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Embedded contact homology and open book decompositions

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APPENDIX WRITTEN JOINTLY WITH YUAN YAO

Given a closed oriented contact 3-manifold M , we prove an equivalence between the embedded contact homology of M and a version of embedded contact homology “relative to the boundary”, defined on the complement of a tubular neighborhood of a null-homologous knot. This paper can be viewed as the first of a series of papers devoted to proving the isomorphism between Heegaard Floer homology and embedded contact homology.

The appendix, written jointly with Yuan Yao, gives a complete proof of Morse–Bott gluing for one-level cascades in embedded contact homology.

57K31, 57K33

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1 Introduction

Let M be a closed, oriented, connected 3-manifold and $N \subset M$ the complement of a tubular neighborhood of a null-homologous knot. The goal of this paper is to associate a specific class of contact forms α to N , to introduce relative embedded contact homology groups $\text{ECH}(N, \partial N, \alpha)$ and $\widehat{\text{ECH}}(N, \partial N, \alpha)$, and to prove their isomorphism with the embedded contact homology groups $\text{ECH}(M)$ and $\widehat{\text{ECH}}(M)$.

The embedded contact homology group $\text{ECH}(M)$ of a closed 3-manifold M , due to Hutchings [2002] partially in collaboration with Taubes [Hutchings and Taubes 2007; 2009a], is defined using a contact form α on M and an adapted almost complex structure J on the symplectization $\mathbb{R} \times M$. The variant $\widehat{\text{ECH}}(M)$, called *ECH hat*, is defined as the mapping cone of a U -map (see Section 2.5). There is currently no direct proof of the fact that these groups are invariants of M ; the only known proof, due to Taubes [2007; 2010], is a consequence of the isomorphism between Seiberg–Witten Floer cohomology and embedded contact homology, combined with the invariance of Seiberg–Witten Floer cohomology established by Kronheimer and Mrowka [2007].

Embedded contact homology groups can be defined over the integers, following [Bourgeois and Mohnke 2004] or [Hutchings and Taubes 2009a, Section 9]. All results in this article hold over the integers as explained in Proposition 4.5.5 and Remark 9.9.5, but we will write detailed proofs only over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ for simplicity. Given a compact 3-manifold N with $\partial N \simeq T^2$, let α be a contact form on N which is nondegenerate on $\text{int}(N)$ and negative Morse–Bott on ∂N (see Definition 4.1.1). In particular, the Reeb orbits on ∂N act as sinks for J -holomorphic curves in $\mathbb{R} \times N$, ie no nontrivial J -holomorphic curve in $\mathbb{R} \times N$ can have a positive end at an orbit in ∂N . Then there exist relative embedded contact homology groups $\text{ECH}(N, \partial N, \alpha)$ and $\widehat{\text{ECH}}(N, \partial N, \alpha)$, whose definitions will be given in Section 7. Moreover, there is a chain map U on the complex defining $\text{ECH}(N, \partial N, \alpha)$, and the homology of the cone of U is isomorphic to $\widehat{\text{ECH}}(N, \partial N, \alpha)$.

The embedded contact homology group of a contact manifold (M, ξ) has a natural decomposition as a direct sum of groups $\text{ECH}(M, \xi, A)$ indexed by homology classes¹ $A \in H_1(M)$. This decomposition depends on the contact structure ξ , although very weakly. For this reason we always specify ξ together with the homology class A .

Similarly, $\text{ECH}(N, \partial N, \alpha)$ and $\widehat{\text{ECH}}(N, \partial N, \alpha)$ decompose as direct sums of groups $\text{ECH}(N, \partial N, \alpha, A)$ and $\widehat{\text{ECH}}(N, \partial N, \alpha, A)$ indexed by relative homology classes $A \in H_1(N, \partial N)$. The maps U in both $\text{ECH}(M, \xi)$ and $\text{ECH}(N, \partial N, \alpha)$ preserve the splitting according to homology classes. Taking into account the fact that K is null-homologous, excision and the relative homology long exact sequence give an isomorphism $\varpi: H_1(N, \partial N) \xrightarrow{\cong} H_1(M)$, and the equivalence between ECH and relative ECH is compatible with the corresponding decompositions.

The main result of this paper is the following:

¹Singular homology groups should always be understood over the integers if no coefficient group is explicitly indicated.

Theorem 1.1.1 *Let $N \subset M$ be the complement of a tubular neighborhood $\text{int}(V)$ of a null-homologous knot K , where $V \simeq K \times D^2$, ξ a contact form on M which is transverse to the foliation $K \times \{*\}$ on V , and α a contact form on N for the contact structure $\xi|_N$. If the Reeb vector field R_α of α is nondegenerate on $\text{int}(N)$, negative Morse–Bott on ∂N and foliates ∂N by meridians, and all closed Reeb orbits in N have nonnegative linking number with K , then, for all $A \in H_1(N, \partial N; \mathbb{Z})$,*

- (1) $\text{ECH}(N, \partial N, \alpha, A) \simeq \text{ECH}(M, \xi, \varpi(A))$, and
- (2) $\widehat{\text{ECH}}(N, \partial N, \alpha, A) \simeq \widehat{\text{ECH}}(M, \xi, \varpi(A))$.

Moreover, the first isomorphism is compatible with the U -maps on both sides.

The prototypical situation to which [Theorem 1.1.1](#) applies is the case of an open book decomposition with connected binding. In this case N is the mapping torus of a surface diffeomorphism $h: S \xrightarrow{\cong} S$ and $V = M - \text{int}(N)$ is a tubular neighborhood of the binding. In other words, [Theorem 1.1.1](#) allows us to rewrite the embedded contact homology groups of M in terms of the relative embedded contact homology groups on the complement of the binding. We remark here that [[Yau 2007](#); [Wendl 2008](#); [2005](#)] examined related issues.

[Theorem 1.1.1](#), applied to the open book case, is the first step in the proof of the equivalence of embedded contact homology and Heegaard Floer homology, a Floer homology theory for three-manifolds defined by Ozsváth and Szabó [[2004a](#); [2004b](#)]. Once we express the embedded contact homology of M purely in terms of N using [Theorem 1.1.1](#), it is easier to define chain maps to and from the hat version of Heegaard Floer homology. In fact, the Giroux correspondence [[2002](#)] — the bijection between open book decompositions up to positive stabilization and isotopy classes of contact structures — provides a bridge between the contact forms used in the definition of ECH and the Heegaard splittings used in the definition of Heegaard Floer homology. We remark that the proof of the equivalence between Heegaard Floer homology and ECH is independent of the hard part of the Giroux correspondence (ie the stabilization equivalence of two open book decompositions which support the same contact structure). The rest of the proof of the equivalence has been carried out in [[Colin et al. 2024a](#); [2024b](#); [2024c](#)]; see [[Colin et al. 2011a](#)] for an overview of the strategy.

Remark 1.1.2 An alternative proof of the equivalence of Heegaard Floer and embedded contact homologies, passing through Seiberg–Witten Floer homology, has been given by Kutluhan, Lee and Taubes [[Kutluhan et al. 2020a](#); [2020b](#); [2020c](#); [2020d](#); [2020e](#)].

In [Section 10](#) we present some independent applications of the techniques developed here to the embedded contact homology for sutured manifolds defined in [[Colin et al. 2011b](#)]. More precisely, we prove that ECH of a sutured manifold is invariant of the contact form and the almost complex structure ([Theorem 10.2.2](#)) and we finish the proof of [[Colin et al. 2011b](#), [Theorem 1.6](#)] by showing that $\widehat{\text{ECH}}(M)$, defined as the homology of the cone of the U -map, is isomorphic to the sutured ECH of the complement of a ball

in M (Theorem 10.3.1). Theorem 10.2.2 has been independently proved by Kutluhan, Sivek and Taubes [Kutluhan et al. 2022].

Organization of the paper Section 2 gives a brief review of ECH; in particular, we define the groups $\text{ECH}(M)$ and $\widehat{\text{ECH}}(M)$. We review some technicalities involving direct limits in Section 3 and some Morse–Bott theory in the context of ECH in Section 4. In Section 5 we discuss topological constraints of J -holomorphic curves arising from the positivity of intersections in dimension four. In Section 6 we construct contact forms on $D^2 \times S^1$ and $T^2 \times [1, 2]$ which are used later. Section 7 is devoted to the definitions of certain ECH groups for compact manifolds with torus boundary and in particular the variants $\text{ECH}(N, \partial N, \alpha)$ and $\widehat{\text{ECH}}(N, \partial N, \alpha)$ which appear in Theorem 1.1.1. In Section 8 we calculate some ECH groups of solid tori which are used in the proof of Theorem 1.1.1. Section 9 then completes the proof of Theorem 1.1.1. Finally, Section 10 relates some of the versions of ECH defined in Section 7 to some sutured ECH groups defined in [Colin et al. 2011b].

2 Review of embedded contact homology

In this paper all manifolds will be oriented and connected, unless stated otherwise.

In this section we briefly review the basic definitions of embedded contact homology (from now on abbreviated ECH). For more details the reader is referred to [Hutchings 2002; 2009] or to [Hutchings 2014]. To avoid orienting the moduli spaces, we will work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

2.1 Generators of the ECH chain complex

Let M be a closed, oriented and connected 3-manifold with a contact form α . We will denote by $\xi = \ker \alpha$ the contact structure with contact form α . The Reeb vector field $R = R_\alpha$ is *nondegenerate* if no Reeb orbit² has 1 as eigenvalue of its linearized first return map. This is a generic condition which can be achieved by a generic C^∞ -small perturbation of the contact form; see for example [Colin and Honda 2013, Lemma 7.1]. For the rest of the section we will assume that α is nondegenerate. The linearization of the first return map along a Reeb orbit is a symplectic transformation of the symplectic plane $(\xi, d\alpha)$. This implies that its eigenvalues are $\{\lambda, \lambda^{-1}\}$, where λ is either real or in the unit circle. Then a Reeb orbit is

- *hyperbolic* if the eigenvalues of its linearized first return map are real; or
- *elliptic* if they lie on the unit circle.

These conditions are mutually exclusive because every orbit is assumed to be nondegenerate.

²In this paper we interchangeably use “Reeb orbit”, “closed orbit” and “closed Reeb orbit”. A Reeb orbit which is not necessarily closed will be called a “Reeb trajectory”.

Let $\mathcal{P}$ be the set of simple orbits of the Reeb vector field R_α . The ECH chain complex³ $\text{ECC}(M, \alpha)$, as a vector space, is generated over $\mathbb{F}$ by finite sets $\gamma = \{(\gamma_i, m_i)\}$, called *orbit sets*, where

- $\gamma_i \in \mathcal{P}$ and $\gamma_i \neq \gamma_j$ for $i \neq j$;
- m_i is a positive integer; and
- if γ_i is a hyperbolic orbit, then $m_i = 1$.

We will say that $\text{ECC}(M, \alpha)$ is *constructed from* $\mathcal{P}$. An orbit set γ will also be written multiplicatively as $\prod \gamma_i^{m_i}$, with the convention that $\gamma_i^2 = 0$ whenever γ_i is hyperbolic. The empty orbit set $\emptyset$ will be written multiplicatively as 1.

The homology class of an orbit set γ is

$$[\gamma] = \sum_i m_i [\gamma_i] \in H_1(M).$$

If we want to specify the direct summand generated by orbit sets of class $A \in H_1(M)$, then we write $\text{ECC}(M, \alpha, A)$.

The *action* $\mathcal{A}_\alpha(\gamma_i)$ of an orbit γ_i is given by $\int_{\gamma_i} \alpha$, and the action of an orbit set γ is given by

$$\mathcal{A}_\alpha(\gamma) = \sum_i m_i \mathcal{A}_\alpha(\gamma_i).$$

2.2 Moduli spaces

We choose an almost complex structure J on $\mathbb{R} \times M$, with $\mathbb{R}$ -coordinate s , which is *adapted to the symplectization of α* (or *adapted to α*), ie

- J is s -invariant;
- J takes ξ to itself on each $\{s\} \times Y$;
- J maps ∂_s to R_α ;
- $J|_\xi$ is $d\alpha$ -compatible, ie $d\alpha(\cdot, J\cdot)$ defines a Euclidean metric on ξ .

Let $\gamma = \{(\gamma_i, m_i)\}$ and $\gamma' = \{(\gamma'_i, m'_i)\}$ be orbit sets with $[\gamma] = [\gamma'] \in H_1(M)$. We denote by $\mathcal{M}_J(\gamma, \gamma')$ the set of holomorphic maps

$$u: (F, j) \rightarrow (\mathbb{R} \times M, J),$$

modulo holomorphic reparametrizations, which satisfy

- (1) (F, j) is a closed Riemann surface with a finite number of punctures removed;
- (2) the neighborhoods of the punctures are mapped asymptotically to cylinders over Reeb orbits;

³The ECH differential depends on the choice of an adapted almost complex structure J (see Section 2.2), but the generators only depend on α . Hence we suppress J from the notation for the moment.

- (3) at the positive end of $\mathbb{R} \times M$, u is asymptotic to $\mathbb{R} \times \gamma_i$ with total multiplicity m_i for each pair (γ_i, m_i) (more precisely, if we list the positive ends of u that are asymptotic to some multiple cover of $\mathbb{R} \times \gamma_i$ and the covering degrees are $m_{i1}, \dots, m_{ij_i}$, then $m_i = m_{i1} + \dots + m_{ij_i}$); and
- (4) at the negative end of $\mathbb{R} \times M$, u is asymptotic to $\mathbb{R} \times \gamma'_i$ with total multiplicity m'_i for each pair (γ'_i, m'_i) .

We often refer to an element u of $\mathcal{M}_J(\gamma, \gamma')$ as a *J-holomorphic map (or curve) from γ to γ'* . We stress that, according to our definition, the genus, the number of connected components and the number of punctures of F are not fixed a priori. If $*$ is a property of J -holomorphic curves, we will denote by $\mathcal{M}_J^*(\gamma, \gamma')$ the subset of $\mathcal{M}_J(\gamma, \gamma')$ satisfying $*$. We can similarly define the “pointed” moduli space $\mathcal{M}_J(\gamma, \gamma'; \text{pt})$ as the set of holomorphic maps

$$u: (F, j, p) \rightarrow (\mathbb{R} \times M, J),$$

modulo holomorphic reparametrizations, where $p \in F$.

Definition 2.2.1 We say that J is *regular* if, for all orbit sets γ and γ' and $u \in \mathcal{M}_J(\gamma, \gamma')$ which has no multiply covered components, $\mathcal{M}_J(\gamma, \gamma')$ is transversely cut out near u (ie the linearized $\bar{\partial}$ -operator D_u at u from [Dragnev 2004, Proposition 2.10] is surjective).

Regular adapted almost complex structures form the complement of a first category set (and therefore are dense) in the space of smooth adapted almost complex structures with respect to the C^∞ -topology by a result of Dragnev [2004]. If no component of u is multiply covered and all components of u are transversely cut out, then, in a neighborhood of u , the moduli space $\mathcal{M}_J(\gamma, \gamma')$ has the structure of a finite-dimensional manifold of dimension $\text{ind}(u)$, where the Fredholm index $\text{ind}(u)$ is the formal dimension of the moduli spaces computed as in the next paragraph; see [Dragnev 2004, Corollary 1]. Our convention throughout the paper will be that the Fredholm index takes into account the dimensions of the Deligne–Mumford moduli space and the automorphism group of the domain of the map. In particular, $\text{ind}(u) = \text{ind}(D_u) - 3\chi(F)$, where $\text{ind}(D_u)$ is the Fredholm index of the linearized Cauchy–Riemann operator at u and $\chi(F)$ is the Euler characteristic of the domain of u .

A J -holomorphic map $u: F \rightarrow \mathbb{R} \times M$ from $\gamma = \{(\gamma_i, m_i)\}$ to $\gamma' = \{(\gamma'_i, m'_i)\}$ determines partitions $\{m_{ij}\}$ of m_i and $\{m'_{ij}\}$ of m'_i such that u is positively asymptotic to m_{ij} -fold covers $\gamma_i^{m_{ij}}$ of the simple Reeb orbits γ_i and negatively asymptotic to m'_{ij} -fold covers $(\gamma'_i)^{m'_{ij}}$ of the simple Reeb orbits γ'_i . Let τ be a trivialization of ξ along each orbit in the orbit sets γ, γ' , let $\mu_\tau(\delta)$ denote the Conley–Zehnder index of a cover δ of an orbit in γ or γ' with respect to τ , and let $c_1(u^*\xi, \tau)$ denote the relative first Chern class of $u^*\xi$ with respect to τ . Then the Fredholm index $\text{ind}(u)$ is given by the formula

$$(2-2-1) \quad \text{ind}(u) = -\chi(F) + 2c_1(u^*\xi, \tau) + \sum_{ij} \mu_\tau(\gamma_i^{m_{ij}}) - \sum_{ij} \mu_\tau((\gamma'_i)^{m'_{ij}}).$$

(See the formula in [Dragnev 2004, Theorem 1.8].)

2.3 The ECH index

The index which appears in the definition of ECH is not the Fredholm index, but the ECH index, which is more topological in nature. In this subsection we will review its definition.

Let $\gamma = \{(\gamma_i, m_i)\}$ and $\gamma' = \{(\gamma'_i, m'_i)\}$ be orbit sets. We denote by $H_2(M, \gamma, \gamma')$ the relative homology classes of surfaces $Z \in H_2(M, (\bigcup_i \gamma_i) \cup (\bigcup_{i'} \gamma'_{i'}))$ such that $\partial Z = \sum m_i [\gamma_i] - \sum m'_i [\gamma'_i]$, where

$$\partial: H_2\left(M, \left(\bigcup_i \gamma_i\right) \cup \left(\bigcup_{i'} \gamma'_{i'}\right)\right) \rightarrow H_1\left(\left(\bigcup_i \gamma_i\right) \cup \left(\bigcup_{i'} \gamma'_{i'}\right)\right)$$

is the connecting homomorphism of the relative homology exact sequence. By abuse of notation, Z will also denote an embedded surface with boundary which represents that homology class. We pick a trivialization τ of ξ along each orbit in the orbit sets γ and γ' , and define $c_1(\xi|_Z, \tau)$ as the first Chern class of ξ evaluated on Z , relative to the trivialization τ on ∂Z .

If $\gamma = \{(\gamma_i, m_i)\}_{i=1}^k$ is an orbit set, then we define the “symmetric” Conley–Zehnder index (so called because of its motivation from studying symplectomorphisms of a symmetric product of a surface) as

$$(2-3-1) \quad \tilde{\mu}_\tau(\gamma) = \sum_{i=1}^k \sum_{j=1}^{m_i} \mu_\tau(\gamma_i^j),$$

where γ_i^j is the orbit which multiply covers γ_i with multiplicity j .

We define the *relative intersection pairing* $Q_\tau(Z)$ as follows: Using the trivialization τ , for each simple orbit γ_i of γ or γ' , fix an identification of a sufficiently small neighborhood $N(\gamma_i)$ of γ_i with $\gamma_i \times D^2$, where D^2 has polar coordinates (r, θ) . Let Σ be an oriented surface and $f: \Sigma \rightarrow [-1, 1] \times M$ a smooth map which satisfies the following:

- (1) f maps $\partial \Sigma$ to $\{-1, 1\} \times M$, $f|_{\text{int}(\Sigma)}$ is an embedding, and f is transverse to $\{-1, 1\} \times M$.
- (2) For all $\varepsilon > 0$ sufficiently small, $f(\Sigma) \cap (\{1 - \varepsilon\} \times M)$ consists of m_i disjoint circles of type $\{r = \varepsilon, \theta = \text{const}\}$ in $N(\gamma_i)$ for all i (and similarly for $f(\Sigma) \cap (\{-1 + \varepsilon\} \times M)$).
- (3) The composition of f with the projection $[-1, 1] \times M \rightarrow M$ is a representative of the class $Z \in H_2(M, \gamma, \gamma')$.

We then choose two maps f_1 and f_2 satisfying (1)–(3) above that are disjoint on $\{-1 + \varepsilon, 1 - \varepsilon\} \times M$ and transverse on $[-1 + \varepsilon, 1 - \varepsilon] \times M$. Then $Q_\tau(Z)$ is the signed intersection number of f_1 and f_2 in $[-1 + \varepsilon, 1 - \varepsilon] \times M$.

We are now in a position to define the ECH index.

Definition 2.3.1 [Hutchings 2002, Definition 1.5] The ECH index $I(\gamma, \gamma', Z)$ is given by

$$(2-3-2) \quad I(\gamma, \gamma', Z) = c_1(\xi|_Z, \tau) + Q_\tau(Z) + \tilde{\mu}_\tau(\gamma) - \tilde{\mu}_\tau(\gamma').$$

The ECH index depends only on the relative homology class $Z \in H_2(M, \gamma, \gamma')$ and not on a particular surface representing it. Moreover, the ECH index is independent also of the choice of trivialization. If

$Z' \in H_2(M, \gamma, \gamma')$ is another relative homology class, then, by [Hutchings 2014],

$$I(\gamma, \gamma', Z') - I(\gamma, \gamma', Z) = \left\langle Z' - Z, c_1(\xi) + \text{PD} \left(\sum_i m_i [\gamma_i] \right) \right\rangle,$$

where $\sum_i m_i [\gamma_i]$ is the total homology class of γ in $H_1(M)$ and PD is the Poincaré duality map.

Remark 2.3.2 A finite-energy holomorphic map u with asymptotics γ and γ' defines a relative homology class $Z \in H_2(M, \gamma, \gamma')$. Hence we can write $I(u) = I(\gamma, \gamma', Z)$.

The ECH index and the Fredholm index satisfy the following index inequality, which is one of the basic tools of ECH:

Theorem 2.3.3 [Hutchings 2009, Theorem 4.15] *If u is simply covered, then $\text{ind}(u) \leq I(u)$.*

2.4 The ECH differential

In this subsection we define the differential ∂ for the ECH chain complex, after recalling some properties of J -holomorphic maps with small ECH index. In the following we will say that a map $u: F \rightarrow \mathbb{R} \times M$ is the “disjoint union” of maps $u_i: F_i \rightarrow \mathbb{R} \times M$ (with $1 \leq i \leq k$) if $F = F_1 \sqcup \cdots \sqcup F_k$ and the images are pairwise disjoint. Here each F_i can still be disconnected. A *trivial cylinder* over a (not necessarily simple) orbit γ with period T is the J -holomorphic map $u: \mathbb{R} \times S^2 \rightarrow \mathbb{R} \times M$, $u(s, t) = (Ts, \gamma(Tt))$. By abuse of notation, we will always denote the trivial cylinder over γ by $\mathbb{R} \times \gamma$.

Lemma 2.4.1 [Hutchings and Taubes 2007, Proposition 7.15] *Let J be a regular almost complex structure adapted to α . Then:*

- (1) *A J -holomorphic map u with $I(u) = 0$ is a disjoint union of branched covers of trivial cylinders over simple Reeb orbits. (Such curves are called *connectors*.)*
- (2) *A J -holomorphic map u with $I(u) = 1$ (resp. 2) from γ to γ' is a disjoint union of a connector and an embedding u' with $I(u') = \text{ind}(u') = 1$ (resp. 2).*

In this paper a “branched cover” will always refer to a “branched cover with possibly empty branch locus”.

The ends of a J -holomorphic map u from γ to γ' determine partitions of the multiplicities of the elliptic orbits. It turns out that, when $I(u) = 1$ or $I(u) = 2$, these partitions must coincide with preferred partitions, called the *outgoing* and *incoming* partitions for positive and negative ends, respectively. The incoming and outgoing partitions can be computed from the dynamics of the linearized Reeb flow. For their definition see [Hutchings 2002, Section 4.1] or [Hutchings 2009, Definition 4.14]. For the relation between these partitions and the ECH index, see [Hutchings 2009, Theorem 4.15], for example. In this article we will not need the precise definition of the incoming or the outgoing partition, except for the following fact, which follows directly from [Hutchings 2009, Definition 4.14]:

Fact 2.4.2 Let γ be a simple elliptic orbit and suppose that its linearized Reeb flow rotates by an angle $2\pi\theta$. If $0 < \theta < 1/m$, then the incoming partition of (γ, m) is (m) and the outgoing partition is $(1, \dots, 1)$. On the other hand, if $-1/m < \theta < 0$, then the incoming partition of (γ, m) is $(1, \dots, 1)$ and the outgoing partition is (m) .

The boundary operator in the ECH chain complex is defined by a count of J -holomorphic maps with index $I = 1$ for a regular almost complex structure J . In order to make the dependence on J explicit, we write the complex as $\text{ECC}(M, \alpha, J)$. However, when J is clear from the context, it will be dropped from the notation.

Definition 2.4.3 Let J be a regular almost complex structure adapted to α . Then the boundary map $\partial: \text{ECC}(M, \alpha, J) \rightarrow \text{ECC}(M, \alpha, J)$ is defined as

$$\partial\gamma = \sum_{\gamma'} \langle \partial\gamma, \gamma' \rangle \gamma',$$

where $\langle \partial\gamma, \gamma' \rangle$ is the (mod 2) count of curves $u \in \mathcal{M}_J^{I=1}(\gamma, \gamma')/\mathbb{R}$ such that every connector component of u is a trivial cylinder over a simple orbit.

The map ∂ was shown to satisfy $\partial^2 = 0$ by Hutchings and Taubes [2007; 2009a]. The homology of the chain complex $(\text{ECC}(M, \alpha, J), \partial)$ is the *embedded contact homology* group $\text{ECH}(M, \alpha, J)$. It is independent of the choice of contact form α , the contact structure ξ and adapted almost complex structure J by [Taubes 2010]. Hence we are justified in writing $\text{ECH}(M)$.

2.5 Definition of $\widehat{\text{ECH}}(M)$

In this subsection we define a map $U: \text{ECH}(M) \rightarrow \text{ECH}(M)$ and a variant $\widehat{\text{ECH}}(M)$ of $\text{ECH}(M)$, called the *ECH hat group*, in analogy with well-known constructions in Heegaard Floer homology. An a priori different group, also called $\widehat{\text{ECH}}(M)$, was defined in [Colin et al. 2011b] using sutured ECH (in analogy with the sutured Floer homology of [Juhász 2006]). In Section 10 we will prove that the two approaches yield isomorphic groups.

Definition 2.5.1 Let J be a regular almost complex structure and $z \in \mathbb{R} \times M$ a generic point such that the evaluation map

$$\text{ev}: \mathcal{M}_J^{I=2}(\gamma, \gamma'; \text{pt}) \rightarrow \mathbb{R} \times M, \quad (u, p) \mapsto u(p),$$

is transverse to z . We define the map $U: \text{ECC}(M, \alpha, J) \rightarrow \text{ECC}(M, \alpha, J)$ as

$$U\gamma = \sum_{\gamma'} \langle U\gamma, \gamma' \rangle \gamma',$$

where $\langle U\gamma, \gamma' \rangle$ is the (mod 2) count of elements of $\text{ev}^{-1}(z) \subset \mathcal{M}_J^{I=2}(\gamma, \gamma'; \text{pt})$ such that every connector component of u is a trivial cylinder over a simple Reeb orbit.

The same techniques used to show that $\partial^2 = 0$ also show that U is a chain map; see [Hutchings and Taubes 2009b, Section 2.5] for more details on the U -map. Then $\widehat{\text{ECH}}(M, \alpha, J)$ is defined as the homology of the mapping cone of U .

3 Cobordism maps and direct limits

In this section we review the work of Hutchings and Taubes [2011] on maps on ECH induced by exact symplectic cobordisms, which in turn makes it possible to define continuation maps and take direct limits in ECH.

3.1 Maps induced by cobordisms

Given a contact 3-manifold (M, α) with α nondegenerate, let $\text{ECC}^L(M, \alpha)$ be the subcomplex of $\text{ECC}(M, \alpha)$ generated by orbit sets γ of action $\mathcal{A}_\alpha(\gamma) < L$, and $\text{ECH}^L(M, \alpha)$ be the resulting homology group. Given $L < L'$, the inclusion of chain complexes $\text{ECC}^L(M, \alpha) \subset \text{ECC}^{L'}(M, \alpha)$ induces a map

$$i_{L, L'}: \text{ECH}^L(M, \alpha) \rightarrow \text{ECH}^{L'}(M, \alpha)$$

on the level of homology. An immediate consequence of the definition of a direct limit is

$$\text{ECH}(M, \alpha) = \lim_{L \rightarrow \infty} \text{ECH}^L(M, \alpha).$$

Let (M_1, α_1) and (M_2, α_2) be contact 3-manifolds. An *exact symplectic cobordism* (X, ω) from⁴ (M_1, α_1) to (M_2, α_2) is an exact symplectic manifold with boundary $\partial X = M_1 - M_2$ and symplectic form $\omega = d\alpha$, where α restricts to α_1 on M_1 and α_2 on M_2 .

Given an exact symplectic cobordism (X, ω) , we form its *completion* $(\widehat{X}, \widehat{\omega})$ by attaching the half positive symplectization of (M_1, α_1) along $M_1 \subset \partial X$ and the half negative symplectization of (M_2, α_2) along $M_2 \subset \partial X$.

Definition 3.1.1 Let $(\widehat{X}, \widehat{\omega})$ be the completed symplectic cobordism with an almost complex structure J which is compatible with $\widehat{\omega}$ and is adapted to α_1 and α_2 at the positive and negative ends. Then the image of an embedding

$$\phi: (\mathbb{R} \times U, d(e^s \alpha_0), J_0) \hookrightarrow (\widehat{X}, \widehat{\omega}, J)$$

is called a *product region* if $\phi_*(d(e^s \alpha_0)) = \widehat{\omega}$, $\phi_* J_0 = J$, J_0 is adapted to α_0 and, at the ends of $\mathbb{R} \times U$, $\phi(s, x) = (s + C_i, \phi_i(x))$ for $i = 1, 2$, for some embedding $\phi_i: U \rightarrow M_i$ and constant C_i .

The main technical result of [Hutchings and Taubes 2013] is the following (the first item in (i) is a slight improvement due to Cristofaro-Gardiner [2013, Theorem 5.1]):

⁴This is the convention from symplectic field theory [Eliashberg et al. 2000] and is opposite from the one used in Heegaard Floer homology, for example.

Theorem 3.1.2 [Hutchings and Taubes 2013, Theorem 1.9] Let (M_1, α_1) and (M_2, α_2) be contact 3-manifolds and let (X, ω) be an exact symplectic cobordism from (M_1, α_1) to (M_2, α_2) . Suppose the contact forms α_1 and α_2 are nondegenerate. Then, for each positive real number L , there exists a map

$$\Phi^L(X, \omega): \text{ECH}^L(M_1, \alpha_1) \rightarrow \text{ECH}^L(M_2, \alpha_2)$$

such that:

- (i) Let J be a regular almost complex structure on $\hat{X}$ which is $\hat{\omega}$ -compatible and is adapted to α_i at the positive and negative ends. Then $\Phi^L(X, \omega)$ is induced from a (noncanonical) chain map

$$\hat{\Phi}^L(X, \omega, J): \text{ECC}^L(M_1, \alpha_1, J|_{M_1}) \rightarrow \text{ECC}^L(M_2, \alpha_2, J|_{M_2}),$$

which is supported on the J -holomorphic curves, ie:

- $\langle \hat{\Phi}^L(X, \omega, J)(\gamma), \gamma' \rangle = 0$ if there is no $I = 0$, J -holomorphic building from γ to γ' in $\hat{X}$.
 - If the only J -holomorphic building in $\hat{X}$ from γ to γ' is a union of covers of product cylinders contained in a product region, then $\langle \hat{\Phi}^L(X, \omega, J)(\gamma), \gamma' \rangle = 1$.
- (ii) The map $\Phi^L(X, \omega)$ only depends on L and (X, ω) , and not on any auxiliary almost complex structure J on $(\hat{X}, \hat{\omega})$. Moreover, it depends on ω only through its homotopy class as an exact symplectic form.
- (iii) If $L < L'$, then the following diagram commutes:

$$(3-1-1) \quad \begin{array}{ccc} \text{ECH}^L(M_1, \alpha_1) & \xrightarrow{\Phi^L(X, \omega)} & \text{ECH}^L(M_2, \alpha_2) \\ i_{L, L'} \downarrow & & \downarrow i_{L, L'} \\ \text{ECH}^{L'}(M_1, \alpha_1) & \xrightarrow{\Phi^{L'}(X, \omega)} & \text{ECH}^{L'}(M_2, \alpha_2) \end{array}$$

Hence the maps pass to the direct limit

$$\Phi(X, \omega): \text{ECH}(M_1, \alpha_1) \rightarrow \text{ECH}(M_2, \alpha_2).$$

- (iv) Suppose (X, ω) is the composition of exact symplectic cobordisms (X_1, ω_1) from (M_1, α_1) to (M', α') and (X_2, ω_2) from (M', α') to (M_2, α_2) , and α' is nondegenerate. Then

$$\Phi^L(X, \omega) = \Phi^L(X_2, \omega_2) \circ \Phi^L(X_1, \omega_1).$$

- (v) If $c > 0$, then the diagram

$$(3-1-2) \quad \begin{array}{ccc} \text{ECH}^L(M_1, \alpha_1) & \xrightarrow{\Phi^L(X, \omega)} & \text{ECH}^L(M_2, \alpha_2) \\ s \downarrow & & \downarrow s \\ \text{ECH}^{cL}(M_1, c\alpha_1) & \xrightarrow{\Phi^{cL}(X, c\omega)} & \text{ECH}^{cL}(M_2, c\alpha_2) \end{array}$$

commutes, where s is the canonical rescaling isomorphism.

(vi) If $X = [0, a] \times M$ and $\omega = d(e^s \alpha)$ with α nondegenerate, then

$$\Phi^L(X, \omega): \text{ECH}^L(M, e^a \alpha) \rightarrow \text{ECH}^L(M, \alpha)$$

is equal to the composition

$$\text{ECH}^L(M, e^a \alpha) \xrightarrow{s} \text{ECH}^{e^{-a}L}(M, \alpha) \xrightarrow{i_{e^{-a}L, L}} \text{ECH}^L(M, \alpha).$$

Remark 3.1.3 The maps involved in this result are borrowed from Seiberg–Witten theory via Taubes’s isomorphism, where one counts solutions of the perturbed Seiberg–Witten equations on the cobordism. As we take a perturbation parameter r to be large, these solutions concentrate near a holomorphic building. It is however not known yet how to reconstruct the count of solutions from just knowing the limit holomorphic building. This explains why there is no direct definition of cobordism maps by a count of holomorphic buildings and also why there is no direct proof of invariance for ECH.

Definition 3.1.4 A contact form α is called L -nondegenerate if all Reeb orbits of action less than L are nondegenerate and there is no orbit set of action exactly L .

The action-truncated ECH groups $\text{ECH}^L(M, \alpha)$ make sense for contact forms α which are L -nondegenerate and [Theorem 3.1.2](#)(i), (ii) and (iv) hold for L -nondegenerate contact forms.

All exact cobordisms considered in this paper will be of the following type:

Definition 3.1.5 An *interpolating cobordism* from (M, α_1) to (M, α_0) is an exact symplectic cobordism $([0, 1] \times M, \lambda)$ from α_1 to α_0 such that λ is of the form

$$\lambda = \Phi^*(f\alpha),$$

where α is the pullback to $[0, 1] \times M$ of a 1-form (also called α) on M , $f: [0, 1] \times M \rightarrow \mathbb{R}$ is a positive function with $\partial f / \partial t > 0$, and $\Phi: [0, 1] \times M \xrightarrow{\sim} [0, 1] \times M$ is a diffeomorphism taking $\{i\} \times M$ to itself for $i = 0, 1$.

In this article, interpolating cobordisms are all constructed as follows: Let α_0 and α_1 be isotopic contact forms on M and let $\{\phi_t: M \xrightarrow{\sim} M\}_{t \in [0, 1]}$ be an isotopy such that

- $\phi_t^*(f_t \alpha_0) = \alpha_t$ for all $t \in [0, 1]$;
- $\{f_t\}$ and $\{\alpha_t\}$ are 1-parameter families of functions and 1-forms on M ; and
- $\phi_0 = \text{id}$ and $f_0 = 1$.

Then define $\Phi: [0, 1] \times M \rightarrow [0, 1] \times M$ by $\Phi(t, \mathbf{x}) = \phi_t(\mathbf{x})$, $f: [0, 1] \times M \rightarrow \mathbb{R}$ by $f(t, \mathbf{x}) = f_t(\mathbf{x})$, and $\lambda_\phi := \Phi^*(f\alpha_0)$. If $\partial f_t / \partial t > 0$, then

$$([0, 1] \times M, \lambda_\phi)$$

is an interpolating cobordism. Interpolating cobordisms do not necessarily exist between any two isotopic α_0 and α_1 , but one can always construct them at the small price of scaling one of the two forms by a constant.

Lemma 3.1.6 *Let $([0, 1] \times M, \lambda_\phi)$ and $([0, 1] \times M, \lambda'_\phi)$ be interpolating cobordisms from (M, α_1) to (M, α_0) defined by contact isotopies ϕ and ϕ' , respectively. If the isotopies $\{\phi_t\}$ and $\{\phi'_t\}$ are homotopic relative to the endpoints, then λ_ϕ and λ'_ϕ are homotopic as exact symplectic forms.*

Proof Define $\Phi(t, \mathbf{x}) = \phi_t(\mathbf{x})$ and $\Phi'(t, \mathbf{x}) = \phi'_t(\mathbf{x})$. Without loss of generality we can write $\lambda_0 := \lambda_\phi = \Phi^*(f\alpha)$ and $\lambda_1 := \lambda'_\phi = (\Phi')^*(f'\alpha)$ with the same form α in both definitions. Let $\{\Phi_s\}$ be a homotopy between Φ and Φ' such that

- $\Phi_0 = \Phi$ and $\Phi_1 = \Phi'$;
- $\Phi_s(0, \mathbf{x}) = \mathbf{x}$ and $\Phi_s(1, \mathbf{x}) = \phi_1(\mathbf{x}) = \phi'_1(\mathbf{x})$ for all $s \in [0, 1]$.

Also define $F_s(t, \mathbf{x}) = (1-s)f(t, \mathbf{x}) + sf'(t, \mathbf{x})$. Then

$$\lambda_s = \Phi_s^*(F_s\alpha)$$

is a homotopy of exact symplectic forms because $\partial F_s / \partial t > 0$ for all $s \in [0, 1]$. □

Lemma 3.1.7 *Let α be a contact form, $L, L' > 0$ real numbers, $\phi_t: M \rightarrow M$ for $t \in [0, 1]$ an isotopy such that $\phi_0 = \text{id}$, and $f, f': M \rightarrow \mathbb{R}^+$ smooth functions such that $Lf' < L'f$. If $f\alpha$ and $f'\alpha$ are L - and L' -nondegenerate, respectively, then there is a map*

$$\text{ECH}^L(M, \phi_1^*(f\alpha)) \rightarrow \text{ECH}^{L'}(M, f'\alpha).$$

Moreover, this map depends only on the homotopy class of $\{\phi_t\}$ relative to the endpoints and has the following properties:

- If $f = f'$ and $\phi_t \equiv \text{id}$ for $t \in [0, 1]$, then the map is induced by the inclusion of chain complexes.
- If $L'' > 0$, $f'': M \rightarrow \mathbb{R}^+$ is another function such that $L'f'' < L''f'$, and $\phi_t: M \rightarrow M$ for $t \in [1, 2]$ is an extension of the isotopy, then the following triangle commutes:

$$\begin{array}{ccc} \text{ECH}^L(M, \phi_2^*(f\alpha)) & \xrightarrow{\quad\quad\quad} & \text{ECH}^{L''}(M, f''\alpha) \\ & \searrow \quad \quad \quad \nearrow & \\ & \text{ECH}^{L'}(M, \phi_1^*(f'\alpha)) & \end{array}$$

Proof The inequality $Lf' < L'f$ implies that there is an interpolating cobordism with $L'\phi_1^*(f\alpha)$ at the positive end and $Lf'\alpha$ at the negative end. We define the map $\text{ECH}^L(M, \phi_1^*(f\alpha)) \rightarrow \text{ECH}^{L'}(M, f'\alpha)$

by the composition

$$\begin{array}{ccc} \mathrm{ECH}^L(M, \phi_1^*(f\alpha)) & \longrightarrow & \mathrm{ECH}^{L'L}(M, L'\phi_1^*(f\alpha)) \\ & \downarrow & \\ & \mathrm{ECH}^{L'L}(M, Lf'\alpha) & \longrightarrow \mathrm{ECH}^{L'}(M, f'\alpha), \end{array}$$

where the map $\mathrm{ECH}^{L'L}(M, L'\phi_1^*(f\alpha)) \rightarrow \mathrm{ECH}^{L'L}(M, Lf'\alpha)$ is the map induced by an interpolating cobordism from $L'\phi_1^*(f\alpha)$ to $Lf'\alpha$ and the horizontal maps are rescaling isomorphisms. The resulting map depends only on the homotopy class of $\{\phi_t\}$ relative to the endpoints by [Lemma 3.1.6](#). The properties of these maps are an immediate consequence of [Theorem 3.1.2](#). $\square$

3.2 Direct limits

One consequence of [Theorem 3.1.2](#) is the following theorem, whose statement and proof were communicated to the authors by Michael Hutchings:

Theorem 3.2.1 (Hutchings and Taubes) *Let M be a closed oriented 3-manifold with a nondegenerate contact form α and $\{f_i\}_{i=1}^\infty$ a sequence of smooth positive functions such that $1 \geq f_1 \geq f_2 \geq \dots$ and $f_i\alpha$ is L_i -nondegenerate for an increasing sequence of positive real numbers L_i such that $\lim_{i \rightarrow \infty} L_i = +\infty$. Then there is a canonical isomorphism*

$$\mathrm{ECH}(M, \alpha) \simeq \lim_{i \rightarrow \infty} \mathrm{ECH}^{L_i}(M, f_i\alpha).$$

Proof We have a map

$$f: \mathrm{ECH}(M, \alpha) \rightarrow \lim_{i \rightarrow \infty} \mathrm{ECH}^{L_i}(M, f_i\alpha),$$

obtained by taking the direct limit of the cobordism maps

$$\mathrm{ECH}^{L_i}(M, \alpha) \rightarrow \mathrm{ECH}^{L_i}(M, f_i\alpha).$$

Choose an increasing sequence of natural numbers c_i such that $L_{c_i} f_i > L_i$. Then there are maps

$$\mathrm{ECH}^{L_i}(M, f_i\alpha) \rightarrow \mathrm{ECH}^{L_{c_i}}(M, \alpha)$$

by [Lemma 3.1.7](#). These maps form a directed system, and, taking the direct limit, we obtain a map

$$g: \lim_{i \rightarrow \infty} \mathrm{ECH}^{L_i}(M, f_i\alpha) \rightarrow \mathrm{ECH}(M, \alpha).$$

The verification that the maps f and g are inverse of each other is a straightforward application of [Lemma 3.1.6](#). $\square$

We can now quantify when it makes sense to take direct limits of a sequence of contact forms α_i for isotopic contact structures. In this case we can write $\phi_i^*(\alpha_i) = f_i\alpha$ for some positive function f_i and diffeomorphism ϕ_i isotopic to the identity.

Definition 3.2.2 Let α be a contact form on M . A sequence $\{\alpha_i\}_{i=1}^\infty$ of contact forms on M is *comensurate to α* if there is a constant $0 < c < 1$, diffeomorphisms ϕ_i of M isotopic to the identity, and functions $f_i: M \rightarrow \mathbb{R}^{>0}$ such that $\phi_i^*\alpha_i = f_i\alpha$ and $c < |f_i|_{C^0} < 1/c$.

A corollary of [Theorem 3.2.1](#) is the following:

Corollary 3.2.3 *Let $\{\alpha_i\}$ be a sequence of contact 1-forms on M which is commensurate to α on M with constant $0 < c < 1$. If $L_i \rightarrow \infty$ is a sequence which satisfies $L_{i+1} > L_i/c^3$ for all i , then the groups $\text{ECH}^{L_i}(M, \alpha_i)$ form a directed system with the maps defined in [Lemma 3.1.7](#) and we have*

$$\text{ECH}(M) = \lim_{i \rightarrow \infty} \text{ECH}^{L_i}(M, \alpha_i).$$

Proof Define $L'_i = c^{2i} L_i$ and $g_i = c^{2i} f_i$. Then $\lim_{i \rightarrow \infty} L'_i = +\infty$ and $1 > g_1 > \cdots > g_i > \cdots$, so we can apply [Theorem 3.2.1](#) to the sequences L'_i and g_i . $\square$

4 Morse–Bott theory

In this section we discuss a special case of Morse–Bott theory as it applies to our context. In particular, we explain how to use [Theorem 3.2.1](#) to justify the Morse–Bott arguments which populate this paper. For a more detailed discussion of Morse–Bott theory in contact homology, the reader is referred to Bourgeois [\[2003; 2002\]](#).

4.1 Morse–Bott contact forms

Let α be a *Morse–Bott contact form* on M . For the purposes of this paper, this means that all the orbits either are isolated and nondegenerate, or come in S^1 -families and are nondegenerate in the normal direction. (In general, there is also the case where the Reeb orbits come in two-dimensional families, ie are the fibers of a circle bundle; however, this will not occur here.) We denote a Morse–Bott family of simple orbits by $\mathcal{N}$ and the Morse–Bott torus corresponding to $\mathcal{N}$ by $T_{\mathcal{N}} = \bigcup_{x \in \mathcal{N}} x$.

Let $\{v_1, v_2\}$ be an oriented basis for ξ at some point $p \in T_{\mathcal{N}}$ such that v_1 is transverse to $T_{\mathcal{N}}$ and v_2 is tangent to $T_{\mathcal{N}}$. The derivative of the first return map $\xi_p \rightarrow \xi_p$ of the Reeb flow is given by the matrix $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ with respect to the basis $\{v_1, v_2\}$. (Here a vector $v = a_1 v_1 + a_2 v_2$ is written as a column vector.) The Morse–Bott condition implies that $a \neq 0$.

Definition 4.1.1 $T_{\mathcal{N}}$ is called a *positive* Morse–Bott torus if $a > 0$ and a *negative* Morse–Bott torus if $a < 0$.

Let us identify a sufficiently small neighborhood of a Morse–Bott torus $T_{\mathcal{N}}$ with $T^2 \times [-\nu, \nu]$ with coordinates (θ, t, y) such that the Reeb vector field is a positive constant times ∂_t along $T_{\mathcal{N}} = \{y = 0\}$. For a positive Morse–Bott torus, the Reeb vector field rotates in a counterclockwise manner as y goes from ν to $-\nu$ (ie in the same direction as a positive contact structure), while for a negative Morse–Bott torus it rotates in a clockwise manner.

On each $\mathcal{N} \simeq S^1$, we pick a Morse function $\bar{g}_{\mathcal{N}}: \mathcal{N} \rightarrow \mathbb{R}$ with two critical points. After perturbing α using these functions, each Morse–Bott family gives rise to an elliptic orbit e and a hyperbolic orbit h .

We choose specific α and perturbations α_{ϵ} as follows: Fix a real constant $L > 0$ such that no Reeb orbit of α has α -action equal to L and let $\mathcal{N}_1, \dots, \mathcal{N}_n$ be the Morse–Bott families consisting of simple orbits with α -action less than L . On the small neighborhood $T^2 \times [-\nu, \nu]$ of $T_{\mathcal{N}_i}$, we set

$$(4-1-1) \quad \alpha = C dt + \delta(f dt + y d\theta), \quad \alpha_{\epsilon} = C dt + \delta(f_{\epsilon} dt + y d\theta),$$

where $\epsilon > 0$ and $\delta > 0$ are small, $C > 0$ is the action of the Reeb orbits of $\mathcal{N}_i$, and:

- (P1) $f(y, \theta) = \pm \frac{1}{2}y^2$ and $f_{\epsilon}(y, \theta) = \pm(\frac{1}{2}y^2 + \epsilon\phi(y)\bar{g}_{\mathcal{N}}(\theta))$, where the sign $\pm$ depends on whether we have a negative or positive Morse–Bott torus.
- (P2) $\bar{g}_{\mathcal{N}}: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ is a perfect Morse function with maximum at $\frac{1}{4}$ and minimum at $-\frac{1}{4}$. More specifically, we assume that $\bar{g}'_{\mathcal{N}}(\theta)$ is 0 on $\theta = \pm\frac{1}{4}$, is linear with positive slope on $[-\frac{1}{4}, -\frac{1}{5}]$, is nondecreasing on $[-\frac{1}{5}, -\frac{1}{6}]$, and is equal to 1 on $[-\frac{1}{6}, \frac{1}{6}]$; and $\bar{g}_{\mathcal{N}}(\theta)$ is an odd function about $\theta = 0$.
- (P3) $\phi: [-\nu, \nu] \rightarrow [0, 1]$ is an even bump function with support on $[-a, a]$ and is equal to 1 on $[-b, b]$, where $\nu > a > b > 0$ are sufficiently small.

In particular, (P1) implies:

- (P4) As $\epsilon \rightarrow 0$, $f_{\epsilon} \rightarrow f$ in C^{∞} .

Proposition 4.1.2 *Let α be a Morse–Bott contact form. After a small modification of α near the $\mathcal{N}_i$, which we still call α , for every $L > 0$ there exist $\delta > 0$, $\nu > 0$ and $\epsilon > 0$ small such that:*

- (1) α_{ϵ} is L -nondegenerate and satisfies (4-1-1) and conditions (P1)–(P4).
- (2) Each $\mathcal{N}_i$ is perturbed into a pair of nondegenerate Reeb orbits e_i and h_i of α_{ϵ} -action less than L .
- (3) All multiples e_i^k and h_i^k of α_{ϵ} -action less than L have Conley–Zehnder indices 1 and 0 if $\mathcal{N}_i$ is positive, and -1 and 0 if $\mathcal{N}_i$ is negative.
- (4) All other orbits which are created have α_{ϵ} -action greater than L .

Here the Conley–Zehnder indices are computed with respect to the trivialization τ induced from $\mathcal{N}_i$.

Strictly speaking, we make the modification of α in [Proposition 4.1.2](#) so that it satisfies the conditions of [Lemma A.9.4](#).

Let $\mathcal{P}'$ be the set of simple nondegenerate orbits of R_{α} and let $\mathcal{P}_{\text{MB}} = \mathcal{P}' \cup (\bigcup_i \mathcal{N}_i)$ be the set of all simple Reeb orbits of R_{α} , where $\mathcal{N}_i$ denotes a Morse–Bott family of simple orbits. An orbit set γ for the Morse–Bott contact form α is an orbit set constructed from $\mathcal{P} = \mathcal{P}' \cup (\bigcup_i \{h_i, e_i\})$, where h_i is treated as a hyperbolic orbit (in particular, its multiplicity cannot be greater than one) and e_i is treated as an elliptic orbit.

4.2 Morse–Bott buildings

Let J be an almost complex structure on $\mathbb{R} \times M$ which is adapted to the Morse–Bott contact form α . We also assume the following:

- (*) For each Morse–Bott torus $T_{\mathcal{N}} = T^2$, J is invariant in the s -, t - and θ -directions on $\mathbb{R} \times T^2 \times [-v, v]$ and the projection of $J|_{\ker \alpha}$ to $(\mathbb{R}/\mathbb{Z}) \times [-v, v]$ with coordinates (θ, y) is the standard complex structure $\partial/\partial y \mapsto \partial/\partial \theta$.

(Strictly speaking, we require α and J to satisfy the conditions of [Lemma A.9.4](#); this can be arranged by a small modification near the Morse–Bott torus.)

Remark 4.2.1 In [\[Bourgeois et al. 2003\]](#), Morse–Bott compactness was proved for slightly different perturbations of α , namely for $f_\epsilon \alpha$, but it still holds in our case.

Although the notation is a bit cumbersome, consider the moduli space

$$\mathcal{M}_J(\gamma_1^+, \dots, \gamma_{i_1}^+; \mathcal{N}_1^+, \dots, \mathcal{N}_{i_2}^+; \gamma_1^-, \dots, \gamma_{i_3}^-; \mathcal{N}_1^-, \dots, \mathcal{N}_{i_4}^-),$$

abbreviated $\mathcal{M}_J(\gamma^+, \mathcal{N}^+, \gamma^-, \mathcal{N}^-)$, of J -holomorphic maps u in $\mathbb{R} \times M$ which have positive ends at orbits $\gamma_1^+, \dots, \gamma_{i_1}^+, \tilde{\gamma}_1^+, \dots, \tilde{\gamma}_{i_2}^+$ and negative ends at orbits $\gamma_1^-, \dots, \gamma_{i_3}^-, \tilde{\gamma}_1^-, \dots, \tilde{\gamma}_{i_4}^-$, where $\gamma_i^\pm$ covers a simple orbit in $\mathcal{P}'$ with multiplicity $l_i^\pm \geq 1$ and $\tilde{\gamma}_i^\pm$ covers a simple orbit in the Morse–Bott family $\mathcal{N}_i^\pm$ with multiplicity $k_i^\pm \geq 1$.

We say that J satisfying (*) is *Morse–Bott regular* if, for all data $\gamma^+, \mathcal{N}^+, \gamma^-$ and $\mathcal{N}^-$ and any $u \in \mathcal{M}_J(\gamma^+, \mathcal{N}^+, \gamma^-, \mathcal{N}^-)$ which has no multiply covered components, $\mathcal{M}_J(\gamma^+, \mathcal{N}^+, \gamma^-, \mathcal{N}^-)$ is transversely cut out (and hence is a manifold) near u . Since it suffices to perturb J outside of the sufficiently small neighborhood $\mathbb{R} \times T^2 \times [-v, v]$, a generic J satisfying (*) is regular.

We now give the definition of a Morse–Bott building. See [\[Bourgeois et al. 2003, Section 11.2\]](#) for a similar definition.

Definition 4.2.2 Let γ and γ' be orbit sets constructed from $\mathcal{P}$. A *Morse–Bott building* $\tilde{u}$ consists of a set $\{u_i : F_i \rightarrow \mathbb{R} \times M \mid i = 1, \dots, n\}$ of holomorphic maps with possibly disconnected domains F_i and a set $\{\delta_{i,j} \mid i = 0, \dots, n, j = 1, \dots, j_i\}$ of gradient flow lines in $\bigcup_k \mathcal{N}_k$ such that the following hold:

- (a) For $i = 1, \dots, n-1$, the negative ends $\mathcal{E}_{i,j}^-$ of u_i are paired with positive ends $\mathcal{E}_{i+1,j'}^+$ of u_{i+1} . Paired ends $(\mathcal{E}_{i,j}^-, \mathcal{E}_{i+1,j'}^+)$ are asymptotic to $k_{i,j}$ -fold covers of simple orbits $(\gamma_{i,j}^-, \gamma_{i+1,j'}^+)$ in the same Morse–Bott family and $\delta_{i,j}$ is a gradient flow line from $\gamma_{i,j}^-$ to $\gamma_{i+1,j'}^+$. (Here $\delta_{i,j}$ can be viewed as a $k_{i,j}$ -fold unbranched cover of a cylinder connecting $\gamma_{i,j}^-$ to $\gamma_{i+1,j'}^+$.)
- (b) Positive ends $\mathcal{E}_{1,j}^+$ of u_1 and negative ends $\mathcal{E}_{n,j}^-$ of u_n which are asymptotic to Reeb orbits in $\bigcup_k \mathcal{N}_k$ are augmented by gradient flow lines $\delta_{0,j}$ and $\delta_{n,j}$ connecting the orbit from/to a critical point of the appropriate Morse function $\bar{g}_{\mathcal{N}_k}$ determined by γ or γ' .

- (c) A nondegenerate orbit is considered as a Morse–Bott family consisting of a single point, and in this case the gradient flow line has length zero.

Given two orbit sets γ and γ' constructed from $\mathcal{P}$, the set of Morse–Bott buildings $\tilde{u}$ from γ to γ' will be denoted by $\mathcal{M}_J^{\text{MB}}(\gamma, \gamma')$.

The collection of maps u_i will be called the *holomorphic part* of the building. The restriction of any map u_i to a connected component of its domain will be called an *irreducible holomorphic component* of $\tilde{u}$.

Definition 4.2.3 A Morse–Bott building $\tilde{u}$ from γ to γ' is *simply covered* if every multiply covered irreducible holomorphic component of $\tilde{u}$ is either

- (i) a branched cover of a trivial cylinder over a simple orbit in $\mathcal{P}$; or
- (ii) an unbranched cover of a trivial cylinder over a simple orbit in $\mathcal{P}_{\text{MB}} - \mathcal{P}$.

Note that this definition allows connectors over the orbits e and h of every Morse–Bott torus, but not connectors over any other Morse–Bott orbit, which would necessarily break a gradient flow line. This second type of connector would make gluing more complicated.

4.3 ECH and Fredholm indices

In this subsection we define the ECH and Fredholm indices of a Morse–Bott building.

Definition 4.3.1 The ECH index $I(\gamma, \gamma', Z)$ in the Morse–Bott setting is defined, as in the nondegenerate case, as

$$I(\gamma, \gamma', Z) = c_1(\xi|_Z, \tau) + Q_\tau(Z) + \tilde{\mu}_\tau(\gamma) - \tilde{\mu}_\tau(\gamma'),$$

where the symmetric Conley–Zehnder indices of γ and γ' are computed with the convention that $\mu_\tau(e_i^j) = 1$ for all j and $\mu_\tau(h_i) = 0$ if $\mathcal{N}_i$ is a positive Morse–Bott family and $\mu_\tau(h_i) = 0$ and $\mu_\tau(e_i^j) = -1$ for all j if $\mathcal{N}_i$ is a negative Morse–Bott family. Here $\tau|_{\mathcal{N}_i}$ is the trivialization defined by $\mathcal{N}_i$.

Remark 4.3.2 The ECH index computed with this definition coincides with the limit of ECH indices computed with respect to nondegenerate perturbations α_ϵ of the Morse–Bott contact form α as $\epsilon \rightarrow 1$.

As in the nondegenerate case, a Morse–Bott building $\tilde{u}$ from γ to γ' determines a relative homology class $Z \in H_2(M, \gamma, \gamma')$ which is obtained from projecting the holomorphic part to M and gluing the annuli corresponding to the gradient trajectories. In view of this construction, we will often write $I(\tilde{u})$ for $I(\gamma, \gamma', Z)$.

We can also define the Fredholm index of a Morse–Bott building as follows. To a building $\tilde{u}$ we associate a map $u_\# : F_\# \rightarrow \mathbb{R} \times M$ by cutting the ends of the holomorphic components of u and connecting them with

cylinders corresponding to the gradient trajectories. Then the Fredholm index of a Morse–Bott building $\tilde{u}$ which is positively asymptotic to Reeb orbits $\gamma_i^{m_{ij}}$ and negatively asymptotic to Reeb orbits $(\gamma'_i)^{m'_{ij}}$ is

$$(4-3-1) \quad \text{ind}(\tilde{u}) = -\chi(F_{\#}) + 2c_1(u_{\#}^* \xi, \tau) + \sum_{ij} \mu_{\tau}(\gamma_i^{m_{ij}}) - \sum_{ij} \mu_{\tau}((\gamma'_i)^{m'_{ij}}),$$

with the same convention for the Conley–Zehnder indices of h_i , e_i and their iterates as in [Definition 4.3.1](#). (See [\[Bourgeois 2002, Corollary 5.4\]](#).)

4.4 Morse–Bott chain complex

In this subsection we introduce a Morse–Bott version of the ECH chain complex. Due to technical difficulties concerning non-simply covered Morse–Bott buildings, we will develop an ECH Morse–Bott theory only for special Morse–Bott contact forms, which we call *nice*.

- Definition 4.4.1** (1) A Morse–Bott building $\tilde{u}$ is *nice* if its holomorphic part has at most one irreducible component which is not a connector. This irreducible component will be called the *principal part* of $\tilde{u}$.
- (2) A Morse–Bott building $\tilde{u}$ is *very nice* if it is nice and every irreducible component besides the principal part is an unbranched cover of a trivial cylinder.
- (3) A Morse–Bott contact form α on M is *nice* if, for a generic almost complex structure J , all J -holomorphic Morse–Bott buildings of ECH index $I = 1$ in the symplectization of (M, α) are nice.

Remark 4.4.2 We will consider contact forms on manifolds with torus boundary which are non-degenerate on the interior and Morse–Bott on the boundary. Such contact forms are automatically nice (see [Lemma 7.1.2](#)). It is not clear whether nice contact forms with $\bigcup_i \mathcal{N}_i \neq \emptyset$ exist on closed manifolds.

Now we describe the relation between moduli spaces of J -holomorphic Morse–Bott buildings for a Morse–Bott contact form α and moduli spaces of holomorphic maps for generic perturbations of α following [\[Bourgeois 2002\]](#). Our statement will be weaker than that of [\[Bourgeois 2002\]](#) because we are going to state only what can be proved without resorting to abstract perturbations.

Let J_0 be a Morse–Bott regular almost complex structure on $\mathbb{R} \times M$ adapted to α , and let J_{ϵ} be almost complex structures on $\mathbb{R} \times M$ adapted to the contact forms α_{ϵ} in [Proposition 4.1.2](#) such that:

- (**) For each Morse–Bott torus $T_{\mathcal{N}} = T^2$, J_{ϵ} is invariant in the s - and t -directions on $\mathbb{R} \times T^2 \times [-\nu, \nu]$ and the projection of $J_{\epsilon}|_{\ker \alpha}$ to $(\mathbb{R}/\mathbb{Z}) \times [-\nu, \nu]$ with coordinates (θ, y) is the standard complex structure $\partial/\partial y \mapsto \partial/\partial \theta$.

In particular, $\lim_{\epsilon \rightarrow 0} J_{\epsilon} = J_0$ in the C^{∞} -topology.

Theorem 4.4.3 *Let α be a Morse–Bott contact form on M . Fix $L > 0$. Then there exist δ, ν, ϵ and α_ϵ as in Proposition 4.1.2, J_0 Morse–Bott regular satisfying $(*)$, and α_ϵ -adapted regular J_ϵ satisfying $(**)$ as in the previous paragraph such that, for all orbit sets $\gamma, \gamma' \in \mathcal{P}$ with action less than L , the following holds:*

- (1) *For all sequences $\epsilon_i \rightarrow 0$ and $u_i \in \mathcal{M}_{J_{\epsilon_i}}(\gamma, \gamma')$, there is a subsequence u_{i_k} which converges to a Morse–Bott building in $\mathcal{M}_{J_0}^{\text{MB}}(\gamma, \gamma')$.*
- (2) *If $\tilde{u}$ is a very nice, simply covered Morse–Bott building, then there is a J_ϵ -holomorphic map $u_\epsilon \in \mathcal{M}_{J_\epsilon}(\gamma, \gamma')$ which is “close to breaking” into $\tilde{u}$ and a curve $u_{\mathcal{T}_0^\epsilon}$ corresponding to a gradient trajectory $\mathcal{T}_0^\epsilon$ of f_ϵ along $y = 0$.*
- (3) *If $\text{ind}(\tilde{u}) = 1$, then the mod 2 algebraic count of $[u_\epsilon] \in \mathcal{M}_{J_\epsilon}(\gamma, \gamma')/\mathbb{R}$ that are “close to breaking” into $\tilde{u}$ and $u_{\mathcal{T}_0^\epsilon}$ is one.*
- (4) *If $\text{ind}(\tilde{u}) = 2$ and $\tilde{u}$ passes through a generic point $z \in \mathbb{R} \times M$, then the mod 2 algebraic count of $u_\epsilon \in \mathcal{M}_{J_\epsilon}(\gamma, \gamma')$ that are “close to breaking” into $\tilde{u}$ and $u_{\mathcal{T}_0^\epsilon}$ and passing through z is one.*

Proof (1) follows from Morse–Bott SFT compactness [Bourgeois et al. 2003; Bourgeois 2002]. The proofs of (2) and (3) are given in the appendix; (4) is similar. $\square$

Lemma 4.4.4 *Let J be a Morse–Bott regular almost complex structure and let $\tilde{u} \in \mathcal{M}_J^{\text{MB}}(\gamma, \gamma')$ be a very nice Morse–Bott building with $I(\tilde{u}) = 1$. Then $\tilde{u}$ is simply covered and $\text{ind}(\tilde{u}) = 1$.*

Proof Assume that $\tilde{u}$ has no trivial cylinders. In the general case, removing the trivial cylinders of $\tilde{u}$ might decrease the ECH index by [Hutchings 2009, Theorem 5.1] and positivity of intersection, but the same argument holds.

We first consider the case when the principal part u of $\tilde{u}$ is nonempty. Suppose that u is a k -fold branched cover of a nontrivial simply covered J -holomorphic curve v . Let $\tilde{v}$ be the Morse–Bott building obtained by augmenting v with gradient trajectories. If the functions $\tilde{g}_N$ are chosen generically, then $\text{ind}(\tilde{v}) > 0$ by the regularity of J . Since $\tilde{v}$ is a very nice simply covered J -holomorphic building, by Theorem 4.4.3(2) we can perturb it to a J_ϵ -holomorphic map v_ϵ for ϵ small. Then $I(v_\epsilon) \geq \text{ind}(v_\epsilon) > 0$ by the ECH index inequality (Theorem 2.3.3), so $I(\tilde{v}) > 0$. Consider the J_ϵ -holomorphic curve v_ϵ^k given by k translated copies of v_ϵ . Since both $\tilde{u}$ and v_ϵ^k represent the same relative homology class in $H_2(M, \gamma, \gamma')$, we have $I(\tilde{u}) = I(v_\epsilon^k)$. Since $I(v_\epsilon^k) \geq kI(v_\epsilon)$ by [Hutchings 2009, Theorem 5.1 and Proposition 5.6], it follows that $I(\tilde{u}) \geq k$. Hence $k = 1$ and u is simply covered.

Next let u_ϵ be the simply covered J_ϵ -holomorphic map which corresponds to $\tilde{u}$ under an arbitrarily small generic perturbation of the Morse–Bott contact form by Theorem 4.4.3(2). Clearly $I(u_\epsilon) = I(\tilde{u}) = 1$, so $\text{ind}(u_\epsilon) = 1$ and $\text{ind}(\tilde{u}) = 1$. This implies the lemma when the principal part of $\tilde{u}$ is nonempty.

If the principal part is empty, then $\tilde{u}$ consists of a gradient trajectory on a Morse–Bott family and a gradient trajectory has ECH index one. $\square$

Lemma 4.4.5 *Let α be a nice Morse–Bott contact form. If we fix a regular almost complex structure J_0 adapted to α , then, for any orbit sets γ and γ' and any $\epsilon > 0$ sufficiently small, there is a bijection*

$$\mathcal{M}_{J_0}^{\text{MB}, I=1, \text{vn}}(\gamma, \gamma')/\mathbb{R} \simeq \mathcal{M}_{J_\epsilon}^{I=1, \text{tn}}(\gamma, \gamma')/\mathbb{R}.$$

Here the modifier vn stands for “very nice” and the modifier tn means that all the connectors are trivial cylinders.

Proof By Theorem 4.4.3 and Lemma 4.4.4, every very nice $I = 1$, J_0 -holomorphic Morse–Bott building can be deformed into an $I = 1$, J_ϵ -holomorphic map all of whose connectors are trivial cylinders.

It remains to show that every sequence v_i of J_{ϵ_i} -holomorphic maps with $I(v_i) = 1$ and trivial cylinders as connectors converges to a very nice J_0 -holomorphic Morse–Bott building $\tilde{u}$ as $\epsilon_i \rightarrow 0$, after possibly passing to a subsequence. Suppose without loss of generality that the domains of the maps v_i are connected. (Indeed, since $I(v_i) = 1$, discarding the possible trivial cylinders does not change $I(v_i)$ by [Hutchings 2009, Theorem 5.1] and positivity of intersection.) By Theorem 4.4.3(1), the sequence v_i converges to a Morse–Bott building $\tilde{u}$ with $I(\tilde{u}) = 1$. Since α is a nice Morse–Bott contact form, the holomorphic part of $\tilde{u}$ has at most one irreducible component which is not a connector. Assume there is a nontrivial principal part u_0 ; the case of $u_0 = \emptyset$ is simpler and is left to the reader. We consider the very nice Morse–Bott building $\tilde{u}'$ obtained by augmenting the Morse–Bott ends of u_0 with gradient flow trajectories to the critical points of the Morse functions on the Morse–Bott tori. Then $I(\tilde{u}') = I(\tilde{u}) = 1$ because they represent the same relative homology class, and therefore Lemma 4.4.4 implies that u_0 is simply covered.

We claim that every other irreducible component is a trivial cylinder over an orbit in $\mathcal{P}$. Arguing by contradiction, suppose there are nontrivial connectors that are connected to u_0 by one or more finite-length gradient flow trajectories. We will show that $\text{ind}(\tilde{u}) > 1$, which contradicts the fact that $\tilde{u}$ is the limit of curves v_i with $\text{ind}(v_i) = 1$. To this end we consider the Morse–Bott building $\tilde{u}'$ defined above. We recall that $\tilde{u}'$ is very nice, simply covered and satisfies $I(\tilde{u}') = I(\tilde{u}) = 1$.

The ends of the building $\tilde{u}$ satisfy the incoming/outcoming partitions because $\tilde{u}$ is the limit of J_{ϵ_i} -holomorphic maps v_i , while the ends of the building $\tilde{u}'$ satisfy the incoming/outgoing partitions because $\tilde{u}'$ can be deformed to J_{ϵ_i} -holomorphic maps v'_i for $i \gg 0$ by Theorem 4.4.3(2).

We now make the simplifying hypothesis that u_0 has ends only at one Morse–Bott torus. (The general case is more complicated only in the notation.) Then the ends of $\tilde{u}$ and $\tilde{u}'$ differ only for the multiplicity of e . We denote by n_+ and n_- the positive and negative multiplicities of e in $\tilde{u}$, respectively, and by n'_+ and n'_- the corresponding multiplicities in $\tilde{u}'$. Moreover, we denote by $\mu(e, n_\pm)$ and $\mu(e, n'_\pm)$ the contributions of ends at e to the Fredholm indices of $\tilde{u}$ and $\tilde{u}'$, respectively. (We recall that these contributions are determined by the total multiplicities because $\tilde{u}$ and $\tilde{u}'$ satisfy the incoming/outgoing partition conditions.) We observe that $n_\pm \geq n'_\pm$ and $n_+ - n'_+ = n_- - n'_-$.

Let F be the domain of v_i and F' the domain of v'_i for $i \gg 0$. Then, by the Fredholm index formula (4-3-1),

$$\text{ind}(\tilde{u}) - \text{ind}(\tilde{u}') = -(\chi(F) - \chi(F')) + (\mu(e, n_+) - \mu(e, n_-)) - (\mu(e, n'_+) - \mu(e, n'_-)).$$

The term $\chi(F) - \chi(F')$ is the sum of the Euler characteristics of the connector components of $\tilde{u}$, so $-(\chi(F) - \chi(F')) > 0$ if $\tilde{u}$ is not very nice. We claim that $(\mu(e, n_+) - \mu(e, n_-)) - (\mu(e, n'_+) - \mu(e, n'_-))$ is always nonnegative. To see this, first we compute the contributions of the ends at e to the Fredholm index. If the Morse–Bott torus is positive, then

$$\mu(e, n_+) = n_+, \quad \mu(e, n_-) = \begin{cases} 0 & \text{if } n_- = 0, \\ 1 & \text{if } n_- > 0. \end{cases}$$

On the other end, if the Morse–Bott torus is negative, then

$$\mu(e, n_+) = \begin{cases} 0 & \text{if } n_+ = 0, \\ -1 & \text{if } n_+ > 0, \end{cases} \quad \mu(e, n_-) = -n_-.$$

Similar formulas hold for $\mu(e, n'_\pm)$.

Now we focus on the case of a positive Morse–Bott torus. (The negative case is completely symmetric.) Then $(\mu(e, n_+) - \mu(e, n_-)) - (\mu(e, n'_+) - \mu(e, n'_-)) = n_+ - n'_+ \geq 0$ if $n_-, n'_- > 0$ or $n_- = n'_- = 0$. On the other hand, if $n_- > 0$ but $n'_- = 0$, we have $(\mu(e, n_+) - \mu(e, n_-)) - (\mu(e, n'_+) - \mu(e, n'_-)) = n_+ - n'_+ - 1$. However, in this case, $n_+ - n'_+ = n_- - n'_- > 0$, so $(\mu(e, n_+) - \mu(e, n_-)) - (\mu(e, n'_+) - \mu(e, n'_-)) \geq 0$.

This proves that, if $\tilde{u}$ is not very nice, $\text{ind}(\tilde{u}) > \text{ind}(\tilde{u}')$. This is a contradiction because $\text{ind}(\tilde{u}) = 1$, as $\tilde{u}$ is a limit of $\text{ind} = 1$ maps v_i , and $\text{ind}(\tilde{u}) > 0$ since J_0 is a Morse–Bott regular almost complex structure. $\square$

Definition 4.4.6 Let α be a nice Morse–Bott contact form and J a Morse–Bott regular almost complex structure adapted to the symplectization of α . Then the Morse–Bott chain complex $(\text{ECC}_{\text{MB}}(M, \alpha, J), \partial_{\text{MB}})$ is generated by orbit sets constructed from $\mathcal{P}$ and the differential counts very nice Morse–Bott buildings with $I(\tilde{u}) = 1$. We denote by $\text{ECC}_{\text{MB}}^L(M, \alpha, J)$ the subcomplex generated by orbit sets of action less than L .

Proposition 4.4.7 Let α be a nice Morse–Bott contact form. If no Reeb orbit of α has action equal to a fixed $L > 0$, then there is an isomorphism of chain complexes

$$\text{ECC}^L(M, \alpha_\epsilon, J_\epsilon) \simeq \text{ECC}_{\text{MB}}^L(M, \alpha, J_0)$$

for all sufficiently small $\epsilon > 0$. In particular, $\partial_{\text{MB}}^2 = 0$.

Proof The isomorphism follows from Lemma 4.4.5. $\square$

4.5 Comparison with the nondegenerate case

In this subsection we use a direct limit argument to prove the isomorphism between ECH of a nondegenerate contact form and Morse–Bott ECH of a nice Morse–Bott form α .

Let $L_i \rightarrow \infty$ be an increasing sequence such that each L_i is positive and there is no Reeb orbit of α with action equal to L_i . Let $\mathcal{N}_1, \dots, \mathcal{N}_{n(i)}$ be the Morse–Bott families consisting of simple orbits with α -action $< L_i$. (In many useful cases, $\lim_{i \rightarrow +\infty} n(i) = +\infty$.)

The following lemma provides a sequence of perturbing functions and is an immediate corollary of [Proposition 4.1.2](#) and [Theorem 4.4.3](#):

Lemma 4.5.1 *Let α be a nice Morse–Bott form and let L_i be a sequence of positive constants such that $L_i \rightarrow +\infty$ and no Reeb orbit of α has action equal to L_i . There exist sequences of positive numbers $\epsilon_i \rightarrow 0$ and functions $g_i : M \rightarrow \mathbb{R}^{\geq 0}$ such that $f_i = 1 + g_i$ and:*

- (1) g_i is supported in a union of disjoint neighborhoods of $T_{\mathcal{N}_1}, \dots, T_{\mathcal{N}_{n(i)}}$.
- (2) The support of g_i is disjoint from all nondegenerate Reeb orbits of α of α -action $< L_i$.
- (3) On a sufficiently small neighborhood $T^2 \times [-\epsilon, \epsilon]$ of $T_{\mathcal{N}_j}$ for $j = 1, \dots, n(i)$, there exist precisely two simple orbits of $f_i \alpha$ of action $\leq L_i$, corresponding to elliptic and hyperbolic orbits of the perturbed Morse–Bott family.
- (4) $\lim_{i \rightarrow +\infty} f_i = 1$ in the C^k -topology for $k \gg 0$.
- (5) For every i , the contact form $f_i \alpha$ satisfies conditions (1)–(4) of [Proposition 4.1.2](#) and the conclusion of [Theorem 4.4.3](#) for orbits of action $\leq L_i$.
- (6) $f_i \alpha$ (resp. $f_{i+1} \alpha$) has no Reeb orbits with $f_i \alpha$ -action (resp. $f_{i+1} \alpha$ -action) in $[a_i^{-2} L_i, a_i^2 L_i]$, where $a_i = (1 + \epsilon_i c_0)^2$ for some constant $c_0 > 0$.

Warning 4.5.2 For all i , Morse–Bott theory (and in particular [Proposition 4.4.7](#)) gives injections $\text{ECC}^{L_i}(M, f_i \alpha) \rightarrow \text{ECC}^{L_{i+1}}(M, f_{i+1} \alpha)$. However, the maps induced in homology by these injections a priori could be different from the canonical maps given in [Lemma 3.1.7](#), and it is with respect to the latter that the direct limit must be taken. (A posteriori, they are shown to be the same in the proof of [Theorem 4.5.9](#).)

Observe that $a_i^{-1} f_i < f_{i+1} < a_i f_i$ for all i . Then [Lemma 3.1.7](#) gives maps

$$\begin{aligned} \Phi_+ : \text{ECH}^{L_i}(M, f_i \alpha) &\rightarrow \text{ECH}^{a_i L_i}(M, f_{i+1} \alpha), \\ \Phi_- : \text{ECH}^{a_i L_i}(M, f_{i+1} \alpha) &\rightarrow \text{ECH}^{a_i^2 L_i}(M, f_i \alpha), \\ \Phi'_- : \text{ECH}^{a_i^{-1} L_i}(M, f_{i+1} \alpha) &\rightarrow \text{ECH}^{L_i}(M, f_i \alpha). \end{aligned}$$

Lemma 4.5.3 *The map Φ_+ is an isomorphism.*

Proof By [Theorems 3.1.2\(ii\)](#) and (iv), the composition

$$\text{ECH}^{L_i}(M, a_i f_{i+1} \alpha) \rightarrow \text{ECH}^{L_i}(M, f_i \alpha) \rightarrow \text{ECH}^{L_i}(M, a_i^{-1} f_{i+1} \alpha)$$

is equal to the cobordism map induced by a piece of symplectization. Then, by [Theorem 3.1.2\(vi\)](#), it is a composition of a scaling with an inclusion. From this and [Lemma 4.5.1\(9\)](#), it follows easily that $\Phi_+ \circ \Phi'_- = \text{id}$. Similarly, $\Phi_- \circ \Phi_+ = \text{id}$. Hence Φ_+ is an isomorphism. $\square$

Let $([0, 1] \times M, d\lambda_i)$ be an interpolating cobordism from $f_i\alpha$ to $a_i^{-1}f_{i+1}\alpha$ and $(\mathbb{R} \times M, d\hat{\lambda}_i)$ its completion. Let $\tilde{J}_i$ be a regular almost complex structure on $(\mathbb{R} \times M, d\hat{\lambda}_i)$ which is $d\lambda_i$ -compatible and adapted to the symplectizations of $f_i\alpha$ and $a_i^{-1}f_{i+1}\alpha$ at the ends. We denote the moduli space of $\tilde{J}_i$ -holomorphic buildings in $(\mathbb{R} \times M, d\hat{\lambda}_i)$ from γ to γ' by $\mathcal{M}_{\tilde{J}_i}^b(\gamma, \gamma')$.

The following lemma, stated without proof, is a consequence of the Morse–Bott compactness theorem [Bourgeois 2002] and the triviality of $I < 0$ moduli spaces in symplectizations.

Lemma 4.5.4 *If $\epsilon_i > 0$ is sufficiently small, then there is a regular almost complex structure $\tilde{J}_i$ such that, if γ and γ' have $f_i\alpha$ -actions less than L_i , then $\mathcal{M}_{\tilde{J}_i}^{b, I=0}(\gamma, \gamma')$ and $\mathcal{M}_{\tilde{J}_i}^{b, I=0}(\gamma', \gamma)$ are empty if $\gamma \neq \gamma'$ and consist of branched covers of trivial holomorphic cylinders if $\gamma = \gamma'$.*

By Morse–Bott theory, there is an identification of complexes

$$e: \text{ECC}^{L_i}(M, f_i\alpha, J_i) \xrightarrow{\cong} \text{ECC}^{a_i L_i}(M, f_{i+1}\alpha, J_{i+1}).$$

In fact, $\text{ECC}^{L_i}(M, f_i\alpha)$ and $\text{ECC}^{a_i L_i}(M, f_{i+1}\alpha)$ are generated by the same orbit sets and the moduli spaces of $I = 1$ holomorphic curves (modulo $\mathbb{R}$ -translations) have the same cardinality, by Lemma 4.5.1 and Proposition 4.4.7. Let e_* be the map induced by e on homology.

Proposition 4.5.5 $e_* = \Phi_+$.

Proof Let

$$\hat{\Phi}_+: \text{ECC}^{L_i}(M, f_i\alpha, J_i) \rightarrow \text{ECC}^{a_i L_i}(M, f_{i+1}\alpha, J_{i+1})$$

be a (noncanonical) chain map which induces Φ_+ and is given by Theorem 3.1.2 and Lemma 3.1.7. Theorem 3.1.2(i) and Lemma 4.5.4 imply that $\hat{\Phi}_+$ is a diagonal map. Note that $\text{ECC}^{L_i}(M, f_i\alpha)$ and $\text{ECC}^{a_i L_i}(M, f_{i+1}\alpha)$ are generated by the same orbit sets. The reason why we cannot conclude that $\hat{\Phi}_+ = e$ by Theorem 3.1.2(i) is that some of the $I = 0$ holomorphic cylinders in the interpolating cobordism from $(M, a_i f_i\alpha)$ to $(M, f_{i+1}\alpha)$ are, strictly speaking, not contained in product regions.

For $\mathbb{F}$ -coefficients we can use the following algebraic trick to finish the proof: Identify $\text{ECC}^{a_i L_i}(M, f_{i+1}\alpha)$ with $\text{ECC}^{L_i}(M, f_i\alpha)$ via e^{-1} . Then

$$(e^{-1} \circ \hat{\Phi}_+) \circ (e^{-1} \circ \hat{\Phi}_+) = e^{-1} \circ \hat{\Phi}_+$$

over $\mathbb{F}$. Since Φ_+ and e_* are isomorphisms, it follows that $e_*^{-1} \circ \Phi_+ = \text{id}$ and $\Phi_+ = e_*$. □

Now we give a sketch of the proof of Proposition 4.5.5 which applies to integer coefficients. The uninterested reader can jump directly to Theorem 4.5.9. Given a pair (λ, J) consisting of a nondegenerate contact form λ and a compatible J , Taubes [2010] first perturbs (λ, J) into an L -flat pair (λ', J') before identifying $\text{ECH}^L(\lambda', J')$ with Seiberg–Witten Floer cohomology. A pair (λ', J') is L -flat if

near each Reeb orbit of length $< L$ it satisfies the conditions in [Taubes 2010, equation (4-1)], and L -flat perturbations are constructed in [Taubes 2010, Proposition 2.5 and Appendix]. (See [Taubes 2010, Section 5.c, Part 2] for the reasons for introducing the L -flat condition.)

The following lemma is a slight rephrasing of [Hutchings and Taubes 2013, Lemma 3.4(d)] and will not be proved:

Lemma 4.5.6 *If (λ^t, J^t, L^t) for $t \in [0, 1]$ is a 1-parameter family and (λ^t, J^t) is L^t -flat, λ^t is L^t -nondegenerate, and J^t is L^t -regular (ie Definition 2.2.1 holds for all γ and γ' with $\mathcal{A}_{\lambda^t}(\gamma) < L^t$) for all $t \in [0, 1]$, then the ECH cobordism map*

$$\text{ECH}^{L^0}(M, \lambda^0) \rightarrow \text{ECH}^{L^1}(M, \lambda^1)$$

is induced by the isomorphism

$$\text{ECC}^{L^0}(M, \lambda^0, J^0) \xrightarrow{\sim} \text{ECC}^{L^1}(M, \lambda^1, J^1)$$

given by the canonical bijection of generators.

Setting $\lambda^0 = f_i \alpha$ and $\lambda^1 = f_{i+1} \alpha$, it is easy to find an extension λ^t for $t \in [0, 1]$ of the form $f_{i+1}^t \alpha$, where $f_{i+1}^0 = f_i$, $f_{i+1}^1 = f_{i+1}$, and f_{i+1}^t satisfies the conditions of Lemma 4.5.1 with f_{i+1} replaced by f_{i+1}^t . By choosing f_{i+1}^t to be sufficiently close to 1 and applying Lemma 4.4.5, there exist an extension L^t for $t \in [0, 1]$ of $L^0 = L_i$ and $L^1 = a_i L_i$ and an extension J^t for $t \in [0, 1]$ of $J^0 = J_i$ and $J^1 = J_{i+1}$ such that J^t is adapted to λ^t and is L^t -regular.

Next we fix a Riemannian metric on M , with respect to which we measure distances. Assume for simplicity that there is a unique Morse–Bott torus $T_{\mathcal{N}}$. Let γ_e and γ_h be the elliptic and hyperbolic orbits of λ^t which are obtained by perturbing $T_{\mathcal{N}}$, where we assume that $\gamma_e \sqcup \gamma_h$ is independent of $t \in [0, 1]$. For each $\varepsilon > 0$ sufficiently small, we construct an L^t -flat family ($t \in [0, 1]$) of perturbations $(\lambda^{t,\varepsilon}, J^{t,\varepsilon})$ of (λ^t, J^t) which are supported on an ε -neighborhood of $\gamma_e \sqcup \gamma_h$. Moreover, $(\lambda^{t,\varepsilon}, J^{t,\varepsilon})$ converges (uniformly in $t \in [0, 1]$) to (λ^t, J^t) in the C^0 -topology as $\varepsilon \rightarrow 0$. The proof is a 1-parameter version of the construction of L -flat perturbations in [Taubes 2010, Proposition 2.5 and Appendix] and will be omitted.

Claim 4.5.7 *For $\varepsilon > 0$ sufficiently small, $J^{t,\varepsilon}$ is L^t -regular for all $t \in [0, 1]$.*

Proof We may assume that $J^{t,\varepsilon}$ for $t \in [0, 1]$ is a generic 1-parameter family of almost complex structures. Arguing by contradiction, there exist orbit sets γ and γ' and sequences $\varepsilon_j \rightarrow 0$, $t_j \in [0, 1]$ and $u_j: F_j \rightarrow \mathbb{R} \times M$, where:

- (1) u_j is a somewhere-injective J^{t_j, ε_j} -holomorphic curve from γ to γ' .
- (2) γ and γ' are constructed from the nondegenerate orbits of α together with γ_e and γ_h and $\mathcal{A}_{\lambda^{t_j, \varepsilon_j}}(\gamma), \mathcal{A}_{\lambda^{t_j, \varepsilon_j}}(\gamma') < L^{t_j}$.
- (3) u_j is not a connector and $I(u_j) = \text{ind}(u_j) = 0$.

Claim 4.5.8 After passing to a subsequence, there exists an SFT limit $u_j \rightarrow u_\infty$, where $I(u_\infty) = \text{ind}(u_\infty) = 0$ and u_∞ is not a connector.

A sketch of [Claim 4.5.8](#) is given in [Section 5.2](#). Since u_∞ is a J^{t_0} -holomorphic curve and J^t is L_t -regular for all $t \in [0, 1]$ by [Lemma 4.4.5](#), we have a contradiction. This implies [Claim 4.5.7](#). $\square$

[Claim 4.5.7](#) and [Lemma 4.5.6](#) then imply [Proposition 4.5.5](#) for integer coefficients.

By passing to direct limits, we obtain the main result of Morse–Bott theory.

Theorem 4.5.9 Let α be a nice Morse–Bott form and J a generic almost complex structure adapted to the symplectization of α . Then

$$\text{ECH}_{\text{MB}}(M, \alpha, J) \simeq \text{ECH}(M).$$

Proof Choose sequences of functions $f_i : M \rightarrow \mathbb{R}$ and constants $L_i \rightarrow +\infty$ which satisfy the hypotheses of [Lemmas 4.5.1](#) and [4.5.4](#). Then

$$(4\text{-}5\text{-}1) \quad \text{ECH}(M) = \lim_{i \rightarrow \infty} \text{ECH}^{L_i}(M, f_i \alpha)$$

by [Corollary 3.2.3](#) and

$$(4\text{-}5\text{-}2) \quad \text{ECC}^{L_i}(M, f_i \alpha, J_i) \simeq \text{ECC}_{\text{MB}}^{L_i}(M, \alpha, J)$$

for all i by [Proposition 4.4.7](#). Also, tautologically,

$$\text{ECC}_{\text{MB}}(M, \alpha, J) = \lim_{i \rightarrow \infty} \text{ECC}_{\text{MB}}^{L_i}(M, \alpha, J).$$

In order to take the direct limit on both sides of [\(4-5-2\)](#) on the level of homology, we need the commutativity of the diagram

$$\begin{array}{ccc} \text{ECH}_{\text{MB}}^{L_i}(M, \alpha) & \xrightarrow{\simeq} & \text{ECH}^{L_i}(M, f_i \alpha) \\ \downarrow & & \downarrow \\ \text{ECH}_{\text{MB}}^{L_{i+1}}(M, \alpha) & \xrightarrow{\simeq} & \text{ECH}^{L_{i+1}}(M, f_{i+1} \alpha) \end{array}$$

for all i , where the rightmost vertical arrow is the natural map defined in [Lemma 3.1.7](#) from interpolating cobordisms. This map coincides with Φ_+ followed by the map induced by the inclusion

$$\text{ECC}^{a_i L_i}(M, f_{i+1} \alpha) \hookrightarrow \text{ECC}^{L_{i+1}}(M, f_{i+1} \alpha).$$

Therefore the diagram commutes by [Proposition 4.5.5](#). $\square$

5 Topological constraints on holomorphic curves

5.1 The winding number

In this subsection we recall the *winding number* from [[Hofer et al. 1995](#), page 290]: Given a contact manifold (M, ξ) with $\xi = \ker \alpha$, an α -adapted almost complex structure J on $\mathbb{R} \times M$, and a J -holomorphic

curve $u: F \rightarrow \mathbb{R} \times M$ between orbits sets, the *winding number* $\text{wind}_\pi(u)$ is an algebraic count of the zeros of the section:

$$s: F \rightarrow \text{Hom}_{\mathbb{C}}(TF, u^*\xi).$$

Here s is obtained by composing

$$TF \xrightarrow{u_*} T(\mathbb{R} \times M) \xrightarrow{(\pi_M)_*} TM \xrightarrow{\pi} \xi,$$

where $\pi_M: \mathbb{R} \times M \rightarrow M$ is the projection onto the second factor and π is the projection along the Reeb vector field R_α .

Hofer, Wysocki and Zehnder [Hofer et al. 1995] prove that $\text{wind}_\pi(u)$ is finite. (This is analogous to the elementary complex analysis fact that the number of zeros of a holomorphic function $f: D^2 \subset \mathbb{C} \rightarrow \mathbb{C}$, counted with multiplicities, is equal to the winding number of $f|_{\partial D^2}$.) An immediate corollary is the following lemma:

Lemma 5.1.1 *The map $u_M = \pi_M \circ u$ is transverse to R_α away from a finite number of points on F . In particular, it is an immersion outside a finite number of points on F .*

Throughout the section we will use the notation $u_M = \pi_M \circ u$.

5.2 Blocking lemma

In this subsection we discuss the topological restrictions that a torus foliated by Reeb trajectories imposes on the J -holomorphic curves.

Let α be a contact form on M and $T \subset M$ an oriented torus which is linearly foliated by Reeb trajectories of α . The foliation can either have closed leaves or dense leaves. We denote by $\mathbb{P}^+ H_1(T; \mathbb{R})$ the quotient of $H_1(T; \mathbb{R}) - \{0\}$ by multiplication of positive real numbers. The Reeb flow on T will then have a well-defined “slope” $s \in \mathbb{P}^+ H_1(T; \mathbb{R})$.

Let $\langle \cdot, \cdot \rangle$ be the intersection pairing on $H_1(T; \mathbb{R})$. We then make the following definition:

Definition 5.2.1 If $\delta \in H_1(T; \mathbb{Z})$, then we write $\delta \cdot s > 0$ (resp. $\delta \cdot s = 0$) if $\langle \delta, \gamma \rangle > 0$ (resp. $= 0$) for any representative $\gamma \in H_1(T; \mathbb{R})$ of $s \in \mathbb{P}^+ H_1(T; \mathbb{R})$.

Note that, if $\delta \cdot s = 0$, then δ represents the slope s or $-s$.

Let $T^2 \times [-\varepsilon, \varepsilon]$ be a neighborhood of the Morse–Bott torus $T = T^2 \times \{0\}$ with coordinates (θ, t, y) . We assume that the normal vector to T points in the direction of ∂_y . Let $u: F \rightarrow \mathbb{R} \times M$ be a J -holomorphic curve such that

(C₁) F is a compact Riemann surface with boundary ∂F ; and

(C₂) $u_M(\partial F) \cap (T^2 \times [-\varepsilon, \varepsilon]) = \emptyset$.

Then $u_M(F) \cap T$ only has a finite number of singularities by [Lemma 5.1.1](#) and we denote by $\delta \in H_1(T; \mathbb{Z})$ the homology class of $u_M(F) \cap T$, where the smooth part of $u_M(F) \cap T$ is oriented as the boundary of $u_M(F) \cap (T^2 \times [-\varepsilon, 0])$.

Lemma 5.2.2 (positivity of intersections in dimension three) *Let $T \subset M$ be an oriented torus which is linearly foliated by Reeb trajectories of slope s . If $u: F \rightarrow \mathbb{R} \times M$ is a J -holomorphic curve satisfying (C_1) and (C_2) and $\delta = [u_M(F) \cap T] \in H_1(T; \mathbb{Z})$, then $\delta \cdot s \geq 0$. Moreover, $\delta \cdot s = 0$ if and only if $u_M(F) \cap T = \emptyset$.*

Proof We will prove this lemma in the harder case when T is foliated by orbits of irrational slope, leaving the rational slope case to the reader.

By [Lemma 5.1.1](#), $u_M(F) \cap T$, if nonempty, is the union of a finite set of points and curves which are immersed outside a finite number of singularities.

Assume first that $u_M(F) \cap T$ has a one-dimensional component. By abuse of notation, we do not distinguish between the homology class δ and its representative $u_M(F) \cap T$. A generic finite-length Reeb trajectory γ on T intersects δ in finitely many points away from the singularities and isolated points. In fact, $\delta \cap \gamma = \pi_M(u(F) \cap (\mathbb{R} \times \gamma))$ and $u(F) \cap (\mathbb{R} \times \gamma)$ is a finite set by the intersection theory of holomorphic curves in dimension four; see [\[Micallef and White 1995, Theorem 7.1\]](#). Since all Reeb trajectories are dense in T , we can choose γ arbitrarily long so that its endpoints are close to each other and far away from δ . Hence we can complete γ to a homologically nontrivial closed curve $\bar{\gamma}$ without introducing extra intersection points with δ . Then the positivity of intersections in dimension four implies that $\delta \cdot [\bar{\gamma}] > 0$. In particular, $\delta \neq 0 \in H_1(T; \mathbb{Z})$. Since we can make the slope of $\bar{\gamma}$ as close as we want to s by taking γ sufficiently long, and s is not an integral homology class, we conclude that $\delta \cdot s > 0$. (Recall that, if $\delta \cdot s = 0$, then δ and either s or $-s$ are parallel.)

Assume now that $u_M(F) \cap T$ is a finite set. We claim that $u_M(F) \cap T = \emptyset$. Suppose that $u_M(F) \cap T \neq \emptyset$ by contradiction. Repeating the construction from the previous paragraph with a finite-length Reeb trajectory γ (resp. γ') which passes through a point in $u_M(F) \cap T$ (resp. is disjoint from $u_M(F) \cap T$), we obtain $\bar{\gamma}$ and $\bar{\gamma}'$. Then $[u(F)] \cdot [\mathbb{R} \times \bar{\gamma}] > 0$ by the positivity of intersections, while $[u(F)] \cdot [\mathbb{R} \times \bar{\gamma}'] = 0$ because they are disjoint. Since $\mathbb{R} \times \bar{\gamma}$ and $\mathbb{R} \times \bar{\gamma}'$ are homologous, we have a contradiction. $\square$

The following is an immediate consequence of [Lemma 5.2.2](#):

Lemma 5.2.3 (blocking lemma) *Let $T \subset M$ be an oriented torus which is linearly foliated by Reeb trajectories of slope s and let $u: F \rightarrow \mathbb{R} \times M$ be a finite-energy J -holomorphic map, where F is a closed Riemann surface with a finite number of punctures removed. Then:*

- (1) *If u is homotopic, by a compactly supported homotopy, to a map whose image is disjoint from $\mathbb{R} \times T$, then $u_M(F) \cap T = \emptyset$.*

- (2) If T' is a torus which is parallel to and disjoint from T , u has no end that limits to a Reeb orbit that intersects the half-open region between T and T' which includes T' but not T , and the homology class $[u_M(F) \cap T']$ is nonzero and has slope $\pm s$, then u has an end which is asymptotic to a Reeb orbit in T .

We now sketch the proof of [Claim 4.5.8](#).

Sketch of proof of Claim 4.5.8 The consideration that needs slight care is that as $\varepsilon_j \rightarrow 0$, $J^{t_j, \varepsilon_j} \rightarrow J^{t_\infty}$ only in the C^0 -topology. Let $N_{\varepsilon_j}(\gamma_e \sqcup \gamma_h)$ be an ε_j -neighborhood of $\gamma_e \sqcup \gamma_h$ and let $F'_j = u_j^{-1}(\mathbb{R} \times (M - N_{\varepsilon_j}(\gamma_e \sqcup \gamma_h)))$. By the Gromov–Taubes compactness theorem [[Taubes 1998](#)], which requires no a priori bound on the genus and is local, there exists a limit u_∞ of $u_j|_{F'_j}$ as currents, after passing to a subsequence. This implies that the homology class $[u_j(F_j)]$ can be taken to be independent of j . The argument from [[Hutchings 2002](#), Theorem 10.1] then gives a bound on the genus of F_j .

We can then either appeal to the C^0 -Gromov compactness theorem of Ivashkovich and Shevchishin [[2000](#)] or argue as follows using the blocking lemma. We make the simplifying assumptions that γ and γ' do not contain γ_h and that u_j does not intersect neighborhoods of $\mathbb{R} \times \gamma_h$ and leave the harder general case to the reader.

We claim that $-\chi(F'_j)$ is bounded above. Since we have a genus bound for F_j , it suffices to show that the number $\#\partial F'_j$ of boundary components of F'_j is bounded above. Let $V'_j = N_{\varepsilon_j}(\gamma_e)$ and let $T'_j = \partial V'_j$ with the boundary orientation. Choose an oriented identification $T'_j \simeq \mathbb{R}^2/\mathbb{Z}^2$ such that the meridian has slope 0 and the longitude is determined by the Morse–Bott family and has slope ∞ . We may assume that T'_j is foliated by Reeb orbits of slope $s'_j \gg 0$ and that there exists a torus $T''_j \subset M - V'_j$ which is parallel to T'_j and is foliated by Reeb orbits of rational slope s'' , where s'' is independent of j and $s'_j > s'' > 0$.

Let $V''_j \subset M$ be the solid torus bounded by T''_j and let $F''_j = u_j^{-1}(\mathbb{R} \times (M - V''_j))$. Let $\pi_M: \mathbb{R} \times M \rightarrow M$ be the projection onto the second factor. By [Lemma 5.2.2](#), if C is a component of $\partial F''_j$, then $\pi_M \circ u_j(C) \cdot s' < 0$. Since $[u_j(F_j)]$ is fixed and s'' is rational, $\#\partial F''_j$ must then be bounded above. On the other hand, let $V_j^{(0)} \subset V'_j$ be a sufficiently small neighborhood of γ_e , $T_j^{(0)} = \partial V_j^{(0)}$, and $F_j^{(0)} = u_j^{-1}(\mathbb{R} \times (M - V_j^{(0)}))$. Since $[u_j(F_j)]$ is fixed, $\#\partial F_j^{(0)}$ is also bounded above by the positivity of intersections in dimension four and the asymptotics of u_j near their ends.

To obtain the bound on $\#\partial F'_j$, it then suffices to show that $u_j^{-1}(\mathbb{R} \times (V''_j - V'_j))$ and $u_j^{-1}(\mathbb{R} \times (V'_j - V_j^{(0)}))$ have no disk components D with $\pi_M \circ u_j(\partial D) \subset T'_j$. By [Lemma 5.2.2](#), $\pi_M \circ u_j(\partial D)$ represents a nonzero homology class in T'_j . On the other hand, the inclusion $T^2 \times \{1\} \rightarrow T^2 \times [0, 1]$ induces an isomorphism on homology, which is a contradiction. This proves the bound on $\#\partial F'_j$ and $-\chi(F'_j)$.

We then apply the SFT compactness theorem to $u_j|_{F'_j}$ to obtain $u_\infty: F_\infty \rightarrow \mathbb{R} \times M$. If C is a component of ∂F_∞ , then $u_\infty(C) \subset \gamma_e$, which in turn implies that u_∞ is a constant. Hence $\partial F_\infty = \emptyset$. The punctures of F_∞ are either removable or limit to orbits in γ and γ' . Finally, since $[u_j(F_j)]$ is not the class given by a connector, u_∞ is also not a connector. $\square$

5.3 Trapping lemma

In this subsection we analyze some topological restrictions on J -holomorphic curves with ends at a Morse–Bott torus.

We fix coordinates (θ, t, y) on a neighborhood $T^2 \times [-\varepsilon, \varepsilon]$ of a torus $T = T^2 \times \{0\}$ and consider contact forms of the type $\alpha = g(\theta, t, y) d\theta + f(\theta, t, y) dt$ such that:

- $f(\partial_y g) - (\partial_y f)g > 0$.
- $f|_{y=0} = 1$.
- $\partial_\theta f|_{y=0} = \partial_t f|_{y=0} = \partial_y f|_{y=0} = 0$.
- $\partial_\theta g|_{y=0} = \partial_t g|_{y=0} = 0$ and $\partial_y g|_{y=0} = 1$.
- $\partial_y^2 f|_{y=0} \neq 0$.

These conditions imply that T is a Morse–Bott torus and that the Reeb vector field R is given by ∂_t on T .

We recall that the asymptotic operator of a closed Reeb orbit γ describes the action of the linearized Reeb flow on sections of the (pullback of the) contact structure $\gamma^*\xi$ along the orbit. More precisely, the linearized Reeb flow gives a symplectic connection ∇^R for $\gamma^*\xi$ and the asymptotic operator is $J\nabla^R$, where J is an almost complex structure on ξ ; see [Hofer et al. 1996] for more details on the asymptotic operator and Section 4.1 for the linearized Reeb vector field.

If we choose a generic almost complex structure J adapted to the symplectization of α such that $\partial_t J|_{y=0} = 0$, then there is a unitary trivialization of ξ along T such that the asymptotic operator of an end of a holomorphic map converging to a Reeb orbit on T has the form

$$(5-3-1) \quad A = -J_0 \frac{d}{dt} + J_0 \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix},$$

where $a > 0$ if T is a positive Morse–Bott torus, $a < 0$ if T is a negative Morse–Bott torus, and $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This unitary trivialization is obtained by projecting $(\partial_y, \partial_\theta)$ to ξ along ∂_t .

Lemma 5.3.1 *The eigenvalues of the asymptotic operator A are $\lambda_0 = 0$, $\lambda_a = -a$, and λ_n and λ_{-n} for $n \in \mathbb{N}$, which are the positive and the negative solutions of the equation $\lambda(\lambda + a) = n^2$. The eigenfunctions that correspond to the eigenvalues λ_0 and λ_a are*

$$f_0(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad f_a(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvalues $\lambda_{\pm n}$ for $n \geq 1$ are degenerate with multiplicity 2 and their eigenfunctions have winding number $\pm n$.

Proof The asymptotic operator is sufficiently simple that we can determine its spectrum by an explicit computation: The eigenfunctions ξ of A are the 2π -periodic solutions of the differential equation

$$(5-3-2) \quad \dot{\xi} = \begin{pmatrix} 0 & -\lambda \\ \lambda + a & 0 \end{pmatrix} \xi.$$

If $\lambda = 0$ or $\lambda = -a$, which are the only cases when the matrix in (5-3-2) cannot be diagonalized, the eigenfunctions are

$$f_0(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad f_a(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If $\lambda(\lambda + a) < 0$, then (5-3-2) can be diagonalized over the real numbers, and it is easy to see that it has no periodic solutions. If $\lambda(\lambda + a) > 0$ a direct computation shows that solutions of (5-3-2) are of the form $\xi(t) = \Phi_\lambda(t)\xi_0$, where

$$\Phi_\lambda(t) = \begin{pmatrix} \cos(\sqrt{\lambda(\lambda+a)}t) & -\frac{\lambda}{\sqrt{\lambda(\lambda+a)}} \sin(\sqrt{\lambda(\lambda+a)}t) \\ \frac{\lambda+a}{\sqrt{\lambda(\lambda+a)}} \sin(\sqrt{\lambda(\lambda+a)}t) & \cos(\sqrt{\lambda(\lambda+a)}t) \end{pmatrix}.$$

Then $\Phi_\lambda(2\pi)$ has eigenvalue 1 if and only if $\lambda(\lambda + a) = n^2 \in \mathbb{N}$, in which case $\Phi_\lambda(2\pi)$ is the identity. $\square$

If u is a J -holomorphic map with an end $\mathcal{E}$ which is asymptotic to a Morse–Bott torus T , we say that $\mathcal{E}$ is *one-sided* if its projection to M does not intersect T .

Lemma 5.3.2 (trapping lemma) *Let α be a contact form, T an α -Morse–Bott torus, and $\mathcal{E}$ a one-sided end of a J -holomorphic map which is asymptotic to a Reeb orbit in T . If T is positive (resp. negative), then $\mathcal{E}$ is a positive (resp. negative) end.*

Proof Suppose T is positive. By [Hofer et al. 1996, Theorem 1.3], a positive (resp. negative) end $\mathcal{E}$ of a J -holomorphic curve approaches a Reeb orbit of T along an eigenfunction of the asymptotic operator with negative (resp. positive) eigenvalue. By Lemma 5.3.1, the eigenfunction has a nonpositive eigenvalue if and only if it has nonpositive winding number. On the other hand, if $\mathcal{E}$ is one-sided, then the asymptotic eigenfunction must have winding number zero. Hence $\mathcal{E}$ must be a positive end. The case for T negative is similar. $\square$

6 Construction of contact forms

In this section we construct some contact forms on $T^2 \times [1, 2]$ and $D^2 \times S^1$ which will be used in the proof of the main theorem.

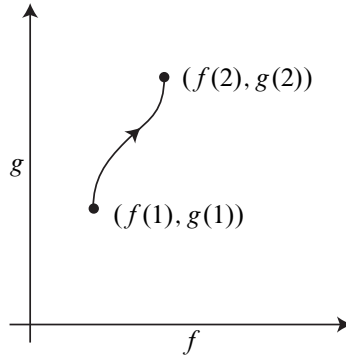
6.1 Contact forms on $T^2 \times [1, 2]$

Let (ϑ, t, y) be coordinates on

$$T^2 \times [1, 2] \simeq (\mathbb{R}^2/\mathbb{Z}^2) \times [1, 2].$$

Slopes of essential curves on T^2 will be measured with respect to (ϑ, t) , ie with respect to the basis of $H_1(T^2)$ given by the homology classes of the curves $\vartheta \mapsto (\vartheta, *)$ and $t \mapsto (*, t)$. Let

$$(6-1-1) \quad \alpha_{f,g} = g(y) d\vartheta + f(y) dt$$

Figure 1: Trajectory of $(f_\delta(y), g_\delta(y))$.

be a contact form on $T^2 \times [1, 2]$, where f and g are functions on $[1, 2]$. We write $f' = df/dy$ and $g' = dg/dy$.

The following is a straightforward calculation:

Lemma 6.1.1 *The form $\alpha_{f,g}$ is a contact form if and only if*

$$(6-1-2) \quad fg' - f'g > 0.$$

The kernel $\ker \alpha_{f,g}$ is spanned by $\{\partial_y, -f'\partial_\vartheta + g'\partial_t\}$. Assuming $\alpha_{f,g}$ is a contact form, the Reeb vector field is given by

$$(6-1-3) \quad R_{\alpha_{f,g}} = \frac{1}{fg' - g'f'}(-f'\partial_\vartheta + g'\partial_t).$$

In words, equation (6-1-2) says that the curve in $\mathbb{R}^2$ parametrized by (f, g) is transverse to the radial rays and rotates in the counterclockwise direction.

Later in the article will need the following family of contact forms on $T^2 \times [1, 2]$:

Example 6.1.2 Given a (small) positive irrational parameter δ we consider pairs of functions (f_δ, g_δ) such that the following hold (see Figure 1):

- (1) (f_δ, g_δ) satisfies (6-1-2).
- (2) $0 \leq f'_\delta(y)/g'_\delta(y) \leq \delta$; $f'_\delta(y)/g'_\delta(y)$ is increasing on $(1, \frac{3}{2})$ and is decreasing on $(\frac{3}{2}, 2)$, and is equal to δ at $y = \frac{3}{2}$.
- (3) $(f_\delta(y), g_\delta(y)) = (f_\delta(1) + (y-1)^2, g_\delta(1) + (y-1))$ near $y = 1$.
- (4) $(f_\delta(y), g_\delta(y)) = (f_\delta(2) - c_\delta(y-2)^2, g_\delta(2) + c_\delta(y-2))$ near $y = 2$.
- (5) $(f_\delta(1), f_\delta(1))$ is independent of δ and all the $(f_\delta(2), f_\delta(2))$ lie on the same line through the origin.
- (6) The constants c_δ are chosen so that any two contact forms α_δ and $\alpha_{\delta'}$ are constant multiples of one another near $y = 2$.

The contact form $\alpha_{f_\delta, g_\delta}$ will also be called α_δ . Its Reeb vector field R_{α_δ} has Morse–Bott tori whose Reeb orbits have rational slope in the interval $[-\infty, -1/\delta]$; each rational slope occurs twice, once on the interval $[1, \frac{3}{2}]$ and once on the interval $[\frac{3}{2}, 2]$. Note that the Reeb orbits in the two Morse–Bott tori of infinite slope have parallel directions and are in “elimination position”, ie assuming that (f_δ, g_δ) is extended slightly to $T^2 \times [1 - \varepsilon, 2 + \varepsilon]$ so that the Reeb orbits have positive slope on $y \in [1 - \varepsilon, 1) \cup (2, 2 + \varepsilon]$, one could deform the pair (f_δ, g_δ) relative to $\{y = 1 - \varepsilon, 2 + \varepsilon\}$ to make the slope of the Reeb vector field always positive; during the deformation we would see the two Morse–Bott tori of infinite slope coming close to each other and finally canceling. Also, by taking δ to be sufficiently small, all the Reeb orbits in $\text{int}(T^2 \times [1, 2])$ can be made to have arbitrarily large action.

6.2 Contact forms on $D^2 \times S^1$

Let (ρ, ϕ, θ) be cylindrical coordinates⁵ on the solid torus

$$D^2 \times S^1 = \{\rho \leq 1\} \times (\mathbb{R}/2\pi\mathbb{Z}).$$

Let $T_{\bar{\rho}} = \{\rho = \bar{\rho}\} \subset D^2 \times S^1$ for $\bar{\rho} \in (0, 1]$.

Convention 6.2.1 Slopes of essential curves on the torus $T_{\bar{\rho}}$ are measured with respect to (θ, ϕ) instead of (ϕ, θ) .

We consider contact forms which can be written as

$$(6-2-1) \quad \alpha_{f,g} = g(\rho) d\theta + f(\rho) d\phi.$$

Here we need to choose $(f(\rho), g(\rho))$ so that $\alpha_{f,g}$ is smooth on all of $D^2 \times S^1$, which means that $f(0) = 0$ and the derivatives of odd degree of both f and g at $\rho = 0$ vanish. We write $f' = df/d\rho$ and $g' = dg/d\rho$. The analog of Lemma 6.1.1 is the following:

Lemma 6.2.2 *The form $\alpha_{f,g}$ is a contact form if and only if*

$$(6-2-2) \quad f'g - fg' > 0 \quad \text{for } \rho > 0, \text{ and}$$

$$(6-2-3) \quad \lim_{\rho \rightarrow 0} \frac{f'g - fg'}{\rho} > 0.$$

The kernel $\ker \alpha_{f,g}$ is spanned by $\{\partial_\rho, -f\partial_\theta + g\partial_\phi\}$. Assuming $\alpha_{f,g}$ is a contact form, the Reeb vector field is given by

$$(6-2-4) \quad R_{\alpha_{f,g}} = \frac{1}{f'g - fg'}(f'\partial_\theta - g'\partial_\phi).$$

In particular, $R_{\alpha_{f,g}}$ is parallel to ∂_θ at $\rho = 0$.

Each torus $T_{\bar{\rho}}$ is linearly foliated by the Reeb flow of $\alpha_{f,g}$.

⁵We are making a distinction between the symbols ϑ and θ .

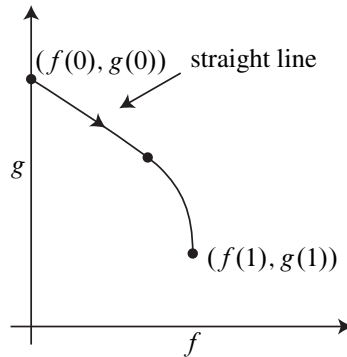


Figure 2: Trajectory of $(f(\rho), g(\rho))$. Here the arrow points in the positive ρ -direction.

Since they will be useful later, we present a pair of constructions of contact forms on $D^2 \times S^1$ of the form given in (6-2-1).

Example 6.2.3 Given $\nu > 0$ and $C > 1$, let $(f(\rho), g(\rho)) = (\nu\rho^2, C - \rho^2)$. This gives a smooth contact form on $D^2 \times S^1$ and the Reeb vector field on T_ρ has slope $-g'/f' = 1/\nu$ for all $\rho > 0$. In particular, if ν is irrational, then the only simple closed orbit of $R_{\alpha_{f,g}}$ is the core curve $\{\rho = 0\}$.

Example 6.2.4 The following contact forms, which generalize those in Example 6.2.3, will be used later in the paper. We define α on $D^2 \times S^1$ so that the following hold:

- (1) (f, g) satisfies (6-2-2).
- (2) $(f(\rho), g(\rho)) = (\rho^2, C - \rho^2)$ near $\rho = 0$, where $C > 0$ is a large constant.
- (3) $(f(\rho), g(\rho)) = (f(1) - (\rho - 1)^2, g(1) - (\rho - 1))$ near $\rho = 1$.⁶
- (4) $-g'/f'$ monotonically increases from 1 to $+\infty$ as ρ goes from 0 to 1.

The profile of the functions (f, g) is shown in Figure 2.

On each torus $T_\rho \subset D^2 \times S^1$, the Reeb vector field R_α gives a foliation by Reeb orbits of slope r in the interval $[1, \infty]$, where there is a unique ρ for each slope $r \in (1, \infty]$.

7 ECH for manifolds with torus boundary

In this section we define several ECH groups on a compact manifold M with torus boundary $T = \partial M$. We fix an oriented identification $T \simeq \mathbb{R}^2/\mathbb{Z}^2$ so that we can refer to slopes of essential curves on T . Let α be a contact form on M such that T is foliated by Reeb orbits of slope r . If r is rational, we assume that T is Morse–Bott. All ECH groups on M and $\text{int}(M)$ are computed using a C^∞ -small perturbation

⁶Here $(f, g)|_{\rho=1} = (f_\delta, g_\delta)|_{y=2}$. This allows us to extend α_δ to $D^2 \times S^1$ for all sufficiently small $\delta > 0$ by writing (f_δ, g_δ) as a suitable constant multiple of $(f_{\delta_0}, g_{\delta_0})$. This is possible because of condition (6) in the definition of α_δ . Observe that the Reeb orbits of α_δ and α_{δ_0} agree on V , modulo parametrization.

of α so that all Reeb orbits in $\text{int}(M)$ are nondegenerate. Let J be a Morse–Bott regular almost complex structure on $\mathbb{R} \times M$ adapted to α .

7.1 Definitions

We introduce several ECH groups:

(1) **ECH(int(M), α)** The ECH chain group $\text{ECC}(\text{int}(M), \alpha)$ is generated by orbit sets whose simple orbits lie in the interior of M . In particular, we are discarding the Morse–Bott family of orbits on T if r is rational. The differential ∂ is the usual one, ie it counts holomorphic curves of ECH index $I(\gamma, \gamma', Z) = 1$ in $\mathbb{R} \times \text{int}(M)$ whose connector components are trivial cylinders. Since $\text{int}(M)$ is not closed, we need to verify that $\text{ECC}(\text{int}(M), \alpha)$ is indeed a chain complex.

Lemma 7.1.1 ∂ is defined and $\partial^2 = 0$.

Proof We claim that the SFT compactness theorem holds in $\mathbb{R} \times \text{int}(M)$. This implies that the arguments used in [Hutchings and Taubes 2007; 2009a] to prove $\partial^2 = 0$ will then carry over to our setting. Let u_n be a sequence of J -holomorphic maps with image in $\mathbb{R} \times \text{int}(M)$. After passing to a subsequence, u_n converges to a building u_∞ such that all its components have image in $\mathbb{R} \times M$. By the blocking lemma, no component of u_∞ can intersect ∂M .

We claim that no component of u_∞ can have an end at a Reeb orbit in ∂M : indeed, if there is a component with a positive (resp. negative) end at a Reeb orbit in ∂M , then there is another component of u_∞ with a negative (resp. positive) end at a Reeb orbit in ∂M . By the trapping lemma, this is impossible if the image of u_∞ is contained in $\mathbb{R} \times M$. $\square$

(2a) **ECH(M , α) for r irrational** This is defined to be $\text{ECH}(\text{int}(M), \alpha)$.

(2b) **ECH(M , α) for r rational** Let $\mathcal{N}$ be the set of simple Reeb orbits on T . The set $\mathcal{N}$ comes with distinguished orbits e, h which become elliptic and hyperbolic after a suitable perturbation. Writing $\mathcal{P}$ for the set of simple orbits in $\text{int}(M)$, $\text{ECC}(M, \alpha)$ is the chain complex which is generated by orbit sets constructed from $\mathcal{P} \cup \{h, e\}$ and whose differential counts Morse–Bott buildings of ECH index 1 in $\mathbb{R} \times M$.

Lemma 7.1.2 If α is nondegenerate on $\text{int}(M)$, then it is nice.

Proof Suppose that ∂M is a negative Morse–Bott torus; the positive case is analogous. Let $\mathcal{N}$ be the Morse–Bott family corresponding to ∂M . If α is not nice, then there is a Morse–Bott building $\tilde{u}$ in $\mathbb{R} \times M$ with ECH index $I(\tilde{u}) = 1$ whose holomorphic part u has more than one nonconnector irreducible component. Assume that there are exactly two nonconnector components u_1 and u_2 (this is mostly to simplify notation; the general case is treated in the same way). By the trapping lemma, the only ends of u_1 and u_2 that limit to ∂M are negative ends. We form the Morse–Bott buildings $\tilde{u}_1$ and $\tilde{u}_2$ by augmenting the ends of u_1 and u_2 at ∂M with gradient flow lines and denote the union of these two buildings by $\tilde{u}'$.

We claim that $I(\tilde{u}) = I(\tilde{u}')$. Indeed, all the ends of u_1 and u_2 that limit to orbits on ∂M are connected to critical points in $\mathcal{N}$ by gradient flow lines, with possible interruptions by connectors. Hence $\tilde{u}$ and $\tilde{u}'$ have the same ends in the ECH sense and define the same relative homology class. This implies that $I(\tilde{u}) = I(\tilde{u}')$.

On the other hand, let u_i for $i = 1, 2$ be a k_i^{th} cover of a J -holomorphic curve v_i , and define very nice, simply covered buildings $\tilde{v}_i$. By [Theorem 4.4.3\(2\)](#), we can perturb $\tilde{v}_1$ and $\tilde{v}_2$ to J_ϵ -holomorphic maps $v_{1,\epsilon}$ and $v_{2,\epsilon}$, respectively. We denote by $v_{i,\epsilon}^{k_i}$ the J_ϵ -holomorphic map made of k_i parallel copies of $v_{i,\epsilon}$. Then, by [\[Hutchings 2009, Theorem 5.1\]](#),

$$I(\tilde{u}) \geq I(v_{1,\epsilon}^{k_1}) + I(v_{2,\epsilon}^{k_2}) \geq k_1 I(v_{1,\epsilon}) + k_2 I(v_{2,\epsilon}).$$

Since $I(v_{i,\epsilon}) > 0$ for $i = 1, 2$, this is a contradiction. $\square$

[Lemma 7.1.2](#) implies that $\partial^2 = 0$, since it guarantees that the Morse–Bott gluing is done at a different end from the gluing of connectors (ie the obstruction bundle gluing of Hutchings and Taubes [\[2007; 2009a\]](#)) and the two kinds of gluing can be done independently.

(3) **ECH^b(M, α)** The chain complex $\text{ECC}^b(M, \alpha)$ is generated by orbit sets which are constructed from $\mathcal{P} \cup \{e\}$. As in the case of $\text{ECC}(M, \alpha)$, if $\mathcal{N}$ is a negative Morse–Bott family, no Morse–Bott building $\tilde{u}$ in $\mathbb{R} \times M$ besides trivial cylinders can have e at the positive end. Hence the differential can be defined by counting Morse–Bott buildings of ECH index 1 in $\mathbb{R} \times M$, whose orbit sets are constructed from $\mathcal{P} \cup \{e\}$.

The verification of $\partial^2 = 0$ needs one extra consideration: an index 2 family of J -holomorphic curves in $\mathbb{R} \times M$ can break into a Morse–Bott building $\tilde{u}$ which involves h at the negative end, followed by a holomorphic cylinder from h to e . (Note that, by the trapping lemma, these holomorphic cylinders are the only nontrivial holomorphic curves which go from h to e and so there are no other cases to consider.)

This type of breaking could be a problem because orbit sets containing h are not in the chain complex $\text{ECC}^b(M, \alpha)$. However, since there are *two* gradient trajectories from h to e with ECH index $I = 1$ and no other holomorphic curve (or building) with a positive end at h , the Morse–Bott building $\tilde{u}$ can be glued onto each of the two gradient trajectories. This proves that families breaking at h always come in pairs, and therefore $\partial^2 = 0$ holds even when we discard orbit sets which contain h .

If $\mathcal{N}$ is a positive Morse–Bott family, then e can only be at the positive end of a J -holomorphic curve in $\mathbb{R} \times M$, and the proof of $\partial^2 = 0$ remains the same with the obvious modifications.

(4) **ECH[#](M, α)** The chain complex $\text{ECC}^\#(M, \alpha)$ is generated by orbit sets which are constructed from $\mathcal{P} \cup \{h\}$, and its differential counts ECH index 1 Morse–Bott buildings which are asymptotic to orbit sets constructed from $\mathcal{P} \cup \{h\}$.

Remark 7.1.3 The differentials of the ECH groups defined in this section preserve the total homology class of the generators. Then we can define subgroups $\text{ECH}(M, \alpha, A)$ for every $A \in H_1(M)$. Similar notation will be used for the variants of this group.

7.2 Well-definition

In this subsection we prove that $\text{ECH}(M, \alpha)$ is independent of the choice of α , provided the slope r is irrational. The verification in the other cases will be omitted; we will be careful to use the invariance of ECH groups for manifolds with torus boundary only in the case where it is proved. The main result proved in this subsection is the following:

Proposition 7.2.1 *Let α_1 and α_2 be contact forms on M which agree on ∂M to first order (and in particular the Reeb vector fields and the characteristic foliations of α_1 and α_2 at ∂M are equal) and define contact structures $\xi_i = \ker \alpha_i$ which are isotopic relative to the boundary. If ∂M is foliated by Reeb orbits of irrational slope, then there is an isomorphism*

$$\text{ECH}(M, \alpha_1) \simeq \text{ECH}(M, \alpha_2).$$

The strategy of the proof is to extend (M, α_i) for $i = 1, 2$ to closed contact manifolds and to use the invariance of ECH for closed manifolds. [Lemma 7.2.3](#) constructs the contact forms which are used to extend (M, α_i) . Then [Lemma 7.2.6](#) shows that, up to some action L , the ECH groups of (M, α_i) are isomorphic to the ECH groups of their extension. Finally, [Lemmas 7.2.7, 7.2.9 and 7.2.10](#) establish some compatibility properties for the continuation maps between the extended forms, so that the proposition can finally be proved by a direct limit argument.

Lemma 7.2.2 *Let $\alpha = g(\rho) d\theta + f(\rho) d\phi$ be a contact form on $D^2 \times S^1$ with cylindrical coordinates (ρ, ϕ, θ) . Write $v(\rho) = (f(\rho), g(\rho))$ and let $|v(\rho)|$ be the norm of $v(\rho)$ and $\zeta(\rho)$ the angle between $v(\rho)$ and $v'(\rho)$, both measured with respect to the standard Euclidean structure on $\mathbb{R}^2$. Then, if the torus T_ρ is foliated by closed Reeb orbits, for every Reeb orbit γ on T_ρ we have*

$$(7-2-1) \quad \mathcal{A}(\gamma) \geq |v(\rho)| |\sin \zeta(\rho)|.$$

Proof Let J be the standard complex structure, $\cdot$ the standard inner product, and $|\cdot|$ the standard Euclidean norm on $\mathbb{R}^2$. For every $\rho \in (0, 1]$ we trivialize the tangent bundle of the torus T_ρ by $(\partial_\phi, \partial_\theta)$ and measure the slope of curves on T_ρ with respect to (ϕ, θ) .⁷

By [Lemma 6.2.2](#), R is tangent to T_ρ for all $\rho \in (0, 1]$ and can be written as

$$R = \frac{(-g', f')}{(-g', f') \cdot (f, g)},$$

with respect to $(\partial_\phi, \partial_\theta)$. If we write $v = (f, g)$, then $Jv' = (-g', f')$ and

$$|R| = \left| \frac{(-g', f')}{(-g', f') \cdot (f, g)} \right| = \left| \frac{Jv'}{Jv' \cdot v} \right| = \frac{1}{|v| |\sin \zeta|},$$

⁷In the proof we are using a different convention from that of [Convention 6.2.1](#).

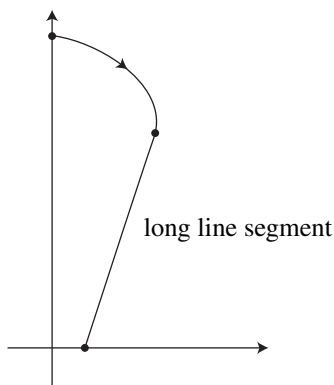


Figure 3: Trajectory of $(f(\rho), g(\rho))$. The arrow is in the direction of increasing ρ .

where $\zeta(\rho)$ is the angle between $v(\rho)$ and $v'(\rho)$. Note that $\text{slope}(R) = \text{slope}(Jv') = -f'/g'$.

Let $\rho \in (0, 1)$ be such that R has rational slope on T_ρ and let w be the shortest integer vector with that slope. Then T_ρ is foliated by Reeb orbits and each Reeb orbit γ has action $\mathcal{A}(\gamma) = |w|/|R|$. Since $|w| \geq 1$, we have the bound

$$\mathcal{A}(\gamma) \geq \frac{1}{|R|} = |v| |\sin \zeta|. \quad \square$$

Lemma 7.2.3 *Given $L > 0$ and $r > 0$ irrational, there is a contact form $\alpha(r, L) = g(\rho) d\theta + f(\rho) d\phi$ on $V = D^2 \times S^1$ with cylindrical coordinates (ρ, ϕ, θ) such that*

- (a) *on ∂V , the Reeb vector field R of $\alpha(r, L)$ has slope $-1/r$ and the characteristic foliation has infinite slope; and*
- (b) *all the closed orbits of R have $\alpha(r, L)$ -action larger than L .*

Proof We describe $\alpha(r, L)$ by describing the vector $v(\rho) = (f(\rho), g(\rho))$. We construct $v(\rho) = (f(\rho), g(\rho))$ “backwards”, starting with larger ρ , subject to the condition $d|v|/d\rho < 0$. The profile of $v(\rho)$ is given in Figure 3.

- (1) For $\rho \in [\frac{3}{4}, 1]$, define $v(\rho)$ so that it parametrizes a “long”⁸ segment and R is constant, has slope $-1/r$, and satisfies $|R| = 1/K$. Since r is irrational, there are no Reeb orbits on T_ρ for $\rho \in [\frac{3}{4}, 1]$.
- (2) Fix an irrational slope $-1/r' > -1/r$ such that all integer vectors with slope between $-1/r'$ and $-1/r$ have norm greater than $2L/K$. For $\rho \in [\frac{1}{2}, \frac{3}{4}]$, define $v(\rho)$ so that $|R(\rho)| < 2/K$ and $\text{slope}(R) = \text{slope}(Jv')$ decreases monotonically from $-1/r'$ to $-1/r$ as ρ increases. One can achieve this by making $v(\rho)$ vary sufficiently slowly for $\rho \in [\frac{1}{2}, \frac{3}{4}]$. Hence, if γ is a Reeb orbit of T_ρ with $\rho \in [\frac{1}{2}, \frac{3}{4}]$, then

$$\mathcal{A}(\gamma) \geq \frac{K}{2} \frac{2L}{K} = L.$$

⁸The segment is chosen so that (7-2-2) from (2) is satisfied.

Let $\bar{\zeta}$ be the clockwise angle from a line of slope $-1/r'$ to a line of slope $-1/r$. By taking the “long” segment to be sufficiently long, we may assume that

$$(7-2-2) \quad |\sin \bar{\zeta}| > KL \left| \sin \left(\zeta \left(\frac{3}{4} \right) \right) \right|.$$

(3) For $\rho \in [\frac{1}{4}, \frac{1}{2}]$, define $v(\rho)$ so that $\text{slope}(Jv')$ decreases monotonically from $1/r'' > 0$ to $-1/r'$ as ρ increases and $|\sin \zeta(\rho)| \geq |\sin \bar{\zeta}|$. We can achieve these properties by changing $v(\rho)$ slowly with respect to the slope of $v'(\rho)$. Then, by (7-2-1) and (7-2-2),

$$(7-2-3) \quad \mathcal{A}(\gamma) \geq |v(\rho)| \cdot KL \left| \sin \left(\zeta \left(\frac{3}{4} \right) \right) \right| \geq KL \left| v \left(\frac{3}{4} \right) \right| \cdot \left| \sin \left(\zeta \left(\frac{3}{4} \right) \right) \right| \geq KL \frac{1}{K} = L,$$

where γ is a Reeb orbit of T_ρ , $\rho \in [\frac{1}{4}, \frac{1}{2}]$.

