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Volume growth of Funk geometry and the flags of polytopes

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We consider the Holmes–Thompson volume of balls in the Funk geometry on the interior of a convex domain. We conjecture that for a fixed radius, this volume is minimized when the domain is a simplex and the ball is centered at the barycenter, or in the centrally symmetric case, when the domain is a Hanner polytope. This interpolates between Mahler's conjecture and Kalai's flag conjecture. We verify this conjecture for unconditional domains.

For polytopal Funk geometries, we study the asymptotics of the volume of balls of large radius, and compute the two highest-order terms. The highest depends only on the combinatorics, namely on the number of flags. The second highest depends also on the geometry, and thus serves as a geometric analogue of the centro-affine area for polytopes.

We then show that for any polytope, the second highest coefficient is minimized by a unique choice of center point, extending the notion of Santaló point. Finally, we show that, in dimension two, this coefficient, with respect to the minimal center point, is uniquely maximized by affine images of the regular polygon.

51K99, 51M10, 52A38, 52A40, 52B05

1 Introduction

We are interested in the volume of metric balls in the Funk geometry — a Finsler geometry with a nonreversible metric defined on the interior of a convex body, closely related to the Hilbert metric. The forward metric balls in this geometry take a particularly simple form: they are merely scaled versions of the body itself. We use the Holmes–Thompson definition of volume.

Our interest stems in part from a connection with a longstanding conjecture in convex geometry. As the radius R of the ball tends to zero, the volume is asymptotic to R^n times the *Mahler volume* (volume product) of the body, with the polar body taken relative to the center of the ball. The Mahler conjecture states that the Mahler volume is minimized when the body is a simplex, or, if the body is required to be centrally symmetric, when it is a Hanner polytope. For a recent survey of the problem, see Fradelizi, Meyer and Zvavitch [8].

We recall that a Hanner (or Hansen–Lima) polytope is any polytope that can be constructed from line segments by taking Cartesian products and polar bodies. One may ask if the same bodies minimize the volume for all $R > 0$.

Conjecture 1.1 *Let K be a centrally symmetric convex body in $\mathbb{R}^n$, and let $R > 0$. Then, the volume of the (forward) ball in the Funk geometry of radius R centered at the origin is minimized when the body is a Hanner polytope. Explicitly,*

$$\mathrm{Vol}_K(B_K(R)) \geq \mathrm{Vol}_H(B_H(R)) = \frac{2^n}{n!}\frac{(\log(2e^R - 1))^n}{\omega_n},$$

where H is any Hanner polytope of dimension n, and ω_n the volume of the n-dimensional Euclidean ball of unit radius.

If K is not assumed to be centrally symmetric, then the centered n-dimensional simplex minimizes $\mathrm{Vol}_K(B_K(R))$.

Our first theorem shows that as the radius becomes large, the asymptotic behavior of the volume of the ball is determined by the number of *flags* of the polytope. Let P be a polytope of dimension n endowed with its Funk geometry, and denote again by $\mathrm{Vol}_P(B_P(R))$ the Holmes–Thompson volume of the ball of radius R about the origin, which we assume to be contained in the interior of P. Also, denote by $\mathrm{Flags}(P)$ the set of flags of P.

Theorem 1.2 *For any polytope P of dimension n,*

$$\lim_{R\to\infty} \frac{\omega_n}{R^n}\mathrm{Vol}_P(B_P(R)) = \frac{|\mathrm{Flags}(P)|}{(n!)^2}.$$

That the asymptotic volume on the left-hand side is proportional to the number of flags can be established by adapting the proof of Vernicos and Walsh [24] from Hilbert geometry, where the same statement holds with a different constant. The proof there makes use of the invariance of the volume under collineations (invertible projective transformations), which holds for any reasonable definition of volume in the Hilbert case. For the Funk metric, however, invariance of the volume under collineations is a special property of the Holmes–Thompson definition, as observed by Faifman [6]. The explicit constant $(n!)^{-2}$ can then be determined, for example, by computing the case of Hanner polytopes, which is done in Section 4.

We will however take a different approach, and deduce this theorem from a comprehensive computation of the first two terms of the asymptotic expansion of $\mathrm{Vol}_P(B_P(R))$, as explained in Theorem 1.5. Let us mention here that volume growth behavior in metric geometry similar to Theorem 1.2 has been observed by Pansu [16] on the universal cover of nilmanifolds, and by Burago and Ivanov [4] in the special case of tori, allowing characterization of flat ones (see also [24] for an analogous rigidity result in Hilbert geometry by the last two authors). However, an explicit second-order term, which we obtain in the present paper, is usually far more complicated, and may not even exist.

In light of Theorem 1.2, our conjecture would imply that, out of all centrally symmetric polytopes of a given dimension, Hanner polytopes have the fewest flags. This statement is known as the *flag conjecture* and has been ascribed in [23, Chapter 22] to Kalai. It appears as a special case of Conjecture C of Kalai [10], although this conjecture was shown to be false in general in Sanyal, Werner and Ziegler [20].

We show that Conjecture 1.1 is true for *unconditional bodies*, and describe all equality cases under that assumption. Recall that an unconditional body is one that is symmetric through reflections in the coordinate hyperplanes.

Theorem 1.3 *Let $K \subset \mathbb{R}^n$ be an unconditional convex body, and let H be any Hanner polytope of the same dimension. Then,*

- $\mathrm{Vol}_K(B_K(R)) \geq \mathrm{Vol}_H(B_H(R))$ *for all* $R > 0$;
- *equality holds if and only if K is a Hanner polytope.*

Letting R tend to infinity and invoking Theorem 1.2, we then deduce the following.

Corollary 1.4 *For an unconditional n-dimensional convex polytope P, and a Hanner polytope H of the same dimension, we have*

$$|\mathrm{Flags}(P)| \geq |\mathrm{Flags}(H)| = 2^n n!.$$

While we lose track of the equality cases when passing to the limit, we do obtain necessary conditions for equality as a corollary of Theorem 1.5 — see Corollary 1.6 below. We remark that recently, the related 3^d conjecture of Kalai concerning the minimal total face number of a symmetric polytope was proved in the unconditional case by Chambers and Portnoy [5], with equality cases established by Sanyal and Winter [21].

Since the highest-order term in the asymptotics of the volume depends only on the combinatorics of the polytope and not on its geometry, it is interesting to look also at the second-highest order term. Let us stress once again that this second-order term is usually not easy to obtain, for instance for the closely related Hilbert metric, a second-order term is not known. In the present setting of Funk geometry, the second-order term is a new affine invariant of convex polytopes.

Recall that polytopes have the following property: if F and F' are two faces of dimension $i-1$ and $i+1$, respectively, and $F \subset F'$, then there are exactly two faces G of dimension i such that $F \subset G \subset F'$. So, for each $i \in \{0, \dots, n-1\}$ and flag $f \in \mathrm{Flags}(P)$, there is a unique flag, which we denote $r_i f$, that differs from f only by having a different face of dimension i. The group $\langle r_0, \dots, r_{n-1} \rangle$ generated by these maps is known as the *monodromy group* or *connection group* of the polytope P. To express the second term in the asymptotics of the volume, we will need a particular element of this group, the *complete flip* $r := r_{n-1} \circ \cdots \circ r_0$.

Theorem 1.5 *The volume of the ball of radius R in the Funk geometry on a polytope P satisfies*

$$\omega_n \mathrm{Vol}_P(B_P(R)) = c_0(P)R^n + c_1(P)R^{n-1} + o(R^{n-1}),$$

where

$$c_0(P) = \frac{|\mathrm{Flags}(P)|}{(n!)^2}$$

and

$$c_1(P) = \frac{n}{(n!)^2} \sum_{f \in \mathrm{Flags}(P)} \log\big(1 - \langle (rf)_{n-1}, f_0 \rangle\big). \tag{1}$$

Here, we are identifying the facet $(rf)_{n-1}$ with the associated vertex of the dual polytope.

Let $c_1(P,x)$ denote the corresponding coefficient with x taking the place of the origin as the center of the ball. Recall from Berck, Bernig and Vernicos [2] and Faifman [6] that for a sufficiently regular convex body $K \subset \mathbb{R}^n$, one has

$$\omega_n \mathrm{Vol}_P(B_K(R)) \sim \frac{1}{n-1} 2^{-\frac{1}{2}(n-1)} \Omega_n(K) e^{\frac{1}{2}(n-1)R},$$

where $\Omega_n(K)$ is the centro-affine surface area of K. Thus $c_1(P)$ can be seen as a geometric analogue for polytopes of the centro-affine surface area. Both $c_1(P,x)$ and the centro-affine surface area can be seen as versions of the Margulis function [13] in the respective settings of Funk geometry.

Corollary 1.6 *For an unconditional n-dimensional convex polytope P, and a Hanner polytope H of the same dimension, assume*

$$|\mathrm{Flags}(P)| = |\mathrm{Flags}(H)|.$$

Then for any flag $f \in \mathrm{Flags}(P)$, we have $-f_0 \in (rf)_{n-1}$.

We remark that all centrally symmetric 2-level polytopes satisfy this condition.

We next consider the dependence of the volume of the metric ball on the choice of its center. Vernicos and Walsh [25] show that, for any $R > 0$, the volume $\mathrm{Vol}_P(B_P(R))$ is minimized by a unique choice of center, denoted by $s_R(P)$, generalizing the notion of the Santaló point of a convex body. Here we study the behavior of this point as R tends to infinity.

Theorem 1.7 *Let P be a polytope. Then, $x \mapsto c_1(P,x)$ is proper and strictly convex, and finite in the interior of P, and hence attains its minimum at a unique point $s_\infty(P)$ in the interior.*

This point can be characterized as follows: $s_\infty(P) = 0$ if and only if the center of mass of the vertices of P° lies at the origin, where each vertex is assigned a weight equal to the number of flags of P° that contain it.

Moreover, we have

$$s_\infty(P) = \lim_{R\to\infty} s_R(P).$$

The characterization of $s_\infty(P)$ in this theorem is analogous to that of the classical Santaló point, namely that the Santaló point is at the origin if and only if the center of mass of the polar body is at the origin.

In dimension two, the second term (1) of Theorem 1.5 reduces to

$$c_1(P) := \frac{1}{2} \sum_{i,j:i\sim j} \log(1 - \langle e_i, v_j \rangle), \tag{2}$$

where the v_j are the vertices of the polygon, the e_i are the edges, which correspond to the vertices of the dual polytope, and $i \sim j$ means that $\langle e_i, v_j \rangle \neq 1$ and $\langle e_i, v_{j+k} \rangle = 1$ for some $k \in \{-1, 1\}$. We consider how the functional c_1 varies as the polygon varies, with the number of vertices being fixed. It is of course $\mathrm{GL}_2(\mathbb{R})$-invariant. We show that every stationary point of c_1 is the image under an element of $\mathrm{GL}_2(\mathbb{R})$ of the regular polygon.

Theorem 1.8 *Consider the space $\mathcal{P}_2^m$ of convex polygons with $m \geq 3$ vertices containing the origin in their interior. If the functional $c_1 : \mathcal{P}_2^m \to \mathbb{R}$ is stationary at P, then P is a linear image of the regular polygon D^m with m vertices, centered at the origin. Moreover, among all $P \in \mathcal{P}_2^m$ with $s_\infty(P)$ at the origin, $c_1(P)$ is uniquely maximized by linear images of D^m.*

This should be compared with Meyer and Reisner [15] (see also Alexander, Fradelizi and Zvavitch [1]), where the regular polygon is shown to uniquely maximize the volume product among all polygons with a fixed number of vertices. One could ask if the regular polygon similarly maximizes $\mathrm{Vol}_P(B_P(R))$ for general $0 < R < \infty$.

1.1 Plan of the paper

After some preliminaries, we prove Theorem 1.3 in Section 4. In Section 5, we decompose the polytope and its polar into flag simplices, and show that the volume of a ball can be expressed as the sum of a certain integral over all pairs consisting of a flag simplex of the polytope and a flag simplex of the polar. We then calculate the two highest-order terms contributed by each of these pairings. In Section 6, we combine this information to prove Theorem 1.5. In Section 7, we prove Theorem 1.7 concerning the convergence of the Funk–Santaló point, and in Section 8 we study two-dimensional Funk geometries, proving Theorem 1.8 and developing an exact expression in dimension two for the growth rate of the volume at any finite radius. We establish a recursive formula for the growth of the volume in the case of the simplex in Section 9. We conclude in Section 10 with some further conjectures, questions and thoughts.

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2 Preliminaries

2.1 Funk geometry

Let K be an open bounded convex body in $\mathbb{R}^n$. Consider a point $p \in K$ and a vector $\vec{v} \in \mathbb{R}^n$. Since K is convex, there exists a unique real number $t > 0$ such that $p + t\vec{v}$ is on the boundary ∂K of K. We define the *Funk weak norm* of $\vec{v}$ at p to be

$$F_K(p, \vec{v}) := \frac{1}{t}.$$

This has all the properties of a norm, except that it is not necessarily homogeneous, only positively homogeneous, and so the associated unit ball is not necessarily symmetric. Some call this the *tautological Finsler structure* associated to K, because at each point p of K the unit ball of the norm F_K is just the convex set K itself centered at p.

Given any two points p and q in K, we can look at all piecewise C^1 paths $\gamma\colon [0,1] \to K$ from p to q, and compute their *Funk length*

$$l_F(\gamma) := \int_0^1 F_K\bigl(\gamma(t), \dot{\gamma}(t)\bigr)\,\mathrm{d}t.$$

If we denote by b the intersection of the half line $[p, q)$ with ∂K, then the infimum of these lengths over all paths from p to q is the *Funk weak distance* and is equal to

$$d_F(p, q) := \log \frac{pb}{qb}.$$

Here, pb/qb is to be understood as the real number λ such that $\vec{pb} = \lambda\,\vec{qb}$. The quantity d_F is a weak distance in the sense that it is nonnegative for any two distinct points, it satisfies the triangle inequality, and $d_F(x, x) = 0$ for every point x. However, it is not symmetric. For simplicity, we will drop the adjective "weak" from now on.

The forward Funk metric ball centered at $p \in K$ of radius R is

$$B_K(p, R) := \{z \in K \mid d_K(p, z) \le R\}.$$

One easily finds that

$$B_K(p, R) = p + (1 - e^{-R})(K - p),$$

that is, the image of K by the dilation of ratio $(1 - e^{-R})$ with respect to p. We shall denote by λ the number $(1 - e^{-R})$ in what follows to simplify the exposition.

