $\mathscr{E}_G(X)$-equivariant rational equivalences between $B_G\mathrm{aut}(X)$ and $B_U\mathrm{aut}(X_{\mathbb{Q}})$.*

Proof Let $r\colon X \to X_{\mathbb{Q}}$ be a rationalization. We may assume that r is a cofibration. This holds in many models for rationalizations (eg cellular rationalization [34, Theorem 9.7] or the $H_*(-;\mathbb{Q})$-localization of Bousfield [22]), but if necessary, one can achieve this by abstract nonsense, eg by factoring r as a cofibration followed by a weak homotopy equivalence. Now consider the pullback square

$$\begin{array}{ccc} \mathrm{aut}(r) & \xrightarrow{\;q\;} & \mathrm{aut}(X_{\mathbb{Q}}) \\ \downarrow & & \downarrow{\scriptstyle r^*} \\ \mathrm{aut}(X) & \xrightarrow{\;r_*\;} & \mathrm{map}(X, X_{\mathbb{Q}})_{\mathrm{re}} \end{array}$$

where $\mathrm{map}(X, X_{\mathbb{Q}})_{\mathrm{re}}$ denotes the space of rational equivalences from X to $X_{\mathbb{Q}}$ and $\mathrm{aut}(r)$ denotes the space of self-equivalences of r viewed as an object in the category of maps. The map r^* is a fibration since r is a cofibration, and it is a weak homotopy equivalence by standard properties of localizations. Hence, the left vertical map is a weak homotopy equivalence as well. The map r_* is in general not a bijection on π_0, but its restriction to each component is a rational equivalence to the component it hits by [44, Theorem II.3.11]. It follows that the top horizontal map q has the same property.

This yields a zig-zag of grouplike monoids

$$\mathrm{aut}(X) \xleftarrow{\sim} \mathrm{aut}(r) \xrightarrow{q} \mathrm{aut}(X_{\mathbb{Q}}), \tag{10}$$

where the left map is a weak equivalence and the right map q induces an isomorphism on $\pi_k(-)\otimes\mathbb{Q}$ for all $k>0$ and may be identified with $\rho\colon \mathscr{E}(X)\to\mathscr{E}(X_{\mathbb{Q}})$ on π_0. This induces a zig-zag

$$B(*,\mathrm{aut}(X),\mathscr{E}(X)/G) \xleftarrow{\sim} B(*,\mathrm{aut}(r),\mathscr{E}(X)/G) \xrightarrow{\psi} B(*,\mathrm{aut}(X_{\mathbb{Q}}),\mathscr{E}(X_{\mathbb{Q}})/U), \tag{11}$$

where ψ is induced by the monoid map $q\colon\mathrm{aut}(r)\to\mathrm{aut}(X_{\mathbb{Q}})$ and the $\mathrm{aut}(r)$-equivariant map $\rho\colon\mathscr{E}(X)/G\to \mathscr{E}(X_{\mathbb{Q}})/U$. The homomorphism $\pi_1(\psi)$ may be identified with $\rho\colon G\to U$, which is a $\mathbb{Q}$-isomorphism by Proposition 3.3. For $k>1$, we may identify $\pi_k(\psi)$ with $\pi_{k-1}(q)\colon \pi_{k-1}(\mathrm{aut}(r))\to\pi_{k-1}(\mathrm{aut}(X_{\mathbb{Q}}))$, which is a $\mathbb{Q}$-isomorphism by the above.

We recognize the leftmost term of (11) as $B_G\mathrm{aut}(X)$ and the rightmost term as $B_U\mathrm{aut}(X_{\mathbb{Q}})$. Moreover, the maps in the zig-zag are clearly $\mathscr{E}_G(X)$-equivariant. We have just shown that ψ induces a $\mathbb{Q}$-isomorphism on all homotopy groups. By Proposition 3.5, the source of ψ is virtually nilpotent and the target is nilpotent, so it follows from Lemma 2.3 that ψ is a rational homology equivalence as well. □

3.4 Equivariant Lie models for covers

As before, X is a simply connected finite CW-complex, $G\leqslant\mathscr{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X;\mathbb{Q})$, and $U\leqslant\mathscr{E}(X_{\mathbb{Q}})$ is the unique minimal unipotent algebraic subgroup of $\mathscr{E}(X_{\mathbb{Q}})$ that contains $\rho(G)$.

We identify $X_{\mathbb{Q}}$ with the nerve of the minimal Quillen model L for X. Recall from Theorem 2.24(iii) that the Lie algebra of $\mathscr{E}(X_{\mathbb{Q}})$ is isomorphic to $H_0(\operatorname{Der}^c L)$, so the Lie algebra of U may be identified with a Lie subalgebra $\mathfrak{u} \leqslant H_0(\operatorname{Der}^c L)$. The following definition can be viewed as a dg Lie algebra companion to Definition 3.1.

Definition 3.11 Let $\operatorname{Der}^c_{\mathfrak{u}} L$ denote the dg Lie subalgebra of $\operatorname{Der}^c L\langle 0\rangle$ defined by the pullback

$$\begin{array}{ccc} \operatorname{Der}^c_{\mathfrak{u}} L & \longrightarrow & \mathfrak{u} \\ \downarrow & & \downarrow \\ \operatorname{Der}^c L\langle 0\rangle & \longrightarrow & H_0(\operatorname{Der}^c L) \end{array}$$

Proposition 3.12 *The dg Lie algebra* $\operatorname{Der}^c_{\mathfrak{u}} L$ *is nilpotent.*

Proof The degree-zero component, $(\operatorname{Der}^c_{\mathfrak{u}} L)_0$, may be identified with the Lie algebra of the preimage $\widetilde{U} = p^{-1}(U)$ under the projection $p\colon \mathscr{A}\mathrm{ut}\, L \to \mathscr{A}\mathrm{ut}^h\, L$. By Theorem 2.24(i), the homomorphism p has unipotent kernel. It follows that $\widetilde{U}$ is an extension of unipotent groups and is hence unipotent; see for instance [55, Section 6.45]. Since $(\operatorname{Der}^c L)_k$ is an algebraic representation of $\widetilde{U}$ for each $k > 0$, it follows that $\widetilde{U}$ acts unipotently on it; see [55, Proposition 14.3]. Hence, $(\operatorname{Der}^c_{\mathfrak{u}} L)_0 = \operatorname{Lie} \widetilde{U}$ is nilpotent and acts nilpotently on $(\operatorname{Der}^c L)_k$ for each $k > 0$. □

Let $\operatorname{Aut}_U(L) \leqslant \operatorname{Aut}(L)$ denote the preimage of the normalizer of $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ under the homomorphism $\operatorname{Aut}(L) \to \mathscr{E}(X_{\mathbb{Q}})$. By design, the group $\operatorname{Aut}_U(L)$ acts on $\mathscr{E}(X_{\mathbb{Q}})$ and on $\operatorname{Der}^c L$ by conjugation, fixing the subgroup U and the Lie subalgebra $\mathfrak{u}$, respectively. If U is normal in $\mathscr{E}(X_{\mathbb{Q}})$ then $\operatorname{Aut}_U(L) = \operatorname{Aut}(L)$.

Proposition 3.13 *There is a weak equivalence of grouplike topological monoids,*

$$|\exp_\bullet(\operatorname{Der}^c_{\mathfrak{u}} L)| \to \operatorname{aut}_U(X_{\mathbb{Q}}), \tag{12}$$

which is equivariant with respect to the conjugation action of $\operatorname{Aut}_U(L)$ *on the domain and codomain. In particular,* $B\operatorname{aut}_U(X_{\mathbb{Q}})$ *is weakly equivalent to the nerve of the dg Lie algebra* $\operatorname{Der}^c_{\mathfrak{u}}(L)$.

Proof The action of the nilpotent dg Lie algebra $\mathfrak{g} = \operatorname{Der}^c_{\mathfrak{u}} L$ on L by outer derivations induces an action of the simplicial group $\exp_\bullet(\mathfrak{g})$ on the simplicial set $\mathsf{MC}_\bullet(L)$ (see [11, Section 3.5]), giving rise to a morphism of grouplike monoids

$$\alpha\colon |\exp_\bullet(\mathfrak{g})| \to \operatorname{aut}(|\mathsf{MC}_\bullet(L)|). \tag{13}$$

The map (13) is known to induce an isomorphism on $\pi_k(-)$ for $k > 0$. To see this, one can apply [12, Proposition 3.7] to the Sullivan model $\Lambda = C^*(L)$. Let us add that this is essentially equivalent to the statement, going back to Tanré [67, VII.4.(4)], that $\operatorname{Der}^c L\langle 1\rangle$ is a Lie model for the simply connected cover of $B\operatorname{aut}(\mathsf{MC}_\bullet(L))$.

By construction, $\pi_0(\mathrm{aut}_U(X_{\mathbb{Q}})) = U$ and $H_0(\mathrm{Der}^c_{\mathfrak{u}} L) = \mathfrak{u}$. By Proposition 2.19, there is an isomorphism

$$\pi_0(\exp_\bullet(\mathrm{Der}^c_{\mathfrak{u}} L)) \cong \exp(H_0(\mathrm{Der}^c_{\mathfrak{u}} L)) = \exp(\mathfrak{u}) = U.$$

One checks that the map induced by (13) on π_0 may be identified with the inclusion of U into $\mathscr{E}(X_{\mathbb{Q}})$. Hence, (13) corestricts to a weak equivalence $|\exp_\bullet(\mathfrak{g})| \to \mathrm{aut}_U(X_{\mathbb{Q}})$. The statement about $\mathrm{Aut}_U(L)$-equivariance is quickly verified by inspection. The last statement follows by applying the classifying space functor to the monoid map (12) and noting that $B|\exp_\bullet(\mathfrak{g})|$ is weakly equivalent to the nerve $|\mathsf{MC}_\bullet(\mathfrak{g})|$. □

Remark 3.14 Proposition 2.19 implies that $B|\exp_\bullet(\mathfrak{g})|$ is nilpotent and rational whenever $\mathfrak{g}$ is a nilpotent dg Lie algebra, so the above gives an alternative proof that the space $B\,\mathrm{aut}_U(X_{\mathbb{Q}})$ is nilpotent and rational.

In particular, forgetting the group actions, we get a Lie model for $B\,\mathrm{aut}_G(X)$.

Corollary 3.15 *The space $B\,\mathrm{aut}_G(X)$ is rationally equivalent to the nerve of the dg Lie algebra* $\mathrm{Der}^c_{\mathfrak{u}}(L)$.

Proof Combine Propositions 3.13 and 3.10. □

Example 3.16 Let $\mathrm{tor}(X)$ denote the *homotopy Torelli monoid*, meaning the submonoid of $\mathrm{aut}(X)$ of those self-equivalences that act trivially on $H_*(X;\mathbb{Z})$. The space $B\,\mathrm{tor}(X)$ is nilpotent by Proposition 3.5. Corollary 3.15 yields a Lie model for $B\,\mathrm{tor}(X)$ described by the following: it agrees with $\mathrm{Der}^c L$ in positive degrees and in degree zero it consists of all derivations θ of L that commute with the differential and are decomposable in the sense that $\theta(L) \subseteq [L,L]$. To see this, note that the action of a derivation $\theta \in \mathrm{Der}\, L$ on the reduced homology of X may be identified with the induced action on the indecomposables $L/[L,L]$.

Remark 3.17 Corollary 3.15 recovers the Lie model for $B\,\mathrm{aut}_G(X)$ of [33, Theorem 0.1]. Indeed, one can check that $\mathrm{Der}^c_{\mathfrak{u}} L$ agrees with the model $\mathrm{Der}^{\mathscr{G}} L \tilde{\times} sL$ of [33, Theorem 0.1], where $\mathscr{G} \leqslant \mathrm{Aut}^h(L)$ corresponds to $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ under the isomorphism $\mathrm{Aut}^h(L) \cong \mathscr{E}(X_{\mathbb{Q}})$. Let us also remark that when U is the unipotent radical of $\mathscr{E}(X_{\mathbb{Q}})$, one can show that the dg Lie algebra $\mathrm{Der}^c_u L$ of Definition 3.11 agrees with the dg Lie algebra $\mathscr{D}er L$ considered in [33, Definition 6.4] if the filtration [33, (23)] is chosen to be a composition series as in (19).

3.5 A lemma on conjugation actions on bar constructions

In this section, we will prove a lemma that will be a key ingredient in the proof of the main results. Before stating it, let us give a few words of motivation. Suppose that

$$1 \to G' \to G \xrightarrow{f} G'' \to 1$$

is a split short exact sequence of groups, giving rise to a homotopy fiber sequence

$$BG' \to BG \xrightarrow{Bf} BG''.$$

It is well-known that the conjugation action of G'' on BG' models the holonomy action of G'' on the homotopy fiber of Bf. On the other hand, the left action of G'' on $G''/\!\!/G = B(G'', G, *)$ also models the holonomy action. The evident map $BG' \to G''/\!\!/G$ is, however, not equivariant; it becomes equivariant if G'' acts by simultaneous left multiplication and conjugation on the target. The lemma below offers a resolution of this seeming incongruity by showing that these two different G''-actions on $G''/\!\!/G$ give weakly equivalent G''-spaces.

Now, let $\mathscr{A}$ be a topological monoid, let $\mathscr{X}$ be a left $\mathscr{A}$-space and $\mathscr{Y}$ a right $\mathscr{A}$-space. Let $\mathscr{A}^\times$ denote the group of invertible elements in $\mathscr{A}$. There is an action of $\mathscr{A}^\times$ on the triple $(\mathscr{Y}, \mathscr{A}, \mathscr{X})$ defined by letting $g \in \mathscr{A}^\times$ act by

$$\mathscr{A} \xrightarrow{c_g} \mathscr{A}, \quad h \mapsto ghg^{-1}, \qquad \mathscr{X} \xrightarrow{\ell_g} c_g^*(\mathscr{X}), \quad x \mapsto gx, \qquad \mathscr{Y} \xrightarrow{r_g} c_g^*(\mathscr{Y}), \quad y \mapsto yg^{-1}.$$

This induces an action of $\mathscr{A}^\times$ on the geometric bar construction $B(\mathscr{Y}, \mathscr{A}, \mathscr{X})$, which we will refer to as the conjugation action.

Lemma 3.18 *There is a natural zig-zag of $\mathscr{A}^\times$-equivariant homotopy equivalences connecting $B(\mathscr{Y}, \mathscr{A}, \mathscr{X})$, with the conjugation action, and the same space with the trivial action.*

Remark 3.19 The lemma implies the fact that, for each individual $g \in \mathscr{A}^\times$, the map $B(r_g, c_g, \ell_g)$ is homotopic to the identity of $B(\mathscr{Y}, \mathscr{A}, \mathscr{X})$, but note that the lemma is not a formal consequence of this fact. Indeed, it is easy to find spaces with a group action where each individual multiplication map is homotopic to the identity but where the action cannot be trivialized via a zig-zag of equivariant homotopy equivalences, eg S^1 with the antipodal action of $\mathbb{Z}/2\mathbb{Z}$.

Proof We exploit the fact that the geometric bar construction extends to topological categories; see for instance [54, Chapter V.2]. Consider the topological category $\mathscr{A} \times [1]$, where $[1]$ is the discrete category associated to the poset $\{0 < 1\}$. Specifying a representation A of $\mathscr{A} \times [1]$, by which we mean a continuous functor from $\mathscr{A} \times [1]$ to the category of spaces, amounts to specifying a morphism of $\mathscr{A}$-spaces

$$A_0 \xrightarrow{f} A_1.$$

We extend $\mathscr{X}$ and $\mathscr{Y}$ to representations $\underline{\mathscr{X}}$ and $\underline{\mathscr{Y}}$ of $\mathscr{A} \times [1]$ by using the respective identity maps. The inclusion functors

$$\mathscr{A} = \mathscr{A} \times \{0\} \xrightarrow{i_0} \mathscr{A} \times [1] \xleftarrow{i_1} \mathscr{A} \times \{1\} = \mathscr{A}$$

together with the identity maps $\mathscr{X} \to i_0^*(\underline{\mathscr{X}})$, $\mathscr{Y} \to i_0^*(\underline{\mathscr{Y}})$, $\mathscr{X} \to i_1^*(\underline{\mathscr{X}})$ and $\mathscr{Y} \to i_1^*(\underline{\mathscr{Y}})$ induce maps on bar constructions

$$B(\mathscr{Y}, \mathscr{A}, \mathscr{X}) \xrightarrow{Bi_0} B(\underline{\mathscr{Y}}, \mathscr{A} \times [1], \underline{\mathscr{X}}) \xleftarrow{Bi_1} B(\mathscr{Y}, \mathscr{A}, \mathscr{X}). \tag{14}$$

The middle term is isomorphic to the cylinder $B(\mathscr{Y}, \mathscr{A}, \mathscr{X}) \times I$ and the maps Bi_0 and Bi_1 may be identified with the bottom and top inclusions. In particular, both Bi_0 and Bi_1 are homotopy equivalences.

The idea is now to define an action of $\mathscr{A}^\times$ on the triple $(\underline{\mathscr{Y}}, \mathscr{A}\times[1], \underline{\mathscr{X}})$ so that the maps (14) become equivariant if the left copy of $B\big(\mathscr{Y}, \mathscr{A}, \mathscr{X}\big)$ is given the trivial action and the right copy the conjugation action. Once we have found such an action, the proof will be complete.

For $g \in \mathscr{A}^\times$, there is a unique continuous functor

$$H_g\colon \mathscr{A}\times[1] \to \mathscr{A}\times[1]$$

such that $H_g^*(A)$ is the $\mathscr{A}{\times}[1]$-representation

$$A_0 \xrightarrow{gf} c_g^*(A_1), \quad a \mapsto gf(a),$$

whenever A is a representation of $\mathscr{A}\times[1]$ as above. The existence and uniqueness of H_g follows from the Yoneda lemma for topologically enriched categories. One checks that $H_1 = 1$ and $H_g H_h = H_{gh}$ for all $g, h \in \mathscr{A}^\times$ so this defines an action of $\mathscr{A}^\times$ on $\mathscr{A}\times[1]$. Next, the commutative square of $\mathscr{A}$-spaces

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\;1\;} & \mathscr{X} \\ \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle g} \\ \mathscr{X} & \xrightarrow{\;g\;} & c_g^*(\mathscr{X}) \end{array}$$

defines a morphism of $\mathscr{A}{\times}[1]$-representations $L_g\colon \underline{\mathscr{X}} \to H_g^*(\underline{\mathscr{X}})$. A morphism of contravariant representations $R_g\colon \underline{\mathscr{Y}} \to H_g^*(\underline{\mathscr{Y}})$ is defined similarly. Verifying that

$$(R_g, H_g, L_g)\colon (\underline{\mathscr{Y}}, \mathscr{A}\times[1], \underline{\mathscr{X}}) \to (\underline{\mathscr{Y}}, \mathscr{A}\times[1], \underline{\mathscr{X}})$$

defines an action of $\mathscr{A}^\times$ with the desired properties is routine and left to the reader. □

3.6 Algebraicity of cohomology and homotopy groups

In this section, we will give Theorem 1.1 a more precise formulation (Theorem 3.22 below) and prove it. Let us first review the notion of simple homotopy groups.

Definition 3.20 For a path-connected space B, we define the *simple homotopy groups* $\pi'_n(B)$ to be the abelian groups

$$\pi'_n(B) = \pi_n(B)/[\pi_1(B), \pi_n(B)]$$

obtained by trivializing the action of $\pi_1(B)$. Explicitly, $\pi'_1(B) = \pi_1(B)^{ab}$ and $\pi'_n(B) = [S^n, B]$ (free homotopy classes of maps) for $n > 1$.

Note that B is a simple space if and only if the canonical homomorphism $\pi_n(B) \to \pi'_n(B)$ is an isomorphism for all n. The simple homotopy groups are functorial for maps that do not necessarily preserve the basepoint. In particular, if a group $\mathscr{E}$ acts on B, then $\pi'_n(B)$ is a representation of $\mathscr{E}$. This is the reason we consider the simple homotopy groups instead of the ordinary homotopy groups.

We now proceed towards the proof of Theorem 3.22.

Notation We fix the following notation for the remainder of the section.