(4) Finally, define $v(\rho)$ for $\rho \in [0, \frac{1}{4}]$ which parametrizes a segment of slope $1/r''$ and satisfies $f(0) = 0$. $\mathcal{A}(\gamma) \geq L$ follows from (7-2-3). $\square$

Remark 7.2.4 We will always assume that, when $L_0 < L_1$, each radial ray in the fg -plane intersects the curve $(f_0(\rho), g_0(\rho))$ defining $\alpha(r, L_0)$ before or at the same time as the curve $(f_1(\rho), g_1(\rho))$ defining $\alpha(r, L_1)$. Then there exist a diffeomorphism $\sigma: D^2 \times S^1 \rightarrow D^2 \times S^1$ such that $\sigma(\rho, \phi, \theta) = (\sigma_0(\rho), \phi, \theta)$ and a function $h: [0, 1] \rightarrow \mathbb{R}^{\geq 0}$ such that $\alpha(r, L_1) = e^{h(\rho)} \sigma^*(r, \alpha(L_0))$.

Let (M, α_i) for $i = 1, 2$ be contact manifolds as in Proposition 7.2.1. We can choose coordinates $(\vartheta, t, y) \in (\mathbb{R}^2/\mathbb{Z}^2) \times [-\varepsilon, 0]$ on a small collar of ∂M such that ∂M corresponds to $T^2 \times \{0\}$ and the contact forms α_i can be written as

$$\alpha_i = g_i(\vartheta, t, y) d\vartheta + f_i(\vartheta, t, y) dt$$

with $\partial f_i / \partial \vartheta = \partial f_i / \partial t = \partial g_i / \partial \vartheta = \partial g_i / \partial t = 0$ at $t = 0$ (ie along ∂M). Note that we have used the assumption that α_1 and α_2 coincide to first order along ∂M to conclude that they can be put in this form with the same choice of coordinates. Moreover, we assume that these coordinates have been chosen so that, on ∂M , the Reeb vector fields of α_1 and α_2 have *negative irrational* slope $-r$ and that the slopes of the characteristic foliations of $\xi_i = \ker \alpha_i$ are nonnegative and sufficiently close to zero.⁹ Here the slope is measured with respect to (ϑ, t) .

For $L' > 0$ sufficiently large, we embed (M, α_i) into a closed contact manifold $(M', \alpha'_i(L'))$ such that

- (1) $M' = M \cup V$, where ∂M and ∂V are glued by the identifications $\rho = 1 - y$, $\phi = 2\pi t$ and $\theta = 2\pi \vartheta$; and
- (2) $\alpha'_i(L')|_M = \alpha_i$ and $\alpha'_i(L')|_V$ is a C^1 -small perturbation of $\alpha(r, L')$ near the boundary.

If the perturbation of the form $\alpha(r, L')$ is small enough in the C^1 -topology, it does not create any closed Reeb orbit of action less than L' . Since the size of the perturbation which is necessary to glue α_i with

⁹Close enough that Claim 7.2.5 applies.

$\alpha(r, L')$ essentially depends on the slope of the characteristic foliation of α_i on ∂M , we can claim the following:

Claim 7.2.5 *All closed Reeb orbits of $(M', \alpha_i(L'))$ of action less than L' are contained in M .*

The next lemma identifies some ECH groups for (M, α_i) with ECH groups for $(M', \alpha'_i(L'))$.

Lemma 7.2.6 *For all $L \leq L'$, if we choose the almost complex structure on the symplectization of $(M', \alpha'_i(L'))$ to extend the almost complex structure picked on the symplectization of (M, α_i) , then there are isomorphisms*

$$\mathrm{ECC}^L(M, \alpha_i) \simeq \mathrm{ECC}^L(M', \alpha'_i(L'))$$

of chain complexes.

Proof By Lemma 7.2.3, there is an isomorphism

$$\mathrm{ECC}^L(M, \alpha_i) \simeq \mathrm{ECC}^L(M', \alpha'_i(L'))$$

as vector spaces. To prove that the isomorphism holds as chain complexes, it suffices to show that every holomorphic curve in $\mathbb{R} \times M'$ which is positively asymptotic to an orbit set of $R_{\alpha'_i(L')}$ of $\alpha'_i(L')$ -action less than L (which is equal to an orbit set of R_{α_i} of α_i -action less than L) has image in $\mathbb{R} \times M$. Let u be a holomorphic map in $\mathbb{R} \times M'$ connecting the orbit set γ of R_{α_i} in M with $\mathcal{A}_{\alpha_i}(\gamma) < L$ to the orbit set γ' of $R_{\alpha'_i(L')}$ in M' . Since $\mathcal{A}_{\alpha_i}(\gamma) < L$, γ' must be contained in M . Hence the homology class of $u_{M'} \cap \partial V$ in $H_1(\partial V)$ is a multiple of the class of the meridian of V . On the other hand, inside V there is a concentric torus V' on which the Reeb orbits are meridians. (This torus corresponds to the vertical tangency of the curve in Figure 3.) Then Lemma 5.2.3 (blocking lemma) implies that u must be asymptotic to some orbits in V' . This is not possible since all the ends of u limit to orbits of action less than L . Hence the image of u is contained in $\mathbb{R} \times M$ by Lemma 5.2.3(1). $\square$

The induced identification

$$\mathrm{ECH}^L(M, \alpha_i) \simeq \mathrm{ECH}^L(M', \alpha'_i(L'))$$

is independent of L' in the following sense: Let $L \leq L_0 \leq L_1$ be positive numbers such that no Reeb orbit in (M, α_i) (for either $i = 1$ or $i = 2$) has action L . By Remark 7.2.4 and Lemma 3.1.7, there are maps

$$\Psi_i^{L, L_0, L_1} : \mathrm{ECH}^L(M', \alpha'_i(L_1)) \rightarrow \mathrm{ECH}^L(M', \alpha'_i(L_0))$$

induced by interpolating cobordisms (W, μ_i) from $(M', \alpha'_i(L_1))$ at the positive end to $(M', \alpha'_i(L_0))$ at the negative end. Then we have the following:

Lemma 7.2.7 *The maps Ψ_i^{L, L_0, L_1} restrict to the identity on $\mathrm{ECH}^L(M, \alpha_i)$.*

Proof The cobordism W is topologically trivial, ie $W \simeq [0, 1] \times M'$, and we can assume that (W, μ_i) restricts to a piece of symplectization on $[0, 1] \times M$. We choose the almost complex structure J to be $\mathbb{R}$ -invariant on $\mathbb{R} \times M$. As before, all orbit sets of $\alpha'_i(L_j)$ -action less than L for $j = 0, 1$ are contained

in M . Then the blocking lemma¹⁰ and the argument of Lemma 7.2.6 imply that all J -holomorphic maps between orbit sets of action less than L are contained in $\mathbb{R} \times M$. If those J -holomorphic maps have ECH index zero, then they are branched covers of trivial cylinders because $([0, 1] \times M, \mu_i|_{[0,1] \times M})$ is a piece of symplectization. Hence the map induced on $\text{ECH}^L(M, \alpha)$ is the identity by Theorem 3.1.2(i). $\square$

We will use the identifications $\text{ECH}^L(M, \alpha_i) \simeq \text{ECH}^L(M', \alpha'_i(L'))$ to define a map

$$\Phi: \text{ECH}(M, \alpha_1) \rightarrow \text{ECH}(M, \alpha_2).$$

This involves two steps: the construction of maps

$$\Phi_L: \text{ECH}^L(M, \alpha_1) \rightarrow \text{ECH}^{\kappa L}(M, \alpha_2)$$

for some $\kappa > 1$ and the taking of direct limits.

Let $f: M \rightarrow \mathbb{R}$ be a smooth positive function such that $\phi^*(\alpha_2) = f\alpha_1$ for some diffeomorphism ϕ of M which is isotopic to the identity and restricts to the identity on ∂M . Then choose $\kappa > 1$ such that $1/\kappa \leq f \leq \kappa$. Given $L' > L$, we consider the contact forms $\alpha'_i(\kappa L')$ for $i = 1, 2$ on M' constructed in Lemma 7.2.3. Then there is an interpolating cobordism $(X, \lambda_{L'})$ from $(M', \alpha'_1(\kappa L'))$ at the positive end to $(M', \kappa^{-1}\alpha'_2(\kappa L'))$ at the negative end. Moreover, we can assume that $(X, \lambda_{L'})$ restricts to a piece of symplectization on a small neighborhood of $[0, 1] \times V$.

We define Φ_L by imposing the commutativity of the diagram

$$(7-2-4) \quad \begin{array}{ccc} \text{ECH}^L(M', \alpha'_1(\kappa L')) & \xrightarrow{\Phi^L} & \text{ECH}^{\kappa L}(M', \alpha'_2(\kappa L')) \\ \simeq \downarrow & & \downarrow \simeq \\ \text{ECH}^L(M, \alpha_1) & \xrightarrow{\Phi_L} & \text{ECH}^{\kappa L}(M, \alpha_2) \end{array}$$

where the vertical maps are the isomorphisms coming from Lemma 7.2.6 and the top map is induced by the interpolating cobordisms $(X, \lambda_{L'})$ via Lemma 3.1.7.

Remark 7.2.8 Using the blocking lemma, one can prove that the map Φ_L is supported, in the sense on Theorem 3.1.2(i), by holomorphic curves in $\mathbb{R} \times M$. See the proof of Lemma 7.2.6 for the details.

Lemma 7.2.9 Φ_L is independent of the choice of L' in diagram (7-2-4).

Proof Suppose $L \leq L_0 \leq L_1$, α_1 has no orbit sets of action L , and α_2 has no orbit sets of action κL . Then the diagram

$$(7-2-5) \quad \begin{array}{ccc} \text{ECH}^L(M', \alpha'_1(\kappa L_1)) & \xrightarrow{\Phi^L} & \text{ECH}^{\kappa L}(M', \alpha'_2(\kappa L_1)) \\ \Psi_1^{L, \kappa L_0, \kappa L_1} \downarrow & & \downarrow \Psi_2^{L, \kappa L_0, \kappa L_1} \\ \text{ECH}^L(M', \alpha'_1(\kappa L_0)) & \xrightarrow{\Phi^L} & \text{ECH}^{\kappa L}(M', \alpha'_2(\kappa L_0)) \end{array}$$

¹⁰This situation is slightly more general than that for which the blocking lemma has been stated and proved because we are in a cobordism. However the lemma is still valid and the proof is unchanged.

commutes by [Theorem 3.1.2](#) since, by [Lemma 3.1.6](#), the compositions of cobordisms $(X, \lambda'_{L_0}) \circ (W, \mu_1)$ and $(W, \mu_2) \circ (X, \lambda'_{L_1})$ are homotopic. The maps $\Psi_i^{L, \kappa L_0, \kappa L_1}$ induce the identity on $\text{ECH}(M, \alpha_i)$ by [Lemma 7.2.7](#), so the maps on the top and bottom of (7-2-5) define the same map $\Phi_L : \text{ECH}^L(M, \alpha_1) \rightarrow \text{ECH}^{\kappa L}(M, \alpha_2)$. $\square$

Lemma 7.2.10 *Let α_1 and α_2 be contact forms as in [Proposition 7.2.1](#). If L_i is an increasing sequence of positive real numbers such that α_1 has no orbit set of action L_i and α_2 has no orbit set of action κL_i for all i ,¹¹ then the maps*

$$\Phi_{L_i} : \text{ECH}^{L_i}(M, \alpha_1) \rightarrow \text{ECH}^{\kappa L_i}(M, \alpha_2)$$

define a morphism of directed systems.

Proof For all $L < L'$ as above, the diagram

$$(7-2-6) \quad \begin{array}{ccc} \text{ECH}^L(M, \alpha_1) & \xrightarrow{\Phi_L} & \text{ECH}^{\kappa L}(M, \alpha_2) \\ \downarrow & & \downarrow \\ \text{ECH}^{L'}(M, \alpha_1) & \xrightarrow{\Phi_{L'}} & \text{ECH}^{\kappa L'}(M, \alpha_2) \end{array}$$

where the vertical arrows are maps induced by the inclusions of chain complexes, commutes by [Lemmas 7.2.7](#) and [7.2.9](#). $\square$

By taking the direct limit of the maps Φ_{L_i} from [Lemma 7.2.10](#), we obtain a linear map

$$\Phi : \text{ECH}(M, \alpha_1) \rightarrow \text{ECH}(M, \alpha_2).$$

Since the roles of α_1 and α_2 are interchangeable, the same arguments can be used to define a map $\Phi' : \text{ECH}(M, \alpha_2) \rightarrow \text{ECH}(M, \alpha_1)$ as a direct limit of maps $\Phi'_{L'_j}$.

Proof of Proposition 7.2.1 We prove that Φ and Φ' are inverses of each other. We identify the composition $\Phi'_{\kappa L} \circ \Phi_L$ (after a proper rescaling) with the map induced by an interpolating cobordism which is homotopic to a piece of symplectization. Then $\Phi'_{\kappa L} \circ \Phi_L = i_{L, \kappa^2 L}$, where $i_{L, \kappa^2 L}$ is the inclusion map. By taking the direct limit, we obtain $\Phi' \circ \Phi = \text{id}$. The proof of $\Phi \circ \Phi' = \text{id}$ is similar. $\square$

Remark 7.2.11 We sketch a possible strategy to prove the invariance of the group $\text{ECH}(M, \alpha)$ when the Reeb vector field of α defines a foliation on ∂M with closed leaves. This result will not be used in the rest of the article.

When ∂M is foliated by closed orbits of the Reeb vector field of α we would like to view $\text{ECH}(M, \alpha)$ as a direct limit of ECH groups of nondegenerate contact forms as in (4-5-1). We pick $L > 0$ and slightly extend (M, α) to $(M_\varepsilon, \alpha_\varepsilon)$ so that

- $M_\varepsilon = M \cup (T^2 \times [0, \varepsilon))$ where $\partial M = T^2 \times \{0\}$;
- $\alpha_\varepsilon|_M = \alpha$;

¹¹This condition can be fulfilled due to the fact that the action spectrum is discrete for a generic contact form.

- ∂M_ε is foliated by Reeb trajectories of α_ε with irrational slope; and
- there are no Reeb orbits of α_ε on $M_\varepsilon - M$ with action $\leq L$.

We now consider the chain complexes $\text{ECC}^L(M_\varepsilon, f_i \alpha_\varepsilon)$, where $f_i: M_\varepsilon \rightarrow \mathbb{R}$ is as in [Lemma 4.5.1](#) for $i \gg 0$. Then

$$(7-2-7) \quad \text{ECC}^L(M, \alpha) \simeq \text{ECC}^L(M_\varepsilon, f_i \alpha_\varepsilon)$$

by [Proposition 4.4.7](#). We then write the ECH group $\text{ECH}(M, \alpha)$ as a direct limit of groups $\text{ECH}^L(M_\varepsilon, f_i \alpha_\varepsilon)$ as in [Corollary 3.2.3](#). We extend $(M_\varepsilon, f_i \alpha_\varepsilon)$ to a closed manifold by using [Lemma 7.2.3](#) and apply the (analogs of the) results of this section to define the ECH cobordism maps.

7.3 Variants of ECH relative to the boundary

The goal of this subsection is to define the homology groups $\text{ECH}(M, \partial M, \alpha)$ and $\widehat{\text{ECH}}(M, \partial M, \alpha)$ which appear in the statement of [Theorem 1.1.1](#). They are variants of $\text{ECH}(M, \alpha)$ and in many ways can be viewed as ECH groups relative to the boundary of M , whence the notation.

Let M be a manifold with $\partial M \simeq T^2$. Let α be a contact form on M which is nondegenerate on $\text{int}(M)$ and such that ∂M is a *negative* Morse–Bott torus. Then the ECH groups introduced in [Section 7.1](#) are defined for (M, α) . In the rest of this section we make the further assumption that there exists a properly embedded oriented surface $(\Sigma, \partial \Sigma) \subset (M, \partial M)$ with connected boundary such that an orbit of the Morse–Bott torus has algebraic intersection number one with Σ .

As before, we pick two orbits on ∂M and label them h and e . There is a perturbation of α near ∂M which makes h hyperbolic and e elliptic; h corresponds to the maximum and e to the minimum of the perturbing Morse function.

Let $\mathcal{P}$ be the set of simple Reeb orbits of α in the interior of M . Let $\text{ECC}_j^b(M, \alpha)$ be the chain complex generated by orbit sets γ constructed from $\mathcal{P} \cup \{e\}$, whose algebraic intersection number $\langle [\gamma], \Sigma \rangle$ is j . By construction, $\text{ECC}_j^b(M, \alpha)$ is a direct summand of $\text{ECC}^b(M, \alpha)$ and its differential is the restriction of the differential for $\text{ECC}^b(M, \alpha)$.

In the same way we write $\text{ECC}_j(M, \alpha)$ for the chain complex generated by orbit sets γ constructed from $\mathcal{P} \cup \{e, h\}$, whose algebraic intersection number $\langle [\gamma], \Sigma \rangle$ is j . By construction, $\text{ECC}_j(M, \alpha)$ is a direct summand of $\text{ECC}(M, \alpha)$ and its differential is the restriction of the differential for $\text{ECC}(M, \alpha)$.

Lemma 7.3.1 *There are inclusions of chain complexes*

$$\text{ECC}_j^b(M, \alpha) \rightarrow \text{ECC}_{j+1}^b(M, \alpha), \quad \text{ECC}_j(M, \alpha) \rightarrow \text{ECC}_{j+1}(M, \alpha)$$

given by the map $\gamma \mapsto e\gamma$, where we are using multiplicative notation for orbit sets.

Proof Let γ be an orbit set in M and u a holomorphic map with image in $\mathbb{R} \times M$ which is positively asymptotic to $e\gamma$. Then u has an irreducible component which is mapped to the trivial cylinder over e . In fact, by the trapping lemma, u cannot have nontrivial positive ends that limit to orbits on ∂M because M is a negative Morse–Bott torus. Also, one can check that, $Z' \in H_2(M, e\gamma, e\gamma')$ is obtained by adding a trivial cylinder over e to $Z \in H_2(M, \gamma, \gamma')$, then $I(e\gamma, e\gamma', Z') = 1$ whenever $I(\gamma, \gamma', Z) = 1$. This is a consequence of [Hutchings 2002, Proposition 7.1], since the associated partitions satisfy the admissibility conditions [Hutchings 2002, Definition 4.7, equations (23) and (24)]. It is crucial in the verification of the admissibility condition that, in the Morse–Bott situation, the outgoing partition for e with multiplicity n is (n) and the incoming partition is $(1, \dots, 1)$ for all n , together with the fact that every J -holomorphic map in $\mathbb{R} \times M$ with a positive end to e is a connector. Hence $\partial^b(e\gamma) = e\partial^b(\gamma)$ and $\partial(e\gamma) = e\partial(\gamma)$. $\square$

The homology of the chain complex $\text{ECC}_j^b(M, \alpha)$ will be written as $\text{ECH}_j^b(M, \alpha)$ and that of the chain complex $\text{ECC}_j(M, \alpha)$ will be written as $\text{ECH}_j(M, \alpha)$.

Definition 7.3.2 We define

$$\text{ECH}(M, \partial M, \alpha) = \lim_{j \rightarrow \infty} \text{ECH}_j^b(M, \alpha), \quad \widehat{\text{ECH}}(M, \partial M, \alpha) = \lim_{j \rightarrow \infty} \text{ECH}_j(M, \alpha).$$

Remark 7.3.3 The groups $\text{ECH}(M, \partial M, \alpha)$ and $\widehat{\text{ECH}}(M, \partial M, \alpha)$ can also be interpreted as the homology of the chain complexes obtained by taking the quotient of the chain complexes $\text{ECC}^b(M, \alpha)$ and $\text{ECC}(M, \alpha)$, respectively, by the subcomplexes generated by all elements of the form $e\gamma - \gamma$, where γ is any orbit set constructed from $\mathcal{P} \cup \{e\}$ in the case of $\text{ECH}(M, \partial M, \alpha)$ or from $\mathcal{P} \cup \{e, h\}$ in the case of $\widehat{\text{ECH}}(M, \partial M, \alpha)$. This alternative definition, unlike Definition 7.3.2, does not need the assumption that the Reeb orbits on the boundary have intersection one with a properly embedded surface.

Remark 7.3.4 The differentials in $\text{ECH}(M, \partial M, \alpha)$ and $\widehat{\text{ECH}}(M, \partial M, \alpha)$ preserve the total relative homology class of the generators. Then we can define subgroups $\text{ECH}(M, \partial M, \alpha, A)$ and $\widehat{\text{ECH}}(M, \partial M, \alpha, A)$ for every $A \in H_1(M, \partial M)$.

8 ECH of the solid torus

8.1 Overview of the computation

In this section we calculate various versions of ECH of the solid torus with certain boundary conditions and specific contact structures. We will write $V = D^2 \times S^1$ and use Convention 6.2.1 to compute the slope of essential curves in ∂V and in boundary-parallel tori contained in V .

The following lemma constructs the contact forms used in the main theorem. Let $V_0 \subset \dots \subset V_i \subset \dots \subset V$ be an exhaustion by concentric solid tori, $T_i = \partial V_i$ and $\mathcal{T} = \bigcup_i T_i$. Let (ρ, ϕ, θ) be the cylindrical coordinates on $V = D^2 \times S^1$ from Section 6.2. We assume that $T_i = \{\rho = \rho_i\}$. We will choose V_i so that the Reeb flow foliates $T_i = \partial V_i$ by orbits of *irrational* slope r_i .

Lemma 8.1.1 *There exists a contact form α_V on $V = D^2 \times S^1$ which is an arbitrarily C^∞ -small perturbation of the contact form α from [Example 6.2.4](#) and which satisfies the following:*

- (a) *The Reeb orbits of α_V in $\text{int}(V)$ are nondegenerate.*
- (b) *α_V and α agree to infinite order along ∂V and along $\mathcal{T}$. In particular, the Reeb flow of α_V foliates the tori T_i by orbits of irrational slope r_i and ∂V by orbits of infinite slope.*
- (c) *For every i , all orbits in $V - V_i$ have slope greater than r_i .*

Proof Let $L_i \rightarrow \infty$ for $i = 1, 2, \dots$ be an increasing sequence of real numbers and let d be a metric on $C^\infty(V)$ inducing the C^∞ -topology.¹² Fix $\varepsilon > 0$ sufficiently small.

We claim that, for $i = 1, 2, \dots$, there exists a function¹³ $f_i: V \rightarrow \mathbb{R}$ such that

- (i) $e^{f_i}\alpha$ is L_i -nondegenerate,
- (ii) $d(f_i, f_{i-1}) < 2^{-i}\varepsilon$, and
- (iii) $\text{supp}(f_i - f_{i-1}) \subset \text{int}(V) - (\mathcal{O}_{i-1} \cup \mathcal{T})$,

where $\mathcal{O}_i$ is the union of all simple Reeb orbits of $e^{f_i}\alpha$ with action less than L_i . Here we are setting $f_0 = 0$, $\mathcal{O}_0 = \emptyset$ and $L_0 = 0$. We define f_i inductively: We choose g_i such that $f_i = f_{i-1} + g_{i-1}$ satisfies (i)–(iii). In fact, as shown for example in the proof of [\[Colin and Honda 2013, Lemma 7.1\]](#), the functions g_i can be chosen arbitrarily close to 0 in the C^∞ -topology and with support in arbitrarily small neighborhoods of the Reeb orbits of action in $[L_{i-1}, L_i]$. The claim then follows. The sequence f_i is a Cauchy sequence, so we define $f = \lim_{i \rightarrow \infty} f_i$ and $\alpha_V = e^f\alpha$. The contact form α_V satisfies (a) and (b).

It remains to prove (c). But this is immediate since the slope in [Example 6.2.4](#) is strictly increasing with the radius on the region $V - V_i$ and we are performing a C^∞ -small perturbation so that this property is preserved. $\square$

Now, ∂V is a positive α_V -Morse–Bott torus. We can perturb α_V so that the Morse–Bott family for ∂V becomes a pair of nondegenerate Reeb orbits e' and h' , where e' is an elliptic orbit corresponding to the maximum of the perturbing function and h' is a hyperbolic orbit corresponding to the minimum. The following is the main result of this section:

Theorem 8.1.2 *Let α_V be a contact form on V as constructed in [Lemma 8.1.1](#). Then:*

- (1) $\text{ECH}(\text{int}(V), \alpha_V) \simeq \mathbb{F}$, generated by $\emptyset$.
- (2) $\text{ECH}^\#(V, \alpha_V) \simeq 0$.
- (3) $\text{ECH}(V, \alpha_V) \simeq 0$.
- (4) $\text{ECH}^b(V, \alpha_V) \simeq \mathbb{F}[e']$, where $\mathbb{F}[e']$ is the polynomial ring generated by e' over $\mathbb{F}$.

¹²For example, we can take $d(f, g) = \sum_{k=0}^{\infty} 2^{-k} \|f - g\|_{C^k} / (1 + \|f - g\|_{C^k})$.

¹³The functions f_i , g_i and f introduced in this proof are, of course, unrelated to the functions f and g defining α in [Example 6.2.4](#).

Remark 8.1.3 Proposition 7.2.1 does not cover contact forms whose Reeb flow has rational slope on ∂V , so we cannot claim that the computation in Theorem 8.1.2 is independent of the contact form. However, the computation for the contact forms α_V constructed in Lemma 8.1.1 will be sufficient for the proof of Theorem 1.1.1.

The proof of (1) proceeds as follows: In Section 8.2 we compute $\text{ECH}(V_i, \alpha_V|_{V_i})$. Since the slope of the Reeb flow of α_V on $T_i = \partial V_i$ is irrational, we can use Proposition 7.2.1 to replace the contact forms $\alpha_V|_{V_i}$ with different forms for which the computation is easy. We also lift the relative grading on the ECH groups given by the ECH index to an absolute grading which is compatible with the maps induced by the interpolating cobordisms. In Section 8.3 we prove that the inclusions $V_i \subset V_{i+1}$ induce inclusions of chain complexes $\text{ECC}(V_i, \alpha_V|_{V_i}) \subset \text{ECC}(V_{i+1}, \alpha_V|_{V_{i+1}})$ as a consequence of the blocking lemma. This implies that

$$\text{ECH}(\text{int}(V), \alpha_V) = \lim_{i \rightarrow \infty} \text{ECH}(V_i, \alpha_V|_{V_i}).$$

We then use the absolute grading to conclude the proof: the degrees of the generators of $\text{ECH}(V_i, \alpha_V|_{V_i})$ that are different from $\emptyset$ go to infinity as $i \rightarrow \infty$, so only $\emptyset$ survives in the direct limit.

The proofs of (2)–(4) are given in Section 8.5 and use (1) and some results on holomorphic curves in $\mathbb{R} \times V$ due to Taubes and Wendl.

8.2 $\text{ECH}(V, \alpha)$ when the Reeb flow has irrational slope on the boundary

In this subsection we compute $\text{ECH}(V, \alpha)$ for contact forms α whose Reeb flow foliates ∂V by orbits of irrational slope and whose underlying contact structure gives the standard contact neighborhood of a transverse knot. For this boundary condition we have proved the invariance of ECH, so, by Proposition 7.2.1, we can choose a particularly simple contact form to do the computation.

Let $r > 0$ be an irrational number. Pick a contact form α_r on $V \simeq D^2 \times S^1$ as in Example 6.2.3 such that the following hold:

- The boundary ∂V and all the concentric tori T_ρ for $\rho \in (0, 1]$ are foliated by Reeb orbits of irrational slope r .
- The contact structure $\ker \alpha_r$ is transverse to all the fibers $\{\text{pt}\} \times S^1$.

There is only one simple closed orbit, namely the core $c = \{0\} \times S^1$. The orbit c is elliptic and all its multiple covers c^n are nondegenerate due to the irrationality of r . Note that $[c^n] = n[S^1] \in H_1(V)$, so we immediately have the following lemma:

Lemma 8.2.1 $\text{ECH}(\text{int}(V), \alpha_r; n[S^1]) \simeq \mathbb{F}$, generated by c^n , if $n \geq 0$ (where $c^n = \emptyset$ if $n = 0$) and $\text{ECH}(\text{int}(V), \alpha_r; n[S^1]) = 0$ if $n < 0$.

In order to plug this computation into the direct limit in the proof of Theorem 8.1.2, we define an absolute grading on the ECH groups of the solid torus in a way which is compatible with the cobordism maps.

For simplicity we will consider only contact forms α which satisfy the following assumption:

($\star$) The core of V is an elliptic Reeb orbit c , all of whose multiple covers are nondegenerate.

The contact forms α_r in [Lemma 8.2.1](#), as well as the contact forms $\alpha_V|_{V_i}$ of [Lemma 8.1.1](#), satisfy this assumption.

Let $\xi = \ker \alpha$. We chose a trivialization τ of ξ such that its restriction to the core orbit e is homotopic to the pullback of a basis of $T_0 D^2$ and the linearized Reeb flow at e is a rotation by angle $2\pi\theta$ with $\theta \in \mathbb{R} - \mathbb{Q}$.

Lemma 8.2.2 *Let α be a contact form on a solid torus V which satisfies ($\star$). Then there is an absolute grading I on $\text{ECC}(\text{int}(V), \alpha)$ such that:*

- (1) $I(c^n) = \sum_{k=1}^n (2\lfloor k\theta \rfloor + 1)$.
- (2) If γ_1 and γ_2 are two orbit sets and Z is a surface from γ_1 to γ_2 , then

$$I(\gamma_1, \gamma_2, Z) = I(\gamma_1) - I(\gamma_2).$$

Proof Given an orbit set γ with $[\gamma] = n[S^2]$, we choose a τ -trivial surface Z from γ to e^n and define

$$(8-2-1) \quad I(\gamma) := \tilde{\mu}_\tau(\gamma) + c_1(\xi|_Z, \tau) + Q_\tau(Z).$$

Since $H_2(V) = 0$, $I(\gamma, c^n, Z)$ is independent of Z by [\[Hutchings 2002, Lemma 2.5\(a\)\]](#). Hence $I(\gamma)$ is well defined.

- (1) follows from the calculation $\tilde{\mu}_\tau(c^n) = \sum_{k=1}^n (2\lfloor k\theta \rfloor + 1)$ using [\[Hutchings 2009, formula 2.3\]](#) and
- (2) follows from the additivity of the ECH index. □

Lemma 8.2.3 *Let α_1 and α_2 be contact forms on V which coincide on ∂V to first order and define contact structures which are isotopic relatively to the boundary. If both α_1 and α_2 satisfy ($\star$) and their Reeb flows foliate ∂V by orbits of irrational slope, then the isomorphism $\text{ECH}(V, \alpha_1) \simeq \text{ECH}(V, \alpha_2)$ from [Proposition 7.2.1](#) preserves the absolute grading I .*

Proof We denote by I_1 and I_2 the absolute grading on the groups $\text{ECH}(V, \alpha_1)$ and $\text{ECH}(V, \alpha_2)$, respectively. We know from [Remark 7.2.8](#) that the isomorphism $\text{ECH}(V, \alpha_1) \xrightarrow{\cong} \text{ECH}(V, \alpha_2)$ is supported by holomorphic buildings in a completed interpolating cobordism $([0, 1] \times V, \lambda)$ from (V, α_1) to (V, α_2) .¹⁴ Moreover, by [\[Cristofaro-Gardiner 2013, Theorem 5.1\]](#), those buildings have total ECH index $I = 0$ for a version of the ECH index in cobordisms; see [\[Hutchings 2009\]](#) for its definition. Then the lemma holds if

$$(8-2-2) \quad I(\gamma_1, \gamma_2, Z) = I_1(\gamma_1) - I_2(\gamma_2)$$

for all surfaces Z in $[0, 1] \times V$ connecting an orbit set γ_1 for α_1 to an orbit set γ_2 for α_2 .

¹⁴To add some confusion, what is called V here corresponds to M in [Section 7.2](#).

Since $H_2(V) = 0$, we can assume that Z is the union of a surface Z_1 from γ_1 to c^n (for some n), the surfaces Z_0^n consisting of n copies of the cylinder Z_0 over the core orbit c , and a surface Z_2 from c^n to γ_2 . Moreover, we can assume that Z_1 and Z_2 project to surfaces in V , so that $I(\gamma_1, c^n, Z_1) = I_1(\gamma_1) - I_1(c^n)$ and $I_2(c^n, \gamma_2, Z_2) = I_2(c^n) - I_2(\gamma_2)$. Then

$$I(\gamma_1, \gamma_2, Z) = I_1(\gamma_1) - I_1(c^n) + I(c^n, c^n, Z_0^n) + I_2(c^n) - I_2(\gamma_2)$$

and consequently (8-2-2) holds if and only if

$$I(c^n, c^n, Z_0^n) = I_1(c^n) - I_2(c_n)$$

for every $n \geq 0$. This is however the case because

$$c_1(T([0, 1] \times V)|_{Z_0^n}, \tau) = Q_\tau(Z_0^n) = 0.$$

□

By combining Proposition 7.2.1 and Lemmas 8.2.1–8.2.3 we obtain:

Lemma 8.2.4 *If α is a contact form on V satisfying $(\star)$ and ∂V is foliated by Reeb orbits of irrational slope $r > 0$, then*

$$\text{ECH}(V, \alpha, n[S^1]) \simeq \begin{cases} \mathbb{F} & \text{in degree } I = \sum_{k=1}^n (2\lfloor kr \rfloor + 1) \text{ for } n > 0, \\ \mathbb{F} & \text{in degree } I = 0 \text{ for } n = 0, \\ 0 & \text{for } n < 0. \end{cases}$$

8.3 Computation of $\text{ECH}(\text{int}(V), \alpha_V)$

The goal of this subsection is to compute $\text{ECH}(\text{int}(V), \alpha_V)$, where α_V is a contact form as constructed in Lemma 8.1.1.

Lemma 8.3.1 *The inclusions $V_i \subset V_j$ for $i < j$ induce inclusions of chain complexes*

$$(8-3-1) \quad \text{ECC}(V_i, \alpha_V|_{V_i}) \rightarrow \text{ECC}(V_j, \alpha_V|_{V_j}).$$

Moreover, the inclusions $V_i \subset V$ induce inclusions of chain complexes

$$\text{ECC}(V_i, \alpha_V|_{V_i}) \rightarrow \text{ECC}(\text{int}(V), \alpha_V).$$

Proof Let γ be an orbit set whose orbits are contained in V_i . We will prove that every J -holomorphic map $u: F \rightarrow \mathbb{R} \times V$ which has γ at its positive end has image in $\mathbb{R} \times V_i$. Let γ' be the orbit set at the negative end of u . We first prove that all the orbits of γ' must be contained in V_i . Arguing by contradiction, suppose $\gamma' = \gamma'_{\text{in}} \gamma'_{\text{out}}$, where the orbits of γ'_{in} are in V_i and the orbits of $\gamma'_{\text{out}} \neq \emptyset$ are in $V - V_i$. The Reeb vector field determines a homology class $s_i \in H_1(T_i; \mathbb{R})$, up to multiplication by a positive constant, which has slope r_i using Convention 6.2.1. We can also regard $[\gamma'_{\text{out}}]$ as a homology class in $H_1(T_i; \mathbb{R})$ and the slope of $[\gamma'_{\text{out}}]$ is larger than r_i because every Reeb orbit in $V - V_i$ has slope larger than r_i by Lemma 8.1.1. This implies that $[\gamma'_{\text{out}}] \cdot s_i > 0$.

Denote by u_V the composition of u with the projection $\mathbb{R} \times V \rightarrow V$ and let $\delta \in H_1(T_i; \mathbb{R})$ be the homology class of the intersection $u_V(F) \cap T_i$, oriented as the boundary of u_V restricted to V_i . (Recall that the

tori T_i are foliated by Reeb orbits of irrational slope, so that u has no ends at T_i .) Then $\delta = -[\gamma'_{\text{out}}]$, so $\delta \cdot s_i < 0$. This contradicts the positivity of intersections in dimension three (Lemma 5.2.2) and therefore all orbits in γ' are contained in V . Hence, Lemma 5.2.3(1) (blocking lemma) implies that $u(F) \subset \mathbb{R} \times V_i$. $\square$

With all these preliminary steps in place, the computation of $\text{ECH}(\text{int}(V), \alpha_V)$ is straightforward.

Proposition 8.3.2 $\text{ECH}(\text{int}(V), \alpha_V) \simeq \mathbb{F}$ and is generated by $\emptyset$.

Proof By Lemma 8.3.1,

$$(8-3-2) \quad \text{ECH}(\text{int}(V), \alpha_V) = \lim_{i \rightarrow \infty} \text{ECH}(V_i, \alpha_V|_{V_i}).$$

Moreover, all the generators of $\text{ECH}(V_i, \alpha_V|_{V_i})$ in Lemma 8.2.4 that are different from $\emptyset$ have degree $I > \lfloor 2r_i \rfloor + 1$. Since $r_i \rightarrow \infty$ and the inclusions

$$\text{ECH}(V_i, \alpha_V|_{V_i}) \rightarrow \text{ECH}(V_j, \alpha_V|_{V_j})$$

are degree-preserving, every generator different from $\emptyset$ eventually is mapped to zero in the directed system. Hence $\text{ECH}(\text{int}(V), \alpha_V) \simeq \mathbb{F}$ and is generated by $\emptyset$. $\square$

8.4 Finite-energy foliations

In this subsection we study finite-energy foliations of $\mathbb{R} \times V$ and $\mathbb{R} \times T^2 \times [1, 2]$ which have been constructed by [Wendl 2008; 2005; Taubes 2002]. Finite-energy foliations were introduced in [Hofer et al. 2003]; here we recall their definition.

Definition 8.4.1 A *finite-energy foliation* of a symplectic cobordism (W, ω) with an adapted almost complex structure J is a codimension two foliation of W such that every leaf is the image of an embedded J -holomorphic map with finite energy.

Here we are using the notion of energy from [Bourgeois et al. 2003, Section 6.1]. The ends of a finite-energy J -holomorphic map in W are asymptotic to cylinders over Reeb orbits.

The purpose of considering finite-energy foliations is twofold: they constrain holomorphic curves by the positivity of intersections and contribute to the ECH differential via the Morse–Bott construction. The foliation on $\mathbb{R} \times V$ will be used in the proof of Theorem 8.1.2(2)–(4) and the foliation on $\mathbb{R} \times T^2 \times [1, 2]$ will be used in the proofs of Lemmas 9.5.3 and 9.9.3.

8.4.1 Automatic transversality For certain moduli spaces of J -holomorphic maps in dimension four, transversality holds for topological reasons and there is no need to perturb the almost complex structure. In this subsection we describe such *automatic transversality* results of Wendl [2010]. We need to discuss automatic transversality, since the finite-energy foliations that we consider are constructed for very symmetric, and therefore nongeneric, almost complex structures.

Let $F = \bar{F} - z$, where $\bar{F}$ is a closed oriented surface and $z = \{z_1, \dots, z_r\}$ is a finite set of punctures. Following Wendl [2010], we fix a partition $\mathcal{P} = \{z_C^+, z_C^-, z_U^+, z_U^-\}$ of z . We use the superscript $+$ (resp. $-$) to indicate the punctures which correspond to the positive (resp. negative) ends and define $z_C = z_C^+ \cup z_C^-$ and $z_U = z_U^+ \cup z_U^-$.

To any puncture $z \in z_C$ we associate an orbit γ_z (which can either be nondegenerate or belong to a Morse–Bott family) and to any puncture $z \in z_U$ we associate a Morse–Bott family $\mathcal{N}_z$. We write

$$\mathcal{M}^{\mathcal{P}} = \mathcal{M}(\{\gamma_z\}_{z \in z_C^+}, \{\mathcal{N}_z\}_{z \in z_U^+}, \{\gamma_z\}_{z \in z_C^-}, \{\mathcal{N}_z\}_{z \in z_U^-})$$

for the moduli space of holomorphic maps $u: (F, j) \rightarrow (\mathbb{R} \times M, J)$, which are positively asymptotic to the orbits γ_z for $z \in z_C^+$ and to the Morse–Bott families $\mathcal{N}_z$ for $z \in z_U^+$ and are negatively asymptotic to the orbits γ_z for $z \in z_C^-$ and to the Morse–Bott families $\mathcal{N}_z$ for $z \in z_U^-$. Here we range over all complex structures j on F and divide by automorphisms of the domain.

Ends which correspond to punctures in z_C are called *constrained ends* and ends which correspond to punctures in z_U are called *unconstrained ends*. The definition of $\mathcal{M}^{\mathcal{P}}$ motivates this terminology: constrained ends are asymptotic to a specific orbit, while unconstrained ends are asymptotic to ends which can move in a Morse–Bott family.

The virtual dimension of $\mathcal{M}^{\mathcal{P}}$ at u will be denoted by $\text{ind}(u, \mathcal{P})$. Fix $\delta > 0$ arbitrarily small. For every puncture $z \in z$, we define

$$c_z = \begin{cases} \delta & \text{if } z \in z_C, \\ -\delta & \text{if } z \in z_U. \end{cases}$$

Choose a symplectic trivialization τ of $\xi|_{\gamma_z}$ which is complex linear with respect to J . Let A_{γ_z} be the asymptotic operator of γ_z . With respect to the trivialization τ , A_{γ_z} can be written in the form $-J d/dt + S(t)$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, t is the direction of γ_z , and $S(t)$ is a loop of symmetric matrices. Also let $A_{\gamma_z}^{\mathcal{P}} = A_{\gamma_z} \pm c_z \text{Id}$ be the *perturbed* asymptotic operator of γ_z , where we choose the positive (resp. negative) sign if $z \in z^+$ (resp. $z \in z^-$). This is equivalent to turning on negative (resp. positive) exponential weights at positive unconstrained (resp. constrained) ends and negative constrained (resp. unconstrained) ends.

The perturbed asymptotic operator $A_{\gamma_z}^{\mathcal{P}}$ yields a path of symplectic matrices $\Phi_z^{\mathcal{P}}$, and we define $\mu_{\tau}(\gamma_z, \mathcal{P}) = \mu(\Phi_z^{\mathcal{P}})$. We say that a puncture z is *even* if $\mu_{\tau}(\gamma_z, \mathcal{P})$ is even and we denote by $\#\Gamma_0(u, \mathcal{P})$ the number of even punctures of $(u, \mathcal{P})$. By the properties of the Conley–Zehnder index the set of even punctures, and therefore $\#\Gamma_0(u, \mathcal{P})$, does not depend on the trivialization τ .

Theorem 8.4.2 [Wendl 2010, equation (1.1) and Remark 1.2] *Let $u: F \rightarrow \mathbb{R} \times M$ be a J -holomorphic map and $\mathcal{P}$ a partition of the ends of u . Then*

$$(8-4-1) \quad \text{ind}(u, \mathcal{P}) = -\chi(F) + 2c_1(u^*\xi, \tau) + \sum_{z \in z^+} \mu_{\tau}(\gamma_z, \mathcal{P}) - \sum_{z \in z^-} \mu_{\tau}(\gamma_z, \mathcal{P}).$$

Moreover, if u is an immersion, then it is a regular point of $\mathcal{M}^{\mathcal{P}}$ if

$$(8-4-2) \quad \text{ind}(u, \mathcal{P}) > 2g(F) - 2 + \#\Gamma_0(u, \mathcal{P}).$$

The following lemma computes $\mu_\tau(\gamma_z, \mathcal{P})$ in terms of the Conley–Zehnder index of a nondegenerate perturbation of the Reeb orbit:

Lemma 8.4.3 *Suppose $\delta > 0$ is sufficiently small.*

- (1) *If γ_z is a nondegenerate orbit, then $\mu_\tau(\gamma_z, \mathcal{P}) = \mu_\tau(\gamma_z)$.*
- (2) *If γ_z belongs to a Morse–Bott family $\mathcal{N}$ and $\gamma_{\min}$ and $\gamma_{\max}$ are the nondegenerate Reeb orbits corresponding to a minimum and a maximum of a Morse function on $\mathcal{N}$, then*
 - $\mu_\tau(\gamma_z, \mathcal{P}) = \mu_\tau(\gamma_{\min})$ *if* $z \in \mathbf{z}_C^+ \cup \mathbf{z}_U^-$, *and*
 - $\mu_\tau(\gamma_z, \mathcal{P}) = \mu_\tau(\gamma_{\max})$ *if* $z \in \mathbf{z}_U^+ \cup \mathbf{z}_C^-$.
- (3) $\#\Gamma_0(u, \mathcal{P})$ *is the total number of*
 - *ends at even nondegenerate orbits,*
 - *constrained positive ends and unconstrained negative ends at positive Morse–Bott tori, and*
 - *unconstrained positive ends and constrained negative ends at negative Morse–Bott tori.*

Proof (1) This is immediate.

(2) Let $T = T_{\mathcal{N}} \subset M$ be the torus corresponding to $\mathcal{N}$ and let $g: M \rightarrow \mathbb{R}$ and $\bar{g}_{\mathcal{N}}: \mathcal{N} \rightarrow \mathbb{R}$ be C^∞ -small functions satisfying (P1)–(P4) from Section 4.1. We denote the Morse–Bott form by α_0 and its Reeb vector field by R_0 . Then the Reeb vector field of the perturbed contact form $(1+g)\alpha_0$ is $R = (1+g)^{-1}R_0 + X$, where $X \in \xi = \ker \alpha_0$ is a solution of

$$i_X d\alpha_0 = (1+g)^{-2}(dg - dg(R_0)\alpha_0).$$

If we choose an almost complex structure J on ξ and a metric h on M which is compatible with J and α_0 in the sense that R_0 is a unit vector field which is orthogonal to ξ and $h|_{\xi \otimes \xi} = d\alpha_0(\cdot, J\cdot)$, then

$$X = -(1+g)^{-2}J(\nabla g - h(\nabla g, R_0)R_0).$$

Let γ be an orbit in $\mathcal{N}$ which corresponds to a critical point of $\bar{g}$, so that γ is also a Reeb orbit for R . We can associate two asymptotic operators to γ : the operator A_γ when we regard γ as a Reeb orbit of R_0 , and the operator A'_γ when we regard γ as a Reeb orbit of R .

Let τ be the period of γ as an orbit of R and assume for simplicity that the period of γ as an orbit of R_0 is 1. Then τ is equal to the value of $1+g$ at any point of γ . If ∇ is a symmetric connection, the asymptotic operators can be written as

$$A_\gamma = -J(\nabla_t - \nabla R_0), \quad A'_\gamma = -J(\nabla_t - \tau \nabla R);$$

see [Wendl 2010, page 370]. Since $dg = 0$ and $\nabla g = 0$ along γ , we have

$$\begin{aligned} \nabla R &= (1+g)^{-1}\nabla R_0 - (1+g)^{-2}\nabla(J(\nabla g - h(\nabla g, R_0)R_0)), \\ \nabla(J(\nabla g - h(\nabla g, R_0)R_0)) &= (\nabla J)(\nabla g - h(\nabla g, R_0)R_0) + J\nabla(\nabla g - h(\nabla g, R_0)R_0) \\ &= J\nabla(\nabla g - h(\nabla g, R_0)R_0) = JHg \end{aligned}$$

along γ , where Hg is the Hessian of g restricted to the ξ -directions. Hence

$$A'_\gamma = -J(\nabla_t - \nabla R_0 + (1+g)^{-1}JHg) = A_\gamma + (1+g)^{-1}Hg.$$

If $\bar{g}$ has a minimum at γ , then $Hg \geq 0$ along γ and A'_γ has the same Conley–Zehnder index as $A_\gamma + \delta$. On the other hand, if $\bar{g}$ has a maximum at γ , then A'_γ has the same Conley–Zehnder index as $A_\gamma - \delta$.

(3) is immediate from (2). $\square$

8.4.2 Foliations on $\mathbb{R} \times V$ and $\mathbb{R} \times T^2 \times [1, 2]$ We first describe the finite-energy foliation on $\mathbb{R} \times V$. The following is proven in [Wendl 2008]; see pages 594–600, especially the removal of singularities argument on page 599; the gist of the proof is to reduce the J -holomorphic curve equation to an ODE [loc. cit., equations (37a) and (37b)].

Proposition 8.4.4 *Let α be a contact form on V as in Example 6.2.4 and J_0 a “cylindrically symmetric” almost complex structure on $\mathbb{R} \times V$ (ie J_0 depends only on the radial coordinate ρ of V) which is adapted to α . Then there is a finite-energy foliation $\mathcal{Z}_0$ of $\mathbb{R} \times V$ such that:*

- (1) $\mathbb{R} \times \text{int}(V)$ is foliated by J_0 -holomorphic planes which are positively asymptotic to the Reeb orbits on ∂V .
- (2) $\mathbb{R} \times \partial V$ is foliated by trivial cylinders over Reeb orbits of ∂V .

Any orbit of ∂V is the limit of a unique 1-dimensional $\mathbb{R}$ -invariant family of noncylindrical leaves and the projections of the leaves to $\text{int}(V)$ foliate $\text{int}(V)$ by meridian disks.

We will use a finite-energy foliation of $\mathbb{R} \times V$ in the proof of Theorem 8.1.2(2)–(4). However, the contact form used there is a small perturbation α_V of α , and for this reason we need to show that $\mathcal{Z}_0$ persists if α and J_0 are deformed.

Proposition 8.4.5 *If α_V is the C^∞ -small perturbation of α from Lemma 8.1.1, then there is a finite-energy foliation $\mathcal{Z}_1$ of $(\mathbb{R} \times V, d(e^s \alpha_V))$ which is isotopic to $\mathcal{Z}_0$ by the lift to $\mathbb{R} \times V$ of an isotopy of V relative to the boundary.*

Proof A leaf u of $\mathcal{Z}_0$, considered as a J_0 -holomorphic map with a *constrained* end, has Fredholm index one and is automatically transverse by Theorem 8.4.2. Indeed, by Lemma 8.4.3, the index of u , as a J_0 -holomorphic map with constrained end, is equal to the index of a J_ε -holomorphic plane u_ε which limits to a hyperbolic orbit h (ie the minimum of the Morse–Bott family) on the boundary for a perturbed contact form. If τ is the trivialization of ξ along h given by $\xi \cap T\partial V$, then $\chi(D^2) = 1$, $c_1(u_\varepsilon^* \xi, \tau) = 1$, $\mu_\tau(h) = 0$, and therefore $\text{ind}(u) = \text{ind}(u_\varepsilon) = 1$. The same leaf u , considered as a J_0 -holomorphic map with an *unconstrained* end, has Fredholm index two and is also automatically transverse.

Let $\mathcal{M}_0$ be the 2-dimensional moduli space of J_0 -holomorphic planes which are leaves of $\mathcal{Z}_0$. By the unconstrained automatic transversality, if we perturb α and the almost complex structure J_0 slightly, then each leaf of $\mathcal{Z}_0$ is deformed to a J -holomorphic curve for the new almost complex structure J and the

space $\mathcal{M}_1$ of deformed J -holomorphic curves is diffeomorphic to $\mathcal{M}_0$. On the other hand, the constrained automatic transversality implies that for each Reeb orbit in ∂V there is exactly one $\mathbb{R}$ -invariant family of J -holomorphic maps in $\mathcal{M}_1$ positively asymptotic to that orbit.

The maps in $\mathcal{M}_1$ are embeddings because embeddedness is an open condition and the exponential decay estimates imply that no self-intersection can be created near infinity. Moreover, the relative intersection number of their images is zero and by the positivity of intersections, their images are pairwise disjoint, so they define a finite-energy foliation $\mathcal{Z}_1$ of $\mathbb{R} \times V$. $\square$

Now we discuss a finite-energy foliation $\mathcal{Z}_2$ on a completed interpolating cobordism $(\mathbb{R} \times T^2 \times [1, 2], \lambda)$ between two contact forms satisfying (6-1-1). In the case of a symplectization, this foliation was constructed by Wendl [2008].

We assume that every slice $\{s\} \times T^2 \times [1, 2]$ is a contact type hypersurface; Then we can write $\lambda = e^s \alpha_s$, where α_s is a contact form on $\{s\} \times T^2 \times [1, 2]$ given by (6-1-1) for pairs of functions (f_s, g_s) which depend on s and y . The forms α_s will define a 2-plane field ξ and a vector field R on $\mathbb{R} \times T^2 \times [1, 2]$ which restrict to the contact structure and the Reeb vector field on each slice $\{s\} \times T^2 \times [1, 2]$. In particular, R is tangent to the tori $\{s\} \times T^2 \times \{y\}$. Moreover, we assume that α_s is constant in s near $\mathbb{R} \times T^2 \times \{1, 2\}$ and that R is parallel to ∂_t when $y = 1, 2$ and not parallel to it otherwise. Finally, we assume that the tori $\{s\} \times T^2 \times \{1\}$ and $\{s\} \times T^2 \times \{2\}$ are foliated by Morse–Bott families $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, for each s , where $\mathcal{N}_1$ is negative and $\mathcal{N}_2$ is positive.

We take an almost complex structure J on $\mathbb{R} \times T^2 \times [1, 2]$ with coordinates (s, ϑ, t, y) such that the following hold:

- J is adapted to λ .
- J is invariant in the s -direction on the cylindrical ends of the cobordism.
- J is invariant in the ϑ, t -directions.
- $J(\partial_s) = R$.
- J sends $\partial_y \in \xi$ to the tangent space to $\{s\} \times T^2 \times \{y\}$.

For the existence of such an almost complex structure we need to verify that the plane distribution generated by ∂_s and R is $d\lambda$ -symplectic, and that ∂_y belongs to its $d\lambda$ -orthogonal. The first property is guaranteed if α_s varies sufficiently slowly in s , while the second property follows from the fact that $\alpha_s(\partial_y) = 0$ everywhere. Finally, the symmetries of J reflect the symmetries of the forms α_s .

Lemma 8.4.6 *Let $(\mathbb{R} \times T^2 \times [1, 2], \lambda)$ be an exact symplectic cobordism with an adapted almost complex structure J as above. Then there is a 2-dimensional family $\mathcal{Z}_2$ of holomorphic cylinders $Z_{s, \vartheta}$ on $\mathbb{R} \times T^2 \times [1, 2]$ for $(s, \vartheta) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ which foliate $\mathbb{R} \times \text{int}(T^2 \times [1, 2])$ and project to cylinders $\vartheta = \text{const}$ in $\text{int}(T^2 \times [1, 2])$. Each cylinder $Z_{s, \vartheta}$ is positively asymptotic to a Reeb orbit in $\mathcal{N}_2$ and negatively asymptotic to a Reeb orbit in $\mathcal{N}_1$.*

Proof Let us write $v = J(\partial_y)$. Our conditions on J and R imply that $\partial_t = a(s, y)v + b(s, y)R$ with $b(s, y) \neq 0$ everywhere and $a(s, y) = 0$ only when $y = 1$ or $y = 2$, in which case $\partial a / \partial y|_{y=0,1} \neq 0$. Then $J(\partial_t) = -a(s, y)\partial_y - b(s, y)\partial_s$. The vector fields ∂_t and $Y(s, y) = a(s, y)\partial_y + b(s, y)\partial_s$ span a J -invariant 2-plane distribution on $\mathbb{R} \times T^2 \times [1, 2]$. Since a and b do not depend on t and ϑ , this distribution is integrable and every integral submanifold in $\mathbb{R} \times T^2 \times [1, 2]$ is the product of $\mathbb{R}/\mathbb{Z}$ with coordinate t and an integral curve of Y on the strip $\mathbb{R} \times [1, 2]$.

The functions a and b are bounded in $\mathbb{R} \times [0, 1]$ because $\partial a / \partial s|_{|s| \gg 0} = \partial b / \partial s|_{|s| \gg 0} = 0$. This implies that Y is complete. Moreover, the maximal integral curves of Y on $\mathbb{R} \times (1, 2)$ project diffeomorphically onto $(1, 2)$ and have vertical asymptotes for $y \rightarrow 1$ and $y \rightarrow 2$ because $a(s, y) \neq 0$ when $y \neq 1, 2$. $\square$

Lemma 8.4.7 *Let $u_{s,\vartheta}: \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times T^2 \times [1, 2]$ be a J -holomorphic map which parametrizes the holomorphic cylinder $Z_{s,\vartheta}$. Then $(u_{s,\vartheta}, \mathcal{P})$ satisfies automatic transversality if at least one end is unconstrained.*

Proof By Theorem 8.4.2,

$$\text{ind}(u_{s,\vartheta}, \mathcal{P}) = \mu_\tau(\gamma_2, \mathcal{P}) - \mu_\tau(\gamma_1, \mathcal{P}),$$

where $\gamma_i \in \mathcal{N}_i$, so $\text{ind}(u_{s,\vartheta}, \mathcal{P}) = 2 - \#\Gamma_0(u_{s,\vartheta}, \mathcal{P})$ by Lemma 8.4.3. Hence the condition for automatic transversality in Theorem 8.4.2 holds if $\#\Gamma_0(u_{s,\vartheta}, \mathcal{P}) < 2$. Both the constrained negative end at $\mathcal{N}_1$ and the constrained positive end at $\mathcal{N}_2$ are even and the lemma follows. $\square$

8.4.3 Constraints on holomorphic curves Finite-energy foliations constrain J -holomorphic maps with the same asymptotics. The following lemma describes an instance of this phenomenon. A similar situation has also been considered in [Wendl 2013].

Lemma 8.4.8 *Let P be a compact oriented surface and α a Morse–Bott contact form on $S^1 \times P$ such that $S^1 \times \partial P$ is a union of Morse–Bott tori and $\{\vartheta\} \times \partial P$ is a union of Reeb orbits for each $\vartheta \in S^1$. If $\mathbb{R} \times S^1 \times P$ has a finite-energy foliation $\mathcal{Z}$ on which $\mathbb{R} \times S^1$ acts freely and transitively and is such that every leaf projects diffeomorphically to $\text{int}(P)$, then every somewhere-injective finite-energy J -holomorphic map $u: F \rightarrow \mathbb{R} \times S^1 \times P$ with no ends at a Reeb orbit in $S^1 \times \text{int}(P)$ is a leaf of $\mathcal{Z}$.*

Proof Let $Z_{s,\vartheta}$ be the leaves of $\mathcal{Z}$ parametrized by $(s, \vartheta) \in \mathbb{R} \times S^1$. Suppose first that there is a leaf Z_{s_0,ϑ_0} such that $u(F) \cap Z_{s_0,\vartheta_0} \neq \emptyset$ and which is asymptotic to different Reeb orbits than u . The intersection points in $u(F) \cap Z_{s_0,\vartheta_0}$ are isolated and positive. However, $u(F) \cap Z_{s'_0,\vartheta_0} = \emptyset$ if s'_0 is sufficiently large, a contradiction. Hence there exists some $\vartheta_0 \in S^1$ such that $u(F) \subset \bigcup_{s \in \mathbb{R}} Z_{s,\vartheta_0}$ and the leaves Z_{s,ϑ_0} are asymptotic to the same Reeb orbits as u . If $u(F)$ is not contained in a leaf, this forces the intersection $u(F) \cap Z_{s_0,\vartheta_0}$ to be one-dimensional for some $s_0 \in \mathbb{R}$. This is too large an intersection, and the unique continuation for J -holomorphic maps [McDuff and Salamon 2004, Theorem 2.3.2] implies that $u(F)$ is a leaf of $\mathcal{Z}$. $\square$

Remark 8.4.9 The proof of [Lemma 8.4.8](#) goes through unchanged for the foliation $\mathcal{Z}_2$ constructed in [Lemma 8.4.6](#), even though the curves $Z_{s,\vartheta}$ and $Z_{s',\vartheta}$ are not translations of one another unless $(\mathbb{R} \times T^2 \times [1, 2], \lambda)$ is a symplectization. In fact, they still project to the same annulus in $T^2 \times [1, 2]$ and, given any point in that annulus, their preimages $x \in Z_{s,\vartheta}$ and $x' \in Z_{s',\vartheta}$ become arbitrarily far apart in the s -coordinate when $|s' - s| \rightarrow +\infty$. These properties of the foliation $\mathcal{Z}_2$ are sufficient to make the proof of [Lemma 8.4.8](#) work.

8.5 Completion of proof of [Theorem 8.1.2](#)

In this subsection we prove (2)–(4) of [Theorem 8.1.2](#).

(2) The inclusion $\text{ECC}(\text{int}(V), \alpha_V) \subset \text{ECC}^\sharp(V, \alpha_V)$ is an inclusion of chain complexes since no J -holomorphic curve in $\mathbb{R} \times V$ with all positive ends in $\text{int}(V)$ can have a negative end on ∂V by the trapping lemma. Moreover, the map

$$\text{ECC}^\sharp(V, \alpha_V) \rightarrow \text{ECC}(\text{int}(V), \alpha_V), \quad \gamma \mapsto 0, \quad h'\gamma \mapsto \gamma,$$

where γ is an orbit set constructed from orbits in $\text{int}(V)$, induces an isomorphism of complexes

$$\text{ECC}^\sharp(V, \alpha_V) / \text{ECC}(\text{int}(V), \alpha_V) \simeq \text{ECC}(\text{int}(V), \alpha_V).$$

This is due to the fact that h' is a hyperbolic orbit and appears with exponent at most one in a generator of $\text{ECC}^\sharp(V, \alpha_V)$. From this we have an exact triangle

$$\begin{array}{ccc} \text{ECH}(\text{int}(V), \alpha_V) & \xrightarrow{\quad} & \text{ECH}(\text{int}(V), \alpha_V) \\ & \nwarrow \quad \nearrow & \\ & \text{ECH}^\sharp(V, \alpha_V) & \end{array}$$

which splits according to homology classes in $H_1(V)$. Then [Proposition 8.3.2](#) implies that

$$\text{ECH}^\sharp(V, \alpha_V, n[S^1]) = 0$$

when $n \neq 0$.

It remains to show that $\text{ECH}^\sharp(V, \alpha_V, n[S^1]) \simeq 0$ for $n = 0$. Its chain complex $\text{ECC}^\sharp(V, \alpha_V, 0)$ is generated by h' and $\emptyset$. We claim that $\partial h' = \emptyset$. By [Proposition 8.4.5](#), there is a finite-energy foliation $\mathcal{Z}_1$ on $(\mathbb{R} \times V, d(e^s \alpha_V))$ whose leaves (in $\mathbb{R} \times \text{int}(V)$) are J -holomorphic planes which are positively asymptotic to the Morse–Bott family on ∂V . This foliation constrains the J -holomorphic curves that limit to orbits on ∂V at the positive ends. Indeed, by [Lemma 8.4.8](#), every holomorphic curve which is positively asymptotic to a simple Reeb orbit on ∂V and has no negative ends must be a plane in $\mathcal{Z}_1$. The leaves of $\mathcal{Z}_1$ also contribute to the differential of $\text{ECC}(V, \alpha_V)$ since they are automatically transverse by [Theorem 8.4.2](#). Hence $\partial h' = \emptyset$, which implies the vanishing of $\text{ECH}^\sharp(V, \alpha_V, 0)$.

(3) We define a filtration $\mathcal{F}$ on $\mathrm{ECC}(V, \alpha_V)$ as follows: Given an orbit set $(e')^m \gamma$, where γ does not have any e' -terms, we set

$$\mathcal{F}((e')^m \gamma) = m.$$

This defines an ascending filtration of chain complexes: since J -holomorphic maps to $\mathbb{R} \times V$ can have only positive ends at e' by the trapping lemma, the differential of $\mathrm{ECC}(V, \alpha_V)$ cannot increase the exponent of e' . The E^1 -term of the associated spectral sequence is isomorphic to $\mathrm{ECH}^\sharp(V, \alpha_V)$ at each filtration level. By (1), $\mathrm{ECH}^\sharp(V, \alpha_V) = 0$, and the spectral sequence converges to 0.

(4) The restriction of $\mathcal{F}$ to $\mathrm{ECC}^b(V, \alpha_V)$ induces a filtration on $\mathrm{ECC}^b(V, \alpha_V)$, which we still denote by $\mathcal{F}$. The E^1 -term of the spectral sequence for $\mathcal{F}$ is isomorphic to

$$\bigoplus_{m=0}^{\infty} \mathrm{ECH}(\mathrm{int}(V), \alpha_V) \cdot (e')^m.$$

Since $\mathrm{ECH}(\mathrm{int}(V), \alpha_V) \simeq \mathbb{F}\{\emptyset\}$ by [Theorem 8.1.2](#), the E^1 -term of the spectral sequence is $\mathbb{F}[e']$. All higher differentials vanish for degree reasons: recall that ECH has a $\mathbb{Z}/2$ grading in which generators with no hyperbolic orbits have even grading. Hence $E^1 = E^\infty$ is the graded group of the induced filtration on $\mathrm{ECH}^b(V, \alpha_V)$. Since the filtration $\mathcal{F}$ on $\mathrm{ECC}^b(V, \alpha_V)$ is bounded below and exhaustive, the spectral sequence converges by [\[Weibel 1994, Theorem 5.5.5\]](#) and therefore $\mathrm{ECH}^b(V, \alpha_V) \simeq \mathbb{F}[e']$.

9 Proof of [Theorem 1.1.1](#)

In this section we prove [Theorem 1.1.1](#). The proof was greatly influenced by Michael Hutchings, who encouraged us to look for an appropriate filtration.

9.1 Intuitive idea behind [Theorem 1.1.1](#)

We briefly explain the intuitive idea behind [Theorem 1.1.1](#). We recall that M denotes a connected, closed, oriented three-manifold and K is a null-homologous knot in M . Suppose for the moment that the contact form α on M , in a neighborhood $V \simeq D^2 \times S^1$ of K , is given by [Example 6.2.3](#). In other words, the concentric tori $T_\rho \subset V$, $\rho \neq 0$, are foliated by Reeb orbits of irrational slope $1/\nu$. We would like to take the limit as $\nu \rightarrow 0$; in the limit ∂V is foliated by Reeb orbits of slope ∞ . Let us write $N = M - \mathrm{int}(V)$. There should be a one-to-one correspondence, modulo $\mathbb{R}$ -translations, between holomorphic curves u in $\mathbb{R} \times M$ of ECH index 1 which intersect the binding k times, and holomorphic curves u' in $\mathbb{R} \times N$ of ECH index 1 which have negative ends at an elliptic orbit e of slope ∞ with total multiplicity k . Also, as we take $\delta \rightarrow 0$, the Conley–Zehnder index of the binding, measured with respect to the longitudinal framing on V , ie the framing given by a Seifert surface Σ for K , goes to ∞ . This suggests that we should be able to effectively ignore the binding if we could take the limit.

The actual proof — at least the one we could find — is considerably more complicated, and uses three ingredients:

- (i) the calculation of ECH on the solid torus from [Section 8](#),
- (ii) some understanding of holomorphic curves that project to a neighborhood of K , and
- (iii) a filtration on $\text{ECC}(M)$.

9.2 Description of the contact forms

We start with a description of the contact forms and their Reeb orbits on M that we use in the proof of [Theorem 1.1.1](#). We fix a neighborhood $V \simeq D^2 \times S^1$ of K and decompose M as

$$M = N \cup (T^2 \times [1, 2]) \cup V.$$

Since K is an oriented null-homologous knot, there is a properly embedded oriented surface $S \subset N$ whose boundary $\partial S \subset \partial V$ is a longitude for K . On V we choose cylindrical coordinates (ρ, ϕ, θ) such that $\partial V = \{\rho = 1\}$ and $\partial S = \{\rho = 1, \phi = \phi_0\}$. On $T^2 \times [1, 2] \simeq (\mathbb{R}^2/\mathbb{Z}^2) \times [1, 2]$ we choose coordinates (ϑ, t, y) such that $(\vartheta, t, 2)$ is identified with $(\rho, \phi, \theta) = (1, 2\pi t, 2\pi\vartheta) \in \partial V$. We identify a neighborhood of ∂N in N with $T^2 \times [0, 1]$ such that $\partial N = T^2 \times \{1\}$ and the coordinates (ϑ, t, y) on $T^2 \times [0, 1]$ extend those on $T^2 \times [1, 2]$; similarly, we identify a neighborhood of ∂V in V with $T^2 \times [2, \frac{5}{2}]$.

We will work with an increasing sequence $L_i \rightarrow +\infty$ and a sequence of Morse–Bott contact forms α_i on M such that

- $\alpha_i|_N$ is a fixed Morse–Bott contact form α which is nondegenerate on $\text{int}(N)$ and its Reeb vector field is positively transverse to S ;
- $\alpha_i|_{T^2 \times [1, 2]}$ is a contact form α_{δ_i} as in [Example 6.1.2](#) which is chosen so that all the Reeb orbits in $T^2 \times (1, 2)$ have action larger than L_i ; and
- $\alpha_i|_V = c_{\delta_i} \alpha_V$ for a fixed contact form α_V constructed as in [Lemma 8.1.1](#) and a decreasing sequence c_{δ_i} which is bounded above by 1 and bounded below by a positive constant.

We also assume the following technical condition:

- There is a decreasing sequence $\epsilon_i \rightarrow 0$ such that α_i agrees with α_{i+1} on $N \cup (T^2 \times [1, 1 + \epsilon_i])$ and with a constant positive multiple of α_{i+1} on $V \cup (T^2 \times [2 - \epsilon_i, 2])$.

We will refer to $T^2 \times (1, 2)$ as the *no man's land*.

The contact form α on N can be constructed using the techniques developed in [\[Colin and Honda 2005; Colin et al. 2011b\]](#). The construction is described in [Section 9.3.1](#) in the special case where K is the binding of an open book decomposition of M and N is the mapping torus of a diffeomorphism of S .

The contact forms α_i are Morse–Bott and all the Morse–Bott tori are of the form $T^2 \times \{y\}$ with $y \in [1, 2]$. In particular, $\partial N = T^2 \times \{1\}$ is foliated by a *negative* Morse–Bott family $\mathcal{N}_1$ and $\partial V = T^2 \times \{2\}$ by a *positive* Morse–Bott family $\mathcal{N}_2$. Both families have infinite slope, ie the Reeb orbits on both tori are meridians of K .

We construct L_i -nondegenerate contact forms $\alpha'_i = f_i \alpha_i$, where the perturbing functions f_i are as in [Section 4.5](#). We choose f_i so that the Morse–Bott family $\mathcal{N}_1$ corresponding to ∂N is perturbed into an elliptic orbit e and a hyperbolic orbit h , the Morse–Bott family $\mathcal{N}_2$ corresponding to ∂V is perturbed into a hyperbolic orbit h' and an elliptic orbit e' , no new closed orbits with action less than L_i are created, and $f_i \equiv 1$ in a neighborhood of all nondegenerate Reeb orbits of α_i with action less than L_i .

For all i we choose regular almost complex structures J_i adapted to α_i and J'_i adapted to α'_i such that all the J_i are fixed on the contact structure outside $T^2 \times [1 - \epsilon_i, 2 + \epsilon_i]$ and J'_i is an arbitrarily small perturbation of J_i .

We will also consider interpolating cobordisms $(\mathbb{R} \times M, \hat{\lambda}_i)$ from (M, α_i) to a rescaling of (M, α_{i+1}) and $(\mathbb{R} \times M, \hat{\lambda}'_i)$ from (M, α'_i) to a rescaling of (M, α'_{i+1}) . By construction, λ'_i is an arbitrarily small perturbation of λ_i . We fix compatible almost complex structures $\hat{J}_i$ on $(\mathbb{R} \times M, \hat{\lambda}_i)$ and $\hat{J}'_i$ on $(\mathbb{R} \times M, \hat{\lambda}'_i)$ such that they are both regular and $\hat{J}'_i$ is an arbitrarily small perturbation of $\hat{J}_i$.

We assume that the perturbing functions are close enough to 1 that the following hold:

- (MB₁) For $k = 1, 2$, if γ_+ and γ_- are generators of $\text{ECC}^{L_i}(M, \alpha'_i)$ and $u \in \mathcal{M}_{J'_i}^{I=k}(\gamma_+, \gamma_-)$, then there is a corresponding $u_\infty \in \mathcal{M}_{J_i}^{\text{MB}, I=k}(\gamma_+, \gamma_-)$.
- (MB₀) If γ_+ and γ_- are generators of $\text{ECC}^{L_i}(M, \alpha'_i)$ and $\text{ECC}^{L_{i+1}}(M, \alpha'_{i+1})$, respectively, and $u \in \mathcal{M}_{\hat{J}'_i}^{I=0}(\gamma_+, \gamma_-)$, then there is a corresponding $u_\infty \in \mathcal{M}_{\hat{J}_i}^{\text{MB}, I=0}(\gamma_+, \gamma_-)$.

Recall from [Definition 4.2.2](#) that $\mathcal{M}_J^{\text{MB}}(\gamma_+, \gamma_-)$ denotes the set of Morse–Bott J -holomorphic buildings from γ_+ to γ_- .

For reference we enumerate the main properties of the Reeb vector fields of the contact forms α_i and their perturbations α'_i :

- (1) α_i is Morse–Bott and α'_i is L_i -nondegenerate.
- (2) R_{α_i} is positively transverse to $S \subset N$ and the meridian disks in $\text{int}(V)$.
- (3) $\alpha_i|_N = \alpha$ and $\alpha_i|_V = c_{\delta_i} \alpha_V$, where the sequence c_{δ_i} is decreasing, bounded above by 1 and bounded below by a positive constant and the contact form α_V is constructed as in [Lemma 8.1.1](#).
- (4) α_i and α_{i+1} coincide on $N \cup (T^2 \times [1, 1 + \epsilon_i])$ and are constant multiples of one another on $V \cup (T^2 \times [2 - \epsilon_i, 2])$, where $\epsilon_i \rightarrow 0$ is a decreasing sequence.
- (5) The Reeb orbits of α_i in the no man’s land come in Morse–Bott families of large negative slope and their action is bounded below by L_i .
- (6) There are concentric solid tori $V_0 \subset V_1 \subset \dots \subset V$ such that ∂V_j for $j = 0, 1, \dots$ is foliated by dense Reeb orbits of irrational slope $r_j > 0$ with $\lim_{j \rightarrow \infty} r_j = +\infty$ for any contact form α_i .
- (7) ∂N is foliated by a negative Morse–Bott family $\mathcal{N}_1$ of Reeb orbits of α_i of slope ∞ . After perturbation, $\mathcal{N}_1$ becomes a pair of orbits e and h . Their Conley–Zehnder indices with respect to the framing coming from ∂N (given by $T(\partial N) \cap \xi$) are $\mu(e) = -1$ and $\mu(h) = 0$.

- (8) ∂V is foliated by a positive Morse–Bott family $\mathcal{N}_2$ of slope ∞ . After perturbation, $\mathcal{N}_2$ becomes a pair of orbits e' and h' . Their Conley–Zehnder indices with respect to the framing coming from ∂V are $\mu(e') = 1$ and $\mu(h') = 0$.

9.3 Construction of the contact forms

In this subsection we construct the contact forms α_i when K is the binding of an open book decomposition. In this case N is the mapping torus of a diffeomorphism $h: S \rightarrow S$ such that $h|_{\partial S} = \text{id}$. This means that

$$N = (S \times [0, 1]) / (x, 1) \sim (h(x), 0),$$

where $x \in S$ and t is the coordinate for $[0, 1]$. Using the coordinates (θ, ϕ) from [Section 9.2](#) we identify the isotopy classes of simple closed curves in ∂N (and in all parallel tori) with rational numbers so that the meridian has slope ∞ and ∂S has slope 0.

Remark 9.3.1 The above slope convention is the same as the usual surgery convention for performing surgery along the binding.

9.3.1 Construction of the contact form on N We take a 1-form β on S such that $\omega = d\beta$ is a positive area form on S and $\beta = cy \, d\theta$ in a neighborhood $N(\partial S) \subset S$ of ∂S . Here $c > 0$ is a small constant and $N(\partial S)$ is identified with $[1 - \delta, 1] \times \mathbb{R}/\mathbb{Z}$ with coordinates (y, θ) .

We assume that the diffeomorphism $h: S \xrightarrow{\sim} S$ satisfies $h|_{N(\partial S)} = \text{id}$. Let $\text{Symp}(S, \partial S, \omega)$ be the group of symplectomorphisms of (S, ω) which restrict to the identity on a neighborhood of ∂S . By Moser's lemma, there is an isotopy of h relative to ∂S such that the resulting diffeomorphism — also called h by abuse of notation — is in $\text{Symp}(S, \partial S, \omega)$.

Lemma 9.3.2 (Giroux) *Given $h \in \text{Symp}(S, \partial S, \omega)$, there exists an isotopy $\{h_t\}_{t \in [0, 1]}$ in $\text{Symp}(S, \partial S, \omega)$ such that $h_0 = h$ and $h_1^* \beta - \beta = df$ for some positive function f on S .*

Proof Let $\mu = h^* \beta - \beta$ and let Y be the vector field which satisfies $i_Y \omega = -\mu$. By Cartan's formula, we compute that $\mathcal{L}_Y \omega = i_Y d\omega + d(i_Y \omega) = -d\mu = 0$ and $\mathcal{L}_Y \mu = i_Y d\mu + d(i_Y \mu) = 0$. Hence the flow ϕ_t of Y preserves ω and μ . Moreover, ϕ_t is equal to the identity near ∂S , where $\mu = 0$.

Now let $h_t = h \circ \phi_t$. We then compute that

$$\frac{d}{dt} h_t^* \beta = \phi_t^* (\mathcal{L}_Y h^* \beta) = d(\phi_t^* (i_Y h^* \beta)) + \phi_t^* (i_Y d(h^* \beta)) = dg_t + \phi_t^* (i_Y \omega) = dg_t - \phi_t^* \mu = dg_t - \mu,$$

where $g_t = \phi_t^* (i_Y h^* \beta)$. Hence

$$(9-3-1) \quad \frac{d}{dt} h_t^* \beta = dg_t + \beta - h^* \beta.$$

Integrating (9-3-1) yields $h_1^* \beta - \beta = df$, where $f = \int_0^1 g_t \, dt + C$ for a sufficiently large constant C . $\square$

By Lemma 9.3.2, we assume that $h \in \text{Symp}(S, \partial S, \omega)$ satisfies $h^* \beta - \beta = df$. Next we construct a contact form on N whose corresponding Reeb vector field is transverse to the fibers and has first return map h .

Lemma 9.3.3 *Let h be a diffeomorphism in $\text{Symp}(S, \partial S, \omega)$ which satisfies $h^* \beta - \beta = df$ for some function f on S . Then there is a contact form $\alpha = f_t dt + \beta_t$ on N , where f_t is a family of positive functions on S and β_t is a family of 1-forms on S , such that the corresponding Reeb vector field R_α is transverse to all the fibers $S \times \{t\}$ and h is the first return map of R_α .*

For a more complete discussion of the realizability of surface symplectomorphisms as the first return map of a Reeb vector field, we refer the reader to [Colin et al. 2008].

Proof Consider the 1-form $\alpha = f_t dt + \beta_t$ on $S \times [0, 1]$, where f_t is to be determined, $\beta_0 = \beta$, $\beta_1 = h^* \beta$, and

$$\beta_t = \chi(t)\beta_1 + (1 - \chi(t))\beta_0$$

interpolates between β_0 and β_1 . Here we take $\chi: [0, 1] \rightarrow [0, 1]$ such that $\chi(0) = 0$, $\chi(1) = 1$, $d\chi(t)/dt = \dot{\chi}(t) \geq 0$ and χ is constant near 0 and 1.

Using the condition $h^* \beta - \beta = d_S f$, we verify that the 1-form $\dot{\beta}_t$ is exact on S :

$$\dot{\beta}_t = \dot{\chi}(t)(\beta_1 - \beta_0) = \dot{\chi}(t)(d_S f) = d_S(\dot{\chi}(t)f).$$

Here d_S is the exterior derivative on S . We then take $f_t = \dot{\chi}(t)f + c$, where c is an arbitrary positive constant such that $f_t > 0$ (and is different from the c in $\beta = cy d\theta$ from the beginning of Section 9.3.1). Then $\dot{\beta}_t = d_S f_t$. Since χ is constant near $t = 0$ and $t = 1$, f_t is also constant, and so is β_t . In particular, $h^* f_1 = f_0$.

We now compute that

$$d\alpha = d_S f_t \wedge dt + d_S \beta_t + dt \wedge \dot{\beta}_t = d_S f_t \wedge dt + \omega + dt \wedge d_S f_t = \omega.$$

Hence α is a contact form, its Reeb vector field is parallel to ∂_t on $S \times [0, 1]$, and its first return map is h . $\square$

Now we make a slight modification to α so that ∂N becomes a *negative* Morse–Bott family — one that behaves like a sink for J -holomorphic maps in $\mathbb{R} \times N$.

On $T_1 = \partial N$, the germ of α is given by $f(y) dt + g(y) d\theta$, where $f(y) = C$ and $g(y) = cy$. Here $c > 0$ is a small constant and $C > 0$ is a large constant. We extend α to $T^2 \times [1, 1 + \varepsilon]$ by extending $(f(y), g(y))$ to $y \in [1, 1 + \varepsilon]$ as follows:

- (1) $(f(y), g(y))$ satisfies (6-1-2).
- (2) $(f(y), g(y))$ for $y \in [1, 1 + \varepsilon]$ is close to $(f(1), g(1))$.
- (3) $(f(y), g(y)) = (f(1 + \varepsilon) + (y - (1 + \varepsilon))^2, g(1 + \varepsilon) + (y - (1 + \varepsilon)))$ near $y = 1 + \varepsilon$.

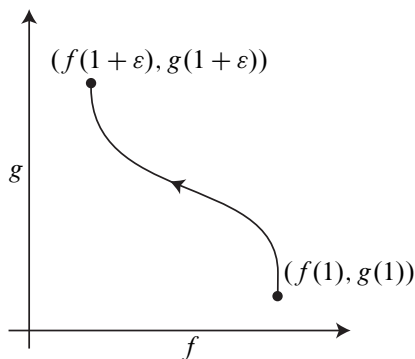


Figure 4: Trajectory of $(f(y), g(y))$. The f -axis and g -axis do not necessarily intersect at $(0, 0)$ in this figure.

See Figure 4. In particular, condition (3) implies that $(f'(1 + \varepsilon), g'(1 + \varepsilon))$ is parallel to $(0, 1)$. Hence $T_{1+\varepsilon}$ is foliated by a Morse–Bott family of Reeb orbits of slope ∞ . We write α for the extension of α to $N \cup (T^2 \times [1, 1 + \varepsilon])$.

We now consider the deformation retract

$$\phi: N \cup (T^2 \times [1, 1 + \varepsilon]) \xrightarrow{\sim} N,$$

obtained by flowing along the vector field $X = -a(y)\partial_y$, where $a(y) = 1$ on $T^2 \times [1, 1 + \varepsilon]$ and damps out to zero on $T^2 \times [1 - \varepsilon, 1]$. Finally, we perturb $\phi_*\alpha$ on N so that all Reeb orbits in $\text{int}(N)$ become nondegenerate, while keeping ∂N Morse–Bott. *The resulting form will be called α in the rest of the paper.*

9.3.2 Extension to M The contact form α has the form

$$\alpha = (b + (y - 1)) d\theta + (a + (y - 1)^2) dt$$

in some collar $T^2 \times [1 - \epsilon, 1]$ of ∂N . Here ϵ is different from the ε in Section 9.3.1.

Choose a decreasing sequence of irrational numbers $\delta_i \rightarrow 0$ and a contact form α_{δ_i} on $T^2 \times [1, 2]$ for each i as in Example 6.1.2 with $f(1) = a$ and $g(1) = b$. Then α on N and α_{δ_i} on $T^2 \times [1, 2]$ glue to a smooth contact form on $N \cup (T^2 \times [1, 2])$. Moreover, there is an increasing sequence $L_i \rightarrow +\infty$ such that all Reeb orbits of α_{δ_i} in $T^2 \times (1, 2)$ have action greater than L_i .

Fix a contact form $\alpha_{f,g}$ on $V \simeq D^2 \times S^1$ as in Example 6.2.4. For each i , a multiple of $\alpha_{f,g}$ glues smoothly to the contact form α_{δ_i} on $T^2 \times [1, 2]$. Let c_{δ_i} be the scaling factor. Then α_{δ_i} glues smoothly also to $c_{\delta_i}\alpha_V$, where α_V is the contact form obtained by applying the construction of Lemma 8.1.1 to $\alpha_{f,g}$. By putting all three pieces together we obtain the contact forms α_i on M .

9.4 The filtrations $\mathcal{F}_i$

For each i we define a filtration $\mathcal{F}_i$ on $\text{ECC}^{L_i}(M, \alpha'_i)$. We first identify $\text{ECC}^{L_i}(M, \alpha'_i)$, as a vector space, with a subspace of

$$\text{ECC}(V, \alpha_V) \otimes \text{ECC}(N, \alpha).$$

This is possible because the Reeb orbits of α'_i in the no man's land have actions greater than L_i and those in V coincide with the Reeb orbits of α_V , up to reparametrization. The generators of $\text{ECC}^{L_i}(V, \alpha'_i)$ will be denoted by $\gamma \otimes \Gamma$, where $\gamma \in \text{ECC}(V, \alpha_V)$ and $\Gamma \in \text{ECC}(N, \alpha)$. Choose an identification

$$\eta: H_1(V; \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$$

so that the homology class of the null-homologous knot K is mapped to 1. Then we define the ascending filtration $\mathcal{F}_i: \text{ECC}^{L_i}(M, \alpha'_i) \rightarrow \mathbb{Z}^{\geq 0}$ as

$$\mathcal{F}_i\left(\sum_n \gamma_n \otimes \Gamma_n\right) = \max_n \eta([\gamma_n]).$$

We define $\mathcal{F}_i^p$ as $\mathcal{F}_i^p = \{x \in \text{ECC}^{L_i}(M, \alpha'_i) : \mathcal{F}_i(x) \leq p\}$. Note that these filtrations are uniformly bounded below because $\mathcal{F}_i^p = 0$ for $p < 0$.

Lemma 9.4.1 *Let $u: F \rightarrow \mathbb{R} \times M$ be a J'_i -holomorphic map which is asymptotic to $\gamma \otimes \Gamma$ at the positive end and to $\gamma' \otimes \Gamma'$ at the negative end. Then*

$$\mathcal{F}_i(\gamma \otimes \Gamma) \geq \mathcal{F}_i(\gamma' \otimes \Gamma').$$

Proof By (MB₁) there is a J_i -holomorphic Morse–Bott building from $\gamma \otimes \Gamma$ to $\gamma' \otimes \Gamma'$. Let $\bar{u}: \bar{F} \rightarrow \mathbb{R} \times M$ be the holomorphic part of this building—which may be disconnected because α_i is not necessarily nice—and denote the projection to M by $\bar{u}_M$.

We will use the tori $T_n = \partial V_n$ in V from Lemma 8.1.1 to constrain the ends of $\bar{u}$. We recall that T_n is foliated by dense Reeb orbits of irrational slope r_n with $r_n \rightarrow +\infty$. Let δ_n be the homology class of $\bar{u}_M(\bar{F}) \cap T_n$, oriented as the boundary of $\bar{u}_M(\bar{F}) \cap V_n$. If n is sufficiently large, then all the orbits in γ and γ' that are not in the Morse–Bott family on ∂V are contained in V_n . Hence the sequence δ_n is constant for $n \gg 0$ and $\eta(\delta_n) = \eta(\gamma') - \eta(\gamma)$.

Regarding both δ_n and r_n as homology classes in $H_1(T_n; \mathbb{R})$ and orienting T_n as the boundary of V_n , for $n \gg 0$ we obtain $\delta_n \cdot r_n \geq 0$ by the positivity of intersections in dimension three (Lemma 5.2.2). Taking the limit $n \rightarrow \infty$ and using the fact that the sequence r_n converges to the slope of the Reeb vector field on ∂V , we obtain $\eta(\delta_n) \leq 0$ for $n \gg 0$. This implies $\mathcal{F}_i(\gamma \otimes \Gamma) \geq \mathcal{F}_i(\gamma' \otimes \Gamma')$. $\square$

Corollary 9.4.2 *The differential of $\text{ECC}^{L_i}(M, \alpha'_i)$ respects the filtration $\mathcal{F}_i$.*

For each i the filtration $\mathcal{F}_i$ induces a spectral sequence $E^r(\mathcal{F}_i)$ which converges to $\text{ECH}^{L_i}(M, \alpha'_i)$. The terms $E^0(\mathcal{F}_i)$ correspond to the graded complexes associated to $\mathcal{F}_i$ and can be identified (as vector spaces) with subspaces of $\text{ECC}(V, \alpha_V) \otimes \text{ECC}(N, \alpha_N)$. The differential ∂_0 on $E^0(\mathcal{F}_i)$ is the filtration-preserving component of the differential on $\text{ECC}^{L_i}(M, \alpha'_i)$. Every sheet $E^r(\mathcal{F}_i)$ has a grading coming from $\mathcal{F}_i$, and the component in degree p of $E^r(\mathcal{F}_i)$ will be denoted by $E_p^r(\mathcal{F}_i)$.

9.5 Description of the differential on $E^0(\mathcal{F}_i)$

In this subsection we compute the differential ∂_0 on $E^0(\mathcal{F}_i)$ using Morse–Bott techniques. This is possible, in spite of the fact that the contact forms α_i are not necessarily nice, because of the following lemma:

Lemma 9.5.1 *Let $\tilde{u}$ be a Morse–Bott building from $\gamma \otimes \Gamma$ to $\gamma' \otimes \Gamma'$ in the symplectization of (M, α_i) and let u be its holomorphic part. If u has a positive end at ∂N or a negative end at ∂V , then $\mathcal{F}_i(\gamma' \otimes \Gamma') < \mathcal{F}_i(\gamma \otimes \Gamma)$.*

Proof We recall that K denotes the core of V and that $S \subset N$ is a properly embedded surface such that ∂S defines a longitude of K . In the case of an open book decomposition, K is the binding and S is a page.

Let $\bar{U} \simeq T^2 \times [1 - \epsilon, 2 + \epsilon] \subset M$ be a small neighborhood of the no man’s land $T^2 \times [1, 2]$ such that u has no ends at Reeb orbits intersecting $\bar{U}$, except at orbits in $\mathcal{N}_1$ or $\mathcal{N}_2$. Assume without loss of generality that the ends of u limit to distinct orbits $\eta_1, \dots, \eta_n$. Then we let U_k be a small tubular neighborhood of η_k for $k = 1, \dots, n$ and let $U = \bar{U} - (U_1 \cup \dots \cup U_n)$. Let $B_k = -\partial U_k$ for $k = 1, \dots, n$, $B_0 = (\partial \bar{U}) \cap V$ and $B_{n+1} = (\partial \bar{U}) \cap N$. We orient each B_k , for $k = 0, \dots, n + 1$, using the boundary orientation of U .

On each B_k for $k = 0, \dots, n + 1$, we choose an oriented basis of curves (μ_k, ν_k) as follows: On B_0 and B_{n+1} we choose μ_0 and μ_{n+1} so that they are longitudes of K coming from S and ν_0 and ν_{n+1} so that they are meridians of K . On each B_k for $k = 1, \dots, n$, we choose ν_k so that it is the longitude of the Reeb orbit in U_k induced by the Morse–Bott torus (which is either ∂N or ∂V) and μ_k so that it is a meridian of U_k . The curves ν_k for $k = 0, \dots, n + 1$ are oriented by the vector field ∂_t and the curves μ_k for $k = 0, \dots, n + 1$ are oriented by $\mu_k \cdot \nu_k = 1$.

By abuse of notation, we identify the oriented curves μ_k and ν_k with their homology classes in $H_1(U; \mathbb{Z})$. With this convention, $\nu_0 = \nu_1 = \dots = \nu_{n+1}$ and $\mu_0 + \mu_1 + \dots + \mu_{n+1} = 0$. Moreover, these relations generate the kernel of the map

$$\bigoplus_{k=0}^{n+1} H_1(B_k; \mathbb{Z}) \rightarrow H_1(U; \mathbb{Z})$$

induced by the inclusion. Let $C = \text{Im}(u_M) \cap U$. Then $\partial C = \delta_0 + \dots + \delta_{n+1}$, where $\delta_k \subset B_k$ is given the orientation induced by C . We will view δ_k either as an element of $H_1(B_k; \mathbb{Z})$ or as an element of $H_1(U; \mathbb{Z})$. Then $\delta_0 + \dots + \delta_{n+1} = 0$ in $H_1(U; \mathbb{Z})$. For each k we write $\delta_k = a_k \mu_k + b_k \nu_k$.

By the trapping lemma and the positivity of intersections in dimension three, we have $\delta_k \cdot \nu_k \geq 0$ for $k = 1, \dots, n$, because the curves ν_k can be represented by Reeb orbits. (Here we are using a variation of [Lemma 5.2.2](#) which is an immediate consequence of the positivity of intersections in dimension four.) Then, for all $k = 1, \dots, n$, $a_k \geq 0$; moreover, if δ_k corresponds either to a positive end at T_1 or to a

negative end at T_2 , then $a_k > 0$. The relations in $H_1(U; \mathbb{Z})$ among the curves μ_k and ν_k imply that $a_0 = \cdots = a_{n+1}$, so $a_0 > 0$ if u has either a positive end at ∂N or a negative end at ∂V . Then

$$a_0 = \eta(\gamma) - \eta(\gamma') = \mathcal{F}_i(\gamma \otimes \Gamma) - \mathcal{F}_i(\gamma' \otimes \Gamma') > 0. \quad \square$$

Corollary 9.5.2 *Let $\gamma \otimes \Gamma$ and $\gamma' \otimes \Gamma'$ be generators of $\text{ECC}^{L_i}(M, \alpha'_i)$. If $\mathcal{F}_i(\gamma \otimes \Gamma) = \mathcal{F}_i(\gamma' \otimes \Gamma')$ and $\tilde{u}$ is a Morse–Bott building with $I(\tilde{u}) = 1$ in the symplectization of (M, α_i) from $\gamma \otimes \Gamma$ to $\gamma' \otimes \Gamma'$, then the holomorphic part of $\tilde{u}$ has at most one nontrivial irreducible component.*

Proof Let u be the holomorphic part of $\tilde{u}$. By Lemma 9.5.1, all ends of u at ∂N are negative and all ends of u at ∂V are positive. Then the structure of $\tilde{u}$ is simple enough that the argument of Lemma 7.1.2 implies that u has a unique irreducible component which is not a connector. $\square$

Corollary 9.5.2 implies that, for the purpose of computing the differential ∂_0 of $E^0(\mathcal{F}_i)$, we can use Morse–Bott theory as if the contact forms α_i were nice.

In order to describe the differential concisely we introduce the following notation. Given two orbit sets $\gamma' = \prod \gamma_i^{m'_i}$ and $\gamma = \prod \gamma_i^{m_i}$ (in multiplicative notation), we set $\gamma/\gamma' = \prod \gamma_i^{m_i - m'_i}$ if $m'_i \leq m_i$ for all i ; otherwise we set $\gamma/\gamma' = 0$. We also call $T_1 = \partial N$ and $T_2 = \partial V$.

We now prove the following lemma, which describes the differential ∂_0 on E^0 in some detail:

Lemma 9.5.3 *After identifying $E_0(\mathcal{F}_i)$, as a vector space, with a subspace of $\text{ECC}(V, \alpha_V) \otimes \text{ECC}(N, \alpha)$, the differential ∂_0 is given by*

$$(9\text{-}5\text{-}1) \quad \partial_0(\gamma \otimes \Gamma) = (\partial_V \gamma) \otimes \Gamma + (\gamma/e') \otimes h\Gamma + (\gamma/h') \otimes e\Gamma + \gamma \otimes (\partial_N \Gamma).$$

Here γ is an orbit set of V ; if h divides Γ , then $h\Gamma$ is understood to be 0; and ∂_X is the differential on the subset $X \subset M$.

Proof Corollary 9.5.2 and Proposition 4.4.7 imply that ∂_0 on $E^0(\mathcal{F}_i)$ can be computed by counting $I = 1$ very nice Morse–Bott buildings in the symplectization of (M, α_i) which do not decrease the filtration level.

The differential ∂_0 does not count holomorphic curves which cross $\mathbb{R} \times T_1 = \mathbb{R} \times \partial N$ or $\mathbb{R} \times T_2 = \mathbb{R} \times \partial V$: Indeed, if u is a holomorphic curve which contributes to ∂_0 and u_M its projection to M , then the homology classes $[\text{Im}(u_M) \cap T_{1\pm\epsilon}] \in H_1(T_{1\pm\epsilon})$ and $[\text{Im}(u_M) \cap T_{2\pm\epsilon}] \in H_1(T_{2\pm\epsilon})$ (for $\epsilon > 0$ small) have slope ∞ , and we apply the blocking lemma (Lemma 5.2.3(2)). This still allows for the possibility of curves that are negatively asymptotic to orbits of T_1 or positively asymptotic to orbits in T_2 . (Curves which are positively asymptotic to orbits of T_1 or negatively asymptotic to orbits of T_2 are ruled out by Lemma 9.5.1 because they have been shown to decrease the filtration level.) Such curves are contained in $\mathbb{R} \times V$, $\mathbb{R} \times T^2 \times [1, 2]$, or $\mathbb{R} \times N$ by a combination of the trapping lemma (Lemma 5.3.2) and the blocking lemma (Lemma 5.2.3).

Curves in $\mathbb{R} \times V$ contribute to the term $(\partial_V \gamma) \otimes \Gamma$, while curves in $\mathbb{R} \times N$ contribute to $\gamma \otimes \partial_N(\Gamma)$. Note that there are two cylinders from e' to h' and two cylinders from h to e corresponding to gradient trajectories on $\mathcal{N}_2$ and $\mathcal{N}_1$; these give $\partial_0(e' \otimes 1) = 0$ and $\partial_0(1 \otimes h) = 0$.

Next we consider curves in $\mathbb{R} \times \text{int}(T^2 \times [1, 2])$. By [Lemma 8.4.8](#), the only somewhere-injective curves in $\mathbb{R} \times \text{int}(T^2 \times [1, 2])$ are the cylinders $Z_{s,\theta}$ defined in [Lemma 8.4.6](#). (Remember that we are ignoring the curves which are asymptotic to the orbits in $\text{int}(T^2 \times [1, 2])$ because they have action larger than L_i .) By [Lemma 8.4.7](#), the cylinders $Z_{s,\theta}$ satisfy automatic transversality as long as at least one of the ends is treated as unconstrained. Branched covers of $Z_{s,\theta}$ of degree > 1 are not counted in the differential since they have $I > 1$ (after augmenting them with cylinders corresponding to gradient trajectories). Modulo translations in the s -direction, there is a unique $I = 1$ Morse–Bott building from h' to e , which gives the term $(\gamma/h') \otimes e\Gamma$, and a unique $I = 1$ Morse–Bott building from e' to h , which gives the term $(\gamma/e') \otimes h\Gamma$ (adding trivial cylinders to these buildings does not change their ECH index because they satisfy the admissibility conditions from [\[Hutchings 2002, Proposition 7.1, equations \(23\) and \(24\)\]](#)). $\square$

9.6 Direct limit

In this subsection we use a direct limit argument to exclude the Reeb orbits in the no man’s land from the complex computing $\text{ECH}(M)$. The limit will be compatible with the filtrations $\mathcal{F}_i$, so the end result will be a spectral sequence E^r converging to $\text{ECH}(M)$. The following lemma is immediate from [Corollary 3.2.3](#) and the construction of the contact forms α'_i :

Lemma 9.6.1 *For an appropriate choice of contact forms α'_i and action thresholds L_i , we have*

$$\text{ECH}(M) = \lim_{i \rightarrow \infty} \text{ECH}^{L_i}(M, \alpha'_i).$$

The direct limit is taken with respect to maps

$$\Phi_i : \text{ECH}^{L_i}(M, \alpha'_i) \rightarrow \text{ECH}^{L_{i+1}}(M, \alpha'_{i+1})$$

induced by interpolating cobordisms via [Lemma 3.1.7](#).

Lemma 9.6.2 *The map Φ_i is induced by a noncanonical chain map*

$$\hat{\Phi}_i : \text{ECC}^{L_i}(M, \alpha'_i) \rightarrow \text{ECC}^{L_{i+1}}(M, \alpha'_{i+1}), \quad \gamma \otimes \Gamma \mapsto \gamma \otimes \Gamma + \mathbf{r}(\gamma \otimes \Gamma),$$

where $\mathcal{F}_{i+1}(\mathbf{r}(\gamma \otimes \Gamma)) < \mathcal{F}_{i+1}(\gamma \otimes \Gamma)$.

Proof The map Φ_i is induced by an interpolating cobordism from α'_i to (a rescaling of) α'_{i+1} . We degenerate this cobordism into a two-level cobordism such that the top level interpolates from $\alpha'_i = f_i \alpha_i$ to $f_{i+1} \alpha_i$ and the bottom level interpolates from $f_{i+1} \alpha_i$ to $\alpha'_{i+1} = f_{i+1} \alpha_{i+1}$. Then $\Phi_i = \Phi'_i \circ \Phi''_i$ by [Theorem 3.1.2](#), where

$$\Phi''_i : \text{ECH}^{L_i}(M, \alpha'_i) \rightarrow \text{ECH}^{L_i}(M, f_{i+1} \alpha_i), \quad \Phi'_i : \text{ECH}^{L_i}(M, f_{i+1} \alpha_i) \rightarrow \text{ECH}^{L_{i+1}}(M, \alpha'_{i+1}).$$

The maps Φ'_i and Φ''_i are induced by noncanonical chain maps $\hat{\Phi}'_i$ and $\hat{\Phi}''_i$. By [Proposition 4.5.5](#) we can assume that $\hat{\Phi}''_i$ is the identity map.

Next we claim that the filtration-nondecreasing part of $\hat{\Phi}'_i$ only counts trivial cylinders. Let $([0, 1] \times M, \lambda'_i)$ be an interpolating cobordism from $f_{i+1}\alpha_i$ to α'_{i+1} and $(\mathbb{R} \times M, \hat{\lambda}'_i)$ its completion. By [Theorem 3.1.2](#), $\hat{\Phi}'_i$ is “supported” on the $I = 0$ holomorphic buildings of $(\mathbb{R} \times M, \hat{\lambda}'_i)$. We are assuming that $\hat{\lambda}'_i$ is sufficiently close to $\hat{\lambda}_i$, where $([0, 1] \times M, \lambda_i)$ is an interpolating cobordism from α_i to α_{i+1} and $(\mathbb{R} \times M, \hat{\lambda}_i)$ is its completion. Hence, by (MB₀), if $\langle \hat{\Phi}'_i(\gamma \otimes \Gamma), \gamma' \otimes \Gamma' \rangle \neq 0$, then there is a Morse–Bott building in $(\mathbb{R} \times M, \hat{\lambda}_i)$ connecting $\gamma \otimes \Gamma$ to $\gamma' \otimes \Gamma'$. Since the 2-form $d\lambda_i$ agrees with a symplectization on a neighborhood of $\mathbb{R} \times (N \cup V)$, we can repeat the argument of [Lemma 9.4.1](#) to show that $\mathcal{F}_i(\gamma \otimes \Gamma) \geq \mathcal{F}_{i+1}(\gamma' \otimes \Gamma')$. Moreover, if $\mathcal{F}_i(\gamma \otimes \Gamma) = \mathcal{F}_{i+1}(\gamma' \otimes \Gamma')$, then the holomorphic buildings in $(\mathbb{R} \times M, \hat{\lambda}_i)$ cannot cross the no man’s land by [Lemma 8.4.6](#) and [Remark 8.4.9](#). Therefore they are contained in the part of the cobordism $(\mathbb{R} \times M, \hat{\lambda}_i)$ which is diffeomorphic to a symplectization. This implies the claim. $\square$

Lemma 9.6.3 *The chain maps $\hat{\Phi}_i: \text{ECC}^{L_i}(M, \alpha'_i) \rightarrow \text{ECC}^{L_{i+1}}(M, \alpha'_{i+1})$ induce chain maps $E^r(\mathcal{F}_i) \rightarrow E^r(\mathcal{F}_{i+1})$. The direct limits*

$$E^r(\mathcal{F}) = \lim_{i \rightarrow \infty} E^r(\mathcal{F}_i)$$

form a spectral sequence converging to $\text{ECH}(M)$. The page $E^0(\mathcal{F})$ can be identified, as a vector space, with $\text{ECC}(V, \alpha) \otimes \text{ECC}(N, \alpha)$ and the differential ∂_0 on $E^0(\mathcal{F})$ is described by (9-5-1).

Proof By [Lemma 9.6.2](#), the continuation maps $\hat{\Phi}_i$ are morphisms of chain complexes. Since the construction of the spectral sequence associated to a filtered complex is functorial (see [\[Weibel 1994, Proposition 5.9.2\]](#)), the maps $\hat{\Phi}_i$ induce a morphism of spectral sequences

$$E^r(\mathcal{F}_i) \rightarrow E^r(\mathcal{F}_{i+1}).$$

We define $E^r(\mathcal{F}) = \lim_{i \rightarrow \infty} E^r(\mathcal{F}_i)$. Since direct limit is an exact functor from the category of directed systems of abelian groups to the category of abelian groups (see for example [\[Weibel 1994, Theorem 2.6.15\]](#)), the limits $E^r(\mathcal{F})$ still form a spectral sequence.

We claim now that $E^\infty(\mathcal{F}) = \lim_{i \rightarrow \infty} E^\infty(\mathcal{F}_i)$. First we recall the definition of the E^∞ term of a spectral sequence: on E^1 there is a sequence of subgroups

$$\{0\} = B^1 \subset B^2 \cdots \subset B^r \subset \cdots \subset Z^r \subset \cdots \subset Z^2 \subset Z^1 = E^1$$

such that $E^r \simeq Z^r / B^r$; then we define

$$Z^\infty = \bigcap_{r \geq 1} Z^r, \quad B^\infty = \bigcup_{r \geq 1} B^r \quad \text{and} \quad E^\infty = Z^\infty / B^\infty.$$

By going through the construction of the spectral sequence, one can see that

$$B^r(\mathcal{F}) = \lim_{i \rightarrow \infty} B^r(\mathcal{F}_i) \quad \text{and} \quad Z^r(\mathcal{F}) = \lim_{i \rightarrow \infty} Z^r(\mathcal{F}_i)$$

because the direct limit is an exact functor. (The description of B^r and Z^r given in [Weibel 1994, Exercise 5.9.1] can be useful to prove this.)

Then, in order to prove the claim, it is enough to prove that

$$(9-6-1) \quad \lim_{i \rightarrow \infty} \left(\bigcup_{r \geq 1} B^r(\mathcal{F}_i) \right) = \bigcup_{r \geq 1} \left(\lim_{i \rightarrow \infty} B^r(\mathcal{F}_i) \right),$$

$$(9-6-2) \quad \lim_{i \rightarrow \infty} \left(\bigcap_{r \geq 1} Z^r(\mathcal{F}_i) \right) = \bigcap_{r \geq 1} \left(\lim_{i \rightarrow \infty} Z^r(\mathcal{F}_i) \right).$$

Equation (9-6-1) is not problematic because direct limits commute with countable unions. In fact countable unions can themselves be seen as direct limits, and direct limits commute as a consequence of their universal property [Lang 2002, Exercise 20]. On the other hand, in general, direct limits do not commute with infinite intersections, so we need more work to prove (9-6-2).

The spectral sequence of a filtered complex has a grading coming from the filtration: we can decompose $E^r(\mathcal{F}_i) = \bigoplus E_p^r(\mathcal{F}_i)$, $B^r(\mathcal{F}_i) = \bigoplus B_p^r(\mathcal{F}_i)$ and $Z^r(\mathcal{F}_i) = \bigoplus Z_p^r(\mathcal{F}_i)$. Since $\mathcal{F}_i^p = 0$ if $p < 0$, it follows from the construction of the spectral sequence that $Z_p^\infty(\mathcal{F}_i) = Z_p^r(\mathcal{F}_i)$ provided that $r \geq p$. (Again [Weibel 1994, Exercise 5.9.1] can be useful here.) Taking the direct limit, we obtain that $\lim_{i \rightarrow \infty} E_p^\infty(\mathcal{F}_i) = E_p^\infty(\mathcal{F})$ and this proves the claim.

The filtrations $\mathcal{F}_i$ induce filtrations on $\text{ECH}^{L_i}(M, \alpha'_i)$; taking direct limits we obtain a filtration on $\text{ECH}(M)$ whose the graded group is the limit of the graded groups of the filtrations on $\text{ECH}^{L_i}(M, \alpha'_i)$ (again because direct limit is an exact functor). Since the filtrations $\mathcal{F}_i$ are bounded below and exhaustive, the classical convergence theorem [Weibel 1994, Theorem 5.5.5] implies that $E^r(\mathcal{F}_i)$ converges to $\text{ECH}^{L_i}(M, \alpha'_i)$ (ie $E^\infty(\mathcal{F}_i)$ is isomorphic to the graded group of $\text{ECH}^{L_i}(M, \alpha'_i)$). Taking a direct limit, we then conclude that $E^r(\mathcal{F})$ converges to $\text{ECH}(M)$. $\square$

Here the notation $E^r(\mathcal{F})$ does not mean that the spectral sequence comes from some filtration $\mathcal{F}$, but only remembers the fact that it is the direct limit of the spectral sequences induced by the filtrations $\mathcal{F}_i$ — in fact $E^r(\mathcal{F})$ is a spectral sequence of a filtration because a direct limit of filtered complexes is a filtered complex; however, the limit complex defining $E^r(\mathcal{F})$ is too abstract to be useful. This notation will be useful in the next section, when we will introduce another spectral sequence.

We now rewrite the differential ∂_0 in a way which highlights the roles played by the orbits h and h' ; this will be used extensively in the following subsections. By factoring out the terms h' and h , we can write the differentials ∂_V and ∂_N as

$$(9-6-3) \quad \partial_V \gamma = \partial_V^b \gamma, \quad \partial_V(h' \gamma) = h' \partial_V^b \gamma + \partial'_V(h' \gamma) \quad \text{and} \quad \partial_N \Gamma = \partial_N^b \Gamma + h \partial'_N \Gamma, \quad \partial_N(h \Gamma) = h \partial_N^b \Gamma,$$

where $\gamma \in \text{ECC}^b(V, \alpha_V)$, $\Gamma \in \text{ECC}^b(N, \alpha)$, ∂_V^b and ∂_N^b are the differentials for the chain complexes $\text{ECC}^b(V, \alpha_V)$ and $\text{ECC}^b(N, \alpha)$, and the terms $\partial'_V(h' \gamma)$ and $\partial'_N \Gamma$ do not contain h' .

9.7 The map σ_*

In this subsection we define an explicit map

$$\sigma_*: \text{ECH}(N, \partial N, \alpha) \rightarrow \text{ECH}(M)$$

and in the next one we will prove that it is an isomorphism. It will be easy to see that σ_* preserves the decomposition by (relative) homology classes; namely, if $\varpi: H_1(N, \partial N) \rightarrow H_1(M)$ is the isomorphism described in the introduction, σ_* maps $\text{ECH}(N, \partial N, A)$ to $\text{ECH}(M, \varpi(A))$ for every $A \in H_1(N, \partial N)$.

We introduce the following notation, which will be used in this and in the following sections. Given a set of Reeb orbits $e_1, \dots, e_n, h_1, \dots, h_m$, where $e_1, \dots, e_n$ are elliptic and $h_1, \dots, h_m$ are hyperbolic, we define

$$\mathcal{R}[e_1, \dots, e_n, h_1, \dots, h_m] := \mathbb{F}[e_1, \dots, e_n, h_1, \dots, h_m]/(h_1^2, \dots, h_m^2);$$

ie in $\mathcal{R}[e_1, \dots, e_n, h_1, \dots, h_m]$ the elliptic orbits are free variables and the hyperbolic orbits are nilpotent variables of order two. Whenever we use the notation $\mathcal{R}[e_1, \dots, e_n, h_1, \dots, h_m]$ in this paper, we will assume $\{e_1, \dots, e_n\} \subseteq \{e, e'\}$ and $\{h_1, \dots, h_m\} \subseteq \{h, h'\}$.

Define $\text{ECC}^{\natural}(N, \alpha)$ as $\mathcal{R}[h'] \otimes \text{ECC}^b(N, \alpha)$ with differential

$$\partial^{\natural}(\gamma \otimes \Gamma) = \gamma \otimes \partial^b \Gamma + \gamma/h' \otimes (1 + e)\Gamma.$$

Lemma 9.7.1 $\text{ECH}^{\natural}(N, \alpha) \simeq \text{ECH}(N, \partial N, \alpha).$

Proof $\text{ECC}^{\natural}(N, \alpha)$ can be identified with the cone of the multiplication map $\cdot(1 + e)$ on $\text{ECC}^b(N, \alpha)$. Hence there is an exact triangle

$$(9-7-1) \quad \begin{array}{ccc} \text{ECH}^b(N, \alpha) & \xrightarrow{\cdot(1+e)} & \text{ECH}^b(N, \alpha) \\ & \nwarrow \quad \nearrow & \\ & \text{ECH}^{\natural}(N, \alpha) & \end{array}$$

The map $\cdot(1 + e)$ is injective on homology since Γ and $e\Gamma$ belong to different singular homology classes for all $\Gamma \in \text{ECC}^b(N, \alpha)$. Then the exact triangle implies that

$$\text{ECH}^{\natural}(N, \alpha) \simeq \frac{\text{ECH}^b(N, \alpha)}{(1 + e) \text{ECH}^b(N, \alpha)} \simeq \text{ECH}(N, \partial N, \alpha). \quad \square$$

We denote by $\text{ECC}_{\leq k}^{\natural, L}(N, \alpha)$ the subcomplex of $\text{ECC}^{\natural}(N, \alpha)$ generated by orbit sets $\gamma \otimes \Gamma$ which have linking number less than or equal to k with K and action less than L . We fix an increasing sequence $L'_k \rightarrow +\infty$ and let $c = \sup_k \mathcal{A}_{\alpha'_k}(e')$. Then, for every k , we choose i_k so that $L_{i_k} \geq kL'_k + ck^2$.

In the following, we will rename $L_{i_k} = L_k$, $\alpha'_{i_k} = \alpha'_k$ and $\mathcal{F}_{i_k} = \mathcal{F}_k$. Also, the composition $\widehat{\Phi}_{i_{k+1}-1} \circ \cdots \circ \widehat{\Phi}_{i_k}$ will be renamed as

$$\widehat{\Phi}_k: \text{ECC}^{L_k}(M, \alpha'_k) \rightarrow \text{ECC}^{L_{k+1}}(M, \alpha'_{k+1}).$$

For any integer k , we define

$$\sigma_k: \text{ECC}_{\leq k}^{\natural, L'_k}(N, \alpha) \rightarrow \text{ECC}^{L_k}(M, \alpha'_k), \quad \gamma \otimes \Gamma \mapsto \sum_{i=0}^{\infty} (e')^i \gamma \otimes (\partial'_N)^i \Gamma,$$

where ∂'_N is defined by (9-6-3) and $\gamma = 1$ or h' .

These maps are well defined because the map ∂'_N is nilpotent. In fact, ∂'_N decreases the linking number with the binding, so $(\partial'_N)^{k+1} = 0$ on $\text{ECC}_{\leq k}^b(N, \alpha)$.

Remark 9.7.2 This and the analogous construction in Section 9.9 are the only places where we use the hypothesis that the Reeb flow is transverse to a fixed Seifert surface for K . In fact, while we could deduce the nilpotency of ∂'_N from an action argument, by choosing to work with the action we would lose the estimate on the nilpotency order of ∂'_N and, consequently, on the action of $\sigma_k(\gamma \otimes \Gamma)$. However, in view of the heuristic argument described in Section 9.1, we suspect that this hypothesis is actually not necessary.

Lemma 9.7.3 *The maps σ_k are chain maps and form a directed system, ie the following diagram commutes:*

$$(9-7-2) \quad \begin{array}{ccc} \text{ECC}_{\leq k}^{\natural, L'_k}(N, \alpha) & \xrightarrow{\sigma_k} & \text{ECC}^{L_k}(M, \alpha'_k) \\ \downarrow \iota_k & & \downarrow \widehat{\Phi}_k \\ \text{ECC}_{\leq k+1}^{\natural, L'_{k+1}}(N, \alpha) & \xrightarrow{\sigma_{k+1}} & \text{ECC}^{L_{k+1}}(M, \alpha'_{k+1}) \end{array}$$

Here ι_k is the inclusion.

Proof (1) We first show that σ_k is a chain map. Since σ_k takes values in the lowest level for the filtration $\mathcal{F}_k$ (recall $\gamma = 1$ or h'), we have $\partial(\sigma_k(\Gamma)) = \partial_0(\sigma_k(\Gamma))$, where ∂_0 is given by (9-5-1). Using the decomposition of ∂_N in (9-6-3) and $\partial_V((e')^i \gamma) = (e')^i \gamma / h'$ for $\gamma = 1, h'$, we obtain

$$\begin{aligned} \partial_0(\sigma_k(\gamma \otimes \Gamma)) &= \partial_0 \left(\sum_{i=0}^{\infty} (e')^i \gamma \otimes (\partial'_N)^i \Gamma \right) \\ &= \sum_{i=0}^{\infty} (e')^i \gamma / h' \otimes (\partial'_N)^i \Gamma + \sum_{i=0}^{\infty} (e')^i \gamma \otimes (\partial_N^b (\partial'_N)^i \Gamma + h (\partial'_N)^{i+1} \Gamma) \\ &\quad + \sum_{i=1}^{\infty} (e')^{i-1} \gamma \otimes h (\partial'_N)^i \Gamma + \sum_{i=0}^{\infty} (e')^i \gamma / h' \otimes e (\partial'_N)^i \Gamma. \end{aligned}$$

Rearranging the sum and using the fact that ∂'_N commutes with ∂_N^b and with the multiplication by $1 + e$ gives

$$\partial_0(\sigma_k(\gamma \otimes \Gamma)) = \sum_{i=0}^{\infty} ((e')^i \gamma \otimes (\partial'_N)^i \partial_N^b \Gamma + (e')^i \gamma / h' \otimes (\partial'_N)^i ((1 + e)\Gamma)).$$

Hence $\partial(\sigma_k(\gamma \otimes \Gamma)) = \partial_0(\sigma_k(\gamma \otimes \Gamma)) = \sigma_k(\partial^{\natural}(\gamma \otimes \Gamma))$.

(2) Diagram (9-7-2) commutes because we have shown in Lemma 9.6.2 that the continuation maps are induced by the identity at the chain level on the lowest filtration level. $\square$

Taking homology first and then direct limits in diagram (9-7-2), we obtain a map

$$\sigma_*: \text{ECH}(N, \partial N, \alpha) \simeq \text{ECH}^{\natural}(N, \alpha) \rightarrow \text{ECH}(M).$$

The maps σ_k also induce maps

$$\sigma^0: \text{ECC}^{\natural}(N, \alpha) \rightarrow E^0(\mathcal{F}), \quad \gamma \otimes \Gamma \mapsto \sum_{i=0}^{\infty} (e')^i \gamma \otimes (\partial'_N)^i \Gamma,$$

and

$$\sigma^r: \text{ECH}(N, \partial N) \simeq \text{ECH}^{\natural}(N, \alpha) \rightarrow E^r(\mathcal{F}), \quad r > 0.$$

9.8 Computation of $E^1(\mathcal{F})$

In this subsection we compute the term $E^1(\mathcal{F})$ of the spectral sequence that converges to $\text{ECH}(M)$ and prove the first half of Theorem 1.1.1.

Recall from Lemma 9.6.3 that $E^0(\mathcal{F}) \simeq \text{ECC}(V, \alpha) \otimes \text{ECC}(N, \alpha)$ as a vector space and the differential ∂_0 is given by (9-5-1) and (9-6-3). If we write

$$C_{k,k'} = (h')^{k'} \text{ECC}^b(V, \alpha) \otimes h^k \text{ECC}^b(N, \alpha),$$

then

$$E^0(\mathcal{F}) \simeq \text{ECC}(V, \alpha) \otimes \text{ECC}(N, \alpha) = C_{0,0} \oplus C_{0,1} \oplus C_{1,0} \oplus C_{1,1}.$$

We can organize all components of the differential ∂_0 besides $\partial_V^b \otimes 1$ and $1 \otimes \partial_N^b$ in the diagram

$$(9-8-1) \quad \begin{array}{ccc} C_{0,1} & \xrightarrow{1 \otimes h \partial'_N + \cdot / e' \otimes h} & C_{1,1} \\ \partial'_V \otimes 1 + \cdot / h' \otimes e \downarrow & & \downarrow \partial'_V \otimes 1 + \cdot / h' \otimes e \\ C_{0,0} & \xrightarrow{1 \otimes h \partial'_N + \cdot / e' \otimes h} & C_{1,0} \end{array}$$

9.8.1 The filtration $\mathcal{G}$ We introduce a filtration $\mathcal{G}$ of length 3 on

$$(E^0(\mathcal{F}), \partial_0) = (\text{ECC}(V, \alpha) \otimes \text{ECC}(N, \alpha), \partial_0),$$

which is defined as

$$\mathcal{G}^0 = C_{1,0}, \quad \mathcal{G}^1 = C_{0,0} \oplus C_{1,1}, \quad \mathcal{G}^2 = C_{0,1}.$$

This filtration induces a spectral sequence $E^r(\mathcal{G})$ which converges to $E^1(\mathcal{F})$. The groups $E^r(\mathcal{G})$ have two gradings: one inherited from the grading on $E^0(\mathcal{F})$ (which, in turn, is induced by the filtrations $\mathcal{F}_i$) and one induced by the filtration $\mathcal{G}$. We will denote the homogeneous components of $E^r(\mathcal{G})$ by $E_{pq}^r(\mathcal{G})$, where p is the degree inherited from $E^0(\mathcal{F})$ and q is the degree induced by $\mathcal{G}$. We also write $E_p^r(\mathcal{G})$, in which case p is the degree inherited from $E^0(\mathcal{F})$.

9.8.2 Determination of $(E^1(\mathcal{G}), \partial_{01})$ The graded complex associated to $\mathcal{G}$ is

$$(E^0(\mathcal{G}), \partial_{00}) \simeq (\mathcal{R}[h', h] \otimes \text{ECC}^b(V, \alpha) \otimes \text{ECC}^b(N, \alpha), 1 \otimes \partial_V^b \otimes 1 + 1 \otimes 1 \otimes \partial_N^b).$$

Then $(E^0(\mathcal{G}), \partial_{00})$ is a product complex and its homology can be computed by the Künneth formula:

$$E^1(\mathcal{G}) = \mathcal{R}[h', h] \otimes \text{ECH}^b(V, \alpha) \otimes \text{ECH}^b(N, \alpha).$$

Taking into account the grading inherited from $E^0(\mathcal{F})$ and the computation of $\text{ECH}^b(V, \alpha)$ from [Theorem 8.1.2](#)(4), we obtain

$$E_p^1(\mathcal{G}) \simeq \begin{cases} \mathcal{R}[e', h', h] \otimes \text{ECH}^b(N, \alpha) & \text{when } p = 0, \\ 0 & \text{when } p > 0. \end{cases}$$

Then $E_p^1(\mathcal{F}) = 0$ for $p > 0$ and standard properties of spectral sequences immediately imply the following lemma:

Lemma 9.8.1 *There is an isomorphism $E_0^1(\mathcal{F}) \simeq \text{ECH}(M)$ which is induced by the direct limit of the inclusion maps $E_0^0(\mathcal{F}_i) \hookrightarrow \text{ECC}^{L_i}(M, \alpha'_i)$.*

The differential ∂_{01} on $E^1(\mathcal{G})$ is induced by the components of ∂_0 between consecutive filtration levels. By [Proposition 8.4.5](#) and [Lemma 8.4.8](#), the only J -holomorphic map in $\mathbb{R} \times V$ with an end at h' is a disk in the foliation $\mathcal{Z}_1$, which has ECH index $I = 1$. Therefore $\partial_V'(h'(e')^i) = (e')^i$. Then the differential ∂_{01} on $E_{0,\bullet}^1(\mathcal{G})$ is described by the commutative diagram

$$(9-8-2) \quad \begin{array}{ccc} h' \mathcal{R}[e'] \otimes \text{ECH}^b(N, \alpha) & \xrightarrow{1 \otimes h \partial_N' + \cdot / e' \otimes h} & h' \mathcal{R}[e'] \otimes h \text{ECH}^b(N, \alpha) \\ \cdot / h' \otimes (1+e) \downarrow & & \downarrow \cdot / h' \otimes (1+e) \\ \mathcal{R}[e'] \otimes \text{ECH}^b(N, \alpha) & \xrightarrow{1 \otimes h \partial_N' + \cdot / e' \otimes h} & \mathcal{R}[e'] \otimes h \text{ECH}^b(N, \alpha) \end{array}$$

9.8.3 Homological algebra lemma The following elementary lemma in homological algebra will be used in the proof of [Theorem 9.8.3](#):

Lemma 9.8.2 *Let A be an abelian group and $f, g: A \rightarrow A$ commuting morphisms. Consider the chain complex*

$$\mathcal{C}_\bullet = (0 \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0) = (0 \rightarrow A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} A^2 \xrightarrow{(g, -f)} A \rightarrow 0).$$

If f has a right inverse $s: A \rightarrow A$ (ie $f \circ s = \text{id}$) such that $g \circ s = s \circ g$, then

$$H_2(\mathcal{C}_\bullet) \simeq \ker f \cap \ker g, \quad H_1(\mathcal{C}_\bullet) \simeq \ker f / g(\ker f), \quad H_0(\mathcal{C}_\bullet) = 0.$$

Proof $H_2(\mathcal{C}_\bullet) \simeq \ker f \cap \ker g$ is immediate and $H_0(\mathcal{C}_\bullet) = 0$ follows from the surjectivity of f .

Next consider $H_1(\mathcal{C}_\bullet)$. By definition, we have $\ker \partial_1 = \{(x, y) \in A^2 \mid g(x) = f(y)\}$ and $\text{Im}(\partial_2) = \{(f(z), g(z)) \in A^2 \mid z \in A\}$. If we define the map

$$\phi: A \rightarrow A^2, \quad x \mapsto (x, g \circ s(x)) = (f \circ s(x), g \circ s(x)),$$

then we can write $\text{Im}(\partial_2) = \text{Im}(\phi) \oplus g(\ker f)$ and $\ker(\partial_1) = \text{Im}(\phi) \oplus \ker f$. The details are left to the reader. Hence $H_1(\mathcal{C}_\bullet) \simeq \ker f / g(\ker f)$. $\square$

9.8.4 Completion of proof of Theorem 1.1.1(1) We use a comparison theorem for spectral sequences (eg [Eisenbud 1995, Exercise A3.41]) to prove Theorem 9.8.3, establishing Theorem 1.1.1(1).

Theorem 9.8.3 *The map $\sigma_*: \text{ECH}(N, \partial N, \alpha) \rightarrow \text{ECH}(M)$ is an isomorphism.*

Proof Since σ_k takes values in the lowest level of the filtration $\mathcal{F}_k$, σ_* factors through the map

$$\sigma^1: \text{ECH}(N, \partial N, \alpha) \simeq \text{ECH}^{\natural}(N, \alpha) \rightarrow E_0^1(\mathcal{F}).$$

By Lemma 9.8.1, it suffices to show that σ^1 is an isomorphism.

Recall the filtration $\mathcal{G}$ on $E^0(\mathcal{F})$ from Section 9.8.1. On $\text{ECC}^{\natural}(N, \alpha)$ we define an analogous filtration $\mathcal{G}^{\natural}$ such that

$$\mathcal{G}^{\natural}(\gamma \otimes \Gamma) = \begin{cases} 2 & \text{if } \gamma = h', \\ 1 & \text{if } \gamma = \emptyset. \end{cases}$$

This filtration induces a spectral sequence $E^r(\mathcal{G}^{\natural})$ such that $E_q^1(\mathcal{G}^{\natural}) \simeq \text{ECH}^b(N, \alpha)$ for $q = 1, 2$ and d_1 is the multiplication by $1 + e$. This is simply a reformulation of exact triangle (9-7-1) in the language of spectral sequences. The map σ^0 is compatible with the filtrations $\mathcal{G}^{\natural}$ and $\mathcal{G}$ and induces a map

$$\bar{\sigma}: E^1(\mathcal{G}^{\natural}) \rightarrow E^1(\mathcal{G}).$$

We now compute the homology of $(E^1(\mathcal{G}), \partial_{01})$ using Lemma 9.8.2. We set

$$A = \mathcal{R}[e'] \otimes \text{ECH}^b(N, \alpha), \quad f = 1 \otimes \partial'_N + \cdot / e' \otimes 1 \quad \text{and} \quad g = 1 \otimes (1 + e),$$

where $fg = gf$ by diagram (9-8-2). Define the map

$$s: \mathcal{R}[e'] \otimes \text{ECH}^b(N, \alpha) \rightarrow \mathcal{R}[e'] \otimes \text{ECH}^b(N, \alpha), \quad (e')^k \otimes \Gamma \mapsto (e')^k \sum_{i=1}^{\infty} (e')^i \otimes (\partial'_N)^{i-1} \Gamma,$$

where Γ denotes an element of $\text{ECH}^b(N, \alpha)$ and not an orbit set as usual. Then s is well defined since ∂'_N is nilpotent. Moreover, $fs = \text{id}$ and $gs = sg$. Then $E_{00}^2(\mathcal{G}) = E_{02}^2(\mathcal{G}) = 0$ because the map g is injective. Next consider $E_{01}^2(\mathcal{G}) = \ker f / g(\ker f)$. An element of $\ker f$ has the form

$$(e')^n \otimes \Gamma_n + (e')^{n-1} \otimes \Gamma_{n-1} + \cdots + 1 \otimes \Gamma_0,$$

where $\Gamma_i \in \text{ECH}^b(N, \alpha)$ and $\Gamma_{i+1} = \partial'_N \Gamma_i$ for $i = 0, 1, \dots$. Hence the map

$$\bar{\sigma}: \text{ECH}^b(N, \alpha) \rightarrow \mathcal{R}[e'] \otimes \text{ECH}^b(N, \alpha), \quad \Gamma \mapsto \sum_{i=0}^{\infty} (e')^i \otimes (\partial'_N)^i \Gamma,$$

is an isomorphism with $\ker f$. The diagram

$$\begin{array}{ccc} \text{ECH}^b(N, \alpha) & \xrightarrow{\bar{\sigma}} & \ker f \\ \cdot(1+e) \downarrow & & \downarrow \cdot(1+e)=g \\ \text{ECH}^b(N, \alpha) & \xrightarrow{\bar{\sigma}} & \ker f \end{array}$$

commutes because $\partial'_N(e\Gamma) = e\partial'_N(\Gamma)$ for all $\Gamma \in \text{ECH}^b(N, \alpha)$ by the trapping lemma. Hence $\bar{\sigma}$ induces an isomorphism

$$E^2(\mathcal{G}^{\natural}) \simeq \text{ECH}^b(N, \alpha)/(\Gamma + e\Gamma) \xrightarrow{\sim} E^2(\mathcal{G}) \simeq \ker f / g(\ker f).$$

By the comparison theorem for spectral sequences, σ^1 is an isomorphism. $\square$

9.9 The U -map

In this subsection we prove that σ_* intertwines the map U on $\text{ECH}(M)$ with the map induced by ∂'_N on $\text{ECH}(N, \partial N, \alpha)$. This will allow us to deduce [Theorem 1.1.1\(2\)](#) from algebraic considerations. Let L_k and L'_k be as in [Section 9.7](#).