The Holmes–Thompson, sometimes also called symplectic, volume in Funk geometry can be defined as follows. Given a Lebesgue measure μ on $\mathbb{R}^n$, there exists a unique dual Lebesgue measure μ^* on the dual of $\mathbb{R}^n$ such that, for any basis $e = (e_1, \dots, e_n)$ and its dual one $e^* = (e_1^*, \dots, e_n^*)$, we get

$$\mu^*(e_1^* \wedge \cdots \wedge e_n^*) \cdot \mu(e_1 \wedge \cdots \wedge e_n) = 1.$$

Denote by K^p the polar body of K with respect to $p \in K$. Then, for any Borel set U in K, its Holmes–Thompson measure is given by

$$\mathrm{Vol}_K(U) := \int_U \frac{\mu^*(K^p)}{\omega_n} \,\mathrm{d}\mu(p),$$

where ω_n is the volume of the n-dimensional Euclidean unit ball.

Typically in computations, once coordinates $p = (p_1, \dots, p_n)$ are fixed, one takes $\mathrm{d}\mu(p) = \mathrm{d}\,\mathrm{Leb}_n(p) = dp_1 \cdots dp_n$, where $dp_i := p_i^*$, hence

$$\mathrm{Vol}_K(U) = \omega_n^{-1} \int_U |K^p| \,\mathrm{d}p \quad \text{with } |K^p| = \int_{K^p} dp_1^* \cdots dp_n^*.$$

2.2 Faces and flags

A *face* of a convex polytope P is the intersection of the polytope with a closed halfspace such that no point of the interior of the polytope lies on the boundary of the halfspace. Note that the polytope itself is a face, as is the empty set. The dimension of a face is the dimension of the affine subspace it generates. As usual, $(n-1)$-dimensional faces are called *facets*, 0-dimensional ones *vertices* and 1-dimensional ones *edges*.

The set of faces of a polytope is a partially ordered set with the inclusion relation. In fact, it forms a lattice, called the *face lattice*.

A *flag* of P is a sequence $f = (f_0, f_1, \dots, f_n)$ of faces such that, for all $i \in \{0, \dots, n\}$, the face f_i is an i-dimensional face, and, for $i \in \{0, \dots, n-1\}$, we have $f_i \subset f_{i+1}$. We denote the set of flags of P by $\mathrm{Flags}(P)$.

Polytopes have the following so-called *diamond property*: if F and F' are two faces of dimensions $i-1$ and $i+1$, respectively, and $F \subset F'$, then there are exactly two faces G of dimension i such that $F \subset G \subset F'$. For each $i \in \{0, \dots, n-1\}$, define the map $r_i : \mathrm{Flags}(P) \to \mathrm{Flags}(P)$ in the following way. For each flag f of P, and $j \in \{-1, \dots, n\}$, let

$$(r_i f)_j := \begin{cases} G' & \text{if } j = i, \\ f_j & \text{otherwise,} \end{cases}$$

where G' is the unique face of dimension i distinct from f_i satisfying $f_{i-1} \subset G' \subset f_{i+1}$.

The two flags $r_i f$ and f are said to be *i-adjacent*. The group $\langle r_0, \dots, r_{n-1} \rangle$ generated by these maps is known as the *monodromy group* or *connection group* of the polytope P.

2.3 Taking the polar with respect to different points

We have the following relationship between the polar of a convex body at the origin and its polar at another point.

Lemma 2.1 *Let K be a convex body, and x a point in* $\mathrm{int}\, K$. *Then,*

$$K^x := (K - x)^\circ = \left\{ \frac{w}{1 - \langle w, x \rangle} \;\middle|\; w \in K^\circ \right\}.$$

3 Properties of the Holmes–Thompson volume in Funk geometry

While the Funk geometry is only an affine invariant, many of its aspects are more naturally considered within projective geometry, as was observed in [6]. In particular, the associated Holmes–Thompson volume exhibits two projective invariance properties. Firstly, it is invariant under collineations.

Proposition 3.1 *If $g\colon \mathbb{RP}^n \to \mathbb{RP}^n$ is a collineation, and $K \subset \mathbb{R}^n$ is a convex body in an affine chart such that $gK \subset \mathbb{R}^n$, then $\mathrm{Vol}_K(U) = \mathrm{Vol}_{gK}(gU)$ for any Borel subset $U \subset K$.*

Secondly, the Holmes–Thompson volume in Funk geometry is invariant under projective polarity [6, Proposition 7.2]. This can be stated as follows.

Proposition 3.2 *If K and L are convex bodies in $\mathbb{R}^n$, with $K \subset \mathrm{int}(L)$, and $p \in \mathrm{int}(K)$, then $\mathrm{Vol}_L(K) = \mathrm{Vol}_{K^p}(L^p)$.*

We now establish a multiplicative property of the volume in Funk geometry.

Lemma 3.3 *If $K \subset \mathbb{R}^k$ and $L \subset \mathbb{R}^l$ are convex bodies, then*

$$(k+l)!\,\omega_{k+l}\mathrm{Vol}_{K\times L}\big(B_{K\times L}((a,b),R)\big) = \big(k!\,\omega_k \mathrm{Vol}_K(B_K(a,R))\big)\big(l!\,\omega_l \mathrm{Vol}_L(B_L(b,R))\big). \tag{3}$$

Proof For convex bodies $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^l$, we define the convex body

$$A \times_1 B = \{(tx,(1-t)y) \mid x \in A, y \in B, 0 \le t \le 1\} \subset \mathbb{R}^{k+l}.$$

Assuming $0 \in A$ and $0 \in B$, it has Lebesgue volume

$$|A \times_1 B| = \frac{k!\,l!}{(k+l)!}|A||B|.$$

Write $\lambda = 1 - e^{-R}$, and assume for simplicity that $a = b = 0$. It holds, for all (x,y) in the interior of $K \times L$, that $(K\times L)^{(x,y)} = K^x \times_1 L^y$, and so

$$\begin{aligned}\omega_{k+l}\mathrm{Vol}_{K\times L}\big(B_{K\times L}((a,b),R)\big) &= \int_{\lambda(K\times L)} |(K\times L)^{(x,y)}|\,\mathrm{d}x\,\mathrm{d}y = \int_{\lambda K\times\lambda L} |K^x \times_1 L^y|\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{k!\,l!}{(k+l)!}\int_{\lambda K}|K^x|\,\mathrm{d}x \int_{\lambda L}|L^y|\,\mathrm{d}y,\end{aligned}$$

concluding the proof. □

We can now calculate the volume of Funk balls in Hanner polytopes.

Lemma 3.4 *Let H be a Hanner polytope in $\mathbb{R}^n$ centered at the origin, and let $R > 0$ and $\lambda = 1 - e^{-R}$. Then,*

$$\mathrm{Vol}_H(B_H(R)) = \frac{2^n}{n!\,\omega_n}\left(\log\frac{1+\lambda}{1-\lambda}\right)^n = \frac{2^n}{n!\,\omega_n}(\log(2e^R - 1))^n.$$

Proof An elementary calculation shows that, for $Q = [-1, 1] \subset \mathbb{R}$, we have

$$\omega_1 \operatorname{Vol}_Q(B_Q(0, R)) = 2 \log \frac{1+\lambda}{1-\lambda}.$$

The conclusion follows on applying Proposition 3.2 and Lemma 3.3, since any Hanner polytope can be constructed from intervals by taking products and polars. □

4 The Mahler inequality for unconditional bodies

It is convenient to extend the Funk Holmes–Thompson volume to functions, as was done in [6]. For a convex function $\phi\colon V \to \mathbb{R} \cup \{+\infty\}$ defined on $V = \mathbb{R}^n$, and $R > 0$, set

$$\widetilde{M}_\lambda(e^{-\phi}) = \frac{\lambda^n}{n!} \int_{V \times V^*} e^{-\phi(x) - \mathscr{L}\phi(\xi) + \lambda\langle x, \xi\rangle}\, \mathrm{d}x\, \mathrm{d}\xi,$$

where $\lambda = 1 - e^{-R}$, and $\mathscr{L}$ is the *Legendre transform* given by

$$\mathscr{L}\phi(\xi) = \sup_{x \in V} (\langle x, \xi\rangle - \phi(x)).$$

Clearly $\widetilde{M}_\lambda(e^{-\phi}) = \widetilde{M}_\lambda(e^{-\mathscr{L}\phi})$. Note also that $\mathbb{1}_K^\infty := -\log \mathbb{1}_K$ is a convex function when K is convex.

The following was verified in [6, Lemma 8.5].

Lemma 4.1 *For a convex body K and $0 < \lambda < 1$, one has*

$$\omega_n \operatorname{Vol}_K(\lambda K) = \widetilde{M}_\lambda(\mathbb{1}_K) = \widetilde{M}_\lambda(e^{-h_K}),$$

where h_K is the support function of K.

Proof The first equality is [6, Lemma 8.5]. Since $\mathscr{L}(\mathbb{1}_K^\infty) = h_K$, we get the second one. □

Theorem 4.2 *Let $K \subset \mathbb{R}^n$ be an unconditional convex body, and $H \subset \mathbb{R}^n$ a Hanner polytope centered at the origin. Then, for any $0 < R < \infty$, we have*

$$\operatorname{Vol}_K(B_K(R)) \geq \operatorname{Vol}_H(B_H(R)).$$

Equality is uniquely attained by Hanner polytopes.

Proof Consider an unconditional convex function $\phi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

Unconditional here means again that it is invariant under reflections in the coordinate hyperplanes. Define, for each $j \geq 0$,

$$I_{2j}(e^{-\phi}) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \langle x, \xi\rangle^{2j}\, e^{-\phi(x) - \mathscr{L}\phi(\xi)}\, \mathrm{d}x\, \mathrm{d}\xi.$$

Due to the unconditionality of ϕ, we have

$$I_{2j}(e^{-\phi}) = \sum_{|I|=j} \binom{2j}{2I} \int_{\mathbb{R}^n} x^{2I} e^{-\phi(x)}\,\mathrm{d}x \int_{\mathbb{R}^n} \xi^{2I} e^{-\mathscr{L}\phi(\xi)}\,\mathrm{d}\xi$$
$$= 4^n \sum_{|I|=j} \binom{2j}{2I} \int_{\mathbb{R}^n_+} x^{2I} e^{-\phi(x)}\,\mathrm{d}x \int_{\mathbb{R}^n_+} \xi^{2I} e^{-\mathscr{L}\phi(\xi)}\,\mathrm{d}\xi,$$

where the sum is over all n-tuples $I = (i_1, \dots, i_n)$ of nonnegative integers with $|I| := \sum_{k=1}^n i_k = j$. We are using here the notation $x^{2I} := \prod_{k=1}^n x_k^{2i_k}$ and

$$\binom{2j}{2I} := \frac{(2j)!}{(2i_1)!\cdots(2i_n)!}.$$

By the inequality of Fradelizi and Meyer [7, Theorem 6], we have

$$\int_{\mathbb{R}^n_+} x^{2I}\, e^{-\phi(x)}\,\mathrm{d}x \int_{\mathbb{R}^n_+} \xi^{2I}\, e^{-\mathscr{L}\phi(\xi)}\,\mathrm{d}\xi \geq \prod_{k=1}^n \frac{\Gamma(2i_k+2)}{(2i_k+1)^2}, \tag{4}$$

where Γ denotes Euler's gamma function. Equality, for any given j and I, occurs (nonuniquely) when $\phi(x) = \|x\|_{\ell_1^n} := \sum |x_k|$, the support function of the cube $[-1,1]^n$.

Denoting $\lambda = 1 - e^{-R}$, we have $B_K(R) = \lambda K$. Since $K = -K$, we have by Lemma 4.1,

$$\omega_n \operatorname{Vol}_K(\lambda K) = \widetilde{M}_\lambda(\mathbb{1}_K) = \frac{\lambda^n}{n!} \int_{K \times \mathbb{R}^n} e^{-h_K(\xi) + \lambda\langle x, \xi\rangle}\,\mathrm{d}x\,\mathrm{d}\xi$$
$$= \frac{\lambda^n}{n!} \int_{K \times \mathbb{R}^n} e^{-h_K(\xi)} \cosh(\lambda\langle x, \xi\rangle)\,\mathrm{d}x\,\mathrm{d}\xi$$
$$= \frac{1}{n!} \sum_{j=0}^\infty \frac{1}{(2j)!} I_{2j}(e^{-h_K}) \lambda^{n+2j}.$$

The last equality here is obtained by expanding the hyperbolic cosine as a power series, and using that $I_{2j}(\exp(-\mathbb{1}_K^\infty)) = I_{2j}(\exp(-h_K))$.

Thus all summands in the series for $\operatorname{Vol}_K(\lambda K)$ are simultaneously minimized when h_K is the support function of the cube. That is, among all unconditional convex bodies in $\mathbb{R}^n$, the cube minimizes $\operatorname{Vol}_K(\lambda K)$. Applying Lemma 3.4, we see that the minimum is also attained by any Hanner polytope.

For the equality cases, it follows from the proof that equality must hold in equation (4) for every I, in particular when $I = (0, \dots 0)$, so that

$$\int_{\mathbb{R}^n_+} e^{-\mathbb{1}_K^\infty(x)}\,\mathrm{d}x \int_{\mathbb{R}^n_+} e^{-h_K(\xi)}\,\mathrm{d}\xi = 1,$$

that is,

$$\frac{n!}{4^n}|K||K^\circ| = 1.$$

The minimizers of the volume product among unconditional convex bodies were shown to coincide with the Hanner polytopes by Meyer [14] and Reisner [17]. This concludes the proof. □

5 A decomposition of the volume

In this section and the next, we determine the two highest-order terms of growth of the balls in the Funk geometry on a polytope P. Our strategy will be to decompose both P and its polar into flag simplices, and to consider separately each pairing of a flag simplex with a dual flag simplex.

To this end, for each face F of P, we make a choice of point $p(F)$ in the relative interior of F. When F is a vertex, this choice of course can only be the vertex itself. When $F = P$, we choose $p(P)$ to be the origin. Our choice of points leads to a flag decomposition of P: to each flag f of P is associated a flag simplex, which is the convex hull of the points $p(f_0), \dots, p(f_n)$. The interiors of these simplices are pairwise disjoint, and their union equals P.

Similarly, for each face G of P°, we make a choice of point $q(G)$ in the relative interior of G. Again, this leads to a flag decomposition of P°.

For each flag f of P, denote by $\Delta^f := \operatorname{conv}\{p(f_0), \dots, p(f_n)\}$ the associated flag simplex. We also define, for each flag f and $\tau \in (0,1)$, the truncated simplex $\Delta^f_\tau := (1-\tau)\Delta^f$. Recall here that $p(P) = 0$. For each dual flag $g \in \operatorname{Flags}(P^\circ)$ and $i \in \{0, \dots, n-1\}$, let $x^g_i(\cdot) := 1 - \langle q(g_i), \cdot \rangle$. The functions x^g_i provide a set of coordinates on $\mathbb{R}^n$, for any given dual flag g. We denote by Leb_n the n-dimensional Lebesgue measure.