- X is a simply connected finite CW-complex.
- L is the minimal Quillen model of X.
- $G \leqslant \mathscr{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X;\mathbb{Q})$.
- $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ is the minimal unipotent algebraic subgroup with $\rho(G) \leqslant U$.
- $\mathfrak{u}$ is the Lie algebra of U.

Recall that the existence and uniqueness of U is guaranteed by Proposition 3.3 and that $\mathfrak{u}$ may be viewed as a subalgebra of $H_0(\mathrm{Der}^c\, L)$ by Theorem 2.24(iii).

We also remind the reader that $\mathscr{E}_U(X_{\mathbb{Q}})$ denotes the group of automorphisms of the $\mathscr{E}(X_{\mathbb{Q}})$-set $\mathscr{E}(X_{\mathbb{Q}})/U$ and that this group may be identified with $N_{\mathscr{E}(X_{\mathbb{Q}})}(U)/U$.

Recall that $\mathrm{Aut}_U(L)$ denotes the preimage of the normalizer $N_{\mathscr{E}(X_{\mathbb{Q}})}(U)$ under the homomorphism $\mathrm{Aut}(L) \to \mathscr{E}(X_{\mathbb{Q}})$. There is a surjective homomorphism

$$p\colon \mathrm{Aut}_U(L) \to \mathscr{E}_U(X_{\mathbb{Q}}).$$

By construction, the space

$$B_U\mathrm{aut}(X_{\mathbb{Q}}) = B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathscr{E}(X_{\mathbb{Q}})/U)$$

carries an action of $\mathscr{E}_U(X_{\mathbb{Q}})$ and hence an action of $\mathrm{Aut}_U(L)$ via p. On the other hand, the group $\mathrm{Aut}_U(L)$ acts on $\mathrm{Der}^c_{\mathfrak{u}}\, L$ by conjugation, and hence it acts on the nerve of $\mathrm{Der}^c_{\mathfrak{u}}\, L$.

Lemma 3.21 *The $\mathrm{Aut}_U(L)$-space $p^* B_U\mathrm{aut}(X_{\mathbb{Q}})$ is weakly equivalent to the nerve of the dg Lie algebra $\mathrm{Der}^c_{\mathfrak{u}}\, L$, on which $\mathrm{Aut}_U(L)$ acts by conjugation.*

Proof The rationalization $X_{\mathbb{Q}}$ may be identified with $|\mathsf{MC}_\bullet(L)|$. By Proposition 3.13, $B|\exp_\bullet(\mathrm{Der}^c_{\mathfrak{u}}\, L)|$ is weakly equivalent to $B\,\mathrm{aut}_U(X_{\mathbb{Q}})$ as an $\mathrm{Aut}_U(L)$-space, where $\mathrm{Aut}_U(L)$ acts by conjugation. Now consider the weak equivalence

$$B\,\mathrm{aut}_U(X_{\mathbb{Q}}) = B(*, \mathrm{aut}_U(X_{\mathbb{Q}}), *) \to B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathscr{E}(X_{\mathbb{Q}})/U), \tag{15}$$

induced by the inclusion of $\mathrm{aut}_U(X_{\mathbb{Q}})$ into $\mathrm{aut}(X_{\mathbb{Q}})$ and the map $* \to \mathscr{E}(X_{\mathbb{Q}})/U$ that selects the coset U.

The space $B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathscr{E}(X_{\mathbb{Q}})/U)$ has two commuting actions of $\mathrm{Aut}_U(L)$: the conjugation action as in Lemma 3.18 via the homomorphism $\mathrm{Aut}_U(L) \to \mathrm{aut}(X_{\mathbb{Q}})^\times$, and the action by automorphisms on the left $\mathrm{aut}(X_{\mathbb{Q}})$-space $\mathscr{E}(X_{\mathbb{Q}})/U$ via the homomorphism $\mathrm{Aut}_U(L) \to \mathscr{E}_U(X_{\mathbb{Q}})$. The weak equivalence (15) is $\mathrm{Aut}_U(L)$-equivariant if $B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathscr{E}(X_{\mathbb{Q}})/U)$ is given the diagonal action. (It is not equivariant with respect to the conjugation action, because the coset $U \in \mathscr{E}(X_{\mathbb{Q}})/U$ is not fixed by the left action of $\mathrm{Aut}_U(L)$.) Lemma 3.18 gives a zig-zag of $\mathrm{Aut}_U(L)$-equivariant weak equivalences connecting $B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathscr{E}(X_{\mathbb{Q}})/U)$, with the diagonal action of $\mathrm{Aut}_U(L)$, and same space where $\mathrm{Aut}_U(L)$ only acts on $\mathscr{E}(X_{\mathbb{Q}})/U$. □

Lemma 3.21 does not give full information about the homotopy type of the $\mathscr{E}_U(X_{\mathbb{Q}})$-space $B_U\operatorname{aut}(X_{\mathbb{Q}})$, but it is enough for proving that the cohomology groups and the simple homotopy groups are algebraic representations.

Theorem 3.22 (i) *$\mathscr{E}_U(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$-points of an affine algebraic group.*

(ii) *The homomorphism $\mathscr{E}_G(X) \to \mathscr{E}_U(X_{\mathbb{Q}})$ has finite kernel and image an arithmetic subgroup.*

(iii) *The representations of $\mathscr{E}_G(X)$ in the rational cohomology groups and the simple rational homotopy groups of the space $B\operatorname{aut}_G(X)$ are restrictions of algebraic representations of $\mathscr{E}_U(X_{\mathbb{Q}})$.*

Proof By Theorem 2.24(ii), $\mathscr{E}(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$-points of the algebraic group $\mathscr{G} = \mathscr{A}\mathrm{ut}(L)/\mathscr{A}\mathrm{ut}_h(L)$. By hypothesis, U corresponds to the $\mathbb{Q}$-points of a unipotent algebraic subgroup $\mathscr{U}$ of $\mathscr{G}$. It follows from Lemmas 2.10 and 2.9 that $\mathscr{E}_U(X_{\mathbb{Q}}) = N_{\mathscr{E}(X_{\mathbb{Q}})}(U)/U$ may be identified with the $\mathbb{Q}$-points of the algebraic group $N_{\mathscr{G}}(\mathscr{U})/\mathscr{U}$. This proves the first claim.

The kernel of $N_{\mathscr{E}(X)}(G)/G \to N_{\mathscr{E}(X_{\mathbb{Q}})}(U)/U$ injects into the set $\rho^{-1}(U)/G$, which is finite by Proposition 3.3. Lemma 3.9 implies that there is an inclusion

$$\rho(N_{\mathscr{E}(X)}(G)) \leqslant N_{\mathscr{E}(X_{\mathbb{Q}})}(U) \cap \rho(\mathscr{E}(X)). \tag{16}$$

The latter group is an arithmetic subgroup of $N_{\mathscr{E}(X_{\mathbb{Q}})}(U)$ by Theorem 2.26 and Lemma 2.11, so if we can show that (16) is the inclusion of a finite-index subgroup, then it follows that $\rho(N_{\mathscr{E}(X)}(G))$ is an arithmetic subgroup of $N_{\mathscr{E}(X_{\mathbb{Q}})}(U)$. It then follows from Lemma 2.12 that the image of $\rho(N_{\mathscr{E}(X)}(G))$ in $\mathscr{E}_U(X_{\mathbb{Q}}) = N_{\mathscr{E}(X_{\mathbb{Q}})}(U)/U$ is arithmetic, and then we are done because this agrees with the image of $\mathscr{E}_G(X) = N_{\mathscr{E}(X)}(G)/G$. To show that the index is finite, observe that (16) is obtained by applying ρ to the inclusion

$$N_{\mathscr{E}(X)}(G) \leqslant \rho^{-1}(N_{\mathscr{E}(X_{\mathbb{Q}})}(U)). \tag{17}$$

The latter group acts on the set of subgroups of $\rho^{-1}(U)$ by conjugation. The stabilizer of G is $N_{\mathscr{E}(X)}(G)$ and the orbit of G consists of subgroups of the form gGg^{-1}, where $g \in \mathscr{E}(X)$ satisfies $g\rho^{-1}(U)g^{-1} = \rho^{-1}(U)$. Note that all subgroups of this form have the same (finite) index in $\rho^{-1}(U)$. There are only finitely many subgroups of a given finite index in a finitely generated group, so the orbit must be finite. It follows that the index of the inclusion (17) is finite and in turn that the same holds for (16).

The action of $\mathscr{E}_G(X)$ on the homology of $B\operatorname{aut}_G(X)$ may be computed as the induced action on the homology of the $\mathscr{E}_G(X)$-space $B_G\operatorname{aut}(X)$. The latter is connected to $B_U\operatorname{aut}(X_{\mathbb{Q}})$ by a sequence of $\mathscr{E}_G(X)$-equivariant rational homology equivalences by Proposition 3.10, so it suffices to show that the cohomology groups of $B_U\operatorname{aut}(X_{\mathbb{Q}})$ are algebraic representations of $\mathscr{E}_U(X_{\mathbb{Q}})$.

Since the cohomology of $B\exp_\bullet(\mathfrak{g})$ is naturally isomorphic to the Chevalley–Eilenberg cohomology of $\mathfrak{g}$ (see Proposition 2.21), Lemma 3.21 implies that there is an $\mathrm{Aut}_U(L)$-equivariant isomorphism

$$H^k(B_U\operatorname{aut}(X_{\mathbb{Q}})) \cong H^k_{\mathrm{CE}}(\mathrm{Der}^c_{\mathfrak{u}} L) \tag{18}$$

for every k.

Note that $\mathrm{Aut}_U(L) = \mathscr{A}\mathrm{ut}_{\mathscr{U}}(L)(\mathbb{Q})$ and that there is an isomorphism of algebraic groups $\mathscr{A}\mathrm{ut}_{\mathscr{U}}(L)/\mathscr{W} \to N_{\mathscr{G}}(\mathscr{U})/\mathscr{U}$, where $\mathscr{A}\mathrm{ut}_{\mathscr{U}}(L)$ and $\mathscr{W}$ are the preimages of $N_{\mathscr{G}}(\mathscr{U})$ and $\mathscr{U}$ under $\pi\colon \mathscr{A}\mathrm{ut}(L) \to \mathscr{G}$. Since π has unipotent kernel, it follows that $\mathscr{W}$ is unipotent. The right-hand side of (18) is manifestly an algebraic representation of the algebraic group $\mathscr{A}\mathrm{ut}_{\mathscr{U}}(L)$. The action of $\mathscr{W}(\mathbb{Q})$ is trivial, because the action of $\mathscr{A}\mathrm{ut}_{\mathscr{U}}(L)(\mathbb{Q}) = \mathrm{Aut}_U(L)$ on the left-hand side factors through $\mathscr{E}_U(X_{\mathbb{Q}})$, which may be identified with $\mathscr{A}\mathrm{ut}_U(L)(\mathbb{Q})/\mathscr{W}(\mathbb{Q})$. Since $\mathscr{W}(\mathbb{Q})$ is dense in $\mathscr{W}$ (by eg [55, Theorem 17.93]), it follows that the representation is trivial as an algebraic representation of $\mathscr{W}$. Thus, it is an algebraic representation of $\mathscr{A}\mathrm{ut}_U(L)/\mathscr{W} \cong N_{\mathscr{G}}(\mathscr{U})/\mathscr{U}$. This shows that the cohomology groups of $B_U\mathrm{aut}(X_{\mathbb{Q}})$ are algebraic representations of $\mathscr{E}_U(X_{\mathbb{Q}})$.

To show algebraicity of the simple homotopy groups, one argues as above using the $\mathrm{Aut}_U(L)$-equivariant isomorphism

$$\pi'_{k+1}(B_U\mathrm{aut}(X_{\mathbb{Q}})) \cong H_k(\mathrm{Der}^c\, L)/[H_0(\mathrm{Der}^c\, L), H_k(\mathrm{Der}^c\, L)],$$

which can be deduced from Lemma 3.21 and Proposition 2.19. □

For later use, we record the following fact that was observed in the above proof.

Lemma 3.23 *The kernel of $p\colon \mathrm{Aut}_U(L) \to \mathscr{E}_U(X_{\mathbb{Q}})$ is unipotent.* □

Before we proceed to the proof of Theorem 1.2, let us discuss two interesting applications of Theorem 3.22.

3.6.1 Arithmeticity of higher homotopy groups As discussed in the introduction, Sullivan and Wilkerson proved that the homomorphism from $\mathscr{E}(X) = \pi_1 B\,\mathrm{aut}(X)$ to $\mathscr{E}(X_{\mathbb{Q}}) = \pi_1 B\,\mathrm{aut}(X_{\mathbb{Q}})$ has finite kernel and image an arithmetic subgroup. It is natural to ask whether there is a similar statement for the higher homotopy groups of $B\,\mathrm{aut}(X)$. The following shows that these are "arithmetic representations" of $\mathscr{E}(X)$, in the sense that they each admit a map, with finite kernel, onto a lattice inside an algebraic representation of the group $\mathscr{E}(X_{\mathbb{Q}})$.

Corollary 3.24 *Let X be a simply connected finite CW-complex. The representations of $\mathscr{E}(X)$ in the rational cohomology and homotopy groups of the universal cover of $B\,\mathrm{aut}(X)$ are restrictions of finite-dimensional algebraic representations of the algebraic group $\mathscr{E}(X_{\mathbb{Q}})$.*

Proof Apply Theorem 3.22 to the trivial group $G = 1$. Since $B\,\mathrm{aut}_1(X)$ is simply connected, the simple homotopy groups agree with the homotopy groups. □

3.6.2 Homotopy Torelli spaces Kupers and Randal-Williams [49] recently established algebraicity and nilpotence results for Torelli groups of the manifolds W_g, ie groups of diffeomorphisms that act trivially on integral homology. By analogy, we can consider the homotopy Torelli monoid of the space X, ie the monoid $\mathrm{tor}(X)$ of self-homotopy equivalences of X that act trivially on $H_*(X;\mathbb{Z})$. The following shows that a homotopical counterpart of the main result of [49] is valid for *arbitrary* simply connected finite complexes.

Note that the homotopy Torelli space sits in a homotopy fiber sequence

$$B\,\mathrm{tor}(X) \to B\,\mathrm{aut}(X) \to B\,\mathsf{GL}^{\mathscr{E}(X)}(H_*(X;\mathbb{Z})),$$

where $\mathsf{GL}^{\mathscr{E}(X)}(H_*(X;\mathbb{Z}))$ is the image of the homomorphism $\mathscr{E}(X) \to \mathsf{GL}(H_*(X;\mathbb{Z}))$.

Corollary 3.25 *Let X be a simply connected finite CW-complex.*

(i) *The group $\mathsf{GL}^{\mathscr{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$ of isomorphisms of $H_*(X;\mathbb{Q})$ that are realizable by a self-homotopy equivalence of $X_{\mathbb{Q}}$ is a linear algebraic group.*

(ii) *The homomorphism $\mathsf{GL}^{\mathscr{E}(X)}(H_*(X;\mathbb{Z})) \to \mathsf{GL}^{\mathscr{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$ has finite kernel and image an arithmetic subgroup.*

(iii) *The representations of $\mathsf{GL}^{\mathscr{E}(X)}(H_*(X;\mathbb{Z}))$ in the rational* (*co*)*homology and rational simple homotopy groups of the space $B\,\mathrm{tor}(X)$ are restrictions of algebraic representations of the linear algebraic group $\mathsf{GL}^{\mathscr{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$.*

Proof Clearly, $\mathrm{tor}(X) = \mathrm{aut}_G(X)$, where $G \leqslant \mathscr{E}(X)$ is the kernel of the action on $H_*(X;\mathbb{Z})$. Let $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ be the kernel of the algebraic representation $H_*(X;\mathbb{Q})$. Then U is unipotent by Theorem 2.24(iv). The cokernel of $G \to \rho^{-1}(U)$ may be identified with the kernel of

$$\mathsf{GL}^{\mathscr{E}(X)}(H_*(X;\mathbb{Z})) \to \mathsf{GL}^{\mathscr{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q})),$$

which is clearly finite. Hence, Proposition 3.3 implies that U is the minimal algebraic unipotent subgroup of $\mathscr{E}(X_{\mathbb{Q}})$ containing G, so Theorem 3.22 applies. □

3.7 Space-level algebraicity

In this section, we will prove Theorem 1.2. We retain the notation of the previous section. The following two lemmas are key.

Lemma 3.26 *If the algebraic group $\mathscr{E}_U(X_{\mathbb{Q}})$ is reductive, then the homomorphism*

$$p\colon \mathrm{Aut}_U(L) \to \mathscr{E}_U(X_{\mathbb{Q}})$$

admits a section σ that is a morphism of algebraic groups.

Proof The kernel of p is unipotent by Lemma 3.23, so if $\mathscr{E}_U(X_{\mathbb{Q}})$ is reductive, then p may be identified with the map from $\mathrm{Aut}_U(L)$ to its maximal reductive quotient and an algebraic section exists by Theorem 2.6. □

Lemma 3.27 *If the homomorphism*

$$p\colon \mathrm{Aut}_U(L) \to \mathscr{E}_U(X_{\mathbb{Q}})$$

admits a section σ, then there is a zig-zag of $\mathscr{E}_G(X)$-equivariant rational equivalences that connects $B_G\mathrm{aut}(X)$ to the nerve of the dg Lie algebra $\sigma^*\,\mathrm{Der}^c_{\mathfrak{u}}(L)$. Moreover, if σ is a morphism of algebraic groups, then the graded components of $\sigma^*\,\mathrm{Der}^c_{\mathfrak{u}}(L)$ are algebraic representations of $\mathscr{E}_U(X_{\mathbb{Q}})$.

Proof By Proposition 3.10, the spaces $B_G\mathrm{aut}(X)$ and $B_U\mathrm{aut}(X_{\mathbb{Q}})$ are connected by a zig-zag of $\mathscr{E}_G(X)$-equivariant rational equivalences. That σ is a section of p means that $p\sigma = 1$, so by Lemma 3.21, we get weak equivalences of $\mathscr{E}_U(X_{\mathbb{Q}})$-spaces

$$B_U\mathrm{aut}(X_{\mathbb{Q}}) = \sigma^* p^* B_U\mathrm{aut}(X_{\mathbb{Q}}) \sim \sigma^* B|\exp_\bullet(\mathrm{Der}^c_{\mathfrak{u}}\, L)| = B|\exp_\bullet(\sigma^*\,\mathrm{Der}^c_{\mathfrak{u}}\, L)|.$$

The graded components of $\mathrm{Der}^c_{\mathfrak{u}}(L)$ are manifestly algebraic representations of the algebraic group $\mathrm{Aut}_U(L)$. Hence, if σ is a morphism of algebraic groups, then the graded components of $\sigma^*\,\mathrm{Der}^c_{\mathfrak{u}}(L)$ are algebraic representations of $\mathscr{E}_U(X_{\mathbb{Q}})$. □

We now prove Theorem 1.2. First, we need to specify the cover $B\,\mathrm{aut}_u(X)$ and the groups $\Gamma(X)$ and $R(X)$.

Definition 3.28 We define

- $B\,\mathrm{aut}_u(X) = B\,\mathrm{aut}_G(X)$,
- $\Gamma(X) = \mathscr{E}_G(X)$,
- $R(X) = \mathscr{E}_U(X_{\mathbb{Q}})$,

where U is the unipotent radical of $\mathscr{E}(X_{\mathbb{Q}})$ and G is the preimage of U under the homomorphism $\rho\colon \mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}})$.

Proposition 3.29 *$R(X)$ is a reductive algebraic group and $\Gamma(X)$ may be identified with an arithmetic subgroup of $R(X)$.*

Proof The unipotent radical U is the maximal normal unipotent algebraic subgroup of $\mathscr{E}(X_{\mathbb{Q}})$. In particular, U is normal in $\mathscr{E}(X_{\mathbb{Q}})$ and $G = \rho^{-1}(U)$ is normal in $\mathscr{E}(X)$. Hence, $\Gamma(X) = \mathscr{E}(X)/G$ and $R(X) = \mathscr{E}(X_{\mathbb{Q}})/U$. We recognize the latter as the maximal reductive quotient of $\mathscr{E}(X_{\mathbb{Q}})$ (cf Section 2.2.1). Since $G = \rho^{-1}(U)$, we may identify $\Gamma(X) = \mathscr{E}(X)/G$ with the image of $\mathscr{E}(X)$ in $R(X) = \mathscr{E}(X_{\mathbb{Q}})/U$. This is an arithmetic subgroup of $R(X)$ by Theorem 3.22(ii). □

By that, the first part of Theorem 1.2 has been verified. Next, we need to construct the dg Lie algebra $\mathfrak{g}(X)$.

