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Invariant prime ideals in equivariant Lazard rings

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Let A be an abelian compact Lie group. We compute the spectrum of invariant prime ideals of the A -equivariant Lazard ring, or equivalently the spectrum of points of the moduli stack of A -equivariant formal groups. We further show that this spectrum is homeomorphic to the Balmer spectrum of compact A -spectra, with the comparison map induced by equivariant complex bordism homology.

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1 Introduction

Let us pose the question: what algebraic input do we need to develop equivariant versions of chromatic homotopy theory?

Chromatic homotopy theory studies stable homotopy theory through the lens of formal groups, building on Quillen’s identification [1969; 1971] of the complex bordism ring $\pi_*\text{MU}$ with the Lazard ring. Around the same time, tom Dieck [1970] introduced for every compact Lie group A an equivariant analog of MU , the homotopical A -equivariant complex bordism MU_A . Letting A be abelian, Cole, Greenlees and Kriz [Cole et al. 2000] many years later found the correct notion of an A -equivariant formal group law. Recently, the first author generalized work of Hanke and Wiemeler [2018] and showed that $\pi_*^A\text{MU}_A$ is indeed the universal ring for A -equivariant formal group laws, thus establishing an equivariant analog of Quillen’s theorem for the equivariant Lazard ring L_A .

Many structural features of stable homotopy theory can be explained through the chromatic perspective. The central notion of chromatic homotopy theory is that of *height*. Honda classified formal groups over a separably closed field of characteristic p in terms of the height $0 \leq n \leq \infty$. Thus, the points of the moduli stack of formal groups \mathcal{M}_{FG} correspond to pairs (p, n) with $n = 0$ if and only if $p = 0$. Hopkins and Smith [1998] showed that the same classification pertains to thick subcategories of finite spectra: Given a finite spectrum X , its MU-homology MU_*X defines a coherent sheaf over \mathcal{M}_{FG} . Taking the support of MU_*X in the Zariski spectrum $|\mathcal{M}_{\text{FG}}|$ of points, we obtain a support theory on compact spectra. The thick subcategory theorem states that this support theory is the universal one. In other words, the induced map $|\mathcal{M}_{\text{FG}}| \rightarrow \text{Spec}(Sp^c)$ to the Balmer spectrum [2005; 2010] of compact spectra is a homeomorphism.

We show the following equivariant generalization (a more precise statement of which we give as Theorem 1.6):

Theorem 1.1 *Let A be an abelian compact Lie group. Then the spectrum of points of the moduli stack $\mathcal{M}_{\text{FG}}^A$ of A -equivariant formal groups is homeomorphic to the Balmer spectrum of compact A -spectra, with the comparison map induced by a support theory based on complex bordism homology $(\text{MU}_A)_*$.*

This establishes MU_A and the theory of equivariant formal groups as fundamental tools for building equivariant versions of chromatic homotopy theory.

For abelian groups as above, the Balmer spectrum of finite A -spectra has been computed completely by Balmer and Sanders [2017] (the case $A = C_p$), Barthel, Hausmann, Naumann, Nikolaus, Noel and Stapleton [Barthel et al. 2019] (the finite abelian case) and Barthel, Greenlees and Hausmann [Barthel et al. 2020] (the general abelian case). In a surprising turn of history, it was the algebraic counterpart which had not been computed before. As a set, both $|\mathcal{M}_{\text{FG}}^A|$ and $\text{Spec}(Sp_A^c)$ decompose as the disjoint union of one copy of $|\mathcal{M}_{\text{FG}}| \cong \text{Spec}(Sp^c)$ for every closed subgroup of A . Thus, the correct notion of height of an A -equivariant formal group F over a field of characteristic p consists of a pair: a height n of a nonequivariant formal group and a closed subgroup $B \subset A$ such that F is induced along the zig-zag $A \rightarrow A/B \leftarrow \{1\}$.

The more subtle information lies in the topology of the spectrum, which encodes on the algebraic level how heights can deform and on the homotopical level the chromatic interdependence of the various geometric fixed points $\Phi^B X$ of a compact A -spectrum X .

We will detail our results below in the language of invariant prime ideals. Crucially, we exhibit equivariant lifts v_n of the classical v_n and show that in many cases they provide a sequence of generators of invariant prime ideals. The nonequivariant v_n play an important role in many of the highlights of chromatic homotopy theory, like the greek-letter construction [Ravenel 1986], the construction of the Morava K-theories or the periodicity theorem [Hopkins and Smith 1998], and we hope that our equivariant lifts open the prospect of generalizing these to the equivariant context.

1.1 Invariant prime ideals and statement of results

As indicated above, the main theorem can also be stated in terms of invariant prime ideals of the equivariant Lazard ring L_A , as we now explain. Similarly to the nonequivariant case, L_A is the ground ring of a flat Hopf algebroid (L_A, S_A) , classifying A -equivariant formal group laws and their strict isomorphisms. By [Hausmann 2022, Theorem 5.52] and the discussion thereafter, this is isomorphic to the graded Hopf algebroid $(\pi_*^A \text{MU}_A, \pi_*^A \text{MU}_A \otimes \text{MU}_A)$ of cooperations. The associated stack is the moduli stack of A -equivariant formal groups. Hence, the category of graded (L_A, S_A) -comodules is equivalent to the category of quasicoherent sheaves over $\mathcal{M}_{\text{FG}}^A$.

Recall that an ideal I of L_A is called *invariant* (in the sense of Hopf algebroids) if it is a subcomodule, ie if $\eta_L(I)S_A = \eta_R(I)S_A$ for the left and right unit $\eta_L, \eta_R: L_A \rightarrow S_A$. Every invariant *prime* ideal \mathfrak{p} gives rise to a point of the moduli stack of prime ideals via the quotient field of L_A/\mathfrak{p} . This defines a map from the set of invariant prime ideals $\text{Spec}^{\text{inv}}(L_A)$ to $|\mathcal{M}_{\text{FG}}^A|$, which we show to be a homeomorphism in Theorem 4.7.

For the nonequivariant Lazard ring, Morava, as well as Landweber [1973, Theorem 2.7], showed that the invariant prime ideals are precisely the ideals $I_{p,n} = (v_0, \dots, v_{n-1})$ for a prime p and $n \in \mathbb{N} \cup \{\infty\}$ (with $I_{p,0}$ being the 0-ideal for all p).

To describe the invariant prime ideals in the equivariant case we recall that L_A contains universal Euler classes e_V for all characters $V \in A^*$, and that equivariant Lazard rings are contravariantly functorial in continuous group homomorphisms. In particular, all equivariant Lazard rings are algebras over the nonequivariant Lazard ring.

Then, given a nonequivariant invariant prime ideal $I_{p,n}$ and a closed subgroup B of A , we obtain an invariant prime ideal $I_{B,p,n}^A \subseteq L_A$ as the kernel of the composite

$$L_A \xrightarrow{\text{res}_B^A} L_B \rightarrow \Phi^B L \rightarrow \Phi^B L \otimes_L L/I_{p,n}.$$

Here, $\Phi^B L$ is defined as the localization of L_B away from all the Euler classes of nontrivial characters for B . The ring $\Phi^B L$ is an algebraic version of geometric fixed points and indeed agrees with the coefficient ring of the B -geometric fixed points of MU_B .

Theorem 1.2 (Theorem 4.7) *For every abelian compact Lie group A , the map*

$$\text{Sub}(A) \times \text{Spec}^{\text{inv}}(L) \rightarrow \text{Spec}^{\text{inv}}(L_A), \quad (B, I_{p,n}) \mapsto I_{B,p,n}^A,$$

is a bijection.

Here, $\text{Sub}(A)$ is the set of all closed subgroups of A . Hence, as for the Balmer spectrum, the invariant prime ideals of L_A decompose as a set as one copy of $\text{Spec}^{\text{inv}}(L)$ for every closed subgroup of A . And similarly to the Balmer spectrum, the main work then lies in understanding the Zariski topology,

in particular in determining the containments between invariant prime ideals associated to different subgroups.

We obtain the following:

Theorem 1.3 (Theorem 5.1) *There is an inclusion $I_{B,p,n}^A \subseteq I_{B',q,n'}^A$ if and only if*

- (1) B' is a subgroup of B ,
- (2) $p = q$ or $n = 0$ (in which case $I_{B,p,0}^A = I_{B,q,0}^A$), the components $\pi_0(B/B')$ are a p -group and $n' \geq n + \text{rank}_p(\pi_0(B/B'))$.

Comparing with [Barthel et al. 2020] we see that these correspond precisely to the inclusions in the Balmer spectrum, but with roles reversed: There is an inclusion $I_{B,p,n}^A \subseteq I_{B',q,n'}^A$ if and only if there is an inclusion $P_{B',q,n'}^A \subseteq P_{B,p,n}^A$. Here, $P_{B,p,n}^A = \{X \in \text{Sp}_A^c \mid K(n)_*(\Phi^B X) = 0\}$ are the thick subcategories of Sp_A^c with $K(n)$ being Morava K-theory at the prime p .

To show that $I_{B,p,n}^A$ indeed includes into $I_{B',q,n'}^A$ when conditions (1) and (2) are satisfied, one can reduce to the case $A = T$ the circle group where it is straightforward to describe explicit generators for the invariant prime ideals. The main step in ruling out further inclusions is the construction of equivariant refinements $\bar{v}_{n-1} \in L_{C_p^n}$ of the elements $v_{n-1} \in L$ which exhibit maximal height shifts (Definition 5.21, Proposition 5.24). Roughly speaking, \bar{v}_{n-1} is of height -1 at the top group C_p^n (ie it lies in the ideal $I_{C_p^n,p,0}^{C_p^n}$) while it is of height $n-1$ at the trivial group (ie it lies in the ideal $I_{\{1\},p,n}^{C_p^n}$ but not in $I_{\{1\},p,n-1}^{C_p^n}$). This is the algebraic analog of the existence of finite C_p^n -spectra of underlying type n whose C_p^n -geometric fixed points are rationally nontrivial as in [Barthel et al. 2019, Section 4] and [Kuhn and Lloyd 2024, Section 7]. More precisely, \bar{v}_{n-1} is canonically defined only modulo a certain smaller ideal (analogously to v_{n-1} only being defined uniquely up to the ideal I_{n-1} and up to a unit). More details are given in Section 5.3.

We further show that — at least over elementary abelian p -groups — the elements \bar{v}_i give rise to generators of the invariant prime ideals:

Theorem 1.4 (Theorem 6.1) *For all primes p and $n \in \mathbb{N}$ the elements*

$$p_1^* \bar{v}_0, p_2^* \bar{v}_1, \dots, p_{n-1}^* \bar{v}_{n-2}, \bar{v}_{n-1}$$

generate the ideal $I_{C_p^n,p,0}^{C_p^n}$. Here, $p_i : C_p^n \rightarrow C_p^i$ denotes the projection to the first i coordinates.

Suitable restrictions of the \bar{v}_n then form generators for the ideals $I_{C_p^n,p,m}^{C_p^n}$ at higher height m ; see Section 6. We emphasize that in contrast to the nonequivariant situation, the sequence of the $p_i^* \bar{v}_{i-1}$ is *not* a regular sequence. In fact, since $I_{C_p^n,p,0}^{C_p^n}$ consists precisely of the Euler-class-power torsion, it does not contain a nonzero divisor and hence cannot be generated by a regular sequence (unless $n = 0$). The torsion in the ring $L_{C_p^n}$ is closely linked to the torsion in the group of characters $(C_p^n)^*$. Hence one might hope that

$I_{B,n}^A$ is generated by a regular sequence whenever A is a torus, and indeed that is the case in all the cases we understand; see Remark 6.7.

Finally, to describe the Zariski topology we need one additional ingredient. When A is infinite, the set of closed subgroups $\text{Sub}(A)$ carries a nontrivial metric topology, turning it into a totally disconnected compact Hausdorff space. Together with the inclusions between the invariant prime ideals, this topology determines the Zariski topology on $\text{Spec}^{\text{inv}}(L_A)$.

Theorem 1.5 (Theorem 7.5) *The Zariski topology on $\text{Spec}^{\text{inv}}(L_A)$ has as basis the closed subsets C which are*

- (i) *closed under upward inclusions (ie if $I_{B',q,n'}^A \in C$ and $I_{B',q,n'}^A \subseteq I_{B,p,n}^A$, then $I_{B,p,n}^A \in C$), and*
- (ii) *are locally constant on $\text{Sub}(A)$ in the sense that every $B \in \text{Sub}(A)$ has a neighborhood U such that, for every n and p , either $I_{B',p,n}^A \in C$ for all $B' \in U$ or $I_{B',p,n}^A \notin C$ for all $B' \in U$.*

Comparing with [Barthel et al. 2020], we see that this description precisely matches the computation of the topology on the Balmer spectrum, with $I_{B,p,n}^A$ replaced by $P_{B,p,n}^A$. Hence the assignment

$$I_{B,p,n}^A \mapsto P_{B,p,n}^A$$

yields a homeomorphism from $\text{Spec}^{\text{inv}}(L_A)$ (and hence $|\mathcal{M}_{\text{FG}}^A|$) to $\text{Spec}(\text{Sp}_A^c)$. In the last section we explain that this comparison map can be obtained less ad hoc via MU_A -homology:

Theorem 1.6 (Section 8.1) *Let X be a compact A -spectrum and $I_{B,p,n}^A$ an invariant prime ideal. Then the localization $(\text{MU}_A)_{I_{B,p,n}^A} \wedge X$ is nontrivial if and only if the B -geometric fixed points $\Phi^B X$ are of type $\leq n$ at p , ie if and only if $P_{B,p,n}^A$ is in the Balmer support of X .*

This shows that

$$X \mapsto \text{supp}((\text{MU}_A)_* X) \subseteq \text{Spec}^{\text{inv}}(L_A) \cong |\mathcal{M}_{\text{FG}}^A|$$

defines a universal support theory on compact A -spectra and thus a homeomorphism $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$. Here, $(\text{MU}_A)_* X$ is the Mackey functor recording $(\text{MU}_B)_* \text{res}_B^A X$ for all closed subgroups B of A , and $\text{supp}((\text{MU}_A)_* X)$ is defined as the set of invariant prime ideals at which the localization of this Mackey functor is nontrivial.

Knowing that $X \mapsto \text{supp}((\text{MU}_A)_* X)$ is a (not necessarily universal) support theory with a characterization in terms of geometric fixed points as above already yields a continuous bijection $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$. This reproves one half of the main theorems of [Barthel et al. 2019] and [Barthel et al. 2020]. This half has been dubbed the chromatic Smith fixed point theorem in [Kuhn 2021] and [Balderrama and Kuhn 2024], where other proofs are given. To establish that our support theory is universal (and hence that $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$ is also open), we need additional topological input from [Barthel et al. 2019; 2020]; see also [Kuhn and Lloyd 2024].

We show in Proposition 8.11 that the support theory from Theorem 1.6 can alternatively be built by viewing $(\underline{\mathrm{MU}}_A)_*X$ as defining quasicoherent sheaves on $\mathcal{M}_{\mathrm{FG}}^B$ for every closed $B \subseteq A$; the union of the supports of these sheaves agrees with $\mathrm{supp}((\underline{\mathrm{MU}}_A)_*X)$ under the homeomorphism $\mathrm{Spec}^{\mathrm{inv}}(L_A) \cong |\mathcal{M}_{\mathrm{FG}}^A|$. To show this, we establish in Proposition 8.8 how the adjunction between Sp^A and $\mathrm{Sp}^{A/B}$ defined by geometric fixed points and pullback corresponds on the algebraic level to pullback and pushforward along the open immersion $\mathcal{M}_{\mathrm{FG}}^{A/B} \subseteq \mathcal{M}_{\mathrm{FG}}^A$ from Proposition 3.12.

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2 Equivariant formal groups

The aim of this section is to recall some basic definitions and properties about equivariant formal groups and equivariant formal group laws from [Cole et al. 2000; Greenlees 2001; Strickland 2011; Hausmann 2022]. To make our paper more self-contained, we also replicate some of the proofs in our language, and we provide some small extensions of known results. Our treatment of equivariant formal groups is far from exhaustive, and especially [Strickland 2011] contains a wealth of results we do not touch upon.

2.1 Basic definitions

In this subsection, we will recall the notions of an equivariant formal group and an equivariant formal group law over a commutative ring k . The definition of an equivariant formal group law is due to Cole, Greenlees and Kriz [Cole et al. 2000], and our definition of an equivariant formal group will be a variant of that of Strickland [2011].

For us, a *formal k -algebra* is a complete linearly topologized commutative k -algebra with a countable system of open ideals generating the topology. We define a functor F from formal k -algebras to the pro-category of commutative k -algebras, sending a formal k -algebra R to the pro-system (R/I) with $I \subseteq R$ running over all open ideals. The essential image is contained in the full subcategory $\mathrm{Pro}^{\mathrm{cont}, \mathrm{surj}}(\mathrm{CRing})$ on those pro-objects indexed on a countable directed set with surjective transition maps and actually equals this subcategory by the discussion after Definition 5.4 in [Yasuda 2009]. On this subcategory, taking the limit of the pro-system defines a right adjoint to F , which is also a one-sided inverse. Thus, F is fully faithful and defines an equivalence onto $\mathrm{Pro}^{\mathrm{cont}, \mathrm{surj}}(\mathrm{CRing})$.

For us, the category of *formal k -schemes* is the opposite of that of formal k -algebras. Using the above, it can be viewed as the category of ind-objects in affine k -schemes, indexed by a countable directed set with closed immersions as transition maps. We will sometimes use the notation $\text{Spec } R$ or $\text{Spf } R$ for the formal k -scheme associated to a formal k -algebra R , and \mathbb{O}_X for the formal k -algebra associated to a formal k -scheme X . The product on affine k -schemes induces one on formal k -schemes, and this corresponds to the completed tensor product on formal k -algebras.

We set $S = \text{Spec } k$. Given a countable set M , we view $S \times M$ as a formal scheme, namely as the colimit over all $S \times N$ with $N \subseteq M$ finite. This corresponds to giving k^M the product topology. If we just write M , we will apply this construction to $\text{Spec } \mathbb{Z}$ instead of S .

For a compact Lie group A , we will denote by $A^* = \text{Hom}(A, \mathbf{T})$ its Pontryagin dual, which is always a discrete group. We use ϵ for the unit element in A^* .

Definition 2.1 Given a compact abelian Lie group A , an *A -equivariant formal group over k* consists of a commutative group object X in formal k -schemes together with a group homomorphism $\varphi: S \times A^* \rightarrow X$ of formal k -schemes satisfying the following two conditions:

- (1) For the composite $\varphi_\epsilon: S \xrightarrow{\text{id}_S \times \epsilon} S \times A^* \xrightarrow{\varphi} X$, the augmentation ideal $I_\epsilon = \ker(R \rightarrow k)$ of the induced map is fpqc-locally on k a free R -module of rank 1, where $R = \mathbb{O}_X$.
- (2) The topology on R is generated by products of the ideals $I_V = \ker(R \xrightarrow{\varphi_V^*} k)$ for $V \in A^*$ and $\varphi_V: S \xrightarrow{\text{id}_S \times V} S \times A^* \xrightarrow{\varphi} X$.

Remark 2.2 In the above definition, one can easily replace $\text{Spec } k$ by an arbitrary quasicompact scheme S , with formal S -schemes being a suitable subcategory of the ind-category of Aff_S , the category of schemes affine over S .

Our definition differs in two aspects from that put forward in [Strickland 2011, Definition 2.15]. First, Strickland restricts to finite A^* . Second, Strickland asks $\ker(\mathbb{O}_X \rightarrow k)$ to be free of rank 1 instead of locally free [loc. cit., Proposition 2.10]. We changed it so that our definition satisfies descent. We could have asked, equivalently, that the augmentation ideal is Zariski locally on k a free R -module of rank 1 because line bundles satisfy fpqc-descent.

Remark 2.3 If we leave out the second condition in Definition 2.1, we get a notion that we call an *A -equivariant group*. The category of A -equivariant formal groups embeds into that of A -equivariant groups and this inclusion has a right adjoint, called *completion*. Concretely, this replaces R in the notation in Definition 2.1 by the formal k -algebra $\lim_{V_1, \dots, V_n \in A^*} R / (I_{V_1} \cdots I_{V_n})$. We will only use A -equivariant groups to complete them to A -equivariant formal groups.

If we fix a trivialization of the augmentation ideal I_ϵ in Definition 2.1, we can spell out the definition in more algebraic terms and obtain the notion of an equivariant formal group law.

Definition 2.4 An A -equivariant formal group law over k is a quadruple

$$(R, \Delta, \theta, y(\epsilon))$$

of a formal k -algebra R , a continuous comultiplication $\Delta: R \rightarrow R \hat{\otimes} R$, a map of k -algebras $\theta: R \rightarrow k^{A^*}$ and an orientation $y(\epsilon) \in R$, such that

- (i) the comultiplication is a map of k -algebras which is cocommutative, coassociative and counital for the augmentation $\theta_\epsilon: R \rightarrow k$,
- (ii) the map θ is compatible with the coproduct, and the topology on R is generated by finite products of the kernels of the component functions $\theta_V: R \rightarrow k$ for $V \in A^*$, and
- (iii) the element $y(\epsilon)$ is regular and generates the kernel of θ_ϵ .

Remark 2.5 If we want to remember the base of an equivariant formal group law, we sometimes also write it as a quintuple $(k, R, \Delta, \theta, y(\epsilon))$.

Lemma 2.6 An A -equivariant formal group $(\text{Spec } R, \varphi)$ over k together with an R -linear isomorphism $I_\epsilon \cong R$ of the augmentation ideal is equivalent datum to an A -equivariant formal group law over k .

Proof The maps Δ and θ are induced by the multiplication on $\text{Spec } R$ and φ , respectively. The element $y(\epsilon)$ corresponds to the trivialization of $I_\epsilon = \ker(R \rightarrow k)$. \square

Given any equivariant formal group $G = (\varphi: A^* \rightarrow X)$ over $S = \text{Spec } k$, we obtain for every $V \in A^*$ a morphism $\varphi_V: S \xrightarrow{\text{id}_S \times V} S \times A^* \xrightarrow{\varphi} X$. If $R = \mathbb{O}_X$, this corresponds to the morphism $\theta_V: R \rightarrow k$. Moreover, φ composed with left multiplication defines an A^* -action on X ; for every $V \in A^*$ this gives a map $l_V: R \rightarrow R$. In terms of the data of an equivariant formal group law $F = (k, R, \Delta, \theta, y(\epsilon))$, this can explicitly be written as

$$l_V: R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{\theta_V \hat{\otimes} \text{id}_R} R.$$

Given $V \in A^*$, we set

$$y(V) = l_V(y(\epsilon)) \in R,$$

which generates the kernel of $\theta_{V^{-1}}$. If A is trivial, we have $R \cong k[[y]]$. We want to describe an analog for general A . A *complete flag* for A is a sequence of characters $f = V_1, V_2, \dots \in (A^*)^{\mathbb{N}}$ such that every character appears infinitely often. Given such a flag and $n \in \mathbb{N}$, we set $W_n = V_n \oplus V_{n-1} \oplus \dots \oplus V_1$ and

$$y(W_n) = y(V_n)y(V_{n-1}) \cdots y(V_1).$$

Then every element x of R can be written uniquely as

$$(2.7) \quad x = \sum_{n \in \mathbb{N}} a_n^f y(W_n)$$

for coefficients $a_n^f \in k$ [Cole et al. 2000, Lemma 13.2]. Hence, as a k -module, R is isomorphic to a countable infinite product of copies of k .

We refer to [Cole et al. 2000] and [Hausmann 2022] for more information about equivariant formal group laws.

2.2 Lazard rings

Our aim in this subsection is to recall the definition of the universal ring for equivariant formal group laws and to clarify its universal property. Let us begin by considering a very strict form of morphisms of equivariant formal group laws.

Definition 2.8 A *morphism* between A -equivariant formal group laws $(k_1, R_1, \Delta_1, \theta_1, y(\varepsilon)_1)$ and $(k_2, R_2, \Delta_2, \theta_2, y(\varepsilon)_2)$ is a pair of maps $f: k_1 \rightarrow k_2$ and $g: R_1 \rightarrow R_2$ which are compatible with both the comultiplications Δ and the augmentations θ , and which send $y(\varepsilon)_1$ to $y(\varepsilon)_2$.

This leads to a category $A\text{-FGL}$ of A -equivariant formal group laws. In [Cole et al. 2000, Section 14] it is shown that this category has an initial object F^{uni} , the ground ring of which is called the *A-equivariant Lazard ring* and denoted by L_A . In fact, the category of A -equivariant formal group laws is equivalent to the category of commutative rings under L_A . To discuss this, note first that the forgetful functor

$$A\text{-FGL} \rightarrow \text{CRing}, \quad (k, R, \Delta, \theta, y(\varepsilon)) \mapsto k,$$

into the category of commutative rings is cofibered in groupoids. Concretely this boils down to the following two observations:

- Every morphism of A -equivariant formal group laws whose first component $f: k_1 \rightarrow k_2$ is the identity map is an isomorphism. One observes indeed that the diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\quad} & R_2 \\ & \searrow \cong & \swarrow \cong \\ & \prod_{\mathbb{N}} k_1 = \prod_{\mathbb{N}} k_2 & \end{array}$$

obtained from equation (2.7) commutes. We call such an isomorphism living over the identity a *very strict isomorphism* between A -equivariant formal group laws, in order to distinguish from other kinds of isomorphisms.

- Given a morphism $f: k_1 \rightarrow k_2$ and an A -equivariant formal group law F over k_1 , one can define a pushforward f_*F over k_2 with the usual universal property. Its underlying k_2 -algebra is given by a completion of $R \otimes_{k_1} k_2$, where R is the underlying k_1 -algebra of F ; see [Greenlees 2001, Section 2.E], [Hausmann 2022, Section 2.3].

Note that the only automorphism of an A -equivariant formal group law over k which is also a very strict isomorphism is the identity map. Hence, given two A -equivariant formal group laws over k , there either exists a unique very strict isomorphism between them or none at all. For this reason it is usually harmless to identify two very strictly isomorphic A -equivariant formal group laws, and we will often do so.