Lemma 5.1 *The Holmes–Thompson volume of the ball $B_P(0,R)$ in the Funk geometry is given, for $\tau := \exp(-R)$, by*

$$\operatorname{Vol}(B_P(0,R)) = \frac{1}{n!\,\omega_n} \sum_{f \in \operatorname{Flags}(P)} \sum_{g \in \operatorname{Flags}(P^\circ)} \int_{\Delta^f_\tau} \frac{\mathrm{d}x^g_0 \cdots \mathrm{d}x^g_{n-1}}{x^g_0 \cdots x^g_{n-1}}.$$

Proof The volume of the ball is

$$\operatorname{Vol}(B_P(0,R)) = \frac{1}{\omega_n} \int_{B_P(0,R)} |P^y| \,\mathrm{dLeb}_n(y),$$

where P^y denotes the polar of P with respect to the point y. Observe that $B_P(0,R)$ is the union of the simplices Δ^f_τ, and that the interiors of these simplices do not intersect. We conclude that

$$\operatorname{Vol}(B_P(0,R)) = \frac{1}{\omega_n} \sum_{f \in \operatorname{Flags}(P)} \int_{\Delta^f_\tau} |P^y| \,\mathrm{dLeb}_n(y). \tag{5}$$

The n-dimensional Lebesgue volume of a simplex with vertices $v_0, \dots, v_{n-1}, 0$ is $|v_0 \wedge \cdots \wedge v_{n-1}|/n!$. So, we have

$$|P^\circ| = \frac{1}{n!} \sum_{g \in \operatorname{Flags}(P^\circ)} |q(g_0) \wedge \cdots \wedge q(g_{n-1})|.$$

As we vary y over int P, the shape of the polar P^y changes according to Lemma 2.1. Note that the combinatorics stay the same. In fact, the change of shape is always by a projective map. For each face G of P° and point y in int P, let

$$q^y(G) := \frac{q(G)}{1-\langle q(G), y\rangle}.$$

Then, the $q^y(G)$ define a flag decomposition of P^y. So, we may write

$$\begin{aligned} |P^y| &= \frac{1}{n!} \sum_{g \in \mathrm{Flags}(P^\circ)} |q^y(g_0) \wedge \cdots \wedge q^y(g_{n-1})| \\ &= \frac{1}{n!} \sum_{g \in \mathrm{Flags}(P^\circ)} \frac{|q(g_0) \wedge \cdots \wedge q(g_{n-1})|}{(1-\langle q(g_0), y\rangle) \cdots (1-\langle q(g_{n-1}), y\rangle)}. \end{aligned} \tag{6}$$

Changing to the coordinates x_i^g changes the measure in the following way:

$$\mathrm{d}x_0^g \cdots \mathrm{d}x_{n-1}^g = |q(g_0) \wedge \cdots \wedge q(g_{n-1})| \, \mathrm{dLeb}_n.$$

Combining this with (5) and (6) gives the result. □

So we see that the Funk volume can be computed by calculating an integral of a relatively simple function over a truncated flag simplex with respect to a coordinate system coming from the dual flag simplex. This is done for every combination of flag simplex and dual flag simplex, and the results are added.

We will see that the highest-order term as R tends to infinity is determined by the flag simplices combined with their duals; the second-highest order term is determined by these pairs as well as pairs where the flag simplex and the dual flag simplex are nearest neighbors, in the sense that the associated flags differ by exactly one face; in the following sections we calculate the contributions of the relevant pairs.

Let us fix notation that will be used throughout this section. Recall that $f(x) = O(x)$ means that $f(x)/x$ is bounded, whereas $f(x) = o(x)$ means that $f(x)/x$ converges to zero as x tends to infinity.

Recall that we have the following elementary integral: for $a \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\int \frac{(\log x + a)^k}{x} \, \mathrm{d}x = \frac{1}{k+1} (\log x + a)^{k+1} + C. \tag{7}$$

For each $k \in \mathbb{N}$, define the function $F^k : \mathbb{R}^n_{>0} \to \mathbb{R}$ by

$$F^k(x) := \frac{(\log x_n - \log \tau)^k}{x_1 \cdots x_n}.$$

Here $\mathbb{R}_{>0}$ is the set of positive real numbers. We use the convention that $\mathbb{N}$, the set of natural numbers, contains 0.

5.1 Combining a flag simplex and its own dual

We start off by considering the case where the flag simplex and the dual flag simplex are associated to the same flag. These pairs will be seen to contribute to both the R^n and R^{n-1} terms.

Let

$$\Delta := \operatorname{conv}\{v_0, \dots, v_n\} \tag{8}$$

be an n-dimensional simplex with vertices

$$v_0 := (0, 0, \dots, 0), \quad v_1 := (v_{1,1}, 0, \dots, 0), \quad \dots, \quad v_n := (v_{n1}, v_{n2}, \dots, v_{nn}).$$

All nonzero coordinates are assumed positive.

Denote by Δ_τ the set $\Delta \backslash (\mathbb{R}^{n-1} \times [0, \tau))$, in other words, the subset of Δ where the last coordinate is greater than or equal to τ.

We are interested in the growth of the integral of F^k over Δ_τ, as τ decreases. It is actually the case $k = 0$ that we will use later, but it is important for the inductive proof to consider more general k.

We will need the following two lemmas. The first shows that, by scaling the coordinates, we may fit the simplex Δ into the "ordered simplex".

Lemma 5.2 *Let Δ be a simplex of the form (8). Then, there exist positive real numbers a_i such that*

$$0 \le a_n x_n \le \dots \le a_1 x_1 \le 1 \quad \text{for all } x \in \Delta.$$

Proof First, choose $a_1 > 0$ so that $a_1 v_{i1} \le 1$ for all i. Then, choose $a_2 > 0$ so that $a_2 v_{i2} \le a_1 v_{i1}$ for all i. Continue in this way until all the a_i are chosen. It is clear that any convex combination of the v_i then satisfies the required relations. □

The next lemma shows that, when we restrict our attention to a region where one of the coordinates is bounded below, we get a bound on the growth rate of the integral of F^k.

Lemma 5.3 *Let $U(R)$ be the integral of F^k over the intersection of Δ_τ with the region $\{x_j \ge \beta\}$, where $j \in \{1, \dots, n-1\}$ and $\beta > 0$, with $\tau = \exp(-R)$. Then, $U(R) = O(R^{n+k-j})$.*

Proof Using Lemma 5.2, we see that the domain of integration is contained in

$$[\alpha, \gamma]^j \times [\delta\tau, \gamma]^{n-j-1} \times [\tau, \gamma],$$

with δ, α and γ positive real numbers. The integrand can be integrated easily over this set with the help of (7), and thus we get the following bound:

$$U(R) \le (\log\gamma - \log\alpha)^j (\log\gamma - \log\delta + R)^{n-j-1} \frac{(\log\gamma + R)^{k+1}}{k+1}. \qquad \square$$

Lemma 5.4 *Let Δ be of the form (8), and let Δ_τ be defined as above. For each $R > 0$ and $k \in \mathbb{N}$, write*

$$V(R) := \frac{1}{k!}\int_{\Delta_\tau} F^k \,\mathrm{dLeb}_n,$$

where $\tau = \exp(-R)$. Then, for all $k \in \mathbb{N}$,

$$V(R) = \frac{R^{n+k}}{(n+k)!} + \frac{R^{n+k-1}}{(n+k-1)!}\left(\sum_{i=1}^{n}\log v_{i,i} - \sum_{i=1}^{n-1}\log v_{i+1,i}\right) + o(R^{n+k-1}).$$

Proof We use induction on the dimension n. When $n = 1$, the integral can be evaluated exactly using (7), and we get that

$$V(R) = \frac{(\log v_{1,1} + R)^{k+1}}{(k+1)!}.$$

So, we see that the statement of the lemma is true in this case.

Assume that $n \geq 2$ and that the statement is true when the dimension is $n-1$. Choose $\alpha > 0$ such that $\alpha < v_{i,1}$ for all $1 \leq i \leq n$. Decompose the integral into two parts, where $x_1 \geq \alpha$, and where $x_1 < \alpha$. Consider the simplices $\Delta' = \Delta \cap \{x_1 = \alpha\}$ and $\Delta'_\tau = \Delta_\tau \cap \{x_1 = \alpha\}$. The vertices of Δ' are

$$v'_1 := (\alpha, 0, \dots, 0), \quad \dots, \quad v'_n := \left(\alpha, \alpha\frac{v_{n,2}}{v_{n,1}}, \dots, \alpha\frac{v_{n,n}}{v_{n,1}}\right).$$

Let $\widehat{\Delta}'$ and $\widehat{\Delta}'_\tau$ be the projections of Δ' and Δ'_τ, respectively, to the last $n-1$ coordinates.

Consider the part of Δ_τ with $x_1 \leq \alpha$. This part can be written

$$\left\{(x_1, x_2, \dots, x_n) \;\middle|\; 0 \leq x_1 \leq \alpha,\ \left(\frac{x_2\alpha}{x_1}, \dots, \frac{x_n\alpha}{x_1}\right) \in \widehat{\Delta}',\ x_n \geq \tau\right\}.$$

We introduce new coordinates $y_j := x_j\alpha/x_1$ for $j \in \{2, \dots, n\}$. In the coordinates $x_1, y_2, \dots, y_n$, the above set takes the form

$$\Delta_\tau \cap \{x_1 \leq \alpha\} = \left\{\frac{\tau\alpha}{y_n} \leq x_1 \leq \alpha,\ y \in \widehat{\Delta}'_\tau\right\}.$$

Thus, we can write the contribution to $V(R)$ of this piece as

$$\frac{1}{k!}\int_{\widehat{\Delta}'_\tau}\int_{\tau\alpha/y_n}^{\alpha}\left(\frac{x_1}{\alpha}\right)^{n-1} F^k\left(x_1, \frac{x_1}{\alpha}y_2, \dots, \frac{x_1}{\alpha}y_n\right) \mathrm{d}x_1 \,\mathrm{dLeb}_{n-1}$$
$$= \frac{1}{k!}\int_{\widehat{\Delta}'_\tau}\int_{\tau\alpha/y_n}^{\alpha} \frac{(\log y_n + \log x_1 - \log\alpha - \log\tau)^k}{x_1 y_2 \cdots y_n}\,\mathrm{d}x_1 \,\mathrm{dLeb}_{n-1}.$$

We do the inner integral using (7), and then use the induction hypothesis:

$$\frac{1}{(k+1)!}\int_{\widetilde{\Delta}'_\tau} \frac{(\log y_n + \log x_1 - \log\alpha - \log\tau)^{k+1}}{y_2\cdots y_n}\bigg|_{\tau\alpha/y_n}^{\alpha} \mathrm{dLeb}_{n-1}$$
$$= \frac{1}{(k+1)!}\int_{\widetilde{\Delta}'_\tau} \frac{(\log y_n - \log\tau)^{k+1}}{y_2\cdots y_n}\,\mathrm{dLeb}_{n-1}$$
$$= \frac{R^{n+k}}{(n+k)!} + \frac{R^{n+k-1}}{(n+k-1)!}\bigg(\sum_{i,j\ge 2: i=j} \log v'_{ij} - \sum_{i,j: i=j+1} \log v'_{ij}\bigg) + o(R^{n+k-1}).$$

The sum of the logarithm terms within the parentheses can be re-expressed as

$$\log\frac{\alpha}{v_{2,1}} + \sum_{i,j\ge 2: i=j} \log v_{ij} - \sum_{i,j\ge 2: i=j+1} \log v_{ij}.$$

Here we have used that $v'_{ij} = \alpha v_{ij}/v_{i1}$ for all i and j with $i \ge 2$.

Now we turn our attention to the part of Δ_τ with $x_1 \ge \alpha$. Let $\widehat{\Delta}$ be the projection of Δ onto the last $n-1$ coordinates. Choose small $\epsilon > 0$. By Lemma 5.3, we may neglect any region outside of the set

$$K_\beta := \{x \in \mathbb{R}^n \mid x_j \le \beta \text{ for } 2 \le j \le n-1\},$$

where β is an arbitrary positive number. The definition of Δ in (8) implies that, by taking β small enough, we can ensure that

$$P_1 \subset K_\beta \cap \Delta \cap \{x \in \mathbb{R}^n \mid \alpha \le x_1\} \subset P_2,$$

where

$$P_1 := K_\beta \cap ([\alpha, v_{1,1} - \epsilon] \times \widehat{\Delta}) \quad\text{and}\quad P_2 := K_\beta \cap ([\alpha, v_{1,1} + \epsilon] \times \widehat{\Delta}).$$

Denote by P_1^τ, P_2^τ and $\widehat{\Delta}_\tau$ the intersections of the set $\{x \in \mathbb{R}^n \mid x_n \ge \tau\}$ with, respectively, P_1, P_2 and $\widehat{\Delta}$.

Let $K'_\beta := \{(x_j)_{j=2}^n \in \mathbb{R}^{n-1} \mid x_j \le \beta \text{ for } 2 \le j \le n-1\}$ be the projection of K_β onto the last $n-1$ coordinates.

Using the induction hypothesis together with Lemma 5.3,

$$\frac{1}{k!}\int_{P_1^\tau} F^k(x)\,\mathrm{dLeb}_n = \bigg(\int_{[\alpha, v_{1,1}-\epsilon]} \frac{1}{x_1}\,\mathrm{d}x_1\bigg)\bigg(\frac{1}{k!}\int_{K'_\beta\cap\widehat{\Delta}_\tau} \frac{(\log x_n - \log\tau)^k}{x_2\cdots x_n}\,\mathrm{dLeb}_{n-1}\bigg)$$
$$= \Big(\log(v_{1,1}-\epsilon) - \log\alpha\Big)\bigg(\frac{R^{n+k-1}}{(n+k-1)!} + O(R^{n+k-2})\bigg).$$

When the integral is over P_2^τ instead of P_1^τ, a similar equation holds with the sign of ϵ changed. Since ϵ is arbitrarily small, we conclude that

$$\frac{1}{k!}\int_{K_\beta\cap\Delta_\tau\cap\{x\in\mathbb{R}^n\mid\alpha\le x_1\}} F^k(x)\,\mathrm{dLeb}_n = (\log v_{1,1} - \log\alpha)\frac{R^{n+k-1}}{(n+k-1)!} + o(R^{n+k-1}).$$

Combining this equation with the previous calculation, we get the result. □

The log terms in the statement of Lemma 5.4 follow the following pattern, shown here for $n=3$:

$$\begin{pmatrix} 0 & 0 & 0 \\ + & 0 & 0 \\ - & + & 0 \\ \cdot & - & + \end{pmatrix}.$$

The plus and minus signs represent the sign of the log term in the expression; a dot means that the term is not present, and 0 stands for a null coordinate. Note that when the vertices of Δ are of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ a & b & 0 \\ a & b & c \end{pmatrix},$$

the proof of the Lemma 5.4 can be adapted to give an exact formula for $V(R)$:

$$V(R)=\frac{(R+\log c)^{n+k}}{(n+k)!}.$$

5.2 Combining a flag simplex with the dual of a neighbor

In this section, we take Δ to be of the same form as in equation (8), except with a single additional positive entry on the upper diagonal. That is, the entry $v_{i,i+1}$ is now positive for some $i\in\{0,\dots,n-1\}$. As before, let Δ_τ be the set $\Delta\backslash(\mathbb{R}^{n-1}\times[0,\tau))$ and, for each $R>0$ and $k\in\mathbb{N}$, write

(9) $$V_\Delta(R):=\frac{1}{k!}\int_{\Delta_\tau}F^k(x)\,\mathrm{dLeb}_n,$$

where $\tau=\exp(-R)$. Sometimes we write simply $V(R)$ when the simplex is clear from context.