Definition 3.30 Define

$$\mathfrak{g}(X) = \sigma^*\,\mathrm{Der}^c_{\mathfrak{u}}(L),$$

where $\mathfrak{u}$ is the Lie algebra of the unipotent radical U of $\mathscr{E}(X_{\mathbb{Q}})$, and where

$$\sigma\colon R(X) \to \mathrm{Aut}_U(L)$$

is a section of p as in Lemma 3.26, which exists since $R(X)$ is reductive.

Theorem 3.31 *The dg Lie algebra $\mathfrak{g}(X)$ is nilpotent, its graded components are algebraic representations of $R(X)$, and there is a zig-zag of $\Gamma(X)$-equivariant rational equivalences that connects $B\operatorname{aut}_u(X)$ to the nerve of $\mathfrak{g}(X)$.*

Proof Nilpotence is shown in Proposition 3.12. The rest is immediate from Lemmas 3.26 and 3.27. □

Corollary 3.32 *The space $B\operatorname{aut}(X)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{g}(X)\rangle_{h\Gamma(X)}$.*

Proof We can recover $B\operatorname{aut}(X)$ as the homotopy orbit space

$$B\operatorname{aut}(X) \sim B\operatorname{aut}_u(X)_{h\Gamma(X)}.$$

Applying homotopy orbits to the zig-zag in Theorem 3.31 yields the result. □

This finishes the proof of Theorem 1.2.

Remark 3.33 It is natural to ask whether the rational homotopy type of the $\mathscr{E}_G(X)$-space $B_G\operatorname{aut}(X)$ can be represented by a dg Lie algebra of algebraic representations of $\mathscr{E}_U(X_{\mathbb{Q}})$ for arbitrary subgroups $G \leqslant \mathscr{E}(X)$ that act nilpotently on $SH_*(X;\mathbb{Q})$. The main problem is to rectify the action up to homotopy of $\mathscr{E}_U(X_{\mathbb{Q}})$ on $\operatorname{Der}_{\mathrm{u}}^c L$. As we have seen, this can be done if $\mathscr{E}_U(X_{\mathbb{Q}})$ is reductive. One can show that $\mathscr{E}_U(X_{\mathbb{Q}})$ is reductive if and only if U is the unipotent radical of a parabolic subgroup of $\mathscr{E}(X_{\mathbb{Q}})$, but $\mathscr{E}_U(X_{\mathbb{Q}})$ will not be reductive in general. Still, even if the action cannot be rectified, one can ask whether a suitable modification of $\operatorname{Der}_{\mathrm{u}}^c L$ could accommodate an action of $\mathscr{E}_U(X_{\mathbb{Q}})$. Since our main applications are in the case when $\mathscr{E}_U(X_{\mathbb{Q}})$ is reductive, we will not pursue this question further in this paper. We leave it as a challenge to the interested reader.

3.8 A commutative cochain algebra model for the classifying space

In this section, we will use Theorem 1.2 to construct a commutative differential graded algebra model for $B\operatorname{aut}(X)$. Before we can formulate the result, we need to review the definition of the commutative cochains of a simplicial set with local coefficients, following Halperin [42].

3.8.1 Commutative cochains with local coefficients For a simplicial set K, we let $(\Delta\downarrow K)$ denote the simplex category. An object is a simplex $\sigma\colon\Delta^n\to K$ and a morphism from σ to τ is a commutative diagram

$$\begin{array}{ccc} \Delta^n & & \\ {\scriptstyle\varphi}\downarrow & \searrow^{\sigma} & \\ \Delta^m & \xrightarrow{\tau} & K \end{array}$$

Definition 3.34 [42, pages 150–151] A local system on a simplicial set K with values in a category $\mathscr{C}$ is a functor

$$F\colon(\Delta\downarrow K)^{\mathrm{op}}\to\mathscr{C}.$$

The global sections of a local system F on K is defined as the inverse limit

$$F(K)=\varprojlim_{\sigma\in(\Delta\downarrow K)} F_\sigma,$$

provided this limit exists in $\mathscr{C}$.

Remark 3.35 To compare this with the notion of a local system of coefficients defined in terms of representations of the fundamental groupoid, one should observe that the groupoidification of the category $(\Delta \downarrow K)$ is a model for the fundamental groupoid of K; see [38, Section III.1]. This means that the category of local systems F such that $F(\varphi)$ is invertible for every morphism φ in $(\Delta \downarrow K)$ is equivalent to the category of representations of the fundamental groupoid. However, since we do not require invertibility of $F(\varphi)$ in general, the notion of a local system is more general.

Example 3.36 (i) If Γ is a discrete group and M is an object of $\mathscr{C}$ with an action of Γ, then there is an associated local system on the nerve $B\Gamma$, also denoted by M, where $M_\sigma = M$ for all $\sigma = (\gamma_1, \dots, \gamma_n) \colon \Delta^n \to B\Gamma$ and $d_i \colon M_\sigma \to M_{d_i\sigma}$ is multiplication by γ_n for $i = n$ and the identity otherwise.

(ii) A simplicial object $X_\bullet \colon \Delta^{\mathrm{op}} \to \mathscr{C}$ determines a local system on every simplicial set K by restriction along $(\Delta \downarrow K)^{\mathrm{op}} \to \Delta^{\mathrm{op}}$. In particular, the simplicial cdga $\Omega_\bullet$ determines a local system Ω^* of cdgas on every simplicial set K. The global sections $\Omega^*(K)$ is the usual model for Sullivan's cdga of polynomial differential forms on K, also denoted by $A^*_{PL}(K)$.

(iii) More generally, if F is a local system of cochain complexes on K, then $\Omega^*(K; F)$ may be defined as the global sections of the local system $\Omega^* \otimes F$; see [42, Definition 13.10]. If A is a local system of cochain algebras, then $\Omega^*(K; A)$ is a cochain algebra, which is commutative if A is.

Proposition 3.37 *Let K be a simplicial set and let A be a local system of $\mathbb{Q}$-cochain complexes on K. The cochain complex $\Omega^*(K; A)$ is naturally quasi-isomorphic to $C^*(K; A)$. If A is a local system of dg algebras, then $\Omega^*(K; A)$ and $C^*(K; A)$ are quasi-isomorphic as dg algebras.*

Proof Integration provides a natural quasi-isomorphism

$$\Omega^*(K; A) \to C^*(K; A),$$

see [42, Theorem 14.18]. The integration map is not multiplicative on the cochain level, but the argument in [34, Section 10(d)] goes through with coefficients A, providing a zig-zag of dg algebra quasi-isomorphisms. □

In particular, for a discrete group Γ and a cochain complex M of $\mathbb{Q}[\Gamma]$-modules, the cohomology of $\Omega^*(\Gamma, M) = \Omega^*(B\Gamma; M)$ agrees with group cohomology $H^*(\Gamma, M)$ (as defined in eg [25, VII.5]). If A is a commutative cochain algebra over $\mathbb{Q}$ with an action of Γ, then $\Omega^*(\Gamma, A)$ is a commutative cochain algebra model for $C^*(\Gamma, A)$. The following is a commutative cochain algebra version, in characteristic zero, of well-known results for homology and singular chains; see eg [25, Section VII.7] or [53, Section 13].

Proposition 3.38 *Let X be a space with a Γ-action. The commutative cochain algebras $\Omega^*(X_{h\Gamma})$ and $\Omega^*(\Gamma, \Omega^*(X))$ are quasi-isomorphic.*

Proof The local system F of the fibration $X_{h\Gamma} \to B\Gamma$ in the sense of [42, page 248] may be identified up to quasi-isomorphism with the local system associated to the Γ-cdga $\Omega^*(X)$. Granted this, the combination of [42, Lemma 19.21] and [42, Theorem 13.12] applied to the local system F shows that

$$\Omega^*(X_{h\Gamma}) \cong F(B\Gamma) \sim \Omega^*(B\Gamma; F) \sim \Omega^*(B\Gamma; \Omega^*(X)). \qquad \square$$

Now we are ready to formulate and prove Corollary 1.3 from the introduction.

Theorem 3.39 *The commutative dg algebra $\Omega^*(B\operatorname{aut}(X))$ is quasi-isomorphic to the commutative dg algebra $\Omega^*(\Gamma(X), C^*_{\mathrm{CE}}(\mathfrak{g}(X)))$.*

Proof It follows from Corollary 3.32 that $\Omega^*(B\operatorname{aut}(X))$ is quasi-isomorphic to $\Omega^*(\langle\mathfrak{g}(X)\rangle_{h\Gamma(X)})$. By Proposition 3.38, the latter commutative cochain algebra is quasi-isomorphic to $\Omega^*(\Gamma(X), \Omega^*(\langle\mathfrak{g}(X)\rangle))$. We then use the natural quasi-isomorphism $C^*_{\mathrm{CE}}(\mathfrak{g}(X)) \to \Omega^*(\langle\mathfrak{g}(X)\rangle)$ of Proposition 2.21 to complete the proof. $\square$

Let us compare with the approach of Gómez-Tato, Halperin and Tanré [39]. They study the fiberwise rationalization of $B\operatorname{aut}(X) \to B\mathscr{E}(X)$ by using local systems, in the sense of Definition 3.34, of cdgas over $B\mathscr{E}(X)$. As discussed in Remark 3.35, such local system are more general, and more complicated, than cdgas with an action of $\mathscr{E}(X)$. Our approach is to instead consider the fiberwise rationalization of $B\operatorname{aut}(X) \to B\Gamma(X)$. This has two significant advantages: Firstly, it lets us work with cdgas with an action of $\Gamma(X)$ rather than local systems of cdgas over $B\Gamma(X)$. Secondly, semisimplicity of algebraic $\Gamma(X)$-representations lets us construct minimal models using familiar methods. Indeed, the counterpart of Problem 3 of [39, Section 7.3] in our setting admits the following solution:

Theorem 3.40 *The minimal Sullivan model $\mathfrak{M}(X)$ for $B\operatorname{aut}_u(X)$ admits an algebraic action of $R(X)$ such that the spatial realization $\langle\mathfrak{M}(X)\rangle$ is rationally equivalent to $B\operatorname{aut}_u(X)$ as a $\Gamma(X)$-space, and $\Omega^*(B\operatorname{aut}(X))$ is quasi-isomorphic to $\Omega^*(\Gamma(X), \mathfrak{M}(X))$ as a commutative cochain algebra.*

Proof The minimal model is unique up to isomorphism, so it is enough to construct a particular minimal model that admits an algebraic $R(X)$-action with the desired properties. Let $\mathfrak{g}(X)$ be the Lie model for $B\operatorname{aut}_u(X)$ in Theorem 3.31. By semisimplicity of $\operatorname{Rep}_{\mathbb{Q}}(R(X))$, we can find a contraction of chain complexes of algebraic $R(X)$-representations

$$h \circlearrowright \mathfrak{g}(X) \underset{g}{\overset{f}{\rightleftarrows}} H_*(\mathfrak{g}(X)).$$

Applying [9, Theorem 1.3], we get a contraction of cocommutative chain coalgebras in $\operatorname{Rep}_{\mathbb{Q}}(R(X))$,

$$H' \circlearrowright C^{\mathrm{CE}}_*(\mathfrak{g}(X)) \underset{G'}{\overset{F'}{\rightleftarrows}} (\Lambda^c(V), d'),$$

where V is the suspension of $H_*(\mathfrak{g}(X))$. The differential d' is given by the explicit formula

$$d' = FtG + FtHtG + Ft(Ht)^2G + \cdots,$$

and F', G' and H' are given by similar formulas. Here, t is the quadratic part of the differential of $C_*^{\mathrm{CE}}(\mathfrak{g}(X))$ that encodes the Lie bracket, F and G are the morphisms of dg coalgebras induced by f and g, and H is the "symmetrized tensor trick homotopy" [9, Section 5]. The dual of $(\Lambda^c(V), d')$ is a minimal model $\mathfrak{M}(X)$ for $B\operatorname{aut}_u(X)$. By construction, it has an algebraic action of $R(X)$ and the dual of G' is a quasi-isomorphism of cochain algebras of $R(X)$-representations $C^*_{\mathrm{CE}}(\mathfrak{g}(X)) \to \mathfrak{M}(X)$. This induces a quasi-isomorphism of commutative cochain algebras $\Omega^*(\Gamma(X), C^*_{\mathrm{CE}}(\mathfrak{g}(X))) \to \Omega^*(\Gamma(X), \mathfrak{M}(X))$, so it follows from Theorem 3.39 that $\Omega^*(\Gamma(X), \mathfrak{M}(X))$ is a cdga model for $\Omega^*(B\operatorname{aut}(X))$. Applying spatial realization, we get an $R(X)$-equivariant weak equivalence $\langle \mathfrak{M}(X)\rangle \to \langle C^*_{\mathrm{CE}}(\mathfrak{g}(X))\rangle \cong \langle \mathfrak{g}(X)\rangle$, and the latter is a $\Gamma(X)$-equivariant model for $B\operatorname{aut}_u(X)$ by Theorem 3.31.

We stress that this construction is completely explicit once the contraction between $\mathfrak{g}(X)$ and its homology has been fixed. □

3.9 The cohomology ring of the classifying space

The following result, which was stated as Corollary 1.4 in the introduction, is perhaps the most striking consequence of Theorem 3.39.

Theorem 3.41 *There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}(X);\mathbb{Q}) \cong H^*(\Gamma(X), H^*_{\mathrm{CE}}(\mathfrak{g}(X))).$$

Proof The Chevalley–Eilenberg complex $C^*_{\mathrm{CE}}(\mathfrak{g}(X))$ is a cochain complex of finite-dimensional algebraic representations of $R(X)$. Since $R(X)$ is reductive, every such representation is semisimple, and thus the cochain complex $C^*_{\mathrm{CE}}(\mathfrak{g}(X))$ is split, which means that there is a contraction of cochain complexes in $\operatorname{Rep}_{\mathbb{Q}}(R(X))$,

$$h \circlearrowright C^*_{\mathrm{CE}}(\mathfrak{g}(X)) \underset{\nabla}{\overset{p}{\rightleftarrows}} H^*_{\mathrm{CE}}(\mathfrak{g}(X)).$$

That is, p and ∇ are chain maps such that $p(z) = [z]$ whenever $z \in C^*_{\mathrm{CE}}(\mathfrak{g}(X))$ is a cocycle, $p\nabla = \mathrm{id}$, and h satisfies $\mathrm{id} - \nabla p = dh + hd$, see eg [16, Lemma B.1].

Applying the dg functor $\Omega^*(\Gamma(X); -)$ yields a new contraction

$$h_* \circlearrowright \Omega^*(\Gamma(X), C^*_{\mathrm{CE}}(\mathfrak{g}(X))) \underset{\nabla_*}{\overset{p_*}{\rightleftarrows}} \Omega^*(\Gamma(X), H^*_{\mathrm{CE}}(\mathfrak{g}(X))).$$

We now use the homotopy transfer theorem (see eg [9]) to produce a C_∞-algebra structure $\{\mu_n\}_{n\geq 2}$ on

$$A = \Omega^*(\Gamma(X); H^*_{\mathrm{CE}}(\mathfrak{g}(X)))$$

and an extension of ∇_* to a C_∞-quasi-isomorphism. The multiplication in this C_∞-algebra structure is given by $\mu_2 = p_* m_2(\nabla_* \otimes \nabla_*)$, where m_2 is the multiplication on $\Omega^*(\Gamma(X); C^*_{\mathrm{CE}}(\mathfrak{g}(X)))$, and one checks that this agrees with the multiplication on $\Omega^*(\Gamma(X); H^*_{\mathrm{CE}}(\mathfrak{g}(X)))$, coming from viewing $H^*_{\mathrm{CE}}(\mathfrak{g}(X))$ as a $\Gamma(X)$-cdga with trivial differential. The cdga (A, μ_2) is of course not equivalent to the C_∞-algebra $(A, \mu_2, \mu_3, \ldots)$ in general, but they have isomorphic cohomology rings. □

4 The ingredients of the algebraic model

In this section, we will give more concrete descriptions of the ingredients $R(X)$, $\Gamma(X)$, $\mathfrak{g}(X)$, $B\,\mathrm{aut}_u(X)$ of Theorem 1.2 and we will state counterparts of our main results for finite Postnikov stages. In Section 5, the results of the present section will be used to make explicit calculations.

4.1 The groups $R(X)$ and $\Gamma(X)$

Let $H_*(X;\mathbb{Q})^{\mathrm{ss}}$ denote the "semisimplification" of $H_*(X;\mathbb{Q})$ as a representation of the algebraic group $\mathscr{E}(X_{\mathbb{Q}})$. Recall from Section 2.2.1 that this is the semisimple, unique up to isomorphism, $\mathscr{E}(X_{\mathbb{Q}})$-representation

$$H_*(X;\mathbb{Q})^{\mathrm{ss}} = \bigoplus_{i=1}^{n} V_i/V_{i-1},$$

where

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = H_*(X;\mathbb{Q}) \tag{19}$$

is any choice of composition series. For a group G and a representation V of G, let $\mathsf{GL}^G(V)$ denote the image of the homomorphism $G \to \mathsf{GL}(V)$.

Theorem 4.1 *There are group isomorphisms*

$$R(X) \cong \mathsf{GL}^{\mathscr{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q})^{\mathrm{ss}}) \quad \textit{and} \quad \Gamma(X) \cong \mathsf{GL}^{\mathscr{E}(X)}(H_*(X;\mathbb{Q})^{\mathrm{ss}}).$$

In other words, $R(X)$ (resp. $\Gamma(X)$) may be identified with the group of automorphisms of $H_*(X;\mathbb{Q})^{\mathrm{ss}}$ that are induced by a self-homotopy equivalence of $X_{\mathbb{Q}}$ (resp. X).

Proof By definition, the group $R(X)$ is the maximal reductive quotient of $\mathscr{E}(X_{\mathbb{Q}})$. By Theorem 2.24(iv), the rational homology $H_*(X;\mathbb{Q})$ is an algebraic representation of $\mathscr{E}(X_{\mathbb{Q}})$ with unipotent kernel. Hence by Lemma 2.7, the maximal reductive quotient $R(X)$ of $\mathscr{E}(X_{\mathbb{Q}})$ may be identified with the image of $\mathscr{E}(X_{\mathbb{Q}})$ in $\mathsf{GL}(H_*(X;\mathbb{Q})^{\mathrm{ss}})$. By definition, the group $\Gamma(X)$ is the image of $\mathscr{E}(X)$ in $R(X)$. In view of the above, this agrees with $\mathsf{GL}^{\mathscr{E}(X)}(H_*(X;\mathbb{Q})^{\mathrm{ss}})$. □

Often, in fact in all cases we consider in Section 5 except the last, the equivalent conditions in the corollary below are satisfied, which simplifies the descriptions.

Corollary 4.2 *The following are equivalent*:

(i) *The $\mathscr{E}(X_{\mathbb{Q}})$-representation $H_*(X;\mathbb{Q})$ is semisimple.*

(ii) *There is an isomorphism of algebraic groups*

$$R(X) \cong \mathsf{GL}^{\mathscr{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q})).$$

(iii) *The algebraic group $\mathsf{GL}^{\mathscr{E}(X_{\mathbb{Q}})}(H_*(X;\mathbb{Q}))$ is reductive.*

If the above conditions hold, then $\Gamma(X)$ may be identified with the group of automorphisms of $H_*(X;\mathbb{Q})$ that are induced by self-homotopy equivalences of X.