Now, given a map $f: L_A \rightarrow k$ there is an induced A -equivariant formal group law f_*F^{uni} over k obtained by pushing forward the universal A -equivariant formal group law. Given an A -equivariant formal group law F over k , we can apply this to the first component $f: L_A \rightarrow k$ of the unique map of A -equivariant formal group laws $F^{\text{uni}} \rightarrow F$. The resulting morphism $f_*F^{\text{uni}} \rightarrow F$ is necessarily a very strict isomorphism. So, as claimed above, we obtain:

Corollary 2.9 *The functor*

$$\text{CAlg}_{L_A} \rightarrow A\text{-FGL}, \quad (f: L_A \rightarrow k) \mapsto f_*F^{\text{uni}},$$

from commutative L_A -algebras to A -equivariant formal group laws is an equivalence of categories. An inverse is given by sending an A -equivariant formal group law $F = (k, R, \Delta, \theta, y(\varepsilon))$ to the first component $f: L_A \rightarrow k$ of the unique morphism $F^{\text{uni}} \rightarrow F$.

Remark 2.10 The above proof is an instance of a general characterization of initial objects X in categories cofibered in groupoids $\mathcal{C} \xrightarrow{F} \mathcal{D}$, namely that pushforward defines an equivalence of $\mathcal{D}_{F(X)/-}$ with \mathcal{C} .

Remark 2.11 Nonequivariantly the k -algebra R is often fixed to be the power series ring $k[[y(\varepsilon)]]$ rather than a ring only isomorphic to it. With this convention, the category of formal group laws is *isomorphic* (not merely equivalent) to the category of commutative rings under L . In other words, every very strict isomorphism of formal group laws is the identity. Equivariantly one needs to be a little more careful: The statement “ A -equivariant formal group laws are represented by the A -equivariant Lazard ring” is only true up to this notion of very strict isomorphism.

2.3 Global functoriality

In this subsection, we will discuss both a covariant and a contravariant functoriality of the category of A -equivariant formal groups in A .

Definition 2.12 Let $\alpha: B \rightarrow A$ be a group homomorphism and let $G = (B^* \rightarrow X)$ be a B -equivariant formal group. We define the *corestriction* α_*G to be the A -equivariant formal group which is the completion of the A -equivariant group $(A^* \xrightarrow{\alpha^*} B^* \rightarrow X)$.

Proposition 2.13 *For every injective group homomorphism $\alpha: B \rightarrow A$, the functor α_* from B -equivariant formal groups to A -equivariant formal groups is fully faithful. The essential image consists of those A -equivariant formal groups where the homomorphism from A^* factors through B^* .*

Proof For every B -equivariant formal group G , by construction α_*G is the same group object as G in formal schemes with the structure morphism $A^* \rightarrow B^* \rightarrow G$ (since $A^* \rightarrow B^*$ is surjective, no completion is necessary). This implies full faithfulness. \square

Upon choosing coordinates, Definition 2.12 corresponds to the construction in Section 2.4 of [Hausmann 2022]: given a B -equivariant formal group law $F = (k, R, \Delta, \theta, y(\varepsilon))$ and a group homomorphism $\alpha: B \rightarrow A$, there is an induced A -equivariant formal group law α_*F over the same ring k , given by completing R at products of the ideals $I_V = \ker(R \xrightarrow{\theta_V} k)$ for those $V \in B^*$ which are in the image of $\alpha^*: A^* \rightarrow B^*$. This defines a functor α_* from B -equivariant formal group laws to A -equivariant formal group laws, which induces a map $\alpha^*: L_A \rightarrow L_B$ on Lazard rings. Hence we obtain a functor

$$L: (\text{abelian compact Lie groups})^{\text{op}} \rightarrow \text{commutative rings},$$

which we call the *global Lazard ring*. As shown in [Hausmann 2022], L_A is isomorphic to $\pi_*^A \text{MU}_A$ and our map α^* corresponds to the restriction map on that level, explaining our terminology.

Remark 2.14 By Pontryagin duality, the opposite category of abelian compact Lie groups is equivalent to the category of finitely generated abelian groups. Therefore, everything in this paper could alternatively be phrased in terms of finitely generated abelian groups rather than abelian compact Lie groups, and the more algebraically minded reader might prefer to do so.

In addition to this covariant functoriality, there is also a contravariant functoriality.

Definition 2.15 Let $\alpha: A \rightarrow C$ be a surjective group homomorphism and let $G = (C^* \rightarrow X)$ be a C -equivariant group. We define the *coinduction* α^*G to be the A -equivariant group $(A^* \rightarrow X \times_{C^*} A^*)$, where the target denotes the quotient of $X \times A^*$ by the antidiagonal C^* -action.

Lemma 2.16 For $\alpha: A \rightarrow C$ a surjective group homomorphism and G a C -equivariant formal group, α^*G is an A -equivariant formal group, ie needs no additional completion.

Proof If $G = (C^* \rightarrow \text{Spec } R)$ is a C -equivariant formal group, then

$$\text{Spec } R \times_{C^*} A^* \cong \text{Spec } \text{Map}_{C^*}(A^*, R).$$

Any choice of coset representatives defines an isomorphism $\text{Map}_{C^*}(A^*, R) \cong \prod_{A^*/C^*} R$ of formal k -algebras, with respect to the product topology on the latter. As products of complete rings are complete, the claim follows. □

Proposition 2.17 Let $\alpha: A \rightarrow C$ be a surjective group homomorphism. As functors between C -equivariant formal groups and A -equivariant formal groups, α^* is the left adjoint of α_* .

Proof For an A -equivariant group $G = (A^* \rightarrow X)$, define $\tilde{\alpha}_*G$ as $C^* \rightarrow A^* \rightarrow X$. Then α^* and $\tilde{\alpha}_*$ are adjoints between C -equivariant groups and A -equivariant groups in the sense of Remark 2.3. Since completion is a right adjoint, the result follows from the previous lemma. □

2.4 Euler classes

Given an A -equivariant formal group law $F = (R, \Delta, \theta, y(\epsilon))$ over k , we can define Euler classes in k . Recall that for $V \in A^*$, we set $y(V) = l_V(y(\epsilon)) \in R$. The corresponding Euler class is

$$e_V = \theta_\epsilon(y(V)) = \theta_V(y(\epsilon)) \in k.$$

In terms of the associated equivariant formal group $G = (\varphi: A^* \rightarrow X)$, we have

$$S \times_{\varphi_\epsilon, X, \varphi_V} S \cong \text{Spec}(k \otimes_{\theta_\epsilon, R, \theta_V} k) \cong \text{Spec}(k/e_V)$$

with $S = \text{Spec } k$ and φ_V being the composite $S \cong S \times \{V\} \subseteq S \times A^* \xrightarrow{\varphi} X$. This implies:

Lemma 2.18 *For a given A -equivariant formal group law with notation as above:*

- (1) *The Euler class e_V is invertible if and only if $S \times_{\varphi_\epsilon, X, \varphi_V} S = \emptyset$.*
- (2) *The Euler class e_V is zero if and only if $\varphi_V = \varphi_\epsilon$.*

Thus, the vanishing or invertibility of Euler classes does not depend on chosen coordinates. This allows us to generalize these concepts to A -equivariant formal groups in the following way:

Definition 2.19 *For an A -equivariant formal group $G = (\varphi: A^* \rightarrow X)$, we say that the Euler class e_V is invertible if $S \times_{\varphi_\epsilon, X, \varphi_V} S = \emptyset$ and that e_V is zero if $\varphi_V = \varphi_\epsilon$.*

Informally, e_V is invertible if and only if the images of $S \times \{\epsilon\}$ and $S \times \{V\}$ in X are disjoint.

Example 2.20 Let $\mathbb{G}_m = \text{Spec } \mathbb{Z}[x^{\pm 1}]$ be the multiplicative group over $S = \text{Spec } \mathbb{Z}$. We choose the group homomorphism $\varphi: C_2 = (C_2)^* \rightarrow \mathbb{G}_m$ picking the units $\{\pm 1\}$ in \mathbb{Z} . This defines the structure of a C_2 -equivariant group, and its completion is a C_2 -equivariant formal group we call $\widehat{\mathbb{G}}_m^{C_2}$. Let $V \in (C_2)^*$ be the unique nontrivial character. One computes

$$\text{Spec } \mathbb{Z} \times_{\varphi_\epsilon, \widehat{\mathbb{G}}_m^{C_2}, \varphi_V} \text{Spec } \mathbb{Z} \cong \text{Spec } \mathbb{Z} \times_{\varphi_\epsilon, \mathbb{G}_m, \varphi_V} \text{Spec } \mathbb{Z} \cong \text{Spec } \mathbb{Z}/2.$$

Thus, $e_V = \pm 2$, depending on the choice of coordinate. See also [Strickland 2011, Section 7] and [Greenlees 2001, Section 7] for more information on this and related examples. Note in particular that our example is the pushforward of the true C_2 -equivariant multiplicative formal group (given by a completion of $\text{Spec } \mathbb{Z}[(C_2 \times T)^*]$) along the map $\mathbb{Z}[C_2^*] \rightarrow \mathbb{Z}$ classifying ± 1 .

We will use the following lemma several times, taken from [Hausmann 2022, Corollary 2.8] and the explanation thereafter:

Lemma 2.21 *Let $B \subseteq A$ be a subgroup. Then the restriction map $L_A \rightarrow L_B$ is surjective, with kernel I_A^B generated by the Euler classes e_V where V runs over a generating set of $\ker(A^* \rightarrow B^*)$.*

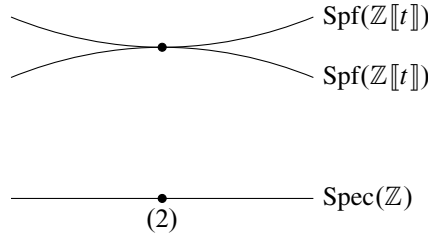


Figure 1: A schematic picture of $\widehat{\mathbb{G}}_m^{C_2}$ from Example 2.20.

Similarly, the following holds on the level of equivariant formal groups.

Proposition 2.22 *Let G be an A -equivariant formal group.*

- (1) *Let $\alpha: A \rightarrow C$ be a surjective group homomorphism. Assume that for $V \notin \text{im}(C^* \rightarrow A^*)$, the Euler class e_V is invertible. Then $\alpha^* \alpha_* G \rightarrow G$ is an isomorphism.*
- (2) *Let $\alpha: B \rightarrow A$ be an injective group homomorphism. Assume that for $V \in \text{ker}(A^* \rightarrow B^*)$, we have $e_V = 0$. Then G is in the essential image of α_* .*

Proof Let $G = (\varphi: A^* \rightarrow X)$. Fixing a coordinate $y(\epsilon)$ Zariski-locally and choosing a complete flag, X is defined by the directed system $(\text{Spec } R/(y(V_1) \cdots y(V_n)))_n$ and each of these terms has underlying space $\bigcup_{i=1}^n \text{im}(\varphi_{V_i}) \subseteq \text{Spec } R$.

In the first item, Lemma 2.18 implies that $\text{im}(\varphi_V)$ and $\text{im}(\varphi_W)$ intersect each other in $\text{Spec } \mathbb{C}_X$ only if $[V] = [W] \in A^*/C^*$. Thus, the underlying space of $(\text{Spec } R/(y(V_1) \cdots y(V_n)))_n$ decomposes into closed subspaces $\bigsqcup_{V \in A^*/C^*} \bigcup_{V_i \in V} \text{im}(\varphi_{V_i})$, of which only finitely many are nonempty. This induces decompositions of the schemes $\text{Spec } R/(y(V_1) \cdots y(V_n))$ and we obtain thus an A^* -equivariant isomorphism $G \cong \bigsqcup_{A^*/C^*} \alpha_* G$ on the level of formal schemes.

By construction, $\alpha^* \alpha_* G$ decomposes in the same way. On the unit copy, the map $\alpha^* \alpha_* G \rightarrow G$ is an isomorphism since $\alpha_* \alpha^* \alpha_* G \rightarrow \alpha_* G$ is one by Proposition 2.17. For the other copies, this follows by the A^* -equivariance of the map $\alpha^* \alpha_* G \rightarrow G$.

For the second item: by definition, the structure morphism $\varphi: A^* \rightarrow X$ of G factors as $A^* \rightarrow B^* \xrightarrow{\varphi'} X$. The result follows from Proposition 2.13. □

The second part is also true in the setting of equivariant formal group laws, as shown in [Hausmann 2022, Lemma 2.7]. The analog of the first part becomes more complicated as a coordinate on \widetilde{G} does not determine a canonical coordinate on $\alpha^* \widetilde{G}$ and hence α^* does not define a functor from C -equivariant formal group laws to A -equivariant formal group laws without additional choices; we will discuss this further in Proposition 2.25.

Corollary 2.23 *Let $\alpha: A \rightarrow B$ be a surjective group homomorphism. Then α^* is a fully faithful embedding from the category of B -equivariant formal groups to that of A -equivariant formal groups. The image consists of those A -equivariant formal groups such that e_V is invertible for V not in the image of $B^* \rightarrow A^*$.*

Proof If G is a B -equivariant formal group, then α^*G has the property that e_V is invertible for V not in the image of $B^* \rightarrow A^*$ by construction. By the preceding proposition, invertibility of these Euler classes characterizes the image of α^* . Moreover, α^* is fully faithful since $G \rightarrow \alpha_*\alpha^*G$ is an isomorphism by construction. □

The following proposition essentially provides a classification of equivariant formal groups over fields. The same statement already appears in [Strickland 2011, Corollary 8.3].

Proposition 2.24 *Let k be a field and G be an A -equivariant formal group over k . Denote by A/B the Pontryagin dual of the subgroup $\{V \in A^* \mid e_V = 0\} \subseteq A^*$. Denote further by $A \xrightarrow{q} A/B \xleftarrow{i} \{1\}$ the obvious morphisms. Then $G \cong q^*i_*p_*G$, where p_*G is the **nonequivariant** formal group defined by $p: A \rightarrow \{1\}$.*

Proof Assume $e_V = 0$, ie $\varphi_\epsilon = \varphi_V$. By the A^* -action, this implies $\varphi_W = \varphi_{VW}$ for every $W \in A^*$. Setting $W = V^{-1}$, this implies $e_{V^{-1}} = 0$. Moreover, if $\varphi_\epsilon = \varphi_W$, this implies $\varphi_\epsilon = \varphi_{VW}$. Thus, $\{V \in A^* \mid e_V = 0\} \subseteq A^*$ is indeed a subgroup.

Since k is a field, $e_V = 0$ if and only if e_V is not invertible. By Proposition 2.22, we thus have $G \cong q^*i_*\Gamma$ for some nonequivariant formal group Γ over k . One computes $\Gamma \cong p_*q^*i_*\Gamma \cong p_*G$. □

For any A , let $\Phi^A L = L_A[e_V^{-1}]$ be the localization of L_A away from all Euler classes e_V for $V \neq \epsilon$. Our results above let us compute $\Phi^A L$ quite explicitly; cf [Greenlees 2001, Corollary 6.4] and [Hausmann 2022, Proposition 2.11].

Proposition 2.25 *There is a ring isomorphism of the form*

$$\Phi^A L \cong L[(b_0^V)^{\pm 1}, b_i^V \mid i > 0, V \in A^* - \{\epsilon\}],$$

where the b_i^V are independent polynomial variables.

Proof The ring $\Phi^A L$ classifies A -equivariant formal group laws $F = (R, \Delta, \theta, y(\epsilon))$ such that e_V is invertible for all $V \neq \epsilon$. By Proposition 2.22, $R \cong \text{map}(A^*, \hat{R})$, where \hat{R} is the completion of R at the augmentation ideal. The structure of F determines the structure of a nonequivariant formal group law p_*F on \hat{R} , where $p: A \rightarrow \{1\}$ is the projection; in particular, we obtain an isomorphism $\hat{R} \cong k[[y]]$, where y is the image of $y(\epsilon)$. Conversely, Δ and θ are determined by p_*F . In particular, θ_ϵ is the composite $\text{map}(A^*, \hat{R}) \xrightarrow{\text{ev}_\epsilon} \hat{R} \rightarrow k$. Thus we see that F is determined by p_*F , plus a choice of $y(\epsilon)$ mapping to y under ev_ϵ . Such $y(\epsilon)$ are exactly those elements $(y^V) \in \text{map}(A^*, k[[y]])$ such that $y^\epsilon = y$ and $y^V = b_0^V + b_1^V y + \dots$ with $b_0^V \in k^\times$. This gives the result. □

Remark 2.26 The elements $b_i^V \in \Phi^A L$ in the previous proposition already come from elements $\gamma_i^V \in L_A$, which are uniquely determined by the property that

$$e_{\epsilon \otimes \tau} = \gamma_0^V + \gamma_1^V e_{V^{-1} \otimes \tau} + \gamma_2^V (e_{V^{-1} \otimes \tau})^2 + \cdots + \gamma_n^V (e_{V^{-1} \otimes \tau})^n \in L_{A \times T} / (e_{V^{-1} \otimes \tau})^{n+1}$$

for all $n \in \mathbb{N}$. Here, $\tau \in T^*$ denotes the tautological character for the circle group T , and we view $L_{A \times T}$ as an algebra over L_A via restriction along the projection $A \times T \rightarrow A$. In particular, γ_0^V equals the Euler class e_V . The elements γ_i^V are natural in the sense that $\alpha^* \gamma_i^V = \gamma_i^{\alpha^* V}$ for every group homomorphism $\alpha: B \rightarrow A$. We refer to [Hausmann 2022, Section 2.7] for more details on this construction.

2.5 The relationship between Lazard rings at different groups and their completions

While not needed for our *classification* of invariant prime ideals, it will be necessary for our study of *containments* between invariant prime ideals to have a deeper look upon how Lazard rings at different groups relate. These properties are all based on the identification of the global Lazard ring with equivariant complex bordism in [Hausmann 2022].

Proposition 2.27 [Hausmann 2022, Proposition 5.50, Corollary 5.33, Lemma 5.28] (1) For every A and every nontorsion character $V \in A^*$, the sequence

$$0 \rightarrow L_A \xrightarrow{e_V} L_A \xrightarrow{\text{res}_{\ker(V)}^A} L_{\ker(V)} \rightarrow 0$$

is exact. In particular, all Euler classes $e_V \in L_A$ for nontorsion characters V are nonzero divisors.

(2) For every A , the complete L_A -Hopf algebra R of the universal A -equivariant formal group law is canonically isomorphic to the completion

$$\lim_{n \in \mathbb{N}, V_1, \dots, V_n \in A^*} (L_{A \times T}) / I_{V_1} \cdots I_{V_n},$$

where I_{V_j} is the kernel of the restriction map $(\text{id}, V_j)^*: L_{A \times T} \rightarrow L_A$. More generally, $R^{\widehat{\otimes} n}$ is a completion of $L_{A \times T^n}$. Under this identification,

- (a) the comultiplication $R \rightarrow R^{\widehat{\otimes} 2}$ and the augmentations $\theta_V: R \rightarrow L_A$ are induced by the maps $(\text{id}_A, m)^*: L_{A \times T} \rightarrow L_{A \times T \times T}$ and $(\text{id}_A, V)^*: L_{A \times T} \rightarrow L_A$ on completion, and
- (b) the elements $y(V)$ are the image of $e_{V \otimes \tau} \in L_{A \times T}$ under the completion map, where $\tau \in T^*$ is the tautological character.

The special case of (2) for A the trivial group is particularly important: Completing L_{T^n} at the kernel I of the augmentation $L_{T^n} \rightarrow L$ yields a power series ring on n generators $L[[y_1, y_2, \dots, y_n]]$, where y_i is the Euler class of the i^{th} projection $T^n \rightarrow T$. Moreover, the isomorphism

$$(L_{T^n})_I^{\widehat{\otimes}} \cong L[[y_1, y_2, \dots, y_n]]$$

is natural in T^n , where

- the functoriality of $(L_{T^n})_I^{\widehat{\otimes}}$ is induced by the global functoriality of L_{T^n} , and
- the functoriality of $L[[y_1, y_2, \dots, y_n]]$ is through the universal formal group law over L .

For example, for any $n \in \mathbb{N}$ the square

$$\begin{CD} L_{\mathbf{T}} @>c>> L[[y]] \\ @V[n]^*VV @VV[n]_FV \\ L_{\mathbf{T}} @>c>> L[[y]] \end{CD}$$

commutes, where $[n]: \mathbf{T} \rightarrow \mathbf{T}$ is the n^{th} power map, and $[n]_F: L[[y]] \rightarrow L[[y]]$ sends y to the n -series $[n]_F(y)$ with respect to the universal formal group law. This implies that the Euler class $e_{\tau^n} \in L_{\mathbf{T}}$ for the n^{th} power map on \mathbf{T} is sent to the n -series $[n]_F(y)$ under the completion map. Similar statements hold for the collection of $L_{A \times \mathbf{T}^n}$ for fixed A and varying \mathbf{T}^n .

We further record, where again $\tau: \mathbf{T} \rightarrow \mathbf{T}$ denotes the identity character:

Corollary 2.28 *Let $x \in L_{\mathbf{T}}$ be an element whose image in $L[[y]]$ is of the form $\lambda y^k + \text{higher-order terms}$. Then x is uniquely divisible by e_{τ}^k , and the quotient x/e_{τ}^k restricts to λ at the trivial group.*

Proof First we note that $e_{\tau} \in L_{\mathbf{T}}$ is a regular element by Proposition 2.27. Hence division by e_{τ} is always unique if possible. By induction on k the corollary then follows from the following facts, for an element $z \in L_{\mathbf{T}}$ and its image $c(z) \in L_{\mathbf{T}}$:

- (1) z is divisible by e_{τ} if and only if $\text{res}_1^{\mathbf{T}}(z) = 0$.
- (2) The leading coefficient of $c(z)$ is equal to $\text{res}_1^{\mathbf{T}}(z)$.
- (3) If $\text{res}_1^{\mathbf{T}}(z) = 0$ and hence z is divisible by e_{τ} , then $c(z/e_{\tau}) = c(z)/y$. □

The global functoriality makes every equivariant Lazard ring L_A into an algebra over the nonequivariant Lazard ring L such that all restriction maps $\alpha^*: L_A \rightarrow L_B$ are L -algebra maps. The following property is shown in [Hausmann 2022, Corollary 5.50] as a consequence of work of Comezaña [1996].

Proposition 2.29 *For every abelian compact Lie group A , the equivariant Lazard ring L_A is free as a module over L .*

Corollary 2.30 *The exact sequences of L -modules in Proposition 2.27(1) are split exact. In particular, they remain exact after applying any additive functor.*

The following special case is of particular importance to us:

Corollary 2.31 *Let $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $I_n \subseteq L$ be the ideal generated by (v_0, \dots, v_{n-1}) . Then for every A and every nontorsion character $V \in A^*$, the sequence*

$$0 \rightarrow L_A/(I_n \cdot L_A) \xrightarrow{e_V} L_A/(I_n \cdot L_A) \xrightarrow{\text{res}_{\ker(V)}^A} L_{\ker(V)}/(I_n \cdot L_{\ker(V)}) \rightarrow 0$$

is exact. In particular, the Euler classes $e_V \in L_A$ for nontorsion characters $V \in A^$ remain nonzero divisors in L_A/I_n . In the terminology of [Hausmann 2022], the assignment*

$$A \mapsto L_A/(I_n \cdot L_A)$$

(together with the image of e_{τ} under $L_{\mathbf{T}} \rightarrow L_{\mathbf{T}}/(I_n \cdot L_{\mathbf{T}})$) is a regular global group law.

3 The Lazard Hopf algebroid and its associated stack

3.1 Strict isomorphisms and the Lazard Hopf algebroid

In this subsection, we will introduce one of our main objects of study, the Hopf algebroid (L_A, S_A) for equivariant formal group laws. There is a hierarchy of notions of isomorphisms between (equivariant) formal group laws, namely

- isomorphisms, which do not need to respect the coordinate and are thus really isomorphisms between the underlying (equivariant) formal groups;
- strict isomorphisms, which respect the coordinate up to quadratic terms;
- very strict isomorphisms as in Section 2.2, which respect the coordinate strictly.

Already classically, strict isomorphisms are especially relevant since the Hopf algebroid modeled on them gives (MU_*, MU_*MU) .

Definition 3.1 A *strict isomorphism* between two A -equivariant formal group laws

$$(k, R_1, \Delta_1, \theta_1, y(\varepsilon)_1) \quad \text{and} \quad (k, R_2, \Delta_2, \theta_2, y(\varepsilon)_2)$$

over the same ground ring k is a k -linear isomorphism

$$\varphi: R_1 \xrightarrow{\cong} R_2$$

of Hopf algebras over k^{A^*} such that $y(\varepsilon)_1$ is sent to $y(\varepsilon)_2$ modulo I_ε^2 , where I_ε is the augmentation ideal in R_2 . Explicitly, this means that $(\varphi \otimes \varphi) \circ \Delta_1 = \Delta_2 \circ \varphi$, $\theta_2 \circ \varphi = \theta_1$ and $\varphi(y(\varepsilon)_1) = x \cdot y(\varepsilon)_2$ for some unit $x \in R_2$ which augments to $1 \in k$.

By definition, strict isomorphisms need not preserve the coordinate, hence they are generally not morphisms of formal group laws in the sense of Definition 2.8. On the other hand, every very strict isomorphism is both a strict isomorphism and an isomorphism in the category of A -equivariant formal group laws.

Let SI be the category of strict isomorphisms of A -equivariant formal group laws. More precisely, its objects are quadruples (k, F^1, F^2, φ) consisting of a commutative ring k , two A -equivariant formal group laws F^1 and F^2 over k and a strict isomorphism φ between them. Morphisms between two such quadruples $(k_1, F_1^1, F_1^2, \varphi_1)$ and $(k_2, F_2^1, F_2^2, \varphi_2)$ are given by a pair of morphisms $f_1: F_1^1 \rightarrow F_2^1$ and $f_2: F_1^2 \rightarrow F_2^2$ in the sense of Definition 2.8, with the same underlying map $k_1 \rightarrow k_2$, such that

$$\begin{array}{ccc} R_1^1 & \xrightarrow{\varphi_1} & R_1^2 \\ f_1 \downarrow & & \downarrow f_2 \\ R_2^1 & \xrightarrow{\varphi_2} & R_2^2 \end{array}$$

commutes.