Lemma 5.5 *If the extra positive entry is $v_{0,1}$, then, for all $k\in\mathbb{N}$,*

(10) $$V(R)=\frac{R^{n+k-1}}{(n+k-1)!}\cdot\left|\log\frac{v_{0,1}}{v_{1,1}}\right|+o(R^{n+k-1}).$$

If, on the other hand, the extra positive entry is $v_{i,i+1}$, with $i\in\{1,\dots,n-1\}$, then

(11) $$V(R)=\frac{R^{n+k-1}}{(n+k-1)!}\cdot\left|\log\left(\frac{v_{i,i}}{v_{i,i+1}}\frac{v_{i+1,i+1}}{v_{i+1,i}}\right)\right|+o(R^{n+k-1}).$$

Proof Again, we use induction on the dimension n. When $n=1$, we can calculate the integral exactly using (7), and get that, for sufficiently small τ,

$$V(R)=\frac{1}{(k+1)!}|(\log v_{0,1}+R)^{k+1}-(\log v_{1,1}+R)^{k+1}|.$$

This implies the statement of the lemma, in the case where $n=1$.

Now, assume that $n\geq 2$ and that the statement is true when the dimension is $n-1$. There are two possibilities to consider.

Case 1 ($v_{0,1} > 0$) For simplicity, we assume that $v_{0,1} > v_{1,1}$; the proof is similar when the inequality is in the other direction. The simplex Δ can be expressed as the difference (up to a set of measure zero) of the two simplices

$$\widetilde{\Delta} := \operatorname{conv}\{0, v_0, v_2, \dots, v_n\} \quad \text{and} \quad \widetilde{\Delta}' := \operatorname{conv}\{0, v_1, v_2, \dots, v_n\}.$$

Both simplices are of the form assumed in Lemma 5.4. Applying this lemma to each of them and subtracting the results gives (10).

Case 2 ($v_{i,i+1} > 0$ for some $i \geq 1$) As in the proof of Lemma 5.4, choose $\alpha > 0$ such that α is less than each of the first coordinates of the nonzero vertices of Δ.

Let Δ' be the intersection of the hyperplane $\{x_1 = \alpha\}$ with the simplex Δ. Observe that the vertices of Δ' are exactly the vertices of Δ (apart from the origin) rescaled so as to have the first coordinate equal to α. Denote these vertices by v'_i, where $i \in \{1, \dots, n\}$. Also, let Δ'_τ be the intersection of $\{x_1 = \alpha\}$ with the set $\Delta_\tau := \Delta \backslash \{x_n < \tau\}$, which is not necessarily a simplex.

Look at the part Δ_α^- of Δ_τ on the same side of the hyperplane as the origin. Just as in the proof of Lemma 5.4, we do a change of coordinates and perform a one-dimensional integration to get that the contribution of this part is

$$\frac{1}{k!} \int_{\Delta_\alpha^-} F^k(x) \, \mathrm{dLeb}_n = \frac{1}{(k+1)!} \int_{\Delta'_\tau} \frac{(\log x_n - \log \tau)^{k+1}}{x_2 \cdots x_n} \, \mathrm{dLeb}_{n-1}.$$

Now we use the induction hypothesis. If $v'_{1,2} > 0$, we get that the contribution is

$$\frac{R^{n+k-1}}{(n+k-1)!} \cdot \left| \log \frac{v'_{1,2}}{v'_{2,2}} \right| + o(R^{n+k-1}).$$

Otherwise, we get

$$\frac{R^{n+k-1}}{(n+k-1)!} \cdot \left| \log \left(\frac{v'_{i,i}}{v'_{i,i+1}} \frac{v'_{i+1,i+1}}{v'_{i+1,i}} \right) \right| + o(R^{n+k-1}),$$

where the extra positive entry is $v'_{i,i+1}$. Recalling that $v'_{i,j} = \alpha v_{i,j} / v_{i,1}$ for each i and j, we get in both cases that the right-hand side of (11) is a lower bound on $V(R)$.

Now we want to show that the same expression is also an upper bound. Choose $\beta > 0$ such that β is greater than each of the first coordinates of the nonzero vertices of Δ. Let Δ'' be the projection of Δ onto the hyperplane $\{x_1 = \beta\}$ along rays passing through the origin. In other words, Δ'' is the intersection of that hyperplane with the cone over Δ'. Observe that the vertices of Δ'' are exactly the points $(\beta/\alpha) v_i$ with $i \in \{1, \dots, n\}$. Another important thing to note is that $\operatorname{conv}(\{0\} \cup \Delta'')$ contains Δ. So we get an upper bound on $V(R)$ by calculating the integral in (9) with Δ_τ replaced by $\operatorname{conv}(\{0\} \cup \Delta'') \cap \{x_n \geq \tau\}$. This calculation is formally the same as the one we have just done, with α replaced by β, and it has the same answer. □

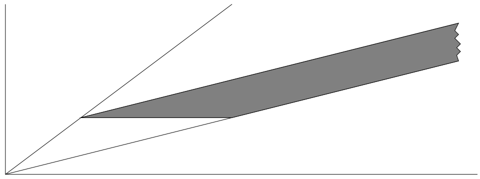

Figure 1: The integral of $1/xy$ over the infinite gray region is finite.

Let $\Phi_\tau : \mathbb{R}^n \to \mathbb{R}^n$ be the map $\Phi_\tau(x) := (1-\tau)x + \tau v_n$ that shrinks by a factor $1-\tau$ towards the point v_n. Write $\widetilde{\Delta}_\tau := \Phi_\tau(\Delta)$. For each $R > 0$ and $k \in \mathbb{N}$, write

$$\widetilde{V}_\Delta(R) := \frac{1}{k!} \int_{\widetilde{\Delta}_\tau} F^k(x)\,\mathrm{dLeb}_n,$$

where $\tau = \exp(-R)$. Again we omit the Δ when the simplex is clear from context. Observe that if the n^{th} coordinate of v_n is 1, and the n^{th} coordinate of every other v_i is zero, then $\Delta_\tau = \widetilde{\Delta}_\tau$ and so $V_\Delta(R) = \widetilde{V}_\Delta(R)$ for all $R > 0$.

The case when the extra positive entry is the coordinate $v_{n-1,n}$ presents a special difficulty, because in this case the region we must integrate over to compute the Funk volume is not Δ_τ but instead $\widetilde{\Delta}_\tau$. We show however that, for our purposes, the difference between the two integrals is unimportant because it is of order R^{n-2}. We first consider the two-dimensional case.

Lemma 5.6 *Let m_1, m_2 and τ be positive real numbers, with $m_1 < m_2$. Define the region*

$$A_\tau := \Big\{(x,y) \subset \mathbb{R}^2 \;\Big|\; \tau \le y,\, m_1 x \le y \text{ and } m_1\Big(x - \frac{\tau}{m_2}\Big) \ge y - \tau\Big\}. \tag{12}$$

Then, the integral over A_τ of $1/xy$ against the Lebesgue measure is finite. Moreover, it depends on m_1 and m_2, but not on τ.

The region A_τ is illustrated in Figure 1.

Proof Observe that the vertical distance between the two parallel lines bounding A_τ is $h := \tau(1 - m_1/m_2)$. Since $y \ge m_1 x$ for $(x,y) \in A_\tau$, and the integrand is decreasing in y, we have the following bound:

$$\int_{A_\tau} \frac{1}{xy}\,\mathrm{d}y\,\mathrm{d}x \le \int_{\tau/m_2}^{\infty} \int_{m_1 x}^{m_1 x + h} \frac{1}{m_1 x^2}\,\mathrm{d}y\,\mathrm{d}x = \frac{-h}{m_1 x}\Big|_{\tau/m_2}^{\infty} = \frac{m_2}{m_1} - 1.$$

That the integral is independent of τ follows from the invariance of the integrand under the map $(x,y) \mapsto (\alpha x, \alpha y)$ mapping A_τ to $A_{\tau/\alpha}$, for any $\alpha > 0$. □

We now consider the n-dimensional case. Note that $\widetilde{\Delta}_\tau \subset \Delta_\tau$ for all τ.

Lemma 5.7 *Assume that Δ is of the form (8), except that the entry $v_{n-1,n}$ is positive. Also, assume that $v_n = (1,\dots,1)$. Then,*

$$V_\Delta^*(R) := \int_{\Delta_\tau \setminus \widetilde{\Delta}_\tau} \frac{\mathrm{dLeb}_n}{x_1 \cdots x_n} = O(R^{n-2}).$$

Proof We treat the case where $v_{n-1,n-1} \geq v_{n-1,n}$. The other case can be handled similarly.

Let $x \in \Delta_\tau \backslash \widetilde{\Delta}_\tau$. Since $x \in \Delta_\tau$ we have

$$x_n \geq \tau \quad \text{and} \quad \frac{x_n}{v_{n-1,n}} \geq \frac{x_{n-1}}{v_{n-1,n-1}}.$$

Since $x \in \Delta \setminus \widetilde{\Delta}_\tau$, we have

$$\frac{x_n - \tau}{v_{n-1,n}} \leq \frac{x_{n-1} - \tau}{v_{n-1,n-1}}.$$

Let $W \subset \mathbb{R}^2$ be the set of pairs (x_{n-1}, x_n) such that these three inequalities hold.

Project Δ to the coordinate hyperplane $x_{n-1} = 0$ to obtain the simplex $\overline{\Delta}$. We may then apply Lemma 5.2 to $\overline{\Delta}$ and find positive constants $a_1, \dots, a_{n-2}, a_n$ such that, for all $x \in \Delta$, we have

$$a_n x_n \leq a_{n-2} x_{n-2} \leq \cdots \leq a_1 x_1 \leq 1.$$

In particular, for all $x \in \Delta_\tau$, we have

$$\tau \leq \frac{a_{n-2}}{a_n} x_{n-2} \leq \frac{a_{n-3}}{a_n} x_{n-3} \leq \cdots \leq \frac{a_1 x_1}{a_n} \leq \frac{1}{a_n}.$$

Let $\Delta'_\tau \subset \mathbb{R}^{n-2}$ be the set of points $(x_1, \dots, x_{n-2})$ satisfying this inequality. Observe that this set is a simplex.

Since $\Delta_\tau \backslash \widetilde{\Delta}_\tau$ is a subset of $\Delta'_\tau \times W$, we have

$$V^*_\Delta(R) \leq \int_{\Delta'_\tau} \frac{\mathrm{dLeb}_{n-2}}{x_1 \cdots x_{n-2}} \cdot \int_W \frac{\mathrm{dLeb}_2}{x_{n-1} x_n}.$$

The first factor grows like $O(R^{n-2})$ according to Lemma 5.4, and the second is bounded, by Lemma 5.6. □

5.3 Combining a flag simplex with the dual of a nonneighbor

Let the n-dimensional vectors v_i for $i \in \{0, \dots, n\}$ be such that $v_{ij} > 0$ for all $j \leq i$. Here j indexes the coordinates and ranges from 1 to n. Assume that $\Delta := \operatorname{conv}\{v_0, \dots, v_n\}$ is a nondegenerate n-dimensional simplex.

On $\mathbb{R}^n$ we use $\leq$ to denote the coordinatewise partial order, that is, $x \leq y$ if $x_j \leq y_j$ for all $j \in \{1, \dots, n\}$. If $A \subset \mathbb{R}^n$, we write $|A|$ for its Lebesgue volume.

Lemma 5.8 *Let Δ_1 and Δ_2 be nondegenerate simplices in $\mathbb{R}^n_+$ with vertices u_i for $i \in \{0, \dots, n\}$, and v_i for $i \in \{0, \dots, n\}$, respectively. Let $R > 0$, and take $k = 0$. If $u_i \leq v_i$ for all $i \in \{0, \dots, n\}$, then*

$$\frac{\widetilde{V}_{\Delta_1}(R)}{|\Delta_1|} \geq \frac{\widetilde{V}_{\Delta_2}(R)}{|\Delta_2|}.$$

Proof There exists an affine transformation T taking u_i to v_i for all i. Each $x \in \Delta_1$ can be expressed as a convex combination of the vertices $\{u_i\}$, and so Tx is a convex combination of the $\{v_i\}$ with the same coefficients. We conclude that $Tx \geq x$ for all $x \in \Delta_1$. Noting that F^0 is monotone decreasing and that T takes $\widetilde{\Delta}_1^\tau := \Phi_\tau(\Delta_1)$ to $\widetilde{\Delta}_2^\tau := \Phi_\tau(\Delta_2)$, we get

$$\frac{|\widetilde{\Delta}_1^\tau|}{|\widetilde{\Delta}_2^\tau|}\int_{\widetilde{\Delta}_2^\tau} F^0(x)\,\mathrm{dLeb}_n = \int_{\widetilde{\Delta}_1^\tau} F^0(Tx)\,\mathrm{dLeb}_n \leq \int_{\widetilde{\Delta}_1^\tau} F^0(x)\,\mathrm{dLeb}_n.$$

It just remains to observe that Φ^τ shrinks Lebesgue volume by the constant factor $(1-\tau)^n$. □

Lemma 5.9 *Let Δ be a nondegenerate simplex in $\mathbb{R}^n_+$ with vertices u_i for $i \in \{0,\dots,n\}$, such that $u_{i,j} > 0$ when $i \geq j$. Assume that there are two distinct vertices v_{p-1} and v_{q-1} of Δ, with $p, q \in \{1,\dots,n\}$, such that $v_{p-1,p}$ and $v_{q-1,q}$ are positive. Take $k = 0$. Then,*

$$\widetilde{V}_\Delta(R) = o(R^{n-1}).$$

Proof Let $\epsilon := \min\{v_{ij} \mid v_{ij} > 0\}$ be the smallest strictly positive coordinate amongst the vertices of Δ. By Lemma 5.8, it is enough to show that the result is true for the simplex obtained from Δ by replacing the lower triangular coordinates, that is, v_{ij} with $i \geq j$, by ϵ, the two diagonal coordinates $v_{p-1,p}$ and $v_{q-1,q}$ by $\epsilon/2$, and all the other coordinates by zero. We verify that this simplex is nondegenerate by subtracting v_0 from each of the other vectors, and calculating the determinant of the resulting matrix, that is, $\det(v_{ij} - v_{0j})_{1\leq i,j\leq n}$. This determinant equals $\epsilon^n/4$, as can be easily verified by separately considering the cases when $0 \in \{q-1, p-1\}$ and $0 \notin \{q-1, p-1\}$.