Proof If $H_*(X;\mathbb{Q})$ is semisimple, then it is isomorphic to $H_*(X;\mathbb{Q})^{\mathrm{ss}}$ so Theorem 4.1 identifies $R(X)$ with $\mathsf{GL}^{\mathscr{E}(X_\mathbb{Q})}(H_*(X;\mathbb{Q}))$. The second condition implies the third since $R(X)$ is reductive by construction. If $\mathsf{GL}^{\mathscr{E}(X_\mathbb{Q})}(H_*(X;\mathbb{Q}))$ is reductive, then $H_*(X;\mathbb{Q})$ is a semisimple representation of it by Theorem 2.5, but then it is also semisimple as a representation of $\mathscr{E}(X_\mathbb{Q})$. □

Remark 4.3 If the equivalent conditions in Corollary 4.2 are satisfied and in addition $H_*(X;\mathbb{Z})$ is torsion-free, then $\Gamma(X)$ may be identified with the group of automorphisms of $H_*(X;\mathbb{Z})$ that are induced by a self-homotopy equivalence.

An obvious necessary condition for an automorphism of $H_*(X;\mathbb{Q})$ to be induced by a self-homotopy equivalence is that it preserves the coproduct in homology, so there is an inclusion of algebraic groups

(20) $$\mathsf{GL}^{\mathscr{E}(X_\mathbb{Q})}(H_*(X;\mathbb{Q})) \leq \mathrm{Aut}_{\mathrm{coalg}}(H_*(X;\mathbb{Q})),$$

where the latter is the group of automorphisms of the homology coalgebra. This may also be identified with the group $\mathrm{Aut}_{\mathrm{alg}}(H^*(X;\mathbb{Q}))$ of automorphisms of the cohomology algebra.

Proposition 4.4 *The inclusion (20) is an equality if and only if X is formal.*

Proof This follows from [66, Theorem 12.7]. □

Corollary 4.5 *If X is formal, then there is an isomorphism of algebraic groups*

$$R(X) \cong \mathrm{Aut}_{\mathrm{alg}}(H^*(X;\mathbb{Q}))$$

if and only if $\mathrm{Aut}_{\mathrm{alg}}(H^*(X;\mathbb{Q}))$ is reductive.

Proof Combine Proposition 4.4 and Corollary 4.2. □

4.2 Normal unipotent fibrations

In this section we will show that $B\,\mathrm{aut}_u(X)$ can be characterized as the classifying space for what we call *normal unipotent fibrations*, defined below.

Consider a fibration of path-connected spaces

(21) $$X \to E \xrightarrow{p} B.$$

The fibration is classified by a homotopy class of maps $f\colon B \to B\,\mathrm{aut}(X)$, which induces a homomorphism on fundamental groups $\pi_1(B) \to \mathscr{E}(X)$ and an action of $\pi_1(B)$ on the homology of X.

Definition 4.6 The fibration (21) is called a (*normal*) *unipotent fibration* if the image of $\pi_1(B) \to \mathscr{E}(X_\mathbb{Q})$ is contained in a (normal) unipotent algebraic subgroup.

Proposition 4.7 *Consider a fibration as in (21).*

(1) *The fibration is unipotent if and only if for each n, the $\pi_1(B)$-module $H_n(X;\mathbb{Q})$ is nilpotent, ie there exists a filtration of $\pi_1(B)$-submodules*
$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r = H_n(X;\mathbb{Q})$$
such that F_i/F_{i-1} is a trivial $\pi_1(B)$-module for every i.

(2) *The fibration is normal unipotent if and only if, for each n, the $\pi_1(B)$-module $H_n(X;\mathbb{Q})$ is nilpotent and, moreover, a filtration as above can be chosen to be a filtration of algebraic $\mathscr{E}(X_{\mathbb{Q}})$-representations.*

Proof For the first statement, apply Proposition 3.2 to the image of the homomorphism $\pi_1(B) \to \mathscr{E}(X)$. For the second statement, note that if the fibration is normal unipotent, then the image of $\pi_1(B) \to \mathscr{E}(X_{\mathbb{Q}})$ is contained in the unipotent radical, which may be described as the kernel of the action of $\mathscr{E}(X_{\mathbb{Q}})$ on the semisimplification $H_*(X;\mathbb{Q})^{\mathrm{ss}}$ by Lemma 2.7 applied to the algebraic $\mathscr{E}(X_{\mathbb{Q}})$-representation $H_*(X;\mathbb{Q})$. In particular, $\pi_1(X)$ acts trivially on $H_*(X;\mathbb{Q})^{\mathrm{ss}}$, which is the associated graded with respect to a filtration of $H_*(X;\mathbb{Q})$ by algebraic $\mathscr{E}(X_{\mathbb{Q}})$-representations. Conversely, if $\pi_1(B)$ acts trivially on the associated graded of a filtration of $H_*(X;\mathbb{Q})$ by algebraic $\mathscr{E}(X_{\mathbb{Q}})$-representations, then the image of $\pi_1(B) \to \mathscr{E}(X_{\mathbb{Q}})$ is contained in the unipotent radical, because any such filtration can be refined to a composition series. □

Remark 4.8 As in Proposition 3.2, one can replace $H_n(X;\mathbb{Q})$ by spherical homology $SH_n(X;\mathbb{Q})$ or rational homotopy $\pi_n(X) \otimes \mathbb{Q}$ in the proposition above. The latter can be used to show that a fibration (21) is unipotent if and only if its fiberwise rationalization is nilpotent in the sense of [44, page 67] or [24, II.4.3].

Proposition 4.9 *The fibration (21) is normal unipotent if and only if the classifying map factors over $B\operatorname{aut}_u(X)$ up to homotopy:*

$$\begin{array}{ccc} & & B\operatorname{aut}_u(X) \\ & \nearrow & \downarrow \\ B & \longrightarrow & B\operatorname{aut}(X) \end{array}$$

Thus, $B\operatorname{aut}_u(X)$ may be interpreted as the classifying space for normal unipotent fibrations with fiber X.

Proof A lift exists if and only if the image of $\pi_1(B) \to \pi_1 B\operatorname{aut}(X) = \mathscr{E}(X)$ is contained in $\pi_1 B\operatorname{aut}_u(X)$. The latter group is equal to the preimage of the unipotent radical of $\mathscr{E}(X_{\mathbb{Q}})$ under $\mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}})$ by definition of $B\operatorname{aut}_u(X)$. □

4.3 The dg Lie algebra $\mathfrak{g}(X)$

We will now give a more concrete description of the dg Lie algebra $\mathfrak{g}(X)$ of Definition 3.30. Recall the notion of a nilradical of a dg Lie algebra from Definition 2.16.

Proposition 4.10 *The dg Lie algebra* $\mathfrak{g}(X)$ *agrees with the nilradical* $\operatorname{nil}\operatorname{Der}^c L$ *and its graded components are given by*

$$\mathfrak{g}(X)_n = \begin{cases} (\operatorname{Der}^c L)_n & \text{if } n > 0, \\ \operatorname{Lie}(\mathscr{A}\mathrm{ut}_u L) & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Here, $\operatorname{Lie}(\mathscr{A}\mathrm{ut}_u L)$ *stands for the Lie algebra of the unipotent radical of* $\mathscr{A}\mathrm{ut}\, L$ *and this Lie algebra may be identified with* $\operatorname{nil}_{QL} Z_0(\operatorname{Der} L)$, *the maximal ideal of* $Z_0(\operatorname{Der} L)$ *of derivations that act nilpotently on* $QL = L/[L,L]$.

Proof We first verify the statement about the graded components of $\mathfrak{g}(X)$. Suppressing the action of $R(X)$, we have that $\mathfrak{g}(X) = \operatorname{Der}^c_{\mathfrak{u}} L$ is the preimage of $\mathfrak{u} \subseteq H_0(\operatorname{Der}^c L) = \operatorname{Lie}(\mathscr{A}\mathrm{ut}^h L)$ under the morphism of dg Lie algebras $q\colon \operatorname{Der}^c L\langle 0\rangle \to H_0(\operatorname{Der}^c L)$, where $\mathfrak{u}$ is the Lie algebra of the unipotent radical $\mathscr{A}\mathrm{ut}^h_u L$ of $\mathscr{A}\mathrm{ut}^h L$. In degree zero, q is the morphism $Z_0(\operatorname{Der}^c L) \to H_0(\operatorname{Der}^c L)$, which may be identified with $\operatorname{Lie}(p)\colon \operatorname{Lie}(\mathscr{A}\mathrm{ut}\, L) \to \operatorname{Lie}(\mathscr{A}\mathrm{ut}^h L)$, where $p\colon \mathscr{A}\mathrm{ut}\, L \to \mathscr{A}\mathrm{ut}^h L$ is the quotient map. Since p has unipotent kernel (see Theorem 2.24), the preimage $p^{-1}(\mathscr{A}\mathrm{ut}^h_u L)$ must be equal to $\mathscr{A}\mathrm{ut}_u L$, the unipotent radical of $\mathscr{A}\mathrm{ut}\, L$. Hence, $\mathfrak{g}(X)_0 = \operatorname{Lie}(p)^{-1}(\operatorname{Lie}(\mathscr{A}\mathrm{ut}^h_u L)) = \operatorname{Lie}(p^{-1}(\mathscr{A}\mathrm{ut}^h_u L)) = \operatorname{Lie}(\mathscr{A}\mathrm{ut}_u L)$. Since the Lie algebra $H_0(\operatorname{Der}^c L)$ is concentrated in degree zero, it follows that $\mathfrak{g}(X)_n = \operatorname{Der}^c L\langle 0\rangle_n$ for $n \neq 0$.

Since $\mathscr{A}\mathrm{ut}^h_u L$ is normal in $\mathscr{A}\mathrm{ut}^h L$, the Lie algebra $\mathfrak{u}$ is an ideal in $H_0(\operatorname{Der}^c L)$. It follows that $\operatorname{Der}^c_{\mathfrak{u}} L = q^{-1}(\mathfrak{u})$ is an ideal in $\operatorname{Der}^c L\langle 0\rangle$. Moreover, $\operatorname{Der}^c_{\mathfrak{u}} L$ is nilpotent by Proposition 3.12. By definition, $\operatorname{nil}\operatorname{Der}^c L$ is the maximal nilpotent ideal, so it follows that $\operatorname{Der}^c_{\mathfrak{u}} L \subseteq \operatorname{nil}\operatorname{Der}^c L$.

We now show the reverse inclusion $\operatorname{nil}\operatorname{Der}^c L \subseteq \operatorname{Der}^c_u L$. The components in degrees $\neq 0$ are clearly equal, so we only need to show that $(\operatorname{nil}\operatorname{Der}^c L)_0 \subseteq \operatorname{Lie}(\mathscr{A}\mathrm{ut}_u L)$. Note that since L is finitely generated as a graded Lie algebra and of finite type, there is some n such that the finite-dimensional graded vector space $L_{\leqslant n} = \bigoplus_{i=1}^n L_i$ is a faithful algebraic representation of $\mathscr{A}\mathrm{ut}\, L$. Thus, by Lemma 2.8, $\operatorname{Lie}(\mathscr{A}\mathrm{ut}_u L)$ is precisely the maximal ideal of $\operatorname{Lie}(\mathscr{A}\mathrm{ut}\, L) = Z_0(\operatorname{Der}^c L)$ consisting of derivations of L that are nilpotent when restricted to $L_{\leqslant n}$. Since $(\operatorname{nil}\operatorname{Der}^c L)_0$ is an ideal in $Z_0(\operatorname{Der}^c L)$ which acts nilpotently on $(\operatorname{Der}^c L)_k \cong (\operatorname{Der} L)_k \oplus L_{k-1}$ for each $k > 0$, it in particular acts nilpotently on $L_{\leq n}$. Hence, $(\operatorname{nil}\operatorname{Der}^c(L))_0 \subseteq \operatorname{Lie}(\mathscr{A}\mathrm{ut}_u L)$.

Applying Lemma 2.8 to $V = L/[L,L]$, the Lie algebra of $\mathscr{A}\mathrm{ut}_u(L)$ is described as $\operatorname{nil}_{QL} Z_0(\operatorname{Der}^c L)$. □

Corollary 4.11 *If* $\operatorname{Aut} L$ *is reductive, then*

$$R(X) = \operatorname{Aut} L \quad \text{and} \quad \mathfrak{g}(X) = \operatorname{Der}^c L\langle 1\rangle,$$

and the action of $R(X)$ *on* $\mathfrak{g}(X)$ *is the conjugation action.*

Proof Each of the homomorphisms $\operatorname{Aut} L \to \mathscr{E}(X_{\mathbb{Q}}) \to R(X)$ has unipotent kernel. If $\operatorname{Aut} L$ is reductive, it has no normal unipotent subgroups. This forces $\operatorname{Aut} L = \mathscr{E}(X_{\mathbb{Q}}) = R(X)$. The description of $\mathfrak{g}(X)$ follows directly from Proposition 4.10. □

4.4 Finite Postnikov stages

The results we have stated have analogs for simply connected finite Postnikov stages X of finite type. We state the results and briefly indicate the necessary modifications of the proofs.

The minimal Sullivan model Λ is finitely generated and we may identify $\mathscr{E}(X_{\mathbb{Q}})$ with the $\mathbb{Q}$-points of the algebraic group $\mathscr{A}\mathrm{ut}^h(\Lambda)$ by Theorem 2.25(ii).

As before, let $G \leqslant \mathscr{E}(X)$ be a subgroup that acts nilpotently on $SH_*(X;\mathbb{Q})$ and let $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ be the minimal unipotent algebraic subgroup that contains $\rho(G)$ as in Proposition 3.3. By Theorem 2.25(iii), we may regard the Lie algebra $\mathfrak{u}$ of U as a Lie subalgebra of $H_0(\mathrm{Der}\,\Lambda)$. Analogously to Definition 3.11, we define the dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}\,\Lambda$ by declaring that there is a pullback square

$$\begin{array}{ccc} \mathrm{Der}_{\mathfrak{u}}\,\Lambda & \longrightarrow & \mathfrak{u} \\ \downarrow & & \downarrow \\ \mathrm{Der}\,\Lambda\langle 0\rangle & \longrightarrow & H_0(\mathrm{Der}\,\Lambda) \end{array}$$

After identifying $X_{\mathbb{Q}}$ with the spatial realization of Λ, Proposition 3.13 admits the following analog, where $\mathrm{Aut}_U(\Lambda)$ denotes the preimage of the normalizer of $U \leqslant \mathscr{E}(X_{\mathbb{Q}})$ under the homomorphism $\mathrm{Aut}(\Lambda) \to \mathscr{E}(X_{\mathbb{Q}})$.

Proposition 4.12 *The dg Lie algebra* $\mathrm{Der}_{\mathfrak{u}}\,\Lambda$ *is nilpotent and there is a weak equivalence of topological monoids*

$$|\exp_\bullet(\mathrm{Der}_{\mathfrak{u}}\,\Lambda)| \longrightarrow \mathrm{aut}_U(X_{\mathbb{Q}}) \tag{22}$$

that is equivariant with respect to the conjugation action of $\mathrm{Aut}_U(\Lambda)$ *on the domain and codomain.*

Proof The nilpotence of $\mathrm{Der}\,\Lambda$ is proved as in Proposition 3.12. The map in (22) is defined as in [12, Proposition 3.7]: the action of the nilpotent dg Lie algebra $\mathfrak{g} = \mathrm{Der}_{\mathfrak{u}}\,\Lambda$ on Λ by derivations induces an action of the simplicial nilpotent group

$$\exp_\bullet(\mathfrak{g}) = \exp(Z_0(\Omega_\bullet \otimes \mathfrak{g}))$$

on $\Omega_\bullet \otimes \Lambda$ by $\Omega_\bullet$-linear automorphisms, which in turn induces an action of $\exp_\bullet(\mathfrak{g})$ on the simplicial set

$$\mathrm{Hom}_{\mathrm{cdga}(\Omega_\bullet)}(\Omega_\bullet \otimes \Lambda, \Omega_\bullet),$$

the geometric realization of which is the spatial realization of Λ. The rest of the proof is entirely analogous to Proposition 3.13. We omit the details. □

Corollary 4.13 *The space* $B\,\mathrm{aut}_G(X)$ *has dg Lie model* $\mathrm{Der}_{\mathfrak{u}}(\Lambda)$. □

A dg Lie algebra $\mathfrak{g}(X)$ as in Theorem 3.31 can be described in terms of Sullivan models as well. Since $\mathrm{Aut}(\Lambda) \to \mathscr{E}(X_{\mathbb{Q}})$ has unipotent kernel, the group $R(X)$ may be identified with the maximal reductive quotient of $\mathrm{Aut}(\Lambda)$ and the quotient map $\mathrm{Aut}(\Lambda) \to R(X)$ admits a splitting by Theorem 2.6.

Theorem 4.14 *There is a zig-zag of $\Gamma(X)$-equivariant rational equivalences that connects $B_u\mathrm{aut}(X)$ to the nerve of the dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}(\Lambda)$, on which $\Gamma(X)$ acts through any choice of splitting of $\mathrm{Aut}(\Lambda) \to R(X)$. Here, $\mathfrak{u} \leqslant H_0(\mathrm{Der}\,\Lambda)$ is the Lie algebra of the unipotent radical of $\mathscr{E}(X_{\mathbb{Q}})$.*

The following is an analog of Proposition 4.10 for Sullivan models. It gives a more concrete description of the dg Lie algebra in Theorem 4.14.

Proposition 4.15 *If $\mathfrak{u}$ is the Lie algebra of the unipotent radical of $\mathscr{E}(X_{\mathbb{Q}})$, then the dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}\,\Lambda$ agrees with the nilradical $\mathrm{nil}\,\mathrm{Der}\,\Lambda$. The graded components of this dg Lie algebra are given by*

$$(\mathrm{nil}\,\mathrm{Der}\,\Lambda)_n = \begin{cases} (\mathrm{Der}\,\Lambda)_n & \text{if } n > 0, \\ \mathrm{Lie}(\mathscr{A}\mathrm{ut}_u\,\Lambda) & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Here, $\mathrm{Lie}(\mathscr{A}\mathrm{ut}_u\,\Lambda)$ stands for the Lie algebra of the unipotent radical of $\mathscr{A}\mathrm{ut}\,\Lambda$, and this Lie algebra may be identified with $\mathrm{nil}_H\,Z_0(\mathrm{Der}\,\Lambda)$, the maximal ideal of $Z_0(\mathrm{Der}\,\Lambda)$ of derivations that act nilpotently on $H = H^*(\Lambda)$.

Proof The proof is entirely analogous to the proof of Proposition 4.10, so we omit most of it. The only thing that requires a different argument is the inclusion $\mathrm{nil}\,\mathrm{Der}\,\Lambda \subseteq \mathrm{Der}_{\mathfrak{u}}\,\Lambda$. For this, we again only need to consider the degree zero component. We have that $(\mathrm{nil}\,\mathrm{Der}\,\Lambda)_0$ is an ideal in $Z_0(\mathrm{Der}\,\Lambda)$ that acts nilpotently on each graded component $(\mathrm{Der}\,\Lambda)_k$ for $k > 0$. On the other hand, $(\mathrm{Der}_{\mathfrak{u}}\,\Lambda)_0$ may be identified with the Lie algebra of the unipotent radical of $\mathscr{A}\mathrm{ut}\,\Lambda$, so the desired inclusion will follow from Lemma 2.8 as soon as we prove that $(\mathrm{Der}\,\Lambda)_{\leq n}$ is a faithful representation of $\mathscr{A}\mathrm{ut}\,\Lambda$ for n sufficiently large.

We are assuming that Λ is finitely generated as an algebra. Fix algebra generators $x_1, \dots, x_k$ and pick n so that $n \geq |x_i|$ for all i. Suppose that $\phi \in \mathrm{Aut}\,\Lambda$ acts trivially on $(\mathrm{Der}\,\Lambda)_{\leq n}$. In particular, we then have an equality in $\mathrm{Der}\,\Lambda$ for every i,

$$\phi^{-1} \circ \frac{\partial}{\partial x_i} \circ \phi = \frac{\partial}{\partial x_i}.$$

Evaluating on x_j, we get equalities in Λ for all i, j,

$$\phi^{-1}\left(\frac{\partial \phi(x_j)}{\partial x_i}\right) = \delta_{ij},$$

Applying the algebra automorphism ϕ, we obtain

$$\frac{\partial \phi(x_j)}{\partial x_i} = \delta_{ij}.$$

This can only happen if $\phi(x_i) = x_i$ for all i, which means that $\phi = 1$. □

Corollary 4.16 *If $\mathrm{Aut}\,\Lambda$ is reductive, then*

$$R(X) = \mathrm{Aut}\,\Lambda \quad \text{and} \quad \mathrm{nil}\,\mathrm{Der}\,\Lambda = \mathrm{Der}\,\Lambda\langle 1\rangle,$$

and the action of $R(X)$ on $\mathrm{nil}\,\mathrm{Der}\,\Lambda$ is the conjugation action. □

5 Case studies

In this section we offer a few case studies. In addition to showcasing how the main results can be used in practice, they illustrate certain general points:

(i) Determining $\Gamma(X)$ typically entails some nontrivial integral homotopy theory, but is often easier than determining $\mathscr{E}(X)$. In many cases, $\Gamma(X)$ is the group of automorphisms of $H_*(X;\mathbb{Z})$ that are induced by self-homotopy equivalences. In the literature, this group often appears as a stepping stone for computing $\mathscr{E}(X)$ or other groups of automorphisms of X.

