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A Weyl's law for singular Riemannian foliations with applications to invariant theory

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We prove a version of Weyl's law for the basic spectrum of a closed singular Riemannian foliation $(M, \mathcal{F})$ with basic mean curvature. In the special case of $M = \mathbb{S}^n$, this gives an explicit formula for the volume of the leaf space $\mathbb{S}^n/\mathcal{F}$ in terms of the algebra of basic polynomials. In particular, $\mathrm{Vol}(\mathbb{S}^n/\mathcal{F})$ is a rational multiple of $\mathrm{Vol}(\mathbb{S}^m)$, where $m = \dim(\mathbb{S}^n/\mathcal{F})$.

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1 Introduction

Given a compact, connected n-dimensional Riemannian manifold M, the Laplacian operator Δ is a self-adjoint operator with a discrete spectrum $0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$. While the eigenvalues themselves strongly depend on the geometry of M, the celebrated Weyl's theorem states that the *growth* of the eigenvalues only depends on the dimension and volume of M: in other words, letting $S(t) = \max\{k \mid \tilde{\lambda}_k < t\}$ denote the counting function for the eigenvalues of Δ, we have that

$$S(t) \sim \frac{\omega_n}{(2\pi)^n} \mathrm{Vol}(M) t^{n/2},$$

where ω_n denotes the volume of the unit n-ball, and $f \sim g$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$. Weyl's law was generalized by Brüning and Heintze [1978/79] and Donnelly [1978] to the context of compact manifolds equipped with an isometric action by a compact Lie group G; in their version, the growth of the counting function for the G*-invariant* eigenfunctions of Δ depends on the volume and dimension of the *orbit space* M/G.

Singular Riemannian foliations $(M, \mathcal{F})$ are, roughly speaking, partitions of a Riemannian manifold M into connected submanifolds (called *leaves*) locally equidistant to one another; see Section 2.1.1 for a more detailed definition. They generalize the decomposition into orbits of isometric actions of connected groups, which we also refer to as *homogeneous* singular Riemannian foliations. Singular Riemannian foliations also appear as natural generalizations of isoparametric foliations in symmetric spaces, Riemannian submersions, and the dual foliation in noncompact manifolds with nonnegative curvature. Recently, the understanding of singular Riemannian foliations has seen rapid development, and despite providing a

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proper generalization of group actions (that is, there exist infinitely many nonhomogeneous examples, even in spheres; see [Radeschi 2014]), it has been shown that they retain a lot of the rigid properties that isometric actions enjoy; see [Galaz-Garcia and Radeschi 2015; Ge and Radeschi 2015; Mendes and Radeschi 2019]. This is especially true when M is a round sphere $\mathbb{S}^n$, which appears as an "infinitesimal model" for general singular Riemannian foliations around a point; see [Lytchak and Radeschi 2018; Mendes and Radeschi 2020b].

Given a singular Riemannian foliation $(M, \mathcal{F})$ with closed leaves, one can define a *leaf space* $M/\mathcal{F}$ which is an analogue to the orbit space, and the algebra of *basic functions* defined as the functions which are constants along the leaves. If $\mathcal{F}$ has *basic mean curvature*, in the sense that the mean curvature vectors of the leaves of $\mathcal{F}$ project to a vector field in $M/\mathcal{F}$, then the Laplace operator restricts to a self-adjoint operator on the space of basic functions, with a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$; see Section 2.1.3. The first main result can then be described as Weyl's law for closed singular Riemannian foliations:

Theorem A *Let M be a compact Riemannian manifold and $\mathcal{F}$ a singular Riemannian foliation of M with closed leaves and basic mean curvature. Let $N(t) = \max\{i \mid \lambda_i < t\}$ be the counting function for the basic spectrum, and $X = M/\mathcal{F}$ be the leaf space. Then*

$$N(t) \sim \frac{\operatorname{vol}(X)\omega_m}{(2\pi)^m} t^{m/2} \quad \text{as } t \to \infty,$$

where $m = \dim X$ and $\omega_m = \operatorname{vol} \mathbb{D}^m$.

The condition of basic mean curvature is always satisfied when $\mathcal{F}$ is homogeneous or when M is the Euclidean space, a round sphere [Alexandrino and Radeschi 2015], as well as complex and quaternionic projective spaces [Alexandrino and Radeschi 2016, page 2205].

In the homogeneous case, Theorem A follows as a special case of Corollary 3.5 of Brüning and Heintze [1978/79] and the Main Theorem of Donnelly [1978]. In the *regular case*, namely when all leaves have the same dimension, this result was proved by Richardson [2010] without the assumptions of closed leaves and basic mean curvature.

A surprising result we prove using Weyl's law is the following:

Theorem B *If $(\mathbb{S}^n, \mathcal{F})$ is a closed singular Riemannian foliation (resp. $G \to O(n+1)$ is an orthogonal representation) with m-dimensional leaf space (resp. orbit space), then $\operatorname{Vol}(\mathbb{S}^n/\mathcal{F})$ (resp. $\operatorname{Vol}(\mathbb{S}^n/G)$) is a rational multiple of $\operatorname{Vol}(\mathbb{S}^m)$.*

To the best of the authors' knowledge, Theorem B is novel even in the homogeneous case, even though it is clearly true when G is finite. Since the quotients $\mathbb{S}^n/\mathcal{F}$ can be thought as singular spaces all of whose geodesics are closed (in the sense of quotient geodesics from [Mendes and Radeschi 2020a, Section 3.4]), Theorem B is similar in spirit to [Weinstein 1977, Theorem 3] which applies to *manifolds* all of whose geodesic are closed.

In the case of foliations, Theorem B follows immediately from Theorem C below, in which the rational number $\mathrm{Vol}(\mathbb{S}^n/\mathcal{F})/\,\mathrm{Vol}(\mathbb{S}^m)$ is computed explicitly in terms of the *algebraic structure* of $(\mathbb{S}^n,\mathcal{F})$. Recall that a singular Riemannian foliation $(\mathbb{S}^n,\mathcal{F})$ extends to an *infinitesimal foliation* $(\mathbb{R}^{n+1},\mathcal{F})$, ie a singular Riemannian foliation with the origin as a leaf. In this case, it was shown in [Mendes and Radeschi 2020a] that the algebra A of basic polynomials is a so-called *Laplacian algebra* (ie a graded algebra preserved by the multiplication by $r^2=\sum_i x_i^2$ and by the Laplacian $\Delta=\sum_i \partial^2/\partial x_i^2$) which completely determines $\mathcal{F}$. Furthermore, there is a dictionary between the geometric properties of an infinitesimal foliation and the algebraic properties of its Laplacian algebra A. The next main theorem adds a line to this dictionary:

Theorem C *Let $(V,\mathcal{F})$, with $\dim V=n+1$, be an infinitesimal foliation with closed leaves, and let $A=\bigoplus_i A_i\subset\mathbb{R}[V]$ be the associated Laplacian algebra of basic polynomials. Denote its Hilbert series by $H(z)=\sum_i(\dim A_i)z^i$. Let $\mathbb{S}V$ denote the unit sphere in V, and m the dimension of the leaf space $X=\mathbb{S}V/\mathcal{F}$.*

Then:

(a) *The Laurent series of $H(z)$ at $z=1$ has the form*
$$H(z)=\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(\mathbb{S}^m)}(1-z)^{-m-1}+(\cdots)(1-z)^{-m}+\cdots.$$

(b) *A is a **free module over a free subalgebra**, ie $A=\bigoplus_{i=1}^{\ell}\mathbb{R}[f_0,\dots,f_m]g_i$ for some homogeneous $f_0,\dots,f_m,g_1,\dots,g_\ell\in A$, and*
$$\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(\mathbb{S}^m)}=\frac{\ell}{\deg f_0\cdots\deg f_m}.$$

The analogue of Theorem C for representations of disconnected groups also holds; see Proposition 40.

Let us comment on the main aspects of the proofs. Theorem A follows a framework similar to the homogeneous version by Brüning and Heintze [1978/79]. In their proof of Weyl's law, however, results from transformation groups and representation theory were used in a fundamental way in many places, which are not available in the foliated case. Some of these obstacles have been overcome using recent results in the literature of singular Riemannian foliations, most notably the averaging operator introduced by Lytchak and Radeschi [2018], and the foliated slice theorem of Mendes and Radeschi [2019]. Others require new uniform estimates on the geometry of leaves (see Section 3), which culminate in the following result:

Theorem D *Given a compact connected Riemannian manifold M and a singular Riemannian foliation $\mathcal{F}$ with closed leaves, there exists a constant $C=C(M,\mathcal{F})$ such that, for every $p\in M$, and every $s>0$, one has*
$$\int_{L_p}\exp\left(-\frac{d_M(p,q)^2}{s}\right)dq\le Cs^{\frac{1}{2}\dim L_p},$$
where L_p denotes the leaf through p.

The proof of Theorem C relies on the observation that the Hilbert series $H(z)$ for the Laplacian algebra A of basic polynomials of $(V, \mathcal{F})$ contains, via the theory of spherical harmonics, the same information as the basic spectrum for the Laplacian of $\mathbb{S}V$. Using well-known techniques from enumerative combinatorics, we relate the pole of $H(z)$ at $z=1$ with the growth of the coefficients of $H(z)$, hence to the growth of basic eigenvalues of $(\mathbb{S}V, \mathcal{F})$, which is given by the foliated Weyl's law (Theorem A).

1.1 Remarks, open questions and generalizations

The results of Theorem A lead to possible generalizations. We list them here as questions for the interested reader.

Given a singular Riemannian foliation $(M, \mathcal{F})$, regardless of whether the mean curvature is basic or not, there always is a basic Laplacian $\Delta_b\colon L^2(M)^b \to L^2(M)^b$ defined as $\Delta_b = dd^* + d^*d$, where $d\colon L^2(M) \to L^2(\Omega^1(M))^b$ is the restriction of the differential to the space of basic functions (see [Richardson 2010, Section 1]) and d^* is its adjoint. If the mean curvature of $\mathcal{F}$ is not basic, however, Δ_b does not coincide with the restriction of the usual Laplacian to the space $L^2(M)^b$.

Question 1 *Given a singular Riemannian foliation $(M, \mathcal{F})$ with closed leaves on a compact manifold M, denoting $N_b(t)$ the counting function for the eigenvalues of Δ_b, does the Weyl's law*

$$N_b(t) \sim \frac{\omega_m}{(2\pi)^m} \operatorname{Vol}(M/\mathcal{F}) t^{m/2}, \quad \text{where } m = \dim M/\mathcal{F},$$

hold?

In the regular case (ie when all the leaves have the same dimension) this was proved by Richardson [2010].

Similarly, one could consider foliations with nonclosed leaves:

Question 2 *Given a singular Riemannian foliation $(M, \mathcal{F})$ with nonclosed leaves, does a Weyl's law still hold?*

Given a singular Riemannian foliation $(M, \mathcal{F})$ with nonclosed leaves, Molino's conjecture (proved by Alexandrino and Radeschi [2017]) states that replacing the leaves of $\mathcal{F}$ with their closures gives rise to a new singular Riemannian foliation $\overline{\mathcal{F}}$, with the same basic functions as $\mathcal{F}$. Therefore, a positive answer to Question 1 would imply that, even in this case, the counting function N_b for the basic Laplacian would have asymptotics

$$N_b(t) \sim \frac{\omega_{\bar{m}}}{(2\pi)^{\bar{m}}} \operatorname{Vol}(M/\overline{\mathcal{F}}) t^{\bar{m}/2}, \quad \text{where } \bar{m} = \dim M/\overline{\mathcal{F}}.$$

One may also consider a variant of Question 2, where $\mathcal{F}$ is assumed to have basic mean curvature. It is unclear to the authors whether this implies that $\overline{\mathcal{F}}$ has basic mean curvature.