We define the map

$$U^{\natural}: \text{ECC}^{\natural}(N, \alpha) \rightarrow \text{ECC}^{\natural}(N, \alpha), \quad \gamma \otimes \Gamma \mapsto \gamma \otimes \partial'_N \Gamma.$$

Since $U^{\natural}(\text{ECC}_{\leq k}^{\natural, L'_k}(N, \alpha)) \subseteq \text{ECC}_{\leq k}^{\natural, L'_k}(N, \alpha)$, we can define

$$U_k^{\natural}: \text{ECC}_{\leq k}^{\natural, L'_k}(N, \alpha) \rightarrow \text{ECC}_{\leq k}^{\natural, L'_k}(N, \alpha)$$

as the restriction of $U^{\natural}$ to $\text{ECC}_{\leq k}^{\natural, L'_k}(N, \alpha)$.

We also define the chain complex

$$\widehat{\text{ECC}}^{\natural}(N, \alpha) = \mathcal{R}[h'] \otimes \text{ECC}(N, \alpha)$$

with differential

$$\hat{\partial}^{\natural}(\gamma \otimes \Gamma) = \gamma \otimes \partial_N \Gamma + \gamma/h' \otimes (1+e)\Gamma.$$

The following lemma is similar to [Lemma 9.7.1](#) and its proof will be omitted:

Lemma 9.9.1

$$\widehat{\text{ECH}}^{\natural}(N, \alpha) \simeq \widehat{\text{ECH}}(N, \partial N, \alpha).$$

The decomposition of the differential ∂_N described in [\(9-6-3\)](#) implies the following lemma:

Lemma 9.9.2 $\widehat{\text{ECC}}^{\natural}(N, \alpha)$ is isomorphic to the cone of $U^{\natural}$. If $L'_k \rightarrow \infty$ is an increasing sequence and $\widehat{\text{ECC}}^{\natural, L'_k}_{\leq k}(N, \alpha)$ is the cone of $U_k^{\natural}$, then

$$\lim_{k \rightarrow \infty} \widehat{\text{ECC}}^{\natural, L'_k}_{\leq k}(N, \alpha) \simeq \widehat{\text{ECC}}^{\natural}(N, \alpha).$$

Let z be a generic point in the interior of $\mathbb{R} \times V$. We denote by U_k the U -map on $\text{ECC}^{L_k}(M, \alpha'_k)$ defined with respect to z .

Lemma 9.9.3 The map U_k preserves the filtration $\mathcal{F}_k$ for each k . On the lowest filtration level, generated by orbit sets $\gamma \otimes \Gamma$ such that $\gamma \in \mathcal{R}[e', h']$, U_k is given by

$$(9-9-1) \quad U_k(\gamma \otimes \Gamma) = \gamma/e' \otimes \Gamma.$$

Proof Fix k . By Lemma 9.4.1, the map U_k preserves the filtration $\mathcal{F}_k$. Moreover, by Lemma 9.5.1 (see also Corollary 9.5.2), curves which contribute to U_k and do not decrease the filtration level do not cross $\mathbb{R} \times T_i$ (for $i = 1, 2$). This implies that $U_k(\gamma \otimes \Gamma) = U_k(\gamma) \otimes \Gamma$ when $\gamma \in \mathcal{R}[e', h']$, and $U_k(\gamma)$ counts index $I = 2$ curves in V passing through z . We will use the ECH index and the Fredholm index to constrain such curves.

Let u be an $I = 2$, J'_k -holomorphic map in $\mathbb{R} \times V$ with $\gamma_+ = (e')^{a_+}(h')^{b_+}$ at the positive end and $\gamma_- = (e')^{a_-}(h')^{b_-}$ at the negative end; of course $b_{\pm} \in \{0, 1\}$. If we denote by $D_{e'}$ and $D_{h'}$ the meridian disks of V with boundary on e' and h' , respectively, and by $Z \in H_2(V, \gamma_+, \gamma_-)$ the relative homology class determined by u , we have $Z = (\alpha_+ - \alpha_-)[D_{e'}] + (\beta_+ - \beta_-)[D_{h'}]$.

We compute $I(\gamma_+, \gamma_-, Z)$ using (2-3-2). On e' and h' we consider the trivialization τ induced by ∂V . The Conley–Zehnder indices are $\mu_{\tau}((e')^i) = 1$ for $i = 1, \dots, k$ and $\mu_{\tau}(h') = 0$ by Definition 4.3.1, because they are on a slight perturbation of a positive Morse–Bott torus. The relative Chern class is $c_1(\xi|_{[D_{e'}]}, \tau) = c_1(\xi|_{[D_{h'}]}, \tau) = 1$. Putting everything together,

$$I(\gamma_+, \gamma_-, Z) = 2(a_+ - a_-) + (b_+ - b_-).$$

Then $I(\gamma_+, \gamma_-, Z) = 2$ implies $e_+ - a_- = 1$ and $b_+ - b_- = 0$, because $b_+ - b_- \in \{-1, 0, 1\}$. We call $b = b_+ = b_-$.

Negative ends at e' cannot be contained in $\mathbb{R} \times V$ by the trapping lemma, Lemma 5.3.2. (While the trapping lemma was proved for orbits on a Morse–Bott torus, it still holds for e' which is a slight elliptic perturbation.) Therefore u consists of a cover of a trivial cylinder over e' of degree a_- , together with a J'_k -holomorphic map $u: F \rightarrow \mathbb{R} \times V$ with positive asymptotics to $e'(h')^b$ and negative asymptotics to $(h')^b$ and representing the relative homology class $[D_{e'}]$. Since $\text{ind}(u) = 2$, the index formula (2-2-1) implies that $\chi(F) = 1$. This leaves only two possibilities: either u consists of a Fredholm index 2 plane which is positively asymptotic to e' together with a trivial cylinder over h' , or it consists of a Fredholm index one cylinder from e' to h' together with a Fredholm index one plane which is positively asymptotic

to h' . The second configuration cannot pass through a generic point z and therefore has to be discarded. The problem of computing U_k in the lowest filtration level is thus reduced to the count of J'_k -holomorphic planes in $\mathbb{R} \times V$ asymptotic to e' and passing through a generic point.

If we degenerate the contact forms α'_k toward the Morse–Bott contact forms α_k and the almost complex structures J'_k toward the almost complex structures J_k , the J'_k -holomorphic curves described above converge to very nice J_k -holomorphic Morse–Bott buildings because the topology of the domain does not allow the creation of branched covers of trivial cylinders (with nonempty branch locus) connected to Morse trajectories. Then, by [Theorem 4.4.3\(4\)](#), the count of $I = 2$, J'_k -holomorphic planes on $\mathbb{R} \times V$ which are positively asymptotic to e' and pass through a generic point z is the same as the count of Morse–Bott buildings consisting of a J_k -holomorphic plane on $\mathbb{R} \times V$ which passes through a generic point z and is positively asymptotic to an orbit of ∂V , augmented by a Reeb trajectory from e' to that orbit.

By [Lemma 8.4.8](#), the principal part of such a Morse–Bott building must be a leaf of the finite-energy foliation $\mathcal{Z}_1$. Since there is a unique leaf through any point, this proves that $U_k(\gamma \otimes \Gamma) = \gamma/e' \otimes \Gamma$. $\square$

Corollary 9.9.4 *The following diagram commutes for each k :*

$$(9-9-2) \quad \begin{array}{ccc} \mathrm{ECC}_{\leq k}^{\mathfrak{h}, L'_k}(N, \alpha) & \xrightarrow{\sigma_k} & \mathrm{ECC}^{L_k}(M, \alpha'_k) \\ U_k^{\mathfrak{h}} \downarrow & & \downarrow U_k \\ \mathrm{ECC}_{\leq k}^{\mathfrak{h}, L'_k}(N, \alpha) & \xrightarrow{\sigma_k} & \mathrm{ECC}^{L_k}(M, \alpha'_k) \end{array}$$

Proof Since σ_k takes values in the lowest level of the filtration $\mathcal{F}_k$, we can use [\(9-9-1\)](#) to compute $U_k \circ \sigma_k$. Then, for $\gamma \otimes \Gamma \in \mathrm{ECC}_{\leq k}^{\mathfrak{h}, L'_k}(N, \alpha)$, we have

$$\begin{aligned} U_k(\sigma_k(\gamma \otimes \Gamma)) &= U_k\left(\sum_{i=0}^{\infty} (e')^i \gamma \otimes (\partial'_N)^i \Gamma\right) = \sum_{i=1}^{\infty} (e')^{i-1} \gamma \otimes (\partial'_N)^i \Gamma, \\ \sigma_k(U_k^{\mathfrak{h}}(\gamma \otimes \Gamma)) &= \sigma_k(\gamma \otimes \partial'_N \Gamma) = \sum_{i=0}^{\infty} (e')^i \gamma \otimes (\partial'_N)^{i+1} \Gamma. \end{aligned}$$

Hence $U_k \circ \sigma_k = \sigma_k \circ U_k^{\mathfrak{h}}$. $\square$

Proof of Theorem 1.1.1(2) By [Lemma 9.9.2](#), diagram [\(9-9-2\)](#), and the naturality property of mapping cones, there is a chain map

$$\widehat{\sigma}_k : \widehat{\mathrm{ECH}}_{\leq k}^{\mathfrak{h}, L'_k}(N, \alpha) \rightarrow \widehat{\mathrm{ECC}}^{L_k}(M, \alpha'_k)$$

for each k . Taking homology (with the help of [Lemma 9.9.1](#)) and direct limits over k , we obtain a map

$$\widehat{\sigma}_* : \widehat{\mathrm{ECH}}(N, \partial N, \alpha) \simeq \widehat{\mathrm{ECH}}^{\mathfrak{h}}(N, \alpha) \rightarrow \widehat{\mathrm{ECH}}(M).$$

This map fits into the U -map exact sequences by properties of mapping cones:

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{U^\natural} & \text{ECH}(N, \partial N) & \longrightarrow & \widehat{\text{ECH}}(N, \partial N) & \longrightarrow & \text{ECH}(N, \partial N) \xrightarrow{U^\natural} \cdots \\
 \sigma_* \downarrow & & \sigma_* \downarrow & & \hat{\sigma}_* \downarrow & & \sigma_* \downarrow \\
 \cdots & \xrightarrow{U} & \text{ECH}(M) & \longrightarrow & \widehat{\text{ECH}}(M) & \longrightarrow & \text{ECH}(M) \xrightarrow{U} \cdots
 \end{array}$$

The five lemma then implies that $\hat{\sigma}_*$ is an isomorphism. Moreover, $\hat{\sigma}_*$ preserves the decompositions of $\widehat{\text{ECH}}(N, \partial N, \alpha)$ and $\widehat{\text{ECH}}(M)$ according to (relative) homology classes. $\square$

Remark 9.9.5 Embedded contact homology can be defined over the integers by choosing a coherent orientation system for the moduli spaces. For its definition or construction we refer to [Bourgeois and Mohnke 2004; Hutchings and Taubes 2009a, Section 9]. Different choices of coherent orientation systems yield isomorphic chain complexes.

All results of this article carry over with integer coefficients, and with the same proofs, if there is a coherent orientation system such that

- the holomorphic plane with positive asymptotics at h' and the holomorphic plane with positive asymptotics at e' and passing through a generic point count positively;
- the holomorphic cylinders from e' to h and from h' to e count positively; and
- the holomorphic cylinders from e' to h' and from h to e have opposite signs, so that they cancel each other in the differentials.

The first two items can be easily obtained by automorphisms of the complexes adjusting the signs of the generators e' , h' , e and h , and the third item follows from the identification of orientations of moduli spaces of Morse trajectories with orientations of the corresponding moduli spaces of holomorphic maps, as sketched in the first paragraph of the proof of [Bourgeois 2002, Lemma 7.6].

10 Applications to sutured ECH

In this section we apply Theorem 1.1.1 to sutured ECH.

10.1 Sutured ECH

In this subsection we briefly review sutured ECH, referring the reader to [Colin et al. 2011b] for more details.

A *sutured manifold* is a pair (M, Γ) , where M is a 3-manifold with boundary and corners, $\Gamma \subset \partial M$ is a possibly disconnected 1-manifold,¹⁵ $N(\Gamma)$ is an annular neighborhood of Γ , and ∂M admits the decomposition into two-dimensional strata

$$\partial M = R_+(\Gamma) \cup R_-(\Gamma) \cup N(\Gamma)$$

¹⁵In this section, Γ will denote a suture, not an orbit set.

as in [Colin et al. 2011b, Definition 2.7]. Note that our definition does not allow for “torus sutures” as in Gabai’s original definition [1983, Definition 2.6].

A *sutured contact form* $\bar{\alpha}$ on (M, Γ) ¹⁶ (see [Colin et al. 2011b, Definition 2.8]) is, roughly speaking, a contact form $\bar{\alpha}$ on M whose Reeb vector field $R_{\bar{\alpha}}$ is positively transverse to $R_+(\Gamma)$, negatively transverse to $R_-(\Gamma)$, and tangent to $N(\Gamma)$, and such that the trajectories of $R_{\bar{\alpha}}|_{N(\Gamma)}$ are arcs from $\partial R_-(\Gamma)$ to $\partial R_+(\Gamma)$. One can easily verify that (M, Γ) admits a sutured contact form if and only if it is *balanced*, ie $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$. A sutured contact manifold $(M, \Gamma, \bar{\alpha})$ admits a *completion* $(M^*, \bar{\alpha}^*)$; see [Colin et al. 2011b, Section 2.4].

Let $(M, \Gamma, \bar{\alpha})$ be a sutured contact manifold. We now describe the *sutured ECH group* $\text{ECH}(M, \Gamma, \bar{\alpha}, J)$. Its chain group¹⁷ $\text{ECC}(M, \Gamma, \bar{\alpha}, J)$ is generated by orbit sets constructed from simple Reeb orbits in $\text{int}(M)$ and the differential counts ECH index one J -holomorphic maps in the symplectization of $(M^*, \bar{\alpha}^*)$ for an almost complex structure J which is adapted to the symplectization and satisfies properties (A₀)–(A₂) from [Colin et al. 2011b, Section 3.1]. Almost complex structures of this type are said to be *tailored to* $(M, \Gamma, \bar{\alpha})$.

Completions are not necessary in dimension three by the following lemma:

Lemma 10.1.1 *Let J be tailored to $(M, \Gamma, \bar{\alpha})$. Then all J -holomorphic curves in $(M^*, \bar{\alpha}^*)$ which are asymptotic to closed Reeb orbits in $\text{int}(M)$ are contained in $\mathbb{R} \times \text{int}(M)$.*

Proof This follows from the proofs of [Colin et al. 2011b, Lemma 5.6 and Corollary 5.7], and relies on the fact that $R_+(\Gamma)$ and $R_-(\Gamma)$ automatically admit Stein structures. $\square$

We finish this review of sutured ECH by recalling a useful result from [Colin et al. 2011b] and sketching a simpler proof in dimension three.

Definition 10.1.2 [Colin et al. 2011b, Section 9] Let $(M, \Gamma, \bar{\alpha})$ be a sutured contact manifold. An *interval-fibered extension* is a contact embedding

$$(M, \Gamma, \bar{\alpha}) \hookrightarrow (M', \Gamma', \bar{\alpha}')$$

such that $M' - \text{int}(M) = W \times [0, 1]$, where

- W is a cobordism from Γ' to Γ , and
- $\bar{\alpha}'|_{W \times [0, 1]} = c \, dt + \beta$ for a Liouville form β on W and $c > 0$.

Lemma 10.1.3 [Colin et al. 2011b, Theorem 9.1] *Let $(M, \Gamma, \bar{\alpha}) \hookrightarrow (M', \Gamma', \bar{\alpha}')$ be an interval-fibered extension. Then there is a canonical isomorphism of chain complexes between $\text{ECC}(M, \Gamma, \bar{\alpha})$ and $\text{ECC}(M', \Gamma', \bar{\alpha}')$.*

¹⁶We use $\bar{\alpha}$ to denote an unspecified sutured contact form because α is reserved, in Section 9, to the contact form on N . Such contact form will appear again later in this section.

¹⁷We will often write $\text{ECC}(M, \Gamma, \bar{\alpha})$ and $\text{ECH}(M, \Gamma, \bar{\alpha})$ for simplicity.

Proof All closed Reeb orbits in M' are contained in M because all Reeb trajectories in $M' - \text{int}(M)$ go from $R_-(\Gamma')$ to $R_+(\Gamma')$. Moreover, J -holomorphic curves in $\mathbb{R} \times M'$ between orbit sets in $\text{int}(M)$ are contained in $\mathbb{R} \times M$. In fact, if a J -holomorphic curve nontrivially intersects $\mathbb{R} \times (M' - M) = \mathbb{R} \times W \times [0, 1]$, then its projection to W is surjective by the positivity of intersections with the Reeb vector field. This implies that the curve touches $\mathbb{R} \times \partial M'$, which is impossible by [Lemma 10.1.1](#). $\square$

10.2 Topological invariance of sutured ECH

In this subsection we pay off a debt from [\[Colin et al. 2011b\]](#), namely we sketch a proof that sutured ECH depends only on the sutured manifold and the contact structure. A more detailed proof can be found in [\[Kutluhan et al. 2022\]](#). In view of [\[Colin et al. 2011b, Conjecture 1.5\]](#), we expect sutured ECH to be independent also of the contact structure.

Lemma 10.2.1 *Let $(M, \Gamma, \bar{\alpha})$ be a sutured contact manifold such that Γ is connected. Then, for every $L \gg 0$, we can embed $(M, \Gamma, \bar{\alpha})$ into a closed contact manifold $(\tilde{M}, \tilde{\alpha}^L)$ such that*

$$\text{ECH}^{L'}(M, \Gamma, \bar{\alpha}) \simeq \text{ECH}^{L'}(\tilde{M}, \tilde{\alpha}^L)$$

for every $L' \leq L$. Moreover, $\tilde{M}$, up to diffeomorphism, depends only on (M, Γ) and, if $\bar{\alpha}_0$ and $\bar{\alpha}_1$ define isotopic contact structures on (M, Γ) , then $\tilde{\alpha}_0^L$ and $\tilde{\alpha}_1^L$ define isotopic contact structures on $\tilde{M}$.

Proof Since (M, Γ) is balanced and Γ is connected, $R_+(\Gamma)$ and $R_-(\Gamma)$ have the same genus and are diffeomorphic. We identify $\partial R_+(\Gamma)$ and $\partial R_-(\Gamma)$ by a diffeomorphism $\partial h_0: \partial R_+(\Gamma) \xrightarrow{\sim} \partial R_-(\Gamma)$, which is defined by the Reeb flow on $N(\Gamma)$, and fix a diffeomorphism $h_0: R_+(\Gamma) \xrightarrow{\sim} R_-(\Gamma)$ which extends ∂h_0 . Let us write $\beta_+ = \bar{\alpha}|_{R_+(\Gamma)}$ and $\beta_- = \bar{\alpha}|_{R_-(\Gamma)}$. Then the contact form $\bar{\alpha}$, on a neighborhood $R_+(\Gamma) \times [1 - \epsilon, 1]$ or $R_-(\Gamma) \times [-1, -1 + \epsilon]$ of $R_\pm(\Gamma) = R_\pm(\Gamma) \times \{\pm 1\}$ with coordinates (x, t) , has the form $c dt + \beta_\pm$ for some $c > 0$ (see [\[Colin et al. 2011b, Definition 2.8\]](#)). Here $\epsilon > 0$ is small.

By Moser's theorem and [Lemma 9.3.2](#), there is a diffeomorphism $h: R_+(\Gamma) \xrightarrow{\sim} R_-(\Gamma)$ isotopic to h_0 relative to ∂h_0 , such that $h^*\beta_- - \beta_+ = df$ for some function $f: R_+(\Gamma) \rightarrow \mathbb{R}$ which is constant near $\partial R_+(\Gamma)$.

Let us write $R = R_+(\Gamma)$. By repeating the proof of [Lemma 9.3.3](#), we construct a contact form $f_t dt + \beta_t$ on $R \times [1, 2]$ such that $f_t > 0$, $f_t dt + \beta_t = c dt + \beta_+$ on $R \times [1, 1 + \epsilon]$, and $f_t dt + \beta_t = c dt + h^*\beta_-$ on $R \times [2 - \epsilon, 2]$. Pick a bump function $\varphi: [1, 2] \rightarrow [1, 2]$ and consider the contact forms $(f_t + C_L \varphi(t)) dt + \beta_t$ on $R \times [1, 2]$ for some large positive constant C_L to be determined later.

We obtain the manifold M' by gluing $R \times \{1\}$ to $R_+(\Gamma)$ by the identity and $R \times \{2\}$ to $R_-(\Gamma)$ by h . The contact forms $\bar{\alpha}$ on M and $(f_t + C_L \varphi(t)) dt + \beta_t$ on $R \times [1, 2]$ match near the gluing region, so they define a contact form on M' . Finally, we obtain $\tilde{M}$ by gluing a solid torus V to M' along the boundary so that a meridian of the solid torus is identified with a Reeb orbit on $\partial M'$. The contact form on M' can be extended to a contact form $\tilde{\alpha}^L$ on $\tilde{M}$ by taking the contact form on V as in [Example 6.2.4](#).

By taking C_L sufficiently large, we ensure that Reeb trajectories from $R_+(\Gamma)$ to $R_-(\Gamma)$ and closed Reeb orbits in V have action larger than L ; for Reeb orbits in V this is a simpler application of the arguments in the proof of [Lemma 7.2.3](#). Hence $\text{ECC}^{L'}(M, \Gamma, \bar{\alpha}) = \text{ECC}^{L'}(\tilde{M}, \tilde{\alpha}^L)$ as abelian groups if $L' \leq L$. Any tailored almost complex structure J on $\mathbb{R} \times M$ can be extended to an almost complex structure J on $\mathbb{R} \times \tilde{M}$ which is adapted to the symplectization of $\tilde{\alpha}^L$.

Next we claim that a J -holomorphic map $u: F \rightarrow \mathbb{R} \times \tilde{M}$ which is asymptotic to orbit sets in M has image in $\mathbb{R} \times M$. These orbit sets have trivial linking number with the core of V , so $\text{Im}(u) \subset \mathbb{R} \times M'$ by the blocking lemma. On the other hand, $\text{Im}(u) \cap (\mathbb{R} \times R \times [1, 2]) = \emptyset$: Observe that $R_{\pm}(\Gamma)$ can be lifted to a family v_s for $s \in \mathbb{R}$ of J -holomorphic maps in $\mathbb{R} \times M'$ which foliate $\mathbb{R} \times R_{\pm}(\Gamma)$. By the positivity of intersections, if u intersects some v_s , then it intersects all v_s . However $\text{Im}(u_{M'}) \cap R_{\pm}(\Gamma)$ is compact and u cannot intersect v_s for $s \gg 0$, a contradiction. Hence $\text{Im}(u) \subset \mathbb{R} \times M$.

The remaining claims in the statement are straightforward. $\square$

Theorem 10.2.2 *Let $\bar{\alpha}_1$ and $\bar{\alpha}_2$ be sutured contact forms on a sutured three-manifold (M, Γ) and let J_1 and J_2 be almost complex structures on $\mathbb{R} \times M$ such that J_i is tailored to $(M, \Gamma, \bar{\alpha}_i)$ for $i = 1, 2$. If $\xi_1 = \ker \bar{\alpha}_1$ and $\xi_2 = \ker \bar{\alpha}_2$ are isotopic through contact structures adapted to the sutures, then*

$$\text{ECH}(M, \Gamma, \bar{\alpha}_1, J_1) \simeq \text{ECH}(M, \Gamma, \bar{\alpha}_2, J_2).$$

Moreover, this isomorphism preserves the decomposition of the sutured ECH groups as direct sums of subgroups indexed by homology classes in $H_1(M)$.

Proof We may assume that Γ is connected, since otherwise we can make Γ connected by gluing an interval-fibered extension, which does not change the sutured ECH groups by [Lemma 10.1.3](#). We extend $(M, \Gamma, \bar{\alpha}_i)$ to $(\tilde{M}, \tilde{\alpha}_i^L)$ as in [Lemma 10.2.1](#) and follow the proof of [Proposition 7.2.1](#) step by step. The statement about the decomposition according to homology classes follows from the fact that the isomorphism is supported on holomorphic buildings contained in $\mathbb{R} \times M$ in the sense of [Theorem 3.1.2\(i\)](#). $\square$

10.3 Applications

If M is a closed 3-manifold and $B \subset M$ is an embedded open 3-ball, we define the sutured manifold

$$M(1) = (M - B, \Gamma_0),$$

where Γ_0 is a connected simple closed curve in $\partial(M - B)$. If $K \subset M$ is a knot and $N(K)$ is an open tubular neighborhood of K , we define the sutured manifold

$$M(K) = (M - N(K), \Gamma_K),$$

where Γ_K consists of two disjoint copies of a meridian of K . When considering $M(1)$, we will assume that $K \setminus B$ is connected and goes from $R_-(\Gamma)$ to $R_+(\Gamma)$. If $\bar{\alpha}$ is a contact form on $M - B$ or $M - N(K)$ satisfying the conditions in [\[Colin et al. 2011b, Definition 2.8\]](#), then the sutured ECH groups $\text{ECH}(M(1), \bar{\alpha})$ and $\text{ECH}(M(K), \bar{\alpha})$ are defined.

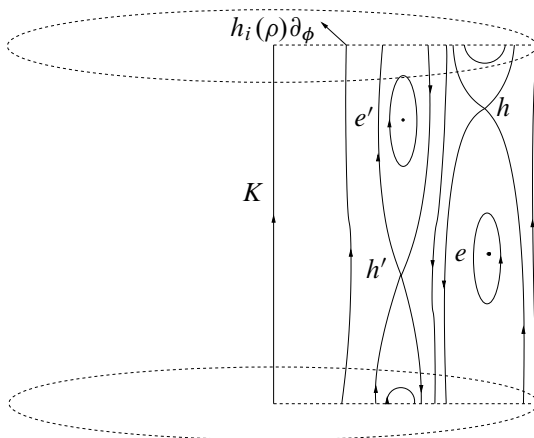


Figure 5: The Reeb vector field R'_i on $N_0(K) = (T^2 \times [1, 2]) \cup V$. The top and the bottom are identified.

Theorem 10.3.1 $\widehat{\text{ECH}}(M) \simeq \text{ECH}(M(1), \bar{\alpha})$.

This theorem concludes the proof of [Colin et al. 2011b, Theorem 1.6].

Proof Let ξ be a contact structure on M extending $\bar{\xi} = \ker \bar{\alpha}$ such that $K \subset M$ is a ξ -transverse knot. Recall the decomposition

$$M = N \cup (T^2 \times [1, 2]) \cup V$$

from previous sections, where we take $N_0(K) = (T^2 \times [1, 2]) \cup V$ to be a neighborhood of K .

There exists a sequence of contact forms α'_i for $i = 0, 1, \dots$ for ξ (up to isotopy) and associated Reeb vector fields R'_i , satisfying properties (1)–(8) of Section 9.2. Figure 5 depicts R'_i on $N_0(K) \simeq D^2(2) \times S^1$ with cylindrical coordinates (ρ, ϕ, θ) , where $D^2(\rho_0) = \{\rho \leq \rho_0\}$ and $V \simeq D^2(1) \times S^1$. The Reeb vector field R'_i is ∂_ϕ -invariant and of the form $R'_i = Y + h_i(\rho)\partial_\phi$, where Y is tangent to the slices $\{\phi = \text{const}\}$ as given in Figure 5 and $h_i(\rho) > 0$ for $\rho > 0$.

Choose almost complex structures J'_i adapted to α'_i as in Section 9.2 so that J'_i is ∂_ϕ -invariant on $N_0(K)$ and is close to the almost complex structure J_0 from Proposition 8.4.4 on V .

We describe a concave ball B in M whose complement is $M(1)$; see Figure 6. Let D be a meridian disk in V which bounds e' and is the projection to V of an $I = 2$, J'_i -holomorphic plane u asymptotic to e' at the positive end. The plane u corresponds to a leaf of the finite-energy foliation $\mathcal{Z}_0$ of $\mathbb{R} \times V$ from Proposition 8.4.4. Let $N(e')$ be a neighborhood of e' whose boundary is tangent to R'_i . We then set $B = N(D) \cup N(e')$, where $N(D)$ is a small neighborhood of D , chosen so that ∂B decomposes into three parts:

- two disks $R_\pm(\Gamma_0)$ transverse to R'_i that are parallel copies of a small retract of D ; and
- an annulus $N(\Gamma_0) \subset \partial N(e')$ tangent to R'_i .

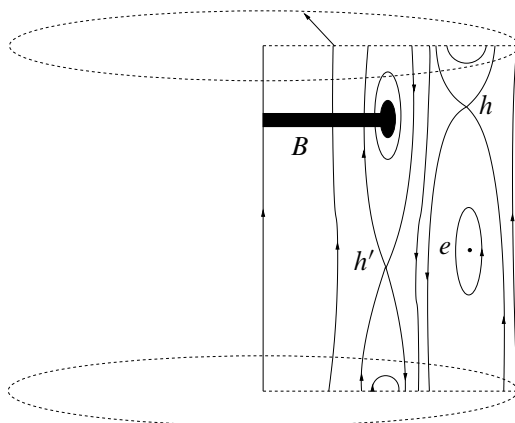


Figure 6: The concave ball B , obtained by rotating the shaded region about the vertical axis.

We assume that the $I = 1$, J'_i -holomorphic plane asymptotic to h' has image in $\mathbb{R} \times (V - B)$ and that $R_{\pm}(\Gamma_0)$ are also chosen to be restrictions of projections to M of $I = 2$, J'_i -holomorphic planes asymptotic to e' . The trajectories of R_i flow from one boundary component of $N(\Gamma_0)$ to the other.

The manifold $(M(1), \Gamma_0, \alpha'_i)$ is a sutured contact manifold and, by [Theorem 10.2.2](#), $\text{ECH}(M(1), \Gamma_0, \alpha'_i)$ is isomorphic to $\text{ECH}(M(1), \Gamma_0, \bar{\alpha})$. By construction, the orbit e' does not belong to $M(1)$ and all the orbits in V are now chords from $\partial M(1)$ to $\partial M(1)$. The Reeb orbits of R'_i that are contained in $M(1)$ are

- (1) all Reeb orbits in N ;
- (2) e , h and h' ; and
- (3) orbits longer than L_i in the no man's land.

By taking direct limits as in [Section 9.6](#), we can discard orbits in the no man's land. The use of direct limits in this context is justified by [Theorem 10.2.2](#).

By our choice of J'_i , if u is a holomorphic curve in $\mathbb{R} \times M$ between orbit sets constructed from orbits of type (1) and (2) in $M(1)$, then $\text{Im}(u) \subset \mathbb{R} \times M(1)$. (The orbits of type (1) and (2) have the lowest $\mathcal{F}_i$ -filtration level and we can use the blocking and trapping lemmas.) In particular, there are exactly two $I = 1$ curves that limit to h' at the positive end, as it is in $\mathbb{R} \times N_0(K)$: one plane from h' to $\emptyset$ and one cylinder from h' to e . Therefore we obtain an identification

$$\lim_{i \rightarrow \infty} (\text{ECC}^{L_i}(M(1), \Gamma_0, \alpha'_i), \partial) \simeq (\widehat{\text{ECC}}^{\natural}(N, \alpha), \widehat{\partial}^{\natural}),$$

which, in view of [Lemma 9.9.1](#) and [Theorem 1.1.1\(2\)](#), implies the theorem. □

If the contact form $\bar{\alpha}$ is chosen carefully, a null-homologous knot $K \subset M$ induces a filtration on the chain complex $\text{ECC}(M(1), \bar{\alpha})$ and the associated graded group is $\text{ECC}(M(K), \bar{\alpha})$. This construction was described in [[Colin et al. 2011b](#), Section 7.2]. If $N = M - N_0(K)$ as above, there is a filtration $\mathcal{E}$ on $\widehat{\text{ECC}}(N, \partial N, \alpha)$ defined as follows: Let $\mathcal{P}$ be the set of simple Reeb orbits in $\text{int}(N)$. The generators of

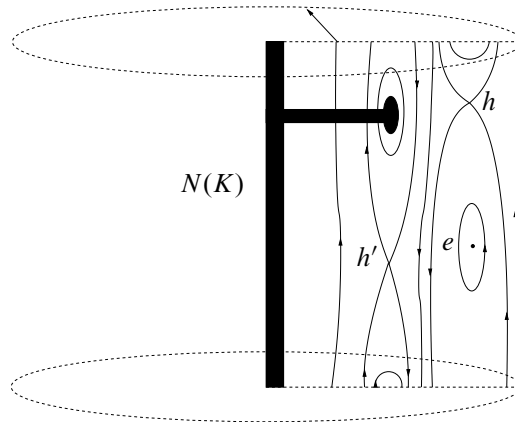


Figure 7: Construction of the concave neighborhood $(N(K), \Gamma_K)$, obtained by rotating the shaded region about the vertical axis

$\widehat{\text{ECC}}(N, \partial N, \alpha)$ are equivalence classes of orbit sets Γ constructed from $\mathcal{P} \cup \{h, e\}$, up to the equivalence relation $\Gamma \sim e\Gamma$. To the equivalence class of Γ we can uniquely associate an orbit set Γ' constructed from $\mathcal{P} \cup \{h\}$. Then we define $\mathcal{E}(\Gamma)$ as the algebraic intersection of Γ' with a Seifert surface of K . The differential of $\widehat{\text{ECC}}(N, \partial N, \alpha)$ preserves $\mathcal{E}$ by the trapping lemma and it is easy to identify the graded group of this filtration with $\text{ECH}^\#(N, \alpha)$.

Theorem 10.3.2 *If $K \subset M$ is a null-homologous knot, then there is a contact form $\bar{\alpha}$ on M for which the isomorphism in Theorem 10.3.1 preserves the filtrations and induces an isomorphism*

$$\text{ECH}(M(K), \bar{\alpha}) \simeq \text{ECH}^\#(N, \alpha).$$

Proof Let $K \subset M$ be a null-homologous knot and Σ a genus-minimizing Seifert surface for K . Following [Colin and Honda 2005], we construct a family of contact forms α'_i on M as in the proof of Theorem 10.3.1 on $N_0(K)$, with the additional property that the Reeb vector fields R'_i are positively transverse to $\text{int}(\Sigma)$. The construction is done in two steps: first on N by a direct application of [Colin and Honda 2005], where we use Σ as the first decomposing surface of a taut sutured hierarchy of N , and then on $N_0(K)$, where we extend the form by the explicit model already described in Section 9.3.2.

We obtain a concave neighborhood $(N(K), \Gamma_K)$ of K by taking $N(K) = B \cup N_\epsilon(K)$, where $N_\epsilon(K)$ is a very small neighborhood of K whose boundary is tangent to R'_i , as in Figure 7, and B is the ball constructed in the proof of Theorem 10.3.1.

The suture Γ_K corresponds to the core curves of the two annuli in $\partial N(K)$ tangent to R'_i . At this point, $(M - N(K), \Gamma_K)$ is not yet a convex sutured manifold, because $\partial N(K)$ is not convex for the dividing set given by the two curves of Γ_K . In fact, on the component A of $\partial N(K)$ coming from $N_\epsilon(K)$, $\ker \alpha'_i|_A$ is negatively transverse to the core of A (oriented as the boundary of $R_+(\Gamma_K)$). To correct this, we glue a collar of the form $(A \times [a, b], dt + f(y) dx)$, $\partial f / \partial y < 0$, to $(M - N(K), \alpha'_i)$ along $A = A \times \{a\}$, where

$A \times [a, b] = [0, 1] \times S^1 \times [a, b]$ has coordinates (t, x, y) . Then the Reeb vector field remains ∂_t while the contact plane rotates until $\ker \alpha'_i|_{A \times \{b\}}$ is positively transverse to the core of A .

The positive transversality of the Reeb vector fields with the Seifert surface Σ ensures that the isomorphism of [Theorem 10.3.1](#) preserves the filtrations given by the linking number with K .

Passing from $M(1)$ to $M(K)$ has the effect of killing the “meridian” holomorphic disk from h' which passes through $\mathbb{R} \times K$. After passing to direct limits, we obtain the desired isomorphism. $\square$

Appendix Morse–Bott gluing

by Vincent Colin, Paolo Ghiggini, Ko Honda and Yuan Yao

The goal of this appendix is to prove Parts (2) and (3) of [Theorem 4.4.3](#). The proof of Part (4) is similar and will be omitted. The proof involves working out Morse–Bott gluing in a special case, which easily generalizes to one-level cascades in ECH. Yao [\[2022b; 2022a\]](#) will prove the general ECH Morse–Bott gluing theorem in the presence of Morse–Bott tori and multiple-level cascades. There are slight differences in packaging, but our strategy and the one from [\[Yao 2022b; 2022a\]](#) for 1-level cascades are essentially equivalent.

For simplicity we assume there is only one Morse–Bott torus $T_{\mathcal{N}}$ and that it is a negative Morse–Bott torus. It is generally acknowledged that the proof of Morse–Bott gluing in [\[Bourgeois 2002\]](#) is incomplete, but instead of fixing this, we carry out a different preguing with a smaller error term. At first we will use a stable Hamiltonian structure whose hyperplane distribution is integrable near the Morse–Bott torus to simplify the gluing estimates in various ways. *In [Section A.9](#) we will explain how to derive a similar statement for contact structures from [Theorem A.2.1](#).*

A.1 Stable Hamiltonian structures, almost complex structures and moduli spaces

Let $[-1, 1] \times T^2 = [-1, 1] \times (\mathbb{R}^2/\mathbb{Z}^2)$ be a neighborhood of the negative Morse–Bott torus $T_{\mathcal{N}}$ with coordinates $(y, (\theta, t))$ such that $T_{\mathcal{N}} = \{0\} \times T^2$, and let $\mathcal{N}$ be the Morse–Bott family of simple orbits of the form $\{y = 0, \theta = \text{const}\}$. Also let $A_{y_0} = [-y_0, y_0] \times \mathbb{R}/\mathbb{Z}$ be an annulus with coordinates (y, θ) .

Morse–Bott perturbation of the stable Hamiltonian vector field The construction will depend on parameters c, a, b_0, b and ϵ which will be made more specific during the course of this appendix and when we make specific choices they will be indicated by $(\dagger_0)$ – $(\dagger_3)$. The parameters c, a and b_0 (chosen in this order) will describe the data of the problem and will be chosen once and for all at the beginning to satisfy

$$0 < 4b_0 < a < c < 1.$$

The constant c depends on the action level, a depends on the Morse–Bott moduli spaces we want to glue — morally speaking it determines the region where the first nonconstant term in the Fourier expansion of the negative end is not dominated by the higher-order terms; see $(\dagger_1)$ — and b_0 is arbitrary, as long as it

is sufficiently smaller than a . The perturbation of the Morse–Bott Reeb vector field and pregluing will depend on the parameters $b \in (0, \frac{1}{2}b_0)$ and $\epsilon > 0$ (chosen in this order). The parameter b will determine the support of the perturbation and the parameter ϵ the size. Then, by the usual contraction mapping argument, we will prove that, for every b sufficiently small and every ϵ sufficiently small compared to b , the preglued curve can be deformed to a holomorphic curve.

On $[-c, c] \times T^2$ consider the stable Hamiltonian structure consisting of the 1-form dt and the 2-form $\omega_H = dH \wedge dt + dy \wedge d\theta$, where $H: A_c \rightarrow \mathbb{R}$ is a function of (y, θ) (and is independent of t). The stable Hamiltonian vector field R_H is then

$$(A-1-1) \quad R_H = \frac{\partial}{\partial t} + X_H, \quad \text{where } i_{X_H} dy \wedge d\theta = dH.$$

Let J_H be the adapted almost complex structure on $\mathbb{R} \times [-c, c] \times T^2$ which sends $\partial/\partial s \mapsto R_H$, $R_H \mapsto -\partial/\partial s$, $\partial/\partial y \mapsto \partial/\partial \theta$ and $\partial/\partial \theta \mapsto -\partial/\partial y$, where s is the $\mathbb{R}$ -coordinate.

We specialize the smooth function H to

$$(A-1-2) \quad f(y, \theta) = \frac{1}{2}y^2 \quad \text{or} \quad f_\epsilon(y, \theta) = \frac{1}{2}y^2 + \epsilon\phi(y)\bar{g}_N(\theta),$$

where $\epsilon > 0$ is small, the domain of $\bar{g}_N(\theta)$ is S^1 viewed as the interval $[-\frac{1}{2}, \frac{1}{2}]$ with the endpoints identified, and the following hold:

- (P2) $\bar{g}_N: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ is a perfect Morse function with maximum at $\frac{1}{4}$ and minimum at $-\frac{1}{4}$. More specifically, we assume that $\bar{g}'_N(\theta) = 0$ on $\theta = \pm\frac{1}{4}$, is linear with positive slope on $[-\frac{1}{4}, -\frac{1}{5}]$, is nondecreasing on $[-\frac{1}{5}, -\frac{1}{6}]$, and is equal to 1 on $[-\frac{1}{6}, \frac{1}{6}]$; and $\bar{g}_N(\theta)$ is an odd function about $\theta = 0$.
- (P3') $\phi: [-c, c] \rightarrow [0, 1]$ is an even function which has support on $[-2b_0, 2b_0]$ and is equal to 1 on $[-b_0, b_0]$.

Here (P2) is exactly the same as (P2) from [Section 4.1](#) and (P3') is a tweaking of (P3). We observe that $f_\epsilon \rightarrow f$ in C^∞ as $\epsilon \rightarrow 0$.

The torus T_N is a negative Morse–Bott torus with respect to R_f . After perturbing to R_{f_ϵ} , the Morse–Bott family of stable Hamiltonian orbits becomes a pair e and h of stable Hamiltonian orbits over $(0, -\frac{1}{4})$ and $(0, \frac{1}{4})$ in A_c . See [Figure 8](#).

The Morse–Bott perturbation is performed below a fixed action L , which is later sent to infinity by a direct limit process. The action of a stable Hamiltonian orbit in $[-c, c] \times T^2$ depends on how many times it intersects an annulus $A_c \times \{t\}$ and therefore, instead of working below an action level L , we work below an intersection number N .

- (†₀) The constant c is chosen so that $0 < c < 1$ and all closed orbits of R_f in $A_c \times S^1$ that intersect $A_c \times \{t\}$ at most N times are covers of orbits in T_N . The constant $\epsilon > 0$ will always be small enough that all closed orbits of R_{f_ϵ} in $A_c \times S^1$ that intersect $A_c \times \{t\}$ at most N times are covers of e and h .

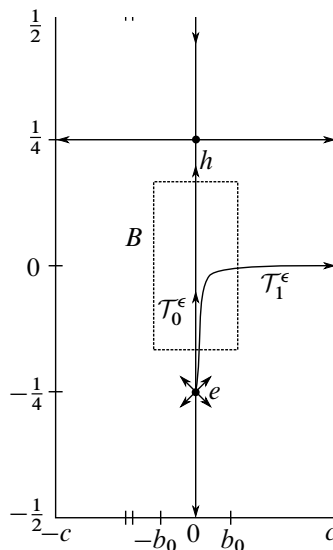


Figure 8: The annulus $A_c = [-c, c] \times \mathbb{R}/\mathbb{Z}$ with some gradient trajectories of f_ϵ . The top and the bottom are identified. The dotted rectangle is the boundary of $B = [-b_0, b_0] \times [-\frac{1}{6}, \frac{1}{6}]$, on which $\phi(y) = 1$ and $\tilde{g}'_{\mathcal{N}}(\theta) = 1$.

The next lemma follows from the explicit constructions in [Section A.9](#) and [Claim A.9.3](#).

Lemma A.1.1 *There exist stable Hamiltonian structures (α, ω) and $(\alpha, \omega_\epsilon)$ on M and almost complex structures J_f and J_{f_ϵ} on $\mathbb{R} \times M$ such that:*

- (1) *On $[-a, a] \times T^2$, $(\alpha, \omega) = (dt, \omega_f)$ and $(\alpha, \omega_\epsilon) = (dt, \omega_{f_\epsilon})$.*
- (2) *On $M - ([-a, a] \times T^2)$, $\omega = \omega_\epsilon$ is a multiple of $d\alpha$ by a positive function (and therefore α is a contact form).*
- (3) *On $M - ([-c, c] \times T^2)$, $\omega = \omega_\epsilon = d\alpha$.*
- (4) *J_f and J_{f_ϵ} are adapted to (α, ω_f) and $(\alpha, \omega_{f_\epsilon})$, respectively, and $J_f = J_{f_\epsilon}$ outside of $\mathbb{R} \times [-a, a] \times T^2$.*

Simplification A.1.2 From now on we will consider only the case $N = 1$ because it contains already all the relevant ideas.

Moduli spaces Let

$$\mathcal{M}^{\text{MB}} := \mathcal{M}_{J_f}^{\text{MB}} := \mathcal{M}_{J_f}^{I=\text{ind}=1}(\gamma; \mathcal{N})$$

be the moduli space of (finite-energy) J_f -holomorphic maps $u_+ : (\dot{F}, j) \rightarrow \mathbb{R} \times M$ modulo domain automorphisms, where

- (C0) $(\dot{F}, j)$ is a closed Riemann surface with a finite number of punctures removed and we are ranging over all complex structures j with a fixed topological type $\dot{F}$;
- (C1) u_+ limits to the orbit set γ at the positive end, where γ does not involve any orbits of the Morse–Bott family $\mathcal{N}$;

- (C2) u_+ limits to some orbit in the Morse–Bott family $\mathcal{N}$ at the negative end; and
- (C3) u_+ has “unconstrained” Fredholm and ECH index 1 (the negative end is unconstrained); see [Section 8.4.1](#) for more details.

By (C3) we mean that, if we concatenate u_+ with a cylinder corresponding to an upward gradient trajectory that starts at $(0, -\frac{1}{4})$ so that we have a map C from γ to e , then the Fredholm and ECH indices of C are 1. (C3) implies that curves of $\mathcal{M}^{\text{MB}}$ are isolated modulo $\mathbb{R}$ -translation and are embedded.

Next let

$$\mathcal{M}_\epsilon := \mathcal{M}_{J_{f_\epsilon}} := \mathcal{M}_{J_{f_\epsilon}}^{I=\text{ind}=1}(\gamma, e)$$

be the moduli space of J_{f_ϵ} -holomorphic maps $u: (\dot{F}, j) \rightarrow \mathbb{R} \times M$ modulo domain automorphisms, where (C0), (C1) (with u instead of u_+) and the following hold:

- (C2') u limits to the negative elliptic orbit e obtained by perturbing the Morse–Bott family.
- (C3') u has Fredholm and ECH index 1.

We also remark that the moduli spaces $\mathcal{M}^{\text{MB}}$ and $\mathcal{M}_\epsilon$ can be made Morse–Bott regular or regular by perturbing J_f and J_{f_ϵ} outside of $[-c, c] \times T^2$.

Holomorphic curves near the Morse–Bott torus

Claim A.1.3 The equation $\bar{\partial}_{J_{f_\epsilon}} u = 0$ for a map¹⁸

$$u: [\tilde{s}_0, \tilde{s}_1] \times S^1 \rightarrow \mathbb{R} \times S_t^1 \times A_c, \quad u(s, t) = (s, t, \eta(s, t)),$$

is equivalent to the equation

$$(A-1-3) \quad \mathcal{D}_\epsilon \eta := \frac{\partial \eta}{\partial s} + j_0 \frac{\partial \eta}{\partial t} - \nabla f_\epsilon(\eta) = 0,$$

where $j_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard almost complex structure on A_c .

Proof We apply $\bar{\partial}_{J_{f_\epsilon}} = \partial_s + J_{f_\epsilon} \partial_t$ to $(s, t, \eta(s, t))$ to obtain

$$(A-1-4) \quad \begin{pmatrix} 1 \\ 0 \\ \partial \eta / \partial s \end{pmatrix} + J_{f_\epsilon} \begin{pmatrix} 0 \\ 1 \\ \partial \eta / \partial t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \partial \eta / \partial s \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ j_0 \partial \eta / \partial t - \nabla f_\epsilon(\eta) \end{pmatrix}.$$

This is because $J_{f_\epsilon}(\partial_t) = -\partial_s - j_0 X_{f_\epsilon}$ and $j_0 X_{f_\epsilon} = \nabla f_\epsilon$ (recall the sign in (A-1-1)). Hence

$$J_{f_\epsilon} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -\nabla f_\epsilon \end{pmatrix}. \quad \square$$

¹⁸We abuse notation and use coordinates (s, t) for both the cylindrical part of the domain and $\mathbb{R} \times S^1$. We also change the order of the coordinates from (y, θ, t) to (t, y, θ) . This has no effect on the orientations of M and A_c .

The claim holds also for $\epsilon = 0$: the equation $\bar{\partial}_{J_{f_\epsilon}} u = 0$ for a map $u(s, t) = (s, t, \eta(s, t))$ as above is equivalent to $\mathcal{D}_0 \eta = 0$, where

$$\mathcal{D}_0 \eta = \frac{\partial \eta}{\partial s} + j_0 \frac{\partial \eta}{\partial t} - \nabla f(\eta).$$

Remark A.1.4 To treat the case $N > 1$ we need to consider maps $u: [\tilde{s}_0, \tilde{s}_1] \times S^1 \rightarrow \mathbb{R} \times S_t^1 \times A_c$ which wind k times around S_t^1 for $k \leq N$. In that case we should write $u(s, t) = (ks, kt, \eta(s, t))$, but all estimates on η remain unchanged.

The following easy consequence of [Claim A.1.3](#) provides the link between gradient trajectories and holomorphic curves:

Lemma A.1.5 Every gradient trajectory $\mathcal{T}$ of f_ϵ (here we are allowing $\epsilon = 0$ and $f_0 = f$) admits a unique lift to a simply covered J_{f_ϵ} -holomorphic cylinder $u_{\mathcal{T}}$ whose projection to A_c is $\mathcal{T}$ modulo reparametrization of the domain and $\mathbb{R}$ -translations of $u_{\mathcal{T}}$.

Proof If $\eta: [\tilde{s}_0, \tilde{s}_1] \rightarrow A_c$ is a parametrization of $\mathcal{T}$ satisfying $d\eta/ds = \nabla f_\epsilon(\eta)$, then $u_{\mathcal{T}}(s, t) := (s, t, \eta(s))$ satisfies $\bar{\partial}_{J_{f_\epsilon}} u_{\mathcal{T}} = 0$ by [Claim A.1.3](#). On the other hand, one can immediately check that a simply covered map to $\mathbb{R} \times S^1 \times A_c$ that projects to $\mathcal{T}$ must be of the form $(s, t) \mapsto (s, t, \eta(s))$ for some η up to reparametrizations and translations. $\square$

A.2 Main result

The main result of the appendix is the following:

Theorem A.2.1 If $\mathcal{M}_{J_f}^{\text{MB}}$ is Morse–Bott regular, then, for $a, b_0 > 0$ sufficiently small, there exist

- J'_f that agrees with J_f on $[-c, c] \times T^2$ and is arbitrarily close to J_f on $M - ([-c, c] \times T^2)$,
- $\epsilon > 0$ that is sufficiently small, and
- J'_{f_ϵ} that agrees with J_{f_ϵ} on $[-c, c] \times T^2$ and with J'_f on $M - ([-c, c] \times T^2)$

such that $\mathcal{M}_{J'_f}^{\text{MB}}$ is Morse–Bott regular, $\mathcal{M}_{J'_{f_\epsilon}}$ is regular, and there is a bijection between $\mathcal{M}_{J'_f}^{\text{MB}}$ and $\mathcal{M}_{J'_{f_\epsilon}}$.

Remark A.2.2 In the case where $\mathcal{M}_{J_f}^{\text{MB}}$ satisfies (C0), (C1), and the unconstrained end is replaced by a constrained end in (C2) and (C3), ie the negative end limits to a hyperbolic orbit after perturbation, we can simply glue in a trivial cylinder at the said end, since having constrained index means not including $\tilde{\partial}_\theta$ in [\(A-5-15\)](#) and Morse–Bott gluing then reduces to standard gluing.

Brief discussion on regularity We will not prove that, for all $\epsilon > 0$ sufficiently small, $\mathcal{M}_\epsilon$ is regular if $\mathcal{M}^{\text{MB}}$ is Morse–Bott regular, although that is true. It suffices for our purposes to know that “for some $\epsilon > 0$ small and some J'_f and J'_{f_ϵ} , there is a bijection between $\mathcal{M}_{J'_f}^{\text{MB}}$ which is Morse–Bott regular and $\mathcal{M}_{J'_{f_\epsilon}}$ which is regular”.

We will explain the existence of J'_f and J'_{f_ϵ} such that $\mathcal{M}_{J'_f}^{\text{MB}}$ is Morse–Bott regular and $\mathcal{M}_{J'_{f_\epsilon}}$ is regular: Since J_f is Morse–Bott regular for $\mathcal{M}_{J_f}^{\text{MB}}$, the same holds for all J'_f that are sufficiently close to J_f on $M - ([-c, c] \times T^2)$ and agree with J_f on $[-c, c] \times T^2$. Next, we perturb J_{f_ϵ} to J'_{f_ϵ} on $M - ([-c, c] \times T^2)$ so that $\mathcal{M}_{J'_{f_\epsilon}}$ is regular. This is possible because the only Reeb orbits of $(\alpha, \omega_{J_\epsilon})$ inside $[-c, c] \times T^2$ come from the perturbation of the Morse–Bott torus, and therefore every holomorphic curve in $\mathcal{M}_{J'_{f_\epsilon}}$ intersects $M - ([-c, c] \times T^2)$, except for the two curves corresponding to the two flow lines on the Morse–Bott family, whose regularity can be easily checked by hand.

Let us fix an $\mathbb{R}$ -invariant Riemannian metric on $\mathbb{R} \times M$ that agrees with the flat metric $ds^2 + dt^2 + dy^2 + d\theta^2$ on $\mathbb{R} \times [-1, 1] \times T^2$. All distances will be measured with respect to this metric.

Definition A.2.3 Let $\kappa > 0$. A curve $u: \dot{F} \rightarrow \mathbb{R} \times M$ in $\mathcal{M}_\epsilon$ is κ -close to breaking into $u_+: \dot{F} \rightarrow \mathbb{R} \times M$ in $\mathcal{M}^{\text{MB}}$ and $u_{\mathcal{T}^\epsilon}: (-\infty, 0] \times S^1 \rightarrow \mathbb{R} \times M$, where $\mathcal{T}^\epsilon$ is an upward gradient trajectory of f_ϵ , if

- (i) on the complement of a negative cylindrical end $(-\infty, 0] \times S^1$ of $\dot{F}$, the maps u and u_+^* (obtained from u_+ by a suitable translation in the domain if $\dot{F}$ is a cylinder and a suitable $\mathbb{R}$ -translation in the target) are a distance $\leq \kappa$ apart;
- (ii) on $(-\infty, 0] \times S^1$, the maps u and $u_{\mathcal{T}^\epsilon}^*$ (obtained from $u_{\mathcal{T}^\epsilon}$ by a suitable $\mathbb{R}$ -translation in the target) are a distance $\leq \kappa$ apart.

Let $u_+: (\dot{F}, j) \rightarrow \mathbb{R} \times M$ be an element of $\mathcal{M}^{\text{MB}}$. In what follows, we may assume without loss of generality that:

- (C4) u_+ limits to the Morse–Bott orbit $\mathfrak{o}$ over the point $(0, 0)$ from the positive y -direction at the negative end.

This is justified as follows: The quotient $\mathcal{M}^{\text{MB}}/\mathbb{R}$ by $\mathbb{R}$ -translations in the target is a finite set by (C3). Let $\mathcal{E}: \mathcal{M}^{\text{MB}}/\mathbb{R} \rightarrow \mathcal{N}$ be the map that sends $[u]$ to the orbit of $\mathcal{N}$ that u limits to at the negative end. Since the image of $\mathcal{E}$ is a finite set, we can parametrize $\mathcal{N} \cong \mathbb{R}/\mathbb{Z}$ such that $\mathcal{E}([u]) \in [-\frac{1}{6}, \frac{1}{6}]$ for all $u \in \mathcal{M}^{\text{MB}}$. Since our proof works in the same way as long as $\mathcal{E}([u])$ is in the interior of the interval $\{\theta \in \mathbb{R}/\mathbb{Z} \mid \bar{g}'_{\mathcal{N}}(\theta) = 1\}$ (refer to (P2) for the definition of $\bar{g}_{\mathcal{N}}$), we normalize $\mathcal{E}([u]) = 0$. Moreover, approaching $\theta = 0$ from the positive y -direction and the negative y -direction can be treated in the same way.

Notation A.2.4 Let $\mathcal{T}_0^\epsilon$ denote the (upward) gradient trajectory of f_ϵ that goes from $(0, -\frac{1}{4})$ to $(0, 0)$.

Theorem A.2.1 is an immediate consequence of the following theorems, which are proved in Sections A.7 and A.8, together with the above discussion on regularity:

Theorem A.2.5 Suppose $a, b_0 > 0$ are small. If $\mathcal{M}^{\text{MB}}$ is Morse–Bott regular, then, for all $\epsilon > 0$ sufficiently small, there exists $u \in \mathcal{M}_\epsilon$ that is κ -close to breaking into u_+ and $u_{\mathcal{T}_0^\epsilon}$.

Theorem A.2.6 Suppose $a, b_0 > 0$ are small. If $\mathcal{M}^{\text{MB}}$ is Morse–Bott regular and $\mathcal{M}_\epsilon$ is regular, then there exists $\kappa > 0$ such that, for all $\epsilon > 0$ sufficiently small and $u, v \in \mathcal{M}_\epsilon$ that are κ -close to breaking into u_+ and $u_{\mathcal{T}_0^\epsilon}$, $u = v$ modulo $\mathbb{R}$ -translation in the target and domain translation if the domain is $\mathbb{R} \times S^1$.

Remark A.2.7 The assumptions

- (i) there is only one Morse–Bott torus $T_{\mathcal{N}}$ and it is negative, and
- (ii) u_+ limits to γ at the positive end and $\mathcal{N}$ at the negative end,

are only to make the notation simpler, since gluing each pair of ends can be done more or less independently. This is due to the fact that the magnitude of the error that comes from a pair $\mathfrak{P}$ of glued ends and needs to be inverted in the Newton iteration decays exponentially with respect to the distance to the gluing region of $\mathfrak{P}$.

A.3 Asymptotic operator

On $B := [-b_0, b_0] \times [-\frac{1}{6}, \frac{1}{6}] \subset A_c$, we have $\bar{g}'_{\mathcal{N}}(\theta) = 1$ by (P2) and $\phi(y) = 1$ by (P3'). Then $\nabla f_\epsilon = y\partial_y + \epsilon\partial_\theta$ and the equation $\mathcal{D}_\epsilon\eta = 0$ becomes the linear equation

$$\frac{\partial\eta}{\partial s} + j_0 \frac{\partial\eta}{\partial t} - \begin{pmatrix} \eta_1 \\ \epsilon \end{pmatrix} = 0,$$

or

$$(J_{f_\epsilon}) \quad \frac{\partial\eta}{\partial s} - A\eta = \begin{pmatrix} 0 \\ \epsilon \end{pmatrix}, \quad A\eta = -j_0 \frac{\partial\eta}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \eta,$$

where $j_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\eta = (\eta_1, \eta_2)$, and A is the asymptotic operator for the negative end of u_+ that goes to the Morse–Bott family $\mathcal{N}$. Here we are regarding S^1 as $[-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$, so that $\eta \subset A_c$ is regarded in $\mathbb{R}^2$ and the matrix multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ makes sense.

Similarly, a J_f -holomorphic map $(s, t) \mapsto (s, t, \eta(s, t))$ with $\eta(s, t) \in A_c$ is equivalent to

$$(J_f) \quad \mathcal{D}_0\eta = \frac{\partial\eta}{\partial s} - A\eta = 0.$$

Remark A.3.1 In the region where $\mathcal{D}_\epsilon\eta = 0$ is equivalent to (J_{f_ϵ}) , a solution of (J_f) can be converted to a solution of (J_{f_ϵ}) by adding $\begin{pmatrix} 0 \\ \epsilon_{s+C} \end{pmatrix}$.

From now on we will write the components of η as row vectors if there is no confusion.

Claim A.3.2 The eigenfunctions of A can be arranged as

$$\dots, g_{-2}, g_{-1}, g_0 = (0, 1), g_1 = (1, 0), g_2, \dots,$$

normalized to have unit L^2 norm, with corresponding eigenvalues

$$\dots \leq \lambda_{-2} \leq \lambda_{-1} < \lambda_0 = 0 < \lambda_1 = 1 < \lambda_2 \leq \dots,$$

where, if λ is any of $\lambda_{2n} = \lambda_{2n+1}$ or $\lambda_{-2n} = \lambda_{-2n+1}$, then $\lambda(\lambda - 1) = (2\pi n)^2$ and

$$\begin{aligned} g_{2n} & \text{ is a multiple of } \left(\frac{2\pi n}{\lambda_{2n}-1} \cos(2\pi n t), \sin(2\pi n t) \right), \\ g_{2n+1} & \text{ is a multiple of } \left(\frac{2\pi n}{\lambda_{2n}-1} \sin(2\pi n t), -\cos(2\pi n t) \right). \end{aligned}$$

Proof If λ is an eigenvalue of A , then

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y'(t) \\ \theta'(t) \end{pmatrix} + \begin{pmatrix} y(t) \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} y(t) \\ \theta(t) \end{pmatrix},$$

which is equivalent to $\theta'(t) = (\lambda - 1)y(t)$, $-y'(t) = \lambda\theta(t)$. Hence $\theta''(t) = (1 - \lambda)\lambda\theta(t)$. If $\theta(t)$ is to be 1-periodic, $\lambda = 0, 1$, $\lambda > 1$ or $\lambda < 0$. In the latter two cases, $\theta(t)$ is a translate of $\sin(2\pi n t)$ and $y(t) = 2\pi n/(\lambda - 1)$ times a translate of $\cos(2\pi n t)$ and $(2\pi n)^2 = \lambda(\lambda - 1)$. $\square$

We can then write a solution $\eta(s, t)$ of (J_f) as a Fourier series

$$(A-3-1) \quad \eta(s, t) = \sum_{i=-\infty}^{\infty} c_i e^{\lambda_i s} g_i(t).$$

A clarification of the meaning of (A-3-1) is in order: the eigenfunctions g_i take values in $T_{(0,0)}A_c \cong \mathbb{R}^2$, while η takes values in $A_c \cong [-c, c] \times S^1$. Thus, in the equality we have tacitly identified a neighborhood of $(0, 0)$ in $T_{(0,0)}A_c$ with a neighborhood of $(0, 0)$ in A_c using the identification of S^1 with a quotient of $[-\frac{1}{2}, \frac{1}{2}]$ fixed at the beginning of the appendix.

A.4 Pregluing

Let $u_+ : \dot{F} \rightarrow \mathbb{R} \times M$ be a J_f -holomorphic map representing an element of $\mathcal{M}^{\text{MB}}$. We fix a cylindrical end $(-\infty, s_0] \times S^1$ of $\dot{F}$ corresponding to the orbit $\mathfrak{o}$ on which u_+ takes the form $u_+(s, t) = (s, t, \eta_+(s, t))$, $\eta_+(s, t) \in A_c$. In view of (C4) we can write

$$(A-4-1) \quad \eta_+(s, t) = \sum_{i=1}^{\infty} c_i e^{\lambda_i s} g_i(t),$$

where $c_1 > 0$. The condition $c_1 \neq 0$ holds for a generic J_f because the moduli space $\mathcal{M}^{\text{MB}}$ is one-dimensional. This is proved in the same way as [Hutchings and Taubes 2009a, Theorem 4.1], which treats the contact case. We further assume that $c_1 > 0$ since the $c_1 < 0$ case can be treated in the same way. Finally, we can assume that (A-4-1) has no $i = 0$ term because we assumed that $\mathfrak{o}$ is the orbit over $(0, 0)$.

Definition A.4.1 Let $T_0 = T_0(a)$ and $T_1 = T_1(b)$ be real numbers such that

$$(A-4-2) \quad c_1 e^{-\lambda_1 T_0} g_1 = (\tfrac{1}{2}a, 0), \quad c_1 e^{-\lambda_1 T_1} g_1 = (b, 0).$$

Note that $T_1(b) \rightarrow \infty$ as $b \rightarrow 0$.

(†₁) We choose $a, b_0 > 0$, with $b_0 < \frac{1}{4}a$, to be sufficiently small that $T_0 > 0$ and, for all $b < b_0$,

$$\eta_+|_{s \leq -T_0} \subset A_a, \quad \eta_+|_{-T_0 \leq s \leq s_0} \subset A_c \cap \{y > 2b_0\} \quad \text{and} \quad \eta_+|_{s \leq -T_1} \subset B.$$

The choice of a is made possible by the fact that $\sum_{i=2}^{\infty} c_i e^{\lambda_i s} g_i(t)$ decays exponentially at a rate which is faster than $c_1 e^{\lambda_1 s} g_1(t)$. From now on, a and T_0 are fixed constants, while b and $T_1(b)$ are, for the moment, still allowed to vary and will be fixed at a later time.

Remark A.4.2 Since the perturbation of f , and therefore of R_f , given in (A-1-2) depends on a and b_0 by conditions (P2) and (P3'), it is important that $\mathcal{M}^{\text{MB}}$ is finite, so that we can find a and b_0 which satisfy $(\dagger_1)$ for every $u_+ \in \mathcal{M}^{\text{MB}}$.

Let $\eta_-^\epsilon: \mathbb{R} \rightarrow A_a$ be a parametrization of the gradient flow trajectory of f_ϵ from $(0, -\frac{1}{4})$ to $(0, \frac{1}{4})$ solving the Cauchy problem

$$\begin{cases} \frac{d\eta_-^\epsilon}{ds} = \nabla f_\epsilon(\eta_-^\epsilon), \\ \eta_-^\epsilon(-T_1) = (0, 0), \end{cases}$$

and let $u_-^\epsilon(s, t) = (s, t, \eta_-^\epsilon(s))$. We trivially extend η_-^ϵ to a function $\eta_-^\epsilon: \mathbb{R} \times S^1 \rightarrow A_a$ by $\eta_-^\epsilon(s, t) = \eta_-^\epsilon(s)$.

Definition A.4.3 Let $T_2 = T_2(\epsilon)$ be a real number such that $T_1 < T_2$ and $\eta_-^\epsilon|_{-T_2 \leq s \leq -T_1} \subset B$.

Note that $T_2(\epsilon) \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

Let $\beta: \mathbb{R} \rightarrow [0, 1]$ be a nondecreasing function such that $\beta(s) = 0$ if $s \leq 0$ and $\beta(s) = 1$ if $s \geq 1$. The pregluing $u_*^{\epsilon, \alpha}$ (note the Fraktur symbol α is different from the parameter a) will depend on ϵ and an extra real parameter $\alpha \in [-\alpha_0, \alpha_0]$, where α_0 is independent of b and ϵ , and small enough that $\eta_-^\epsilon(-T_1 + \alpha/\epsilon)$ is contained in B , where ∇f_ϵ is constant. Then we define

$$(A-4-3) \quad u_*^{\epsilon, \alpha}(s, t) := \begin{cases} u_+(s, t) & \text{on } \dot{F} - (-\infty, -T_0] \times S^1, \\ (s, t, \eta_*^{\epsilon, \alpha}(s, t)) & \text{on } (-\infty, -T_0] \times S^1, \end{cases}$$

where

$$(A-4-4) \quad \eta_*^{\epsilon, \alpha}(s, t) = \begin{cases} \eta_+(s, t) + \beta\left(\frac{s+T_0}{-T_1+T_0}\right)(0, \epsilon(s+T_1)) + \beta(-s-T_0)(0, \alpha) & \text{on } [-T_1, -T_0] \times S^1, \\ \eta_-^\epsilon(s + \alpha/\epsilon, t) + \beta(s+T_2) \cdot \eta_+(s, t) & \text{on } (-\infty, -T_1] \times S^1. \end{cases}$$

Observe that $\eta_-^\epsilon(s + \alpha/\epsilon, t) = (0, \epsilon(s + T_1) + \alpha)$ on $[-T_2, -T_1] \times S^1$ since $\nabla f_\epsilon(y, \theta) = (y, \epsilon)$ and $\eta_-^\epsilon(-T_1, t) = (0, 0)$. Hence the two definitions agree along $s = -T_1$. Therefore $u_*^{\epsilon, \alpha}$ coincides with u_+ for $s \geq -T_0$, with the lift of a gradient trajectory of f_ϵ for $s \leq -T_2$, and interpolates between the two for $s \in [-T_2, -T_0]$. The interpolation is performed in three steps: for $s \in [-T_0 - 1, -T_0]$ the holomorphic curve is pushed in the θ -direction (ie along the Morse–Bott family) by a small amount α ; for $s \in [-T_1, -T_0]$ a perturbation corresponding to the gradient trajectory is slowly turned on and added to u_+ ; for $s \in [-T_2 + 1, -T_1]$ the preglued curve $u_*^{\epsilon, \alpha}$ is the sum of u_+ and the lift of a gradient trajectory of f_ϵ ; and for $s \in [-T_2, -T_2 + 1]$ the contribution of u_+ is turned off.

$(\dagger_2)$ We choose $\epsilon = \epsilon(b) > 0$ such that

$$\lim_{b \rightarrow 0} \epsilon(b) e^{T_1(b)} T_1(b) = 0.$$

Note that T_0 has become a constant after we fixed a , while T_1 depends on b and T_2 depends on ϵ .

Lemma A.4.4 If $u_{*}^{\epsilon, \alpha}$ is defined by (A-4-3) and (A-4-4), then $\bar{\partial}_{J_{f_{\epsilon}}} u_{*}^{\epsilon, \alpha}$ is supported on

- (1) $([-T_2, -T_2 + 1] \times S^1) \cup ([-T_1, -T_0] \times S^1)$; and
- (2) the “thick” parts of the domain of u_+ , ie $\dot{F} - (-\infty, T_0] \times S^1$, where the curve may still enter the region $y \in [-2b_0, 2b_0]$.¹⁹

Proof Note that $\bar{\partial}_{J_{f_{\epsilon}}} u_{*}^{\epsilon, \alpha} = 0$

- (a) on $\dot{F} - (-\infty, -T_0] \times S^1$, away from the region described in (2), where $u_{*}^{\epsilon, \alpha} = u_+$ and $f_{\epsilon} = f$, and
- (b) on $(-\infty, -T_2] \times S^1$, where $u_{*}^{\epsilon, \alpha}$ coincides with the $\bar{\partial}_{J_{f_{\epsilon}}}$ -holomorphic lift of a gradient trajectory of f_{ϵ} .

(For (a), note that f_{ϵ} and f differ only when $y \in [-2b_0, 2b_0]$ by (A-1-2) and (P3'), but we are assuming $(\dagger_1)$, which ensures that $\eta_+(T_0, t)$ has y -coordinate $> 2b_0$.) Therefore $\bar{\partial}_{J_{f_{\epsilon}}} u_{*}^{\epsilon, \alpha}$ is supported in $[-T_2, -T_0] \times S^1$, where $\bar{\partial}_{J_{f_{\epsilon}}} u_{*}^{\epsilon, \alpha} = 0$ is equivalent to $\mathcal{D}_{\epsilon} \eta_{*}^{\epsilon, \alpha} = 0$. We claim that $\mathcal{D}_{\epsilon} \eta_{*}^{\epsilon, \alpha} = 0$ on $[-T_2 + 1, -T_1] \times S^1$. In fact, in that region,

$$\eta_{*}^{\epsilon, \alpha} = \eta_{-}^{\epsilon, \alpha} + \eta_+$$

by (A-4-4) and the definition of β . Moreover, $\eta_{*}^{\epsilon, \alpha}$ takes values in B by condition $(\dagger_1)$ and Definition A.4.3, and in B we have

$$\mathcal{D}_{\epsilon} = \mathcal{D}_0 + (0, \epsilon),$$

where $\mathcal{D}_0$ is linear by $(J_{f_{\epsilon}})$. Thus,

$$\mathcal{D}_{\epsilon}(\eta_{*}^{\epsilon, \alpha}) = \mathcal{D}_0(\eta_{-}^{\epsilon, \alpha}) + \mathcal{D}_0(\eta_+) + (0, \epsilon) = \mathcal{D}_{\epsilon}(\eta_{-}^{\epsilon, \alpha}) + \mathcal{D}_0(\eta_+) = 0$$

because $\mathcal{D}_{\epsilon}(\eta_{-}^{\epsilon, \alpha}) = \mathcal{D}_0(\eta_+) = 0$. □

A.5 Function spaces

Let us introduce the notation

$$(A-5-1) \quad \eta_{-}^{\epsilon, \alpha}(s, t) = \eta_{-}^{\epsilon}(s + \alpha/\epsilon, t), \quad u_{-}^{\epsilon, \alpha}(s, t) = (s, t, \eta_{-}^{\epsilon, \alpha}(s, t)).$$

In this subsection we describe the linearized $\bar{\partial}$ -operators D_+ and $D_{-}^{\epsilon, \alpha}$ for u_+ and $u_{-}^{\epsilon, \alpha}$.

Since we are assuming that the ECH and Fredholm indices of u_+ and $u_{-}^{\epsilon, \alpha}$ are both 1, they are embedded and admit normal bundles. Let N_+ be a J_f -invariant normal bundle to u_+ in $\mathbb{R} \times M$ such that $N_+ = TA_a$ on $(-\infty, -T_0] \times S^1$, let $N_{-}^{\epsilon, \alpha} = TA_a$ be the normal bundle to $u_{-}^{\epsilon, \alpha}$ in $\mathbb{R} \times [-a, a] \times T^2$, and let $N_{*}^{\epsilon, \alpha}$ be the normal bundle to $u_{*}^{\epsilon, \alpha}$ that agrees with N_+ on $\dot{F} - (-\infty, -T_0] \times S^1$ and with TA_a on $(-\infty, -T_0] \times S^1$.

A.5.1 Exponential maps Let $\mathbb{D}_{\kappa} N_+$ denote the disk bundle of N_+ of radius $\kappa > 0$, measured with respect to g . Writing an element of N_+ as $(x, \xi(x))$, where $x \in \dot{F}$ and $\xi(x) \in N_+(u_+(x))$, for $\kappa > 0$ small we choose an exponential map

$$\exp_{u_+} : \mathbb{D}_{\kappa} N_+ \rightarrow \mathbb{R} \times M$$

¹⁹In this case the error is extremely small, of total size $C\epsilon$, and we will not mention it further.

such that $\exp_{u_+}(x, 0) = u_+(x)$, $d_{(x,0)} \exp_{u_+}(0, \zeta) = \zeta(u_+(x))$ for a section ζ of N^+ , and

$$\exp_{u_+}(x, \xi(x)) = (s(x), t(x), \eta_+(x) + \xi(x))$$

when $u_+(x) = (s(x), t(x), \eta_+(x))$ and $x \in (-\infty, -T_0] \times S^1$. We also define

$$\exp_{u_{\underline{\epsilon}, \alpha}}: \mathbb{D}_\kappa N_{\underline{\epsilon}, \alpha}^{\epsilon, \alpha} \rightarrow \mathbb{R} \times [-a, a] \times T^2, \quad (x, \xi(x)) \mapsto (s(x), t(x), \eta_{\underline{\epsilon}, \alpha}^{\epsilon, \alpha}(x) + \xi(x)).$$

Finally, we define $\exp_{u_*^{\epsilon, \alpha}}$ on $\mathbb{D}_\kappa N_*^{\epsilon, \alpha}$ so that it agrees with $\exp_{u_+}$ on $\dot{F} - (-\infty, -T_0] \times S^1$ and satisfies

$$(x, \xi(x)) \mapsto (s(x), t(x), \eta_*^{\epsilon, \alpha}(x) + \xi(x))$$

on $(-\infty, -T_0] \times S^1$. In particular, $\exp_{u_*^{\epsilon, \alpha}}$ coincides with $\exp_{u_{\underline{\epsilon}, \alpha}}$ on $(-\infty, -T_2) \times S^1$.

A.5.2 Normal $\bar{\partial}$ -equations Instead of using the full $\bar{\partial}$ -operator on sections of $u_+^* T(\mathbb{R} \times M)$ and $(u_{\underline{\epsilon}, \alpha}^*)^* T(\mathbb{R} \times M)$, following [Hutchings and Taubes 2009a] we will use the normal $\bar{\partial}$ -operators which act on sections of N_+ and $N_{\underline{\epsilon}, \alpha}^{\epsilon, \alpha}$. *The primary purpose of using the normal $\bar{\partial}$ -operators, assuming the curves are embedded, is to simplify the notation*, since the Teichmüller space parameters are automatically taken care of. More precisely, let L be the total linearized $\bar{\partial}$ -operator — this includes the Teichmüller space parameters — and let L_N be the normal linearized $\bar{\partial}$ -operator L_N . Then $\text{coker } L \simeq \text{coker } L_N$ and $\ker L_N \simeq (\ker L)/V$, where V is the subspace generated by the infinitesimal generators of the reparametrizations of the domain.

By standard local existence results of holomorphic disks, for $\kappa > 0$ small there exists a foliation of $\exp_{u_+}(\mathbb{D}_\kappa N_+)$ by J_f -holomorphic disks such that the holomorphic disk passing through $u_+(x)$ is tangent to $N_+(u_+(x))$. We can therefore adjust the map $\exp_{u_+}$ so that the fibers of $\mathbb{D}_\kappa N_+$ are mapped to holomorphic disks, use local coordinates (σ, τ, ξ) on $\exp_{u_+}(\mathbb{D}_\kappa N_+)$, where $\sigma + i\tau$ are holomorphic coordinates on $\dot{F}$ and ξ is the fiber coordinate, and write

$$J_f(\sigma, \tau, \xi) = \begin{pmatrix} \tilde{j}(\sigma, \tau, \xi) & 0 \\ X(\sigma, \tau, \xi) & j_0 \end{pmatrix},$$

where $\tilde{j}(\sigma, \tau, 0) = j_0$ and $X(\sigma, \tau, 0) = 0$. Since $\tilde{j}^2 = -I$, we have

$$\tilde{j}(\sigma, \tau, \xi) = \begin{pmatrix} a(\sigma, \tau, \xi) & c(\sigma, \tau, \xi) \\ b(\sigma, \tau, \xi) & -a(\sigma, \tau, \xi) \end{pmatrix}$$

and $\det \tilde{j} = 1$. Also $X\tilde{j} + j_0X = 0$.

We derive the normal $\bar{\partial}$ -equation for a section ξ of N_+ such that

$$(A-5-2) \quad \bar{\partial}_{J_f} \exp_{u_+} \xi = 0.$$

We recall that $\bar{\partial}_{J_f} u = du + J_f \circ du \circ j$, where j is a complex structure on the domain of u , and therefore (A-5-2) is an equation for a pair (j, ξ) , where j is a complex structure on $\dot{F}$. Then solving (A-5-2) is

equivalent to solving for $A(\sigma, \tau, \xi)$, $B(\sigma, \tau, \xi)$, and $\xi(\sigma, \tau)$ in

$$(A-5-3) \quad \left(\frac{\partial}{\partial \sigma} + J_f(\sigma, \tau, \xi) \left(A(\sigma, \tau, \xi) \frac{\partial}{\partial \sigma} + B(\sigma, \tau, \xi) \frac{\partial}{\partial \tau} \right) \right) \begin{pmatrix} \sigma \\ \tau \\ \xi \end{pmatrix} = 0.$$

Here the adjustment of the domain complex structure is equivalent to solving for $A(\sigma, \tau, \xi)$ and $B(\sigma, \tau, \xi)$. One easily verifies that

$$A(\sigma, \tau, \xi) = a(\sigma, \tau, \xi) \quad \text{and} \quad B(\sigma, \tau, \xi) = b(\sigma, \tau, \xi)$$

are the unique functions such that the (σ, τ) -component of (A-5-3) holds. Then the ξ -component of (A-5-3) is the normal $\bar{\partial}$ -equation for the section ξ of N_+ :

$$(A-5-4) \quad \bar{\partial}_{N_+, f} \xi := \frac{\partial \xi}{\partial \sigma} + j_0 \left(a(\sigma, \tau, \xi) \frac{\partial \xi}{\partial \sigma} + b(\sigma, \tau, \xi) \frac{\partial \xi}{\partial \tau} \right) + X(\sigma, \tau, \xi) \begin{pmatrix} a(\sigma, \tau, \xi) \\ b(\sigma, \tau, \xi) \end{pmatrix} = 0$$

such that

$$(A-5-5) \quad a(\sigma, \tau, 0) = 0, \quad b(\sigma, \tau, 0) = 1, \quad \text{and} \quad X(\sigma, \tau, 0) = 0.$$

Next we derive the normal $\bar{\partial}$ -equation for the section ξ of $N_-^{\epsilon, a}$ such that

$$\bar{\partial}_{J_{f_\epsilon}} \exp_{u_-^{\epsilon, a}} \xi = 0.$$

Recall that we write $\exp_{u_-^{\epsilon, a}} \xi = (s, t, \eta_-^{\epsilon, a} + \xi)$ on $\mathbb{R} \times [-a, a] \times T^2$. Since $\bar{\partial}_{J_{f_\epsilon}} u_-^{\epsilon, a} = 0$, the normal $\bar{\partial}$ -equation for ξ has the explicit expression

$$(A-5-6) \quad \begin{aligned} \bar{\partial}_{N_-^{\epsilon, a}, f_\epsilon} \xi &:= \mathcal{D}_\epsilon(\eta_-^{\epsilon, a} + \xi) = \mathcal{D}_\epsilon(\eta_-^{\epsilon, a}) + \mathcal{D}_\epsilon(\xi) \\ &= \frac{\partial \xi}{\partial s} + j_0 \frac{\partial \xi}{\partial t} - \nabla f_\epsilon(\eta_-^{\epsilon, a} + \xi) + \nabla f_\epsilon(\eta_-^{\epsilon, a}) = 0, \end{aligned}$$

by Claim A.1.3.

Finally, $\bar{\partial}_{N_*^{\epsilon, a}, f_\epsilon} \xi$ for the section ξ of $N_*^{\epsilon, a}$ agrees with $\bar{\partial}_{N_+, f} \xi$ on $\dot{F} - (-\infty, -T_0] \times S^1$ and with $\mathcal{D}_\epsilon(\eta_* + \xi)$ on $(-\infty, -T_0] \times S^1$ by Claim A.1.3.

The linearized operators for $\bar{\partial}_{N_+, f}$, $\bar{\partial}_{N_-^{\epsilon, a}, f_\epsilon}$, and $\bar{\partial}_{N_*^{\epsilon, a}, f_\epsilon}$ will be denoted by D_+ , $D_-^{\epsilon, a}$, and $D_*^{\epsilon, a}$. Next we will describe the proper function-theoretic setup for these operators.

A.5.3 Morrey spaces The function spaces that we use are *Morrey spaces*, following [Hutchings and Taubes 2009a, Section 5.5]. Let $u: \dot{F} \rightarrow \mathbb{R} \times M$ be a finite-energy holomorphic curve. On $\dot{F}$ we choose a Riemannian metric such that the ends are isometric to $\mathbb{R}/\mathbb{Z} \times [0, \infty)$ with the product metric. On $\mathbb{R} \times M$ we continue to use the $\mathbb{R}$ -invariant Riemannian metric from before.

The *Morrey space* $\mathcal{H}_0(\dot{F}, \Lambda^{0,1} N_+)$ is the Banach space which is the completion of the compactly supported sections of $\Lambda^{0,1} N_+$ with respect to the norm

$$(A-5-7) \quad \|\xi\| = \left(\int_{\dot{F}} |\xi|^2 \right)^{1/2} + \left(\sup_{x \in \dot{F}} \sup_{\rho \in (0, 1]} \rho^{-1/2} \int_{B_\rho(x)} |\xi|^2 \right)^{1/2},$$

where $B_\rho(x) \subset \dot{F}$ is the ball of radius ρ about x . Similarly, $\mathcal{H}_1(\dot{F}, N_+)$ is the completion of the compactly supported sections of N_+ with respect to

$$(A-5-8) \quad \|\xi\|_* = \|\nabla \xi\| + \|\xi\|.$$

Although Morrey spaces are not used as frequently as Sobolev spaces, they satisfy the analog of the usual Sobolev embedding theorem ([Lemma A.5.1](#)) and have the advantage that *we only need to do elementary L^2 -type estimates instead of more complicated L^p -type estimates.*

The analog of the usual Sobolev embedding theorem is the following:²⁰

Lemma A.5.1 *There is a bounded linear map*

$$\mathcal{H}_1(\dot{F}, N_+) \rightarrow C^0(\dot{F}, N_+) \cap L^\infty(\dot{F}, N_+), \quad \xi \mapsto \xi.$$

Proof If $\xi \in \mathcal{H}_1(\dot{F}, N_+)$ and $K \subset \dot{F}$ is a subdomain, then let us define

$$|\xi|_{C^{0,1/4},K} = \sup_{x \neq y \in K} \frac{|\xi(x) - \xi(y)|}{|x - y|^{1/4}}.$$

The lemma is a consequence of [[Morrey 1966](#), Theorem 3.5.2], which implies²¹ that, for any compact subdomain $K \subset \dot{F}$, there exists C_K such that

$$(A-5-9) \quad |\xi|_{C^{0,1/4},K} \leq C_K \left(\sup_{x \in \dot{F}} \sup_{\rho \in (0,1]} \rho^{-1/2} \int_{B_\rho(x)} |\nabla \xi|^2 \right)^{1/2} \leq C_K \|\xi\|_*.$$

This implies that any $\xi \in \mathcal{H}_1(\dot{F}, N_+)$ is continuous.

Since $\dot{F}$ has cylindrical ends, we can write $\dot{F} = K_0 \cup K_1 \cup K_2 \cup \dots$, where all the K_i are compact connected subdomains and $K_1, K_2, \dots$ are annuli of the form $\mathbb{R}/\mathbb{Z}$ times a unit interval. For each K_i and $x \neq y \in K_i$, we have

$$(A-5-10) \quad |\xi(x) - \xi(y)| \leq C_{K_i} \|\xi\|_* |x - y|^{1/4} \leq C(C')^{1/4} \|\xi\|_*,$$

where $C = \max\{C_{K_0}, C_{K_1}\}$, since $C_{K_1} = C_{K_2} = \dots$ and C' is the supremum of the diameters of K_i for $i = 0, 1, \dots$. Since ξ is continuous, on each K_i there exists x_i such that

$$(A-5-11) \quad |\xi(x_i)| = \|\xi|_{K_i}\|_{L^2/\text{vol}(K_i)} \leq \|\xi\|_*/C'',$$

where $C'' = \inf_i \text{vol}(K_i) > 0$. Inequalities (A-5-10) and (A-5-11) together imply that there exists a constant $c > 0$ which is independent of ξ and such that $|\xi(x)| \leq c \|\xi\|_*$ for all $x \in \dot{F}$. $\square$

²⁰The lemma is stated slightly differently from [[Hutchings and Taubes 2009a](#), Lemma 5.3].

²¹Morrey's theorem is stated for a Euclidean ball of radius R , but applies equally well to our setting. We take $p = 2$, $\nu = 2$ and $\mu = \frac{1}{4}$ in the theorem.

Given $\delta > 0$ sufficiently small, we define a smooth weight function

$$(A-5-12) \quad g_\delta: \dot{F} \rightarrow \mathbb{R}^+, \quad \begin{cases} g_\delta(x) = 1 & \text{on } \dot{F} - (-\infty, -T_0 + 1] \times S^1, \\ g_\delta(s, t) = e^{\delta|s+T_0|} & \text{for } s \leq -T_0. \end{cases}$$

Also define the smooth weight function

$$(A-5-13) \quad h_\delta: \mathbb{R} \times S^1 \rightarrow \mathbb{R}^+, \quad (s, t) \mapsto e^{-\delta(s+T_0)}.$$

Note that h_δ agrees with g_δ for $s \leq -T_0$. We recall that T_0 has been fixed once and for all in [Definition A.4.1](#) and [\(†0\)](#). For our purposes we define $\lambda := \min(\lambda_1, |\lambda_{-1}|)$ and take δ such that $5\delta < \lambda$.

We also define the weighted Morrey spaces $\mathcal{H}_{1,g_\delta}(\dot{F}, N_+)$ and $\mathcal{H}_{0,g_\delta}(\dot{F}, \Lambda^{0,1}N_+)$ as the spaces of sections ξ (of the respective bundles) such that the weighted Morrey norms

$$(A-5-14) \quad \|\xi\|_{*,g_\delta} := \|\xi \cdot g_\delta\|_*, \quad \|\xi\|_{g_\delta} := \|\xi \cdot g_\delta\|$$

are finite. Observe that, since we are using normal bundles, it is not necessary to use weights except at the end which limits to the Morse–Bott orbit. The Morrey spaces for $N_-^{\epsilon,a}$ and $N_*^{\epsilon,a}$ (with and without weights) are defined analogously.

A.5.4 Linearized operators Let $\tilde{\partial}_\theta$ be a smooth section of N_+ which is equal to $\beta(-s - T_0)\partial_\theta$ on $s \leq -T_0$ and is zero elsewhere. We view D_+ as a bounded linear operator

$$(A-5-15) \quad D_+^\delta: \mathcal{H}_{1,g_\delta}(\dot{F}, N_+) \oplus \mathbb{R}\langle \tilde{\partial}_\theta \rangle \rightarrow \mathcal{H}_{0,g_\delta}(\dot{F}, \Lambda^{0,1}N_+), \quad (\xi, c\tilde{\partial}_\theta) \mapsto D_+(\xi + c\tilde{\partial}_\theta).$$

The term $\mathbb{R}\langle \tilde{\partial}_\theta \rangle$ is included since D_+^δ is the linearized operator for the Morse–Bott family $\mathcal{M}^{\text{MB}} = \mathcal{M}_{J_f}^{I=\text{ind}=1}(\mathcal{Y}; \mathcal{N})$ with an unconstrained negative end but the infinitesimal deformations parallel to the Morse–Bott family do not belong to the Morrey space with weights.

We write

$$(A-5-16) \quad \mathcal{H}_{+,\delta} = \mathcal{H}_{1,g_\delta}(\dot{F}, N_+) \quad \text{and} \quad \mathcal{H}'_{+,\delta} = D_+^\delta(\mathcal{H}_{+,\delta}) \subset \mathcal{H}_{0,g_\delta}(\dot{F}, \Lambda^{0,1}N_+),$$

and let $\bar{D}_+^\delta: \mathcal{H}_{+,\delta} \rightarrow \mathcal{H}'_{+,\delta}$ be the map induced by D_+^δ by restriction.

Let us define

$$(A-5-17) \quad \nu := -\beta'(-s - T_0)\partial_\theta.$$

Then ν has compact support in $\{-T_0 - 1 \leq s \leq -T_0\}$ and satisfies

$$D_+^\delta(0, 1) = D_+(\tilde{\partial}_\theta) = \nu.$$

Also observe that $\nu \notin \mathcal{H}'_{+,\delta}$ because D_+^δ is surjective and $\text{ind}(D_+^\delta) = 1$ with $\ker D_+^\delta \subset \mathcal{H}_{+,\delta}$, and $\nu \neq 0$. Then we can define the projection

$$(A-5-18) \quad \Pi: \mathcal{H}_{0,g_\delta}(\dot{F}, \Lambda^{0,1}N_+) \rightarrow \mathcal{H}'_{+,\delta}$$

with $\nu \in \ker \Pi$. Note that ν has compact support in $\{-T_0 - 1 \leq s \leq -T_0\}$, where it can be written as $\nu = -\beta'(-s - T_0)\partial_\theta$.

Remark A.5.2 The domain of D_+^δ is the tangent space to the Banach manifold

$$\mathcal{H}_{1,g_\delta}(\dot{F}, \mathbb{R} \times M) := \{\exp_u(\xi) \mid u \in \mathcal{C}, \xi \in \mathcal{H}_{1,g_\delta}(\dot{F}, N_+^u)\},$$

where $\mathcal{C}$ is the space of smooth embeddings $u: \dot{F} \rightarrow \mathbb{R} \times M$ that agree with holomorphic maps parametrizing trivial holomorphic half-cylinders near each of the punctures; the positive ends of u and u_+ agree and the negative end of u limits to $\mathcal{N}$; and N_+^u is the J_f -invariant normal bundle to u .

Linearizing (A-5-6), we obtain

$$(A-5-19) \quad D_-^{\epsilon,a} \xi = \frac{\partial \xi}{\partial s} + j_0 \frac{\partial \xi}{\partial t} - (H f_\epsilon)(\eta_-^{\epsilon,a}) \xi,$$

where $H f_\epsilon$ is the Hessian of f_ϵ .

We view $D_-^{\epsilon,a}$ as a bounded linear operator

$$(A-5-20) \quad D_-^{\epsilon,a}: \mathcal{H}_1(\mathbb{R} \times S^1, N_-^{\epsilon,a}) \rightarrow \mathcal{H}_0(\mathbb{R} \times S^1, \Lambda^{0,1} N_-^{\epsilon,a}).$$

Since the normal bundles $N_-^{\epsilon,a}$ are trivialized, we can identify the domains and codomains of $D_-^{\epsilon,a}$ for different values of ϵ and a . We abbreviate $\mathcal{H}_- = \mathcal{H}_1(\mathbb{R} \times S^1, N_-^{\epsilon,a})$ and $\mathcal{H}'_- = \mathcal{H}_0(\mathbb{R} \times S^1, \Lambda^{0,1} N_-^{\epsilon,a})$. Both D_+^δ and $D_-^{\epsilon,a}$ are Fredholm of index 1.

We consider also operators

$$D_-^{\epsilon,a,\delta}: \mathcal{H}_{1,h_\delta}(\mathbb{R} \times S^1, N_-^{\epsilon,a}) \rightarrow \mathcal{H}_{0,h_\delta}(\mathbb{R} \times S^1, \Lambda^{0,1} N_-^{\epsilon,a})$$

which have the same expression as $D_-^{\epsilon,a}$ but act on the Morrey spaces with weights. We abbreviate $\mathcal{H}_{-, \delta} = \mathcal{H}_{1,h_\delta}(\mathbb{R} \times S^1, N_-^{\epsilon,a})$ and $\mathcal{H}'_{-, \delta} = \mathcal{H}_{0,h_\delta}(\mathbb{R} \times S^1, \Lambda^{0,1} N_-^{\epsilon,a})$.

Remark A.5.3 Sections $\xi \in \mathcal{H}_{-, \delta}$ can diverge as $s \rightarrow +\infty$ and therefore $\exp_{u_-^{\epsilon,a}}(\xi)$ may not be well defined. This makes the spaces $\mathcal{H}_{-, \delta}$ unsuitable for the nonlinear analysis of the moduli space containing $u_-^{\epsilon,a}$. However, they can still be used in the proof of [Theorem A.2.1](#) because, for the purposes of gluing, what happens near the positive end of $u_-^{\epsilon,a}$ is irrelevant. The reason we are using the operators $D_-^{\epsilon,a,\delta}$ is so that we can take the limit of $D_-^{\epsilon,a,\delta}$ as $\epsilon \rightarrow 0$ and obtain a Fredholm operator $D_-^{0,a,\delta}$ of the same index in the limit. This would not be true if we worked without weights, as the operators $D_-^{\epsilon,a}$ converge, for $\epsilon \rightarrow 0$, to an operator which is not Fredholm.

Lemma A.5.4 If $\delta, a_0 = a_0(\delta)$ and $\epsilon_0 = \epsilon_0(\delta, a_0)$ are sufficiently small subject to $0 \leq \epsilon_0 < \delta$, and $(a, \epsilon) \in [-a_0, a_0] \times [0, \epsilon_0]$, then the operators $D_-^{\epsilon,a,\delta}$ are invertible. Moreover, for a fixed δ , the norms of the inverse operators $(D_-^{\epsilon,a,\delta})^{-1}$ are uniformly bounded on $[-a_0, a_0] \times [0, \epsilon_0]$.

Proof The operators $D_-^{\epsilon,a,\delta}: \mathcal{H}_{-, \delta} \rightarrow \mathcal{H}'_{-, \delta}$ (including for $\epsilon = a = 0$, which is well defined because H_f is constant on $\{y = 0\}$) are conjugated to the operators

$$\tilde{D}_-^{\epsilon,a,\delta} = D_-^{\epsilon,a} + \delta \text{Id}: \mathcal{H}_- \rightarrow \mathcal{H}'_-.$$

The operator

$$\tilde{D}_{-}^{0,0,\delta} = \frac{\partial \eta}{\partial s} + j_0 \frac{\partial \eta}{\partial t} + \begin{pmatrix} -1+\delta & 0 \\ 0 & \delta \end{pmatrix}$$

is Fredholm because, for δ small, its asymptotic operators are invertible. Moreover, $\tilde{D}_{-}^{0,0,\delta}$ has no spectral flow, and therefore $\text{ind}(\tilde{D}_{-}^{0,0,\delta}) = 0$. Hence $D_{-}^{0,0,\delta}$ is also a Fredholm operator of index zero.

By elliptic regularity all elements of $\ker D_{-}^{0,0,\delta}$ are smooth solutions of (J_f) , and from the Fourier series expansion (A-3-1) we see that no such solution has the correct growth for $s \rightarrow \pm\infty$ to belong to $\mathcal{H}_{-,\delta}$. Then $D_{-}^{0,0,\delta}$ is injective and therefore, having index zero, is invertible. Since

$$\|D_{-}^{\epsilon,\alpha,\delta}\xi - D_{-}^{0,0,\delta}\xi\|_{h_\delta} \leq \epsilon C \|\xi\|_{*,h_\delta}$$

for a constant C which is independent of ϵ and α and invertibility is an open condition, for a fixed δ , all operators $D_{-}^{\epsilon,\alpha,\delta}$ are invertible when the conditions of the lemma are met. The uniform bound on the norms of $(D_{-}^{\epsilon,\alpha,\delta})^{-1}$ then follows by the continuity of taking the inverse. $\square$

A.6 Setting up the gluing

The gluing setup will follow [Bao and Honda 2023], which in turn is based on [Hutchings and Taubes 2009a].

Define smooth cutoff functions

$$(A-6-1) \quad \beta_+, \beta_- : \mathbb{R} \rightarrow [0, 1]$$

such that $\beta_+ + \beta_- = 1$ and $\beta_+(s) = 0$ for $s \leq -T_1$ and $\beta_+(s) = 1$ for $s \geq -T_0 - 1$. The cutoff functions $\beta_\pm$ will depend on the parameter b and will be denoted by $\beta_\pm^b$ when we want to make the dependence explicit. Let us write $-T_1(b)$ for $-T_1$ viewed as a function of b . Then:

(†₃) If $a > 0$ is fixed but we take $b \rightarrow 0$, then $-T_1(b) \rightarrow -\infty$ and we take $\beta_\pm^b$ such that $|(\beta^b)'_\pm|_{C^0} \rightarrow 0$ as $b \rightarrow 0$.

Let ψ_+ and $\psi_-^{\epsilon,\alpha}$ be sections in $\mathcal{H}_{+,\delta}$ and $\mathcal{H}_{-,\delta}$ of sufficiently small norm. The goal is to deform the pregluing $u_{*}^{\epsilon,\alpha}$ to

$$(A-6-2) \quad u^{\epsilon,\alpha} = \exp_{u_{*}^{\epsilon,\alpha}}(\beta_+\psi_+ + \beta_-\psi_-^{\epsilon,\alpha}),$$

and solve for ψ_+ and $\psi_-^{\epsilon,\alpha}$ in the equation $\Pi \partial_{N_{*,f_\epsilon}^{\epsilon,\alpha}}(\beta_+\psi_+ + \beta_-\psi_-^{\epsilon,\alpha}) = 0$ when ϵ is sufficiently small. (Recall the identifications of the normal bundles made at the beginning of Section A.5 that justify writing $\beta_+\psi_+ + \beta_-\psi_-^{\epsilon,\alpha}$.) The solutions will determine functions $p_\epsilon : [-\alpha_0, \alpha_0] \rightarrow \mathbb{R}$ such that

$$(A-6-3) \quad \partial_{N_{*,f_\epsilon}}(\beta_+\psi_+ + \beta_-\psi_-^{\epsilon,\alpha}) = p_\epsilon(\alpha)v.$$

Finally, we will solve the equation $p_\epsilon(\alpha) = 0$.

In the following lemmas we will repeatedly use Taylor expansions of the form

$$(I) \quad \phi(\mathbf{x}) = \phi(\mathbf{0}) + \sum_i \ell_i(\mathbf{x})x_i,$$

$$(II) \quad \phi(\mathbf{x}) = \phi(\mathbf{0}) + \sum_i \partial_i \phi(\mathbf{0})x_i + \sum_{j,k} q_{j,k}(\mathbf{x})x_jx_k$$

for a smooth function $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$.

Lemma A.6.1 Over the domain $(-\infty, -T_0] \times S^1$, we can expand

$$(A-6-4) \quad \bar{\partial}_{N_*^{\epsilon,a}, f_\epsilon}(\beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a}) = \mathcal{D}_\epsilon \eta_*^{\epsilon,a} + \beta_+(D_+ \psi_+ + \mathcal{L}_+(\psi_+, \psi_-^{\epsilon,a}) + \mathcal{Q}_+(\psi_+, \psi_-^{\epsilon,a})) \\ + \beta_-(D_-^{\epsilon,a} \psi_-^{\epsilon,a} + \mathcal{L}_-(\psi_+, \psi_-^{\epsilon,a}) + \mathcal{Q}_-(\psi_+, \psi_-^{\epsilon,a})),$$

where:

$$(1) \quad D_+ \psi_+ = \partial \psi_+ / \partial s + j_0 \partial \psi_+ / \partial t - \mathbf{H}f(\eta_+) \psi_+.$$

$$(2) \quad D_-^{\epsilon,a} \psi_-^{\epsilon,a} = \partial \psi_-^{\epsilon,a} / \partial s + j_0 \partial \psi_-^{\epsilon,a} / \partial t - \mathbf{H}f_\epsilon(\eta_-^{\epsilon,a}) \psi_-^{\epsilon,a}.$$

$$(3) \quad \mathcal{L}_\pm(\psi_+, \psi_-^{\epsilon,a}) \text{ are linear in } \psi_+ \text{ and } \psi_-^{\epsilon,a} \text{ with coefficients which are smooth coefficients of } \psi_+ \text{ and } \psi_-^{\epsilon,a}, \text{ are supported in } [-T_1, -T_0] \times S^1 \text{ and } (-\infty, -T_0 - 1] \times S^1, \text{ respectively, and satisfy}$$

$$(A-6-5) \quad |\mathcal{L}_\pm(\psi_+(x), \psi_-^{\epsilon,a}(x))| < (c_1(a)\epsilon + c_2(b)) \cdot (|\psi_+(x)| + |\psi_-^{\epsilon,a}(x)|),$$

at every point x of the domain, $c_1(a)$ is a constant which depends only on a , $c_2(b)$ depends only on b , and $\lim_{b \rightarrow 0} c_2(b) = 0$.

$$(4) \quad \mathcal{Q}_\pm \text{ are quadratic functions of } \psi_+ \text{ and } \psi_-^{\epsilon,a} \text{ with coefficients which are smooth functions of } \psi_+ \text{ and } \psi_-^{\epsilon,a}, \text{ and there exists } C > 0 \text{ such that}$$

$$(A-6-6) \quad |\mathcal{Q}_\pm(\psi_+(x), \psi_-^{\epsilon,a}(x))| < C(|\psi_+(x)|^2 + |\psi_-^{\epsilon,a}(x)|^2)$$

at every point x of the domain.

$$(5) \quad \mathcal{L}_+ = 0 \text{ and } \mathcal{Q}_+ = 0 \text{ for } s \leq -T_1 \text{ and } \mathcal{L}_-, \mathcal{Q}_- \text{ can be extended smoothly to } \mathcal{L}_- = 0 \text{ and } \mathcal{Q}_- = 0 \text{ for } s \geq -T_0.$$

Proof Over the domain $(-\infty, -T_0] \times S^1$, we have

$$\bar{\partial}_{N_*^{\epsilon,a}, f_\epsilon}(\beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a}) = \mathcal{D}_\epsilon(\eta_*^{\epsilon,a} + \beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a})$$

by Claim A.1.3. Writing $\eta_*^{\epsilon,a} = \eta_*^{\epsilon,a} + \beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a}$ we expand

$$\mathcal{D}_\epsilon \eta_*^{\epsilon,a} = \frac{\partial}{\partial s}(\eta_*^{\epsilon,a} + \beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a}) + j_0 \frac{\partial}{\partial t}(\eta_*^{\epsilon,a} + \beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a}) - \nabla f_\epsilon(\eta_*^{\epsilon,a} + \beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a}).$$

Using the Taylor expansion of type (II), we write

$$\nabla f_\epsilon(\eta_*^{\epsilon,a} + \beta_+ \psi_+ + \beta_- \psi_-^{\epsilon,a}) = \nabla f_\epsilon(\eta_*^{\epsilon,a}) + \beta_+ \mathbf{H}f_\epsilon(\eta_*^{\epsilon,a}) \psi_+ + \beta_- \mathbf{H}f_\epsilon(\eta_*^{\epsilon,a}) \psi_-^{\epsilon,a} - \mathcal{Q}(\beta_+ \psi_+, \beta_- \psi_-^{\epsilon,a}),$$

where $\mathcal{Q}$ is a quadratic function of $\beta_+\psi_+, \beta_-\psi_-^{\epsilon, \alpha}$ with coefficients which are smooth functions of $\beta_+\psi_+, \beta_-\psi_-^{\epsilon, \alpha}$. Then

$$\begin{aligned}
 \text{(A-6-7)} \quad \mathcal{D}_\epsilon \eta^{\epsilon, \alpha} &= \left(\frac{\partial \eta_*^{\epsilon, \alpha}}{\partial s} + j_0 \frac{\partial \eta_*^{\epsilon, \alpha}}{\partial t} - \nabla f_\epsilon(\eta_*^{\epsilon, \alpha}) \right) \\
 &\quad + \beta_+ \left(\frac{\partial \psi_+}{\partial s} + j_0 \frac{\partial \psi_+}{\partial t} - \mathbf{H}f(\eta_+) \psi_+ \right) + \beta_- \left(\frac{\partial \psi_-^{\epsilon, \alpha}}{\partial s} + j_0 \frac{\partial \psi_-^{\epsilon, \alpha}}{\partial t} - \mathbf{H}f_\epsilon(\eta_-^{\epsilon, \alpha}) \psi_-^{\epsilon, \alpha} \right) \\
 &\quad + \beta_+ (\mathbf{H}f(\eta_+) - \mathbf{H}f_\epsilon(\eta_*^{\epsilon, \alpha})) \psi_+ + \beta_- (\mathbf{H}f_\epsilon(\eta_-^{\epsilon, \alpha}) - \mathbf{H}f_\epsilon(\eta_*^{\epsilon, \alpha})) \psi_-^{\epsilon, \alpha} \\
 &\quad + \beta'_+(s) \psi_+ + \beta'_-(s) \psi_-^{\epsilon, \alpha} \\
 &\quad + \mathcal{Q}(\beta_+ \psi_+, \beta_- \psi_-^{\epsilon, \alpha}).
 \end{aligned}$$

The right-hand side of the first line is $\mathcal{D}_\epsilon \eta_*$ and the second line is $\beta_+ D_+ \psi_+ + \beta_- D_-^{\epsilon, \alpha} \psi_-^{\epsilon, \alpha}$. Let us define $g(y, \theta) = \phi(y) \bar{g}_N(\theta)$. We write

$$\text{(A-6-8)} \quad c_1 = |\mathbf{H}g|_{C^0} \quad \text{and} \quad c_2(b) = \max\{|\beta'_+|_{C^0}, |\beta'_-|_{C^0}\}.$$

Using the fact that $\eta_*^{\epsilon, \alpha}$, η_+ and $\eta_-^{\epsilon, \alpha}$ take values in A_a , where $\mathbf{H}f$ is constant, for $s \leq -T_0$, the terms of the third and fourth lines can be bounded as follows:

$$\begin{aligned}
 &|\beta'_+(s) \psi_+(x) + \beta'_-(s) \psi_-^{\epsilon, \alpha}(x)| \leq c_2(b) (|\psi_+(x)| + |\psi_-^{\epsilon, \alpha}(x)|) \\
 \text{(A-6-9)} \quad &|\beta_+ (\mathbf{H}f(\eta_+) - \mathbf{H}f_\epsilon(\eta_*^{\epsilon, \alpha})) \psi_+(x)| \leq \epsilon c_1 \beta_+ |\psi_+(x)|, \\
 &|\beta_- (\mathbf{H}f_\epsilon(\eta_-^{\epsilon, \alpha}) - \mathbf{H}f_\epsilon(\eta_*^{\epsilon, \alpha})) \psi_-^{\epsilon, \alpha}(x)| \leq 2\epsilon c_1 \beta_- |\psi_-^{\epsilon, \alpha}(x)|.
 \end{aligned}$$

We then set

$$\text{(A-6-10)} \quad \mathcal{L}_+^{(1)}(\psi_+, \psi_-^{\epsilon, \alpha}) = (\mathbf{H}f(\eta_+) - \mathbf{H}f_\epsilon(\eta_*^{\epsilon, \alpha})) \psi_+ + (\beta'_+(s) \psi_+ + \beta'_-(s) \psi_-^{\epsilon, \alpha}),$$

$$\text{(A-6-11)} \quad \mathcal{L}_-^{(1)}(\psi_+, \psi_-^{\epsilon, \alpha}) = (\mathbf{H}f_\epsilon(\eta_-^{\epsilon, \alpha}) - \mathbf{H}f_\epsilon(\eta_*^{\epsilon, \alpha})) \psi_-^{\epsilon, \alpha} + (\beta'_+(s) \psi_+ + \beta'_-(s) \psi_-^{\epsilon, \alpha}),$$

and $\beta_\pm \mathcal{L}_\pm^{(1)}(\psi_+, \psi_-^{\epsilon, \alpha})$ satisfies inequality (A-6-5). The terms $\mathcal{L}_+^{(1)}$ and $\mathcal{L}_-^{(1)}$ are not necessarily supported in $[-T_1, -T_0] \times S^1$ and $(-\infty, -T_0 - 1] \times S^1$, respectively, and therefore we rearrange

$$\beta_+ \mathcal{L}_+^{(1)} + \beta_- \mathcal{L}_-^{(1)} = \beta_+ (\beta_+ + \beta_-)^2 \mathcal{L}_+^{(1)} + \beta_- (\beta_+ + \beta_-)^2 \mathcal{L}_-^{(1)} = \beta_+ \mathcal{L}_+(\psi_+, \psi_-^{\epsilon, \alpha}) + \beta_- \mathcal{L}_-(\psi_+, \psi_-^{\epsilon, \alpha}),$$

where

$$\begin{aligned}
 \text{(A-6-12)} \quad \mathcal{L}_+(\psi_+, \psi_-^{\epsilon, \alpha}) &= \beta_+^2 \mathcal{L}_+^{(1)} + 2\beta_+ \beta_- \mathcal{L}_+^{(1)} + \beta_+ \beta_- \mathcal{L}_-^{(1)}, \\
 \mathcal{L}_-(\psi_+, \psi_-^{\epsilon, \alpha}) &= \beta_+ \beta_- \mathcal{L}_+^{(1)} + 2\beta_+ \beta_- \mathcal{L}_-^{(1)} + \beta_-^2 \mathcal{L}_-^{(1)},
 \end{aligned}$$

the same inequalities hold for $\mathcal{L}_\pm^{(1)}$ and $\mathcal{L}_\pm$ and (5) holds for $\mathcal{L}_\pm$. However, the constants c_1 and c_2 in the statement are closely related to the constants c_1 and c_2 in (A-6-9) but not exactly the same. Finally we can decompose and rearrange so that

$$\mathcal{Q}(\beta_+ \psi_+, \beta_- \psi_-^{\epsilon, \alpha}) = \beta_+ \mathcal{Q}_+(\psi_+, \psi_-^{\epsilon, \alpha}) + \beta_- \mathcal{Q}_-(\psi_+, \psi_-^{\epsilon, \alpha}),$$

inequality (A-6-6) holds, and (5) holds for $\mathcal{Q}_\pm$. □

In particular, equation (A-6-3) is satisfied on $(-\infty, -T_0] \times S^1$ if the following pair of equations hold:

$$(A-6-13) \quad \mathcal{D}_\epsilon \eta_{*}^{\epsilon, a} + D_+ \psi_+ + \mathcal{L}_+(\psi_+, \psi_-^{\epsilon, a}) + \mathcal{Q}_+(\psi_+, \psi_-^{\epsilon, a}) = p_\epsilon(a)v,$$

$$(A-6-14) \quad \mathcal{D}_\epsilon \eta_{*}^{\epsilon, a} + D_-^{\epsilon, a} \psi_-^{\epsilon, a} + \mathcal{L}_-(\psi_+, \psi_-^{\epsilon, a}) + \mathcal{Q}_-(\psi_+, \psi_-^{\epsilon, a}) = av.$$

The term av in the second equation is legitimate because v is supported on $[-T_0 - 1, -T_0] \times S^1$ where $\beta_- = 0$. It was chosen to make $\mathcal{D}_\epsilon \eta_{*}^{\epsilon, a} - av$ small (independent of a) in the sense of estimate (A-7-2).

Remark A.6.2 More in line with the obstruction gluing of [Hutchings and Taubes 2007], equation (A-6-3) can be split into

$$\bar{D}_+^\delta \psi_+ + \Pi(\mathcal{D}_\epsilon \eta_{*}^{\epsilon, a} + \mathcal{L}_+(\psi_+, \psi_-^{\epsilon, a}) + \mathcal{Q}_+(\psi_+, \psi_-^{\epsilon, a})) = 0,$$

$$(1 - \Pi)(\mathcal{D}_\epsilon \eta_{*}^{\epsilon, a} + \mathcal{L}_+(\psi_+, \psi_-^{\epsilon, a}) + \mathcal{Q}_+(\psi_+, \psi_-^{\epsilon, a})) = p_\epsilon(a)v,$$

$$D_-^{\epsilon, a, \delta} \psi_-^{\epsilon, a} + (\mathcal{D}_\epsilon \eta_{*}^{\epsilon, a} - av) + \mathcal{L}_-(\psi_+, \psi_-^{\epsilon, a}) + \mathcal{Q}_-(\psi_+, \psi_-^{\epsilon, a}) = 0,$$

where $\Pi: \mathcal{H}_{0, g_\delta}(\dot{F}, \Lambda^{0,1} N_+) \rightarrow \mathcal{H}'_{+, \delta}$ is the projection from Section A.5.4 and the second equation is always satisfied.

We say that $\mathcal{Q}(\psi_+)$ is *type 1 quadratic* if it can be written as

$$\mathcal{Q}(\psi_+) = P(\psi_+) + Q(\psi_+) \cdot \nabla \psi_+,$$

where there exists a constant $C > 0$ such that $|P(\psi_+(x))| < C|\psi_+(x)|^2$ and $|Q(\psi_+(x))| \leq C|\psi_+(x)|$ at every point x of the domain.

Remark A.6.3 The reason for the different treatment of the term $\tilde{\partial}_\theta$ compared to the other infinitesimal deformations of the map u_+ is that the term $\beta'(s)\tilde{\partial}_\theta$ which would appear in (A-6-9) cannot be made small in $\mathcal{H}_{0, g_\delta}(\dot{F}, \Lambda^{0,1} N_+)$ by choosing b and ϵ small.

Lemma A.6.4 Over the domain $\dot{F} - (-\infty, -T_0) \times S^1$, we can expand

$$(A-6-15) \quad \bar{\partial}_{N_{*}^{\epsilon, a}, f_\epsilon}(\beta_- \psi_-^{\epsilon, a} + \beta_+ \psi_+) = D_+ \psi_+ + \mathcal{Q}(\psi_+),$$

where $\mathcal{Q}(\psi_+)$ is type 1 quadratic, and (A-6-4) and (A-6-15) agree along $s = -T_0$.

Proof Over the domain $\dot{F} - (-\infty, -T_0) \times S^1$, $\beta_+ = 1$, $\beta_- = 0$, $u_{*}^{\epsilon, a} = u_+$ and $u^{\epsilon, a} = \exp_{u_{*}^{\epsilon, a}} \psi_+ = \exp_{u_+} \psi_+$. Hence ψ_+ satisfies (A-5-4) with ψ_+ instead of ξ and $(\sigma, \tau) = (s, t)$. Equation (A-6-15) then follows from (A-5-4) together with (A-5-5) by applying the Taylor expansion of type (I) to a , b and X . The agreement of (A-6-4) and (A-6-15) along $s = -T_0$ is a consequence of the definition of $\exp_{u_+}$ for $s \leq -T_0$. $\square$

A.7 Proof of Theorem A.2.5

In this subsection and the next, we use the convention that constants such as C , c_1 , $c_2(b)$ may change from line to line when making estimates. Recall that

$$(A-7-1) \quad \lambda := \min(\lambda_1, |\lambda_{-1}|) > 5\delta.$$

Lemma A.7.1 *There exists a constant $C > 0$ such that*

$$(A-7-2) \quad \|\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}} - \mathfrak{a} \nu\|_{g_\delta} \leq C(e^{(\delta-\lambda)T_2(\epsilon)} + \epsilon T_1 e^{\delta T_1}).$$

Proof By Lemma A.4.4, it suffices to estimate $\|\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}}\|_{g_\delta}$ on $[-T_2, -T_2 + 1] \times S^1$ and $[-T_1, -T_0] \times S^1$. We will use the simple fact that the Morrey norm of a continuous function on a compact domain is dominated by the C^0 norm.

First we estimate $\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}} - \mathfrak{a} \nu$ on $[-T_2, -T_2 + 1] \times S^1$, where $\nu = 0$. By the definition of $\eta_*^{\epsilon, \mathfrak{a}}$ (equation (A-4-4)) and (J_{f_ϵ}) , and with the understanding that all maps and norms are restricted to $[-T_2, -T_2 + 1] \times S^1$ (on which $f_\epsilon(y, \theta) = \frac{1}{2}y^2 + \epsilon \bar{g}_N(\theta)$, $\nabla f_\epsilon = y \partial_y + \epsilon \partial_\theta$ and $\mathcal{D}_0$ is linear), we have

$$\mathcal{D}_\epsilon(\eta_*^{\epsilon, \mathfrak{a}}) = \mathcal{D}_\epsilon(\eta_-^{\epsilon, \mathfrak{a}}) + \beta(s + T_2)\mathcal{D}_0(\eta_+) + \beta'(s + T_2)\eta_+ = \beta'(s + T_2)\eta_+$$

and therefore

$$(A-7-3) \quad \|\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}}\|_{g_\delta} \leq C \left\| \sum_{i=1}^{\infty} c_i e^{\lambda_i s} g_i(t) \right\|_{g_\delta} \leq C e^{(\delta-\lambda)(T_2(\epsilon)-T_0)} \leq C e^{(\delta-\lambda)T_2(\epsilon)}.$$

Next we estimate $\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}} - \mathfrak{a} \nu$ on $[-T_1, -T_0] \times S^1$. The restriction of $\eta_*^{\epsilon, \mathfrak{a}}$ to $[-T_1, -T_0] \times S^1$ takes values in the region where ∇f_ϵ no longer has the simple expression which leads to (J_{f_ϵ}) , but from (A-1-2) we obtain

$$(A-7-4) \quad \mathcal{D}_\epsilon(\eta_*^{\epsilon, \mathfrak{a}}) = \mathcal{D}_0(\eta_*^{\epsilon, \mathfrak{a}}) - \epsilon \nabla g(\eta_*^{\epsilon, \mathfrak{a}}),$$

where $g(y, \theta) = \phi(y) \bar{g}_N(\theta)$. By the definition of $\eta_*^{\epsilon, \mathfrak{a}}$ and ν , equation (A-7-4), and with the understanding that all maps and norms are restricted to $[-T_1, -T_0] \times S^1$, we have

$$(A-7-5) \quad \|\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}} + \beta'(-s - T_0) \cdot (0, \mathfrak{a})\|_{g_\delta} \leq \left\| \frac{\partial}{\partial s} \left(\beta \left(\frac{s + T_0}{-T_1 + T_0} \right) (0, \epsilon(s + T_1)) \right) \right\|_{g_\delta} + \epsilon \|\nabla g(\eta_*^{\epsilon, \mathfrak{a}})\|_{g_\delta} \\ \leq C \epsilon T_1 e^{\delta T_1}.$$

Estimates (A-7-3) and (A-7-5) imply estimate (A-7-2). $\square$

Remark A.7.2 Since $\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}} + \beta'(-s - T_0) \cdot (0, \mathfrak{a})$ is supported in $(-\infty, -T_0] \times S^1$, where g_δ and h_δ coincide, we can also regard it as an element of $\mathcal{H}'_{-, \delta}$ with norm

$$\|\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}} + \beta'(-s - T_0) \cdot (0, \mathfrak{a})\|_{h_\delta} = \|\mathcal{D}_\epsilon \eta_*^{\epsilon, \mathfrak{a}} + \beta'(-s - T_0) \cdot (0, \mathfrak{a})\|_{g_\delta}.$$

Let $(\bar{D}_+^\delta)^{-1}$ be the inverse of $\bar{D}_+^\delta$, viewed as a map to the orthogonal complement $\mathcal{H}_{+, \delta}^\perp$ of $\ker \bar{D}_+^\delta$, and let $(D_-^{\epsilon, \mathfrak{a}, \delta})^{-1}$ be the inverse of $D_-^{\epsilon, \mathfrak{a}, \delta}$. Recall that the norm of $(D_-^{\epsilon, \mathfrak{a}, \delta})^{-1}$ is uniformly bounded in ϵ and $\mathfrak{a}$ by Lemma A.5.4. Let

$$(A-7-6) \quad \mathcal{B}_+ = \text{closed ball of radius } \tilde{\epsilon} \text{ in } \mathcal{H}_{+, \delta}^\perp,$$

$$(A-7-7) \quad \mathcal{B}_- = \text{closed ball of radius } \tilde{\epsilon} \text{ in } \mathcal{H}_{-, \delta},$$

where the small constant $\tilde{\epsilon} > 0$ is to be determined more precisely later.

Let $\mathcal{I}_+ : \mathcal{B}_+ \times \mathcal{B}_- \rightarrow \mathcal{H}_{+, \delta}^\perp$ and $\mathcal{I}_- : \mathcal{B}_+ \times \mathcal{B}_- \rightarrow \mathcal{H}_{-, \delta}$ be maps given by

$$(A-7-8) \quad \mathcal{I}_+(\psi_+, \psi_-^{\epsilon, a}) = -(\bar{D}_+^\delta)^{-1} \Pi(\mathcal{F}_+(\psi_+, \psi_-^{\epsilon, a})),$$

$$(A-7-9) \quad \mathcal{I}_-(\psi_+, \psi_-^{\epsilon, a}) = -(D_-^{\epsilon, a, \delta})^{-1} (\mathcal{F}_-(\psi_+, \psi_-^{\epsilon, a})),$$

where

$$\mathcal{F}_+(\psi_+, \psi_-^{\epsilon, a}) := \begin{cases} \mathcal{D}_\epsilon \eta_*^{\epsilon, a} + \mathcal{L}_+(\psi_+, \psi_-^{\epsilon, a}) + \mathcal{Q}_+(\psi_+, \psi_-^{\epsilon, a}) & \text{on } (-\infty, -T_0] \times S^1, \\ \mathcal{Q}(\psi_+) & \text{on } \dot{F} - (-\infty, -T_0] \times S^1, \end{cases}$$

$$\mathcal{F}_-(\psi_+, \psi_-^{\epsilon, a}) := \begin{cases} \mathcal{D}_\epsilon \eta_*^{\epsilon, a} - a\nu + \mathcal{L}_-(\psi_+, \psi_-^{\epsilon, a}) + \mathcal{Q}_-(\psi_+, \psi_-^{\epsilon, a}) & \text{on } (-\infty, -T_0] \times S^1, \\ 0 & \text{on } [-T_0, \infty) \times S^1. \end{cases}$$

Here the definitions of $\mathcal{F}_+(\psi_+, \psi_-^{\epsilon, a})$ agree on $\{-T_0\} \times S^1$ by [Lemma A.6.4](#) and the definitions of $\mathcal{F}_-(\psi_+, \psi_-^{\epsilon, a})$ agree on $\{-T_0\} \times S^1$ by [Lemmas A.6.1\(5\) and A.4.4](#).

Solving [\(A-6-13\)](#) and [\(A-6-14\)](#) is then equivalent to solving the equations

$$(A-7-10) \quad \psi_+ = \mathcal{I}_+(\psi_+, \psi_-^{\epsilon, a}),$$

$$(A-7-11) \quad \psi_-^{\epsilon, a} = \mathcal{I}_-(\psi_+, \psi_-^{\epsilon, a}).$$

The following two lemmas provide the necessary estimates to apply the contraction mapping theorem to [\(A-7-10\)](#) and [\(A-7-11\)](#).

Notation A.7.3 We will sometimes write

$$(A-7-12) \quad \|\cdot\|_{*, \delta} := \|\cdot\|_{*, g_\delta} \quad \text{or} \quad \|\cdot\|_{*, h_\delta},$$

depending on the context.

Lemma A.7.4 If $(\psi_+, \psi_-^{\epsilon, a}) \in \mathcal{B}_+ \times \mathcal{B}_-$, then

$$(A-7-13) \quad \|\mathcal{I}_\pm(\psi_+, \psi_-^{\epsilon, a})\|_{*, \delta} \leq C(e^{(\delta-\lambda)T_2(\epsilon)} + \epsilon T_1(b)e^{\delta T_1(b)} + (c_1\epsilon + c_2(b))(\|\psi_+\|_{*, g_\delta} + \|\psi_-^{\epsilon, a}\|_{*, h_\delta}) + C(\|\psi_+\|_{*, g_\delta}^2 + \|\psi_-^{\epsilon, a}\|_{*, h_\delta}^2)),$$

where c_1 is constant and $\lim_{b \rightarrow 0} c_2(b) = 0$.

Proof We will carry out estimates on the $(-\infty, -T_0] \times S^1$ portion, with the understanding that the norms are restricted to $(-\infty, -T_0] \times S^1$, where $g_\delta = h_\delta$. (This justifies the use of the weight g_δ throughout the proof, even where one should expect h_δ .) The estimates on the $\dot{F} - (-\infty, -T_0] \times S^1$ portion, which involve only $\mathcal{I}_+$ and ψ_+ , are straightforward and are left to the reader.

By the definitions of $\mathcal{I}_\pm(\psi_+, \psi_-^{\epsilon, a})$,

$$(A-7-14) \quad \|\mathcal{I}_\pm(\psi_+, \psi_-^{\epsilon, a})\|_{*, g_\delta} \leq C(\|\mathcal{D}_\epsilon \eta_* - a\nu\|_{g_\delta} + \|\mathcal{L}_\pm(\psi_+, \psi_-^{\epsilon, a})\|_{g_\delta} + \|\mathcal{Q}_\pm(\psi_+, \psi_-^{\epsilon, a})\|_{g_\delta}),$$

since $(\bar{D}_+^\delta)^{-1}$ is bounded, $(D_-^{\epsilon, a, \delta})^{-1}$ are uniformly bounded, and $\Pi(\mathcal{D}_\epsilon \eta_*) = \Pi(\mathcal{D}_\epsilon \eta_* - a\nu)$.

We will make frequent use of the estimate

$$(A-7-15) \quad |\zeta|_{C^0} \leq C \|\zeta\|_{*,g_\delta},$$

which follows from Lemma A.5.1 and $\|\zeta\|_* \leq \|\zeta\|_{*,g_\delta}$ since $g_\delta > 1$.

By (A-7-2),

$$(A-7-16) \quad \|\mathcal{D}_\epsilon \eta_*^{\epsilon,a} - \mathfrak{av}\|_{g_\delta} \leq C(e^{(\delta-\lambda)T_2(\epsilon)} + \epsilon T_1 e^{\delta T_1}).$$

Next, since $\mathcal{L}_\pm$ satisfies estimate (A-6-5),

$$(A-7-17) \quad \|\mathcal{L}_\pm(\psi_+, \psi_-^{\epsilon,a})\|_{g_\delta} \leq (c_1\epsilon + c_2(b))(\|\psi_+\|_{g_\delta} + \|\psi_-^{\epsilon,a}\|_{g_\delta}).$$

Finally, since $\mathcal{Q}_\pm$ satisfies estimate (A-6-6),

$$(A-7-18) \quad \begin{aligned} \|\mathcal{Q}_\pm(\psi_+, \psi_-^{\epsilon,a})\|_{g_\delta} &\leq C(|\psi_+|_{C^0}\|\psi_+\|_{g_\delta} + |\psi_-^{\epsilon,a}|_{C^0}\|\psi_-^{\epsilon,a}\|_{g_\delta}) \\ &\leq C(\|\psi_+\|_{*,g_\delta}\|\psi_+\|_{g_\delta} + \|\psi_-^{\epsilon,a}\|_{*,g_\delta}\|\psi_-^{\epsilon,a}\|_{g_\delta}), \end{aligned}$$

using estimate (A-7-15). We explain the first line of estimate (A-7-18). The first term of the Morrey norm is the weighted L^2 norm, and we can bound

$$\begin{aligned} \int_{\dot{F}} g_\delta^2 |\mathcal{Q}_+(\psi_+, \psi_-^{\epsilon,a})|^2 &\leq \int_{\dot{F}} g_\delta^2 C^2 (|\psi_+|^4 + |\psi_-^{\epsilon,a}|^4) \\ &\leq C^2 |\psi_+|_{C_0}^2 \int_{\dot{F}} g_\delta^2 |\psi_+|^2 + C^2 |\psi_-^{\epsilon,a}|_{C_0}^2 \int_{\dot{F}} g_\delta^2 |\psi_-^{\epsilon,a}|^2, \\ \left(\int_{\dot{F}} g_\delta^2 |\mathcal{Q}_+(\psi_+, \psi_-^{\epsilon,a})|^2 \right)^{1/2} &\leq C |\psi_+|_{C_0} \left(\int_{\dot{F}} g_\delta^2 |\psi_+|^2 \right)^{1/2} + C |\psi_-^{\epsilon,a}|_{C_0} \left(\int_{\dot{F}} g_\delta^2 |\psi_-^{\epsilon,a}|^2 \right)^{1/2}. \end{aligned}$$

The bound for the second term of the Morrey norm is similar, since it is of L^2 type.