(ii) The Lie algebra cohomology $H^*_{\mathrm{CE}}(\mathfrak{g}(X))$ is sometimes explicitly computable, sometimes not. For elliptic spaces, such as products of spheres, one can often compute it explicitly. On the other hand, a complete computation for the manifolds $W_{g,1}$ would entail the computation of the homology of Kontsevich's Lie graph complex, which is a hard problem.

(iii) Even in cases where $H^*_{\mathrm{CE}}(\mathfrak{g}(X))$ is explicitly computable, a complete calculation of the cohomology $H^*(\Gamma(X), H^*_{\mathrm{CE}}(\mathfrak{g}(X)))$ is in general out of reach, due to the difficulty of computing cohomology of arithmetic groups. However, in some cases the cohomology, or parts of it, can be understood via automorphic forms. A paradigmatic example is the Eichler–Shimura isomorphism.

(iv) By contrast, all that is left modulo nilpotent elements is the invariant ring

$$H^0(\Gamma(X), H^*_{\mathrm{CE}}(\mathfrak{g}(X))) = H^*_{\mathrm{CE}}(\mathfrak{g}(X))^{\Gamma(X)},$$

and this is more tractable. By employing structural results for affine algebraic groups over $\mathbb{Q}$ and density results [18], this can often be reduced to classical invariant theory for finite or reductive groups, which is well understood.

(v) For a graded representation H of a reductive group G and an arithmetic subgroup Γ of $G(\mathbb{Q})$, the split exact sequence

$$0 \to H^G \to H \to H/H^G \to 0$$

gives rise to a split exact sequence

$$0 \to H^*(\Gamma, \mathbb{Q}) \otimes H^G \to H^*(\Gamma, H) \to H^*(\Gamma, H/H^G) \to 0.$$

In many cases of interest, stability and vanishing results for the cohomology of arithmetic groups with coefficients in nontrivial algebraic representations, as in Borel's work [19; 20], show that the cokernel vanishes in a range of degrees. Thus, in this "stable range" the cohomology of $B\mathrm{aut}(X)$ is isomorphic to

$$H^*(\Gamma(X), \mathbb{Q}) \otimes H^*_{\mathrm{CE}}(\mathfrak{g}(X))^{R(X)}.$$

However, how nontrivial this "stable range" is depends on the group $\Gamma(X)$ and the representations $H^*_{\mathrm{CE}}(\mathfrak{g}(X))$.

5.1 Products of spheres

Consider the n-fold product of a d-dimensional sphere,

$$S^{d\times n} = S^d \times \cdots \times S^d.$$

We begin by describing the reductive group $R(S^{d\times n})$. The minimal Quillen model can be described explicitly (see [67, V.2.(3)]), but the minimal Sullivan model is finitely generated and even easier to describe in this case, so we will work with the latter.

Proposition 5.1 *The automorphism group of the minimal Sullivan model Λ for $S^{d\times n}$ is given by*

$$\operatorname{Aut}\Lambda \cong \begin{cases} \mathsf{GL}_n(\mathbb{Q}) & \text{if } d \text{ is odd,} \\ \Sigma_n \ltimes (\mathbb{Q}^\times)^n & \text{if } d \text{ is even.} \end{cases}$$

In particular, $\operatorname{Aut}\Lambda$ *is reductive*, *whence* $R(S^{d\times n}) \cong \operatorname{Aut}\Lambda$.

Proof For d odd, the minimal model Λ is an exterior algebra $\Lambda(x_1,\dots,x_n)$ on generators of degree d with zero differential. Clearly, $\operatorname{Aut}\Lambda \cong \mathsf{GL}_n(\mathbb{Q})$. For d even, the minimal model has the form

$$\Lambda = (\Lambda(x_1,\dots,x_n,y_1,\dots,y_n),d),$$

with $|x_i| = d$, $|y_i| = 2d-1$ and $dx_i = 0$, $dy_i = x_i^2$. Let φ be an automorphism of Λ. Then

$$\varphi(x_i) = \sum_j a_{ij}x_j$$

for some $A = (a_{ij}) \in \mathsf{GL}_n(\mathbb{Q})$. In cohomology, the equality

$$0 = \varphi(x_i^2) = \varphi(x_i)^2 = \sum_{j<k} 2a_{ij}a_{ik}x_jx_k$$

implies $a_{ij}a_{ik} = 0$ for all $j \neq k$, whence exactly one entry in each row $(a_{i1},\dots,a_{in})$ must be nonzero. Therefore, $\varphi(x_i) = \lambda_i x_{\sigma(i)}$ for some permutation σ and some $\lambda_i \in \mathbb{Q}^\times$. Since $d\varphi(y_i) = \varphi(dy_i) = \lambda_i^2 x_{\sigma(i)}^2$, the only possibility is $\varphi(y_i) = \lambda_i^2 y_{\sigma(i)}$. This shows that every automorphism φ of Λ is of the form

$$\varphi(x_i) = \lambda_i x_{\sigma(i)} \quad \text{and} \quad \varphi(y_i) = \lambda_i^2 y_{\sigma(i)}$$

for some $\sigma \in \Sigma_n$ and $\lambda_i \in \mathbb{Q}^\times$. One checks that this yields an isomorphism $\operatorname{Aut}\Lambda \cong \Sigma_n \ltimes (\mathbb{Q}^\times)^n$. The description of $R(S^{d\times n})$ follows from Corollary 4.16. □

Remark 5.2 The group $\Sigma_n \ltimes (\mathbb{Q}^\times)^n$ may be identified with the group of "monomial matrices", ie invertible matrices with exactly one nonzero entry in each row. This is an example of a disconnected reductive group. The identity component is the torus $(\mathbb{Q}^\times)^n$ and the group of components is Σ_n.

Remark 5.3 The preceding result, as well as the remainder of this section, goes through even for $d = 1$. The cautious reader will object on the grounds that the space $S^{1\times n} = (S^1)^n$ is not simply connected. It is, however, nilpotent (indeed a topological group), and therefore amenable to analysis by our methods.

We now turn to the determination of the group $\Gamma(S^{d\times n})$. For this, we first need to work out some elementary homotopy theory of maps between products of spheres.

Definition 5.4 Let us call an integer vector $(a_1,\dots,a_n)\in\mathbb{Z}^n$ *realizable* if there is a map

$$S^d\times\cdots\times S^d\to S^d$$

such that the restriction to the i^{th} factor is a degree a_i self-map of S^d. This is equivalent to asking the n-fold higher-order Whitehead product

$$[a_1\iota_d,\dots,a_n\iota_d]\subseteq\pi_{nd-1}(S^d)$$

to be defined and contain 0, where $\iota_d\in\pi_d(S^d)$ is the class of the identity map.

We would be surprised if the following has not been observed before, but we have not found a reference (except for the simplest case $n=2$, which is discussed in eg [2, Example 5.1]), so we supply a proof.

Proposition 5.5 (i) *For $d=1,3,7$, every integer vector $(a_1,\dots,a_n)$ is realizable.*

(ii) *For d odd $\neq 1,3,7$, an integer vector $(a_1,\dots,a_n)$ is realizable if and only if at most one a_i is odd.*

(iii) *For d even, an integer vector $(a_1,\dots,a_n)$ is realizable if and only if at most one a_i is nonzero.*

Lemma 5.6 *If $(a_1,\dots,a_n)$ is realizable, then so are the vectors*

$$(a_{\sigma_1},\dots,a_{\sigma_k}),\quad (\lambda_1a_1,\dots,\lambda_na_n)\quad\text{and}\quad(a_1,\dots,a_n,0)$$

for all injective maps $\sigma\colon\{1,\dots,k\}\to\{1,\dots,n\}$ and all $\lambda_1,\dots,\lambda_n\in\mathbb{Z}$.

Proof Precompose the given realizable map $S^d\times\cdots\times S^d\to S^d$ with the map that includes the k-fold product of S^d according to σ and inserts the basepoint in the other factors, or with the map $\lambda_1\times\cdots\times\lambda_n$, or with the projection onto the first n factors, respectively. □

Lemma 5.7 *If $(1,a_2,\dots,a_n)$ and $(b_1,\dots,b_n)$ are realizable, then so is*

$$(b_1,b_2+a_2,\dots,b_n+a_n).$$

Proof By hypothesis, the matrices on the left-hand side of the equation

$$\begin{pmatrix}1&a_2&\cdots&a_n\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}\begin{pmatrix}b_1&b_2&\cdots&b_n\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}=\begin{pmatrix}b_1&b_2+a_2&\cdots&b_n+a_n\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}$$

can be realized as self-maps of $S^{d\times n}$. It follows that the same is true of the matrix on the right-hand side. In particular, its first row is realizable. □

Lemma 5.8 *If $(1, a)$ is realizable, then so is $(1, a, \dots, a) \in \mathbb{Z}^n$ for every $n \geq 2$.*

Proof Assume by induction that $(1, a, \dots, a) \in \mathbb{Z}^{n-1}$ is realizable. Then both $(1, a \dots, a, 0) \in \mathbb{Z}^n$ and $(1, 0, \dots, 0, a) \in \mathbb{Z}^n$ are realizable by Lemma 5.6, and hence $(1, a, \dots, a) \in \mathbb{Z}^n$ is realizable by Lemma 5.7. □

Proof of Proposition 5.5 For $d = 1, 3, 7$, the fact that S^d is an H-space means precisely that $(1, 1)$ is realizable. It follows from Lemma 5.8 that $(1, \dots, 1) \in \mathbb{Z}^n$ is realizable and then from Lemma 5.6 that $(a_1, \dots, a_n)$ is realizable for all $a_1, \dots, a_n \in \mathbb{Z}$.

For d odd $\neq 1, 3, 7$, it is well known that the Whitehead product

$$[\iota_d, \iota_d] \in \pi_{2d-1}(S^d) \tag{23}$$

is a nonzero class of order 2 (this can be seen, for instance, by inspecting the EHP sequence). As noted above, (a_1, a_2) is realizable precisely when the Whitehead product $[a_1\iota_d, a_2\iota_d]$ is trivial. Since the binary Whitehead product is bilinear, this happens if and only if $a_1 a_2$ is even. By the first part of Lemma 5.6, this implies that $(a_1, \dots, a_n)$ is realizable only if $a_i a_j$ is even for all $i \neq j$, which implies that at most one a_i is odd. Conversely, we have that $(1, 2)$ is realizable since $[\iota_d, 2\iota_d] = 0$. Hence so is $(1, 2, \dots, 2)$ by Lemma 5.8. Now one can use Lemma 5.6 to deduce that $(a_1, \dots, a_n)$ is realizable if at most one of the entries is odd.

For d even, the Whitehead product (23) is a nonzero element of infinite order. As above, this implies that $(a_1, \dots, a_n)$ is realizable only if at most one a_i is nonzero. Conversely, Lemma 5.6 shows that $(a_1, \dots, a_n)$ is realizable if at most one entry is nonzero. □

Now we are ready to compute $\Gamma(S^{d\times n})$.

Proposition 5.9 *We have*

$$\Gamma(S^{d\times n}) \cong \begin{cases} \mathsf{GL}_n(\mathbb{Z}) & \text{if } d = 1, 3, 7, \\ \mathsf{GL}_n^{\Sigma}(\mathbb{Z}) & \text{if } d \text{ is odd and } d \neq 1, 3, 7, \\ \Sigma_n^{\pm} & \text{if } d \text{ is even,} \end{cases}$$

where $\mathsf{GL}_n^{\Sigma}(\mathbb{Z})$ denotes the group of invertible $n \times n$ integer matrices with exactly one odd entry in each row, and $\Sigma_n^{\pm}$ denotes the group of $n \times n$ signed permutation matrices.

Proof By Corollary 4.2, we may identify $\Gamma(S^{d\times n})$ with the group of automorphisms of $H_*(S^{d\times n}; \mathbb{Z})$ that are realizable by a self-homotopy equivalence.

Given $A \in \mathsf{GL}_n(\mathbb{Z})$, it is clear how to write down a map

$$S^d \vee \dots \vee S^d \to S^d \times \dots \times S^d$$

that realizes A on $H_d(-; \mathbb{Z})$. This extends to the product if and only if the projection to each factor S^d does, which is precisely the condition that each row in A is realizable. When an extension exists, it follows from the Whitehead theorem that it is a homotopy equivalence. Thus, $\Gamma(S^{d\times n})$ may be identified with the group of invertible $n \times n$ integer matrices in which each row is realizable. To finish the proof, invoke Proposition 5.5. (In the case d odd $\neq 1, 3, 7$, note that invertibility of the matrix implies that at least one entry in each row of must be odd. Similarly, in the case d even, invertibility of the matrix implies that there is a unique nonzero entry in each row and that this must be a unit.) $\square$

Remark 5.10 The group $\mathsf{GL}_n^{\Sigma}(\mathbb{Z})$ may be identified with the semidirect product,

$$\mathsf{GL}_n^{\Sigma}(\mathbb{Z}) \cong \Sigma_n \ltimes \mathsf{GL}_n(\mathbb{Z}, 2),$$

where $\mathsf{GL}_n(\mathbb{Z}, 2) \leqslant \mathsf{GL}_n(\mathbb{Z})$ denotes the principal level 2 congruence subgroup, ie the kernel of the homomorphism $\mathsf{GL}_n(\mathbb{Z}) \to \mathsf{GL}_n(\mathbb{Z}/2\mathbb{Z})$ that reduces the entries mod 2, and where the symmetric group Σ_n acts by simultaneous permutation of the rows and columns.

The signed permutation group $\Sigma_n^{\pm}$, a.k.a. the hyperoctahedral group, admits a similar decomposition

$$\Sigma_n^{\pm} \cong \Sigma_n \ltimes D_n(\mathbb{Z}),$$

where $D_n(\mathbb{Z}) \cong (\mathbb{Z}^{\times})^n$ is the group of diagonal matrices in $\mathsf{GL}_n(\mathbb{Z})$.

Remark 5.11 The group of homology isomorphisms that are realizable by a self-homotopy equivalence of $S^{d \times n}$ has also been determined by Basu and Farrell [5, Section 2]. The proof given here is simpler because we do not need to argue using generators for the groups involved.

Work of Lucas and Saeki [51] shows that $\Gamma(S^{d \times n})$ also agrees with the group of homology isomorphisms of $S^{d \times n}$ that are realizable by a diffeomorphism.

The group $\Gamma(S^{d \times 2})$ agrees with the group G_d that is used as a stepping stone in the computation of the group of self-homotopy equivalences of $S^d \times S^d$ due to Baues [6, Section 6].

Remark 5.12 If we let $\Gamma^+(S^{d \times n})$ denote the image in $R(S^{d \times n})$ of the orientation-preserving self-homotopy equivalences, then it is easily seen that $\Gamma^+(S^{d \times n}) = \Gamma(S^{d \times n}) \cap \mathsf{SL}_n(\mathbb{Z})$ for d odd, and that $\Gamma^+(S^{d \times n}) \leqslant \Sigma_n \ltimes (\mathbb{Z}^{\times})^n$ is the subgroup of all (σ, λ) such that $\lambda_1 \dots \lambda_n = 1$ for d even.

Next, we determine an algebraic Lie model for $B\operatorname{aut}_u(S^{d \times n})$.

Proposition 5.13 *Let d be odd. The space $B\operatorname{aut}_u(S^{d \times n})$ admits an algebraic Lie model of the form*

$$V_n^*[-d],$$

which is the dual of the standard representation $V_n = \mathbb{Q}^n$ concentrated in homological degree d. The differential and the Lie bracket are trivial.

Proof When d is odd, the Sullivan minimal model of $S^{d \times n}$ is the exterior algebra $\Lambda = \Lambda(x_1, \dots, x_n)$ with zero differential and with x_i in cohomological degree d. By Proposition 5.1, the group $\operatorname{Aut}\Lambda \cong \mathsf{GL}_n(\mathbb{Q})$ is reductive, so by Corollary 4.16, an algebraic Lie model is given by $\operatorname{Der}\Lambda\langle 1\rangle$, with the conjugation action of $R(S^{d\times n}) = \operatorname{Aut}\Lambda$. Recall that we are using the convention that cohomological degrees are regarded as negative homological degrees. The only derivations of Λ of positive homological degree are

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}. \tag{24}$$

Thus, $\operatorname{Der}\Lambda\langle 1\rangle$ is an abelian dg Lie algebra with basis (24) in degree d and zero differential. As a representation of GL_n, it is dual to the standard representation. □

Remark 5.14 The Chevalley–Eilenberg cochain algebra of the abelian Lie algebra $V_n^*[-d]$ may be identified with $\operatorname{Sym}(V_n[d+1])$, the polynomial algebra on the standard $\mathsf{GL}_n(\mathbb{Q})$-representation $V_n = \mathbb{Q}^n$ concentrated in cohomological degree $d+1$ with trivial differential. Hence, there are isomorphisms of graded algebras of $\Gamma(S^{d\times n})$-modules,

$$H^*(B\operatorname{aut}_u(S^{d\times n});\mathbb{Q}) \cong H^*_{\mathrm{CE}}(\mathfrak{g}(S^{d\times n})) \cong \operatorname{Sym}(V_n[d+1]).$$

This can be given a more geometric interpretation. The above implies that the evident map

$$B\operatorname{aut}_u(S^d)\times\cdots\times B\operatorname{aut}_u(S^d) \to B\operatorname{aut}_u(S^d\times\cdots\times S^d)$$

is a rational equivalence. This may be interpreted as a splitting principle of sorts: every normal unipotent $S^{d\times n}$-fibration is rationally equivalent to the "Whitney sum" of n normal unipotent S^d-fibrations. Since $H^*(B\operatorname{aut}_u(S^d);\mathbb{Q})$ is a polynomial ring in the Euler class, we can say that the ring of rational characteristic classes of normal unipotent $S^{d\times n}$-fibrations is a polynomial ring in the Euler classes $e_1,\dots,e_n$ of the associated S^d-fibrations.

By combining Propositions 5.13 and 5.9 and Corollary 1.4, we obtain:

Theorem 5.15 *For d odd, there is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}^+(S^{d\times n});\mathbb{Q}) \cong H^*\big(\Gamma^+(S^{d\times n}); \operatorname{Sym}^\bullet(V_n[d+1])\big), \tag{25}$$

where $V_n = \mathbb{Q}^n$ is the standard representation of $\mathsf{GL}_n(\mathbb{Q})$, where

$$\Gamma^+(S^{d\times n}) = \begin{cases} \mathsf{SL}_n(\mathbb{Z}) & \text{if } d = 1,3,7, \\ \mathsf{SL}_n^{\Sigma}(\mathbb{Z}) & \text{if } d \neq 1,3,7, \end{cases}$$

and where $\mathsf{SL}_n^{\Sigma}(\mathbb{Z}) \leqslant \mathsf{SL}_n(\mathbb{Z})$ is the subgroup of matrices with exactly one odd entry in each row. □

For $n=2$ the right-hand side of (25) can be computed in terms of modular forms via the Eichler–Shimura isomorphism, as we now will discuss.