Thus we assume that Δ is of this new form. Now scale Δ so that all coordinates of all vertices lie in $\{0, \frac{1}{2}, 1\}$. Notice that $\widetilde{V}_\Delta$ does not change, as the integrand is invariant under diagonal linear maps.

Since we now have $v_{nn} = 1$, the inclusion $\widetilde{\Delta}_\tau \subset \Delta_\tau$ holds for each $\tau \in (0,1)$, and so

$$\widetilde{V}_\Delta(R) \leq V_\Delta(R) \quad \text{for all } R > 0. \tag{13}$$

By switching labels if necessary, we can assume that $q > p$. Let v' be the vector having the same coordinates as v_q, except for the q^{th} coordinate, which we set equal to 0. So,

$$v' = (1,\dots,1,0,0,\dots,0), \quad v_{q-1} = \left(1,\dots,1,\tfrac{1}{2},0,\dots,0\right), \quad v_q = (1,\dots,1,1,0,\dots,0).$$

Observe that the simplex Δ can be expressed as the difference (up to a set of measure zero) of the two simplices

$$\Delta_1 := \mathrm{conv}\{v_0,\dots,v_{q-2},v',v_{q-1},v_{q+1},\dots,v_n\},$$
$$\Delta_2 := \mathrm{conv}\{v_0,\dots,v_{q-2},v',v_q,v_{q+1},\dots,v_n\}.$$

Here Δ_1 is the simplex obtained from Δ by replacing the vertex v_q with the vertex v', and Δ_2 is obtained similarly by replacing v_{q-1} with v'. Both of these simplices are of the form assumed in Lemma 5.5. We apply this lemma to each of them, with $k = 0$, and see that the term of order $n-1$ is the same in each case. Subtracting, we get that $V_\Delta(R) = o(R^{n-1})$. That the same holds for $\widetilde{V}_\Delta(R)$ follows from (13). □

6 The contribution of a flag simplex

Recall that a flag decomposition of a polytope involves, for each face F of the polytope, a choice of point $p(F)$ in the relative interior of F. The polytope P is itself a face, and we choose $p(P)$ to be the origin, which we have assumed to be in the interior of P. We also take a flag decomposition of the dual polytope, and so for each face F we also have a dual point $q(F)$ in the relative interior of the dual face of F. To simplify notation, we write

$$L(F,G) := \log\bigl(1 - \langle q(F), p(G)\rangle\bigr)$$

for faces F and G of the polytope P. Observe that $L(F,G)$ is finite as long as G is not a subset of F. For a face F of P, by F° we mean the face of P° that is dual to F.

We will make use of the following basic fact from linear algebra: the space of linear antisymmetric 2-forms on V, which can be thought of as the space $(\wedge^2 V)^*$ of linear functionals on the exterior square of V, is naturally isomorphic to the exterior square $\wedge^2(V^*)$ of the dual. The corresponding pairing $\wedge^2 V^* \times \wedge^2 V \to \mathbb{R}$ is given on simple 2-vectors by

$$\langle v_1 \wedge v_2, u_1 \wedge u_2\rangle = \langle v_1, u_1\rangle\langle v_2, u_2\rangle - \langle v_1, u_2\rangle\langle v_2, u_1\rangle.$$

Lemma 6.1 *Fix $1 \le i \le n-2$. Let f be a flag of P, and $g := r_i f$. For $j \in \{i-1, i, i+1\}$ choose points $p(f_j)$ in the relative interior of f_j, and $q(g_j)$ in the relative interior of g_j°. Write*

$$u_j := p(f_j) - p(f_{j-1}) \quad \text{for } j \in \{i, i+1\},$$
$$v'_j := q(g_j) - q(g_{j+1}) \quad \text{for } j \in \{i, i-1\}.$$

Then $\langle v'_i \wedge v'_{i-1}, u_i \wedge u_{i+1}\rangle \ge 0$.

Proof Let E be the 2-dimensional affine plane through $q(g_j)$, where $j \in \{i-1, i, i+1\}$. It lies in the affine span of $g_{i-1}^\circ = f_{i-1}^\circ$, and since it contains $q(f_{i+1}^\circ)$, it intersects the face f_i° along an interval I. This interval cannot lie inside $f_{i+1}^\circ = g_{i+1}^\circ$, for in this case E would lie inside the affine span of g_i°, and so would not contain $q(f_{i-1}^\circ)$. Therefore, I passes through the relative interior of f_i°. Choose any point $q(f_i)$ in the interior of I, and define $v_i := q(f_i) - q(g_{i+1})$ and $v_{i-1} := q(g_{i-1}) - q(f_i)$. Since $\{v_i, v_{i-1}\}$ and $\{v'_i, v'_{i-1}\}$ are bases of the linear space of E having opposite orientations, $v_i \wedge v_{i-1} = c v'_i \wedge v'_{i-1}$ for some $c < 0$. Thus it remains to check that $\langle v_i \wedge v_{i-1}, u_i \wedge u_{i+1}\rangle \le 0$.

Note that, since $\langle v_i, u_i\rangle = 0$, we have

$$\langle v_i \wedge v_{i-1}, u_i \wedge u_{i+1}\rangle = -\langle v_{i-1}, u_i\rangle\langle v_i, u_{i+1}\rangle.$$

Now

$$\langle v_{i-1}, u_i\rangle = \langle q(f_{i-1}) - q(f_i), p(f_i) - p(f_{i-1})\rangle = \langle q(f_{i-1}), p(f_i)\rangle - 1 \le 0,$$

and similarly $\langle v_i, u_{i+1}\rangle \le 0$. This concludes the proof. □

Lemma 6.2 *The volume of the Funk outward ball of radius R inside a polytope P satisfies*

$$\omega_n \mathrm{Vol}_P(B_P(R)) = c_0(P)R^n + c_1(P)R^{n-1} + o(R^{n-1}), \tag{14}$$

where $c_0(P) = |\mathrm{Flags}(P)|/(n!)^2$ and

$$c_1(P) = \frac{1}{n!(n-1)!} \sum_{f \in \mathrm{Flags}(P)} \biggl(\sum_{i=0}^{n-1} L((r_i f)_i, f_i) - \sum_{i=0}^{n-1} L((r_i f)_i, f_{i+1}) \biggr). \tag{15}$$

Proof Lemma 5.1 tells us that to find the contribution of a flag f to the volume, we must consider it paired with each of the dual flag simplices. We use the results of the previous section to determine the contribution of each of these pairings. This contribution depends on the degree of proximity of the dual flag to f.

Indeed, Lemma 5.9 shows that only the dual of f and its nearest neighbors are important, when paired with f, while all other dual flags only contribute to the $o(R^{n-1})$ error term. Recall that a nearest neighbor of a flag is one that differs in exactly one face. Moreover, by Lemma 5.5, only the pairing of the flag with its own dual contributes to the R^n term. Together, those observations confirm the asymptotic formula (14), and it remains to compute the values of $c_0(P)$ and $c_1(P)$.

We apply Lemma 5.4 to the simplex having vertices v_i for $0 \le i \le n$, given by

$$v_{ij} = x^f_{j-1}(p(f_i)) = 1 - \langle q(f_{j-1}), p(f_i) \rangle \quad \text{for } 1 \le j \le n,$$

taking $k = 0$. The contribution of f, paired with its dual, to the leading term is $(1/n!)R^n$. Together with Lemma 5.1, the value for $c_0(P)$ follows. The contribution to $c_1(P)$ of f paired with its own dual is

$$\sum_{i=0}^{n-1} L(f_i, f_{i+1}) - \sum_{i=0}^{n-2} L(f_i, f_{i+2}).$$

When we consider a nearest neighbor having a different face of dimension i, all that changes is the $(i+1)^{\text{th}}$ coordinate of each vertex. Using Lemmas 5.5 and 5.7, we get that the contribution to c_1 of f paired with this neighboring dual flag simplex is, up to sign,

$$L((r_i f)_i, f_i) + L(f_{i-1}, f_{i+1}) - L(f_{i-1}, f_i) - L((r_i f)_i, f_{i+1})$$

when $i \in \{1, \dots, n-1\}$, while for $i = 0$ we get

$$L((r_0 f)_0, f_0) - L((r_0 f)_0, f_1).$$

We will now confirm that we have chosen the sign correctly, namely that these expressions are positive in all cases. For $i = 0$, this is straightforward to check, so we assume $i \ge 1$.

Write $f' := r_i f$. We must check that

$$(1 - \langle q(f'_i), p(f_i) \rangle)(1 - \langle q(f_{i-1}), p(f_{i+1}) \rangle) \ge (1 - \langle q(f_{i-1}), p(f_i) \rangle)(1 - \langle q(f'_i), p(f_{i+1}) \rangle).$$

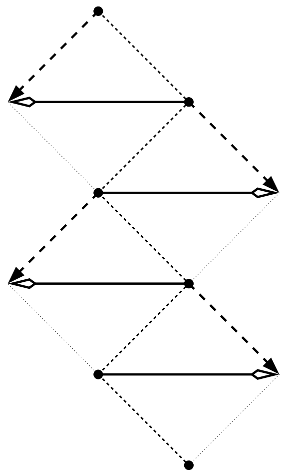

Figure 2: The contribution of a single flag simplex.

First consider the case $i = n-1$. Then $p(f_{i+1}) = 0$, and it remains to check that

$$\langle q(f_{i-1}), p(f_i)\rangle \geq \langle q(f_i'), p(f_i)\rangle \iff \langle q(f_{i-1}) - q(f_i'), p(f_i)\rangle \geq 0,$$

which holds since $q(f_{i-1}) - q(f_i') = c(q(f_i) - q(f_{i-1}))$ for some $c > 0$.

For $1 \leq i \leq n-2$, set $u_j = p(f_j) - p(f_{j-1})$ for $j \in \{i, i+1\}$, and $v_j' = q(f_j') - q(f_{j+1}')$ for $j \in \{i-1, i\}$. One computes

$$\begin{aligned}
\langle q(f_i'), p(f_i)\rangle &= 1 + \langle v_i', u_i\rangle, \\
\langle q(f_{i-1}), p(f_{i+1})\rangle &= 1 + \langle v_i', u_i\rangle + \langle v_i', u_{i+1}\rangle + \langle v_{i-1}', u_i\rangle + \langle v_{i-1}', u_{i+1}\rangle, \\
\langle q(f_{i-1}), p(f_i)\rangle &= 1 + \langle v_i', u_i\rangle + \langle v_{i-1}', u_i\rangle, \\
\langle q(f_i'), p(f_{i+1})\rangle &= 1 + \langle v_i', u_i\rangle + \langle v_i', u_{i+1}\rangle.
\end{aligned}$$

Then the desired inequality is easily seen to be equivalent to

$$\langle v_i', u_i\rangle\langle v_{i-1}', u_{i+1}\rangle \geq \langle v_{i-1}', u_i\rangle\langle v_i', u_{i+1}\rangle,$$

which holds by Lemma 6.1.

Adding up all these contributions, recalling that $L(f_{n-1}, f_n) = 0$, we get the stated formula for $c_1(P)$. □

This is illustrated in Figure 2. The dotted line represents the flag, connecting the full polytope face at the top to the empty set at the bottom. The arrows represent the terms of the above expression. An arrow starting at a face F and ending at face G represents the term $\pm\log(1 - \langle q(G), p(F)\rangle)$, with a positive sign if the arrow is full, and a negative sign if the arrow is dashed.

It will be convenient to express the sum in (15) in a different way.

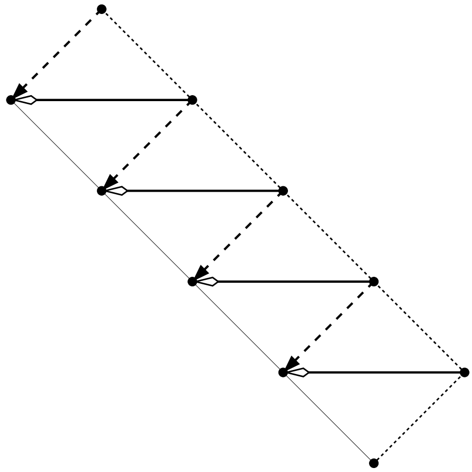

Figure 3: The contribution of a flag simplex after the sum is rearranged.

Lemma 6.3 *Assume we are given a polytope P with a flag decomposition, and a flag decomposition of its dual. Then, the sum over the flags in (15) is equal to*

$$\sum_{f\in \mathrm{Flags}(P)} \Big(\sum_{i=0}^{n-1} L((rf)_i, f_i) - \sum_{i=0}^{n-1} L((rf)_i, f_{i+1}) \Big).$$

Proof Fix i, and observe that $r_{i-1}\cdots r_0$ is a permutation of the flags. If g and f are two flags such that $g = r_{i-1}\cdots r_0 f$, then

$$g_i = f_i, \quad g_{i+1} = f_{i+1} \quad \text{and} \quad (r_i g)_i = (rf)_i.$$

Therefore,

$$\sum_{g\in \mathrm{Flags}(P)} L((r_i g)_i, g_i) = \sum_{f\in \mathrm{Flags}(P)} L((rf)_i, f_i),$$

$$\sum_{g\in \mathrm{Flags}(P)} L((r_i g)_i, g_{i+1}) = \sum_{f\in \mathrm{Flags}(P)} L((rf)_i, f_{i+1}).$$

Summing over all i gives the result. □

These terms are illustrated in Figure 3 (for a single flag f) using the same convention as before.

In the following lemma, we are using the convention that faces are closed. We use aff F to denote the affine hull of a set F, and by $X - Y$ we mean the Minkowski sum of X and $-Y$.

Lemma 6.4 *Let F and G be faces of P such that $F \subset G$ and $\dim G = \dim F + 1$. If $p, p' \in G$ and $q, q' \in F^\circ$, then*

(16) $$(1 - \langle q, p\rangle)(1 - \langle q', p'\rangle) = (1 - \langle q, p'\rangle)(1 - \langle q', p\rangle).$$

Proof Because the dimensions of F and G differ by exactly 1, we can write $p = s + \alpha v$ and $p' = s' + \alpha' v$, with $s, s' \in \operatorname{aff} F$ and $\alpha, \alpha' \in \mathbb{R}$, and where v is some element of $(\operatorname{aff} G \setminus \operatorname{aff} F) - \operatorname{aff} F$. Likewise, we can write $q = r + \beta u$ and $q' = r' + \beta' u$, with $r, r' \in \operatorname{aff} G^\circ$ and $\beta, \beta' \in \mathbb{R}$, and where u is some element of $(\operatorname{aff} F^\circ \setminus \operatorname{aff} G^\circ) - \operatorname{aff} G^\circ$.