Proposition 3.2 *The category SI has an initial object, whose underlying ring S_A is a localization of an infinite polynomial ring $L_A[a_1^f, a_2^f, \dots]$ over the Lazard ring L_A .*

By Remark 2.10, we can equivalently say that there is an object $(S_A, F^{\text{uni}}, \tilde{F}^{\text{uni}}, \varphi) \in \text{SI}$ such that the functor

$$\text{CAlg}_{S_A} \rightarrow \text{SI}, \quad (f : S_A \rightarrow k) \mapsto (k, f_* F^{\text{uni}}, f_* \tilde{F}^{\text{uni}}, f_* \varphi),$$

is an equivalence of categories. Here, we use that SI is again cofibered in groupoids over commutative rings.

Before we prove the proposition, it will be good to review two general results about the maps $\theta_V : R \rightarrow k^{A^*} \rightarrow k$ for an A -equivariant formal group law $(k, R, \Delta, \theta, y(\varepsilon))$. Recall that after choosing a complete flag f we can write every element $x \in R$ uniquely as

$$x = \sum_{n \in \mathbb{N}} a_n^f y(W_n).$$

Given $V \in A^*$, we have

$$\theta_V(x) = \sum_{n \in \mathbb{N}} a_n^f \theta_V(y(W_n)) = \sum_{n \in \mathbb{N}} a_n^f e_{V W_n},$$

where $e_{V W_n}$ is defined as the product $e_{V \cdot V_n} e_{V \cdot V_{n-1}} \cdots e_{V \cdot V_1}$. This is a finite sum, since $e_{V W_n} = 0$ if there exists some $i \leq n$ where $V_i = V^{-1}$. We obtain:

Lemma 3.3 *The augmentation θ_V is a linear combination of a_n^f whose coefficients are products of Euler classes.*

Moreover, we have the following:

Lemma 3.4 *Let $F = (k, R, \Delta, \theta, y(\varepsilon))$ be an A -equivariant formal group law. Then an element $x \in R$ is a unit if and only if $\theta_V(x)$ is a unit in k for all $V \in A^*$.*

Proof See [Hausmann 2022, Lemma 2.3]. □

Proof of Proposition 3.2 For every object $(k, F^1, F^2, \varphi) \in \text{SI}$ we can define a new A -equivariant formal group law \tilde{F}^2 for which the components k, R, Δ and θ agree with those of F^1 , but $\tilde{y}(\varepsilon)_2$ is defined as $\varphi^{-1}(y(\varepsilon)_2)$, the preimage of the coordinate of F^2 under φ . Then we obtain a new object $(k, F^1, \tilde{F}^2, \text{id}_{R_1})$, which is isomorphic in SI to (k, F^1, F^2, φ) via the commutative square

$$\begin{array}{ccc} F^1 & \xrightarrow{\text{id}} & \tilde{F}^2 \\ \text{id} \downarrow & & \downarrow \varphi \\ F^1 & \xrightarrow{\varphi} & F^2 \end{array}$$

Note that the two vertical maps are in fact very strict isomorphisms of A -equivariant formal group laws. In summary, every object of SI is isomorphic to one of the form (k, F_1, F_2, id) , where F_1 and F_2 are given by the same k^{A^*} -augmented k -Hopf algebra and the strict isomorphism is the identity. This is the

same data as a single A -equivariant formal group law F together with a second choice of coordinate $y(\varepsilon)_2 = x \cdot y(\varepsilon)_1$ for some unit $x \in R$ which augments to 1. Up to very strict isomorphism we can further assume that F is the pushforward along a map $L_A \rightarrow k$. We claim that the functor sending an A -equivariant formal group law to the set of all units $x \in R$ augmenting to 1 is representable by an L_A -algebra S_A . Indeed, a presentation for S_A is given by

$$(3.5) \quad S_A = L_A[a_1^f, a_2^f, \dots][P_V(1, a_1^f, a_2^f, \dots)^{-1} \mid V \in A^* - \{\varepsilon\}],$$

where f is a complete flag starting with ε and P_V is the linear combination expressing θ_V in terms of the coefficients with respect to f ; see Lemmas 3.3 and 3.4. Here we use that the elements augmenting to 1 are precisely the elements of the form $1 + \sum_{n \in \mathbb{N}^+} a_n^f y(W_n)$, with $y(W_n)$ as in the end of Section 2.1.

To conclude, let F^{uni} be the pushforward of the universal equivariant formal group law along $L_A \rightarrow S_A$, and let \tilde{F}^{uni} be the same formal group with the coordinate replaced with $(1 + \sum_{n \in \mathbb{N}^+} a_n^f y(W_n)) \cdot y(\varepsilon)$. The functor

$$\text{CAlg}_{S_A} \rightarrow \text{SI}, \quad (f : S_A \rightarrow k) \mapsto (k, f_* F^{\text{uni}}, f_* \tilde{F}^{\text{uni}}, \text{id}),$$

is fully faithful by Section 2.2, and we have demonstrated it to be essentially surjective. Thus, it is an equivalence of categories, as was to be shown. \square

Remark 3.6 The same proof shows that the category of all (not necessarily strict) isomorphisms also has an initial object, whose underlying ring is $S_A[(a_0^f)^{\pm 1}]$. In fact, the only difference in the proof is that the equation for the unit x is now of the form $\sum_{n \in \mathbb{N}} a_n^f y(W_n)$ so that the presentation for the analog of S_A becomes

$$L_A[a_0^f, a_1^f, a_2^f, \dots][(a_0^f)^{-1}, P_V(a_0^f, a_1^f, a_2^f, \dots)^{-1} \mid V \in A^* - \{\varepsilon\}].$$

Since $P_V(a_0^f, a_1^f, a_2^f, \dots) = a_0^f P_V(1, a_1^f/a_0^f, a_2^f/a_0^f, \dots)$, this ring is indeed isomorphic to

$$L_A\left[(a_0^f)^{\pm 1}, \frac{a_1^f}{a_0^f}, \frac{a_2^f}{a_0^f}, \dots\right]\left[P_V\left(1, \frac{a_1^f}{a_0^f}, \frac{a_2^f}{a_0^f}, \dots\right)^{-1} \mid V \in A^* - \{\varepsilon\}\right] \cong S_A[(a_0^f)^{\pm 1}].$$

There are functors

$$s, t : \text{SI} \rightarrow A\text{-FGL}$$

sending a strict isomorphism to its source and target, respectively, as well as an “identity” functor

$$A\text{-FGL} \rightarrow \text{SI},$$

an “inverse” functor

$$i : \text{SI} \rightarrow \text{SI}$$

and a “composition”

$$c : \text{SI} \times \text{SI} \rightarrow \text{SI},$$

which restrict to the full subcategory of those objects of SI where the isomorphism φ is given by the identity. By representability we obtain analogous source and target maps $s, t : L_A \rightarrow S_A$, an identity map $S_A \rightarrow L_A$, an inverse map $i : S_A \rightarrow S_A$ and a composition map $c : S_A \rightarrow S_A \otimes_{L_A} S_A$.

Corollary 3.7 *The pair (L_A, S_A) together with the above structure defines a Hopf algebroid ie a cogroupoid object in commutative rings. The associated functor*

$$\text{CRing} \rightarrow \text{Groupoids}$$

is equivalent to the one sending a commutative ring k to the groupoid of A -equivariant formal group laws over k and strict isomorphisms between them.

Remark 3.8 The Hopf algebroid (L_A, S_A) has a natural grading. This can be constructed in three equivalent ways:

- Hausmann [2022, Corollary 5.6] shows that the global Lazard ring admits a unique grading that defines a graded global group law. By the same arguments, the “universal global group law with a strict n -tuple of coordinates” $L^{(n)}$ [loc. cit., Section 5.8] also carries a unique grading for every n such that the coordinates have degree -2 . The source–target maps $s, t: L \rightarrow L^{(2)}$, the identity map $L^{(2)} \rightarrow L$, the inverse map $i: L^{(2)} \rightarrow L^{(2)}$ and the composition map $c: L^{(2)} \rightarrow L^{(3)}$ all preserve this grading, since they are defined through their effect on the respective coordinates. Evaluating these maps at a group A yields the Hopf algebroids (L_A, S_A) , which hence inherit a grading compatible with all restriction and inflation maps.
- The groupoid-valued functor represented by (L_A, S_A) admits a \mathbb{G}_m -action from multiplying the coordinate of the equivariant formal group law by a unit u and acting on strict isomorphisms by multiplying a_n^f by u^n . This corresponds to an even grading on (L_A, S_A) , putting eg a_n^f in degree $2n$.
- By its interpretation as representing the groupoid of A -equivariant formal group laws and isomorphisms between them (see Remark 3.6), $(L_A, S_A[(a_0^f)^{-1}])$ obtains the structure of a Hopf algebroid. An element s in S_A is of degree $2n$ if applying the composition map $S_A[(a_0^f)^{-1}] \rightarrow S_A[(a_0^f)^{-1}] \otimes_{L_A} S_A[(a_0^f)^{-1}]$ to s gives $(a_0^f)^n c(s)$, where $c: S_A \rightarrow S_A \otimes_{L_A} S_A$ is the composition map of (L_A, S_A) .

One can show that this is the same grading coming from the isomorphism

$$(L_A, S_A) \cong (\pi_*^A \text{MU}_A, \pi_*^A \text{MU}_A \wedge \text{MU}_A)$$

from [loc. cit., Theorem E].

Given a strict isomorphism $\varphi: F_1 \cong F_2$ of B -equivariant formal group laws and a group homomorphism $\alpha: B \rightarrow A$, we obtain an induced strict isomorphism $\alpha_* \varphi: \alpha_* F_1 \cong \alpha_* F_2$ by completion. This assignment is compatible with composition of strict isomorphisms. Therefore, the functor L from Section 2.3 extends to a functor

$$L: (\text{abelian compact Lie groups})^{\text{op}} \rightarrow \text{Hopf algebroids},$$

which we call the *global Lazard Hopf algebroid*.

3.2 The moduli stack of equivariant formal groups

Throughout this subsection we let A be an abelian compact Lie group. Our goal is to discuss the stack $\mathcal{M}_{\text{FG}}^A$ of equivariant formal groups and its relationship to the Lazard Hopf algebroid. Our stacks will always be stacks for the fpqc-topology on commutative rings. More precisely, we will consider stacks as the full sub-2-category of the 2-category of (pseudo)functors from commutative rings to groupoids on those (pseudo)functors satisfying fpqc-descent on objects and morphisms (the usual stack conditions).¹

Let us define $\mathcal{M}_{\text{FG}}^A$, leaving the verification that it is indeed an fpqc-stack to Proposition 3.10.

Definition 3.9 Let $\mathcal{M}_{\text{FG}}^A$ be the (pseudo)functor sending a commutative ring to the groupoid of A -equivariant formal groups over it (with morphisms arbitrary isomorphisms between them).

We have discussed above that the Hopf algebroid (L_A, S_A) represents the functor sending a commutative ring to the groupoid of A -equivariant formal group laws and strict isomorphisms between them. As discussed in Remark 3.8, (L_A, S_A) is naturally a graded Hopf algebroid. On the other hand, as we have discussed in Remark 3.6, the ungraded Hopf algebroid $(L_A, S_A[a_0^{\pm 1}])$ classifying A -equivariant formal group laws and all isomorphisms between them. It is easy to see that this is precisely the ungraded Hopf algebroid associated to the graded Hopf algebroid (L_A, S_A) in the sense of [Meier and Ozornova 2020, Section 4.1].² We will follow [loc. cit., Definition 4.1] by defining the stack associated to a graded Hopf algebroid as the fpqc-stackification of the groupoid-valued functor corepresented by its associated ungraded Hopf algebroid. In particular, the stack associated to the graded Hopf algebroid (L_A, S_A) is the same as the stack associated to the ungraded Hopf algebroid $(L_A, S_A[a_0^{\pm 1}])$. Equivalently, it is the quotient of the stack associated to (L_A, S_A) by the \mathbb{G}_m -action induced by the grading.

Proposition 3.10 (1) *The groupoid-valued presheaf $\mathcal{M}_{\text{FG}}^A$ from Definition 3.9 is an fpqc-stack.*

(2) *Sending an A -equivariant formal group law to its underlying A -equivariant formal group defines an equivalence from the stack associated to the graded Hopf algebroid (L_A, S_A) to $\mathcal{M}_{\text{FG}}^A$.*

Proof Let us first show that (1) implies (2): denote the stack associated to the graded Hopf algebroid (L_A, S_A) by \mathcal{X}_A ; equivalently, this is the stack associated to the ungraded Hopf algebroid $(L_A, S_A[a_0^{\pm 1}])$. Since the augmentation ideal of every A -equivariant formal group (X, φ) is by definition fpqc-locally trivial, (X, φ) comes after fpqc-base change from an A -equivariant formal group law (see Lemma 2.6). Thus, $\mathcal{X}_A \rightarrow \mathcal{M}_{\text{FG}}^A$ is essentially surjective as a functor of stacks. Moreover, it is fully faithful since

¹We write (pseudo)functors since stacks are typically not strict functors into the 2-category of groupoids. All 2-morphisms in our setting are invertible; thus we can also decide to work in the setting of $(\infty, 1)$ -categories and look at all functors from commutative rings to the $(\infty, 1)$ -category of groupoids. In this setting, the “pseudoness” is implicit; which is why we set “pseudo” in brackets.

²Meier and Ozornova [2020] use an algebraic grading convention, while we use a topological one; thus one has to double all degrees.

isomorphisms between A -equivariant formal group laws are precisely isomorphisms of the underlying A -equivariant formal groups. Thus, it remains to show (1), ie that \mathcal{M}_{FG}^A satisfies fpqc-descent on morphisms and objects.

Given two formal k -schemes X and Y , the functor

$$k\text{-algebras} \rightarrow \text{Set}, \quad K \mapsto \text{Hom}_K(X \times_{\text{Spec } k} \text{Spec } K, Y \times_{\text{Spec } k} \text{Spec } K),$$

is an fpqc-sheaf. Indeed, if we view X and Y as ind-objects (X_i) and (Y_j) , we can rewrite this Hom as $\lim_i \text{colim}_j \text{Hom}_K(X_i \times_{\text{Spec } k} \text{Spec } K, Y_j \times_{\text{Spec } k} \text{Spec } K)$ and equalizers commute with filtered colimits in sets. This easily implies that \mathcal{M}_{FG}^A satisfies fpqc-descent on morphisms.

Let $\text{Fin}(A^*)$ denote the directed set of finite multi subsets³ of A^* , ordered by inclusion. Sending an A -equivariant formal group (X, φ) over k to the system $(\text{Spec } \mathbb{O}_X/\mathcal{I})$, where \mathcal{I} runs over all finite products of the $\ker(\mathbb{O}_X \xrightarrow{\phi(V)^*} k)$ for $V \in A^*$, defines a functor from A -equivariant formal groups over k to the subcategory $\mathcal{C}(k)$ of $\text{Fun}(\text{Fin}(A^*), \text{Aff}_k)$ where all transition maps are closed immersions. The associated object in $\text{Ind}(\text{Aff}_k)$ defines the underlying formal scheme of X . Moreover, the functors $\mathcal{M}_{FG}^A(k) \rightarrow \mathcal{C}(k)$ assemble into a natural transformation $\mathcal{M}_{FG}^A \rightarrow \mathcal{C}$ of (pseudo)functors from commutative rings to groupoids.

Given a descent datum D for A -equivariant formal groups for an fpqc-cover $k \rightarrow K$, we thus obtain a descent diagram for \mathcal{C} along $k \rightarrow K$. This in turn defines a $\text{Fin}(A^*)$ -diagram of descent data for Aff_k along $k \rightarrow K$. As Aff and closed immersions satisfy fpqc-descent, we obtain a $\text{Fin}(A^*)$ -diagram in Aff_k such that all transition maps are closed immersions. This defines a formal k -scheme X . Denoting the functor

$$(\text{formal schemes over } k) \rightarrow (\text{descent data for formal schemes along } k \rightarrow K)$$

by \mathcal{D} , one checks that $\mathcal{D}(X)$ is the underlying descent datum of formal schemes for D . Moreover, \mathcal{D} preserves products. By descent for morphisms between formal k -schemes, X obtains a group structure and also a group homomorphism $\varphi: S \times A^* \rightarrow X$. Conditions (1) and (2) for an A -equivariant formal group are fulfilled by construction. □

Remark 3.11 As every equivariant formal group comes Zariski-locally from an equivariant formal group law, in our case only a Zariski-stackification was necessary to pass from $(L_A, S_A[u^{\pm 1}])$ to \mathcal{M}_{FG}^A .

Proposition 3.12 *Let $B \subseteq A$ be a subgroup of an abelian group A . Denote by $\alpha: B \rightarrow A$ the inclusion and by $q: A \rightarrow A/B$ the projection.*

- (i) *The functor q^* induces an open immersion $\mathcal{M}_{FG}^{A/B} \rightarrow \mathcal{M}_{FG}^A$, whose image is the common nonvanishing locus of the Euler classes e_V for all $V \notin \text{im}((A/B)^* \rightarrow A^*)$.*
- (ii) *The functor α_* induces a closed immersion $\mathcal{M}_{FG}^B \rightarrow \mathcal{M}_{FG}^A$, inducing an equivalence of \mathcal{M}_{FG}^B to the common vanishing locus of the e_V for all $V \in \ker(A^* \rightarrow B^*)$.*

³A *multiset* is a version of a set where elements can occur more than once. Formally speaking, a *multi subset* of a set S is a function $S \rightarrow \mathbb{Z}_{\geq 0}$, indicating which element occurs how often. It is *finite* if the function has finite support. A multi subset F of S is *contained* in another multi subset G of S if and only if $F(s) \leq G(s)$ for all $s \in S$.



Figure 2: Decompositions of $\mathcal{M}_{FG}^{C_4}$ into open and closed substacks, using lighter shading for open and darker shading for closed. Every ellipse stands for the image of an immersion $\mathcal{M}_{FG} \rightarrow \mathcal{M}_{FG}^{C_4}$, as in Remark 3.14.

Proof The first part is a reformulation of Corollary 2.23. The immersion is open since it is open after pullback to $\text{Spec } L_A$.

For the second, note that by Proposition 2.22 every A -equivariant formal group such that $e_V = 0$ for all $V \in \ker(A^* \rightarrow B^*)$ is of the form α_*G for G a B -equivariant formal group. Moreover, by construction, this vanishing of Euler classes is true for all A -equivariant formal groups of the form α_*G and thus characterizes the image of α_* . The substack given by this image is closed since it is closed after pullback to $\text{Spec } L_A$. Moreover, α_* is fully faithful by Proposition 2.13. \square

Remark 3.13 With notation as in the preceding proposition, we have $H \cong q_*q^*H$ for every A/B -equivariant formal group H . Thus, the inclusion of substacks $\mathcal{M}_{FG}^{A/B} \rightarrow \mathcal{M}_{FG}^A$ in the preceding proposition allows for a retraction (up to isomorphism) in the 2-category of stacks.

Remark 3.14 Let A be a compact abelian Lie group. Proposition 3.12 allows us to define for every subgroup $B \subseteq A$ an immersion $\mathcal{M}_{FG} \rightarrow \mathcal{M}_{FG}^A$ as the composition of the open immersion $\mathcal{M}_{FG} \rightarrow \mathcal{M}_{FG}^{B/B}$ and the closed immersion $\mathcal{M}_{FG}^B \rightarrow \mathcal{M}_{FG}^A$. Proposition 4.4 allows us to view the collection of all such immersions as a stratification of \mathcal{M}_{FG}^A .

Example 3.15 Proposition 3.12 gives closed immersions of \mathcal{M}_{FG} and $\mathcal{M}_{FG}^{C_2}$ into $\mathcal{M}_{FG}^{C_4}$. The first is the common vanishing locus of all Euler classes (which equals the vanishing locus of the Euler class of one of the two generators of $(C_4)^*$, see Proposition 2.24). Its complement is the open substack given by the nonvanishing locus of the Euler class of the generators and hence equivalent to $\mathcal{M}_{FG}^{C_4/C_2}$. The second, ie the closed immersion of $\mathcal{M}_{FG}^{C_2}$, equals the vanishing locus of $e_{[2]}$ for $[2]: C_4 \xrightarrow{[2]} C_4 \hookrightarrow \mathbf{T}$. Its complement is an open substack equivalent to $\mathcal{M}_{FG}^{C_4/C_4}$, the nonvanishing locus of all Euler classes of nontrivial characters.

For a general group A the situation is more complicated, and we cannot expect that the complement of the closed substack \mathcal{M}_{FG}^B in \mathcal{M}_{FG}^A can be expressed as a single open substack $\mathcal{M}_{FG}^{A/C}$ in general and vice versa. In general, the complement of \mathcal{M}_{FG}^B in \mathcal{M}_{FG}^A can be written as the union of the open substacks $\mathcal{M}_{FG}^{A/C}$ where C runs over the minimal subgroups of A not contained in B . We have indicated the situation for $A = C_2 \times C_2$ in Figure 3.

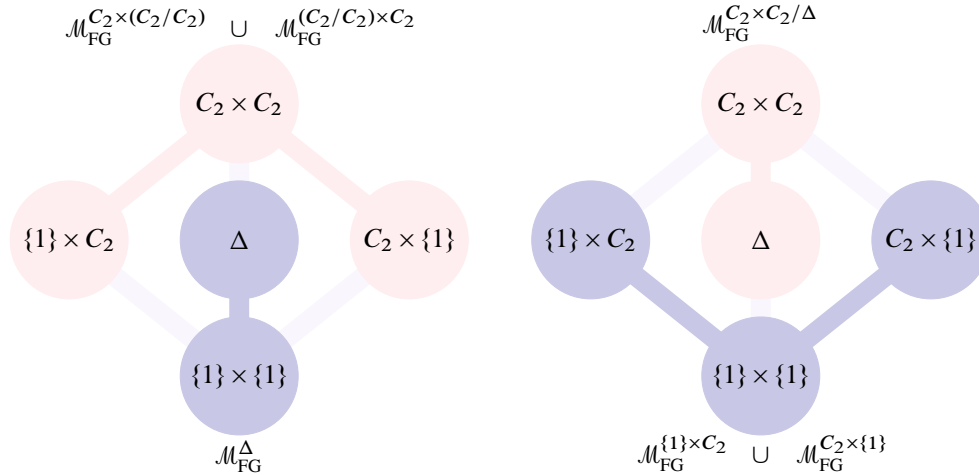


Figure 3: Decompositions of $\mathcal{M}_{FG}^{C_2 \times C_2}$ into open and closed substacks, Δ being the diagonal subgroup. Every circle stands for the image of an immersion $\mathcal{M}_{FG} \rightarrow \mathcal{M}_{FG}^{C_2 \times C_2}$, as in Remark 3.14.

4 Points of the moduli stack of equivariant formal groups and invariant prime ideals

The goal of this section is to classify the points of \mathcal{M}_{FG}^A and the invariant prime ideals of (L_A, S_A) . Although the latter could be done without the former, we feel that both questions are of the same importance and the stack point of view makes some issues more transparent.

4.1 The space associated to a stack

As mentioned above, given a graded flat Hopf algebroid (A, Γ) , we can associate an ungraded Hopf algebroid $(A, \Gamma[u^{\pm 1}])$ to it; see eg [Meier and Ozornova 2020, Section 4.1].⁴ The category of comodules over the latter is equivalent to that of graded comodules over (A, Γ) . The stack \mathcal{X} associated to (A, Γ) is by definition the stack associated to $(A, \Gamma[u^{\pm 1}])$, ie the fpqc-stackification of the presheaf of groupoids represented by $(A, \Gamma[u^{\pm 1}])$ on the category of all schemes. We denote the resulting morphism $\text{Spec } A \rightarrow \mathcal{X}$ by π .

Definition 4.1 Let (A, Γ) be a Hopf algebroid with units η_L and η_R . An ideal $I \subseteq A$ is called *invariant* if $\eta_L(I)\Gamma = \eta_R(I)\Gamma$. If (A, Γ) is graded, we will assume that I is also graded, ie generated by homogeneous elements.

It is easy to check that invariant ideals in a graded Hopf algebroid (A, Γ) correspond exactly to graded subcomodules of A and thus to ideal sheaves on \mathcal{X} . Here, we use that $\pi^* : \text{QCoh}(\mathcal{X}) \rightarrow \text{QCoh}(\text{Spec } A) \simeq \text{Mod}_A$

⁴Standard conventions force us to use A as part of the notation of a general Hopf algebroid, while A stands for a compact abelian Lie group in most of this article. We trust that this does not cause confusion.

refines to an equivalence from quasicoherent $\mathcal{O}_{\mathcal{X}}$ -modules to graded (A, Γ) -comodules [Meier and Ozornova 2020, Proposition 4.3].

Following [Laumon and Moret-Bailly 2000, Section 5] and [Stacks 2005–, Tag 04XL] in the case of Artin stacks, we can associate a topological space to \mathcal{X} . To that purpose, we define a morphism $\mathcal{U} \rightarrow \mathcal{X}$ of stacks to be an *open immersion* if the pullback $\mathcal{U} \times_{\mathcal{X}} \text{Spec } A \rightarrow \text{Spec } A$ is an open immersion.

Definition 4.2 For \mathcal{X} as above, define the underlying set of $|\mathcal{X}|$ to consist of equivalence classes of morphisms $x: \text{Spec } K \rightarrow \mathcal{X}$ for K a field; the equivalence relation is generated by isomorphisms and $x \sim (\text{Spec } L \rightarrow \text{Spec } K \xrightarrow{x} \mathcal{X})$, where L is a field extension of K . We call a subset of $|\mathcal{X}|$ *open* if it is the image of $|\mathcal{U}| \rightarrow |\mathcal{X}|$ for an open immersion $\mathcal{U} \rightarrow \mathcal{X}$.