Regarding the Weyl's law, a natural question is whether more geometric information on a singular Riemannian foliation $(M, \mathcal{F})$ (with, say, closed leaves and basic mean curvature) can be obtained from finer information about the eigenvalues of the basic Laplacian — more specifically, from the asymptotics of the *Laplace transform* $n(s) := s\int_0^\infty e^{-st}N(t)\,dt$ of the basic counting function $N(t)$, as $s\to 0$. The proof of Theorem A hinges on the estimate that

$$n(s)\sim s^{-m/2}\frac{\Gamma(m/2+1)\omega_m}{(2\pi)^m}\operatorname{Vol}(M/\mathcal{F})$$

(see Proposition 27), but a more precise estimate could yield more profound connections with the geometry of the leaf space:

Question 3 *What are the next terms in the asymptotics for $n(s)$? Can they be described in terms of geometric quantities of $M/\mathcal{F}$?*

Estimates of this type were obtained by Richardson [2010] for regular Riemannian foliations. In the special case of closed leaves, he obtained (Theorem 3.5):

$$n(s)=\frac{1}{(4\pi s)^{m/2}}\left(\operatorname{Vol}(M/\mathcal{F})+\frac{\sqrt{\pi}}{2}\operatorname{Vol}(\partial M/\mathcal{F})s^{1/2}+C(M/\mathcal{F})s+O(s^{3/2})\right),$$

where $C(M/\mathcal{F})$ depends on the total curvature, as well as the codimension-2 strata of $M/\mathcal{F}$. Similar estimates were obtained by Brüning and Heintze [1984] in the homogeneous case.

In the special case of a round sphere $M=\mathbb{S}^n$, given the correspondence between singular Riemannian foliations $(\mathbb{S}(V),\mathcal{F})$ and Laplacian algebras $A\subseteq\mathbb{R}[V]$, one can ask whether the terms in the asymptotics of $n(s)$ correspond somehow to the terms in the Laurent expansion for the Hilbert series $H(t)$ of A, at $t=1$:

Question 4 *Let A be the Laplacian algebra of a singular Riemannian foliation $(\mathbb{S}^n,\mathcal{F})$, and $H(z)$ be the Hilbert series of A. If the Laurent series of $H(z)$ at $z=1$ is of the form $H(z)=(1-z)^{-m-1}\sum_k a_k z^k$, what is (if any) the correspondence between the coefficients a_k and the terms in the asymptotics of $n(s)$?*

We end this section with:

Question 5 *In the context of Theorem B, what rational numbers appear as*

$$\frac{\operatorname{Vol}(\mathbb{S}^n/\mathcal{F})}{\operatorname{Vol}(\mathbb{S}^m)}\quad\textit{or}\quad\frac{\operatorname{Vol}(\mathbb{S}^n/G)}{\operatorname{Vol}(\mathbb{S}^m)}?$$

Are they all the reciprocal of a natural number?

1.2 Structure of the paper

The paper is structured as follows: in Section 2, we recall the main definitions and important results that we will use throughout the paper. Section 3 has a more geometric flavor and is devoted to the proof of Theorem D. In Section 4, which has a more analytic flavor, we prove Theorem A. Finally, the more algebraic Section 5 contains the proof of Theorem C.

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2 Preliminaries

2.1 Singular Riemannian foliations

2.1.1 Basic definitions and facts The main objects of study in the present paper are *singular Riemannian foliations*, first defined by Molino [1988]. By definition, a singular Riemannian foliation is a partition $\mathcal{F}$ of a complete Riemannian manifold M into connected, injectively immersed smooth submanifolds (called *leaves*), which is generated by a set of smooth vertical fields, and such that every geodesic of M that meets one leaf orthogonally is orthogonal to every other leaf that it meets. The last condition is equivalent to the leaves being locally at a constant distance apart [Radeschi 2025, Proposition 2.3].

The main examples are fibers of a Riemannian submersion, leaves of a (regular) Riemannian foliation, the parallel and focal sets of an isoparametric submanifold of a sphere, and orbits of an isometric group action (sometimes called "homogeneous" singular Riemannian foliations).

We say $\mathcal{F}$ is *closed* if all the leaves are closed. In this case, and assuming M complete, the condition involving normal geodesics is equivalent to (global) equidistance of every pair of leaves [Radeschi 2025, Remark 2.5].

We say $\mathcal{F}$ is *infinitesimal* when M is isometric to some Euclidean space $\mathbb{R}^{n+1}$, and the origin $\{0\}$ is a leaf. In this case any sphere centered at the origin is a union of leaves (in other words, is "saturated" by $\mathcal{F}$) and, by Molino's homothetic transformation lemma [1988, Lemma 6.2], $\mathcal{F}$ is determined by its restriction to any such sphere. In fact, the map $\mathcal{F} \mapsto \mathcal{F}|_{\mathbb{S}^n}$ is a bijection between infinitesimal singular Riemannian foliations of $\mathbb{R}^{n+1}$ and singular Riemannian foliations of $\mathbb{S}^n$.

Given a singular Riemannian foliation $(M, \mathcal{F})$, the set of leaves is called the *leaf space*, and is denoted by $M/\mathcal{F}$. When the leaves are closed, the leaf space $M/\mathcal{F}$ has a natural metric structure, with respect to which the natural projection map $\sigma\colon M \to M/\mathcal{F}$ is a "manifold submetry". For more information on submetries and manifold submetries, see [Kapovitch and Lytchak 2022; Mendes and Radeschi 2020a].

A function defined on M is *$\mathcal{F}$-basic* (or just *basic*, if $\mathcal{F}$ is understood from context) when it is constant on the leaves of $\mathcal{F}$. Equivalently, when it is the pullback by σ of a function defined on the "base" $M/\mathcal{F}$. We sometimes denote sets of basic functions with the superscript ^{b}, for example, $C^\infty(M)^b$ for the set of all smooth basic (real-valued) functions on M.

2.1.2 Invariant theory If $(V, \mathcal{F})$ is an infinitesimal closed singular Riemannian foliation, the algebra of all basic *polynomials*, denoted by $A = \mathcal{B}(\mathcal{F}) = \mathbb{R}[V]^b$, is large enough that it separates leaves, and moreover, it is a finitely generated algebra. This is known as the *algebraicity theorem* [Lytchak and Radeschi 2018]. When $(V, \mathcal{F})$ is homogeneous, that is, when the leaves are given as the orbits of an orthogonal representation of a compact group, this algebra A is called the "algebra of invariants", and the algebraicity theorem reduces to classical facts in invariant theory.

2.1.3 Principal part Given a singular Riemannian foliation $(M, \mathcal{F})$ on a complete manifold M, there exists an open, dense, saturated, full-measure subset $M_0 \subset M$, called the *principal part*, such that $M_0/\mathcal{F} \subset M/\mathcal{F}$ is a smooth Riemannian manifold, and $\sigma|_{M_0} \colon M_0 \to M_0/\mathcal{F}$ is a Riemannian submersion; see [Lytchak and Radeschi 2018, page 3]. (To justify the name, note that when $\mathcal{F}$ is homogeneous, M_0 is the union of the principal orbits.) When M is compact and $\mathcal{F}$ is closed, the map $\sigma|_{M_0} \colon M_0 \to M_0/\mathcal{F}$ is proper, hence a locally trivial fiber bundle by Ehresmann's lemma. In particular, the function $h \colon M_0/\mathcal{F} \to \mathbb{R}$ given by $h(x) = \mathrm{vol}(\sigma^{-1}(x))$, and its pullback $h \circ \sigma|_{M_0} \colon M_0 \to \mathbb{R}$, given by $p \mapsto \mathrm{vol}(L_p)$, are smooth.

We denote by $H(p) \in T_pM$ the mean curvature vector of the leaf L_p as a submanifold of M, for every $p \in M_0$. Note that H is a smooth vector field on M_0. We say $\mathcal{F}$ has *basic mean curvature* when H is σ-related to a (smooth) vector field $\underline{H}$ on $M_0/\mathcal{F}$. By [Alexandrino and Radeschi 2015, Proposition 3.1], this is automatically the case when M is a round sphere, or M is Euclidean space.

2.1.4 Tubular neighborhoods and the slice theorem Let M be a closed Riemannian manifold, and $\mathcal{F}$ a closed singular Riemannian foliation. Fix a leaf L, and $\epsilon > 0$ be smaller than the normal injectivity radius of L. Denote by νL the normal bundle of L, by $\nu L^{\le\epsilon}$ (resp. $\nu L^{<\epsilon}$) the set of all vectors in νL of length at most ϵ (resp. less than ϵ), by $B_\epsilon(L)$ the (open) tube of radius ϵ around L, that is, the set of all points at distance less than ϵ from L, and by $\exp \colon \nu L \to M$ the normal exponential map. Note that the restriction of exp to $\nu L^{<\epsilon}$ is a diffeomorphism onto $B_\epsilon(L)$. We denote by $\pi \colon B_\epsilon(L) \to L$ the nearest-point projection, and note that $\pi \circ \exp$ coincides with the natural projection $\nu L \to L$ on $\nu L^{<\epsilon}$. Note also that the restriction of π to any leaf $L' \subset B_\epsilon(L)$ is a submersion $L' \to L$. This follows from [Alexandrino et al. 2013, Theorem 2.2(b)], and see also [Mendes and Radeschi 2019, Proposition 13].

Let $p \in L$. The *disconnected slice foliation* (see [Mendes and Radeschi 2019, Section 3.3]) of $\mathcal{F}$ at p, denoted by $\mathcal{F}^p$, is a partition of ν_pL obtained by first intersecting the leaves of $\mathcal{F}$ with $\exp \nu_pL^{\le\epsilon}$, then pulling back this decomposition by exp, and finally extending it via homothetic transformations to all of ν_pL. This partition $\mathcal{F}^p$ of the Euclidean space ν_pL is a "disconnected infinitesimal foliation" in the sense of [Mendes and Radeschi 2019, Definition 8], and its restriction to any sphere centered at the origin is a "disconnected singular Riemannian foliation" in the sense of Definition 15 below. By the slice theorem [Mendes and Radeschi 2019, Theorem A], the disconnected slice foliation $\mathcal{F}^q$ at any other $q \in L$ is equivalent to $\mathcal{F}^p$ in the sense that there exists an isometry $\nu_p L \to \nu_q L$ sending leaves onto leaves.

2.1.5 Averaging operator Given a singular Riemannian foliation $(M, \mathcal{F})$ with compact leaves, and an L^2 function $f: M \to \mathbb{R}$, we define its average $\mathrm{Av}(f): M \to \mathbb{R}$ over $\mathcal{F}$ by

$$\mathrm{Av}(f)(p) = \frac{1}{\mathrm{vol}(L_p)} \int_{q \in L_p} f(q)\, d\mathrm{vol}^{L_p}(q),$$

where $d\mathrm{vol}^{L_p}$ denotes the Riemannian volume form of the compact Riemannian manifold L_p and $\mathrm{vol}(L_p)$ its volume. We will frequently write dq instead of $d\mathrm{vol}^{L_p}(q)$ to simplify notation. When $\mathcal{F}$ has basic mean curvature, the averaging operator Av takes smooth functions to smooth functions, and commutes with the Laplace operator on the principal part M_0; see [Lytchak and Radeschi 2018, Theorem 3.3 and Lemma 3.2], and see also [Park and Richardson 1996] for similar considerations in the case of regular Riemannian foliations.

2.2 Hörmander's estimate

We briefly recall Hörmander's estimate on spectral functions, which is one of the main analytical ingredients in the proof of Weyl's law for singular Riemannian foliations.

Let $\mathcal{H}$ be a Hilbert space with inner product $(\ ,\ )_{\mathcal{H}}$ and let A be a self-adjoint, possibly unbounded linear operator on $\mathcal{H}$. Given any characteristic function $\chi_{\mathcal{U}}$ on a Borel measurable set $\mathcal{U}$ in $\mathbb{R}$, one can use Borel functional calculus to define an orthogonal projection $\chi_{\mathcal{U}}(A)$ on $\mathcal{H}$; see [Reed and Simon 1972, page 262].

Given any $\lambda \in \mathbb{R}$, define $E(\lambda) := \chi_{(-\infty,\lambda)}(A)$. The family of projections $\{E(\lambda)\}$ is called the *spectral resolution* of the self-adjoint linear operator A.