Note that $\langle r, s \rangle = 1$ and $\langle r, v \rangle = \langle u, s \rangle = 0$. We calculate that $1 - \langle q, p \rangle = -\alpha\beta\langle u, v \rangle$, and the other factors in (16) can be calculated similarly. The conclusion then follows. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5 Let f be a flag of P. For each $i \in \{0, \dots, n-2\}$, we apply Lemma 6.4 with $F := (rf)_i$ and $G := f_{i+1}$. Observe that $q((rf)_{i+1})$ and $q((rf)_i)$ are in F°, and that $p(f_{i+1})$ and $p(f_0)$ are in G. From the lemma,

$$L((rf)_{i+1}, f_0) + L((rf)_i, f_{i+1}) = L((rf)_{i+1}, f_{i+1}) + L((rf)_i, f_0).$$

It is clear that no proper face of the flag f is a subset of any proper face of rf. Hence, all four of these terms are finite. Summing over $i \in \{0, \dots, n-2\}$, and using that $L((rf)_{n-1}, f_n) = 0$ (this is because $p(P)$ is the origin), we get

$$L((rf)_{n-1}, f_0) + \sum_{i=0}^{n-1} L((rf)_i, f_{i+1}) = \sum_{i=0}^{n-1} L((rf)_i, f_i).$$

Summing over all flags, and comparing with Lemmas 6.2 and 6.3, we get the result. □

Proof of Corollary 1.6 Let Q be a cube of the same dimension as P. Both P and Q have $2^n n!$ flags. Using Theorem 1.5 we may write

$$\omega_n \mathrm{Vol}_P(B_P(R)) = \frac{2^n}{n!} R^n + \frac{n}{(n!)^2} \sum_{f \in \mathrm{Flags}(P)} \log\bigl(1 - \langle (rf)_{n-1}, f_0 \rangle\bigr) R^{n-1} + o(R^{n-1}).$$

We have by Theorem 1.3 that

$$\mathrm{Vol}_P(B_P(R)) \ge \mathrm{Vol}_Q(B_Q(R)) \quad \text{for all } R > 0.$$

As the leading term in the asymptotics for the volume is the same for Q and P, it must hold that

$$\sum_{f \in \mathrm{Flags}(P)} \log\bigl(1 - \langle (rf)_{n-1}, f_0 \rangle\bigr) \ge \sum_{f \in \mathrm{Flags}(Q)} \log\bigl(1 - \langle (rf)_{n-1}, f_0 \rangle\bigr). \tag{17}$$

For each $f \in \mathrm{Flags}(Q)$, one easily verifies that $\langle (rf)_{n-1}, f_0 \rangle = -1$. Thus the right-hand side of (17) equals $2^n n! \log 2$.

For each $f \in \mathrm{Flags}(P)$, it holds by the central symmetry of P that $\langle (rf)_{n-1}, f_0 \rangle \ge -1$, and so none of the summands on the left-hand side of (17) exceed $\log 2$. As there are $2^n n!$ summands, it follows that each summand must equal $\log 2$, that is, $\langle (rf)_{n-1}, f_0 \rangle = -1$ for all $f \in \mathrm{Flags}(P)$. Thus $\langle (rf)_{n-1}, -f_0 \rangle = 1$, or equivalently, $-f_0 \in (rf)_{n-1}$ for all $f \in \mathrm{Flags}(P)$, concluding the proof. □

7 The limit of the Funk–Santaló point for polytopal Funk geometries as the radius tends to infinity

Santaló [19] studied the dependence of the Mahler volume of a convex body on the point in the body through which the polar is taken. He showed that there is a unique point in the interior of the body where this quantity is minimized. This point is now known as the *Santaló point* of the body.

This notion is generalized in [25], where, instead of the Mahler volume, the Holmes–Thompson volume of metric balls of the Funk geometry was considered. The following theorem appears there.

Theorem 7.1 [25] *Let K be a convex body, and let $R>0$. Then, the function $f_{K,R}\colon K\to\mathbb{R}$, defined by*

$$f_{K,R}(x) := \mathrm{Vol}_K(B_K(x,R)),$$

is proper and strictly convex, and hence attains its infimum at a unique point $s_R(K)$ in the interior of K.

Proper means that the function converges to infinity as the boundary of K is approached. We call the point $s_R(K)$ the *Funk–Santaló point* of the convex body K, for radius R.

It is shown in [25] that the Funk–Santaló point of a convex body converges to the Santaló point as the radius tends to zero. Here, we study the behavior of the Funk–Santaló point as the radius becomes large in the case of polytopes. We will see that this behavior is governed by the second-highest order term of the volume growth at infinity.

Recall that if a sequence f_j of convex functions converges pointwise on a dense subset D of an open convex set C in $\mathbb{R}^n$, then the limit function f exists on C, and the convergence is locally uniform on C; see [18, Theorem 10.8].

The following lemma concerning the convergence of infima of convex functions is established in [25].

Lemma 7.2 *Let f_j be a sequence of lower semicontinuous convex functions defined on a convex body K in a finite-dimensional vector space, taking values in $\mathbb{R}\cup\{\infty\}$. Assume that f_j converges pointwise on* int K *to a lower semicontinuous convex function f that is finite on* int K. *Then,* inf f_j *converges to* inf f. *Moreover, if for each j, the function f_j attains its infimum at x_j, and this sequence of points has a limit point x, then f attains its infimum at x.*

Proof of Theorem 1.7 By Theorem 1.5 and Lemma 2.1, we have

$$c_1(P,x) = \frac{1}{n!\,(n-1)!}\sum_{f\in\mathrm{Flags}(P)} \log\left(1-\left\langle \frac{(rf)_{n-1}}{1-\langle (rf)_{n-1},x\rangle}, f_0 - x\right\rangle\right) \tag{18}$$

$$= C - \frac{1}{n!\,(n-1)!}\sum_{F\in F_{n-1}(P)} |\mathrm{Flags}(F)|\log(1-\langle q(F),x\rangle),$$

where C is a constant that does not depend on x, $F_{n-1}(P)$ denotes the set of facets of P, and $q(F)$ is the vertex of P° corresponding to the facet F.

Notice that each nonconstant term is minus the logarithm of an affine function, and hence convex. In fact, the level sets of any one of these terms are parallel hyperplanes, and the term is strictly convex along line segments that do not lie in one of these hyperplanes. Since the vertices of P°, that is, the set of dual vectors $q(F)$ with $F \in F_{n-1}(P)$, affinely generate the whole dual space, these points separate $\mathbb{R}^n$, and so there is no nondegenerate line segment lying in a level set of every one of the terms. We conclude that $c_1(P,x)$ is strictly convex.

The function $c_1(P,x)$ is clearly continuous, and finite in $\operatorname{int} P$. If x is on the boundary of P, then there is at least one facet F for which $x \in F$, or equivalently $\langle q(F), x\rangle = 1$. Since none of the terms can be $-\infty$, we deduce that $c_1(P,x) = \infty$. This establishes that $c_1(P,x)$ is proper.

It follows immediately that the infimum of $c_1(P,x)$ is attained at a unique point, which we denote by $s_\infty(P)$.

As $c_1(P,x)$ is smooth in x, $s_\infty(P)$ must be its unique stationary point. Thus

$$\sum_{F \in F_{n-1}(P)} |\mathrm{Flags}(F)| \frac{1}{1 - \langle q(F), s_\infty(P)\rangle} q(F) = 0,$$

and assuming $s_\infty(P) = 0$ we obtain the asserted characterization of $s_\infty(P)$.

For each $R > 0$, define a function of $x \in \operatorname{int} P$ by

$$f_{P,R}(x) := \frac{1}{R^{n-1}} \left(\omega_n \mathrm{Vol}(B_P(x,R)) - \frac{1}{(n!)^2} |\mathrm{Flags}(P)| R^n \right).$$

By Theorem 7.1, this function is a proper convex function for all $R > 0$, and its infimum is attained at $s_R(P)$. By Theorem 1.5 it holds that

$$\lim_{R\to\infty} f_{P,R}(x) = c_1(P,x) \quad \text{for all } x \in \operatorname{int} P.$$

Thus the conditions of Lemma 7.2 are satisfied, and it follows upon applying that lemma that $s_R(P)$ converges to $s_\infty(P)$ as $R \to \infty$. □

8 Stationary points of the second-highest-order-term functional in 2D

In this section, we prove Theorem 1.8, that is, that, in dimension two, the only stationary points of the second-highest order term $c_1(P)$ in the expansion of the volume of balls are the regular polytopes, up to linear transformations. We show moreover that they uniquely maximize $c_1(P, s_\infty(P))$.

The proof will rely on the following lemma concerning polygons where c_1 remains stationary under certain perturbations of a single vertex. Recall that $\mathcal{P}_2^m$ denotes the space of convex polygons in $\mathbb{R}^2$ with m vertices that contain the origin in their interior.

Lemma 8.1 *Let v_1, v_2, v_3 be consecutive vertices of a polygon $K \in \mathcal{P}_2^m$. Denote by e_2, e_3 and e_4 the consecutive vertices of K° such that*

$$\langle e_2, v_1\rangle = \langle e_2, v_2\rangle = 1 = \langle e_3, v_2\rangle = \langle e_3, v_3\rangle = \langle e_4, v_3\rangle.$$

For small $t \in \mathbb{R}$, let K^t be the polygon obtained from K by moving v_2 by an amount $t(v_2 - v_1)$. If t is small enough, then K^t is a convex polygon. Assume that

$$\frac{d}{dt} c_1(K^t)\Big|_{t=0} = 0.$$

Then, there is an element γ of $\mathrm{GL}_2(\mathbb{R})$ such that $\gamma(v_1) = e_4$, $\gamma(v_2) = e_3$ and $\gamma(v_3) = e_2$.

Proof As t varies, so do v_2 and e_3. So there are four terms in the expression for $c_1(K^t)$ that change with t. More precisely, by equation (2),

$$2c_1(K^t) = \log(1 - \langle e_1, v_2(t)\rangle) + \log(1 - \langle e_4, v_2(t)\rangle) + \log(1 - \langle e_3(t), v_1\rangle) + \log(1 - \langle e_3(t), v_4\rangle) + C,$$

where C is a constant independent of t. Here, v_4 and e_1 are the vertices of K and K°, respectively, such that v_2, v_3 and v_4 are consecutive, and e_1, e_2 and e_3 are consecutive. So

$$2\frac{d}{dt} c_1(K^t)\Big|_{t=0} = -\frac{\langle e_1, v_2 - v_1\rangle}{1 - \langle e_1, v_2\rangle} - \frac{\langle e_4, v_2 - v_1\rangle}{1 - \langle e_4, v_2\rangle} - \frac{\langle \dot{e}_3, v_1\rangle}{1 - \langle e_3, v_1\rangle} - \frac{\langle \dot{e}_3, v_4\rangle}{1 - \langle e_3, v_4\rangle}.$$

Here, $\dot{e}_3$ denotes the derivative of e_3 with respect to t at $t = 0$.

To simplify calculations, we choose (e_3, e_2) as a coordinate basis for $\mathbb{R}^2$. With respect to this basis

$$v_1 = (x, 1), \quad v_2 = (1, 1) \quad \text{and} \quad v_3 = (1, y),$$

with x and y in $(-\infty, 1)$. Moreover, with respect to the dual basis,

$$e_2 = (0, 1), \quad e_3 = (1, 0) \quad \text{and} \quad e_4 = (w, z),$$

where w and z satisfy

$$w + yz = 1. \tag{19}$$

Since $\langle e_4, v_3 - v_2\rangle > 0$, we have $z < 0$. A short calculation shows that

$$\dot{e}_3 = \frac{1 - x}{1 - y}(y, -1).$$

Denote the coordinates of v_4 by $v_4 = (a, b)$. Since $\langle e_4, v_3\rangle = \langle e_4, v_4\rangle = 1$, we have $w + yz = aw + bz = 1$, and hence

$$\frac{b - y}{1 - a} = \frac{w}{z}.$$

Using these coordinates, and that $\langle e_1, v_1\rangle = 1$, we get

$$2\frac{d}{dt} c_1(K^t)\Big|_{t=0} = 1 - \frac{(1 - x)w}{1 - w - z} + \frac{1 - xy}{1 - y} - \frac{1 - x}{1 - y} \cdot \frac{ay - b}{1 - a}.$$

Observing that

$$-\frac{ay-b}{1-a} = \frac{y(1-a)+(b-y)}{1-a} = y + \frac{w}{z},$$

we have

$$2\frac{d}{dt}c_1(K^t)\Big|_{t=0} = 1 + \frac{(1-x)w}{(1-y)z} + \frac{1-xy}{1-y} + \frac{1-x}{1-y}\Big(y+\frac{w}{z}\Big)$$
$$= \frac{2}{1-y}\Big((1-x)\frac{w}{z} + 1 - xy\Big).$$

By assumption, this expression is zero, so we obtain a linear relation between w and z, which combined with (19) gives us a pair of equations having the unique solution

$$w = \frac{1-xy}{1-y} \quad \text{and} \quad z = -\frac{1-x}{1-y}.$$

The linear map defined by the matrix

$$\gamma := \frac{1}{1-y}\begin{pmatrix} -y & 1 \\ 1 & -1 \end{pmatrix}$$

satisfies $\gamma(v_1) = e_4$, $\gamma(v_2) = e_3$ and $\gamma(v_3) = e_2$. □

We will also need to know that the Funk–Santaló point depends smoothly on P.

Lemma 8.2 *The map $s_\infty \colon \mathcal{P}_2^m \to \mathbb{R}^2$ is C^∞-smooth.*

Proof As s_∞ commutes with translations, it suffices to consider the open subset $U \subset \mathcal{P}_2^m$ consisting of polygons containing 0 in their interior. For $P \in U$, let e_i for $1 \le i \le m$ be a cyclical ordering of the vertices of P°. By Theorem 1.7 and equation (18), $s_\infty(P)$ is the unique minimum of the following strictly convex smooth function defined on the interior of P:

$$f_P(z) = -\sum_{i=1}^{m} \log(1 - \langle e_i, z\rangle).$$

It follows that $s_\infty(P)$ is the unique solution of the equation $H(P,\cdot) = 0$, where

$$H(P,z) := \nabla f_P(z) = \sum_{i=1}^{m} \frac{1}{1-\langle e_i, z\rangle} e_i.$$

Let us verify that $\partial H/\partial z$ has rank 2, so that by the implicit function theorem $z = s_\infty(P)$ depends smoothly on P.