Equivalently, we can characterize the opens as the images of those opens in $\text{Spec } A$ that are invariant, ie have the same preimage along both the left and right unit $\text{Spec } \Gamma[u^{\pm 1}] \rightarrow \text{Spec } A$. Indeed: since morphisms of stacks satisfy descent along the fpqc-cover $\text{Spec } A \rightarrow \mathcal{X}$ and $\text{Spec } A \times_{\mathcal{X}} \text{Spec } A \simeq \text{Spec } \Gamma[u^{\pm 1}]$, an open immersion $\mathcal{U} \rightarrow \mathcal{X}$ corresponds to an open immersion $\mathcal{V} \rightarrow \text{Spec } A$ with an isomorphism $\mathcal{V} \times_{\text{Spec } A} \text{Spec } \Gamma[u^{\pm 1}] \cong \text{Spec } \Gamma[u^{\pm 1}] \times_{\text{Spec } A} \mathcal{V}$ over $\text{Spec } \Gamma[u^{\pm 1}]$ satisfying a cocycle condition. But the category of open immersions into some X with isomorphisms over X between them is equivalent to the discrete category of open subsets of X , yielding the equivalence we claimed. Since invariant opens in $\text{Spec } A$ form a topology, we deduce that the opens in $|\mathcal{X}|$ form a topology. One further checks that the map induced by any morphism of stacks is continuous.

Our definition of the topological space associated to a stack coincides with that of Laumon and Moret-Bailly [2000] and [Stacks 2005–] (who do it for Artin stacks) in the intersection of their domains, eg when \mathcal{X} is an affine scheme, where we get the usual topology.

Proposition 4.3 *Let (A, Γ) be a graded Hopf algebroid with associated stack \mathcal{X} . Then:*

- (1) *For every invariant prime ideal $I \subseteq A$, the image of $V(I) \subseteq |\text{Spec } A|$ is closed in $|\mathcal{X}|$ and $\eta_I = |\pi|(\eta)$ for η the generic point of $V(I)$ (ie the point in $|\text{Spec } A|$ corresponding to I) is generic, that is, $\overline{\{\eta_I\}} = |\pi|(V(I))$.*
- (2) *Mapping I to η_I defines an injection from the set of invariant prime ideals $\text{Spec}^{\text{inv}}(A)$ in A to $|\mathcal{X}|$. Equipping $\text{Spec}^{\text{inv}}(A)$ with the subspace topology from $\text{Spec } A$, this map is continuous; if it is a bijection, it is a homeomorphism.*

Proof For the first point, observe first that for every invariant ideal I , the set $V(I)$ is invariant and hence the complement of $V(I)$ defines an invariant open. Thus the image in $|\mathcal{X}|$ is open. Moreover, $|\pi|^{-1}(|\pi|(V(I))) = V(I)$. Thus $|\pi|(V(I))$ is the complement of $|\pi|(|\text{Spec } A| \setminus V(I))$ and thus closed.

Assume now that I is an invariant prime ideal. Let η_I be the image of the generic point η of $V(I)$. Then $\overline{\{\eta_I\}} \subseteq |\pi|(V(I)) = |\pi|\{\eta\} \subseteq \overline{\{\eta\}}$ and hence $\overline{\{\eta_I\}} = |\pi|(V(I))$.

For the injectivity of $\text{Spec}^{\text{inv}}(A) \rightarrow |\mathcal{X}|$, let $I, J \subseteq A$ be two invariant prime ideals with $\eta_I = \eta_J$. By the first point, this implies that $|\pi|(V(I)) = |\pi|(V(J))$ and hence $V(I) = V(J)$. Thus, $I = J$.

The continuity of $\text{Spec}^{\text{inv}}(A) \rightarrow |\mathcal{X}|$ follows from that of $\text{Spec } A \rightarrow |\mathcal{X}|$. An arbitrary closed set of $\text{Spec}^{\text{inv}}(A)$ is of the form $V(I) \cap \text{Spec}^{\text{inv}}(A)$ for some ideal $I \subseteq A$. Set $I' = \bigcap_{I \subseteq J} J$, where the $J \subseteq A$ run over all invariant ideals containing I . Then $V(I) \cap \text{Spec}^{\text{inv}}(A) = V(J) \cap \text{Spec}^{\text{inv}}(A)$. If $|\pi|: \text{Spec}^{\text{inv}}(A) \rightarrow |\mathcal{X}|$ is a bijection, $|\pi|^{-1}(|\pi|(V(J))) = V(J)$ implies $|\pi|(V(J) \cap \text{Spec}^{\text{inv}}(A)) = |\pi|(V(J))$, and this is closed by the first part. \square

We warn the reader that in general, the preimage of an irreducible closed subset of $|\mathcal{X}|$ won't be irreducible in $|\text{Spec } A|$ and thus does not correspond to an invariant prime ideal.

We recall that $|\mathcal{M}_{\text{FG}}|$ has been computed by Honda: its points are classified by a pair (p, n) , where $p \geq 0$ is the characteristic of the field and $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the height of the formal group, with $n = 0$ if and only if $p = 0$. More precisely, Honda defined for each prime field of characteristic p the Honda formal group law of height n , and showed that any formal group law over a separably closed field is isomorphic to the pushforward of a Honda formal group law. As any field embeds into a separably closed field, the classification of points of \mathcal{M}_{FG} follows. We will often identify points of \mathcal{M}_{FG} with pairs (p, n) .

Proposition 4.4 *Let A be a compact abelian Lie group. We obtain a bijection of sets*

$$|\mathcal{M}_{\text{FG}}^A| \rightarrow |\mathcal{M}_{\text{FG}}| \times \text{Sub}(A),$$

with $\text{Sub}(A)$ being the set of closed subgroups of A , as follows: For a given A -equivariant formal group G over a field, the associated point in $|\mathcal{M}_{\text{FG}}|$ is defined by the pushforward of G along $p: A \rightarrow \{e\}$, and the associated subgroup of A is Pontryagin dual to $A^*/\{V \in A^* \mid e_V = 0\}$.

Proof Given a closed subgroup B of A and a nonequivariant formal group Γ over a field k , we obtain an A -equivariant formal group via $q^*i_*\Gamma$, where $A \xrightarrow{q} A/B \xleftarrow{i} \{1\}$. By Proposition 2.24, every A -equivariant formal group G over a field k is isomorphic to one of this form. Here, the projection $A \rightarrow A/B$ is Pontryagin dual to the subgroup inclusion $\{V \in A^* \mid e_V = 0\} \subseteq A^*$, and thus B is uniquely defined by G . Moreover, necessarily $\Gamma \cong p_*G$. This implies that $|\mathcal{M}_{\text{FG}}^A| \rightarrow |\mathcal{M}_{\text{FG}}| \times \text{Sub}(A)$ is surjective and that two A -equivariant formal groups G_1 and G_2 defined over the same field k are isomorphic if and only if they have the same associated closed subgroup and their completions p_*G_1 and p_*G_2 are isomorphic as nonequivariant formal groups.

To show that $|\mathcal{M}_{\text{FG}}^A| \rightarrow |\mathcal{M}_{\text{FG}}| \times \text{Sub}(A)$ is injective, assume that two points

$$G_i: \text{Spec } K_i \rightarrow \mathcal{M}_{\text{FG}}^A$$

for $i = 1, 2$ define the same element of $|\mathcal{M}_{\text{FG}}| \times \text{Sub}(A)$ for separable closed fields K_i . By the above we know that $G_1 \cong q^*i_*p_*G_1$ and $G_2 \cong q^*i_*p_*G_2$, where i and q are defined with respect to the

same subgroup B of A . Moreover, the heights of p_*G_1 and p_*G_2 and the characteristics of K_1 and K_2 agree by assumption. Let Γ be a formal group law of the same height as the p_*G_i , but defined over the common prime field K of the K_i . Both p_*G_i are isomorphic to the pushforward of Γ to the K_i . Thus, the A -equivariant formal group $q^*i_*\Gamma$ defines the same element in $|\mathcal{M}_{FG}^A|$ as the $G_i \cong q^*i_*p_*G_i$. Hence G_1 and G_2 define the same element of $|\mathcal{M}_{FG}^A|$. \square

Remark 4.5 By Proposition 3.12, there is an open immersion $\mathcal{M}_{FG}^{A/B} \rightarrow \mathcal{M}_{FG}^A$ for any closed subgroup $B \subseteq A$ and hence $|\mathcal{M}_{FG}^{A/B}|$ is homeomorphic to an open subset of $|\mathcal{M}_{FG}^A|$. Given $C \subseteq A/B$, the point corresponding to (C, p, n) is $(q^{-1}(C), p, n)$ for $q: A \rightarrow A/B$ the projection. Indeed, for an A/B -equivariant formal group G over a field corresponding to (C, p, n) , we have

$$\{V \in A^* \mid e_V = 0\} = q^*(\{V \in (A/B)^* \mid e_V = 0\}) = q^*((A/B)/C)^*.$$

The Pontryagin dual of $A^*/q^*((A/B)/C)^*$ is precisely $q^{-1}(C)$ since $A/q^{-1}(C) \cong (A/B)/C$.

4.2 Invariant prime ideals of (L_A, S_A)

Our goal in this section is to classify the invariant prime ideals of the Hopf algebroid (L_A, S_A) . For that purpose let us introduce and recall some notation.

As before, let $\Phi^B L = L_B[e_V^{-1}]$ be the localization of L_B away from all Euler classes e_V for $V \neq \epsilon$ and let $\Phi^B S = S_B \otimes_{L_B} \Phi^B L$. We denote by v_n the coefficient of x^{p^n} in the p -series $[p]_{F^{\text{uni}}}(x)$ of the universal formal group law F^{uni} over L . We further denote by $I_{p,n}$ the ideal $(v_0 = p, v_1, v_2, \dots, v_{n-1}) \subseteq L$, ie the unique invariant prime ideal at height n containing p . We also include the case of $I_{p,0} = 0$. In this section, we will denote this ideal by $I_{0,0}$ to uniformize notation with respect to the residue characteristic.⁵

Construction 4.6 For every triple (B, p, n) of a closed subgroup B of A , a prime p and $n \in \bar{\mathbb{N}}$ we define an invariant ideal $I_{B,p,n}^A \subseteq L_A$ as the preimage of $\Phi^B L \cdot I_{p,n} \subseteq \Phi^B L$ along the composite map of Hopf algebroids

$$(L_A, S_A) \rightarrow (L_B, S_B) \rightarrow (\Phi^B L, \Phi^B S).$$

The $\Phi^B L \cdot I_{p,n}$ are indeed prime ideals since by Proposition 2.25 the quotient ring $\Phi^B L / \Phi^B L \cdot I_{p,n}$ is of the form $(L/I_{p,n})[(b_0^V)^{\pm 1}, b_i^V \mid i > 0, V \in A^* - \{\epsilon\}]$ and hence an integral domain. Thus, the $I_{B,p,n}^A$ are prime as well.

To simplify notation, we will from now on often write $\Phi^B L / I_{p,n}$ instead of $\Phi^B L / \Phi^B L \cdot I_{p,n}$ and likewise in similar situations.

⁵Our choice of v_n is different from the more commonly chosen Araki generators v'_n , which satisfy $[p](x) = \sum_n^{F^{\text{uni}}} v'_n x^{p^n}$; see for instance [Ravenel 1986, A.2.2.4]. Thus, modulo $I'_{p,n} = (v'_0 = p, v'_1, \dots, v'_{n-1})$, the p -series is of the form $v'_n x^{p^n} + \dots$. Therefore, $v_n \equiv v'_n \pmod{I'_{p,n}}$. Inductively, we see that $I_{p,n} = I'_{p,n}$.

Theorem 4.7 *The assignment*

$$\text{Sub}(A) \times |\mathcal{M}_{\text{FG}}| \rightarrow \text{Spec}^{\text{inv}}(L_A), \quad (B, p, n) \mapsto I_{B,p,n}^A,$$

is a bijection. In other words, the ideals $I_{B,p,n}^A$ are pairwise different and constitute all the invariant prime ideals in L_A .

The map $\text{Spec}^{\text{inv}}(L_A) \rightarrow |\mathcal{M}_{\text{FG}}^A|$ from Proposition 4.3 is a homeomorphism.

Proof By Proposition 4.3, we know that $\text{Spec}^{\text{inv}}(L_A)$ injects into $|\mathcal{M}_{\text{FG}}^A|$ and the latter we computed to be $\text{Sub}(A) \times |\mathcal{M}_{\text{FG}}|$ as a set. Thus, it suffices to show that the element of $\text{Sub}(A) \times |\mathcal{M}_{\text{FG}}|$ associated to $I_{B,p,n}^A \in \text{Spec}^{\text{inv}}(L_A)$ is precisely $(B, (p, n))$, where p is a prime number if $n > 0$ and 0 if $n = 0$.

To spell this out concretely, let k be the field of fractions of $L_A/I_{B,p,n}^A$. We denote by F the pushed forward A -equivariant formal group law over k and by G the corresponding A -equivariant formal group. Since $\text{Spec } k \rightarrow \text{Spec } L_A$ hits the point corresponding to the prime ideal $I_{B,p,n}^A$, the corresponding point in $|\mathcal{M}_{\text{FG}}^A|$ is represented by $\text{Spec } k \xrightarrow{G} \mathcal{M}_{\text{FG}}^A$. By the classification in Proposition 4.4, we need to show three things:

- (1) the set of $V \in A^*$ such that $e_V = 0$ in k is precisely $\ker(A^* \rightarrow B^*)$,
- (2) k has characteristic p (which is clear), and
- (3) the pushforward p_*G along $A \xrightarrow{p} \{1\}$ has height n .

For the first, recall from Lemma 2.21 that $I_B^A = \ker(\text{res}: L_A \rightarrow L_B)$ is generated by the Euler classes e_V for all $V \in \ker(A^* \rightarrow B^*)$. These Euler classes must vanish in k since $L_A \rightarrow k$ factors through L_A/I_B^A . If V is not in $\ker(A^* \rightarrow B^*)$, then $e_V \neq 0$ in L_B and hence also in $\Phi^B L/I_{p,n}$ (as else $\Phi^B L/I_{p,n} = 0$). Since $L_A/I_{B,p,n}^A$ injects into $\Phi^B L/I_{p,n}$, the Euler class e_V is actually nonzero in $L_A/I_{B,p,n}^A$ and hence invertible in k . This shows the first point.

The pushforward p_*G is classified by the composite

$$g: L \rightarrow L_A \rightarrow L_A/I_{B,p,n}^A \rightarrow k.$$

The ideal $I_{p,n} \cdot L_A$ maps to 0 in $\Phi^B L/I_{p,n}$ and is hence contained in $I_{B,p,n}^A$. Therefore, g factors through $L/I_{p,n}$ and the height of p_*G is at least n . If $n = \infty$, we are done. If $n < \infty$, it remains to show that v_n is nonzero in $L_A/I_{B,p,n}^A$ and hence in k . By definition of $I_{B,p,n}^A$, the map $L_A \rightarrow \Phi^B L/I_{p,n}$ factors over $L_A/I_{B,p,n}^A$. By Proposition 2.25, $\Phi^B L/I_{p,n}$ is an integral domain of the form $(L/I_{p,n})[(b_0^V)^{\pm 1}, b_i^V]$. In particular, v_n is nontrivial in $\Phi^B L/I_{p,n}$ and hence in $L_A/I_{B,p,n}^A$. This shows that p_*G is of height n as desired, which finishes the proof. □

Unraveling the definition, we obtain the following description of the ideal $I_{B,p,n}^A$, where I_B^A still denotes the kernel of the restriction map $L_B \rightarrow L_A$.

Lemma 4.8 An element $x \in L_A$ lies in $I_{B,p,n}^A$ if and only if there exists an A -representation W with $W^B = 0$, such that

$$x \cdot e_W \in (I_B^A, I_{p,n}).$$

Proof By definition, $I_{B,p,n}^A$ is the kernel of the composition

$$L_A \xrightarrow{\text{res}_B^A} L_B \rightarrow L_B/L_B \cdot I_{p,n} \rightarrow \Phi^B L/\Phi^B L \cdot I_{p,n}.$$

The composition $L_A \rightarrow L_B \rightarrow L_B/I_{p,n}$ is surjective with kernel $(I_B^A, I_{p,n})$, and the map $L_B/I_{p,n} \rightarrow \Phi^B L/I_{p,n}$ inverts all Euler classes $e_{\bar{W}}$ for B -representations \bar{W} with $\bar{W}^B = 0$. The product of Euler classes is an Euler class again. Hence, if $x \in L_A$ is contained in $I_{B,p,n}^A$, its image \bar{x} in $L_B/I_{p,n}$ must be annihilated by such an Euler class $e_{\bar{W}}$. We can extend \bar{W} to an A -representation W and find that $x \cdot e_W$ is contained in $(I_B^A, I_{p,n})$, as desired.

For the opposite direction, if we assume that $x \cdot e_W \in (I_B^A, I_{p,n})$ for some A -representation W with $W^B = 0$, then $\bar{x} \cdot e_{\text{res}_B^A W} = 0$ in $L_B/I_{p,n}$. Since $\text{res}_B^A W$ has trivial B -fixed points, $e_{\text{res}_B^A W}$ becomes invertible in $\Phi^B L/I_{p,n}$ and hence \bar{x} is taken to 0 there. Therefore, x is contained in $I_{B,p,n}^A$. \square

When B is a torus, the ideal $I_{B,p,n}^A$ is easy to describe explicitly:

Corollary 4.9 If B is a torus, then $I_{B,p,n}^A = (I_B^A, I_{p,n})$.

Proof When B is a torus, every nontrivial character is nontorsion and thus the map

$$L_B/I_{p,n} \rightarrow \Phi^B L/I_{p,n}$$

is injective (Corollary 2.31). Hence, $I_{B,p,n}^A$ is equal to the kernel of

$$L_A \rightarrow L_A/I_{p,n} \rightarrow L_B/I_{p,n},$$

which is generated by I_B^A and $I_{p,n}$. \square

We note the following useful corollary:

Corollary 4.10 Let A be a torus, $I_{p,n} \subseteq L$ an invariant prime ideal with $n < \infty$, and consider the augmentation ideal

$$I = \ker(L_A/I_{p,n} \rightarrow L/I_{p,n}),$$

ie the ideal generated by all the Euler classes. Then the intersection $J = \bigcap_{k \in \mathbb{N}} I^k \subseteq L_A/I_{p,n}$ equals the 0-ideal.

For example, this shows that no element in the T -equivariant Lazard ring L_T is infinitely often divisible by the Euler class e .

Proof Since I is an invariant ideal of $L_A/I_{p,n}$, so are all its powers and the intersection thereof. Moreover, J equals the kernel of the completion map (see Section 2.5)

$$L_A/I_{p,n} \rightarrow L/I_{p,n}[[y_1, \dots, y_r]],$$

where r is the rank of A and the y_i are the images of the Euler classes ranging through a basis of A^* . Since $L/I_{p,n}[[y_1, \dots, y_r]]$ is an integral domain, J must be prime and hence an invariant prime ideal.

Therefore J must be of the form $I_{B,p,m}^A$ (or rather its image under the projection $L_A \rightarrow L_A/I_{p,n}$) for some subgroup B and height $m \geq n$. Note that v_n is not contained in I , hence in particular not in J . This means that we must have $m = n$. Moreover, given a character $V = V_1^{\otimes k_1} \otimes \dots \otimes V_r^{\otimes k_r} \in A^*$ expressed in the chosen basis V_1, \dots, V_r above, the image of e_V under the completion map is given by

$$F([k_1]_F(y_1), \dots, [k_r]_F(y_r)) \in L/I_{p,n}[[y_1, \dots, y_r]],$$

where F is the universal formal group law pushed forward to $L/I_{p,n}$. Since we assumed n to be a finite height, the $[k]$ -series of F is nontrivial whenever k is nonzero. It follows that for any nontrivial V the image of e_V is nontrivial in $L/I_{p,n}[[y_1, \dots, y_r]]$. Hence J contains no Euler class e_V and we must have $B = A$, ie

$$J = I_{A,p,n}^A = 0 \subseteq L_A/I_{p,n},$$

as desired. □

5 Inclusions between invariant prime ideals

Nonequivariantly the ideals $I_{p,n} \subseteq L$ form ascending towers

$$(0) = I_{p,0} \subseteq I_{p,1} \subseteq I_{p,2} \subseteq \dots,$$

essentially by definition. Except for the overlap at (0), there are no inclusions between the towers for different primes p . For invariant prime ideals in the equivariant Lazard ring L_A we saw that we have one tower

$$I_{B,p,0}^A \subseteq I_{B,p,1}^A \subseteq I_{B,p,2}^A \subseteq \dots$$

for every pair of a closed subgroup B and prime p . Again, there will be no interplay between the towers associated to different primes (except for the overlap at height 0). However, there are additional inclusions connecting the towers for different subgroups B, B' at the same prime p . This relationship between the heights at different subgroups is one of the essential properties of equivariant formal groups. It is closely related to the blue-shift phenomenon in stable homotopy theory. We say more about this in Section 8.

To see that there is no inclusion between towers associated with different primes, note that $p = v_0 \in I_{B,p,n}^A$ whenever $n \geq 1$. It is easy to see that p maps nontrivially under $L_A \rightarrow \Phi^{B'} L/I_{q,n'}$ for $q \neq p$ (since the target is free over $L/I_{q,n}$ by Proposition 2.25). Hence there cannot be an inclusion $I_{B,p,n}^A \subseteq I_{B',q,n'}^A$ for $n \geq 1$ and $p \neq q$. Moreover, if $n = 0$ we have $I_{B,p,0}^A = I_{B,q,0}^A$. Hence we can reduce to studying containments between invariant prime ideals associated to the same prime p .

New convention For this reason and to simplify notation, from now on and for the rest of the paper we implicitly localize at a fixed prime p . That is, we consider the p -localized Lazard ring L_A and denote its invariant prime ideals simply by $I_{B,n}^A$, omitting the chosen prime p . We further sometimes abbreviate $I_{A,n}^A$ to $I_{A,n}$.

Hence our goal is to understand for which pairs of subgroups B, B' and natural numbers n, n' there is an inclusion

$$I_{B,n}^A \subseteq I_{B',n'}^A.$$

We will show the following:

Theorem 5.1 *There is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$ if and only if the following conditions are satisfied:*

- (1) B' is a subgroup of B and $\pi_0(B/B')$ is a p -group.
- (2) We have $n' \geq n + \text{rank}_p(\pi_0(B/B'))$.

Hence, for example there are inclusions $I_{\mathbb{T},n}^{\mathbb{T}} \subseteq I_{1,n}^{\mathbb{T}}$ and $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$, but $I_{C_p^k,n}^{C_p^k}$ is not contained in $I_{1,n+k-1}^{C_p^k}$. In fact the theorem can be formally reduced to checking those three special cases, as we will see below. The theorem says, in particular, that given a chain of inclusions $B' \subseteq B \subseteq A$, the question of whether $I_{B,n}^A$ is contained in $I_{B',n'}^A$ does not depend on the ambient group A , but only on B, B', n and n' .

Theorem 5.1 can be interpreted as a statement about the heights of geometric fixed points of localizations of L_A , in the following way. Recall from [Hovey and Strickland 2005, Definition 4.1] that the *height* $\text{ht}(R)$ of an L -algebra R is the maximal n such that $R/(I_n \cdot R) \neq 0$; equivalently, it is the minimal number n such that $I_{n+1} \cdot R = R$. If there is no such n , the height is understood to be infinite. If $R = 0$, the height is -1 . Denoting by $\Phi^A((L_A)_{I_{B,n}^A})$ the localization of L_A inverting all Euler classes and all elements outside of $I_{B,n}^A$, we have the following:

Corollary 5.2 *Let $B \subseteq A$ be a closed subgroup, and let $n \in \mathbb{N}$. If $\pi_0(A/B)$ is not a p -group, or if $\pi_0(A/B)$ is a p -group but $\text{rank}_p(\pi_0(A/B)) > n$, then the geometric fixed points $\Phi^A((L_A)_{I_{B,n}^A})$ are trivial. Otherwise, their height is given by*

$$\text{ht}(\Phi^A((L_A)_{I_{B,n}^A})) = n - \text{rank}_p(\pi_0(A/B)).$$

Proof We have $I_m \cdot \Phi^A((L_A)_{I_{B,n}^A}) = \Phi^A((L_A)_{I_{B,n}^A})$ if and only if there exists an element of L_A not contained in $I_{B,n}^A$ which is mapped to $I_m \cdot \Phi^A L_A$ under the geometric fixed point map. Since $I_{A,m}$ is defined precisely as the preimage of $I_m \cdot \Phi^A L_A$, this in turn is equivalent to $I_{A,m}^A$ not being contained in $I_{B,n}^A$.

By Theorem 5.1, if $\pi_0(A/B)$ is not a p -group or if $\pi_0(A/B)$ is a p -group but $\text{rank}_p(\pi_0(A/B)) > n$, then $I_{A,0}^A$ is not contained in $I_{B,n}^A$. Since $I_0 = (0)$, this implies that the geometric fixed points $\Phi^A((L_A)_{I_{B,n}^A}) = I_0 \cdot \Phi^A((L_A)_{I_{B,n}^A})$ are trivial.

If $\pi_0(A/B)$ is a p -group and $r = \text{rank}_p(\pi_0(A/B)) \leq n$, then the theorem tells us that $I_{A,n-r}^A \subseteq I_{B,n}^A$ and $I_{A,n-r+1}^A \not\subseteq I_{B,n}^A$. Hence we have $I_{n-r} \cdot \Phi^A((L_A)_{I_{B,n}^A}) \neq \Phi^A((L_A)_{I_{B,n}^A})$ and $I_{n-r+1}^A \cdot \Phi^A((L_A)_{I_{B,n}^A}) = \Phi^A((L_A)_{I_{B,n}^A})$, as claimed. \square

Remark 5.3 The techniques of this paper can be used to compute the height of geometric fixed points for many complex oriented theories. We give one example of this in Proposition 8.6 below.

Remark 5.4 There is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$ if and only if in $|\mathcal{M}_{FG,(p)}^A|$, the point corresponding to $I_{B',n'}^A$ lies in the closure of the point corresponding to $I_{B,n}^A$. Thus, Theorem 5.1 can be interpreted as a result about the topology of $|\mathcal{M}_{FG,(p)}^A|$.

The proof of Theorem 5.1 takes up the remainder of this section.