Let $V = \mathbb{C}^2$ denote the standard representation of $\mathsf{GL}_2(\mathbb{C})$ and let $\widetilde{\Gamma}$ be a congruence subgroup of $\mathsf{GL}_2(\mathbb{Z})$ that contains $-I$ and that strictly contains $\Gamma = \widetilde{\Gamma}\cap\mathsf{SL}_2(\mathbb{Z})$, so that we have an exact sequence

$$1 \to \Gamma \to \widetilde{\Gamma} \xrightarrow{\det} \mathbb{Z}^\times \to 1.$$

This gives rise to an action of $\mathbb{Z}^\times$ on $H^*(\Gamma, \mathrm{Sym}^\bullet(V))$, ie an involution, such that

$$H^*(\widetilde{\Gamma}, \mathrm{Sym}^\bullet(V)) \cong H^*(\Gamma, \mathrm{Sym}^\bullet(V))_+,$$

where $+$ indicates the $+1$ eigenspace of the involution.

Lemma 5.16 *There is an isomorphism of graded vector spaces with involution*,

$$H^*(\Gamma, \mathrm{Sym}^\bullet(V)) \cong \mathbb{C}[0]^+ \oplus M_{\bullet+2}(\Gamma)[1]^- \oplus S_{\bullet+2}(\Gamma)[1]^+, \tag{26}$$

where $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the spaces of modular forms and cusp forms of weight k for Γ, and where $\mathbb{C}[0]$ is $\mathbb{C}$ concentrated in $\bullet = 0$ and $* = 0$. A superscript $\pm$ indicates how the involution acts. In particular, *extraction of the $+1$ eigenspace yields*

$$H^*(\widetilde{\Gamma}, \mathrm{Sym}^\bullet(V)) \cong \mathbb{C}[0] \oplus S_{\bullet+2}(\Gamma)[1]. \tag{27}$$

Proof The Eichler–Shimura isomorphism gives an isomorphism

$$\mathrm{ES}\colon M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\cong} H^1(\Gamma, \mathrm{Sym}^{k-2}(V)), \tag{28}$$

where $M_k(\Gamma)$ is the space of modular forms of weight k for Γ and $\overline{S_k(\Gamma)}$ is the space of antiholomorphic cusp forms of weight k; see eg [70, Section 6]. Identifying the involution on the left-hand side of (28) requires a little care.

The space of modular forms decomposes as $M_k(\Gamma) = E_k(\Gamma) \oplus S_k(\Gamma)$, where $E_k(\Gamma)$ denotes the Eisenstein space. Letting r denote the automorphism of the upper half-plane given by $r(z) = -\overline{z}$, one can check that the composite

$$E_k(\Gamma) \oplus S_k(\Gamma) \oplus S_k(\Gamma) \xrightarrow{\varphi} M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\mathrm{ES}} H^1(\Gamma, \mathrm{Sym}^{k-2}(V)),$$

where

$$\varphi(e, f, g) = (e + f + g, fr - gr)$$

is an isomorphism of vector spaces with involution, where the involution on the left-hand side is given by $(e, f, g) \mapsto (-e, -f, g)$. This explains the isomorphism (26) for $* = 1$. Restriction to the $+1$ eigenspace yields an isomorphism

$$S_k(\Gamma) \to H^1(\Gamma, \mathrm{Sym}^{k-2}(V))_+$$

given by $g \mapsto \mathrm{ES}(g, -gr)$.

For the other cohomological degrees, one checks that

$$H^0(\Gamma, \mathrm{Sym}^{k-2}(V)) = \mathrm{Sym}^{k-2}(V)^\Gamma \cong \mathbb{C},$$

and $H^i(\Gamma, \mathrm{Sym}^{k-2}(V)) = 0$ for $i > 1$, because the virtual cohomological dimension of any finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is 1. □

Assembly of the above considerations yields:

Theorem 5.17 *For d odd, there is an isomorphism of graded vector spaces with involution*

$$\widetilde{H}^*\big(B\operatorname{aut}^+(S^d\times S^d);\mathbb{Q}\big)\cong\bigoplus_k\big(M_k(\Gamma)^-\oplus S_k(\Gamma)^+\big)[(k-2)(d+1)+1],$$

where $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the spaces of modular forms and cusp forms of weight k for Γ, and where $\Gamma=\mathsf{SL}_2(\mathbb{Z})$ for $d=1,3,7$, and $\Gamma=\mathsf{SL}_2^{\Sigma}(\mathbb{Z})$ for $d\neq 1,3,7$.

In particular, since the reduced cohomology is concentrated in odd degrees, $B\operatorname{aut}^+(S^d\times S^d)$ is formal with trivial cohomology ring. □

Define the Poincaré series of a graded vector space H^* with involution to be the formal power series in z with coefficients in $\mathbb{Z}[\epsilon]/(\epsilon^2-1)$ given by

$$\sum_{k\geq 0}(\dim(H_+^k)+\dim(H_-^k)\epsilon)z^k.$$

There are well-known dimension formulas for spaces of modular forms, so as a corollary we get a formula for the Poincaré series of the cohomology.

Corollary 5.18 *The Poincaré series of the graded vector space with involution*

$$H^*(B\operatorname{aut}^+(S^d\times S^d);\mathbb{Q})$$

is given by the following formulas, where we set $\ell=d+1$.

For $d=1,3,7$:

$$1+z^{2\ell+1}\frac{\epsilon(1+z^{4\ell}-z^{6\ell})+z^{8\ell}}{(1-z^{4\ell})(1-z^{6\ell})}.$$

For d odd $\neq 1,3,7$:

$$1+z\frac{\epsilon(1+z^{2\ell}-z^{4\ell})+z^{6\ell}}{(1-z^{2\ell})(1-z^{4\ell})}.$$

Proof As is well known, the ring of modular forms $M(\mathsf{SL}_2(\mathbb{Z}))$ is freely generated by the Eisenstein series E_4 and E_6, and the space of cusp forms $S(\mathsf{SL}_2(\mathbb{Z}))$ is the principal ideal generated by the discriminant Δ, which is of weight 12. In particular, the Poincaré series are given by

$$\sum_{k\geq 0}\dim M_k(\mathsf{SL}_2(\mathbb{Z}))t^k=\frac{1}{(1-t^4)(1-t^6)}\quad\text{and}\quad\sum_{k\geq 0}\dim S_k(\mathsf{SL}_2(\mathbb{Z}))t^k=\frac{t^{12}}{(1-t^4)(1-t^6)}.$$

The group $\mathsf{SL}_2^{\Sigma}(\mathbb{Z})$ is an index 3 subgroup of $\mathsf{SL}_2(\mathbb{Z})$, sometimes referred to as the "theta group". It is generated by the two matrices

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&2\\0&1\end{pmatrix}.$$

The Poincaré series of the modular forms and the cusp forms for the theta group are

$$\sum_{k\geq 0}\dim M_k(\mathsf{SL}_2^{\Sigma}(\mathbb{Z}))t^k=\frac{1}{(1-t^2)(1-t^4)}\quad\text{and}\quad\sum_{k\geq 0}\dim S_k(\mathsf{SL}_2^{\Sigma}(\mathbb{Z}))t^k=\frac{t^8}{(1-t^2)(1-t^4)}.$$

This can be seen from the dimension formulas in [47, Proposition 1], or alternatively by observing that the matrix $\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)$ conjugates $\mathsf{SL}_2^{\Sigma}(\mathbb{Z})$ to the congruence subgroup $\Gamma_0(2)=\Gamma_1(2)$; dimension formulas for the latter can be found in [29, page 108]. Theorem 5.17 together with the above formulas yield the desired result after some manipulations with generating functions. □

Remark 5.19 Computations of $H^*(B\,\mathrm{Diff}^+(T^2);\mathbb{Q})$ have been carried out in [56] and [36] (the latter using the Eichler–Shimura isomorphism). The inclusion of $\mathrm{Diff}^+(T^2)$ into $\mathrm{aut}^+(T^2)$ is a weak homotopy equivalence, so we recover this computation by setting $d=1$ and $\epsilon=1$ in the above.

The cohomology of $\mathsf{GL}_3(\mathbb{Z})$ and $\mathsf{SL}_3(\mathbb{Z})$ with coefficients in irreducible algebraic representations has recently been computed in many cases [1]. Let $V=\mathbb{C}^3$ be the standard representation of $\mathsf{GL}_3(\mathbb{C})$. The combination of [1, Corollary 18] and [1, Theorem 16] specialized to $\mathcal{M}_{k,0}=\mathrm{Sym}^k(V)$ (which is not self-dual for $k>0$) shows

$$H^q(\mathsf{SL}_3(\mathbb{Z}),\mathrm{Sym}^k(V))\cong\begin{cases}S_{k+2} & \text{for } q=3 \text{ and } k>0 \text{ even},\\ S_{k+3}\oplus\mathbb{C} & \text{for } q=2 \text{ and } k \text{ odd},\\ \mathbb{C} & \text{for } q=0 \text{ and } k=0,\\ 0 & \text{otherwise},\end{cases}$$

where S_k denotes the space of cusp forms of weight k for $\mathsf{SL}_2(\mathbb{Z})$. If we take the liberty of writing M_{k+3} for the isomorphic vector space $S_{k+3}\oplus\mathbb{C}$ in the second case above, we can summarize the calculation as an isomorphism of bigraded vector spaces with involution:

$$H^*(\mathsf{SL}_3(\mathbb{Z}),\mathrm{Sym}^\bullet(V))\cong\mathbb{C}[0]^+\oplus M_{\bullet+3}[2]^-\oplus S_{\bullet+2}[3]^+. \tag{29}$$

The action of the involution on the right-hand side is implicit in [1, Lemma 17]; this lemma asserts that

$$H^*(\mathsf{GL}_3(\mathbb{Z}),\mathrm{Sym}^\bullet(V))\cong\mathbb{C}[0]\oplus S_{\bullet+2}[3]. \tag{30}$$

As before, this leads to:

Theorem 5.20 *For $d=1,3,7$, there is an isomorphism of graded vector spaces with involution*

$$\widetilde{H}^*(B\,\mathrm{aut}^+(S^{d\times 3});\mathbb{Q})\cong\bigoplus_k M_k^-[(k-3)(d+1)+2]\oplus S_k^+[(k-2)(d+1)+3)],$$

and the Poincaré series is given by

$$z^{\ell+2}\frac{\epsilon(1+z^{2\ell}-z^{6\ell})+z^{9\ell+1}}{(1-z^{4\ell})(1-z^{6\ell})}.$$ □

Remark 5.21 It seems plausible that computations similar to those of [1] can be carried out for the group $\mathsf{SL}_3^{\Sigma}(\mathbb{Z})$, but this is beyond the scope of this paper. However, even if we currently lack complete calculations, certain qualitative conclusions can be drawn. By Theorem 3.39, the cdga $\Omega^*(B\,\mathrm{aut}(X))$ is quasi-isomorphic to

$$\Omega^*(\Gamma(X),C^*_{\mathrm{CE}}(\mathfrak{g}(X))).$$

If $C^*_{\mathrm{CE}}(\mathfrak{g}(X))$ is formal as a cdga in $\mathrm{Rep}_{\mathbb{Q}}(R(X))$, then the above is quasi-isomorphic to

$$\Omega^*(\Gamma(X); H^*_{\mathrm{CE}}(\mathfrak{g}(X)))$$

as a cdga. An application of the homotopy transfer theorem for C_∞-algebras yields a C_∞-algebra structure $\{m_n\}$ on $H^*(\Gamma(X); H^*_{\mathrm{CE}}(\mathfrak{g}(X)))$ that is bigraded in the sense that m_n has bidegree $(2-n, 0)$, and that is C_∞-quasi-isomorphic to $\Omega^*(B\,\mathrm{aut}(X))$.

If the cohomology of $\Gamma(X)$ with coefficients in algebraic representations is concentrated in a small range of degrees, this can force these C_∞-operations to be trivial, implying that the cdga $\Omega^*(B\,\mathrm{aut}(X))$ is formal. For example, this happens if there is an r such that $\widetilde{H}^i(\Gamma(X), V) = 0$ unless $r \leq i \leq 3r-2$, for all $V \in \mathrm{Rep}_{\mathbb{Q}}(R(X))$.

For example, if $\Gamma \leqslant \mathsf{SL}_3(\mathbb{Z})$ is a finite-index subgroup, then $H^1(\Gamma, V) = 0$ for any finite-dimensional $\mathbb{Q}$-vector space V with an action of Γ by [4], and $H^i(\Gamma, V) = 0$ for $i > 3$, because Γ has virtual cohomological dimension 3.

These considerations lead to:

Theorem 5.22 *For d odd, the space $B\,\mathrm{aut}^+(S^d \times S^d \times S^d)$ is formal and the rational cohomology has trivial ring structure.* □

Another remark is that the invariant ring

$$H^*(B\,\mathrm{aut}_u(S^{d\times n}); \mathbb{Q})^{\Gamma(S^{d\times n})}$$

is the trivial ring $\mathbb{Q}$ in this case, because $\mathrm{Sym}^k(V_n)^{\Gamma(S^{d\times n})}$ is easily seen to be trivial. In particular, $H^*(B\,\mathrm{aut}\,(S^{d\times n}); \mathbb{Q})$ consists entirely of nilpotent elements.

For $n > 3$, calculations become increasingly difficult, but one can at least say something about the limit as $n \to \infty$. Consider the maps

$$\sigma\colon B\,\mathrm{aut}(S^{d\times n}) \to B\,\mathrm{aut}(S^{d\times(n+1)}) \quad \text{and} \quad \pi\colon B\,\mathrm{aut}(S^{d\times n}) \to B\Gamma(S^{d\times n}),$$

where σ is induced by extending a self-homotopy equivalence of $S^{d\times n}$ by the identity on the second factor of $S^{d\times(n+1)} = S^{d\times n} \times S^d$, and π is the evident map. It is a consequence of Borel's results [19; 20] on the stable cohomology of arithmetic groups that these maps induce an isomorphism in $H^i(-;\mathbb{Q})$ for $n \gg i$. The explicit ranges stated below rely on more recent results due to Kupers, Miller and Patzt [48] and Putman [58].

Theorem 5.23 *Let d be odd. The map*

$$\sigma^*\colon H^i(B\,\mathrm{aut}(S^{d\times(n+1)}); \mathbb{Q}) \to H^i(B\,\mathrm{aut}(S^{d\times n}); \mathbb{Q})$$

is an isomorphism for $n \geq i+1$ if $d = 1, 3, 7$, and for $n \geq 2i+6$ for all odd d. The map

$$\pi^*\colon H^i(\Gamma(S^{d\times n}), \mathbb{Q}) \to H^i(B\,\mathrm{aut}(S^{d\times n}); \mathbb{Q}).$$

is an isomorphism for $n \geq i-d+1$ if $d = 1, 3, 7$, and for $n \geq 2i-2d+6$ for all odd d.

Proof By Theorem 5.15, the map π^* is injective and its cokernel is isomorphic to

$$\bigoplus_{\substack{j+k(d+1)=i\\ k>0}} H^j(\Gamma(S^{d\times n}), \mathrm{Sym}^k(V_n)). \tag{31}$$

By Borel's vanishing theorem [20, Theorem 4.4], the summands vanish for n large. Explicitly, [48, Theorem 7.6] implies vanishing for $n \geq j+k+1$ for $d=1,3,7$, and [58, Theorem C] implies vanishing for $n \geq 2j+2k+6$ for $d \neq 1,3,7$. It follows that (31) vanishes for $n \geq i-d+1$ for $d=1,3,7$ and for $n \geq 2i-2d+6$ for $d \neq 1,3,7$.

The statement about σ^* now follows from the corresponding statement for

$$H^i(\Gamma(S^{d\times(n+1)}), \mathbb{Q}) \to H^i(\Gamma(S^{d\times n}), \mathbb{Q}).$$

By Borel's work [19] this is an isomorphism for n large. According to [48, Theorem A] it is an isomorphism for $n \geq i+1$ for $d=1,3,7$, and [58, Theorem C] implies it is an isomorphism for $n \geq 2i+6$ for $d \neq 1,3,7$. □

By Borel's calculation [19], the stable cohomology may be identified with an exterior algebra

$$\varprojlim_n H^*(B\operatorname{aut}(S^{d\times n}); \mathbb{Q}) \cong \varprojlim_n H^*(\Gamma(S^{d\times n}), \mathbb{Q}) \cong \Lambda[x_5, x_9, x_{13}, \dots].$$

We now turn to the case d even, which behaves very differently.

Proposition 5.24 *For d even, the dg Lie algebra $\mathfrak{g}(S^{d\times n})$ is formal with homology the abelian Lie algebra with basis*

$$\alpha_1, \dots, \alpha_n, \quad \beta_{ij} \quad \text{for } 1 \leq i \neq j \leq n,$$

in degrees $|\alpha_i| = 2d-1$ and $|\beta_{ij}| = d-1$. The action of

$$(\sigma, \lambda) \in R(S^{d\times n}) = \Sigma_n \ltimes (\mathbb{Q}^\times)^n$$

is given by

$$(\sigma,\lambda)\cdot\alpha_i = \lambda_i^{-2}\alpha_{\sigma(i)}, \quad (\sigma,\lambda)\cdot\beta_{ij} = \lambda_i\lambda_j^{-2}\beta_{\sigma(i)\sigma(j)}.$$

Proof Let Λ denote the minimal model. Since $\operatorname{Aut}\Lambda$ is reductive, the positive truncation $\operatorname{Der}\Lambda\langle 1\rangle$ is a Lie model for $B\operatorname{aut}_u(X)$ by Corollary 4.16. This is spanned by derivations of the form

$$\frac{\partial}{\partial x_i}, \quad x_i\frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial y_i},$$

and the only nontrivial differential is given by

$$\left[d, \frac{\partial}{\partial x_i}\right] = -2x_i\frac{\partial}{\partial y_i}.$$

The action of the group $\operatorname{Aut}\Lambda$ is easily computed, eg if φ corresponds to $(\sigma,\lambda) \in \Sigma_n \ltimes (\mathbb{Q}^\times)^n$, then

$$\varphi\left(x_i\frac{\partial}{\partial y_j}\right) = \lambda_i\lambda_j^{-2}x_{\sigma(i)}\frac{\partial}{\partial y_{\sigma(j)}}.$$

A basis for the homology is represented by the cycles $\alpha_i = \partial/\partial y_i$ and $\beta_{ij} = x_i\,\partial/\partial y_i$ for $i \neq j$. These span an abelian dg Lie subalgebra stable under the action of $\operatorname{Aut}\Lambda$, and the inclusion into $\operatorname{Der}\Lambda\langle 1\rangle$ is a quasi-isomorphism. □

Corollary 5.25 *For d even, the space $B\operatorname{aut}(S^{d\times n})$ is formal and there is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}(S^{d\times n});\mathbb{Q}) \cong \mathbb{Q}[a_i, b_{ij}]^{\Gamma(S^{d\times n})},$$

where $|a_i| = 2d$, $|b_{ij}| = d$ and $(\sigma,\lambda) \in \Gamma(S^{d\times n}) = \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ acts by

$$(\sigma,\lambda)\cdot a_i = a_{\sigma(i)} \quad \text{and} \quad (\sigma,\lambda)\cdot b_{ij} = \lambda_i b_{\sigma(i)\sigma(j)}.$$

Similarly, the space $B\operatorname{aut}^+(S^{d\times n})$ is formal and there is an isomorphism of algebras

$$H^*(B\operatorname{aut}^+(S^{d\times n});\mathbb{Q}) \cong \mathbb{Q}[a_i, b_{ij}]^{\Gamma^+(S^{d\times n})},$$

where $\Gamma^+(S^{d\times n}) \leqslant \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ is the subgroup consisting of all (σ,λ) such that $\lambda_1\ldots\lambda_n = 1$.

Proof Clearly, $C^*_{\mathrm{CE}}(\mathfrak{g}(S^{d\times n})) = \mathbb{Q}[a_i, b_{ij}]$, where a_i and b_{ij} are the dual 1-cochains of α_i and β_{ij}, and the differential is trivial since $\mathfrak{g}(S^{d\times n})$ is abelian with trivial differential. In particular, $C^*_{\mathrm{CE}}(\mathfrak{g}(S^{d\times n}))$ is formal, so Remark 5.21 shows that the C_∞-algebra structure on $H^*(\Gamma(S^{d\times n}); H^*_{\mathrm{CE}}(\mathfrak{g}(S^{d\times n})))$ respects the bigrading. But since the group $\Gamma(S^{d\times n})$ is finite, the cohomology reduces to the invariants concentrated in bidegree $(0,*)$, whence the C_∞-operations m_n vanish for $n > 2$. □

The computation of invariant subrings $\mathbb{Q}[V]^G$ for finite-dimensional representations V of finite or reductive groups G is classical and well-understood in principle; see eg [28]. The invariant subring is a Cohen–Macaulay ring, which means that one can find a regular sequence $\theta_1,\ldots,\theta_m$ of invariant polynomials (the "primary invariants") such that $\mathbb{Q}[V]^G$ is a finitely generated free module over $\mathbb{Q}[\theta_1,\ldots,\theta_m]$ on certain invariant polynomials $\eta_1,\ldots,\eta_t$ (the "secondary invariants").