We write $z = (z_1, z_2)$ and $e_i = (e_i^1, e_i^2)$, for each i. Observe that the Jacobian is

$$\frac{\partial H_j}{\partial z_k} = \sum_{i=1}^{m} \frac{e_i^j e_i^k}{(1-\langle e_i, z\rangle)^2} \quad \text{for } j \in \{1,2\}.$$

For clarity we write $x(e_i)$, $y(e_i)$ for the coordinates of e_i. So, the determinant is given by

$$\det \frac{\partial H}{\partial z} = \sum_{i=1}^{m} \frac{x(e_i)^2}{(1-\langle e_i, z\rangle)^2} \sum_{j=1}^{m} \frac{y(e_j)^2}{(1-\langle e_j, z\rangle)^2} - \left(\sum_{i=1}^{m} \frac{x(e_i)y(e_i)}{(1-\langle e_i, z\rangle)^2}\right)^2$$
$$= \sum_{1\le i<j\le m} \frac{\big(x(e_i)y(e_j)-x(e_j)y(e_i)\big)^2}{(1-\langle e_i, z\rangle)^2(1-\langle e_j, z\rangle)^2}.$$

If $\partial H/\partial z$ were not of rank 2, then we would have $x(e_i)y(e_j) = x(e_j)y(e_i)$ for all i and j, that is, all the points e_i would lie on a line, which is impossible. □

Lemma 8.3 *The function $f(P) = c_1(P, s_\infty(P))$ has a global maximum on $\mathcal{P}_2^m$ for each $m \ge 3$.*

Proof Let X_m denote the set of m-sided convex polygons $P \subset \mathbb{R}^2$ in John's position. Recall that P is said to be in John's position if the John ellipse of P — the unique maximal volume ellipse it contains — is the unit Euclidean ball centered at the origin.

Since $f(P)$ is invariant under affine transformations of P, and any $P \in \mathcal{P}_2^m$ has an affine image in John's position, it suffices to show that $f(P)$ attains a global maximum on X_m.

Let B denote the Euclidean unit disc around the origin. It follows from John's theorem that each polygon $P \in X_m$ is contained in $2B$. Also, as P contains B, it holds that the polar body P° lies inside B. We conclude that for all $v \in P$ and $e \in P^\circ$, one has $|\langle e, v\rangle| \le 2$. It follows that each summand in the expression in (2) for $c_1(P) = c_1(P, 0)$ is bounded from above by $\frac{1}{2}\log 3$.

Let $v_1, \dots, v_m$ by the consecutive vertices of P, with cyclical index, and let $e_1, \dots, e_m$ be the vertices of P° such that $\langle e_i, v_i\rangle = \langle e_i, v_{i-1}\rangle = 1$ for all i.

By Blaschke's selection theorem, X_m is precompact within the space $\mathcal{K}(\mathbb{R}^2)$ of convex bodies equipped with the Hausdorff distance. Its closure $\overline{X}_m$ in $\mathcal{K}(\mathbb{R}^2)$ is $X_m \cup \partial X_m$, where $\partial X_m := X_{m-1} \cup \cdots \cup X_3$ is the geometric boundary of X_m.

Suppose now that $P \in X_m$ approaches the boundary ∂X_m. This implies that there is some i for which either $|v_i - v_{i+1}|$ or $|e_i - e_{i+1}|$ tends to zero. If the former is the case, then

$$1 - \langle e_i, v_{i+1}\rangle = \langle e_i, v_i - v_{i+1}\rangle$$

tends to zero. Similarly, in the latter case, $1 - \langle e_{i+1}, v_{i-1}\rangle$ tends to zero.

We conclude that at least one of the summands in (2) for $c_1(P)$ tends to $-\infty$. Since we have seen that each of the summands is bounded above, we conclude that $c_1(P)$ tends to $-\infty$, as P approaches the boundary. It follows that $f(P)$ also tends to the same limit as $P \to \partial X_m$, since

$$f(P) = c_1(P, s_\infty(P)) \le c_1(P, 0) = c_1(P),$$

by the definition of $s_\infty(P)$.

Using this, and that the function $f(P)$ is continuous on X_m, we deduce that it attains a global maximum in X_m, and thus in $\mathcal{P}^m$, concluding the proof. □

Proof of Theorem 1.8 Let the function c_1 be stationary at a polygon $K \in \mathcal{P}_2^m$. Let $v_1, \dots, v_m$ be the consecutive vertices of K with cyclical index, and $e_1, \dots, e_m$ the consecutive vertices of K° as in Lemma 8.1, so that $\langle e_i, v_i \rangle = \langle e_i, v_{i-1} \rangle = 1$ for all i. By that lemma, there is an element γ of $\mathrm{GL}_2(\mathbb{R})$ such that the triple of vertices (v_1, v_2, v_3) is mapped to the triple (e_4, e_3, e_2) of dual vertices. We apply the same lemma again, this time to K°, to get that there exists another element γ' of $\mathrm{GL}_2(\mathbb{R})$ such that the triple (e_2, e_3, e_4) is mapped to (v_4, v_3, v_2). Composing these maps, we get an element γ'' of $\mathrm{GL}_2(\mathbb{R})$ taking the triple of points (v_1, v_2, v_3) to (v_2, v_3, v_4). Similarly, we have an element of $\mathrm{GL}_2(\mathbb{R})$ taking the triple (v_2, v_3, v_4) to (v_3, v_4, v_5). Note that this is actually the same map, because it agrees with γ'' on v_2 and v_3. Proceeding inductively, we get that γ'' takes v_j to v_{j+1} for all j. It follows that $(\gamma'')^m$ is the identity map. The iterates $(\gamma'')^k$ for $k \in \{0, \dots, m-1\}$ form a finite subgroup of $\mathrm{GL}_2(\mathbb{R})$, and are therefore contained in some maximal compact subgroup conjugate to the orthogonal group $\mathrm{O}_2(\mathbb{R})$. Thus, we may write $\gamma'' = ghg^{-1}$ for some $g \in \mathrm{GL}_2(\mathbb{R})$ and $h \in \mathrm{O}_2(\mathbb{R})$. So, $K = gD_m$, where D_m is the regular polygon $D_m := \mathrm{conv}\{h^k(g^{-1}v_1) \mid k \in \{0, \dots, m-1\}\}$.

It remains to show that $P = D_m$ maximizes $c_1(P, s_\infty(P))$ among all $P \in \mathcal{P}_2^m$ with $s_\infty(P)$ at the origin, uniquely up to linear transformations.

By Lemma 8.2, $s_\infty \colon \mathcal{P}_2^m \to \mathbb{R}^2$ is smooth. As it commutes with translations, it is everywhere submersive. It follows that $Y_m := \{P \in \mathcal{P}_2^m \mid s_\infty(P) = 0\}$ is a smooth submanifold of $\mathcal{P}_2^m$.

By Lemma 8.3, the function $c_1(P, s_\infty(P))$ has a global maximum on $\mathcal{P}_2^m$. In fact, since it is affinely invariant, the maximum, which we denote by P_0, can be taken to be in Y_m. Note that on Y_m, we have $c_1(P, s_\infty(P)) = c_1(P)$.

We can write the tangent space at P_0 as

$$T_{P_0}\mathcal{P}_2^m = T_{P_0}Y_m \oplus T_{P_0}(GP_0),$$

where $G = \mathbb{R}^2$ is the group of translations of $\mathbb{R}^2$.

It follows from the definition of s_∞ that P_0 is a local minimum of $c_1(P)$ in the directions of $T_{P_0}(GP_0)$, while by the above it is a local maximum in the directions of $T_{P_0}Y_m$. Thus, P_0 is a stationary point of $c_1(P)$ in $\mathcal{P}_2^m$. By the first part of the theorem, P_0 coincides with D_m up to a linear transformation. The assertion of the theorem readily follows. □

Next we consider the derivative of $V_K(\lambda) := \omega_n \mathrm{Vol}_P(B_K(R))$, where the radius R is related to λ by $R = -\log(1-\lambda)$. In general, this derivative can be written as an integral over the boundary of the ball. For $K = P \in \mathcal{P}_2^m$, this integral can be computed explicitly, as follows.

Let $v_1, v_2, \dots, v_m$ be the consecutive vertices of P with cyclical index, and $e_1, \dots, e_m$ the consecutive vertices of P°, so that $\langle e_i, v_i \rangle = \langle e_i, v_{i-1} \rangle = 1$ for all i. For each i and j, write $G_{ij} := 1 - \lambda \langle e_j, v_j \rangle$, understood as a function of λ.

Proposition 8.4 *It holds that*

$$\frac{dV_K(\lambda)}{d\lambda} = \sum_{i,j} \lambda \cdot \left|\log \frac{G_{i,j}\, G_{i+1,j+1}}{G_{i,j+1}\, G_{i+1,j}}\right| \cdot \frac{|\langle e_i, v_j\rangle\langle e_{i+1}, v_{j+1}\rangle - \langle e_i, v_{j+1}\rangle\langle e_{i+1}, v_j\rangle|}{|G_{i,j}\, G_{i+1,j+1} - G_{i,j+1}\, G_{i+1,j}|}$$
$$= \sum_{i,j} \left|\log \frac{G_{i,j}\, G_{i+1,j+1}}{G_{i,j+1}\, G_{i+1,j}}\right| \cdot \left|\lambda - \frac{\langle e_i, v_j\rangle + \langle e_{i+1}, v_{j+1}\rangle - \langle e_i, v_{j+1}\rangle - \langle e_{i+1}, v_j\rangle}{\langle e_i, v_j\rangle\langle e_{i+1}, v_{j+1}\rangle - \langle e_i, v_{j+1}\rangle\langle e_{i+1}, v_j\rangle}\right|^{-1}.$$

Proof Observe that the proof of Lemma 5.1 can be repeated to show that

$$\omega_n \mathrm{Vol}(B_P(0,R)) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{\lambda\Delta(0,v_i,v_{i+1})} \frac{\mathrm{d}y_j\, \mathrm{d}y_{j+1}}{y_j\, y_{j+1}},$$

where $y_j = 1 - \langle e_j, \cdot\rangle$ and $\Delta(0, v_i, v_{i+1})$ is the triangle with the listed vertices. We then parametrize $p = s(1-t)v_i + st v_{i+1}$, with $0 \le s \le \lambda$ and $0 \le t \le 1$, and use the fundamental theorem of calculus to deduce the stated formula. □

Let us finish this section by pointing out that for the regular polygon D_m we have a very simple formula for the second-order term:

$$c_1(D_m) = 2m \log\left(2 \sin\left(\frac{\pi}{m}\right)\right) = 2m \log\left(\frac{l}{r}\right),$$

where l is the length of an edge, and r is the radius of the circumscribed circle.

9 Simplices

Theorem 9.1 *Consider the Funk geometry of the simplex $\Delta_n \subset \mathbb{R}^n$. The Holmes–Thompson volume of the forward ball $\Delta_n(R)$ of radius R centered at the barycenter is $1/\omega_n$ times*

$$V_{\Delta_n}(R) = \frac{n+1}{n!} \int_{\Delta_n(R)} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}. \tag{20}$$

We also have a recursive formula for $n \ge 2$,

$$V'_{\Delta_n}(R) = \frac{n+1}{n} \frac{1}{1 - \frac{e^{-R}}{n+1}} V_{\Delta_{n-1}}\left(R + \log\left(1 + \frac{1}{n} - \frac{1}{n}e^{-R}\right)\right). \tag{21}$$

For all $n \ge 1$ it holds for $R \to \infty$ that

$$V_{\Delta_n}(R) = \frac{(n+1)!}{(n!)^2} R^n \big(1 + n\log(n+1) R^{-1} + o(R^{-1})\big), \tag{22}$$

while for $R \to 0$ we have

$$V_{\Delta_n}(R) = \frac{(n+1)^{n+1}}{(n!)^2} \big(R^n - \tfrac{1}{2} n R^{n+1} + o(R^{n+1})\big). \tag{23}$$

Proof Consider the standard n-dimensional simplex Δ_n, with vertices $C_0 = 0, C_1 = e_1, \dots, C_n = e_n$, where e_i is the standard basis. We will consider a point $C = \sum_{i=0}^n \alpha_i C_i \in \Delta_n$, with $\alpha_i \geq 0$ and $\sum \alpha_i = 1$, whose barycentric coordinates are therefore $(\alpha_0, \dots, \alpha_n)$. Then in standard Euclidean coordinates,

$$C = (\alpha_1, \dots, \alpha_n).$$

Notice that $\alpha_0 = 0$ corresponds to the face F_0 belonging to the hyperplane $\sum_{i=1}^n x_i = 1$, and for $1 \leq i \leq n$, $\alpha_i = 0$ corresponds to the face F_i belonging to the hyperplane $x_i = 0$. It follows that the vectors

$$V_1 = (-\alpha_1, 0, \dots, 0), \quad V_2 = (0, -\alpha_2, 0, \dots, 0), \quad \dots, \quad V_n = (0, \dots, 0, -\alpha_n),$$

are unit vectors with respect to the Funk metric at C. The same is true for the vector

$$V_0\left(\frac{1 - \sum_{i=1}^n \alpha_i}{n}, \dots, \frac{1 - \sum_{i=1}^n \alpha_i}{n}\right).$$

Hence for $i = 1, \dots, n$, the hyperplanes $X_i = -\alpha_i$ are the faces of the tangent unit ball, and so is the hyperplane $\sum_{i=1}^n X_i = 1 - \sum_{i=1}^n \alpha_i = \alpha_0$. Then the dual simplex can be seen as the convex hull of the following vectors in the dual space:

$$W_1 = \left(\frac{-1}{\alpha_1}, 0, \dots, 0\right), \quad \dots, \quad W_n = \left(0, \dots, \frac{-1}{\alpha_n}\right) \quad \text{and} \quad W_0 = \frac{1}{\alpha_0}(1, \dots, 1).$$

The volume of this simplex is the sum of the volumes of the simplices defined by 0 and any n of this points. For instance, the volume defined by $0, W_1, \dots, W_n$ is exactly

$$\frac{1}{n!}\frac{1}{\alpha_1 \cdots \alpha_n},$$

hence by adding all these terms we get the Holmes–Thompson density in the form

$$\beta^*_{\Delta_n}(A) = \frac{1}{n!}\sum_{i=0}^n \frac{1}{\alpha_0 \cdots \widehat{\alpha}_i \cdots \alpha_n} = \frac{1}{n!}\frac{1}{\alpha_0 \cdots \alpha_n}, \tag{24}$$

where the $\widehat{\ }$ means that the term is removed). Recall that the Funk ball $\Delta_n(R)$ is the image of Δ_n under the dilation with ratio $(1 - e^{-R})$ centered at the barycenter $(1/(n+1))(1, \dots, 1)$. Then

$$V_{\Delta_n}(R) = \frac{1}{n!}\sum_{i=0}^n \int_{\Delta_n(R)} \frac{1}{\alpha_0 \cdots \widehat{\alpha}_i \cdots \alpha_n}\, \mathrm{d}x.$$

If for $i = 1, \dots, n$ we use the change of variable

$$F_i(x) = \left(x_1, \dots, x_{i-1}, 1 - \sum_{k=1}^n x_k, x_{i+1}, \dots, x_n\right),$$

we get that

$$\int_{\Delta_n(R)} \frac{1}{\alpha_0 \cdots \widehat{\alpha}_i \cdots \alpha_n}\, \mathrm{d}x = \int_{\Delta_n(R)} \frac{1}{\alpha_1 \cdots \alpha_n}\, \mathrm{d}x,$$

which allows us to get the formula

$$V_{\Delta_n}(R) = \frac{n+1}{n!} \int_{\Delta_n(R)} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_n} \tag{25}$$

for the volume of the forward ball of radius R centered at the barycenter of the simplex in euclidean coordinates.