5.1 Formal reduction to the p -toral case

Our first step is the following:

Lemma 5.5 *If there is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$, then B' is a subgroup of B and $n' \geq n$.*

Proof When B' is not a subgroup of B we can choose a character $V \in A^*$ which is trivial when restricted to B but nontrivial when restricted to B' . Its Euler class e_V then restricts to 0 in L_B , in particular it is contained in $I_{B,n}^A$ for all n . On the other hand, its restriction to B' becomes an invertible element in the nontrivial ring $\Phi^{B'}L/I_{n'}$ and is hence nontrivial. Therefore e_V is not an element of $I_{B',n'}^A$. It follows that $I_{B,n}^A$ cannot be contained in $I_{B',n'}^A$.

If $n' < n$, then $v_{n'}$ is contained in I_n but not in $I_{n'}$. This implies that $v_{n'}$ (now thought of as an element of L_A) is contained in $I_{B,n}^A$ but not in $I_{B',n'}^A$, since $\Phi^{B'}L/I_{n'}$ is a nontrivial free module over $L/I_{n'}$ by Proposition 2.25. Hence, again, $I_{B,n}^A$ cannot be contained in $I_{B',n'}^A$. \square

Lemma 5.6 *Let $B' \subseteq B$ be an inclusion of subgroups of A and $n, n' \in \mathbb{N}$. Then there is an inclusion*

$$I_{B,n}^A \subseteq I_{B',n'}^A$$

if and only if there is an inclusion

$$I_{B,n}^B \subseteq I_{B',n'}^B$$

if and only if there is an inclusion

$$I_{B/B',n}^{B/B'} \subseteq I_{B'/B',n'}^{B/B'}$$

Proof The first two statements are equivalent since the restriction map $\text{res}_B^A: L_A \rightarrow L_B$ identifies L_B with a quotient of L_A , and the ideals $I_{B,n}^A$ and $I_{B',n'}^A$ are the preimages of the ideals $I_{B,n}^B$ and $I_{B',n'}^B$ under the quotient projection.

Phrased differently, the projection $L_A \rightarrow L_B$ induces a closed embedding of the stack of B -equivariant formal groups into the stack of A -equivariant formal groups. On spectra, the image consists precisely of these $I_{B'',n}^A$ with $B'' \subseteq B$. This implies the desired equivalence by Remark 5.4.

For the second equivalence, recall the open embedding $|\mathcal{M}_{\text{FG}}^{B/B'}| \subseteq |\mathcal{M}_{\text{FG}}^B|$ from Remark 4.5, sending the point $(B''/B', n)$ to (B'', n) for every $B' \subseteq B'' \subseteq B$. Since the closure relation among points in a subspace can be detected in the subspace, Remark 5.4 gives the result. \square

Taken together, the previous two lemmas allow us to reduce to the case $B = A$ and $B' = 1$ and understand under what conditions there is an inclusion

$$I_{A,n}^A \subseteq I_{1,n'}$$

or in other words whether the restriction map $L_A \rightarrow L$ maps $I_{A,n}^A$ into the ideal $I_{n'}$.

Our next goal is to show that we can further reduce to the case where $\pi_0 A$ is a p -group. For this we choose a prime q and consider the Euler class $e_{\tau^q} \in L_T$, ie the pullback of $e_{\tau} \in L_T$ along the q^{th} power map $[q]: T \rightarrow T$. The Euler class e_{τ^q} restricts to 0 at the trivial group and is hence uniquely divisible by e_{τ} . We set $v_0^{(q)} \in L_T$ to be the unique element satisfying $e_{\tau^q} = v_0^{(q)} \cdot e_{\tau}$ (the reason for this choice of notation will become clear in Section 5.2).

Under the completion map $L_T \rightarrow L[[e_{\tau}]]$, the Euler class e_{τ^q} is sent to the q -series $[q]_F(e_{\tau})$ of the universal formal group law. Hence, $v_0^{(q)}$ is sent to the quotient $[q]_F(e_{\tau})/e_{\tau}$, whose leading coefficient equals q . Since the restriction of any element in L_T to the trivial group equals the leading coefficient of its image in $L[[e_{\tau}]]$, we see that $\text{res}_1^T v_0^{(q)} = q \in L$. We further set $\bar{v}_0^{(q)} \in L_{C_q}$ to be the restriction of $v_0^{(q)}$, and find that it satisfies

$$\bar{v}_0^{(q)} \cdot e_{\bar{\tau}} = 0, \quad \text{res}_1^{C_q} \bar{v}_0^{(q)} = q.$$

Here, $\bar{\tau}$ denotes the restriction of the tautological character $\tau \in T^*$ to C_q . We are now ready to show:

Lemma 5.7 *If $\pi_0 A$ is not a p -group, then there is no inclusion of the form $I_{A,n}^A \subseteq I_{1,n'}$.*

Proof If $\pi_0 A$ is not a p -group we can choose a surjection $f: A \rightarrow C_q$ with $q \neq p$ a prime. Then the element $f^* \bar{v}_0^{(q)} \in L_A$ satisfies the equation $f^* \bar{v}_0^{(q)} \cdot e_{f^* \bar{\tau}} = 0$. Since f is surjective, the character $f^* \bar{\tau} \in A^*$ is nontrivial. Hence, $f^* \bar{v}_0^{(q)}$ is an element of $I_{A,0}^A$ and hence also of $I_{A,n}^A$. Its restriction to the trivial group equals that of $\bar{v}_0^{(q)}$, which is q and hence a unit in the (p -localized) ring $L/I_{n'}$. In other words, $f^* \bar{v}_0^{(q)}$ is not an element of $I_{1,n'}$. Hence $I_{A,0}^A$ does not include into $I_{1,n'}$. \square

Combined with Lemmas 5.5 and 5.6 we obtain:

Corollary 5.8 *If there is an inclusion $I_{B,n}^A \subseteq I_{B',n'}$, then B' is a subgroup of B , $n' \geq n$ and the quotient B/B' is p -toral, ie a product of a p -group and a torus.*

5.2 Proof of inclusions

The next goal is to prove the “if” part of Theorem 5.1, ie to show that if the conditions on B, B', n and n' stated there are satisfied, we do have an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$. We start with the easiest case:

Lemma 5.9 *Let $B' \subseteq B$ be an inclusion of subgroups of A such that B/B' is a torus. Then there is an inclusion $I_{B,n}^A \subseteq I_{B',n}^A$ for all n .*

Proof By Lemma 5.6 we can reduce to $B = A$ and $B' = 1$ the trivial group. Hence A is a torus. Corollary 4.9 then implies that $I_{1,n}^A$ is equal to the ideal generated by I_n and the augmentation ideal I_1^A , whereas $I_{A,n}^A$ is generated by I_n only. Clearly the latter is contained in the former. \square

We now turn to showing that there are inclusions $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$. We will show that $I_{C_{p^k},n}^{C_{p^k}}$ is generated by I_n plus one additional element which reduces to v_n under the restriction map $L_{C_{p^k}}/I_n \rightarrow L/I_n$. We start with the case $k = 1$ and recall again that the Euler class e_{τ^p} is sent to the p -series $[p]_F(e_{\tau})$ under the completion map $L_{\mathcal{T}} \rightarrow L[[e_{\tau}]]$. Modulo I_n , this p -series is of the form $v_n e_{\tau}^{p^n} +$ higher-order terms. Hence Corollary 2.28 implies that there exists a unique element $\psi_p^{(n)} \in L_{\mathcal{T}}/I_n$ such that $e_{\tau^p} = \psi_p^{(n)} e_{\tau}^{p^n}$. This element satisfies $\text{res}_1^{\mathcal{T}}(\psi_p^{(n)}) = v_n$, ie it is the coefficient of x^{p^n} in the p -series of the universal formal group law.

We then set

$$\psi_{p^k}^{(n)} = [p^{k-1}]^*(\psi_p^{(n)}) \in L_{\mathcal{T}}/I_n$$

for all $k \geq 2$, where $[p^{k-1}]$ is the multiplication-by- p^{k-1} map on the circle. By functoriality, $\psi_{p^k}^{(n)}$ also restricts to v_n at the trivial group. In fact, it already restricts to v_n at $L_{C_{p^{k-1}}}/I_n$. This is because $C_{p^{k-1}}$ is the kernel of $[p^{k-1}]$ and hence the restriction map factors through the trivial group. Applying $[p^{k-1}]^*$ to the defining equation for $\psi_p^{(n)}$, we also obtain $e_{p^k} = \psi_{p^k}^{(n)} e_{p^{k-1}}^{p^n}$; here and in the following, we will often abbreviate e_{τ^m} to e_m .

Proposition 5.10 *For every $k \geq 1$ and $n \in \mathbb{N}$ the element $\psi_{p^k}^{(n)}$ generates the kernel of*

$$\phi_{C_{p^k}}^{\mathcal{T}} : L_{\mathcal{T}}/I_n \rightarrow L_{C_{p^k}}/I_n \rightarrow \Phi^{C_{p^k}} L/I_n.$$

Hence, $I_{C_{p^k},n}^{\mathcal{T}}$ is generated by the regular sequence $v_0, v_1, \dots, v_{n-1}, \psi_{p^k}^{(n)}$.

Corollary 5.11 *The ideal $I_{C_{p^k},n}^{C_{p^k}}$ is generated by I_n and the restriction $\bar{\psi}_{p^k}^{(n)}$ of $\psi_{p^k}^{(n)}$ to $L_{C_{p^k}}/I_n$.*

Proof of Proposition 5.10 Let $x \in L_{\mathcal{T}}/I_n$ be an element mapping to 0 in $\Phi^{C_{p^k}} L/I_n$, ie an element in the image of $I_{C_{p^k},n}^{\mathcal{T}}$ under the reduction map $L_{\mathcal{T}} \rightarrow L_{\mathcal{T}}/I_n$. By Lemma 4.8 we know that we have an equation of the form

$$(5.12) \quad x \cdot e_1^{b_1} \cdots e_{p^{k-1}}^{b_{p^{k-1}}} = y \cdot e_{p^k} = y' \cdot \psi_{p^k}^{(n)}$$

for some $b_i \in \mathbb{N}$, $y \in L_{\mathcal{T}}/I_n$ and $y' = y \cdot (e_{p^k}/\psi_{p^k}^{(n)})$ since $I_{C_{p^k}}^{\mathcal{T}} = (e_{p^k})$ by Lemma 2.21.

If l is coprime to p , then $e_{p^i l}$ and e_{p^i} become multiples of one another modulo e_{p^k} : indeed, the corresponding characters in $(C_{p^k})^*$ generate the same subgroup and thus Lemma 2.21 implies that $e_{p^i l}$ and e_{p^i} generate the same ideal in $L_{C_{p^k}}$. It follows that equation (5.12) gives rise to an equation

$$x \cdot e_1^{a_0} \cdot e_p^{a_1} \cdots e_{p^{k-1}}^{a_{k-1}} = y'' \cdot \psi_{p^k}^{(n)},$$

with only Euler classes for powers of p appearing on the left-hand side. We claim that y'' must be divisible by the entire product $e_1^{a_0} \cdot e_p^{a_1} \cdots e_{p^{k-1}}^{a_{k-1}}$. Since $L_{\mathbf{T}}/I_n$ is an integral domain, this implies that x is divisible by $\psi_{p^k}^{(n)}$, as desired. To see the claim, recall that $\psi_{p^k}^{(n)}$ restricts to v_n in $L_{C_{p^l}}/I_n$ for all $l < k$. Hence if $a_l > 0$ we have that

$$0 = \text{res}_{C_{p^l}}^{\mathbf{T}}(y'' \cdot \psi_m^{(n)}) = \text{res}_{C_{p^l}}^{\mathbf{T}}(y'') \cdot v_n \in L_{C_{p^l}}/I_n.$$

Since v_n is a regular element in $L_{C_{p^l}}/I_n$ by Proposition 2.29, this implies that $\text{res}_{C_{p^l}}^{\mathbf{T}}(y'') = 0$ and hence y'' is (uniquely) divisible by e_{p^l} . This argument can be iterated by replacing y'' by y''/e_{p^l} , and the statement follows. \square

Remark 5.13 One can show that more generally there exist elements $\psi_m^{(n)} \in L_{\mathbf{T}}/I_n$ for all $m \in \mathbb{N}$ uniquely determined by the equations

$$(5.14) \quad e_m = \prod_{t|m} (\psi_t^{(n)})^{p^{v_p(m/t) \cdot n}},$$

where $v_p(-)$ denotes the p -adic valuation of a natural number. The element $\psi_m^{(n)}$ generates the kernel of

$$\phi_{C_m}^{\mathbf{T}} : L_{\mathbf{T}}/I_n \rightarrow L_{C_m}/I_n \rightarrow \Phi^{C_m} L/I_n,$$

and its restriction to the trivial group is given by

$$\text{res}_1^{\mathbf{T}}(\psi_m^{(n)}) = \begin{cases} 0 & \text{if } m = 1, \\ v_n & \text{if } m = p^l \text{ and } l > 0, \\ q & \text{if } m = q^l \text{ with } q \neq p \text{ prime and } l > 0, \\ 1 & \text{otherwise.} \end{cases}$$

We note also that every $\psi_m^{(n)}$ is prime, since the geometric fixed points $\Phi^{C_m} L/I_n$ are integral domains. The elements $\psi_m^{(0)}$ were previously considered in [Hausmann 2022, Proposition 5.46], denoted by ψ_m there.

Remark 5.15 The proof of Proposition 5.10 applies in a more general context. Let X be a global group law in the sense of [Hausmann 2022, Definition 5.1]. As the global Lazard ring L is the initial global group law, there is a unique map $L \rightarrow X$. Assume that the map $L = L(1) \rightarrow X(1)$ sends I_n to 0, and that for every $l = 0, \dots, k$ the Euler classes e_{p^l} in $X(\mathbf{T})$ are regular elements and v_n is a regular element in $X(C_{p^l}) = X(\mathbf{T})/e_{p^l}$ (for example this is the case if v_n is regular in $X(\mathbf{T})$ and e_{p^l} remains a regular element modulo v_n). Then the image of $\psi_{p^k}^{(n)}$ in $X(\mathbf{T})$ generates the kernel of the composition

$$\Phi_{C_{p^k}}^{\mathbf{T}} : X(\mathbf{T}) \rightarrow X(C_{p^k}) \rightarrow \Phi^{C_{p^k}} X.$$

For example, this applies to the coefficients of many Borel-equivariant complex oriented spectra, which can be used to compute their blue-shift numbers. We make use of this in Proposition 8.6.

Corollary 5.16 *We have an inclusion $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$.*

Proof By Corollary 5.11, $I_{C_{p^k},n}^{C_{p^k}}$ is generated by I_n and $\bar{\psi}_{p^k}^{(n)}$. Since I_n is clearly contained in $I_{1,n+1}^{C_{p^k}}$ (even in $I_{1,n}^{C_{p^k}}$), we can reduce modulo I_n and need to show that $\bar{\psi}_{p^k}^{(n)}$ is taken to 0 under the composition

$$L_{C_{p^k}}/I_n \xrightarrow{\text{res}_1^T} L/I_n \rightarrow L/I_{n+1}.$$

But this is clear, since $\bar{\psi}_{p^k}^{(n)}$ restricts to v_n at the trivial group and v_n lies inside I_{n+1} . □

We now have all the ingredients to prove the “if” direction in Theorem 5.1:

Corollary 5.17 *Let $B' \subseteq B$ be an inclusion of subgroups of A such that B/B' is p -toral, and $n', n \in \mathbb{N}$ such that $n' \geq n + \text{rank}_p(\pi_0(B/B'))$. Then there is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$.*

Proof By Lemma 5.6 we can assume that $A = B$ and that $B' = 1$ is the trivial subgroup. Hence, A is a p -toral group. Let A^0 denote the path component of the identity. We have $I_{A^0,n'}^A \subseteq I_{1,n'}^A$ by Lemma 5.9. Hence it suffices to show that $I_{A,n}^A$ is contained in $I_{A^0,n'}^A$. For this, making use of Lemma 5.6 once more, we can assume that $A^0 = 1$ and hence A is a finite abelian p -group. We write $m = \text{rank}_p(A)$ and choose a filtration of subgroups

$$1 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_m = A$$

such that every subquotient A_i/A_{i-1} is a cyclic p -group. By Corollary 5.16 and Lemma 5.6 we see that

$$I_{A,n}^A \subseteq I_{A_{m-1},n+1}^A \subseteq I_{A_{m-2},n+2}^A \subseteq \dots \subseteq I_{1,n+m}^A \subseteq I_{1,n'}^A.$$

The final inclusion follows from the assumption $n' \geq n + \text{rank}_p(\pi_0(B/B')) = n + m$. This finishes the proof. □

5.3 Proof of noninclusions

For the “only if” direction we need to rule out further inclusions between prime ideals by constructing elements whose restrictions exhibit a large “height shift”. We will abbreviate $I_{A,n}^A$ to $I_{A,n}$ in the following, for various groups A . The goal is for every $n \in \mathbb{N}$ to construct an element $x \in L_{C_p^{n+1}}$ which lies in the ideal $I_{C_p^{n+1},0}$ and whose restriction to the trivial group lies outside of $I_n \subseteq L$.

It turns out to be more natural to define such an element modulo a subideal of $I_{C_p^{n+1},0}$, namely the ideal generated by the inflation of the ideal $I_{C_p^n,0}$ along the projection $p_{C_p^n}: C_p^{n+1} \cong C_p^n \times C_p \rightarrow C_p^n$. That is, we construct an element

$$\bar{v}_n \in I_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0}.$$

By Corollary 5.17 the restriction map $L_{C_{p^n}} \rightarrow L$ takes $I_{C_p^n,0}$ into I_n . Therefore, the restriction map $L_{C_p^{n+1}} \rightarrow L$ takes $p^*I_{C_p^n,0}$ into I_n and we obtain an induced restriction map

$$L_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0} \rightarrow L / I_n.$$

We will see that, under this restriction, \bar{v}_n is sent to v_n . Later in Section 6 we will show that \bar{v}_n in fact forms a generator of the quotient $I_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0}$ and that suitable inflations and restrictions of these elements generate all the invariant prime ideals at elementary abelian p -groups.

We now turn to the construction of the element \bar{v}_n . We set $A = C_p^n$ and first define an element v_n in the ring $L_{A \times T} / p_A^* I_{A,0}$, whose restriction to $L_{A \times C_p} / p_A^* I_{A,0}$ then yields \bar{v}_n .

For every character $V \in A^*$, Proposition 2.27 yields a short exact sequence

$$0 \rightarrow L_{A \times T} \xrightarrow{e_{V \otimes \tau}} L_{A \times T} \xrightarrow{(\text{id}, V^{-1})^*} L_A \rightarrow 0,$$

where we use that $(\text{id}, V^{-1}): A \rightarrow A \times T$ identifies A with the kernel of $(V \otimes \text{id}): A \times T \rightarrow T$. The inflation map $L_A \rightarrow L_{A \times T}$ provides an L_A -linear splitting of this exact sequence if we view $L_{A \times T}$ as an L_A -module via the same inflation map. Thus, we obtain a short exact sequence

$$(5.18) \quad 0 \rightarrow L_{A \times T} / p_A^* I_{A,0} \xrightarrow{e_{V \otimes \tau}} L_{A \times T} / p_A^* I_{A,0} \xrightarrow{(\text{id}, V^{-1})^*} L_A / I_{A,0} \rightarrow 0$$

of $L_A / I_{A,0}$ -modules.

Remark 5.19 We will see below in Section 6 that the ideal in $L_{A \times T}$ generated by $p_A^* I_{A,0}$ equals the invariant prime ideal $I_{A \times T,0}$. In particular, $L_{A \times T} / p_A^* I_{A,0}$ is again an integral domain. At this point it is only clear that $p_A^* I_{A,0}$ is contained in $I_{A \times T,0}$.

We now consider the Euler class $e_{\epsilon \otimes \tau^p} \in L_{A \times T} / I_{A,0}$. We have $(\text{id}, V^{-1})^*(e_{\epsilon \otimes \tau^p}) = e_{V^{-p}} = 0$, since A is an elementary abelian p -group and hence every character is p -torsion. By exactness, it follows that $e_{\epsilon \otimes \tau^p}$ is divisible by $e_{V \otimes \tau}$ for all $V \in A^*$. We want to define \bar{v}_n as the quotient of $e_{\epsilon \otimes \tau^p}$ by the product over all the $e_{V \otimes \tau}$. For this we first need to check that the different $e_{V \otimes \tau}$ are coprime. To understand this, we consider the following:

Lemma 5.20 *Let $x, y \in L_{A \times T} / p_A^* I_{A,0}$, and assume that y restricts to a nonzero element under the map $L_{A \times T} / p_A^* I_{A,0} \xrightarrow{(\text{id}, V^{-1})^*} L_A / I_{A,0}$. Then x is divisible by $e_{V \otimes \tau}$ if and only if $x \cdot y$ is divisible by $e_{V \otimes \tau}$.*

Proof By exactness of (5.18), x is divisible by $e_{V \otimes \tau}$ if and only if $(\text{id}, V^{-1})^* x = 0$ in $L_A / I_{A,0}$. Since $(\text{id}, V^{-1})^*(y)$ is nonzero by assumption and $L_A / I_{A,0}$ is an integral domain, this is the case if and only if

$$(\text{id}, V^{-1})^*(x \cdot y) = (\text{id}, V^{-1})^* x \cdot (\text{id}, V^{-1})^* y = 0,$$

which in turn is equivalent to $x \cdot y$ being divisible by $e_{V \otimes \tau}$. □

Given a character $V_2 \neq V$, the restriction of $e_{V_2 \otimes \tau}$ along $(\text{id}, V^{-1})^*$ equals the nontrivial Euler class $e_{V_2 V^{-1}} \in L_A/I_{A,0}$. Hence the lemma applies to y being any product of Euler classes of the form $e_{V' \otimes \tau}$ with $V' \neq V$. We find that in the quotient $L_{A \times T}/p_A^* I_{A,0}$, the Euler class $e_{\epsilon \otimes \tau^p}$ is uniquely divisible by the product $\prod_{V \in A^*} e_{V \otimes \tau}$. To summarize:

Definition 5.21 Let $n \in \mathbb{N}$ and $A = C_p^n$. We define $\mathbf{v}_n \in L_{A \times T}/p_A^* I_{A,0}$ to be the unique element satisfying

$$e_{\epsilon \otimes \tau^p} = \mathbf{v}_n \cdot \prod_{V \in A^*} e_{V \otimes \tau} \in L_{A \times T}/p_A^* I_{A,0},$$

and we set

$$\bar{\mathbf{v}}_n = \text{res}_{A \times C_p}^{A \times T}(\mathbf{v}_n) \in L_{A \times C_p}/p_A^* I_{A,0}.$$

Remark 5.22 The elements $\mathbf{v}_0 \in L_T$ and $\bar{\mathbf{v}}_0 \in L_{C_p}$ agree with the elements $\mathbf{v}_0^{(p)}$ and $\bar{\mathbf{v}}_0^{(p)}$ respectively from Section 5.1.

Furthermore, the element $\psi_p^{(n)} \in L_T/I_n$ introduced before Proposition 5.10 is equal to the restriction of $\mathbf{v}_n \in L_{A \times T}/p^* I_{A,0}$ to L_T/I_n . (Note that the latter makes sense since we know from Corollary 5.17 that the restriction of $I_{A,0} \subseteq L_{C_p^n}$ to the nonequivariant Lazard ring L lands in I_n .) Indeed, the defining relation for \mathbf{v}_n above restricts to the equation

$$e_{\tau^p} = \text{res}_T^{A \times T}(\mathbf{v}_n) \cdot \prod_{V \in A^*} \text{res}_T^{A \times T}(e_{V \otimes \tau}) = \text{res}_T^{A \times T}(\mathbf{v}_n) \cdot e_{\tau}^{p^n} \in L_T/I_n,$$

since each $e_{V \otimes \tau}$ restricts to e_{τ} at the circle group. Since $\psi_p^{(n)}$ was defined as the unique element of L_T/I_n satisfying $e_{\tau^p} = \psi_p^{(n)} \cdot e_{\tau}^{p^n}$, the claim follows. Consequently, $\bar{\psi}_p^{(n)} \in L_{C_p}/I_n$ agrees with the restriction of $\bar{\mathbf{v}}_n \in L_{A \times C_p}/p^* I_{A,0}$ to L_{C_p}/I_n .

Remark 5.23 Lemma 5.20 also applies in the ring $L_{A \times T}$ itself (ie before quotienting by $p_A^* I_{A,0}$) if we demand that $(\text{id}, V)^* y$ is a regular element, rather than just being nonzero. These two conditions are equivalent in $L_A/I_{A,0}$ since it is an integral domain. However, the Euler classes are not regular elements in L_A , hence the lemma does not apply for y a product of the $e_{V' \otimes \tau}$. In fact $e_{\epsilon \otimes \tau^p}$ is not divisible by $\prod_{V \in A^*} e_{V \otimes \tau}$ before dividing out $p_A^* I_{A,0}$, even though it is divisible by each individual $e_{V \otimes \tau}$. One can see this by restricting from $A \times T$ to T : If $e_{\epsilon \otimes \tau^p}$ was divisible by the product of all the $e_{V \otimes \tau}$, this would imply that its restriction $e_{\tau^p} \in L_T$ was divisible by $e_{\tau}^{p^n}$, since every $e_{V \otimes \tau}$ restricts to e_{τ} . But $e_{\tau^p} \in L_T$ is divisible by e_{τ} precisely once, since e_{τ^p}/e_{τ} restricts to p at the trivial group; see Section 5. It is for this reason that it is most natural to define \mathbf{v}_n and $\bar{\mathbf{v}}_n$ in this quotient. As we will see now, this matches nicely with the fact that \mathbf{v}_n is most naturally defined in the quotient L/I_n .

Proposition 5.24 (1) The element $\bar{\mathbf{v}}_n$ defines a class in the ideal $I_{A \times C_p,0}/p_A^* I_{A,0}$, ie it is sent to 0 under the map $L_{A \times C_p,0}/p_A^* I_{A,0} \rightarrow \Phi^{A \times C_p} L$.

(2) The restriction map

$$L_{A \times C_p, 0} / p_A^* I_{A, 0} \rightarrow L / I_n$$

takes \bar{v}_n to v_n .