Example 6 Suppose that $A: \mathcal{H} \to \mathcal{H}$ is a positive, self-adjoint linear operator with discrete spectrum

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \cdots$$

and an orthonormal eigenbasis basis $\{u_i\}$. By the spectral theorem in projection-valued measure form [Reed and Simon 1972, Theorem VIII.6], the spectral resolution of A is simply the projection to the sum of eigenspaces whose corresponding eigenvalues are less than λ. In other words,

$$E(\lambda)(f) = \chi_{(-\infty,\lambda)}(A)(f) = \sum_{\lambda_i < \lambda} (f, u_i)_{\mathcal{H}} \cdot u_i. \tag{1}$$

To define spectral functions, let Ω be a smooth, m-dimensional manifold with a smooth density μ. Let $\widehat{P}: L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ be a formally positive self-adjoint extension of an elliptic operator P on $C^\infty(\Omega)$. Let $\{E(\lambda)\}$ be the spectral resolution of $\widehat{P}$. It is a standard result [Hörmander 1955, Section 3.7] that the ellipticity of P implies that for each t, there exists a unique kernel $\theta(x,y;t) \in C^\infty(\Omega \times \Omega)$ such that for any $f \in L^2(\Omega,\mu)$,

$$E(t)(f)(x) = \int_\Omega \theta(x,y;t) f(y)\, d\mu(y). \tag{2}$$

The kernels $\theta(x,y;t)$ are called *spectral functions*.

Hörmander proved the following asymptotics for the spectral functions:

Theorem 7 [Hörmander 1968] *Let p be the principal symbol of an elliptic differential operator P of order k and let $d\xi$ be the measure induced by $d\mu$ on each fiber of the cotangent bundle $T^*\Omega$. Define*

$$R(x,\lambda) := \lambda^{-m/k}\theta(x,x;\lambda) - (2\pi)^{-m}\int_{B_x} d\xi,$$

where $B_x = \{\xi \in T_x^*\Omega \mid p(\xi) < 1\}$. Then on every compact subset of Ω, $R(x,\lambda) = O(\lambda^{-1/m})$ uniformly as $\lambda \to \infty$.

2.3 Basic Laplacian

Let (M,g) be a compact Riemannian manifold, $\mathcal{F}$ a closed singular Riemannian foliation with basic mean curvature, and let $\Delta(f) := -\mathrm{div}\nabla(f)$ be the Laplace operator on functions.

Recall from Section 2.1.5 that the averaging operator Av commutes with the Laplace operator Δ on $C^\infty(M)$. Therefore, the Laplace operator Δ sends basic smooth functions to basic smooth functions. We then extend the restriction $\Delta|_{C^\infty(M)^b}$ of Δ on the space of basic functions, to obtain a self-adjoint linear operator $R\colon L^2(M)^b \to L^2(M)^b$ with domain $\mathfrak{D}(R) = H^2(M)^b$.

Let M_0 be the principal part of M and let $X_0 := \sigma(M_0)$ be the manifold part of X such that $\sigma: M_0 \to X_0$ is a Riemannian submersion. Given any point $x \in X_0$, define $h(x)$ to be the volume of the leaf $L_{\sigma^{-1}(x)}$, and recall from Section 2.1.3 that h is a smooth function. By Fubini's theorem for Riemannian submersions, the pullback of σ is a Hilbert space isometry $\sigma^*\colon L^2(X_0,h) \to L^2(M_0)^b$, where $L^2(X_0,h)$ is the set of square-integrable functions on X_0 with respect to the measure $h(x)\, d\mathrm{vol}_{X_0}$.

Let σ_* be the inverse of σ^*. Define $T\colon L^2(X_0,h) \to L^2(X_0,h)$ to be

$$T := \sigma_* \circ R \circ \sigma^*,$$

with the domain $\mathfrak{D}(T) = \sigma_*(\mathfrak{D}(R))$. A simple calculation proves the following lemma that relates the operator T to the Laplace operator on X_0:

Lemma 8 *For any $f \in C_c^\infty(X_0)$, we have*

$$T(f) = \Delta_{X_0} f + \underline{H}\, f,$$

where $\underline{H}$ is the projection of the mean curvature vector field H through σ.

By Lemma 8, the operator T is a self-adjoint extension of an elliptic differential operator, whose principal symbol is the same as that of the Laplace operator on X_0.

Remark 9 As M is compact, the Laplace operator has discrete spectrum:

$$0 = \widetilde{\lambda}_0 < \widetilde{\lambda}_1 \le \widetilde{\lambda}_2 \cdots \le \widetilde{\lambda}_k \le \cdots .$$

Since $T\colon L^2(X_0,h) \to L^2(X_0,h)$ is equivalent to $R\colon L^2(M)^b \to L^2(M)^b$, it must have a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_i \le \cdots ,$$

and we can define the counting function

$$N(t) = \max\{k \mid \lambda_k < t\}.$$

Furthermore, since the Laplace operator commutes with Av, there exists a smooth, orthonormal basis of eigenfunctions $\{\phi_k\}$ for $L^2(M)$ with a subsequence $\{\phi_{k_i}\}$ that generates the vector subspace $L^2(M)^b$. Define $\varphi_i := \sigma_*(\phi_{k_i})$. Then $\{\varphi_i\}$ form a smooth orthonormal basis of T-eigenfunctions for $L^2(X_0,h)$.

Proposition 10 *Let $\theta^T(x,y;t)$ and $\theta^\Delta(x,y;t)$ be the spectral function of T and Δ, respectively, and let $\{\phi_k\}$ and $\{\varphi_i\}$ be the smooth orthonormal bases in Remark 9. Then for each $t \in \mathbb{R}$, $p,q \in M$ and $x,y \in X_0$ we have the following*:

(a) $\theta^T(x,y;t) = \sum_{\lambda_i<t} \varphi_i(x)\varphi_i(y)$ *and* $\theta^\Delta(p,q;t) = \sum_{\widetilde{\lambda}_k<t} \phi_k(p)\phi_k(q)$.

(b) *For each pair of $p,q \in M$ and each $t \in \mathbb{R}$,*

$$\theta^T(\sigma(p),\sigma(q);t) = (\mathrm{Av} \otimes \mathrm{Av})\theta^\Delta(p,q;t).$$

Proof By (1), $\sum_{\lambda_i<t} \varphi_i(x)\varphi_i(y)$ and $\sum_{\widetilde{\lambda}_k<t} \phi_k(p)\phi_k(q)$ are two smooth functions satisfying (2). Hence (a) follows from the uniqueness of spectral functions.

To prove (b), simply note that

$$\begin{aligned}\theta^T(\sigma(p),\sigma(q);t) &= \sum_{\lambda_i<t} \varphi_i(\sigma(p))\varphi_i(\sigma(q)) = \sum_{\lambda_i<t} \sigma^*\varphi_i(p)\sigma^*\varphi_i(q)\\ &= \sum_{\widetilde{\lambda}_k<t} \mathrm{Av}\,\phi_k(p)\,\mathrm{Av}\,\phi_k(q) = (\mathrm{Av}\otimes\mathrm{Av})\theta^\Delta(p,q;t).\end{aligned}$$ □

By applying Theorem 7 to the spectral function of T, we have the following:

Proposition 11 *Let $\theta^T(x,y;t)$ be the spectral function of the elliptic differential operator that generates T. Then for each $x \in X_0$,*

$$\theta^T(x,x;t) \sim t^{m/2}\frac{\omega_m}{h(x)(2\pi)^m}$$

as $t \to \infty$, where ω_m is the volume of unit balls in $\mathbb{R}^m$.

2.4 Heat kernels

Let $\phi_i(x)$ be the orthonormal eigenbasis for Δ constructed in Remark 9. A well-known result by Minakshisundaram and Pleijel [1949] (see also [Buser 2010, Theorem 7.2.18]) states that, for each $\sigma > \dim(M)$, the Dirichlet series $\sum_i \lambda_i^{-\sigma}\phi_i(p)\phi_i(q)$ converges uniformly and absolutely on $M \times M$. Combining with (a) in Proposition 10, we see that the spectral function $\theta^\Delta(p,q;t)$ has uniform polynomial growth, in the sense that there exists positive constants C and N such that for all $p,q \in M$,

$$|\theta^\Delta(p,q;t)| < Ct^N.$$

Using Proposition 10(a) and (b), we see that $\theta^T(x,y;t)$ also has uniform polynomial growth. Hence, for any $s > 0$, $x,y \in X_0$, and $p,q \in M_0$, one can take the Laplace transform of θ^Δ and θ^T to define

$$\Gamma_s^T(x,y) := s\int_0^\infty e^{-st}\theta^T(x,y;t)\,dt \quad \text{and} \quad \Gamma_s^\Delta(p,q) := s\int_0^\infty e^{-st}\theta^\Delta(p,q;t)\,dt. \tag{3}$$

The functions $\Gamma_s^T(x,y)$ and $\Gamma_s^\Delta(p,q)$ are called the heat kernels and they are the Schwartz kernels for the operators e^{-sT} and $e^{-s\Delta}$ respectively; see [Brüning and Heintze 1978/79, page 181]. Note that $\Gamma_s^\Delta(p,q)$ coincides with the fundamental solution to the heat equation on M. The parametrix construction for the fundamental solutions of heat equations, which was first introduced in [Minakshisundaram and Pleijel 1949], immediately implies the following estimate:

Proposition 12 *There exists positive constants C_1 and C_2 such that for any pair of points $p,q \in M^n$,*

$$|\Gamma_s^\Delta(p,q)| \le C_1 s^{-n/2}\, e^{-C_2\, d_M^2(p,q)/s}.$$

Proof The proposition directly follows from equation (45) on page 154 of [Chavel 1984]. □

2.5 Tauberian and abelian theorems

Given two real-valued functions $f(t)$ and $g(t)$. We write $f(t) \sim g(t)$ as $t \to \infty$ (resp. $t \to 0$) if $\lim_{t\to\infty} f(t)/g(t) = 1$ (resp. $\lim_{t\to 0} f(t)/g(t) = 1$). We state the Tauberian theorem and abelian theorem together below.

Theorem 13 [Feller 1971, Theorem 2 on page 445] *Let $F\colon [0,\infty) \to \mathbb{R}$ be a function with bounded variation such that the Laplace transform $f(s) = \int_0^\infty e^{-st}F(t)\,dt$ exists. Then for fixed $\alpha \in [0,\infty)$ and $C \in \mathbb{R}$, each of the relations*

$$sf(s) \sim \frac{C}{s^\alpha} \qquad \text{as } s \to 0, \tag{4}$$

$$F(t) \sim \frac{C}{\Gamma(\alpha+1)}t^\alpha \quad \text{as } t \to \infty, \tag{5}$$

implies the other.

Historically, the implication from (5) to (4) is called an abelian theorem, and the converse is called a Tauberian theorem [Feller 1971, page 445].

3 Geometric estimate

The main goal of this section is to prove Theorem D by induction on the dimension of M. For such an inductive argument to work, we will, in fact, prove a more general statement (Theorem 18 below), that applies to *disconnected* singular Riemannian foliations. First, we need the following definition.

Definition 14 Given a closed singular Riemannian foliation $(M, \mathcal{F})$, an isometry of $M/\mathcal{F}$ is called *liftable* if it is induced by an isometry of M that permutes the leaves of $\mathcal{F}$.

Definition 15 Let $(M, \mathcal{F}_0)$ be a closed singular Riemannian foliation with canonical projection $\sigma_0\colon M \to M/\mathcal{F}_0$, and let Γ be a finite group acting on $M/\mathcal{F}_0$ by liftable isometries. The *disconnected singular Riemannian foliation* (with closed leaves) on M induced by $(\mathcal{F}_0, \Gamma)$ is the partition $\mathcal{F}$ of M into sets of the form $\sigma_0^{-1}(\Gamma \cdot x)$, for $x \in M/\mathcal{F}_0$, which are called the *leaves of* $\mathcal{F}$.