Corollary 5.26 *For d even, $H^*(B\operatorname{aut}^+(S^{d\times n});\mathbb{Q})$ is a Cohen–Macaulay ring of Krull dimension n^2, concentrated in degrees that are multiples of d.* □

Let us examine the case $n = 2$ closer to illustrate. The invariant theory calculation can be carried out in two steps using

$$\mathbb{Q}[a_1,a_2,b_{12},b_{21}]^{\Gamma(S^{d\times 2})} = (\mathbb{Q}[a_1,a_2,b_{12},b_{21}]^{(\mathbb{Z}^\times)^2})^{\Sigma_2}.$$

The invariant ring with respect to the action of $(\mathbb{Z}^\times)^2$ is easily seen to be a polynomial ring in $a = a_1$, $b = b_{12}^2$, $c = b_{21}^2$, $d = a_2$, so we are left to identify the invariant ring

$$\mathbb{Q}[a,b,c,d]^{\Sigma_2},$$

where the nontrivial element of Σ_2 acts as the permutation $(ad)(bc)$. By using Molien's theorem (specialized to permutation representations as in [64, Proposition 4.3.4]), the Poincaré series with respect to word-length in the generators can be computed to be

$$\frac{1+t^2}{(1-t)^2(1-t^2)^2}.$$

The invariant polynomials

$$a+d, \quad b+c, \quad ad, \quad bc,$$

form a regular sequence of length equal to the Krull dimension, so these form a set of primary invariants. A choice of secondary invariants is given by the two polynomials

$$1, \quad ab+cd.$$

The ring structure is determined by writing $(ab+cd)^2$ in the basis; one has

$$(ab+cd)^2 = \big((a+d)(b+c)\big)\cdot(ab+cd) - \big((a+d)^2bc + ad(b+c)^2 - 4(ad)(bc)\big)\cdot 1.$$

The calculation for $\Gamma^+(X_2)$ is similar. We omit the details; the only difference is that there is one additional invariant $\alpha_0 = b_{12}b_{21}$, which squares to $bc = b_{12}^2 b_{21}^2$. Writing α_1, α_2, β_1, β_2, η for $a+d$, $b+c$, ad, bc, $ab+cd$, respectively, these considerations lead to:

Theorem 5.27 *For d even, there is an isomorphism of graded algebras with involution*

$$H^*(B\operatorname{aut}^+(S^d\times S^d);\mathbb{Q}) \cong \mathbb{Q}[\alpha_0,\alpha_1,\alpha_2,\beta_1,\beta_2,\eta]/I,$$

where the generators $\alpha_0,\alpha_1,\alpha_2$ are of degree $2d$, and β_1,β_2,η are of degree $4d$, where I is the ideal generated by the two elements

$$\eta^2 - \alpha_1\alpha_2\eta + (\alpha_1^2\beta_2 + \beta_1\alpha_2^2 - 4\beta_1\beta_2) \quad \text{and} \quad \alpha_0^2 - \beta_2,$$

and where the involution acts by $\alpha_0 \mapsto -\alpha_0$ and trivially on the other generators. The Poincaré series of this graded vector space with involution is given by

$$(1+\epsilon z^{2d})\frac{1+z^{4d}}{(1-z^{2d})^2(1-z^{4d})^2}.$$

A presentation for the cohomology ring of $B\operatorname{aut}(S^d\times S^d)$ is obtained by removing the generator α_0 and the relation $\alpha_0^2-\beta_2$ from the above presentation. The Poincaré series is obtained by removing the term involving ϵ. □

The case $n>2$ can in principle be treated similarly, but we stop here.

5.2 Highly connected even-dimensional manifolds

The rational homotopy theory of the topological monoid of self-homotopy equivalences of highly connected even-dimensional manifolds has been thoroughly studied in [16, Section 5]. We will now discuss how the methods of the present paper lead to simplifications of certain arguments in [16], as well as to some new results.

Let M be an $(n-1)$-connected $2n$-dimensional manifold with $n>1$. We will consider the space $\operatorname{aut}^+(M)$ of orientation-preserving self-homotopy equivalences. Let $\Gamma^+(M)$ denote the image of $\mathscr{E}^+(M) = \pi_0\operatorname{aut}^+(M)$ in $R(M)$, and let $\operatorname{Aut}^+\mathbb{L}$ denote the group of automorphisms of the minimal Quillen model $\mathbb{L}$ of M that represent orientation-preserving self-homotopy equivalences.

As discussed in [16, Section 3.5], the minimal Quillen model of M can be presented as

$$\mathbb{L} = (\mathbb{L}(\alpha_1,\dots,\alpha_r,\gamma), \delta\gamma = \omega),$$

where $\alpha_1,\dots,\alpha_r$ is a basis for $H_n(M;\mathbb{Q})[1-n]$, where γ corresponds to the fundamental class of M, and where ω is dual to the cup product pairing

$$\langle\ ,\ \rangle\colon H^n(M;\mathbb{Q})\otimes H^n(M;\mathbb{Q})\to\mathbb{Q},\quad \langle x,y\rangle = \langle x\smile y,[M]\rangle.$$

Explicitly,

$$\omega = \frac{1}{2}\sum_i [\alpha_i^\#,\alpha_i],$$

where $\alpha_i^\#$ is the dual basis with respect to the intersection form.

Proposition 5.28 *Let M be an $(n-1)$-connected $2n$-dimensional manifold, where $n>1$. The group $\mathrm{Aut}^+\mathbb{L}$ is isomorphic to the group of automorphisms of $H^n(M;\mathbb{Q})$ that preserve the cup product pairing. In particular, $\mathrm{Aut}^+\mathbb{L}$ is reductive, whence $\mathrm{Aut}^+\mathbb{L}\cong\mathscr{E}^+(M_{\mathbb{Q}})\cong R^+(M)$.*

Proof An automorphism of $\mathbb{L}$ represents an orientation-preserving self-homotopy equivalence if and only if it fixes γ. Using this, one sees that $\mathrm{Aut}^+\mathbb{L}$ is isomorphic to the group of automorphisms of the space spanned by $\alpha_1,\dots,\alpha_r$ that fix ω. Since ω is dual to the cup product pairing, this is in turn isomorphic to $\mathrm{Aut}(H^n(M;\mathbb{Q}),\langle\ ,\ \rangle)$. The group of automorphisms of a nondegenerate symmetric or skew-symmetric bilinear form is well known to be reductive. □

Corollary 4.2 implies that the group $\Gamma^+(M)$ may be identified with the group of automorphisms of $H_*(M;\mathbb{Z})$ that are induced by an orientation-preserving homotopy equivalence of M. This group is known; see [6, Theorem 8.14], [15, Theorem 2.12] or [16, Section 5.1]. To describe it, recall the cohomology operation $\psi\colon H^n(M)\to\pi_{2n-1}(S^n)$ defined by Kervaire–Milnor [46, Section 8]; the class $\psi(x)$ is the obstruction for the existence of a map $f\colon M\to S^n$ such that $f^*(s)=x$, where s is a generator for $H^n(S^n)$.

Proposition 5.29 *Let M be an $(n-1)$-connected $2n$-dimensional manifold, where $n>1$. The group $\Gamma^+(M)$ may be identified with the group of automorphisms of $H^n(M)$ that preserve the cup product pairing*

$$\langle -,-\rangle\colon H^n(M)\otimes H^n(M)\to\mathbb{Z}$$

and the Kervaire–Milnor cohomology operation

$$\psi\colon H^n(M)\to\pi_{2n-1}(S^n).$$ □

For example, for the manifold $W_g = \#^g\, S^n\times S^n$, the group $\Gamma^+(W_g)$ coincides with the group denoted by Γ_g in [16]. As explained in [16, Example 5.5] it may be described by

$$\Gamma_g \cong \begin{cases} \mathsf{O}_{g,g}(\mathbb{Z}) & \text{for } n \text{ even},\\ \mathsf{Sp}_{2g}(\mathbb{Z}) & \text{for } n=1,3,7,\\ \mathsf{Sp}_{2g}^q(\mathbb{Z}) & \text{for } n \text{ odd} \neq 1,3,7,\end{cases}$$

where $\mathsf{Sp}^q_{2g}(\mathbb{Z})$ denotes the group of block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{Sp}_{2g}(\mathbb{Z})$$

such that the diagonal entries of the $g \times g$ matrices $C^t A$ and $D^t B$ are even.

The next result should be viewed as an upgrade of [16, Theorem 5.8], which asserts that the Lie model for the space $B\,\mathrm{aut}_\circ(M)$ is formal. The novelty is that the formality can be made equivariant.

Proposition 5.30 *Let M be an $(n-1)$-connected $2n$-dimensional manifold, where $n > 1$, such that* $\dim H^n(M;\mathbb{Q}) > 2$. *The algebraic Lie model $\mathfrak{g}(M)$ for $B\,\mathrm{aut}_u^+(M)$ is formal as a dg Lie algebra of algebraic representations of $R^+(M)$, and its homology is the graded Lie algebra*

$$H_*(\mathfrak{g}(M)) \cong \mathrm{Der}\, L/\mathrm{ad}\, L\langle 1\rangle,$$

where $L = \pi_*(\Omega M) \otimes \mathbb{Q}$ is the rational homotopy Lie algebra of M. This Lie algebra admits the presentation

$$L \cong \mathbb{L}(\alpha_1, \dots, \alpha_r)/(\omega),$$

with $\alpha_1, \dots, \alpha_r$ and ω as above. The action of $R^+(M)$ is induced by the action on $H_*(M;\mathbb{Q})$.

Proof Since $\mathrm{Aut}^+\,\mathbb{L}$ is reductive, we can show that $\mathrm{Der}^c\,\mathbb{L}\langle 1\rangle$ is an $R^+(M)$-algebraic Lie model for $B\,\mathrm{aut}_u^+(M)$ as in Corollary 4.11. Given this, it is straightforward to inspect that the zig-zag of quasi-isomorphisms that connects this dg Lie algebra with its homology given in the proof of [16, Theorem 5.9] can be made into a zig-zag of algebraic $R^+(M)$-representations. □

The action of $\Gamma^+(M)$ on the rational homotopy groups was also identified in [16], but the methods of [16] were insufficient for proving the following result, which is a direct consequence of Corollary 1.4 and Proposition 5.30.

Theorem 5.31 *Let M be an $(n-1)$-connected $2n$-dimensional manifold, with $n > 1$, and suppose that* $\dim H^n(M;\mathbb{Q}) > 2$. *There is an isomorphism of graded algebras*

$$H^*(B\,\mathrm{aut}^+(M);\mathbb{Q}) \cong H^*(\Gamma^+(M), H^*_{\mathrm{CE}}(\mathfrak{g})),$$

where $\Gamma^+(M)$ is the group $\mathrm{Aut}\big(H^n(M;\mathbb{Z}), \langle\ ,\ \rangle, \psi\big)$, $\mathfrak{g}$ is $\mathrm{Der}\, L/\mathrm{ad}\, L\langle 1\rangle$, and L is the graded Lie algebra $\pi_*(\Omega M) \otimes \mathbb{Q}$. □

When adapted to the space $\mathrm{aut}_\partial(W_{g,1})$ of self-homotopy equivalences of

$$W_{g,1} = W_g \setminus \mathrm{int}\, D^{2n}$$

relative to the boundary, our methods yield a significant simplification of the computation of the stable cohomology of $B\,\mathrm{aut}_\partial(W_{g,1})$ of [16]. By applying our methods to the homotopy fiber sequence

$$\Gamma_g /\!\!/ \mathrm{aut}_\partial(W_{g,1}) \to B\,\mathrm{aut}_\partial(W_{g,1}) \to B\Gamma_g, \tag{32}$$

one can upgrade [16, Theorem 3.12] to show that the Γ_g-space $\Gamma_g /\!\!/ \operatorname{aut}_\partial(W_{g,1})$ has R_g-algebraic Lie model

$$\mathfrak{g}_g = \operatorname{Der}_\omega \mathbb{L}\langle 1\rangle,$$

where $\operatorname{Der}_\omega \mathbb{L}$ denotes the graded Lie algebra of derivations of the free graded Lie algebra $\mathbb{L} = \mathbb{L}(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ that annihilate the element

$$\omega = [\alpha_1, \beta_1] + \cdots + [\alpha_g, \beta_g].$$

As above, this implies:

Theorem 5.32 *There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}_\partial(W_{g,1}); \mathbb{Q}) \cong H^*(\Gamma_g, H^*_{\mathrm{CE}}(\mathfrak{g}_g)). \tag{33}$$ □

Remark 5.33 In the stable range, ie for g large compared to the cohomological degree, the right-hand side can be simplified. Let $H = H^*_{\mathrm{CE}}(\mathfrak{g}_g)$. By semisimplicity of $\operatorname{Rep}_{\mathbb{Q}}(R_g)$ there is a split exact sequence

$$0 \to H^{R_g} \to H \to H/H^{R_g} \to 0.$$

This gives rise to a ring homomorphism

$$H^*(\Gamma_g, \mathbb{Q}) \otimes H^{R_g} \cong H^*(\Gamma_g, H^{R_g}) \to H^*(\Gamma_g, H),$$

whose cokernel may be identified with

$$H^*(\Gamma_g, H/H^{R_g}).$$

Since H/H^{R_g} is a direct sum of nontrivial irreducible algebraic representations of R_g, the cokernel vanishes in degrees that are small compared to g by Borel's vanishing theorem [20, Theorem 4.4]. Note that this entails the statement that $H^{R_g} = H^{\Gamma_g}$ for g large (set $* = 0$). Thus, in the stable range, we obtain an isomorphism

$$H^*(B\operatorname{aut}_\partial(W_{g,1}); \mathbb{Q}) \cong H^*(\Gamma_g, \mathbb{Q}) \otimes H^{\Gamma_g}.$$

This essentially recovers Theorem 1.3 of [16].

Remark 5.34 The isomorphism (33), and in particular the collapse of the spectral sequence of (32), is only obtained in the stable range (cohomological degrees below $g/2-2$) in [16]. Moreover, the argument in [16] depends on several extraneous ingredients. One step in the argument is to show that the map

$$B\operatorname{aut}_\partial(W_{g,1}) \to B\Gamma_g$$

is injective on indecomposables in rational cohomology in the stable range [16, Theorem 8.6]. This is proved by establishing the stronger statement [16, Theorem 8.2] that

$$B\operatorname{Diff}_\partial(W_{g,1}) \to B\Gamma_g$$

is injective on indecomposables in stable rational cohomology, which in turn relies on deep results on the stable cohomology of $B\operatorname{Diff}_\partial(W_{g,1})$ due to Galatius and Randal-Williams [37], as well as on nonvanishing results for the coefficients of the Hirzebruch L-polynomials due to Berglund and Bergström [14]. Our proof of Theorem 5.32 does not depend on these extraneous ingredients. In fact, Theorem 5.32 implies a strengthening of [16, Theorem 8.6]; see Corollary 5.35 below.

Furthermore, many arguments of [16] rely on deep results on almost algebraicity of finite-dimensional representations of Γ_g due to Bass, Milnor and Serre [4, Section 16]; see also [62, Section 1.3(9)] and [16, Appendix A]. The existence of algebraic Lie models shows that the representations in question are in fact algebraic, which in particular removes the necessity of invoking such results.

Corollary 5.35 *The ring homomorphism*

$$H^*(\Gamma_g, \mathbb{Q}) \to H^*(B\,\mathrm{aut}_\partial(W_{g,1}); \mathbb{Q})$$

is split injective for all g. In particular, the induced map on indecomposables

$$QH^*(\Gamma_g, \mathbb{Q}) \to QH^*(B\,\mathrm{aut}_\partial(W_{g,1}); \mathbb{Q})$$

is injective for all g. □

5.3 Highly connected odd-dimensional manifolds

The geometry and rational homotopy theory of $(n-1)$-connected $(2n+1)$-dimensional manifolds has also been thoroughly studied (eg in Wall's classification [69]), and we make some remarks on the odd-dimensional case in this section.

Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold, where $n > 8$. Using the techniques of [16, Section 3.5], one can show that the minimal Quillen model of M is given by

$$\mathbb{L} = (\mathbb{L}(\alpha_1, \dots, \alpha_r, \alpha_1^\#, \dots, \alpha_r^\#, \gamma), \delta\gamma = \omega),$$

where the α_i are a basis for $s^{-1}\widetilde{H}_n(M; \mathbb{Q})$, the $\alpha_i^\#$ are the dual basis of the space $s^{-1}\widetilde{H}_{n+1}(M; \mathbb{Q})$ with respect to the intersection form, and

$$\omega = \sum_i [\alpha_i^\#, \alpha_i]$$

is dual to the intersection form. (So up to rational equivalence, such manifolds are completely classified by the torsion-free rank of $H_n(M)$.) We use the notation of the preceding section freely.

Lemma 5.36 *Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold with $n > 1$. The group $\mathrm{Aut}^+ \mathbb{L}$ is isomorphic to the group $\mathsf{GL}(H^n(M; \mathbb{Q}))$ of linear automorphisms of $H^n(M; \mathbb{Q})$. In particular, it is reductive, wherefore $\mathrm{Aut}^+ \mathbb{L} \cong \mathscr{E}^+(M_\mathbb{Q}) \cong R^+(M)$.*

Proof An automorphism of $\mathbb{L}$ is orientation-preserving precisely when it fixes γ. Thus an automorphism $\phi \in \mathrm{Aut}^+ \mathbb{L}$ is uniquely determined by its action on $\mathbb{L}_{n-1} = \langle \alpha_1, \dots, \alpha_r \rangle$: it preserves the element $\omega = \delta\gamma$, so its action on the $\alpha_i^\#$ is ω-dual to the action on the α_i. □

The identification of the group $\Gamma^+(M)$ is somewhat more involved than in the preceding case, and we will not elaborate on it here. We remark that the closely related group

$$\Gamma^+_{\mathbb{Z}}(M) = \mathrm{Aut}^{\mathscr{E}^+(M)}(H_*(M;\mathbb{Z}))$$

of orientation-preserving automorphisms of *integral* homology that are realizable by homotopy automorphisms of M, which has been studied by several authors, provides a ready substitute for $\Gamma^+(M)$ in our results: the space $B\,\mathrm{aut}^+(M)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{g}(M)\rangle_{\mathrm{h}\Gamma^+_{\mathbb{Z}}(M)}$. The group $\Gamma^+_{\mathbb{Z}}(M)$ has been determined in many cases by Floer [35] in the course of his classification of the homotopy types of $(n-1)$-connected $(2n+1)$-dimensional Poincaré complexes. See also Barden [3] and Baues and Buth [7] for the case $n = 2$, and Crowley and Nordström [27] for the case $n = 3$.

Just like in the case of even-dimensional manifolds, $\mathfrak{g}(M)$ enjoys a close connection to the derivations of the homotopy Lie algebra $\pi_*(\Omega M) \otimes \mathbb{Q}$. The following proposition is an easy consequence of the methods of [16, Section 5.4], especially Theorem 5.9 therein.

Proposition 5.37 *Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold, with $n > 1$, such that $\dim H^n(M;\mathbb{Q}) > 1$. The algebraic Lie model $\mathfrak{g}(M)$ for the space $B\,\mathrm{aut}_u(M)$ is formal as a dg Lie algebra of algebraic representations of $R(M)$ and its homology is the graded Lie algebra*

$$H_*(\mathfrak{g}(M)) \cong \mathrm{Der}\, L/\mathrm{ad}\, L\langle 1\rangle$$

where $L = \pi_*(\Omega M) \otimes \mathbb{Q}$ is the rational homotopy Lie algebra of M, which admits the presentation

$$L \cong \mathbb{L}(\alpha_1, \dots, \alpha_r, \alpha_1^\#, \dots, \alpha_r^\#)/(\omega)$$

with $\alpha_i, \alpha_i^\#$ and ω as above. The action of $R(M)$ is induced by the action on $H_*(M;\mathbb{Q})$.