Proof (1) The equation

$$e_{\epsilon \otimes \tau^p} = v_n \cdot \prod_{V \in A^*} e_{V \otimes \tau}$$

reduces to the equation

$$0 = \bar{v}_n \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}$$

in $L_{A \times C_p} / p_A^* I_{A, 0}$, where $\bar{\tau}$ denotes the restriction of $\tau \in T^*$ to C_p . Note that each $(A \times C_p)$ -character of the form $V \otimes \bar{\tau}$ is nontrivial. Hence \bar{v}_n forms Euler-power torsion and therefore maps to 0 in the geometric fixed points.

(2) By Remark 5.22, the intermediate restriction $\text{res}_T^{A \times T} v_n \in L_T / I_n$ equals the element $\psi_p^{(n)}$ constructed before Proposition 5.10. Since $\text{res}_1^T \psi_p^{(n)} = v_n$, this finishes the proof. \square

Corollary 5.25 *If x_n is a preimage of \bar{v}_n under the projection $L_{C_p^{n+1}} \rightarrow L_{C_p^{n+1}} / p_{C_p^n}^* I_{C_p^n, 0}$ and $B \subseteq C_p^{n+1}$ is a subgroup of rank $0 \leq m \leq n + 1$, then*

$$x_n \in I_{B, n+1-m}^{C_p^{n+1}} - I_{B, n-m}^{C_p^{n+1}}.$$

Proof By the previous proposition, x_n is an element of $I_{C_p^{n+1}, 0}^{C_p^{n+1}}$. As A/B has rank $(n + 1 - m)$, we know by Corollary 5.17 that x_n must lie in $I_{B, n+1-m}^{C_p^{n+1}}$.

If x_n were an element of $I_{B, n-m}^{C_p^{n+1}}$, then applying Corollary 5.17 to the inclusion of the trivial group into B shows that x_n is also an element of $I_{1, n}^{C_p^{n+1}}$. This contradicts the fact that, modulo I_n , we have $\text{res}_1^{C_p^{n+1}}(x_n) = \text{res}_1^{C_p^{n+1}}(\bar{v}_n) = v_n$. \square

Corollary 5.26 *If $B' \subseteq B$ is a p -toral inclusion of subgroups of A (that is, B/B' is p -toral) and if $n' < n + \text{rank}_p(\pi_0(B/B'))$, then $I_{B, n}^A$ does not include into $I_{B', n'}^A$.*

Proof By Lemma 5.6 we can reduce to the case $A = B$ and $B' = 1$. Let $r = \text{rank}_p(\pi_0(A))$, and $q: A \rightarrow C_p^r$ be a surjection. Let $x_{n+r-1} \in L_{C_p^{n+r}}$ as in Corollary 5.25; then the restriction $\text{res}_{C_p^r}^{C_p^{n+r}}(x_{n+r-1})$ is an element of $I_{C_p^r, n}^{C_p^r}$ but not an element of $I_{1, n+r-1}^{C_p^r}$. Therefore

$$x = q^*(\text{res}_{C_p^r}^{C_p^{n+r}}(x_{n+r-1}))$$

is an element of $I_{A, n}^A$ whose restriction to the trivial group is not contained in I_{n+r-1} . In other words, x is an element of $I_{A, n}^A$ but not an element of $I_{1, n+r-1}^A$. Since by assumption we have $n' \leq n + r - 1$ and hence $I_{1, n'}^A \subseteq I_{1, n+r-1}^A$, this proves that $I_{A, n}^A$ does not include into $I_{1, n'}^A$. \square

Combined with Corollaries 5.8 and 5.17, this proves Theorem 5.1.

6 Generators for invariant prime ideals

In this section we show that over elementary abelian p -groups the elements \bar{v}_n — together with the Euler classes — generate all invariant prime ideals under restriction and inflation maps. More precisely, we show the following theorem:

Theorem 6.1 (1) For every torus B and $n \in \mathbb{N}$, the ideal $I_{C_p^n \times B, 0} = I_{C_p^n \times B, 0}^{C_p^n \times B}$ is generated by the elements

$$p_1^*(\bar{v}_0), p_2^*(\bar{v}_1), \dots, p_{n-1}^*(\bar{v}_{n-2}), p_n^*(\bar{v}_{n-1}),$$

where $p_i: C_p^n \times B \rightarrow C_p^i$ is the projection to the first i factors.

(2) For every $m \in \mathbb{N}$ and every inclusion $i: C_p^n \rightarrow C_p^{n+m}$, the restriction map

$$(i \times B)^*: L_{C_p^{n+m} \times B} \rightarrow L_{C_p^n \times B}$$

maps $I_{C_p^{n+m} \times B, 0}$ surjectively onto $I_{C_p^n \times B, m}$.

Remark 6.2 Implicit in the statement of the theorem is that each $p_i^*(\bar{v}_{i-1})$ is well-defined modulo the ideal generated by $p_1^*(\bar{v}_0), \dots, p_{i-1}^*(\bar{v}_{i-2})$. By definition, $p_i^*(\bar{v}_{i-1})$ is an element of the quotient by the subideal generated by $p_{i-1}^* I_{C_p^{i-1}, 0}$. Applying the theorem to rank $n - 1$ and $B = 0$ we see that this ideal is indeed generated by $p_1^*(\bar{v}_0), \dots, p_{i-1}^*(\bar{v}_{i-2})$, so the sequence of elements makes sense. Hence, the theorem and sequence should be interpreted in an inductive manner.

Combining both parts it follows that $I_{C_p^n \times B, m}$ is generated by

$$(i \times B)^* p_1^*(\bar{v}_0), (i \times B)^* p_2^*(\bar{v}_1), \dots, (i \times B)^* p_{n+m-1}^*(\bar{v}_{n+m-2}), (i \times B)^* \bar{v}_{n+m-1},$$

where $i: C_p^n \rightarrow C_p^{n+m}$ is any inclusion. The choice of inclusion will generally affect the resulting generators. For example, setting $n = m = 1$ and B the trivial group: If we choose $i_1: C_p \rightarrow C_p^2$ to be the inclusion into the first factor, the composite $p_1 \circ i_1$ becomes the identity. Hence we obtain that $I_{C_p, 1}$ is generated by the elements \bar{v}_0 and $i_1^*(\bar{v}_1)$. If we alternatively use the inclusion $i_2: C_p \rightarrow C_p^2$ into the second factor the composite $p_1 \circ i_2$ becomes the constant map, yielding the generators $v_0 = p$ and $i_2^*(\bar{v}_1)$ (ie the same ones as in Corollary 5.11, as $\bar{\psi}_p^{(1)}$ equals $i_2^*(\bar{v}_1)$). Furthermore, it follows that generators for ideals of the form $I_{C_p^n \times B, m}^A$ with B a torus can be obtained as the union of Euler classes $(e_V)_{V \in \mathcal{B}}$ for a basis \mathcal{B} of $\ker(A^* \rightarrow (C_p^n \times B)^*)$ together with a choice of generators for $I_{C_p^n \times B, m}^{C_p^n \times B}$.

We prove both parts of Theorem 6.1 by induction on n , alongside the following statement:

(★) For every torus B and $m \in \mathbb{N}$ the ideal $I_{C_p^n \times B, m}$ is generated by $p_{C_p^n}^* I_{C_p^n, m}$.

For (1), the induction base case $n = 0$ is the statement that $I_{B, 0} = I_{B, 0}^B$ is the 0-ideal for any torus B (Corollary 4.9); this also shows (★) for $n = 0$. For (2) we need to see that the restriction $L_{C_p^m \times B} \rightarrow L_B$

maps $I_{C_p^m \times B, 0}$ surjectively onto $I_{B, m}$, which we know is generated by I_m by Corollary 4.9. For $i = 0, \dots, m - 1$ we can consider the elements $\bar{v}_i \in I_{C_p^{i+1}, 0} / p_i^* I_{C_p^i, 0}$, which reduce to $v_i \in L / I_{i-1}$. It follows that the inflation of \bar{v}_i to C_p^m via any choice of surjection $C_p^m \times B \rightarrow C_p^{i+1}$ gives an element of a quotient of $I_{C_p^m \times B, 0}$ which reduces to v_i in L_B / I_{i-1} . Since I_i / I_{i-1} is generated by v_i , the claim follows.

We now assume that Theorem 6.1 holds for an elementary abelian p -group A of rank n and show it also holds for $A \times C_p$. For any $m \in \mathbb{N}$ we consider the surjection

$$L_{A \times T \times B} / p_A^* I_{A, m} \rightarrow L_{A \times C_p \times B} / p_A^* I_{A, m},$$

with kernel generated by e_{τ^p} for τ the tautological T -character pulled back to $A \times T \times B$. We first claim that if $V \in A^*$ is nontrivial, then the Euler class $p_A^*(e_V)$ is a nonzero divisor in $L_{A \times C_p \times B} / p_A^* I_{A, m}$. To see this, we use that since $T \times B$ is a torus we can apply the induction hypothesis to $L_{A \times T \times B}$. In particular, we know that the statement (\star) holds for $A \cong C_p^n$ and hence $p_A^* I_{A, m}$ generates the ideal $I_{A \times T \times B, m}$. Therefore, $L_{A \times T \times B} / p_A^* I_{A, m}$ is an integral domain. So we have to show that $p_A^*(e_V)$ still acts regularly modulo e_{τ^p} . Since both Euler classes are regular, this is equivalent to showing that e_{τ^p} is regular modulo $p_A^*(e_V)$. We have an isomorphism

$$L_{A \times T \times B} / (p_A^* I_{A, m}, p_A^*(e_V)) \cong L_{\ker(V) \times T \times B} / p_{\ker(V)}^* (\text{res}_{\ker(V)}^A I_{A, m}).$$

By the induction hypothesis for (2), we know that $I_{A, m}$ restricts onto $I_{\ker(V), m+1}$, hence the latter quotient identifies with $L_{\ker(V) \times T \times B} / p_{\ker(V)}^* I_{\ker(V), m+1}$. Again we know by the induction hypothesis that this quotient is an integral domain, and e_{τ^p} is clearly a nontrivial element. So the claim follows and we have shown that $p_A^* I_{A, m}$ generates the Euler power torsion in $L_{A \times C_p \times B}$ (at height m) for characters inflated up from A .

Hence to understand the full ideal $I_{A \times C_p \times B, m}$ it suffices to further divide by the Euler-power torsion for the remaining torsion characters in $(A \times C_p \times B)^*$ (there is no Euler-power torsion for nontorsion characters by Proposition 2.27). These torsion characters are of the form $V \otimes \bar{\tau}^k$, where $V \in A^*$, $\bar{\tau}$ is the restriction of $\tau \in T^*$ to C_p and $k \in \{1, \dots, p - 1\}$. Furthermore we can assume that $k = 1$: Any $V \otimes \bar{\tau}^k$ has some power of the form $V' \otimes \bar{\tau}$ and hence $e_{V' \otimes \bar{\tau}}$ is a multiple of $e_{V \otimes \bar{\tau}^k}$. Thus, $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$ is generated by Euler-power torsion for characters of the form $V \otimes \bar{\tau}$.

Again it is beneficial to pass to the integral domain $L_{A \times T \times B} / p_A^* I_{A, m}$ to understand the Euler-power torsion for these characters. We have the following:

Lemma 6.3 *Let $A = C_p^n$ be an elementary abelian p -group, B a torus and $m \in \mathbb{N}$. Further, let $x \in L_{A \times T \times B} / p_A^* I_{A, m}$ be an element satisfying*

$$x \cdot \prod_{V \in A^*} e_{V \otimes \tau}^{n_V} = y \cdot e_{\tau^p}$$

for some y and collection of natural numbers n_V . Then x lies in the ideal generated by

$$p_{A \times T}^* \text{res}_{A \times T}^{A \times C_p^m \times T} (\mathbf{v}_{n+m}).$$

Proof By applying $p_{A \times C_p^m \times T}^*$ to the defining property of \mathbf{v}_{n+m} (Definition 5.21) we obtain the equation

$$e_{\tau^p} = p_{A \times C_p^m \times T}^*(\mathbf{v}_{n+m}) \cdot \prod_{V \in (A \times C_p^m)^*} e_{V \otimes \tau}$$

in $L_{A \times C_p^m \times T \times B} / p_{A \times C_p^m}^* I_{A \times C_p^m, 0}$. Restricting from $A \times C_p^m$ to $A = C_p^n$ yields

$$(6.4) \quad e_{\tau^p} = p_{A \times T}^* \operatorname{res}_{A \times T}^{A \times C_p^m \times T}(\mathbf{v}_{n+m}) \cdot \prod_{V \in A^*} e_{V \otimes \tau}^{p^m}$$

in the quotient $L_{A \times T \times B, m} / p_A^* I_{A, m}$. This uses that every character of A extends to p^m different characters of $A \times C_p^m$, and that the restriction of $I_{A \times C_p^m, 0} \subseteq L_{A \times C_p^m}$ lands in the ideal $I_{A, m} \subseteq L_A$. For the rest of the proof we write z for the element $p_{A \times T}^* \operatorname{res}_{A \times T}^{A \times C_p^m \times T}(\mathbf{v}_{n+m})$. With x as in the statement of the lemma, we hence obtain an equation of the form

$$(6.5) \quad x \cdot \prod_{V \in A^*} e_{V \otimes \tau}^{n_V} = z \cdot \prod_{V \in A^*} e_{V \otimes \tau}^{p^m} \cdot y,$$

and we need to show that x is a multiple of z . The Euler classes $e_{V \otimes \tau}$ fit into short exact sequences of the form

$$0 \rightarrow L_{A \times T \times B} / p_A^* I_{A, m} \xrightarrow{e_{V \otimes \tau}} L_{A \times T \times B} / p_A^* I_{A, m} \xrightarrow{((\operatorname{id}, V^{-1}) \times B)^*} L_{A \times B} / p_A^* I_{A, m} \rightarrow 0,$$

analogously to equation (5.18). The induction hypothesis implies that the quotient $L_{A \times B} / p_A^* I_{A, m}$ is an integral domain. We know that z restricts to $v_{n+m} \in L / I_{n+m}$ at the trivial group. In particular it must restrict nontrivially under each $((\operatorname{id}, V^{-1}) \times B)^*$. Hence the above short exact sequence implies: If an Euler class $e_{V \otimes \tau}$ divides an element of the form $z \cdot \alpha$, then $e_{V \otimes \tau}$ divides α . Applying this iteratively to equation (6.5) (and using that $L_{A \times T \times B} / p_A^* I_{A, m}$ is an integral domain as well by the induction hypothesis) we see that $\prod_{V \in A^*} e_{V \otimes \tau}^{n_V}$ must divide the term $\prod_{V \in A^*} e_{V \otimes \tau}^{p^m} \cdot y$. Dividing on both sides shows that x is a multiple of z , as desired. \square

Corollary 6.6 *The quotient*

$$I_{A \times C_p \times B, m} / p_A^* I_{A, m}$$

is generated by the element

$$\operatorname{res}_{A \times C_p \times B}^{A \times C_p^m \times C_p \times B} (p_{A \times C_p^m \times C_p}^* \bar{\mathbf{v}}_{n+m}) = p_{A \times C_p}^* (\operatorname{res}_{A \times C_p}^{A \times C_p^m \times C_p} \bar{\mathbf{v}}_{n+m}).$$

Proof We saw above that the quotient $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$ is generated by Euler-power torsion for characters of the form $V \otimes \bar{\tau}$. An element \bar{x} of $L_{A \times C_p \times B} / p_A^* I_{A, m}$ is such a torsion element if and only if it is the reduction of an element $x \in L_{A \times T \times B} / p_A^* I_{A, m}$ satisfying the conditions of the lemma. Since the reduction of $p_{A \times T}^* \operatorname{res}_{A \times T}^{A \times C_p^m \times T}(\mathbf{v}_{n+m})$ equals $p_{A \times C_p}^* \operatorname{res}_{A \times C_p}^{A \times C_p^m \times C_p}(\bar{\mathbf{v}}_{n+m})$, it follows that \bar{x} lies in the ideal generated by the latter.

As $p_{A \times C_p}^* \operatorname{res}_{A \times C_p}^{A \times C_p^m \times C_p}(\bar{\mathbf{v}}_{n+m})$ is Euler-power torsion itself, it forms a generator of $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$. \square

Finishing the proof of Theorem 6.1 Setting $m = 0$ in the corollary shows that \bar{v}_n generates the quotient $I_{A \times C_p \times B, 0} / p_A^* I_{A, 0}$. By the induction hypothesis, $I_{A, 0}$ is generated by $p_1^*(\bar{v}_0), \dots, p_{n-1}^*(\bar{v}_{n-2}), \bar{v}_{n-1}$. Combined this proves (1) for the group $A \times C_p$.

For (2) and general m , we first note that it suffices to show the statement for any choice of injection $i: A \times C_p \rightarrow C_p^{n+m+1}$ since any two only differ by postcomposition with an automorphism of C_p^{n+m+1} . We can hence pick the canonical inclusion $A \times C_p \rightarrow A \times C_p^m \times C_p$ avoiding C_p^m . By the induction hypothesis we know that $I_{A \times C_p^m, 0}$ surjects onto $I_{A, m}$. From the diagram

$$\begin{array}{ccc} I_{A \times C_p^m, 0} & \xrightarrow{\text{res}} & I_{A, m} \\ p_{A \times C_p^m}^* \downarrow & & \downarrow p_A^* \\ I_{A \times C_p^m \times C_p \times B, 0} & \xrightarrow{\text{res}} & I_{A \times C_p \times B, m} \end{array}$$

we see that $p_A^*(I_{A, m})$ is contained in the image of the lower-horizontal arrow. Furthermore, Corollary 6.6 implies that $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$ is generated by the restriction of an element of $I_{A \times C_p^m \times C_p \times B, m}$. Finally, we need to see that (\star) holds true for $C_p^{n+1} \cong A \times C_p$. This is a direct consequence of Corollary 6.6, as by definition both $p_A^* I_{A, m}$ and $p_{A \times C_p}^*(\text{res}_{A \times C_p}^{A \times C_p^m \times C_p} \bar{v}_{n+m})$ lie in the image of $p_{A \times C_p}^* I_{A \times C_p, m}$. \square

Remark 6.7 Unlike the sequence v_0, \dots, v_{n-1} , the sequence

$$p_1^*(\bar{v}_0), p_2^*(\bar{v}_1), \dots, p_{n-1}^*(\bar{v}_{n-2}), \bar{v}_{n-1}$$

isn't regular. In fact, the ideal $I_{C_p^n, 0}$ generated by these elements is precisely that of Euler-power torsion.

This can be corrected by passing to a torus: The ideal $I_{C_p^n, 0}^{\mathbf{T}^n}$ is generated by the sequence

$$p_1^*(\mathbf{v}_0), p_2^*(\mathbf{v}_1), \dots, p_{n-1}^*(\mathbf{v}_{n-2}), \mathbf{v}_{n-1}.$$

Here, each $p_{i+1}^*(\mathbf{v}_i)$ is the element of

$$(6.8) \quad L_{\mathbf{T}^n} / (p_1^*(\mathbf{v}_0), p_2^*(\mathbf{v}_1), \dots, p_i^*(\mathbf{v}_{i-1})) \cong L_{C_p^i \times \mathbf{T}^{n-i}} / I_{C_p^i \times \mathbf{T}^{n-i}, 0}$$

obtained as the inflation of $\mathbf{v}_i \in L_{C_p^i \times \mathbf{T}} / I_{C_p^i \times \mathbf{T}, 0}$ along the projection to the first coordinate of \mathbf{T}^{n-i} . Since each successive quotient $L_{C_p^i \times \mathbf{T}^{n-i}} / I_{C_p^i \times \mathbf{T}^{n-i}, 0}$ is an integral domain, the regularity of the sequence is clear once we have demonstrated the isomorphism claimed in (6.8). Similarly one shows that $I_{C_p^n, m}^{\mathbf{T}^n}$ is generated by a regular sequence of length $n + m$.

To establish the isomorphism, first note that $(e_{\tau^p}) \subseteq (p_{C_p^{i-1} \times \mathbf{T}}^*(\mathbf{v}_{i-1}))$ in the quotient ring

$$L_{C_p^{i-1} \times \mathbf{T}^{n-i+1}} / p_{C_p^{i-1}}^* I_{C_p^{i-1}, 0}$$

by Lemma 6.3. Thus, Corollary 6.6 with $m = 0$ gives

$$L_{\mathbf{T}^n} / (p_1^*(\mathbf{v}_0), p_2^*(\mathbf{v}_1), \dots, p_i^*(\mathbf{v}_{i-1})) \cong L_{C_p^i \times \mathbf{T}^{n-i}} / (p_1^*(\bar{v}_0), p_2^*(\bar{v}_1), \dots, p_i^*(\bar{v}_{i-1})),$$

and the claimed isomorphism becomes Theorem 6.1.

7 The Zariski topology on the spectrum of invariant prime ideals

The goal of this section is to describe the Zariski topology on $\text{Spec}^{\text{inv}}(L_A)$, or equivalently the topology on the space $|\mathcal{M}_{\text{FG}}^A|$ (Theorem 4.7). By definition, the closed subsets of $\text{Spec}^{\text{inv}}(L_A)$ are the subsets of the form

$$V(X) = \{I_{B,n}^A \in \text{Spec}^{\text{inv}}(L_A) \mid X \subseteq I_{B,n}^A\}$$

for some subset X of L_A . Hence we need to determine the collections of invariant prime ideals that arise as $V(X)$ for some X . We now fix a subset X . Since a containment $X \subseteq I_{B,n}^A$ automatically implies $X \subseteq I_{B,n'}^A$ for all $n' \geq n$, it suffices to understand, for every closed subgroup B of A , the smallest value of $n \in \mathbb{N} \cup \{\infty\}$ such that $X \subseteq I_{B,n}^A$. In other words we need to determine the function

$$\text{ht}_X : \text{Sub}(A) \rightarrow \bar{\mathbb{N}}_- = \{-1\} \cup \mathbb{N} \cup \{\infty\} \quad \text{defined by} \quad \text{ht}_X(B) = \sup\{n \mid X \not\subseteq I_{B,n}^A\},$$

where we set $\text{sup}(\emptyset) = -1$. We note that with this definition the height function of the image of $v_n \in L$ in L_A (thought of as a one-element set) is constantly n . This follows from the fact that $\Phi^B L$ is a free L -module by Proposition 2.25; cf. also the proof of Theorem 4.7. Moreover, we have:

Example 7.1 Let $x_n \in L_{C_p^{n+1}}$ be a lift of $\bar{v}_n \in L_{C_p^{n+1}}/p_{C_p^n}^* I_{C_p^n,0}$. Then Corollary 5.25 implies that $\text{ht}_{x_n}(B) = n - \text{rk}(B)$.

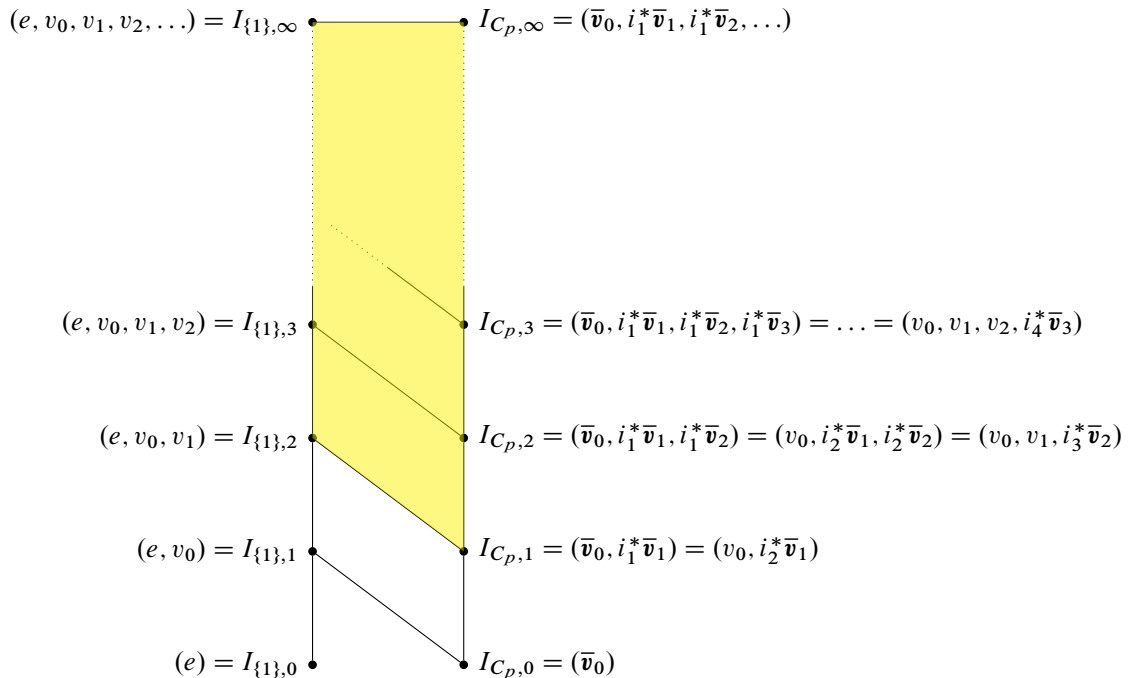


Figure 4: A picture of $\text{Spec}^{\text{inv}}(L_{C_p})$, localized at p , including different choices of generators, with $i_j : C_p \rightarrow C_p^k$ denoting the j^{th} canonical inclusion (generators arising from the further inclusions $C_p \rightarrow C_p^k$ are omitted). The shaded area depicts the closure of the point $I_{C_p,1}$.

Our goal is to understand which functions $\text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ arise as such height functions. In the previous section we showed that there are inclusions between invariant prime ideals associated to different subgroups of A . These translate to conditions between the different values of ht_X : If $B' \subseteq B$ is an inclusion of subgroups of A such that B/B' is p -toral, and X is contained in $I_{B,n}^A$, then X is also contained in $I_{B',n+\text{rank}_p(\pi_0(B/B'))}^A$. In terms of the height function this translates to the inequality

$$\text{ht}_X(B') \leq \text{ht}_X(B) + \text{rank}_p(\pi_0(B/B')).$$

This leads us to the following definition:

Definition 7.2 A function $f : \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ is called *admissible* if it satisfies the inequality

$$f(B') \leq f(B) + \text{rank}_p(\pi_0(B/B'))$$

for every p -toral inclusion $B' \subseteq B$ of closed subgroups of A . Here, $B' \subseteq B$ is p -toral if B/B' is a product of a torus and a p -group.