Given a singular Riemannian foliation with disconnected leaves as above, we observe that its leaf space is $M/\mathcal{F} = (M/\mathcal{F}_0)/\Gamma$, and the canonical projection $\sigma\colon M \to M/\mathcal{F}$ is the composition of $\sigma_0\colon M \to M/\mathcal{F}_0$ with the quotient map $M/\mathcal{F}_0 \to (M/\mathcal{F}_0)/\Gamma$.

Example 16 Given a compact disconnected Lie group G acting on a Riemannian manifold M by isometries, the partition into G-orbits is an example of disconnected singular Riemannian foliation. In this case $\mathcal{F}_0$ is the orbit decomposition into G_0-orbits, where G_0 is the identity component of G, $\Gamma = G/G_0$ and the action of G/G_0 on M/G_0 is the natural one.

Remark 17

- While the isometries in Γ are required to be liftable, we do not assume that the entire action of Γ lifts to an action on M. This is in general not the case even in the example above.
- Definition 15 is very similar to [Mendes and Radeschi 2019, Definition 8], with the exceptions that the latter allows for $\mathcal{F}$ and $\mathcal{F}_0$ to have nonclosed leaves, and the total space is assumed to be Euclidean space.
- The key fact we will use below is that any slice foliations of any closed singular Riemannian foliation is a disconnected singular Riemannian foliation in the sense of Definition 15; see Section 2.1.4.

The following theorem generalizes Theorem D:

Theorem 18 *Given a compact connected Riemannian manifold M and a disconnected singular Riemannian foliation $\mathcal{F}$ with closed leaves, there exists a constant $C = C(M, \mathcal{F})$ such that, for every $p \in M$, and every $s > 0$, one has*

$$\int_{L_p} \exp\left(-\frac{d_M(p,q)^2}{s}\right) dq \le C s^{\frac{1}{2}\dim L_p}, \tag{6}$$

where the integral is computed with respect to the Riemannian volume form of L_p.

3.1 Reductions

To make the proof of Theorem 18 shorter, we prove a couple of reductions as separate lemmas, starting with the reduction to the case where the leaves are connected:

Lemma 19 *Let $M, \mathcal{F}$ be as in Theorem 18, and $\mathcal{F}_0, \Gamma$ as in Definition 15 . Suppose the conclusion of Theorem 18 holds for $(M, \mathcal{F}_0)$. Then it holds for $(M, \mathcal{F})$.*

Proof We are assuming the existence of $C = C(M, \mathcal{F}_0)$ such that, for all $p \in M$, and all $s > 0$,

$$\int_{L_p} \exp\left(-\frac{d_M(p,q)^2}{s}\right) dq \le C s^{\frac{1}{2}\dim L_p}, \tag{7}$$

where L_p denotes the $\mathcal{F}_0$-leaf through p. We claim that, for $(M, \mathcal{F})$, we can take $C(M, \mathcal{F}) = C|\Gamma| 2^{\dim M}$.

Indeed, let $s > 0$, $p \in M$, and let L denote the $\mathcal{F}$-leaf through p. Then L is the disjoint union of $\mathcal{F}_0$-leaves $L_1, \dots, L_k$, where $k \le |\Gamma|$, and we may assume $p \in L_1$. Note that each L_j is the image of L_1 under an isometry of M, so, in particular, they all have the same dimension, which we may denote by $\dim L$. Let $p_1 = p$, and p_j be a point of L_j that is closest to p. Then, for any $q \in L_j$, we have, by the triangle inequality,

$$d_M(q,p) \ge \frac{d_M(q,p) + d_M(p,p_j)}{2} \ge \frac{d_M(q,p_j)}{2}.$$

Therefore

$$\int_L \exp\left(-\frac{d_M(p,q)^2}{s}\right) dq \le \sum_{j=1}^{k} \int_{L_j} \exp\left(-\frac{d_M(p_j,q)^2}{4s}\right) dq \le |\Gamma| C (4s)^{\frac{1}{2}\dim L},$$

where in the last inequality we have used (7) for each L_j, and $k \le |\Gamma|$. Since $(4s)^{\frac{1}{2}\dim L} \le 2^{\dim M} s^{\frac{1}{2}\dim L}$, this finishes the proof of the claim. □

The next lemma corresponds to the case in Theorem 18 where $\mathcal{F}$ has a single leaf:

Lemma 20 *Let M be a compact connected Riemannian manifold. Then, there exists a constant C such that, for every $p \in M$, and every $s > 0$, one has*

$$\int_M \exp\left(-\frac{d_M(p,q)^2}{s}\right) dq \le C s^{\frac{1}{2}\dim M}.$$

Proof See [Brüning and Heintze 1978/79, Lemma 4.5]. □

Remark 21 (scaling-invariance) If Theorem 18 holds for a given manifold (M, g), then it holds for $(M, \lambda^2 g)$, for any $\lambda > 0$, with the *same* constant C.

3.2 Sasaki metrics

We will need to put a special metric on the normal bundle of a leaf L in order to prove Lemma 24 below, so we collect here the definition and some basic properties.

Definition 22 Let L be a closed Riemannian manifold, and $\pi\colon E \to L$ be a vector bundle equipped with an inner product $\langle\ ,\ \rangle_x$ on the fiber $E_x = \pi^{-1}(x)$ for each $x \in L$, and a metric connection ∇. Parallel translation determines a choice of "horizontal space" $H_{(x,v)}$ at each $(x,v) \in E$ (where $x \in L$ and $v \in E_x$), and we define the *Sasaki metric* on E by declaring $H_{(x,v)}$ to be orthogonal to the "vertical space" $V_{(x,v)} = \ker d\pi_{(x,v)} = T_{(x,v)}E_x = E_x$, using the given inner product on $V_{(x,v)} = E_x$, and pulling back the given inner product on T_xL to $H_{(x,v)}$.

The construction above is also described in [Besse 1987, Example 9.60].

Proposition 23 *In the notation of the definition above, endow E with the Sasaki metric. Then*:

(a) $\pi\colon E \to L$ *is a Riemannian submersion.*

(b) *For each $x \in L$, the fiber E_x is flat and totally geodesic.*

(c) *E is complete.*

(d) *There exists $r > 0$ such that, for all $x \in L$, one has $d_E = d_{E_x}$ on the set $E_x^{\leq r} = \{v \in E_x \mid \|v\| \leq r\}$.*

Proof (a) Follows immediately from the definition.

(b) For each $x \in L$, the Riemannian metric on E_x is flat by definition, being isometric to the Euclidean space $(E_x, \langle\ ,\ \rangle_x)$. For the second claim, see [Besse 1987, Theorem 9.59 and Example 9.60].

(c) See [Besse 1987, Theorem 9.59 and Example 9.60].

(d) Identify L with the zero section in E, and note that, for every $r > 0$, the closed tube of radius r around L in E coincides with $\bigcup_{x \in L} E_x^{\leq r}$. Since E is complete, the injectivity radius function on E is continuous [Chavel 2006, Theorem III.2.1], and thus, by compactness of L, there exists $r > 0$ such that the injectivity radius is larger than $2r$ on $\bigcup_{x \in L} E_x^{\leq r}$.

Let $x \in L$, and $v, w \in E_x^{\leq r}$. Then $\|v - w\| \leq 2r$, and, in particular, $d_E((x,v),(x,w)) \leq 2r$. Thus there is a unique $\xi \in T_{(x,w)}E$ with $\|\xi\| \leq 2r$ which exponentiates to (x,v). Since $v - w$ is another vector with the same property (because E_x is totally geodesic), we obtain $\xi = v - w$. Therefore $d_E((x,v),(x,w)) = \|\xi\| = \|v - w\| = d_{E_x}(v,w)$. □

3.3 Uniform estimates

The next two technical lemmas will be used in the proof of Theorem 18, by providing uniform information about the geometry of leaves around a fixed one. We start with an analogue of [Brüning and Heintze 1978/79, Lemma 4.7].

Lemma 24 *Let L be a leaf of the closed singular Riemannian foliation $\mathcal{F}$ on the closed Riemannian manifold M. Assume L has codimension at least one. Denote by $\pi\colon B_\epsilon(L) \to L$ the closest-point projection, where ϵ is smaller than the normal injectivity radius of L. Let $E = \nu L$ denote the normal bundle of L in M, and* exp *the normal exponential map $\nu L \to M$. Then there exist $C_1 > 0$ and $r \in (0, \epsilon)$ such that, for every $y \in L$ and $p \in B_r(L)$, there exists $v = v(p, y) \in \nu_y L$ such that $\exp(v) \in L_p$ and, for all $q \in \pi^{-1}(y) \cap L_p$, one has*

$$C_1 d_M^2(p,q) \geq \begin{cases} d_L^2(\pi(p),\pi(q)) + d_{\mathrm{sph}}^2(v, \exp^{-1}(q)) & \text{if } \operatorname{codim}(L) \geq 2, \\ d_L^2(\pi(p),\pi(q)) & \text{if } \operatorname{codim}(L) = 1. \end{cases}$$

Here d_{sph} denotes the distance in the round sphere in $\nu_y L$ of radius equal to the distance between L and L_p (that is, the leaf through p).

Proof To simplify notation, denote by ψ the inverse of the diffeomorphism

$$\exp|_{E^{<\epsilon}}\colon E^{<\epsilon} \to B_\epsilon(L).$$

Fix a metric connection on E (eg the normal connection), and endow E with the associated Sasaki metric; see Section 3.2. Since ϵ is strictly smaller than the normal injectivity radius, there exists $K > 0$ such that, for all $\eta \leq \epsilon$, the map $\exp|_{E^{<\eta}}\colon E^{<\eta} \to B_\eta(L)$ is K-bi-Lipschitz.

Choose $r > 0$ that is smaller than the r given in Proposition 23(d), and such that $2r < \epsilon$.

We claim that, for all $p, q \in B_r(L)$,

$$d_M(p,q) \geq C d_E(\psi(p), \psi(q)),$$

where $C = (2r + \operatorname{diam}(L))^{-1} K^{-1} r$. We consider two cases.

Case 1 $(d_M(p,q) \geq r)$ Using the triangle inequality, we obtain

$$d_{B_r(L)}(p,q) \leq 2r + \operatorname{diam}(L).$$

Thus

$$\begin{aligned} d_M(p,q) &\geq r \cdot \frac{2r + \operatorname{diam}(L)}{2r + \operatorname{diam}(L)} \geq \frac{r}{2r + \operatorname{diam}(L)} d_{B_r(L)}(p,q) \\ &\geq \frac{r}{2r + \operatorname{diam}(L)} \frac{1}{K} d_{E^{\leq r}}(\psi(p), \psi(q)) \\ &\geq \frac{r}{2r + \operatorname{diam}(L)} \frac{1}{K} d_E(\psi(p), \psi(q)) = C d_E(\psi(p), \psi(q)). \end{aligned}$$

Case 2 $(d_M(p,q) < r)$ Then $q \in B_r(p) \subset B_\epsilon(L)$, so that

$$\begin{aligned} d_M(p,q) = d_{B_\epsilon(L)}(p,q) &\geq K^{-1} d_{E^{\leq \epsilon}}(\psi(p), \psi(q)) \\ &\geq K^{-1} d_E(\psi(p), \psi(q)) \geq C d_E(\psi(p), \psi(q)), \end{aligned}$$

which finishes the proof of the claim.

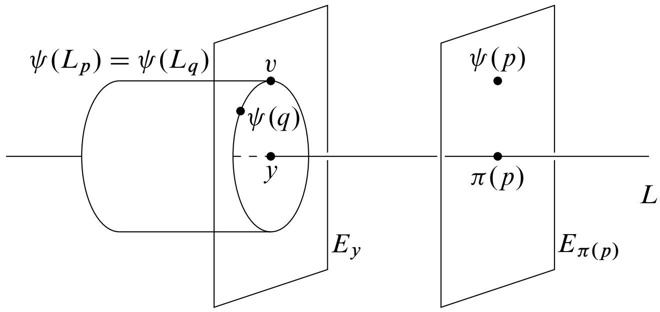

Figure 1

The claim above reduces the proof of the statement of the lemma to a similar statement, where $d_M(p,q)$ is replaced with $d_E(\psi(p),\psi(q))$.