Estimates (A-7-14) and (A-7-16)–(A-7-18), together with estimate (A-7-14), give estimate (A-7-13). Here we are using the trivial observation $\|\cdot\|_{g_\delta} \leq \|\cdot\|_{*,g_\delta}$. $\square$

Lemma A.7.5 *If $(\psi_+, \psi_-^{\epsilon,a}), (\bar{\psi}_+, \bar{\psi}_-^{\epsilon,a}) \in \mathcal{B}_+ \times \mathcal{B}_-$, then*

$$(A-7-19) \quad \|\mathcal{I}_\pm(\psi_+, \psi_-^{\epsilon,a}) - \mathcal{I}_\pm(\bar{\psi}_+, \bar{\psi}_-^{\epsilon,a})\|_{*,\delta} \leq (c_1\epsilon + c_2(b) + C\tilde{\epsilon})(\|\psi_+ - \bar{\psi}_+\|_{*,g_\delta} + \|\psi_-^{\epsilon,a} - \bar{\psi}_-^{\epsilon,a}\|_{*,h_\delta}).$$

Proof Again we carry out the estimate on the $(-\infty, -T_0] \times S^1$ portion, with the understanding that the norms are restricted to $(-\infty, -T_0] \times S^1$, and leave the estimate on the $\dot{F} - (-\infty, -T_0] \times S^1$ portion to the reader.

In the following equation $\mathfrak{D}_\pm$ stands for either $\bar{D}_+^\delta$ or $D_-^{\epsilon,a,\delta}$. We have

$$\begin{aligned} \mathcal{I}_\pm(\psi_+, \psi_-^{\epsilon,a}) - \mathcal{I}_\pm(\bar{\psi}_+, \bar{\psi}_-^{\epsilon,a}) \\ = -\mathfrak{D}_\pm^{-1}((\mathcal{L}_\pm(\psi_+, \psi_-^{\epsilon,a}) - \mathcal{L}_\pm(\bar{\psi}_+, \bar{\psi}_-^{\epsilon,a})) + (\mathcal{Q}_\pm(\psi_+, \psi_-^{\epsilon,a}) - \mathcal{Q}_\pm(\bar{\psi}_+, \bar{\psi}_-^{\epsilon,a}))). \end{aligned}$$

By (A-6-10), (A-6-11), (A-6-12), as well as an analog of estimate (A-6-5), we have

$$\|\mathcal{L}_\pm(\psi_+, \psi_-^{\epsilon,a}) - \mathcal{L}_\pm(\bar{\psi}_+, \bar{\psi}_-^{\epsilon,a})\|_{g_\delta} \leq (c_1\epsilon + c_2(b))(\|\psi_+ - \bar{\psi}_+\|_{g_\delta} + \|\psi_-^{\epsilon,a} - \bar{\psi}_-^{\epsilon,a}\|_{g_\delta}).$$

By Lemma A.6.1(4), $\mathcal{Q}_\pm(\psi_+, \psi_-^{\epsilon, \alpha})$ is a quadratic function of $\psi_+, \psi_-^{\epsilon, \alpha}$ with uniformly bounded coefficients which are smooth functions of $\psi_+, \psi_-^{\epsilon, \alpha}$. Therefore

$$\begin{aligned} \|\mathcal{Q}_\pm(\psi_+, \psi_-^{\epsilon, \alpha}) - \mathcal{Q}_\pm(\bar{\psi}_+, \bar{\psi}_-^{\epsilon, \alpha})\|_{g_\delta} \\ \leq C(|\psi_+ - \bar{\psi}_+|_{C^0} + |\psi_-^{\epsilon, \alpha} - \bar{\psi}_-^{\epsilon, \alpha}|_{C^0})(\|\psi_+\|_{g_\delta} + \|\psi_-^{\epsilon, \alpha}\|_{g_\delta} + \|\bar{\psi}_+\|_{g_\delta} + \|\bar{\psi}_-^{\epsilon, \alpha}\|_{g_\delta}). \end{aligned}$$

Combining the two estimates and using (A-7-15) and $\|\cdot\|_{g_\delta} \leq \|\cdot\|_{*, g_\delta}$, we obtain estimate (A-7-19). $\square$

Proposition A.7.6 *There exists $\tilde{\epsilon} > 0$ sufficiently small that, for all b, α_0 and $\epsilon_0 = \epsilon_0(\alpha_0, b)$ sufficiently small (in particular satisfying $(\dagger_2)$) and for all $(\alpha, \epsilon) \in [-\alpha_0, \alpha_0] \times (0, \epsilon_0]$, there exists a unique $(\psi_+, \psi_-^{\epsilon, \alpha}) \in \mathcal{B}_+ \times \mathcal{B}_-$ satisfying*

$$(A-7-20) \quad \mathcal{I}_+(\psi_+, \psi_-^{\epsilon, \alpha}) = \psi_+, \quad \mathcal{I}_-(\psi_+, \psi_-^{\epsilon, \alpha}) = \psi_-^{\epsilon, \alpha}.$$

Moreover, the solutions of (A-7-20) satisfy the estimate

$$(A-7-21) \quad \|\psi_+\|_{*, g_\delta} + \|\psi_-^{\epsilon, \alpha}\|_{*, h_\delta} \leq C(e^{(\delta-\lambda)T_2(\epsilon)} + \epsilon T_1(b))e^{\delta T_1(b)}.$$

Proof Let $\mathcal{I} = (\mathcal{I}_+, \mathcal{I}_-): \mathcal{B}_+ \times \mathcal{B}_- \rightarrow \mathcal{H}_{+, \delta}^\perp \times \mathcal{H}_{-, \delta}$. Lemmas A.7.4 and A.7.5 imply that, for sufficiently small $\tilde{\epsilon}$, there are sufficiently small constants b, α_0 and $\epsilon_0 = \epsilon_0(\alpha_0)$ such that, for all $(\alpha, \epsilon) \in [-\alpha_0, \alpha_0] \times (0, \epsilon_0]$, we have estimates

$$(A-7-22) \quad \|\mathcal{I}_\pm(\psi_+, \psi_-^{\epsilon, \alpha})\|_{*, \delta} \leq C(e^{(\delta-\lambda)T_2(\epsilon)} + \epsilon T_1(b))e^{\delta T_1(b)} + \frac{1}{4}(\|\psi_+\|_{*, g_\delta} + \|\psi_-^{\epsilon, \alpha}\|_{*, h_\delta}),$$

$$(A-7-23) \quad \|\mathcal{I}_\pm(\psi_+, \psi_-^{\epsilon, \alpha}) - \mathcal{I}_\pm(\bar{\psi}_+, \bar{\psi}_-^{\epsilon, \alpha})\|_{*, \delta} \leq \frac{1}{4}(\|\psi_+ - \bar{\psi}_+\|_{*, g_\delta} + \|\psi_-^{\epsilon, \alpha} - \bar{\psi}_-^{\epsilon, \alpha}\|_{*, h_\delta})$$

and $\mathcal{I}$ is a contraction of $\mathcal{B}_+ \times \mathcal{B}_-$ (for the metric induced by the sum of the norms). Then the contraction mapping theorem implies that there is a unique pair $(\psi_+, \psi_-^{\epsilon, \alpha}) \in \mathcal{B}_+ \times \mathcal{B}_-$ such that $\mathcal{I}(\psi_+, \psi_-^{\epsilon, \alpha}) = (\psi_+, \psi_-^{\epsilon, \alpha})$. Finally, estimate (A-7-21) is obtained by plugging (A-7-20) in (A-7-22), rearranging the terms, and renaming the constant C . $\square$

Proposition A.7.6 produces, provided b, α_0 and $\epsilon_0 = \epsilon_0(\alpha_0)$ are sufficiently small, a map

$$u^{\epsilon, \alpha} = \exp_{u_*^{\epsilon, \alpha}}(\beta_+ \psi_+ + \beta_- \psi_-^{\epsilon, \alpha})$$

for all $(\alpha, \epsilon) \in [-\alpha_0, \alpha_0] \times (0, \epsilon_0]$ and a continuous function $p_\epsilon: [-\alpha_0, \alpha_0] \rightarrow \mathbb{R}$ such that $\bar{\partial}_{J_{f_\epsilon}} u^{\epsilon, \alpha} = p_\epsilon(\alpha) v$ for all $\epsilon \in (0, \epsilon_0]$. Moreover, by (A-7-2) and (A-7-21),

$$\begin{aligned} |p_\epsilon(\alpha) - \alpha| &\leq C(\|p_\epsilon(\alpha) v - \alpha v\|_{g_\delta}) \leq C(\|\bar{\partial}_{J_{f_\epsilon}} u^{\epsilon, \alpha} - \bar{\partial}_{J_{f_\epsilon}} u_*^{\epsilon, \alpha}\|_{g_\delta} + \|\bar{\partial}_{J_{f_\epsilon}} u_*^{\epsilon, \alpha} - \alpha v\|_{g_\delta}) \\ &\leq C(\|\psi_+\|_{*, g_\delta} + \|\psi_-^{\epsilon, \alpha}\|_{*, h_\delta}) + C(\|\mathcal{D}_\epsilon \eta_*^{\epsilon, \alpha} - \alpha v\|_{g_\delta}) \\ &\leq C(e^{(\delta-\lambda)T_2(\epsilon)} + \epsilon T_1(b))e^{\delta T_1(b)}. \end{aligned}$$

In order to bound $\|\bar{\partial}_{J_{f_\epsilon}} u^{\epsilon, \alpha} - \bar{\partial}_{J_{f_\epsilon}} u_*^{\epsilon, \alpha}\|_{g_\delta}$ we used Lemmas A.6.1 and A.6.4 and (A-7-21).

If ϵ_0 is sufficiently small, then $p_\epsilon(-\alpha_0) < 0$ and $p_\epsilon(\alpha_0) > 0$, and therefore p_ϵ has an odd number of zeros in the interval $[-\alpha_0, \alpha_0]$.

A.8 Proof of Theorem A.2.6

From now on, until the end of the appendix, we fix a b such that Proposition A.7.6 holds. Therefore, from now on, b and $T_1(b)$ are to be considered constants.

Remark A.8.1 In [Yao 2022b], the strategy for the proof of Theorem A.2.6 is slightly different: one can actually differentiate the 1-parameter family of functions $\psi_-^{\epsilon, \mathfrak{a}}$ with respect to $\mathfrak{a}$ to show that $p_\epsilon(\mathfrak{a})$ is C^1 -close to $\mathfrak{a}$ and hence that the zero is unique.

Arguing by contradiction, suppose there are sequences $\{\kappa_i\}_{i=1}^\infty$, $\{\epsilon_i\}_{i=1}^\infty$, $\{u^{\epsilon_i}\}_{i=1}^\infty$ and $\{v^{\epsilon_i}\}_{i=1}^\infty$ (with a , b , c and ϵ' small but fixed) such that

- (1) $\kappa_i \rightarrow 0$ and $\epsilon_i \rightarrow 0$,
- (2) $u^{\epsilon_i}, v^{\epsilon_i} : (\dot{F}, j_i) \rightarrow \mathbb{R} \times M$ are $\bar{\partial}_{J_{f_{\epsilon_i}}}$ -holomorphic and are not related by $\mathbb{R}$ -translations in the target (and possibly the domain), and
- (3) u^{ϵ_i} and v^{ϵ_i} are κ_i -close to breaking into u_+ and a cylinder over $\mathcal{T}_0^{\epsilon_i}$.

After translating the u^{ϵ_i} and v^{ϵ_i} in the target and possibly in the domain, we can find $T_1 > 0$ such that $u^{\epsilon_i}|_{(-\infty, -T_1]}$ and $v^{\epsilon_i}|_{(-\infty, -T_1]}$ have image in $\mathbb{R} \times [-b - \epsilon', b + \epsilon'] \times T^2$ and $u^{\epsilon_i}|_{s=-T_1}$ and $v^{\epsilon_i}|_{s=-T_1}$ have image in $\mathbb{R} \times [b, b + \epsilon'] \times T^2$. On $(-\infty, -T_1] \times S^1$ we write

$$u^{\epsilon_i}(s, t) = (s, t, \eta^{u^{\epsilon_i}}(s, t)) \quad \text{and} \quad v^{\epsilon_i}(s, t) = (s, t, \eta^{v^{\epsilon_i}}(s, t)).$$

Recall that $\eta^{u^{\epsilon_i}}$ and $\eta^{v^{\epsilon_i}}$ satisfy (A-1-3), which we repeat here:

$$\frac{\partial \eta}{\partial s} + j_0 \frac{\partial \eta}{\partial t} - \nabla f_\epsilon(\eta) = 0.$$

If we restrict $\eta^{u^{\epsilon_i}}$ and $\eta^{v^{\epsilon_i}}$ to any cylinder $[-T_1'(\epsilon_i), -T_1] \times S^1$ such that their images are in B , then (A-1-3) specializes to $(J_{f_{\epsilon_i}})$ and their difference $\zeta^{\epsilon_i}(s, t) = \eta^{v^{\epsilon_i}}(s, t) - \eta^{u^{\epsilon_i}}(s, t)$ satisfies the linear equation (J_f) :

$$(A-8-1) \quad \frac{\partial \zeta^{\epsilon_i}}{\partial s} - A \zeta^{\epsilon_i} = 0.$$

Next, by the definition of κ_i -close to breaking and estimates on derivatives in the proof of Gromov–Hofer compactness, after applying the relevant translations in the domain or in the target and passing to a subsequence, we can choose a sequence $T_2'(\epsilon_i) \rightarrow \infty$ such that $-T_2'(\epsilon_i) < -T_1$ and there are rough initial estimates

$$(A-8-2) \quad \|\zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L^2} \leq c\kappa_i, \quad \|\zeta_-^{\epsilon_i}(-T_2'(\epsilon_i))\|_{L^2} \leq c\kappa_i,$$

where $c > 0$ is independent of ϵ_i or κ_i .

Normalization We normalize u^{ϵ_i} so that, at $s = -T_1$, the g_1 term of the Fourier series of $\eta^{u^{\epsilon_i}}$ is equal to $(b, 0)$ and the g_0 term is equal to $(0, h_{\epsilon_i})$, where $h_{\epsilon_i} \rightarrow 0$ as $i \rightarrow \infty$. This is possible because $\eta^{u^{\epsilon_i}}(-T_1)$, before normalization, is close to $(b, 0)$ and the g_0 term in the Fourier series for the negative end of u_+

vanishes. Similarly, we normalize v^{ϵ_i} by translating slightly in the target $\mathbb{R}$ -direction, so that the g_1 term of ζ^{ϵ_i} is zero.

Definition A.8.2 An element of $\ker D_+^\delta$ or $\ker D_-^{\epsilon_i, \alpha, \delta}$ (from Section A.5.4) is a *nontranslation* element if it is nonzero and does not correspond to an $\mathbb{R}$ -translation of the domain or target.

A sufficient condition for detecting a nontranslation element of $\ker D_+^\delta$ is given in Lemma A.8.6 in terms of the coefficient of g_1 in the Fourier expansion.

Idea of proof The idea of the proof is to start with $v^{\epsilon_i} - u^{\epsilon_i}$ for $\epsilon_i > 0$ small and construct a nontranslation element of $\ker D_+^\delta$ or $\ker D_-^{\epsilon_i, 0, \delta}$ (taking $\alpha = 0$ suffices) by damping out and inverting the error. The damping out occurs on a long neck region $[-T_3(\epsilon_i), -T_1] \times S^1$ (with $-T_3(\epsilon_i)$ defined later) that is mapped to B by $\eta^{u^{\epsilon_i}}$ and $\eta^{v^{\epsilon_i}}$. The D_+^δ and $D_-^{\epsilon_i, 0, \delta}$ cases respectively correspond to Cases 1 and 2 below. (There is a slight complication in the $D_-^{\epsilon_i, 0, \delta}$ case, which will be explained in Case 2.) The existence of a nontranslation element is a contradiction.

By (A-8-1), $\zeta^{\epsilon_i}|_{[-T_3(\epsilon_i), -T_1] \times S^1}$ can be written as a Fourier series

$$(A-8-3) \quad \zeta^{\epsilon_i}(s, t) = \sum_{j=-\infty}^{\infty} d_j^{\epsilon_i} e^{\lambda_j s} g_j(t).$$

We write $(\cdot)_-, (\cdot)_0$ and $(\cdot)_+$ for the L^2 -projections of $(\cdot) = \zeta^{\epsilon_i}$ etc to the negative, null and positive eigenspaces of A , and write $(\cdot)(s_0)$ for $(\cdot)|_{s=s_0}$. By our normalization we may assume that $d_1^{\epsilon_i} = 0$.

Lemma A.8.3 Fix $T'_1 \in [T_1 + 1, T'_2(\epsilon_i)]$. For all $s \in [-T'_1, -T_1]$,

$$(A-8-4) \quad \|\zeta_+^{\epsilon_i}(s)\|_{L_1^2} \leq \|\zeta_+^{\epsilon_i}(-T_1)\|_{L_1^2} \cdot e^{\lambda(s+T_1)},$$

$$(A-8-5) \quad \|\zeta_-^{\epsilon_i}(s)\|_{L_1^2} \leq \|\zeta_-^{\epsilon_i}(-T'_1)\|_{L_1^2} \cdot e^{-\lambda(s+T'_1)},$$

where $\lambda = \min(\lambda_1, |\lambda_{-1}|)$ and L_1^2 refers to the L^2 -Sobolev space with one derivative.

Proof We prove the first inequality. By the Fourier expansion (A-8-3) and Parseval's identity, we have

$$\|\zeta_+^{\epsilon_i}(s)\|_{L_1^2}^2 = \sum_{j \geq 1} d_j^2 (1 + \lambda_j^2) e^{2\lambda_j s}, \quad \|\zeta_+^{\epsilon_i}(-T_1)\|_{L_1^2}^2 \cdot e^{2\lambda(s+T_1)} = \sum_{j \geq 1} d_j^2 (1 + \lambda_j^2) e^{2(\lambda - \lambda_j)T_1 + 2\lambda s}.$$

Then (A-8-4) follows from the inequality

$$e^{\lambda_j s} \leq e^{(\lambda - \lambda_j)T_1 + \lambda s},$$

which holds for $j > 0$. To prove this inequality we divide the second term by the first and observe that $e^{(\lambda - \lambda_j)(T_1 + s)} \geq 1$ because $\lambda - \lambda_j \leq 0$ and $T_1 + s \leq 0$.

Now we prove the second inequality. We have

$$\|\zeta_-^{\epsilon_i}(s)\|_{L_1^2}^2 = \sum_{j<0} d_j^2(1+\lambda_j^2)e^{2\lambda_j s}, \quad \|\zeta_-^{\epsilon_i}(-T_1')\|_{L_1^2}^2 \cdot e^{-2\lambda(s+T_1')} = \sum_{j<0} d_j^2(1+\lambda_j^2)e^{-2(\lambda_j+\lambda)T_1'-2\lambda s}.$$

Then (A-8-4) follows from the inequality

$$e^{\lambda_j s} \leq e^{-(\lambda_j+\lambda)T_1'-\lambda s},$$

which holds for $j < 0$ because $\lambda + \lambda_j \leq 0$ and $s + T_1' \geq 0$. $\square$

There are two cases to consider:

- (1) $\|\zeta_0^{\epsilon_i}(-T_1 - 1) + \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} \geq \|\zeta_-^{\epsilon_i}(-T_2'(\epsilon_i))\|_{L_1^2}$ holds for infinitely many indices i , or
- (2) $\|\zeta_0^{\epsilon_i}(-T_1 - 1) + \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} \leq \|\zeta_-^{\epsilon_i}(-T_2'(\epsilon_i))\|_{L_1^2}$ holds for infinitely many indices i .

Note that the two cases are not mutually exclusive.

Case 1 Up to extracting a subsequence, we assume that, for every i ,

$$(A-8-6) \quad \|\zeta_0^{\epsilon_i}(-T_1 - 1) + \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} \geq \|\zeta_-^{\epsilon_i}(-T_2'(\epsilon_i))\|_{L_1^2}.$$

By (A-8-5) with $T_1' = T_2'(\epsilon_i)$ and $s = -T_1 - 1$,

$$(A-8-7) \quad \begin{aligned} \|\zeta_-^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} &\leq \|\zeta_-^{\epsilon_i}(-T_2'(\epsilon_i))\|_{L_1^2} \cdot e^{\lambda(-T_2'(\epsilon_i)+T_1+1)} \\ &\leq \|\zeta_0^{\epsilon_i}(-T_1 - 1) + \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} \cdot e^{\lambda(-T_2'(\epsilon_i)+T_1+1)}. \end{aligned}$$

Let $\xi^{\epsilon_i} = (\xi_1^{\epsilon_i}, \xi_2^{\epsilon_i}) \in \mathcal{H}_{1,g_\delta}(\dot{F}, N_+) \oplus \mathbb{R}\langle \tilde{\partial}_\theta \rangle$ be such that

- on $\dot{F} - (-\infty, -T_1] \times S^1$, $\xi^{\epsilon_i} = \eta^{v^{\epsilon_i}} - \eta^{u^{\epsilon_i}}$, where $u^{\epsilon_i} = \exp_{u_+} \eta^{u^{\epsilon_i}}$, $v^{\epsilon_i} = \exp_{u_+} \eta^{v^{\epsilon_i}}$, and $\eta^{u^{\epsilon_i}}$ and $\eta^{v^{\epsilon_i}}$ are viewed as sections of the normal bundle N_+ to u_+ ;
- on $(-\infty, -T_1] \times S^1$, $\xi^{\epsilon_i} = \beta(s + T_1 + 1)\zeta_-^{\epsilon_i}(s, t) + \zeta_0^{\epsilon_i}(s, t) + \zeta_+^{\epsilon_i}(s, t)$ and $\xi_2^{\epsilon_i} = \zeta_0^{\epsilon_i}$.

Recall on the negative end of u_+ we are identifying $N_+ \simeq TA_a \simeq \mathbb{R}^2$ with coordinates y, θ . As before, $\beta: \mathbb{R} \rightarrow [0, 1]$ is a nondecreasing function such that $\beta(s) = 0$ if $s \leq 0$ and $\beta(s) = 1$ if $s \geq 1$. Informally, we are damping out the $\zeta_-^{\epsilon_i}$ term to zero for $s < -T_1$, under the condition that it is much smaller than $\zeta_0^{\epsilon_i} + \zeta_+^{\epsilon_i}$ at $s = -T_1$.

Notation A.8.4 We denote the norm on $\mathbb{R}\langle \tilde{\partial}_\theta \rangle$ by $\|\cdot\|_\circ$ and the norm on $\mathcal{H}_{1,g_\delta}(\dot{F}, N_+) \oplus \mathbb{R}\langle \tilde{\partial}_\theta \rangle$ by $\|\xi^{\epsilon_i}\|_\bullet = \|\xi_1^{\epsilon_i}\|_{*,g_\delta} + \|\xi_2^{\epsilon_i}\|_\circ$.

Lemma A.8.5 *There exist constants $C_i > 0$ with $\lim_{i \rightarrow +\infty} C_i = 0$ such that*

$$\|D_+^\delta \xi^{\epsilon_i}\|_{g_\delta} \leq C_i (\|\xi_1^{\epsilon_i}\|_{*,g_\delta} + \|\xi_2^{\epsilon_i}\|_\circ) = C_i \|\xi^{\epsilon_i}\|_\bullet.$$

Proof On $(-\infty, -T_1] \times S^1$, we use the fact that $D_+^\delta \zeta_*^{\epsilon_i} = 0$ for $*$ = $-, 0, +$, to bound the contribution to $\|D_+^\delta \xi^{\epsilon_i}\|_{g_\delta}$ from above as follows:

$$\begin{aligned}
 \text{(A-8-8)} \quad & \|D_+^\delta (\beta(s + T_1 + 1) \zeta_-^{\epsilon_i}(s, t))\|_{g_\delta} \\
 &= \|\beta'(s + T_1 + 1) \zeta_-^{\epsilon_i}(s, t)\|_{g_\delta} \\
 &\leq C \left(\sup_{s \in [-T_1-1, -T_1]} g_\delta(s) \|\zeta_-^{\epsilon_i}(s)\|_{L^2} + \sup_{s \in [-T_1-1, -T_1]} g_\delta(s) |\zeta_-^{\epsilon_i}(s)|_{C^0} \right) \\
 &\leq C \sup_{s \in [-T_1-1, -T_1]} g_\delta(s) \|\zeta_-^{\epsilon_i}(s)\|_{L_1^2} \\
 &\leq C g_\delta(-T_1 - 1) \|\zeta_-^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} \\
 &\leq C g_\delta(-T_1) e^{\lambda(-T_2'(\epsilon_i) + T_1 + 1)} \|\zeta_0^{\epsilon_i}(-T_1 - 1) + \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} \\
 &\leq C e^{\lambda(-T_2'(\epsilon_i) + T_1)} (\|\xi_1^{\epsilon_i}\|_{*, g_\delta} + \|\xi_2^{\epsilon_i}\|_{\circ}).
 \end{aligned}$$

The first line to the second follows from the definition of $\|\cdot\|$ (equation (A-5-7)) and an easy C^0 -bound of the right-hand term of the definition of $\|\cdot\|$; the second line to the third uses a standard Sobolev inequality (ie there is a bounded inclusion map $L_1^2(S^1) \rightarrow C^0(S^1)$); the third line to the fourth follows from (A-8-5) applied to $T_1' = T_1 + 1$; and the fourth line to the fifth uses (A-8-7). The fifth line to the sixth follows from

$$\begin{aligned}
 \|\xi_1^{\epsilon_i}\|_{*, g_\delta} &\geq \|\xi_1^{\epsilon_i}|_{[-T_1-1, -T_1] \times S^1}\|_{*, g_\delta} \geq \|\zeta_+^{\epsilon_i}|_{[-T_1-1, -T_1] \times S^1}\|_{*, g_\delta} \\
 &\geq C \left(\int_{[-T_1-1, -T_1]} \int_{S^1} g_\delta^2 |\zeta_+^{\epsilon_i}|^2 \right)^{1/2} + C \left(\int_{[-T_1-1, -T_1]} \int_{S^1} g_\delta^2 |\nabla \zeta_+^{\epsilon_i}|^2 \right)^{1/2} \\
 &\geq C g_\delta(-T_1) (\|\zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L^2} + \|\nabla \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L^2}).
 \end{aligned}$$

On the other hand, writing $\|\cdot\|'_{g_\delta}$ and $\|\cdot\|''_{g_\delta}$ for the restrictions of $\|\cdot\|_{g_\delta}$ to $\dot{F} - (-\infty, -T_1] \times S^1$ and $[-T_1, -T_0] \times S^1$, writing $v^{\epsilon_i} = \exp_{u^{\epsilon_i}}(P^{-1} \xi^{\epsilon_i})$ on $\dot{F} - (-\infty, -T_1] \times S^1$, where P is the parallel transport of the appropriate bundles from u^{ϵ_i} to u_+ , and using the fact that

$$\tilde{\xi}^{\epsilon_i} = \xi^{\epsilon_i} + \mathcal{B}(\eta^{u^{\epsilon_i}}, \xi^{\epsilon_i}) + \mathcal{Q}(\xi^{\epsilon_i}),$$

where $\mathcal{B}(\eta^{u^{\epsilon_i}}, \xi^{\epsilon_i})$ is bilinear in $\eta^{u^{\epsilon_i}}$ and ξ^{ϵ_i} and $\mathcal{Q}(\xi^{\epsilon_i})$ is quadratic in ξ^{ϵ_i} , both with coefficients which are smooth coefficients of $\eta^{u^{\epsilon_i}}$ and ξ^{ϵ_i} , the contribution to $\|D_+^\delta \xi^{\epsilon_i}\|_{g_\delta}$ on F' is bounded above by

$$\begin{aligned}
 \|D_+^\delta \xi^{\epsilon_i}\|'_{g_\delta} &= \|D_+^\delta \xi^{\epsilon_i} - P \bar{\partial}_{J_{f_{\epsilon_i}}} \exp_{u^{\epsilon_i}}(P^{-1} \tilde{\xi}^{\epsilon_i})\|'_{g_\delta} \\
 &\leq \|D_+^\delta \xi^{\epsilon_i} - P D_{u^{\epsilon_i}}^\delta P^{-1} \xi^{\epsilon_i}\|'_{g_\delta} + \|P(D_{u^{\epsilon_i}}^\delta P^{-1} \xi^{\epsilon_i} - \bar{\partial}_{J_{f_{\epsilon_i}}} \exp_{u^{\epsilon_i}} P^{-1} \xi^{\epsilon_i})\|'_{g_\delta} \\
 &\quad + \|P(\bar{\partial}_{J_{f_{\epsilon_i}}} \exp_{u^{\epsilon_i}} P^{-1} \xi^{\epsilon_i} - \bar{\partial}_{J_{f_{\epsilon_i}}} \exp_{u^{\epsilon_i}} P^{-1} \tilde{\xi}^{\epsilon_i})\|'_{g_\delta} \\
 &\leq C_i \|\xi^{\epsilon_i}\|'_{*, g_\delta} \leq C_i (\|\xi_1^{\epsilon_i}\|_{*, g_\delta} + \|\xi_2^{\epsilon_i}\|''_{*, g_\delta}) \leq C_i (\|\xi_1^{\epsilon_i}\|_{*, g_\delta} + e^{\delta(T_1 - T_0)} \|\xi_2^{\epsilon_i}\|_{\circ}),
 \end{aligned}$$

where $\lim_{i \rightarrow \infty} C_i = 0$. (Recall that b and $T_1(b)$ are constants that were fixed at the beginning of Section A.8.)

The two estimates together imply the lemma. $\square$

In view of [Lemma A.8.5](#), inverting the error using $(D_+^\delta)^{-1}$ (as before the image of $(D_+^\delta)^{-1}$ is L^2 -orthogonal to $\ker D_+^\delta$) yields

$$(\xi')^{\epsilon_i} = \xi^{\epsilon_i} - (D_+^\delta)^{-1}(D_+^\delta \xi^{\epsilon_i}) \in \ker D_+^\delta,$$

so that $\|\xi^{\epsilon_i}\|_\bullet \gg \|(D_+^\delta)^{-1}(D_+^\delta \xi^{\epsilon_i})\|_\bullet$, which implies that $(\xi')^{\epsilon_i} \neq 0$. We define

$$\bar{\xi}^{\epsilon_i} = \frac{\xi^{\epsilon_i}}{\|(\xi')^{\epsilon_i}\|_\bullet}, \quad (\bar{\xi}')^{\epsilon_i} = \frac{(\xi')^{\epsilon_i}}{\|(\xi')^{\epsilon_i}\|_\bullet}.$$

Lemma A.8.6 *There exists a nontranslation element of $\ker D_+^\delta$.*

Proof So far we have constructed sequences $\{\bar{\xi}^{\epsilon_i}\}$ and $\{(\bar{\xi}')^{\epsilon_i}\}$ such that

- (1) the Fourier coefficient relative to g_1 is $\bar{d}_1^{\epsilon_i} = 0$ for all $\bar{\xi}^{\epsilon_i}$;
- (2) $\|(\bar{\xi}')^{\epsilon_i}\|_\bullet = 1$;
- (3) $(\bar{\xi}')^{\epsilon_i} \in \ker D_+^\delta$; and
- (4) $\|(\bar{\xi}')^{\epsilon_i} - \bar{\xi}^{\epsilon_i}\|_\bullet \rightarrow 0$ as $i \rightarrow \infty$.

Since $\ker D_+^\delta$ is finite-dimensional, the unit ball of $\ker D_+^\delta$ is compact and, after possibly passing to a subsequence $(\bar{\xi}')^{\epsilon_i}$, converges to a nonzero $\xi' \in \ker D_+^\delta$. Then (4) implies that $\|\bar{\xi}^{\epsilon_i} - \xi'\|_\bullet \rightarrow 0$ and therefore, from [Lemma A.5.1](#), we obtain $\bar{\xi}^{\epsilon_i}(-T_1) \rightarrow \xi'(-T_1)$ in C^0 . This in turn implies that $\bar{d}_1^{\epsilon_i} \rightarrow d'_1$, where $\bar{d}_1^{\epsilon_i}$ and d'_1 are the Fourier coefficients of g_1 in $\bar{\xi}^{\epsilon_i}$ and ξ' ; this is because the Fourier coefficients can be extracted by integration. Hence $d'_1 = 0$.

Finally we explain why $d'_1 = 0$ implies that ξ' is a nontranslation element: Recall that $u_+(s, t) = (s, t, \sum_{i=1}^\infty c_i e^{\lambda_i s} g_i(t))$ with $c_1 > 0$ at the negative end (see the beginning of [Section A.4](#)). Let u_+^σ be the translate of u_+ by $\sigma \in \mathbb{R}$ in the symplectization direction. Then, at the negative end,

$$u_+^\sigma(s, t) = \left(s + \sigma, t, \sum_{i=1}^\infty c_i e^{\lambda_i s} g_i(t)\right),$$

or, after the change of coordinates $(s + \sigma, t) \mapsto (s, t)$ at the negative end of $\dot{F}$,

$$u_+^\sigma(s, t) = \left(s, t, \sum_{i=1}^\infty c_i e^{\lambda_i(s-\sigma)} g_i(t)\right).$$

Then a translation element is a nontrivial multiple of the projection of $\partial u_+^\sigma / \partial \sigma|_{\sigma=0}$ to the normal bundle N_+ , ie $-\sum_{i=1}^\infty c_i \lambda_i e^{\lambda_i s} g_i(t)$, and has nontrivial $g_1(t)$ -coefficient. $\square$

The existence of a nontranslation element of $\ker D_+^\delta$ is a contradiction.

Case 2 Up to extracting a subsequence, we assume that, for every i ,

$$(A-8-9) \quad \|\zeta_0^{\epsilon_i}(-T_1 - 1) + \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L_1^2} \leq \|\zeta_-^{\epsilon_i}(-T_2'(\epsilon_i))\|_{L_1^2}.$$

Let $-T_4(\epsilon_i) < -T_3(\epsilon_i) < -T_2'(\epsilon_i)$ be such that $T_3(\epsilon_i) - T_2'(\epsilon_i), T_4(\epsilon_i) - T_3(\epsilon_i) \rightarrow \infty$ as $i \rightarrow \infty$ and

$$\operatorname{Im} u^{\epsilon_i}|_{s=-T_3(\epsilon_i)} \subset \left\{-\frac{1}{6} \leq \theta \leq -\frac{1}{6} + \kappa_i\right\}, \quad \operatorname{Im} u^{\epsilon_i}|_{s=-T_4(\epsilon_i)} \subset \left\{-\frac{1}{5} - \kappa_i \leq \theta \leq -\frac{1}{5}\right\},$$

where Im denotes the image. Using (A-8-5) with $T'_1 = T_3(\epsilon_i)$ and $s = -T_2(\epsilon_i)$, we have

$$(A-8-10) \quad \|\zeta_0^{\epsilon_i}(-T_1 - 1) + \zeta_+^{\epsilon_i}(-T_1 - 1)\|_{L^2_1} \leq C e^{\lambda(-T_3(\epsilon_i) + T'_2(\epsilon_i))} \cdot \|\zeta_-^{\epsilon_i}(-T_3(\epsilon_i))\|_{L^2_1}.$$

Complication There is one complication. By (A-1-3), the $\bar{\partial}$ -operator $\mathcal{D}_{\epsilon_i}$ is linear on $-\frac{1}{4} \leq \theta \leq -\frac{1}{5}$, with respect to $(y, \theta + \frac{1}{4})$, because $\bar{g}'_{\mathcal{N}}(\theta) = C(\theta + \frac{1}{4})$, and each of $\eta = \eta^{v^{\epsilon_i}}$, $\eta^{u^{\epsilon_i}}$ satisfies the equation

$$\mathcal{D}_{\epsilon_i} \eta = \frac{\partial \eta}{\partial s} + j_0 \frac{\partial \eta}{\partial t} - \left(\epsilon_i C \left(\eta_2 + \frac{1}{4} \right) \right) = 0,$$

where $\eta = (\eta_1, \eta_2)$, ie η_1 is the y -coordinate and η_2 the θ -coordinate of η . Hence $\zeta^{\epsilon_i}(s, t)$ admits a Fourier expansion at the negative end whose leading term has the form $(k_{i1}e^s, k_{i2}e^{\epsilon_i C s})$ for constants k_{i1} and k_{i2} . However, a section with growth rate $e^{\epsilon_i C s}$ as $s \rightarrow \infty$ is not in $\mathcal{H}_{1, h_\delta}(\mathbb{R} \times S^1, N_-^{\epsilon_i, 0})$ since we have been assuming that $0 < \epsilon_i C < \delta$; in fact, $\epsilon_i \rightarrow 0$ while δ is constant. To circumvent this difficulty we switch to

$$D_-^{\epsilon_i, 0, -\delta} : \mathcal{H}_{1, h_\delta}(\mathbb{R} \times S^1, N_-^{\epsilon_i, 0}) \rightarrow \mathcal{H}_{1, h_\delta}(\mathbb{R} \times S^1, \Lambda^{0,1} N_-^{\epsilon_i, 0}),$$

where $\delta > 0$ is sufficiently small. The analog of Lemma A.5.4 also holds for h_δ , ie the operators $D_-^{\epsilon_i, 0, -\delta}$ are invertible with bounded inverses that are uniform with respect to ϵ_i .

Let ξ^{ϵ_i} be the section of the normal bundle $N_-^{\epsilon_i, 0} = TA_a$ to $u_-^{\epsilon_i, 0}$ such that

- $\xi^{\epsilon_i} = \zeta^{\epsilon_i}$ on $(-\infty, -T_3(\epsilon_i)] \times S^1$;
- $\xi^{\epsilon_i} = (1 - \beta(s + T_1 + 1))\zeta_-^{\epsilon_i}(s, t) + (1 - \beta(s + T_3(\epsilon_i))) (\zeta_0^{\epsilon_i}(s, t) + \zeta_+^{\epsilon_i}(s, t))$ on $[-T_3(\epsilon_i), -T_1] \times S^1$ (here we write ζ_+ , ζ_0 , ζ_- for the L^2 -projections of ζ to the positive, null, and negative eigenspaces of A); and
- $\xi^{\epsilon_i} = 0$ on $[-T_1, \infty) \times S^1$.

Informally, we damp out $\zeta_-^{\epsilon_i}$ to zero for $s \geq -T_1$ and $\zeta_0^{\epsilon_i} + \zeta_+^{\epsilon_i}$ to zero for $s \geq -T_3(\epsilon_i) + 1$ so that the damped-out $\zeta_-^{\epsilon_i}$ dominates. By the previous paragraph, $\xi^{\epsilon_i} \in \mathcal{H}_{1, h_\delta}(\mathbb{R} \times S^1, N_-^{\epsilon_i, 0})$. Also $D_-^{\epsilon_i, 0, -\delta} \xi^{\epsilon_i}$ has support on

$$(A-8-11) \quad \{-T_4(\epsilon_i) \leq s \leq -T_3(\epsilon_i) + 1\} \cup \{-T_1 - 1 \leq s \leq -T_1\}.$$

One can compute, using (A-8-11), estimate (A-8-10), the method of estimating (A-8-8), and the error estimate between $D_-^{\epsilon_i, 0, -\delta}$ and the actual normal $\bar{\partial}_{J_{f_{\epsilon_i}}}$ -operator, that

$$\|D_-^{\epsilon_i, 0, -\delta} \xi^{\epsilon_i}\|_{h_\delta} \ll \|\xi^{\epsilon_i}\|_{*, h_\delta}.$$

Hence inverting the error using $(D_-^{\epsilon_i, 0, -\delta})^{-1}$ yields

$$(\xi')^{\epsilon_i} = \xi^{\epsilon_i} - (D_-^{\epsilon_i, 0, -\delta})^{-1} (D_-^{\epsilon_i, 0, -\delta} \xi^{\epsilon_i}) \in \ker D_-^{\epsilon_i, 0, -\delta},$$

so that $\|\xi^{\epsilon_i}\|_{*, h_\delta} \gg \|(D_-^{\epsilon_i, 0, -\delta})^{-1} (D_-^{\epsilon_i, 0, -\delta} \xi^{\epsilon_i})\|_{*, h_\delta}$, which implies that $(\xi')^{\epsilon_i} \neq 0$.

The existence of a nontrivial element of $\ker D_-^{\epsilon_i, 0, -\delta}$ contradicts Lemma A.5.4. This completes the proof of Theorem A.2.6.

A.9 How to recover the contact case

In this subsection we explain how to recover the contact case from the stable Hamiltonian case. The brief idea is to start with the stable Hamiltonian case for which Morse–Bott gluing holds, perturb it to the contact case, and use the bifurcation method to establish Morse–Bott gluing in the contact case.

Let a and c be the positive numbers satisfying $0 < b < a < c < 1$ introduced in [Section A.1](#) and subject to the conditions [\(†₀\)](#) and [\(†₁\)](#). Recall the smooth functions

$$f, f_\epsilon: [-1, 1] \times T^2 \rightarrow \mathbb{R}$$

given by [\(A-1-2\)](#).

Warning A.9.1 The following have different meanings in this subsection from the previous subsections of the appendix: the real parameter δ in this subsection is unrelated to the weight appearing in the Morrey norms, and the functions g, h, g_δ and h_δ appearing in this subsection are unrelated to the functions with the same names appearing in the previous subsections of the appendix.

We then define smooth functions

$$g, h: [-1, 1] \rightarrow \mathbb{R},$$

such that

- (i) g is odd,
- (ii) $g(y) = 0$ on $[-a, a]$,
- (iii) $g'(y) > 0$ on $(a, 1]$ and $[-1, -a)$,
- (iv) $g(y) = y$ on $y \geq c$ and $y \leq -c$,
- (v) $h(0) = 0$, and
- (vi) $h'(y) = g'(y) \partial f / \partial y = g'(y)y$.

In particular, $h(y) = 0$ on $[-a, a]$.

We define differential forms

$$(A-9-1) \quad \alpha = dt + h(y) dt + g(y) d\theta, \quad \omega = df \wedge dt + dy \wedge d\theta, \quad \omega_\epsilon = df_\epsilon \wedge dt + dy \wedge d\theta$$

on $[-1, 1] \times T^2$. (Here without loss of generality we are suppressing some constants that appeared in [\(4-1-1\)](#).)

Claim A.9.2 *The pairs (α, ω) and $(\alpha, \omega_\epsilon)$ on $[-1, 1] \times T^2$ are stable Hamiltonian structures.*

Proof It is immediate that $d\omega = d\omega_\epsilon = 0$.

Next we show that $\ker d\alpha \supset \ker \omega$ and $\ker d\alpha \supset \ker \omega_\epsilon$. We have

$$d\alpha = h'(y) dy \wedge dt + g'(y) dy \wedge d\theta = g'(y) \frac{\partial f}{\partial y} dy \wedge dt + g'(y) dy \wedge d\theta.$$

On $-a \leq y \leq a$, $d\alpha = 0$ and hence $\ker d\alpha \supset \ker \omega$. Outside of $-a \leq y \leq a$, $g'(y) \neq 0$ and

$$\ker d\alpha = \mathbb{R} \left\langle \frac{\partial}{\partial t} - \frac{\partial f}{\partial y} \frac{\partial}{\partial \theta} \right\rangle = \ker \omega.$$

Moreover, outside of $-a \leq y \leq a$, $f_\epsilon = f$ and $\omega = \omega_\epsilon$.

Finally,

$$\alpha \wedge \omega = \alpha \wedge \omega_\epsilon = dt \wedge dy \wedge d\theta > 0$$

on $-a \leq y \leq a$ and

$$\alpha \wedge \omega = \alpha \wedge \omega_\epsilon = (1 + h(y) - g(y) \frac{\partial f}{\partial y}) dt \wedge dy \wedge d\theta = (1 + h(y) - g(y)y) dt \wedge dy \wedge d\theta$$

outside of $-a \leq y \leq a$. By (vi), $(h(y) - g(y)y)' = -g(y)$. Since $|g(y)| < 1$ except when $y = 1$, $1 + h(y) - g(y)y > 0$.

Hence (α, ω) and $(\alpha, \omega_\epsilon)$ are both stable Hamiltonian structures. □

Claim A.9.3 (A1) On $[-a, a] \times T^2$,

$$(\alpha, \omega) = (dt, df \wedge dt + dy \wedge d\theta), \quad (\alpha, \omega_\epsilon) = (dt, df_\epsilon \wedge dt + dy \wedge d\theta).$$

(A2) On $([-1, -a] \cup (a, 1]) \times T^2$, the stable Hamiltonian structures (α, ω) and $(\alpha, \omega_\epsilon)$ agree and $d\alpha$ is a positive function $g'(y)$ times $\omega = \omega_\epsilon$.

(A3) On $([-1, -c] \cup (c, 1]) \times T^2$, $g'(y) = 1$, and (α, ω) and $(\alpha, \omega_\epsilon)$ are contact.

Proof This is immediate from the definitions and the proof of [Claim A.9.2](#). □

In view of [Claim A.9.3](#), there exists an extension of $(\alpha, \omega) = (\alpha, \omega_\epsilon)$ to $(\alpha, d\alpha)$ on $M - ([-1, 1] \times T^2)$. (In practice, we start with a contact form α on all of M and modify it to the stable Hamiltonian structures (α, ω) and $(\alpha, \omega_\epsilon)$ on $[-1, 1] \times T^2$.)

Let J_f and J_{f_ϵ} be almost complex structures on $\mathbb{R} \times M$ such that

(A4) on the complement of $\mathbb{R} \times [-a, a] \times T^2$, J_f and J_{f_ϵ} agree and are adapted to the same contact structure;

(A5) on $\mathbb{R} \times [-1, 1] \times T^2$, J_f and J_{f_ϵ} are adapted to the stable Hamiltonian structures (α, ω) and $(\alpha, \omega_\epsilon)$;

(A6) J_f is Morse–Bott regular, J_{f_ϵ} is regular (at least for the moduli spaces that are involved in the Morse–Bott gluing), and the pair satisfies Morse–Bott gluing (ie [Theorem 4.4.3\(2\)–\(3\)](#)).

The existence of such J_f and J_{f_ϵ} follows from [Theorem A.2.1](#).

The key point is the following lemma, which allows us to perturb to the contact case:

Lemma A.9.4 *There exist almost complex structures $J'_{f,\delta}$ and $J'_{f_\epsilon,\delta}$ that are C^1 -close to J_f and J_{f_ϵ} and contact forms α'_δ and $\alpha'_{\epsilon,\delta}$ that are C^1 -close to α (here the size of the perturbations depend on $\delta > 0$) such that:*

- (1) *On the complement of $\mathbb{R} \times [-c, c] \times T^2$, J_f , J_{f_ϵ} , $J'_{f,\delta}$ and $J'_{f_\epsilon,\delta}$ all agree.*
- (2) *$J'_{f,\delta}$ and $J'_{f_\epsilon,\delta}$ are adapted to α'_δ and $\alpha'_{\epsilon,\delta}$, respectively.*
- (3) *The stable Hamiltonian vector fields corresponding to J_f and $J'_{f,\delta}$ (and those corresponding to J_{f_ϵ} and $J'_{f_\epsilon,\delta}$) are parallel.*
- (4) *On $\mathbb{R} \times [-a, a] \times T^2$, α'_δ and $\alpha'_{\epsilon,\delta}$ are as given in (4-1-1) and (P1)–(P4) in Section 4.1 with $C = 1$ and $J'_{f,\delta}$ and $J'_{f_\epsilon,\delta}$ satisfy (*) in Section 4.2 and (**) in Section 4.4.*

Proof For $\delta > 0$ small, let $g_\delta: [-1, 1] \rightarrow \mathbb{R}$ be a smooth function which is a perturbation of g such that

- (i) g_δ is odd,
- (ii') $g_\delta(y) = \delta y$ on $[-a, a]$,

and (iii) and (iv) still hold. We define

$$\begin{aligned} \alpha'_\delta &= dt + \delta(f dt + y d\theta) & \text{if } y \in [-a, a], \\ \alpha'_{\epsilon,\delta} &= dt + \delta(f_\epsilon dt + y d\theta) & \text{if } y \in [-a, a], \\ \alpha'_\delta &= \alpha'_{\epsilon,\delta} = dt + h_\delta(y) dt + g_\delta(y) d\theta & \text{if } y \notin [-a, a], \end{aligned}$$

and $h'_\delta(y) = g'_\delta(y) \partial f / \partial y$ for $y \notin [-a, a]$ and $h_\delta(\pm a) = \delta f(\pm a)$. If $c \gg a$, then it is not hard to see that we can choose g_δ such that $h_\delta(\pm c) = h(\pm c)$. Then (α'_δ, ω) and $(\alpha'_{\epsilon,\delta}, \omega_\epsilon)$ are stable Hamiltonian structures corresponding to contact structures, and are close to (α, ω) and $(\alpha, \omega_\epsilon)$. (Strictly speaking, $d\alpha'_\delta = \phi_1 \omega$ and $d\alpha'_{\epsilon,\delta} = \phi_2 \omega_\epsilon$ for some functions ϕ_1 and ϕ_2 .)

We verify the contact property for α'_δ and $\alpha'_{\epsilon,\delta}$: For $y \in [-a, a]$,

$$\alpha'_\delta \wedge d\alpha'_\delta = (1 + \delta f) dt \wedge \delta dy d\theta + \delta y d\theta \wedge \delta df dt > 0,$$

since we are assuming that $\delta > 0$ is small. Similarly, $\alpha'_{\epsilon,\delta}$ is contact by replacing f by f_ϵ on $[-a, a]$. For $y \notin [-a, a]$, $\alpha'_\delta = \alpha'_{\epsilon,\delta}$ and

$$\alpha'_\delta \wedge d\alpha'_\delta = (1 + h_\delta) dt \wedge g'_\delta(y) dy d\theta + g_\delta(y) d\theta \wedge h'_\delta(y) dy dt = g'_\delta(y) [(1 + h_\delta) - g_\delta(y)y] dt dy d\theta > 0$$

as in the proof of Claim A.9.2.

Let $J'_{f,\delta}$ and $J'_{f_\epsilon,\delta}$ be the corresponding adapted almost complex structures that are close to J_f and J_{f_ϵ} and subject to the condition that the projections of $J'_{f,\delta}|_{\ker \alpha'}$ and $J'_{f_\epsilon,\delta}|_{\ker \alpha'_\epsilon}$ to $[-1, 1] \times (\mathbb{R}/\mathbb{Z})$ with coordinates (y, θ) is the standard complex structure $\partial/\partial y \mapsto \partial/\partial \theta$.

The C^1 -closeness and (1)–(4) are immediate from the construction. □

For the next lemma we introduce the following notation:

Notation A.9.5 If $\mathcal{M}$ is a moduli space of J -holomorphic curves in a symplectization for a cylindrical almost complex structure J , we denote by $\widehat{\mathcal{M}} := \mathcal{M}/\mathbb{R}$ the quotient of $\mathcal{M}$ by translations in the symplectization direction.

Lemma A.9.6 *There exist $\delta > 0$ sufficiently small and $\epsilon_0 = \epsilon_0(\delta) > 0$ such that Theorem 4.4.3(2)–(3) hold for any ϵ satisfying $0 < \epsilon < \epsilon_0$, with J_0 and J_ϵ replaced by $J'_{f,\delta}$ and $J'_{f_\epsilon,\delta}$.*

Proof Consider the $I = \text{ind} = 1$, unconstrained, Morse–Bott regular moduli space $\mathcal{M}_{J_f}^{I=\text{ind}=1}(\gamma; \mathcal{N})$ from Section A.1. Then $\widehat{\mathcal{M}}_{J_f}^{I=\text{ind}=1}(\gamma; \mathcal{N})$ consists of a finite number of holomorphic maps u_+ . If $\delta > 0$ is small, then $J'_{f,\delta}$ is also Morse–Bott regular since it is close to J_f and

$$(A-9-2) \quad \widehat{\mathcal{M}}_{J_f}^{I=\text{ind}=1}(\gamma; \mathcal{N}) \simeq \widehat{\mathcal{M}}_{J'_{f,\delta}}^{I=\text{ind}=1}(\gamma; \mathcal{N}),$$

where $\simeq$ indicates a bijection. (If signs were done carefully, they would be preserved by the bijection.) Next, there exists $\epsilon > 0$ small such that J_{f_ϵ} is regular (after possibly perturbing J_f) and there exists $\delta = \delta(\epsilon) > 0$ such that $J'_{f_\epsilon,\delta}$ is close to J_{f_ϵ} and hence is regular and

$$(A-9-3) \quad \widehat{\mathcal{M}}_{J_{f_\epsilon}}^{I=\text{ind}=1}(\gamma, e) \simeq \widehat{\mathcal{M}}_{J'_{f_\epsilon,\delta}}^{I=\text{ind}=1}(\gamma, e),$$

where e is the negative elliptic orbit obtained by perturbing the Morse–Bott family. Also, for the same $\epsilon > 0$ small, the Morse–Bott gluing theorem in the stable Hamiltonian case (Theorem A.2.1) gives a bijection

$$(A-9-4) \quad \widehat{\mathcal{M}}_{J_f}^{I=\text{ind}=1}(\gamma; \mathcal{N}) \simeq \widehat{\mathcal{M}}_{J_{f_\epsilon}}^{I=\text{ind}=1}(\gamma, e).$$

Combining (A-9-2), (A-9-3) and (A-9-4) gives

$$(A-9-5) \quad \widehat{\mathcal{M}}_{J'_{f,\delta}}^{I=\text{ind}=1}(\gamma; \mathcal{N}) \simeq \widehat{\mathcal{M}}_{J'_{f_\epsilon,\delta}}^{I=\text{ind}=1}(\gamma, e)$$

for some $\epsilon > 0$ small and $\delta = \delta(\epsilon) > 0$ small. The difficulty is that we want ϵ to depend on δ , not the other way around.

To remedy this we start with $\epsilon_1 > 0$ small, choose $\delta = \delta(\epsilon_1) > 0$ small such that (A-9-3) holds with $\epsilon = \epsilon_1$, and apply the bifurcation method to the 1-parameter family $\{J'_{f_\epsilon,\delta}\}_{\epsilon \in (0, \epsilon_1]}$. We may assume that $J'_{f_{\epsilon_1},\delta}$ is regular and that $\{J'_{f_\epsilon,\delta}\}_{\epsilon \in (0, \epsilon_1]}$ is regular as a family. By (A-9-2)–(A-9-5), $\widehat{\mathcal{M}}_{J'_{f_{\epsilon_1},\delta}}^{I=\text{ind}=1}(\gamma, e)$ consists of a finite number of holomorphic maps (up to translation in the target) that are close to breaking and is in bijection with $\widehat{\mathcal{M}}_{J_f}^{I=\text{ind}=1}(\gamma; \mathcal{N})$.

We claim that, for ϵ_1 and δ sufficiently small,

$$\#\widehat{\mathcal{M}}_{J'_{f_\epsilon,\delta}}^{I=\text{ind}=1}(\gamma, e) \equiv \#\widehat{\mathcal{M}}_{J'_{f_{\epsilon_1},\delta}}^{I=\text{ind}=1}(\gamma, e) \pmod{2}$$

for all $\epsilon \in (0, \epsilon_1]$. To this end we consider the 1-dimensional parametric moduli space which, slightly abusing notation, we denote by $\coprod_{\epsilon \in (0, \epsilon_1]} \widehat{\mathcal{M}}_{J'_{f_\epsilon,\delta}}^{I=\text{ind}=1}(\gamma, e)$. Note that the Reeb orbits do not vary as ϵ varies by Lemma A.9.4(2). The claim is a consequence of the following claim: For ϵ_1 and δ small there is no $u_{\tilde{\epsilon}} \in \partial(\coprod_{\epsilon \in (0, \epsilon_1]} \widehat{\mathcal{M}}_{J'_{f_\epsilon,\delta}}^{I=\text{ind}=1}(\gamma, e))$, where $u_{\tilde{\epsilon}}$ is a limit $J'_{f_{\tilde{\epsilon}},\delta}$ -holomorphic curve/building for some

$\tilde{\epsilon} \in (0, \epsilon_1)$. Arguing by contradiction, suppose there exist sequences $\{(\delta_i, \tilde{\epsilon}_i)\}_{i=1}^\infty$ and $\{u_{\tilde{\epsilon}_i}\}_{i=1}^\infty$ such that $(\delta_i, \tilde{\epsilon}_i) \rightarrow (0, 0)$ and $u_{\tilde{\epsilon}_i}$ converges to a J_f -holomorphic limit curve u which is

- (i) a 2-level holomorphic building $\tilde{u}_1 \cup \tilde{u}_2$, one of whose components — say $\tilde{u}_1$ — satisfies $I(\tilde{u}_1) = \text{ind}(\tilde{u}_1) = 0$; or
- (ii) a multiple cover of a holomorphic map $\tilde{v}$ with $I(\tilde{v}) = \text{ind}(\tilde{v}) = 0$;

and neither can occur since u is J_f -holomorphic and J_f is regular. □

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A landscape of contact manifolds via rational SFT

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We define a hierarchy functor from the exact symplectic cobordism category to a totally ordered set from a BL_∞ (Bi-Lie) formalism of the rational symplectic field theory (RSFT). The hierarchy functor consists of three levels of structures, namely *algebraic planar torsion*, *order of semidilation* and *planarity*, all taking values in $\mathbb{N} \cup \{\infty\}$, where algebraic planar torsion can be understood as the analogue of the algebraic torsion of Latschev and Wendl (2011) in the context of RSFT. The hierarchy functor is well-defined through a partial construction of RSFT and is within the scope of established virtual techniques. We develop computational tools for those functors and prove that all three of them are surjective. In particular, the planarity functor is surjective in all dimension ≥ 3 . Then we use the hierarchy functor to study the existence of exact cobordisms. We discuss examples including iterated planar open books, spinal open books, affine varieties with uniruled compactification and links of singularities.

[53D10](#), [53D35](#), [53D42](#)

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1 Introduction

One central subject in symplectic and contact topology is the study of symplectic cobordisms. Unlike the usual cobordism relation in differential topology, a symplectic cobordism is asymmetric; the collection of such cobordisms endows the collection of contact manifolds with a structure similar to a partial order. The fundamental dichotomy between *overtwisted* contact structures and *tight* contact structures discovered by Eliashberg [28] in dimension 3, and Borman, Eliashberg and Murphy [7] in higher dimensions, is reflected

by the fact that overtwisted contact structures behave like least elements. (A least element in a poset is an element that is smaller than any other elements; a minimal element in a poset is an element such that there is no smaller element.) Namely, there is always an exact cobordism from an overtwisted contact manifold to any other contact manifold in dimension 3 (see Etnyre and Honda [32]), and the same holds for higher dimensions when the obvious topological obstructions vanish (see Eliashberg and Murphy [30]). Moreover, the overtwisted contact 3-manifold behaves like minimal elements in the Weinstein cobordism category, as any contact 3-manifold that is Weinstein cobordant to an overtwisted contact manifold is overtwisted by Wand [79]. To explore the realm of the more mysterious class of tight contact structures, the hierarchy imposed by the existence of symplectic cobordisms is a useful guiding principle, as the complexity of contact topology should not decrease in a cobordism. In dimension 3, a further hierarchy in the world of tight contact manifolds was discovered by Giroux [40] and Wendl [82]. In higher dimensions, the notion of Giroux torsion was generalized by Massot, Niederkrüger and Wendl [58].

On the other hand, since contact manifolds and (exact) symplectic cobordisms form a natural category, which we will refer to as the (exact) symplectic cobordism category $\mathfrak{Con}$, one natural approach to study $\mathfrak{Con}$ is by understanding functors from $\mathfrak{Con}$ to some algebraic category, a.k.a. a field theory. *Symplectic field theory* (SFT), as proposed by Eliashberg, Givental and Hofer in [29], is a very general framework for defining such functors, and many invariants of contact manifolds and symplectic cobordisms can be defined via suitable counts of punctured holomorphic curves which approach Reeb orbits at their punctures. The formidable algebraic richness of the general theory, together with the serious technical difficulties arising in building its analytical foundations, conspire to make explicit computations a complicated matter. Therefore, rather than focusing on computing the full SFT invariant, one could focus on extracting simpler invariants from the general theory whose computation is in principle approachable via currently available techniques. An example of this philosophy is the notion of *algebraic torsion* introduced by Latschev and Wendl in [51], which associates to every contact manifold a number in $\mathbb{N} \cup \{\infty\}$ and can be viewed as the algebraic interpretation of the geometric concept of *planar torsion* defined by Wendl [82].

In this paper, we follow the same methodology of Latschev and Wendl to study the structure of $\mathfrak{Con}$. Instead of the full SFT, we use the rational SFT (RSFT), ie we only consider genus 0 curves, to construct a functor from $\mathfrak{Con}$ to a totally ordered set. Our main theorem is the following.

Theorem A *We have the following monoidal functors:*

- (1) **Algebraic planar torsion** (Definition 3.12) $\text{APT}: \mathfrak{Con} \rightarrow \mathbb{N} \cup \{\infty\}$, where the monoidal structure on $(\mathbb{N} \cup \{\infty\}, \leq)$ is given by $a \otimes b := \min\{a, b\}$ (Proposition 3.13).
- (2) **Planarity** (Definition 3.22) $\text{P}: \mathfrak{Con} \rightarrow \mathbb{N} \cup \{\infty\}$, where the monoidal structure on $(\mathbb{N} \cup \{\infty\}, \leq)$ is given by $a \otimes b := \max\{a, b\}$ if $a, b \neq 0$ and $0 \otimes a = a \otimes 0 = 0$ (Proposition 3.25).
- (3) **Order of semidilation** (Equation (4-2)) $\text{SD}: \text{P}^{-1}(1) \rightarrow (\mathbb{N} \cup \{\infty\}, \leq)$, where the monoidal structure on $\mathbb{N} \cup \{\infty\}$ is given by $a \otimes b := \max\{a, b\}$ (Proposition 4.5).

When APT is finite, it is necessary to have $P = 0$, ie APT and SD are refinements of the cases with $P = 0, 1$. Therefore we can assemble all three functors into a functor H_{cx} measuring the contact complexity,

$$H_{\text{cx}}: \mathfrak{Con} \rightarrow \mathcal{H} := \underbrace{\{0^{\text{APT}} < 1^{\text{APT}} < \dots < \infty^{\text{APT}}\}}_{0^P} < \underbrace{\{0^{\text{SD}} < 1^{\text{SD}} < \dots < \infty^{\text{SD}}\}}_{1^P} < 2^P < \dots < \infty^P\}.$$

(There is a (unique) morphism $a \rightarrow b$ if and only if $a \leq b$, $a, b \in \mathcal{H}$.) Here,

- k^{APT} stands for $P = 0$ and $\text{APT} = k$,
- k^{SD} stands for $P = 1$ and $\text{SD} = k$,
- k^P stands for $P = k$.

APT can be viewed as the analogue of algebraic torsion in the context of RSFT. In particular, finiteness of $\text{APT}(Y)$ implies that Y has no strong filling just like algebraic torsion. However, H_{cx} goes well beyond nonfillable contact manifolds, ie SD and P provide measurements for fillable contact manifolds. Roughly speaking, APT looks for rational curves without negative punctures and P looks for rational curves with a point constraint in symplectizations. And SD is defined using the $\mathbb{Q}[U]$ -module structure on linearized contact homology introduced by Bourgeois and Oancea [13]. APT measures the obstruction to augmentations of RSFT, while SD and P can be phrased in the linearized theory, and hence require the existence of augmentations. To make SD and P independent of the augmentation, we need to define SD and P via traversing the set of all possible augmentations of the RSFT.

Of course, [Theorem A](#) as stated could be trivial, as the true content of the claim is contained in the algebraic construction. The following results endow the functors with geometric content.

Theorem B *The functors above have the following properties.*

- (1) If Y has planar k -torsion (Wendl [82]), then $\text{APT}(Y) \leq k$ (Latschev and Wendl [51, Theorem 6], [Theorem 3.17](#)).
- (2) If Y is overtwisted then $\text{APT}(Y) = 0$ (Bourgeois and van Koert [10]).
- (3) If Y has (higher-dimensional) Giroux torsion (Massot, Niederkrüger and Wendl [58]), then $\text{APT}(Y) \leq 1$ (Moreno [64, Theorem 1.7], [Theorem 3.21](#)).
- (4) If $\text{APT}(Y) < \infty$, then Y is not strongly fillable ([Corollary 3.15](#)). If Y admits an exact filling then $P(Y) \geq 1$ ([Proposition 3.14](#)).
- (5) If Y is an iterated planar open book (Acu [2]) where the initial page has k -punctures, then $P(Y) \leq k$ ([Theorem 6.3](#)).
- (6) If Y has an exact filling that is not k -uniruled (McLean [60]), then $P(Y) \geq k + 1$ ([Theorem 5.13](#)).
- (7) APT, SD, P are all surjective (Latschev and Wendl [51, Theorem 4], [Theorem 3.18](#), [Theorem 7.28](#), [Corollary 6.8](#)). In particular, P is surjective in all odd dimensions ≥ 3 ([Corollary 6.8](#)).

1.1 Rational SFT

The original algebraic formalism of SFT in [29] packaged the full SFT into a super Weyl algebra with a distinguished odd-degree Hamiltonian H such that $H \star H = 0$. Cieliebak and Latschev [22] reformulated the algebra into a BV_∞ algebra, which was used in the definition of algebraic torsion in Latschev and Wendl [51]. The BV_∞ algebra structure was further refined to an IBL_∞ (*Involutive Bi-Lie infinity*) algebra by Cieliebak, Fukaya and Latschev [21], which, roughly speaking, is precisely the boundary combinatorics for the SFT compactification in Bourgeois, Eliashberg, Hofer, Wysocki and Zehnder [9]. For rational SFT, the original algebraic formalism was a Poisson algebra with a distinguished odd-degree Hamiltonian h such that $\{h, h\} = 0$. Analogous reformulations of the algebraic structure of RSFT can be found in Hutchings' " q -variable only RSFT" [44], and an L_∞ formalism of RSFT by Siegel [76]. In this paper, we introduce a notion of BL_∞ (*Bi-Lie infinity*) algebra to describe RSFT, which precisely describes the boundary combinatorics for rational curves in the SFT compactification and is a specialization of the IBL_∞ formalism. By building functors from the category of BL_∞ algebras to totally ordered sets, we can build the hierarchy functor in Theorem A by a composition

$$(1-1) \quad H_{cx} : \mathfrak{Con} \xrightarrow{\text{RSFT}} BL_\infty(\text{with additional structures up to homotopy}) \rightarrow \mathcal{H}.$$

On the other hand, the general holomorphic curve theory in manifolds with contact boundaries faces serious analytical challenges, which makes a complete construction of the first functor in (1-1) a difficult task. To obtain a construction of SFT/RSFT, one needs to deploy more powerful virtual techniques, eg either *polyfold* approaches by Fish and Hofer [33] and Hofer, Wysocki and Zehnder [43], *implicit atlases and virtual fundamental cycles* by Pardon [71; 72], or *Kuranishi* approaches by Ishikawa [46]. However, for the purpose of defining H_{cx} , it is sufficient to build RSFT partially. In particular, we do not need to discuss compositions and homotopies for morphisms of BL_∞ algebras as $\mathcal{H}$ is a totally ordered set, where there is no ambiguity for compositions and homotopy equivalences. This greatly simplifies our demands for virtual machinery, as homotopies in SFT is a subtle subject. Moreover, the combinatorics for a BL_∞ algebra is "tree-like", which is very similar to the combinatorics for contact homology. As a consequence, we can use Pardon's construction [72] of contact homology to provide all the analytic foundation of the functor H_{cx} . In particular, Theorem A is well-posed without any hidden hypotheses on virtual machinery (except for Theorem A(3), for which we give a sketch and details will appear in a future work). Moreover, it is expected that any other virtual technique will suffice for Theorem A. We will also explain how to obtain another construction of H_{cx} from a small part of the polyfold construction of SFT by Fish and Hofer [33].

In general, a full computation of RSFT and SFT is very difficult, as we need to understand many moduli spaces. On the other hand, the hierarchy functor H_{cx} extracts partial information from BL_∞ algebras, so only partial knowledge of the moduli spaces is needed. In particular, H_{cx} is relatively computable. It is a nontrivial question whether H_{cx} is independent of the choice of virtual technique. However, since every virtual technique has the property that we can count a compactified moduli space geometrically if it is cut out transversely in the classical sense, Theorem B does not depend on the choice of virtual technique.

1.2 Foundational claims

For the sake of clarity, let us here make precise what our claims are, pertaining to the foundations of RSFT. In particular, to avoid misunderstandings, we will clarify which parts of the necessary foundational work are carried out in this paper, and which ones are not. The following claims are proven in [Theorem 3.11](#).

- For a fixed choice of geometric data on a symplectization, there is a nonempty set of perturbation data in the context of Pardon's VFC package, which allow to define (virtual) counts of rigid rational curves which satisfy a quadratic relation. RSFT can then be implemented as an instance of what we call a BL_∞ -algebra, where the differential is given by these counts.
- For a fixed choice of geometric data on an exact cobordism (compatible with choices at the ends) there is a nonempty set of perturbation data (compatible with the perturbation data at the ends), which allow one to define virtual counts of rigid rational curves in the cobordism satisfying a compatibility relation with the previously defined counts at the ends. This gives a morphism from the BL_∞ algebra of the convex end, to that of the concave end, ie this establishes functoriality of the theory.
- In both cases above, there are versions of virtual counts of rational curves with a point constraint, again satisfying compatibility relations with the above differentials and morphisms.
- We make no claims about homotopies between morphisms obtained from different choices of geometric data and/or perturbation data. In particular, we do not claim to prove independence of the resulting *full* algebraic structures from the auxiliary choices (contact form, almost complex structure, perturbations).
- However, our invariants algebraic planar torsion and planarity, which are defined purely at the algebraic level, ie for BL_∞ algebras, are well-defined for RSFT. In other words, invariance under homotopies is not needed for their implementation. These invariants are, moreover, independent of the choice of virtual perturbation scheme for applications in this paper, provided very mild requirements are satisfied by the scheme (namely, that virtual counts coincide with geometric counts if transversality holds). All the results pertaining to these invariants are therefore rigorous.
- A full implementation of the order of semidilation requires a rigorous implementation of the U -map in linearized contact homology (cf Bourgeois and Oancea [\[13\]](#)), eg within the framework of Pardon's VFC package, or other virtual technique. We give a heuristic approach at the end of [Section 4.1](#), which we defer to later work, and precisely state what is needed to define the order of semidilation (see [Claim 4.3](#)). All results pertaining to this invariant are therefore conditional on the unproven [Claim 4.3](#). On the other hand, implementation of homotopies is also not necessary for this invariant to be well-defined.

1.3 Applications

Since H_{cx} is a measurement of the complexity of contact topology, the main application of H_{cx} is obstructing the existence of exact cobordisms. The following theorem answers a conjecture of Wendl [\[82\]](#) affirmatively, although the invariant we use is P , whereas the original conjecture used algebraic torsion.

Theorem C (Theorem A + Corollary 6.8) *For any dimension ≥ 3 , there exists an infinite sequence of contact manifolds $Y_1, Y_2, \dots$ such that there is an exact cobordism from Y_i to Y_{i+1} , but there is no exact cobordism from Y_{i+1} to Y_i .*

The above result was obtained in dimension 3 in Latschev and Wendl [51]. In fact, there are many examples of Y_i , the simplest example being the boundary of the product of n copies of S_k -2-spheres with k disks removed, as we will show in Section 6 that $P(\partial(S_k)^n) = k$ for $n \geq 2$. There are many more examples for which Theorem C holds; see eg Theorem L below.

Following the definition of P , it is easy to see that if Y admits a contact structure without Reeb orbits, then $P(Y) = \infty$. Therefore, as a corollary, we have the following.

Corollary D *If $P(Y) < \infty$, then the Weinstein conjecture holds for Y .*

In other words, counterexamples to the Weinstein conjecture (if any) should be looked for in the highest complexity level ∞^P . In particular, the combination of (5) in Theorem B and Corollary D yields a proof of the Weinstein conjecture for iterated planar open books, which was previously obtained for dimension 3 in Abbas, Cieliebak and Hofer [1], and higher dimensions in Acu [2] and Acu and Moreno [5]. In some sense, the proof of Theorem B(5) endows the ruling holomorphic curve in [1; 2; 5] with a homological meaning, ie the ruling curve defines a map that is visible on homology; in particular, such a curve cannot be eliminated by perturbing the contact form. On the other hand, not every contact manifold with finite planarity is iterated planar: for example $P(T^3, \xi_{\text{std}}) = 2$ by Corollary 6.8, while (T^3, ξ_{std}) is not supported by a planar open book, by Etnyre [31] (it is, however, supported by a planar *spinal* open book, by Wendl [81] and Lisi, Van Horn-Morris and Wendl [57]). By functoriality, if there is an exact cobordism from Y to Y' with $P(Y') < \infty$, then the Weinstein conjecture holds for Y .

The study of planar open books in dimension 3 has a very long history, since they enjoy nice properties like equivalence of weak fillability and Weinstein fillability; see Niederkrüger and Wendl [69] and Wendl [81]. We refer readers to the introduction of Acu, Capovilla-Searle, Gąbka, Marinković, Murphy, Starkston and Wu [3] for a comprehensive summary on the subject. Obstructions to planar open book structures were obtained in Etnyre [31] and Ozsváth, Stipsicz and Szabó [70]. In higher dimensions, obstructions to supporting an iterated planar open book were found in Acu and Moreno [5]. By (5) of Theorem B, infinite planarity is an obstruction to an iterated planar structure. In particular, we answer Question 1.14 of Acu, Etnyre and Ozbagci [4] negatively, by the following general result.

Corollary E (Corollary 7.37) *In all dimensions ≥ 5 , consider (Y, J) an almost contact manifold which has an exactly fillable contact representative (Y, ξ) . Then there is a contact structure ξ' , in the homotopy class of J , such that (Y, ξ') is not iterated planar.*

In particular, since every simply connected almost contact 5-manifold is almost Weinstein fillable by Geiges [38], there is a contact structure in each homotopy class of almost complex structures that is not iterated planar for every simply connected 5-manifold.

1.4 Examples

In addition to [Theorem B](#), there are many situations where we can compute or estimate H_{cx} . By [\(6\)](#) of [Theorem B](#), it is natural to look at affine varieties with a uniruled projective compactification. One special case is affine varieties with a $\mathbb{CP}^n$ compactification.

Theorem F ([Theorem 7.6](#)) *Let D be k generic hyperplanes in $\mathbb{CP}^n$ for $n \geq 2$. Then:*

- (1) $P(\partial D^c) \geq k + 1 - n$ for $k > n + 1$, where ∂D^c is the contact boundary of the affine variety $D^c := \mathbb{CP}^n \setminus D$. (See [Section 7.1](#).)
- (2) $P(\partial D^c) = k + 1 - n$ for $n + 1 < k < \frac{1}{2}(3n - 1)$ and n odd.
- (3) $P(\partial D^c) = 2$ for $k = n + 1$.
- (4) $H_{\text{cx}}(\partial D^c) = 0^{\text{SD}}$ for $k \leq n$.

The condition of n being odd (also for [Theorems H](#) and [I](#) below) is not essential. We use it to obtain automatic closedness of a chain in the computation of planarity for any augmentation. In [Remark 7.23](#), we explain how one can drop this condition using polyfold techniques in Zhou [\[86\]](#). On the other hand, the role of $k < \frac{1}{2}(3n - 1)$ is more mysterious. Although it is unlikely to be optimal, whether an upper bound is necessary is unclear. An extreme case is when $n = 1$, then ∂D^c is a disjoint union of circles, then we have $P(\partial D^c) = \infty$ and unlike the situation in [Theorem F](#) there is no obstruction to exact cobordisms for different k when $n = 1$. One difficulty of computing P and obtaining cobordism obstructions is that we need to carry out computation for all hypothetical “fillings”, ie augmentations. Indeed, different choices of augmentations will affect the computation dramatically. For example, there exists an affine variety with a $\mathbb{CP}^n$ compactification whose contact boundary has infinite planarity; cf [Theorem 7.14](#). However, if we use the augmentation from the affine variety, then the planarity is finite.

Remark 1.1 Based on the notion of asymptotically dynamically convex manifolds introduced by Lazarev [\[52\]](#), the work of Zhou [\[90; 88\]](#) exploited the uniqueness of $\mathbb{Z}$ -graded (dga) augmentations to the contact homology to obtain some cobordism obstructions. However, to maintain the effectiveness of the $\mathbb{Z}$ -grading, one needs to make additional topological assumptions (vanishing of first Chern class, injectivity of the fundamental groups etc) for the argument in [\[90; 88\]](#) to work. Dropping the $\mathbb{Z}$ -grading condition will almost certainly result in multiple augmentations (if an augmentation exists). Moreover, having a unique $\mathbb{Z}$ -graded BL_∞ augmentation requires an index assumption much more restrictive than asymptotically dynamical convexity, which, in the context of flexibly fillable contact manifolds with vanishing first Chern class, requires that the flexible filling has only 0, 1, 2, 3 handles. The strategy in this paper is very different. Instead of using the uniqueness of certain augmentations, we search for examples and structures on the RSFT that are independent of all augmentations (which are not unique). Showing such independence is the main challenge in the proof of [Theorem F](#).

On the other hand, D_k^c embeds exactly into D_{k+1}^c , which follows from a general construction, as follows. Let $\mathcal{L}$ be a very ample line bundle over a smooth projective variety X . Then for any nonzero holomorphic

section $s \in H^0(\mathcal{L})$, $X \setminus s^{-1}(0)$ is an affine variety whose contact boundary is denoted by Y_s . The projective space $\mathbb{P}H^0(\mathcal{L})$ should be stratified¹ by the singularity type of $s^{-1}(0)$, with the top stratum corresponding to the case where $s^{-1}(0)$ is smooth with multiplicity 1. We say that there is a morphism from stratum A to stratum B if we can change $s^{-1}(0)$ from A to B by an arbitrarily small perturbation of the section s , ie A is contained in the closure of B . Moreover, one obtains an exact cobordism from Y_s to $Y_{s'}$, where s' is the perturbed section. Then we have a natural functor from the category of strata to $\mathfrak{Con}$. As a concrete example, consider $\mathcal{L} = \mathcal{O}(2)$ on $\mathbb{CP}^2$, then the category of stratification is the graph $A_1 \rightarrow A_2 \rightarrow A_3$, where A_1, A_2, A_3 correspond to a double line, two generic lines and a smooth quadratic curve as the divisor, respectively. The corresponding affine varieties as exact domains are $\mathbb{C}^2, \mathbb{C} \times T^*S^1, T^*\mathbb{RP}^2$, which clearly have the exact embedding relations as claimed.

In view of this, when we view one of the k hyperplanes in D_k as having multiplicity 2 (a “double” hyperplane), we can get an exact cobordism from ∂D_k^c to ∂D_{k+1}^c , by perturbing the double hyperplane to two distinct hyperplanes. Then [Theorem F](#) asserts that a reversed exact cobordism cannot be found if $n \leq k < \frac{1}{2}(3n-1)$ and n is odd. Note that the natural inclusion $D_{k+1}^c \subset D_k^c$ is symplectic; hence we always have a strong cobordism from ∂D_{k+1}^c to ∂D_k^c , which shows the essential difference between these two notions of cobordisms and the obstruction from P is not topological. When $k \leq n$, D_k^c is in fact subcritical, and they can be embedded exactly into each other regardless of k .

As a concept closely related to $\mathfrak{Con}$, we introduce $\mathfrak{Con}_*$ as the under category of $\mathfrak{Con}$ under $\emptyset$, ie the objects of $\mathfrak{Con}_*$ are pairs of contact manifolds with exact fillings and morphisms are exact embeddings. Then SD and P can be defined² on $\mathfrak{Con}_*$ using the augmentation from the given exact filling. Moreover, we recall another functor U , called the *order of uniruledness*, which is defined to be the minimal k such that an exact domain W is k -uniruled in the sense of McLean [60]. That U is a functor from $\mathfrak{Con}_* \rightarrow \mathbb{N}_+ \cup \{\infty\}$ was proven in [60]. By (6) of [Theorem B](#), $P(Y)$ is bounded below by $U(W)$ for an exact filling W of Y . The functor U measures the complexity of exact domains and serves as an exact embedding obstruction. An interesting aspect of U is that the well-definedness and basic properties of U do not depend on any Floer theory. As a byproduct of the proof of [Theorem F](#), we have following for any $n \geq 1$.

Theorem G ([Theorem 7.21](#)) *Let D_k denote the divisor of k generic hyperplanes in $\mathbb{CP}^n$ for $n \geq 1$, and let D_k^c denote the complement affine variety. Then $U(D_k^c) = \max\{1, k+1-n\}$. In particular, D_{k+1}^c cannot be embedded into D_k^c exactly for $k \geq n$.*

Remark 1.2 The same embedding question is studied independently by Ganatra and Siegel [36], where more general normal crossing divisors in $\mathbb{CP}^n$ are studied. The planarity for exact domains mentioned

¹It is quite a nontrivial task to make this stratification precise, as in general we do not have a classification of the possible singularities of the divisor.

²The functorial property of P requires a full construction of RSFT, including compositions and homotopies. In particular, it makes more demands for virtual constructions than explained in this paper. On the other hand, the functorial property of SD follows from the Viterbo transfer map of S^1 -equivariant symplectic cohomology and the isomorphism between the positive S^1 -equivariant symplectic cohomology and the linearized contact homology in Bourgeois and Oancea [13].

above is equivalent to $\mathbb{G}\langle p \rangle$ in [36]. The authors of [36] also consider holomorphic curves with local tangent constraints to define functors $\mathbb{G}\langle T^m p \rangle$ on $\mathfrak{Con}_*$. In view of the local tangent constraints, one can define an analogous order of uniruledness with local tangent conditions, the well-definedness and functorial property of such invariants follows from the same argument as [60]. Such a functor can also serve as an embedding obstruction as U in Theorem G. It is an interesting question as to whether those geometric invariants are the same as the algebraic invariants (defined via RSFT in [36]), which is the case for Theorem G.

In view of (6) of Theorem B, one can also consider affine varieties with uniruled compactification, in particular those affine varieties with Fano hypersurfaces as compactification. In general, we have:

Theorem H (Theorem 7.24) *Let X be a smooth degree m hypersurface in $\mathbb{CP}^{n+1}$, with m and n such that $2 \leq m < \frac{1}{2}(n+1) \leq n$, and let D be $k \geq n$ generic hyperplanes, ie $D = (H_1 \cup \dots \cup H_k) \cap X$, where the H_i are hyperplanes in $\mathbb{CP}^{n+1}$ in generic position with each other and X . Then $P(\partial D^c) = k + m - n$ for n odd and $k + m < \frac{1}{2}(3n+1)$.*

The following results provide affine variety examples with nontrivial SD.

Theorem I (Theorem 7.22) *Assume D_s is a smooth degree- k hypersurface in $\mathbb{CP}^n$ for $n \geq 3$ odd and with $2 \leq k < n$. Then $(k-1)^{\text{SD}} \leq H_{\text{cx}}(\partial D_s^c) \leq (2k-2)^{\text{SD}}$. When n is even and $2 \leq k < \frac{1}{2}(n+1)$, then we have $H_{\text{cx}}(\partial D_s^c) \leq (2k-2)^{\text{SD}}$.*

As explained before, the difficulty of computing P and SD is from enumerating through all possible augmentations. The strategy of proving Theorems F, H and I is finding the curve responsible for P and SD with low energy, so that there is no room for the dependence on augmentations. This is the reason why the results require that the hypersurfaces to have low degrees. In particular, a degree $m \leq n$ smooth hypersurface in $\mathbb{CP}^{n+1}$ is uniruled by degree 1 curves. Our proof also uses somewhere injectivity of degree 1 curves to obtain transversality in various places, hence this low degree condition is needed for technical reasons as well. It is interesting to look at the case of degree $n+1$ smooth hypersurfaces in $\mathbb{CP}^{n+1}$, which is uniruled but not by degree 1 curves. A more systematic way to study H_{cx} is to derive formulas for RSFT of affine varieties with normal crossing divisor complement using log/relative Gromov–Witten invariants similar to the formula for symplectic cohomology in Diogo and Lisi [27].

Another rich class of contact manifolds comes from links of isolated singularities. They provide examples with every order of semidilation based on computations in Zhou [90].

Theorem J (Theorem 7.28) *We use $\text{LB}(k, n)$ to denote the contact link of the Brieskorn singularity $x_0^k + \dots + x_n^k = 0$. Then $H_{\text{cx}}(\text{LB}(k, n))$ is*

- (1) $(k-1)^{\text{SD}}$ if $k < n$, and $\geq (k-1)^{\text{SD}}$ if $k = n$;
- (2) $> 1^{\text{P}}$ if $k = n+1$;
- (3) ∞^{P} if $k > n+1$.

Another type of singularity is the quotient singularity, whose contact links are not exact fillable in many cases, by Zhou [87]. In fact, the symplectic aspect of the proof in [87] can be restated as a computation of H_{cx} as follows.

Theorem K (Theorem 7.30) *Let Y be the quotient $(S^{2n-1}/\mathbb{Z}_k, \xi_{\text{std}})$ by the diagonal action of $e^{2\pi i/k}$ for $n \geq 2$.*

- (1) *If $n > k$, we have $H_{\text{cx}}(Y) = 0^{\text{SD}}$.*
- (2) *If $n \leq k$, we have $0^{\text{SD}} \leq H_{\text{cx}}(Y) \leq (n-1)^{\text{SD}}$. When $n = k$, we have $H_{\text{cx}}(Y) \geq 1^{\text{SD}}$.*

The second case of the above theorem is another situation where the computation depends on the augmentation. Roughly speaking, $H_{\text{cx}}(Y) = 0^{\text{SD}}$ means that any exact filling of Y has vanishing symplectic cohomology. And if there is a (possibly strong) filling with vanishing symplectic cohomology, then the order of semidilation using the induced augmentation from the filling³ is 0. The natural prequantization bundle filling provides augmentations such that the symplectic cohomology vanishes [73]. On the other hand, there are other augmentations with positive orders of semidilation. For example the exact filling T^*S^2 of $(\mathbb{R}P^3, \xi_{\text{std}})$ has order of semidilation 1, such a phenomenon was also explained in [87, Remark 2.16].

Theorem L *Let V be an exact domain with $c_1(V) = 0$ and S_k be the k -punctured sphere. Then*

- (1) $P(\partial(V \times S_k)) \leq k$ (Theorem 6.6).
- (2) *If V is an affine variety that is not $(k-1)$ -uniruled, then $P(\partial(V \times S_k)) = k$ (Corollary 6.8).*
- (3) $H_{\text{cx}}(\partial(V \times \mathbb{D})) = 0^{\text{SD}}$ (Theorem 7.33).

In particular, (2) in Theorem L provides many examples to Theorem C, and (3) is a reformulation of the symplectic step in Zhou [91] for obtaining uniqueness results on fillings of $\partial(V \times \mathbb{D})$.

Organization of the paper We introduce the concept of BL_∞ algebra in Section 2 and then define algebraic planar torsion as well as planarity at the level of algebra. In Section 3, we implement Pardon's VFC [72] to define APT and P. We recall in Section 4 the $\mathbb{Q}[U]$ module structure on linearized contact homology following Bourgeois and Oancea [13] to define SD and finish the proof of Theorem A. We give a lower bound for P in Section 5, and an upper bound for P in Section 6. We discuss examples, applications, and finish the proof of Theorem B in Section 7.

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³Assuming the filling is monotone, so that we can evaluate $T = 1$ in the Novikov coefficients to get back to $\mathbb{Q}$ -coefficients.

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2 L_∞ algebras and BL_∞ algebras

In this section, we recall the basics of L_∞ algebras and introduce BL_∞ algebras, which serve as the underlying algebraic structures for rational symplectic field theory. The algebraic formalism here is essentially the q -variable-only reformulation in [44] and the L_∞ algebra formalism on contact homology algebra in [76], but we make the compatibility of the algebraic structure on the contact homology algebra with the L_∞ structure more precise and define such an object as a BL_∞ algebra, which is a specialization of the IBL_∞ algebra in [21] and the homotopic version of bi-Lie algebras, with (co)-curvature. The algebraic relations in BL_∞ algebra are precisely the boundary combinatorics of the moduli spaces of rational curves in the SFT compactification. We then introduce algebraic planar torsion and planarity at the algebraic level.

2.1 L_∞ algebras

Throughout this section, we assume $\mathbf{k}$ is a field with characteristic 0 for simplicity, although the discussion works for any commutative ring. Let V be a $\mathbb{Z}_2$ -graded $\mathbf{k}$ -vector space. Then we have the $\mathbb{Z}_2$ -graded symmetric algebra $SV := \bigoplus_{k \geq 0} S^k V$ and the nonunital symmetric algebra $\bar{S}V = \bigoplus_{k \geq 1} S^k V$, where $S^k V = \bigotimes^k V / \text{Sym}_k$ in the graded sense. In particular, we have

$$ab = (-1)^{|a||b|}ba$$

for homogeneous elements a, b in $SV, \bar{S}V$. Therefore $S^k V$ is spanned by vectors of the form $v_1 \cdots v_k$ with $v_i \in V$. However, to introduce the L_∞ algebra, we will view SV and $\bar{S}V$ as coalgebras by the following co-product operation:

$$\Delta(v_1 \cdots v_k) = \sum_{i=1}^{k-1} \sum_{\sigma \in \text{Sh}(i, k-i)} (-1)^\diamond (v_{\sigma(1)} \cdots v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \cdots v_{\sigma(k)}),$$

where $\text{Sh}(i, k-i)$ is the subset of permutations σ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$ and

$$\diamond = \sum_{1 \leq i < j \leq k, \sigma(i) > \sigma(j)} |v_i| |v_j|.$$

Then both SV and $\bar{S}V$ satisfy the coassociativity property $(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$, and the cocommutativity property $\mathbf{R} \circ \Delta = \Delta$, where $\mathbf{R}: SV \otimes SV \rightarrow SV \otimes SV$ is given by $\mathbf{R}(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for homogeneous elements x, y . A coderivation of the coalgebra $(\bar{S}V, \Delta)$ is a $\mathbf{k}$ -linear

map $\delta: \bar{S}V \rightarrow \bar{S}V$ satisfying the co-Leibniz rule $\Delta \circ \delta = (\delta \otimes \text{id}) \circ \Delta + (\text{id} \otimes \delta) \circ \Delta$. Here we use the Koszul–Quillen sign convention that $(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y)$ for $x, y \in V, W$ and $f, g \in V^\vee, W^\vee$.

Definition 2.1 We use sV to denote $V[1]$. An L_∞ algebra structure on V is a degree 1 coderivation $\hat{\ell}$ on $\bar{S}sV$ satisfying $\hat{\ell}^2 = 0$.

Note that we have a well-defined degree -1 map $s: V \rightarrow sV$. The coderivation property of $\hat{\ell}$ implies that it is determined by maps $\ell^k: S^k sV \rightarrow sV$ defined by the composition $S^k sV \hookrightarrow \bar{S}sV \xrightarrow{\hat{\ell}} \bar{S}sV \rightarrow sV$, where the first map is the natural inclusion and the last map is the natural projection, satisfying the quadratic relation

$$(2-1) \quad \sum_{k=1}^n \sum_{\sigma \in \text{Sh}(k, n-k)} (-1)^{\diamond'} \ell^{n-k+1}(\ell^k(sv_{\sigma(1)} \cdots sv_{\sigma(k)})sv_{\sigma(k+1)} \cdots sv_{\sigma(n)}) = 0,$$

where

$$\diamond' = \sum_{1 \leq i < j \leq k, \sigma(i) > \sigma(j)} (|v_i| - 1)(|v_j| - 1).$$

$(\bar{S}sV, \hat{\ell})$ is called the reduced bar complex. The word-length filtration $\bar{B}^1 sV \subset \bar{B}^2 sV \subset \cdots \subset \bar{S}sV$ is compatible with the differential, where $\bar{B}^k sV := \bigoplus_{j=1}^k S^j sV$.

Definition 2.2 An L_∞ homomorphism from (V, ℓ) to (V', ℓ') is a degree 0 coalgebra map $\hat{\phi}: \bar{S}sV \rightarrow \bar{S}sV'$ such that $\hat{\phi} \circ \hat{\ell} = \hat{\ell}' \circ \hat{\phi}$.

Given $\phi_i: S^{k_i} sV \rightarrow sV'$ for $1 \leq i \leq n$ and $m = \sum_{i=1}^n k_i$, we define $\phi_1 \cdots \phi_n: S^m sV \rightarrow S^n sV'$ by sending $sv_1 \cdots sv_m$ to⁴

$$\pi \left(\sum_{\sigma} \frac{(-1)^{\diamond'}}{k_1! \cdots k_n!} (\phi_1 \otimes \cdots \otimes \phi_n)((sv_{\sigma(1)} \cdots sv_{\sigma(k_1)}) \otimes \cdots \otimes (sv_{\sigma(m-k_n+1)} \cdots sv_{\sigma(m)})) \right).$$

Here π is the natural map $\bigotimes^k sV \rightarrow S^k sV$. By the coalgebra property, if $\hat{\phi}$ is an L_∞ morphism, we know that $\hat{\phi}$ is determined by

$$\{\phi^k: S^k sV \rightarrow \bar{S}sV \xrightarrow{\hat{\phi}} \bar{S}sV' \rightarrow sV'\}_{k \geq 1}.$$

More explicitly, $\hat{\phi}$ is defined by the formula

$$\hat{\phi}(sv_1 \cdots sv_n) = \sum_{k \geq 1, i_1 + \cdots + i_k = n} \frac{1}{k!} (\phi^{i_1} \cdots \phi^{i_k})(sv_1 \cdots sv_n).$$

⁴The factorial here is a consequence of the redundant summing over all permutations. However, the construction works any coefficient ring; this is easier to see using the tree description in [Section 2.3](#).

The relation $\hat{\phi} \circ \hat{\ell} = \hat{\ell}' \circ \hat{\phi}$ can be written as

$$\sum_{p+q=n+1} \sum_{\sigma \in \text{Sh}(q, n-q)} (-1)^{\diamond'} \phi^p(\ell^q(sv_{\sigma(1)} \cdots sv_{\sigma(q)})sv_{\sigma(q+1)} \cdots sv_{\sigma(n)}) \\ = \sum_{k \geq 1, i_1 + \cdots + i_k = n} \frac{1}{k!} \ell^k(\phi^{i_1} \cdots \phi^{i_k})(sv_1 \cdots sv_n).$$

In particular, $\hat{\phi}$ preserves the word-length filtration. The composition of L_∞ homomorphism is the naive composition $\hat{\phi} \circ \hat{\psi}$, which is clearly a coalgebra chain map. Unwrapping the definition, we have

$$(\phi \circ \psi)^n = \sum_{k \geq 1, i_1 + \cdots + i_k = n} \frac{1}{k!} \phi^k((\psi^{i_1} \cdots \psi^{i_k})(sv_1 \cdots sv_n)).$$

2.2 BL_∞ algebras

In this section, we define the BL_∞ (bi-Lie-infinity) algebra structure on a $\mathbb{Z}_2$ graded vector space V , which will govern the rational symplectic field theory. Let EV denote $\bar{S}SV$. Given a linear operator $p^{k,l}: S^k V \rightarrow S^l V$ for $k \geq 1$ and $l \geq 0$, we will define a map $\hat{p}^{k,l}: S^k SV \rightarrow SV$. To emphasize the differences between products on two symmetric algebras, we use $\odot$ for the product on the outside symmetric product $\bar{S}$ and $*$ for the product on the inside symmetric product S when it cannot be abbreviated. We will first describe the definition using formulas and then introduce a graph description, which is very convenient to describe BL_∞ algebras as well as various related structures and also governs all the signs and coefficients. Let $w_1, \dots, w_k \in SV$. Then $\hat{p}^{k,l}$ is defined by the following properties.

- (1) $\hat{p}^{k,l}|_{\odot^k V \subset S^k SV}$ is defined by $p^{k,l}$.
- (2) If $w_i \in \mathbf{k}$, then $\hat{p}^{k,l}(w_1 \odot \cdots \odot w_k) = 0$.
- (3) $\hat{p}^{k,l}$ satisfies the Leibniz rule in each argument, ie we have

$$(2-2) \quad \hat{p}^{k,l}(w_1 \odot \cdots \odot w_k) = \sum_{j=1}^m (-1)^{\square} v_1 \cdots v_{j-1} \hat{p}^{k,l}(w_1 \odot \cdots \odot v_j \odot \cdots \odot w_k) v_{j+1} \cdots v_m.$$

Here $w_i = v_1 \cdots v_m$, and

$$(2-3) \quad \square = \sum_{s=1}^{i-1} |w_s| \cdot \sum_{s=1}^{j-1} |v_s| + \sum_{s=1}^{j-1} |v_s| |p^{k,l}| + \sum_{s=i+1}^n |w_s| \cdot \sum_{s=j+1}^m |v_s|.$$

It is clear from the definition that $\hat{p}^{k,l}$ is determined uniquely by the above three conditions. More explicitly, $\hat{p}^{k,l}$ is defined by

$$(2-4) \quad w_1 \odot \cdots \odot w_k \mapsto \sum_{(i_1, \dots, i_k), 1 \leq i_j \leq n_j} (-1)^{\bigcirc} p^{k,l}(v_{i_1}^1 \odot \cdots \odot v_{i_k}^k) \check{w}_1 \cdots \check{w}_k,$$

where $w_j = v_1^j \cdots v_{n_j}^j$, $\check{w}_j = v_1^j \cdots \check{v}_{i_j}^j \cdots v_{n_j}^j$ and $w_1 \cdots w_k = (-1)^{\bigcirc} v_{i_1}^1 \cdots v_{i_k}^k \check{w}_1 \cdots \check{w}_k$. Then we

define $\hat{p}^k: S^k SV \rightarrow SV$ by $\bigoplus_{l \geq 0} \hat{p}^{k,l}$. To ensure it is well-defined, we need to assume that for any $v_1, \dots, v_k \in V$, there are at most finitely many l such that $p^{k,l}(v_1 \odot \dots \odot v_k) \neq 0$. Then we can define $\hat{p}: EV \rightarrow EV$ by

$$(2-5) \quad w_1 \odot \dots \odot w_n \mapsto \sum_{k=1}^n \sum_{\sigma \in \text{Sh}(k, n-k)} (-1)^\diamond \hat{p}^k(w_{\sigma(1)} \odot \dots \odot w_{\sigma(k)}) \odot w_{\sigma(k+1)} \odot \dots \odot w_{\sigma(n)},$$

ie following the same rule of $\hat{\ell}$ from ℓ^k .

Definition 2.3 $(V, \{p^{k,l}\}_{k \geq 1, l \geq 0})$ is a BL_∞ algebra if $\hat{p} \circ \hat{p} = 0$ and $|\hat{p}| = 1$.

To explain the terminology, assume $p^{1,0} = 0, p^{2,0} = 0$. Then $p^{1,1}$ defines a differential on V , such that $p^{2,1}$ defines a Lie bracket on the homology of $(V, p^{1,1})$ and $p^{1,2}$ defines a Lie cobracket on the homology. The compatibility is that $p^{1,2} \circ p^{2,1} = 0$ on the homology level. The main difference with the IBL_∞ algebra [21, Definition 2.3] is that we will not consider the compatibility condition on $p^{2,1} \circ p^{1,2} = 0$, which will increase genus.⁵ A direct consequence of the definition is that $(SV, \hat{p}^1)$ is a chain complex and the $\hat{p}^k$ define an L_∞ structure on $(SV)[-1]$. As noted in [76, Remark 3.12], SV carries a natural commutative algebra structure, the Leibniz rule in the definition of $\hat{p}^{k,l}$ implies the L_∞ structure is compatible with the algebra structure, and $(SV)[-1]$ should be some version of a G_∞ algebra. Definition 2.3 can be viewed as one method of making the compatibility precise.

Remark 2.4 (shift vs no shift) The degree shift in Section 2.1 is the classical sign convention introduced by Stasheff [77, page 133], as L_∞ algebra is a higher generalization of Lie algebra, where the Lie bracket is skew-symmetric and has degree 0. The IBL_∞ formalism in [21] kept such a tradition of shifting degrees by 1; as a result, in the SFT context, generators are graded by the SFT degree shifted by 1 [21, Section 7] to cancel the shift in the definition. Here, we choose to drop the degree shift in Definition 2.3, so that our generators will be graded by the SFT degree in the context of SFT, since operations from counting holomorphic curves are naturally supersymmetric with respect to the SFT degrees, due to the orientation scheme in SFT. As a consequence, $((SV)[-1], \{\hat{p}^k\})$ and $(V[-1], \{p^{k,1}\})$ (assuming all other $p^{k,l}$ is zero in the latter case) are L_∞ algebras in the sense of Definition 2.1. If we use a degree-shifted version of Definition 2.3, $(V, \{p^{k,1}\})$ (under the same vanishing assumption) is an L_∞ algebra, while neither SV nor SsV are L_∞ algebras, ie extra shift is inevitable. All shifts can go away if one is willing to adopt a version of L_∞ algebra without degree shift.

Remark 2.5 BL_∞ algebra is not a “direct” specialization of the IBL_∞ algebra as introduced in [21]. However, there is an equivalent reformulation of the IBL_∞ relations,⁶ from which one can see that an IBL_∞ algebra contains a BL_∞ algebra, as well as algebras with any genus upper bound; see [67, Section 5.2, Proposition 5.10 and Corollary 5.12] for details.

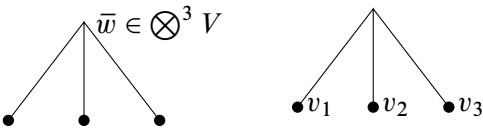
⁵The other difference is that the IBL_∞ algebra in [21] describes the algebra for linearized SFT, where $p^{k,0,g} = 0$ for any number of positive punctures k and genus g .

⁶In the special case of setting $\tau = 1$ in [21, Definition 2.3].

2.3 The rules for tree calculus

A useful way to explain the combinatorics of operations is the following description using graphs, which appeared in [76, Section 3.4.2]. The combinatorics is also relevant in the virtual technique setup; see Section 3.6. The main advantage of such graphical language is in freeing us from the book-keeping of signs and explicit components of compositions, eg in (2-2), which are governed by graphs.

Let $w \in S^k V$. We can represent w by an element $\bar{w}$ in $\bigotimes^k V$, that is, $\bar{w} = \sum_{i=1}^N c_i v_1^i \otimes \cdots \otimes v_k^i$ for $c_i \in \mathbf{k}$ and $v_*^* \in V$, such that $\pi(\bar{w}) = w$ for $\pi: \bigotimes^k V \rightarrow S^k V$. We represent it by a rooted tree with k leaves (represented by $\bullet$) labeled by $\bar{w}$. The leaves are ordered from left to right to indicate the k copies of V in $\bigotimes^k V$. When $\bar{w} = v_1 \otimes \cdots \otimes v_k$, we may label the leaves by $v_1, \dots, v_k$ to mean the same thing. We can view a general labeled tree as a formal linear combination of such trees with leaves labeled.

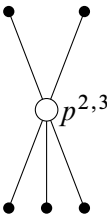


Now let $s \in S^k S V$. We can represent s by $\bar{s} \in \boxtimes^k T V$, where $T V = \bigoplus_{k \in \mathbb{N}} (\bigotimes^k V)$. Here we use $\boxtimes$ to differentiate it from the inner tensor $\otimes$. We write

$$\bar{s} = \sum_{i=1}^N c_i \bar{w}_1^i \boxtimes \cdots \boxtimes \bar{w}_k^i, \quad \text{where } c_i \in \mathbf{k}, \bar{w}_*^* \in \bigotimes^{m_*^*} V.$$

We represent $\bar{w}_1^i \boxtimes \cdots \boxtimes \bar{w}_k^i$ by an ordered forest of labeled trees as in Figure 1. Then $\bar{s}$ is a formal linear combination of such forests.

We represent the operation $p^{k,l}: S^k V \rightarrow S^l V$ by a graph with $k + l + 1$ vertices, k top input vertices, l bottom output vertices and one middle vertex $\bigcirc$ labeled by $p^{k,l}$ representing the operation type:



So far the discussion is completely formal without any actual content, the real content is in the following interpretation of a glued graph, whose definition will be clear from one example.

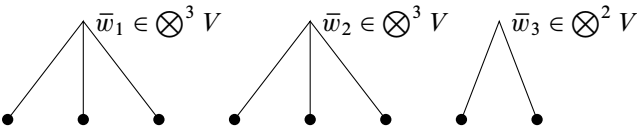


Figure 1: Forest of labeled trees.

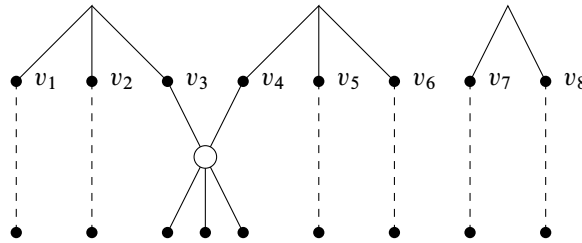
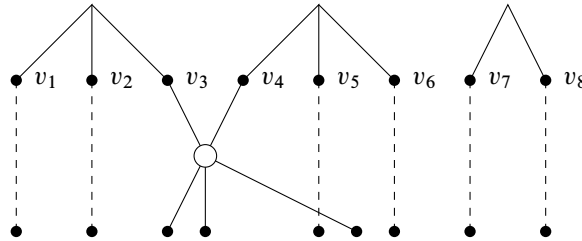
Figure 2: Gluing forests $\iff$ applying operations.

Figure 3: If we switch the output order, it still represents the same glued forest as Figure 2.

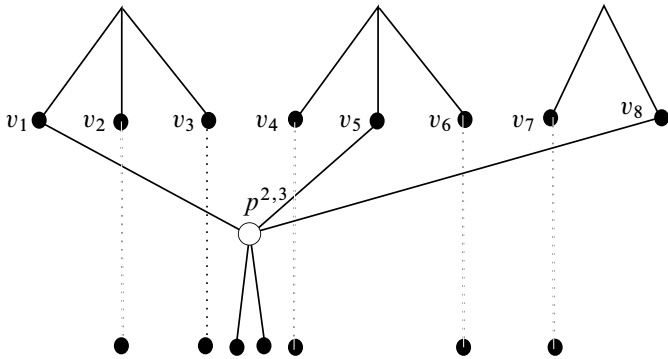
The glued graph in Figure 2 represents a forest: we first fix a representative $\bar{p}$ of $p^{2,3}(v_3v_4)$ in $\bigotimes^3 V$. The glued forest in Figure 2 represents $\pm(v_1 \otimes v_2 \otimes \bar{p} \otimes v_5 \otimes v_6) \boxtimes (v_7 \otimes v_8)$. In the gluing, we do not create cycles in the glued graph, each dashed line represents the identity map, and each connected component represents a tree in the output.

The trees and forests are considered as abstract trees and forests, the inclusion into the plane/space is not part of the information. Drawing the input element as a forest of ordered trees with ordered leaves means that we are choosing representatives from the tensor product, not the symmetric product.

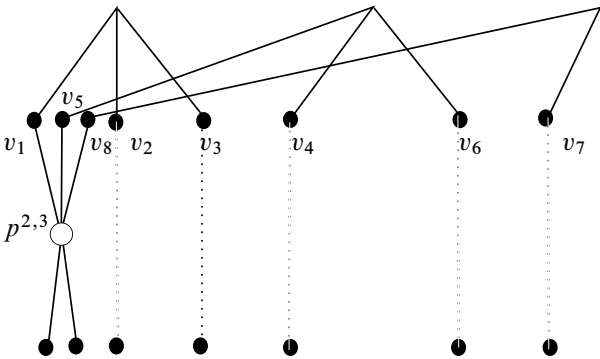
Finally, when we draw the glued graph as in Figure 2, ie choosing an order of the trees and leaves (hence edges will cross over each other if we draw it on a plane), this will determine a representative in the tensor product. In other words, we view different orders as equivalent up to the obvious sign change. For example, Figure 3 is an equivalent gluing as that in Figure 2 but with an extra sign when viewing it in the tensor product. The extra sign is $(-1)^{|v_5||b_3|}$ if the representative $\bar{p}$ is $b_1 \otimes b_2 \otimes b_3$.

The sign is determined similarly to (2-3); in the case shown in Figure 2, the sign is $(-1)^{(|v_1|+|v_2|)|p^{2,3}|}$. In a formal description, we apply order changes to the input forest (edges can cross when it is drawn in a row), then we glue $p^{k,l}$ so that there is no new edge crossing (this corresponds to that $p^{k,l}$ acts on k consecutive leaves), and finally, we change the output order back to the chosen one (eg an order prescribed in Figure 3); the final sign is given by the product of the sign changes of the two order changes and the sign of the composition using the Koszul–Quillen convention.

Example 2.6 In the following, we work out an explicit example which involves the features explained above. We consider the following glued graph representing a component of $p^{2,3}$ acting on $v_1v_2v_3 \odot v_4v_5v_6 \odot v_7v_8$:



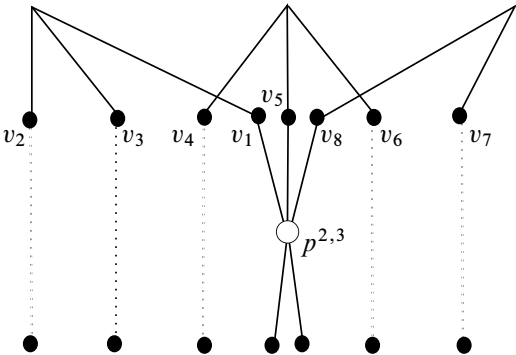
We first switch the input leaves of the input forest, so that the insertion of $p^{2,3}$ will not create crossings at the input edges of $p^{2,3}$ as follows, this will pick up a sign $(-1)^{|v_5|(|v_2|+|v_3|+|v_4|)+|v_8|(|v_2|+|v_3|+|v_4|+|v_6|+|v_7|)}$.



Now the composition with $p^{2,3}$ has no sign from the Koszul–Quillen convention, the output is a forest, which is a single tree here, representing

(2-6) $(-1)^{|v_5|(|v_2|+|v_3|+|v_4|)+|v_8|(|v_2|+|v_3|+|v_4|+|v_6|+|v_7|)} p^{2,3}(v_1 v_5 v_8) v_2 v_3 v_4 v_6 v_7.$

Alternatively, we can do the following switch of the input leaves



resulting in a sign change by

$$(-1)^{|v_1|(|v_2|+|v_3|+|v_4|)+|v_8|(|v_6|+|v_7|)}.$$

Now the composition will pick up a sign $(-1)^{p^{2,3}(|v_2|+|v_3|+|v_4|)}$ by the Koszul–Quillen convention, so the output is

$$(-1)^{|v_1|(|v_2|+|v_3|+|v_4|)+|v_8|(|v_6|+|v_7|)+p^{2,3}(|v_2|+|v_3|+|v_4|)} v_2 v_3 v_4 p^{2,3} (v_1 v_5 v_8) v_6 v_7,$$

which is

$$(-1)^{|v_1|(|v_2|+|v_3|+|v_4|)+|v_8|(|v_6|+|v_7|)+p^{2,3}(|v_2|+|v_3|+|v_4|)+(p^{2,3}+|v_1|+|v_5|+|v_8|)(|v_2|+|v_3|+|v_4|)} \cdot p^{2,3} (v_1 v_5 v_8) v_2 v_3 v_4 v_6 v_7.$$

This expression is exactly (2-6), as there are sign cancellations. And indeed they are the same glued tree/forest, as the different ordering is just choosing a different representative in the tensor product. There is more freedom in terms of choosing representatives, eg switching the leaves in the input for $p^{k,l}$, and switching as in Figure 3. They all result in choosing different representatives in the same equivalence class of the same glued forests. We can also have part of the forest not interacting with $p^{k,l}$ (although after choosing an order, it can have crossings with the part interacting with $p^{k,l}$), but this will not change the discussion. In other words, since everything is graded and supercommutative, the application of the Koszul–Quillen convention guarantees that everything is well-defined as equivalence classes in the symmetric product.

Writing the forest using a glued graph as in Figure 2 contains slightly more refined information than just labeling the forest as in Figure 1, namely we keep track of which leaves are from $p^{k,l}$ in a representative. From the discussion in Example 2.6, the following observation is tautological.

Proposition 2.7 *The output of a glued forest is well-defined in EV .*

To enumerate all admissible gluings, each output leaf and tree are considered as different. However, we do not differentiate between the input leaves of $p^{k,l}$. Therefore when we glue a $p^{k,l}$ component, we pick k trees — this is $\text{Sh}(k, n-k)$ in (2-5) — from the forest and then one leaf from each chosen tree — that is, $1 \leq i_j \leq n_j$ in (2-4) — to glue to $p^{k,l}$. For example, in the situation of Figure 2, we have $3 * 3 + 3 * 2 + 3 * 2 = 21$ direct ways to glue $p^{2,3}$. The ambiguity from choosing a representative of the input is then eliminated by summing over all possible gluings, by the following tautological observation.

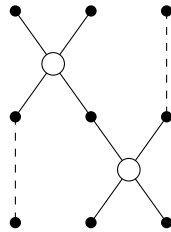
Proposition 2.8 *When summed over all possible gluings of one $p^{k,l}$, the output is independent of the choice of representatives of the input forest.*

Combining the above two propositions, we see that gluing forests corresponds to operations on EV . Indeed, the language of trees and forests is just packaging the signs and components in (2-2), (2-4) and (2-5) by providing geometric intuition. The translation into forests makes it easier to understand algebraic relations; for example, many relations come from interpreting the same glued forests in two different ways. Table 1 is the dictionary of the algebraic formulae in Section 2.2 in terms of forests, where the signs can be compared directly from the rule convention before Example 2.6 and signs in Section 2.2.

$\hat{p}^{k,l}$ on $\odot^k V$	(unique) gluing of $p^{k,l}$ to a forest of k trees of single leaf
$\hat{p}^{k,l}$ in (2-4)	sum of gluing of $p^{k,l}$ to a forest of k trees to get a tree
$\hat{p}^k$	sum of gluing of $p^{k,*}$ to a forest of k trees to get a tree
$\hat{p}$ in (2-5)	sum of gluing of $p^{*,*}$ to a forest to get a forest
$\hat{p}^{k,l}(w_1 \odot \cdots \odot w_k) = 0$ if $w_i \in \mathbf{k}$	no way to glue $p^{k,l}$ to a forest with a tree without leaf to get a tree

Table 1

We use $p_2^{k,l}: S^k V \rightarrow S^l V$ for $k \geq 1$ and $l \geq 0$ to denote the sum of all *connected* graphs with two levels of $\odot$ vertices, k input vertices and l output vertices as follows:

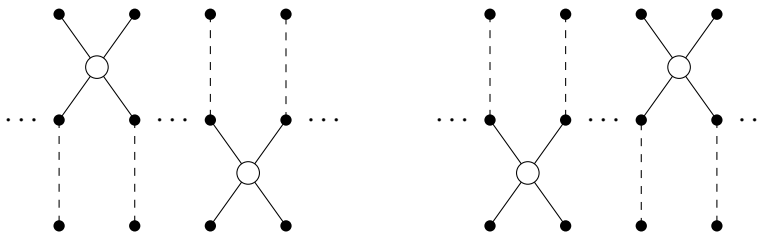


In terms of a formula, we have that $p_2^{k,l} = \pi_{1,l} \circ \hat{p}^2|_{\odot^k V}$, where $\pi_{1,l}$ denotes the projection $EV \rightarrow S^1 S^l V \rightarrow S^l V$. This follows from that elements in $\odot^k V$ are represented by forests consisting of trees of a single leaf, to get a single tree after applying $\hat{p}^2$, we must have the two $p^{*,*}$ components connected to each other directly. Note that, in the applications we have in mind, ie rational SFT, $p_2^{k,l}$ can be viewed as the codimension-one boundary of the rational SFT moduli space. The following proposition shows that the BL_∞ algebra structure captures exactly such combinatorics.

Proposition 2.9 *The set $\{p^{k,l}\}_{k \geq 1, l \geq 0}$ forms a BL_∞ algebra if and only if $p_2^{k,l} = 0$ for $k \geq 1$ and $l \geq 0$.*

Proof Since $p_2^{k,l} = \pi_{1,l} \circ \hat{p}^2|_{\odot^k V}$, if $\{p^{k,l}\}_{k \geq 1, l \geq 0}$ forms a BL_∞ algebra then $p_2^{k,l} = 0$ for $k \geq 1$ and $l \geq 0$. Now assume $p_2^{k,l} = 0$ for $k \geq 1$ and $l \geq 0$. In the glued forests representing $\hat{p}^2$ acting on an element in EV , there are two cases:

- (1) The two $p^{*,*}$ -components are glued to each other directly, those are zero because $p_2^{k,l} = 0$.
- (2) The two $p^{*,*}$ components are not glued to each other (but they could be in the same tree after gluing), by switching the levels of those two $p^{*,*}$ -components, we see that they pair up and cancel with each other as $|p^{*,*}| = 1$:



□

2.4 BL_∞ morphisms

In the following, we define morphisms between BL_∞ algebras. Given a family of operators

$$\{\phi^{k,l}: S^k V \rightarrow S^l V'\}_{k \geq 1, l \geq 0}$$

of degree $0 \in \mathbb{Z}_2$ such that for any $v_1 \cdots v_k \in S^k V$, there are at most finitely many l such that $\phi^{k,l}(v_1 \cdots v_k) \neq 0$. To explain the map $\hat{\phi}: EV \rightarrow EV'$, we will use the description of graphs. To represent $\phi^{k,l}$, we use a graph similar to the one representing $p^{k,l}$ but replace $\bigcirc$ by $\bullet$ to indicate that they are maps of different roles.

Then $\hat{\phi}$ is represented by the sum of all possible gluing of a whole layer or $\phi^{k,l}$ such that no cycles are created and every leaf of the input forest is glued. Unlike the definition of $\hat{p}$, where we need to glue exactly one $p^{k,l}$ graph, it is possible that we do not glue in any $\phi^{k,l}$ graphs. This is the case when the input is in $\odot^m k$, ie the input is represented by trees without leaves. In particular, $\hat{\phi}$ is the identity in such a case, ie $\hat{\phi}(1 \odot \cdots \odot 1) = 1 \odot \cdots \odot 1$. All the rules, like orders, signs, and the well-definedness on EV , are similar to the $\hat{p}$ case.

In terms of formulae, we first define $\hat{\phi}^k: S^k SV \rightarrow SV'$. It is determined by the following:

- (1) $\hat{\phi}^{k+1}(w_1 \odot \cdots \odot w_k \odot 1) = 0$ for $k \geq 1$ and $\hat{\phi}^1(1) = 1$.
- (2) $\hat{\phi}^k: \odot^k V \subset S^k SV \rightarrow SV'$ is defined by $\sum_{l \geq 0} \phi^{k,l}$.
- (3) Let $\{i_j\}_{1 \leq j \leq k}$ be a sequence of positive integers. We define $N := \sum_{j=1}^k i_j$ and $N_i := \sum_{j=1}^i i_j$. Let $w_i = v_{N_{i-1}+1} \cdots v_{N_i}$. The following sum is over all partitions $J_1 \sqcup \cdots \sqcup J_b = \{1, \dots, N\}$ such that the graph with $k + b + N$ vertices $A_1, \dots, A_k, B_1, \dots, B_b, v_1, \dots, v_N$, with A_i connected to $v_{N_{i-1}+1}, \dots, v_{N_i}$ and B_i connected to v_j if and only if $j \in J_i$, is connected and has no cycles:

$$\hat{\phi}^k(w_1 \odot \cdots \odot w_k) = \sum_{\substack{\text{admissible partitions} \\ J_1 \sqcup \cdots \sqcup J_b}} \frac{(-1)^{\bigcirc}}{b!} \sum_{l_1, \dots, l_b=0}^{\infty} \phi^{|J_1|, l_1}(v^{J_1}) * \cdots * \phi^{|J_b|, l_b}(v^{J_b}),$$

where $w_1 \cdots w_k = (-1)^{\bigcirc} v^{J_1} \cdots v^{J_b}$. There is no extra sign as we assume $\phi^{k,l}$ has degree $0 \in \mathbb{Z}_2$. The reduction by $b!$ is because a different order of the partition does not count as a different gluing, but the order will affect the sign when viewing it as elements in the tensor product. The appearance of such a factor is precisely the reason why we want to use tree descriptions, as algebraic formulae (when not phrased in an optimal form) might give us the *wrong* impression that such structure can only be defined over a coefficient ring where $b! \neq 0$. The number of such partitions divided by $b!$ is exactly the number of ways of gluing a layer of $\sum_{l=0}^{\infty} \phi^{*,l}$.

Then we define $\hat{\phi}$ from $\hat{\phi}^k$ just like the L_∞ morphism $\hat{\phi}$ built from ϕ^k . [Table 2](#) is a dictionary between algebraic description and tree description.

$\hat{\phi}^{k+1}(w_1 \odot \cdots \odot w_k \odot 1) = 0$	no way to glue a forest with at least two trees with one of them having no leaves, to a single tree
$\hat{\phi}^1(1) = 1$	trivial/empty gluing to a leafless tree
$\hat{\phi}^k$ on $\odot^k V$	sum of gluings of $\phi^{k,*}$ to a forest of k trees of single leaf, one gluing for each $* \in \mathbb{N}$
$\hat{\phi}^k$	sum of gluing of $\phi^{k,*}$ to a forest of k trees to get a tree
$\hat{\phi}$	sum of gluing of $\phi^{*,*}$ to a forest to get a forest

Table 2

Definition 2.10 The collection $\{\phi^{k,l}\}_{k \geq 1, l \geq 0}$ is a BL_∞ morphism from (V, p) to (V', p') if $\hat{\phi} \circ \hat{p} = \hat{p}' \circ \hat{\phi}$ and $|\hat{\phi}| = 0$.

The composition of $\phi: V \rightarrow V'$ and $\psi: V' \rightarrow V''$ is defined by

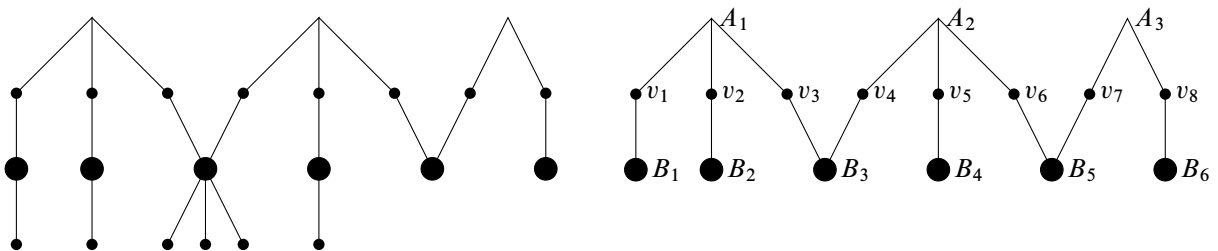
$$(\psi \circ \phi)^{k,l} = \pi_{1,l} \circ \hat{\psi} \circ \hat{\phi}|_{\odot^k V}.$$

The more explicit algebraic description is as follows. Let $I = \{1, \dots, k\}$ and $I_1 \sqcup \cdots \sqcup I_a$ be any partition of I (any partition is admissible as the input is a forest of single-leaf trees), then

$$(\psi \circ \phi)^{k,l}(v_1 \cdots v_k) = \pi_l \left(\sum_{\text{partitions } I_1 \sqcup \cdots \sqcup I_a} \hat{\psi}^a \left(\frac{(-1)^{\circ}}{a!} \sum_{l_1, \dots, l_a=0}^{\infty} \phi^{l_1, l_1}(v^{I_1}) \odot \cdots \odot \phi^{l_a, l_a}(v^{I_a}) \right) \right),$$

where $v_1 \cdots v_k = (-1)^{\circ} v^{I_1} \cdots v^{I_a}$ and π_l is the projection $SV'' \rightarrow S^l V''$. It is clear that the graph representing $\hat{\psi} \circ \hat{\phi}$ has no cycle, and that $(\psi \circ \phi)^{k,l}$ is represented by connected graphs without cycles glued from one level from ϕ and one level from ψ . It is clear from the graph description that $\hat{\psi} \circ \hat{\phi} = \hat{\psi} \circ \hat{\phi}$. It is simply two ways to interpret the same gluing of two layers.

Similar to the definition of $p^{k,2}$, we define $(\phi \circ p)^{k,l} := \pi_{1,l} \circ \hat{\phi} \circ \hat{p}|_{\odot^k V}$, where $\pi_{1,l}$ denotes the projection $EV \rightarrow S^1 SV \rightarrow S^l V$, and we define $(p \circ \phi)^{k,l} := \pi_{1,l} \circ \hat{p} \circ \hat{\phi}|_{\odot^k V}$. Therefore, $(\phi \circ p)^{k,l}$ is the sum of all *connected* graphs with first a level of a single $\bigcirc$ vertex followed by a level of $\bullet$ vertices, similarly for $(p \circ \phi)^{k,l}$. Similar to Proposition 2.9, we have the following result, which allows us to only consider degeneration from connected curves in an analytical setup.


Figure 4: A component of $\hat{\phi}$, left, and an admissible partition, right.

Proposition 2.11 $\{\phi^{k,l}\}_{k \geq 1, l \geq 0}$ forms a BL_∞ morphism from $\{V, \{p^{k,l}\}_{k \geq 1, l \geq 0}\}$ to $\{U, \{q^{k,l}\}_{k \geq 1, l \geq 0}\}$ if and only if $(\phi \circ p)^{k,l} = (q \circ \phi)^{k,l}$ for $k \geq 1, l \geq 0$.

To motivate the proof below, we first work out a simple example by applying $\hat{\phi} \circ \hat{p}$ and $\hat{q} \circ \hat{\phi}$ to $v_1 v_2 \odot u \in S^2 V \odot S^1 V$ for $v_1, v_2, u \in V$.

$$\begin{aligned}
 (2-7) \quad \hat{\phi} \circ \hat{p}(v_1 v_2 \odot u) &= (-1)^{|v_1|} \left(\sum_{l=0}^{\infty} \phi^{1,l}(v_1) \sum_{l=0}^{\infty} (\phi \circ p)^{2,l}(v_2 \odot u) \right) \\
 &\quad + (-1)^{|v_2||u|} \left(\sum_{l=0}^{\infty} (\phi \circ p)^{2,l}(v_1 \odot u) \sum_{l=0}^{\infty} \phi^{1,l}(v_2) \right) \\
 &\quad + \left(\sum_{l=0}^{\infty} (\phi \circ p)^{1,l}(v_1) \sum_{l=0}^{\infty} \phi^{1,l}(v_2) \right) \odot \sum_{l=0}^{\infty} \phi^{1,l}(u) \\
 &\quad + (-1)^{|v_1|} \left(\sum_{l=0}^{\infty} \phi^{1,l}(v_2) \sum_{l=0}^{\infty} (\phi \circ p)^{1,l}(v_2) \right) \odot \sum_{l=0}^{\infty} \phi^{1,l}(u) \\
 &\quad + (-1)^{|v_1|+|v_2|} \left(\sum_{l=0}^{\infty} \phi^{1,l}(v_1) \sum_{l=0}^{\infty} \phi^{1,l}(v_2) \right) \odot \sum_{l=0}^{\infty} (\phi \circ p)^{1,l}(u).
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 (2-8) \quad \hat{q} \circ \hat{\phi}(v_1 v_2 \odot u) &= (-1)^{|v_1|} \left(\sum_{l=0}^{\infty} \phi^{1,l}(v_1) \sum_{l=0}^{\infty} (q \circ \phi)^{2,l}(v_2 \odot u) \right) \\
 &\quad + (-1)^{|v_2||u|} \left(\sum_{l=0}^{\infty} (q \circ \phi)^{2,l}(v_1 \odot u) \sum_{l=0}^{\infty} \phi^{1,l}(v_2) \right) \\
 &\quad + \left(\sum_{l=0}^{\infty} (q \circ \phi)^{1,l}(v_1) \sum_{l=0}^{\infty} \phi^{1,l}(v_2) \right) \odot \sum_{l=0}^{\infty} \phi^{1,l}(u) \\
 &\quad + (-1)^{|v_1|} \left(\sum_{l=0}^{\infty} \phi^{1,l}(v_2) \sum_{l=0}^{\infty} (q \circ \phi)^{1,l}(v_2) \right) \odot \sum_{l=0}^{\infty} \phi^{1,l}(u) \\
 &\quad + (-1)^{|v_1|+|v_2|} \left(\sum_{l=0}^{\infty} \phi^{1,l}(v_1) \sum_{l=0}^{\infty} \phi^{1,l}(v_2) \right) \odot \sum_{l=0}^{\infty} (q \circ \phi)^{1,l}(u).
 \end{aligned}$$

Therefore, we have that $(\phi \circ p)^{k,l} = (q \circ \phi)^{k,l}$ for $k = 1, 2$ and $l \geq 0$ implies that

$$\hat{\phi} \circ \hat{p}(v_1 v_2 \odot u) = \hat{q} \circ \hat{\phi}(v_1 v_2 \odot u).$$

Here we emphasize that $\phi^{1,l}(v_1/v_2/u)$ in (2-7) is represented by a dashed line (identity map) in the upper level glued with some $\phi^{1,l}$ in $\hat{\phi} \circ \hat{p}$, while $\phi^{1,l}(v_1/v_2/u)$ in (2-8) is represented by some $\phi^{1,l}$ glued with a level of dashed lines in $\hat{q} \circ \hat{\phi}$. In terms of forests, they are identified through shifting the isolated $\phi^{1,l}$ components (in the sense of not being connected to the tree representing $(\phi \circ p)^{k,l}$) to the upper level and replacing $(\phi \circ p)^{k,l}$ by $(q \circ \phi)^{k,l}$.