By the above considerations, any height function ht_X is admissible. When the group A is finite, it turns out that the converse also holds: Any admissible function is realized by a height function ht_X . For positive dimensional A there is an additional condition on top of admissibility. To state this condition, we recall that choosing an invariant Riemannian metric d on A also equips the set of closed subgroups $\text{Sub}(A)$ with the Hausdorff metric, the underlying topology of which does not depend on the chosen metric on A . This turns $\text{Sub}(A)$ into a compact totally disconnected metric space, in which a sequence $(B_i)_{i \in \mathbb{N}}$ of closed subgroups converges to another closed subgroup $B \in \text{Sub}(A)$ if and only if almost all B_i are subgroups of B and for every element $b \in B$ the distance function $d(b, B_i)$ converges to zero; see [tom Dieck 1979, Section 5.6]. If $B_i \rightarrow B$, we have the following two implications about representations:

- (1) Let W be a (finite-dimensional) representation of B with $W^B = 0$. Thus, writing W as a sum of characters V_i , none of the V_i is trivial. For sufficiently large i , no $\ker(V_i)$ contains B_i (since $\ker(V_i) \subseteq B$ is a closed proper subgroup) and thus $W^{B_i} = 0$.
- (2) Let V and W be two characters of B such that $\text{res}_{B_i}^B V = \text{res}_{B_i}^B W$ for all sufficiently large i . If $V \neq W$, then $(V \cdot W^{-1})^B = 0$, in contradiction with the previous point. Thus $V = W$.

We have the following:

Proposition 7.3 For every *finite* subset $X \subseteq L_A$, the height function ht_X is a locally constant function on $\text{Sub}(A)$.

Proof We start by noting that

$$\text{ht}_X(B) = \max(\text{ht}_{\{x\}}(B) \mid x \in X).$$

The maximum of a finite number of locally constant functions is again locally constant. Hence it suffices to understand that $\text{ht}_{\{x\}}$ is locally constant for any element x of L_A .

Now let $(B_i)_{i \in \mathbb{N}}$ denote a sequence of subgroups of A converging to a subgroup B . We need to show that $\text{ht}_{\{x\}}(B_i) = \text{ht}_{\{x\}}(B)$ for almost all i . Without loss of generality we can assume that the B_i are subgroups of B . Replacing x by $\text{res}_B^A(x)$ if necessary we can further restrict to the case $A = B$.

We first assume that $x \in I_{A,n}^A$ for some n , and show that then also $x \in I_{B_i,n}^A$ for almost all B_i . If $x \in I_{A,n}^A$, there exists an A -representation W with $W^A = 0$, such that $e_W \cdot x$ lies in the ideal $L_A \cdot I_n$; see Lemma 4.8. For all i large enough we have $W^{B_i} = 0$, meaning that for these i the restriction $e_{\text{res}_{B_i}^A} W$ becomes invertible in $\Phi^{B_i} L$. It follows that $\text{res}_{B_i}^A x$ maps to the ideal generated by I_n in $\Phi^{B_i} L$, in other words x is contained in $I_{B_i,n}^A$.

For the other direction, we assume that x is not contained in $I_{A,n}^A$ for some n , and show that then also $x \notin I_{B_i,n}^A$ for almost all B_i . For this we recall from Remark 2.26 the construction of elements γ_j^V for every character $V \in G^*$ and $j \in \mathbb{N}$ satisfying the following three properties:

- (1) $\alpha^*(\gamma_j^V) = \gamma_j^{\alpha^*(V)}$ for every group homomorphism $\alpha: G' \rightarrow G$,
- (2) $\gamma_0^V = e_V$, and
- (3) The canonical map $L[e_V^{\pm 1}, \gamma_j^V \mid V \in G^* - \{\epsilon\}, j > 0] \rightarrow \Phi^G L$ is an isomorphism for all abelian compact Lie groups G .

Now, if x is not contained in $I_{A,n}^A$, it maps to a nontrivial element in

$$\Phi^A L / I_n = L / I_n [e_V^{\pm 1}, \gamma_j^V \mid V \in A^* - \{\epsilon\}, j > 0].$$

In other words, there exists an A -representation W with $W^A = 0$ and pairwise different nontrivial characters V_1, \dots, V_k such that $e_W \cdot x \in L_A$ is a polynomial over L in the classes $\gamma_j^{V_l}$ for $l = 1, \dots, k$ and $j \geq 0$, not all of whose coefficients are contained in I_n . For all i large enough we have that

- (i) $W^{B_i} = 0$, and
- (ii) all the characters V_1, \dots, V_k restrict to pairwise different and nontrivial characters of B_i .

It then follows that for these i the element $e_{\text{res}_{B_i}^A} W \cdot \text{res}_{B_i}^A x$ equals the corresponding polynomial in the classes $\gamma_j^{\text{res}_{B_i}^A V_l}$, implying that it maps to a nontrivial element in $\Phi^{B_i} L / I_n$. In other words, x is not contained in $I_{B_i,n}^A$ for i large enough. This finishes the proof. □

Together with admissibility, this property characterizes the height functions of finite subsets of L_A :

Proposition 7.4 *Given a function $f: \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$, the following are equivalent:*

- (1) *There exists a finite subset $X \subseteq L_A$ such that $f = \text{ht}_X$.*
- (2) *The function f is admissible and locally constant.*

Proof We have already shown the implication 1. \Rightarrow 2.

It remains to show that given a locally constant admissible function f , there exists a finite subset $X \subseteq L_A$ with $f = \text{ht}_X$. We start with the following claim: If f is admissible, then given any pair of subgroups $B, B' \subseteq A$, there exists an element $x_{B,B'} \in L_A$ such that $\text{ht}_{x_{B,B'}}(B) = f(B)$ and $\text{ht}_{x_{B,B'}}(B') \leq f(B')$.

To see this, we distinguish between three cases:

- (i) If B is not a subgroup of B' , we can choose a character $V \in A^*$ which restricts to the trivial character over B' but to a nontrivial character over B . Then $x_{B,B'} = e_V \cdot v_{f(B)}$ has the desired properties, since $\text{ht}_{e_V \cdot v_n}(B) = n$ and $\text{ht}_{e_V \cdot v_n}(B') = -1$. Here and in the following, we set $v_{-1} = 0$, $v_0 = p$ and $v_\infty = 1$.
- (ii) If B is a subgroup of B' with $\pi_0(B'/B)$ not a p -group, we choose a prime $q \neq p$ dividing the order of $\pi_0(B'/B)$ and a surjection $g: B' \rightarrow C_q$ containing B in the kernel. Then we set $y = v_{f(B)} \cdot g^*(\bar{v}_0^{(q)}) \in L_{B'}$, where $\bar{v}_0^{(q)} \in L_{C_q}$ is the element introduced in Section 5.1. Then $\bar{v}_0^{(q)}$ is an element of $I_{C_q,0}^{C_q}$ and its restriction to the trivial group is given by q and hence a unit. It follows that $\text{ht}_y(B') = -1$, since $\text{ht}_{g^*(\bar{v}_0^{(q)})}(B') = -1$. Moreover, $\text{ht}_y(B) = f(B)$, since $\text{res}_B^{B'}(y) = v_{f(B)} \cdot q$. Hence, we can set $x_{B,B'}$ to be any lift of y to an element of L_A .
- (iii) The remaining case is when B is a subgroup of B' with $\pi_0(B'/B)$ a p -group. Let r be the minimum of the p -rank of $\pi_0(B'/B)$ and the number $f(B) + 1$, and choose a surjection $g: B'/B \rightarrow C_p^r$. By Corollary 5.25 we know for $f(B) < \infty$ that there exist an element $x_{f(B)} \in L_{C_p^{f(B)+1}}$ such that $\text{ht}_{x_{f(B)}}(1) = f(B)$ and $\text{ht}_{x_{f(B)}}(C_p^{f(B)+1}) = -1$. (We set $x_{-1} = 0$.) We can choose an embedding $C_p^r \rightarrow C_p^{f(B)+1}$ and restrict $x_{f(B)}$ to an element $y \in L_{C_p^r}$. Then we have $\text{ht}_y(1) = f(B)$ and $\text{ht}_y(C_p^r) = f(B) - r$; see Corollary 5.25. If $f(B) = \infty$, choose $y = 1$. It follows that $\text{ht}_{g^*(y)}(B) = f(B)$ and $\text{ht}_{g^*(y)}(B') \leq f(B) - r \leq f(B')$, since f is admissible. Hence, $g^*(y)$ has the desired properties.

Now given any such pair (B, B') there exists an open neighborhood $U_{B'}$ of B' on which both $\text{ht}_{x_{B,B'}}$ and f are constant. The $U_{B'}$ for varying B' form an open cover of the compact space $\text{Sub}(A)$. Let $U_{B'_1}, \dots, U_{B'_k}$ be a finite subcover. We then set

$$x_B = x_{B,B'_1} \cdot x_{B,B'_2} \cdots x_{B,B'_k}$$

to be the product of the corresponding elements. For any closed subgroup B' we have

$$\text{ht}_{x_B}(B') = \min(\text{ht}_{x_{B,B'_j}}(B') \mid j = 1, \dots, k).$$

For $B' = B$ this gives $\text{ht}_x(B) = f(B)$, since $\text{ht}_{x_{B,B'_j}}(B) = f(B)$ for all j . Any B' is contained in $U_{B'_i}$ for some i , yielding

$$\text{ht}_{x_B}(B') \leq \text{ht}_{x_{B,B'_i}}(B') = \text{ht}_{x_{B,B'_i}}(B'_i) \leq f(B'_i) = f(B').$$

In summary, the height function ht_{x_B} is less than or equal to f everywhere and agrees with f at B itself. Once more we can apply that ht_{x_B} and f are locally constant to find that ht_{x_B} and f in fact agree on a neighborhood V_B of B . Letting B vary, this yields an open cover of $\text{Sub}(A)$, for which we can choose a finite subcover V_{B_1}, \dots, V_{B_l} . Hence, for every $B \in \text{Sub}(A)$, there exists some $i \in \{1, \dots, l\}$ such that $f(B) = \text{ht}_{x_{B_i}}$. Finally, we define X to be the set $\{x_{B_1}, \dots, x_{B_l}\}$. Since ht_X is given by the maximum of the functions ht_{B_i} , it follows that X has the desired property. □

For the following theorem, we remind the reader of our convention that everything is implicitly localized at a fixed prime p .

Theorem 7.5 *The Zariski topology on $\text{Spec}^{\text{inv}}(L_A)$ has as a basis the closed sets*

$$V_f = \{I_{B,n} \mid n > f(B)\}$$

for all locally constant, admissible functions $f : \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$.

Proof A basis for the Zariski topology is given by the sets $V(X)$ for all finite subsets X of L_A . As we saw above, $V(X)$ is determined by its height function ht_X as

$$V(X) = \{I_{B,n} \mid n > \text{ht}_X(B)\}.$$

By Proposition 7.4, the functions $\text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ that occur as height functions of finite subsets are precisely the locally constant admissible functions, which finishes the proof. \square

8 Comparison with A -spectra

In this final section, we discuss the relationship of the algebraic results of the previous sections with the theory of A -spectra. Throughout, we will fix a prime p and implicitly localize at it.

8.1 The universal support theory via MU_A -homology

We begin by comparing our classification of invariant prime ideals with the Balmer spectrum of compact p -local A -spectra. We recall from [Balmer 2005] that a prime ideal \mathfrak{p} of a tensor-triangulated category \mathcal{C} is defined to be a thick tensor-ideal with the additional property that if $X \otimes Y$ is contained in \mathfrak{p} , then $X \in \mathfrak{p}$ or $Y \in \mathfrak{p}$. The set of all prime ideals assembles to a topological space $\text{Spec}(\mathcal{C})$, the Balmer spectrum, with the topology generated by the closed sets $\text{supp}(X) = \{\mathfrak{p} \mid X \notin \mathfrak{p}\}$ for all objects $X \in \mathcal{C}$.

Here, the support function $\text{supp}(-)$, assigning a closed set of the Balmer spectrum to every object of \mathcal{C} , is the *universal support theory* in the sense of Balmer. This means that it is terminal among pairs (T, σ) of a topological space T and a function

$$\sigma : \text{ob}(\mathcal{C}) \rightarrow \{\text{closed subsets of } T\}$$

satisfying

$$\sigma(0) = \emptyset, \quad \sigma(1) = T, \quad \sigma(X \oplus Y) = \sigma(X) \cup \sigma(Y), \quad \sigma(\Sigma X) = \sigma(X) \quad \text{and} \quad \sigma(X \otimes Y) = \sigma(X) \cap \sigma(Y)$$

for all $X, Y \in \text{ob}(\mathcal{C})$, and $\sigma(X) \subseteq \sigma(Y) \cup \sigma(Z)$ whenever there exists a distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$. See [Balmer 2005] for more details.

In the case of compact p -local A -spectra Sp_A^c for an abelian compact Lie group A , the Balmer spectrum was computed in [Balmer and Sanders 2017; Barthel et al. 2019; 2020]. Given a closed subgroup B and $n \in \mathbb{N} \cup \{\infty\}$, one defines a prime ideal

$$P_{B,n}^A = \{X \in \mathrm{Sp}_A^c \mid K(n)_*(\Phi^B X) = 0\},$$

where $K(n)$ denotes the n^{th} Morava K -theory, and $\Phi^B X$ is the B -geometric fixed point spectrum of X , a compact p -local spectrum.

Theorem 8.1 [Balmer and Sanders 2017; Barthel et al. 2019; 2020] *The map*

$$\mathrm{Sub}(A) \times \overline{\mathbb{N}} \rightarrow \mathrm{Spec}(\mathrm{Sp}_A^c), \quad (B, n) \mapsto P_{B,n}^A,$$

defines a bijection. Moreover, the topology on $\mathrm{Spec}(\mathrm{Sp}_A^c)$ has a basis given by the closed sets defined by

$$\{P_{B,n}^A \mid n \geq f(B)\},$$

where f ranges through all admissible functions $\mathrm{Sub}(A) \rightarrow \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Here, ‘‘admissible’’ is meant in the sense of Definition 7.2. Hence, comparing to Theorem 7.5, we see that the assignment $I_{B,n}^A \mapsto P_{B,n}^A$ defines a homeomorphism from $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ to $\mathrm{Spec}(\mathrm{Sp}_A^c)$.⁶ The computation of $\mathrm{Spec}(\mathrm{Sp}_A^c)$ is also analogous to the one of $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ in the way that computing the underlying set is relatively straightforward (and is in fact known for all compact Lie groups), with most work going into understanding the topology.

Hence, we can view the universal support theory of Sp_A^c to take values in the invariant prime ideals of L_A . The goal of the remainder of this section is to construct this universal support theory more intrinsically using MU_A -homology and the structure of equivariant formal group laws described in this paper. The idea is the following: Given a compact p -local A -spectrum X , we can consider its equivariant complex bordism homology $(\underline{\mathrm{MU}}_A)_* X$. Here, underlining MU_A indicates that we take the A -Mackey functor valued homology of X , ie we record the collection of $(\mathrm{MU}_B)_*(\mathrm{res}_B^A X)$ for all closed subgroups B of A , together with restriction and transfer maps between them. We reiterate that we will always work at the fixed prime p and p -localize everything implicitly.

Since the coefficients $\pi_*^A \mathrm{MU}_A$ are isomorphic to the Lazard ring L_A and moreover the cooperations $\pi_*^A \mathrm{MU}_A \wedge \mathrm{MU}_A$ agree with S_A (see [Hausmann 2022, Theorem 5.52] and the discussion thereafter), the groups $(\mathrm{MU}_A)_* X$ form a graded A -Mackey functor in (L_A, S_A) -comodules. As such, we can take its support in the invariant prime ideals

$$\mathrm{supp}((\underline{\mathrm{MU}}_A)_* X) = \{I_{B,n}^A \mid ((\underline{\mathrm{MU}}_A)_* X)_{I_{B,n}^A} \neq 0\} \subseteq \mathrm{Spec}^{\mathrm{inv}}(L_A).$$

⁶The reader may have noticed that in this section our admissible functions f take values in $\overline{\mathbb{N}}$, while in the last section they took values in \mathbb{N} . Likewise, we consider the condition $n \geq f(B)$ here, and considered $n > f(B)$ before. A shift by one shows that the topologies agree. The shift is caused by wanting v_n to have height n in the last section.

Remark 8.2 In general, $\text{supp}((\underline{\text{MU}}_A)_*X)$ is different from $\text{supp}((\text{MU}_A)_*X)$. Take for example $A = C_2$, $p = 2$ and $X = S^\sigma$, the circle with action given by reflection at a line. We have

$$\text{MU}_*(\text{res}_1^{C_2} S^\sigma)_{I_{1,0}} = \text{MU}_*(S^1)_{\mathbb{Q}} \neq 0.$$

The module $(\text{MU}_{C_2})_*(S^\sigma)_{I_{1,0}^{C_2}}$ is rational as well since $2 \notin I_{1,0}^{C_2}$. Rationally, $(\text{MU}_{C_2})_*(S^\sigma)$ splits into the coinvariants $(\text{MU}_{C_2})_*(S^\sigma)_{C_2} = 0$ and the geometric fixed points $(\text{MU}_{C_2})_*(S^\sigma)[e^{-1}]$, for e the Euler class of the unique nontrivial character. The element \bar{v}_0 from Definition 5.21 becomes zero in the geometric fixed points, but restricts to 2 and is thus not in $I_{1,0}^{C_2}$; thus $(\text{MU}_{C_2})_*(S^\sigma)[e^{-1}]$ becomes zero as well after localization at $I_{1,0}^{C_2}$, and $(\text{MU}_{C_2})_*(S^\sigma)_{I_{1,0}^{C_2}} = 0$.

Remark 8.3 (transfer maps) The isomorphisms $L_A \cong \pi_*^A \text{MU}_A$ imply that on top of the contravariant restriction maps along group homomorphisms there also exist transfer maps $L_B \rightarrow L_A$ for inclusions $B \subseteq A$ of finite index. In other words, the collection of all equivariant Lazard rings L_A forms a “global Green functor” on the family of abelian compact Lie groups, in the sense of [Schwede 2018, Definition 5.1.3]. Strickland [2011, Section 23] defines transfers for equivariant formal groups and shows that they agree with the topological transfers for complex oriented equivariant cohomology theories, in particular MU_A .

Using the methods from this paper, transfers on equivariant Lazard rings can be determined as follows: By Frobenius reciprocity, it suffices to compute $\text{tr}_B^A: L_B \rightarrow L_A$ on $1 \in L_B$ since the restriction is surjective. Inductively, we can further assume that $A/B \cong C_p$. Furthermore,

$$\text{tr}_B^A(1) = q^* \text{tr}_{\{1\}}^{C_p}(1) \quad \text{for } q: A \rightarrow A/B \cong C_p,$$

since transfers are compatible with inflation maps; see eg [Schwede 2018, Theorem 4.2.6 ff.]. Hence it suffices to identify $\text{tr}_{\{1\}}^{C_p}(1)$. We claim that it equals $\bar{v}_0 \in L_{C_p}$. Indeed, any transfer maps to zero in the geometric fixed points and is thus an element of $I_{C_p,0}$. We know that \bar{v}_0 generates $I_{C_p,0}$. Hence, we can write $\text{tr}_{\{1\}}^{C_p}(1) = x \cdot \bar{v}_0$ for $x \in L_{C_p}$. Writing x as $x_0 + x' \cdot e$ for e a nontrivial Euler class and $x_0 \in L$, we get $\text{tr}_{\{1\}}^{C_p}(1) = x_0 \cdot \bar{v}_0$ since $e \cdot \bar{v}_0 = 0$ by the definition of \bar{v}_0 . We obtain $x_0 \cdot p = \text{res}_{\{1\}}^{C_p} \text{tr}_{\{1\}}^{C_p}(1) = p$ since \bar{v}_0 restricts to p , and thus $x_0 = 1$ and $\text{tr}_{\{1\}}^{C_p}(1) = \bar{v}_0$.

We will now see that our notion of support is another model for the universal support theory on compact A -spectra.

We first note a major inconvenience: It is unclear whether $(\text{MU}_A)_*X$ is a finitely generated L_A -module, even for compact X . This is in contrast with the nonequivariant situation, where finite generation of MU_*X for compact X follows from the fact that L is a polynomial ring and hence coherent. An analogous statement is unknown for equivariant Lazard rings. In particular, it is a priori unclear that $\text{supp}((\underline{\text{MU}}_A)_*X)$ is indeed a *closed* subset of $\text{Spec}^{\text{inv}}(L_A)$ and we have to prove this by hand; see Proposition 8.5 below.

The following proposition gives the relationship between the MU_A -homology support theory described above and geometric fixed points.

Proposition 8.4 *Let B be a subgroup of A , $n \in \overline{\mathbb{N}}$ and X a compact A -spectrum. Then the following are equivalent:*

- (i) *The Mackey functor $((\underline{\text{MU}}_A)_*(X))_{I_{B,n}^A}$ is trivial.*
- (ii) *The $(\text{MU}_B)_*$ -module $((\text{MU}_B)_*(\text{res}_B^A X))_{I_{B,n}^B}$ is trivial.*
- (iii) *The B -geometric fixed points $\Phi^B(X)$ are of chromatic type $\geq n + 1$.*

We give two proofs of this proposition below, one using the results of [Balmer and Sanders 2017; Barthel et al. 2019; 2020] and one using the methods from this paper. The latter one is complicated by the fact that we don't know whether $(\text{MU}_A)_* X$ is always finitely generated.

We have the following corollary:

Proposition 8.5 *Let X be a compact A -spectrum. Then its support $\text{supp}((\underline{\text{MU}}_A)_* X)$ is a closed subset of $\text{Spec}^{\text{inv}}(L_A)$.*

Moreover, the assignment

$$\text{ob}(\text{Sp}_A^c) \rightarrow \text{Spec}^{\text{inv}}(L_A), \quad X \mapsto \text{supp}((\underline{\text{MU}}_A)_* X),$$

is a support theory on compact A -spectra.

Proof We consider the type function

$$\text{type}_X : \text{Sub}(A) \rightarrow \overline{\mathbb{N}}, \quad X \mapsto \text{type}(\Phi^B X),$$

which is locally constant by [Barthel et al. 2020, Proposition 4.3]. By Proposition 8.4, $I_{B,n}$ is an element of $\text{supp}((\underline{\text{MU}}_A)_* X)$ if and only if $\text{type}_X(B) < n + 1$. Since the support $\text{supp}((\underline{\text{MU}}_A)_* X)$ is closed under inclusion, Theorem 5.1 implies that $I_{B',n} \in \text{supp}((\underline{\text{MU}}_A)_* X)$ if $I_{B,n-\text{rk}_p(\pi_0(B/B'))} \in \text{supp}((\underline{\text{MU}}_A)_* X)$ for every p -toral inclusion $B' \subseteq B$. Thus type_X is admissible in the sense of Definition 7.2. Theorem 7.5 implies that $\text{supp}((\underline{\text{MU}}_A)_* X)$ is closed, as desired.

For the second part, all required properties of a support theory follow easily from exactness of localization $(-)_I_{B,n}^A$ except for the one on the interplay with smash products. This in turn follows from the third characterization in Proposition 8.4, since the type of a smash product of two compact spectra is the maximum of the two types. □

By the universal property of the Balmer spectrum, we obtain a continuous map

$$\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c).$$

Proposition 8.4 makes it clear that this map sends $I_{B,n}^A$ to $P_{B,n}^A$ and is hence bijective. Therefore we can conclude from the results of this paper that the topology on $\text{Spec}(\text{Sp}_A^c)$ is at least as coarse as the one on $\text{Spec}^{\text{inv}}(L_A)$. In other words, our proof of the existence of inclusions $I_{B,n} \subseteq I_{B',n'}$ gives another

proof of the analogous inclusion $P_{B',n'}^A \subseteq P_{B,n}^A$ on the topological side. The fact that there are no further topological inclusions requires additional arguments, namely the existence of compact A -spectra with “maximal type shifting behavior”; see [Barthel et al. 2019, Section 4] or [Kuhn and Lloyd 2024, Section 7]. Knowing this, we see that $X \mapsto \text{supp}((\underline{\text{MU}}_A)_* X)$ is a universal support theory on compact A -spectra.

It remains to give the proof of Proposition 8.4, which we will do in three steps:

(i) \implies (ii) The L_A -action on $(\text{MU}_B)_*(X)$ factors through $\text{res}_B^A: L_A \rightarrow L_B$ and $I_{B,n}^A = (\text{res}_B^A)^{-1}(I_{B,n}^B)$, so it follows that we have an isomorphism $((\text{MU}_B)_*(X))_{I_{B,n}^A} \cong ((\text{MU}_B)_*(X))_{I_{B,n}^B}$. In particular, the vanishing of the entire Mackey-functor $(\underline{\text{MU}}_* (X))_{I_{B,n}^A}$ implies the vanishing at the subgroup B as a special case.

(ii) \iff (iii) Note that $I_{B,n}^B$ contains none of the nontrivial Euler classes for B , hence the nontrivial Euler classes act invertibly on $((\text{MU}_B)_*(X))_{I_{B,n}^B}$. It follows that we have an isomorphism

$$((\text{MU}_B)_*(X))_{I_{B,n}^B} \cong ((\Phi^B \text{MU}_B)_* \Phi^B X)_{I_{B,n}^B} \cong \Phi^B ((\text{MU}_B)_{I_{B,n}^B})_* \Phi^B X.$$

Modulo I_n , the ring $\Phi^B((\text{MU}_B)_{I_{B,n}^B})_*$ embeds into the field of fractions of $\Phi^B L/I_n$, which is nontrivial. Moreover, v_n is not contained in $I_{B,n}^B$ and hence is invertible in the localization. It follows that the localization $(\Phi^B \text{MU}_B)_{I_{B,n}^B}$ is an MU -algebra of height n , ie its vanishing detects compact spectra of type $\geq n + 1$.