If $\operatorname{codim}(L)=1$, the desired inequality follows from the fact (see Proposition 23(a)) that $\pi\colon E\to L$ is a Riemannian submersion. Assume $\operatorname{codim}(L)\geq 2$.

Let $y\in L$, $p\in B_r(L)$. Choose a point v of $E_y\cap\psi(L_p)$ minimizing $d_E(v,\psi(p))$. Let $q\in\pi^{-1}(y)\cap L_p$, which is equivalent to $\psi(q)\in E_y\cap\psi(L_p)$. See Figure 1.

By the triangle inequality,

$$d_E(\psi(q),v)\leq d_E(\psi(q),\psi(p))+d_E(\psi(p),v).$$

By the choice of v,

$$d_E(\psi(p),v)\leq d_E(\psi(q),\psi(p)).$$

Using Proposition 23(d) and some Euclidean geometry, we obtain

$$d_E(\psi(q),v)=d_{E_y}(\psi(q),v)\geq \frac{2}{\pi}d_{\mathrm{sph}}(\psi(q),v).$$

Combining the last three inequalities, we obtain

$$d_E(\psi(q),\psi(p))\geq \frac{1}{\pi}d_{\mathrm{sph}}(\psi(q),v). \tag{8}$$

On the other hand, since $\pi\colon E\to L$ is a Riemannian submersion (see Proposition 23(a)), we have

$$d_E(\psi(q),\psi(p))\geq d_L(y,\pi(p)).$$

Squaring this inequality and adding the square of (8), we obtain

$$2d_E^2(\psi(q),\psi(p))\geq d_L^2(y,\pi(p))+\frac{1}{\pi^2}d_{\mathrm{sph}}^2(\psi(q),v),$$

from which the desired inequality follows (with $C_1=2\pi^2/C^2$). □

The closest-point projection to a leaf, when restricted to a nearby leaf, is known to be a submersion; see Section 2.1.4. The next lemma states that it is, in fact, not too far from being a *Riemannian* submersion; see [Lytchak 2024, Proposition 1.3] for a more precise and general statement. It will be used to control the Jacobian appearing in a coarea formula inside the proof of Theorem 18 below.

Lemma 25 *Let L be a leaf of the closed singular Riemannian foliation $\mathcal{F}$ on the closed Riemannian manifold M. Assume L has codimension at least one, and denote by $\pi\colon B_r(L) \to L$ the closest-point projection, where $r > 0$ is strictly smaller than the normal injectivity radius of L. For each $q \in B_r(L)$, let L_q denote the leaf through q, and $d\pi'_q\colon T_qL_q \to T_{\pi(q)}L$ the restriction of $d\pi_q$ to T_qL_q. Let $f(q)$ be the smallest eigenvalue of the self-adjoint linear map*

$$d\pi'_q \circ (d\pi'_q)^*\colon T_{\pi(q)}L \to T_{\pi(q)}L.$$

Then $C_2 = \inf_{q \in B_r(L)} f(q)$ is positive.

Proof Denote by R the normal injectivity radius of L. Recall from Section 2.1.4 that, for every $q \in B_R(L)$, the map $\pi|_{L_q}\colon L_q \to L$ is a submersion; in particular $f(q) > 0$.

Assume, for a contradiction, that $C_2 = 0$. Then, there exists a sequence $q_i \in B_r(L)$ with $\lim f(q_i) = 0$. Choose $v_i \in T_{\pi(q_i)}L$ to be a unit eigenvector of $d\pi'_{q_i} \circ (d\pi'_{q_i})^*$ associated to the eigenvalue $f(q_i)$.

By compactness, we may assume, after passing to a subsequence, that q_i converges to $q \in \overline{B_r(L)} \subset B_R(L)$, and that v_i converge to $v \in T_{\pi(q)}L$.

Since $d\pi'_q$ is surjective, let $w \in T_qL_q$ such that $d\pi'_q(w) = v$. Since $\mathcal{F}$ is a singular Riemannian foliation, it is generated by smooth vector fields. In particular, there exist vectors $w_i \in T_{q_i}L_{q_i}$ converging to w, so that $\lim d\pi_{q_i}(w_i) = d\pi_q(w) = v$. Then

$$\lim \langle (d\pi'_{q_i})^* v_i, w_i \rangle = \lim \langle v_i, d\pi_{q_i} w_i \rangle = \langle v, v \rangle = 1.$$

On the other hand,

$$\lim \|(d\pi'_{q_i})^* v_i\|^2 = \lim \langle d\pi'_{q_i} \circ (d\pi'_{q_i})^* v_i, v_i \rangle = \lim f(q_i) = 0$$

and $\lim \|w_i\| = \|w\| < \infty$, so that, by the Cauchy–Schwarz inequality,

$$\lim \langle (d\pi'_{q_i})^* v_i, w_i \rangle = 0,$$

a contradiction. Therefore $C_2 > 0$. □

3.4 Proof of Theorem 18

Proof of Theorem 18 Let $\mathcal{F}_0$ and Γ as in Definition 15.

We use induction on $\dim M$. If $\dim M = 1$, then either $\mathcal{F} = \mathcal{F}_0$ has a single leaf, in which case we can apply Lemma 20, or $\mathcal{F}_0$ is the trivial foliation by points, for which the conclusion is trivial, so we can apply Lemma 19. Assume $\dim M > 1$.

By Lemma 19, we may assume that $\mathcal{F} = \mathcal{F}_0$, that is, $\mathcal{F}$ is a singular Riemannian foliation (with connected, closed leaves).

By Lemma 20, we may assume that M is not a single leaf. Since M is connected, that means every leaf has codimension at least one.

Since M is compact, it is enough to prove that, for every leaf L, there exists $r, C_L > 0$ such that (6) holds for all p in the tube $B_r(L)$ and all $s > 0$, with C_L in place of C. Indeed, M can be covered by finitely many such tubes, so one can take C to be the maximum of the corresponding constants C_L. Therefore, we fix a leaf L.

Take $r > 0$ and C_1 given by Lemma 24 (in particular r is smaller than the normal injectivity radius of L), and C_2 given in Lemma 25. Let $p \in B_r(L)$. The restriction of π to L_p is a submersion $L_p \to L$ (see Section 2.1.4), with Jacobian

$$\operatorname{Jac}(q) = \sqrt{\det(d\pi'_q \circ (d\pi'_q)^*)},$$

which, by Lemma 25, satisfies $\operatorname{Jac}(q) \geq C_2^{\dim L/2}$ for all $q \in L_p$. To simplify notation, let

$$g_s(q) = \exp\Big(-\frac{d_M(p,q)^2}{s}\Big).$$

Then

$$\int_{q \in L_p} g_s(q)\, d\mathrm{vol}^{L_p}(q) \leq C_2^{-\frac{1}{2}\dim L} \int_{q \in L_p} g_s(q) \operatorname{Jac}(q)\, d\mathrm{vol}^{L_p}(q).$$

Applying the coarea formula [Chavel 2006, page 160, Exercise III.12(c)], the integral on the right becomes

$$\int_{y \in L} \Big[\int_{q \in L_p \cap \pi^{-1}(y)} g_s(q)\, d\mathrm{vol}^{L_p \cap \pi^{-1}(y)}(q) \Big] d\mathrm{vol}^L(y). \tag{9}$$

The next step is to use the inequality in Lemma 24, so we split in two cases: $\operatorname{codim}(L) = 1$ and $\operatorname{codim}(L) \geq 2$.

Case 1 ($\operatorname{codim}(L) = 1$) By Lemma 24, $g_s(q) \leq \exp(-d_L^2(\pi(p), \pi(q))/C_1 s)$. The leaves L and L_p have the same dimension and, for each $y \in L$, the set $L_p \cap \pi^{-1}(y) \subseteq \{\exp_y x \mid x \in \nu_y L, \|x\| = d_M(y, p)\}$ consists of at most two points. Thus (9) is less than or equal to

$$2 \int_{y \in L} \exp\Big(\frac{-d_L^2(\pi(p), y)}{C_1 s}\Big)\, d\mathrm{vol}^L(y),$$

which, by Lemma 20, is at most

$$2C_3 (C_1 s)^{\frac{1}{2}\dim L} = 2C_3 C_1^{\frac{1}{2}\dim L_p} s^{\frac{1}{2}\dim L_p}$$

for some C_3 that depends only on L. Therefore,

$$\int_{L_p} \exp\Big(-\frac{d_M(p,q)^2}{s}\Big)\, dq \leq \big[C_2^{-\frac{1}{2}\dim L} 2C_3 C_1^{\frac{1}{2}\dim L_p}\big] s^{\frac{1}{2}\dim L_p},$$

finishing the proof of (6) in Case 1.

Case 2 (codim$(L) \geq 2$) By Lemma 24,

$$g_s(q) \leq \exp\biggl(\frac{-d_L^2(\pi(p),\pi(q))}{C_1 s}\biggr)\exp\biggl(\frac{-d_{\mathrm{sph}}^2(v,\exp^{-1}(q))}{C_1 s}\biggr),$$

where $v = v(p, y)$ depends only on p and y.

Thus (9) is at most

$$\int_{y\in L} e^{-d_L^2(\pi(p),y)/C_1 s}\biggl[\int_{q\in L_p\cap\pi^{-1}(y)} e^{-d_{\mathrm{sph}}^2(v,\exp^{-1}(q))/C_1 s}\,d\mathrm{vol}^{L_p\cap\pi^{-1}(y)}(q)\biggr]\,d\mathrm{vol}^L(y). \tag{10}$$

Next we make a change of variables in the inner integral above, using the normal exponential map $\exp\colon \nu_y L \to \pi^{-1}(y)$. This is a diffeomorphism when restricted to the ball of radius r in $\nu_y L$, and, since L is compact, it is bi-Lipschitz with a constant depending only on L, r. Since $d\mathrm{vol}^{L_p\cap\pi^{-1}(y)}$ is the Riemannian volume form of the Riemannian manifold $L_p \cap \pi^{-1}(y)$ endowed with the induced Riemannian structure from $\pi^{-1}(y)$, there exists C_4, depending only on L and r, such that the inner integral in (10) is at most

$$C_4 \int_{z\in L_v} e^{-d_{\mathrm{sph}}^2(v,z)/C_1 s}\,d\mathrm{vol}^{L_v}(z). \tag{11}$$

Here $L_v = \exp^{-1}(L_p \cap \pi^{-1}(y))$ is the leaf through v of the (disconnected) slice foliation $\mathcal{F}^y$ of the Euclidean space $\nu_y L$; see Section 2.1.4.

We apply the inductive hypothesis to the restriction of $\mathcal{F}^y$ to the unit sphere $\mathbb{S}(\nu_y L)$ and conclude that there exists C_5 such that, for all $w \in \mathbb{S}(\nu_y L)$,

$$\int_{z\in L_w} e^{-d_{\mathrm{sph}}^2(w,z)/C_1 s}\,d\mathrm{vol}^{L_w}(z) \leq C_5(C_1 s)^{\frac{1}{2}\dim L_w}. \tag{12}$$

By Remark 21, the same conclusion holds with w replaced with any $v \in \nu_y L$ (and with the same constant C_5), because homothetic transformations of $\nu_y L$ take leaves of $\mathcal{F}^y$ to leaves of $\mathcal{F}^y$; see the second paragraph after [Mendes and Radeschi 2019, Definition 8]. In principle, the constant C_5 depends on $y \in L$, but in fact that it can be chosen independently of y, because the slice foliation does not depend on y; see Section 2.1.4.