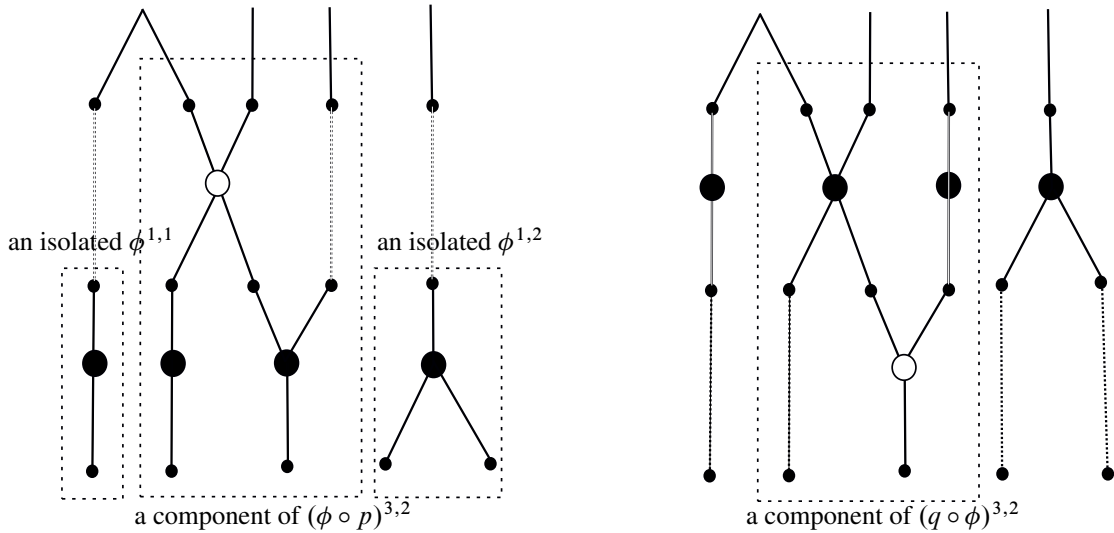


Figure 5: Components of $\hat{\phi} \circ \hat{p}$ and $\hat{q} \circ \hat{\phi}$ and the pairing by shifting up isolated $\phi^{*,*}$ components and replacing the $(\phi \circ p)^{k,l}$ by $(q \circ \phi)^{k,l}$.

Proof If $\{\phi^{k,l}\}_{k \geq 1, l \geq 0}$ forms a BL_∞ morphism, then

$$(\phi \circ p)^{k,l} - (q \circ \phi)^{k,l} = \pi_{1,l} \circ (\hat{\phi} \circ \hat{p} - \hat{q} \circ \hat{\phi})|_{\odot^k V} = 0.$$

Conversely, we assume that $(\phi \circ p)^{k,l} = (q \circ \phi)^{k,l}$. In the glued forests representing $\hat{\phi} \circ \hat{p}$, by looking at the levels containing $p^{*,*}$ and $\phi^{*,*}$, there is exactly one connected graph representing a component in $(\phi \circ p)^{k,l}$, and many $\phi^{*,*}$ components in the second level. Fixing the $(\phi \circ p)^{k,l}$ component, by shifting the remaining isolated $\phi^{*,*}$ components up to the upper level and replacing the $(\phi \circ p)^{k,l}$ component by $(q \circ \phi)^{k,l}$, we find a correspondence between components in $\hat{\phi} \circ \hat{p}$ involving $(\phi \circ p)^{k,l}$ and components in $\hat{q} \circ \hat{\phi}$ involving $(q \circ \phi)^{k,l}$ as in the concrete example before the proof (or Figure 5). They evaluate to the same value as $(\phi \circ p)^{k,l} = (q \circ \phi)^{k,l}$ and there is no extra sign from this shifting (ie changing the order of applying the operations) since $|\phi^{*,*}| = 0$. $\square$

Remark 2.12 There are different notions of homotopies between BL_∞ morphisms if we wish to define notions of homotopy equivalences of BL_∞ algebras. In practice, we cannot associate a canonical BL_∞ algebra to a contact manifold but one depends on various choices and is only well-defined up to homotopy. However, for the purpose of this paper, as we are constructing functors from $\mathfrak{Con}$ to a totally ordered set, homotopy invariance is not needed. The homotopy in SFT is one of most subtle aspects of the theory both algebraically and analytically. Nevertheless, we have the following brief remarks on homotopy.

- (1) One can define a notion of homotopy, which is a homotopy on the bar/cobar complex. That is, one can define a map by counting rigid but *disconnected* curves in a one-parameter family. One advantage of such definition is that it is easier to construct as we will neglect the structures from each connected component. Any homological structure on the level of bar/cobar complex will be

an invariant. For example, the contact homology in [72] used this notion of homotopy. However, homotopic augmentations in this sense do not give rise to homotopic linearized theory.

- (2) Another notion of homotopy is defined through the notion of path objects, eg [21, Definition 4.1], see also [76, Definition 2.9] for the homotopy in the L_∞ context with a specific path object model. This definition is the right one to discuss linearized theory but is more involved to get in the construction of SFT. In particular, homotopic augmentations give rise to homotopic linearized theories with such a notion of homotopy. Such a homotopy is expected to be derived from the homotopy used in [29, Section 2.4]. However, from the curve counting point of view, such a construction is more subtle.

2.5 Augmentations

When $V = \{0\}$, it has a unique trivial BL_∞ algebra structure by $p^{k,l} = 0$. We use $\mathbf{0}$ to denote this trivial BL_∞ algebra. Note that $\mathbf{0}$ is the initial object in the category of BL_∞ algebras, with $\mathbf{0} \rightarrow V$ defined by $\phi^{k,l} = 0$.

Definition 2.13 A BL_∞ augmentation is a BL_∞ morphism $\epsilon: V \rightarrow \mathbf{0}$, ie a family of operators $\epsilon^k: S^k V \rightarrow \mathbf{k}$ satisfying Definition 2.10.

For a BL_∞ algebra V , we define $E^k V = \bar{B}^k S V$, which is a filtration on EV compatible with the differential $\hat{p}$. Note that $E\mathbf{0} = \mathbf{k} \oplus S^2 \mathbf{k} \oplus \cdots$ with $\hat{p} = 0$, and we have $H_*(E\mathbf{0}) = E\mathbf{0}$. Similarly we have $H_*(E^k \mathbf{0}) = E^k \mathbf{0}$ for all $k \geq 1$. We define $1_{\mathbf{0}}$ be the generator in $E^1 \mathbf{0}$, so $1_{\mathbf{0}} \neq 0 \in H_*(E^k \mathbf{0})$ for all $k \geq 1$. Then we define $1_V \in H_*(E^k V)$ to be the image of $1_{\mathbf{0}}$ under the chain map $E^k \mathbf{0} \rightarrow E^k V$ induced by the trivial BL_∞ morphism $\mathbf{0} \rightarrow V$.

Proposition 2.14 If there is a $k \geq 1$ such that $1_V \in H_*(E^k V)$ is zero, then V has no BL_∞ augmentation.

Proof If there is an augmentation $\epsilon: V \rightarrow \mathbf{0}$, then the sequences of BL_∞ morphisms $\mathbf{0} \rightarrow V \xrightarrow{\epsilon} \mathbf{0}$ induce a chain morphism $E^k \mathbf{0} \mapsto E^k V \mapsto E^k \mathbf{0}$. It is direct to check the composition is identity by definition. If $1_V \in H^*(E^k V)$ is zero, then we have a contradiction since $1_{\mathbf{0}} \neq 0 \in H^*(E^k \mathbf{0})$. $\square$

Definition 2.15 We define the torsion of a BL_∞ algebra V to be

$$T(V) := \min\{k - 1 \mid 1_V = 0 \in H^*(E^k V), k \geq 1\}.$$

Here the minimum of an empty set is defined to be ∞ .

By definition, we have that $T(V) = 0$ if and only if $1_V \in H^*(SV, \hat{p}^1)$ is zero. Since $H^*(SV, \hat{p}^1)$ is an algebra with 1_V a unit, we have $H^*(SV, \hat{p}^1) = 0$. In the context of SFT, $T(V) = 0$ if and only if the contact homology vanishes, ie algebraically overtwisted [12].

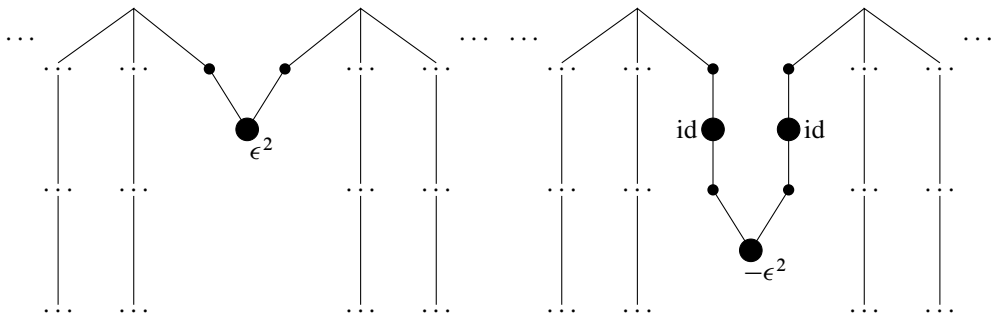
Since a BL_∞ morphism preserves the word filtration on the bar complex, we know that if there is a BL_∞ morphism from V to V' then $T(V) \geq T(V')$. Therefore we have the following obvious property, which is crucial for the invariant property for our applications in symplectic topology.

Proposition 2.16 *If there are BL_∞ morphisms between V and V' in both directions, then we have $T(V) = T(V')$.*

Given a BL_∞ augmentation ϵ , we can linearize with respect to ϵ by the following procedure. More precisely, there is a change of coordinate to kill off all constant terms $p^{k,0}$. We define $F_\epsilon^{1,1} = \mathrm{id}_V$ and $F_\epsilon^{k,0} = \epsilon^k$ and all other $F_\epsilon^{k,l} = 0$. Then following the recipe of constructing $\hat{\phi}$ from $\phi^{k,l}$, we can define $\hat{F}_\epsilon$ on EV . Then $\hat{F}_\epsilon$ preserves the word-length filtration and on the diagonal $\pi_k \circ \hat{F}_\epsilon|_{S^k S V}$ is $\odot^k \hat{F}_\epsilon^1$, where $\hat{F}_\epsilon^1$ is an algebra isomorphism determined by $\hat{F}_\epsilon^1(x) = x + \epsilon^1(x)$ and π_k is the projection $EV \rightarrow S^k S V$. Indeed the inverse is given by the following proposition.

Proposition 2.17 *Let $\hat{F}_{-\epsilon}$ denote the map on EV defined by $F_{-\epsilon}^{1,1} = \mathrm{id}_V$ and $F_{-\epsilon}^{k,0} = -\epsilon^{k,0}$ and all other $F_{-\epsilon}^{k,l} = 0$, then $\hat{F}_{-\epsilon}$ is the inverse of $\hat{F}_\epsilon$.*

Proof In the gluing of forests representing $\hat{F}_{-\epsilon} \circ \hat{F}_\epsilon(x)$, the ϵ and $-\epsilon$ components are not connected directly as they have no outputs. Therefore we can shift up a $-\epsilon$ in the second layer to ϵ , which is still an admissible gluing, or reversely, eg the figure below:



We can group all gluings into those that can be related by these moves. Then there is one group containing the unique gluing containing id only. All other groups have 2^N gluings, where $N > 0$ is the number of $\pm\epsilon$ involved. It is clear that we pair up gluings in a group such that they cancel with each other by a move at a chosen position, eg the figure above. Therefore we have $\hat{F}_{-\epsilon} \circ \hat{F}_\epsilon = \mathrm{id}$, similarly for $\hat{F}_\epsilon \circ \hat{F}_{-\epsilon} = \mathrm{id}$. $\square$

We use $\hat{F}_\epsilon$ as a change of coordinate on EV and consider $\hat{p}_\epsilon := \hat{F}_\epsilon \circ \hat{p} \circ \hat{F}_{-\epsilon}: EV \rightarrow EV$, so $\hat{p}_\epsilon^2 = 0$. We can define

$$p_\epsilon^{k,l} := \pi_{1,l} \circ \hat{F}_\epsilon \circ \hat{p} \circ \hat{F}_{-\epsilon}|_{\odot^k V},$$

where $\pi_{1,l}$ is the projection $EV \rightarrow S^1 S^l V$. We claim that $p_\epsilon^{k,l} = \pi_{1,l} \circ \hat{F}_\epsilon \circ \hat{p}|_{\odot^k V}$. To see this, note that $\hat{F}_{-\epsilon}$ restricted on $\odot^k V$ is the identity map plus extra terms landing in $(\odot^m V) \odot (\odot^l k)$ for $l + m \leq k$ and $l > 0$. When we apply $\hat{p}$ on the extra term, we must have that the output is in $\mathrm{im} \hat{p}|_{\odot^m V} \odot (\odot^l k)$.

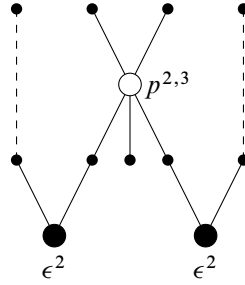


Figure 6: A component of $p_\epsilon^{4,1}$.

The image after applying $\pi_{1,l} \circ \hat{F}_\epsilon$ must be zero since it contains at least two $\odot$ components. In other words, those extra terms represent disconnected graphs in the description of $\hat{F}_\epsilon \circ \hat{p} \circ \hat{F}_{-\epsilon}|_{\odot^k V}$, hence is projected to zero by $\pi_{1,l}$. Then $p_\epsilon^{k,l}$ is represented by the sum of connected graphs without cycle with k inputs, l outputs, one $\bigcirc$ component and possibly several components from ϵ ; see Figure 6.

Proposition 2.18 *The map $\hat{p}_\epsilon$ is determined by $p_\epsilon^{k,l}$ following the same recipe as for $\hat{p}$ from $p^{k,l}$. Moreover, we have $p_\epsilon^{k,0} = 0$ for all k .*

Proof In the graph representation of $\hat{F}_\epsilon \circ \hat{p} \circ \hat{F}_{-\epsilon}$ on $S^{i_1} V \odot \cdots \odot S^{i_k} V$, there is exactly one component containing a $p_\epsilon^{k,l}$ as a subgraph. All the other $\pm\epsilon$ components (ie those not in $p_\epsilon^{k,l}$) are not connected to the $p^{*,*}$ component directly (they could be in the same tree), we call them isolated $\pm\epsilon$ components. Similar to the proof of Proposition 2.17, we get collections of gluings that can be related by moving up/down the isolated $\pm\epsilon$ components. Hence those with isolated $\pm$ components in $\hat{F}_\epsilon \circ \hat{p} \circ \hat{F}_{-\epsilon}$ sum to zero, that is, $\hat{F}_\epsilon \circ \hat{p} \circ \hat{F}_{-\epsilon} = \hat{p}_\epsilon$ is determined by gluing only $p_\epsilon^{k,l}$.

Finally, because ϵ is an augmentation, we have $\pi^{1,0} \circ \hat{F}_\epsilon \circ \hat{p} = \hat{\epsilon} \circ \hat{p} = 0$. Therefore $p_\epsilon^{k,0} = \pi^{1,0} \circ \hat{F}_\epsilon \circ \hat{p}|_{\odot^k V}$ for all k . $\square$

As a corollary of Proposition 2.18, $\ell_\epsilon^k := p_\epsilon^{k,1}$ defines an L_∞ structure on $V[-1]$. Next we introduce the structure which will be relevant to the definition of planarity. Let $p_\bullet^{k,l}: S^k V \rightarrow S^l V$ for $k \geq 1$ and $l \geq 0$ be a family of linear maps. We can define $\hat{p}_\bullet^{k,l}$ and $\hat{p}_\bullet$ just like $\hat{p}^{k,l}$ and $\hat{p}$ (see (2-2)), ie by componentwise Leibniz rule and co-Leibniz rule with the modification that $|p_\bullet^{k,l}|$ is not necessarily $1 \in \mathbb{Z}_2$.

Definition 2.19 We say that $\{p_\bullet^{k,l}\}$ is a pointed map if $\hat{p}_\bullet \circ \hat{p} = (-1)^{|\hat{p}_\bullet|} \hat{p} \circ \hat{p}_\bullet$.

Proposition 2.20 *The map $\{p_\bullet^{k,l}\}$ is a pointed map if and only if*

$$\pi_{1,l} \circ (\hat{p}_\bullet \circ \hat{p} - (-1)^{|\hat{p}_\bullet|} \hat{p} \circ \hat{p}_\bullet)|_{\odot^k V} = 0 \quad \text{for all } k \geq 1 \text{ and } l \geq 0.$$

Proof The proof is identical to the proof of Proposition 2.9; the extra sign is consistent with the fact that $|p^{*,*}| = 1$ in view of the pairing from switching levels. $\square$

The relation $\pi_{1,l} \circ (\hat{p}_\bullet \circ \hat{p} - (-1)^{|\hat{p}_\bullet|} \hat{p} \circ \hat{p}_\bullet)|_{\odot^k V} = 0$ only involves degeneration from connected curves in an analytical setup. In applications, $p_\bullet^{k,l}$ will come from counting holomorphic curves with one interior marked point subject to a constraint from $H_*(Y)$. The degree of $p_\bullet$ is the same as the degree of the constraint. Typically we will only consider a point constraint, then the degree is 0. Note that it does not define BL_∞ morphisms as the combinatorics for packaging $\hat{p}_\bullet$ is different from $\hat{\phi}$. Nevertheless, $\hat{p}_\bullet$ still defines a morphism on the bar complex and preserves the word-length filtration.

Then by the same argument in Proposition 2.18, we can define $\hat{p}_{\bullet,\epsilon} := \hat{F}_\epsilon \circ \hat{p}_\bullet \circ \hat{F}_{-\epsilon}$ and $\hat{p}_{\bullet,\epsilon}$ is determined by $p_{\bullet,\epsilon}^{k,l}$, which is defined similarly to $p_\epsilon^{k,l}$. Note that we also have $\hat{p}_\epsilon \circ \hat{p}_{\bullet,\epsilon} = \hat{p}_{\bullet,\epsilon} \circ \hat{p}_\epsilon$. However, it is not necessarily true that $p_{\bullet,\epsilon}^{k,0} = 0$. In fact, the failure of this property on the homological level will be another hierarchy that we are interested in. We define $\ell_{\bullet,\epsilon}^{k,0}$ by $p_{\bullet,\epsilon}^{k,0}$. Then $\hat{\ell}_{\bullet,\epsilon} := \sum_{k \geq 0} \ell_{\bullet,\epsilon}^{k,0}$ defines a chain morphism $(\bar{S}V, \hat{\ell}_\epsilon) \rightarrow k$. That $\hat{\ell}_{\bullet,\epsilon} \circ \hat{\ell}_\epsilon = 0$ follows from $0 = \pi_k \circ \hat{p}_\epsilon \circ \hat{p}_{\bullet,\epsilon} = \pi_k \circ \hat{p}_{\bullet,\epsilon} \circ \hat{p}_\epsilon$ restricted to $\bar{S}V = \bar{S}S^1V \subset EV$, and π_k is the projection from EV to $k \subset S^1SV \subset EV$.

Definition 2.21 Given a BL_∞ augmentation and a pointed map $p_\bullet$, the $(\epsilon, p_\bullet)$ order of V is defined to be

$$O(V, \epsilon, p_\bullet) := \min\{k \mid 1 \in \text{im } \hat{\ell}_{\bullet,\epsilon}|_{H_*(\bar{B}^k V, \hat{\ell}_\epsilon)}\},$$

where the minimum of an empty set is defined to be ∞ .

Next, we need to compare the construction under BL_∞ morphisms. Given a BL_∞ morphism $\phi: (V, p) \rightarrow (V', q)$ and a family of morphisms $\phi_\bullet^{k,l}: S^k V \rightarrow S^l V'$, we can define $\hat{\phi}_\bullet: EV \rightarrow EV'$ by the same rule of $\hat{\phi}$ with exactly one $\phi_\bullet^{k,l}$ component and all the others $\phi^{k,l}$ components. Since we must have one $\phi_\bullet^{k,l}$ component, we have $\hat{\phi}_\bullet(1 \odot \cdots \odot 1) = 0$.

In terms of formulae, we define $\hat{\phi}_\bullet^k: S^k SV \rightarrow S^k V'$, which corresponds to the tree component of $\hat{\phi}$ containing $\phi_\bullet^{*,*}$. It is determined by the following.

- (1) $\hat{\phi}_\bullet^k(w_1 \odot \cdots \odot w_{k-1} \odot 1) = 0$ for $k \geq 0$.
- (2) $\hat{\phi}_\bullet^k: \odot^k V \subset S^k SV \rightarrow S^k V'$ is defined by $\sum_{l \geq 0} \phi_\bullet^{k,l}$.
- (3) Let $\{i_j\}_{1 \leq j \leq k}$ be a sequence of positive integers. We define $N := \sum_{j=1}^k i_j$ and $N_i := \sum_{j=1}^i i_j$. Let $w_i = v_{N_{i-1}+1} \cdots v_{N_i}$. The following sum is over all partitions $J_1 \sqcup \cdots \sqcup J_b = \{1, \dots, N\}$ such that the graph with $k + b + N$ vertices $A_1, \dots, A_k, B_1, \dots, B_b, v_1, \dots, v_N$ with A_i connected to $v_{N_{i-1}+1}, \dots, v_{N_i}$ and B_i connected to v_j if and only if $j \in J_i$, is connected and has no cycles:

$$\hat{\phi}_\bullet^k(w_1 \odot \cdots \odot w_k) = \sum_{\substack{\text{admissible partitions} \\ J_1 \sqcup \cdots \sqcup J_b}} \frac{(-1)^{\odot}}{(b-1)!} \sum_{l_1, \dots, l_b=0}^{\infty} \phi_\bullet^{|J_1|, l_1}(v^{J_1}) * \phi^{|J_2|, l_2}(v^{J_2}) * \cdots * \phi^{|J_b|, l_b}(v^{J_b}),$$

where $w_1 \cdots w_k = (-1)^{\odot} v^{J_1} \cdots v^{J_b}$. There is no extra sign as we assume $\phi_\bullet^{k,l}$ has degree $0 \in \mathbb{Z}_2$. The reduction by $(b-1)!$ is because a different order of the partition for the $\phi_\bullet^{*,*}$ components does not count as a different gluing. As a consequence, $\hat{\phi}_\bullet^k$ is the sum of glued trees from a forest of k trees glued with one $\phi_\bullet^{*,*}$ component and multiple $\phi_\bullet^{*,*}$ components.

We define $\hat{\phi}_\bullet: EV \rightarrow EV'$ similarly to $\hat{\phi}$, but with exactly one $\hat{\phi}_\bullet^*$ term. In terms of formulae, we have

$$\hat{\phi}_\bullet(w_1 \odot \cdots \odot w_n) = \sum_{\substack{k \geq 1 \\ i_1 + \cdots + i_k = n}} \frac{1}{(k-1)!} (\hat{\phi}_\bullet^{i_1} \hat{\phi}^{i_2} \cdots \hat{\phi}^{i_k})(w_1 \odot \cdots \odot w_n),$$

where $(\hat{\phi}_\bullet^{i_1} \hat{\phi}^{i_2} \cdots \hat{\phi}^{i_k})$ is defined after [Definition 2.2](#), with sV replaced by SV here.

Definition 2.22 Assume that $p_\bullet$ and $q_\bullet$ are two pointed maps of (V, p) and (V', q) , respectively, of the same degree, and ϕ is a BL_∞ morphism from (V, p) to (V', q) . We say that $p_\bullet$, $q_\bullet$ and ϕ are compatible if there is a family of $\phi_\bullet^{k,l}$ such that

$$\hat{q}_\bullet \circ \hat{\phi} - (-1)^{|\hat{q}_\bullet|} \hat{\phi} \circ \hat{p}_\bullet = \hat{q} \circ \hat{\phi}_\bullet - (-1)^{|\hat{\phi}_\bullet|} \hat{\phi}_\bullet \circ \hat{p} \quad \text{and} \quad |\hat{\phi}_\bullet| = |\hat{p}_\bullet| + 1.$$

Proposition 2.23 The maps $p_\bullet$, $q_\bullet$ and ϕ are compatible using $\phi_\bullet$ if and only if

$$\pi_{1,l}(\hat{q}_\bullet \circ \hat{\phi} - (-1)^{|\hat{q}_\bullet|} \hat{\phi} \circ \hat{p}_\bullet - \hat{q} \circ \hat{\phi}_\bullet + (-1)^{|\hat{\phi}_\bullet|} \hat{\phi}_\bullet \circ \hat{p})|_{\odot^k V} = 0$$

for all $k \geq 1$ and $l \geq 0$.

Proof The proof is similar to the proof of [Proposition 2.11](#) with one difference: in the middle two levels presenting

$$\hat{q}_\bullet \circ \hat{\phi} - (-1)^{|\hat{q}_\bullet|} \hat{\phi} \circ \hat{p}_\bullet - \hat{q} \circ \hat{\phi}_\bullet + (-1)^{|\hat{\phi}_\bullet|} \hat{\phi}_\bullet \circ \hat{p},$$

the $p^{*,*}$ and $q^{*,*}$ components may not glue to the $\phi_\bullet^{*,*}$ components. However, those pieces correspond to $(\phi \circ p)^{k,l}$ and $(q \circ \phi)^{k,l}$ along with one $\phi_\bullet^{*,*}$ and multiple $\phi^{*,*}$ components. Since $(\phi \circ p)^{k,l} = (q \circ \phi)^{k,l}$ from [Proposition 2.11](#), those extra terms sum to zero as $|p^{*,*}| = 1$. $\square$

In practice, $\phi_\bullet^{k,l}$ is defined by counting connected rational holomorphic curves in the cobordism with a marked point passing through a cobordism between the constraints in the definition of $p_\bullet$ and $q_\bullet$. In our typical case of point constraint, the cobordism will be a path connecting the point constraints, where we have $|\hat{\phi}_\bullet| = 1$. In principle, we can consider the category consisting of pairs $(p, p_\bullet)$ with morphisms given by pairs $(\phi, \phi_\bullet)$ with a suitable definition of composition. Then the definition of orders is functorial. For our purpose, we only need the following property without the precise definition of a composition.⁷

Proposition 2.24 Assume that $p_\bullet$ and $q_\bullet$ are two pointed maps of degree 0 of (V, p) and (V', q) , respectively, and that ϕ is a BL_∞ -morphism from (V, p) to (V', q) such that $p_\bullet$, $q_\bullet$ and ϕ are compatible. Then for any BL_∞ augmentation ϵ of V' , we have $O(V, \epsilon \circ \phi, p_\bullet) \geq O(V', \epsilon, q_\bullet)$.

Proof By the definition of compatibility, we compose $\hat{F}_\epsilon$ in the front and $\hat{F}_{-\epsilon \circ \phi}$ in the bottom to have

$$(2-9) \quad \hat{q}_{\bullet, \epsilon} \circ \hat{F}_\epsilon \circ \hat{\phi} \circ \hat{F}_{-\epsilon \circ \phi} - \hat{F}_\epsilon \circ \hat{\phi} \circ \hat{F}_{-\epsilon \circ \phi} \circ \hat{p}_{\bullet, \epsilon \circ \phi} = \hat{F}_\epsilon \circ \hat{\phi}_\bullet \circ \hat{F}_{-\epsilon \circ \phi} \circ \hat{p}_{\epsilon \circ \phi} + \hat{q}_\epsilon \circ \hat{F}_\epsilon \circ \hat{\phi}_\bullet \circ \hat{F}_{-\epsilon \circ \phi}.$$

⁷It is easy to spell out the composition using the graph description.

We have a diagram (not commutative)

$$\begin{array}{ccc} \bar{S}V & \xrightarrow{\hat{\ell}_{\bullet, \epsilon \circ \phi}} & \mathbf{k} \\ \downarrow \hat{\phi}_{\epsilon}^1 & & \downarrow \text{id} \\ \bar{S}V' & \xrightarrow{\hat{\ell}_{\bullet, \epsilon}} & \mathbf{k} \end{array}$$

where $\hat{\phi}_{\epsilon}^1$ is determined by the L_{∞} morphism $\phi_{\epsilon}^{k,1}$ since $\phi_{\epsilon}^{k,0} = 0$. In other words,

$$\hat{\phi}_{\epsilon}^1|_{\odot^k V} = \sum_{l \geq 1} \pi_{l,1} \circ \hat{\phi}_{\epsilon}|_{\odot^k V},$$

where $\pi_{l,1}$ is the projection from EV to $\odot^l V$. We claim that the diagram is commutative up to homotopy $\hat{\phi}_{\bullet, \epsilon}^0: \bar{S}V \rightarrow \mathbf{k}$ defined by $\sum_{k \geq 1} \phi_{\bullet, \epsilon}^{k,0}$, where $\phi_{\bullet, \epsilon}^{k,0} = \pi_{\mathbf{k}} \circ \hat{F}_{\epsilon} \circ \hat{\phi}_{\bullet} \circ \hat{F}_{-\epsilon \circ \phi}|_{\odot^k V}$. Indeed, this homotopy relation is exactly (2-9) restricted to $\bar{S}V = \bigoplus_{k=1}^{\infty} \odot^k V$ and then projected to $\mathbf{k}$. It is clear that $\hat{\phi}_{\epsilon}^1, \hat{\phi}_{\bullet, \epsilon}^0$ preserve the length filtration, therefore $1 \in \text{im } \hat{\ell}_{\bullet, \epsilon \circ \phi}|_{H_*(\bar{B}^k V, \hat{\ell}_{\epsilon \circ \phi})}$ implies that $1 \in \text{im } \hat{\ell}_{\bullet, \epsilon}|_{H_*(\bar{B}^k V', \hat{\ell}_{\epsilon})}$. Hence $O(V, \epsilon \circ \phi, p_{\bullet}) \geq O(V', \epsilon, q_{\bullet})$. $\square$

Remark 2.25 There are various generalizations of $O(V, \epsilon, p_{\bullet})$; some of the associated spectral invariants are Siegel's higher symplectic capacities with multiple point constraints [76]. See [67, Section 5.1] for details of the construction and relation.

3 Rational symplectic field theory

In this section, we explain the construction of rational symplectic field theory (RSFT) as BL_{∞} algebras. RSFT was originally packaged into a Poisson algebra with a distinguished odd-degree class $\mathbf{h}$ such that $\{\mathbf{h}, \mathbf{h}\} = 0$ in [29, Section 2.1]. However for the purpose of building hierarchy functors from contact manifolds, it is useful to reformulate RSFT as BL_{∞} algebras. It is important to note that we will use the same moduli spaces of holomorphic curves as the original RSFT but reinterpret the relations as other algebraic structures.

3.1 Notations on symplectic topology

We first briefly recall the basics of symplectic and contact topology. A (co-oriented) contact manifold (Y, ξ) is a $(2n-1)$ -dimensional manifold with a (co-oriented) hyperplane distribution ξ such that there is a one-form α with $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^{n-1} \neq 0$ and α induces the co-orientation on ξ . Such a one-form α is called a contact form and we will call (Y, α) a strict contact manifold. The manifold Y will always be assumed to be closed. Given a contact form α , the Reeb vector field R_{α} is characterized by $\alpha(R_{\alpha}) = 1$ and $\iota_{R_{\alpha}} d\alpha = 0$. We say a contact form α is nondegenerate if and only if all Reeb orbits are nondegenerate. Any contact form can be perturbed into a nondegenerate contact form and in particular, every contact manifold admits nondegenerate contact forms. Throughout this paper, (Y, α) is always assumed to be a strict contact manifold with a nondegenerate contact form unless specified otherwise.

Definition 3.1 A compact symplectic manifold (X, ω) with $\partial W = Y_- \sqcup Y_+$ is

- (1) a strong cobordism from (Y_-, ξ_-) to (Y_+, ξ_+) if and only if $\omega = d\lambda_{\pm}$ near $Y_{\pm}$ with $\xi_{\pm} = \ker \lambda_{\pm}$ such that, if we define $V_{\pm}$ by $\iota_{V_{\pm}} \omega = \lambda_{\pm}$, then V_+ points outwards along Y_+ and V_- points inwards along Y_- ;
- (2) an exact cobordism from (Y_-, ξ_-) to (Y_+, ξ_+) if, moreover, $\omega = d\lambda$ on X (the vector field V defined by $\iota_V \omega = \lambda$ is called the Liouville vector field);
- (3) a Weinstein cobordism from (Y_-, ξ_-) to (Y_+, ξ_+) if, moreover, the Liouville vector field is gradient-like for some Morse function ϕ with $Y_{\pm}$ as the regular level sets of maximum/minimum value for ϕ .

We say a cobordism (X, ω) from (Y_-, α_-) to (Y_+, α_+) is strict if and only if $\lambda_{\pm}|_{Y_{\pm}} = \alpha_{\pm}$. It is clear from definition that we can glue strict cobordisms to get a strict cobordism. In general, given two exact cobordisms W_1 and W_2 from Y_1 and Y_2 to Y_2 and Y_3 , respectively, the composition $W_2 \circ W_1$ from Y_1 to Y_3 is not uniquely defined, but up to homotopies of Liouville structures [20, Section 11.2], it is well-defined. The central geometric object of our interests is the following cobordism category.

Definition 3.2 The exact cobordism category of contact manifolds $\mathfrak{Con}$ is defined to be the category whose objects are contact manifolds and morphisms are exact cobordisms up to homotopy. The composition is given by gluing cobordisms. We will use $\mathfrak{Con}^{2k-1}$ to denote the subcategory of $(2k-1)$ -dimensional contact manifolds. Similarly, we use $\mathfrak{Con}_W$ to denote the Weinstein cobordism category and $\mathfrak{Con}_S$ to denote the strong cobordism category.

All of the categories above have monoidal structures given by the disjoint union. It is clear that we have natural functors $\mathfrak{Con}_W \rightarrow \mathfrak{Con} \rightarrow \mathfrak{Con}_S$, which are identities on the object level.

Remark 3.3 There is a forgetful functor from $\mathfrak{Con}$ to the cobordism category of almost contact manifolds, where the cobordisms are almost symplectic cobordisms. In the case of $\mathfrak{Con}_W$, there is a forgetful map to the almost Weinstein cobordism category of almost contact manifolds. These are purely topological objects; the latter was studied thoroughly in [17; 18].

Roughly speaking, the principle in the symplectic cobordism category is that the complexity of contact geometry increases in the direction of the cobordism. In view of this, we can introduce the following category, which only remembers if there exists a cobordism.

Definition 3.4 We define $\mathfrak{Con}_{\leq}$ to be the category of contact manifolds, such that there is at most one arrow between two contact manifolds and the arrow exists if and only if there is an exact cobordism. Similarly, we can define $\mathfrak{Con}_{\leq, W}$ and $\mathfrak{Con}_{\leq, S}$.

It is a natural question to ask whether $\mathcal{C}\text{on}_{\leq}$ is a poset. It is clear that we only need to prove that $Y_1 \leq Y_2$ and $Y_2 \leq Y_1$ implies that $Y_1 = Y_2$. Unfortunately, this is not the case, as we may take Y_1 and Y_2 to be two different three-dimensional overtwisted contact manifolds [32] or suitable flexibly fillable contact manifolds. One extreme case is that Y_1 can be different from Y_2 (as *smooth* manifolds) even if the cobordisms are inverse to each other [25]. However, we can mod out this ambiguity to get a poset. It is clear that the existence of (exact) cobordisms between Y_1 and Y_2 in both directions defines an equivalence relation. We denote by $[Y]$ the corresponding equivalence class of the contact manifold Y .

Definition 3.5 We define $\overline{\mathcal{C}\text{on}}_{\leq}$ to be the poset whose objects are the equivalence classes of contact manifolds with respect to the above equivalence relation, and there is a morphism $[Y_1] \leq [Y_2]$ if and only if there is an exact cobordism from Y_1 to Y_2 . Similarly, we can define the posets $\overline{\mathcal{C}\text{on}}_{\leq, W}$ and $\overline{\mathcal{C}\text{on}}_{\leq, S}$.

Under this condition, all three-dimensional overtwisted contact manifolds become the same least object in $\overline{\mathcal{C}\text{on}}_{\leq}$; see [32]. In higher dimensions, overtwisted contact manifolds are least objects up to topological constraints [30]. It is clear that we have functors $\mathcal{C}\text{on} \rightarrow \mathcal{C}\text{on}_{\leq} \rightarrow \overline{\mathcal{C}\text{on}}_{\leq}$. The theme of this paper is to construct functors from $\mathcal{C}\text{on}$ to some totally ordered set. Since it always descends to $\overline{\mathcal{C}\text{on}}_{\leq}$, results in this paper can be understood as some structures on the poset $\overline{\mathcal{C}\text{on}}_{\leq}$. It is also a natural question as to whether $\overline{\mathcal{C}\text{on}}_{\leq}$ is a totally ordered set,⁸ which is addressed in the negative [39] using exact orbifolds.

An exact (Weinstein, strong) cobordism from $\emptyset$ to Y is called an exact (Weinstein, strong) filling of Y . We also introduce a category $\mathcal{C}\text{on}_*$ as the under category under the empty set.

Definition 3.6 The objects of $\mathcal{C}\text{on}_*$ are pairs (Y, W) , where W is an exact filling of Y up to homotopy. A morphism from (Y_1, W_1) to (Y_2, W_2) is an exact cobordism X from Y_1 to Y_2 such that $X \circ W_1 = W_2$ up to homotopy, or equivalently an exact embedding of W_1 into W_2 up to homotopy.

Example 3.7 The symplectic cohomology is a functor from $\mathcal{C}\text{on}_*$ to the category of BV algebras, where the functoriality follows from the Viterbo transfer map. The S^1 -equivariant symplectic cochain complex is also a functor $\mathcal{C}\text{on}_*$ to the homotopy category of S^1 -cochain complexes. The order of dilation and the order of semidilation in [90, Corollary D] are functors from $\mathcal{C}\text{on}_*$ to $\mathbb{N} \cup \{\infty\}$.

3.2 Geometric setups for holomorphic curves

As usual, an almost complex structure J on the symplectization $(\mathbb{R}_s \times Y, d(e^s \alpha))$ is said to be admissible if and only if

- (1) J is invariant under s -translation and restricts to a tame almost complex structure on $(\xi = \ker \alpha, d\alpha)$,
- (2) J sends ∂_s to the Reeb vector R_α .

⁸For the total order, we will not consider $\emptyset$ as an object in $\overline{\mathcal{C}\text{on}}_{\leq}$, as overtwisted contact manifolds and $\emptyset$ are not comparable in $\overline{\mathcal{C}\text{on}}_{\leq}$, by well-known results.

Let (W, λ) be an exact filling and (X, λ) an exact cobordism. An almost complex structure J on completions $(\widehat{W}, \widehat{\lambda})$ or $(\widehat{X}, \widehat{\lambda})$ is admissible if and only if

- (1) J is tame for $d\widehat{\lambda}$,
- (2) J is admissible on cylindrical ends.

Occasionally, we will also consider strong fillings (W, ω) , where the definition of admissible almost complex structure on $\widehat{W}$ is similar. For each Reeb orbit γ , we can fix a basepoint b_γ on the image. Now fix an admissible J , and consider two collections of Reeb orbits $\gamma_1^+, \dots, \gamma_{s^+}^+$ and $\gamma_1^-, \dots, \gamma_{s^-}^-$, possibly with duplicates. A pseudoholomorphic map in the symplectization $\mathbb{R} \times Y$ or completions $\widehat{W}, \widehat{X}$ with positive asymptotics $\gamma_1^+, \dots, \gamma_{s^+}^+$ and negative asymptotics $\gamma_1^-, \dots, \gamma_{s^-}^-$ consists of:

- (1) a sphere Σ , with a complex structure denoted by j ,
- (2) a collection of pairwise distinct points $z_1^+, \dots, z_{s^+}^+, z_1^-, \dots, z_{s^-}^- \in \Sigma$, each equipped with an asymptotic marker, ie a direction in the tangent sphere bundle $S_{z_i^\pm} \Sigma$,
- (3) a map $\dot{\Sigma} \rightarrow \mathbb{R} \times Y, \widehat{W}, \widehat{X}$ satisfying $du \circ j = J \circ du$, where $\dot{\Sigma}$ denotes the punctured Riemann surface $\Sigma \setminus \{z_1^+, \dots, z_{s^+}^+, z_1^-, \dots, z_{s^-}^-\}$,
- (4) for each z_i^+ with corresponding polar coordinates (r, θ) around z_i^+ such that the asymptotic marker corresponds to $\theta = 0$, we have

$$(3-1) \quad \lim_{r \rightarrow 0} (\pi_{\mathbb{R}} \circ u)(re^{i\theta}) = +\infty \quad \text{and} \quad u_{z_i^+}(\theta) := \lim_{r \rightarrow 0} (\pi_Y \circ u)(re^{i\theta}) = \gamma_i^+ \left(-\frac{1}{2\pi} T_i^+ \theta \right),$$

where T_i^+ is the period of the parametrized orbit γ_i^+ and $\gamma_i^+(0) = b_{\gamma_i^+}$,

- (5) for each z_i^- with corresponding polar coordinates (r, θ) compatible with asymptotic marker, we have

$$(3-2) \quad \lim_{r \rightarrow 0} (\pi_{\mathbb{R}} \circ u)(re^{i\theta}) = -\infty \quad \text{and} \quad u_{z_i^-}(\theta) := \lim_{r \rightarrow 0} (\pi_Y \circ u)(re^{i\theta}) = \gamma_i^- \left(\frac{1}{2\pi} T_i^- \theta \right),$$

where T_i^- is the period of the parametrized orbit γ_i^- and $\gamma_i^-(0) = b_{\gamma_i^-}$.

A holomorphic curve is an equivalence of holomorphic maps modulo biholomorphisms of Σ commuting with all the data. Throughout this paper, we will work with $\mathbb{Z}_2$ -grading unless specified otherwise. Let $\Gamma^+ = \{\gamma_1^+, \dots, \gamma_{s^+}^+\}$ and $\Gamma^- = \{\gamma_1^-, \dots, \gamma_{s^-}^-\}$ be two ordered sets of Reeb orbits possibly with duplicates. Choosing trivializations of ξ over orbits in Γ^+ and Γ^- , we can assign the Conley–Zehnder index $\mu_{CZ}(\gamma_i^\pm)$ to each orbit. With such a trivialization, we have a relative first Chern class

$$c_1: H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \rightarrow \mathbb{Z},$$

and similarly for W and X . Let A be a relative homology class representing the curve u . Then the Fredholm index of the Cauchy Riemann operator at u minus the dimension of the automorphism group

(ie biholomorphisms of Σ commuting with all the data) is

$$\text{ind}(u) = (n-3)(2-s^+-s^-) + \sum_{i=1}^{s^+} \mu_{\text{CZ}}(\gamma_i^+) - \sum_{i=1}^{s^-} \mu_{\text{CZ}}(\gamma_i^-) + 2c_1(A).$$

In this paper, we will consider the following moduli spaces.

- (1) $\mathcal{M}_{Y,A}(\Gamma^+, \Gamma^-)$ is the moduli space of rational holomorphic curves in the symplectization, modulo automorphism and the $\mathbb{R}$ translation. The expected dimension is $\text{ind}(u) - 1$.
- (2) $\mathcal{M}_{W,A}(\Gamma^+, \emptyset)$ (resp. $\mathcal{M}_{X,A}(\Gamma^+, \Gamma^-)$) are the moduli spaces of rational holomorphic curves in the filling (resp. cobordism), modulo automorphism. The expected dimension is $\text{ind}(u)$.
- (3) $\mathcal{M}_{Y,A,o}(\Gamma^+, \Gamma^-)$ is the moduli space of rational holomorphic curves with one interior marked point in the symplectization modulo automorphisms. Here the marked point is required to be mapped to $(0, o) \in \mathbb{R} \times Y$ for a point $o \in Y$. The expected dimension is $\text{ind}(u) + 2 - 2n$.
- (4) $\mathcal{M}_{W,A,o}(\Gamma^+, \emptyset)$ is the moduli space of rational holomorphic curves with one interior marked point in the filling modulo automorphism. And the marked point is required to be mapped to $o \in W$. The expected dimension is $\text{ind}(u) + 2 - 2n$.
- (5) $\mathcal{M}_{X,A,\gamma}(\Gamma^+, \Gamma^-)$ is the moduli space of rational curves with one interior marked point in the exact cobordism X modulo automorphism. The marked point is required to go through a path $\hat{\gamma}$, which is the completion of a path γ from a point in Y_+ to a point in Y_- , ie extension by constant maps in each slice in the cylindrical ends. The expected dimension is $\text{ind}(u) + 3 - 2n$.

Another fact that is important for our later proof is that

$$(3-3) \quad \int_{\Gamma^+} \alpha - \int_{\Gamma^-} \alpha \geq 0$$

whenever $\mathcal{M}_{Y,A}(\Gamma^+, \Gamma^-)$ or $\mathcal{M}_{Y,A,o}(\Gamma^+, \Gamma^-)$ are not empty. All of the moduli spaces above have a SFT building compactification by [9], denoted by $\overline{\mathcal{M}}$. The orientation convention follows [11], and we need to require that all asymptotic Reeb orbits are good [84, Definition 11.6] to orient $\overline{\mathcal{M}}$. One property of this convention is that if we switch two orbits γ_1, γ_2 that are next to each other in Γ^+ or Γ^- , the induced orientation is changed by $(-1)^{|\mu_{\text{CZ}}(\gamma_1)+n-3| \cdot |\mu_{\text{CZ}}(\gamma_2)+n-3|}$; see [84, Section 11.2]. In the following, we will count zero-dimensional moduli spaces to define coefficients in the structural maps. First of all, this requires an orientation, hence we can only count when all asymptotic Reeb orbits are good.⁹ Next we need transversality, where the count is an honest count of orbifold points, or a virtual machinery, where the count is a count of weighted orbifold points in perturbed moduli spaces [33; 46] or an algebraic count after fixing some auxiliary data [72]. For simplicity, we will just use $\#\overline{\mathcal{M}}$ to denote the count.

⁹Alternatively, the count is evaluated in the fixed space of an orientation line with a group action, the appearance of a bad orbit is exactly when the group action is not trivial; see [72, Remark 4.15] for details.

3.3 Contact homology algebra

We will first recall the definition of the contact homology algebra. Let V_α denote the free $\mathbb{Q}$ -module generated by formal variables q_γ for each good orbit γ of (Y, α) . We grade q_γ by $\mu_{\text{CZ}}(\gamma) + n - 3$, which should be understood as a well-defined $\mathbb{Z}_2$ grading in general. The contact homology algebra $\text{CHA}(Y)$ is the free symmetric algebra SV_α . The differential is defined as

$$(3-4) \quad \partial_l(q_\gamma) = \sum_{||[\Gamma]||=l} \# \overline{\mathcal{M}}_{Y,A}(\{\gamma\}, \Gamma) \frac{1}{\mu_\Gamma \kappa_\Gamma} q^\Gamma,$$

where the number $\# \overline{\mathcal{M}}$ should be understood as a virtual counting once the virtual machinery is chosen, the same applies to the discussion on RSFT in the next subsection. The sum is over all multisets $[\Gamma]$, ie sets with duplicates, of size l . And Γ is an ordered representation of $[\Gamma]$, eg

$$\Gamma = \{\underbrace{\eta_1, \dots, \eta_1}_{i_1}, \dots, \underbrace{\eta_m, \dots, \eta_m}_{i_m}\}$$

is an ordered set of good orbits with $\eta_i \neq \eta_j$ for $i \neq j$ and $\sum i_j = l$. We write $\mu_\Gamma = i_1! \cdots i_m!$, and $\kappa_\Gamma = \kappa_{\eta_1}^{i_1} \cdots \kappa_{\eta_m}^{i_m}$ is the product of multiplicities, and $q^\Gamma = q_{\eta_1} \cdots q_{\eta_m}$. We mod out μ_Γ as we should count holomorphic curves with unordered punctures, and mod out κ_Γ to compensate for having κ_γ different ways to glue when we have a breaking at γ . The orientation property of $\overline{\mathcal{M}}_{Y,A}(\{\gamma\}, \Gamma)$ implies that (3-4) is independent of the representative Γ . The differential on a single generators is defined by

$$\partial(q_\gamma) = \sum_{l=0}^{\infty} \partial_l(q_\gamma),$$

which is always a finite sum by (3-3). Then the differential on $\text{CHA}(Y)$ is defined by the Leibniz rule

$$\partial(q_{\gamma_1} \cdots q_{\gamma_l}) = \sum_{j=1}^l (-1)^{|q_{\gamma_1}| + \cdots + |q_{\gamma_{j-1}}|} q_{\gamma_1} \cdots q_{\gamma_{j-1}} \partial(q_{\gamma_j}) q_{\gamma_{j+1}} \cdots q_{\gamma_l}.$$

The relation $\partial^2 = 0$ follows from the boundary configuration of $\overline{\mathcal{M}}_{Y,A}(\{\gamma\}, \Gamma)$ with virtual dimension 1 appropriately interpreted in the chosen virtual machinery.

Given an exact cobordism (X, λ) from Y_- to Y_+ , we have an algebra map ϕ from $\text{CHA}(Y_+)$ to $\text{CHA}(Y_-)$, which on generators is defined by

$$\phi(q_\gamma) = \sum_{l=0}^{\infty} \sum_{||[\Gamma]||=l} \# \overline{\mathcal{M}}_{X,A}(\{\gamma\}, \Gamma) \frac{1}{\mu_\Gamma \kappa_\Gamma} q^\Gamma,$$

where Γ is a collection of good orbits of Y_- . The boundary configuration of $\overline{\mathcal{M}}_{X,A}(\{\gamma\}, \Gamma)$ with virtual dimension 1 gives the relation $\partial \circ \phi = \phi \circ \partial$. Then we have a functor from $\mathcal{C}\text{on}$ to the category of supercommutative differential graded algebras.

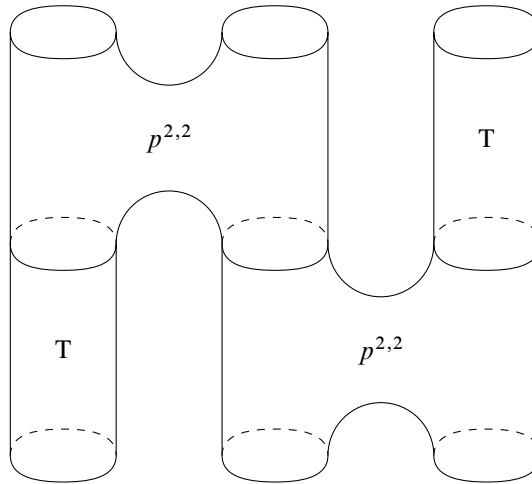


Figure 7: Showing $\hat{p} \circ \hat{p} = 0$, where T stands for a trivial cylinder.

Theorem 3.8 [72, functor (1.22)] *The homology $H_*(\text{CHA}(Y))$ above realized in VFC gives a monoidal functor from $\mathfrak{Con}$ to the category of (super)commutative algebras.*

Remark 3.9 Using semiglobal Kuranishi charts, Bao and Honda [6] gave an alternative definition of contact homology enjoying the same invariance and functorial properties.

3.4 Rational SFT as BL_∞ algebras

To assign BL_∞ algebras to strict contact manifolds, we need to consider moduli spaces with multiple inputs and multiple outputs. In the following, we give an informal description of the BL_∞ structure arising from counting holomorphic curves, neglecting any transversality issues. We use $\overline{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-)$ to denote the compactified moduli space of rational curves in class A with positive asymptotics Γ^+ and negative asymptotics Γ^- in the symplectization $\mathbb{R} \times Y$. Then we can define $p^{k,l}$ by¹⁰

$$(3-5) \quad p^{k,l}(q^{\Gamma^+}) = \sum_{||\Gamma^-||=l} \# \overline{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-) \frac{1}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} q^{\Gamma^-}.$$

Here $[\Gamma^-]$ is a multiset with Γ^- an ordered set representative and $|\Gamma^+| = k$. In particular, the orientation property of $\overline{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-)$ implies that $p^{k,l}$ is a map from $S^k V_\alpha$ to $S^l V_\alpha$. The count $\# \overline{\mathcal{M}}$ is a virtual count, which will be made precise after the virtual setup in Section 3.6. For simplicity, we pretend the moduli spaces are cut out transversely and $\# \overline{\mathcal{M}}$ is the geometric count of oriented orbifold points. When transversality and gluing holds, the boundary of the one-dimensional moduli spaces $\overline{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-)$ yields that $\{p^{k,l}\}$ is a BL_∞ algebra $\text{RSFT}(Y)$ (showed schematically by the pictures); see Theorem 3.11 for details.

¹⁰A previous version of this paper made a mistake with an extra $1/\mu_{\Gamma^+}$ in the coefficient.

Remark 3.10 In the original formalism of the full SFT given by Eliashberg, Givental and Hofer [33, Section 2.2.3], the Hamiltonian H is defined as

$$H = \sum_{A, [\Gamma_+], [\Gamma_-]} \frac{\hbar^{g-1}}{\mu_{\Gamma_+} \mu_{\Gamma_-} \kappa_{\Gamma_+} \kappa_{\Gamma_-}} \# \bar{\mathcal{M}}_{Y,g,A}(\Gamma_+, \Gamma_-) q^{\Gamma_-} p^{\Gamma_+}$$

in the Weyl algebra $\mathcal{W}$ of power series in the variables $\hbar$ and p_γ , with coefficients that are polynomial in the variables q_γ . $\bar{\mathcal{M}}_{Y,g,A}(\Gamma_+, \Gamma_-)$ is the genus g analogue of $\bar{\mathcal{M}}_{Y,A}(\Gamma_+, \Gamma_-)$. $\mathcal{W}$ is equipped with the associative product $*$ in which all variables supercommute according to their gradings except for the variables p_γ and q_γ corresponding to the same Reeb orbit γ , for which we have

$$p_\gamma * q_\gamma - (-1)^{|p_\gamma||q_\gamma|} q_\gamma * p_\gamma = \kappa_\gamma \hbar.$$

In the BV_∞ reformalism of the full SFT introduced by Cieliebak and Latschev [22, Section 6], the differential D_{SFT} on $SV_\alpha[[\hbar]]$ is defined as

$$D_{\text{SFT}}(s) = \sum_{A, [\Gamma_+], [\Gamma_-]} \frac{\hbar^{g-1}}{\mu_{\Gamma_+} \mu_{\Gamma_-} \kappa_{\Gamma_+} \kappa_{\Gamma_-}} \# \bar{\mathcal{M}}_{Y,g,A}(\Gamma_+, \Gamma_-) q^{\Gamma_-} \prod_{\gamma_i \in \Gamma_+} \left(\kappa_{\gamma_i} \hbar \frac{\partial}{\partial q_{\gamma_i}} \right) s.$$

In view of the relation between the Weyl algebra formalism and the IBL_∞ formalism by Cieliebak, Fukaya and Latschev following [21, (7.4)], the operation $p^{k,l,g}$ is defined by

$$\begin{aligned} p^{k,l,g}(q^{\Gamma_+}) &= \sum_{A, [\Gamma_-]} \frac{1}{\mu_{\Gamma_+} \mu_{\Gamma_-} \kappa_{\Gamma_+} \kappa_{\Gamma_-}} \# \bar{\mathcal{M}}_{Y,g,A}(\Gamma_+, \Gamma_-) q^{\Gamma_-} \prod_{\gamma_i \in \Gamma_+} \left(\kappa_{\gamma_i} \hbar \frac{\partial}{\partial q_{\gamma_i}} \right) q^{\Gamma_+} \\ &= \sum_{A, [\Gamma_-]} \frac{1}{\mu_{\Gamma_-} \kappa_{\Gamma_-}} \# \bar{\mathcal{M}}_{Y,g,A}(\Gamma_+, \Gamma_-) q^{\Gamma_-}. \end{aligned}$$

In view of [67, Corollary 5.12], since IBL_∞ algebras restrict to BL_∞ algebras, our rational SFT formalism in (3-5) has consistent coefficients with [22; 21; 33]. Heuristically, $\mu_{\Gamma_-} \kappa_{\Gamma_-}$ is the order of the “isotropy group” of the output orbit set Γ_- .

Similarly, for a strict exact cobordism X from Y_- to Y_+ , by considering the moduli spaces $\bar{\mathcal{M}}_{X,A}(\Gamma^+, \Gamma^-)$ of rational curves in X , we can define a BL_∞ morphism from $\text{RSFT}(Y_+)$ to $\text{RSFT}(Y_-)$ by

$$(3-6) \quad \phi^{k,l}(q^{\Gamma^+}) = \sum_{|[\Gamma^-]|=l} \# \bar{\mathcal{M}}_{X,A}(\Gamma^+, \Gamma^-) \frac{1}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} q^{\Gamma^-},$$

where $|\Gamma^+| = k$. Then the boundary of the one-dimensional moduli spaces $\bar{\mathcal{M}}_{X,A}(\Gamma^+, \Gamma^-)$ yields that $\{\phi^{k,l}\}$ is a BL_∞ morphism $\text{RSFT}(Y_+) \rightarrow \text{RSFT}(Y_-)$. In Figure 8, C indicates the cobordism level.

If we fix a point o in Y , then by considering moduli spaces $\bar{\mathcal{M}}_{Y,A,o}(\Gamma^+, \Gamma^-)$ we can define a pointed morphism $p_\bullet$ by

$$(3-7) \quad p_\bullet^{k,l}(q^{\Gamma^+}) = \sum_{|[\Gamma^-]|=l} \# \bar{\mathcal{M}}_{Y,A,o}(\Gamma^+, \Gamma^-) \frac{1}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} q^{\Gamma^-}.$$

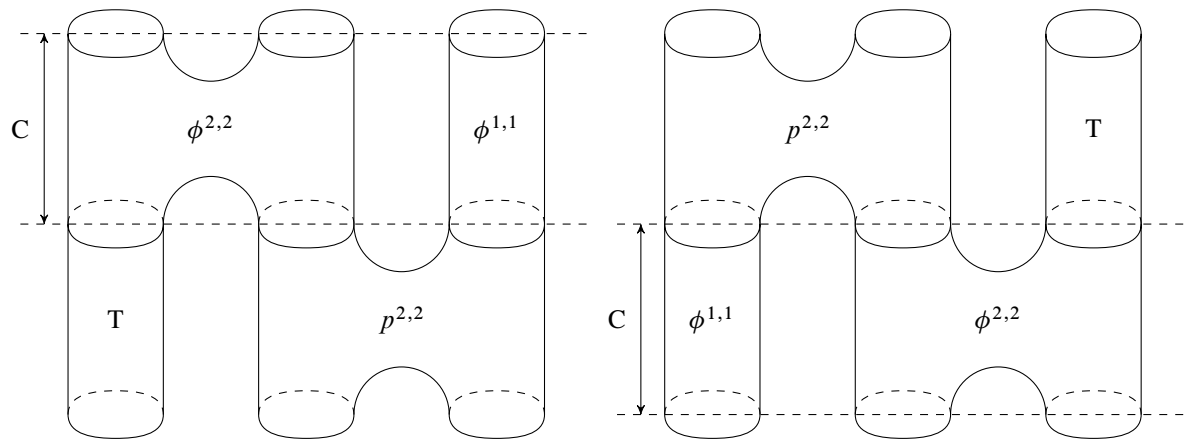


Figure 8: Showing $\hat{\phi} \circ \hat{p} = \hat{p} \circ \hat{\phi}$.

Then the boundary of the one-dimensional moduli spaces $\overline{\mathcal{M}}_{Y,A,o}(\Gamma^+, \Gamma^-)$ yields that $\{p_{\bullet}^{k,l}\}$ is a pointed morphism of degree 0. Note that $\overline{\mathcal{M}}_{Y,A,o}(\Gamma^+, \Gamma^-)$ consists of holomorphic curves with a point constraint in the symplectization with a s -independent J , therefore in the level containing an element of $\overline{\mathcal{M}}_{Y,A,o}(\Gamma^+, \Gamma^-)$ in a rigid breaking, there is only one nontrivial component.

For a strict exact cobordism X from Y_- to Y_+ , if we choose a path γ from $o_- \in Y_-$ to $o_+ \in Y_+$, then we can complete the path γ to a proper $\hat{\gamma}$ path in $\hat{X}$ by constants in the cylindrical ends. Then we claim that the pointed morphisms $p_{\bullet}, q_{\bullet}$ determined by o_-, o_+ and the BL_{∞} morphism ϕ are compatible, with $\phi_{\bullet}$ given by

(3-8)
$$\phi_{\bullet}^{k,l}(q^{\Gamma^+}) = \sum_{||\Gamma^-||=l} \# \overline{\mathcal{M}}_{X,A,\gamma}(\Gamma^+, \Gamma^-) \frac{1}{\mu_{\Gamma^+} \kappa_{\Gamma^-}} q^{\Gamma^-}.$$

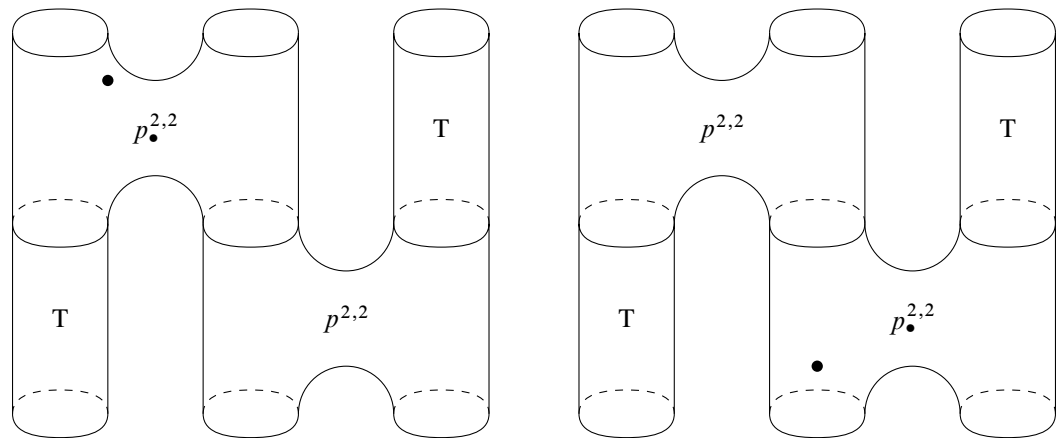


Figure 9: Showing $\hat{p}_{\bullet} \circ \hat{p} = \hat{p} \circ \hat{p}_{\bullet}$.

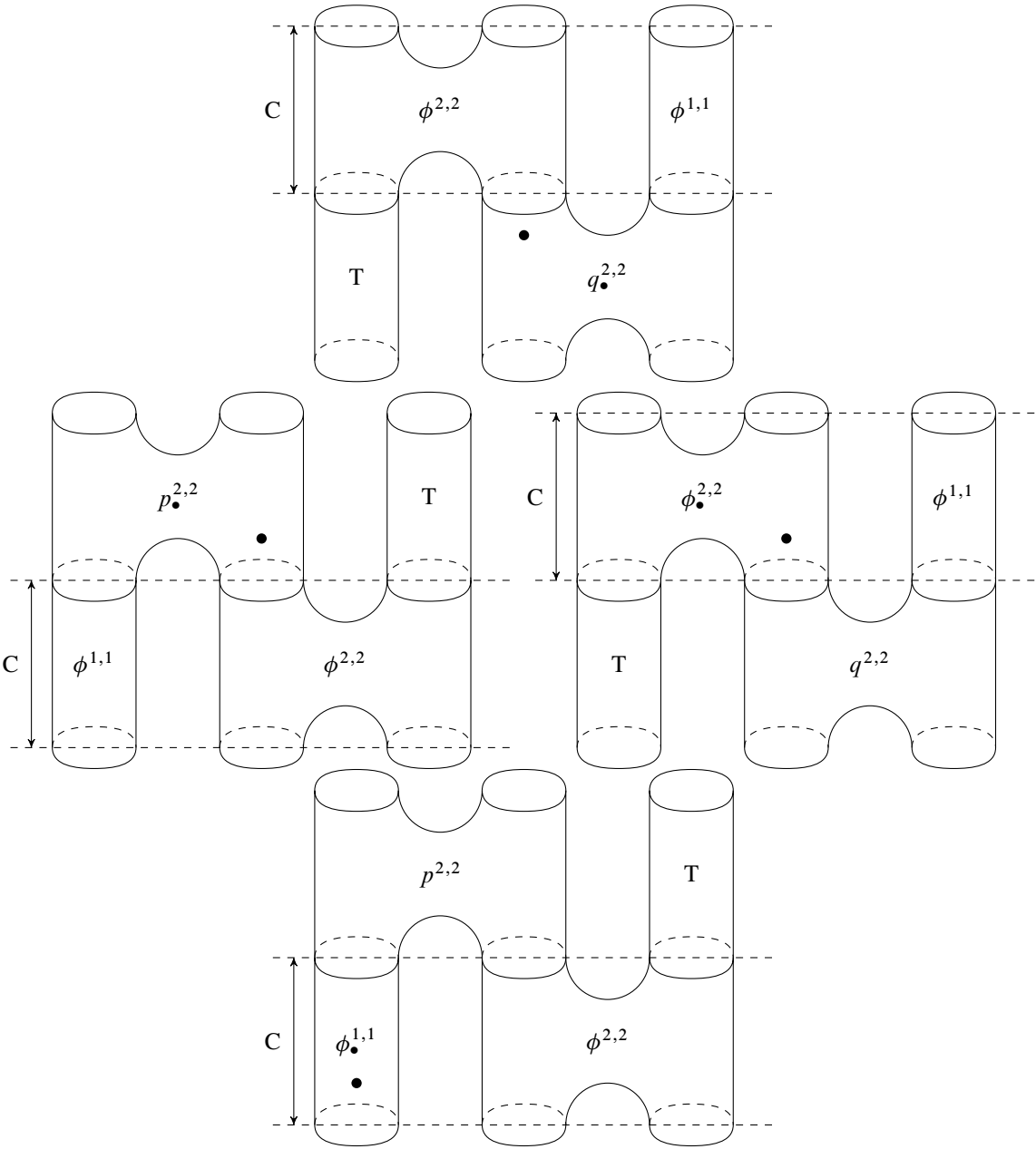


Figure 10: Showing $\hat{q}_{\bullet} \circ \hat{\phi} - \hat{\phi} \circ \hat{p}_{\bullet} = \hat{\phi}_{\bullet} \circ \hat{p} + \hat{q} \circ \hat{\phi}_{\bullet}$.

In order to turn the above informal discussion into a rigorous construction, we need to make sense of $\#\overline{\mathcal{M}}$ such that they have the desired relations. The main theorem of this section is that after fixing auxiliary choices depending on the choice of virtual machinery, we almost have a functor from strict contact cobordism category (with auxiliary choices) to the category of BL_{∞} algebras (the composition is not discussed, nor is it needed for our applications).

Theorem 3.11 *Let (Y, α) be a strict contact manifold with a nondegenerate contact form, then we have the following.*

- (1) *There exists a nonempty set of auxiliary data Θ such that for each $\theta \in \Theta$ we have a BL_∞ algebra p_θ (Definition 2.3) on V_α .*
- (2) *For any point $o \in Y$, there exists a set of auxiliary data Θ_o with a surjective map $\Theta_o \rightarrow \Theta$, such that for any $\theta_o \in \Theta_o$, we have a pointed map $p_{\bullet, \theta_o}$ (Definition 2.19) for p_θ , where θ is the image of θ_o in $\Theta_o \rightarrow \Theta$.*
- (3) *Assume there is a strict exact cobordism X from (Y', α') to (Y, α) . Let Θ and Θ' be the sets of auxiliary data for α and α' . Then there exists a set of auxiliary data Ξ , with a surjective map $\Xi \rightarrow \Theta \times \Theta'$, such that for $\xi \in \Xi$, there is a BL_∞ morphism ϕ_ξ (Definition 2.10) from (V_α, p_θ) to $(V_{\alpha'}, p_{\theta'})$, where (θ, θ') is the image of ξ under $\Xi \rightarrow \Theta \times \Theta'$.*
- (4) *Assume in addition that we fix a point $o' \in Y'$ that is in the same component of o in X . Then for any compatible auxiliary data $\theta, \theta', \theta_o, \theta_{o'}, \xi$, we have that $p_{\bullet, \theta_o}, p_{\bullet, \theta_{o'}}, \phi_\xi$ are compatible (Definition 2.22).*
- (5) *For compatible auxiliary data θ, θ_o , there exists compatible auxiliary data $k\theta, k\theta_o$ for $(Y, k\alpha)$ for $k \in \mathbb{R}_+$ such that $p_{k\theta}$ and $p_{\bullet, k\theta_o}$ are identified with p_θ and $p_{\bullet, \theta_o}$ by the canonical identification between V_α and $V_{k\alpha}$.*

To make sense of $\#\overline{\mathcal{M}}$ we need to fix a choice of virtual machinery, and the meaning of auxiliary data also depends on the choice. If one adopts the perturbative scheme in [33; 46], Theorem 3.11 is a special case of their main constructions. On the other hand, since we only consider rational curves, the combinatorics is not essentially different from the construction of differentials and morphisms in [72]. In particular, Pardon's VFC works in a verbatim account. We will explain the VFC construction to prove Theorem 3.11 and discuss other virtual techniques in Section 3.6.

3.5 Augmentations and linearized theories

Definition 3.12 For a strict contact manifold (Y, α) , we fix an auxiliary choice $\theta \in \Theta$, then we define algebraic planar torsion $\mathrm{APT}(Y, \alpha, \theta)$ to be the torsion of the BL_∞ algebra (V_α, p_θ) over $\mathbb{Q}$.

As a consequence of Proposition 2.16 and Theorem 3.11, we have $\mathrm{APT}(Y, \alpha, \theta)$ is an invariant for Y in the following sense.

Proposition 3.13 *$\mathrm{APT}(Y, \alpha, \theta)$ is independent of α and θ , hence can be abbreviated as $\mathrm{APT}(Y)$. Moreover, $\mathrm{APT}: \mathcal{C}\mathrm{on} \rightarrow \mathbb{N} \cup \{\infty\}$ is a monoidal functor, where the monoidal structure on $\mathbb{N} \cup \{\infty\}$ is given by $a \otimes b = \min\{a, b\}$.*

Proof By (5) of Theorem 3.11, $(V_\alpha, p_\theta) = (V_{k\alpha}, p_{k\theta})$ for any $k \in \mathbb{R}_+$. Let α' be another contact form, and θ' corresponding auxiliary data. Then there exist k_1 and k_2 such that there are strict cobordisms from $(Y, k_1\alpha)$ and (Y, α') to (Y, α') and $(Y, k_2\alpha)$, respectively. Then by (3) of Theorem 3.11 and

Proposition 2.16, $\text{APT}(Y, \alpha, \theta) = \text{APT}(Y, \alpha', \theta')$. For $(Y_1, \alpha_1, \theta_1)$ and $(Y_2, \alpha_2, \theta_2)$, the BL_∞ algebra for the disjoint union $(Y_1 \sqcup Y_2, \alpha_1 \sqcup \alpha_2, \theta_1 \sqcup \theta_2)$ is given by $(V_{\alpha_1} \oplus V_{\alpha_2}, \{p_{\theta_1}^{k,l} \oplus p_{\theta_2}^{k,l}\})$, ie there are no mixed structural maps. Then, clearly, $\text{APT}(V_{\alpha_1} \oplus V_{\alpha_2}, \{p_{\theta_1}^{k,l} \oplus p_{\theta_2}^{k,l}\}) = \min\{\text{APT}(V_{\alpha_1}, p_{\theta_1}), \text{APT}(V_{\alpha_2}, p_{\theta_2})\}$. That APT is a functor follows from (3) of **Theorem 3.11**. $\square$

When $\text{APT}(Y) = 0$, it is equivalent to $H_*(\text{CHA}(Y)) = 0$, which is also known as algebraically overtwisted [12] or 0-algebraic torsion [51], and is implied by overtwistedness [10; 85]. Since finite order of torsion is an obstruction to augmentations, finite algebraic planar torsion is an obstruction to symplectic fillings in view of the following.

Proposition 3.14 *Let (Y, α) be a strict contact manifold with an auxiliary data θ . If $(W, d\lambda)$ is a strict exact filling, then there is a BL_∞ augmentation of (V_α, p_θ) over $\mathbb{Q}$.*

Proof We define

$$\epsilon^k(q^{\Gamma^+}) := \sum_A \# \bar{\mathcal{M}}_{W,A}(\Gamma^+, \emptyset)$$

for $|\Gamma^+| = k$. Then by the third claim of **Theorem 3.11**, $\{\epsilon^k\}_{k \geq 1}$ defines a BL_∞ augmentation. $\square$

Corollary 3.15 *If $\text{APT}(Y) < \infty$, then Y has no strong filling.*

Proof **Proposition 3.14** implies that Y has no exact filling. For the case of strong filling, we consider BL_∞ algebras over $V_\alpha \otimes_{\mathbb{Q}} \Lambda$, where Λ is the Novikov field

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{Q}, \lim_{i \rightarrow \infty} \lambda_i = +\infty \right\}.$$

The structural maps in (3-5) need an additional factor of $T^{\int u d\alpha}$ for $u \in \bar{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-)$; we use $p_{\Lambda}^{k,l}$ to denote such structural maps. If $\text{APT}(Y) < \infty$, we have $x \in E^k V$, such that $\hat{p}(x) = 1$. Let F denote the map $V_\alpha \rightarrow V_\alpha \otimes_{\mathbb{Q}} \Lambda$ sending q_γ to $T^{-\int \gamma^* \alpha} q_\gamma$. Then we have an induced map $E^k F: E^k V \rightarrow E^k(V \otimes_{\mathbb{Q}} \Lambda)$. Since $\int u^* \alpha = \sum_{\gamma \in \Gamma_+} \int \gamma^* \alpha - \sum_{\gamma \in \Gamma_-} \int \gamma^* \alpha$, it is straightforward to check that $\hat{p}_\Lambda(E^k F(x)) = 1$.

Hence $(V_\alpha \otimes_{\mathbb{Q}} \Lambda, p_{\Lambda}^{k,l})$ has no augmentations. On the other hand, given a strict strong filling (W, ω) , by a similar argument of the filling case of **Theorem 3.11**,

$$\epsilon^k(q^{\Gamma^+}) = \sum_A \# \bar{\mathcal{M}}_{W,A}(\Gamma^+, \emptyset) T^{\int u^* \bar{\omega}}$$

defines an augmentation of $(V_\alpha \otimes_{\mathbb{Q}} \Lambda, p_{\Lambda}^{k,l})$, where $|\Gamma^+| = k$ and $\bar{\omega}$ is the two-form on $\hat{W}$ that is ω on W and $d\alpha$ on the cylindrical end. As a consequence, there is no strong filling. $\square$

Remark 3.16 Λ is equipped with a decreasing filtration $\Lambda_{\rho_1} \subset \Lambda_{\rho_2}$ if $\rho_1 \geq \rho_2$, where Λ_ρ consists of those elements with $\lambda_i \geq \rho$. Λ is complete with respect to such a filtration. We view elements in V_α as having filtration 0. Then $V \otimes_{\mathbb{Q}} \Lambda$, $SV \otimes_{\mathbb{Q}} \Lambda$ and $EV \otimes_{\mathbb{Q}} \Lambda$ all have induced filtrations, and we use $\bar{V} \otimes_{\mathbb{Q}} \Lambda$, $\overline{SV} \otimes_{\mathbb{Q}} \Lambda$ and $\overline{EV} \otimes_{\mathbb{Q}} \Lambda$ to denote the completions. In the context of SFT, due to the feature

of the compactness result, the structural maps $p^{k,l}$ and $\phi^{k,l}$ may only be well-defined on the completion. It is necessary to use the completion to describe Maurer–Cartan elements [22], and functoriality for strong cobordisms [67], as well as the SFT for a stable Hamiltonian structure [51]. The situation in Corollary 3.15 is a rather special case of strong cobordisms where a naive version of Novikov coefficient without completion is sufficient.

Roughly speaking the algebraic planar torsion looks for rigid curves with multiple positive punctures and no negative puncture. One particular situation, where we can infer information of algebraic planar torsion, is the planar torsion introduced by Wendl [82, Definition 2.13], which generalizes overtwisted contact structures and the Giroux torsion in dimension 3. The following two results were essentially proven in [51].

Theorem 3.17 *If Y is a three-dimensional contact manifold with planar torsion of order k , then $\text{APT}(Y) \leq k$.*

Proof This follows from the same argument of [51, Theorem 6] based on a precise description of low-energy curves in [51, Proposition 3.6]; see also [82]. In fact, we do not need the genus > 0 assertion from the fifth property of [51, Proposition 3.6], as we do not consider higher-genus curves. $\square$

Theorem 3.18 *For any $k \in \mathbb{N}$, there exists a three-dimensional contact manifold Y with $\text{APT}(Y) = k$.*

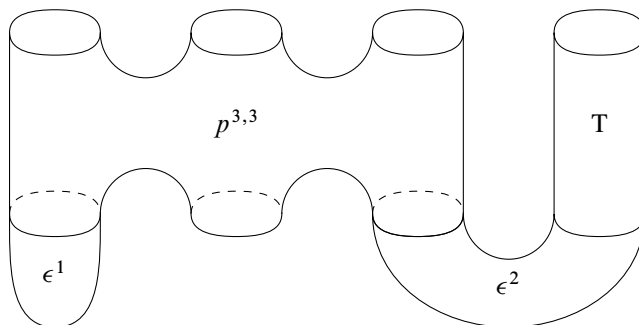
Proof This follows from the same argument of [51, Theorem 4]. In fact, we only need the genus-zero part of [51, Lemma 4.15] to get a lower bound. $\square$

Remark 3.19 As follows from [51, Corollary 1], there are examples of three-dimensional contact manifolds Y_i with planar torsion of order k such that there is an exact cobordism from Y_i to Y_{i+1} but no exact cobordism from Y_{i+1} to Y_i . On the other hand, there is always a strong cobordism from Y_{i+1} to Y_i by [83, Theorem 1]. We will see similar phenomena in higher dimensions in Sections 6 and 7.

Remark 3.20 It is an interesting question to understand the relations between algebraic planar torsion and algebraic torsion. The BV_∞ reformulation of SFT is recalled in Remark 3.10. Following [51, Definition 1.1], Y has algebraic torsion k if k is the smallest number such that $\hbar^k$ is 0 in the homology of $(SV_\alpha[[\hbar]], D_{\text{SFT}})$.

Let us consider the simplest case with an algebraic planar 1-torsion, in which there are two generators q_1 and q_2 such that $p^{2,0}(q_1 q_2) = 1$, $p^{2,l}(q_1 q_2) = 0$ for all $l > 0$, and $p^{1,l}(q_i) = 0$ for all $l \geq 0$ and $i = 1, 2$. The natural candidate for algebraic torsion is $q_1 q_2$, and we compute

$$\begin{aligned} D_{\text{SFT}} q_1 q_2 &= \hbar + \sum_{g=1}^{\infty} \sum_{A, [\Gamma]} \frac{\hbar^{g+1}}{\mu_{\Gamma} \kappa_{\Gamma}} \# \overline{\mathcal{M}}_{Y,g,A}(\{\gamma_1, \gamma_2\}, \Gamma) q^{\Gamma} \\ &\quad + \sum_{g=1}^{\infty} \sum_{A, [\Gamma]} \frac{\hbar^g}{\mu_{\Gamma} \kappa_{\Gamma}} (\# \overline{\mathcal{M}}_{Y,g,A}(\{\gamma_1\}, \Gamma) q^{\Gamma} q_{\gamma_2} + (-1)^{|q_{\gamma_1}|} \# \overline{\mathcal{M}}_{Y,g,A}(\{\gamma_2\}, \Gamma) q_{\gamma_1} q^{\Gamma}). \end{aligned}$$

Figure 11: A component of ℓ_ϵ^4 .

Since we have no knowledge of $\overline{\mathcal{M}}_{Y,g,A}$ for $g > 0$ in RSFT, one should not expect that $q_1 q_2$ is a primitive of $\hbar$ under D_{SFT} . We note here that the above consideration is a very special case and in general algebraic planar k -torsion is not equivalent to $\hbar^k$ being the image of the genus 0 term of D_{SFT} unless $k = 0$. In fact, algebraic torsion and algebraic planar torsion can be viewed as two “independent” axes in a grid of torsions, which make different requirements on holomorphic curves; see [67, Section 5.2] for details.

The notion of Giroux torsion was generalized to higher dimensions in [58]; the following theorem is a reformulation of [64, Theorem 1.7].

Theorem 3.21 *If Y has Giroux torsion, then $\text{APT}(Y) \leq 1$.*

Now we assume (V_α, p_θ) does have a BL_∞ augmentation ϵ over $\mathbb{Q}$. Then $\text{APT}(Y)$ is ∞ . In view of Section 2 and Theorem 3.11, a point $o \in Y$ and an auxiliary data θ_o give rise to a pointed morphism $p_{\bullet, \theta_o}$. Hence we can define the order $O(V_\alpha, \epsilon, p_{\bullet, \theta_o})$. In the following, we use $\text{Aug}_{\mathbb{Q}}(V_\alpha)$ to denote the set of BL_∞ augmentations of V_α over $\mathbb{Q}$.

Definition 3.22 For a strict contact manifold (Y, α) with auxiliary data θ , we define

$$O(Y, \alpha, \theta) := \max\{O(V_\alpha, \epsilon, p_{\bullet, \theta_o}) \mid \epsilon \in \text{Aug}_{\mathbb{Q}}(V_\alpha), o \in Y, \theta_o \in \Theta_o\},$$

where the maximum of an empty set is defined to be zero.

Proposition 3.23 $O(Y, \alpha, \theta)$ is independent of α and θ .

Proof We first show that if there is a strict exact cobordism X from (Y_-, α_-) to (Y_+, α_+) , then $O(Y_+, \alpha_+, \theta_+) \geq O(Y_-, \alpha_-, \theta_-)$ for any θ_+, θ_- . For any $o_- \in Y_-$, there exists a point $o_+ \in Y_+$ such that there is a path in X connecting o_+ and o_- . Then by (4) of Theorem 3.11 and Proposition 2.24, for any augmentation ϵ to V_{α_-} and auxiliary data θ_{o_-} , there exists an auxiliary data $\xi \in \Xi$ and θ_{o_+} such that $O(V_{\alpha_+}, \epsilon \circ \phi_\xi, p_{\bullet, \theta_{o_+}}) \geq O(V_{\alpha_-}, \epsilon, p_{\bullet, \theta_{o_-}})$. Hence $O(Y, \alpha_+, \theta_+) \geq O(Y, \alpha_-, \theta_-)$. Then by the same argument in Proposition 3.13, $O(Y, \alpha, \theta)$ is independent of α and θ . $\square$

Definition 3.24 The planarity $P(Y)$ of a contact manifold Y is defined to be $O(Y, \alpha, \theta)$.

Proposition 3.25 $P: \mathfrak{Con} \rightarrow \mathbb{N} \cup \{\infty\}$ is a monoidal functor, where the monoidal structure on $\mathbb{N} \cup \{\infty\}$ is given by $0 \otimes a = 0$ for all a , and $a \otimes b = \max\{a, b\}$ for all $a, b \geq 1$.

Proof That P is a functor is proven in Proposition 3.23. To prove the monoidal structure, we first note that $(V_1 \oplus V_2, p_1 \oplus p_2)$ has an augmentation if and only if V_1 and V_2 both have augmentations, since the natural inclusion $V_1 \rightarrow V_1 \oplus V_2$ defines a BL_∞ morphism. This verifies the case for $0 \otimes a$. When both Y_1 and Y_2 have augmentations, it follows from definition that $P(Y_1 \sqcup Y_2) = \max\{P(Y_1), P(Y_2)\}$. $\square$

Since finite algebraic planar torsion is an obstruction to BL_∞ augmentations, we have that $\mathrm{APT}(Y) \leq \infty$ implies that $P(Y) = 0$. Since $P(Y) = 0$ corresponds precisely to those contact manifolds without augmentations, algebraic planar torsion is the inner hierarchy inside $P(Y) = 0$. However it is still possible (at least on the algebraic level, eg by exploiting the nonclosedness of $\mathbb{Q}$ algebraically) that $P(Y) = 0$ but $\mathrm{APT}(Y) = \infty$, ie there is no augmentation, nor is there finite algebraic planar torsion.

3.6 Implementation of virtual techniques

In the following, we will explain how to get the algebraic count of moduli spaces in Theorem 3.11 using virtual techniques. Any choice of virtual machinery should give a construction of P and APT with the claimed properties, although it is not clear whether different virtual techniques give rise to the same P and APT . However, the geometric results, examples and applications in this paper, do not depend on the choice, as we have the following axiom for virtual machinery, which holds for any of the virtual techniques mentioned in this paper.

Axiom 3.26 A virtual implementation of a holomorphic curve theory has the property that the virtual count of a compactified moduli space equals the geometric count, when transversality holds for that moduli space.

In the following, we will finish the proof of Theorem 3.11 by implementing Pardon's implicit atlas and virtual fundamental cycles [71; 72]. The construction is essentially the constructions of contact homology algebra and morphisms in [72]. As explained in [72, Section 1.8], the only difference that one needs to pay attention to is the underlying combinatorics for holomorphic curves. One needs to show that an implicit atlas with cell-like stratification still exists for RSFT, in particular the space of gluing parameters has a cell-like stratification. However, the combinatorics for RSFT is also “tree-like” like contact homology, hence the construction is a verbatim account of [72]. More precisely, the only places we need to pay attention to because of the differences of the combinatorics are that the gluing parameter spaces are cell-like stratified and virtual fundamental cycles gives rise to the right coefficient in BL_∞ coefficient. In the following, we give a brief description of the construction.

3.6.1 $\mathcal{R}$ -modules We first introduce a category $\mathcal{R}$ which will play the same role of $\mathcal{S}_I$ in [72, Section 2.1] to govern the combinatorics of rational holomorphic curves in the symplectization. The objects of $\mathcal{R}$ are connected nonempty directed graphs without cycles, such that each vertex has at least one incoming edge.

Edges with missing source, ie input edges, and edges with missing sink, ie output edges, are allowed. Those edges are called external edges and all other edges are called interior edges. The graph T is equipped with decorations as follows.

- (1) For each edge $e \in E(T)$, a Reeb orbit γ_e .
- (2) For each vertex $v \in V(T)$, a relative homology class $\beta_v \in H_2(Y, \{\gamma_{e^+}\}_{e^+ \in E^+(v)} \sqcup \{\gamma_{e^-}\}_{e^- \in E^-(v)})$, where we denote by $E^+(v)$ the set of incoming edges at v and $E^-(v)$ the set of outgoing edges at v , which can be empty.
- (3) For each external edge $e \in E^{\text{ext}}(T)$, a basepoint $b_e \in \text{im } \gamma_e$.

A morphism $\pi: T \rightarrow T'$ in $\mathcal{R}$ consists first of a contraction of the underlying graph of T to T' by collapsing some of the interior edges of T . The decorations have the following property.

- (1) For each noncontracted edge $e \in E(T)$, we have $\gamma_{\pi(e)} = \gamma_e$.
- (2) For each vertex $v' \in V(T')$, we have $\beta_{v'} = \sum_{\pi(v)=v'} \beta_v$.

Finally, we specify for each external edge $e \in E^{\text{ext}}(T) = E^{\text{ext}}(T')$ a path along $\text{im } \gamma_e$ between the basepoints b_e and b'_e modulo the relation that identifies two such paths if and only if their difference lifts γ_e . In particular, there are exactly κ_{γ_e} different equivalence classes of paths. Then the automorphism group of T with a single vertex is a product of cyclic groups and symmetric groups with cardinality $\mu_{\Gamma^+} + \mu_{\Gamma^-} - \kappa_{\Gamma^+} + \kappa_{\Gamma^-}$. For $T \rightarrow T'$, we use $\text{Aut}(T/T')$ to denote the subgroup of $\text{Aut}(T)$ compatible with $T \rightarrow T'$.

A concatenation in $\mathcal{R}$ consists of a finite nonempty collection of objects $T_i \in \mathcal{R}$ along a matching between some pairs of output edges and input edges with matching orbit label, such that the resulting gluing is a directed graph without cycles, along with a choice of paths between the basepoints for each pair of matching edges. Given a concatenation $\{T_i\}_i$ in $\mathcal{R}$, there is a resulting object $\#_i T_i \in \mathcal{R}$. A morphism of concatenations $\{T_i\}_i \rightarrow \{T'_i\}_i$ means a collection of morphisms $T_i \rightarrow T'_i$ covering a bijection of index sets. Then a morphism $\{T_i\}_i \rightarrow \{T'_i\}_i$ induces a morphism $\#_i T_i \rightarrow \#_i T'_i$. If $\{T_i\}_i$ is a concatenation and $T_i = \#_j T_{ij}$ for some concatenation $\{T_{ij}\}_{ij}$, then there is a resulting composite concatenation $\{T_{ij}\}_{ij}$ with natural isomorphisms $\#_{ij} T_{ij} = \#_i \#_j T_{ij} = \#_i T_i$. We use $\text{Aut}(\{T_i\}_i / \#_i T_i)$ to represent the group of automorphisms of $\{T_i\}_i$ acting trivially on $\#_i T_i$, ie the product $\prod_e \mathbb{Z}_{\kappa_{\gamma_e}}$ over junction edges.

The key concept to organize the moduli spaces, implicit atlases, and virtual fundamental cycles is the following $\mathcal{R}$ -module.

Definition 3.27 [72, Definition 4.5] A $\mathcal{R}$ -module X valued in a symmetric monoidal category $\mathcal{C}^{\otimes}$ consists of the following data.

- (1) A functor $X: \mathcal{R} \rightarrow \mathcal{C}$.

(2) For every concatenation $\{T_i\}_i$ in $\mathcal{R}$, a morphism

$$\bigotimes_i X(T_i) \rightarrow X(\#_i T_i)$$

such that the diagrams

$$\begin{array}{ccc} \bigotimes_i X(T_i) & \longrightarrow & X(\#_i T_i) \\ \downarrow & & \downarrow \\ \bigotimes_i X(T'_i) & \longrightarrow & X(\#_i T'_i) \end{array} \quad \begin{array}{ccc} & \nearrow & \searrow \\ & \bigotimes_i X(T_i) & \\ \bigotimes_{i,j} X(T_{ij}) & \longrightarrow & X(\#_{ij} T_{ij}) \end{array}$$

commute for any morphism of concatenations and composition of concatenations.

Example 3.28 A holomorphic building of type $T \in \mathcal{R}$ consists of the following data.

- (1) For every vertex v , a closed, connected nodal genus-zero Riemann surface C_v , along with distinct points $\{p_{v,e} \in C_v\}_e$ indexed by the edges incident at v and a J holomorphic map $u_v: C_v \setminus \{p_{v,e}\}_e \rightarrow \mathbb{R} \times Y$ up to $\mathbb{R}$ -translation.
- (2) Each map u_v converges to γ_{e^+} near p_{v,e^+} in the sense of (3-1) for $e^+ \in E^+(v)$ and converges to γ_{e^-} near p_{v,e^-} in the sense of (3-2) for $e^- \in E^-(v)$. We use $(u_v)_{p_{v,e}}: S^1 \rightarrow Y$ to denote the Y -component of the limit map at the puncture $p_{v,e}$, ie the corresponding orbit.
- (3) For every input/output edge e , an asymptotic marker $L_e \in S_{p_{v,e}} C_v$, which is mapped to the basepoint b_e by $(u_v)_{p_{v,e}}$.
- (4) For every interior edge $v \xrightarrow{e} v'$, a matching isomorphism $m_e: S_{p_{v,e}} C_v \rightarrow S_{p_{v',e}} C_{v'}$ intertwining $(u_v)_{p_{v,e}}$ and $(u_{v'})_{p_{v',e}}$.

An isomorphism between two buildings is a collection of isomorphisms between C_v commuting with all the data. Then we define $\mathcal{M}(T)$ to be the set of isomorphism classes of holomorphic buildings of type T . Note that $\text{Aut}(T)$ acts on $\mathcal{M}(T)$ by changing markings. Then we define

$$\overline{\mathcal{M}}(T) := \bigsqcup_{T' \rightarrow T} \mathcal{M}(T') / \text{Aut}(T'/T).$$

The union is over the set of isomorphism classes in the over category $\mathcal{R}/_T$. Moreover, $\overline{\mathcal{M}}(T)$ is endowed with the Gromov topology and is a compact Hausdorff space [72, Sections 2.9 and 2.10]. Note that here for each $v \in V(T)$, we view u_v as a curve in its own copy of the symplectization. In particular, we have no level structure and the topology is slightly different from the buildings in [9] by forgetting all trivial cylinders. However this poses no difference for the compactness. In particular, there is a surjective map from the compactification in [9] to $\overline{\mathcal{M}}(T)$ by collapsing the boundary configurations containing levels with multiple disconnected nontrivial curves into corners. The functor $\overline{\mathcal{M}}$ is an $\mathcal{R}$ -module in the category of compact Hausdorff spaces with disjoint union as the monoidal structure. The natural map $\overline{\mathcal{M}}(T) \rightarrow \mathcal{R}/_T$ is a stratification in the sense of [72, Definition 2.15]. We define $\text{vdim}(T)$ as $\sum_{v \in V(T)} (\text{ind}(u_v) - 1)$ and $\text{codim}(T'/T)$ is the number of interior edges collapsed in $T' \rightarrow T$. Then we have $\text{codim}(T'/T) + \text{vdim}(T') = \text{vdim}(T)$.

Example 3.29 Definition 2.46 of [72] constructs, for each nondegenerate Reeb orbit γ (good or bad) and basepoint $b \in \text{im } \gamma$, a canonical $\mathbb{Z}_2$ graded line $\mathfrak{o}_{\gamma,b}$ with grading $\mu_{\text{CZ}}(\gamma) + n - 3 \bmod 2$. Any path $b \rightarrow b'$ gives rise to a functorial isomorphism $\mathfrak{o}_{\gamma,b} \rightarrow \mathfrak{o}_{\gamma,b'}$, two paths induces the same isomorphism if the difference is a lift of γ . As a consequence, $\mathbb{Z}_{\kappa_\gamma}$ acts on $\mathfrak{o}_{\gamma,b}$. Then γ is good if and only if the action is trivial. Let T be a tree, then we have the determinant line $\mathfrak{o}_T^\circ|_v$ of the linearized Cauchy–Riemann operator at the vertex v and a canonical isomorphism from $\mathfrak{o}_T^\circ|_v$ to $\bigotimes_{e^+ \in E^+(v)} \mathfrak{o}_{\gamma_{e^+}, b_{e^+}} \otimes \bigotimes_{e^- \in E^-(v)} \mathfrak{o}_{\gamma_{e^-}, b_{e^-}}^\vee$. Moreover, $\mathfrak{o}^\circ$ is an $\mathcal{R}$ -module [71, Example 4.7]. We define $\mathfrak{o}_T$ by $\mathfrak{o}_T^\circ \otimes (\mathfrak{o}_\mathbb{R}^\vee)^{V(T)}$, where the line $\mathfrak{o}_\mathbb{R}$ comes from linearizing the $\mathbb{R}$ translation; $\mathfrak{o}_T$ will be canonically isomorphic to the orientation sheaf of $\overline{\mathcal{M}}(T)$. Moreover, for $T' \rightarrow T$, there is an induced isomorphism $\mathfrak{o}_{T'} \rightarrow \mathfrak{o}_T$ by [72, (2.61)] from an orientation of the gluing parameter space.

Example 3.30 In the construction of the implicit atlas [72, Definition 3.2] for moduli spaces $\overline{\mathcal{M}}(T)$, we need to construct a thickened moduli space, which roughly speaking consists of solutions to pseudo-holomorphic curve equation up to a finite-dimensional error. The choices involved in defining implicit atlas, called thickening data as defined in [72, Definition 3.9], work verbatim for our purpose. Then we have the set of thickening data $A(T)$, and we may define

$$\bar{A}(T) := \bigsqcup_{T' \subseteq T} A(T'),$$

where the disjoint union is over all connected subgraphs that are in $\mathcal{R}$. Then clearly $\bar{A}$ is an $\mathcal{R}^{\text{op}}$ -module to the category of sets.

Proposition 3.31 $\overline{\mathcal{M}}(T)$ is equipped with an implicit atlas $\bar{A}(T)$ with oriented cell-like stratification.

Proof First of all, we have the space of gluing parameters $G_{T/}$ that associates to each interior edge a number in $(0, \infty]$. Since there are no cycles in T , there are no relations among those gluing parameters. In particular, $G_{T/}$ has a cell-like stratification [72, Definition 3.1] over $\mathcal{R}_{T/}$ like $(G_I)_{T/}$ in [72, Lemma 3.5]. Then the claim follows from the same proof of [72, Theorem 3.23]. The analogues of [72, Theorems 3.31 and 3.32] hold for our setup since we only glue one puncture at a time, hence the gluing analysis in [72, Section 5] applies verbatim. $\square$

With the existence of an implicit atlas with cell-like stratification, the machinery of virtual fundamental cycles induces a pushforward map

$$(3-9) \quad C_{\text{vir}}^{*+\text{vdim}(T)}(\overline{\mathcal{M}}(T) \text{ rel } \partial; \bar{A}(T)) \rightarrow C_{-*}(E; \bar{A}(T)),$$

where E is part of the data in $\bar{A}(T)$ and $C_{-*}(E; \bar{A}(T))$ is quasi-isomorphic to the $\mathbb{Q}$ concentrated in degree 0, and $H_{\text{vir}}^*(\overline{\mathcal{M}}(T) \text{ rel } \partial; \bar{A}(T))$ is isomorphic to $\check{H}^*(\overline{\mathcal{M}}(T), \mathfrak{o}_{\text{rel } \partial})$ with $\mathfrak{o}_{\text{rel } \partial} = j_! \mathfrak{o}$ for the orientation bundle $\mathfrak{o}$ and $j: \overline{\mathcal{M}}(T) \setminus \partial \overline{\mathcal{M}}(T) \rightarrow \overline{\mathcal{M}}(T)$.¹¹ Heuristically, the pushforward map can be viewed as integration of a compactly supported degree- $\text{vdim}(T)$ cohomology class on $\overline{\mathcal{M}}(T)$.

¹¹Here $\partial \overline{\mathcal{M}}(T)$ is the preimage of $\mathcal{R}^{<T}$ in the stratification $\overline{\mathcal{M}}(T) \rightarrow \mathcal{R}_{/T}$.

The construction of virtual fundamental cycles for BL_∞ algebra requires combining single pushforward/virtual fundamental class with respect to the combinatorics of $\mathcal{R}$. The proof hinges on an induction argument, which in particular requires certain finiteness. An object $T \in \mathcal{R}$ is called effective if and only if $\overline{\mathcal{M}}(T) \neq \emptyset$. Then for any morphism $T \rightarrow T'$, if T is effective, so is T' . For any concatenation $\{T_i\}_i$, every T_i is effective if and only if $\#_i T_i$ is effective. In the following, $\mathcal{R}$ will mean the full subcategory spanned by effective objects, which depends on J . Then $\mathcal{R}$ has the following properties, which allows one to apply inductive constructions.

- (1) Every T can be written as a concatenation of maximal elements $\#_i T_i$, where an element T_i is maximal if and only if every morphism mapping out of T_i is an isomorphism. That is, T_i has only one interior vertex.
- (2) Let $T, T' \in \mathcal{R}$. We say $T' \preceq T$ if and only if there is a morphism $\#_i T_i \rightarrow T$ with some T_i isomorphic to T' . Then there are no infinite strictly decreasing sequences (in the effective version). This is a consequence of compactness or positivity of contact energy (3-3) in the exact cobordism setting of this paper.

As a consequence, a lot of the constructions can be built inductively from the minimal elements in $(\mathcal{R}, \preceq)$. Note that maximal T is not necessarily maximal in $\preceq$, ie commodification of the moduli spaces of curves in $\hat{Y}$ might involve breaking. But minimal elements of $(\mathcal{R}, \preceq)$ are necessarily maximal, ie compactified moduli spaces without breaking are necessarily without breaking. The induction will typically start at minimal elements of $(\mathcal{R}, \preceq)$ and the induction process for any tree T will terminate since there are no infinite strictly decreasing sequence.

Following the same procedure of [72, Definition 4.19], there is a canonical construction of $\mathcal{R}$ -module $C_{\mathrm{vir}}^{*+\mathrm{vdim}}(\overline{\mathcal{M}} \operatorname{rel} \partial)$ by homotopy colimit in the category of cochain complexes such that

$$C_{\mathrm{vir}}^{*+\mathrm{vdim}(T)}(\overline{\mathcal{M}}(T) \operatorname{rel} \partial) \text{ is quasi-isomorphic to } C_{\mathrm{vir}}^{*+\mathrm{vdim}(T)}(\overline{\mathcal{M}}(T) \operatorname{rel} \partial; \overline{A}(T)).$$

Similarly, by the homotopy colimit as in [72, Definition 4.20], there is an $\mathcal{R}$ -module $C_*(E)$ and (3-9) leads to a canonical map of $\mathcal{R}$ -modules $C_{\mathrm{vir}}^{*+\mathrm{vdim}}(\overline{\mathcal{M}} \operatorname{rel} \partial) \rightarrow C_{-*}(E)$. Similar to [72, Definition 4.14], there is an $\mathcal{R}$ -module $\mathbb{Q}[\mathcal{R}]$ governing the boundary information,

$$\mathbb{Q}[\mathcal{R}](T) := \mathbb{Q}[\mathcal{R}/_T] = \bigoplus_{T' \rightarrow T} \circ_{T'}[\mathrm{vdim}(T')],$$

with the differential given by the sum of all codimension-one maps $T'' \rightarrow T'$ in $\mathcal{R}/_T$ of boundary map $\circ_{T'} \rightarrow \circ_{T''}$ in Example 3.29.¹² The $\mathcal{R}$ -module structure on $\mathbb{Q}[\mathcal{R}]$ is described in [72, (4.37) and (4.38)]. Then a virtual fundamental class recording the combinatorics of $\mathcal{R}$ is simply an $\mathcal{R}$ -module map $\mathbb{Q}[\mathcal{R}] \rightarrow \mathbb{Q}$. Namely, for every T , the image of $\circ_T[\mathrm{vdim} T] \subset \mathbb{Q}[\mathcal{R}](T) \rightarrow \mathbb{Q}$ determines an element of $(\circ_T^\vee)^{\mathrm{Aut}(T)}$ (the $\mathrm{Aut}(T)$ invariant part¹³ of $\circ_T^\vee$) if $\mathrm{vdim}(T) = 0$. On the other hand, it is not hard to

¹²It should be viewed as multiplying an additional $1/|\mathrm{Aut}(T''/T')| = 1$.

¹³The $\mathrm{Aut}(T)$ -invariance forces the vanishing of the virtual fundamental cycle at T if one of the input/output edges of T is labeled with a bad Reeb orbit.

believe that the right virtual fundamental cycle should be the pushforward of 1. By [72, Lemma 4.23], $\mathrm{Hom}_{\mathcal{R}/T}(\mathbb{Q}[\mathcal{R}], C_{\mathrm{vir}}^{*+\mathrm{vdim}}(\overline{\mathcal{M}} \mathrm{rel} \partial))$ is an $\mathcal{R}^{\mathrm{op}}$ module of complexes whose cohomology is isomorphic to the $\mathcal{R}^{\mathrm{op}}$ module $T \rightarrow \check{H}^*(\overline{\mathcal{M}}(T))$. In particular, by the same argument of [72, Lemma 4.31], there is an $\mathcal{R}$ -module map $\mathbb{Q}[\mathcal{R}] \rightarrow C_{\mathrm{vir}}^{*+\mathrm{vdim}}(\overline{\mathcal{M}} \mathrm{rel} \partial)$ representing $1 \in H^0(\overline{\mathcal{M}}(T))$ for all T .

In order to build an $\mathcal{R}$ module map $\mathbb{Q}[\mathcal{R}] \rightarrow \mathbb{Q}$ following $\mathbb{Q}[\mathcal{R}] \rightarrow C_{\mathrm{vir}}^{*+\mathrm{vdim}}(\overline{\mathcal{M}} \mathrm{rel} \partial) \rightarrow C^{-*}(E)$, we need to find a quasi-isomorphism of $\mathcal{R}$ -module from $C_*(E)$ to the trivial $\mathcal{R}$ -module $\mathbb{Q}$ to evaluate at the chain level. Instead of building a direct quasi-isomorphism, this can be done with a cofibrant replacement, ie a diagram of quasi-isomorphisms of $\mathcal{R}$ modules,

$$C_*(E) \xleftarrow{\sim} C_*^{\mathrm{cof}}(E) \xrightarrow{\sim} \mathbb{Q},$$

where $C_*^{\mathrm{cof}}(E)$ is cofibrant in the sense of [72, Definition 4.24]. The construction of $C_*^{\mathrm{cof}}(E)$ follows from the same recipe as for [72, Definition 4.28] by induction on $\preceq$. $\mathbb{Q}[\mathcal{R}]$ is again cofibrant in the sense of [72, Definition 4.24] by the argument of [72, Lemma 4.26]. Now we can introduce the auxiliary data in the VFC setup of RSFT.

Definition 3.32 Given α and J , an element of $\Theta(\alpha, J)$ is a commuting diagram of $\mathcal{R}$ -modules

$$(3-10) \quad \begin{array}{ccccc} \mathbb{Q}[\mathcal{R}] & \xrightarrow{\tilde{w}_*} & C_{-*}^{\mathrm{cof}}(E) & \xrightarrow{p_*} & \mathbb{Q} \\ \downarrow w_* & & \downarrow \sim & & \\ C_{\mathrm{vir}}^{*+\mathrm{vdim}}(\overline{\mathcal{M}} \mathrm{rel} \partial) & \longrightarrow & C_{-*}(E) & & \end{array}$$

satisfying the following properties:

- (1) p_* induces the canonical isomorphism $H_*^{\mathrm{cof}}(E) = H_*(E) = \mathbb{Q}$,
- (2) w_* is such that for any $T \in \mathcal{R}$, $w_* \in \mathrm{Hom}_{\mathcal{R}/T}(\mathbb{Q}[\mathcal{R}], C_{\mathrm{vir}}^{*+\mathrm{vdim}}(\overline{\mathcal{M}} \mathrm{rel} \partial))$ on cohomology level represents the constant function $1 \in \check{H}^0(\overline{\mathcal{M}}(T))$ under the identification in [72, Lemma 4.23].

Proof of Theorem 3.11(1) In the context of VFC, $\Theta(\alpha) = \bigsqcup_J \Theta(J, \Theta(\alpha, J))$. Moreover, $\Theta(\alpha, J)$ is not empty. The existence of p_* follows from [72, Lemma 4.30], the existence of w_* follows from [72, Lemma 4.31], and the existence of lifting $\tilde{w}_*$ follows from the cofibrant property and induction on $\preceq$ as in [72, Proposition 4.34].

Given a diagram (3-10), we have an $\mathcal{R}$ -module map $p_* \circ \tilde{w}_*: \mathbb{Q}[\mathcal{R}] \rightarrow \mathbb{Q}$ that assigns to each T with $\mathrm{vdim}(T) = 0$ an element $\#\overline{\mathcal{M}}(T)^{\mathrm{vir}} \in (\mathfrak{o}_T^\vee)^{\mathrm{Aut}(T)}$ which, after fixing a trivialization of $\mathfrak{o}_{\gamma,b}$ for every Reeb orbits, is a rational number. If an exterior edge of T is labeled by a bad orbit, then being $\mathrm{Aut}(T)$ -invariant implies that $\#\overline{\mathcal{M}}(T)^{\mathrm{vir}} = 0$. $\#\overline{\mathcal{M}}(T)^{\mathrm{vir}}$ is the virtual count of the moduli space of holomorphic curves modeled by the tree T with labels, ie of $\overline{\mathcal{M}}(T)$. If T is a maximal tree (without interior edges), then $\#\overline{\mathcal{M}}(T)^{\mathrm{vir}}$ (after choosing invariant trivializations of $\mathfrak{o}_{\gamma,b}$) is interpreted as $\#\overline{\mathcal{M}}_{Y,A}(\Gamma_+, \Gamma_-)$ in (3-5),

where the marking of T is determined by A , Γ_+ and Γ_- . Finally, being an $\mathcal{R}$ -module implies that

$$(3-11) \quad \begin{aligned} 0 &= \sum_{\text{codim}(T'/T)=1} \frac{1}{|\text{Aut}(T'/T)|} \# \bar{\mathcal{M}}(T')^{\text{vir}} \\ &= \sum_{\text{codim}(T'/T)=1} \# \bar{\mathcal{M}}(T')^{\text{vir}}, \quad \text{since } \text{Aut}(T'/T) = 1, \end{aligned}$$

$$(3-12) \quad \# \bar{\mathcal{M}}(\#_i T_i)^{\text{vir}} = \frac{1}{|\text{Aut}(\{T_i\}_i / \#T_i)|} \prod_i \# \bar{\mathcal{M}}(T_i)^{\text{vir}}.$$

Let T be a tree with one interior vertex labeled by homology class β , with k input edges labeled by Γ^+ , and with l output edges labeled by Γ^- . If $\text{vdim}(T) = 0$, then we define q^{Γ^-} coefficient of $p^{k,l}(q^{\Gamma^+})$, ie

$$\langle p^{k,l}(q^{\Gamma^+}), q^{\Gamma^-} \rangle = \sum_T \frac{\mu_{\Gamma^+}^k \mu_{\Gamma^-}^l}{|\text{Aut}(T)|} \# \bar{\mathcal{M}}(T)^{\text{vir}} = \sum_T \frac{1}{\mu_{\Gamma^+}^k \mu_{\Gamma^-}^l} \# \bar{\mathcal{M}}(T)^{\text{vir}},$$

where the sum is over isomorphism classes of such maximal trees with $\text{vdim} = 0$.

In view of [Proposition 2.9](#), we need to prove that $\langle p_2^{k,l}(q^{\Gamma^+}), q^{\Gamma^-} \rangle$ is zero for any multisets Γ^+ and Γ^- with $|\Gamma^+| = k$ and $|\Gamma^-| = l$. We claim that

$$(3-13) \quad \langle p_2^{k,l}(q^{\Gamma^+}), q^{\Gamma^-} \rangle = \frac{1}{\mu_{\Gamma^+}^k \mu_{\Gamma^-}^l} \sum_T \sum_{\text{codim}(T'/T)=1} \# \bar{\mathcal{M}}(T')^{\text{vir}} = 0,$$

where the first sum is over the isomorphism classes of maximal T with $\text{vdim}(T) = 1$ with marking given by Γ^+ , Γ^- and all possible homology classes. Recall from [Section 2.3](#), when we apply $p^{k,l}$ as gluing trees, we choose a representative of input by ordering the vertices, while the output is understood as an equivalence class, or unordered. More precisely, by the definition of $p^{k,l}$ before [Proposition 2.9](#), we have

$$\langle p^{k,l}(q^{\Gamma^+}), q^{\Gamma^-} \rangle = \sum \pm \langle p^{|\Gamma_1^+|, |\Gamma_1^-|+1}(q^{\Gamma_1^+}), q^{\Gamma_1^-} q_\gamma \rangle \cdot \langle p^{|\Gamma_2^+|+1, |\Gamma_2^-|}(q^\gamma q^{\Gamma_2^+}), q^{\Gamma_2^-} \rangle,$$

where the sum is over divisions of Γ^+ as an ordered set into $\Gamma_1^+ \cup \Gamma_2^+$ with $|\Gamma_1^+| \geq 1$ (hence $2^{|\Gamma^+|} - 1$ such divisions), divisions of $[\Gamma^-]$ as a unordered multiset into $[\Gamma_1^-] \cup [\Gamma_2^-]$ and good Reeb orbits γ and a choice of output vertices in $p^{|\Gamma_1^+|, |\Gamma_1^-|+1}$ marked with γ . Here the sign is determined by the rule in [Section 2.3](#) from switching orders. In other words, $\langle p^{k,l}(q^{\Gamma^+}), q^{\Gamma^-} \rangle$ is

$$(3-14) \quad \sum_{\substack{\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^+ \\ [\Gamma^-] = [\Gamma_1^-] \cup [\Gamma_2^-], \gamma}} \pm \langle p^{|\Gamma_1^+|, |\Gamma_1^-|+1}(q^{\Gamma_1^+}), q^{\Gamma_1^-} q_\gamma \rangle \cdot n(\gamma, \Gamma_1^-) \cdot \langle p^{|\Gamma_2^+|+1, |\Gamma_2^-|}(q^\gamma q^{\Gamma_2^+}), q^{\Gamma_2^-} \rangle,$$

where $n(\gamma, \Gamma_1^-)$ is the number of γ in Γ_1^- plus 1, which is the number of choices for γ -vertices. They have the same sign when $|q_\gamma| = 0$, and when $|q_\gamma| = 1$ the claim is tautological as $q_\gamma^2 = 0$.

Now assume

$$\Gamma^+ := \{\overbrace{\gamma_1, \dots, \gamma_1}^{k_1}, \dots, \overbrace{\gamma_m, \dots, \gamma_m}^{k_m}\}, \quad \Gamma^- := \{\overbrace{\eta_1, \dots, \eta_1}^{l_1}, \dots, \overbrace{\eta_s, \dots, \eta_s}^{l_s}\}$$

for $\sum_{i=1}^m k_i = k$, $\sum_{i=1}^s l_i = l$ and $\gamma_i \neq \gamma_j$, $\eta_i \neq \eta_j$ for $i \neq j$. Note that every T' with $\text{codim}(T'/T) = 1$

is determined by ordered divisions $\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^+$ and $\Gamma^- = \Gamma_1^- \cup \Gamma_2^-$ and one connecting interior edge marked with γ . Every unordered division $[\Gamma^-] = [\Gamma_1^-] \cup [\Gamma_2^-]$ appears $\prod_{i=1}^s \binom{l_i}{l'_i}$ times in the ordered divisions, where

$$[\Gamma_2^-] = [\{\overbrace{\eta_1, \dots, \eta_1}^{l'_1}, \dots, \overbrace{\eta_s, \dots, \eta_s}^{l'_s}\}]$$

for $l'_i \geq 0$. We can cut out the interior edge to obtain T'_1 and T'_2 with the interior edge turning into an output edge for T'_1 marked with γ , ie $T' = T'_1 \# T'_2$. Note that by (3-12), we have

$$\#\bar{\mathcal{M}}(T')^{\text{vir}} = \frac{1}{\kappa_\gamma} \#\bar{\mathcal{M}}(T'_1)^{\text{vir}} \#\bar{\mathcal{M}}(T'_2)^{\text{vir}}.$$

Therefore we have

$$\begin{aligned} (3-15) \quad \frac{1}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} \sum_{\text{codim}(T'/T)=1} \#\bar{\mathcal{M}}(T')^{\text{vir}} &= \frac{1}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} \sum_{\substack{\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^+ \\ \Gamma^- = \Gamma_1^- \cup \Gamma_2^-, \gamma}} \#\bar{\mathcal{M}}(T')^{\text{vir}} \\ &= \sum_{\substack{\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^+ \\ \Gamma^- = \Gamma_1^- \cup \Gamma_2^-, \gamma}} \frac{1}{\mu_{\Gamma^-} \kappa_{\Gamma^-} \kappa_\gamma} \#\bar{\mathcal{M}}(T'_1)^{\text{vir}} \#\bar{\mathcal{M}}(T'_2)^{\text{vir}} \\ &= \sum_{\substack{\Gamma^+ = \Gamma_1^+ \cup \Gamma_2^+ \\ [\Gamma^-] = [\Gamma_1^-]^- \cup [\Gamma_2^-]^-, \gamma}} \frac{1}{\mu_{\Gamma^-} \kappa_{\Gamma^-} \kappa_\gamma} \prod_{i=1}^s \binom{l_i}{l'_i} \#\bar{\mathcal{M}}(T'_1)^{\text{vir}} \#\bar{\mathcal{M}}(T'_2)^{\text{vir}}. \end{aligned}$$

Since

$$\frac{1}{\mu_{\Gamma^-}} \prod_{i=1}^s \binom{l_i}{l'_i} = \frac{n(\gamma, \Gamma_-^1)}{\mu_{\Gamma_1^- \cup \{\gamma\}} \mu_{\Gamma_2^-}},$$

equations (3-14) and (3-15) imply the claim (3-13) as the extra signs from switching order in (3-14) are encoded in the operations on $(\mathfrak{o}_T^\vee)^{\text{Aut}(T)}$ to write (3-11) and (3-12). Hence $p^{k,l}$ gives a BL_∞ structure. $\square$

The next proposition follows from [72, Proposition 4.33]. It is [Axiom 3.26](#) in the context of VFC.

Proposition 3.33 *If $\bar{\mathcal{M}}(T)$ is cut out transversely with $\text{vdim}(T) = 0$, then $\#\bar{\mathcal{M}}(T)^{\text{vir}} = \#\bar{\mathcal{M}}(T) = \#\mathcal{M}(T)$ for any $\theta \in \Theta(\alpha, J)$.*

3.6.2 $\mathcal{R}_{\text{II}}$, $\mathcal{R}^\bullet$ and $\mathcal{R}_{\text{II}}^\bullet$ modules In the following, we introduce $\mathcal{R}_{\text{II}}$, $\mathcal{R}^\bullet$ and $\mathcal{R}_{\text{II}}^\bullet$ to govern moduli spaces as well virtual fundamental cycles for BL_∞ morphisms, pointed maps and the homotopy in [Definition 2.22](#).

- (1) The category $\mathcal{R}_{\text{II}}$ is the analogue of $\mathcal{S}_{\text{II}}$ in [72, Section 2.1]. The objects of $\mathcal{R}_{\text{II}}$ are graphs without cycles as before, but now each edge $e \in E(T)$ is labeled with a symbol $*(e) \in \{0, 1\}$ such that all input edges are labeled with 0 and all output edges are labeled with 1. For each vertex $v \in V(T)$, we associate it with a pair of symbols $*^\pm(v) \in \{0, 1\}$ such that $*^+(v) \leq *^-(v)$ and $*(e^\pm(v)) = *^\pm(v)$.

If $*^+(v) = *^-(v)$, then v is called a symplectization vertex and if $*^+(v) < *^-(v)$, then v is called a cobordism vertex. Given an exact cobordism X from Y_- to Y_+ , for every $T \in \mathcal{R}_\Pi$, we can similarly define the moduli space $\mathcal{M}_\Pi(T)$, where the curve attached to a symplectization vertex v with $*^\pm(v) = 0$ is a holomorphic curve in $\mathbb{R} \times Y_+$ modulo $\mathbb{R}$ -translation, the curve attached to a symplectization vertex v with $*^\pm(v) = 1$ is a holomorphic curve in $\mathbb{R} \times Y_-$ modulo $\mathbb{R}$ -translation, and the curve attached to a cobordism vertex v is a holomorphic curve in $\hat{X}$. Then we have the analogous compactification $\bar{\mathcal{M}}_\Pi(T)$ using the over category over T , which is an $\mathcal{R}_\Pi$ -module.

- (2) The category $\mathcal{R}^\bullet$ is similar to $\mathcal{R}$ but with exactly one vertex labeled by $\bullet$. The morphisms in $\mathcal{R}^\bullet$ consist again of contractions of graphs such that the $\bullet$ vertex is mapped to the $\bullet$ vertex. For every $T \in \mathcal{R}^\bullet$, we can associate a moduli space $\mathcal{M}^\bullet(T)$, which is defined similarly as $\mathcal{M}(T)$ but the map associated to $\bullet$ vertex is a holomorphic curve with a marked point mapped to the fixed point $(0, o) \in \mathbb{R} \times Y$. We can similarly define the compactified moduli spaces $\bar{\mathcal{M}}^\bullet(T)$, which is an $\mathcal{R}^\bullet$ -module.
- (3) The category $\mathcal{R}_\Pi^\bullet$ is the combination of $\mathcal{R}_\Pi$ and $\mathcal{R}^\bullet$, ie the objects are the same as $\mathcal{R}_\Pi$ but one of the vertices is marked with $\bullet$. In the definition of $\mathcal{M}_\Pi^\bullet(T)$, the curve attached to the $\bullet$ vertex is a curve in the symplectization with a point constraint if the vertex is a symplectization vertex, and is a curve in the cobordism with a path constraint if the vertex is a cobordism vertex.

Proof of the rest of Theorem 3.11 We need to argue that $\bar{\mathcal{M}}_\Pi(T)$, $\bar{\mathcal{M}}^\bullet(T)$ and $\bar{\mathcal{M}}_\Pi^\bullet(T)$ are equipped with implicit atlases with oriented cell-like stratification. For this, we only need to argue that the gluing parameter spaces are cell-like, the remaining of the argument is the same as [72, Theorem 3.23]. The gluing parameter space $(G^\bullet)_{T/}$ for $\mathcal{R}_{T/}^\bullet$ is same as the $G_{T/}$, ie $(0, \infty]^{E^{\text{int}}(T)}$, since there are no relations among gluing parameters. The gluing parameter space $(G_\Pi)_{T/}$ for $(\mathcal{R}_\Pi)_{T/}$ is defined as a subset of

$$\{(\{g_e\}_e, \{g_v\}_v) \in (0, \infty]^{E^{\text{int},0}(T)} \times [-\infty, 0)^{E^{\text{int},1}(T)} \times (0, \infty]^{V_{00}(T)} \times [-\infty, 0)^{V_{11}(T)}\}$$

subject to the constraints

$$\begin{aligned} g_v &= g_e + g_{v'} & \text{for } v \xrightarrow{e} v' \text{ with } *(e) = 0, \\ g_{v'} &= g_e + g_v & \text{for } v \xrightarrow{e} v' \text{ with } *(e) = 1. \end{aligned}$$

where g_v is interpreted as 0 if $v \in V_{01}(T)$, while $V_{ij}(T)$ is the set of vertices with $*^+(v) = i$ and $*^-(v) = j$, and $E^{\text{int},i}(T)$ is the set of interior edges e such that $*(e) = i$. Then g_v can be viewed as the height of the vertex v for $v \in V_{ij}(T)$, where the heights of all cobordism vertices are 0, as all of them are placed in the same level. Following the argument of [72, Lemma 3.6], it is sufficient to prove $(G_\Pi)_{T/}$ is a topological manifold with boundary. We can perform the same change of coordinates $h = e^{-g} \in [0, 1)$ for $v \in V_{00}(T)$, $e \in E^{\text{int},0}(T)$, and $h = e^g \in [0, 1)$ for $v \in V_{11}(V)$, $e \in E^{\text{int},1}(T)$. We allow $h \in [0, \infty)$ for convenience. Then the relation becomes $h_v = h_e h_{v'}$ for $*(e) = 0$ and $h_{v'} = h_e h_v$ for $*(e) = 1$. Now the difference with [72, Lemma 3.6] is that we do not have $v_{\max}$, which in contact homology corresponds to

the vertex with the input edge. In our case, the subgraph generated $V_{00}(T)$ is a disjoint union of graphs $\{T_i^0\}_{i \in I^0}$ and the subgraph generated $V_{11}(T)$ is a disjoint union of graphs $\{T_i^1\}_{i \in I^1}$. We pick vertices v_i^0 and v_i^1 in T_i^0 and T_i^1 , respectively. Since T_i^0 has no cycles and we can view v_i^0 as a root, we can parametrize the gluing parameters associated to T_i^0 by $h_{v_i^0} \in [0, \infty)$, $q_e = h_e^2 - h_{v'}^2 \in \mathbb{R}$ if e is in the same direction with the direction pointed away from the root v_i^0 , and $h_e \in [0, \infty)$ if e is the opposite direction with the tree direction. In the same-direction case, $h_v \in [0, \infty)$ and $q_e = h_e^2 - h_{v'}^2 \in \mathbb{R}$ determine $h_e, h_{v'} \in [0, \infty)$ as in [72, Lemma 3.6]. It is clear that such a change of coordinates parametrizes the gluing parameters by $[0, \infty) \times \mathbb{R}^{|\{e \mid \text{in same direction}\}|} \times [0, \infty)^{|\{e \mid \text{in opposite direction}\}|}$. Similarly we parametrize the gluing parameters on T_i^1 by $h_{v_i^1} \in [0, \infty)$, $h_e \in [0, \infty)$ if e is in the same direction with the tree direction, and $q_e = h_e^2 - h_{v'}^2 \in \mathbb{R}$ if e is the opposite direction with the tree direction. As a consequence $(G_{\Pi})_{T/}$ is the product of the such parametrization of T_i^0 and T_i^1 , which is a topological manifold with boundary, and the top stratum corresponds to the interior. The gluing parameter space $(G_{\Pi}^{\bullet})_{T/}$ is the same as $(G_{\Pi})_{T/}$. Therefore $\overline{\mathcal{M}}_{\Pi}(T)$, $\overline{\mathcal{M}}^{\bullet}(T)$ and $\overline{\mathcal{M}}_{\Pi}^{\bullet}(T)$ are equipped with implicit atlases with oriented cell-like stratification.

The virtual fundamental cycles for BL_{∞} morphisms, pointed maps and homotopies are module morphisms $\mathbb{Q}[\mathcal{R}_{\Pi}] \rightarrow \mathbb{Q}$, $\mathbb{Q}[\mathcal{R}^{\bullet}] \rightarrow \mathbb{Q}$ and $\mathbb{Q}[\mathcal{R}_{\Pi}^{\bullet}] \rightarrow \mathbb{Q}$, respectively, that are derived from diagrams like (3-10). The nonemptiness of such diagrams and surjectivity of the projections of admissible auxiliary data follows from [72, Proposition 4.34]. Combined with Propositions 2.11, 2.20 and 2.23, it follows from the same proof as that of Theorem 3.11(1) that module morphisms $\mathbb{Q}[\mathcal{R}_{\Pi}] \rightarrow \mathbb{Q}$, $\mathbb{Q}[\mathcal{R}^{\bullet}] \rightarrow \mathbb{Q}$ and $\mathbb{Q}[\mathcal{R}_{\Pi}^{\bullet}] \rightarrow \mathbb{Q}$ give rise counts to BL_{∞} morphisms, pointed maps and homotopies. For (5) of Theorem 3.11, it is clear the whole construction for α can be identified with the construction for $k\alpha$ as long as we use the same admissible almost complex structure J for $k > 0$. $\square$

3.6.3 Polyfold approach The polyfold construction of SFT [33], which is described in [34], will imply Theorem 3.11 as well. However, we cannot use the “tree-like” compactification as in $\overline{\mathcal{M}}(T)$ because the gluing parameter space is only topological manifold with boundary. For the analytic requirement in a polyfold, it is important to use the building compactification in [9] so that all gluing parameters are independent and form a smooth manifold with boundary and corners. To implement the polyfold construction for our purpose, it is sufficient to build polyfold strong bundles with sc-Fredholm sections for the SFT building compactification, which is sketched in [34].

Since we will not need to discuss more subtle cases like neck-stretching and homotopies, the abstract theory of polyfold developed in [43] suffices to provide transverse perturbations by the similar induction on $(\mathcal{R}, \preceq)$ starting from minimal elements in $\preceq$, which are polyfolds without boundaries. The nonempty set Θ in Theorem 3.11 now consists pairs (J, σ) , where J is an admissible almost complex structure and σ is a family of compatible sc^+ -multisections in general position. Then (3-11) and (3-12) follow from the Stokes’ theorem in [43], where the coefficients can be explained as being the discrepancies of isotropy among polyfolds with their boundary polyfolds and boundary polyfolds with product polyfolds.

To verify [Axiom 3.26](#), we first note that classical transversality implies polyfold transversality by definition. If $\overline{\mathcal{M}}(T)$ is cut out transversely for $\text{vdim}(T) = 0$, we may still need to perturb the sc-Fredholm section on the associated polyfold, because we construct perturbations by induction. Even though we know that the section is transverse on the boundary polyfolds, but the section can be nontransverse on some factor of the boundary, which will be perturbed before we construct perturbations for $\overline{\mathcal{M}}(T)$. However, we can choose our perturbations small enough to get the local invariance of $\# \overline{\mathcal{M}}(T)$ when $\text{vdim}(T) = 0$. In other words, [Axiom 3.26](#) holds if we choose sufficiently small perturbations. This matches with [Proposition 3.33](#), as $\Theta(\alpha, J)$ in VFC can be understood as “infinitesimal” perturbations.

Remark 3.34 (Kuranishi approach) The Kuranishi approach of SFT [\[46\]](#) would also imply [Theorem 3.11](#). [Axiom 3.26](#) should follow from the same argument above for small enough perturbations in a reasonable measurement.

4 Semidilations

In this section, we introduce an inner hierarchy called the order of semidilation for the $P = 1$ case.

Remark 4.1 As explained in [Section 1.2](#), a full implementation of the order of semidilation is dependent on the unproven [Claim 4.3](#) below, as it requires a rigorous implementation of the U -map, which we defer to later work.

If $P(Y) = 1$, then $\text{RSFT}(Y)$ admits BL_∞ augmentations and for any BL_∞ augmentation ϵ , then the order is 1 for any point in Y . Note that $(\bar{B}^1 V_\alpha, \hat{\ell}_\epsilon)$ is the chain complex $(V_\alpha, \ell_\epsilon^1)$ for the linearized contact homology. Since $P(Y) = 1$, for any point in Y , we have a class $x \in H_*(V_\alpha, \ell_\epsilon^1)$ such that $\ell_{\bullet, \epsilon}^1(x) = 1$.

If the augmentation ϵ_W is from an exact filling W , then by [\[13; 16\]](#), the linearized contact homology $\text{LCH}_*(Y, \epsilon_W) = \text{LCH}_*(W) := H_*(V_\alpha, \ell_{\epsilon_W}^1)$ is isomorphic to the equivariant symplectic (co)homology, which as an S^1 -equivariant theory carries a $H^*(BS_1) = \mathbb{Q}[U]$ -module structure and fits into the following Gysin sequences:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{SH}_+^{2n-3-k}(W) & \longrightarrow & \text{LCH}_k(W) & \xrightarrow{U} & \text{LCH}_{k-2}(W) \longrightarrow \text{SH}_+^{2n-2-k}(W) \longrightarrow \cdots \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \cdots & \longrightarrow & \text{SH}_+^{2n-3-k}(W) & \longrightarrow & \text{SH}_{+, S^1}^{2n-3-k}(W) & \xrightarrow{U} & \text{SH}_{+, S^1}^{2n-1-k}(W) \longrightarrow \text{SH}_+^{2n-2-k}(W) \longrightarrow \cdots \end{array}$$

As we use homological convention in this paper, U has degree -2 in the case with a $\mathbb{Z}$ grading for LCH. And U has degree 2 for the S^1 -equivariant symplectic cohomology, which is graded by $n - \mu_{\text{CZ}}$.