(iii) \implies (i) Let X be a compact A -spectrum such that $\Phi^B(X)$ is of type $> n$. By the previous paragraph we know that $((\text{MU}_B)_*(X))_{I_{B,n}^A} = 0$, and we have to show that $((\text{MU}_{\tilde{B}})_*(X))_{I_{B,n}^A} = 0$ for all other closed subgroups \tilde{B} , too. We first assume that \tilde{B} does not contain B . Then there exists a character $V \in A^*$ which restricts to the trivial character for \tilde{B} but to a nontrivial character for B . It follows that e_V is not contained in $I_{B,n}^A$ and hence acts invertibly on $(\text{MU}_{\tilde{B}})_{I_{B,n}^A}$. On the other hand e_V restricts to 0 in $\text{MU}_{\tilde{B}}$, so it follows that $(\text{MU}_{\tilde{B}})_{I_{B,n}^A} = 0$ and in particular $((\text{MU}_{\tilde{B}})_*(X))_{I_{B,n}^A} = 0$, as desired. Hence we can assume that \tilde{B} contains B as a subgroup. Since the statement then no longer depends on the ambient group, we can reduce to the case $\tilde{B} = A$.

Hence we need to show that $(\text{MU}_A)_*(X)_{I_{B,n}^A} = 0$. By induction on the pair (dimension, cardinality of $\pi_0(A/B)$) it follows that all the localizations at smaller intermediate groups $B \subseteq \tilde{B} \subseteq A$ vanish, and hence the homotopy groups of $(\text{MU}_A)_{I_{B,n}^A} \wedge X$ are concentrated at A . In particular this implies that the map $((\text{MU}_A)_{I_{B,n}^A})_* X \rightarrow ((\Phi^A \text{MU}_A)_{I_{B,n}^A})_* \Phi^A X$ is an isomorphism and all Euler classes e_V for nontrivial characters act invertibly on $((\text{MU}_A)_{I_{B,n}^A})_* X$. Now, if V restricts to the trivial character in B^* (such a V always exists since we can assume that B is a proper subgroup of A), its Euler class e_V lies in the maximal ideal $I_{B,n}^A$ of $((\text{MU}_A)_{I_{B,n}^A})_*$.

If we knew that $((\text{MU}_A)_{I_{B,n}^A})_* X$ is finitely generated over $(\text{MU}_A)_*$ we could apply Nakayama’s lemma to see directly that $((\text{MU}_A)_{I_{B,n}^A})_* X = 0$, as desired. Since we do not know this, we have to argue differently: By Corollary 5.2, we know that $\Phi^A((\text{MU}_A)_{I_{B,n}^A})_*$ is trivial if $\pi_0(A/B)$ is not a p -group, and is of height $n - \text{rank}_p(\pi_0(A/B))$ otherwise (where negative heights again mean that the theory is trivial). Hence,

what we want to show is that if $\Phi^B X$ is of type $\geq n + 1$, then $\Phi^A X$ is of type $\geq n + 1 - \text{rank}_p(\pi_0(A/B))$. Indeed, this implies that $((\text{MU}_A)_{I_{B,n}^A})_* X \cong ((\Phi^A \text{MU}_A)_{I_{B,n}^A})_* \Phi^A X$ is trivial.

This precise statement about $\Phi^A X$ is one of the main results of [Barthel et al. 2020], building on [Balmer and Sanders 2017] and [Barthel et al. 2019]. Hence, using these results, Proposition 8.4 follows. Alternatively, rather than importing we can reprove the above statement using the methods from this paper. Note that by induction on the rank of A/B and by replacing X by the compact A/B -spectrum $\Phi^B X$ it suffices to show two special cases:

- (1) If X is a compact C_{p^k} -spectrum of underlying type $\geq n + 1$, then the type of $\Phi^{C_{p^k}} X$ is at least n .
- (2) If X is a compact T -spectrum of underlying type $\geq n + 1$, then the type of $\Phi^T X$ is also $\geq n + 1$.

For (1) it suffices to find a complex oriented theory E of height n (meaning here that E_* vanishes on a compact spectrum X if and only if X is of type $\geq n + 1$) such that $\Phi^{C_{p^k}} \underline{E}$ is of height $\geq n - 1$, where $\underline{E} = F(EG_+, E)$ denotes the Borel theory associated to E (see the proofs of [Barthel et al. 2019, Corollary 3.12] or [Barthel et al. 2020, Proposition 6.10] for details on this argument). In [Barthel et al. 2019] it was shown that Morava E -theory $E = E_n$ has this property, building on results of Hopkins, Kuhn and Ravenel [Hopkins et al. 2000] on the p -divisible group associated to E_n and extending earlier work of Greenlees, Hovey and Sadofsky. Using Remark 5.15 we obtain similar results more generally:

Proposition 8.6 *Let E be any complex oriented ring spectrum of height n which is Landweber exact over $\text{MU}/I_{n-1} = \text{MU}/(v_0, \dots, v_{n-2})$, ie I_{n-1} acts trivially on $\pi_* E$, $v_{n-1} \in \pi_* E$ is a regular element and v_n is a unit modulo v_{n-1} . Then $\Phi^{C_{p^k}} \underline{E}$ is of height $n - 1$.*

Proof We apply Remark 5.15 and check that its assumptions are satisfied. We have $(\underline{E}^T)_* = E_*[[e]]$, with Euler classes e_n given by the n -series $[n]_F(e)$ for the formal group law associated to E . If n is a power of p , the leading term of this Euler class is a power of v_{n-1} , which we assumed to be regular. Modulo v_{n-1} , the leading term becomes v_n which is a unit since E/v_{n-1} is of height n . Hence by Remark 5.15 we find that the pushforward of the element $\psi_{p^k}^{(n-1)}$ to $E_*[[e]]$, ie the element $\sum_{i=0}^\infty a_i [p^{k-1}]_F(e)$ for $\sum_{i=0}^\infty a_i e^i = [p]_F(e)/e^{p^{n-1}}$, generates the kernel of the composite

$$E_*[[e]] \rightarrow E_*[[e]]/[p^k]_F(e) = (\underline{E}^{C_{p^k}})_* \rightarrow (\Phi^{C_{p^k}} \underline{E})_*.$$

The leading term of $\psi_{p^k}^{(n-1)}$ equals v_{n-1} , which is not a unit. Hence $\Phi^{C_{p^k}} \underline{E}$ is nontrivial and since I_{n-1} acts trivially it must be of height $\geq n - 1$. Furthermore, reducing modulo v_{n-1} the leading coefficient of $\psi_{p^k}^{(n-1)}$ equals a power of v_n . Since v_n is a unit modulo v_{n-1} , it follows that so is $\psi_{p^k}^{(n-1)}$. Therefore $(\Phi^{C_{p^k}} \underline{E})_*/v_{n-1}$ is trivial as an algebra over $E_*[[e]]/(v_{n-1}, \psi_{p^k}^{(n-1)}) = 0$, and hence $\Phi^{C_{p^k}} \underline{E}$ is of height exactly $n - 1$. □

Example 8.7 Given any Landweber exact complex oriented ring spectrum E of height n , its quotient E/I_{n-1} satisfies the assumptions of the previous proposition. It follows that also in this case the height of $\Phi^{C_{p^k}} E$ equals $n - 1$. For example this applies to Johnson–Wilson spectra, to $\text{MU}[v_n^{-1}]$ or to Morava E -theories.

Similarly, for (2) one needs to find a complex oriented theory E of height n such that $\Phi^T \underline{E}$ is also of height n . This is more elementary and satisfied by any p -local complex oriented theory E of height n . To see this, note again that the Euler classes e_n are given by the n -series $[n]_F(e) \in E_*[[e]]$. Writing $n = p^k \cdot m$ with m coprime to p , we find that $[n]_F(e)$ is a unit multiple of $[p^k]_F(e)$. Modulo I_n , for $k > 0$ the leading term of $[p^k]_F(e)$ is a power of v_n , which by assumption is a unit in E_*/I_n . It follows that, modulo I_n , the coefficients of $\Phi^T \underline{E}$ are given by $E_*/I_n((e))$, which is always nontrivial when E_*/I_n is.

8.2 Change of groups and the structure of \mathcal{M}_{FG}^A

For any A -spectrum X , the groups $MU_{2*}^A(X)$ come equipped with the structure of a graded (L_A, S_A) -comodule. Since the stack \mathcal{M}_{FG}^A is the stack associated to this graded Hopf algebroid, we obtain an associated quasicohherent sheaf $\mathcal{F}^A(X)$ on \mathcal{M}_{FG}^A ; see [Meier and Ozornova 2020, Proposition 4.3]. This is the 0th graded piece of a $\text{QCoh}(\mathcal{M}_{FG}^A)$ -valued homology theory on A -spectra, whose i th piece $\mathcal{F}_i^A(X)$ is given by $MU_{2*+i}^A(X)$. We end this section with a closer look at how the structure of \mathcal{M}_{FG}^A relates to this homology theory.

Recall (Proposition 3.12) that for a closed subgroup $B \subseteq A$, there is an open immersion $j : \mathcal{M}_{FG}^{A/B} \rightarrow \mathcal{M}_{FG}^A$ and a closed immersion $i : \mathcal{M}_{FG}^B \rightarrow \mathcal{M}_{FG}^A$. We obtain corresponding adjunctions

$$\text{QCoh}(\mathcal{M}_{FG}^A) \begin{matrix} \xleftarrow{j^*} \\ \xrightarrow{j_*} \end{matrix} \text{QCoh}(\mathcal{M}_{FG}^{A/B}) \quad \text{and} \quad \text{QCoh}(\mathcal{M}_{FG}^A) \begin{matrix} \xleftarrow{i^*} \\ \xrightarrow{i_*} \end{matrix} \text{QCoh}(\mathcal{M}_{FG}^B).$$

Believing that the structure of \mathcal{M}_{FG}^A dictates the structure of the ∞ -category Sp_A of A -spectra, we expect a relation to the adjunctions

$$\text{Sp}_A \begin{matrix} \xleftarrow{\underline{\Phi}^B} \\ \xrightarrow{P_{A/B}^*} \end{matrix} \text{Sp}_{A/B} \quad \text{and} \quad \text{Sp}_A \begin{matrix} \xleftarrow{\text{res}_B^A} \\ \xrightarrow{\text{coind}_B^A} \end{matrix} \text{Sp}_B.$$

Here, denoting by $\mathcal{F}[B]$ the family of subgroups of A not containing B and by $q : A \rightarrow A/B$ the projection, we define $\underline{\Phi}^B(X) = (\tilde{E}\mathcal{F}[B] \wedge X)^B$ and $P_{A/B}^*(Y) = \tilde{E}\mathcal{F}[B] \wedge q^*X$. Note that the definition of $\underline{\Phi}^B$ is made so that its underlying spectrum is Φ^B and more generally $\Phi^{C/B} \underline{\Phi}^B \simeq \Phi^C$ for every $B \subseteq C \subseteq A$. For more details on the first adjunction, see [Lewis et al. 1986, Section II.9], [Hill 2012, Section 4.1] and [Meier et al. 2023, Section 2.2]. For the benefit of the reader, we give a brief sketch of its basic properties: The adjunction between q^* and $(-)^H$ induces maps

$$\text{id} \rightarrow \underline{\Phi}^B P_{A/B}^* \quad \text{and} \quad P_{A/B}^* \underline{\Phi}^B \rightarrow \tilde{E}\mathcal{F}[B] \wedge (-),$$

which can be checked to be equivalences on geometric fixed points. The inverses

$$\underline{\Phi}^B P_{A/B}^* \xrightarrow{\simeq} \text{id} \quad \text{and} \quad \text{id} \rightarrow \tilde{E}\mathcal{F}[B] \wedge (-) \xrightarrow{\simeq} P_{A/B}^* \underline{\Phi}^B$$

form the counit and unit of the adjunction. In particular, $P_{A/B}^*$ is fully faithful and its image agrees with that of $\tilde{E}\mathcal{F}[B] \wedge$. Since smashing with $\tilde{E}\mathcal{F}[H]$ is idempotent and hence symmetric monoidal, $P_{A/B}^*$ is symmetric monoidal as well and so is $\underline{\Phi}^B$.

Proposition 8.8 For every abelian compact Lie group A and closed subgroup $B \subseteq A$, the diagram

$$\begin{array}{ccc}
 \mathrm{Sp}_A & \xleftarrow{\Phi^B} & \mathrm{Sp}_{A/B} \\
 \downarrow \mathcal{F}^A & \mathcal{P}_{A/B}^* & \downarrow \mathcal{F}^{A/B} \\
 \mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}^A) & \xleftarrow{j^*} & \mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}^{A/B}) \\
 & j_* &
 \end{array}$$

commutes, ie there are natural isomorphisms $j^* \mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(\Phi^B X)$ for $X \in \mathrm{Sp}_A$ and $j_* \mathcal{F}^{A/B}(Y) \cong \mathcal{F}^A(\mathcal{P}_{A/B}^* Y)$ for $Y \in \mathrm{Sp}_{A/B}$.

Moreover, there is a natural isomorphism $i_* \mathcal{F}^B(Z) \cong \mathcal{F}^A(\mathrm{coind}_B^A Z)$ for $Z \in \mathrm{Sp}_B$.

No such isomorphism can be expected for i^* and res_B^A in general. One reason is that i^* is not flat, but even a spectral sequence relating i^* and res_B^A seems not to exist for A not a torus for reasons related to Remark 8.2.

Before proving the proposition we need a lemma, in which we will use the Hopf algebroid $(\Phi^B L_A, \Phi^B S_A)$. Here $\Phi^B L_A$ is obtained from L_A by inverting e_V for all $V \notin \mathrm{im}((A/B)^* \rightarrow A^*)$, and the ring $\Phi^B S_A$ is defined as $\Phi^B L_A \otimes_{L_A} S_A \otimes_{L_A} \Phi^B L_A$. This classifies strict isomorphism between equivariant formal group laws where the relevant Euler classes are invertible on source and target. Since the invertibility of Euler classes only depends on the underlying equivariant formal group and not on the choice of coordinate, this simplifies to $S_A \otimes_{L_A} \Phi^B L_A$.

Lemma 8.9 For $Y \in \mathrm{Sp}_B$ and $q: A \rightarrow A/B$ the projection, there are natural isomorphisms

$$\Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_*(q^* Y) \xrightarrow{\cong} \Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_*(\mathcal{P}_{A/B}^* Y) \xrightarrow{\cong} (\Phi^B \mathrm{MU}_A)_*(Y)$$

and

$$\Phi^B L_A \otimes_{L_{A/B}} (\mathrm{MU}_{A/B})_*(Y) \xrightarrow{\cong} (\Phi^B \mathrm{MU}_A)_*(Y)$$

of graded $(\Phi^B L_A, \Phi^B S_A)$ -comodules, where the map $L_{A/B} \rightarrow \Phi^B L_A$ is defined as the composite $L_{A/B} \xrightarrow{q^*} L_A \rightarrow \Phi^B L_A$.

Proof A model for $\tilde{E}\mathcal{F}[B]$ is given by $S^{\infty W}$ for W the sum of all characters $V \in \mathcal{V}$, for \mathcal{V} the set of characters V not restricting to 1 in B or, equivalently, $V \notin \mathrm{im}((A/B)^* \rightarrow A^*)$. Indeed, $W^C = 0$ for $C \subseteq A$ if and only if $B \subseteq C$, as this is equivalent to none of the $V \in \mathcal{V}$ restricting to 1 in C .

In other words, smashing with $\tilde{E}\mathcal{F}[B]$ is the same as inverting all the maps $S^0 \rightarrow S^V$ for $V \in \mathcal{V}$. For an MU_A -module, this is equivalent to inverting e_V for $V \in \mathcal{V}$.

We have

$$\begin{aligned}
 \Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_*(q^* Y) &\cong (\mathrm{MU}_A)_*(q^* Y)[e_V^{-1} : V \in \mathcal{V}] \cong \pi_*(\mathrm{MU}_A \wedge q^* Y \wedge \tilde{E}\mathcal{F}[B])^A \\
 &\cong \pi_*(\Phi^B(\mathrm{MU}_A \wedge \mathcal{P}_{A/B}^* Y))^{A/B} \cong \pi_*(\Phi^B \mathrm{MU}_A \wedge Y)^{A/B} \\
 &\cong (\Phi^B \mathrm{MU}_A)_*(Y).
 \end{aligned}$$

Replacing q^*Y by $P_{A/B}^*Y$ in the chain of isomorphisms above yields a similar chain of isomorphisms. All the isomorphisms are isomorphisms of comodules since the isomorphisms are natural in the MU_A -variable and we can plug into this variable the left and right unit $MU_A \rightarrow MU_A \wedge MU_A$.

To construct the second isomorphism, note that as part of the global structure of equivariant MU , there is a ring map $q^*MU_{A/B} \rightarrow MU_A$; see [Linskens et al. 2025]. Applying $\underline{\Phi}^B = (- \wedge \tilde{E}\mathcal{F}[H])^B$ yields a ring map $MU_{A/B} \rightarrow \underline{\Phi}^B MU_A$. This induces a morphism

$$\underline{\Phi}^B L_A \otimes_{L_{A/B}} (MU_{A/B})^*(Y) \rightarrow (\underline{\Phi}^B MU_A)^*(Y).$$

It is enough to show that this is an isomorphism for finite Y and hence for $Y = (A/B)/(B'/B)_+ \cong A/B'_+$ for $B \subseteq B' \subseteq A$. In this case, the map becomes

(8.10)
$$\underline{\Phi}^B L_A \otimes_{L_{A/B}} L_{B'/B} \rightarrow (\underline{\Phi}^B MU_A)^*(A/B'_+).$$

The natural map

$$L_A \otimes_{L_{A/B}} L_{B'/B} \rightarrow L_{B'}$$

is an isomorphism since $L_A \rightarrow L_{B'}$ is a surjection with kernel generated by the Euler classes e_V for those $V \in A^*$ restricting trivially to B' ; these are exactly the images of the Euler classes e_W for those $W \in (A/B)^*$ restricting trivially to B'/B .

Thus, (8.10) becomes

$$\underline{\Phi}^B L_A \otimes_{L_A} MU_A^*(A/B'_+) \cong \underline{\Phi}^B L_A \otimes_{L_A} L_{B'} \rightarrow (\underline{\Phi}^B MU_A)^*(A/B'_+),$$

which is an isomorphism by the first part. Similar to the first part, all isomorphisms are isomorphisms of comodules again. □

Proof of Proposition 8.8 We establish first the isomorphism $j^*\mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(\underline{\Phi}^B X)$ for A -spectra X . Consider the commutative diagram

$$\begin{array}{ccccc} \text{Spec } L_A & \longleftarrow & \text{Spec } \underline{\Phi}^B L_A & \longrightarrow & \text{Spec } L_{A/B} \\ & & \searrow \varphi & & \swarrow \\ \mathcal{M}_{\text{FG}}^A & \xleftarrow{j} & \mathcal{M}_{\text{FG}}^{A/B} & & \end{array}$$

where the down-right arrow comes from applying q_* to an A -equivariant formal group classified by a morphism to $\text{Spec } \underline{\Phi}^B L_A$, and the right-pointing horizontal morphisms come from the composition $L_{A/B} \xrightarrow{q^*} L_A \rightarrow \underline{\Phi}^B L_A$. The square is a pullback square by Proposition 3.12. Thus φ is faithfully flat and hence φ^* induces an equivalence of $\text{QCoh}(\mathcal{M}_{\text{FG}}^{A/B})$ to graded $(\underline{\Phi}^B L_A, \underline{\Phi}^B S_A)$ -comodules.

The comodule corresponding to $j^*\mathcal{F}_X^A$ is $\underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_{2*} X$. As in (the proof of) the first isomorphism in Lemma 8.9, we observe that this is isomorphic to

$$\begin{aligned} \underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_{2*}(X \wedge \tilde{E}\mathcal{F}[B]) &\cong \underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_{2*}(P_{A/B}^* \underline{\Phi}^B X) \\ &\cong (\underline{\Phi}^B MU_A)_{2*}(\underline{\Phi}^B X). \end{aligned}$$

By the second isomorphism in Lemma 8.9, this is isomorphic to the comodule corresponding to $\mathcal{F}_{\Phi^B X}^{A/B}$, ie to $\underline{\Phi}^B L_A \otimes_{L_{A/B}} (\underline{\text{MU}}_{A/B})_{2*}(\Phi^B X)$. This establishes the first claimed isomorphism of sheaves.

Since the counit $\underline{\Phi}^B P_{A/B}^* \rightarrow \text{id}_{\text{Sp}_{A/B}}$ is an equivalence, we can assume for the proof of $j_* \mathcal{F}^{A/B}(Y) \cong \mathcal{F}^A(P_{A/B}^* Y)$ for $Y \in \text{Sp}_{A/B}$ that $Y = \underline{\Phi}^B X$ for some $X \in \text{Sp}_A$, and we obtain from the first isomorphism in this case a natural isomorphism

$$j^* j_* \mathcal{F}^{A/B}(Y) \cong j^* j_* j^* \mathcal{F}^A(X) \cong j^* \mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(Y) \cong j^* \mathcal{F}^A(P_{A/B}^* Y).$$

The Euler classes for characters $V \notin \text{im}((A/B)^* \rightarrow A^*)$ act invertibly on $\mathcal{F}^A(P_{A/B}^* Y)$, and Proposition 3.12 implies that $\mathcal{F}^A(P_{A/B}^* Y)$ is in the image of j_* . Since the counit

$$j^* j_* \rightarrow \text{id}_{\text{QCoh}(\mathcal{M}_{\text{FG}}^{A/B})}$$

is an isomorphism, the claimed isomorphism follows.

Lastly, we show $i_* \mathcal{F}^B(Z) \cong \mathcal{F}^A(\text{coind}_B^A Z)$ for $Z \in \text{Sp}_B$. This follows from the chain

$$\begin{aligned} (\underline{\text{MU}}_A)_{2*}(\text{coind}_B^A Z) &\cong \pi_{2*}^A(\underline{\text{MU}}_A \otimes \text{coind}_B^A Z) \cong \pi_{2*}^A(\text{coind}_B^A(\underline{\text{MU}}_B \otimes Z)) \\ &\cong \pi_{2*}^B(\underline{\text{MU}}_B \otimes Z) \cong \underline{\text{MU}}_{2*}^B(Z) \end{aligned}$$

of isomorphisms of (L_A, S_A) -comodules. □

We end this section by connecting the stacky point of view with support theories. For a compact A -spectrum X , we may consider the support $\text{supp } \mathcal{F}_i^A(X) \subseteq |\mathcal{M}_{\text{FG}}^A|$, corresponding to those $x: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}}^A$ such that $x^* \mathcal{F}_i^A(X)$ is nontrivial. As remarked above, in general the sheaves $\mathcal{F}_i^A(X)$ do not play well with restrictions. Thus, for an A -spectrum X , one should consider all $\mathcal{F}_i^B(\text{res}_B^A X) \in \text{QCoh}(\mathcal{M}_{\text{FG}}^B)$ for $B \subseteq A$ in conjunction. Using the closed embeddings $\mathcal{M}_{\text{FG}}^B \hookrightarrow \mathcal{M}_{\text{FG}}^A$, the correct notion of support of $\mathcal{F}_*^A(X)$ becomes thus $\text{supp } \mathcal{F}_*(X) = \bigcup_{B \subseteq A} \text{supp } \mathcal{F}_*^B(\text{res}_B^A X)$.

Proposition 8.11 *For any compact A -spectrum X , the subsets*

$$\text{supp } \mathcal{F}_*(X) \subseteq |\mathcal{M}_{\text{FG}}^A| \quad \text{and} \quad \text{supp}((\underline{\text{MU}}_A)_* X) \subseteq \text{Spec}^{\text{inv}}(L_A)$$

correspond to each other under the bijection from Theorem 4.7. Thus, $\text{supp } \mathcal{F}_(-)$ is a universal support theory as well.*

Proof Consider a point $[x: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}}^A] \in |\mathcal{M}_{\text{FG}}^A|$ corresponding to a pair (B, n) . Then by Proposition 3.12 and Remark 4.5, x factors as $\bar{x}: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}} = \mathcal{M}_{\text{FG}}^{B/B}$ composed with $\mathcal{M}_{\text{FG}}^{B/B} \xrightarrow{j} \mathcal{M}_{\text{FG}}^B \xrightarrow{i} \mathcal{M}_{\text{FG}}^A$. By Proposition 8.8, the pullback functor j^* corresponds to $\underline{\Phi}^B$. Thus, $x \in \text{supp } \mathcal{F}_*^B(\text{res}_B^A X)$ if and only if $\bar{x}^* \mathcal{F}_*^{\{1\}}(\Phi^B X) \neq 0$, ie if $\Phi^B X$ has chromatic type at least n . We have seen in Proposition 8.4, this happens if and only if $I_{B,n}^A \in \text{supp}((\underline{\text{MU}}_A)_* X)$.

Now suppose $I_{B,n}^A \notin \text{supp}((\underline{\text{MU}}_A)_* X)$ and let $B \subseteq \tilde{B} \subseteq A$. Thus $(\underline{\text{MU}}_{\tilde{B}})_*(\text{res}_{\tilde{B}}^A X)_{I_{B,n}^{\tilde{B}}}$ vanishes. Since $[x] \in |\mathcal{M}_{\text{FG}}^{\tilde{B}}|$ is the image of the prime ideal $I_{B,n}^{\tilde{B}} \in \text{Spec } L_{\tilde{B}} \cong \text{Spec}(\underline{\text{MU}}_{\tilde{B}})_*$, we see by Lemma 2.6 that $x: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}}^{\tilde{B}}$ factors over $\tilde{x}: \text{Spec } k \rightarrow \text{Spec}(L_{\tilde{B}})_{I_{B,n}^{\tilde{B}}}$. Thus, we can identify $x^* \mathcal{F}_*^{\tilde{B}}(X)$ with $\tilde{x}^*(\underline{\text{MU}}_{\tilde{B}})_*(\text{res}_{\tilde{B}}^A X)_{I_{B,n}^{\tilde{B}}} = 0$, and $x \notin \text{supp } \mathcal{F}_*(X)$. □

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
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