Combining the inequality (9) $\leq$ (10) with the estimates (11) and (12), and noting that $\dim L_v + \dim L = \dim L_p$, we obtain

$$\int_{q\in L_p} g_s(q)\,d\mathrm{vol}^{L_p}(q) \leq C_2^{-\frac{1}{2}\dim L} C_4 C_5 (C_1 s)^{\frac{1}{2}(\dim L_p - \dim L)} \int_{y\in L} e^{-d_L^2(\pi(p),y)/C_1 s}\,d\mathrm{vol}^L(y).$$

Finally, by Lemma 20, there exists C_3 (depending only on L) such that the integral on the right is at most $C_3(C_1 s)^{\frac{1}{2}\dim L}$. Thus

$$\int_{q\in L_p} g_s(q)\,d\mathrm{vol}^{L_p}(q) \leq (C_2^{-\frac{1}{2}\dim L} C_4 C_5 C_3 C_1^{\frac{1}{2}\dim L_p}) s^{\frac{1}{2}\dim L_p},$$

which concludes the proof of (6) in Case 2. □

4 Weyl's law

Throughout this section, we will use the terminology introduced in Section 2.

We follow the approach in [Brüning and Heintze 1978/79, Section 3] to prove Weyl's law (Theorem A). The proof is roughly organized as follows. First, we derive a trace formula for the operator e^{-sT}, where $T\colon L^2(X_0,h)\to L^2(X_0,h)$ is the operator defined in Section 2.3. This formula relates the Laplace transform of the counting function $N(t)$ of T to an integral of its heat kernel. At each point in X_0, one can apply Hörmander's estimate (Theorem 7) and the abelian theorem (Theorem 13) to obtain the asymptotics for the heat kernel. Theorem D allows us to integrate the asymptotics of the heat kernel. The asymptotics for $N(t)$ then follows from the Tauberian theorem and the trace formula.

4.1 A trace formula

The counting function $N(t)$ and Γ_s^T are related via the following trace formula for e^{-sT}:

Proposition 26 *The function $N(t)$ has polynomial growth as $t\to\infty$. Furthermore,*

$$s\int_0^\infty e^{-st}N(t)\,dt=\int_{X_0}\Gamma_s^T(x,x)h(x)\,dx, \tag{13}$$

where $\Gamma_s^T(x,y)$ is the heat kernel of T defined by (3).

Proof Let $S(t)$ be the number of eigenvalues of the Laplace operator Δ on M that are less than t. The standard Weyl's law for the Laplace operator on compact Riemannian manifolds implies that $S(t)$ must have polynomial growth as $t\to\infty$. Since the growth rate of $N(t)$ cannot exceed that of $S(t)$, $N(t)$ must have polynomial growth as $t\to\infty$.

To prove the integral formula, let $\{\varphi_i\}$ be the orthonormal basis for $L^2(X_0,h)$ in Remark 9 such that $T(\varphi_i)=\lambda_i\varphi_i$. Then by Proposition 10(a),

$$N(t)=\sum_{\lambda_i<t}\|\varphi_i(x)\|^2_{L^2(X_0,h)}=\int_{X_0}\theta^T(x,x,t)h(x)\,dx. \tag{14}$$

Hence,

$$s\int_0^\infty e^{-st}N(t)\,dt=s\int_0^\infty e^{-st}\int_{X_0}\theta^T(x,x,t)h(x)\,dx\,dt. \tag{15}$$

Since $\theta^T(x,x,t)$ has uniform polynomial growth (see Section 2.4) we may use Fubini's theorem to interchange the integral and obtain

$$s\int_0^\infty e^{-st}N(t)\,dt=\int_{X_0}\int_0^\infty se^{-st}\theta^T(x,x,t)\,dt\,h(x)\,dx=\int_{X_0}\Gamma_s^T(x,x)h(x)\,dx, \tag{16}$$

which proves the integral formula. □

Proposition 11 and Theorem 13 directly imply the following asymptotic estimate for the heat kernel:

Proposition 27 *For each point $x \in X_0$, as $s \to 0$ we have*

$$\Gamma_s^T(x,x) \sim s^{-m/2}\Gamma\left(\frac{m}{2}+1\right)\frac{\omega_m}{h(x)(2\pi)^m}.$$

4.2 Uniform estimates for the heat kernel

Proposition 27 in the previous section can be restated by saying that the function $h(x)s^{m/2}\Gamma_s^T(x,x)$, as a function of s and x, converges pointwise to a constant as $s \to 0$. Such convergence however might not be uniform, as X_0 is not compact in general. Nevertheless, the main result in this section uses Theorem D to show that this function is uniformly bounded. This will allow us to use the Dominated Convergence Theorem which, together with the Tauberian Theorem, will give Weyl's law.

Theorem 28 *Let M^n be a compact n-dimensional manifold and let $\mathcal{F}$ be a closed singular Riemannian foliation with basic mean curvature. Then there exists a constant C such that for all $x \in X_0$,*

$$|s^{m/2}\Gamma_s^T(x,x)h(x)| < C,$$

where $m = \dim(X_0)$.

To prove Theorem 28, we first relate the heat kernel of Δ to that of T.

Lemma 29 *Let Γ_s^T and Γ_s^Δ be the heat kernel of T and Δ constructed by* (3)*. Then for each pair of points $p, q \in M_0$, we have*

$$\Gamma_s^T(\sigma(p),\sigma(q)) = (\mathrm{Av}\otimes\mathrm{Av})\Gamma_s^\Delta(p,q).$$

Proof Let $\{\phi_k\}$ and $\{\varphi_i\}$ be the orthonormal basis of $L^2(M_0)$ and $L^2(X_0,h)$ constructed in Remark 9 and let $\theta^T(x,y;t)$ and $\theta^\Delta(x,y;t)$ be the spectral functions of T and Δ respectively.

Since $\theta^T(x,y;t)$ has uniform polynomial growth, Proposition 10 and Fubini's theorem imply

$$\begin{aligned}\Gamma_s^T(\sigma(p),\sigma(q)) = s\int_0^\infty e^{-st}\theta^T(\sigma(p),\sigma(q);t)\,dt &= s\int_0^\infty e^{-st}(\mathrm{Av}\otimes\mathrm{Av})\theta^\Delta(p,q;t)\,dt\\ &= (\mathrm{Av}\otimes\mathrm{Av})\Gamma_s^\Delta(p,q).\end{aligned}$$

□

Proof of Theorem 28 Let $\mathcal{F}$ be a closed singular Riemannian foliation with basic mean curvature and $\dim(X_0) = m$. Let $x \in X_0$ and $p \in \sigma^{-1}(x)$. By Lemma 29 and Proposition 12,

$$\begin{aligned}|s^{m/2}\Gamma_s^T(x,x)h(x)| &= \left|\frac{s^{m/2}}{\mathrm{vol}(L_p)}\int_{L_p}\int_{L_p}\Gamma_s^\Delta(p',q')\,dp'\,dq'\right|\\ &\le \sup_{q\in L_p} s^{m/2}\int_{L_p}|\Gamma_s^\Delta(p',q)|\,dp' \le C_1\sup_{q\in L_p} s^{(m-n)/2}\int_{L_p} e^{-C_2 d_M^2(p',q)/s}\,dp'.\end{aligned}$$

Since $\dim(L_p) = n-m$, the theorem follows from Theorem D. □

4.3 Proof of Weyl's law

Proof of Theorem A Since X_0 has finite volume, Proposition 27, Theorem 28 and the dominated convergence theorem imply that

$$\int_{X_0} \Gamma_s^T(x,x)h(x)\,dx \sim s^{-m/2}\Gamma\left(\frac{m}{2}+1\right)\frac{\omega_m}{(2\pi)^m}\operatorname{vol}(X_0).$$

Theorem A then follows from Proposition 26 (trace formula) and Theorem 13 (the Tauberian theorem). □

5 Foliations of the sphere and invariant theory

In this section we consider the special case of closed singular Riemannian foliations $(M,\mathcal{F})$ where M is a round sphere. The main goal is to relate the volume of the leaf space to algebraic invariants of the algebra of basic polynomials (see Section 2.1.2), culminating in the proof of Theorem C. This is an application of Weyl's law (Theorem A) and the theory of spherical harmonics.

5.1 The Cohen–Macaulay property and Hilbert series

Let V be an $(n+1)$-dimensional real vector space with inner product, and let $\mathbb{R}[V]$ denote the $\mathbb{R}$-algebra of real-valued polynomial functions on V. Fix an orthonormal basis for V, via which the algebra $\mathbb{R}[V]$ is isomorphic to $\mathbb{R}[x_0,\dots,x_n]$.

Following [Mendes and Radeschi 2020a, Definition 4]:

Definition 30 A subalgebra $A\subset\mathbb{R}[V]$ is called *Laplacian* if it contains $r^2=\sum_i x_i^2$ and is preserved by the Laplace operator $\Delta=\sum_i \partial^2/\partial x_i^2$.

The typical examples of Laplacian algebras are rings of invariants $A=\mathbb{R}[V]^G$, where G is a group acting on V by linear isometries; and the ring of basic polynomials $A=\mathcal{B}(\mathcal{F})$, where $\mathcal{F}$ is an infinitesimal singular Riemannian foliation of V. More generally, by Mendes and Radeschi [2020a; 2023], Laplacian algebras are in one-to-one correspondence with infinitesimal manifold submetries of V.

Definition 31 [Derksen and Kemper 2015, Definition 2.5.6] Let $A\subset\mathbb{R}[V]$ be a graded subalgebra. A subset $\{f_0,\dots,f_m\}\subset A$ of homogeneous elements is called a *homogeneous system of parameters* if they are algebraically independent, and A is a finitely generated module over the subalgebra $\mathbb{R}[f_0,\dots,f_m]$.

We note that finitely generated graded subalgebras $A\subset\mathbb{R}[V]$ always have a homogeneous system of parameters, by the Noether normalization lemma; see [Derksen and Kemper 2015, Corollary 2.5.8].

Definition 32 [Derksen and Kemper 2015, Proposition 2.6.3] Let $A\subset\mathbb{R}[V]$ be a finitely generated graded subalgebra. We say A is *Cohen–Macaulay* if, for every homogeneous system of parameters $f_0,\dots,f_m$, the algebra A is a *free* module over $\mathbb{R}[f_0,\dots,f_m]$.

We note that, in the above definition, the quantifier "for every" can be replaced with "there exists", resulting in an equivalent definition; see [Derksen and Kemper 2015, Proposition 2.6.3].

Our first result is the observation that the celebrated Hochster–Roberts theorem [Derksen and Kemper 2015, Theorem 2.6.5] generalizes from rings of invariants to arbitrary Laplacian algebras:

Proposition 33 *Suppose that $A \subset \mathbb{R}[V]$ is a Laplacian algebra. Then A is graded, finitely generated, and Cohen–Macaulay.*

Proof By [Mendes and Radeschi 2020a, Corollary 22], the Laplacian algebra A is graded, and by [Mendes and Radeschi 2020a, Lemma 24], it is finitely generated. By [Derksen and Kemper 2015, Theorem 2.6.11], it suffices to show that, for every ideal $I \subset A$, one has $(I\mathbb{R}[V]) \cap A = I$.

By [Mendes and Radeschi 2020a, Theorem 23], there exists a Reynolds operator $\Pi\colon \mathbb{R}[V] \to A$. This means Π is a linear projection onto A, and $\Pi(fg) = f\,\Pi(g)$ for all $f \in A$ and $g \in \mathbb{R}[V]$. Let $I \subset A$ be an ideal. The inclusion $I \subset (I\mathbb{R}[V]) \cap A$ is clear. To prove the reverse inclusion, let $f \in (I\mathbb{R}[V]) \cap A$. Then $f = \sum_i g_i f_i$ for $f_i \in I$ and $g_i \in \mathbb{R}[V]$. Applying Π to this equation, and using the assumption that $f \in A$, so that $\Pi(f) = f$, we obtain $f = \sum_i f_i \Pi(g_i)$, which proves that $f \in I$, because I is an ideal of A, and $\Pi(g_i) \in A$. Therefore $(I\mathbb{R}[V]) \cap A = I$, and A is Cohen–Macaulay. □

We note that the proof that $(I\mathbb{R}[V]) \cap A = I$ above is analogous to the proof of [Derksen and Kemper 2015, Lemma 2.6.10].