Denote by

$$A_0 = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1} \right)$$

the barycenter of Δ_n. Then for $1 \le k \le n$,

$$A_k(R) = A_0 + (1 - e^{-R}) \left(\frac{1}{n-k} \sum_{j=1}^{n-k} e_j - A_0 \right)$$

are the vertices of a simplex $S_n(R)$ in the barycentric subdivision of $\Delta_n(R)$.

Explicitly,

$$A_1(R) = \left(\frac{1}{n} - \frac{e^{-R}}{n(n+1)}, \dots, \frac{1}{n} - \frac{e^{-R}}{n(n+1)}, \frac{e^{-R}}{n+1} \right), \tag{26}$$

$$\vdots$$

$$A_k(R) = \Big(\underbrace{\frac{1}{n-k+1} - \frac{k}{(n+1)(n-k+1)} e^{-R}}_{n-k \text{ times}}, \underbrace{\frac{e^{-R}}{n+1}}_{k \text{ times}} \Big), \tag{27}$$

$$\vdots$$

$$A_n(R) = \left(\frac{e^{-R}}{n+1}, \dots, \frac{e^{-R}}{n+1} \right).$$

Therefore, because of the symmetries involved, using the right equality in (24) with $X = x_1 + \cdots + x_n$, we have

$$\int_{\Delta_n(R)} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{(1-X)x_1 \cdots x_n} = (n+1)! \int_{S_n(R)} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{(1-X)x_1 \cdots x_n}. \tag{28}$$

Let us focus on

$$W_n(R) = \int_{S_n(R)} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{(1 - x_1 - \cdots - x_n)x_1 \cdots x_n}, \tag{29}$$

so that

$$V_{\Delta_n}(R) = (n+1) W_n(R).$$

We can rewrite $W_n(R)$ using Fubini's theorem, and writing $s = x_n$,

$$W_n(R) = \int_{e^{-R}/(n+1)}^{1/(n+1)} \frac{\mathrm{d}s}{s} \int_{S_n(-\log(s(n+1))) \cap \{x_n = s\}} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}}{((1-s) - x_1 - \cdots - x_{n-1})x_1 \cdots x_{n-1}}.$$

Now let us proceed with the change of variable $t_i = x_i/(1-s)$ for $i = 1, \dots, n-1$ in the inner integral. This gives

(30)
$$W_n(R) = \int_{e^{-R}/(n+1)}^{1/(n+1)} \frac{ds}{s(1-s)} \int_{S_{n-1}(-\log(sn/(1-s)))} \frac{dt_1 \cdots dt_{n-1}}{(1-t_1-\cdots-x_{n-1})\cdot t_1\cdots t_{n-1}}$$
$$= \int_{e^{-R}/(n+1)}^{1/(n+1)} \frac{ds}{s(1-s)} W_{n-1}\left(-\log\frac{s\cdot n}{1-s}\right).$$

Hence it holds that

$$W_n'(R) = \frac{1}{1-\frac{e^{-R}}{n+1}} W_{n-1}\left(-\log\frac{n}{(n+1)e^R-1}\right),$$

from which we get the following differential equation by multiplying by $(n+1)$:

(31)
$$V'_{\Delta_n}(R) = \frac{n+1}{n}\frac{1}{1-\frac{e^{-R}}{n+1}} V_{\Delta_{n-1}}\left(R+\log\left(1+\frac{1}{n}-\frac{1}{n}e^{-R}\right)\right),$$

as claimed.

Next consider the asymptotic behavior as $R \to 0$. The one-dimensional case follows from Lemma 3.4, yielding

$$V_{\Delta_1}(R) = 2\log(2e^R-1) = 2R + 2\log 2 + 2\log\left(1-\tfrac{1}{2}e^{-R}\right) = 4R - 2R^2 + o(R^2).$$

Next we proceed by induction, utilizing the recursive formula. Writing

$$V_{\Delta_n}(R) = a_n R^n + b_n R^{n+1} + o(R^{n+1}),$$

and taking into account that

$$R + \log\left(1+\frac{1}{n+1}-\frac{1}{n+1}e^{-R}\right) = \frac{n+2}{n+1}R\left(1-\frac{R}{2(n+1)}+o(R)\right),$$

we get that

(32)
$$W_{\Delta_n}(R) := V_{\Delta_n}\left(R+\log\left(1+\frac{1}{n+1}-\frac{1}{n+1}e^{-R}\right)\right)$$
$$= \left(\frac{n+2}{n+1}\right)^n a_n R^n + \left(\frac{n+2}{n+1}\right)^n \left(\frac{n+2}{n+1}b_n - a_n\frac{n}{2(n+1)}\right) R^{n+1} + o(R^{n+1}).$$

Now equation (31) implies

$$V'_{\Delta_{n+1}}(R) = \left(\frac{n+2}{n+1}\right)^2 \left(1-\frac{R}{n+1}+o(R)\right) W_{\Delta_n}(R),$$

to obtain

$$V'_{\Delta_{n+1}}(R) = \left(\frac{n+2}{n+1}\right)^{n+2} a_n R^n \left(1-\frac{R}{n+1}+o(R)\right) \times \left(1+\frac{1}{a_n}\left(\frac{n+2}{n+1}b_n - a_n\frac{n}{2(n+1)}\right)R + o(R)\right),$$

which finally gives us

$$V'_{\Delta_{n+1}}(R) = \left(\frac{n+2}{n+1}\right)^{n+2} a_n R^n \left(1 + \frac{n+2}{a_n(n+1)}\left(b_n - \tfrac{1}{2}a_n\right)R + o(R)\right). \tag{33}$$

After integration, we get

$$a_{n+1} = \left(\frac{n+2}{n+1}\right)^{n+2} \frac{1}{n+1} a_n \quad \text{and} \quad b_{n+1} = \left(\frac{n+2}{n+1}\right)^{n+2} \frac{1}{n+1}\left(b_n - \tfrac{1}{2}a_n\right).$$

Taking into account that $a_1 = 4$ and $b_1 = -2$, we easily deduce by induction that for all $n \geq 2$,

$$a_n = \frac{(n+1)^{n+1}}{(n!)^2} \quad \text{and} \quad b_n = -\frac{n}{2}\frac{(n+1)^{n+1}}{((n)!)^2},$$

as stated.

Finally, the asymptotics as $R \to \infty$ follow at once from Theorem 1.5, or they can be deduced from the recursive formula as in the $R \to 0$ regime. □

10 Reflections and questions

10.1 Flags of polytopes and comparison of volume growth

Another conjecture related to flags and Mahler volume, and which may be related to Funk volume, is:

Conjecture 10.1 *Let P be a centrally symmetric polytope in dimension n. Then,*

$$|\mathrm{Flags}(P)| \geq \frac{(n!)^2}{2^n}|P||P^\circ|.$$

Kalai [11] talks about this conjecture in a YouTube video, around 32 minutes in. He ascribes it to conversations with Freij, Henze, Schmitt and Ziegler, about a decade ago, based on computer experimentation. We might be so bold as to make the following conjecture.

Conjecture 10.2 *Let Ω be a centrally symmetric convex body, and let H be a Hanner polytope of the same dimension. Then, for any r and R positive real numbers with $r < R$,*

$$\frac{\mathrm{Vol}_\Omega(B_\Omega(R))}{\mathrm{Vol}_\Omega(B_\Omega(r))} \geq \frac{\mathrm{Vol}_H(B_H(R))}{\mathrm{Vol}_H(B_H(r))}.$$

This is a strengthening of Conjecture 10.1. Indeed, we recover that conjecture from Conjecture 10.2 by letting r approach zero, and R approach infinity.

Conjecture 10.2 reminds us of the Bishop–Gromov theorem in Riemannian geometry, but with the inequality reversed. The inequality in Funk geometry directly corresponding to the Bishop–Gromov theorem would be

$$\frac{\mathrm{Vol}_\Omega(B_\Omega(R))}{\mathrm{Vol}_\Omega(B_\Omega(r))} \le \frac{\mathrm{Vol}_E(B_E(R))}{\mathrm{Vol}_E(B_E(r))},$$

where E is a centered ellipsoid. This inequality is false: considering small radii, it would imply the inequality

$$\int_{\Omega\times\Omega^\circ} \langle x, \xi\rangle^2 \,\mathrm{d}x\,\mathrm{d}\xi \le \frac{n}{(n+2)^2}|\Omega|\,|\Omega^\circ|,$$

which was shown to be false for large n by Klartag [12, Proposition 1.5], even for unconditional bodies.

10.2 Some further questions and remarks

The paper [9] studies Hansen polytopes of split graphs. This class contains polytopes that are only slightly worse than the Hanner polytopes with respect to the Mahler conjecture, the flag conjecture, and also Kalai's 3^d conjecture concerning the minimal number of faces of a centrally symmetric polytope. It would be interesting to see how closely these polytopes come to minimizing the volume of Funk balls.

In [22], it was shown that the number of flags of a polytope determines the asymptotics of the volume of its floating body. This phenomenon was later studied in greater generality in [3]. Since for smooth bodies the asymptotics is governed by the affine surface area, the authors consider the flag number to be an analogue of the affine surface area for polytopes. One could ask if it is possible to obtain a refinement that depends on the geometry of the polytope, similarly to how we have the second-highest order term for the Funk volume.

It was conjectured in [6] that, if one considers the volume of a ball in a Funk geometry centered about the centroid of the body, then this quantity is maximized when the body is an ellipsoid. This is an upper bound counterpart to Conjecture 1.1. It likewise interpolates between two inequalities, this time that of Blaschke–Santaló when the radius goes to zero, and the centro-affine isoperimetric inequality when the radius tends to infinity; see [25]. This upper bound conjecture was established [6] in the case of unconditional bodies.

References

[1] **M Alexander**, **M Fradelizi**, **A Zvavitch**, *Polytopes of maximal volume product*, Discrete Comput. Geom. 62 (2019) 583–600 MR

[2] **G Berck**, **A Bernig**, **C Vernicos**, *Volume entropy of Hilbert geometries*, Pacific J. Math. 245 (2010) 201–225 MR

[3] **F Besau**, **C Schütt**, **E M Werner**, *Flag numbers and floating bodies*, Adv. Math. 338 (2018) 912–952 MR

[4] **D Burago**, **S Ivanov**, *On asymptotic volume of tori*, Geom. Funct. Anal. 5 (1995) 800–808 MR

[5] **G R Chambers**, **E Portnoy**, *A note on Kalai's 3^d Conjecture*, preprint (2022) arXiv 2211.09215

[6] **D Faifman**, *A Funk perspective on billiards, projective geometry and Mahler volume*, J. Differential Geom. 127 (2024) 161–212 MR

[7] **M Fradelizi**, **M Meyer**, *Increasing functions and inverse Santaló inequality for unconditional functions*, Positivity 12 (2008) 407–420 MR

[8] **M Fradelizi**, **M Meyer**, **A Zvavitch**, *Volume product*, from "Harmonic analysis and convexity" (A Koldobsky, A Volberg, editors), Adv. Anal. Geom. 9, De Gruyter (2023) 163–222 MR

[9] **R Freij**, **M Henze**, **M W Schmitt**, **G M Ziegler**, *Face numbers of centrally symmetric polytopes produced from split graphs*, Electron. J. Combin. 20 (2013) art. id. 32 MR

[10] **G Kalai**, *The number of faces of centrally-symmetric polytopes*, Graphs Combin. 5 (1989) 389–391 MR

[11] **G Kalai**, *Challenges in discrete geometry, I: Convex polytopes*, recorded lecture, Fields Institute (2021)

[12] **B Klartag**, *Isotropic constants and Mahler volumes*, Adv. Math. 330 (2018) 74–108 MR

[13] **G A Margulis**, *Certain applications of ergodic theory to the investigation of manifolds of negative curvature*, Funkcional. Anal. i Priložen. 3 (1969) 89–90 MR In Russian; translated in Funct. Anal. Appl. 3 (1969) 335–336

[14] **M Meyer**, *Une caractérisation volumique de certains espaces normés de dimension finie*, Israel J. Math. 55 (1986) 317–326 MR

[15] **M Meyer**, **S Reisner**, *On the volume product of polygons*, Abh. Math. Semin. Univ. Hambg. 81 (2011) 93–100 MR

[16] **P Pansu**, *Croissance des boules et des géodésiques fermées dans les nilvariétés*, Ergodic Theory Dynam. Systems 3 (1983) 415–445 MR

[17] **S Reisner**, *Minimal volume-product in Banach spaces with a 1-unconditional basis*, J. London Math. Soc. 36 (1987) 126–136 MR

[18] **R T Rockafellar**, *Convex analysis*, Princeton Mathematical Series 28, Princeton Univ. Press (1970) MR

[19] **L A Santaló**, *An affine invariant for convex bodies of n-dimensional space*, Portugal. Math. 8 (1949) 155–161 MR In Spanish

[20] **R Sanyal**, **A Werner**, **G M Ziegler**, *On Kalai's conjectures concerning centrally symmetric polytopes*, Discrete Comput. Geom. 41 (2009) 183–198 MR

[21] **R Sanyal**, **M Winter**, *Kalai's 3^d conjecture for unconditional and locally anti-blocking polytopes*, Proc. Amer. Math. Soc. 153 (2025) 279–290 MR

[22] **C Schütt**, *The convex floating body and polyhedral approximation*, Israel J. Math. 73 (1991) 65–77 MR

[23] **M Senechal** (editor), *Shaping space: exploring polyhedra in nature, art, and the geometrical imagination*, Springer (2013) MR

[24] **C Vernicos**, **C Walsh**, *Flag-approximability of convex bodies and volume growth of Hilbert geometries*, Ann. Sci. Éc. Norm. Supér. 54 (2021) 1297–1314 MR

[25] **C Vernicos**, **C Walsh**, *Asymptotic Santaló points of centro-affine and Funk geometry*, in preparation (2025)

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