Strictly speaking, the papers [\[13; 16\]](#) make several transversality assumptions which limit the collection of exact fillings where the isomorphism is rigorously established. However, the geometric ideas behind [\[13; 16\]](#) should work when implemented using a suitable virtual machinery to prove isomorphisms for

general cases. We first carry out the discussion neglecting the foundational issues for now. The U map is defined on the linearized contact homology $H_*(V_\alpha, \ell_\epsilon^1)$ for any augmentation ϵ . And for any element $x \in H_*(V_\alpha, \ell_\epsilon^1)$ there exists $k \in \mathbb{N}_+$ such that $U^k(x) = 0$. In the following, we first recall the definition of U -map for linearized contact homology.

4.1 $H_*(V_\alpha, \ell_\epsilon^1)$ as a $\mathbb{Q}[U]$ -module

To explain the U -map, we recall the following two moduli spaces from [13, Section 7.2]. In some sense, the following moduli spaces should be viewed as a version of cascade moduli spaces.

4.1.1 $\mathcal{M}_{Y,A}^1(\gamma^+, \gamma^-, \Gamma^-)$ Let γ^+ and γ^- be two good Reeb orbits and Γ^- be an ordered multiset of good Reeb orbits of cardinality $k \geq 0$. Then an element in $\mathcal{M}_{Y,A}^1(\gamma^+, \gamma^-, \Gamma^-)$ consists of the following data.

- (1) A sphere (Σ, j) , with one positive puncture z^+ and $1 + k$ negative punctures $z^-, z_1^-, \dots, z_k^-$. We pick an asymptotic marker on z^+ , then by choosing a global polar coordinate on $\Sigma \setminus \{z^+, z^-\}$, there is a canonically induced asymptotic marker on z^- by requiring it to have the same angle as the asymptotic marker at z^+ in the polar coordinate. We also pick free asymptotic markers on z_i^- for $1 \leq i \leq k$.
- (2) A map $u: \dot{\Sigma} \rightarrow \mathbb{R} \times Y$ such that $du \circ j = J \circ du$ and $[u] = A$ modulo automorphism and the $\mathbb{R}$ -translation, where $\dot{\Sigma}$ is the $(2+k)$ -punctured sphere.
- (3) u is asymptotic to γ^+, γ^- and Γ^- near z^+, z^- and $\{z_i^-\}_i$ with respect to the asymptotic markers and the chosen marked points on the image of Reeb orbits as before.

We use $\overline{\mathcal{M}}_{Y,A}^1(\gamma^+, \gamma^-, \Gamma^-)$ to denote the compactification of $\mathcal{M}_{Y,A}^1(\gamma^+, \gamma^-, \Gamma^-)$. Then for an exact cobordism X , we can similarly define $\overline{\mathcal{M}}_{X,A}^1(\gamma^+, \gamma^-, \Gamma^-)$, where we do not mod out by the $\mathbb{R}$ translation.

4.1.2 $\mathcal{M}_{Y,A}^2(\gamma^+, \gamma, \gamma^-, \Gamma_1^-, \Gamma_2^-)$ Let γ^+ and γ^- be two good Reeb orbits, let γ be a Reeb orbit that could be bad,¹⁴ and let Γ_1^- and Γ_2^- be two ordered multisets of good Reeb orbits of cardinality $k_1, k_2 \geq 0$. Then an element in $\mathcal{M}_{Y,A}^2(\gamma^+, \gamma, \gamma^-, \Gamma_1^-, \Gamma_2^-)$ consists of the following data.

- (1) Two spheres (Σ, j) and $(\tilde{\Sigma}, j)$, each with one positive puncture z^+ and $\tilde{z}^+$, and respectively with $1 + k_1$ negative punctures $z^-, z_1^-, \dots, z_{k_1}^-$, and $1 + k_2$ negative punctures $\tilde{z}^-, \tilde{z}_1^-, \dots, \tilde{z}_{k_2}^-$. Each puncture is equipped with an asymptotic marker.
- (2) Two holomorphic curves u and $\tilde{u}$ from $\dot{\Sigma}$ and $\dot{\tilde{\Sigma}}$ to $\mathbb{R} \times Y$ modulo automorphism and $\mathbb{R}$ -translations, such that $[u] \#_\gamma [\tilde{u}] = A$.
- (3) The curve u is asymptotic to γ^+, γ and Γ_1^- , and $\tilde{u}$ is asymptotic to γ, γ^- and Γ_2^- .

¹⁴The necessity of γ being potentially bad is explained in [15, Section 2].

- (4) Let L_- and L_+ be two asymptotic markers on z^- and $\tilde{z}^+$ that are induced from the chosen asymptotic markers on z^+ and $\tilde{z}^-$ by global polar coordinates.¹⁵ We can define $\text{ev}_{L_-}(u)$ and $\text{ev}_{L_+}(\tilde{u})$ to be the limit point in the Y component evaluated along the asymptotic markers L_- and L_+ . Then we require $(b_\gamma, \text{ev}_{L_-}(u), \text{ev}_{L_+}(\tilde{u}))$ to be the natural order on $\text{im } \gamma$, where b_γ is the chosen marked point on $\text{im } \gamma$.

We use $\overline{\mathcal{M}}_{Y,A}^2(\gamma^+, \gamma, \gamma^-, \Gamma_1^-, \Gamma_2^-)$ to denote the compactification. Note that we need to add in the stratum corresponding to the collision of $(b_\gamma, \text{ev}_{L_-}(u), \text{ev}_{L_+}(\tilde{u}))$ in addition to usual building structures. We can similarly define $\overline{\mathcal{M}}_{X,A}^{2,\uparrow}(\gamma^+, \gamma, \gamma^-, \Gamma_1^-, \Gamma_2^-)$ and $\overline{\mathcal{M}}_{X,A}^{2,\downarrow}(\gamma^+, \gamma, \gamma^-, \Gamma_1^-, \Gamma_2^-)$ for an exact cobordism X . The difference is that the former one has u in $\hat{X}$ and the latter one has $\tilde{u}$ in $\hat{X}$.

Given a dga augmentation ϵ^1 to $\text{CHA}(Y)$, ie a map $\epsilon^1: V_\alpha \rightarrow \mathbb{Q}$ which extends to an algebra map $\hat{\epsilon}^1: \text{CHA}(Y) \rightarrow \mathbb{Q}$ such that $\hat{\epsilon}^1 \circ \hat{p}^1 = 0$, the map $U: V_\alpha \rightarrow V_\alpha$ is defined by

$$(4-1) \quad U(q_{\gamma^+}) := \sum_{\gamma^-, [\Gamma^-]} \frac{1}{\kappa_{\gamma^-} \mu_{\Gamma^-} \kappa_{\Gamma^-}} \# \overline{\mathcal{M}}_{Y,A}^1(\gamma^+, \gamma^-, \Gamma^-) \prod_{\gamma' \in \Gamma^-} \epsilon(\gamma') q_{\gamma^-} \\ + \sum_{\substack{\gamma^-, \gamma, \\ [\Gamma_1^-], [\Gamma_2^-]}} \frac{1}{N} \# \overline{\mathcal{M}}_{Y,A}^2(\gamma^+, \gamma, \gamma^-, \Gamma_1^-, \Gamma_2^-) \prod_{\gamma' \in \Gamma_1^- \cup \Gamma_2^-} \epsilon(\gamma') q_{\gamma^-},$$

where $N = \kappa_{\gamma^-} \kappa_\gamma \mu_{\Gamma_1^-} \mu_{\Gamma_2^-} \kappa_{\Gamma_1^-} \kappa_{\Gamma_2^-}$.

Remark 4.2 The $\mathcal{M}_{Y,A}^2$ in [13, Section 7.2] requires modding by an equivalence

$$(L_-, L_+) \simeq \left(L_- + \frac{2\pi}{\kappa_\gamma}, L_+ + \frac{2\pi}{\kappa_\gamma} \right),$$

ie the moduli space of “glued” two-level buildings. Here we do not introduce the equivalence, the discrepancy is just the extra $1/\kappa_\gamma$ in (4-1) compared to [13, (85)].¹⁶

The reason that U is a chain map from $(V_\alpha, \ell_\epsilon^1)$ to itself follows from the boundary of one-dimensional $\overline{\mathcal{M}}_{Y,A}^1$ and $\overline{\mathcal{M}}_{Y,A}^2$. More precisely, the codimension-one boundary of $\overline{\mathcal{M}}_{Y,A}^1$ consists of

- (1) a level breaking where the lower level does not contain z_- , and
- (2) a level breaking where the lower level contains z_- .

For case (1), such a contribution is zero when capping Γ^- off with ϵ^1 by the relation $\hat{\epsilon}^1 \circ \hat{p}^1 = 0$. For case (2), the contribution will cancel with the codimension-one boundary part of $\overline{\mathcal{M}}_{Y,A}^2$ corresponding to the collision of $\text{ev}_{L_-}(u)$ and $\text{ev}_{L_+}(\tilde{u})$. The other parts of the codimension-one boundary of $\overline{\mathcal{M}}_{Y,A}^2$ consists of:

- (1) A level breaking of u where the lower level does not contain z_- . This is again killed by the capping off with ϵ^1 .

¹⁵In particular, they may be different from the chosen asymptotic markers on z^- and $\tilde{z}^+$.

¹⁶The extra coefficient μ_Γ comes from considering Γ as an ordered set, and μ_Γ is the size of the isotropy coming from permutation.

- (2) A level breaking of u where the lower level contains z_- . This corresponds to a component of $\ell_\epsilon^1 \circ U$, where the U part is contributed by a $\overline{\mathcal{M}}_{Y,A}^2$.
- (3) Similar level breakings for $\tilde{u}$.
- (4) The collision of b_γ and $\text{ev}_{L-}(u)$. This corresponds to a component of $\ell_\epsilon^1 \circ U$, where the U part is contributed by a $\overline{\mathcal{M}}_{Y,A}^1$, similarly, the collision of b_γ and $\text{ev}_{L+}(\tilde{u})$ is the remaining part of $U \circ \ell_\epsilon^1$.

Similarly, given an exact cobordism, we can show that the chain morphism $\phi_\epsilon^{1,1}: V_\alpha \rightarrow V_{\alpha'}$ commutes with U up to homotopy, where the homotopy is defined by $\overline{\mathcal{M}}_{X,A}^1$, $\overline{\mathcal{M}}_{X,A}^{2,\uparrow}$, and $\overline{\mathcal{M}}_{X,A}^{2,\downarrow}$ by a formula similar to (4-1) with a similar argument.

In order to define the order of semidilation, we need to define the U -map to the following extent.

Claim 4.3 *Let (Y, α) be a nondegenerate contact manifold and θ be an auxiliary datum which is used in defining a BL_∞ structure p_θ .*

- (1) *There is an auxiliary datum θ_U for the definition of U such that for any BL_∞ augmentation ϵ of (V_α, p_θ) , we have a map $U_{\theta_U}: H_*(V_\alpha, \ell_\epsilon^1) \rightarrow H_{*-2}(V_\alpha, \ell_\epsilon^1)$, and for any $x \in H_*(V_\alpha, \ell_\epsilon^1)$ there exists k such that $U_{\theta_U}^k(x) = 0$. (This nilpotent property follows from the fact that U decreases the contact action for nondegenerate α .)*
- (2) *When there is a strict exact cobordism X from (Y', α') to (Y, α) with admissible auxiliary data $\theta, \theta', \theta_U, \theta'_U$ for α, α' and their U -maps, respectively, then there exists an auxiliary datum ξ such that the map $\phi_{\xi, \epsilon}^{1,1}: H_*(V_\alpha, \ell_{\epsilon \circ \phi_\xi}^1) \rightarrow H_*(V_{\alpha'}, \ell_\epsilon^1)$ commutes with the U -maps for any BL_∞ augmentation ϵ for $(V_{\alpha'}, p_{\theta'})$.*
- (3) *For any $k \in \mathbb{R}_+$, there exists a $k\theta_U$ such that $U_{k\theta_U}$ is canonically identified with U_{θ_U} .*
- (4) *When the augmentation is from an exact filling W , then there is an isomorphism $H_*(V_\alpha, \ell_\epsilon^1) \rightarrow \text{SH}_{+, S^1}^{2n-3-*}(W)$ preserving the U -map.*

Remark 4.4 In Claim 4.3, we are not claiming that linearized contact homology for all possible augmentations forms a contact invariant, which requires establishing the homotopy property of contact homology.

Assuming Claim 4.3, let Y be a contact manifold with $P(Y) = 1$. Define the order of semidilation $\text{SD}(Y)$ by

$$(4-2) \quad \text{SD}(Y) := \max\{\min\{k \mid U^{k+1}(x) = 0, x \in H_*(V_\alpha, \ell_\epsilon^1), \ell_{\bullet, \epsilon}^1(x) = 1\} \mid o \in Y, \epsilon \in \text{Aug}_{\mathbb{Q}}(V_\alpha)\}.$$

Proposition 4.5 *For those contact manifolds Y with $P(Y) = 1$, the assignment $\text{SD}(Y)$ is well-defined and is a monoidal functor from the full subcategory $\text{P}^{-1}(1)$ of $\mathcal{C}\text{on}$ to $\mathbb{N} \cup \{\infty\}$, where the monoidal structure on $\mathbb{N} \cup \{\infty\}$ is defined by $a \otimes b = \max\{a, b\}$.*

Proof That $\text{SD}(Y)$ is independent of all choices follows from the same argument of Proposition 3.13. The monoidal structure follows from $H_*(V \oplus V', \ell_\epsilon^1 \oplus \ell_{\epsilon'}^1) = H_*(V, \ell_\epsilon^1) \oplus H_*(V', \ell_{\epsilon'}^1)$ as $\mathbb{Q}[U]$ -modules. $\square$

Proof of Theorem A It follows from Propositions 3.13, 3.23 and 4.5. $\square$

Although Claim 4.3 is expected to hold in any virtual implementation of SFT, it does not follow from a direct generalization of Pardon's construction [72] of (linearized) contact homology. One of the main issues is the different perspectives of the contact homology, namely as a quotient theory in [72] versus as an equivariant theory in [13]. More precisely, in the definition of contact homology or RSFT, the count $\#\bar{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-)$ does not depend on the specific choice of basepoint b_γ for $\gamma \in \Gamma^+ \cup \Gamma^-$. This is clear if $\bar{\mathcal{M}}_{Y,A}(\Gamma^+, \Gamma^-)$ is geometrically cut out transversely and it is also true for the VFC construction by the definition of $\mathcal{R}$ -module (different choices of b_γ give rise to isomorphic objects in $\mathcal{R}$). On the other hand, the definition of $\bar{\mathcal{M}}_{Y,A}^1$ and $\bar{\mathcal{M}}_{Y,A}^2$ is actually sensitive to the choice of b_γ both geometrically and in virtual constructions. For example, as we explained, the collision of b_γ with $\text{ev}_{L-}(u)$ should correspond to a component of $\ell_\epsilon^1 \circ U$, where the U part is contributed by a $\bar{\mathcal{M}}_{Y,A}^1$, with one caveat that we must have $\text{ev}_{L+}(\tilde{u}) \neq b_\gamma$, for otherwise such a degeneration would be from a corner component instead of a boundary component of the compactification of $\mathcal{M}_{Y,A}^2$. If $\tilde{u}$ is cut out transversely, we can certainly arrange this by choosing a slightly different $b_{\gamma-}$. This shows the crucial dependence on basepoints, and we cannot count $\tilde{u}$ from $\#\bar{\mathcal{M}}_{Y,A}(\gamma, \{\gamma^-\} \cup \Gamma_2^-)^{\text{vir}}$ defined in [72], which is independent of the choice of basepoints. In particular, the virtual implementation of U -map cannot be built directly on the contact homology algebra in [72].

Instead of making sense of $\#\bar{\mathcal{M}}_{Y,A}^1$ and $\#\bar{\mathcal{M}}_{Y,A}^2$ directly and facing the difficulties mentioned above, we can follow an alternative way using the methods in [13]. More precisely, we can first define a positive S^1 -equivariant symplectic cohomology for algebraic augmentations using VFC, where the U -map is more natural. Then there is an isomorphism from the positive S^1 -equivariant symplectic cohomology to the linearized contact homology following the idea in [13], and we can transfer the U -map on the positive S^1 -equivariant symplectic cohomology to the linearized contact homology. Indeed, this is how the U -map on linearized contact homology in [13, Section 7.2] is motivated. By taking this detour, we do not need to modify the construction of contact homology, while the construction of the positive S^1 -equivariant symplectic cohomology and the isomorphism do not exceed the techniques provided in [71; 72]. For the sake of simplicity, we will defer this alternative approach to Claim 4.3 for later work.

4.2 Order of semidilation for fillings

The notions of k -dilation and k -semidilation were introduced in [90] as structures on S^1 -equivariant symplectic cohomology, which are generalizations of the symplectic dilation of Seidel and Solomon [75]. More precisely, an exact domain W carries a k -dilation if and only if there is a class $x \in \text{SH}_{+,S^1}^*(W)$ such that x is sent to 1 by $\text{SH}_{+,S^1}^*(W) \rightarrow H_{S^1}^{*+1}(W)$ and $U^{k+1}(x) = 0$, where

$$H_{S^1}^*(W) := H^*(W) \otimes (\mathbb{Q}[U, U^{-1}]/[U]).$$

W carries a k -semidilation if and only if x is sent to 1 in $\text{SH}_{+,S^1}^*(W) \rightarrow H_{S^1}^{*+1}(W) \rightarrow H^0(W)$ and $U^{k+1}(x) = 1$. Under the isomorphism $\text{SH}_{+,S^1}^*(W) = \text{LCH}_{2n-3-*}(W)$ of [13; 16] stated in Theorem 4.6

below, the element we are looking for in (4-2) is exactly the k -semidilation in [90]. It is natural to expect that examples with nontrivial SD come from examples with nontrivial k -(semi)dilation found in [90]. Indeed, it is the case, and we will show in Section 7 that SD is surjective. The only extra thing we need to argue, compared to [90, Definition 3.4], is that the computation is independent of the augmentation. To make the connection, we restate the main theorem of [13] as follows. We use $\mathrm{LCH}_*^{<A}(Y, \epsilon)$ to denote the truncated linearized contact homology generated by Reeb orbits of period smaller than A .

Theorem 4.6 [13] *Let W be a strict exact filling of (Y, α) . Assume $\mathrm{LCH}_*^{<A}(Y, \epsilon_W)$ is defined using a generic J , where ϵ_W is the augmentation from W (for instance assume either the conditions in [15], or that Reeb orbits with period smaller than A are simple). Then $\mathrm{LCH}_*^{<A}(Y, \epsilon_W)$ is isomorphic to the positive S^1 -equivariant symplectic cohomology $\mathrm{SH}_{+, S^1}^{*, <A}(W)$ of W using a Hamiltonian of slope A as $\mathbb{Q}[U]$ -modules. Assume Y is connected, under this isomorphism. Then $\ell_{\bullet, \epsilon_W}^1 : \mathrm{LCH}_*^{<A}(Y, \epsilon_W) \rightarrow \mathbb{Q}$ is isomorphic to $\mathrm{SH}_{+, S^1}^{*, <A}(W) \rightarrow H_{S^1}^{*+1}(W) = H^{*+1}(W) \otimes (\mathbb{Q}[U, U^{-1}]/U) \rightarrow H^0(W) = \mathbb{Q}$.*

Then item (4) of Claim 4.3 implies that we can use the U -map on positive S^1 -equivariant symplectic cohomology to estimate SD in (4-2).

Remark 4.7 From the Viterbo transfer map, a functor SD from $\mathcal{C}\mathrm{on}_* \rightarrow \mathbb{N} \cup \{\infty\}$ was defined in [90, Corollary D] using S^1 -equivariant symplectic cohomology. In the context of this paper, the order of semidilation in [90] is (4-2), using an augmentation from the exact filling. One can similarly define planarity of an exact filling. However, to establish the well-definedness and functoriality, we need to introduce the notation of homotopy between BL_∞ augmentations for linearized theories, which is beyond the scope of this paper. The trick in Propositions 3.13, 3.23 and 4.5 cannot help dropping the dependence on contact forms or auxiliary data, since it requires comparing the composition of the morphism from an exact cobordism X and the augmentation from an exact filling W to an augmentation from $X \circ W$. However this is essentially a BL_∞ homotopy from neck-stretching.

5 Lower bounds for planarity

As explained in Section 3, the curve responsible for finiteness of planarity is a curve with multiple positive punctures and a point constraint. Since planarity does not depend on the choice of the point, one should expect that finiteness of planarity implies uniruledness. In this section, we will prove such an implication and a lower bound for planarity. We first recall the notion of uniruledness from [60].

5.1 Order of uniruledness

Definition 5.1 [60, Section 2] Let (W, λ) be an exact domain. A $d\lambda$ -compatible almost complex structure J on W is convex if and only if there is a function ϕ such that

- (1) ϕ attains its maximum on ∂W and ∂W is a regular level set, and
- (2) $\lambda \circ J = d\phi$ near ∂W .

Definition 5.2 [60, Definition 2.2] Let $k > 0$ be an integer and $\Lambda > 0$ a real number. We say that an exact domain (W, λ) is (k, Λ) -uniruled if, for every convex almost complex structure J on W and every $p \in W^\circ$ (the interior of W) where J is integrable near p , there is a proper J -holomorphic map $u: S \rightarrow W^\circ$ passing through p , and the following holds:

- (1) S is a genus-0 Riemann surface and $\text{rank}(H_1(S; \mathbb{Q})) \leq k - 1$,
- (2) $\int_S u^* d\lambda \leq \Lambda$.

We say W is k -uniruled if W is (k, Λ) -uniruled for some $\Lambda > 0$.

The number Λ depends on the Liouville form λ ; this is not relevant for our purposes. However, the number k only depends on the Liouville structure up to homotopy.

Definition 5.3 Let W be an exact domain. We define the order of uniruledness by

$$U(W) := \min\{k \mid W \text{ is } k\text{-uniruled}\}.$$

The following was (inexplicitly) proven by McLean [60].

Proposition 5.4 U is a functor from $\mathcal{C}\text{on}_*$ to $\mathbb{N}_+ \cup \{\infty\}$.

Proof Let $V \subset W$ be an exact subdomain. Then $U(V) \leq U(W)$ by [60, Proposition 3.1]. It is clear from definition that $U(V, \lambda) = U(V, t\lambda)$ for $t > 0$. Since for any Liouville structure θ on V that is homotopic to λ , we have exact embeddings $(V, t^{-1}\lambda) \subset (V, \theta) \subset (V, t\lambda)$ for $t \gg 0$, it follows that U is a well-defined functor on $\mathcal{C}\text{on}_*$. $\square$

Remark 5.5 A point worth noting is that the definition and functorial property of U do not depend on any Floer theory. However, U gives a measurement of “complexity” of exact domains. By [90, Theorem 3.27], the existence of a k -(semi)dilation implies that the order of uniruledness is 1. Hence the order of (semi)dilation in [90, Corollary D] is a refined hierarchy in $U = 1$.

For an affine variety V , we define the order of algebraically uniruledness $AU(V)$ to be the minimal number k such that V is algebraically k uniruled, ie through every generic $p \in V$ there is a polynomial map $S \rightarrow V$ passing through p , where S is a punctured $\mathbb{CP}^1$ with at most k punctures.

Proposition 5.6 [60, Theorem 2.5] Let V be an affine variety. Then $U(V) \geq AU(V)$.

Example 5.7 Let S_k be the sphere with k disjoint disks removed. Then $U(S_k) = k$. Let $\Sigma_{g,k}$ be the genus $g \geq 1$ surface with k disjoint disks removed. Then $U(\Sigma_{g,k}) = \infty$. It is clear that S_k embeds exactly into S_{k+1} . However, S_{k+1} can only be embedded in S_k symplectically, not exactly.

In general, we have the following.

Theorem 5.8 We have $U((S_k)^n) = k$ and $U((\Sigma_{g,k})^n) = \infty$ for $g \geq 1$. In particular, U is a surjective functor in any dimension ≥ 2 .

Proof S_k^n has a projective compactification $(\mathbb{CP}^1)^n$, so we may assume the symplectic form is a product of the same symplectic form on $\mathbb{CP}^1$. Then, for any compatible almost complex structure J on $(\mathbb{CP}^1)^n$, there is a holomorphic curve passing through any fixed point, in the class $[\mathbb{CP}^1 \times \{\text{pt}\} \times \cdots \times \{\text{pt}\}]$ and intersecting each divisor $\{p_i\} \times (\mathbb{CP}^1)^{n-1}$ exactly once for $1 \leq i \leq k$, where p_i is the i^{th} puncture. We may assume J is an extension of a convex almost complex structure on S_k^n (which is not necessarily split). Therefore $U((S_k)^n) \leq k$, by neck-stretching.

On the other hand, the affine varieties corresponding to $(S_k)^n$ and $(\Sigma_{g,k})^n$ are $(\mathbb{CP}^1 \setminus \{p_1, \dots, p_k\})^n$ and $(\Sigma_g \setminus \{p_1, \dots, p_k\})^n$. A rational algebraic curve in $(\mathbb{CP}^1 \setminus \{p_1, \dots, p_k\})^n$ and $(\Sigma_g \setminus \{p_1, \dots, p_k\})^n$ projects to each factor as a rational algebraic curve. Then every rational curve must have at least k punctures. Therefore $AU((S_k)^n) = k$ and $AU((\Sigma_{g,k})^n) = \infty$, and the claim follows from [Proposition 5.6](#). $\square$

As a consequence, we find in each dimension a nested sequence of exact domains $V_1 \subset V_2 \cdots$ such that V_i cannot be embedded into V_j exactly if $i > j$. Sequences with such a property in dimensions ≥ 10 were also obtained in [\[54, Corollary 1.5\]](#).

Remark 5.9 In [Section 6](#), we will show that $P(\partial(S_k)^n) = k$ if $n \geq 2$. Therefore not only is there no exact embedding from $(S_{k+1})^n$ to $(S_k)^n$, but also there is no exact cobordism from $\partial(S_{k+1})^n$ to $\partial(S_k)^n$.

Remark 5.10 From U on $\mathcal{C}\text{on}_*$, we can build a functor U_∂ on $\mathcal{C}\text{on}$ via

$$U_\partial(Y) := \max\{U(W) \mid W \text{ is an exact filling of } Y\},$$

where the maximum of the empty set is defined as zero. Then [Corollary 5.15](#) below implies that $U_\partial \leq P$. The equality does not always hold. For example, $U_\partial(\mathbb{RP}^{2n-1}, \xi_{\text{std}}) = 0$ for $n \neq 2^k$ by [\[87\]](#), but $P(\mathbb{RP}^{2n-1}, \xi_{\text{std}}) = 1$ when $n \geq 3$ by [Theorem 7.30](#). Those discrepancies come from the difference between fillings and augmentations. It is possible to generalize the notions of order of uniruledness U and U_∂ to strong fillings or even weak fillings, but we will not pursue this in this paper.

In the following, we introduce an alternative definition of k -uniruledness but on the completion $\widehat{W}$, which is suitable to be related to SFT.

Definition 5.11 Let (W, λ) be an exact filling with a nondegenerate contact boundary. We say that the completion $\widehat{W}$ is k -uniruled if there exists $\Lambda > 0$ such that for every $p \in W^\circ$ and every admissible almost complex structure J that is integrable near p , there is a rational holomorphic curve passing through p with at most k positive punctures and contact energy of the curve is at most Λ .

Proposition 5.12 *An exact filling (W, λ) is k -uniruled if and only if $\widehat{W}_\epsilon$ is k -uniruled, where W_ϵ is Liouville homotopic to W with a nondegenerate contact boundary.*

Proof We first show that (W, λ) is k -uniruled implies $\widehat{W}_\epsilon$ is k -uniruled. Without loss of generality, we can take $W_\epsilon \subset W$, since we can rescale W_ϵ . By assumption there is a $\Lambda > 0$ such that for any $p \in W_\epsilon^\circ$ and any J integrable near p and convex near ∂W , there is a J -rational curve $u: S \rightarrow W$ with $\int_S u^* d\lambda < \Lambda$ and $H_1(S; \mathbb{Q}) \leq k - 1$. In particular, we can choose J to be cylindrical convex near ∂W_ϵ . Then by applying neck-stretching along ∂W_ϵ , we must have a rational holomorphic curve $u: S_\epsilon \rightarrow \widehat{W}_\epsilon$ passing through p with contact energy smaller than Λ . We know that S_ϵ is a punctured sphere, as ∂W_ϵ is nondegenerate. It is sufficient to prove $\text{rank } H_1(S_\epsilon; \mathbb{Z}) = \text{rank } H_1(S_\epsilon; \mathbb{Q}) \leq k - 1$. Assume otherwise. Then we know that $H_1(S_\epsilon; \mathbb{Z}) \rightarrow H_1(S; \mathbb{Z})$ is not injective, for if not, we have $\text{rank } H_1(S; \mathbb{Q}) \geq \text{rank } H_1(S_\epsilon; \mathbb{Q}) \geq k$. Therefore we find a class $[\gamma] \in H_1(S_\epsilon; \mathbb{Z})$ such that $[\gamma]$ is represented by a disjoint union γ of possibly multiply covered loops around punctures of S_ϵ , and there is an immersed surface A in $S \setminus S_\epsilon$ whose boundary is γ . Then in the fully stretched case, $u|_A$ corresponds to a holomorphic building with only negative punctures, which is impossible for energy reasons.

Now assume $\widehat{W}_\epsilon$ is k -uniruled. Without loss of generality, assume $W \subset W_\epsilon$. By [88, Proposition 5.3], any convex almost complex structure on W can be extended to an admissible almost complex structure on $\widehat{W}_\epsilon$. By assumption, there is a rational curve $u: S \rightarrow \widehat{W}_\epsilon$ passing through the chosen point $p \in W$ with S an at most k -punctured sphere and the contact energy of u is at most Λ . Let S' be the connected component of $u^{-1}(W^\circ)$ containing the point mapped to p . It is clear that the area of $u|_{S'}$ is bounded by Λ . We claim that $H_1(S'; \mathbb{Z}) \rightarrow H_1(S; \mathbb{Z})$ is injective. For otherwise, there is a class $A \in H_2(S, S'; \mathbb{Z})$ mapped to a nontrivial element by $H_2(S, S'; \mathbb{Z}) \rightarrow H_1(S'; \mathbb{Z})$. Then we can find an $S'' \subset S'$ such that $\lambda \circ J = d\phi$ on $u|_{S' \setminus S''}$, where ϕ is the function in Definition 5.1. Then by excision, we have A represented by an immersed surface in $S \setminus S''$ not contained completely in $S' \setminus S''$ with boundary in $S' \setminus S''$. Let $\hat{\phi}$ be the extension of ϕ on $\widehat{W}_\epsilon$ by [88, Proposition 5.3]. In particular, the maximum principle holds for $\hat{\phi}$. Then we arrive at a contradiction, since $\hat{\phi}(u)|_{\partial A} < \max \hat{\phi}(u)|_A$. Since $\mathbb{Q}$ is flat, we know that $H_1(S'; \mathbb{Q}) \rightarrow H_1(S; \mathbb{Q})$ is also injective, hence $\text{rank } H_1(S'; \mathbb{Q}) \leq \text{rank } H_1(S; \mathbb{Q}) \leq k - 1$. $\square$

5.2 Uniruledness and planarity

The main theorem of this section is the following.

Theorem 5.13 *If $P(Y) = k$, then any exact filling of Y is k -uniruled.*

An exact filling W gives rise to a BL_∞ augmentation ϵ_W over $\mathbb{Q}$. As a consequence we have a chain morphism $\hat{\ell}_{\bullet, \epsilon_W}: \bar{B}V \rightarrow \mathbb{Q}$ after fixing a point o in Y and auxiliary data. We can define a different map $\eta_W: \bar{B}V \rightarrow \mathbb{Q}$ by

$$(5-1) \quad \eta_W(q^{\Gamma^+}) = \sum_A \# \bar{\mathcal{M}}_{W, A, p}(\Gamma^+, \emptyset)$$

for a fixed point $p \in W$ such that p and o are in the same connected component of W , and $|\Gamma^+| = k$.

Proposition 5.14 *The map η_W is a chain morphism and is homotopic to $\hat{\ell}_{\bullet, \epsilon_W}$ with appropriate choices of auxiliary data, where ϵ_W is the augmentation from W . Moreover, η_W is compatible with the word-length filtration, ie for $k \geq 1$, the following diagram is commutative up to homotopy H :*

$$\begin{array}{ccc} (\bar{B}^k V, \hat{\ell}_{\epsilon_W}) & \xrightarrow{\hat{\ell}_{\bullet, \epsilon_W}} & (\mathbb{Q}, 0) \\ \downarrow \text{id} & \searrow H & \downarrow \text{id} \\ (\bar{B}^k V, \hat{\ell}_{\epsilon_W}) & \xrightarrow{\eta_W} & (\mathbb{Q}, 0) \end{array}$$

Proof That η_W defines a chain morphism, ie $\eta_W \circ \hat{\ell}_{\epsilon_W} = 0$, follows from the boundary of one-dimensional $\bar{\mathcal{M}}_{W, A, p}(\Gamma^+, \emptyset)$. Let γ be a path in W connecting p to o and use $\hat{\gamma}$ to denote the completion ray of γ in $\hat{W}$. Then the homotopy $H: \bar{B}V \rightarrow \mathbb{Q}$ is defined by

$$H(q^{\Gamma^+}) = \sum_A \# \bar{\mathcal{M}}_{W, A, \gamma}(\Gamma^+, \emptyset),$$

where $\bar{\mathcal{M}}_{W, A, \gamma}(\Gamma^+, \emptyset)$ is defined similarly to (5) before (3-3). The realization of those operators using virtual techniques is similar to $\phi_{\bullet}$ in (4) of Theorem 3.11 in Section 3.6. The homotopy relation, $\hat{\ell}_{\bullet, \epsilon_W} - \eta_W = H \circ \hat{\ell}_{\epsilon_W}$, comes from the boundary of one-dimensional $\bar{\mathcal{M}}_{W, A, \gamma}(\Gamma^+, \emptyset)$. It is clear that both η_W and H are compatible with the word-length filtration. $\square$

Proof of Theorem 5.13 The theory of BL_{∞} algebras considered for contact manifolds is equipped with a filtration by the contact action, where the action $\mathcal{A}(q_{\gamma})$ of a generator is $\int \gamma^* \alpha$. Then the action can be extended to EV and SV by declaring that the action of a monomial in SV , or a monomial of monomials in EV using the $\{q_{\gamma}\}$ as basis, is the sum of the actions, and the action of an element using those monomials as basis is the maximum of the actions of the monomials with nonzero coefficients. Then all of the operators for contact manifolds and exact cobordisms will decrease the action. It may not be true that the spectral invariant for $P(Y) = k$ is bounded for all BL_{∞} augmentations, but for an augmentation ϵ_W coming from an exact filling W , we have that the spectral invariant is bounded, ie there is a $\Lambda > 0$ and an $x \in \bar{B}^k V$ with $\mathcal{A}(x) \leq \Lambda$, $\hat{\ell}_{\epsilon_W}(x) = 0$, and $\hat{\ell}_{\bullet, \epsilon_W}(x) = 1$. Then by Proposition 5.14, we have $\eta_W(x) = 1$. We must have that the unperturbed/geometric $\bar{\mathcal{M}}_{W, A, p}(\Gamma_+, \emptyset)$ is not empty for some Γ_+ with $|\Gamma_+| \leq k$ and $\sum_{\gamma \in \Gamma_+} \int \gamma^* \alpha \leq \Lambda$, for otherwise we would have $\eta_W(x) = 0$ by Axiom 3.26. It is clear from the proof of Proposition 5.14 that p can be any point in W . This shows that $\hat{W}$ is (k, Λ) uniruled; by Proposition 5.12, W is k -uniruled. $\square$

Theorem 5.13 provides a lower bound for P .

Corollary 5.15 *Let W be an exact filling of Y . Then $P(Y) \geq U(W)$. If W is an affine variety, then $P(Y) \geq \text{AU}(W)$.*

6 Upper bounds for planarity

In this section, we will obtain upper bounds of P for the following two cases.

6.1 Iterated planar open books

Definition 6.1 [2, Definition 2.2] An *iterated planar Lefschetz fibration* $f: (W^{2n}, \omega) \rightarrow \mathbb{D}$ on a $2n$ -dimensional Weinstein domain (W^{2n}, ω) is an exact symplectic Lefschetz fibration satisfying the following properties:

- (1) There exists a sequence of exact symplectic Lefschetz fibrations $f_i: (W^{2i}, \omega_i) \rightarrow \mathbb{D}$ for $i = 2, \dots, n$ with $f = f_n$.
- (2) The total space (W^{2i}, ω_i) of f_i is a regular fiber of f_{i+1} for $i = 2, \dots, n-1$.
- (3) $f_2: (W^4, \omega_2) \rightarrow \mathbb{D}$ is a planar Lefschetz fibration, ie the regular fiber of f_2 is a genus-zero surface with nonempty boundary, which we denote by W^2 .

Definition 6.2 [2, Definitions 2.3 and 2.4] An *iterated planar open book decomposition* of a contact manifold (Y^{2n+1}, ξ) is an open book decomposition for Y whose page W admits an iterated planar Lefschetz fibration which supports the contact structure ξ in the sense of Giroux. We say that (Y, ξ) is iterated planar (IP).

If the number of boundary components of W^2 in the above definition is k , we say that (Y, ξ) is *k-iterated planar* or *k-IP*. We remark that the collection of IP contact manifolds is already a large class of examples, as for instance the fundamental group is not an obstruction in any fixed dimension at least 5; see [4, Theorem 1.4].

Theorem 6.3 Let (Y, ξ) be a *k-IP* contact manifold. Then $P(Y) \leq k$.

The strategy for obtaining an upper bound $P(Y) \leq k$ on the planarity of a contact manifold (Y, ξ) is via the following algebrogeometric condition.

Lemma 6.4 Let (Y, ξ) be a contact manifold. Assume the following holds.

- ($\star$)_k There exists a point $o \in \mathbb{R} \times Y$, a contact form α for ξ , a choice of α -compatible cylindrical almost complex structure J on $\mathbb{R} \times Y$, and some collection $\Gamma = (\gamma_1, \dots, \gamma_k)$ of precisely k distinct, nondegenerate and simply covered α -Reeb orbits, for which the following holds.
- (1)_k If $\Gamma^+ \subseteq \Gamma$ and $\Gamma^- \neq \emptyset$, then $\mathcal{M}_{Y,A}(\Gamma^+, \Gamma^-) = \emptyset$ for every homology class A .
 - (2)_k The moduli space $\mathcal{M}_{Y,A,o}(\Gamma, \emptyset)$ is transversely cut out for every A with expected dimension 0.
 - (3)_k For some choice of coherent orientations, the algebraic count of the k -punctured spheres in $\bigcup_{A, \text{vdim}=0} \mathcal{M}_{Y,A,o}(\Gamma, \emptyset)$ is nonzero.

Then $P(Y) \leq k$.

Proof By the first property, we have $\bar{\mathcal{M}}_{Y, \mathcal{A}, o}(\Gamma^+, \Gamma^-) = \emptyset$ for any $\Gamma^+ \subseteq \Gamma$ and $\Gamma^- \neq \emptyset$. Therefore by [Axiom 3.26](#), the second and third conditions imply that $\hat{\ell}_{\bullet, \epsilon}(q^{\Gamma^+}) \neq 0$ for any augmentation ϵ (if there is no augmentation, then $P(Y) = 0$ by definition). Moreover, by the first property, we have that q^{Γ^+} is closed in $(\bar{S}V, \hat{\ell}_\epsilon)$ for any augmentation ϵ . Then the claim follows. $\square$

Proof of Theorem 6.3 We proceed by induction on dimension.

If $\dim Y = 3$, then an IP contact 3-manifold is simply a planar contact 3-manifold. Fix a choice of planar open book supporting the contact structure, with page a sphere with k -disks removed, and so with binding consisting of k circles. One then constructs an adapted Giroux form so that each component of the binding is a nondegenerate and simply covered orbit, and a *holomorphic* open book as for instance in [\[80\]](#). This provides a Fredholm-regular foliation of $\mathbb{R} \times Y$ whose leaves are either trivial cylinders over the binding, or holomorphic Fredholm-regular k -punctured spheres projecting to pages and asymptotic to the binding. One can prove via standard four-dimensional arguments coming from Siefring intersection theory (the same as in higher-dimensions, as used below), that any curve whose positive asymptotics are a subset of the binding is a leaf of this foliation. The argument is as follows: given such a curve v , let u be a leaf in the foliation which is not a trivial cylinder (and therefore has no negative ends). Denote the set of positive asymptotics of v by Γ_v^+ . By [\[51, Lemma 4.9\]](#), the Siefring intersection between u and v is given by

$$(6-1) \quad u * v = \sum_{\gamma_\xi \in \Gamma_v^+} u * (\mathbb{R} \times \gamma_\xi).$$

In a suitable trivialization, we have that the Conley–Zehnder index of every binding component is $\mu_{\text{CZ}}(\gamma) = 1$. Moreover, we have the relation $\mu_{\text{CZ}}(\gamma) = 2\alpha_-(\gamma) + p(\gamma)$, where $\alpha_-(\gamma)$ is the largest winding number of an eigenfunction of negative eigenvalue for the asymptotic operator of γ , and $p(\gamma)$ is the parity of the Conley–Zehnder index. We conclude that $\alpha_-(\gamma) = 0$. This coincides with the asymptotic winding number of u about each γ , which is therefore an extremal winding number. This means that there are no intersections between u and $\mathbb{R} \times \gamma$ coming from infinity, ie $u * (\mathbb{R} \times \gamma_\xi) = 0$ for each binding component γ . Combining with (6-1), we conclude that $u * v = 0$. On the other hand, if v were not a leaf in the foliation, by positivity of intersections we would have $u * v > 0$ for some leaf u ; a contradiction.

While this a priori holds for an almost complex structure which is compatible with a SHS deforming the contact form, one may perturb this SHS to nearby contact data without changing the isotopy class of the contact form, and for which the binding still consists of closed Reeb orbits. After perturbing the original almost complex structure J_0 to a J which is compatible with this nearby contact data and generic, the curves in the foliation survive by Fredholm regularity (an open condition). Moreover, the uniqueness statement still holds if the perturbation is small enough, ie every holomorphic curve whose positive asymptotics are a subset of the binding is a perturbation of a leaf in the original foliation. Indeed, if v_k is a sequence of such J_k -holomorphic curves with $J_k \rightarrow J_0$, by SFT compactness we obtain a building v_∞ as a limit configuration. Applying uniqueness for J_0 to the topmost floor, we conclude that

v_∞ is a noncylindrical leaf in the foliation. By Fredholm regularity, it follows that v_k is a perturbed leaf of the foliation for J_0 if k is sufficiently large.

In particular, $(1)_k$ and $(2)_k$ in Lemma 6.4 hold for the perturbed J . In this case the geometric (and hence the algebraic) count of these curves with a point constraint is 1 for any generic point o , and so $(3)_k$ is also satisfied. We fix such an o for which we have this uniqueness property.

If $\dim Y \geq 5$, we fix an IP open book $\pi: Y \setminus B \rightarrow S^1$ supporting ξ , with binding $B \subset Y$, a codimension-2 contact submanifold. Since B is also k -IP if Y is, we may assume by induction that $(\star)_k$ holds for B . We may then extend the Giroux contact form α_B on B for which $(\star)_k$ holds to a Giroux contact form α on Y , in such a way that all k Reeb orbits $\Gamma = (\gamma_1, \dots, \gamma_k)$ from the induction step are still nondegenerate orbits in Y . Indeed, one can do this in a local model near the binding: take a collar neighborhood $B \times \mathbb{D}$ of B , and let $\alpha = (1 - r^2)(\alpha_B + r^2 d\theta)$, where (r, θ) are polar coordinates on the $\mathbb{D}^2$ -factor; then α is extended to the mapping torus piece as in Giroux's construction. The Reeb vector field of α coincides with that of α_B along B , and so each γ_i is still an orbit for α . Moreover, the linearized Reeb flow for each γ_i splits into components tangent and normal to B . The first component is nondegenerate by assumption; the second one is too, since the Hessian of the function $1 - r^2$ is nondegenerate at the critical value $r = 0$.

On the other hand, the holomorphic open book construction can also be done in arbitrary dimensions — again, after deforming the Giroux form away from B to a stable Hamiltonian structure; cf [19, Appendix A] and [64; 65]. The choice of almost complex structure can be taken to agree with the one from the inductive step along $H_B := \mathbb{R} \times B$, which is then a holomorphic submanifold. The leaves of the resulting codimension-2 foliation are now either H_B , or a codimension-2 holomorphic submanifold which is a copy of the Liouville completion of the page, and which is asymptotic to H_B at infinity in the sense of [66]. We let $\mathcal{F}$ denote this codimension-2 foliation on $\mathbb{R} \times Y$. The moduli space $\mathcal{M}_B$ of k -punctured spheres defined on $\mathbb{R} \times B$ in the inductive step extends to a moduli space $\mathcal{M}_Y$ on $\mathbb{R} \times Y$, consisting of curves having the same positive asymptotics as curves in $\mathcal{M}_B$. An application of Siefring intersection theory as in [65; 66] — this is *exactly* the same argument as in the three-dimensional case done above — shows that *any* holomorphic curve u whose positive asymptotics are a subset of Γ either completely lies in H_B , or its image lies completely in a noncylindrical leaf H of $\mathcal{F}$. In the first case, u cannot have negative ends by $(1)_k$ applied to B . In the second case, since H has no negative ends, neither does u , and so $(1)_k$ holds on $\mathbb{R} \times Y$. This also shows that $\mathcal{M}_{Y,A,o}(\Gamma, \emptyset) = \mathcal{M}_{B,A,o}(\Gamma, \emptyset)$ for every $o \in H_B$. If moreover we take $o \in H_B$ to be the point given by the inductive step, for which (by the base case) we may assume that we have the uniqueness property that curves in $\mathcal{M}_{B,A,o}(\Gamma; \emptyset)$ are necessarily elements in $\mathcal{M}_B$ (for every A), and in particular transversely cut out inside H_B , then the same analysis as carried out in [64, Lemma 4.13] (by splitting the normal linearized CR-operator into tangent and normal components with respect to H_B , and using automatic transversality on the normal summand) shows that curves in $\mathcal{M}_B$ are transversely cut out in $\mathbb{R} \times Y$. Then $(2)_k$ holds, and $(3)_k$ also (and in fact the geometric count is 1 for our particular choice of o). Note that all of these conditions still hold after perturbing to nearby

contact data, via the same argument as above, ie using SFT compactness and using uniqueness on the topmost level together with Fredholm regularity. An appeal to [Lemma 6.4](#) finishes the proof. $\square$

It is clear from the definition that the Weinstein conjecture holds for contact manifolds with finite planarity. In the case of iterated planar open books, the Weinstein conjectures was proven for dimension 3 in [\[1\]](#) and higher dimensions in [\[2; 5\]](#). In view of the proof of Weinstein conjecture, [Theorem 6.3](#) is of the same spirit as the proofs in [\[1; 2; 5\]](#). However, more importantly, [Theorem 6.3](#) endows the holomorphic curve with SFT meaning. We further remark that the proof of the above theorem in the 5-dimensional case actually provides a foliation, as opposed to a homological one, as shown in [\[65\]](#).

[Theorem 6.3](#) can be viewed as a special case of the following conjecture.

Conjecture 6.5 *Let Y be an open book whose page is W . Then $P(Y) \leq P(W)$ and $SD(Y) \leq SD(W)$.*

In the context of semidilations in symplectic cohomology, [\[90, Proposition 3.31\]](#) proved such a claim for Lefschetz fibrations. The geometric intuition behind the conjecture is clear and was used in [Theorem 6.3](#), the difficulty lies in making the virtual machinery compatible with the geometry for general Y and W .

6.2 Trivial planar SOBDs

We now consider a related example as to the ones considered above. Fix $(S_k, d\lambda)$ a sphere with k -disks removed together with a Liouville form λ , and let $(M, d\alpha)$ be any $2n$ -dimensional Liouville domain. Define $(V := S_k \times M, \omega = d(\lambda + \alpha))$, endowed with the product Liouville domain structure. Let $(Y = \partial V, \xi = \ker(\alpha + \lambda))$ be the contact manifold filled by V .

Theorem 6.6 *If $c_1(M) = 0$ or M supports a perfect exhausting Morse function, then we have $P(Y) \leq k$.*

We first note that Y admits a supporting (trivial) SOBD, as considered in [\[57; 64\]](#). In other words, we have a decomposition

$$Y = Y_S \cup Y_P,$$

where $Y_S = \partial S_k \times M$ is the *spine* and $Y_P = S_k \times \partial M$ is the *paper*, and we have trivial fibrations

$$\pi_S: Y_S \rightarrow M \quad \text{and} \quad \pi_P: Y_P \rightarrow \partial M.$$

We view the first one as a contact fibration over a Liouville domain, and the second as a Liouville fibration over a contact manifold (its fibers are called the *pages*). By choosing a small Morse function H on the *vertebrae* M (the base of the spine) which vanishes near ∂M and has a unique maximum and no minimum (as critical points), we may perturb the contact form along Y_S to $e^H(\alpha + \lambda)$.

As explained in [\[64, Section 3.1\]](#), each critical point $p \in M$ of H gives rise to k nondegenerate Reeb orbits of the form $\gamma_{p,i} = S^1 \times \{p\} \subset Y_S$, where S^1 is the boundary of the i^{th} puncture of S_k . One then

deforms the contact form to a stable Hamiltonian structure which coincides with $\mathcal{H} = (\alpha, d(\alpha + \lambda))$ on Y_P , and so its kernel there is $TS_k \oplus \ker \alpha$, tangent to S_k . After this, one can construct a compatible cylindrical almost complex structure J_0 which preserves the splitting $T(\mathbb{R} \times Y_P) = (T\mathbb{R} \oplus \langle R_\alpha \rangle) \oplus TS_k \oplus \ker \alpha$. We first prove the following proposition, which constrains the possible holomorphic curves:

Proposition 6.7 *Let $\Gamma = \{\gamma_{p_1, j_1}, \dots, \gamma_{p_s, j_s}\}$ be a set of simple Reeb orbits such that p_i is a critical point of H and $1 \leq j_1 < j_2 < \dots < j_s \leq k$. Then there exists an almost complex structure J such that if H is chosen sufficiently C^2 -small, then any rational J -holomorphic curve u in $\mathbb{R} \times Y$ whose positive asymptotics form a subset of Γ is one of the following.*

- (1) *A trivial cylinder over $\gamma_{p,i}$.*
- (2) *A cylinder u from $\gamma_{p,j}$ to $\gamma_{q,j}$ in $\mathbb{R} \times S^1 \times M \subset \mathbb{R} \times Y_S$, where S^1 is the boundary of the j^{th} puncture of S_k . Moreover, there is a one-to-one correspondence from such cylinders to negative gradient flow lines of H from p to q .*
- (3) *$\Gamma = \{\gamma_{p_1,1}, \dots, \gamma_{p_k,k}\}$ and u has no negative puncture.*

Proof We first prove the claim for the stable Hamiltonian structure $\mathcal{H}$. First of all, one can achieve that all orbits below a fixed action threshold correspond to critical points of the Morse function. For energy reasons, and using that the number of positive ends of u is a priori bounded by k , one can choose the action threshold large enough (depending on k) so that the negative ends of u also necessarily correspond to critical points, and the number of negative ends (counted with covering multiplicity) is bounded above by the number of positive ends; cf [64, Lemma 3.8]. We then separate into two cases: either the image of u is fully contained in $\mathbb{R} \times Y_S$ (case A), or it is not (case B).

Case A Since Y_S has k connected components, by the assumptions on Γ , u has precisely one positive end, and by the above discussion, we have that u has at most one negative end. Since orbits $\gamma_{p,i}$ are not contractible in Y_S , we then see that u has precisely one negative end, which is simply covered.

We now argue via holomorphic cascades. In the degenerate case, where $H \equiv 0$, the trivial projection $\pi_S: \mathbb{R} \times Y_S \rightarrow M$ is actually J_0 -holomorphic. Therefore $v = \pi_S \circ u$ is a holomorphic curve in M . The asymptotics of u project to points in M , so v extends to a holomorphic map on a closed surface. But M is exact, and so v is constant, and we deduce that u is a trivial cylinder.

If we take $H_t = tH$, we denote by J_t the corresponding almost complex structure (which only differs in that it intertwines the varying R_t with the $\mathbb{R}$ -direction). If we assume we have a sequence $\{u_n\}$ of J_{t_n} -holomorphic maps with $t_n \rightarrow 0$, with one positive and negative simply covered orbits corresponding to critical points $p_\pm$, then we obtain a stable holomorphic cascade u_∞^H as a limiting object; see [8]. Since the positive end of u_n is simply covered, for energy reasons as explained above, every Reeb orbit appearing in u_∞^H is simply covered, and therefore each of its holomorphic map components cannot be multiply covered. Stability of the cascade means that it does not have trivial cylinder components. We conclude that the

space of holomorphic cascades which glue to curves as in our hypothesis consists solely of flow-lines, which are regular by the Morse–Smale condition, and come in an $(\text{ind}_{p_+}(H) - \text{ind}_{p_-}(H) - 1)$ -dimensional family. Note that such flow-line cylinders can *always* be glued to another flow-line cylinder. The implicit function theorem then implies that u is a flow-line cylinder if H is sufficiently small. Indeed, we can argue as follows. Consider the parametric moduli space

$$\overline{\mathcal{M}} := \overline{\mathcal{M}}(\mathbb{R} \times M_Y, \{J_t\}_{t \in [0,1]}; \gamma_{p_+}, \gamma_{p_-}) = \{(t, u) \mid t \in [0, 1], u \in \overline{\mathcal{M}}(\mathbb{R} \times M_Y, J_t; \gamma_{p_+}, \gamma_{p_-})\},$$

where $\overline{\mathcal{M}}(\mathbb{R} \times M_Y, J_t; \gamma_{p_+}, \gamma_{p_-})$ denotes the Gromov-compactified moduli space of J_t -holomorphic curves in $\mathbb{R} \times M_Y$ (of any genus) which have simply covered positive asymptotic γ_{p_+} , and negative asymptotic γ_{p_-} , and where the compactification at $t = 0$ corresponds to stable holomorphic cascades. Since the Reeb orbits are simply covered, curves in the parametric moduli are somewhere injective. Then, by the Morse–Smale condition and the implicit function theorem, the parametric moduli space is, for sufficiently small H , an $(\text{ind}_{p_+}(H) - \text{ind}_{p_-}(H))$ -dimensional compact manifold whose boundary contains $\overline{\mathcal{M}}(\mathbb{R} \times M_Y, J^0; \gamma_{p_+}, \gamma_{p_-})$, only consisting of flow line cylinders. The flow-line parametric moduli space

$$\begin{aligned} \overline{\mathcal{M}}_{\text{flow-line}} &:= \overline{\mathcal{M}}_{\text{flow-line}}(\mathbb{R} \times M_Y, \{J_t\}_{t \in [0,1]}; \gamma_{p_+}, \gamma_{p_-}) \\ &:= \{(t, u) \mid t \in [0, 1], u \text{ corresponds to a flow-line in } \overline{\mathcal{M}}(\mathbb{R} \times M_Y, J_t; \gamma_{p_+}, \gamma_{p_-})\} \end{aligned}$$

is, a priori, a submanifold of $\overline{\mathcal{M}}(\mathbb{R} \times M_Y, \{J_t\}_{t \in [0,1]}; \gamma_{p_+}, \gamma_{p_-})$, which shares the boundary component $\overline{\mathcal{M}}(\mathbb{R} \times M_Y, J^0; \gamma_{p_+}, \gamma_{p_-})$, and has its same dimension. By thinking of $\overline{\mathcal{M}}$ as a collar neighborhood of this boundary component, we obtain

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}_{\text{flow-line}},$$

from which our uniqueness follows. Observe that since there are finitely many critical points, we can take H (or $\epsilon > 0$) uniformly small.

In what follows, we start with a $J = J_t$ corresponding to t sufficiently small.

Case B We argue via energy considerations. Denote by $Z^\pm \subset \Sigma$ the positive/negative punctures of u , where Σ is a closed Riemann surface, and $\dot{\Sigma} = \Sigma \setminus (Z^+ \cup Z^-)$ is the domain of u . Since orbits associated to Z^+ correspond to critical points of H , so does every orbit associated to Z^- (which may a priori be multiply covered). For $z \in Z^\pm$, denote by $\gamma_{p_z}^{\kappa_z}$ the corresponding orbit, where $p_z \in \text{crit}(H)$, and $\kappa_z \geq 1$ is the covering multiplicity ($\kappa_z = 1$ for all $z \in Z^+$). By assumption, $\#Z^+ \leq k = \#\pi_0(Y_S)$. We now estimate the Ω -energy of u , where $\mathcal{H} = (\beta, \Omega)$ is the SHS, and Ω is exact, the primitive being $e^H(\alpha + d\theta)$ along Y_S . Using Stokes' theorem, we obtain

$$E(u) := \int_{\dot{\Sigma}} u^* \Omega = 2\pi \left(\sum_{z \in Z^+} e^{H(p_z)} - \sum_{z \in Z^-} \kappa_z e^{H(p_z)} \right) \leq 2\pi \|e^H\|_{C^0 \# Z^+} \leq 2\pi k \|e^H\|_{C^0}.$$

We now bound $E(u)$ from below. The projection $p: \mathbb{R} \times Y_P \rightarrow S_k$ is also holomorphic. Moreover, the same is true in a small closed ϵ -neighborhood of $\mathbb{R} \times Y_P$ inside $\mathbb{R} \times Y$, which intersects $\mathbb{R} \times Y_S$ along a collar

of the form $\mathbb{R} \times \partial S_k \times [0, \epsilon] \times \partial M$, along which β picks up a $d\theta$ component. Let $S := S_k \cup_{\partial} \partial S_k \times [0, \epsilon]$ this small extension of S_k . For each $s \in S$, $E_s := \mathbb{R} \times \{s\} \times \partial M$ is a holomorphic hypersurface, the fiber of p over s . By assumption, u has nontrivial intersection with $\mathbb{R} \times Y_P$, which is foliated by these holomorphic hypersurfaces E_s . By Sard's theorem, we may shrink ϵ if necessary so that u is transverse to $\mathbb{R} \times \partial S \times M$. Therefore $\Sigma' := u^{-1}(\mathbb{R} \times S \times M)$ is a smooth (possibly disconnected) surface with boundary. Note that since the asymptotics of u are away from S , the intersection of u with any of the E_s consists of a finite collection of points in the domain of u , which are away from the punctures. We then see that the algebraic intersection $\deg(u) := u \cdot E_s$ is *positive*, independent of s , and in fact is the degree of $F := p \circ u: \Sigma' \rightarrow S$, which is a holomorphic branched cover.

Write $\partial \Sigma' = \bigcup_{i=1}^l C_i$, oriented with the boundary orientation, where C_i is a simple closed curve whose image under F wraps around one of the boundary components of S with winding number $n_i \neq 0$ (measured with respect to $d\theta$). By holomorphicity of F , one easily sees that $n_i > 0$. Since u intersects E_s for every $s \in \partial S$, we have $l \geq k$. By counting preimages of a point $s \in \partial S$, we obtain

$$\int_{\partial \Sigma'} u^* d\theta = 2\pi \sum_{i=1}^l n_i = 2\pi k \deg(u).$$

Using that $\Omega = d\alpha + d\lambda$ along S , that $\lambda = e^t d\theta$ near ∂S where $t \in [0, \epsilon]$, that $u^* d\alpha \geq 0$ by choice of J_0 , and Stokes' theorem, we see that

$$E(u) \geq \int_{\Sigma'} u^* \Omega = \int_{\Sigma'} u^* (d\alpha + d\lambda) \geq \int_{\Sigma'} u^* d\lambda = e^\epsilon \int_{\partial \Sigma'} u^* d\theta = 2\pi k e^\epsilon \deg(u).$$

Combining with the previous upper bound for $E(u)$, we obtain

$$2\pi k \leq 2\pi k e^\epsilon \deg(u) \leq 2\pi \left(\sum_{z \in Z^+} e^{H(p_z)} - \sum_{z \in Z^-} \kappa_z e^{H(p_z)} \right) \leq 2\pi \|e^H\|_{C^0 \# Z^+} \leq 2\pi k \|e^H\|_{C^0}.$$

Since H can be taken arbitrarily close to 0, we see that $\#Z^+ = k$, $\#Z^- = 0$ and $\deg(u) = 1$. This proves the proposition for the SHS $\mathcal{H}$ and J_0 . It is clear the same holds for any nearby contact structure and almost complex structure. $\square$

Proof of Theorem 6.6 If Γ^+ is set of Reeb orbits in form of $\gamma_{p,i}$ of cardinality at most k , then Proposition 6.7 implies that $\mathcal{M}_Y(\Gamma^+, \Gamma^-)$ is empty if $\Gamma^- \neq \emptyset$, unless $\Gamma^+ = \{\gamma_{p,i}\}$ and $\Gamma^- = \{\gamma_{q,i}\}$ with $\text{ind } p > \text{ind } q$. In particular, this implies that $\hat{\ell}_\epsilon(q^{\Gamma^+})$ is independent of ϵ . Moreover, we have $\text{vdim } \mathcal{M}_{Y,o}(\{\gamma_{p,i}\}, \{\gamma_{q,i}\}) = \text{ind}(p) - \text{ind}(q) - 2n < 0$, hence $\hat{\ell}_{\bullet, \epsilon}(q^{\Gamma^+})$ is also independent of ϵ . As a result, it suffices to find one ϵ and one Γ^+ such that $\hat{\ell}_\epsilon(q^{\Gamma^+}) = 0$ and $\hat{\ell}_{\bullet, \epsilon}(q^{\Gamma^+}) \neq 0$. In view of Proposition 5.14, the latter condition is satisfied if $\eta_{M \times S_k}(q^{\Gamma^+}) \neq 0$. In particular, we can work with the nice exact filling $M \times S_k$.

Assume we start with a Liouville form on S_k such that the simple Reeb orbits on each boundary component have period 1 and hence the area of S_k is k . Following [91, Section 2.1], by choosing the Liouville form

on M sufficiently large, ie the Reeb orbits on ∂M have sufficiently large period, we may assume all Reeb orbits below a period threshold are of the form $\gamma_{p,i}^m$, ie the m^{th} multiple cover of the Reeb orbit over the critical point p of H and the i^{th} boundary of S_k , so that the period of $\gamma_{p,i}^m$ is approximately m . On homology level (both in $H_1(\partial(M \times S_k))$ and $H_1(M \times S_k)$), the only relations are

$$[\gamma_{p,i}^m] = m[\gamma_{p,i}], \quad [\gamma_{p,i}] = [\gamma_{q,i}] \quad \text{and} \quad \sum_{i=1}^k [\gamma_{p,i}] = 0.$$

Using the existence of symplectic caps (see [24, Theorem 1.6], [53, Corollary 1.14] and [56, Theorem 3.2]), we know that there exists an exact embedding of M into W , where the contact boundary ∂W is the contact boundary of some affine variety. Then we consider a projective compactification of the affine variety by adding simple normal crossing divisors D , but instead we add the divisors D to W to obtain $\overline{W}$. Then $\overline{W}$ is a closed symplectic manifold whose symplectic form is Poincaré dual to D . Now we consider holomorphic curves in $\overline{W} \times \mathbb{CP}^1$, which contains $M \times S_k$ as a domain with contact boundary; here the symplectic area of $\mathbb{CP}^1$ is slightly bigger than k so as to contain S_k . We use $\overline{\mathcal{M}}_{J,o}$ to denote the compactified moduli spaces of holomorphic spheres passing through $o \in M \times S_k$ with homology class $A + [* \times \mathbb{CP}^1]$ for $A \in H_2(M)$ with $c_1(A) = 0$. We can form such a compactification, since M is exact; in particular, those curves have uniformly bounded symplectic area. If we pick J to be the split almost complex structure on $\overline{W} \times \mathbb{CP}^1$, then $\overline{\mathcal{M}}_{J,o}$ is cut out transversely and consists of exactly one point. Then we deform the almost complex structure so that it coincides with the almost complex structure in Proposition 6.7 near $\partial(M \times S_k)$, and also apply neck-stretching along $\partial(M \times S_k)$. We claim transversality holds for this process for generic choices. First we will prove that the moduli space of curves in class $A + [* \times \mathbb{CP}^1]$ for $A \in H_2(M)$ (hence A has zero symplectic area) with $c_1(A) = 0$ has no bubble degeneration in the compactification. For this, we pick an almost complex structure that makes $D \times \mathbb{CP}^1$ holomorphic and is in split form near $D \times \mathbb{CP}^1$. Suppose there is a bubble degeneration with a component in class $B + k[* \times \mathbb{CP}^1]$ for $B \in H_2(\overline{W})$. If B has negative symplectic area, ie the intersection with $D \times \mathbb{CP}^1$ is negative, by the positivity of intersection, we have that the component is contained in $D \times \mathbb{CP}^1$. Since the almost complex structure is split, the projection to D has negative symplectic area, a contradiction. Therefore all components in a hypothetical bubble degeneration are of the form $B + k[* \times \mathbb{CP}^1]$ where the symplectic area of B is zero. Now since $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^1)$ cannot be decomposed into multiple components with positive symplectic area, we see that there is no bubble degeneration. Note that the region where we make assumptions on the almost complex structure is outside the contact boundary that we apply neck-stretching to. In the fully stretched case, due to action and first homology class reasons, we must have that the bottom curve, the one with the point constraint, has asymptotics in the form of $\Gamma^+ = \{\gamma_{p_1,1}, \dots, \gamma_{p_k,k}\}$. Moreover, since all relevant Reeb orbits are simple, we may assume the full-stretched moduli space is also cut out transversely.

For $\Gamma^+ = \{\gamma_{p_1,1}, \dots, \gamma_{p_k,k}\}$, we first note that $\text{vdim } \mathcal{M}_{M \times S_k, A, o}(\Gamma^+, \emptyset)$ is

$$(6-2) \quad 2\langle c_1(M), A \rangle + \sum_{i=1}^k \text{ind}(p_i) - 2nk,$$

where we view the relative homology class A as in $H_2(M)$. To see this, we first consider S_k . Let γ_i be the Reeb orbit on the i^{th} boundary, using a trivialization of the symplectic vector bundle over S_k , the Conley–Zehnder indices satisfy

$$\sum_{i=1}^k \mu_{\text{CZ}}(\gamma_i) = 2(2-k).$$

Following [88, Proof of Theorem 6.3], in the product $M \times S_k$, the normal direction picks up an extra Conley–Zehnder index of $\text{ind}(p_i) - n$, ie using the trivialization over the bounding surface S_k , we have

$$\sum_{i=1}^k \mu_{\text{CZ}}(\gamma_{p_i,i}) = \sum_{i=1}^k (\text{ind}(p_i) - n) + 2(2-k).$$

Hence

$$\text{vdim } \mathcal{M}_{M \times S_k, A, o} = 2\langle c_1(M), A \rangle + (n-2)(2-k) + \sum_{i=1}^k \mu_{\text{CZ}}(\gamma_{p_i,i}) - 2n = (6-2).$$

Next we separate the proof into two cases.

- (1) If $c_1(M) = 0$, then $\text{vdim } \mathcal{M}_{M \times S_k, A, o}(\Gamma^+, \emptyset)$ is independent of the relative homology class A . Moreover, the expected dimension in (6-2) is nonpositive and is zero if and only if $p_1 = \dots = p_k$ is the unique maximum $p_{\max}$ of H . Then the above neck-stretching argument implies that

$$\bigcup_A \mathcal{M}_{M \times S_k, A, o}(\Gamma^+, \emptyset) = \bigcup_A \bar{\mathcal{M}}_{M \times S_k, A, o}(\Gamma^+, \emptyset),$$

which has nontrivial algebraic count for $\Gamma^+ = \{\gamma_{p_{\max},1}, \dots, \gamma_{p_{\max},k}\}$. In particular, $\eta_{M \times S_k}(q^{\Gamma^+}) \neq 0$. By Proposition 6.7, we have

$$\hat{\ell}_\epsilon(q^{\Gamma^+}) = \sum_{i=1}^k \sum_{\text{ind } q = 2n-1} \# \bar{\mathcal{M}}_Y(\{\gamma_{p_{\max},i}\}, \{\gamma_{q,i}\}) q_{\gamma_{p_{\max},1}} \cdots q_{\gamma_{q,i}} \cdots q_{\gamma_{p_{\max},k}} = 0,$$

since $p_{\max}$ is the unique maximum. Hence we have $P(Y) \leq k$.

- (2) If M supports a perfect exhausting Morse function, then we can assume the perturbation H is perfect. Since $c_1(M)$ is not necessarily 0, the neck-stretching argument above only gives us

$$\bigcup_{2c_1(A) + \sum_{i=1}^k \text{ind}(p_i) - 2nk = 0} \mathcal{M}_{M \times S_k, A, o}(\Gamma^+, \emptyset) = \bigcup_{2c_1(A) + \sum_{i=1}^k \text{ind}(p_i) - 2nk = 0} \bar{\mathcal{M}}_{M \times S_k, A, o}(\Gamma^+, \emptyset),$$

with nontrivial algebraic count for some $\Gamma^+ = \{\gamma_{p_1,1}, \dots, \gamma_{p_k,k}\}$. Since H is perfect, ie the count of rigid gradient flow lines between every pair of critical points is zero, Proposition 6.7 implies that $\hat{\ell}_\epsilon(q^{\Gamma^+}) = 0$. Therefore we have $P(Y) \leq k$. $\square$

Corollary 6.8 *Let V be an affine variety with $c_1(V) = 0$ such that $\text{AU}(V) \geq k$. Then $P(\partial(S_k \times V)) = k$.*

Proof By Theorem 6.6, $P(\partial(S_k \times V)) \leq k$. On the other hand it is easy to see that $\text{AU}(S_k \times V) = k$, as every algebraic curve in $V \times S_k$ projects to an algebraic curve in both V and S_k . Then the claim follows from Corollary 5.15. $\square$

7 Examples and applications

In this section, we will discuss two more classes of examples, where we can compute the hierarchy functors. The first case is smooth affine varieties with a $\mathbb{CP}^n$ compactification, or more generally a Fano hypersurface compactification. The second case is links of singularities, including links of Brieskorn singularities and quotient singularities by the diagonal action of cyclic groups. In particular, we will finish the proof of [Theorem B](#).

7.1 Affine varieties

Let V be a smooth affine variety. Then V is naturally a Weinstein manifold by viewing $V \subset \mathbb{C}^N$, and the function $|x - x_0|^2$ on $\mathbb{C}^N$ restricted to V is a Morse function with finitely many critical points for a generic $x_0 \in \mathbb{C}^N$; see [\[63, Section 6\]](#). In particular, we obtain a contact manifold by taking the intersection of V with the boundary of a large enough ball. We will use ∂V to denote the contact boundary of the intersection; the interior of the intersection is called the associated Liouville domain. Both notions up to homotopy are independent of the size of the large ball.

An alternative way of associating a Weinstein structure to V is by using a smooth projective compactification $\bar{V}$, with an ample line bundle $\mathcal{L}$ with a holomorphic section s such that $s^{-1}(0)$ is a normal crossing divisor such that $V = \bar{V} \setminus s^{-1}(0)$. We choose a metric on $\mathcal{L}$ such that the curvature is a Kähler form ω on $\bar{V}$. Then by [\[74, Lemma 4.3\]](#), $h = -\log |s|$ and $-d^{\mathbb{C}}h$ defines a Weinstein structure (possibly after a compactly supported perturbation) on V . The equivalence of these two definitions can be found in [\[60\]](#).

We first give a description of the embedding relations of affine varieties with the same projective compactification.

Lemma 7.1 *Let X be a smooth projective variety with a very ample line bundle $\mathcal{L}$ endowed with a Hermitian metric. For $s \in H^0(\mathcal{L})$, we use V_s to denote the Liouville domain associated to the affine variety $X \setminus s^{-1}(0)$. Then for $s \neq 0 \in H^0(\mathcal{L})$, there exists $\epsilon > 0$ such that for all $t \in H^0(\mathcal{L})$ with $|s - t| < \epsilon$, V_s embeds exactly into V_t .*

Proof With the very ample line bundle $\mathcal{L}$, X can be embedded in $\mathbb{P}H^0(\mathcal{L})$ such that every $s \in H^0(\mathcal{L})$ corresponds to a hyperplane $H_s \subset \mathbb{P}H^0(\mathcal{L})$ and $s^{-1}(0) = X \cap H_s$. We can view the Liouville domain V_s as the intersection of X with a large ball in the identification of $\mathbb{C}^N$ with $\mathbb{P}H^0(\mathcal{L}) \setminus H_s$. Then for t sufficiently close to s , ie H_t sufficiently close to H_s , the Liouville form of V_t restricted to $V_s \cap S_R$ is a contact form, where S_R is the radius $R \gg 0$ sphere in $\mathbb{C}^N$. The Gray stability theorem implies that all of them induce the same contact structure on $V_s \cap S_R$ for t sufficiently close to s , and the Liouville form of V_t restricted to $V_s \cap B_R$ is homotopic to the Liouville structure on V_s ; hence V_s embeds exactly into V_t . $\square$

Remark 7.2 In principle, the exact embedding from V_s to V_t should be built from a Weinstein cobordism. Hence one expects a more precise description of the Weinstein cobordism, which depends on the deformation from s to t . Some results in this direction can be found in [3; 68].

Roughly speaking, we should have a stratification on $\mathbb{P}H^0(\mathcal{L})$ indexed by the singularity type of $s^{-1}(0)$. The index set forms a category by declaring there to be a morphism from stratum A to stratum B if the closure of B contains A . Then Lemma 7.1 implies that we have a functor from the index set (which should be a poset) to $\mathfrak{Con}_*$. Making such description precise is not easy, as we do not have a classification of singularities of $s^{-1}(0)$ in general. However, we can describe some subcategory of the index set. The following lemma is also very useful in understanding the embedding relations of affine varieties which arise from different line bundles.

Lemma 7.3 [74, Lemma 4.4] *Assume that the smooth affine variety V has a smooth projective compactification $\bar{V}$. Assume there are two ample line bundles $\mathcal{L}_i$ with sections s_i such that $s_1^{-1}(0) = s_2^{-1}(0) = \bar{V} \setminus V$ is normal crossings, but possibly with different multiplicities. Then the Liouville structures on V defined by the s_i are homotopic.*

Example 7.4 $\mathbb{CP}^n$ minus k generic hyperplanes can be viewed as the complement of $(s_1 \otimes \cdots \otimes s_k)^{-1}(0)$ for generic sections s_i of $\mathcal{O}(1)$. On the other hand, $\mathbb{CP}^n$ minus $k-1$ generic hyperplanes can be viewed as the complement of $(s_1 \otimes s_1 \otimes s_3 \otimes \cdots \otimes s_k)^{-1}(0)$ by Lemma 7.3. As a consequence of Lemma 7.1, we have an exact embedding of $\mathbb{CP}^n$ minus $k-1$ generic hyperplanes to $\mathbb{CP}^n$ minus k generic hyperplanes. As a simple example, $\mathbb{CP}^2$ minus a line is $\mathbb{C}^2$, $\mathbb{CP}^2$ minus two generic lines is $\mathbb{C} \times T^*S^1$, and $\mathbb{CP}^2$ minus three generic lines is T^*T^2 . It is clear that we have the embedding relations. Moreover, some of the relations cannot be reversed, eg T^*T^2 cannot be embedded exactly into $\mathbb{C}^2$ or $\mathbb{C} \times T^*S^1$. But $\mathbb{C} \times T^*S^1$ can be embedded back into $\mathbb{C}^2$ by adding a 2-handle corresponding to the positive Dehn twist in the trivial open book for $\partial(\mathbb{C} \times T^*S^1)$. More generally, $\mathbb{CP}^n$ minus k generic hyperplanes is $\mathbb{C}^{n+1-k} \times T^*T^{k-1}$ for $k \leq n$, and they can be embedded into each other exactly.

Example 7.5 $\mathbb{CP}^2$ minus 3 hyperplanes passing through the same point is $\mathbb{C} \times S_3$, where S_3 is the thrice punctured sphere. Since $\mathbb{CP}^2$ minus 2 hyperplanes can still be viewed as a further degeneration, we have that $\mathbb{C} \times T^*S^1$ embeds to $\mathbb{C} \times S_3$, which is obviously true. On the other hand, $\mathbb{CP}^2$ minus 3 generic hyperplanes, ie T^*T^2 , contains $\mathbb{C} \times S_3$ as an exact subdomain. Moreover, $\mathbb{CP}^2$ minus a smooth degree-2 curve is $T^*\mathbb{RP}^2$, which is obtained from attaching a 2-handle to $\mathbb{C} \times T^*S^1$, ie $\mathbb{CP}^2$ minus 2 generic lines. $\mathbb{CP}^2$ minus a smooth degree-3 curve can be described as attaching three 2-handles to T^*T^2 ; see [3] for details. It is not obvious if the complement of a smooth degree-2 curve embeds exactly into the complement of a smooth degree-3 curve. However, the former embeds exactly into the complement of a smooth degree-4 curve by Lemmas 7.1 and 7.3.

Let D be a divisor. We use D^c to denote the complement affine variety. Our main theorem in this section is the following.

Theorem 7.6 *Let D be k generic hyperplanes in $\mathbb{CP}^n$ for $n \geq 2$. Then we have the following.*

- (1) $P(\partial D^c) \geq k + 1 - n$ for $k > n + 1$.
- (2) $P(\partial D^c) = k + 1 - n$ for $n + 1 < k < \frac{1}{2}(3n - 1)$ and n odd.
- (3) $P(\partial D^c) = 2$ for $k = n + 1$.
- (4) $H_{\text{cx}}(\partial D^c) = 0^{\text{SD}}$ for $k \leq n$.

The strategy to obtain [Theorem 7.6](#) is to first prove $P(\partial D^c) \geq \max\{1, k + 1 - n\}$ by index computations. Then we obtain that the planarity of the affine variety D^c is at most $\max\{1, k + 1 - n\}$ by looking at the affine variety D_s^c , where D_s is the smoothing of D , ie a smooth degree- k hypersurface. Finally, we use index computations to show that the relevant portion of the computation of planarity is independent of the BL_∞ augmentation for $\text{RSFT}(\partial D^c)$ when $n + 1 < k < \frac{1}{2}(3n - 1)$. The benefit of considering the smoothing is twofold. First, the relevant holomorphic curve theory for the pair $(\mathbb{CP}^n, D_s)$ is easier both in terms of analysis and computation. In particular, we can use the computation of relative Gromov–Witten invariants in [\[37\]](#) to supply the holomorphic curve we need. Second, ∂D^c carries more complicated Reeb dynamics than ∂D_s^c , which makes it harder to prove the independence of augmentations by looking at ∂D^c alone. More precisely, for any augmentation of $\text{RSFT}(\partial D_s^c)$ given by the composition of the BL_∞ morphism from $\text{RSFT}(\partial D_s^c)$ to $\text{RSFT}(\partial D^c)$ with an augmentation of $\text{RSFT}(\partial D^c)$, we will show that the planarity for $\text{RSFT}(\partial D_s^c)$ of that augmentation is $\max\{1, k + 1 - n\}$. Then, planarity of $\text{RSFT}(\partial D^c)$ is $\max\{1, k + 1 - n\}$ for any augmentation by functoriality. The condition of n being odd is to obtain automatic closedness of a suitable chain in $\bar{S}V_{\partial D_s^c}$ for any augmentation, and is expected to be irrelevant. However, to drop this constraint, we need to use stronger transversality properties supplied by [\[86\]](#); see [Remark 7.23](#) for more discussion. The $n + 1 < k < \frac{1}{2}(3n - 1)$ condition is likely not optimal but possibly necessary. It is a difficult task to compute planarity for all augmentations. There are many affine varieties with a $\mathbb{CP}^n$ compactification such that the contact boundary has infinite planarity, while the planarity of the affine variety, ie using the augmentation from the affine variety, is finite; see [Theorem 7.14](#). In particular, different augmentations do make a difference in general. Therefore it is a subtle question to determine which affine variety has finite planarity. In general, we need to develop a computation method of RSFT from log/relative Gromov–Witten invariants like the symplectic (co)homology computation in [\[27\]](#).

7.1.1 Reeb dynamics on the divisor complement In this part, we describe the Reeb dynamics on the boundary of a tubular neighborhood of a simple normal crossing divisor. The general description was obtained in [\[59, Section 5\]](#), but see also [\[35, Section 2.1\]](#), [\[61\]](#), [\[78\]](#) or [\[36, Section 2.4\]](#). In the following, we state the special cases for [Theorem 7.6](#).

Case 1 (a smooth degree- k hypersurface in $\mathbb{CP}^n$ for $n \geq 2$) Let D be a smooth degree- k hypersurface in $\mathbb{CP}^n$. Then the contact boundary is the concave boundary of the $\mathcal{O}(k)$ line bundle over D , which

carries a natural Morse–Bott contact form whose Reeb flow is the S^1 action on the circle bundle. Consider the hypersurface $r = f$ in $\mathcal{O}(k)$, where r is the radial coordinate. Given an action threshold C , one can choose a C^2 -small and positive Morse function f such that all Reeb orbits in this hypersurface, of action less than C , have the following properties.

- (1) There is a simple Reeb orbit γ_p over every critical point p of f and these are all of the simple Reeb orbits. We use γ_p^m to denote the m^{th} cover of γ_p . All of the Reeb orbits are good and nondegenerate. The period of γ_p^m is greater than the period of γ_q^l if and only if $m = l$ and $f(p) < f(q)$ or $m > l$. (Note that f is the r -coordinate of the perturbed hypersurface in the symplectic cap $\mathcal{O}(k)$, hence larger value of f means smaller period of the S^1 -fiber.) As the contact structure is the Boothby–Wang contact structure, computation can be found in eg van Koert’s note [49]. It is also a special case of the more general normal crossing case in [36, Theorem 2.7].
 - (2) Using the obvious disk cap¹⁷ bounded by γ_q^m in the symplectic cap $\mathcal{O}(k)|_D$ that intersects D once with order m , which induces a trivialization of $\det_{\mathbb{C}} \xi$, the Conley–Zehnder index satisfies
- $$(7-1) \quad n - 3 - \mu_{\text{CZ}}(\gamma_p^m) = 2m - 2 + \text{ind}(p),$$
- where $\text{ind}(p)$ is the Morse index of p . This is a special case of [36, Proposition 2.10].
- (3) The class $[\gamma_p] \in H_1(D^c)$ is a generator of order k , as the intersection of a generic line $A \simeq \mathbb{CP}^1 \subset \mathbb{CP}^n$ with D^c shows that $k[\gamma_p]$ is null-homologous. On the other hand, if $l[\gamma_p]$ is null-homologous in D^c for $l < k$, by gluing with the obvious caps, we get a second homology class in $\mathbb{CP}^n$ whose intersection number with D is l , a contradiction.

One way to understand (7-1) is following: $n - 3 - \mu_{\text{CZ}}(\gamma_p^m)$ is the virtual dimension of the moduli space of holomorphic disks in $\mathcal{O}(k)$ with the same relative homology class as the bounding disk. Then $n - 3 - \mu_{\text{CZ}}(\gamma_p^m) + 2 - 2m$ is the virtual dimension of the moduli spaces of holomorphic disks in the bounding disk homology class, with one marked point intersecting D with order m . In the Morse–Bott case, ie when the contact form is Boothby–Wang and f is used in a cascades model, then the moduli space mentioned above is cut out transversely and is identified with the stable manifold of p , whose dimension is $\text{ind}(p)$. When $p_{\min}$ is the unique minimum of f , by the argument in [14] and transversality for the cascade model, we know that the moduli space of holomorphic disks in the cap $\mathcal{O}(k)$ asymptotic to $\gamma_{p_{\min}}^m$ with one marked point intersecting D of order m and homology class the bounding disk is cut out transversely, compact, with algebraic count ± 1 .

Case 2 (k generic hyperplanes in $\mathbb{CP}^n$ for $k \geq n + 1$) Let $D_1, \dots, D_k$ denote the k hyperplanes. Let $I \subset \{1, \dots, k\}$ be a set of cardinality at most n . We define D_I to be the intersection $\bigcap_{i \in I} D_i$, which is a copy of $\mathbb{CP}^{n-|I|}$. We define $\check{D}_I$ by $D_I \setminus (D_I \cap \bigcup_{i \notin I} D_i)$. Let N_i be the normal disk bundle over D_i . Then $\bigoplus_{i \in I} \partial N_i|_{D_I}$ is a $T^{|I|}$ bundle over D_I . Then the contact boundary is topologically decomposed as

¹⁷That is, map from $\mathbb{CP}^1 \setminus \mathbb{D}$ and view the boundary as a negative end. In other words, it is the unit disk $\mathbb{D}$ mapped to $\mathcal{O}(k)|_D$, such that the induced boundary map with the usual boundary orientation is $-\gamma_q^m$.

$\bigcup_{I \subset \{1, \dots, k\}} \bigoplus_{i \in I} \partial N_i|_{\check{D}_I}$. We pick an exhausting Morse function f_I on each $\check{D}_I$, ie the gradient of f_I points out along $\partial \check{D}_I$. The Reeb dynamics has the following properties.

- (1) For each critical point p of f_I and any function $t: \{1, \dots, k\} \rightarrow \mathbb{N}$ with $\text{supp } t = I$, we have a $T^{|I|-1}$ Morse–Bott family of Reeb orbits γ_p^t . The homology class of γ_p^t is given by the image of $t \in H_1(T^{|I|}) = \mathbb{Z}^{|I|}$ in $H_1(D^c)$ by inclusion, where $T^{|I|}$ is the $T^{|I|}$ fiber of $\bigoplus_{i \in I} \partial N_i|_{\check{D}_I}$ over p . This follows from [59, Section 5] or [78, Theorem 2.17], which is summarized in [36, Theorem 2.7].
- (2) $H_1(D^c)$ is generated by the simple circles $[\beta_i]$ wrapping around D_i once (ie the oriented boundary of small disks that intersect with D_i negatively once) subject to the relation $\sum_{i=1}^k [\beta_i] = 0$; see [55, Proposition 2.3]. The homology class $[\gamma_p^t]$ over $\check{D}_I$ is $\sum_{i \in I} t(i)[\beta_i]$; see [36, Theorem 2.7(5)]. Moreover, there is a natural disk cap with boundary γ_p^t intersecting D_i of order $t(i)$ (by cap instead of disk/filling, we emphasize the boundary map with induced orientation is $-\gamma_p^t$).
- (3) The generalized Conley–Zehnder index, using the obvious disk cap whose intersection number with D_i is $t(i)$ for $i \in I$, is given by¹⁸

$$n - 3 - \mu_{\text{CZ}}(\gamma_p^t) = 2 \sum_{i \in I} t(i) - 2 + \text{ind}(p) + \frac{1}{2}(|I| - 1).$$

In the following, we use $\sum t$ as a shorthand for $\sum_{i \in I} t(i)$. After a perturbation, the $T^{|I|-1}$ family of Reeb orbits degenerate to $2^{|I|-1}$ nondegenerate orbits corresponding to generators of $H_*(T^{|I|-1})$, and the Conley–Zehnder indices span the region

$$(7-2) \quad n - 3 - \mu_{\text{CZ}} \in \left[2 \sum t - 2 + \text{ind}(p), 2 \sum t - 2 + \text{ind}(p) + |I| - 1 \right].$$

This follows from [36, Proposition 2.10]. We use $\check{\gamma}_p^t$ to denote the orbit with $n - 3 - \mu_{\text{CZ}}(\check{\gamma}_p^t) = 2 \sum t - 2 + \text{ind}(p)$, and $\hat{\gamma}_p^t$ to denote the orbit with $n - 3 - \mu_{\text{CZ}}(\hat{\gamma}_p^t) = 2 \sum t - 2 + \text{ind}(p) + |I| - 1$.

Remark 7.7 Following [59, Lemmas 5.17 and 5.18], the period of γ_p^t is close to (smaller than) $\sum_{i \in I} t(i)$ with a small discrepancy depending on the symplectic size of the neighborhood of D that is removed and a smaller discrepancy from the perturbation from f , with the property that the period of γ_p^t is smaller than that of γ_q^t if and only if $f(p) > f(q)$. In particular, $\sum t$ can be thought as the period of the Morse–Bott family with the “ideal” case of only removing the divisor D . The period will be further perturbed after we perfect the $T^{|I|-1}$ family into nondegenerate orbits. Those newly created orbits have periods arbitrarily close to the period of γ_p^t and the period of $\check{\gamma}_p^t$ is larger than the period of $\hat{\gamma}_p^t$. In the general case, for $s \in H^0(\mathcal{L})$ with $s^{-1}(0)$ normal crossings, and $s^{-1}(0) = \sum_{i=1}^k a_i D_i$ as a divisor, when we use $-\text{d}^{\mathbb{C}} \log |s|$ as the Liouville structure on $X \setminus s^{-1}(0)$, after the deformation as in [59, Lemma 5.17] to organize the Liouville form nicely near the boundary, we have a similar description of Reeb orbits as the above, and

¹⁸The extra $\frac{1}{2}(|I| - 1)$ is from the $T^{|I|-1}$ Morse–Bott family, which after perturbation spans the region of Morse indices of the Morse function $T^{|I|-1}$ as in (7-2). It particular, it is analogous to Case 1 for smooth divisors. An analogous situation can be found in [62, Theorem 5.16]. The two situations are different in the sense that [62] considered the symplectic filling by normal crossing divisors instead of a symplectic cap and used a preferred global trivialization of a power of the contact distribution.

the period of the Reeb orbits with intersection order $t: \{1, \dots, k\} \rightarrow \mathbb{N}$ with $\text{supp } t = I$ over $\check{D}_I$ is given by $\sum_{i=1}^k a_i t(i)$ minus a (arbitrarily) small discrepancy.

7.1.2 Lower bound of $P(\partial D^c)$

Proposition 7.8 *Let $D = D_1 \cup \dots \cup D_k$ denote the $k > n + 1$ generic hyperplanes in $\mathbb{C}\mathbb{P}^n$ for $n \geq 2$.*

- (1) *For any Reeb orbits set $\Gamma := \{\gamma_1, \dots, \gamma_r\}$ for $r < k + 1 - n$ with $\sum[\gamma_i] = 0 \in H_1(D^c)$, the virtual dimension of the moduli space $\overline{\mathcal{M}}_{D^c, A, o}(\Gamma, \emptyset)$ is less than 0 for any A .*
- (2) *For any Reeb orbits set $\Gamma := \{\gamma_1, \dots, \gamma_{k+1-n}\}$ with $\sum[\gamma_i] = 0 \in H_1(D^c)$ and such that the virtual dimension of the moduli space $\overline{\mathcal{M}}_{D^c, A, o}(\Gamma, \emptyset)$ is nonnegative, there is a partition of $\{1, \dots, k\}$ into subsets $I_1, \dots, I_{k+1-n}$ such that $\Gamma = \{\check{\gamma}_{p_i, \min}^{\sigma_{I_i}}\}_i$, where $p_{i, \min}$ is the minimum on $\check{D}_{I_i}$ and σ_{I_i} is the indication function supported on I_i , ie $\sigma_{I_i}(j) = 1$ if $j \in I_i$ and is otherwise zero.*

Proof Since $c_1(D^c) = 0$, the virtual dimension does not depend on A , and we will suppress it from the notation in the following discussion (the same applies everywhere in this subsection). Given a curve u in the same homotopy class of a curve in $\mathcal{M}_{D^c, o}(\Gamma, \emptyset)$, we use u_i to denote the natural disk cap of γ_i . Then we have

$$\begin{aligned} \text{ind}(u) + \sum_{i=1}^r (n - 3 - \mu_{\text{CZ}}(\gamma_i)) &= 2c_1\left(u \# \#_{i=1}^r u_i\right) + 2(n - 3) - 2n + 2 \\ &= 2c_1\left(u \# \#_{i=1}^r u_i\right) - 4. \end{aligned}$$

Here, note that the $-2n + 2$ comes from the point constraint. We assume γ_i is of the form $\gamma_{p_i}^{t_i}$ after perturbations. Since $\sum_{i=1}^r [\gamma_i] = 0$ in homology, if we view each t_i as a k -dimensional integral vector, we have $\sum_{i=1}^r t_i = (N, \dots, N)$ for some $N \in \mathbb{N}_+$. Since t_i keeps track of the intersection of the natural disk u_i with D , we know that $[u \# \#_{i=1}^r u_i] \cap D$ is Nk points. In particular, $[u \# \#_{i=1}^r u_i]$ is N times the generator in $H_2(\mathbb{C}\mathbb{P}^n)$, and $c_1(u \# \#_{i=1}^r u_i) = N(n + 1)$. Then

$$(7-3) \quad \text{ind}(u) \leq 2N(n + 1) - 4 - \sum_{i=1}^r \left(2 \sum t_i - 2 + \text{ind}(p_i)\right)$$

$$\begin{aligned} (7-4) \quad &\leq 2N(n + 1) - 4 - 2 \sum_{i=1}^r \sum t_i + 2r \\ &= 2N(n + 1 - k) - 4 + 2r \\ &= 2(N - 1)(n + 1 - k) + 2(r + n - k - 1) < 0 \quad \text{when } r < k + 1 - n. \end{aligned}$$

If $r = k + 1 - n$, to have $\text{ind}(u) \geq 0$, we must have $N = 1$. In this case, both inequalities (7-3) and (7-4) must be equalities. In particular, $\text{ind}(p_i) = 0$ and γ_i must be a check orbit $\check{\gamma}_{p_i}$, ie the claim holds. $\square$

Then by Proposition 5.14, we have the following.

Corollary 7.9 (lower bound on $P(\partial D^c)$) *If D denotes k generic hyperplanes for $k > n + 1$ and $n \geq 2$, then $P(\partial D^c) \geq k + 1 - n$. The same holds for ∂V_s if s is a perturbation of such a D .*

Proof Let D be k generic hyperplanes as in the statement. Then η_{D^c} on $\bar{B}^r V_{\partial D^c}$ in Proposition 5.14 is zero for $r < k + 1 - n$ by Proposition 7.8. Therefore $P(\partial D^c) \geq k + 1 - n$. The remainder of the claim follows from Lemma 7.1 and the functorial property of P . $\square$

In the following, we will separate the proof of the upper bound of $P(\partial D^c)$ into two steps, namely, we will first show the existence of a holomorphic curve that is responsible for the finite planarity, and then we will argue that the phenomenon is independent of augmentations for certain k .

7.1.3 Step one for the upper bound of $P(\partial D^c)$: source of holomorphic curves

Proposition 7.10 *Let D_s be a smooth degree- k hypersurfaces in $\mathbb{CP}^n$. Then the following holds.*

- (1) *If $k \leq n$, for a point $o \in D_s^c$, there is a Reeb orbit γ_p^k with $\text{ind}(p) = 2(n - k)$ and an admissible complex structure such that $\bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_p^k\}, \emptyset)$ is cut out transversely and $\#\bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_p^k\}, \emptyset) \neq 0$.*
- (2) *If $k \geq n + 1$, for a point $o \in D_s^c$, there are two Reeb orbits $\gamma_{p_{\min}}^n$ and $\gamma_{p_{\min}}$, with $p_{\min}$ the unique minimum on D_s , and an admissible almost complex structure such that*

$$\bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \underbrace{\gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}}_{k-n}\}, \emptyset)$$

is cut out transversely with nontrivial algebraic count.

Proof This follows from applying neck-stretching to $\mathbb{CP}^n$ along ∂D_s^c . We denote by

$$\text{GW}_{0, k, (s_1, \dots, s_l), A}^{\mathbb{CP}^n, D_s}(C_1, \dots, C_k, E_1, \dots, E_l)$$

the relative Gromov–Witten invariant that counts genus-zero holomorphic curves in class A with k marked points going through $C_1, \dots, C_k \in H_*(\mathbb{CP}^n)$ and l marked points going through $E_1, \dots, E_l \in H_*(D_s)$ and which intersect D_s with multiplicity at least $s_1, \dots, s_l$, respectively; see [45]. The source of holomorphic curves is the nonvanishing relative Gromov–Witten invariants

$$\begin{aligned} &\text{GW}_{0, 1, (k), A}^{\mathbb{CP}^n, D_s}([\text{pt}], [D_s] \cap^{n-k} [H]) \quad \text{for } k \leq n, \\ &\text{GW}_{0, 1, (n, 1, \dots, 1), A}^{\mathbb{CP}^n, D_s}([\text{pt}], [D_s], \dots, [D_s]) \quad \text{for } k > n, \end{aligned}$$

respectively, from [37], where $H \in H_{2n-2}(\mathbb{CP}^n)$ is the hyperplane class and A is the generator of $H_2(\mathbb{CP}^n)$. More precisely, when $k > n$, by the divisor axiom we have

$$\text{GW}_{0, 1, (n, 1, \dots, 1), A}^{\mathbb{CP}^n, D_s}([\text{pt}], [D_s], \dots, [D_s]) = (k)^{k-n} \text{GW}_{0, 1, (n), A}^{\mathbb{CP}^n, D_s}([\text{pt}], [D_s]).$$

By applying [37, Theorem 2.6] $n - 1$ times,¹⁹ we have

$$\begin{aligned} \mathrm{GW}_{0,1,(n),\mathcal{A}}^{\mathbb{CP}^n,D_s}([\mathrm{pt}], [D_s]) &= \int_{\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^n, \mathcal{A})} \mathrm{ev}_1^* \mathrm{PD}([\mathrm{pt}]) \wedge \mathrm{ev}_2^* \mathrm{PD}([D_s]) \wedge \prod_{i=1}^{n-1} (\mathrm{ev}_2^* \mathrm{PD}([D_s]) + i\psi) \\ &= \int_{\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^n, \mathcal{A}, \mathrm{pt})} \mathrm{ev}_2^* \mathrm{PD}([D_s]) \wedge \prod_{i=1}^{n-1} (\mathrm{ev}_2^* \mathrm{PD}([D_s]) + i\psi), \end{aligned}$$

where $\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^n, \mathcal{A})$ is the compactified moduli space of holomorphic spheres in class $\mathcal{A} \in H_2(\mathbb{CP}^n)$ with two marked points (ev_1 and ev_2 are two evaluation maps), and $\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^n, \mathcal{A}, \mathrm{pt})$ is the one where the first marked point is subject to a point constraint, while ψ is the psi class [37, page 183] in descendant Gromov–Witten invariants [48, Section 4.5.5]. Note that $\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^n, \mathcal{A}, \mathrm{pt}) \simeq \mathbb{CP}^n$ and ψ is the first Chern class of the tautological line bundle. As a consequence, we have

$$\mathrm{GW}_{0,1,(n),\mathcal{A}}^{\mathbb{CP}^n,D_s}([\mathrm{pt}], [D_s]) = \prod_{i=1}^n (k - i) > 0.$$

Similarly, when $k < n$, we have

$$\begin{aligned} \mathrm{GW}_{0,1,(k),\mathcal{A}}^{\mathbb{CP}^n,D_s}([\mathrm{pt}], [D_s] \cap^{n-k} [H]) &= \int_{\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^n, \mathcal{A}, \mathrm{pt})} \mathrm{ev}_2^* \mathrm{PD}([D_s] \cap^{n-k} [H]) \wedge \prod_{i=1}^{k-1} (\mathrm{ev}_2^* \mathrm{PD}([D_s]) + i\psi) \\ &= k!. \end{aligned}$$

Since a curve in class $\mathcal{A}$ is necessarily somewhere injective and not contained in D_s because we can choose the $[\mathrm{pt}]$ class in D_s^c , one can assume transversality in the process of neck-stretching. In the fully stretched picture, each connected component of the bottom curve has at most $\max\{1, k + 1 - n\}$ positive punctures, for otherwise genus has to be created. If the component of the bottom curve with the point constraint has $0 < r < \max\{1, k + 1 - n\}$ positive punctures, in particular, $k \geq n + 1$, we assume the positive asymptotics are $\Gamma^+ = \{\gamma_{p_i}^{d_i}\}_{1 \leq i \leq r}$. Since $\sum [\gamma_{p_i}^{d_i}] = 0$ in homology, and $[\gamma_{p_i}]$ is the generator of $H_1(D_s^c)$ of order k , we have $\sum d_i = km$ for some m . Then the expected dimension of $\overline{\mathcal{M}}_{D_s^c, o}(\Gamma^+, \emptyset)$ is given by

$$\begin{aligned} \mathrm{ind}(u) &= 2m(n + 1) - 4 - \sum_{i=1}^r (n - 3 - \mu_{\mathrm{CZ}}(\gamma_{p_i}^{d_i})) = 2m(n + 1) - 4 - \sum_{i=1}^r (2d_i - 2 + \mathrm{ind}(p_i)) \\ &\leq 2m(n + 1) - 4 - 2 \sum_{i=1}^r d_i + 2r = 2m(n + 1 - k) + 2r - 4 \\ &= 2(m - 1)(n + 1 - k) + 2(r + n - k - 1) < 0. \end{aligned}$$

Therefore the component of the bottom curve with the point constraint must have $\max\{1, k + 1 - n\}$ positive punctures. Moreover, from the above computation, we separate the proof into three cases.

¹⁹ $D_{\alpha,k}(X, \beta)$ in [37, Theorem 2.6], ie those bubble trees with a subtree contained in the divisor, is empty, as we are considering degree 1 curves.

Case 1 ($k > n + 1$) To have $\text{ind}(u) \geq 0$, we must have that p_i is the minimum $p_{\min}$ and $m = 1$. That is, the positive asymptotics of the bottom curve are $\{\gamma_{p_{\min}}^{d_i}\}_{1 \leq i \leq k+1-n}$ with $\sum d_i = k$. Next, we consider the curves in the symplectic cap. Because a curve in the symplectic cap must intersect D_s , there are at most $k + 1 - n$ connected components of the top curve. If there are less than $k + 1 - n$ connected components of the top curve, genus must be created because the bottom curve component with a point constraint has $\max\{1, k + 1 - n\}$ positive punctures. As a consequence, one component intersects D_s with order n , and $k - n$ components intersect D_s with order 1. Note that the component v in the symplectic cap that intersects D_s with order n must be asymptotic to $\{\gamma_{q_1}^{n_1}, \dots, \gamma_{q_l}^{n_l}\}$ for critical points q_* with $\sum_{i=1}^l n_i = n$. Assume, otherwise, that the sum of multiplicity is $n + km$ for $m \geq 1$. Then the relative homology class of v is the same as the sum of the natural disk of γ_q^{n+km} and $-mA$ for the positive generator $A \in H_2(D_s)$ that is mapped to the generator of $H_2(\mathbb{CP}^n)$. Then the symplectic area of v is the sum of the area of the natural disk and $-m\omega_{\mathbb{CP}^n}(A)$. Since the symplectic area of the natural disk can be arbitrarily small if we only remove a sufficiently small neighborhood of the divisor, the symplectic area of v is negative, a contradiction. Similarly, the component intersecting D_s with order 1 must be a sphere with one negative puncture asymptotic to γ_q for a critical point q . The component intersecting D_s with order n must be a sphere with one negative puncture asymptotic to γ_q^n for a (potentially different) critical point q . Assume otherwise; then to glue to a sphere in $\mathbb{CP}^n$, we must have at least another component in the bottom level. Since the total symplectic area of the curves outside the symplectic cap is approximately k times the period of γ_q , which is approximately the symplectic area of the bottom curve with a point constraint, there is no action room for another bottom-level curve. Therefore the curves in the middle symplectization level must be cylinders, we must have $(d_1, \dots, d_{k+1-n}) = (n, 1, \dots, 1)$. Moreover, since $\gamma_{p_{\min}}^n$ and $\gamma_{p_{\min}}$ have the maximum period in their respective homology classes, there is no room for nontrivial curves in the symplectization level. The bottom curve moduli space

$$\mathcal{M}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \underbrace{\gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}}_{k-n}\}, \emptyset)$$

consists of somewhere injective curves; for otherwise, assuming $u \in \mathcal{M}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}, \emptyset)$ is a branched cover over u' , we can cap off u' with natural disks to obtain a homology class A in $H_2(\mathbb{CP}^2)$ with $A \cap D_s < k$, which is a contradiction. It is direct to check that the holomorphic disks in the symplectic cap (ie disk fibers) are cut out transversely (see the discussion after (7-1)), hence transversality holds for the fully stretched situation. Therefore we have

$$\#\overline{\mathcal{M}}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}, \emptyset) \neq 0.$$

Case 2 ($k = n + 1$) To have $\text{ind}(u) \geq 0$, we must have that p_i is the minimum of $p_{\min}$ but $m \geq 1$. By the same area argument as for the cap, the total contact action of negative asymptotics of curves in the symplectic cap is close to k . On the other hand, the total contact action of $\{\gamma_{p_i}^{d_i}\}_{1 \leq i \leq 2}$ is close to mk . Hence we must have $m = 1$ and curves in the symplectic cap must be once punctured spheres that are asymptotic to γ_q^n and $\gamma_{q'}$. The remainder of the argument is the same as before.

Case 3 ($k \leq n$) Since the bottom level with the point constraint has one positive puncture, it is asymptotic to γ_q^{km} . By the same area and action argument, we have $m = 1$ and the both top and bottom level have one component with one puncture. Assume the negative asymptotics of the disk cap is γ_p^k . Then we must have $\text{ind}(p) \geq 2(n - k)$, to have nonnegative expected dimension for the disk. On the other hand, for the bottom curve, we must have $\text{ind}(q) \leq 2(n - k)$, to have nonnegative expected dimension. Therefore if $p \neq q$, the expected dimension of the cylinders in the symplectization is negative. Hence we have $p = q$ with $\text{ind}(p) = 2(n - k)$. Then we know that there is at least one critical point p with $\text{ind}(p) = 2(n - k)$ such that $\#\overline{\mathcal{M}}_{D_s^c, o}(\{\gamma_p^k\}, \emptyset) \neq 0$ and the unstable manifold of p represents multiples of $[D_s] \cap^{n-k} [H]$. $\square$

Remark 7.11 (algebraic Gromov–Witten invariants vs symplectic Gromov–Witten invariants) The proof of Proposition 7.10 makes an inexplicit assumption that the algebraically defined Gromov–Witten invariants used in [37] are the same as the symplectic version, where we only use a compatible almost complex structure. Without such an assumption, the above argument only shows that moduli spaces in Proposition 7.10 are not empty by compactness, but the algebraic count might be zero. Such equivalence is expected, but not established. However, for the very special case of degree-one curves in Proposition 7.10, such an equivalence is easier to establish. Here we only mention two strategies to establish this special equivalence to make Proposition 7.10 completely self-contained:

- (1) Translate the proofs in [37] into the symplectic version.
- (2) Interpret the algebraic Gromov–Witten invariants in Proposition 7.10 as Euler classes of an obstruction bundle over $\overline{\mathcal{M}}_{0,2}(\mathbb{CP}^n, A, \text{pt}) \simeq \mathbb{CP}^n$, then establish the equivalence with the symplectic version.

We will not pursue the details of those arguments in this paper.

Corollary 7.12 Let D_s be the smooth degree- k hypersurface in $\mathbb{CP}^n$ for $k > n + 1$ and $n \geq 2$. Then $\eta_{D_s^c}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}) \neq 0$ and $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}$ is closed in $(\overline{B}^{k+1-n} V_{\partial D^c}, \widehat{\ell}_{\epsilon_{D_s^c}})$.

Proof We may assume the Morse function on D_s is perfect; this follows from a direct check for $n = 2$, from [42] for $n = 3$, and the Lefschetz hyperplane theorem and the h -cobordism theorem for $n \geq 4$. Then Proposition 7.10 implies that $\eta_{D_s^c}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}) \neq 0$. It suffices to prove that $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}$ is closed. Since the parity of the SFT grading is the same as the Morse index, and $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}$ has even grading, we only need to consider if $\widehat{\ell}_{\epsilon_{D_s^c}}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}})$ contains any $q_{\gamma_p^m}$ with $\text{ind}(p) = n - 1$ for n even. As a consequence, we need to consider

$$\overline{\mathcal{M}}_{\partial D_s^c}(\Gamma^+, \Gamma^-) \quad \text{for } \Gamma^+ \subset \{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\} \quad \text{and} \quad \Gamma^- = \{\gamma_p^m, \gamma_{p_1}^{d_1}, \dots, \gamma_{p_s}^{d_s}\},$$

then we close off $\{\gamma_{p_i}^{d_i}\}_{1 \leq i \leq s}$ and a subset of the complement of Γ^+ by the augmentation from D_s^c . On the other hand, for homology reasons, we know the sum of multiplicities of Γ^- equals the sum of

multiplicities of Γ^+ . As a consequence, there is no subset of $\{\gamma_{p_i}^{d_i}\}_{1 \leq i \leq s}$ whose sum with a subset of the complement of Γ^+ represents a null-homologous class in D_s^c . In particular, there is no room for augmentation from D_s^c to apply, and we only need to consider $\overline{\mathcal{M}}_{\partial D_s^c}(\Gamma^+, \{\gamma_p^m\})$, where m is the sum of the multiplicities of Γ^+ . It is direct to check that the expected dimension of this moduli space is $(n-2) + 2|\Gamma^+| - 2$, which is strictly positive whenever $n \geq 3$. When $n = 2$, it is a direct check that the only cases with expected dimension 0 are $\overline{\mathcal{M}}_{\partial D_s^c}(\{\gamma_{p_{\min}}^2\}, \{\gamma_p^2\})$ and $\overline{\mathcal{M}}_{\partial D_s^c}(\{\gamma_{p_{\min}}\}, \{\gamma_p\})$, each of them corresponding to a moduli space of gradient trajectories from minimum $p_{\min}$ to the index-one critical point p , whose algebraic count is zero, as our Morse function is perfect. Therefore $q_{\gamma_{p_{\min}}}^n q_{\gamma_{p_{\min}}}^{k-n}$ is closed in $(\overline{B}^{k+1-n} V_{\partial D_s^c}, \hat{\ell}_{\epsilon_{D_s^c}})$. $\square$

Remark 7.13 Corollary 7.12 and Proposition 7.8 essentially imply that the planarity of D_s^c is $k+1-n$ for $k > n+1$. In the proof of Corollary 7.12, we use the topology of the filling D_s^c to get some restrictions on the augmentation; in particular, the augmentation respects the homology classes of orbits. However, we cannot run such an argument for general augmentations to obtain Theorem F.

7.1.4 Step two for the upper bound of $P(\partial D^c)$: independence of augmentations So far, we have proved that $P(D_s^c) \leq k+1-n$ (the planarity using the augmentation from the exact filling D_s^c) for a smooth degree $k > n+1$ divisor D_s . Even if we assume the functoriality of P for exact domains in Remark 4.7 was proven, and then we have that $P(D^c) \leq k+1-n$ for D the k generic hyperplanes, we still need to argue that the computation is independent of augmentations. In the following, we first show that the independence of augmentation is not tautological.

Theorem 7.14 Let D_s be a smooth degree $k > 2n-3$ hypersurface in $\mathbb{CP}^n$. Then $P(\partial D_s^c) = \infty$.

Proof We claim $\#\overline{\mathcal{M}}_{\partial D_s^c}(\Gamma^+, \emptyset) = 0$ and $\#\overline{\mathcal{M}}_{\partial D_s^c, o}(\Gamma^+, \emptyset) = 0$. For this we use a cascades model (but only the compactness), ie we consider the Boothby–Wang contact form on ∂D_s^c . Following the compactness argument in [14], if we degenerate the contact form on D_s (as perturbed by the Morse function) to the Boothby–Wang contact form, the curves in $\overline{\mathcal{M}}_{\partial D_s^c}(\Gamma^+, \emptyset)$, $\overline{\mathcal{M}}_{\partial D_s^c, A, o}(\Gamma^+, \emptyset)$ degenerate to cascades. But since $|\Gamma^+| \neq \emptyset$, there is one level containing nontrivial holomorphic curves in the symplectization of the Boothby–Wang contact form, which projects to a holomorphic sphere in D_s . However since $k > 2n-3$, there is no holomorphic sphere in D_s . Hence the claim follows. Now $\#\overline{\mathcal{M}}_{\partial D_s^c}(\Gamma^+, \emptyset) = 0$ implies that $\epsilon^k = 0$ for all $k \geq 1$ form a BL_∞ augmentation. Then $\#\overline{\mathcal{M}}_{\partial D_s^c, o}(\Gamma^+, \emptyset) = 0$ implies that the planarity is ∞ using such augmentation. $\square$

If one applies neck-stretching to the curve found in Proposition 7.10, we will get an SFT building, which might contain curves with negative punctures subject to a point constraint and augmentation curves in the filling. Theorem 7.14 is a situation where the augmentation from the natural filling and the trivial algebraic augmentation yield different computations. The following proposition singles out the module spaces for ∂D_s^c that might influence the computation of planarity for different augmentations.

Proposition 7.15 Let D_s be a smooth degree $k \geq n + 1$ hypersurface in $\mathbb{CP}^n$. Assume Γ^+ is a subset of

$$\{\gamma_{p_{\min}}^n, \underbrace{\gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}}_{k-n}\}.$$

Then for $\Gamma^- \neq \emptyset$, one has $\#\bar{\mathcal{M}}_{\partial D_s^c, o}(\Gamma^+, \Gamma^-) = 0$ unless $\Gamma^+ = \{\gamma_{p_{\min}}^n\}$ and $\Gamma^- = \{\gamma_{p_{\max}}^n\}$, or $\Gamma^+ = \{\gamma_{p_{\min}}\}$ and $\Gamma^- = \{\gamma_{p_{\max}}\}$.

Proof We can assume $\Gamma^- = \{\gamma_{p_1}^{d_1}, \dots, \gamma_{p_r}^{d_r}\}$ with $\sum d_i$ is the total multiplicity of Γ^+ (which is at most k) by homology and action reasons, as the total multiplicity of Γ^- is at most the total multiplicity of Γ^+ with the difference a multiple of k . Then we can run the Morse–Bott compactness argument as in Theorem 7.14 for $\bar{\mathcal{M}}_{\partial D_s^c, o}(\Gamma^+, \Gamma^-)$, and in the limit cascades moduli space, the holomorphic curve part necessarily has zero energy and hence is constant. Therefore due to the generic point constraint o , we must have $p_1 = \dots = p_r = p_{\max}$. Then the expected dimension of such moduli space is computed by

$$2k - 2|\Gamma^+| + \text{vdim } \bar{\mathcal{M}}_{\partial D_s^c, A, o}(\Gamma^+, \Gamma^-) - 2|\Gamma^-| - 2k = -4.$$

Hence $\text{vdim } \bar{\mathcal{M}}_{\partial D_s^c, A, o}(\Gamma^+, \Gamma^-) = 2|\Gamma^+| + 2|\Gamma^-| - 4$, which is zero if and only if $|\Gamma^+| = |\Gamma^-| = 1$. The claim follows. $\square$

Remark 7.16 In the case considered in Proposition 7.15, the only nonempty moduli spaces contributing to the pointed map are $\bar{\mathcal{M}}_{\partial D_s^c, o}(\gamma_{p_{\min}}^n, \gamma_{p_{\max}}^n)$ and $\bar{\mathcal{M}}_{\partial D_s^c, o}(\gamma_{p_{\min}}, \gamma_{p_{\max}})$. Moreover, the algebraic count is not zero as the gradient trajectories from $p_{\min}$ to $p_{\max}$ traverse the whole manifold. This follows from a cascades construction with gluing as in [14].

Theorem 7.14 along with Corollary 7.12 shows that computation of planarity can depend on the augmentations. In the special case of the contact boundary of smooth divisor complements, Proposition 7.15 isolates how this dependence works, ie BL_∞ augmentations to $q_{\gamma_{p_{\max}}^n} q_{\gamma_{p_{\min}}^n}^{k-n}$ and $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\max}}^n} q_{\gamma_{p_{\min}}^n}^{k-n-1}$ determine whether

$$\{\gamma_{p_{\min}}^n, \underbrace{\gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}}_{k-n}\}$$

contributes to finite planarity. Indeed applying neck-stretching to the moduli space for $\eta_{D_s^c}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^n}^{k-n}) \neq 0$ in Corollary 7.12 with the point constraint picked from the contact boundary along a copy of the contact boundary that is pushed in a bit, Proposition 7.15 and the proof of Theorem 7.14 imply that we must have nontrivial augmentations²⁰ to $q_{\gamma_{p_{\max}}^n} q_{\gamma_{p_{\min}}^n}^{k-n}$ or $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\max}}^n} q_{\gamma_{p_{\min}}^n}^{k-n-1}$ using the standard filling²¹ D_s^c . However, in Theorem 7.14, we choose the trivial augmentation, which kills the planarity.

²⁰Indeed, this is case. Roughly speaking, the augmentation to $q_{\gamma_{p_{\max}}^n} q_{\gamma_{p_{\min}}^n}^{k-n}$.

²¹Indeed, the augmentation to $q_{\gamma_{p_{\max}}^n} q_{\gamma_{p_{\min}}^n}^{k-n}$ (up to a multiple related to the multiplicity of Reeb orbits) counts degree-1 curves in $\mathbb{CP}^n$ passing through a fixed point of the divisor D_s with multiplicity n and $k - n$ marked points passing through D_s , and using the divisor axiom and [23, Proposition 3.4], such counting should be $(n - 1)!k^{k-n} \neq 0$.

Let D be k generic hyperplanes. To prove the upper bounds for [Theorem 7.6](#), we need to

- (1) find a collection of Reeb orbits on ∂D^c that bounds a rational curve with a point constraint;
- (2) show that the collection of Reeb orbits represents a closed class in $(\bar{B}^*V, \hat{\ell}_\epsilon)$ for any augmentation;
- (3) show that the nontrivial planarity also does not depend on augmentation.

Although there is an obvious candidate for step (1), to establish results in the spirit of [Corollary 7.12](#), we need to understand curves in the symplectic cap/neighborhood of the normal crossing divisors. In principle, one should be able to set up a relation between RSFT curves in an affine variety (complement of simple normal crossing divisors) with the log Gromov–Witten invariants. However, this is technically much harder than relating the relative Gromov–Witten invariants with RSFT curves of the complement of a smooth divisor in [Corollary 7.12](#). For steps (2) and (3), as the contact boundary ∂D^c has a much more complicated Reeb dynamics compared to the smooth case, it is highly nontrivial to establish them. Therefore the strategy to obtain the upper bounds in [Theorem 7.6](#) is to take advantage of simple Reeb dynamics on ∂D_s^c and functoriality, where D_s is a degree- k smooth hypersurface. More precisely, we use X to denote the exact cobordism from ∂D^c to ∂D_s^c from [Lemma 7.1](#). From the discussion above the planarity of ∂D_s^c depends on augmentations, and it is still possible for the planarity for ∂D_s^c to be finite if we only use augmentation in the form of $\epsilon \circ \phi$, where ϵ is the augmentation of $\text{RSFT}(\partial D^c)$ and ϕ is the BL_∞ morphism from X ; this is the content of [Proposition 7.19](#). Then we use the functoriality in [Proposition 2.24](#) and argue that the computation we did with the filling D_s^c in [Corollary 7.12](#) is in the form $\epsilon_{D^c} \circ \phi$, where ϵ_{D^c} is the augmentation of $\text{RSFT}(\partial D^c)$ from D^c . In principle, this involves a homotopy argument by neck-stretching. To avoid the overhead of introducing homotopies of BL_∞ morphisms, we show that the formula can be identified on the nose, due to the fact that when transversality in neck-stretching holds, we can identify a fully stretched moduli space with a sufficiently stretched moduli space by classical gluing. This is the content of [Proposition 7.18](#). In the following, we first prove a property explaining the role of $k < \frac{1}{2}(3n-1)$.

Proposition 7.17 *Let X be the cobordism from ∂D^c to ∂D_s^c as above. If $\Gamma^+ = \{\gamma_{p_{\max}}^{k_1}, \gamma_{p_{\min}}^{k_2}, \dots, \gamma_{p_{\min}}^{k_s}\}$ for $\sum_{i=1}^s k_i \leq k$, we have $\text{vdim } \mathcal{M}_X(\Gamma^+, \Gamma^-) < 0$, or $\bar{\mathcal{M}}_X(\Gamma^+, \Gamma^-) = \emptyset$ if $\Gamma^- \neq \emptyset$ and $s < \frac{1}{2}(n+1)$.*

Proof Let $\Gamma^- = \{\gamma_r^-\}_{r \in R}$, where $\{\gamma_r^-\}_{r \in R}$ are perturbations from $\{\gamma_r^{t_r}\}_{r \in R}$ such that $\bar{\mathcal{M}}_X(\Gamma^+, \Gamma^-) \neq \emptyset$. Then we have $\sum_{r \in R} \sum t_r = \sum_{i=1}^s k_i \pmod k$ by homology reasons. On the other hand, as explained in [Remark 7.7](#), the total contact action of Γ^- is close to $\sum_{r \in R} \sum t_r$. If we choose the smoothing D_s contained in the neighborhood of D that is removed to get the nice contact boundary to construct the exact cobordism X , the total contact energy of Γ^+ is approximately $\sum_{i=1}^s k_i$ and hence $\sum_{r \in R} \sum t_r \leq \sum_{i=1}^s k_i$. Therefore $\sum_{r \in R} \sum t_r = \sum_{i=1}^s k_i$. Then the expected dimension of $\mathcal{M}_{X,A}(\Gamma^+, \Gamma^-)$, ie $\text{ind}(u)$ for $u \in \mathcal{M}_{X,A}(\Gamma^+, \Gamma^-)$, satisfies

$$2n-2 + \sum_{i=1}^s 2(k_i-1) + \text{ind}(u) + \sum_{r=1}^R (\mu_{\text{CZ}}(\gamma_r^-) + n-3) = 2n-6.$$

Since $(\mu_{\text{CZ}}(\gamma_r^-) + n - 3) \geq 2n - 3 - 2 \sum t_r - \text{ind}(p_r) - |\text{supp } t_r|$ and $\check{D}_{\text{supp } t_r}$ is Weinstein by $k \geq n + 1$, we have $\text{ind}(p_r) \leq n - |\text{supp } t_r|$ and $(\mu_{\text{CZ}}(\gamma_r^-) + n - 3) \geq (n - 3) - 2 \sum t_r$. As a consequence, we have

$$\text{ind}(u) \leq -4 - |R|(n - 3) + 2s.$$

In particular, $\text{ind}(u) < 0$ if $|R| \neq 0$ and $s < \frac{1}{2}(n + 1)$. $\square$

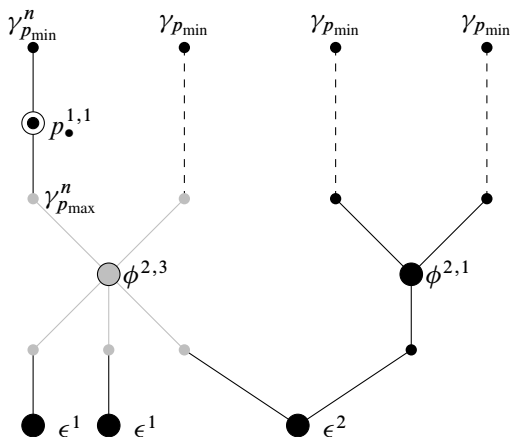
Proposition 7.18 *Let ϕ denote the BL_∞ morphism from the cobordisms X , and let $\epsilon_{D_s^c}$ and ϵ_{D^c} denote the augmentations from D_s^c and D^c , respectively. Then we have*

$$\begin{aligned} \hat{\ell}_{\bullet, \epsilon_{D_s^c}}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}) &= \hat{\ell}_{\bullet, \epsilon_{D^c} \circ \phi}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}) \\ &= \hat{\ell}_{\bullet, \epsilon_{D^c}} \circ \hat{\phi}_{\epsilon_{D^c}}^1(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}) \quad \text{by Proposition 2.24} \\ &\neq 0, \end{aligned}$$

where $\hat{\phi}_{\epsilon_{D^c}}^1$ is defined in Proposition 2.24, is the map on the bar complex for the linearized L_∞ morphism from $(V_{\partial D_s^c}, \{\ell_{\epsilon_{D^c} \circ \phi}^k\}_{k \geq 1})$ to $(V_{\partial D^c}, \{\ell_{\epsilon_{D^c}}^k\}_{k \geq 1})$ induced by ϕ .

Proof We will apply a neck-stretching for $\bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}, \emptyset)$ in Proposition 7.10(2) along ∂D^c for $o \in D^c$. We first claim that every curve in $\bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}, \emptyset)$ is somewhere injective. For otherwise, assume $u \in \bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}, \emptyset)$ is a branched cover over u' , then we can cap off u' with natural disks to obtain a homology class A in $H_2(\mathbb{C}\mathbb{P}^2)$ with $A \cap D_s < k$, which is a contradiction. Therefore it is safe to assume $\bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}, \emptyset)$ is cut out transversely for the stretching J_t . In the fully stretched picture, the bottom level containing the marked point o must have $k + 1 - n$ positive punctures. This is because we must have the number of positive punctures no larger than $k + 1 - n$ for otherwise genus has to be created. If there are fewer punctures, then by Proposition 7.8, the curve cannot exist by dimension reasons. By the dimension computation in Proposition 7.8, the only possible bottom level is described in Proposition 7.8. Then by the same capping argument, we know that the bottom curve is necessarily somewhere injective. As a consequence, all the levels above the bottom level must be unions of cylinders because of the number of positive punctures. Then by considering homology of the cobordism X , the positive asymptotics of the bottom level must be of the form $\check{\gamma}_{p_{I, \min}}^{\sigma_I} \cup \{\gamma_{p_{i, \min}}\}_{i \in I^c}$, where $I \subset \{1, \dots, k\}$ is a subset of size n , and $p_{I, \min}, p_{i, \min}$ are minimums. Next we still have multiple symplectization levels of cylinders for ∂D^c and one level of cylinders in X . Since $\check{\gamma}_{p_{I, \min}}^{\sigma_I} \cup \{\gamma_{p_{i, \min}}\}_{i \in I^c}$ have the maximal period in their respective homology classes, there is no action room for the symplectization levels. The top level cylinders in X , ie $\bar{\mathcal{M}}_X(\{\gamma_{p_{\min}}^n\}, \{\check{\gamma}_{p_{I, \min}}^{\sigma_I}\})$ and $\bar{\mathcal{M}}_X(\{\gamma_{p_{\min}}\}, \{\gamma_{p_{i, \min}}\})$ for $i \in I^c$, are rigid and cut out transversely, as the negative asymptotic orbits are simple. Therefore, the fully stretched moduli space is cut out transversely. The transversality of neck-stretching implies that this two-level breaking can be identified with $\bar{\mathcal{M}}_{D_s^c, o}(\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}, \emptyset)$ for sufficiently stretched J_t . By Axiom 3.26, we can count them to obtain that

$$\eta_{D_s^c}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}) = \eta_{D^c} \circ \hat{\phi}_{\epsilon_{D^c}}^1(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}^{k-n}}).$$



$\hat{\ell}_{\bullet, \epsilon \circ \phi}(q_{\gamma_{p_{\min}}}^n q_{\gamma_{p_{\min}}}^{k-n})$ with influence from ϵ . The light gray part has negative dimension.

Then we can use [Proposition 5.14](#) to relate η back to $\hat{\ell}_{\bullet, \epsilon}$, since by [Corollary 7.12](#) $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}}^{k-n}$ is closed in $(\bar{B}^{k+1-n} V_{\partial D_s^c}, \hat{\ell}_{\epsilon D_s^c})$. The nonvanishing follows from [Corollary 7.12](#). $\square$

Proposition 7.19 *If $n + 1 \leq k < \frac{1}{2}(3n - 1)$, then $\hat{\ell}_{\bullet, \epsilon \circ \phi}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}}^{k-n}) \neq 0$ is independent of the augmentation ϵ of $\text{RSFT}(\partial D^c)$.*

Proof When $k < \frac{1}{2}(3n - 1)$, we have $1 + k - n < \frac{1}{2}(n + 1)$. By Proposition 7.15, a component to $\hat{\ell}_{\bullet, \epsilon \circ \phi}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}}^{k-n})$ with influence from ϵ is described in the graph below (Figure 12), which does not exist by dimension reasons by Proposition 7.17. Therefore $\hat{\ell}_{\bullet, \epsilon \circ \phi}(q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}}^{k-n})$ is independent of ϵ . The nonvanishing then follows from Proposition 7.18 by taking $\epsilon \in \mathcal{D}^c$. $\square$

Proof of Theorem 7.6 If $k \leq n$, then $D^c = T^*T^{k-1} \times \mathbb{C}^{n-k+1}$, then $H_{\text{cx}}(\partial D^c) = 0^{\text{SD}}$ by Theorem 7.33. If $k = n + 1$, then $P(\partial D^c) = 2$ by Corollary 6.8. For $k > n + 1$, the lower bound follows from Corollary 7.9. When $k < \frac{1}{2}(3n - 1)$ and n is odd, for any augmentation ϵ of $\text{RSFT}(\partial D^c)$, $q_{\gamma_{p_{\min}}}^n q_{\gamma_{p_{\min}}}^{k-n}$ represents a closed class in $(\bar{B}^{k+1-n} V_{\partial D_s^c}, \hat{\ell}_{\epsilon \circ \phi})$, as the SFT grading of $\text{RSFT}(\partial D_s^c)$ is even for all generators. In particular, $\hat{\phi}_\epsilon^1(q_{\gamma_{p_{\min}}}^n q_{\gamma_{p_{\min}}}^{k-n})$ is closed in $(\bar{B}^{k+1-n} V_{\partial D^c}, \hat{\ell}_\epsilon)$ for any ϵ . Then by Proposition 7.19, $\hat{\ell}_{\bullet, \epsilon \circ \phi}(q_{\gamma_{p_{\min}}}^n q_{\gamma_{p_{\min}}}^{k-n}) \neq 0$ for any ϵ , and we conclude that $P(\partial D^c) = k + 1 - n$ if $n + 1 < k < \frac{1}{2}(3n - 1)$ by Proposition 2.24. $\square$

Remark 7.20 Our computation method above can be summarized as finding a curve contributing to the planarity by relative Gromov–Witten invariants and then arguing independence of augmentation by index computations. The trick we use is arguing closedness in the smooth divisor, where generators are simpler, proving the upper bounds using the functoriality, and arguing that everything interesting about the functoriality happens purely in X (ie does not depend on augmentation for $\text{RSFT}(\partial D^c)$).