Definition 34 Given any graded algebra $A = \bigoplus_{k=0}^{\infty} A_k$, we define its Hilbert series by

$$H(z) = \sum_{k=0}^{\infty} \dim(A_k) z^k.$$

The next lemma is an application of Weyl's law (Theorem A) together with the theory of spherical harmonics to relate the volume of the leaf space to the growth of coefficients of a Hilbert series.

Lemma 35 *Let $\mathcal{F}$ be a closed infinitesimal singular Riemannian foliation of V, and let $X = \mathbb{S}V/\mathcal{F}$ be the leaf space. Let $A \subset \mathbb{R}[V]$ be the associated Laplacian algebra of basic polynomials, and let $H(z)$ denote its Hilbert series. Then the coefficients of the power series*

$$(1+z)H(z) = \sum_{k=0}^{\infty} b_k z^k$$

satisfy

$$b_k \sim \frac{\operatorname{vol}(X)\omega_m}{(2\pi)^m} k^m \quad \text{as } k \to \infty,$$

where $m = \dim X$ and $\omega_m = \operatorname{vol} \mathbb{D}^m$.

Proof The singular Riemannian foliation $\mathcal{F}$ has basic mean curvature; see Section 2.1.3. By Theorem A, the counting function $N(t)$ for the basic spectrum satisfies

$$N(t) \sim \frac{\operatorname{vol}(X)\omega_m}{(2\pi)^m} t^{m/2} \quad \text{as } t \to \infty. \tag{17}$$

By the theory of spherical harmonics (see eg [Howe and Tan 1992, Exercise 12(e), page 118]), the eigenvalues of the sphere $\mathbb{S}V$ are $\{k(k+n-1) \mid k = 0, 1, \dots\}$, with corresponding eigenspaces the restrictions to $\mathbb{S}V$ of the space $\mathcal{H}_k \subset \mathbb{R}[V]_k$ of all homogeneous harmonic polynomials of degree k. Moreover, one has the direct sum decomposition

$$\mathbb{R}[V]_k = \mathcal{H}_k \oplus r^2\mathbb{R}[V]_{k-2} \quad \text{for all } k \geq 2.$$

Since A is Laplacian, we obtain the analogous decomposition

$$A_k = (A \cap \mathcal{H}_k) \oplus r^2 A_{k-2}$$

see [Mendes and Radeschi 2023, Lemma 4(b)]. Note that the basic eigenvalues are $k(k+n-1)$, with multiplicity $\dim(A \cap \mathcal{H}_k)$. Denoting by $G(z)$ the corresponding generating function

$$G(z) = \sum_{k=0}^{\infty} \dim(A \cap \mathcal{H}_k) z^k,$$

we obtain the formula $G(z) = (1 - z^2)H(z)$, and thus

$$(1 + z + z^2 + \cdots)G(z) = (1 + z)H(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Note that

$$b_k = \sum_{j=0}^{k} \dim(A \cap \mathcal{H}_k) = N(k(k+n-1)).$$

Using (17), we obtain the desired asymptotic. □

The next lemma is purely algebraic, and will be used in combination with Lemma 35 in the proof of Theorem C. Moreover, part (b) gives algebraic constraints on the Hilbert series of the algebra of basic polynomials of an infinitesimal singular Riemannian foliation. These constraints seem to be new, and may be of independent interest. We remind the reader that here, and throughout this article, $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t\to\infty} f(t)/g(t) = 1$.

Lemma 36 *Let $A \subset \mathbb{R}[V]$ be a graded, finitely generated, Cohen–Macaulay algebra, and denote its Hilbert series by $H(z)$. Let $f_0, \dots, f_m \in A$ be a homogenous system of parameters (so $m+1$ is the Krull dimension of A). Assume that the coefficients of the power series*

$$(1 + z)H(z) = \sum_{k=0}^{\infty} b_k z^k$$

satisfy

$$b_k \sim C k^{m'} \quad \text{as } k \to \infty \tag{18}$$

for some $C \neq 0$ and $m' \in \mathbb{N}$.

Then:

(a) $m = m'$.

(b) $H(z)$ *is a rational function whose poles are roots of unity, such that* $z = 1$ *is a pole of order* $m+1$, $z = -1$ *is (possibly) a pole of order at most* $m+1$*. All other poles have order at most* m.

(c) *The Laurent series of* $H(z)$ *at* $z = 1$ *has the form*

$$\frac{Cm!}{2}(1-z)^{-m-1} + (\cdots)(1-z)^{-m} + \cdots .$$

(d) *The constant* C *appearing in* (18) *is given by*

$$C = \frac{2\dim_{\mathbb{R}[f_0,\dots,f_m]} A}{m!\deg f_0 \cdots \deg f_m},$$

where $\dim_{\mathbb{R}[f_0,\dots,f_m]} A$ *is the rank of* A *as a free, finitely generated module over* $\mathbb{R}[f_0,\dots,f_m]$.

Proof Since the algebra A is Cohen–Macaulay, it is a finite free module over $\mathbb{R}[f_0,\dots,f_m]$. Let $g_1,\dots,g_\ell$ be a basis consisting of homogeneous elements of A, so that $\ell = \dim_{\mathbb{R}[f_0,\dots,f_m]} A$. Thus A has the presentation, sometimes called a "Hironaka decomposition",

$$A = \mathbb{R}[f_0,\dots,f_m]\,g_1 \oplus \cdots \oplus \mathbb{R}[f_0,\dots,f_m]\,g_\ell,$$

which implies that the Hilbert series $H(z)$ has the form

$$H(z) = \frac{z^{\deg(g_1)} + \cdots + z^{\deg(g_\ell)}}{(1-z^{\deg(f_0)})\cdots(1-z^{\deg(f_m)})}. \tag{19}$$

The first sentence of (b) follows immediately from this formula, that is: 1 is a pole of order $m+1$ and all poles have order at most $m+1$. Moreover, factoring $(1-z)^{-m-1}$ out of the above expression, and plugging $z=1$ in the remaining term, we obtain the Laurent series of $H(z)$ at $z=1$, namely

$$\frac{Dm!}{2}(1-z)^{-m-1} + (\cdots)(1-z)^{-m} + \cdots, \quad \text{where } D = \frac{2\dim_{\mathbb{R}[f_0,\dots,f_m]} A}{m!\deg f_0 \cdots \deg f_m}.$$

To prove (c) and (d), it remains to prove that $D = C$, where C is the constant that appears in the hypothesis (18) above.

In order to make use of (18), we consider the partial fraction decomposition of $(1+z)H(z)$, and separate the terms associated to poles of maximum possible order, namely $m+1$. These terms will dominate the behavior of b_k as $k \to \infty$.

More precisely, let us denote by $\Omega' \subset \mathbb{C}$ the (finite) set of all poles of $(1+z)H(z)$, and writing $d = \gcd(\deg(f_0),\dots,\deg(f_m))$, let $\Omega \subset \mathbb{C}$ denote the set of all d^{th} roots of unity, that is, the "potential" poles of order $m+1$. For each $\omega \in \Omega'$, denote by $d(\omega)$ the order of ω as a pole of $(1+z)H(z)$. Note that every $\omega \in \Omega' \setminus \Omega$ is a root of unity with order $d(\omega) \le m$.

The partial fraction decomposition of $(1+z)H(z)$ is

$$(1+z)H(z) = P(z) + \sum_{\omega\in\Omega} \alpha_\omega(\omega - z)^{-m-1} + \sum_{\omega\in\Omega'}\sum_{j=1}^{d_\omega} \beta_{j,\omega}(\omega - z)^{-j}, \tag{20}$$

where $P(z)$ is the polynomial part of $(1+z)H(z)$ and $d_\omega = \min(d(\omega), m)$. Thus the first sum above corresponds to poles of order $m+1$, and part of our goal is to show that $\alpha_\omega = 0$ for all $\omega \in \Omega \setminus \{1\}$. Using the binomial series, we obtain

$$(\omega - z)^{-j} = \omega^{-j}(1-(\omega^{-1}z))^{-j} = \omega^{-j}\sum_{k=0}^{\infty}\binom{k+j-1}{k}\omega^{-k}z^k. \tag{21}$$

Putting equations (20) and (21) together, the formula for $(1+z)H(z)$ becomes

$$P(z) + \sum_k\biggl(\sum_{\omega\in\Omega}\alpha_\omega\omega^{-k-m-1}\binom{k+m}{k} + \sum_{\omega\in\Omega'}\sum_{j=1}^{d_\omega}\beta_{j,\omega}\omega^{-k-j}\binom{k+j-1}{k}\biggr)z^k$$

For fixed j, the binomial $\binom{k+j-1}{k}$ is a polynomial in k with leading term $k^{j-1}/(j-1)!$ so we get

$$b_k \sim \biggl(\sum_{\omega\in\Omega}\frac{1}{m!}\alpha_\omega\omega^{-k-m-1}\biggr)k^m + \text{(lower-order terms)} \quad \text{as } k\to\infty.$$

We note that the sum multiplying k^m is periodic in k. If it vanished identically, that is,

$$\sum_{\omega\in\Omega}\frac{1}{m!}\alpha_\omega\omega^{-k-m-1} = 0 \quad \text{for all } k,$$

then, averaging over all k from 1 to d, we would get

$$0 = \frac{1}{d}\sum_{k=1}^{d}\sum_{\omega\in\Omega}\frac{1}{m!}\alpha_\omega\omega^{-k-m-1} = \frac{1}{d}\sum_{\omega\in\Omega}\frac{1}{m!}\alpha_\omega\omega^{-m-1}\biggl(\sum_{k=1}^{d}\omega^{-k}\biggr) = \frac{1}{m!}\alpha_1,$$

where the last equality follows from $\sum_{k=1}^{d}\omega^{-k} = 0$ for $\omega \in \Omega\setminus\{1\}$. Since $\alpha_1 \neq 0$ (because 1 is a pole of order $m+1$), we thus know that the asymptotic behavior of b_k as $k\to\infty$ is dominated by the term

$$\frac{1}{m!}\biggl(\sum_{\omega\in\Omega}\alpha_\omega\omega^{-m-1}\omega^{-k}\biggr)k^m \quad \text{as } k\to\infty.$$

Our assumption that

$$b_k \sim Ck^{m'} \quad \text{as } k\to\infty$$

implies that $m = m'$ (proving (a)), and that the periodic function

$$\frac{1}{m!}\biggl(\sum_{\omega\in\Omega}\alpha_\omega\omega^{-m-1}\omega^{-k}\biggr)$$

is constant (hence equal to its average) and equal to C:

$$\frac{1}{m!}\alpha_1 = \frac{1}{m!}\sum_{\omega\in\Omega}\alpha_\omega\omega^{-m-1}\omega^{-k} = C \quad \text{for all } k. \tag{22}$$

Using the partial fraction decomposition (20), this implies $C = D$, which finishes the proof of (c) and (d). Similarly, fixing $\omega_0 \in \Omega\setminus\{1\}$, multiplying (22) with ω_0^k, and adding for $k = 1, \dots, d$ yields $\alpha_{\omega_0} = 0$, thus proving (b). □

Remark 37 The arguments in the proof above are standard in enumerative combinatorics; see for example [Wilf 2006, Chapter 5] for a general reference, and [Israel 2012] for the specific inspiration.

Remark 38 The proof of Lemma 36 could be considerably shortened, in the case of Laplacian algebras, if we knew that a degree-2 generator (eg r^2) were part of a system of homogeneous parameters. We believe this is the case, but cannot prove it. Furthermore, we point out that the conclusions of Lemma 36 do not hold for general Cohen–Macaulay algebras. For example in $\mathbb{R}[t^3]$ a system of parameters is just $f_0 = t^3$ (hence $m = 0$) but in the series $(1+z)H(z) = \sum_k b_k t^k$, the limit b_k/k^m oscillates between 0 and 1 — thus $b_k \not\sim Ck^{m'}$ for any m'. The authors do not know, however, whether every Cohen–Macaulay algebra containing r^2 satisfies the conclusions of the lemma.