Thus, we obtain the following consequence of Corollary 1.4.

Theorem 5.38 *Let M be an $(n-1)$-connected $(2n+1)$-dimensional manifold with $\dim H^n(M;\mathbb{Q}) > 1$. There is an isomorphism of graded algebras*

$$H^*(B\,\mathrm{aut}^+(M);\mathbb{Q}) \cong H^*\big(\Gamma^+_{\mathbb{Z}}(M), H^*_{\mathrm{CE}}(\mathfrak{g}(M))\big),$$

where $\mathfrak{g} = \mathrm{Der}\, L/\mathrm{ad}\, L\langle 1\rangle$ is the truncated dg Lie algebra of outer derivations of $L = \pi_*(\Omega M) \otimes \mathbb{Q}$, and $\Gamma^+_{\mathbb{Z}}(M) = \mathrm{Aut}^{\mathscr{E}^+(M)}(H^*(M;\mathbb{Z}))$. □

We conclude this section by making some remarks about the manifolds

$$Z_g = \#^g\, S^n \times S^{n+1} \quad \text{and} \quad Z_{g,1} = Z_g \setminus \mathrm{int}\, D^{2n+1} \quad \text{for } n > 1.$$

By Lemma 5.36, we have that

$$R^+(Z_g) \cong \mathsf{GL}(H_n(Z_g;\mathbb{Q})) \cong \mathsf{GL}_g(\mathbb{Q}).$$

Let

$$\Gamma_\partial(Z_{g,1}) \cong \mathsf{GL}^{\operatorname{aut}_\partial(Z_{g,1})}(H_*(Z_{g,1};\mathbb{Q}))$$

be the group of automorphisms of the rational homology of $Z_{g,1}$ which can be realized by a boundary-preserving homotopy automorphism of $Z_{g,1}$. Each such homotopy automorphism can be extended by the identity on the interior of the removed disc to an orientation-preserving homotopy automorphism of Z_g, yielding an injection

$$\Gamma_\partial(Z_{g,1}) \hookrightarrow \Gamma^+(Z_g). \tag{34}$$

Proposition 5.39 *The homomorphism (34) is an isomorphism, and the group $\Gamma^+(Z_g)$ corresponds to the subgroup $\operatorname{Aut}(H_n(Z_g;\mathbb{Z}))$ of the group*

$$\mathsf{GL}(H_n(Z_g;\mathbb{Q})) \cong R^+(Z_g).$$

Proof It is evident that $\Gamma^+(Z_g)$ injects into $\operatorname{Aut}(H_n(Z_g;\mathbb{Z}))$. Both claims can be established simultaneously by showing that every automorphism of the group $H_n(Z_g;\mathbb{Z}) \cong \mathbb{Z}^g$ may be realized by a homotopy automorphism of Z_g that fixes the given embedded disc $D^{2n+1} \subset Z_g$ pointwise. This is an easy consequence of the Hilton–Milnor theorem once one notes that $Z_{g,1} \simeq \bigvee^g S^n \vee \bigvee^g S^{n+1}$ and that Z_g is obtained from $Z_{g,1}$ by attaching a $(2n+1)$-cell along

$$\omega = \sum_{i=1}^{g} [\alpha_i, \alpha_i^\#] \in \pi_{2n}(Z_{g,1}),$$

where $\alpha_i \in \pi_n(Z_{g,1})$ are the classes of the S^n-factors of $Z_{g,1}$ and $\alpha_i^\# \in \pi_{n+1}(Z_{g,1})$ are the classes of the S^{n+1}-factors, suitably indexed. See [40, Section 5], where the more general case of connected sums of products of spheres of varying dimensions is investigated. □

When applied to the homotopy fiber sequence

$$\Gamma_\partial(Z_{g,1}) /\!\!/ \operatorname{aut}_\partial(Z_{g,1}) \to B\operatorname{aut}_\partial(Z_{g,1}) \to B\Gamma_\partial(Z_{g,1}),$$

our methods show that the $\Gamma_\partial(Z_{g,1})$-space $\Gamma_\partial(Z_{g,1}) /\!\!/ \operatorname{aut}_\partial(Z_{g,1})$ admits an $R^+(Z_g)$-algebraic Lie model

$$\mathfrak{g}_\partial(Z_{g,1}) = \big(\operatorname{Der}_\omega \mathbb{L}(\alpha_1,\dots,\alpha_g,\alpha_1^\#,\dots,\alpha_g^\#)\big)\langle 1\rangle,$$

where $\operatorname{Der}_\omega$ indicates derivations that annihilate the element $\omega = \sum_i [\alpha_i,\alpha_i^\#]$. The action of $R^+(Z_g)$ (and thus by restriction also the action of $\Gamma_\partial(Z_{g,1})$) comes from the action on $H_*(Z_g;\mathbb{Q})$. Thus we obtain:

Theorem 5.40 *There is an isomorphism of graded algebras*

$$H^*(B\operatorname{aut}_\partial(Z_{g,1});\mathbb{Q}) \cong H^*(\Gamma_\partial(Z_{g,1}); H^*_{\mathrm{CE}}(\mathfrak{g}_\partial(Z_{g,1}))). \qquad \square$$

Using this result, the stable cohomology of $B\operatorname{aut}_\partial(Z_{g,1})$ is computed by Stoll [65].

5.4 A counterexample

In all the examples discussed so far, the space X was always formal, the group $\mathscr{E}(X_{\mathbb{Q}})$ was reductive (hence equal to $R(X)$) and isomorphic to the group of algebra automorphisms of $H^*(X;\mathbb{Q})$, the group $\Gamma(X)$ was the group of automorphisms of $H_*(X;\mathbb{Q})$ that are induced by a self-homotopy equivalence of X, and the dg Lie algebra $\mathfrak{g}(X)$ was formal. In this section, we will discuss an example of a space X that has none of these properties. In fact, X will be the space that is discussed in [39, Example 6.7].

Let X be the space $F \times K(\mathbb{Z},3) \times K(\mathbb{Z},6)$, where F is the homotopy fiber of the cup product

$$K(\mathbb{Z},3) \times K(\mathbb{Z},3) \xrightarrow{\smile} K(\mathbb{Z},6).$$

This is evidently a finite Postnikov section with finitely generated homotopy groups, and its Sullivan model is the cdga

$$\Lambda = (\Lambda(x,y,z,u,w), d),$$

whose underlying graded commutative algebra is freely generated by cocycles x, y, z in degree 3, a cocycle w in degree 6, and an element u in degree 5 which satisfies $du = yz$.

Remark 5.41 X is indeed not rationally formal: the homomorphism

$$\operatorname{Aut}\Lambda \to \operatorname{Aut}_{\mathrm{alg}} H^*(\Lambda)$$

is not surjective. For any $\lambda \in \mathbb{Q}^\times$, there is a graded algebra automorphism ϕ of $H^*(\Lambda)$ given by $\phi(a) = \lambda^{|a|}a$. But ϕ affords no lift to $\operatorname{Aut}\Lambda$ unless $\lambda = 1$: any such lift $\tilde{\phi}$ would have to have $\tilde{\phi}(u) = \lambda^6 u$ so as to commute with d. But then $\tilde{\phi}(uy) = \lambda^9 uy$, whereas $\phi(uy) = \lambda^8 uy$.

Proposition 5.42 *The group $R(X)$ is isomorphic to*

$$\mathsf{GL}_1^2 \times \mathsf{GL}_2 .$$

A section of the surjection $\operatorname{Aut}\Lambda \to R(X)$ *is given by letting the two* GL_1 *factors act by scalar multiplication on x and on w, respectively, and by letting $A \in \mathsf{GL}_2$ act via the standard representation on the linear span of y and z, and by scalar multiplication by* $\det A$ *on u. The subgroup $\Gamma(X)$ of $R(X)$ corresponds to*

$$(\mathbb{Z}^\times)^2 \times \mathsf{GL}_2^{\Sigma}(\mathbb{Z})$$

under this isomorphism.

Remark 5.43 The authors of [39, Example 6.7] construct two automorphisms $\phi, \psi \in \operatorname{Aut}\Lambda$. Here ϕ is the identity on all generators but w, where $\phi(w) = w + xy$, and ψ is the identity on all generators but x, where $\psi(x) = x + z$. They observe that these two automorphisms span a copy of $\mathbb{Z}^2$ in $\operatorname{Aut}^h \Lambda \cong \mathscr{E}(X_{\mathbb{Q}})$, but there exists no pair (ϕ', ψ') of homotopic automorphisms (or even automorphisms with the same effect on $H^*(\Lambda)$ as ϕ and ψ, respectively) which would already commute in $\operatorname{Aut}\Lambda$. This shows that we cannot in general expect to be able to lift the action of $\mathscr{E}(X_{\mathbb{Q}})$ on $\pi_* X \otimes \mathbb{Q}$ or on $H^*(X;\mathbb{Q})$ to an action on the minimal Sullivan model. It is clear from the proof of Proposition 5.42 that both ϕ and ψ lie in the kernel of $\operatorname{Aut}\Lambda V \to R(X)$, so they do not obstruct the existence of an action of $R(X)$ on Λ.

Proposition 5.44 *The dg Lie algebra $\mathfrak{g}(X)$ is not formal: the Massey product*

$$\left\langle y\frac{\partial}{\partial x}, x\frac{\partial}{\partial u}, z\frac{\partial}{\partial w}\right\rangle$$

is nontrivial.

The rest of the section is devoted to proving Propositions 5.42 and 5.44.

We write V for the graded vector subspace of Λ which is spanned by the generators x, y, z, u and w, and we write ΛV for the underlying graded commutative algebra of Λ. The group $\mathsf{GL}(V)$ of graded linear automorphisms of V evidently injects into the group $\operatorname{Aut}\Lambda V$ of graded commutative algebra automorphisms of ΛV, and we will abuse notation by referring to its image by $\mathsf{GL}(V)$ as well.

Lemma 5.45 *The automorphism group* $\operatorname{Aut}\Lambda V$ *of the underlying graded commutative algebra is a semidirect product*

$$\mathsf{GL}(V) \ltimes \operatorname{Hom}(V^6, \Lambda^2 V^3)$$

where the abelian group $\operatorname{Hom}(V^6, \Lambda^2 V^3)$ *injects into* $\operatorname{Aut}\Lambda V$ *by sending* $f\colon V^6 \to \Lambda^2 V^3$ *to the automorphism that is the identity on all generators except w and that takes w to $w + f(w)$.*

Proof There is an obvious identification of V with the indecomposables $Q\Lambda V = I/I^2$, where $I = (\Lambda V)^{>0}$ is the augmentation ideal. This yields a retraction $\operatorname{Aut}\Lambda V \twoheadrightarrow \mathsf{GL}(Q\Lambda V) \cong \mathsf{GL}(V)$. Let $\operatorname{Aut}_1 \Lambda V$ be the kernel of this retraction. Every $\phi \in \operatorname{Aut}_1 \Lambda V$ fixes x, y, z and u, and is determined by the value $\phi(w) - w \in \Lambda^2 V^3 \subset (\Lambda V)^6$. It can be seen that there is an isomorphism

$$\operatorname{Aut}_1 \Lambda V \cong \operatorname{Hom}(V^6, \Lambda^2 V^3)$$

of groups with a $\mathsf{GL}(V)$-action which sends $\phi \in \operatorname{Aut}_1 \Lambda V$ to the homomorphism $w \mapsto \phi(w) - w$. □

The group $\operatorname{Aut}\Lambda$ of cdga automorphisms of Λ is the isotropy subgroup of the differential d inside $\operatorname{Aut}\Lambda V$. The subgroup $\operatorname{Aut}_1 \Lambda V$ of the preceding lemma clearly commutes with d, so it suffices to determine $\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$.

Lemma 5.46 *Let $W = \langle y, z\rangle \subset V^3$ be the $\mathbb{Q}$-linear span of y and z. An automorphism $\phi \in \mathsf{GL}(V)$ commutes with d if and only if W is ϕ-stable, and $\phi(u) = (\det \phi|_W) \cdot u$.*

Proof Note that the line $\Lambda^2 W$ is precisely the span of yz, and also the image of the differential $d\colon (\Lambda V)^5 \hookrightarrow (\Lambda V)^6$. Hence ϕ commutes with d if and only if it carries $\Lambda^2 W$ to itself and it acts on $\Lambda^2 W$ and $(\Lambda V)^5$ by the same scalar. But $\phi(\Lambda^2 W) = \Lambda^2 \phi(W)$ is equal to $\Lambda^2 W$ if and only if $\phi(W) = W$. If this is the case, then ϕ scales $\Lambda^2 W$ by $\det \phi|_W$. □

Proof of Proposition 5.42 The unipotent radical of $\operatorname{Aut}\Lambda V$, which we identified in Lemma 5.45, is contained in $\operatorname{Aut}\Lambda$, and hence in its unipotent radical. Thus the maximal reductive quotient of $\operatorname{Aut}\Lambda$ agrees with that of $\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$. From Lemma 5.46 it is easily seen that the unipotent radical of

$\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$ consists of the automorphisms that fix all generators except x and that send x to x plus an element of W, and the maximal reductive quotient is as described in the statement of the proposition.

To determine $\Gamma(X)$, recall that X is the homotopy fiber of the 6^{th} integral cohomology class

$$K(\mathbb{Z},3)^3 \times K(\mathbb{Z},6) \to K(\mathbb{Z},6)$$

given by the cup product of the fundamental classes of two of the $K(\mathbb{Z},3)$ factors.

The integral cohomology ring of X agrees up to dimension 6 with

$$\Lambda_{\mathbb{Z}}(\bar{x},\bar{y},\bar{z},\bar{w})/(\overline{yz},2\bar{x}^2,2\bar{y}^2,2\bar{z}^2),$$

where the generators $\bar{x},\bar{y}$ and $\bar{z}$ live in degree 3, $\bar{w}$ lives in degree 6, and each generator maps to the corresponding element of $H^*(\Lambda) \cong H^*(X;\mathbb{Q})$ under the change-of-coefficient homomorphism. Explicitly, $H^3(X) \cong \mathbb{Z}^3$ is spanned by $\bar{x},\bar{y},\bar{z}$, and $H^6(X) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3$ is spanned by $\overline{x}\overline{y}, \overline{xz}, \bar{w}$ and $\bar{x}^2,\bar{y}^2,\bar{z}^2$.

Thus the homotopy class of an endomorphism $f\colon X \to X$ is uniquely determined by the quadruple $(f^*\bar{x}, f^*\bar{y}, f^*\bar{z}, f^*\bar{w}) \in H^3(X)^3 \times H^6(X)$, and a quadruple $(\bar{x}',\bar{y}',\bar{z}',\bar{w}')$ is induced by an endomorphism if $\bar{y}' \smile \bar{z}' = 0$.

Thus an element $(\lambda, A, \mu) \in \mathsf{GL}_1 \times \mathsf{GL}_2 \times \mathsf{GL}_1 \cong R(X)$ can be realized by a homotopy automorphism of X if and only if $\lambda,\mu \in \mathbb{Z}$, the matrix A has integer entries, and each row of A contains at least one even number (since $(a\bar{y}+b\bar{z})(c\bar{y}+d\bar{z}) = ac\bar{y}^2 + bd\bar{z}^2 \in H^6(X)$ is nonzero if ac or bd are odd). □

Proof of Proposition 5.44 We begin by describing in some detail the structure of the dg Lie algebra $\mathfrak{g}(X)$. In positive degrees, $\mathfrak{g}(X)$ is just the full dg Lie algebra of derivations of the free cdga ΛV, and it is easy to list the basis in each degree.

In degree 0, $\mathfrak{g}(X)$ is the Lie algebra of the unipotent radical of $\operatorname{Aut}\Lambda$. This radical sits in a (split) short exact sequence

$$1 \to \operatorname{Hom}(V^6, \Lambda^2 V^3) \to \operatorname{Aut}_u \Lambda \to \operatorname{Hom}(\langle x\rangle, W) \to 1,$$

where the left-hand term is the unipotent radical of $\operatorname{Aut}\Lambda V$ and the right-hand term is the unipotent radical of $\operatorname{Aut}\Lambda \cap \mathsf{GL}(V)$. From this, a basis for $\mathfrak{g}_0(X)$ can be read off. We list bases in all degrees in Table 1.

The differential δ is given by

$$u\frac{\partial}{\partial w} \mapsto yz\frac{\partial}{\partial w}, \qquad \frac{\partial}{\partial y} \mapsto -z\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial z} \mapsto y\frac{\partial}{\partial u};$$

it vanishes on all the other generators.

Note that

$$\left[y\frac{\partial}{\partial x}, x\frac{\partial}{\partial u}\right] = y\frac{\partial}{\partial u} = \delta\left(\frac{\partial}{\partial z}\right)$$

while the other two pairs bracket to zero. Thus the Massey product contains

$$\left[\frac{\partial}{\partial z}, z\frac{\partial}{\partial w}\right] = \frac{\partial}{\partial w}.$$

degree	basis
0	$y\frac{\partial}{\partial x},\ z\frac{\partial}{\partial x},\ xy\frac{\partial}{\partial w},\ xz\frac{\partial}{\partial w},\ yz\frac{\partial}{\partial w}$
1	$u\frac{\partial}{\partial w}$
2	$x\frac{\partial}{\partial u},\ y\frac{\partial}{\partial u},\ z\frac{\partial}{\partial u}$
3	$x\frac{\partial}{\partial w},\ y\frac{\partial}{\partial w},\ z\frac{\partial}{\partial w},\ \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$
5	$\frac{\partial}{\partial u}$
6	$\frac{\partial}{\partial w}$

Table 1

The indeterminacy is equal to the subset

$$\left[y\frac{\partial}{\partial x}, H_6(\mathfrak{g}(X))\right]+\left[x\frac{\partial}{\partial u}, H_4(\mathfrak{g}(X))\right]+\left[z\frac{\partial}{\partial w}, H_3(\mathfrak{g}(X))\right]$$

of $H_6(\mathfrak{g}(X))$. This is easily seen to only contain 0. Thus the Massey product is indeed nontrivial as $\partial/\partial w$ is a nontrivial cycle, obstructing formality of $\mathfrak{g}(X)$. □

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Department of Mathematics, Stockholm University
Stockholm, Sweden

alexb@math.su.se, tomas.zeman@math.su.se

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Jesper Grodal — University of Copenhagen, jg@math.ku.dk

Misha Gromov — IHÉS and NYU, Courant Institute, gromov@ihes.fr

Mark Gross — University of Cambridge, mgross@dpmms.cam.ac.uk

Rob Kirby — University of California, Berkeley, kirby@math.berkeley.edu

Bruce Kleiner — NYU, Courant Institute, bkleiner@cims.nyu.edu

Sándor Kovács — University of Washington, skovacs@uw.edu

Urs Lang — ETH Zürich, urs.lang@math.ethz.ch

Marc Levine — Universität Duisburg-Essen, marc.levine@uni-due.de

Jianfeng Lin — Tsinghua University, linjian5477@mail.tsinghua.edu.cn

Ciprian Manolescu — University of California, Los Angeles, cm@math.ucla.edu

Haynes Miller — Massachusetts Institute of Technology, hrm@math.mit.edu

Aaron Naber — Institute for Advanced Studies, anaber@ias.edu

Peter Ozsváth — Princeton University, petero@math.princeton.edu

Leonid Polterovich — Tel Aviv University, polterov@post.tau.ac.il

Colin Rourke — University of Warwick, gt@maths.warwick.ac.uk

Roman Sauer — Karlsruhe Institute of Technology, roman.sauer@kit.edu

Stefan Schwede — Universität Bonn, schwede@math.uni-bonn.de

Natasa Sesum — Rutgers University, natasas@math.rutgers.edu

Gang Tian — Massachusetts Institute of Technology, tian@math.mit.edu

Nathalie Wahl — University of Copenhagen, wahl@math.ku.dk

Kirsten Wickelgren — Duke University, kirsten.wickelgren@duke.edu

Anna Wienhard — Universität Heidelberg, wienhard@mathi.uni-heidelberg.de

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