A more systematic way of computing planarity is deriving a formula for the BL_∞ algebra as well as the augmentation from the affine variety using log/relative Gromov–Witten invariants. In the context of symplectic (co)homology, such a formula was obtained in [27].

Even though Theorem 7.6 depends on the parity of n and the size of k , things get easier if we only consider the augmentation from the affine variety (Corollary 7.12). The following can be viewed as the geometric reinterpretation of Corollary 7.12, which does not depend on n, k and is sufficient for obstructing exact embeddings.

Theorem 7.21 *Let D be k generic hyperplanes in $\mathbb{CP}^n$ for $n \geq 1$. Then $U(D^c) = \max\{1, k + 1 - n\}$.*

Proof The $n = 1$ case is obvious. For $n \geq 2$, we can use Proposition 7.8 to claim that $\overline{\mathcal{M}}_{D^c, o}(\Gamma^+, \emptyset) = \emptyset$ for generic J as long as $|\Gamma^+| < \max\{1, k + 1 - n\}$. This is because we can obtain the classical transversality of $\overline{\mathcal{M}}_{D^c, o}(\Gamma^+, \emptyset) = \emptyset$, as every curve is a branched cover of a somewhere injective curve with negative expected dimension. Therefore, we have $U(D^c) \geq \max\{1, k + 1 - n\}$ by Proposition 5.12. On the other hand, the nontrivial relative Gromov–Witten invariant used in Proposition 7.10 implies that $U(D^c) \leq \max\{1, k + 1 - n\}$ by neck-stretching. $\square$

7.1.5 Examples with nontrivial SD when k is small

Theorem 7.22 *Assume D_s is a smooth degree $2 \leq k < n$ hypersurface in $\mathbb{CP}^n$ for $n \geq 3$ odd. Then $(k - 1)^{\mathrm{SD}} \leq H_{\mathrm{cx}}(\partial D_s^c) \leq (2k - 2)^{\mathrm{SD}}$. When n is even and $2 \leq k < \frac{1}{2}(n + 1)$, we have $H_{\mathrm{cx}}(\partial D_s^c) \leq (2k - 2)^{\mathrm{SD}}$.*

Proof Let p be the critical point in Proposition 7.10(1). Then we have $\eta_{D_s^c}(q_{\gamma_p^k}) \neq 0$ by the same argument of Corollary 7.12. We can pick the Morse function on D_s to be perfect and self-indexing, similar to [87, Proposition 3.1], and we can choose the perturbation Morse function such that if

$$(7-5) \quad \int \alpha^* \gamma_p^d - \sum_{i=1}^j \int \alpha^* \gamma_{p_i}^{d_i} \geq 0$$

for $d \leq k$ and $\sum d_i = d$, then for every i we have $\mathrm{ind}(p_i) \geq \mathrm{ind}(p)$. This energy constraint will help us exclude certain configurations.

Note that in our setup here, higher $f(p)$ or $\mathrm{ind}(p)$ means smaller contact action, since we apply the perturbation in the cap of the positive prequantization bundle instead of the filling of the negative prequantization bundle. In particular, the order is reversed compared to [87, Section 2.1] and the proof of Theorem 7.30 below.

Claim *The class $q_{\gamma_p^k}$ is closed in $(\bar{B}^1 V_{\partial D_s^c}, \ell_\epsilon^1)$ for any augmentation ϵ for $2 \leq k < \frac{1}{2}(n + 1)$, or n odd with $2 \leq k < n$.*

Proof The parity of the SFT grading of γ_q^d is the same as the parity of $\text{ind}(q)$. As a consequence, we only need to consider $\langle \ell_\epsilon^1(q_{\gamma_p^k}), q_{\gamma_q^d} \rangle$ for $\text{ind}(q) = n - 1$ when n is even. In other words, we need to consider $\mathcal{M} := \bar{\mathcal{M}}_{\partial D_s^c}(\{\gamma_p^k\}, \{\gamma_q^d\} \cup \{\gamma_{q_i}^{d_i}\}_{1 \leq i \leq r})$ for $d + \sum d_i = k$. By (7-5), to have $\mathcal{M} \neq \emptyset$, we must have $\text{ind}(q_i), \text{ind}(q) \geq \text{ind}(p) = 2n - 2k$. By $k < \frac{1}{2}(n + 1)$, we have $\text{ind}(p) = 2n - 2k > \text{ind}(q) = n - 1$ and $\mathcal{M}$ is empty. Hence the claim follows. $\square$

Claim The value of $\ell_{\bullet, \epsilon}^1(q_{\gamma_p^k})$ is independent of ϵ .

Proof It suffices to show that $\bar{\mathcal{M}}_{\partial D_s^c, o}(\{\gamma_p^k\}, \Gamma^-)$ is empty for $\Gamma^- \neq \emptyset$. If $\Gamma^- \neq \emptyset$, then $\Gamma^- = \{\gamma_{p_i}^{d_i}\}_{1 \leq i \leq r}$ with $\sum d_i = k$ and $\text{ind}(p_i) \geq \text{ind}(p)$ for every i . The claim follows from the argument of Proposition 7.15 and $\text{ind}(p) > 0$. $\square$

Claim
$$H_{\text{cx}}(\partial D_s^c) \leq (2k - 2)^{\text{SD}}.$$

Proof We need to show that $U^{2k-1}(q_{\gamma_p^k}) = 0$ for any augmentation. Note that the U map decreases contact action. For homology reasons and the contact action property (7-5), for any augmentation and $d \leq k$, $U(q_{\gamma_q^d})$ can only have nontrivial coefficient for $q_{\gamma_{q'}^{d'}}$ for $d' < d$ and $\text{ind}(q') \geq \text{ind}(q)$ and $\text{ind}(q') = \text{ind}(q) \bmod 2$, or for $q_{\gamma_q^d}$ for $\text{ind}(q') = \text{ind}(q) + 2$. Therefore, $U^{2k-1}(q_{\gamma_p^k}) = 0$ for any augmentation. $\square$

Claim When n is odd, $H_{\text{cx}}(\partial D_s^c) \geq (k - 1)^{\text{SD}}$.

Proof The linearized contact homology/positive S^1 -equivariant symplectic cohomology has a contact action filtration such that the filtered theory around period $\int (\gamma_p^k)^* \alpha = k \int \gamma_p^* \alpha$ is generated by the k^{th} covered orbits. The U map on this filtered theory is represented by multiplying $c_1(\mathcal{O}(k)|_D)$ to the cochain represented by the critical point, ie the Poincaré dual of the unstable manifold. Using (4) of Claim 4.3, we can consider the U map on positive S^1 -equivariant symplectic cohomology. In fact, one can prove the filtered linearized contact homology (up to k^{th} multiples of simple orbits) is isomorphic to the filtered S^1 positive symplectic cohomology following [13] as there is no room for the influence of augmentations and transversality can be achieved. Then the U -map on the filtered S^1 positive symplectic cohomology is given by multiplying the first Chern class by a standard Morse–Bott argument, eg [90, Proposition 5.9]. Therefore, by the argument in the second claim,

$$U^{k-1}(q_{\gamma_p^k}) = k^{k-1} q_{\gamma_{p_{\max}}^k} + \text{terms with lower multiplicities}$$

for the unique maximum $p_{\max}$ with $\text{ind}(p_{\max}) = 2n - 2$. When n is odd, all generators have even SFT degree and represent nontrivial classes in the linearized contact homology, hence $U^{k-1}(q_{\gamma_p^k}) \neq 0$ in homology. $\square$

Remark 7.23 The n being odd condition in Theorems 7.6, 7.14 and 7.22 as well as 7.24 below is not necessary, as one can show $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}}^{k-n}$ is always closed. This is because a differential from $q_{\gamma_{p_{\min}}^n} q_{\gamma_{p_{\min}}}^{k-n}$

involves counting $\overline{\mathcal{M}}_Y(\Gamma^+, \Gamma^-)$, with Γ^+ a subset of $\{\gamma_{p_{\min}}^n, \gamma_{p_{\min}}, \dots, \gamma_{p_{\min}}\}$ and $\Gamma^- = \{\gamma_{p_i}^{d_i}\}_{1 \leq i \leq r}$ for $\sum d_i = k$. Therefore if we use the Morse–Bott contact form and cascades construction, the relevant holomorphic curve must be covers of trivial cylinders. Then the moduli space $\overline{\mathcal{M}}_Y(\Gamma^+, \Gamma^-)$ is the fiber product of $D \times \mathcal{M}$ with unstable/stable manifolds of $p_{\min}$ and p_i , where $\mathcal{M}$ before compactification is the space of meromorphic functions on $\mathbb{CP}^1$ with a pole of order n and $k - n$ simple poles and a zero with order d_i for all $1 \leq i \leq r$ modulo the $\mathbb{R}$ rescaling on meromorphic functions and the automorphism of the punctured Riemann surface. Then by the nontrivial S^1 -action on meromorphic functions, we expect to have $\#\overline{\mathcal{M}}_Y(\Gamma^+, \Gamma^-) = 0$ unless $|\Gamma^+| = |\Gamma^-| = 1$. If $|\Gamma^+| = |\Gamma^-| = 1$, $\overline{\mathcal{M}}_Y(\Gamma^+, \Gamma^-)$ is identified with Morse trajectories (here the S^1 -action on meromorphic function is identical with the S^1 -action in the automorphism group of surface, hence is trivial on the quotient), whose algebraic count is zero, as we assume the Morse function is perfect. To make this precise, one can follow a Morse perturbation of the contact form as before. And using a J that is S^1 -invariant under the rotation in the fiber direction, then applying the S^1 -equivariant transversality for quotients from [86], we can argue that $\#\overline{\mathcal{M}}_Y(\Gamma^+, \Gamma^-) = 0$ unless $|\Gamma^+| = |\Gamma^-| = 1$, similar to Floer’s proof of the isomorphism between Hamiltonian Floer cohomology and Morse cohomology. This argument requires building our functors using polyfolds as in [33].

7.1.6 Generalization to Fano hypersurfaces In the following, we will generalize Theorem 7.6 to some affine varieties contained in a Fano hypersurface in $\mathbb{CP}^{n+1}$.

Theorem 7.24 *Let X be a smooth degree m hypersurface in $\mathbb{CP}^{n+1}$ for $2 \leq m < \frac{1}{2}(n+1) \leq n$ and D be $k \geq n$ generic hyperplanes, ie $D = (H_1 \cup \dots \cup H_k) \cap X$ for H_i is a hyperplane in $\mathbb{CP}^{n+1}$ in generic position with each other and X . Then $P(\partial D^c) = k + m - n$ for n odd and $k + m < \frac{1}{2}(3n+1)$.*

Proof We separate the proof into several steps. The Reeb dynamics on ∂D^c has the same property with the $\mathbb{CP}^n$ case, with the only difference being that the minimal Chern number of X is $n+2-m$, which will enter into the computation of virtual dimensions.

Claim *For any set of Reeb orbits $\Gamma := \{\gamma_1, \dots, \gamma_r\}$ for $r < k + m - n$ with $\sum [\gamma_i] = 0 \in H_1(D^c)$, the virtual dimension of the moduli space $\overline{\mathcal{M}}_{D^c, A, o}(\Gamma, \emptyset)$ is negative for any A .*

Proof This follows from the same argument in Proposition 7.8, with the difference that $c_1(u \# \#_{i=1}^r u_i) = 2N(n+2-m)$. Therefore we have

$$\begin{aligned} \text{ind}(u) &\leq 2N(n+2-m) - 4 - \sum_{i=1}^r \left(2 \sum t_i - 2 + \text{ind}(p_i) \right) \\ &\leq 2N(n+2-m) - 4 - 2kN - 4 + 2r \\ &= 2(N-1)(n+2-m-k) + 2(r+n-m-k) < 0, \end{aligned}$$

since $r < m + k - n$ and $k \geq n, m \geq 2$. This computation also implies the lower bound of $P(\partial D^c)$ is $k + m - n$. $\square$

Claim Assume D_s in the generic intersection of a degree- k hypersurface in $\mathbb{CP}^{n+1}$ with X . Then

$$\eta_{D_s^c}(q_{\gamma_{p_{\min}}^{n+1-m}} q_{\gamma_{p_{\min}}}^{k+m-n-1}) \neq 0,$$

and $q_{\gamma_{p_{\min}}^{n+1-m}} q_{\gamma_{p_{\min}}}^{k+m-n-1}$ is closed in $(\bar{B}^{k+m-n} V_{\partial D_s^c}, \hat{\ell}_{\epsilon_{D_s^c}})$.

Proof That $\eta_{D_s^c}(q_{\gamma_p}^{k+m-n-1} q_{\gamma_p^n}) \neq 0$ follows from the nonvanishing of

$$\text{GW}_{0,1,(n+1-m,1,\dots,1),A}^{X,D}([\text{pt}], \underbrace{[D_s], \dots, [D_s]}_{k+m-n})$$

from [37], for A is the positive generator of $H_2(X)$ that is mapped to the generator of $H_2(\mathbb{CP}^{n+1})$, and the same argument as in Proposition 7.10. The remainder of the argument is exactly the same as Corollary 7.12. To see the nonvanishing of the Gromov–Witten invariants, we have

$$\text{GW}_{0,1,(n+1-m,1,\dots,1),A}^{X,D}([\text{pt}], \underbrace{[D_s], \dots, [D_s]}_{k+m-n}) = (k)^{k+m-n-1} \text{GW}_{0,1,(n+1-m),A}^{X,D}([\text{pt}], [D_s])$$

by the divisor axiom. Then by [37, Theorem 2.6],

$$\text{GW}_{0,1,(n+1-m),A}^{X,D}([\text{pt}], [D_s]) = \int_{[\bar{\mathcal{M}}_{0,2}(X,A,\text{pt})]^{\text{vir}}} \prod_{i=0}^{n+1-m} (\text{ev}_2^* \text{PD}([D_s]) + i\psi).$$

Following the strategy in [37, Corollary 5.7], by taking $\alpha = (1, m)$ in [37, (8)] we have

$$((m-1)\psi + \text{ev}_2^* \text{PD}([X])) \cdot [\bar{\mathcal{M}}_{0,(1,m-1)}(\mathbb{CP}^{n+1}, A, \text{pt})]^{\text{vir}} = [\bar{\mathcal{M}}_{0,2}(X, A, \text{pt})]^{\text{vir}},$$

where the point constraint is in $X \subset \mathbb{CP}^{n+1}$. Then we apply [37, Theorem 2.6] for another $m-2$ times, to get

$$\begin{aligned} \text{GW}_{0,1,(n+1-m),A}^{X,D}([\text{pt}], [D_s]) &= \int_{[\bar{\mathcal{M}}_{0,2}(\mathbb{CP}^{n+1}, A, \text{pt})]^{\text{vir}}} \prod_{i=0}^{n+1-m} (\text{ev}_2^* \text{PD}(k[H]) + i\psi) \prod_{i=1}^{m-1} (\text{ev}_2^* \text{PD}([X]) + i\psi) \\ &= \frac{k!m!}{(k+m-n-1)!} \end{aligned}$$

where H is the hyperplane class in $\mathbb{CP}^{n+1}$. □

Then by the same neck-stretching argument as in Proposition 7.18, we have

$$\begin{aligned} \hat{\ell}_{\bullet, \epsilon_{D_s^c}}(q_{\gamma_{p_{\min}}^{n+1-m}} q_{\gamma_{p_{\min}}}^{k+m-n-1}) &= \hat{\ell}_{\bullet, \epsilon_{D^c} \circ \phi}(q_{\gamma_{p_{\min}}^{n+1-m}} q_{\gamma_{p_{\min}}}^{k+m-n-1}) \\ &= \hat{\ell}_{\bullet, \epsilon_{D^c}} \circ \hat{\phi}_{\epsilon_{D^c}}^1(q_{\gamma_{p_{\min}}^{n+1-m}} q_{\gamma_{p_{\min}}}^{k+m-n-1}) \neq 0. \end{aligned}$$

Next, Propositions 7.15 and 7.17 also hold, as the dimension computation there is essentially for trivial homology class, which does not depend on m . It is important to note that in the proof of Proposition 7.17, we use that $\check{D}_I$ is Weinstein to obtain an upper bound of Morse indices. Such a property also holds here as we assume $k \geq n$. Then the remainder of the proof is the same as Theorem 7.6. □

From the proof above, the source of holomorphic curves is supplied by the degree-1 holomorphic curves in X for $m \leq n$. For $m = n + 1$, the degree-1 curve does not unirule X anymore, but a degree-2 curve unirules X . In the proofs of Theorems 7.6 and 7.24, being degree 1 is used in several places to obtain somewhere injectivity (the capping argument). Indeed, for $m = n + 1$, the situation is different: we will prove $P(\partial D^c) \geq 2$ for D is a generic intersection of X with a hyperplane in $\mathbb{CP}^{n+1}$. For $m \geq n + 2$, X is not uniruled, which implies D^c is not k uniruled for any k by [60]; therefore $P(\partial D^c) = \infty$ by Corollary 5.15.

In view of Theorems 5.13, 7.6, 7.21 and 7.24, we make the following conjecture.

Conjecture 7.25 *V is a k -uniruled affine variety then $P(V) < \infty$ and $P(V) = U(V) = AU(V)$.*

On the other hand, by Theorem 7.14, it is not true that any uniruled affine variety has a contact boundary with finite planarity. It is a subtle question to determine which affine variety with a $\mathbb{CP}^n$ compactification has a finite planarity boundary.

Question 7.26 *Let D be k generic hyperplanes in $\mathbb{CP}^n$. Is $P(\partial D^c)$ always finite?*

Theorems 7.6 and 7.24 along with Lemmas 7.1 and 7.3 imply that there are many sequences of contact manifolds where exact cobordisms only exist in one direction. On the other hand, exact embedding problems in the flavor of Theorem 7.21 are studied in [36]. It is an interesting question to determine whether those embedding obstructions can lift to cobordism obstructions.

7.2 Links of singularities

Another natural source of contact manifolds is links of isolated singularities. In the following, we will consider the Brieskorn singularities and quotient singularities from diagonal cyclic actions on $\mathbb{C}^n$.

7.2.1 Brieskorn singularities A Brieskorn singularity is of the form

$$x_0^{a_0} + \cdots + x_n^{a_n} = 0$$

for $2 \leq a_0 \leq \dots \leq a_n$. We use $\bar{a}$ to denote the sequence, the link $\text{LB}(\bar{a})$ is defined to be the intersection $\text{LB}(\bar{a}) := \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid x_0^{a_0} + \cdots + x_n^{a_n} = 0\} \cap S^{2n+1}$, which is a $(2n-1)$ -dimensional contact manifold. Moreover, $\text{LB}(\bar{a})$ is exactly fillable by the smooth affine variety $x_0^{a_0} + \cdots + x_n^{a_n} = 1$, which is called the Brieskorn variety. We refer readers to [50] for more details on the contact topology of Brieskorn manifolds. We have the following fact about embedding relations for Brieskorn varieties.

Proposition 7.27 [47, Lemma 9.9] *We say $\bar{a} \leq \bar{b}$ if and only if $a_i \leq b_i$ for all i . Then if $\bar{a} \leq \bar{b}$, the Brieskorn variety of $\bar{a}$ embeds exactly into the Brieskorn variety of $\bar{b}$. In particular, $\text{LB}(\bar{a}) \leq \text{LB}(\bar{b})$ in $\overline{\text{Con}}_{\leq}$.*

Many Brieskorn varieties were showed in [90, Theorem A] to support k -dilations. Some of the computation in [90] can be improved to be the computation of H_{cx} , ie independent of BL_∞ augmentations. In particular, we will have either a computation or an estimate of $H_{\text{cx}}(\text{LB}(\bar{a}))$ for any $\bar{a}$ from the theorem below and Proposition 7.27.

Theorem 7.28 Using $\text{LB}(k, n)$ to denote the contact link of the Brieskorn singularity $x_0^k + \cdots + x_n^k = 0$, then $H_{\text{cx}}(\text{LB}(k, n))$ is

- (1) $(k - 1)^{\text{SD}}$ if $k < n$, and is $\geq (k - 1)^{\text{SD}}$ if $k = n$,
- (2) $> 1^{\text{P}}$ if $k = n + 1$,
- (3) ∞^{P} if $k > n + 1$.

Proof If $k = n + 1$, since the log-Kodaira dimension of the corresponding Brieskorn variety V is 0, we know that V is not algebraically 1-uniruled. Hence the planarity is greater than 1 by Corollary 5.15. When $k > n + 1$, the Brieskorn variety admits a compactification that is not uniruled, hence the planarity is infinity by Corollary 5.15.

When $k \leq n$, the associated Brieskorn variety $V(k, n)$ carries a $k - 1$ (semi)dilation by [90, Theorem A] using the definition with S^1 -equivariant symplectic cohomology. Moreover, the $k - 1$ semidilation is observed by the truncated S^1 -equivariant symplectic cohomology generated by simple Reeb orbits. By Theorem 4.6, this means the order of semidilation using a augmentation from the Brieskorn variety is $k - 1$. Then we have at least $\text{SD} \geq k - 1$.

Note that the Brieskorn variety is an affine variety $X \setminus D$, where X is the smooth projective variety $x_0^k + \cdots + x_n^k = x_{n+1}^k$ in $\mathbb{CP}^{n+1}$ and D is the smooth divisor $X \cap \{x_{n+1} = 0\}$. We will adopt the same notation for Reeb orbits on a smooth divisor complement as before. The semidilation is provided by a simple Reeb orbit γ_p with $\text{ind}(p) = 2n - 2k$ by [90, Theorem A].

Next we will argue that the semidilation supplied by γ_p with $\text{ind}(p) = 2n - 2k$ is independent of the augmentation when $k < n$. To see that, we first claim that the q_{γ_p} contribution to $\text{P} = 1$ is independent of augmentation. If not, we have a nonempty moduli space $\overline{\mathcal{M}}_{\text{LB}(k, n), o}(\{\gamma_p\}, \{\gamma_q\})$ whose expected dimension is $\text{ind}(q) - \text{ind}(p) + 2 - 2n = \text{ind}(q) + 2k + 2 - 4n < 0$ when $k < n$ since $\text{ind}(q) \leq 2n - 2$. Therefore planarity of $\text{LB}(k, n)$ is always 1 if $k < n$. Moreover, $U^i(q_{\gamma_p})$ is independent of augmentation as we are at the minimal period; there is no room for U to depend on augmentations. Therefore we have $H_{\text{cx}}(\text{LB}(k, n)) \leq (k - 1)^{\text{SD}}$ if $k < n$. Hence the claim follows. $\square$

Remark 7.29 If Conjecture 6.5 was proven, one can get a better estimate for $H_{\text{cx}}(\text{LB}(\bar{a}))$ by writing $\text{LB}(\bar{a})$ as an open book with a Brieskorn variety page. In the context of symplectic cohomology, computation in such a spirit can be found in [90, Section 5].

Proof of Theorem B This theorem is a combination of Theorem 3.17, Theorem 3.18, Theorem 3.21, Corollary 5.15, Theorem 6.3, Corollary 6.8 and Theorem 7.28. $\square$

7.2.2 Quotient singularities by cyclic groups. Let $\mathbb{Z}_k$ act on $\mathbb{C}^n$ by the diagonal action multiplying by $e^{2\pi i/k}$. Then the link of the quotient singularity $\mathbb{C}^n/\mathbb{Z}_k$ is the quotient contact manifold $(S^{2n-1}/\mathbb{Z}_k, \xi_{\text{std}})$. Such contact manifolds provide many examples of strongly fillable but not exactly fillable contact manifolds [87]. In fact, the symplectic part of [87] is a computation of the hierarchy functor H_{cx} in the context of symplectic cohomology, which will be rephrased as follows.

Theorem 7.30 *Let Y be the quotient $(S^{2n-1}/\mathbb{Z}_k, \xi_{\text{std}})$ by the diagonal action by $e^{2\pi i/k}$ for $n \geq 2$.*

- (1) *If $n > k$, we have $H_{\text{cx}}(Y) = 0^{\text{SD}}$.*
- (2) *If $n \leq k$, we have $0^{\text{SD}} \leq H_{\text{cx}}(Y) \leq (n-1)^{\text{SD}}$. When $n = k$, we have $H_{\text{cx}}(Y) \geq 1^{\text{SD}}$.*

Proof We follow the same setup as in [87, Proposition 3.1]. We have a nondegenerate contact form on ξ_{std} by perturbing with a C^2 -small perfect Morse function f on $\mathbb{CP}^{n-1}$ so that the Reeb orbits are the following.

- (1) Reeb orbits of period smaller than $k+1$ are γ_i^j for $0 \leq i \leq n-1$ and $1 \leq j \leq k$, where γ_i^j is the j -multiple cover of γ_i and γ_i projects to the i^{th} critical point q_i of f with $\text{ind}(q_i) = 2i$. (It is important to note that now we perturb the contact form using that f is the prequantization filling in $\mathcal{O}(-k)$, following the convention in [87]; therefore higher Morse index means larger period, which is reverse to Theorem 7.6.)
- (2) The period of γ_j is $1 + \epsilon_j$.
- (3) $\epsilon_j < \epsilon_{j+1}/k$ and $\epsilon_j \ll 1$.
- (4) The Conley–Zehnder index of γ_i^j with the natural disk in $\mathcal{O}(-k)$ satisfies $\mu_{\text{CZ}}(\gamma_i^j) + n - 3 = 2i + 2j - 2$.

Claim $P(Y) = 1$ for $n \geq 2$ and for all k .

Proof By the same argument as [87, Step 3 of Proposition 3.1], we have $\#\bar{\mathcal{M}}_{Y,o}(\{\gamma_0^k\}, \emptyset) = k$ for $n \geq 2$, which is induced from the holomorphic curve in the symplectization of the standard sphere. When $\Gamma^- \neq \emptyset$, we have $\#\bar{\mathcal{M}}_{Y,o}(\{\gamma_0^k\}, \Gamma^-) = \emptyset$ by action and homology reasons, unless $\Gamma^- = \{\gamma_0^{d_i}\}_{1 \leq i \leq r}$ for $\sum d_i = k$. In this case, a curve in $\bar{\mathcal{M}}_{Y,o}(\{\gamma_0^k\}, \Gamma^-)$ is necessarily a branched cover over a trivial cylinder. In particular, $\bar{\mathcal{M}}_{Y,o}(\{\gamma_0^k\}, \Gamma^-) = \emptyset$ for generic o . Since all Reeb orbits have even SFT degree, $q_{\gamma_0^k}$ is closed in any linearized contact homology, and the planarity is 1 for any augmentation (which exists in abundance as all SFT gradings are even) by $q_{\gamma_0^k}$. $\square$

Claim If $k < n$, then $H_{\text{cx}}(Y) = 0^{\text{SD}}$.

Proof For action reasons, $U(q_{\gamma_0^k})$ can only have nontrivial coefficients in $q_{\gamma_0^d}$ for $d < k$. Note that the filtered linearized contact homology/ S^1 -equivariant symplectic cohomology with action supported

around d is generated by $q_{\gamma_r^d}$ for $0 \leq r \leq n-1$. In particular, the homology is $H^*(\mathbb{CP}^{n-1})$ with the U map is the multiplication by $c_1(\mathcal{O}(k))$. As a consequence, for action reasons,

$$U(q_{\gamma_r^d}) = kq_{\gamma_{r-1}^d} + \sum_{i=1}^{d-1} \sum_{j=0}^r a_{ij} q_{\gamma_j^i}.$$

Therefore for any augmentation, there exist c_{ij} such that $U(q_{\gamma_0^k} + \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} c_{ij} q_{\gamma_j^i}) = 0$ by the same argument as [87, (3.2)]. In order to finish the proof, it is sufficient to prove $\overline{\mathcal{M}}_{Y,o}(\gamma_j^i, \Gamma^-) = \emptyset$ for $i+j \leq k$ and $j > 0$. This follows from the same dimension computation as in [87, Step 7 of Proposition 3.1] and is the place where $n > k$ is essential. Then $q_{\gamma_0^k} + \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} c_{ij} q_{\gamma_j^i}$ also contributes $P = 1$ and is killed by U . In particular, $H_{\text{cx}}(Y) = 0^{\text{SD}}$. $\square$

Claim If $k \geq n$, then $0^{\text{SD}} \leq H_{\text{cx}}(Y) \leq (n-1)^{\text{SD}}$.

Proof First note that all generators have even degree, hence any maps $\{\epsilon^k\}_{k \geq 1}$ form an augmentation. By the argument in Remark 7.16, we have that $\#\overline{\mathcal{M}}_{Y,o}(\{\gamma_{n-1}^d\}, \{\gamma_0^d\}) = 1$ for $1 \leq d \leq k-1$. On the other hand, following the argument to obtain [27, $\langle \partial(\check{p}_{k+}), \hat{p}_{k-} \rangle$ in Theorem 9.1, Lemma 9.4], we know that $\langle U(q_{\gamma_{i+}^k}), q_{\gamma_i^k-} \rangle = (k_+ - k_-)\epsilon^1(q_{\gamma_0^{k_+ - k_-}})$ for augmentation $\{\epsilon^k\}_{k \geq 1}$.²² If for every $1 \leq i \leq k-n$ we have $\epsilon^1(q_{\gamma_0^i}) = 0$, then for $d \leq k$, we have $U(q_{\gamma_0^d}) = \sum_{j=1}^{d-1+n-k} (d-j)\epsilon^1(q_{\gamma_0^{d-j}})q_{\gamma_0^j}$. Therefore we have $U^n(q_{\gamma_0^k}) = 0$. Otherwise, we assume i is the minimum among $\{1, \dots, k-n\}$ such that $\epsilon^1(q_{\gamma_0^i}) \neq 0$. As a consequence, we have planarity 1 contributed by $q_{\gamma_{2n-2}^i}$ by $\#\overline{\mathcal{M}}_{Y,o}(\{\gamma_{n-1}^i\}, \{\gamma_0^i\}) = 1$. Since i is the minimal one with nontrivial augmentation, we know that $U^n(q_{\gamma_{n-1}^i}) = 0$. Hence we have $H_{\text{cx}}(Y) \leq (n-1)^{\text{SD}}$. $\square$

Claim If $k = n$, then $H_{\text{cx}}(Y) \geq 1^{\text{SD}}$.

Proof It suffices to find one augmentation such that the order of semidilation is 1. We choose our augmentation to be $\epsilon^1(q_{\gamma_0}) = -n$ and $\epsilon^k = 0$ in all other cases. We first list the following expected dimensions of various moduli spaces:

- (1) $\text{vdim } \mathcal{M}_{Y,o}(\{\gamma_{p_i}^{nm}\}, \emptyset) = 2i + 2n(m-1) \geq 0$, which is positive unless $i = 0$ and $m = 1$, where we know $\#\overline{\mathcal{M}}_{Y,o}(\{\gamma_0^n\}, \emptyset) = n$.
- (2) For $l > 0$,

$$\text{vdim } \mathcal{M}_{Y,o}(\{\gamma_{p_i}^{mn+l}\}, \{\overbrace{\gamma_{p_0}, \dots, \gamma_{p_0}}^l\}) = 2i + 2l + 2n(m-1).$$

Then it is zero if and only if

- (i) $i = n-1, l = 1$ and $m = 0$, then $\#\overline{\mathcal{M}}_{Y,o}(\{\gamma_{2n-2}\}, \{\gamma_0\}) = 1$; or
- (ii) $i < n-1, l = n-i > 1$ and $m = 0$, then $\overline{\mathcal{M}}_{Y,o}(\{\gamma_{p_{n-l}}^l\}, \{\gamma_{p_0}, \dots, \gamma_{p_0}\}) = \emptyset$ by the argument of Proposition 7.15.

²²Although such a structure originally appears as part of the differential in the symplectic cochain complex, it contributes to the U -map in the S^1 -equivariant symplectic cohomology; see [89, Section 5] for a discussion.

(3) For $l > 0$,

$$\mathrm{vdim} \mathcal{M}_Y^1(\gamma_{p_i}^{nm+l}, \gamma_{p_j}^l, \emptyset) = \mathrm{vdim} \mathcal{M}_Y^2(\gamma_{p_i}^{nm+l}, \gamma_{*}^*, \gamma_{p_j}^l, \emptyset, \emptyset) = 2mn + 2i - 2j - 2.$$

It is zero if and only if

- (i) $m = 0$ and $i = j + 1$, which corresponds to $\langle U(q_{\gamma_{i+1}^l}), q_{\gamma_i^l} \rangle = n$; or
- (ii) $m = 1, i = 0$ and $j = n - 1$, assuming $\langle U(q_{\gamma_0^{n+l}}), q_{\gamma_{n-1}^l} \rangle = a_l$.

(4) For $l, s > 0$,

$$\mathrm{vdim} \mathcal{M}_Y^1(\gamma_{p_i}^{nm+l+s}, \gamma_{p_j}^l, \{\overbrace{\gamma_{p_0}, \dots, \gamma_{p_0}}^s\}) = 2mn + 2s + 2i - 2j - 2.$$

It is zero if and only if $m = 0$ and $j = s + i - 1$. In this case we have $\mathcal{M}^1$ as well as the corresponding $\mathcal{M}^2$ are possibly nonempty if and only if $s = 1$, for otherwise, we have $s + i - 1 > i$. Then we can use the compactness argument in [Proposition 7.15](#) and the fact that the stable manifold of q_i does not intersect with unstable manifold of q_{s+i-1} . When $m = 0, s = 1$ and $j = i$, this corresponds to $\langle U(q_{\gamma_i^{l+1}}), q_{\gamma_i^l} \rangle = \epsilon^1(q_{\gamma_0}) = -n$.

By (1) and (2), to supply for planarity 1, we must have $aq_{\gamma_0^n} + bq_{\gamma_{n-1}}$ with $a - b \neq 0$. Note that $U(aq_{\gamma_0^n} + bq_{\gamma_{n-1}}) = -anq_{\gamma_0^{n-1}} + bnq_{\gamma_{n-2}} \neq 0$. Therefore if the order of semidilation of this augmentation is smaller than 1, then there exists A generated by generators other than $q_{\gamma_0^n}, q_{\gamma_{n-1}}$ such that $U(A) = anq_{\gamma_0^{n-1}} - bnq_{\gamma_{n-2}}$. The only way to eliminate $-bnq_{\gamma_{n-2}}$ is having $bq_{\gamma_{n-2}^2}$ in A , which adds $bnq_{\gamma_{n-3}^2}$ to $U(A)$. The only way to compensate for such a term is add a $bq_{\gamma_{n-3}^3}$ to A . We can keep the argument going, and claim that A must be $b \sum_{i=2}^{n-1} q_{\gamma_{n-i}^i}$; then $U(A) = bnq_{\gamma_0^{n-1}} - bnq_{\gamma_{n-2}} \neq anq_{\gamma_0^{n-1}} - bnq_{\gamma_{n-2}}$, since $a - b \neq 0$. The claim follows. $\square$

When $n \leq k$, there are augmentations with zero order of semidilation. For example, one can use the augmentation from natural prequantization bundle filling, then the order of semidilation is 0 since the symplectic cohomology vanishes [73]. However, we conjecture that $H_{\mathrm{cx}}(Y) \geq 1^{\mathrm{SD}}$ whenever $n \leq k$. It is possible that there are BL_{∞} augmentations that are not from (even singular) fillings. Note that $n > k$ is the region where the quotient singularity is terminal. Hence we ask whether there is a relation between this algebrogeometric property with the contact property of the link via the hierarchy functor H_{cx} .

Conjecture 7.31 For discrete $G \subset U(n)$, if $\mathbb{C}^n/G$ is an isolated singularity then

$$H_{\mathrm{cx}}(S^{2n-1}/G, \xi_{\mathrm{std}}) = 0^{\mathrm{SD}}$$

if the singularity is terminal.

Combining with [Theorem 7.28](#), we can also ask the following question.

Question 7.32 Is the planarity of an isolated terminal singularity always 1? Is it true for terminal hypersurface singularities?

Similarly to Theorem 7.30, [91, Theorem 1.1(2)] can be rephrased as follows.

Theorem 7.33 *Let V be an exact domain, then $H_{\text{cx}}(\partial(V \times \mathbb{D})) = 0^{\text{SD}}$.*

7.3 An obstruction to IP

In dimension 3, obstructions to planar open book decomposition were studied from many different perspectives in [31; 70]. In higher dimensions, obstructions to supporting an iterated planar structure were found in [5]. By Corollary 5.15 and Theorem 6.3, we get the following easy-to-check obstruction to iterated planar structure.

Corollary 7.34 *If a contact manifold Y admits an exact filling that is not k -uniruled for any k , then Y is not iterated planar.*

As an application of this corollary, we have the following.

Corollary 7.35 *Let Q be a hyperbolic manifold of dimension ≥ 3 . Then S^*Q is not iterated planar.*

Proof The claim follows from a result of Viterbo [29, Theorem 1.7.5] that T^*Q is not k -uniruled for any k . $\square$

For other classes of cosphere bundles, by Theorem 7.28, $H_{\text{cx}}(S^*S^n) = 1^{\text{SD}}$ for $n \geq 2$. By Corollary 6.8, $H_{\text{cx}}(S^*T^n) = 2^{\text{P}}$ for $n \geq 2$. By Theorem 4.6, since $\text{SH}^*(T^*Q) \neq 0$ for any Q , we know $H_{\text{cx}}(S^*Q) > 0^{\text{SD}}$ (assuming Claim 4.3). As a consequence, there is no exact cobordism from S^*Q to $\partial(V \times \mathbb{D})$ for any Liouville domain V , which is a generalization of a result of Gromov [41]. By [90, Proposition 5.1], T^*Q admits a k -dilation for some $k \geq 1$ for a rationally inessential n -manifold Q ; ie if $H_n(Q; \mathbb{Q}) \rightarrow H_n(B\pi_1(Q); \mathbb{Q})$ vanishes, then we can update the estimate $H_{\text{cx}}(S^*Q)$ by figuring out k . For a Lagrangian Q that is a $K(\pi, 1)$ space, we have $H_{\text{cx}}(S^*Q) \geq 2^{\text{P}}$, since T^*Q carries no k -semidilation for any k .

Corollary 7.36 *For every $n \geq 3$, there exists a tight S^{2n-1} with the standard almost contact structure that is not iterated planar.*

Proof Note that the contact boundary of the Brieskorn variety $x_0^{n+2} + \dots + x_n^{n+2} = 1$ has planarity order ∞ by Theorem 7.28. By [50, Proposition 3.6], there are $a_i \geq n+2$ such that $Y := \text{LB}(a_0, \dots, a_n)$ is an exotic sphere. Proposition 7.27 implies that $P(Y) = \infty$. Then there exists k such that $\#^k Y$ is the standard smooth sphere, where $\#$ is the contact connected sum. However, the almost contact structure, which can be computed from [50, (19) and (20)], may not be standard. By [26, Theorem 1.2], there exists a Weinstein fillable contact sphere Y' such that $\#^k Y \# Y'$ is the standard almost contact sphere. Then $P(\#^k Y \# Y') \geq P(Y) \otimes P(Y) \otimes \dots \otimes P(Y') = \infty$ as $P(Y') \geq 1$, and the claim follows. $\square$

Corollary 7.37 *In all dimensions ≥ 5 , if (Y, J) is an almost contact manifold which has an exactly fillable contact representation (Y, ξ) , then there is a contact structure ξ' in the homotopy class of J such that (Y, ξ') is not iterated planar. In particular, any almost contact simply connected 5-manifold admits a contact representation which is not iterated planar.*

Proof Let Y' be the tight sphere from Corollary 7.36. Since $P(Y, \xi) > 0$ as (Y, ξ) has an exact filling, $H_{\text{cx}}(Y \# Y') = \infty^P$. By Corollary 7.34, $Y \# Y'$ is not iterated planar. The last claim follows from any almost contact simply connected 5-manifold being almost Weinstein fillable [38]; in particular, there is a contact representation that is Weinstein fillable by [20]. $\square$

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Algebraic models for classifying spaces of fibrations

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We prove new structural results for the rational homotopy type of the classifying space $B\mathrm{aut}(X)$ of fibrations with fiber a simply connected finite CW-complex X .

We first study nilpotent covers of $B\mathrm{aut}(X)$ and show that their rational cohomology groups are algebraic representations of the associated transformation groups. For the universal cover, this yields an extension of the Sullivan–Wilkerson theorem to higher homotopy and cohomology groups. For the cover corresponding to the kernel of the homology representation, this proves algebraicity of the cohomology of the homotopy Torelli space.

For the cover that classifies what we call *normal unipotent fibrations*, we then prove the stronger result that there exists a nilpotent dg Lie algebra $\mathfrak{g}(X)$ in algebraic representations that models its equivariant rational homotopy type. This leads to an algebraic model for the space $B\mathrm{aut}(X)$ and to a description of its rational cohomology ring as the cohomology of a certain arithmetic group $\Gamma(X)$ with coefficients in the Chevalley–Eilenberg cohomology of $\mathfrak{g}(X)$. This has strong structural consequences for the cohomology ring and, in certain cases, allows it to be completely determined using invariant theory and calculations with modular forms. We illustrate these points with concrete examples.

As another application, we significantly improve on certain results on self-homotopy equivalences of highly connected even-dimensional manifolds due to Berglund and Madsen, and we prove parallel new results in odd dimensions.

55P62, 55R40; 11F75, 55R15

1 Introduction

In this paper we prove new structural results about the classifying space $B\mathrm{aut}(X)$ of the topological monoid of self-homotopy equivalences of a simply connected finite CW-complex X from the point of view of rational homotopy theory.

The fundamental group of $B\mathrm{aut}(X)$ may be identified with the group $\mathcal{E}(X)$ of homotopy classes of self-homotopy equivalences of X . By a deep result due to Sullivan [66] and Wilkerson [71], the group $\mathcal{E}(X)$ maps onto an arithmetic group with finite kernel. Our first result extends this in two directions: to more general deck transformation groups and to higher homotopy and (co)homology groups.

Theorem 1.1 *Let $\mathcal{E}_G(X)$ be the deck transformation group of the cover $B\mathrm{aut}_G(X)$ of $B\mathrm{aut}(X)$ associated to a subgroup G of $\mathcal{E}(X)$. If G acts nilpotently on $H_*(X; \mathbb{Q})$, then $\mathcal{E}_G(X)$ maps onto an arithmetic group with finite kernel. Moreover, the representations of $\mathcal{E}_G(X)$ in the rational homology and simple homotopy groups¹ of $B\mathrm{aut}_G(X)$ are algebraic representations of the ambient algebraic group.*

This appears as [Theorem 3.22](#) in the text. For G the trivial group, this is precisely the result of Sullivan and Wilkerson, enhanced by the statement that the representations of $\mathcal{E}(X)$ in the higher homotopy and homology groups of the universal cover of $B\mathrm{aut}(X)$ are algebraic; see [Section 3.6.1](#). For G the kernel of the action of $\mathcal{E}(X)$ on $H_*(X; \mathbb{Z})$, this yields a homotopical counterpart of the algebraicity result of Kupers and Randal-Williams [\[49\]](#) for the cohomology of Torelli groups of the manifolds W_g — for arbitrary simply connected finite complexes X ; see [Section 3.6.2](#).

Our main result extends [Theorem 1.1](#) further to a space-level statement, for a particular choice of cover, and we use this to construct an algebraic model for $B\mathrm{aut}(X)$.

The problem of finding an algebraic model for $B\mathrm{aut}(X)$ has been raised in Gómez-Tato, Halperin and Tanré [\[39, Section 7.3\]](#) and Lazarev [\[50, Section 7.2\]](#). This problem is highly nontrivial, because the space $B\mathrm{aut}(X)$ is not nilpotent in general and therefore not amenable to the classical methods of rational homotopy theory. Algebraic models are known for the universal cover of $B\mathrm{aut}(X)$ (see Sullivan [\[66\]](#) and Tanré [\[67\]](#)) and for certain nilpotent covers (see Félix, Fuentes and Murillo [\[33\]](#)), but there are no general results that incorporate the action of the deck transformation groups into such models, let alone integrate such information to structural results about $B\mathrm{aut}(X)$. In this paper, we offer a solution. Instead of trying to incorporate the action of $\mathcal{E}(X)$ into models for the universal cover, which is the approach suggested in [\[39; 50\]](#), the key to our solution is pass to a cover with better properties.

Theorem 1.2 *There is a normal cover $B\mathrm{aut}_u(X)$ of $B\mathrm{aut}(X)$, with deck transformation group $\Gamma(X)$, such that*

- (1) $\Gamma(X)$ is an arithmetic subgroup of a reductive algebraic group $R(X)$,
- (2) $B\mathrm{aut}_u(X)$ is $\Gamma(X)$ -equivariantly rationally equivalent to the nerve $\langle \mathfrak{g}(X) \rangle$ of a nilpotent dg Lie algebra $\mathfrak{g}(X)$ of algebraic representations of $R(X)$.

The proof is given in [Section 3.7](#). Here, we say that two connected spaces are *rationally equivalent* if they can be connected by a zig-zag of maps that induce rational isomorphisms² on all homotopy and homology groups. The *nerve* is the functor from dg Lie algebras to topological spaces that effects the

¹By the *simple* homotopy groups of a connected space, we mean the homotopy groups modulo the action of the fundamental group; see [Definition 3.20](#).

²We call a group homomorphism a rational isomorphism if its kernel is torsion and if for each element x of the target, the k^{th} power x^k belongs to the image for some $k \neq 0$; see [Definition 2.1](#). For abelian groups, this is equivalent to inducing an isomorphism after tensoring with $\mathbb{Q}$.

equivalence between the homotopy category of nilpotent dg Lie algebras and the homotopy category of rational nilpotent spaces; see [Section 2.4](#). The cover $B\mathrm{aut}_u(X)$ can be characterized as the classifying space for what we call *normal unipotent fibrations*; see [Section 4.2](#).

As a first application, [Theorem 1.2](#) yields the following commutative dg algebra model for $B\mathrm{aut}(X)$.

Corollary 1.3 *The space $B\mathrm{aut}(X)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{g}(X) \rangle_{h\Gamma(X)}$ and there is a zig-zag of quasi-isomorphisms of commutative dg algebras*

$$\Omega^*(B\mathrm{aut}(X)) \sim \Omega^*(\Gamma(X), C_{\mathrm{CE}}^*(\mathfrak{g}(X))).$$

This is proved as [Theorem 3.39](#) in the text. Here $\Omega^*(B)$ stands for Sullivan’s polynomial differential forms on B . It is a commutative dg algebra model for the singular cochains $C^*(B; \mathbb{Q})$. The right-hand side is a commutative dg algebra model for the cochains on the group $\Gamma(X)$ with coefficients in the Chevalley–Eilenberg cochains of $\mathfrak{g}(X)$; see [Section 3.8](#).

[Theorem 1.2](#) can be interpreted as a “space-level” enhancement of [Theorem 1.1](#) in the following sense: not only are the rational cohomology groups of $B\mathrm{aut}_u(X)$ algebraic representations of the ambient algebraic group $R(X)$, but the entire *rational homotopy type* of $B\mathrm{aut}_u(X)$, represented by the dg Lie algebra $\mathfrak{g}(X)$, is an algebraic representation of $R(X)$. This subsumes algebraicity of cohomology, because there is an isomorphism of $\Gamma(X)$ -modules

$$H^*(B\mathrm{aut}_u(X); \mathbb{Q}) \cong H_{\mathrm{CE}}^*(\mathfrak{g}(X)),$$

and the right-hand side is obviously an algebraic representation of $R(X)$ if $\mathfrak{g}(X)$ is. However, space-level algebraicity has much stronger consequences than algebraicity of cohomology. Together with semisimplicity of algebraic representations of reductive groups, it implies the following result, which reduces the computation of the rational cohomology ring of $B\mathrm{aut}(X)$ to the computation of Chevalley–Eilenberg cohomology and cohomology of arithmetic groups with coefficients in algebraic representations.

Corollary 1.4 *There is an isomorphism of graded algebras*

$$(1) \quad H^*(B\mathrm{aut}(X); \mathbb{Q}) \cong H^*(\Gamma(X), H_{\mathrm{CE}}^*(\mathfrak{g}(X))). \quad \square$$

The proof is given in [Section 3.9](#). This result implies, but is stronger than, collapse of the rational Serre spectral sequence of the homotopy fiber sequence

$$(2) \quad B\mathrm{aut}_u(X) \rightarrow B\mathrm{aut}(X) \rightarrow B\Gamma(X).$$

Indeed, collapse of the spectral sequence only implies an isomorphism of algebras after passing to the associated graded algebra of some filtration, whereas [Corollary 1.4](#) says that $H^*(B\mathrm{aut}(X); \mathbb{Q})$ is isomorphic to the E_2 -page of this spectral sequence as a graded algebra. In particular, this algebra admits a bigrading by

$$(3) \quad H^{p,q} = H^p(\Gamma(X), H_{\mathrm{CE}}^q(\mathfrak{g}(X))).$$

Combined with the finiteness of the virtual cohomological dimension of arithmetic groups (see Borel and Serre [21]), this implies the following result, which reduces the computation of the ring $H^*(B\operatorname{aut}(X); \mathbb{Q})$ modulo nilpotent elements to invariant theory.

Corollary 1.5 *The ring homomorphism*

$$(4) \quad H^*(B\operatorname{aut}(X); \mathbb{Q}) \rightarrow H^*(B\operatorname{aut}_u(X); \mathbb{Q})^{\Gamma(X)}$$

is split surjective and its kernel I is a nilpotent ideal such that $I^n = 0$ for all $n > \operatorname{vcd}(\Gamma(X))$. In particular, (4) induces an isomorphism modulo nilradicals. In other words, the algebraic variety of the graded commutative ring $H^*(B\operatorname{aut}(X); \mathbb{Q})$ is isomorphic to that of the invariant ring $H_{\text{CE}}^*(\mathfrak{g}(X))^{\Gamma(X)}$.

Proof Under the isomorphism (1), the homomorphism (4) corresponds to the projection $H^{*,*} \rightarrow H^{0,*}$, which is clearly split. The kernel is the ideal $H^{\geq 1,*}$, the n^{th} power of which is contained in $H^{\geq n,*}$, which vanishes for $n > \operatorname{vcd}(\Gamma(X))$. $\square$

Let us point out another nontrivial facet of Corollary 1.4.

Corollary 1.6 *The ring homomorphism*

$$H^*(\Gamma(X), \mathbb{Q}) \rightarrow H^*(B\operatorname{aut}(X); \mathbb{Q})$$

is split injective.

Proof The homomorphism in question corresponds to the inclusion of the subring $H^{*,0}$ into $H^{*,*}$, which is clearly split. $\square$

This means in particular that all cohomology classes of the arithmetic group $\Gamma(X)$ are faithfully represented as characteristic classes of fibrations with fiber X . This is especially striking in view of the fact that many arithmetic groups can be realized as $\Gamma(X)$ for some X , cf [66, Theorem 10.3(iv)].

We have chosen to state our main result as an existence theorem in order to highlight the strong consequences of the mere existence of a dg Lie model of algebraic representations, but the ingredients $R(X)$, $\Gamma(X)$ and $\mathfrak{g}(X)$ can be given concrete descriptions; see Section 4. In Section 5 we use these concrete descriptions to make several explicit computations. Let us discuss one of these computations here. For the n -fold product of a d -dimensional sphere, $S^{d \times n} = S^d \times \cdots \times S^d$, Corollary 1.4 assumes the following form.

Theorem 1.7 *For d odd, there is an isomorphism of graded algebras*

$$(5) \quad H^*(B\operatorname{aut}(S^{d \times n}); \mathbb{Q}) \cong H^*(\Gamma(S^{d \times n}), \operatorname{Sym}^\bullet(V_n[d+1])),$$

where $V_n[d+1]$ denotes the standard representation of $\operatorname{GL}_n(\mathbb{Q})$ put in degree $d+1$,

$$\Gamma(S^{d \times n}) = \begin{cases} \operatorname{GL}_n(\mathbb{Z}) & \text{if } d = 1, 3, 7, \\ \operatorname{GL}_n^\Sigma(\mathbb{Z}) & \text{if } d \neq 1, 3, 7, \end{cases}$$

and $\operatorname{GL}_n^\Sigma(\mathbb{Z}) \leq \operatorname{GL}_n(\mathbb{Z})$ is the congruence subgroup of matrices with exactly one odd entry in each row.

For $n = 2$ the right-hand side of (5) can be computed in terms of modular forms via the Eichler–Shimura isomorphism. We also carry out computations for $n = 3$, but for larger n the cohomology of $\mathrm{GL}_n(\mathbb{Z})$, or its congruence subgroups, with coefficients in algebraic representations is not fully known. See Section 5.1 for more details and a further discussion.

The above example illustrates that even in cases where $\Gamma(X)$ and $H_{\mathrm{CE}}^*(\mathfrak{g}(X))$ can be described explicitly, a complete computation of the right-hand side of (1) is in general out of reach, due to the difficulty of calculating the cohomology of arithmetic groups. This suggests a different perspective on the isomorphism (1). Rather than interpreting it as a computation of $H^*(B\mathrm{aut}(X); \mathbb{Q})$, it tells us that cohomology classes of arithmetic groups in the right-hand side, say classes constructed using automorphic forms, can in principle be represented as characteristic classes of fibrations. Connections between characteristic classes and automorphic forms have been observed before in special cases; see eg Furusawa, Tezuka and Yagita [36]. Our results show that this is not an isolated phenomenon. This suggests that there is a deep connection between characteristic classes of fibrations and cohomology of arithmetic groups.

Another important application of our results is that they lead to significant simplifications and improvements of certain key results of Berglund and Madsen [16] about the cohomology of $B\mathrm{aut}(W_g)$, and related spaces, for the manifold

$$W_g = \#^g S^d \times S^d.$$

In fact, this was the original motivation for this work. This is discussed in Section 5.2. Our methods also yield parallel results for highly connected odd-dimensional manifolds that were unattainable by the methods of [16]; see Section 5.3. This is used in the work of Stoll [65].

In the final section, Section 5.4, we study an example that, among other things, illuminates the advantage of working over $\Gamma(X)$ rather than $\mathcal{E}(X)$.

1.1 Some comments on related work

The first paragraph on page 314 in Sullivan’s [66] contains, without proof, the idea of modeling $B\mathrm{aut}(|\Lambda|)$, where $|\Lambda|$ is the realization of a minimal Sullivan algebra Λ , by taking the nerve of the maximal nilpotent ideal of $\mathrm{Der} \Lambda$ modulo the action of the reductive part of $\mathrm{Aut} \Lambda$. This idea seems to have been largely overlooked in the subsequent rational homotopy theory literature; we are not aware of any source where this idea and its consequences have been properly developed (and in fact we only became aware of this paragraph in the final stages of writing this paper). Theorem 4.14 could be viewed as giving a precise formulation and proof. The key points of the present paper — the treatment of $B\mathrm{aut}(X)$ for nonrational X , the algebraicity of the cohomology of nilpotent covers of $B\mathrm{aut}(X)$, the existence of algebraic Lie models and its strong consequences for the structure of the cohomology ring of $B\mathrm{aut}(X)$ — are to the best of our knowledge new. Our results could be regarded as a strong vindication of Sullivan’s idea.

We were inspired by Oprea [57] (via Burghlea [26]) for the idea of passing to the maximal reductive quotient of $\mathcal{E}(X_{\mathbb{Q}})$ to rectify homotopy actions on the algebraic models. The idea of studying the fiberwise

rationalization of $B\mathrm{aut}(X) \rightarrow B\Gamma(X)$ as we do here is similar in spirit to studying relative Malcev completions of mapping class groups as done by Hain [41]. The algebraicity result for the cohomology of the Torelli group of W_g of Kupers and Randal-Williams [49] inspired us to study similar questions for $B\mathrm{aut}(X)$.

Lazarev [50, Theorem 5.1] constructs Lie models for the universal cover of $B\mathrm{aut}(X_{\mathbb{Q}})$ and shows that the action of the Lie algebra of $\mathcal{E}(X_{\mathbb{Q}})$ on the higher homotopy groups of $B\mathrm{aut}(X_{\mathbb{Q}})$ can be computed in terms of Chevalley–Eilenberg and Harrison cohomology. However, he does not address algebraicity of the representations and he does not construct group actions on the Lie models; in fact he raises this as a problem [50, Section 7.2].

Félix, Fuentes and Murillo [33] construct a Lie model for the space $B\mathrm{aut}_G(X)$ when $G \leq \mathcal{E}(X)$ is a subgroup that acts nilpotently on $H_*(X; \mathbb{Z})$. We recover this Lie model; see Corollary 3.15 and Remark 3.17. A crucial advantage of our approach is that it lets us incorporate the action of the deck transformation group. This aspect is not addressed in [33]. This is what allows us to construct an algebraic model for the full space $B\mathrm{aut}(X)$ and not only for nilpotent covers of it.

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2 Background and preliminaries

2.1 Localization of nilpotent groups and spaces

We begin by introducing some terminology and recalling some facts about localizations of nilpotent groups and spaces, mainly following [44].

Definition 2.1 We call a group G uniquely divisible, or $\mathbb{Q}$ -local, if the equation $x^k = a$ has a unique solution $x \in G$ for every nonzero integer k and every $a \in G$. For a group homomorphism $f: G \rightarrow H$, we use the following terminology:

- (1) f is $\mathbb{Q}$ -injective if every element of $\ker(f)$ has finite order.
- (2) f is $\mathbb{Q}$ -surjective if for all $x \in H$, we have $x^k \in \mathrm{im}(f)$ for some $k \neq 0$.
- (3) f is a $\mathbb{Q}$ -isomorphism if it is both $\mathbb{Q}$ -injective and $\mathbb{Q}$ -surjective.

Every nilpotent group G admits a $\mathbb{Q}$ -localization $r: G \rightarrow G_{\mathbb{Q}}$, characterized up to isomorphism by the properties that $G_{\mathbb{Q}}$ is a $\mathbb{Q}$ -local nilpotent group and that r is a $\mathbb{Q}$ -isomorphism; see [44, page 7]. A homomorphism $G \rightarrow H$ between nilpotent groups is a $\mathbb{Q}$ -isomorphism if and only if the induced homomorphism $G_{\mathbb{Q}} \rightarrow H_{\mathbb{Q}}$ is an isomorphism. For abelian groups G , one has $G_{\mathbb{Q}} \cong G \otimes \mathbb{Q}$.

Recall that a connected topological space X is called nilpotent if the group $\pi_1(X)$ is nilpotent and if its action on $\pi_n(X)$ is nilpotent for every $n \geq 2$, in the sense that there is a filtration of $\pi_n(X)$ by $\pi_1(X)$ -submodules such that the action on the filtration quotients is trivial.

A nilpotent space X is called $\mathbb{Q}$ -local if the group $\pi_n(X)$ is $\mathbb{Q}$ -local for each $n \geq 1$. Every nilpotent space X admits a $\mathbb{Q}$ -localization, or rationalization, $r: X \rightarrow X_{\mathbb{Q}}$, characterized up to homotopy by the properties that $X_{\mathbb{Q}}$ is a $\mathbb{Q}$ -local nilpotent space and that $\pi_n(r): \pi_n(X) \rightarrow \pi_n(X_{\mathbb{Q}})$ is a $\mathbb{Q}$ -localization for every n .

Definition 2.2 We call a map $f: X \rightarrow Y$ between connected topological spaces a *rational homotopy equivalence* if $\pi_n(f): \pi_n(X) \rightarrow \pi_n(Y)$ is a $\mathbb{Q}$ -isomorphism for all n , and a *rational homology equivalence* if $H_n(f): H_n(X; \mathbb{Q}) \rightarrow H_n(Y; \mathbb{Q})$ is an isomorphism for all n .

We say that f is a *rational equivalence* if it is both a rational homotopy equivalence and a rational homology equivalence.

It is well-known that a map between nilpotent spaces is a rational homotopy equivalence if and only if it is a rational homology equivalence. We will need an extension of this fact to virtually nilpotent spaces. Recall from [30; 31] that a connected space X is called *virtually nilpotent* if $\pi_1(X)$ has a nilpotent subgroup of finite index and for each $n \geq 2$, there is a finite-index subgroup of $\pi_1(X)$ that acts nilpotently on $\pi_n(X)$. Equivalently, X is virtually nilpotent if each Postnikov section $P_n X$ of X admits a finite cover $E \rightarrow P_n X$ such that E is nilpotent.

Lemma 2.3 Let $f: X \rightarrow Y$ be a map from a virtually nilpotent space X to a nilpotent space Y . If f is a rational homotopy equivalence, then f is a rational homology equivalence.

Proof It suffices to show that f induces a rational homology equivalence on each Postnikov section, so we may without loss of generality assume that X admits a finite cover $p: E \rightarrow X$ such that E is nilpotent. Clearly, p is a rational homotopy equivalence. The composite $fp: E \rightarrow Y$ is then a rational homotopy equivalence between nilpotent spaces, so it is a rational homology equivalence. Since p is a finite cover, a transfer argument shows that $H_*(p; \mathbb{Q}): H_*(E; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$ is surjective. We just saw that $H_*(f; \mathbb{Q}) \circ H_*(p; \mathbb{Q}) = H_*(fp; \mathbb{Q})$ is an isomorphism, so $H_*(p; \mathbb{Q})$ must be injective as well. It follows that $H_*(f; \mathbb{Q})$ is an isomorphism. $\square$

Remark 2.4 The converse is true if X is nilpotent, but false in general: the map $\mathbb{R}P^2 \rightarrow *$ is a rational homology equivalence from a virtually nilpotent space to a nilpotent space, but the induced map on π_2 is not a $\mathbb{Q}$ -isomorphism.

Also, one cannot relax nilpotence of Y to virtual nilpotence. The universal cover $S^2 \rightarrow \mathbb{R}P^2$ provides an example of a map from a nilpotent space to a virtually nilpotent space which is a rational homotopy equivalence but not a rational homology equivalence.

2.2 Affine algebraic groups and arithmetic groups

In this section we will collect the basic facts about affine algebraic groups over $\mathbb{Q}$ and arithmetic groups that we will need, following mainly [45; 55; 62]. All algebras, vector spaces, undecorated tensor products, affine schemes, etc, should be taken to be over $\mathbb{Q}$ unless explicitly specified otherwise.

Recall that an affine algebraic group is a group object in the category of affine schemes. A linear algebraic group is an algebraic subgroup of GL_n for some n . It is a well-known fact that every affine algebraic group admits a faithful algebraic representation [55, Corollary 4.10], so the notions of affine algebraic group and linear algebraic group essentially coincide.

2.2.1 Unipotent and reductive groups Recall that a representation V is called unipotent (or nilpotent) if there is a sequence of subrepresentations

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V$$

such that V_i/V_{i-1} is a trivial representation for every i .

An affine algebraic group U is called unipotent if every algebraic representation V of U is unipotent. Equivalently, U is unipotent if and only if it is an algebraic subgroup of the group $\mathbb{U}_n$ of $n \times n$ upper-triangular matrices with ones along the diagonal for some n ; see [55, Theorem 14.5]. The group of $\mathbb{Q}$ -points $U(\mathbb{Q})$ of a unipotent group U is, in particular, nilpotent and uniquely divisible.

Every affine algebraic group G admits a largest normal unipotent subgroup, called the unipotent radical G_u of G ; see [45, Theorem 10.5]. An affine algebraic group G is called reductive if G_u is trivial. The representation theory of reductive groups in characteristic zero is particularly well-behaved:

Theorem 2.5 [55, Corollary 22.43], [45, page 78] *Every finite-dimensional algebraic representation of a reductive group is semisimple.* $\square$

In other words, the category $\mathrm{Rep}_{\mathbb{Q}}(G)$ of finite-dimensional algebraic representations of G is a semisimple abelian category whenever G is reductive.

Another feature of affine algebraic groups in characteristic zero is the existence of Levi decompositions. There is an extension

$$(6) \quad 1 \rightarrow G_u \rightarrow G \rightarrow G/G_u \rightarrow 1$$

of the maximal reductive quotient of G by the unipotent radical.

Theorem 2.6 [45, Theorem 14.2] *The extension (6) is (noncanonically) split and moreover, any two splittings are conjugate in the action of G_u .* $\square$

The following lemma gives a concrete description of the unipotent radical and the maximal reductive quotient. It is presumably well-known, but we include a proof as we have not found the precise statement we give here in the literature.

Recall that a composition series of a representation V is a filtration

$$0 = V_0 \subseteq \cdots \subseteq V_n = V$$

by subrepresentations such that each V_i/V_{i-1} is a simple representation. The “semisimplification” of V is the associated graded representation

$$V^{\text{ss}} = \bigoplus_{i=1}^n V_i/V_{i-1}.$$

It is semisimple by construction and its isomorphism type is independent of the choice of composition series by the Jordan–Hölder theorem. Note that V is isomorphic to V^{ss} if and only if V is semisimple.

Lemma 2.7 *Let G be an affine algebraic group defined over $\mathbb{Q}$. If V is a representation of G with unipotent kernel, then the unipotent radical G_u and the maximal reductive quotient G/G_u may be identified with the kernel and the image, respectively, of the homomorphism $G \rightarrow \text{GL}(V^{\text{ss}})$.*

Proof Let K and N denote the kernel of the action of G on V and V^{ss} , respectively. Clearly, both K and N are normal in G and $K \leq N$. Furthermore, K is unipotent by hypothesis, and N/K is unipotent because V is a faithful unipotent representation of it. Since unipotent groups are closed under extensions (see [55, Corollary 14.7]), it follows that N is unipotent.

To show that N is the unipotent radical, we need to show that every normal unipotent subgroup U of G acts trivially on V^{ss} . For this, it suffices to show that U acts trivially on every simple G -representation W . Since U is unipotent, there is a nonzero $w \in W$ that is fixed by U ; see [55, Proposition 14.1]. Since U is normal in G , it also fixes gw for every $g \in G$. Indeed, for every $u \in U$ we have $g^{-1}ug \in U$, whence $(g^{-1}ug)w = w$ so that $ugw = gw$. Since W is simple, w generates W as a G -module, so U acts trivially on W . $\square$

This has the following consequence:

Lemma 2.8 *Let G be an affine algebraic group defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$. If V is a representation of G with unipotent kernel, then the Lie algebra of the unipotent radical, $\text{Lie } G_u$, may be identified with the maximal ideal*

$$\text{nil}_V \mathfrak{g} \subseteq \mathfrak{g},$$

consisting of elements which act nilpotently on V .

Proof By [55, Section 10.14], the functor $G \mapsto \text{Lie } G$ from affine algebraic groups to Lie algebras commutes with pullbacks, so in particular it preserves kernels of morphisms. Thus by the preceding lemma,

$$(7) \quad \text{Lie } G_u = \ker (\mathfrak{g} \longrightarrow \mathfrak{gl}(V^{\text{ss}})).$$

Thus we have that $\text{Lie } G_u \subseteq \text{nil}_V \mathfrak{g}$.

For the reverse inclusion, it suffices to show that $\mathrm{nil}_V \mathfrak{g}$ is the Lie algebra of a normal unipotent subgroup H of G , as then $H \leq G_u$ and $\mathrm{nil}_V \mathfrak{g} \subseteq \mathrm{Lie} G_u$. Let K be the kernel of the action of G on V , and let $\mathfrak{h}'$ be the image of $\mathrm{nil}_V \mathfrak{g}$ under the action morphism $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$. Then $\mathfrak{h}'$ consists of nilpotent endomorphisms of V , so by Engel's theorem, $\mathfrak{h}'$ is a nilpotent subalgebra of $\mathrm{Lie}(G/K) \subseteq \mathfrak{gl}(V)$. Thus by [55, Theorem 14.37] there is a unipotent subgroup $H' \leq G/K \leq \mathrm{GL}(V)$ such that $\mathfrak{h}' = \mathrm{Lie} H'$. Let H be the preimage of H' in G . Then H is an extension of the unipotent group H' by the unipotent group K , hence unipotent, and $\mathrm{Lie} H$ is therefore nilpotent. Moreover, since the functor Lie commutes with pullbacks, we get that $\mathrm{nil}_V \mathfrak{g} = \mathrm{Lie} H$. Finally, since $\mathrm{nil}_V \mathfrak{g}$ is an ideal of $\mathfrak{g}$, the subgroup H of G is normal. $\square$

The quotient G/H of an affine algebraic group G by a normal algebraic subgroup H always exists [55, Theorem 5.14], but the rational points of the quotient $(G/H)(\mathbb{Q})$ need not agree with $G(\mathbb{Q})/H(\mathbb{Q})$ in general. They do agree, however, if H is unipotent.

Lemma 2.9 *Let G be an affine algebraic group. If H is a normal unipotent algebraic subgroup of G , then the natural homomorphism*

$$G(\mathbb{Q})/H(\mathbb{Q}) \rightarrow (G/H)(\mathbb{Q})$$

is an isomorphism.

Proof This is a consequence of the vanishing of Galois cohomology,

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), H),$$

for unipotent groups H ; see [71, Theorem 9.5]. $\square$

Similarly, the normalizer $N_G(H)$ of an algebraic subgroup H of an affine algebraic group G always exists and its rational points, $N_G(H)(\mathbb{Q})$, are contained in $N_{G(\mathbb{Q})}(H(\mathbb{Q}))$, the normalizer of $H(\mathbb{Q})$ in $G(\mathbb{Q})$ in the ordinary sense [55, Proposition 1.83]. In general, equality need not hold, but it does if H is unipotent.

Lemma 2.10 *If H is a unipotent algebraic subgroup of an affine algebraic group G , then $N_G(H)(\mathbb{Q}) = N_{G(\mathbb{Q})}(H(\mathbb{Q}))$.*

Proof This follows from [55, Proposition 1.84] upon noting that every affine algebraic group over $\mathbb{Q}$ is smooth [55, Theorem 3.23] and that $H(\mathbb{Q})$ is dense in H if H is unipotent (eg by [55, Theorem 17.93]). $\square$

2.2.2 Arithmetic subgroups Given a linear algebraic group $G \leq \mathrm{GL}_n$, we write

$$G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$$

for the integer matrices inside the rational points of G . By definition, an arithmetic subgroup Γ of G is a subgroup of $G(\mathbb{Q})$ which is commensurable with $G(\mathbb{Z})$, ie the intersection $\Gamma \cap G(\mathbb{Z})$ has finite index in both $G(\mathbb{Z})$ and Γ .

Arithmeticity can be thought of as a strong finiteness property. In particular, arithmetic groups are finitely presented [62].

Lemma 2.11 [62, Section 1.1] *Let G be an affine algebraic group over $\mathbb{Q}$ and let $H \leq G$ be an algebraic subgroup. If Γ is arithmetic in $G(\mathbb{Q})$, then $\Gamma \cap H(\mathbb{Q})$ is arithmetic in $H(\mathbb{Q})$.*

Lemma 2.12 [18, Theorem 6] *Let $\varphi: G \rightarrow G'$ be a surjective morphism of affine algebraic groups defined over $\mathbb{Q}$. If Γ is an arithmetic subgroup of $G(\mathbb{Q})$, then $\varphi(\Gamma)$ is an arithmetic subgroup of $G'(\mathbb{Q})$.*

We record the following, presumably well-known, characterization of arithmetic subgroups of unipotent groups for later use. We include a proof for completeness.

Lemma 2.13 *Let U be a unipotent algebraic group defined over $\mathbb{Q}$ and let Γ be a subgroup of $U(\mathbb{Q})$.*

- (1) Γ is Zariski dense in U if and only if the inclusion of Γ into $U(\mathbb{Q})$ is $\mathbb{Q}$ -surjective.
- (2) Γ is an arithmetic subgroup of $U(\mathbb{Q})$ if and only if it is finitely generated and Zariski dense.

Proof We may assume that U is a subgroup of $\mathbb{U}_n$ for some n . Following [61, page 104], consider the Lie subalgebra $\mathcal{L}(\Gamma)$ of the Lie algebra $\mathfrak{u}$ of U given by the linear span of the image of Γ under the bijection $\log: U(\mathbb{Q}) \rightarrow \mathfrak{u}$. The associated unipotent algebraic subgroup $\exp \mathcal{L}(\Gamma) \leq U$ contains Γ , so it must be equal to U if Γ is Zariski dense. The inclusion $\Gamma \rightarrow U(\mathbb{Q})$ is then $\mathbb{Q}$ -surjective by [61, Theorem 2(iv), page 104]. Conversely, if the inclusion of Γ into $U(\mathbb{Q})$ is $\mathbb{Q}$ -surjective, then so is the inclusion of the Zariski closure $\bar{\Gamma}$. This is then a $\mathbb{Q}$ -isomorphism between nilpotent uniquely divisible groups, so it must be an equality. This proves the first statement.

It follows from [61, Exercise 13, page 123] that every finitely generated dense subgroup of $U(\mathbb{Q})$ is arithmetic. Conversely, if Γ is arithmetic, then it is finitely generated, and we will now argue that the inclusion $\Gamma \rightarrow U(\mathbb{Q})$ is $\mathbb{Q}$ -surjective. Inclusions of finite-index subgroups are clearly $\mathbb{Q}$ -isomorphisms, so it suffices to show that the inclusion of $U(\mathbb{Z})$ into $U(\mathbb{Q})$ is $\mathbb{Q}$ -surjective, ie that for every $A \in U(\mathbb{Q})$, there is a positive integer k such that $A^k \in U(\mathbb{Z})$. Since A is a unipotent $n \times n$ matrix, the matrix $N = A - I$ satisfies $N^n = 0$. Pick a positive integer d such that dN^i has integer entries for all $i \geq 1$, and let $k = 1!2! \cdots n!d$. Then the matrix

$$A^k = \sum_{i=0}^{n-1} \binom{k}{i} N^i$$

has integer entries, because each coefficient $\binom{k}{i}$ is divisible by d . □

Since $U(\mathbb{Q})$ is nilpotent and $\mathbb{Q}$ -local when U is unipotent, the group $U(\mathbb{Q})$ will be a $\mathbb{Q}$ -localization of the nilpotent group $\Gamma \leq U(\mathbb{Q})$ whenever Γ is dense in U .

Remark 2.14 One cannot relax unipotence to nilpotence in Lemma 2.13. The multiplicative group is abelian and in particular nilpotent, but the inclusion of $\mathbb{Z}^\times$ into $\mathbb{Q}^\times$ is not $\mathbb{Q}$ -surjective.

2.3 Nilpotent radicals of dg Lie algebras

In this section we discuss the notion of nilradical in the setting of differential graded Lie algebras. Let $(\mathfrak{g}, \delta)$ be a dg Lie algebra, possibly unbounded as a chain complex. We write $\Gamma^k \mathfrak{g}$ for the lower central series of $\mathfrak{g}$, so $\Gamma^1 \mathfrak{g} = \mathfrak{g}$ and $\Gamma^{k+1} \mathfrak{g} = [\Gamma^k \mathfrak{g}, \mathfrak{g}]$. Given an integer k , we write $\mathfrak{g}\langle k \rangle$ for the truncation of $\mathfrak{g}$ given by

$$\mathfrak{g}\langle k \rangle_n = \begin{cases} 0 & \text{if } n < k, \\ \ker(\delta: \mathfrak{g}_n \rightarrow \mathfrak{g}_{n-1}) & \text{if } n = k, \\ \mathfrak{g}_n & \text{if } n > k. \end{cases}$$

We call $\mathfrak{g}$ *connected* if $\mathfrak{g} = \mathfrak{g}\langle 0 \rangle$ and we call $\mathfrak{g}$ *simply connected* if $\mathfrak{g} = \mathfrak{g}\langle 1 \rangle$. We say that $\mathfrak{g}$ is of *finite type* if $\mathfrak{g}_i$ is finite-dimensional for every i .

Definition 2.15 A connected dg Lie algebra $\mathfrak{g}$ is *nilpotent* if the following equivalent conditions are satisfied:

- (i) for every n , we have $(\Gamma^k \mathfrak{g})_n = 0$ for $k \gg n$;
- (ii) $\mathfrak{g}_0$ is a nilpotent Lie algebra which acts nilpotently on $\mathfrak{g}_n$ for every $n > 0$.

Definition 2.16 The *nilradical* of $\mathfrak{g}$, denoted by $\text{nil } \mathfrak{g}$, is the maximal nilpotent ideal of $\mathfrak{g}\langle 0 \rangle$, provided such an ideal exists.

Remark 2.17 Ordinary Lie algebras may be identified with dg Lie algebras concentrated in degree 0. For these, the above definitions specialize to the usual definitions of nilpotence and nilradicals. In particular, the nilradical does not necessarily exist unless certain finiteness conditions are imposed. If the nilradical exists, however, then it is unique: if I, J are nilpotent ideals of $\mathfrak{g}$, then so is $I + J$. Thus if I is maximal, then $J \subseteq I + J = I$.

Lemma 2.18 Suppose that $Z_0(\mathfrak{g})$ is finite-dimensional. Then $\text{nil } \mathfrak{g}$ exists.

Proof We may assume that $\mathfrak{g}$ is connected. Note that the positively graded truncation $\mathfrak{g}\langle 1 \rangle$ is always a nilpotent ideal of $\mathfrak{g}$, so the poset of all nilpotent ideals contains a maximal element if and only if the poset of nilpotent ideals above $\mathfrak{g}\langle 1 \rangle$ does. But the latter poset injects into the poset of nilpotent (or indeed all) ideals of the quotient dg Lie algebra $\mathfrak{g}' = \mathfrak{g}/\mathfrak{g}\langle 1 \rangle$. Clearly $\mathfrak{g}'_i = 0$ for $i > 1$, the differential $\mathfrak{g}'_1 \rightarrow \mathfrak{g}'_0$ is injective, and $\mathfrak{g}'_0 = \mathfrak{g}_0$ is finite-dimensional, so every chain of ideals of $\mathfrak{g}'$ is finite. $\square$

2.4 Geometric realizations of nilpotent dg Lie algebras

To each nilpotent Lie algebra $\mathfrak{g}$ over $\mathbb{Q}$, one can associate a group $\exp(\mathfrak{g})$ with underlying set $\mathfrak{g}$ and with multiplication given by the Baker–Campbell–Hausdorff formula. The association $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ is part of an equivalence of categories between nilpotent Lie algebras over $\mathbb{Q}$ and nilpotent uniquely divisible

groups; see [59, Appendix A]. When $\mathfrak{g}$ is finite-dimensional, $\exp(\mathfrak{g})$ can be given the structure of an affine algebraic group, and the association $\mathfrak{g} \mapsto \exp(\mathfrak{g})$ is part of an equivalence between the category of finite-dimensional nilpotent Lie algebras over $\mathbb{Q}$ and the category of unipotent algebraic groups over $\mathbb{Q}$; see [55, Theorem 14.37].

Now let $\mathfrak{g}$ be a nilpotent dg Lie algebra and consider the simplicial group

$$\exp_{\bullet} \mathfrak{g} = \exp Z_0(\mathfrak{g} \otimes \Omega_{\bullet}),$$

where $\Omega_{\bullet}$ denotes the simplicial commutative differential graded algebra over $\mathbb{Q}$ of polynomial differential forms on the standard simplices.

Proposition 2.19 [13, Theorem 6.2] *For every nilpotent dg Lie algebra $\mathfrak{g}$, there is a natural isomorphism of groups,*

$$(8) \quad \exp H_0(\mathfrak{g}) \rightarrow \pi_0 \exp_{\bullet}(\mathfrak{g}),$$

and a natural isomorphism of abelian groups

$$H_k(\mathfrak{g}) \rightarrow \pi_k \exp_{\bullet}(\mathfrak{g})$$

for every $k > 0$, compatible with the actions of the groups in (8). □

By [13, Theorem 5.2(2)], a natural model for the classifying space $B \exp_{\bullet}(\mathfrak{g})$ is given by the nerve, or Maurer–Cartan space. This is the simplicial set defined by

$$\mathrm{MC}_{\bullet}(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g} \otimes \Omega_{\bullet}),$$

where MC denotes the set of Maurer–Cartan elements in a dg Lie algebra, ie the solutions to the equation

$$\delta(\tau) + \frac{1}{2}[\tau, \tau] = 0.$$

Note that **Proposition 2.19** implies that $|\mathrm{MC}_{\bullet}(\mathfrak{g})|$ is a $\mathbb{Q}$ -local nilpotent space if $\mathfrak{g}$ is a nilpotent dg Lie algebra.

Definition 2.20 A Lie model for a space B is by definition a dg Lie algebra $\mathfrak{g}$ such that B is rationally homology equivalent to $|\mathrm{MC}_{\bullet}(\mathfrak{g})|$.

If X is a simply connected space, then Quillen’s dg Lie algebra $\lambda(X)$ [59] is a Lie model for X in the sense of the above definition (this follows from [13, Theorem 8.1]). Recall (see eg [8]) that every simply connected space X admits a unique, up to noncanonical isomorphism, Lie model of the form $L_X = (\mathbb{L}(V), d)$, where $\mathbb{L}(V)$ denotes the free graded Lie algebra on the graded vector space $V = s^{-1} \tilde{H}_*(X; \mathbb{Q})$ and d is decomposable in the sense that the induced differential on $L_X/[L_X, L_X]$ is trivial. Note that L_X is finitely generated if and only if $\tilde{H}_*(X; \mathbb{Q})$ is finite-dimensional. We will refer to L_X as the minimal Quillen model of X .

Recall from eg [34, Section 22] that the Chevalley–Eilenberg complex of a dg Lie algebra $\mathfrak{g}$ is the dg coalgebra

$$C_*(\mathfrak{g}) = (\Lambda s\mathfrak{g}, d = d_0 + d_1),$$

where d_0 and d_1 are the coderivations characterized by

$$d_0(sx) = -s(dx), \quad d_1(sx_1 \wedge sx_2) = (-1)^{|x_1|} s[x_1, x_2].$$

The Chevalley–Eilenberg cochain algebra is the dual dg algebra $C^*(\mathfrak{g}) = C_*(\mathfrak{g})^\vee$. Denote its cohomology algebra by $H_{\text{CE}}^*(\mathfrak{g})$.

Next, recall that the polynomial differential forms on a simplicial set K is the dg algebra $\Omega^*(K) = \text{Hom}_{\text{sSet}}(K, \Omega_\bullet)$. It is a commutative dg algebra model for the cochains on K ; see Section 3.8.1 below for a further discussion. Also, recall that the spatial realization of a commutative dg algebra Λ is the simplicial set $\langle \Lambda \rangle = \text{Hom}_{\text{cdga}}(\Lambda, \Omega_\bullet)$.

Proposition 2.21 *Let $\mathfrak{g}$ be a nilpotent dg Lie algebra of finite type. There is a natural quasi-isomorphism of commutative dg algebras*

$$C^*(\mathfrak{g}) \rightarrow \Omega^*(\text{MC}_\bullet(\mathfrak{g})).$$

In particular, if $\mathfrak{g}$ is a Lie model for the space B , then $H^*(B; \mathbb{Q}) \cong H_{\text{CE}}^*(\mathfrak{g})$ and $|\text{MC}_\bullet(\mathfrak{g})|$ is a $\mathbb{Q}$ -localization of B .

Proof There is a natural isomorphism of simplicial sets

$$\text{MC}_\bullet(\mathfrak{g}) \xrightarrow{\cong} \langle C^*(\mathfrak{g}) \rangle,$$

see eg [11, Corollary 3.6]. The Chevalley–Eilenberg cochains $C^*(\mathfrak{g})$ is a Sullivan algebra and hence cofibrant (see eg [10, Theorem 2.3]), so the adjoint map $C^*(\mathfrak{g}) \rightarrow \Omega^*(\text{MC}_\bullet(\mathfrak{g}))$ is a quasi-isomorphism as a consequence of [23, Theorem 9.4]. $\square$

2.5 The dg Lie algebra of curved derivations

The main result of Quillen’s theory [59] is that the functor $X \mapsto \lambda(X)$ induces an equivalence between the rational homotopy category of simply connected pointed spaces and the homotopy category of simply connected dg Lie algebras. If one wants to model unpointed spaces, one has to enlarge the set of morphisms of dg Lie algebras. A possible solution is to work with so called curved morphisms of dg Lie algebras; see [52].

Let L be a positively graded dg Lie algebra and let $L_+ = (L * \mathbb{L}(\tau), d^\tau)$ be the dg Lie algebra obtained by freely adjoining a Maurer–Cartan element τ to L and twisting the differential by τ , so $d^\tau(x) = d(x) + [\tau, x]$ for $x \in L$. It is straightforward to check that morphisms from L_+ to a dg Lie algebra L' correspond to curved morphisms from L to L' . This is analogous to the fact that the space of free maps from a pointed space X to another pointed space Y can be recovered as the space of pointed maps from X_+ to Y , where X_+ is the space obtained from X by adding a disjoint basepoint.

The projection $p: L_+ \rightarrow L$ that restricts to the identity on L and sends τ to zero is a morphism of dg Lie algebras. Let $\text{Der}^c(L)$ denote the chain complex of p -derivations from L_+ to L . Its elements are maps $\theta: L_+ \rightarrow L$ that satisfy

$$\theta[x, y] = [\theta(x), p(y)] + (-1)^{|\theta||x|}[p(x), \theta(y)],$$

for all $x, y \in L_+$.

As is well known, the mapping cone of the chain map $\text{ad}: L \rightarrow \text{Der } L$, denoted by $\text{Der } L // \text{ad } L$ or $\text{Der } L \ltimes_{\text{ad}} sL$, admits a dg Lie algebra structure; see eg [67] or [11, page 252]. We now make the observation that this mapping cone may be identified with the chain complex of curved derivations.

Proposition 2.22 *The map $\varphi: \text{Der } L // \text{ad } L \rightarrow \text{Der}^c(L)$, defined by*

$$\varphi(\theta, s\xi) = \theta \circ p + (-1)^{|\xi|} \xi \frac{\partial}{\partial \tau},$$

is an isomorphism of chain complexes, with inverse

$$\varphi^{-1}(v) = (v|_L, (-1)^{|v|+1} s v(\tau)).$$

Proof Straightforward calculation. □

In particular, this implies that $\text{Der}^c(L)$ admits a dg Lie algebra structure and that it acts on L by outer derivations in the sense of [11, Section 3.5]. We do not recall the full definition here, but we point out that the outer action of $\text{Der}^c(L)$ on L gives rise to an (ordinary) action by coderivations on the Chevalley–Eilenberg chains $C_*(L) = (\Lambda sL, d)$. This action may be constructed by noting that sL_+ contains $\mathbb{Q}[0] \oplus sL = \Lambda^{\leq 1} sL$ as a graded subspace, where the copy of $\mathbb{Q}$ in degree 0 is generated by $s\tau$. Hence every curved derivation $\phi \in \text{Der}^c(L)$ determines a map $\Lambda^{\leq 1} sL \rightarrow sL$ by suspension and restriction, and thus it determines a unique coderivation of the cofree coalgebra ΛsL . Explicitly,

$$(9) \quad \phi(sx_1 \wedge \cdots \wedge sx_n) = (-1)^{|\phi|} s\phi(\tau) \wedge sx_1 \wedge \cdots \wedge sx_n + \sum_i \pm sx_1 \wedge \cdots \wedge s\phi(x_i) \wedge \cdots \wedge sx_n.$$

Dually, $\text{Der}^c(L)$ then acts by derivations on the cdga $C^*(L) = C_*(L)^\vee$ of Chevalley–Eilenberg cochains on L .

Remark 2.23 The constructions L_+ and $\text{Der}^c(L)$ essentially agree with the constructions $L\langle\tau\rangle$ and $\text{Der}_\tau(L\langle\tau\rangle)$ considered in [50, pages 44–45].

2.6 Algebraic groups of automorphisms

The following goes back to Sullivan [66, Section 6]. We outline the proof for the reader's convenience.

Theorem 2.24 *Let X be a simply connected finite CW-complex with minimal Quillen model L .*

- (i) *The automorphisms of L form an affine algebraic group $\mathcal{A}\text{ut}(L)$ and the automorphisms homotopic to the identity form a unipotent subgroup $\mathcal{A}\text{ut}_h(L)$.*

- (ii) The group $\mathcal{E}(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$ -points of the quotient affine algebraic group $\mathcal{A}ut^h(L) = \mathcal{A}ut(L)/\mathcal{A}ut_h(L)$.
- (iii) The Lie algebra of the algebraic group $\mathcal{A}ut^h(L)$ is isomorphic to $H_0(\mathrm{Der}^c L)$.
- (iv) The representation of $\mathcal{E}(X_{\mathbb{Q}})$ in $H_*(X; \mathbb{Q})$ extends to an algebraic representation of $\mathcal{A}ut^h(L)$ with unipotent kernel.

Outline of proof Since X is a finite CW-complex, the minimal Quillen model L is finitely generated. We can then define a linear algebraic group $\mathcal{A}ut L$ with functor of points

$$R \mapsto \mathrm{Aut}_{\mathrm{dgl}(R)}(L \otimes R),$$

sending a $\mathbb{Q}$ -algebra R to the group of R -linear automorphisms of the dg Lie algebra $L \otimes R$. To see that this indeed is a linear algebraic group, note that a faithful finite-dimensional representation is given by $L_{\leq N}$, where N is the maximal degree of a generator.

The subgroup $\mathcal{A}ut_h(L)$ of automorphisms homotopic to the identity may be identified with the unipotent algebraic group associated to the nilpotent Lie algebra $B_0 \mathrm{Der} L$ of derivations of L of the form $[d, \theta]$, for some derivation $\theta: L \rightarrow L$ of degree 1; see [66, Section 6] or [17, Theorem 3.4], or the proof of [33, Lemma 8.1].

Since $X_{\mathbb{Q}}$ is simply connected, there is an isomorphism of groups $\mathcal{E}_*(X_{\mathbb{Q}}) \cong \mathcal{E}(X_{\mathbb{Q}})$. Quillen's equivalence [59] between the category of simply connected pointed spaces, localized at the rational homotopy equivalences, and the category of positively graded dg Lie algebras, localized at the quasi-isomorphisms, coupled with the fact that a quasi-isomorphism between minimal dg Lie algebras is an isomorphism, leads to an isomorphism of groups

$$\mathcal{E}_*(X_{\mathbb{Q}}) \cong \mathrm{Aut}(L)/\mathrm{Aut}_h(L),$$

where $\mathrm{Aut}(L)$ and $\mathrm{Aut}_h(L)$ denote the $\mathbb{Q}$ -points of $\mathcal{A}ut(L)$ and $\mathcal{A}ut_h(L)$, respectively. It follows from Lemma 2.9 that the right-hand side may be identified with the $\mathbb{Q}$ -points of the quotient algebraic group $\mathcal{A}ut^h(L) = \mathcal{A}ut(L)/\mathcal{A}ut_h(L)$.

The Lie algebra of $\mathcal{A}ut(L)$ may be identified with $Z_0(\mathrm{Der} L)$, so the Lie algebra of $\mathcal{A}ut^h(L)$ may be identified with $Z_0(\mathrm{Der} L)/B_0(\mathrm{Der} L) = H_0(\mathrm{Der} L)$. Since L is positively graded, $H_0(\mathrm{Der} L) = H_0(\mathrm{Der}^c L)$.

For the last statement, one identifies

$$H_*(X; \mathbb{Q}) = \mathbb{Q} \oplus sL/[L, L],$$

and notes that the right-hand side is an algebraic representation of $\mathcal{A}ut L$ on which $\mathcal{A}ut_h L$ acts trivially. The associated graded gr L of L with respect to the lower central series is isomorphic to the free Lie algebra on $L/[L, L]$ as a representation of $\mathcal{A}ut L$. It follows that the kernel of the representation $L/[L, L]$ acts trivially on gr L , which implies that it acts unipotently on L in each degree. $\square$

There is a parallel statement for finite Postnikov stages. The proof is essentially the same so we omit it.

Theorem 2.25 *Let X be a simply connected Postnikov stage of finite type with minimal Sullivan model Λ .*

- (i) *The automorphisms of Λ form an affine algebraic group $\mathcal{A}ut(\Lambda)$ and the automorphisms homotopic to the identity form a unipotent subgroup $\mathcal{A}ut_h(\Lambda)$.*
- (ii) *The group $\mathcal{E}(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$ -points of the algebraic group*

$$\mathcal{A}ut^h \Lambda = \mathcal{A}ut \Lambda / \mathcal{A}ut_h \Lambda.$$
- (iii) *The Lie algebra of the algebraic group $\mathcal{A}ut^h \Lambda$ is isomorphic to $H_0(\text{Der } \Lambda)$.*
- (iv) *The representation of $\mathcal{E}(X_{\mathbb{Q}})$ in $H_*(X; \mathbb{Q})$ extends to an algebraic representation of $\mathcal{A}ut^h(\Lambda)$ with unipotent kernel.* $\square$

For a simply connected finite complex X of dimension n , there is an isomorphism $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(P_n X_{\mathbb{Q}})$, where $P_n X$ denotes the n^{th} Postnikov stage of X ; see [17, Proposition 3.1(2)]. We therefore have two, a priori different, ways to realize $\mathcal{E}(X_{\mathbb{Q}})$ as the $\mathbb{Q}$ -points of an algebraic group, one using Quillen models and one using Sullivan models. Saleh [60] has recently shown that these algebraic group structures agree. In the rest of the paper we will, somewhat imprecisely, refer to $\mathcal{E}(X_{\mathbb{Q}})$ as an algebraic group, with the understanding that we refer to either of the isomorphic algebraic groups above.

The following goes back to Sullivan [66, Theorem 10.3] and Wilkerson [71, Theorem B]. An elaboration of Sullivan's argument can be found in [68].

Theorem 2.26 *The homomorphism $\rho: \mathcal{E}(X) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ induced by rationalization has finite kernel and image an arithmetic subgroup.* $\square$

3 Proofs of the main results

Throughout this section, we fix a simply connected finite CW-complex X and we let $\text{aut}(X)$ denote the topological monoid of self-homotopy equivalences of X . The group $\mathcal{E}(X)$ will interchangeably be thought of as the group $\pi_0 \text{aut}(X)$ of components of the monoid $\text{aut}(X)$ or as the fundamental group $\pi_1 B \text{aut}(X)$ of the classifying space.

Definition 3.1 For a subgroup $G \leq \mathcal{E}(X)$, let $\text{aut}_G(X)$ denote the union of the components of $\text{aut}(X)$ that belong to G , so that there is a pullback square

$$\begin{array}{ccc} \text{aut}_G(X) & \longrightarrow & G \\ \downarrow & & \downarrow \\ \text{aut}(X) & \longrightarrow & \mathcal{E}(X) \end{array}$$

The cover of $B \text{aut}(X)$ associated to $G \leq \mathcal{E}(X)$ is weakly equivalent to the classifying space $B \text{aut}_G(X)$ of the monoid $\text{aut}_G(X)$, defined eg using the geometric bar construction [53, Section 7], and we will tacitly identify these two spaces.

Here is an outline of the proof of the main results:

We fix a subgroup $G \leq \mathcal{E}(X)$ that acts nilpotently on the rational homology of X . In [Section 3.1](#) we show that G uniquely determines a unipotent algebraic subgroup $U \leq \mathcal{E}(X_{\mathbb{Q}})$ such that the homomorphism $\rho: \mathcal{E}(X) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ restricts to a $\mathbb{Q}$ -isomorphism $G \rightarrow U$, and we give several equivalent characterizations of this U . In [Section 3.2](#) we show that the space $B\text{aut}_G(X)$ is virtually nilpotent. In [Section 3.3](#), we start incorporating the action of the deck transformation group $\mathcal{E}_G(X)$ on $B\text{aut}_G(X)$; in particular, we show that $B\text{aut}_G(X)$ is $\mathcal{E}_G(X)$ -equivariantly rationally equivalent to $B\text{aut}_U(X_{\mathbb{Q}})$. In [Section 3.4](#) we construct a Lie model for the space $B\text{aut}_U(X_{\mathbb{Q}})$. In general, there is no action of the deck transformation group on this Lie model, but there is an action of a larger group defined in terms of the minimal Quillen model of X . A crucial step in the proof is to relate this algebraically defined action to the action of the deck transformation group. The key is a lemma about conjugation actions on bar constructions, which we prove in [Section 3.5](#). In [Section 3.6](#), we use this to prove [Theorem 1.1](#). In [Section 3.7](#), we observe that if U is the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$, then the deck transformation group can be made to act on the Lie model and this leads to the proof of [Theorem 1.2](#).

3.1 Unipotent groups of self-homotopy equivalences

Recall that X is assumed to be a simply connected finite CW-complex. We begin by discussing how the homomorphism induced by rationalization,

$$\rho: \mathcal{E}(X) \rightarrow \mathcal{E}(X_{\mathbb{Q}}),$$

can be used to characterize subgroups of $\mathcal{E}(X)$ that act nilpotently on $H_*(X; \mathbb{Q})$.

Since X is simply connected, $\mathcal{E}(X)$ is isomorphic to the group $\mathcal{E}_*(X)$ of pointed homotopy classes of pointed self-homotopy equivalences. This group acts on the homotopy and homology groups of X . Moreover, the Hurewicz homomorphism $\pi_n(X) \rightarrow H_n(X)$ is $\mathcal{E}(X)$ -equivariant. The *spherical homology* $SH_n(X)$ is by definition the image of the Hurewicz homomorphism. It is a $\mathcal{E}(X)$ -submodule of $H_n(X)$ and a quotient $\mathcal{E}(X)$ -module of $\pi_n(X)$.

Proposition 3.2 *Let $G \leq \mathcal{E}(X)$ be a subgroup and let R be a subring of $\mathbb{Q}$. The following are equivalent:*

- (i) G acts nilpotently on $H_n(X; R)$ for all n .
- (ii) G acts nilpotently on $\pi_n(X) \otimes R$ for all n .
- (iii) G acts nilpotently on $SH_n(X; R)$ for all n .

For $R = \mathbb{Q}$ these conditions are equivalent to the following:

- (iv) $\rho(G)$ is contained in a unipotent algebraic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$.

Proof It is well-known that the first two conditions are equivalent (eg the argument in [43, Theorem 2.1] goes through) and they clearly imply the third. For the converse, assume inductively that G acts nilpotently on the R -homotopy groups of the Postnikov section $P_{n-1}(X)$ and use that the Hurewicz homomorphism sits in an exact sequence

$$H_{n+1}(P_{n-1}(X)) \rightarrow \pi_n(X) \rightarrow H_n(X).$$

This gives rise to an exact sequence

$$H_{n+1}(P_{n-1}(X); R) \rightarrow \pi_n(X) \otimes R \rightarrow SH_n(X; R) \rightarrow 0,$$

where the left and right terms are nilpotent G -modules. It follows from [44, Proposition I.4.3] that $\pi_n(X) \otimes R$ is a nilpotent G -module.

Finally, we prove the equivalence between the first condition and the fourth when $R = \mathbb{Q}$. If G acts nilpotently on $H_*(X; \mathbb{Q})$, then the image of G in $\mathrm{GL}(H_*(X; \mathbb{Q}))$ lies in a unipotent algebraic subgroup U'' . The preimage U' of U'' in $\mathcal{E}(X_{\mathbb{Q}})$ contains $\rho(G)$ and it is a unipotent algebraic subgroup since it is an extension of U'' by the kernel of the $\mathcal{E}(X_{\mathbb{Q}})$ -representation $H_*(X; \mathbb{Q})$, which is unipotent by Theorem 2.24(iv). Conversely, since $H_*(X; \mathbb{Q})$ is an algebraic representation of $\mathcal{E}(X_{\mathbb{Q}})$, any unipotent algebraic subgroup of the latter acts nilpotently on it. $\square$

Proposition 3.3 *If $G \leq \mathcal{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X; \mathbb{Q})$, then there is a unique unipotent algebraic subgroup $U \leq \mathcal{E}(X_{\mathbb{Q}})$ that satisfies the following equivalent conditions:*

- (i) U is minimal among the unipotent algebraic subgroups that contain $\rho(G)$.
- (ii) $\rho(G)$ is a Zariski dense subgroup of U .
- (iii) $\rho(G)$ is an arithmetic subgroup of U .
- (iv) G is a finite-index subgroup of $\rho^{-1}(U)$.
- (v) $\rho(G) \leq U$ and the induced homomorphism $\rho: G \rightarrow U$ is a $\mathbb{Q}$ -isomorphism.

Moreover, G is finitely generated in this situation.

Conversely, if $U \leq \mathcal{E}(X_{\mathbb{Q}})$ is a unipotent algebraic subgroup, then there is a unique commensurability class of subgroups $G \leq \mathcal{E}(X)$ such that the above conditions are satisfied.

Proof If $G \leq \mathcal{E}(X)$ acts nilpotently on $SH_*(X; \mathbb{Q})$, then $\rho(G)$ is contained in a unipotent algebraic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$ by Proposition 3.2. If we let U be the intersection of all unipotent algebraic subgroups of $\mathcal{E}(X_{\mathbb{Q}})$ that contain $\rho(G)$, then U is clearly the unique minimal unipotent algebraic subgroup that contains $\rho(G)$.

If U is minimal among the unipotent algebraic subgroups that contain $\rho(G)$, then the Zariski closure $\overline{\rho(G)}$ is contained in U since the latter is Zariski closed. But then $\overline{\rho(G)}$ must be unipotent and therefore equal to U by minimality of U .

Now assume that $\rho(G)$ is Zariski dense in U . Clearly, $\rho(G)$ is contained in $U \cap \rho(\mathcal{E}(X))$ and the latter is an arithmetic subgroup of U by [Lemma 2.11](#). Arithmetic subgroups of unipotent groups are nilpotent and finitely generated ([Lemma 2.13](#)), and subgroups of finitely generated nilpotent groups are always finitely generated, so it follows that $\rho(G)$ is finitely generated. Since $\rho(G)$ is Zariski dense in U , [Lemma 2.13](#) shows that $\rho(G)$ is an arithmetic subgroup of U . We note in passing that we can use the exact sequence

$$1 \rightarrow \ker(\rho) \cap G \rightarrow G \rightarrow \rho(G) \rightarrow 1$$

to deduce that G is finitely generated as well. Indeed, the kernel is finite by [Theorem 2.26](#) and we have just seen that $\rho(G)$ is finitely generated.

Next, if $\rho(G)$ is an arithmetic subgroup of U , then it must have finite index in $\rho(\rho^{-1}(U)) = U \cap \rho(\mathcal{E}(X))$, because the latter is also an arithmetic subgroup of U . As ρ has finite kernel, this implies that G has finite index in $\rho^{-1}(U)$.

The homomorphism $\rho: G \rightarrow U$ is the composite of the inclusion $G \rightarrow \rho^{-1}(U)$ followed by the homomorphism $\rho: \rho^{-1}(U) \rightarrow U$. If G has finite index in $\rho^{-1}(U)$, then the inclusion is in particular a $\mathbb{Q}$ -isomorphism. The homomorphism $\rho: \rho^{-1}(U) \rightarrow U$ has finite kernel and image an arithmetic subgroup of U , so it is a $\mathbb{Q}$ -isomorphism by [Lemma 2.13](#). This shows that $\rho: G \rightarrow U$ is a $\mathbb{Q}$ -isomorphism if G has finite index in $\rho^{-1}(U)$.

Finally, suppose ρ restricts to a $\mathbb{Q}$ -isomorphism $G \rightarrow U$. If $U' \leq \mathcal{E}(X_{\mathbb{Q}})$ is a unipotent algebraic subgroup such that $\rho(G) \leq U'$, then $G \rightarrow U$ factors as $G \rightarrow U \cap U'$ followed by the inclusion $U \cap U' \rightarrow U$, implying the latter is $\mathbb{Q}$ -surjective. But this is then a $\mathbb{Q}$ -surjective inclusion of uniquely divisible groups, so it must be an equality. This shows that U is minimal among the unipotent algebraic subgroups that contain $\rho(G)$. By that we have gone full circle, showing the five conditions are equivalent.

Conversely, given U and two subgroups G and G' that satisfy the equivalent conditions, the fourth condition shows that G and G' are finite-index subgroups of $\rho^{-1}(U)$. This implies that $G \cap G'$ has finite index in both G and G' . So the commensurability class determined by U is precisely the set of finite-index subgroups of $\rho^{-1}(U)$. $\square$

3.2 Virtual nilpotence of covers

As before, X is a simply connected finite CW-complex and G is a subgroup of $\mathcal{E}(X)$. The purpose of this section is to show that the space $B \operatorname{aut}_G(X)$ is virtually nilpotent if G acts nilpotently on the rational spherical homology of X . This extends a result of Dror and Zabrodsky [[32](#), Theorem D], which says that $B \operatorname{aut}_G(X)$ is nilpotent if G acts nilpotently on the integral homology of X .

Lemma 3.4 *Let G be a group acting on a finitely generated abelian group A . If G acts nilpotently on $A \otimes \mathbb{Q}$, then there is a finite-index subgroup $K \leq G$ that acts nilpotently on A .*

Proof Let K be the kernel of the action of G on the torsion subgroup A_{tor} . Since the group A_{tor} is finite, so is its automorphism group, so the exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow \text{Aut}(A_{\text{tor}})$$

shows that K has finite index in G . If G acts nilpotently on $A \otimes \mathbb{Q}$, then there exists a filtration of G -modules

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = A \otimes \mathbb{Q}$$

such that G acts trivially on V_i/V_{i-1} for each i . Let $W_i \subseteq A$ be the preimage of V_i under the homomorphism $A \rightarrow A \otimes \mathbb{Q}$. This yields a filtration of G -modules

$$W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = A,$$

and it follows that W_i/W_{i-1} has trivial G -action for $1 \leq i \leq n$. Note that $W_0 = A_{\text{tor}}$, so if G acts nilpotently on A_{tor} it follows that it acts nilpotently on A as well. In general, the G -action on A_{tor} need not be nilpotent, but the action of K on it is trivial by definition. Thus, extending the filtration by $W_{-1} = 0$ yields a filtration witnessing that K acts nilpotently on A . $\square$

Proposition 3.5 *If $G \leq \mathcal{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X; \mathbb{Q})$, then the space $B\text{aut}_G(X)$ is virtually nilpotent. It is nilpotent if G acts nilpotently on $SH_*(X; \mathbb{Z})$.*

Proof We will in fact prove the slightly stronger statement that $B\text{aut}_G(X)$ admits a finite cover which is nilpotent. By Lemma 3.4, there is a finite-index subgroup $K \leq G$ that acts nilpotently on $SH_*(X; \mathbb{Z})$. By Proposition 3.2, the group K acts nilpotently on $H_*(X; \mathbb{Z})$. It follows from [32, Theorem D] that the space $B\text{aut}_K(X)$ is nilpotent. The space $B\text{aut}_K(X)$ is weakly equivalent to the finite cover of $B\text{aut}_G(X)$ that corresponds to the finite-index subgroup $K \leq G$. $\square$

Example 3.6 The space $B\text{aut}_G(X)$ is not virtually nilpotent in general. For example, if we take $X = S^2 \vee S^2$ and $G = \mathcal{E}(X)$, then $G \cong \text{GL}_2(\mathbb{Z})$, which is not virtually nilpotent.

Example 3.7 The space $B\text{aut}_G(X)$ need not be nilpotent even if G acts nilpotently on $SH_*(X; \mathbb{Q})$. To see this, consider the Moore space $X = M(\mathbb{Z}/3\mathbb{Z}, 2)$, ie the homotopy cofiber of a degree-3 self-map of S^2 . This space is rationally equivalent to point, so $G = \mathcal{E}(X)$ acts nilpotently on $SH_*(X; \mathbb{Q}) = 0$ for trivial reasons. The group of self-equivalences is not difficult to compute (see eg [63, Theorem 2]); there is an isomorphism

$$\mathcal{E}(X) \cong (\mathbb{Z}/3\mathbb{Z})^\times \ltimes \mathbb{Z}/3\mathbb{Z} \cong \Sigma_3,$$

showing $\pi_1(B\text{aut}(X)) = \mathcal{E}(X)$ is not nilpotent. The criterion in Proposition 3.5 for nilpotency of $B\text{aut}_G(X)$ is not satisfied because the action on $SH_2(X; \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$ is through the projection onto $(\mathbb{Z}/3\mathbb{Z})^\times$ and this action is not nilpotent.

3.3 Equivariant rationalization of covers

In this section, we will show that $B\operatorname{aut}_G(X)$ is rationally equivalent to $B\operatorname{aut}_U(X_{\mathbb{Q}})$ when G and U are as in [Proposition 3.3](#). To keep track of the action of the deck transformation group, we need a model for $B\operatorname{aut}_G(X)$ that is appropriately functorial in $G \leq \mathcal{E}(X)$. Recall that the subgroups of $\mathcal{E}(X)$ form the objects of the *orbit category*, where a morphism $G \rightarrow H$ is given by a morphism of left $\mathcal{E}(X)$ -sets $\mathcal{E}(X)/G \rightarrow \mathcal{E}(X)/H$.

Definition 3.8 We define a functor from the orbit category of $\mathcal{E}(X)$ to the category of spaces over $B\operatorname{aut}(X)$ by sending $G \leq \mathcal{E}(X)$ to the space

$$B_{G\operatorname{aut}}(X) = B(*, \operatorname{aut}(X), \mathcal{E}(X)/G)$$

defined using the geometric bar construction [\[53, Section 7\]](#) of the topological monoid $\operatorname{aut}(X)$ acting on $\mathcal{E}(X)/G$ via the canonical map $\operatorname{aut}(X) \rightarrow \mathcal{E}(X)$.

The space $B_{G\operatorname{aut}}(X)$ can be thought of as a functorial model for the cover of $B\operatorname{aut}(X)$ associated to $G \leq \mathcal{E}(X)$. The map

$$B\operatorname{aut}_G(X) = B(*, \operatorname{aut}_G(X), *) \rightarrow B(*, \operatorname{aut}(X), \mathcal{E}(X)/G) = B_{G\operatorname{aut}}(X),$$

induced by the inclusion of monoids $\operatorname{aut}_G(X) \rightarrow \operatorname{aut}(X)$ and the map of $\operatorname{aut}_G(X)$ -spaces $* \rightarrow \mathcal{E}(X)/G$ that selects the coset G , is a weak equivalence, so both source and target are models for the classifying space of the monoid $\operatorname{aut}_G(X)$. The advantage of $B_{G\operatorname{aut}}(X)$ is that it carries an action of the group $\mathcal{E}_G(X)$ of automorphisms of the $\mathcal{E}(X)$ -set $\mathcal{E}(X)/G$. We remind the reader that this group, the “deck transformation group”, may be identified with $N_{\mathcal{E}(X)}(G)/G$, the normalizer of G in $\mathcal{E}(X)$ modulo G .

In what follows, we fix a subgroup $G \leq \mathcal{E}(X)$ that acts nilpotently on $SH_*(X; \mathbb{Q})$ (or equivalently on $H_*(X; \mathbb{Q})$ or $\pi_*(X) \otimes \mathbb{Q}$) and we let $U \leq \mathcal{E}(X_{\mathbb{Q}})$ be the minimal unipotent algebraic subgroup that contains $\rho(G)$ as in [Proposition 3.3](#).

Lemma 3.9 *The homomorphism $\rho: \mathcal{E}(X) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ carries the normalizer of G to the normalizer of U . In particular, there is an induced group homomorphism $\mathcal{E}_G(X) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}})$.*

Proof Let $g \in N_{\mathcal{E}(X)}(G)$. We need to show that $\rho(g) \in N_{\mathcal{E}(X_{\mathbb{Q}})}(U)$, ie that ${}^gU = \rho(g)U\rho(g)^{-1}$ agrees with U . By assumption, U is the minimal unipotent algebraic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$ that contains $\rho(G)$. Thus ${}^gU$ is a minimal unipotent algebraic subgroup that contains $\rho(g)\rho(G)\rho(g)^{-1} = \rho(gGg^{-1}) = \rho(G)$, so ${}^gU = U$ by the uniqueness part of [Proposition 3.3](#). $\square$

By [Lemma 3.9](#), $\mathcal{E}_G(X)$ acts on $B_{U\operatorname{aut}}(X_{\mathbb{Q}})$ via the induced homomorphism $\rho: \mathcal{E}_G(X) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}})$. We remind the reader that we call a map between connected spaces a *rationale equivalence* if it induces a $\mathbb{Q}$ -isomorphism on all homotopy groups *and* all homology groups; see [Definition 2.2](#).

Proposition 3.10 *There is a zig-zag of $\mathcal{E}_G(X)$ -equivariant rational equivalences between $B_G \text{aut}(X)$ and $B_U \text{aut}(X_{\mathbb{Q}})$.*

Proof Let $r: X \rightarrow X_{\mathbb{Q}}$ be a rationalization. We may assume that r is a cofibration. This holds in many models for rationalizations (eg cellular rationalization [34, Theorem 9.7] or the $H_*(-; \mathbb{Q})$ -localization of Bousfield [22]), but if necessary, one can achieve this by abstract nonsense, eg by factoring r as a cofibration followed by a weak homotopy equivalence. Now consider the pullback square

$$\begin{array}{ccc} \text{aut}(r) & \xrightarrow{q} & \text{aut}(X_{\mathbb{Q}}) \\ \downarrow & & \downarrow r^* \\ \text{aut}(X) & \xrightarrow{r_*} & \text{map}(X, X_{\mathbb{Q}})_{\text{re}} \end{array}$$

where $\text{map}(X, X_{\mathbb{Q}})_{\text{re}}$ denotes the space of rational equivalences from X to $X_{\mathbb{Q}}$ and $\text{aut}(r)$ denotes the space of self-equivalences of r viewed as an object in the category of maps. The map r^* is a fibration since r is a cofibration, and it is a weak homotopy equivalence by standard properties of localizations. Hence, the left vertical map is a weak homotopy equivalence as well. The map r_* is in general not a bijection on π_0 , but its restriction to each component is a rational equivalence to the component it hits by [44, Theorem II.3.11]. It follows that the top horizontal map q has the same property.

This yields a zig-zag of grouplike monoids

$$(10) \quad \text{aut}(X) \xleftarrow{\sim} \text{aut}(r) \xrightarrow{q} \text{aut}(X_{\mathbb{Q}}),$$

where the left map is a weak equivalence and the right map q induces an isomorphism on $\pi_k(-) \otimes \mathbb{Q}$ for all $k > 0$ and may be identified with $\rho: \mathcal{E}(X) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ on π_0 . This induces a zig-zag

$$(11) \quad B(*, \text{aut}(X), \mathcal{E}(X)/G) \xleftarrow{\sim} B(*, \text{aut}(r), \mathcal{E}(X)/G) \xrightarrow{\psi} B(*, \text{aut}(X_{\mathbb{Q}}), \mathcal{E}(X_{\mathbb{Q}})/U),$$

where ψ is induced by the monoid map $q: \text{aut}(r) \rightarrow \text{aut}(X_{\mathbb{Q}})$ and the $\text{aut}(r)$ -equivariant map $\rho: \mathcal{E}(X)/G \rightarrow \mathcal{E}(X_{\mathbb{Q}})/U$. The homomorphism $\pi_1(\psi)$ may be identified with $\rho: G \rightarrow U$, which is a $\mathbb{Q}$ -isomorphism by Proposition 3.3. For $k > 1$, we may identify $\pi_k(\psi)$ with $\pi_{k-1}(q): \pi_{k-1}(\text{aut}(r)) \rightarrow \pi_{k-1}(\text{aut}(X_{\mathbb{Q}}))$, which is a $\mathbb{Q}$ -isomorphism by the above.

We recognize the leftmost term of (11) as $B_G \text{aut}(X)$ and the rightmost term as $B_U \text{aut}(X_{\mathbb{Q}})$. Moreover, the maps in the zig-zag are clearly $\mathcal{E}_G(X)$ -equivariant. We have just shown that ψ induces a $\mathbb{Q}$ -isomorphism on all homotopy groups. By Proposition 3.5, the source of ψ is virtually nilpotent and the target is nilpotent, so it follows from Lemma 2.3 that ψ is a rational homology equivalence as well. $\square$

3.4 Equivariant Lie models for covers

As before, X is a simply connected finite CW-complex, $G \leq \mathcal{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X; \mathbb{Q})$, and $U \leq \mathcal{E}(X_{\mathbb{Q}})$ is the unique minimal unipotent algebraic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$ that contains $\rho(G)$.

We identify $X_{\mathbb{Q}}$ with the nerve of the minimal Quillen model L for X . Recall from [Theorem 2.24\(iii\)](#) that the Lie algebra of $\mathcal{E}(X_{\mathbb{Q}})$ is isomorphic to $H_0(\mathrm{Der}^c L)$, so the Lie algebra of U may be identified with a Lie subalgebra $\mathfrak{u} \leq H_0(\mathrm{Der}^c L)$. The following definition can be viewed as a dg Lie algebra companion to [Definition 3.1](#).

Definition 3.11 Let $\mathrm{Der}_{\mathfrak{u}}^c L$ denote the dg Lie subalgebra of $\mathrm{Der}^c L\langle 0 \rangle$ defined by the pullback

$$\begin{array}{ccc} \mathrm{Der}_{\mathfrak{u}}^c L & \longrightarrow & \mathfrak{u} \\ \downarrow & & \downarrow \\ \mathrm{Der}^c L\langle 0 \rangle & \longrightarrow & H_0(\mathrm{Der}^c L) \end{array}$$

Proposition 3.12 The dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}^c L$ is nilpotent.

Proof The degree-zero component, $(\mathrm{Der}_{\mathfrak{u}}^c L)_0$, may be identified with the Lie algebra of the preimage $\tilde{U} = p^{-1}(U)$ under the projection $p: \mathcal{A}ut L \rightarrow \mathcal{A}ut^h L$. By [Theorem 2.24\(i\)](#), the homomorphism p has unipotent kernel. It follows that $\tilde{U}$ is an extension of unipotent groups and is hence unipotent; see for instance [\[55, Section 6.45\]](#). Since $(\mathrm{Der}^c L)_k$ is an algebraic representation of $\tilde{U}$ for each $k > 0$, it follows that $\tilde{U}$ acts unipotently on it; see [\[55, Proposition 14.3\]](#). Hence, $(\mathrm{Der}_{\mathfrak{u}}^c L)_0 = \mathrm{Lie} \tilde{U}$ is nilpotent and acts nilpotently on $(\mathrm{Der}^c L)_k$ for each $k > 0$. $\square$

Let $\mathrm{Aut}_U(L) \leq \mathrm{Aut}(L)$ denote the preimage of the normalizer of $U \leq \mathcal{E}(X_{\mathbb{Q}})$ under the homomorphism $\mathrm{Aut}(L) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$. By design, the group $\mathrm{Aut}_U(L)$ acts on $\mathcal{E}(X_{\mathbb{Q}})$ and on $\mathrm{Der}^c L$ by conjugation, fixing the subgroup U and the Lie subalgebra $\mathfrak{u}$, respectively. If U is normal in $\mathcal{E}(X_{\mathbb{Q}})$ then $\mathrm{Aut}_U(L) = \mathrm{Aut}(L)$.

Proposition 3.13 There is a weak equivalence of grouplike topological monoids,

$$(12) \quad |\exp_{\bullet}(\mathrm{Der}_{\mathfrak{u}}^c L)| \rightarrow \mathrm{aut}_U(X_{\mathbb{Q}}),$$

which is equivariant with respect to the conjugation action of $\mathrm{Aut}_U(L)$ on the domain and codomain. In particular, $B\mathrm{aut}_U(X_{\mathbb{Q}})$ is weakly equivalent to the nerve of the dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}^c(L)$.

Proof The action of the nilpotent dg Lie algebra $\mathfrak{g} = \mathrm{Der}_{\mathfrak{u}}^c L$ on L by outer derivations induces an action of the simplicial group $\exp_{\bullet}(\mathfrak{g})$ on the simplicial set $\mathrm{MC}_{\bullet}(L)$ (see [\[11, Section 3.5\]](#)), giving rise to a morphism of grouplike monoids

$$(13) \quad \alpha: |\exp_{\bullet}(\mathfrak{g})| \rightarrow \mathrm{aut}(|\mathrm{MC}_{\bullet}(L)|).$$

The map [\(13\)](#) is known to induce an isomorphism on $\pi_k(-)$ for $k > 0$. To see this, one can apply [\[12, Proposition 3.7\]](#) to the Sullivan model $\Lambda = C^*(L)$. Let us add that this is essentially equivalent to the statement, going back to Tanré [\[67, VII.4.\(4\)\]](#), that $\mathrm{Der}^c L\langle 1 \rangle$ is a Lie model for the simply connected cover of $B\mathrm{aut}(\mathrm{MC}_{\bullet}(L))$.

By construction, $\pi_0(\mathrm{aut}_U(X_{\mathbb{Q}})) = U$ and $H_0(\mathrm{Der}_{\mathfrak{u}}^c L) = \mathfrak{u}$. By [Proposition 2.19](#), there is an isomorphism

$$\pi_0(\exp_{\bullet}(\mathrm{Der}_{\mathfrak{u}}^c L)) \cong \exp(H_0(\mathrm{Der}_{\mathfrak{u}}^c L)) = \exp(\mathfrak{u}) = U.$$

One checks that the map induced by (13) on π_0 may be identified with the inclusion of U into $\mathcal{E}(X_{\mathbb{Q}})$. Hence, (13) corestricts to a weak equivalence $|\exp_{\bullet}(\mathfrak{g})| \rightarrow \mathrm{aut}_U(X_{\mathbb{Q}})$. The statement about $\mathrm{Aut}_U(L)$ -equivariance is quickly verified by inspection. The last statement follows by applying the classifying space functor to the monoid map (12) and noting that $B|\exp_{\bullet}(\mathfrak{g})|$ is weakly equivalent to the nerve $|\mathrm{MC}_{\bullet}(\mathfrak{g})|$. $\square$

Remark 3.14 [Proposition 2.19](#) implies that $B|\exp_{\bullet}(\mathfrak{g})|$ is nilpotent and rational whenever $\mathfrak{g}$ is a nilpotent dg Lie algebra, so the above gives an alternative proof that the space $B\mathrm{aut}_U(X_{\mathbb{Q}})$ is nilpotent and rational.

In particular, forgetting the group actions, we get a Lie model for $B\mathrm{aut}_G(X)$.

Corollary 3.15 *The space $B\mathrm{aut}_G(X)$ is rationally equivalent to the nerve of the dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}^c(L)$.*

Proof Combine [Propositions 3.13](#) and [3.10](#). $\square$

Example 3.16 Let $\mathrm{tor}(X)$ denote the *homotopy Torelli monoid*, meaning the submonoid of $\mathrm{aut}(X)$ of those self-equivalences that act trivially on $H_*(X; \mathbb{Z})$. The space $B\mathrm{tor}(X)$ is nilpotent by [Proposition 3.5](#). [Corollary 3.15](#) yields a Lie model for $B\mathrm{tor}(X)$ described by the following: it agrees with $\mathrm{Der}^c L$ in positive degrees and in degree zero it consists of all derivations θ of L that commute with the differential and are decomposable in the sense that $\theta(L) \subseteq [L, L]$. To see this, note that the action of a derivation $\theta \in \mathrm{Der} L$ on the reduced homology of X may be identified with the induced action on the indecomposables $L/[L, L]$.

Remark 3.17 [Corollary 3.15](#) recovers the Lie model for $B\mathrm{aut}_G(X)$ of [\[33, Theorem 0.1\]](#). Indeed, one can check that $\mathrm{Der}_{\mathfrak{u}}^c L$ agrees with the model $\mathrm{Der}^{\mathcal{G}} L \tilde{\times}_s L$ of [\[33, Theorem 0.1\]](#), where $\mathcal{G} \leq \mathrm{Aut}^h(L)$ corresponds to $U \leq \mathcal{E}(X_{\mathbb{Q}})$ under the isomorphism $\mathrm{Aut}^h(L) \cong \mathcal{E}(X_{\mathbb{Q}})$. Let us also remark that when U is the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$, one can show that the dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}^c L$ of [Definition 3.11](#) agrees with the dg Lie algebra $\mathcal{D}er L$ considered in [\[33, Definition 6.4\]](#) if the filtration [\[33, \(23\)\]](#) is chosen to be a composition series as in [\(19\)](#).

3.5 A lemma on conjugation actions on bar constructions

In this section, we will prove a lemma that will be a key ingredient in the proof of the main results. Before stating it, let us give a few words of motivation. Suppose that

$$1 \rightarrow G' \rightarrow G \xrightarrow{f} G'' \rightarrow 1$$

is a split short exact sequence of groups, giving rise to a homotopy fiber sequence

$$BG' \rightarrow BG \xrightarrow{Bf} BG''.$$

It is well-known that the conjugation action of G'' on BG' models the holonomy action of G'' on the homotopy fiber of Bf . On the other hand, the left action of G'' on $G'' \parallel G = B(G'', G, *)$ also models the holonomy action. The evident map $BG' \rightarrow G'' \parallel G$ is, however, not equivariant; it becomes equivariant if G'' acts by simultaneous left multiplication and conjugation on the target. The lemma below offers a resolution of this seeming incongruity by showing that these two different G'' -actions on $G'' \parallel G$ give weakly equivalent G'' -spaces.

Now, let $\mathcal{A}$ be a topological monoid, let $\mathcal{X}$ be a left $\mathcal{A}$ -space and $\mathcal{Y}$ a right $\mathcal{A}$ -space. Let $\mathcal{A}^\times$ denote the group of invertible elements in $\mathcal{A}$. There is an action of $\mathcal{A}^\times$ on the triple $(\mathcal{Y}, \mathcal{A}, \mathcal{X})$ defined by letting $g \in \mathcal{A}^\times$ act by

$$\mathcal{A} \xrightarrow{c_g} \mathcal{A}, \quad h \mapsto ghg^{-1}, \quad \mathcal{X} \xrightarrow{\ell_g} c_g^*(\mathcal{X}), \quad x \mapsto gx, \quad \mathcal{Y} \xrightarrow{r_g} c_g^*(\mathcal{Y}), \quad y \mapsto yg^{-1}.$$

This induces an action of $\mathcal{A}^\times$ on the geometric bar construction $B(\mathcal{Y}, \mathcal{A}, \mathcal{X})$, which we will refer to as the conjugation action.

Lemma 3.18 *There is a natural zig-zag of $\mathcal{A}^\times$ -equivariant homotopy equivalences connecting $B(\mathcal{Y}, \mathcal{A}, \mathcal{X})$, with the conjugation action, and the same space with the trivial action.*

Remark 3.19 The lemma implies the fact that, for each individual $g \in \mathcal{A}^\times$, the map $B(r_g, c_g, \ell_g)$ is homotopic to the identity of $B(\mathcal{Y}, \mathcal{A}, \mathcal{X})$, but note that the lemma is not a formal consequence of this fact. Indeed, it is easy to find spaces with a group action where each individual multiplication map is homotopic to the identity but where the action cannot be trivialized via a zig-zag of equivariant homotopy equivalences, eg S^1 with the antipodal action of $\mathbb{Z}/2\mathbb{Z}$.

Proof We exploit the fact that the geometric bar construction extends to topological categories; see for instance [54, Chapter V.2]. Consider the topological category $\mathcal{A} \times [1]$, where $[1]$ is the discrete category associated to the poset $\{0 < 1\}$. Specifying a representation A of $\mathcal{A} \times [1]$, by which we mean a continuous functor from $\mathcal{A} \times [1]$ to the category of spaces, amounts to specifying a morphism of $\mathcal{A}$ -spaces

$$A_0 \xrightarrow{f} A_1.$$

We extend $\mathcal{X}$ and $\mathcal{Y}$ to representations $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ of $\mathcal{A} \times [1]$ by using the respective identity maps. The inclusion functors

$$\mathcal{A} = \mathcal{A} \times \{0\} \xrightarrow{i_0} \mathcal{A} \times [1] \xleftarrow{i_1} \mathcal{A} \times \{1\} = \mathcal{A}$$

together with the identity maps $\mathcal{X} \rightarrow i_0^*(\underline{\mathcal{X}})$, $\mathcal{Y} \rightarrow i_0^*(\underline{\mathcal{Y}})$, $\mathcal{X} \rightarrow i_1^*(\underline{\mathcal{X}})$ and $\mathcal{Y} \rightarrow i_1^*(\underline{\mathcal{Y}})$ induce maps on bar constructions

$$(14) \quad B(\mathcal{Y}, \mathcal{A}, \mathcal{X}) \xrightarrow{Bi_0} B(\underline{\mathcal{Y}}, \mathcal{A} \times [1], \underline{\mathcal{X}}) \xleftarrow{Bi_1} B(\mathcal{Y}, \mathcal{A}, \mathcal{X}).$$

The middle term is isomorphic to the cylinder $B(\mathcal{Y}, \mathcal{A}, \mathcal{X}) \times I$ and the maps Bi_0 and Bi_1 may be identified with the bottom and top inclusions. In particular, both Bi_0 and Bi_1 are homotopy equivalences.

The idea is now to define an action of $\mathcal{A}^\times$ on the triple $(\mathcal{Y}, \mathcal{A} \times [1], \mathcal{X})$ so that the maps (14) become equivariant if the left copy of $B(\mathcal{Y}, \mathcal{A}, \mathcal{X})$ is given the trivial action and the right copy the conjugation action. Once we have found such an action, the proof will be complete.

For $g \in \mathcal{A}^\times$, there is a unique continuous functor

$$H_g: \mathcal{A} \times [1] \rightarrow \mathcal{A} \times [1]$$

such that $H_g^*(A)$ is the $\mathcal{A} \times [1]$ -representation

$$A_0 \xrightarrow{gf} c_g^*(A_1), \quad a \mapsto gf(a),$$

whenever A is a representation of $\mathcal{A} \times [1]$ as above. The existence and uniqueness of H_g follows from the Yoneda lemma for topologically enriched categories. One checks that $H_1 = 1$ and $H_g H_h = H_{gh}$ for all $g, h \in \mathcal{A}^\times$ so this defines an action of $\mathcal{A}^\times$ on $\mathcal{A} \times [1]$. Next, the commutative square of $\mathcal{A}$ -spaces

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{1} & \mathcal{X} \\ \downarrow 1 & & \downarrow g \\ \mathcal{X} & \xrightarrow{g} & c_g^*(\mathcal{X}) \end{array}$$

defines a morphism of $\mathcal{A} \times [1]$ -representations $L_g: \mathcal{X} \rightarrow H_g^*(\mathcal{X})$. A morphism of contravariant representations $R_g: \mathcal{Y} \rightarrow H_g^*(\mathcal{Y})$ is defined similarly. Verifying that

$$(R_g, H_g, L_g): (\mathcal{Y}, \mathcal{A} \times [1], \mathcal{X}) \rightarrow (\mathcal{Y}, \mathcal{A} \times [1], \mathcal{X})$$

defines an action of $\mathcal{A}^\times$ with the desired properties is routine and left to the reader. $\square$

3.6 Algebraicity of cohomology and homotopy groups

In this section, we will give Theorem 1.1 a more precise formulation (Theorem 3.22 below) and prove it. Let us first review the notion of simple homotopy groups.