5.2 Proof of Theorem C and its representation-theoretic analogue

In this section we prove the following, slightly more precise version of Theorem C:

Theorem 39 *Let $(V, \mathcal{F})$, with $\dim V = n+1$, be an infinitesimal foliation with closed leaves. Let $A = \bigoplus_i A_i \subset \mathbb{R}[V]$ be the associated Laplacian algebra of basic polynomials, and let $H(z) = \sum_i (\dim A_i) z^i$ denote its Hilbert series. Denote by $\mathbb{S}V$ the unit sphere in V, and by m the dimension of the leaf space $X = \mathbb{S}V/\mathcal{F}$.*

Then:

(a) *The Laurent series of $H(z)$ at $z = 1$ has the form*

$$H(z) = \frac{\mathrm{Vol}(X)}{\mathrm{Vol}(\mathbb{S}^m)}(1-z)^{-m-1} + (\cdots)(1-z)^{-m} + \cdots.$$

(b) *A is a Cohen–Macaulay algebra with Krull dimension $m+1$, and, for every homogeneous system of parameters $f_0, \dots, f_m \in A$,*

$$\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(\mathbb{S}^m)} = \frac{\dim_{\mathbb{R}[f_0,\dots,f_m]} A}{\deg f_0 \cdots \deg f_m},$$

where $\dim_{\mathbb{R}[f_0,\dots,f_m]} A$ is the rank of A as a finite free module over its subalgebra $\mathbb{R}[f_0, \dots, f_m]$.

Proof By Proposition 33, the algebra A is graded, finitely generated, and Cohen–Macaulay.

By Lemma 35, the coefficients of the power series

$$(1+z)H(z) = \sum_{k=0}^{\infty} b_k z^k \qquad \text{satisfy} \qquad b_k \sim Ck^m \quad \text{as } k \to \infty,$$

where $C = (\mathrm{vol}(X)\omega_m)/(2\pi)^m$, $m = \dim X$ and $\omega_m = \mathrm{vol}\,\mathbb{D}^m$.

By Lemma 36, the Krull dimension of A is $m+1$, the leading term in the Laurent series of $H(z)$ at $z=1$ is $\frac{1}{2}Cm!(1-z)^{-m-1}$, and

$$C = \frac{2\dim_{\mathbb{R}[f_0,\dots,f_m]} A}{m!\deg f_0 \cdots \deg f_m}.$$

Thus

$$\frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{S}^m)} = \frac{\dim_{\mathbb{R}[f_0,\dots,f_m]} A}{\deg f_0 \cdots \deg f_m} \cdot \frac{2(2\pi)^m}{m!\omega_m \,\mathrm{vol}(\mathbb{S}^m)}.$$

Finally, it remains to show that $\mathrm{vol}(\mathbb{S}^m)\omega_m = 2(2\pi)^m/m!$. This can be done with a straightforward induction argument using the recursive formulas $\omega_{i+1} = \mathrm{vol}(\mathbb{S}^i)/(i+1)$ and $\mathrm{vol}(\mathbb{S}^{i+1}) = 2\pi\omega_i$. □

The analogous result to Theorem C for orthogonal representations is:

Proposition 40 *Let V be the Euclidean vector space of dimension $n+1$, and $G \to O(V)$ an orthogonal representation of the compact group G. Let $A = \mathbb{R}[V]^G \subset \mathbb{R}[V]$ be the associated algebra of invariant polynomials, and let $H(z)$ denote its Hilbert series. Denote by $\mathbb{S}V$ the unit sphere in V, and by m the dimension of the orbit space $X = \mathbb{S}V/G$.*

Then:

(a) *The Laurent series of $H(z)$ at $z=1$ has the form*

$$H(z) = \frac{\mathrm{Vol}(X)}{\mathrm{Vol}(\mathbb{S}^m)}(1-z)^{-m-1} + (\cdots)(1-z)^{-m} + \cdots.$$

(b) *A is a Cohen–Macaulay algebra with Krull dimension $m+1$, and, for every homogeneous system of parameters $f_0,\dots,f_m \in A$,*

$$\frac{\mathrm{Vol}(X)}{\mathrm{Vol}(\mathbb{S}^m)} = \frac{\dim_{\mathbb{R}[f_0,\dots,f_m]} A}{\deg f_0 \cdots \deg f_m},$$

where $\dim_{\mathbb{R}[f_0,\dots,f_m]} A$ is the rank of A as a finite free module over its subalgebra $\mathbb{R}[f_0,\dots,f_m]$.

Proof The proof is completely analogous to the proof of Theorem 39, except that the analogue of Lemma 35 for representations is proved using the main results of [Donnelly 1978; Brüning and Heintze 1978/79] instead of Theorem A. □

Even though Proposition 40 follows from [Donnelly 1978; Brüning and Heintze 1978/79] using classical results, to the best of the authors' knowledge this fact has not been noticed for general (infinite) groups G.

5.3 Examples

In this section we give examples to illustrate Theorem C and its version for representations, Proposition 40.

Example 41 Suppose G is a *finite* group, and $G \to O(V)$ is injective (ie faithful). Then the orbit space $X = \mathbb{S}V/G$ is obtained from a fundamental domain of the action by boundary identifications. In particular, $\dim X = \dim \mathbb{S}V$ and $\mathrm{Vol}(X)/\mathrm{Vol}(\mathbb{S}V) = |G|^{-1}$. Moreover, Proposition 40(a) follows immediately from Molien's formula; see [Neusel and Smith 2002, Proposition 3.1.4].

Specializing even more:

Example 42 Suppose $G \subset O(V)$ is a finite group generated by reflections. Then $A = \mathbb{R}[V]^G$ is free by the Chevalley–Shephard–Todd theorem [Neusel and Smith 2002, Theorem 7.1.4], so that Proposition 40(b) reduces to the well-known fact that

$$|G| = \deg f_0 \cdots \deg f_m$$

for an algebraically independent generating set $f_0, \dots, f_m$ of $\mathbb{R}[V]^G$; see [Neusel and Smith 2002, Proposition 3.1.5].

An example illustrating Theorem C:

Example 43 (Clifford foliations) The assertions and definitions below can be found in [Radeschi 2014].

Let $V = \mathbb{R}^{2l}$ and $C = (P_0, \dots, P_m)$ be a Clifford system on V, which implies $m \le l$. Let $\psi : V \to \mathbb{R}^{m+2}$ be the polynomial map given by

$$\psi(x) = (\psi_1, \dots, \psi_{m+2})(x) = (\|x\|^2, \langle P_0 x, x\rangle, \dots, \langle P_m x, x\rangle).$$

If $m \ne l-1$, the fibers of ψ form an infinitesimal singular Riemannian foliation $\mathcal{F}_C$ of V, called a *Clifford foliation*. The leaf space $X = \mathbb{S}V/\mathcal{F}_C$ is isometric to a hemisphere in $\mathbb{S}^{m+1}\left(\frac{1}{2}\right)$ when $m \le l-2$, and to $\mathbb{S}^m\left(\frac{1}{2}\right)$ when $m = l$. Thus the quotient of volumes $\mathrm{vol}\, X/\mathrm{vol}(\mathbb{S}^{\dim X})$ appearing in Theorem C is 2^{-m-2}, respectively 2^{-m}.

On the other hand, $\psi_1, \dots, \psi_{m+2}$ actually generate the algebra A of basic polynomials of $\mathcal{F}_C$; see [Mendes and Radeschi 2020b, proof of Theorem C]. When $m \le l-2$, they are algebraically independent because the image of ψ has nonempty interior. Each ψ_i has degree two, so this confirms the conclusion of Theorem C(b).

When $m = l$, there is one relation between the generators, namely $\psi_1^2 = \psi_2^2 + \cdots + \psi_{m+2}^2$. The polynomials $\psi_1, \dots, \psi_{m+1}$ form a homogeneous system of parameters, and A is a free module of rank 2 over $\mathbb{R}[\psi_1, \dots, \psi_{m+1}]$. Thus the right-hand side of the equation in Theorem C(b) is $2/2^{m+1} = 2^{-m}$, as expected.

Example 44 (Hopf fibrations) Special cases of the previous example (all with $m = l$) are the infinitesimal singular Riemannian foliations of $\mathbb{R}^4 = \mathbb{C}^2$, $\mathbb{R}^8 = \mathbb{H}^2$ and $\mathbb{R}^{16} = \mathbb{C}\mathrm{a}^2$, whose restrictions to the unit sphere form the fibers of one of the Hopf fibrations $\mathbb{S}^3 \to \mathbb{S}^2\big(\frac{1}{2}\big)$, $\mathbb{S}^7 \to \mathbb{S}^4\big(\frac{1}{2}\big)$, and $\mathbb{S}^{15} \to \mathbb{S}^8\big(\frac{1}{2}\big)$.

Concretely in the complex case, the infinitesimal singular Riemannian foliation of $\mathbb{R}^4 = \mathbb{C}^2$ is given by the orbits under the action of the unit complex numbers $a \in \mathbb{S}^1$ by $a(z, w) = (az, aw)$. The algebra A of invariants is generated by the degree-2 real polynomials $\psi_1 = |z|^2 + |w|^2$, $\psi_2 = |z|^2 - |w|^2$, $\psi_3 = 2\,\mathrm{Re}(z\overline{w})$ and $\psi_4 = 2\,\mathrm{Im}(z\overline{w})$. The only relation is $\psi_1^2 = \psi_2^2 + \psi_3^2 + \psi_4^2$. A possible choice of system of parameters is ψ_1, ψ_2, ψ_3, for which $\dim_{\mathbb{R}[\psi_1,\psi_2,\psi_3]} A = 2$, thus illustrating part (b) of either Theorem C or Proposition 40, because $2/2^3 = \mathrm{vol}\,\mathbb{S}^2\big(\frac{1}{2}\big)/\,\mathrm{vol}\,\mathbb{S}^2(1)$.

6 A note on manifold submetries

An even more general concept than closed singular Riemannian foliations is given by manifold submetries, which are maps $\sigma\colon M \to X$, from a smooth Riemannian manifold to a metric space, whose fibers are smooth and mutually equidistant. This generalizes closed singular Riemannian foliations in essentially two ways, first by not requiring the presence of smooth vector fields generating the tangent spaces of the fibers (although it is conjectured that such vector fields would exist anyway), and by allowing the fibers to be disconnected. The structure of submetries from Riemannian manifolds (and, in particular, manifold submetries) has been studied by Kapovitch and Lytchak [2022]. In the particular case where M is the unit sphere in $\mathbb{R}^n$, it was proved by Mendes and Radeschi [2020a; 2023] that there is a one-to-one correspondence between manifold submetries from M and Laplacian algebras contained in $\mathbb{R}[x_1, \dots, x_n]$.

The statement of Theorem A still makes sense for manifold submetries, and the authors believe it still holds. However, the main obstacle to generalizing our proof to the manifold submetry case is the lack of a slice theorem; see Section 2.1.4. More specifically, given $L = \sigma^{-1}(x)$, the differential $d_p\sigma\colon \mathbb{S}(\nu_p L) \to \Sigma_x X$ (where $\Sigma_x X$ is the space of directions of X at x) might, in principle, depend on the choice of $p \in L$.

The slice theorem was used in a fundamental way in the inductive step to prove the geometric estimate (12) in the proof of Theorem D. In view of the discussion above, proving that the estimate (12) still holds essentially requires a stronger version of Theorem D, applied to a manifold submetry $\sigma\colon M \to X$, where the constant C only depends on X, rather than the specific σ.

After this paper was finished, we were informed by A Lytchak of his new manuscript [Lytchak 2024], in which he obtains a number of results bounding the geometry of leaves in a manifold submetry $\sigma\colon M \to X$, which only depend on the geometry of M and X rather than on σ itself. In particular, he obtains a more general and accurate version of Lemma 25; see Proposition 1.3 and Corollary 7.4 of [Lytchak 2024]. The authors believe that the results in [Lytchak 2024], together with previous results on the structure of transnormal submetries in [Kapovitch and Lytchak 2022], could constitute important ingredients to generalizing Theorem D in the way mentioned above, and hence Theorem A, to the manifold submetry case.

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