Definition 3.20 For a path-connected space B , we define the *simple homotopy groups* $\pi'_n(B)$ to be the abelian groups

$$\pi'_n(B) = \pi_n(B) / [\pi_1(B), \pi_n(B)]$$

obtained by trivializing the action of $\pi_1(B)$. Explicitly, $\pi'_1(B) = \pi_1(B)^{ab}$ and $\pi'_n(B) = [S^n, B]$ (free homotopy classes of maps) for $n > 1$.

Note that B is a simple space if and only if the canonical homomorphism $\pi_n(B) \rightarrow \pi'_n(B)$ is an isomorphism for all n . The simple homotopy groups are functorial for maps that do not necessarily preserve the basepoint. In particular, if a group $\mathcal{E}$ acts on B , then $\pi'_n(B)$ is a representation of $\mathcal{E}$. This is the reason we consider the simple homotopy groups instead of the ordinary homotopy groups.

We now proceed towards the proof of Theorem 3.22.

Notation We fix the following notation for the remainder of the section.

- X is a simply connected finite CW-complex.
- L is the minimal Quillen model of X .
- $G \leq \mathcal{E}(X)$ is a subgroup that acts nilpotently on $SH_*(X; \mathbb{Q})$.
- $U \leq \mathcal{E}(X_{\mathbb{Q}})$ is the minimal unipotent algebraic subgroup with $\rho(G) \leq U$.
- $\mathfrak{u}$ is the Lie algebra of U .

Recall that the existence and uniqueness of U is guaranteed by [Proposition 3.3](#) and that $\mathfrak{u}$ may be viewed as a subalgebra of $H_0(\mathrm{Der}^c L)$ by [Theorem 2.24\(iii\)](#).

We also remind the reader that $\mathcal{E}_U(X_{\mathbb{Q}})$ denotes the group of automorphisms of the $\mathcal{E}(X_{\mathbb{Q}})$ -set $\mathcal{E}(X_{\mathbb{Q}})/U$ and that this group may be identified with $N_{\mathcal{E}(X_{\mathbb{Q}})}(U)/U$.

Recall that $\mathrm{Aut}_U(L)$ denotes the preimage of the normalizer $N_{\mathcal{E}(X_{\mathbb{Q}})}(U)$ under the homomorphism $\mathrm{Aut}(L) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$. There is a surjective homomorphism

$$p: \mathrm{Aut}_U(L) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}}).$$

By construction, the space

$$B_U \mathrm{aut}(X_{\mathbb{Q}}) = B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathcal{E}(X_{\mathbb{Q}})/U)$$

carries an action of $\mathcal{E}_U(X_{\mathbb{Q}})$ and hence an action of $\mathrm{Aut}_U(L)$ via p . On the other hand, the group $\mathrm{Aut}_U(L)$ acts on $\mathrm{Der}_{\mathfrak{u}}^c L$ by conjugation, and hence it acts on the nerve of $\mathrm{Der}_{\mathfrak{u}}^c L$.

Lemma 3.21 *The $\mathrm{Aut}_U(L)$ -space $p^* B_U \mathrm{aut}(X_{\mathbb{Q}})$ is weakly equivalent to the nerve of the dg Lie algebra $\mathrm{Der}_{\mathfrak{u}}^c L$, on which $\mathrm{Aut}_U(L)$ acts by conjugation.*

Proof The rationalization $X_{\mathbb{Q}}$ may be identified with $|\mathrm{MC}_{\bullet}(L)|$. By [Proposition 3.13](#), $B|\exp_{\bullet}(\mathrm{Der}_{\mathfrak{u}}^c L)|$ is weakly equivalent to $B \mathrm{aut}_U(X_{\mathbb{Q}})$ as an $\mathrm{Aut}_U(L)$ -space, where $\mathrm{Aut}_U(L)$ acts by conjugation. Now consider the weak equivalence

$$(15) \quad B \mathrm{aut}_U(X_{\mathbb{Q}}) = B(*, \mathrm{aut}_U(X_{\mathbb{Q}}), *) \rightarrow B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathcal{E}(X_{\mathbb{Q}})/U),$$

induced by the inclusion of $\mathrm{aut}_U(X_{\mathbb{Q}})$ into $\mathrm{aut}(X_{\mathbb{Q}})$ and the map $* \rightarrow \mathcal{E}(X_{\mathbb{Q}})/U$ that selects the coset U .

The space $B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathcal{E}(X_{\mathbb{Q}})/U)$ has two commuting actions of $\mathrm{Aut}_U(L)$: the conjugation action as in [Lemma 3.18](#) via the homomorphism $\mathrm{Aut}_U(L) \rightarrow \mathrm{aut}(X_{\mathbb{Q}})^{\times}$, and the action by automorphisms on the left $\mathrm{aut}(X_{\mathbb{Q}})$ -space $\mathcal{E}(X_{\mathbb{Q}})/U$ via the homomorphism $\mathrm{Aut}_U(L) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}})$. The weak equivalence (15) is $\mathrm{Aut}_U(L)$ -equivariant if $B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathcal{E}(X_{\mathbb{Q}})/U)$ is given the diagonal action. (It is not equivariant with respect to the conjugation action, because the coset $U \in \mathcal{E}(X_{\mathbb{Q}})/U$ is not fixed by the left action of $\mathrm{Aut}_U(L)$.) [Lemma 3.18](#) gives a zig-zag of $\mathrm{Aut}_U(L)$ -equivariant weak equivalences connecting $B(*, \mathrm{aut}(X_{\mathbb{Q}}), \mathcal{E}(X_{\mathbb{Q}})/U)$, with the diagonal action of $\mathrm{Aut}_U(L)$, and same space where $\mathrm{Aut}_U(L)$ only acts on $\mathcal{E}(X_{\mathbb{Q}})/U$. $\square$

Lemma 3.21 does not give full information about the homotopy type of the $\mathcal{E}_U(X_{\mathbb{Q}})$ -space $B_{U\text{aut}}(X_{\mathbb{Q}})$, but it is enough for proving that the cohomology groups and the simple homotopy groups are algebraic representations.

Theorem 3.22 (i) $\mathcal{E}_U(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$ -points of an affine algebraic group.
 (ii) The homomorphism $\mathcal{E}_G(X) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}})$ has finite kernel and image an arithmetic subgroup.
 (iii) The representations of $\mathcal{E}_G(X)$ in the rational cohomology groups and the simple rational homotopy groups of the space $B_{\text{aut}_G}(X)$ are restrictions of algebraic representations of $\mathcal{E}_U(X_{\mathbb{Q}})$.

Proof By [Theorem 2.24\(ii\)](#), $\mathcal{E}(X_{\mathbb{Q}})$ may be identified with the $\mathbb{Q}$ -points of the algebraic group $\mathcal{G} = \mathcal{A}\text{ut}(L)/\mathcal{A}\text{ut}_h(L)$. By hypothesis, U corresponds to the $\mathbb{Q}$ -points of a unipotent algebraic subgroup $\mathcal{U}$ of $\mathcal{G}$. It follows from [Lemmas 2.10](#) and [2.9](#) that $\mathcal{E}_U(X_{\mathbb{Q}}) = N_{\mathcal{E}(X_{\mathbb{Q}})}(U)/U$ may be identified with the $\mathbb{Q}$ -points of the algebraic group $N_{\mathcal{G}}(\mathcal{U})/\mathcal{U}$. This proves the first claim.

The kernel of $N_{\mathcal{E}(X)}(G)/G \rightarrow N_{\mathcal{E}(X_{\mathbb{Q}})}(U)/U$ injects into the set $\rho^{-1}(U)/G$, which is finite by [Proposition 3.3](#). [Lemma 3.9](#) implies that there is an inclusion

$$(16) \quad \rho(N_{\mathcal{E}(X)}(G)) \leq N_{\mathcal{E}(X_{\mathbb{Q}})}(U) \cap \rho(\mathcal{E}(X)).$$

The latter group is an arithmetic subgroup of $N_{\mathcal{E}(X_{\mathbb{Q}})}(U)$ by [Theorem 2.26](#) and [Lemma 2.11](#), so if we can show that (16) is the inclusion of a finite-index subgroup, then it follows that $\rho(N_{\mathcal{E}(X)}(G))$ is an arithmetic subgroup of $N_{\mathcal{E}(X_{\mathbb{Q}})}(U)$. It then follows from [Lemma 2.12](#) that the image of $\rho(N_{\mathcal{E}(X)}(G))$ in $\mathcal{E}_U(X_{\mathbb{Q}}) = N_{\mathcal{E}(X_{\mathbb{Q}})}(U)/U$ is arithmetic, and then we are done because this agrees with the image of $\mathcal{E}_G(X) = N_{\mathcal{E}(X)}(G)/G$. To show that the index is finite, observe that (16) is obtained by applying ρ to the inclusion

$$(17) \quad N_{\mathcal{E}(X)}(G) \leq \rho^{-1}(N_{\mathcal{E}(X_{\mathbb{Q}})}(U)).$$

The latter group acts on the set of subgroups of $\rho^{-1}(U)$ by conjugation. The stabilizer of G is $N_{\mathcal{E}(X)}(G)$ and the orbit of G consists of subgroups of the form gGg^{-1} , where $g \in \mathcal{E}(X)$ satisfies $g\rho^{-1}(U)g^{-1} = \rho^{-1}(U)$. Note that all subgroups of this form have the same (finite) index in $\rho^{-1}(U)$. There are only finitely many subgroups of a given finite index in a finitely generated group, so the orbit must be finite. It follows that the index of the inclusion (17) is finite and in turn that the same holds for (16).

The action of $\mathcal{E}_G(X)$ on the homology of $B_{\text{aut}_G}(X)$ may be computed as the induced action on the homology of the $\mathcal{E}_G(X)$ -space $B_{G\text{aut}}(X)$. The latter is connected to $B_{U\text{aut}}(X_{\mathbb{Q}})$ by a sequence of $\mathcal{E}_G(X)$ -equivariant rational homology equivalences by [Proposition 3.10](#), so it suffices to show that the cohomology groups of $B_{U\text{aut}}(X_{\mathbb{Q}})$ are algebraic representations of $\mathcal{E}_U(X_{\mathbb{Q}})$.

Since the cohomology of $B \exp_{\bullet}(\mathfrak{g})$ is naturally isomorphic to the Chevalley–Eilenberg cohomology of $\mathfrak{g}$ (see [Proposition 2.21](#)), [Lemma 3.21](#) implies that there is an $\text{Aut}_U(L)$ -equivariant isomorphism

$$(18) \quad H^k(B_{U\text{aut}}(X_{\mathbb{Q}})) \cong H_{\text{CE}}^k(\text{Der}_{\mathfrak{u}}^c L)$$

for every k .

Note that $\mathrm{Aut}_U(L) = \mathcal{A}\mathrm{ut}_{\mathcal{U}}(L)(\mathbb{Q})$ and that there is an isomorphism of algebraic groups $\mathcal{A}\mathrm{ut}_{\mathcal{U}}(L)/\mathcal{W} \rightarrow N_{\mathcal{G}}(\mathcal{U})/\mathcal{U}$, where $\mathcal{A}\mathrm{ut}_{\mathcal{U}}(L)$ and $\mathcal{W}$ are the preimages of $N_{\mathcal{G}}(\mathcal{U})$ and $\mathcal{U}$ under $\pi: \mathcal{A}\mathrm{ut}(L) \rightarrow \mathcal{G}$. Since π has unipotent kernel, it follows that $\mathcal{W}$ is unipotent. The right-hand side of (18) is manifestly an algebraic representation of the algebraic group $\mathcal{A}\mathrm{ut}_{\mathcal{U}}(L)$. The action of $\mathcal{W}(\mathbb{Q})$ is trivial, because the action of $\mathcal{A}\mathrm{ut}_{\mathcal{U}}(L)(\mathbb{Q}) = \mathrm{Aut}_U(L)$ on the left-hand side factors through $\mathcal{E}_U(X_{\mathbb{Q}})$, which may be identified with $\mathcal{A}\mathrm{ut}_U(L)(\mathbb{Q})/\mathcal{W}(\mathbb{Q})$. Since $\mathcal{W}(\mathbb{Q})$ is dense in $\mathcal{W}$ (by eg [55, Theorem 17.93]), it follows that the representation is trivial as an algebraic representation of $\mathcal{W}$. Thus, it is an algebraic representation of $\mathcal{A}\mathrm{ut}_U(L)/\mathcal{W} \cong N_{\mathcal{G}}(\mathcal{U})/\mathcal{U}$. This shows that the cohomology groups of $B_{U\mathrm{aut}}(X_{\mathbb{Q}})$ are algebraic representations of $\mathcal{E}_U(X_{\mathbb{Q}})$.

To show algebraicity of the simple homotopy groups, one argues as above using the $\mathrm{Aut}_U(L)$ -equivariant isomorphism

$$\pi'_{k+1}(B_{U\mathrm{aut}}(X_{\mathbb{Q}})) \cong H_k(\mathrm{Der}^c L)/[H_0(\mathrm{Der}^c L), H_k(\mathrm{Der}^c L)],$$

which can be deduced from Lemma 3.21 and Proposition 2.19. □

For later use, we record the following fact that was observed in the above proof.

Lemma 3.23 *The kernel of $p: \mathrm{Aut}_U(L) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}})$ is unipotent.* □

Before we proceed to the proof of Theorem 1.2, let us discuss two interesting applications of Theorem 3.22.

3.6.1 Arithmeticity of higher homotopy groups As discussed in the introduction, Sullivan and Wilkerson proved that the homomorphism from $\mathcal{E}(X) = \pi_1 B\mathrm{aut}(X)$ to $\mathcal{E}(X_{\mathbb{Q}}) = \pi_1 B\mathrm{aut}(X_{\mathbb{Q}})$ has finite kernel and image an arithmetic subgroup. It is natural to ask whether there is a similar statement for the higher homotopy groups of $B\mathrm{aut}(X)$. The following shows that these are “arithmetic representations” of $\mathcal{E}(X)$, in the sense that they each admit a map, with finite kernel, onto a lattice inside an algebraic representation of the group $\mathcal{E}(X_{\mathbb{Q}})$.

Corollary 3.24 *Let X be a simply connected finite CW-complex. The representations of $\mathcal{E}(X)$ in the rational cohomology and homotopy groups of the universal cover of $B\mathrm{aut}(X)$ are restrictions of finite-dimensional algebraic representations of the algebraic group $\mathcal{E}(X_{\mathbb{Q}})$.*

Proof Apply Theorem 3.22 to the trivial group $G = 1$. Since $B\mathrm{aut}_1(X)$ is simply connected, the simple homotopy groups agree with the homotopy groups. □

3.6.2 Homotopy Torelli spaces Kupers and Randal-Williams [49] recently established algebraicity and nilpotence results for Torelli groups of the manifolds W_g , ie groups of diffeomorphisms that act trivially on integral homology. By analogy, we can consider the homotopy Torelli monoid of the space X , ie the monoid $\mathrm{tor}(X)$ of self-homotopy equivalences of X that act trivially on $H_*(X; \mathbb{Z})$. The following shows

that a homotopical counterpart of the main result of [49] is valid for *arbitrary* simply connected finite complexes.

Note that the homotopy Torelli space sits in a homotopy fiber sequence

$$B \operatorname{tor}(X) \rightarrow B \operatorname{aut}(X) \rightarrow B \operatorname{GL}^{\mathcal{E}(X)}(H_*(X; \mathbb{Z})),$$

where $\operatorname{GL}^{\mathcal{E}(X)}(H_*(X; \mathbb{Z}))$ is the image of the homomorphism $\mathcal{E}(X) \rightarrow \operatorname{GL}(H_*(X; \mathbb{Z}))$.

Corollary 3.25 *Let X be a simply connected finite CW-complex.*

- (i) *The group $\operatorname{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q}))$ of isomorphisms of $H_*(X; \mathbb{Q})$ that are realizable by a self-homotopy equivalence of $X_{\mathbb{Q}}$ is a linear algebraic group.*
- (ii) *The homomorphism $\operatorname{GL}^{\mathcal{E}(X)}(H_*(X; \mathbb{Z})) \rightarrow \operatorname{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q}))$ has finite kernel and image an arithmetic subgroup.*
- (iii) *The representations of $\operatorname{GL}^{\mathcal{E}(X)}(H_*(X; \mathbb{Z}))$ in the rational (co)homology and rational simple homotopy groups of the space $B \operatorname{tor}(X)$ are restrictions of algebraic representations of the linear algebraic group $\operatorname{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q}))$.*

Proof Clearly, $\operatorname{tor}(X) = \operatorname{aut}_G(X)$, where $G \leq \mathcal{E}(X)$ is the kernel of the action on $H_*(X; \mathbb{Z})$. Let $U \leq \mathcal{E}(X_{\mathbb{Q}})$ be the kernel of the algebraic representation $H_*(X; \mathbb{Q})$. Then U is unipotent by [Theorem 2.24\(iv\)](#). The cokernel of $G \rightarrow \rho^{-1}(U)$ may be identified with the kernel of

$$\operatorname{GL}^{\mathcal{E}(X)}(H_*(X; \mathbb{Z})) \rightarrow \operatorname{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q})),$$

which is clearly finite. Hence, [Proposition 3.3](#) implies that U is the minimal algebraic unipotent subgroup of $\mathcal{E}(X_{\mathbb{Q}})$ containing G , so [Theorem 3.22](#) applies. $\square$

3.7 Space-level algebraicity

In this section, we will prove [Theorem 1.2](#). We retain the notation of the previous section. The following two lemmas are key.

Lemma 3.26 *If the algebraic group $\mathcal{E}_U(X_{\mathbb{Q}})$ is reductive, then the homomorphism*

$$p: \operatorname{Aut}_U(L) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}})$$

admits a section σ that is a morphism of algebraic groups.

Proof The kernel of p is unipotent by [Lemma 3.23](#), so if $\mathcal{E}_U(X_{\mathbb{Q}})$ is reductive, then p may be identified with the map from $\operatorname{Aut}_U(L)$ to its maximal reductive quotient and an algebraic section exists by [Theorem 2.6](#). $\square$

Lemma 3.27 *If the homomorphism*

$$p: \operatorname{Aut}_U(L) \rightarrow \mathcal{E}_U(X_{\mathbb{Q}})$$

admits a section σ , then there is a zig-zag of $\mathcal{E}_G(X)$ -equivariant rational equivalences that connects $B_{G\operatorname{aut}}(X)$ to the nerve of the dg Lie algebra $\sigma^* \operatorname{Der}_{\mathfrak{u}}^c(L)$. Moreover, if σ is a morphism of algebraic groups, then the graded components of $\sigma^* \operatorname{Der}_{\mathfrak{u}}^c(L)$ are algebraic representations of $\mathcal{E}_U(X_{\mathbb{Q}})$.

Proof By [Proposition 3.10](#), the spaces $B_{G\operatorname{aut}}(X)$ and $B_{U\operatorname{aut}}(X_{\mathbb{Q}})$ are connected by a zig-zag of $\mathcal{E}_G(X)$ -equivariant rational equivalences. That σ is a section of p means that $p\sigma = 1$, so by [Lemma 3.21](#), we get weak equivalences of $\mathcal{E}_U(X_{\mathbb{Q}})$ -spaces

$$B_{U\operatorname{aut}}(X_{\mathbb{Q}}) = \sigma^* p^* B_{U\operatorname{aut}}(X_{\mathbb{Q}}) \sim \sigma^* B|\exp_{\bullet}(\operatorname{Der}_{\mathfrak{u}}^c L)| = B|\exp_{\bullet}(\sigma^* \operatorname{Der}_{\mathfrak{u}}^c L)|.$$

The graded components of $\operatorname{Der}_{\mathfrak{u}}^c(L)$ are manifestly algebraic representations of the algebraic group $\operatorname{Aut}_U(L)$. Hence, if σ is a morphism of algebraic groups, then the graded components of $\sigma^* \operatorname{Der}_{\mathfrak{u}}^c(L)$ are algebraic representations of $\mathcal{E}_U(X_{\mathbb{Q}})$. $\square$

We now prove [Theorem 1.2](#). First, we need to specify the cover $B\operatorname{aut}_{\mathfrak{u}}(X)$ and the groups $\Gamma(X)$ and $R(X)$.

Definition 3.28 We define

- $B\operatorname{aut}_{\mathfrak{u}}(X) = B\operatorname{aut}_G(X)$,
- $\Gamma(X) = \mathcal{E}_G(X)$,
- $R(X) = \mathcal{E}_U(X_{\mathbb{Q}})$,

where U is the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$ and G is the preimage of U under the homomorphism $\rho: \mathcal{E}(X) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$.

Proposition 3.29 $R(X)$ is a reductive algebraic group and $\Gamma(X)$ may be identified with an arithmetic subgroup of $R(X)$.

Proof The unipotent radical U is the maximal normal unipotent algebraic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$. In particular, U is normal in $\mathcal{E}(X_{\mathbb{Q}})$ and $G = \rho^{-1}(U)$ is normal in $\mathcal{E}(X)$. Hence, $\Gamma(X) = \mathcal{E}(X)/G$ and $R(X) = \mathcal{E}(X_{\mathbb{Q}})/U$. We recognize the latter as the maximal reductive quotient of $\mathcal{E}(X_{\mathbb{Q}})$ (cf [Section 2.2.1](#)). Since $G = \rho^{-1}(U)$, we may identify $\Gamma(X) = \mathcal{E}(X)/G$ with the image of $\mathcal{E}(X)$ in $R(X) = \mathcal{E}(X_{\mathbb{Q}})/U$. This is an arithmetic subgroup of $R(X)$ by [Theorem 3.22\(ii\)](#). $\square$

By that, the first part of [Theorem 1.2](#) has been verified. Next, we need to construct the dg Lie algebra $\mathfrak{g}(X)$.

Definition 3.30 Define

$$\mathfrak{g}(X) = \sigma^* \operatorname{Der}_{\mathfrak{u}}^c(L),$$

where $\mathfrak{u}$ is the Lie algebra of the unipotent radical U of $\mathcal{E}(X_{\mathbb{Q}})$, and where

$$\sigma: R(X) \rightarrow \operatorname{Aut}_U(L)$$

is a section of p as in [Lemma 3.26](#), which exists since $R(X)$ is reductive.

Theorem 3.31 *The dg Lie algebra $\mathfrak{g}(X)$ is nilpotent, its graded components are algebraic representations of $R(X)$, and there is a zig-zag of $\Gamma(X)$ -equivariant rational equivalences that connects $B\mathrm{aut}_u(X)$ to the nerve of $\mathfrak{g}(X)$.*

Proof Nilpotence is shown in [Proposition 3.12](#). The rest is immediate from [Lemmas 3.26](#) and [3.27](#). $\square$

Corollary 3.32 *The space $B\mathrm{aut}(X)$ is rationally equivalent to the homotopy orbit space $\langle \mathfrak{g}(X) \rangle_{h\Gamma(X)}$.*

Proof We can recover $B\mathrm{aut}(X)$ as the homotopy orbit space

$$B\mathrm{aut}(X) \sim B\mathrm{aut}_u(X)_{h\Gamma(X)}.$$

Applying homotopy orbits to the zig-zag in [Theorem 3.31](#) yields the result. $\square$

This finishes the proof of [Theorem 1.2](#).

Remark 3.33 It is natural to ask whether the rational homotopy type of the $\mathcal{E}_G(X)$ -space $B_G\mathrm{aut}(X)$ can be represented by a dg Lie algebra of algebraic representations of $\mathcal{E}_U(X_{\mathbb{Q}})$ for arbitrary subgroups $G \leq \mathcal{E}(X)$ that act nilpotently on $SH_*(X; \mathbb{Q})$. The main problem is to rectify the action up to homotopy of $\mathcal{E}_U(X_{\mathbb{Q}})$ on $\mathrm{Der}_{\mathfrak{u}}^c L$. As we have seen, this can be done if $\mathcal{E}_U(X_{\mathbb{Q}})$ is reductive. One can show that $\mathcal{E}_U(X_{\mathbb{Q}})$ is reductive if and only if U is the unipotent radical of a parabolic subgroup of $\mathcal{E}(X_{\mathbb{Q}})$, but $\mathcal{E}_U(X_{\mathbb{Q}})$ will not be reductive in general. Still, even if the action cannot be rectified, one can ask whether a suitable modification of $\mathrm{Der}_{\mathfrak{u}}^c L$ could accommodate an action of $\mathcal{E}_U(X_{\mathbb{Q}})$. Since our main applications are in the case when $\mathcal{E}_U(X_{\mathbb{Q}})$ is reductive, we will not pursue this question further in this paper. We leave it as a challenge to the interested reader.

3.8 A commutative cochain algebra model for the classifying space

In this section, we will use [Theorem 1.2](#) to construct a commutative differential graded algebra model for $B\mathrm{aut}(X)$. Before we can formulate the result, we need to review the definition of the commutative cochains of a simplicial set with local coefficients, following Halperin [\[42\]](#).

3.8.1 Commutative cochains with local coefficients For a simplicial set K , we let $(\Delta \downarrow K)$ denote the simplex category. An object is a simplex $\sigma: \Delta^n \rightarrow K$ and a morphism from σ to τ is a commutative diagram

$$\begin{array}{ccc} \Delta^n & & \\ \varphi \downarrow & \searrow \sigma & \\ \Delta^m & \xrightarrow{\tau} & K \end{array}$$

Definition 3.34 [\[42, pages 150–151\]](#) A local system on a simplicial set K with values in a category $\mathcal{C}$ is a functor

$$F: (\Delta \downarrow K)^{\mathrm{op}} \rightarrow \mathcal{C}.$$

The global sections of a local system F on K is defined as the inverse limit

$$F(K) = \varprojlim_{\sigma \in (\Delta \downarrow K)} F_{\sigma},$$

provided this limit exists in $\mathcal{C}$.

Remark 3.35 To compare this with the notion of a local system of coefficients defined in terms of representations of the fundamental groupoid, one should observe that the groupoidification of the category $(\Delta \downarrow K)$ is a model for the fundamental groupoid of K ; see [38, Section III.1]. This means that the category of local systems F such that $F(\varphi)$ is invertible for every morphism φ in $(\Delta \downarrow K)$ is equivalent to the category of representations of the fundamental groupoid. However, since we do not require invertibility of $F(\varphi)$ in general, the notion of a local system is more general.

Example 3.36 (i) If Γ is a discrete group and M is an object of $\mathcal{C}$ with an action of Γ , then there is an associated local system on the nerve $B\Gamma$, also denoted by M , where $M_\sigma = M$ for all $\sigma = (\gamma_1, \dots, \gamma_n): \Delta^n \rightarrow B\Gamma$ and $d_i: M_\sigma \rightarrow M_{d_i\sigma}$ is multiplication by γ_n for $i = n$ and the identity otherwise.

(ii) A simplicial object $X_\bullet: \Delta^{\text{op}} \rightarrow \mathcal{C}$ determines a local system on every simplicial set K by restriction along $(\Delta \downarrow K)^{\text{op}} \rightarrow \Delta^{\text{op}}$. In particular, the simplicial cdga $\Omega_\bullet$ determines a local system Ω^* of cdgas on every simplicial set K . The global sections $\Omega^*(K)$ is the usual model for Sullivan's cdga of polynomial differential forms on K , also denoted by $A_{PL}^*(K)$.

(iii) More generally, if F is a local system of cochain complexes on K , then $\Omega^*(K; F)$ may be defined as the global sections of the local system $\Omega^* \otimes F$; see [42, Definition 13.10]. If A is a local system of cochain algebras, then $\Omega^*(K; A)$ is a cochain algebra, which is commutative if A is.

Proposition 3.37 Let K be a simplicial set and let A be a local system of $\mathbb{Q}$ -cochain complexes on K . The cochain complex $\Omega^*(K; A)$ is naturally quasi-isomorphic to $C^*(K; A)$. If A is a local system of dg algebras, then $\Omega^*(K; A)$ and $C^*(K; A)$ are quasi-isomorphic as dg algebras.

Proof Integration provides a natural quasi-isomorphism

$$\Omega^*(K; A) \rightarrow C^*(K; A),$$

see [42, Theorem 14.18]. The integration map is not multiplicative on the cochain level, but the argument in [34, Section 10(d)] goes through with coefficients A , providing a zig-zag of dg algebra quasi-isomorphisms. $\square$

In particular, for a discrete group Γ and a cochain complex M of $\mathbb{Q}[\Gamma]$ -modules, the cohomology of $\Omega^*(\Gamma, M) = \Omega^*(B\Gamma; M)$ agrees with group cohomology $H^*(\Gamma, M)$ (as defined in eg [25, VII.5]). If A is a commutative cochain algebra over $\mathbb{Q}$ with an action of Γ , then $\Omega^*(\Gamma, A)$ is a commutative cochain algebra model for $C^*(\Gamma, A)$. The following is a commutative cochain algebra version, in characteristic zero, of well-known results for homology and singular chains; see eg [25, Section VII.7] or [53, Section 13].

Proposition 3.38 Let X be a space with a Γ -action. The commutative cochain algebras $\Omega^*(X_{h\Gamma})$ and $\Omega^*(\Gamma, \Omega^*(X))$ are quasi-isomorphic.

Proof The local system F of the fibration $X_{h\Gamma} \rightarrow B\Gamma$ in the sense of [42, page 248] may be identified up to quasi-isomorphism with the local system associated to the Γ -cdga $\Omega^*(X)$. Granted this, the combination of [42, Lemma 19.21] and [42, Theorem 13.12] applied to the local system F shows that

$$\Omega^*(X_{h\Gamma}) \cong F(B\Gamma) \sim \Omega^*(B\Gamma; F) \sim \Omega^*(B\Gamma; \Omega^*(X)). \quad \square$$

Now we are ready to formulate and prove [Corollary 1.3](#) from the introduction.

Theorem 3.39 *The commutative dg algebra $\Omega^*(B\operatorname{aut}(X))$ is quasi-isomorphic to the commutative dg algebra $\Omega^*(\Gamma(X), C_{\text{CE}}^*(\mathfrak{g}(X)))$.*

Proof It follows from [Corollary 3.32](#) that $\Omega^*(B\operatorname{aut}(X))$ is quasi-isomorphic to $\Omega^*(\langle \mathfrak{g}(X) \rangle_{h\Gamma(X)})$. By [Proposition 3.38](#), the latter commutative cochain algebra is quasi-isomorphic to $\Omega^*(\Gamma(X), \Omega^*(\langle \mathfrak{g}(X) \rangle))$. We then use the natural quasi-isomorphism $C_{\text{CE}}^*(\mathfrak{g}(X)) \rightarrow \Omega^*(\langle \mathfrak{g}(X) \rangle)$ of [Proposition 2.21](#) to complete the proof. $\square$

Let us compare with the approach of Gómez-Tato, Halperin and Tanré [39]. They study the fiberwise rationalization of $B\operatorname{aut}(X) \rightarrow B\mathcal{G}(X)$ by using local systems, in the sense of [Definition 3.34](#), of cdgas over $B\mathcal{G}(X)$. As discussed in [Remark 3.35](#), such local system are more general, and more complicated, than cdgas with an action of $\mathcal{G}(X)$. Our approach is to instead consider the fiberwise rationalization of $B\operatorname{aut}(X) \rightarrow B\Gamma(X)$. This has two significant advantages: Firstly, it lets us work with cdgas with an action of $\Gamma(X)$ rather than local systems of cdgas over $B\Gamma(X)$. Secondly, semisimplicity of algebraic $\Gamma(X)$ -representations lets us construct minimal models using familiar methods. Indeed, the counterpart of Problem 3 of [39, Section 7.3] in our setting admits the following solution:

Theorem 3.40 *The minimal Sullivan model $\mathfrak{M}(X)$ for $B\operatorname{aut}_u(X)$ admits an algebraic action of $R(X)$ such that the spatial realization $\langle \mathfrak{M}(X) \rangle$ is rationally equivalent to $B\operatorname{aut}_u(X)$ as a $\Gamma(X)$ -space, and $\Omega^*(B\operatorname{aut}(X))$ is quasi-isomorphic to $\Omega^*(\Gamma(X), \mathfrak{M}(X))$ as a commutative cochain algebra.*

Proof The minimal model is unique up to isomorphism, so it is enough to construct a particular minimal model that admits an algebraic $R(X)$ -action with the desired properties. Let $\mathfrak{g}(X)$ be the Lie model for $B\operatorname{aut}_u(X)$ in [Theorem 3.31](#). By semisimplicity of $\operatorname{Rep}_{\mathbb{Q}}(R(X))$, we can find a contraction of chain complexes of algebraic $R(X)$ -representations

$$h \begin{array}{c} \hookrightarrow \\ \circlearrowleft \end{array} \mathfrak{g}(X) \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} H_*(\mathfrak{g}(X)).$$

Applying [9, Theorem 1.3], we get a contraction of cocommutative chain coalgebras in $\operatorname{Rep}_{\mathbb{Q}}(R(X))$,

$$H' \begin{array}{c} \hookrightarrow \\ \circlearrowleft \end{array} C_{*}^{\text{CE}}(\mathfrak{g}(X)) \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{G'} \end{array} (\Lambda^c(V), d'),$$

where V is the suspension of $H_*(\mathfrak{g}(X))$. The differential d' is given by the explicit formula

$$d' = FtG + FtHtG + Ft(Ht)^2G + \cdots,$$

and F' , G' and H' are given by similar formulas. Here, t is the quadratic part of the differential of $C_{\text{CE}}^*(\mathfrak{g}(X))$ that encodes the Lie bracket, F and G are the morphisms of dg coalgebras induced by f and g , and H is the “symmetrized tensor trick homotopy” [9, Section 5]. The dual of $(\Lambda^c(V), d')$ is a minimal model $\mathfrak{M}(X)$ for $B\text{aut}_u(X)$. By construction, it has an algebraic action of $R(X)$ and the dual of G' is a quasi-isomorphism of cochain algebras of $R(X)$ -representations $C_{\text{CE}}^*(\mathfrak{g}(X)) \rightarrow \mathfrak{M}(X)$. This induces a quasi-isomorphism of commutative cochain algebras $\Omega^*(\Gamma(X), C_{\text{CE}}^*(\mathfrak{g}(X))) \rightarrow \Omega^*(\Gamma(X), \mathfrak{M}(X))$, so it follows from Theorem 3.39 that $\Omega^*(\Gamma(X), \mathfrak{M}(X))$ is a cdga model for $\Omega^*(B\text{aut}(X))$. Applying spatial realization, we get an $R(X)$ -equivariant weak equivalence $\langle \mathfrak{M}(X) \rangle \rightarrow \langle C_{\text{CE}}^*(\mathfrak{g}(X)) \rangle \cong \langle \mathfrak{g}(X) \rangle$, and the latter is a $\Gamma(X)$ -equivariant model for $B\text{aut}_u(X)$ by Theorem 3.31.

We stress that this construction is completely explicit once the contraction between $\mathfrak{g}(X)$ and its homology has been fixed. $\square$

3.9 The cohomology ring of the classifying space

The following result, which was stated as Corollary 1.4 in the introduction, is perhaps the most striking consequence of Theorem 3.39.

Theorem 3.41 *There is an isomorphism of graded algebras*

$$H^*(B\text{aut}(X); \mathbb{Q}) \cong H^*(\Gamma(X), H_{\text{CE}}^*(\mathfrak{g}(X))).$$

Proof The Chevalley–Eilenberg complex $C_{\text{CE}}^*(\mathfrak{g}(X))$ is a cochain complex of finite-dimensional algebraic representations of $R(X)$. Since $R(X)$ is reductive, every such representation is semisimple, and thus the cochain complex $C_{\text{CE}}^*(\mathfrak{g}(X))$ is split, which means that there is a contraction of cochain complexes in $\text{Rep}_{\mathbb{Q}}(R(X))$,

$$h \left(\begin{array}{c} \curvearrowright \end{array} \right) C_{\text{CE}}^*(\mathfrak{g}(X)) \xrightleftharpoons[\nabla]{p} H_{\text{CE}}^*(\mathfrak{g}(X)).$$

That is, p and ∇ are chain maps such that $p(z) = [z]$ whenever $z \in C_{\text{CE}}^*(\mathfrak{g}(X))$ is a cocycle, $p\nabla = \text{id}$, and h satisfies $\text{id} - \nabla p = dh + hd$, see eg [16, Lemma B.1].

Applying the dg functor $\Omega^*(\Gamma(X); -)$ yields a new contraction

$$h_* \left(\begin{array}{c} \curvearrowright \end{array} \right) \Omega^*(\Gamma(X), C_{\text{CE}}^*(\mathfrak{g}(X))) \xrightleftharpoons[\nabla_*]{p_*} \Omega^*(\Gamma(X), H_{\text{CE}}^*(\mathfrak{g}(X))).$$

We now use the homotopy transfer theorem (see eg [9]) to produce a C_{∞} -algebra structure $\{\mu_n\}_{n \geq 2}$ on

$$A = \Omega^*(\Gamma(X); H_{\text{CE}}^*(\mathfrak{g}(X)))$$

and an extension of ∇_* to a C_{∞} -quasi-isomorphism. The multiplication in this C_{∞} -algebra structure is given by $\mu_2 = p_* m_2 (\nabla_* \otimes \nabla_*)$, where m_2 is the multiplication on $\Omega^*(\Gamma(X); C_{\text{CE}}^*(\mathfrak{g}(X)))$, and one checks that this agrees with the multiplication on $\Omega^*(\Gamma(X); H_{\text{CE}}^*(\mathfrak{g}(X)))$, coming from viewing $H_{\text{CE}}^*(\mathfrak{g}(X))$ as a $\Gamma(X)$ -cdga with trivial differential. The cdga (A, μ_2) is of course not equivalent to the C_{∞} -algebra $(A, \mu_2, \mu_3, \dots)$ in general, but they have isomorphic cohomology rings. $\square$

4 The ingredients of the algebraic model

In this section, we will give more concrete descriptions of the ingredients $R(X)$, $\Gamma(X)$, $\mathfrak{g}(X)$, $B\mathrm{aut}_u(X)$ of [Theorem 1.2](#) and we will state counterparts of our main results for finite Postnikov stages. In [Section 5](#), the results of the present section will be used to make explicit calculations.

4.1 The groups $R(X)$ and $\Gamma(X)$

Let $H_*(X; \mathbb{Q})^{\mathrm{ss}}$ denote the “semisimplification” of $H_*(X; \mathbb{Q})$ as a representation of the algebraic group $\mathcal{E}(X_{\mathbb{Q}})$. Recall from [Section 2.2.1](#) that this is the semisimple, unique up to isomorphism, $\mathcal{E}(X_{\mathbb{Q}})$ -representation

$$H_*(X; \mathbb{Q})^{\mathrm{ss}} = \bigoplus_{i=1}^n V_i / V_{i-1},$$

where

$$(19) \quad 0 = V_0 \subset V_1 \subset \cdots \subset V_n = H_*(X; \mathbb{Q})$$

is any choice of composition series. For a group G and a representation V of G , let $\mathrm{GL}^G(V)$ denote the image of the homomorphism $G \rightarrow \mathrm{GL}(V)$.

Theorem 4.1 *There are group isomorphisms*

$$R(X) \cong \mathrm{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q})^{\mathrm{ss}}) \quad \text{and} \quad \Gamma(X) \cong \mathrm{GL}^{\mathcal{E}(X)}(H_*(X; \mathbb{Q})^{\mathrm{ss}}).$$

In other words, $R(X)$ (resp. $\Gamma(X)$) may be identified with the group of automorphisms of $H_*(X; \mathbb{Q})^{\mathrm{ss}}$ that are induced by a self-homotopy equivalence of $X_{\mathbb{Q}}$ (resp. X).

Proof By definition, the group $R(X)$ is the maximal reductive quotient of $\mathcal{E}(X_{\mathbb{Q}})$. By [Theorem 2.24\(iv\)](#), the rational homology $H_*(X; \mathbb{Q})$ is an algebraic representation of $\mathcal{E}(X_{\mathbb{Q}})$ with unipotent kernel. Hence by [Lemma 2.7](#), the maximal reductive quotient $R(X)$ of $\mathcal{E}(X_{\mathbb{Q}})$ may be identified with the image of $\mathcal{E}(X_{\mathbb{Q}})$ in $\mathrm{GL}(H_*(X; \mathbb{Q})^{\mathrm{ss}})$. By definition, the group $\Gamma(X)$ is the image of $\mathcal{E}(X)$ in $R(X)$. In view of the above, this agrees with $\mathrm{GL}^{\mathcal{E}(X)}(H_*(X; \mathbb{Q})^{\mathrm{ss}})$. $\square$

Often, in fact in all cases we consider in [Section 5](#) except the last, the equivalent conditions in the corollary below are satisfied, which simplifies the descriptions.

Corollary 4.2 *The following are equivalent:*

- (i) *The $\mathcal{E}(X_{\mathbb{Q}})$ -representation $H_*(X; \mathbb{Q})$ is semisimple.*
- (ii) *There is an isomorphism of algebraic groups*

$$R(X) \cong \mathrm{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q})).$$

- (iii) *The algebraic group $\mathrm{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q}))$ is reductive.*

If the above conditions hold, then $\Gamma(X)$ may be identified with the group of automorphisms of $H_*(X; \mathbb{Q})$ that are induced by self-homotopy equivalences of X .

Proof If $H_*(X; \mathbb{Q})$ is semisimple, then it is isomorphic to $H_*(X; \mathbb{Q})^{\text{ss}}$ so [Theorem 4.1](#) identifies $R(X)$ with $\text{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q}))$. The second condition implies the third since $R(X)$ is reductive by construction. If $\text{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q}))$ is reductive, then $H_*(X; \mathbb{Q})$ is a semisimple representation of it by [Theorem 2.5](#), but then it is also semisimple as a representation of $\mathcal{E}(X_{\mathbb{Q}})$. $\square$

Remark 4.3 If the equivalent conditions in [Corollary 4.2](#) are satisfied and in addition $H_*(X; \mathbb{Z})$ is torsion-free, then $\Gamma(X)$ may be identified with the group of automorphisms of $H_*(X; \mathbb{Z})$ that are induced by a self-homotopy equivalence.

An obvious necessary condition for an automorphism of $H_*(X; \mathbb{Q})$ to be induced by a self-homotopy equivalence is that it preserves the coproduct in homology, so there is an inclusion of algebraic groups

$$(20) \quad \text{GL}^{\mathcal{E}(X_{\mathbb{Q}})}(H_*(X; \mathbb{Q})) \leq \text{Aut}_{\text{coalg}}(H_*(X; \mathbb{Q})),$$

where the latter is the group of automorphisms of the homology coalgebra. This may also be identified with the group $\text{Aut}_{\text{alg}}(H^*(X; \mathbb{Q}))$ of automorphisms of the cohomology algebra.

Proposition 4.4 *The inclusion (20) is an equality if and only if X is formal.*

Proof This follows from [\[66, Theorem 12.7\]](#). $\square$

Corollary 4.5 *If X is formal, then there is an isomorphism of algebraic groups*

$$R(X) \cong \text{Aut}_{\text{alg}}(H^*(X; \mathbb{Q}))$$

if and only if $\text{Aut}_{\text{alg}}(H^*(X; \mathbb{Q}))$ is reductive.

Proof Combine [Proposition 4.4](#) and [Corollary 4.2](#). $\square$

4.2 Normal unipotent fibrations

In this section we will show that $B\text{aut}_u(X)$ can be characterized as the classifying space for what we call *normal unipotent fibrations*, defined below.

Consider a fibration of path-connected spaces

$$(21) \quad X \rightarrow E \xrightarrow{p} B.$$

The fibration is classified by a homotopy class of maps $f: B \rightarrow B\text{aut}(X)$, which induces a homomorphism on fundamental groups $\pi_1(B) \rightarrow \mathcal{E}(X)$ and an action of $\pi_1(B)$ on the homology of X .

Definition 4.6 The fibration (21) is called a *(normal) unipotent fibration* if the image of $\pi_1(B) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ is contained in a (normal) unipotent algebraic subgroup.

Proposition 4.7 Consider a fibration as in (21).

- (1) The fibration is unipotent if and only if for each n , the $\pi_1(B)$ -module $H_n(X; \mathbb{Q})$ is nilpotent, ie there exists a filtration of $\pi_1(B)$ -submodules

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r = H_n(X; \mathbb{Q})$$

such that F_i/F_{i-1} is a trivial $\pi_1(B)$ -module for every i .

- (2) The fibration is normal unipotent if and only if, for each n , the $\pi_1(B)$ -module $H_n(X; \mathbb{Q})$ is nilpotent and, moreover, a filtration as above can be chosen to be a filtration of algebraic $\mathcal{E}(X_{\mathbb{Q}})$ -representations.

Proof For the first statement, apply Proposition 3.2 to the image of the homomorphism $\pi_1(B) \rightarrow \mathcal{E}(X)$.

For the second statement, note that if the fibration is normal unipotent, then the image of $\pi_1(B) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ is contained in the unipotent radical, which may be described as the kernel of the action of $\mathcal{E}(X_{\mathbb{Q}})$ on the semisimplification $H_*(X; \mathbb{Q})^{\text{ss}}$ by Lemma 2.7 applied to the algebraic $\mathcal{E}(X_{\mathbb{Q}})$ -representation $H_*(X; \mathbb{Q})$. In particular, $\pi_1(X)$ acts trivially on $H_*(X; \mathbb{Q})^{\text{ss}}$, which is the associated graded with respect to a filtration of $H_*(X; \mathbb{Q})$ by algebraic $\mathcal{E}(X_{\mathbb{Q}})$ -representations. Conversely, if $\pi_1(B)$ acts trivially on the associated graded of a filtration of $H_*(X; \mathbb{Q})$ by algebraic $\mathcal{E}(X_{\mathbb{Q}})$ -representations, then the image of $\pi_1(B) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ is contained in the unipotent radical, because any such filtration can be refined to a composition series. $\square$

Remark 4.8 As in Proposition 3.2, one can replace $H_n(X; \mathbb{Q})$ by spherical homology $SH_n(X; \mathbb{Q})$ or rational homotopy $\pi_n(X) \otimes \mathbb{Q}$ in the proposition above. The latter can be used to show that a fibration (21) is unipotent if and only if its fiberwise rationalization is nilpotent in the sense of [44, page 67] or [24, II.4.3].

Proposition 4.9 The fibration (21) is normal unipotent if and only if the classifying map factors over $B \text{aut}_u(X)$ up to homotopy:

$$\begin{array}{ccc} & & B \text{aut}_u(X) \\ & \nearrow & \downarrow \\ B & \longrightarrow & B \text{aut}(X) \end{array}$$

Thus, $B \text{aut}_u(X)$ may be interpreted as the classifying space for normal unipotent fibrations with fiber X .

Proof A lift exists if and only if the image of $\pi_1(B) \rightarrow \pi_1 B \text{aut}(X) = \mathcal{E}(X)$ is contained in $\pi_1 B \text{aut}_u(X)$. The latter group is equal to the preimage of the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$ under $\mathcal{E}(X) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ by definition of $B \text{aut}_u(X)$. $\square$

4.3 The dg Lie algebra $\mathfrak{g}(X)$

We will now give a more concrete description of the dg Lie algebra $\mathfrak{g}(X)$ of Definition 3.30. Recall the notion of a nilradical of a dg Lie algebra from Definition 2.16.

Proposition 4.10 *The dg Lie algebra $\mathfrak{g}(X)$ agrees with the nilradical $\text{nil Der}^c L$ and its graded components are given by*

$$\mathfrak{g}(X)_n = \begin{cases} (\text{Der}^c L)_n & \text{if } n > 0, \\ \text{Lie}(\mathcal{A}ut_u L) & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Here, $\text{Lie}(\mathcal{A}ut_u L)$ stands for the Lie algebra of the unipotent radical of $\mathcal{A}ut L$ and this Lie algebra may be identified with $\text{nil}_{QL} Z_0(\text{Der } L)$, the maximal ideal of $Z_0(\text{Der } L)$ of derivations that act nilpotently on $QL = L/[L, L]$.

Proof We first verify the statement about the graded components of $\mathfrak{g}(X)$. Suppressing the action of $R(X)$, we have that $\mathfrak{g}(X) = \text{Der}^c_u L$ is the preimage of $\mathfrak{u} \subseteq H_0(\text{Der}^c L) = \text{Lie}(\mathcal{A}ut^h L)$ under the morphism of dg Lie algebras $q: \text{Der}^c L \langle 0 \rangle \rightarrow H_0(\text{Der}^c L)$, where $\mathfrak{u}$ is the Lie algebra of the unipotent radical $\mathcal{A}ut^h_u L$ of $\mathcal{A}ut^h L$. In degree zero, q is the morphism $Z_0(\text{Der}^c L) \rightarrow H_0(\text{Der}^c L)$, which may be identified with $\text{Lie}(p): \text{Lie}(\mathcal{A}ut L) \rightarrow \text{Lie}(\mathcal{A}ut^h L)$, where $p: \mathcal{A}ut L \rightarrow \mathcal{A}ut^h L$ is the quotient map. Since p has unipotent kernel (see [Theorem 2.24](#)), the preimage $p^{-1}(\mathcal{A}ut^h_u L)$ must be equal to $\mathcal{A}ut_u L$, the unipotent radical of $\mathcal{A}ut L$. Hence, $\mathfrak{g}(X)_0 = \text{Lie}(p)^{-1}(\text{Lie}(\mathcal{A}ut^h_u L)) = \text{Lie}(p^{-1}(\mathcal{A}ut^h_u L)) = \text{Lie}(\mathcal{A}ut_u L)$. Since the Lie algebra $H_0(\text{Der}^c L)$ is concentrated in degree zero, it follows that $\mathfrak{g}(X)_n = \text{Der}^c L \langle 0 \rangle_n$ for $n \neq 0$.

Since $\mathcal{A}ut^h_u L$ is normal in $\mathcal{A}ut^h L$, the Lie algebra $\mathfrak{u}$ is an ideal in $H_0(\text{Der}^c L)$. It follows that $\text{Der}^c_u L = q^{-1}(\mathfrak{u})$ is an ideal in $\text{Der}^c L \langle 0 \rangle$. Moreover, $\text{Der}^c_u L$ is nilpotent by [Proposition 3.12](#). By definition, $\text{nil Der}^c L$ is the maximal nilpotent ideal, so it follows that $\text{Der}^c_u L \subseteq \text{nil Der}^c L$.

We now show the reverse inclusion $\text{nil Der}^c L \subseteq \text{Der}^c_u L$. The components in degrees $\neq 0$ are clearly equal, so we only need to show that $(\text{nil Der}^c L)_0 \subseteq \text{Lie}(\mathcal{A}ut_u L)$. Note that since L is finitely generated as a graded Lie algebra and of finite type, there is some n such that the finite-dimensional graded vector space $L_{\leq n} = \bigoplus_{i=1}^n L_i$ is a faithful algebraic representation of $\mathcal{A}ut L$. Thus, by [Lemma 2.8](#), $\text{Lie}(\mathcal{A}ut_u L)$ is precisely the maximal ideal of $\text{Lie}(\mathcal{A}ut L) = Z_0(\text{Der}^c L)$ consisting of derivations of L that are nilpotent when restricted to $L_{\leq n}$. Since $(\text{nil Der}^c L)_0$ is an ideal in $Z_0(\text{Der}^c L)$ which acts nilpotently on $(\text{Der}^c L)_k \cong (\text{Der } L)_k \oplus L_{k-1}$ for each $k > 0$, it in particular acts nilpotently on $L_{\leq n}$. Hence, $(\text{nil Der}^c L)_0 \subseteq \text{Lie}(\mathcal{A}ut_u L)$.

Applying [Lemma 2.8](#) to $V = L/[L, L]$, the Lie algebra of $\mathcal{A}ut_u(L)$ is described as $\text{nil}_{QL} Z_0(\text{Der}^c L)$. $\square$

Corollary 4.11 *If $\text{Aut } L$ is reductive, then*

$$R(X) = \text{Aut } L \quad \text{and} \quad \mathfrak{g}(X) = \text{Der}^c L \langle 1 \rangle,$$

and the action of $R(X)$ on $\mathfrak{g}(X)$ is the conjugation action.

Proof Each of the homomorphisms $\text{Aut } L \rightarrow \mathcal{E}(X_{\mathbb{Q}}) \rightarrow R(X)$ has unipotent kernel. If $\text{Aut } L$ is reductive, it has no normal unipotent subgroups. This forces $\text{Aut } L = \mathcal{E}(X_{\mathbb{Q}}) = R(X)$. The description of $\mathfrak{g}(X)$ follows directly from [Proposition 4.10](#). $\square$

4.4 Finite Postnikov stages

The results we have stated have analogs for simply connected finite Postnikov stages X of finite type. We state the results and briefly indicate the necessary modifications of the proofs.

The minimal Sullivan model Λ is finitely generated and we may identify $\mathcal{E}(X_{\mathbb{Q}})$ with the $\mathbb{Q}$ -points of the algebraic group $\mathcal{A}ut^h(\Lambda)$ by [Theorem 2.25\(ii\)](#).

As before, let $G \leq \mathcal{E}(X)$ be a subgroup that acts nilpotently on $SH_*(X; \mathbb{Q})$ and let $U \leq \mathcal{E}(X_{\mathbb{Q}})$ be the minimal unipotent algebraic subgroup that contains $\rho(G)$ as in [Proposition 3.3](#). By [Theorem 2.25\(iii\)](#), we may regard the Lie algebra $\mathfrak{u}$ of U as a Lie subalgebra of $H_0(\text{Der } \Lambda)$. Analogously to [Definition 3.11](#), we define the dg Lie algebra $\text{Der}_{\mathfrak{u}} \Lambda$ by declaring that there is a pullback square

$$\begin{array}{ccc} \text{Der}_{\mathfrak{u}} \Lambda & \longrightarrow & \mathfrak{u} \\ \downarrow & & \downarrow \\ \text{Der } \Lambda \langle 0 \rangle & \longrightarrow & H_0(\text{Der } \Lambda) \end{array}$$

After identifying $X_{\mathbb{Q}}$ with the spatial realization of Λ , [Proposition 3.13](#) admits the following analog, where $\text{Aut}_U(\Lambda)$ denotes the preimage of the normalizer of $U \leq \mathcal{E}(X_{\mathbb{Q}})$ under the homomorphism $\text{Aut}(\Lambda) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$.

Proposition 4.12 *The dg Lie algebra $\text{Der}_{\mathfrak{u}} \Lambda$ is nilpotent and there is a weak equivalence of topological monoids*

$$(22) \quad |\exp_{\bullet}(\text{Der}_{\mathfrak{u}} \Lambda)| \longrightarrow \text{aut}_U(X_{\mathbb{Q}})$$

that is equivariant with respect to the conjugation action of $\text{Aut}_U(\Lambda)$ on the domain and codomain.

Proof The nilpotence of $\text{Der } \Lambda$ is proved as in [Proposition 3.12](#). The map in (22) is defined as in [\[12, Proposition 3.7\]](#): the action of the nilpotent dg Lie algebra $\mathfrak{g} = \text{Der}_{\mathfrak{u}} \Lambda$ on Λ by derivations induces an action of the simplicial nilpotent group

$$\exp_{\bullet}(\mathfrak{g}) = \exp(Z_0(\Omega_{\bullet} \otimes \mathfrak{g}))$$

on $\Omega_{\bullet} \otimes \Lambda$ by $\Omega_{\bullet}$ -linear automorphisms, which in turn induces an action of $\exp_{\bullet}(\mathfrak{g})$ on the simplicial set

$$\text{Hom}_{\text{cdga}(\Omega_{\bullet})}(\Omega_{\bullet} \otimes \Lambda, \Omega_{\bullet}),$$

the geometric realization of which is the spatial realization of Λ . The rest of the proof is entirely analogous to [Proposition 3.13](#). We omit the details. $\square$

Corollary 4.13 *The space $B\text{aut}_G(X)$ has dg Lie model $\text{Der}_{\mathfrak{u}}(\Lambda)$.* $\square$

A dg Lie algebra $\mathfrak{g}(X)$ as in [Theorem 3.31](#) can be described in terms of Sullivan models as well. Since $\text{Aut}(\Lambda) \rightarrow \mathcal{E}(X_{\mathbb{Q}})$ has unipotent kernel, the group $R(X)$ may be identified with the maximal reductive quotient of $\text{Aut}(\Lambda)$ and the quotient map $\text{Aut}(\Lambda) \rightarrow R(X)$ admits a splitting by [Theorem 2.6](#).

Theorem 4.14 *There is a zig-zag of $\Gamma(X)$ -equivariant rational equivalences that connects $B_{\mathfrak{u}\text{aut}}(X)$ to the nerve of the dg Lie algebra $\text{Der}_{\mathfrak{u}}(\Lambda)$, on which $\Gamma(X)$ acts through any choice of splitting of $\text{Aut}(\Lambda) \rightarrow R(X)$. Here, $\mathfrak{u} \leq H_0(\text{Der } \Lambda)$ is the Lie algebra of the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$.*

The following is an analog of [Proposition 4.10](#) for Sullivan models. It gives a more concrete description of the dg Lie algebra in [Theorem 4.14](#).

Proposition 4.15 *If $\mathfrak{u}$ is the Lie algebra of the unipotent radical of $\mathcal{E}(X_{\mathbb{Q}})$, then the dg Lie algebra $\text{Der}_{\mathfrak{u}} \Lambda$ agrees with the nilradical $\text{nil Der } \Lambda$. The graded components of this dg Lie algebra are given by*

$$(\text{nil Der } \Lambda)_n = \begin{cases} (\text{Der } \Lambda)_n & \text{if } n > 0, \\ \text{Lie}(\mathcal{A}\text{ut}_{\mathfrak{u}} \Lambda) & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Here, $\text{Lie}(\mathcal{A}\text{ut}_{\mathfrak{u}} \Lambda)$ stands for the Lie algebra of the unipotent radical of $\mathcal{A}\text{ut } \Lambda$, and this Lie algebra may be identified with $\text{nil}_H Z_0(\text{Der } \Lambda)$, the maximal ideal of $Z_0(\text{Der } \Lambda)$ of derivations that act nilpotently on $H = H^*(\Lambda)$.

Proof The proof is entirely analogous to the proof of [Proposition 4.10](#), so we omit most of it. The only thing that requires a different argument is the inclusion $\text{nil Der } \Lambda \subseteq \text{Der}_{\mathfrak{u}} \Lambda$. For this, we again only need to consider the degree zero component. We have that $(\text{nil Der } \Lambda)_0$ is an ideal in $Z_0(\text{Der } \Lambda)$ that acts nilpotently on each graded component $(\text{Der } \Lambda)_k$ for $k > 0$. On the other hand, $(\text{Der}_{\mathfrak{u}} \Lambda)_0$ may be identified with the Lie algebra of the unipotent radical of $\mathcal{A}\text{ut } \Lambda$, so the desired inclusion will follow from [Lemma 2.8](#) as soon as we prove that $(\text{Der } \Lambda)_{\leq n}$ is a faithful representation of $\mathcal{A}\text{ut } \Lambda$ for n sufficiently large.

We are assuming that Λ is finitely generated as an algebra. Fix algebra generators $x_1, \dots, x_k$ and pick n so that $n \geq |x_i|$ for all i . Suppose that $\phi \in \text{Aut } \Lambda$ acts trivially on $(\text{Der } \Lambda)_{\leq n}$. In particular, we then have an equality in $\text{Der } \Lambda$ for every i ,

$$\phi^{-1} \circ \frac{\partial}{\partial x_i} \circ \phi = \frac{\partial}{\partial x_i}.$$

Evaluating on x_j , we get equalities in Λ for all i, j ,

$$\phi^{-1} \left(\frac{\partial \phi(x_j)}{\partial x_i} \right) = \delta_{ij},$$

Applying the algebra automorphism ϕ , we obtain

$$\frac{\partial \phi(x_j)}{\partial x_i} = \delta_{ij}.$$

This can only happen if $\phi(x_i) = x_i$ for all i , which means that $\phi = 1$. □

Corollary 4.16 *If $\text{Aut } \Lambda$ is reductive, then*

$$R(X) = \text{Aut } \Lambda \quad \text{and} \quad \text{nil Der } \Lambda = \text{Der } \Lambda \langle 1 \rangle,$$

and the action of $R(X)$ on $\text{nil Der } \Lambda$ is the conjugation action. □

5 Case studies

In this section we offer a few case studies. In addition to showcasing how the main results can be used in practice, they illustrate certain general points:

- (i) Determining $\Gamma(X)$ typically entails some nontrivial integral homotopy theory, but is often easier than determining $\mathcal{E}(X)$. In many cases, $\Gamma(X)$ is the group of automorphisms of $H_*(X; \mathbb{Z})$ that are induced by self-homotopy equivalences. In the literature, this group often appears as a stepping stone for computing $\mathcal{E}(X)$ or other groups of automorphisms of X .
- (ii) The Lie algebra cohomology $H_{\text{CE}}^*(\mathfrak{g}(X))$ is sometimes explicitly computable, sometimes not. For elliptic spaces, such as products of spheres, one can often compute it explicitly. On the other hand, a complete computation for the manifolds $W_{g,1}$ would entail the computation of the homology of Kontsevich's Lie graph complex, which is a hard problem.
- (iii) Even in cases where $H_{\text{CE}}^*(\mathfrak{g}(X))$ is explicitly computable, a complete calculation of the cohomology $H^*(\Gamma(X), H_{\text{CE}}^*(\mathfrak{g}(X)))$ is in general out of reach, due to the difficulty of computing cohomology of arithmetic groups. However, in some cases the cohomology, or parts of it, can be understood via automorphic forms. A paradigmatic example is the Eichler–Shimura isomorphism.
- (iv) By contrast, all that is left modulo nilpotent elements is the invariant ring

$$H^0(\Gamma(X), H_{\text{CE}}^*(\mathfrak{g}(X))) = H_{\text{CE}}^*(\mathfrak{g}(X))^{\Gamma(X)},$$

and this is more tractable. By employing structural results for affine algebraic groups over $\mathbb{Q}$ and density results [18], this can often be reduced to classical invariant theory for finite or reductive groups, which is well understood.

- (v) For a graded representation H of a reductive group G and an arithmetic subgroup Γ of $G(\mathbb{Q})$, the split exact sequence

$$0 \rightarrow H^G \rightarrow H \rightarrow H/H^G \rightarrow 0$$

gives rise to a split exact sequence

$$0 \rightarrow H^*(\Gamma, \mathbb{Q}) \otimes H^G \rightarrow H^*(\Gamma, H) \rightarrow H^*(\Gamma, H/H^G) \rightarrow 0.$$

In many cases of interest, stability and vanishing results for the cohomology of arithmetic groups with coefficients in nontrivial algebraic representations, as in Borel's work [19; 20], show that the cokernel vanishes in a range of degrees. Thus, in this “stable range” the cohomology of $B\text{aut}(X)$ is isomorphic to

$$H^*(\Gamma(X), \mathbb{Q}) \otimes H_{\text{CE}}^*(\mathfrak{g}(X))^{R(X)}.$$

However, how nontrivial this “stable range” is depends on the group $\Gamma(X)$ and the representations $H_{\text{CE}}^*(\mathfrak{g}(X))$.

5.1 Products of spheres

Consider the n -fold product of a d -dimensional sphere,

$$S^{d \times n} = S^d \times \cdots \times S^d.$$

We begin by describing the reductive group $R(S^{d \times n})$. The minimal Quillen model can be described explicitly (see [67, V.2.(3)]), but the minimal Sullivan model is finitely generated and even easier to describe in this case, so we will work with the latter.

Proposition 5.1 *The automorphism group of the minimal Sullivan model Λ for $S^{d \times n}$ is given by*

$$\mathrm{Aut} \, \Lambda \cong \begin{cases} \mathrm{GL}_n(\mathbb{Q}) & \text{if } d \text{ is odd,} \\ \Sigma_n \ltimes (\mathbb{Q}^\times)^n & \text{if } d \text{ is even.} \end{cases}$$

In particular, $\mathrm{Aut} \, \Lambda$ is reductive, whence $R(S^{d \times n}) \cong \mathrm{Aut} \, \Lambda$.

Proof For d odd, the minimal model Λ is an exterior algebra $\Lambda(x_1, \dots, x_n)$ on generators of degree d with zero differential. Clearly, $\mathrm{Aut} \, \Lambda \cong \mathrm{GL}_n(\mathbb{Q})$. For d even, the minimal model has the form

$$\Lambda = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n), d),$$

with $|x_i| = d$, $|y_i| = 2d - 1$ and $dx_i = 0$, $dy_i = x_i^2$. Let φ be an automorphism of Λ . Then

$$\varphi(x_i) = \sum_j a_{ij} x_j$$

for some $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{Q})$. In cohomology, the equality

$$0 = \varphi(x_i^2) = \varphi(x_i)^2 = \sum_{j < k} 2a_{ij}a_{ik}x_jx_k$$

implies $a_{ij}a_{ik} = 0$ for all $j \neq k$, whence exactly one entry in each row $(a_{i1}, \dots, a_{in})$ must be nonzero. Therefore, $\varphi(x_i) = \lambda_i x_{\sigma(i)}$ for some permutation σ and some $\lambda_i \in \mathbb{Q}^\times$. Since $d\varphi(y_i) = \varphi(dy_i) = \lambda_i^2 x_{\sigma(i)}^2$, the only possibility is $\varphi(y_i) = \lambda_i^2 y_{\sigma(i)}$. This shows that every automorphism φ of Λ is of the form

$$\varphi(x_i) = \lambda_i x_{\sigma(i)} \quad \text{and} \quad \varphi(y_i) = \lambda_i^2 y_{\sigma(i)}$$

for some $\sigma \in \Sigma_n$ and $\lambda_i \in \mathbb{Q}^\times$. One checks that this yields an isomorphism $\mathrm{Aut} \, \Lambda \cong \Sigma_n \ltimes (\mathbb{Q}^\times)^n$. The description of $R(S^{d \times n})$ follows from [Corollary 4.16](#). $\square$

Remark 5.2 The group $\Sigma_n \ltimes (\mathbb{Q}^\times)^n$ may be identified with the group of “monomial matrices”, ie invertible matrices with exactly one nonzero entry in each row. This is an example of a disconnected reductive group. The identity component is the torus $(\mathbb{Q}^\times)^n$ and the group of components is Σ_n .

Remark 5.3 The preceding result, as well as the remainder of this section, goes through even for $d = 1$. The cautious reader will object on the grounds that the space $S^{1 \times n} = (S^1)^n$ is not simply connected. It is, however, nilpotent (indeed a topological group), and therefore amenable to analysis by our methods.

We now turn to the determination of the group $\Gamma(S^{d \times n})$. For this, we first need to work out some elementary homotopy theory of maps between products of spheres.

Definition 5.4 Let us call an integer vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$ *realizable* if there is a map

$$S^d \times \dots \times S^d \rightarrow S^d$$

such that the restriction to the i^{th} factor is a degree a_i self-map of S^d . This is equivalent to asking the n -fold higher-order Whitehead product

$$[a_1 \iota_d, \dots, a_n \iota_d] \subseteq \pi_{nd-1}(S^d)$$

to be defined and contain 0, where $\iota_d \in \pi_d(S^d)$ is the class of the identity map.

We would be surprised if the following has not been observed before, but we have not found a reference (except for the simplest case $n = 2$, which is discussed in eg [2, Example 5.1]), so we supply a proof.

Proposition 5.5 (i) For $d = 1, 3, 7$, every integer vector $(a_1, \dots, a_n)$ is realizable.

(ii) For d odd $\neq 1, 3, 7$, an integer vector $(a_1, \dots, a_n)$ is realizable if and only if at most one a_i is odd.

(iii) For d even, an integer vector $(a_1, \dots, a_n)$ is realizable if and only if at most one a_i is nonzero.

Lemma 5.6 If $(a_1, \dots, a_n)$ is realizable, then so are the vectors

$$(a_{\sigma_1}, \dots, a_{\sigma_k}), \quad (\lambda_1 a_1, \dots, \lambda_n a_n) \quad \text{and} \quad (a_1, \dots, a_n, 0)$$

for all injective maps $\sigma: \{1, \dots, k\} \rightarrow \{1, \dots, n\}$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$.

Proof Precompose the given realizable map $S^d \times \dots \times S^d \rightarrow S^d$ with the map that includes the k -fold product of S^d according to σ and inserts the basepoint in the other factors, or with the map $\lambda_1 \times \dots \times \lambda_n$, or with the projection onto the first n factors, respectively. $\square$

Lemma 5.7 If $(1, a_2, \dots, a_n)$ and $(b_1, \dots, b_n)$ are realizable, then so is

$$(b_1, b_2 + a_2, \dots, b_n + a_n).$$

Proof By hypothesis, the matrices on the left-hand side of the equation

$$\begin{pmatrix} 1 & a_2 & \dots & a_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 + a_2 & \dots & b_n + a_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

can be realized as self-maps of $S^{d \times n}$. It follows that the same is true of the matrix on the right-hand side. In particular, its first row is realizable. $\square$

Lemma 5.8 *If $(1, a)$ is realizable, then so is $(1, a, \dots, a) \in \mathbb{Z}^n$ for every $n \geq 2$.*

Proof Assume by induction that $(1, a, \dots, a) \in \mathbb{Z}^{n-1}$ is realizable. Then both $(1, a, \dots, a, 0) \in \mathbb{Z}^n$ and $(1, 0, \dots, 0, a) \in \mathbb{Z}^n$ are realizable by Lemma 5.6, and hence $(1, a, \dots, a) \in \mathbb{Z}^n$ is realizable by Lemma 5.7. $\square$

Proof of Proposition 5.5 For $d = 1, 3, 7$, the fact that S^d is an H -space means precisely that $(1, 1)$ is realizable. It follows from Lemma 5.8 that $(1, \dots, 1) \in \mathbb{Z}^n$ is realizable and then from Lemma 5.6 that $(a_1, \dots, a_n)$ is realizable for all $a_1, \dots, a_n \in \mathbb{Z}$.

For d odd $\neq 1, 3, 7$, it is well known that the Whitehead product

$$(23) \quad [\iota_d, \iota_d] \in \pi_{2d-1}(S^d)$$

is a nonzero class of order 2 (this can be seen, for instance, by inspecting the EHP sequence). As noted above, (a_1, a_2) is realizable precisely when the Whitehead product $[a_1 \iota_d, a_2 \iota_d]$ is trivial. Since the binary Whitehead product is bilinear, this happens if and only if $a_1 a_2$ is even. By the first part of Lemma 5.6, this implies that $(a_1, \dots, a_n)$ is realizable only if $a_i a_j$ is even for all $i \neq j$, which implies that at most one a_i is odd. Conversely, we have that $(1, 2)$ is realizable since $[\iota_d, 2\iota_d] = 0$. Hence so is $(1, 2, \dots, 2)$ by Lemma 5.8. Now one can use Lemma 5.6 to deduce that $(a_1, \dots, a_n)$ is realizable if at most one of the entries is odd.

For d even, the Whitehead product (23) is a nonzero element of infinite order. As above, this implies that $(a_1, \dots, a_n)$ is realizable only if at most one a_i is nonzero. Conversely, Lemma 5.6 shows that $(a_1, \dots, a_n)$ is realizable if at most one entry is nonzero. $\square$

Now we are ready to compute $\Gamma(S^{d \times n})$.

Proposition 5.9 *We have*

$$\Gamma(S^{d \times n}) \cong \begin{cases} \mathrm{GL}_n(\mathbb{Z}) & \text{if } d = 1, 3, 7, \\ \mathrm{GL}_n^{\Sigma}(\mathbb{Z}) & \text{if } d \text{ is odd and } d \neq 1, 3, 7, \\ \Sigma_n^{\pm} & \text{if } d \text{ is even,} \end{cases}$$

where $\mathrm{GL}_n^{\Sigma}(\mathbb{Z})$ denotes the group of invertible $n \times n$ integer matrices with exactly one odd entry in each row, and $\Sigma_n^{\pm}$ denotes the group of $n \times n$ signed permutation matrices.

Proof By Corollary 4.2, we may identify $\Gamma(S^{d \times n})$ with the group of automorphisms of $H_*(S^{d \times n}; \mathbb{Z})$ that are realizable by a self-homotopy equivalence.

Given $A \in \mathrm{GL}_n(\mathbb{Z})$, it is clear how to write down a map

$$S^d \vee \dots \vee S^d \rightarrow S^d \times \dots \times S^d$$

that realizes A on $H_d(-; \mathbb{Z})$. This extends to the product if and only if the projection to each factor S^d does, which is precisely the condition that each row in A is realizable. When an extension exists, it follows from the Whitehead theorem that it is a homotopy equivalence. Thus, $\Gamma(S^{d \times n})$ may be identified

with the group of invertible $n \times n$ integer matrices in which each row is realizable. To finish the proof, invoke [Proposition 5.5](#). (In the case d odd $\neq 1, 3, 7$, note that invertibility of the matrix implies that at least one entry in each row of must be odd. Similarly, in the case d even, invertibility of the matrix implies that there is a unique nonzero entry in each row and that this must be a unit.) $\square$

Remark 5.10 The group $\mathrm{GL}_n^\Sigma(\mathbb{Z})$ may be identified with the semidirect product,

$$\mathrm{GL}_n^\Sigma(\mathbb{Z}) \cong \Sigma_n \ltimes \mathrm{GL}_n(\mathbb{Z}, 2),$$

where $\mathrm{GL}_n(\mathbb{Z}, 2) \leq \mathrm{GL}_n(\mathbb{Z})$ denotes the principal level 2 congruence subgroup, ie the kernel of the homomorphism $\mathrm{GL}_n(\mathbb{Z}) \rightarrow \mathrm{GL}_n(\mathbb{Z}/2\mathbb{Z})$ that reduces the entries mod 2, and where the symmetric group Σ_n acts by simultaneous permutation of the rows and columns.

The signed permutation group $\Sigma_n^\pm$, a.k.a. the hyperoctahedral group, admits a similar decomposition

$$\Sigma_n^\pm \cong \Sigma_n \ltimes D_n(\mathbb{Z}),$$

where $D_n(\mathbb{Z}) \cong (\mathbb{Z}^\times)^n$ is the group of diagonal matrices in $\mathrm{GL}_n(\mathbb{Z})$.

Remark 5.11 The group of homology isomorphisms that are realizable by a self-homotopy equivalence of $S^{d \times n}$ has also been determined by Basu and Farrell [\[5, Section 2\]](#). The proof given here is simpler because we do not need to argue using generators for the groups involved.

Work of Lucas and Saeki [\[51\]](#) shows that $\Gamma(S^{d \times n})$ also agrees with the group of homology isomorphisms of $S^{d \times n}$ that are realizable by a diffeomorphism.

The group $\Gamma(S^{d \times 2})$ agrees with the group G_d that is used as a stepping stone in the computation of the group of self-homotopy equivalences of $S^d \times S^d$ due to Baues [\[6, Section 6\]](#).

Remark 5.12 If we let $\Gamma^+(S^{d \times n})$ denote the image in $R(S^{d \times n})$ of the orientation-preserving self-homotopy equivalences, then it is easily seen that $\Gamma^+(S^{d \times n}) = \Gamma(S^{d \times n}) \cap \mathrm{SL}_n(\mathbb{Z})$ for d odd, and that $\Gamma^+(S^{d \times n}) \leq \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ is the subgroup of all (σ, λ) such that $\lambda_1 \dots \lambda_n = 1$ for d even.

Next, we determine an algebraic Lie model for $B \mathrm{aut}_u(S^{d \times n})$.

Proposition 5.13 *Let d be odd. The space $B \mathrm{aut}_u(S^{d \times n})$ admits an algebraic Lie model of the form*

$$V_n^*[-d],$$

which is the dual of the standard representation $V_n = \mathbb{Q}^n$ concentrated in homological degree d . The differential and the Lie bracket are trivial.

Proof When d is odd, the Sullivan minimal model of $S^{d \times n}$ is the exterior algebra $\Lambda = \Lambda(x_1, \dots, x_n)$ with zero differential and with x_i in cohomological degree d . By [Proposition 5.1](#), the group $\mathrm{Aut} \Lambda \cong \mathrm{GL}_n(\mathbb{Q})$ is reductive, so by [Corollary 4.16](#), an algebraic Lie model is given by $\mathrm{Der} \Lambda \langle 1 \rangle$, with the conjugation

action of $R(S^{d \times n}) = \text{Aut } \Lambda$. Recall that we are using the convention that cohomological degrees are regarded as negative homological degrees. The only derivations of Λ of positive homological degree are

$$(24) \quad \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}.$$

Thus, $\text{Der } \Lambda \langle 1 \rangle$ is an abelian dg Lie algebra with basis (24) in degree d and zero differential. As a representation of GL_n , it is dual to the standard representation. $\square$

Remark 5.14 The Chevalley–Eilenberg cochain algebra of the abelian Lie algebra $V_n^*[-d]$ may be identified with $\text{Sym}(V_n[d+1])$, the polynomial algebra on the standard $\text{GL}_n(\mathbb{Q})$ -representation $V_n = \mathbb{Q}^n$ concentrated in cohomological degree $d+1$ with trivial differential. Hence, there are isomorphisms of graded algebras of $\Gamma(S^{d \times n})$ -modules,

$$H^*(B \text{aut}_u(S^{d \times n}); \mathbb{Q}) \cong H_{\text{CE}}^*(\mathfrak{g}(S^{d \times n})) \cong \text{Sym}(V_n[d+1]).$$

This can be given a more geometric interpretation. The above implies that the evident map

$$B \text{aut}_u(S^d) \times \dots \times B \text{aut}_u(S^d) \rightarrow B \text{aut}_u(S^d \times \dots \times S^d)$$

is a rational equivalence. This may be interpreted as a splitting principle of sorts: every normal unipotent $S^{d \times n}$ -fibration is rationally equivalent to the “Whitney sum” of n normal unipotent S^d -fibrations. Since $H^*(B \text{aut}_u(S^d); \mathbb{Q})$ is a polynomial ring in the Euler class, we can say that the ring of rational characteristic classes of normal unipotent $S^{d \times n}$ -fibrations is a polynomial ring in the Euler classes $e_1, \dots, e_n$ of the associated S^d -fibrations.

By combining Propositions 5.13 and 5.9 and Corollary 1.4, we obtain:

Theorem 5.15 *For d odd, there is an isomorphism of graded algebras*

$$(25) \quad H^*(B \text{aut}^+(S^{d \times n}); \mathbb{Q}) \cong H^*(\Gamma^+(S^{d \times n}); \text{Sym}^\bullet(V_n[d+1])),$$

where $V_n = \mathbb{Q}^n$ is the standard representation of $\text{GL}_n(\mathbb{Q})$, where

$$\Gamma^+(S^{d \times n}) = \begin{cases} \text{SL}_n(\mathbb{Z}) & \text{if } d = 1, 3, 7, \\ \text{SL}_n^\Sigma(\mathbb{Z}) & \text{if } d \neq 1, 3, 7, \end{cases}$$

and where $\text{SL}_n^\Sigma(\mathbb{Z}) \leq \text{SL}_n(\mathbb{Z})$ is the subgroup of matrices with exactly one odd entry in each row. $\square$

For $n = 2$ the right-hand side of (25) can be computed in terms of modular forms via the Eichler–Shimura isomorphism, as we now will discuss.

Let $V = \mathbb{C}^2$ denote the standard representation of $\text{GL}_2(\mathbb{C})$ and let $\tilde{\Gamma}$ be a congruence subgroup of $\text{GL}_2(\mathbb{Z})$ that contains $-I$ and that strictly contains $\Gamma = \tilde{\Gamma} \cap \text{SL}_2(\mathbb{Z})$, so that we have an exact sequence

$$1 \rightarrow \Gamma \rightarrow \tilde{\Gamma} \xrightarrow{\det} \mathbb{Z}^\times \rightarrow 1.$$

This gives rise to an action of $\mathbb{Z}^\times$ on $H^*(\Gamma, \text{Sym}^\bullet(V))$, ie an involution, such that

$$H^*(\tilde{\Gamma}, \text{Sym}^\bullet(V)) \cong H^*(\Gamma, \text{Sym}^\bullet(V))_+,$$

where $+$ indicates the $+1$ eigenspace of the involution.

Lemma 5.16 *There is an isomorphism of graded vector spaces with involution,*

$$(26) \quad H^*(\Gamma, \text{Sym}^\bullet(V)) \cong \mathbb{C}[0]^+ \oplus M_{\bullet+2}(\Gamma)[1]^- \oplus S_{\bullet+2}(\Gamma)[1]^+,$$

where $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the spaces of modular forms and cusp forms of weight k for Γ , and where $\mathbb{C}[0]$ is $\mathbb{C}$ concentrated in $\bullet = 0$ and $\ast = 0$. A superscript $\pm$ indicates how the involution acts. In particular, extraction of the $+1$ eigenspace yields

$$(27) \quad H^*(\tilde{\Gamma}, \text{Sym}^\bullet(V)) \cong \mathbb{C}[0] \oplus S_{\bullet+2}(\Gamma)[1].$$

Proof The Eichler–Shimura isomorphism gives an isomorphism

$$(28) \quad \text{ES}: M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\cong} H^1(\Gamma, \text{Sym}^{k-2}(V)),$$

where $M_k(\Gamma)$ is the space of modular forms of weight k for Γ and $\overline{S_k(\Gamma)}$ is the space of antiholomorphic cusp forms of weight k ; see eg [70, Section 6]. Identifying the involution on the left-hand side of (28) requires a little care.

The space of modular forms decomposes as $M_k(\Gamma) = E_k(\Gamma) \oplus S_k(\Gamma)$, where $E_k(\Gamma)$ denotes the Eisenstein space. Letting r denote the automorphism of the upper half-plane given by $r(z) = -\bar{z}$, one can check that the composite

$$E_k(\Gamma) \oplus S_k(\Gamma) \oplus S_k(\Gamma) \xrightarrow{\varphi} M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\text{ES}} H^1(\Gamma, \text{Sym}^{k-2}(V)),$$

where

$$\varphi(e, f, g) = (e + f + g, fr - gr)$$

is an isomorphism of vector spaces with involution, where the involution on the left-hand side is given by $(e, f, g) \mapsto (-e, -f, g)$. This explains the isomorphism (26) for $\ast = 1$. Restriction to the $+1$ eigenspace yields an isomorphism

$$S_k(\Gamma) \rightarrow H^1(\Gamma, \text{Sym}^{k-2}(V))_+$$

given by $g \mapsto \text{ES}(g, -gr)$.

For the other cohomological degrees, one checks that

$$H^0(\Gamma, \text{Sym}^{k-2}(V)) = \text{Sym}^{k-2}(V)^\Gamma \cong \mathbb{C},$$

and $H^i(\Gamma, \text{Sym}^{k-2}(V)) = 0$ for $i > 1$, because the virtual cohomological dimension of any finite-index subgroup of $\text{SL}_2(\mathbb{Z})$ is 1. □

Assembly of the above considerations yields:

Theorem 5.17 *For d odd, there is an isomorphism of graded vector spaces with involution*

$$\tilde{H}^*(B \operatorname{aut}^+(S^d \times S^d); \mathbb{Q}) \cong \bigoplus_k (M_k(\Gamma)^- \oplus S_k(\Gamma)^+)[(k-2)(d+1)+1],$$

where $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the spaces of modular forms and cusp forms of weight k for Γ , and where $\Gamma = \operatorname{SL}_2(\mathbb{Z})$ for $d = 1, 3, 7$, and $\Gamma = \operatorname{SL}_2^\Sigma(\mathbb{Z})$ for $d \neq 1, 3, 7$.

In particular, since the reduced cohomology is concentrated in odd degrees, $B \operatorname{aut}^+(S^d \times S^d)$ is formal with trivial cohomology ring. $\square$

Define the Poincaré series of a graded vector space H^* with involution to be the formal power series in z with coefficients in $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$ given by

$$\sum_{k \geq 0} (\dim(H_+^k) + \dim(H_-^k)\epsilon) z^k.$$

There are well-known dimension formulas for spaces of modular forms, so as a corollary we get a formula for the Poincaré series of the cohomology.

Corollary 5.18 *The Poincaré series of the graded vector space with involution*

$$H^*(B \operatorname{aut}^+(S^d \times S^d); \mathbb{Q})$$

is given by the following formulas, where we set $\ell = d + 1$.

For $d = 1, 3, 7$:

$$1 + z^{2\ell+1} \frac{\epsilon(1 + z^{4\ell} - z^{6\ell}) + z^{8\ell}}{(1 - z^{4\ell})(1 - z^{6\ell})}.$$

For d odd $\neq 1, 3, 7$:

$$1 + z \frac{\epsilon(1 + z^{2\ell} - z^{4\ell}) + z^{6\ell}}{(1 - z^{2\ell})(1 - z^{4\ell})}.$$

Proof As is well known, the ring of modular forms $M(\operatorname{SL}_2(\mathbb{Z}))$ is freely generated by the Eisenstein series E_4 and E_6 , and the space of cusp forms $S(\operatorname{SL}_2(\mathbb{Z}))$ is the principal ideal generated by the discriminant Δ , which is of weight 12. In particular, the Poincaré series are given by

$$\sum_{k \geq 0} \dim M_k(\operatorname{SL}_2(\mathbb{Z})) t^k = \frac{1}{(1-t^4)(1-t^6)} \quad \text{and} \quad \sum_{k \geq 0} \dim S_k(\operatorname{SL}_2(\mathbb{Z})) t^k = \frac{t^{12}}{(1-t^4)(1-t^6)}.$$

The group $\operatorname{SL}_2^\Sigma(\mathbb{Z})$ is an index 3 subgroup of $\operatorname{SL}_2(\mathbb{Z})$, sometimes referred to as the “theta group”. It is generated by the two matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The Poincaré series of the modular forms and the cusp forms for the theta group are

$$\sum_{k \geq 0} \dim M_k(\operatorname{SL}_2^\Sigma(\mathbb{Z})) t^k = \frac{1}{(1-t^2)(1-t^4)} \quad \text{and} \quad \sum_{k \geq 0} \dim S_k(\operatorname{SL}_2^\Sigma(\mathbb{Z})) t^k = \frac{t^8}{(1-t^2)(1-t^4)}.$$

This can be seen from the dimension formulas in [47, Proposition 1], or alternatively by observing that the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ conjugates $\mathrm{SL}_2^{\Sigma}(\mathbb{Z})$ to the congruence subgroup $\Gamma_0(2) = \Gamma_1(2)$; dimension formulas for the latter can be found in [29, page 108]. Theorem 5.17 together with the above formulas yield the desired result after some manipulations with generating functions. $\square$

Remark 5.19 Computations of $H^*(B \mathrm{Diff}^+(T^2); \mathbb{Q})$ have been carried out in [56] and [36] (the latter using the Eichler–Shimura isomorphism). The inclusion of $\mathrm{Diff}^+(T^2)$ into $\mathrm{aut}^+(T^2)$ is a weak homotopy equivalence, so we recover this computation by setting $d = 1$ and $\epsilon = 1$ in the above.

The cohomology of $\mathrm{GL}_3(\mathbb{Z})$ and $\mathrm{SL}_3(\mathbb{Z})$ with coefficients in irreducible algebraic representations has recently been computed in many cases [1]. Let $V = \mathbb{C}^3$ be the standard representation of $\mathrm{GL}_3(\mathbb{C})$. The combination of [1, Corollary 18] and [1, Theorem 16] specialized to $\mathcal{M}_{k,0} = \mathrm{Sym}^k(V)$ (which is not self-dual for $k > 0$) shows

$$H^q(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Sym}^k(V)) \cong \begin{cases} S_{k+2} & \text{for } q = 3 \text{ and } k > 0 \text{ even,} \\ S_{k+3} \oplus \mathbb{C} & \text{for } q = 2 \text{ and } k \text{ odd,} \\ \mathbb{C} & \text{for } q = 0 \text{ and } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where S_k denotes the space of cusp forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$. If we take the liberty of writing M_{k+3} for the isomorphic vector space $S_{k+3} \oplus \mathbb{C}$ in the second case above, we can summarize the calculation as an isomorphism of bigraded vector spaces with involution:

$$(29) \quad H^*(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Sym}^{\bullet}(V)) \cong \mathbb{C}[0]^+ \oplus M_{\bullet+3}[2]^- \oplus S_{\bullet+2}[3]^+.$$

The action of the involution on the right-hand side is implicit in [1, Lemma 17]; this lemma asserts that

$$(30) \quad H^*(\mathrm{GL}_3(\mathbb{Z}), \mathrm{Sym}^{\bullet}(V)) \cong \mathbb{C}[0] \oplus S_{\bullet+2}[3].$$

As before, this leads to:

Theorem 5.20 For $d = 1, 3, 7$, there is an isomorphism of graded vector spaces with involution

$$\tilde{H}^*(B \mathrm{aut}^+(S^{d \times 3}); \mathbb{Q}) \cong \bigoplus_k M_k^-[(k-3)(d+1)+2] \oplus S_k^+[(k-2)(d+1)+3],$$

and the Poincaré series is given by

$$z^{\ell+2} \frac{\epsilon(1+z^{2\ell}-z^{6\ell})+z^{9\ell+1}}{(1-z^{4\ell})(1-z^{6\ell})}.$$

$\square$

Remark 5.21 It seems plausible that computations similar to those of [1] can be carried out for the group $\mathrm{SL}_3^{\Sigma}(\mathbb{Z})$, but this is beyond the scope of this paper. However, even if we currently lack complete calculations, certain qualitative conclusions can be drawn. By Theorem 3.39, the cdga $\Omega^*(B \mathrm{aut}(X))$ is quasi-isomorphic to

$$\Omega^*(\Gamma(X), C_{\mathrm{CE}}^*(\mathfrak{g}(X))).$$

If $C_{\text{CE}}^*(\mathfrak{g}(X))$ is formal as a cdga in $\text{Rep}_{\mathbb{Q}}(R(X))$, then the above is quasi-isomorphic to

$$\Omega^*(\Gamma(X); H_{\text{CE}}^*(\mathfrak{g}(X)))$$

as a cdga. An application of the homotopy transfer theorem for C_{∞} -algebras yields a C_{∞} -algebra structure $\{m_n\}$ on $H^*(\Gamma(X); H_{\text{CE}}^*(\mathfrak{g}(X)))$ that is bigraded in the sense that m_n has bidegree $(2 - n, 0)$, and that is C_{∞} -quasi-isomorphic to $\Omega^*(B\text{aut}(X))$.

If the cohomology of $\Gamma(X)$ with coefficients in algebraic representations is concentrated in a small range of degrees, this can force these C_{∞} -operations to be trivial, implying that the cdga $\Omega^*(B\text{aut}(X))$ is formal. For example, this happens if there is an r such that $\tilde{H}^i(\Gamma(X), V) = 0$ unless $r \leq i \leq 3r - 2$, for all $V \in \text{Rep}_{\mathbb{Q}}(R(X))$.

For example, if $\Gamma \leq \text{SL}_3(\mathbb{Z})$ is a finite-index subgroup, then $H^1(\Gamma, V) = 0$ for any finite-dimensional $\mathbb{Q}$ -vector space V with an action of Γ by [4], and $H^i(\Gamma, V) = 0$ for $i > 3$, because Γ has virtual cohomological dimension 3.

These considerations lead to:

Theorem 5.22 *For d odd, the space $B\text{aut}^+(S^d \times S^d \times S^d)$ is formal and the rational cohomology has trivial ring structure.* $\square$

Another remark is that the invariant ring

$$H^*(B\text{aut}_u(S^{d \times n}); \mathbb{Q})^{\Gamma(S^{d \times n})}$$

is the trivial ring $\mathbb{Q}$ in this case, because $\text{Sym}^k(V_n)^{\Gamma(S^{d \times n})}$ is easily seen to be trivial. In particular, $H^*(B\text{aut}(S^{d \times n}); \mathbb{Q})$ consists entirely of nilpotent elements.

For $n > 3$, calculations become increasingly difficult, but one can at least say something about the limit as $n \rightarrow \infty$. Consider the maps

$$\sigma: B\text{aut}(S^{d \times n}) \rightarrow B\text{aut}(S^{d \times (n+1)}) \quad \text{and} \quad \pi: B\text{aut}(S^{d \times n}) \rightarrow B\Gamma(S^{d \times n}),$$

where σ is induced by extending a self-homotopy equivalence of $S^{d \times n}$ by the identity on the second factor of $S^{d \times (n+1)} = S^{d \times n} \times S^d$, and π is the evident map. It is a consequence of Borel's results [19; 20] on the stable cohomology of arithmetic groups that these maps induce an isomorphism in $H^i(-; \mathbb{Q})$ for $n \gg i$. The explicit ranges stated below rely on more recent results due to Kupers, Miller and Patzt [48] and Putman [58].

Theorem 5.23 *Let d be odd. The map*

$$\sigma^*: H^i(B\text{aut}(S^{d \times (n+1)}); \mathbb{Q}) \rightarrow H^i(B\text{aut}(S^{d \times n}); \mathbb{Q})$$

is an isomorphism for $n \geq i + 1$ if $d = 1, 3, 7$, and for $n \geq 2i + 6$ for all odd d . The map

$$\pi^*: H^i(\Gamma(S^{d \times n}), \mathbb{Q}) \rightarrow H^i(B\text{aut}(S^{d \times n}); \mathbb{Q}).$$

is an isomorphism for $n \geq i - d + 1$ if $d = 1, 3, 7$, and for $n \geq 2i - 2d + 6$ for all odd d .

Proof By Theorem 5.15, the map π^* is injective and its cokernel is isomorphic to

$$(31) \quad \bigoplus_{\substack{j+k(d+1)=i \\ k>0}} H^j(\Gamma(S^{d \times n}), \text{Sym}^k(V_n)).$$

By Borel's vanishing theorem [20, Theorem 4.4], the summands vanish for n large. Explicitly, [48, Theorem 7.6] implies vanishing for $n \geq j + k + 1$ for $d = 1, 3, 7$, and [58, Theorem C] implies vanishing for $n \geq 2j + 2k + 6$ for $d \neq 1, 3, 7$. It follows that (31) vanishes for $n \geq i - d + 1$ for $d = 1, 3, 7$ and for $n \geq 2i - 2d + 6$ for $d \neq 1, 3, 7$.

The statement about σ^* now follows from the corresponding statement for

$$H^i(\Gamma(S^{d \times (n+1)}), \mathbb{Q}) \rightarrow H^i(\Gamma(S^{d \times n}), \mathbb{Q}).$$

By Borel's work [19] this is an isomorphism for n large. According to [48, Theorem A] it is an isomorphism for $n \geq i + 1$ for $d = 1, 3, 7$, and [58, Theorem C] implies it is an isomorphism for $n \geq 2i + 6$ for $d \neq 1, 3, 7$. $\square$

By Borel's calculation [19], the stable cohomology may be identified with an exterior algebra

$$\varprojlim_n H^*(B\text{aut}(S^{d \times n}); \mathbb{Q}) \cong \varprojlim_n H^*(\Gamma(S^{d \times n}), \mathbb{Q}) \cong \Lambda[x_5, x_9, x_{13}, \dots].$$

We now turn to the case d even, which behaves very differently.

Proposition 5.24 *For d even, the dg Lie algebra $\mathfrak{g}(S^{d \times n})$ is formal with homology the abelian Lie algebra with basis*

$$\alpha_1, \dots, \alpha_n, \quad \beta_{ij} \quad \text{for } 1 \leq i \neq j \leq n,$$

in degrees $|\alpha_i| = 2d - 1$ and $|\beta_{ij}| = d - 1$. The action of

$$(\sigma, \lambda) \in R(S^{d \times n}) = \Sigma_n \ltimes (\mathbb{Q}^\times)^n$$

is given by

$$(\sigma, \lambda) \cdot \alpha_i = \lambda_i^{-2} \alpha_{\sigma(i)}, \quad (\sigma, \lambda) \cdot \beta_{ij} = \lambda_i \lambda_j^{-2} \beta_{\sigma(i)\sigma(j)}.$$

Proof Let Λ denote the minimal model. Since $\text{Aut } \Lambda$ is reductive, the positive truncation $\text{Der } \Lambda \langle 1 \rangle$ is a Lie model for $B\text{aut}_u(X)$ by Corollary 4.16. This is spanned by derivations of the form

$$\frac{\partial}{\partial x_i}, \quad x_i \frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial y_i},$$

and the only nontrivial differential is given by

$$\left[d, \frac{\partial}{\partial x_i} \right] = -2x_i \frac{\partial}{\partial y_i}.$$

The action of the group $\text{Aut } \Lambda$ is easily computed, eg if φ corresponds to $(\sigma, \lambda) \in \Sigma_n \ltimes (\mathbb{Q}^\times)^n$, then

$$\varphi \left(x_i \frac{\partial}{\partial y_j} \right) = \lambda_i \lambda_j^{-2} x_{\sigma(i)} \frac{\partial}{\partial y_{\sigma(j)}}.$$

A basis for the homology is represented by the cycles $\alpha_i = \partial/\partial y_i$ and $\beta_{ij} = x_i \partial/\partial y_i$ for $i \neq j$. These span an abelian dg Lie subalgebra stable under the action of $\text{Aut } \Lambda$, and the inclusion into $\text{Der } \Lambda \langle 1 \rangle$ is a quasi-isomorphism. $\square$

Corollary 5.25 *For d even, the space $B \text{aut}(S^{d \times n})$ is formal and there is an isomorphism of graded algebras*

$$H^*(B \text{aut}(S^{d \times n}); \mathbb{Q}) \cong \mathbb{Q}[a_i, b_{ij}]^{\Gamma(S^{d \times n})},$$

where $|a_i| = 2d$, $|b_{ij}| = d$ and $(\sigma, \lambda) \in \Gamma(S^{d \times n}) = \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ acts by

$$(\sigma, \lambda) \cdot a_i = a_{\sigma(i)} \quad \text{and} \quad (\sigma, \lambda) \cdot b_{ij} = \lambda_i b_{\sigma(i)\sigma(j)}.$$

Similarly, the space $B \text{aut}^+(S^{d \times n})$ is formal and there is an isomorphism of algebras

$$H^*(B \text{aut}^+(S^{d \times n}); \mathbb{Q}) \cong \mathbb{Q}[a_i, b_{ij}]^{\Gamma^+(S^{d \times n})},$$

where $\Gamma^+(S^{d \times n}) \leq \Sigma_n \ltimes (\mathbb{Z}^\times)^n$ is the subgroup consisting of all (σ, λ) such that $\lambda_1 \dots \lambda_n = 1$.

Proof Clearly, $C_{\text{CE}}^*(\mathfrak{g}(S^{d \times n})) = \mathbb{Q}[a_i, b_{ij}]$, where a_i and b_{ij} are the dual 1-cochains of α_i and β_{ij} , and the differential is trivial since $\mathfrak{g}(S^{d \times n})$ is abelian with trivial differential. In particular, $C_{\text{CE}}^*(\mathfrak{g}(S^{d \times n}))$ is formal, so [Remark 5.21](#) shows that the C_∞ -algebra structure on $H^*(\Gamma(S^{d \times n}); H_{\text{CE}}^*(\mathfrak{g}(S^{d \times n})))$ respects the bigrading. But since the group $\Gamma(S^{d \times n})$ is finite, the cohomology reduces to the invariants concentrated in bidegree $(0, *)$, whence the C_∞ -operations m_n vanish for $n > 2$. $\square$

The computation of invariant subrings $\mathbb{Q}[V]^G$ for finite-dimensional representations V of finite or reductive groups G is classical and well-understood in principle; see eg [\[28\]](#). The invariant subring is a Cohen–Macaulay ring, which means that one can find a regular sequence $\theta_1, \dots, \theta_m$ of invariant polynomials (the “primary invariants”) such that $\mathbb{Q}[V]^G$ is a finitely generated free module over $\mathbb{Q}[\theta_1, \dots, \theta_m]$ on certain invariant polynomials $\eta_1, \dots, \eta_t$ (the “secondary invariants”).

Corollary 5.26 *For d even, $H^*(B \text{aut}^+(S^{d \times n}); \mathbb{Q})$ is a Cohen–Macaulay ring of Krull dimension n^2 , concentrated in degrees that are multiples of d .* $\square$

Let us examine the case $n = 2$ closer to illustrate. The invariant theory calculation can be carried out in two steps using

$$\mathbb{Q}[a_1, a_2, b_{12}, b_{21}]^{\Gamma(S^{d \times 2})} = (\mathbb{Q}[a_1, a_2, b_{12}, b_{21}]^{(\mathbb{Z}^\times)^2})^{\Sigma_2}.$$

The invariant ring with respect to the action of $(\mathbb{Z}^\times)^2$ is easily seen to be a polynomial ring in $a = a_1$, $b = b_{12}^2$, $c = b_{21}^2$, $d = a_2$, so we are left to identify the invariant ring

$$\mathbb{Q}[a, b, c, d]^{\Sigma_2},$$

where the nontrivial element of Σ_2 acts as the permutation $(ad)(bc)$. By using Molien’s theorem (specialized to permutation representations as in [\[64, Proposition 4.3.4\]](#)), the Poincaré series with respect to word-length in the generators can be computed to be

$$\frac{1 + t^2}{(1 - t)^2(1 - t^2)^2}.$$

The invariant polynomials

$$a + d, \quad b + c, \quad ad, \quad bc,$$

form a regular sequence of length equal to the Krull dimension, so these form a set of primary invariants. A choice of secondary invariants is given by the two polynomials

$$1, \quad ab + cd.$$

The ring structure is determined by writing $(ab + cd)^2$ in the basis; one has

$$(ab + cd)^2 = ((a + d)(b + c)) \cdot (ab + cd) - ((a + d)^2 bc + ad(b + c)^2 - 4(ad)(bc)) \cdot 1.$$

The calculation for $\Gamma^+(X_2)$ is similar. We omit the details; the only difference is that there is one additional invariant $\alpha_0 = b_{12}b_{21}$, which squares to $bc = b_{12}^2 b_{21}^2$. Writing $\alpha_1, \alpha_2, \beta_1, \beta_2, \eta$ for $a + d, b + c, ad, bc, ab + cd$, respectively, these considerations lead to:

Theorem 5.27 *For d even, there is an isomorphism of graded algebras with involution*

$$H^*(B \operatorname{aut}^+(S^d \times S^d); \mathbb{Q}) \cong \mathbb{Q}[\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, \eta]/I,$$

where the generators $\alpha_0, \alpha_1, \alpha_2$ are of degree $2d$, and β_1, β_2, η are of degree $4d$, where I is the ideal generated by the two elements

$$\eta^2 - \alpha_1 \alpha_2 \eta + (\alpha_1^2 \beta_2 + \beta_1 \alpha_2^2 - 4\beta_1 \beta_2) \quad \text{and} \quad \alpha_0^2 - \beta_2,$$

and where the involution acts by $\alpha_0 \mapsto -\alpha_0$ and trivially on the other generators. The Poincaré series of this graded vector space with involution is given by

$$(1 + \epsilon z^{2d}) \frac{1 + z^{4d}}{(1 - z^{2d})^2 (1 - z^{4d})^2}.$$

A presentation for the cohomology ring of $B \operatorname{aut}(S^d \times S^d)$ is obtained by removing the generator α_0 and the relation $\alpha_0^2 - \beta_2$ from the above presentation. The Poincaré series is obtained by removing the term involving ϵ . $\square$

The case $n > 2$ can in principle be treated similarly, but we stop here.

5.2 Highly connected even-dimensional manifolds

The rational homotopy theory of the topological monoid of self-homotopy equivalences of highly connected even-dimensional manifolds has been thoroughly studied in [16, Section 5]. We will now discuss how the methods of the present paper lead to simplifications of certain arguments in [16], as well as to some new results.

Let M be an $(n-1)$ -connected $2n$ -dimensional manifold with $n > 1$. We will consider the space $\operatorname{aut}^+(M)$ of orientation-preserving self-homotopy equivalences. Let $\Gamma^+(M)$ denote the image of $\mathcal{E}^+(M) = \pi_0 \operatorname{aut}^+(M)$ in $R(M)$, and let $\operatorname{Aut}^+ \mathbb{L}$ denote the group of automorphisms of the minimal Quillen model $\mathbb{L}$ of M that represent orientation-preserving self-homotopy equivalences.

As discussed in [16, Section 3.5], the minimal Quillen model of M can be presented as

$$\mathbb{L} = (\mathbb{L}(\alpha_1, \dots, \alpha_r, \gamma), \delta\gamma = \omega),$$

where $\alpha_1, \dots, \alpha_r$ is a basis for $H_n(M; \mathbb{Q})[1-n]$, where γ corresponds to the fundamental class of M , and where ω is dual to the cup product pairing

$$\langle \cdot, \cdot \rangle: H^n(M; \mathbb{Q}) \otimes H^n(M; \mathbb{Q}) \rightarrow \mathbb{Q}, \quad \langle x, y \rangle = \langle x \smile y, [M] \rangle.$$

Explicitly,

$$\omega = \frac{1}{2} \sum_i [\alpha_i^\#, \alpha_i],$$

where $\alpha_i^\#$ is the dual basis with respect to the intersection form.

Proposition 5.28 *Let M be an $(n-1)$ -connected $2n$ -dimensional manifold, where $n > 1$. The group $\text{Aut}^+ \mathbb{L}$ is isomorphic to the group of automorphisms of $H^n(M; \mathbb{Q})$ that preserve the cup product pairing. In particular, $\text{Aut}^+ \mathbb{L}$ is reductive, whence $\text{Aut}^+ \mathbb{L} \cong \mathcal{E}^+(M_{\mathbb{Q}}) \cong R^+(M)$.*

Proof An automorphism of $\mathbb{L}$ represents an orientation-preserving self-homotopy equivalence if and only if it fixes γ . Using this, one sees that $\text{Aut}^+ \mathbb{L}$ is isomorphic to the group of automorphisms of the space spanned by $\alpha_1, \dots, \alpha_r$ that fix ω . Since ω is dual to the cup product pairing, this is in turn isomorphic to $\text{Aut}(H^n(M; \mathbb{Q}), \langle \cdot, \cdot \rangle)$. The group of automorphisms of a nondegenerate symmetric or skew-symmetric bilinear form is well known to be reductive. $\square$

Corollary 4.2 implies that the group $\Gamma^+(M)$ may be identified with the group of automorphisms of $H_*(M; \mathbb{Z})$ that are induced by an orientation-preserving homotopy equivalence of M . This group is known; see [6, Theorem 8.14], [15, Theorem 2.12] or [16, Section 5.1]. To describe it, recall the cohomology operation $\psi: H^n(M) \rightarrow \pi_{2n-1}(S^n)$ defined by Kervaire–Milnor [46, Section 8]; the class $\psi(x)$ is the obstruction for the existence of a map $f: M \rightarrow S^n$ such that $f^*(s) = x$, where s is a generator for $H^n(S^n)$.

Proposition 5.29 *Let M be an $(n-1)$ -connected $2n$ -dimensional manifold, where $n > 1$. The group $\Gamma^+(M)$ may be identified with the group of automorphisms of $H^n(M)$ that preserve the cup product pairing*

$$\langle -, - \rangle: H^n(M) \otimes H^n(M) \rightarrow \mathbb{Z}$$

and the Kervaire–Milnor cohomology operation

$$\psi: H^n(M) \rightarrow \pi_{2n-1}(S^n).$$

$\square$

For example, for the manifold $W_g = \#^g S^n \times S^n$, the group $\Gamma^+(W_g)$ coincides with the group denoted by Γ_g in [16]. As explained in [16, Example 5.5] it may be described by

$$\Gamma_g \cong \begin{cases} \text{O}_{g,g}(\mathbb{Z}) & \text{for } n \text{ even,} \\ \text{Sp}_{2g}(\mathbb{Z}) & \text{for } n = 1, 3, 7, \\ \text{Sp}_{2g}^q(\mathbb{Z}) & \text{for } n \text{ odd } \neq 1, 3, 7, \end{cases}$$

where $\mathrm{Sp}_{2g}^q(\mathbb{Z})$ denotes the group of block matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$$

such that the diagonal entries of the $g \times g$ matrices $C^t A$ and $D^t B$ are even.

The next result should be viewed as an upgrade of [16, Theorem 5.8], which asserts that the Lie model for the space $B \mathrm{aut}_o(M)$ is formal. The novelty is that the formality can be made equivariant.

Proposition 5.30 *Let M be an $(n-1)$ -connected $2n$ -dimensional manifold, where $n > 1$, such that $\dim H^n(M; \mathbb{Q}) > 2$. The algebraic Lie model $\mathfrak{g}(M)$ for $B \mathrm{aut}_u^+(M)$ is formal as a dg Lie algebra of algebraic representations of $R^+(M)$, and its homology is the graded Lie algebra*

$$H_*(\mathfrak{g}(M)) \cong \mathrm{Der} L / \mathrm{ad} L \langle 1 \rangle,$$

where $L = \pi_*(\Omega M) \otimes \mathbb{Q}$ is the rational homotopy Lie algebra of M . This Lie algebra admits the presentation

$$L \cong \mathbb{L}(\alpha_1, \dots, \alpha_r) / (\omega),$$

with $\alpha_1, \dots, \alpha_r$ and ω as above. The action of $R^+(M)$ is induced by the action on $H_*(M; \mathbb{Q})$.

Proof Since $\mathrm{Aut}^+ \mathbb{L}$ is reductive, we can show that $\mathrm{Der}^c \mathbb{L} \langle 1 \rangle$ is an $R^+(M)$ -algebraic Lie model for $B \mathrm{aut}_u^+(M)$ as in Corollary 4.11. Given this, it is straightforward to inspect that the zig-zag of quasi-isomorphisms that connects this dg Lie algebra with its homology given in the proof of [16, Theorem 5.9] can be made into a zig-zag of algebraic $R^+(M)$ -representations. $\square$

The action of $\Gamma^+(M)$ on the rational homotopy groups was also identified in [16], but the methods of [16] were insufficient for proving the following result, which is a direct consequence of Corollary 1.4 and Proposition 5.30.

Theorem 5.31 *Let M be an $(n-1)$ -connected $2n$ -dimensional manifold, with $n > 1$, and suppose that $\dim H^n(M; \mathbb{Q}) > 2$. There is an isomorphism of graded algebras*

$$H^*(B \mathrm{aut}^+(M); \mathbb{Q}) \cong H^*(\Gamma^+(M), H_{\mathrm{CE}}^*(\mathfrak{g})),$$

where $\Gamma^+(M)$ is the group $\mathrm{Aut}(H^n(M; \mathbb{Z}), \langle \cdot, \cdot \rangle, \psi)$, $\mathfrak{g}$ is $\mathrm{Der} L / \mathrm{ad} L \langle 1 \rangle$, and L is the graded Lie algebra $\pi_*(\Omega M) \otimes \mathbb{Q}$. $\square$

When adapted to the space $\mathrm{aut}_\partial(W_{g,1})$ of self-homotopy equivalences of

$$W_{g,1} = W_g \setminus \mathrm{int} D^{2n}$$

relative to the boundary, our methods yield a significant simplification of the computation of the stable cohomology of $B \mathrm{aut}_\partial(W_{g,1})$ of [16]. By applying our methods to the homotopy fiber sequence

$$(32) \quad \Gamma_g // \mathrm{aut}_\partial(W_{g,1}) \rightarrow B \mathrm{aut}_\partial(W_{g,1}) \rightarrow B \Gamma_g,$$

one can upgrade [16, Theorem 3.12] to show that the Γ_g -space $\Gamma_g // \text{aut}_\partial(W_{g,1})$ has R_g -algebraic Lie model

$$\mathfrak{g}_g = \text{Der}_\omega \mathbb{L}\langle 1 \rangle,$$

where $\text{Der}_\omega \mathbb{L}$ denotes the graded Lie algebra of derivations of the free graded Lie algebra $\mathbb{L} = \mathbb{L}(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ that annihilate the element

$$\omega = [\alpha_1, \beta_1] + \dots + [\alpha_g, \beta_g].$$

As above, this implies:

Theorem 5.32 *There is an isomorphism of graded algebras*

$$(33) \quad H^*(B \text{aut}_\partial(W_{g,1}); \mathbb{Q}) \cong H^*(\Gamma_g, H_{\text{CE}}^*(\mathfrak{g}_g)). \quad \square$$

Remark 5.33 In the stable range, ie for g large compared to the cohomological degree, the right-hand side can be simplified. Let $H = H_{\text{CE}}^*(\mathfrak{g}_g)$. By semisimplicity of $\text{Rep}_{\mathbb{Q}}(R_g)$ there is a split exact sequence

$$0 \rightarrow H^{R_g} \rightarrow H \rightarrow H/H^{R_g} \rightarrow 0.$$

This gives rise to a ring homomorphism

$$H^*(\Gamma_g, \mathbb{Q}) \otimes H^{R_g} \cong H^*(\Gamma_g, H^{R_g}) \rightarrow H^*(\Gamma_g, H),$$

whose cokernel may be identified with

$$H^*(\Gamma_g, H/H^{R_g}).$$

Since H/H^{R_g} is a direct sum of nontrivial irreducible algebraic representations of R_g , the cokernel vanishes in degrees that are small compared to g by Borel's vanishing theorem [20, Theorem 4.4]. Note that this entails the statement that $H^{R_g} = H^{\Gamma_g}$ for g large (set $*$ = 0). Thus, in the stable range, we obtain an isomorphism

$$H^*(B \text{aut}_\partial(W_{g,1}); \mathbb{Q}) \cong H^*(\Gamma_g, \mathbb{Q}) \otimes H^{\Gamma_g}.$$

This essentially recovers Theorem 1.3 of [16].

Remark 5.34 The isomorphism (33), and in particular the collapse of the spectral sequence of (32), is only obtained in the stable range (cohomological degrees below $g/2 - 2$) in [16]. Moreover, the argument in [16] depends on several extraneous ingredients. One step in the argument is to show that the map

$$B \text{aut}_\partial(W_{g,1}) \rightarrow B\Gamma_g$$

is injective on indecomposables in rational cohomology in the stable range [16, Theorem 8.6]. This is proved by establishing the stronger statement [16, Theorem 8.2] that

$$B \text{Diff}_\partial(W_{g,1}) \rightarrow B\Gamma_g$$

is injective on indecomposables in stable rational cohomology, which in turn relies on deep results on the stable cohomology of $B \text{Diff}_\partial(W_{g,1})$ due to Galatius and Randal-Williams [37], as well as on nonvanishing

results for the coefficients of the Hirzebruch L -polynomials due to Berglund and Bergström [14]. Our proof of Theorem 5.32 does not depend on these extraneous ingredients. In fact, Theorem 5.32 implies a strengthening of [16, Theorem 8.6]; see Corollary 5.35 below.

Furthermore, many arguments of [16] rely on deep results on almost algebraicity of finite-dimensional representations of Γ_g due to Bass, Milnor and Serre [4, Section 16]; see also [62, Section 1.3(9)] and [16, Appendix A]. The existence of algebraic Lie models shows that the representations in question are in fact algebraic, which in particular removes the necessity of invoking such results.

Corollary 5.35 *The ring homomorphism*

$$H^*(\Gamma_g, \mathbb{Q}) \rightarrow H^*(B \operatorname{aut}_{\partial}(W_{g,1}); \mathbb{Q})$$

is split injective for all g . In particular, the induced map on indecomposables

$$QH^*(\Gamma_g, \mathbb{Q}) \rightarrow QH^*(B \operatorname{aut}_{\partial}(W_{g,1}); \mathbb{Q})$$

is injective for all g . □

5.3 Highly connected odd-dimensional manifolds

The geometry and rational homotopy theory of $(n-1)$ -connected $(2n+1)$ -dimensional manifolds has also been thoroughly studied (eg in Wall's classification [69]), and we make some remarks on the odd-dimensional case in this section.

Let M be an $(n-1)$ -connected $(2n+1)$ -dimensional manifold, where $n > 8$. Using the techniques of [16, Section 3.5], one can show that the minimal Quillen model of M is given by

$$\mathbb{L} = (\mathbb{L}(\alpha_1, \dots, \alpha_r, \alpha_1^{\#}, \dots, \alpha_r^{\#}, \gamma), \delta\gamma = \omega),$$

where the α_i are a basis for $s^{-1}\tilde{H}_n(M; \mathbb{Q})$, the $\alpha_i^{\#}$ are the dual basis of the space $s^{-1}\tilde{H}_{n+1}(M; \mathbb{Q})$ with respect to the intersection form, and

$$\omega = \sum_i [\alpha_i^{\#}, \alpha_i]$$

is dual to the intersection form. (So up to rational equivalence, such manifolds are completely classified by the torsion-free rank of $H_n(M)$.) We use the notation of the preceding section freely.

Lemma 5.36 *Let M be an $(n-1)$ -connected $(2n+1)$ -dimensional manifold with $n > 1$. The group $\operatorname{Aut}^+ \mathbb{L}$ is isomorphic to the group $\operatorname{GL}(H^n(M; \mathbb{Q}))$ of linear automorphisms of $H^n(M; \mathbb{Q})$. In particular, it is reductive, wherefore $\operatorname{Aut}^+ \mathbb{L} \cong \mathcal{E}^+(M_{\mathbb{Q}}) \cong R^+(M)$.*

Proof An automorphism of $\mathbb{L}$ is orientation-preserving precisely when it fixes γ . Thus an automorphism $\phi \in \operatorname{Aut}^+ \mathbb{L}$ is uniquely determined by its action on $\mathbb{L}_{n-1} = \langle \alpha_1, \dots, \alpha_r \rangle$: it preserves the element $\omega = \delta\gamma$, so its action on the $\alpha_i^{\#}$ is ω -dual to the action on the α_i . □

The identification of the group $\Gamma^+(M)$ is somewhat more involved than in the preceding case, and we will not elaborate on it here. We remark that the closely related group

$$\Gamma_{\mathbb{Z}}^+(M) = \text{Aut}^{\mathcal{E}^+(M)}(H_*(M; \mathbb{Z}))$$

of orientation-preserving automorphisms of *integral* homology that are realizable by homotopy automorphisms of M , which has been studied by several authors, provides a ready substitute for $\Gamma^+(M)$ in our results: the space $B \text{aut}^+(M)$ is rationally equivalent to the homotopy orbit space $(\mathfrak{g}(M))_{\text{h}\Gamma_{\mathbb{Z}}^+(M)}$. The group $\Gamma_{\mathbb{Z}}^+(M)$ has been determined in many cases by Floer [35] in the course of his classification of the homotopy types of $(n-1)$ -connected $(2n+1)$ -dimensional Poincaré complexes. See also Barden [3] and Baues and Buth [7] for the case $n = 2$, and Crowley and Nordström [27] for the case $n = 3$.

Just like in the case of even-dimensional manifolds, $\mathfrak{g}(M)$ enjoys a close connection to the derivations of the homotopy Lie algebra $\pi_*(\Omega M) \otimes \mathbb{Q}$. The following proposition is an easy consequence of the methods of [16, Section 5.4], especially Theorem 5.9 therein.

Proposition 5.37 *Let M be an $(n-1)$ -connected $(2n+1)$ -dimensional manifold, with $n > 1$, such that $\dim H^n(M; \mathbb{Q}) > 1$. The algebraic Lie model $\mathfrak{g}(M)$ for the space $B \text{aut}_u(M)$ is formal as a dg Lie algebra of algebraic representations of $R(M)$ and its homology is the graded Lie algebra*

$$H_*(\mathfrak{g}(M)) \cong \text{Der } L / \text{ad } L \langle 1 \rangle$$

where $L = \pi_*(\Omega M) \otimes \mathbb{Q}$ is the rational homotopy Lie algebra of M , which admits the presentation

$$L \cong \mathbb{L}(\alpha_1, \dots, \alpha_r, \alpha_1^\#, \dots, \alpha_r^\#) / (\omega)$$

with $\alpha_i, \alpha_i^\#$ and ω as above. The action of $R(M)$ is induced by the action on $H_*(M; \mathbb{Q})$.

Thus, we obtain the following consequence of Corollary 1.4.

Theorem 5.38 *Let M be an $(n-1)$ -connected $(2n+1)$ -dimensional manifold with $\dim H^n(M; \mathbb{Q}) > 1$. There is an isomorphism of graded algebras*

$$H^*(B \text{aut}^+(M); \mathbb{Q}) \cong H^*(\Gamma_{\mathbb{Z}}^+(M), H_{\text{CE}}^*(\mathfrak{g}(M))),$$

where $\mathfrak{g} = \text{Der } L / \text{ad } L \langle 1 \rangle$ is the truncated dg Lie algebra of outer derivations of $L = \pi_*(\Omega M) \otimes \mathbb{Q}$, and $\Gamma_{\mathbb{Z}}^+(M) = \text{Aut}^{\mathcal{E}^+(M)}(H^*(M; \mathbb{Z}))$. □

We conclude this section by making some remarks about the manifolds

$$Z_g = \#^g S^n \times S^{n+1} \quad \text{and} \quad Z_{g,1} = Z_g \setminus \text{int } D^{2n+1} \quad \text{for } n > 1.$$

By Lemma 5.36, we have that

$$R^+(Z_g) \cong \text{GL}(H_n(Z_g; \mathbb{Q})) \cong \text{GL}_g(\mathbb{Q}).$$

Let

$$\Gamma_{\partial}(Z_{g,1}) \cong \mathrm{GL}^{\mathrm{aut}_{\partial}(Z_{g,1})}(H_*(Z_{g,1}; \mathbb{Q}))$$

be the group of automorphisms of the rational homology of $Z_{g,1}$ which can be realized by a boundary-preserving homotopy automorphism of $Z_{g,1}$. Each such homotopy automorphism can be extended by the identity on the interior of the removed disc to an orientation-preserving homotopy automorphism of Z_g , yielding an injection

$$(34) \quad \Gamma_{\partial}(Z_{g,1}) \hookrightarrow \Gamma^+(Z_g).$$

Proposition 5.39 *The homomorphism (34) is an isomorphism, and the group $\Gamma^+(Z_g)$ corresponds to the subgroup $\mathrm{Aut}(H_n(Z_g; \mathbb{Z}))$ of the group*

$$\mathrm{GL}(H_n(Z_g; \mathbb{Q})) \cong R^+(Z_g).$$

Proof It is evident that $\Gamma^+(Z_g)$ injects into $\mathrm{Aut}(H_n(Z_g; \mathbb{Z}))$. Both claims can be established simultaneously by showing that every automorphism of the group $H_n(Z_g; \mathbb{Z}) \cong \mathbb{Z}^g$ may be realized by a homotopy automorphism of Z_g that fixes the given embedded disc $D^{2n+1} \subset Z_g$ pointwise. This is an easy consequence of the Hilton–Milnor theorem once one notes that $Z_{g,1} \simeq \bigvee^g S^n \vee \bigvee^g S^{n+1}$ and that Z_g is obtained from $Z_{g,1}$ by attaching a $(2n+1)$ -cell along

$$\omega = \sum_{i=1}^g [\alpha_i, \alpha_i^{\#}] \in \pi_{2n}(Z_{g,1}),$$

where $\alpha_i \in \pi_n(Z_{g,1})$ are the classes of the S^n -factors of $Z_{g,1}$ and $\alpha_i^{\#} \in \pi_{n+1}(Z_{g,1})$ are the classes of the S^{n+1} -factors, suitably indexed. See [40, Section 5], where the more general case of connected sums of products of spheres of varying dimensions is investigated. $\square$

When applied to the homotopy fiber sequence

$$\Gamma_{\partial}(Z_{g,1}) // \mathrm{aut}_{\partial}(Z_{g,1}) \rightarrow B \mathrm{aut}_{\partial}(Z_{g,1}) \rightarrow B \Gamma_{\partial}(Z_{g,1}),$$

our methods show that the $\Gamma_{\partial}(Z_{g,1})$ -space $\Gamma_{\partial}(Z_{g,1}) // \mathrm{aut}_{\partial}(Z_{g,1})$ admits an $R^+(Z_g)$ -algebraic Lie model

$$\mathfrak{g}_{\partial}(Z_{g,1}) = (\mathrm{Der}_{\omega} \mathbb{L}(\alpha_1, \dots, \alpha_g, \alpha_1^{\#}, \dots, \alpha_g^{\#})) \langle 1 \rangle,$$

where Der_{ω} indicates derivations that annihilate the element $\omega = \sum_i [\alpha_i, \alpha_i^{\#}]$. The action of $R^+(Z_g)$ (and thus by restriction also the action of $\Gamma_{\partial}(Z_{g,1})$) comes from the action on $H_*(Z_g; \mathbb{Q})$. Thus we obtain:

Theorem 5.40 *There is an isomorphism of graded algebras*

$$H^*(B \mathrm{aut}_{\partial}(Z_{g,1}); \mathbb{Q}) \cong H^*(\Gamma_{\partial}(Z_{g,1}); H_{\mathrm{CE}}^*(\mathfrak{g}_{\partial}(Z_{g,1}))).$$

$\square$

Using this result, the stable cohomology of $B \mathrm{aut}_{\partial}(Z_{g,1})$ is computed by Stoll [65].

5.4 A counterexample

In all the examples discussed so far, the space X was always formal, the group $\mathcal{E}(X_{\mathbb{Q}})$ was reductive (hence equal to $R(X)$) and isomorphic to the group of algebra automorphisms of $H^*(X; \mathbb{Q})$, the group $\Gamma(X)$ was the group of automorphisms of $H_*(X; \mathbb{Q})$ that are induced by a self-homotopy equivalence of X , and the dg Lie algebra $\mathfrak{g}(X)$ was formal. In this section, we will discuss an example of a space X that has none of these properties. In fact, X will be the space that is discussed in [39, Example 6.7].

Let X be the space $F \times K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 6)$, where F is the homotopy fiber of the cup product

$$K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 3) \xrightarrow{\smile} K(\mathbb{Z}, 6).$$

This is evidently a finite Postnikov section with finitely generated homotopy groups, and its Sullivan model is the cdga

$$\Lambda = (\Lambda(x, y, z, u, w), d),$$

whose underlying graded commutative algebra is freely generated by cocycles x, y, z in degree 3, a cocycle w in degree 6, and an element u in degree 5 which satisfies $du = yz$.

Remark 5.41 X is indeed not rationally formal: the homomorphism

$$\mathrm{Aut} \Lambda \rightarrow \mathrm{Aut}_{\mathrm{alg}} H^*(\Lambda)$$

is not surjective. For any $\lambda \in \mathbb{Q}^\times$, there is a graded algebra automorphism ϕ of $H^*(\Lambda)$ given by $\phi(a) = \lambda^{|a|}a$. But ϕ affords no lift to $\mathrm{Aut} \Lambda$ unless $\lambda = 1$: any such lift $\tilde{\phi}$ would have to have $\tilde{\phi}(u) = \lambda^6 u$ so as to commute with d . But then $\tilde{\phi}(uy) = \lambda^9 uy$, whereas $\phi(uy) = \lambda^8 uy$.

Proposition 5.42 *The group $R(X)$ is isomorphic to*

$$\mathrm{GL}_1^2 \times \mathrm{GL}_2.$$

A section of the surjection $\mathrm{Aut} \Lambda \rightarrow R(X)$ is given by letting the two GL_1 factors act by scalar multiplication on x and on w , respectively, and by letting $A \in \mathrm{GL}_2$ act via the standard representation on the linear span of y and z , and by scalar multiplication by $\det A$ on u . The subgroup $\Gamma(X)$ of $R(X)$ corresponds to

$$(\mathbb{Z}^\times)^2 \times \mathrm{GL}_2^{\Sigma}(\mathbb{Z})$$

under this isomorphism.

Remark 5.43 The authors of [39, Example 6.7] construct two automorphisms $\phi, \psi \in \mathrm{Aut} \Lambda$. Here ϕ is the identity on all generators but w , where $\phi(w) = w + xy$, and ψ is the identity on all generators but x , where $\psi(x) = x + z$. They observe that these two automorphisms span a copy of $\mathbb{Z}^2$ in $\mathrm{Aut}^h \Lambda \cong \mathcal{E}(X_{\mathbb{Q}})$, but there exists no pair (ϕ', ψ') of homotopic automorphisms (or even automorphisms with the same effect on $H^*(\Lambda)$ as ϕ and ψ , respectively) which would already commute in $\mathrm{Aut} \Lambda$. This shows that we cannot in general expect to be able to lift the action of $\mathcal{E}(X_{\mathbb{Q}})$ on $\pi_* X \otimes \mathbb{Q}$ or on $H^*(X; \mathbb{Q})$ to an action on the minimal Sullivan model. It is clear from the proof of Proposition 5.42 that both ϕ and ψ lie in the kernel of $\mathrm{Aut} \Lambda \rightarrow R(X)$, so they do not obstruct the existence of an action of $R(X)$ on Λ .

Proposition 5.44 *The dg Lie algebra $\mathfrak{g}(X)$ is not formal: the Massey product*

$$\left\langle y \frac{\partial}{\partial x}, x \frac{\partial}{\partial u}, z \frac{\partial}{\partial w} \right\rangle$$

is nontrivial.

The rest of the section is devoted to proving Propositions 5.42 and 5.44.

We write V for the graded vector subspace of Λ which is spanned by the generators x, y, z, u and w , and we write ΛV for the underlying graded commutative algebra of Λ . The group $\mathrm{GL}(V)$ of graded linear automorphisms of V evidently injects into the group $\mathrm{Aut} \Lambda V$ of graded commutative algebra automorphisms of ΛV , and we will abuse notation by referring to its image by $\mathrm{GL}(V)$ as well.

Lemma 5.45 *The automorphism group $\mathrm{Aut} \Lambda V$ of the underlying graded commutative algebra is a semidirect product*

$$\mathrm{GL}(V) \ltimes \mathrm{Hom}(V^6, \Lambda^2 V^3)$$

where the abelian group $\mathrm{Hom}(V^6, \Lambda^2 V^3)$ injects into $\mathrm{Aut} \Lambda V$ by sending $f: V^6 \rightarrow \Lambda^2 V^3$ to the automorphism that is the identity on all generators except w and that takes w to $w + f(w)$.

Proof There is an obvious identification of V with the indecomposables $Q\Lambda V = I/I^2$, where $I = (\Lambda V)^{>0}$ is the augmentation ideal. This yields a retraction $\mathrm{Aut} \Lambda V \twoheadrightarrow \mathrm{GL}(Q\Lambda V) \cong \mathrm{GL}(V)$. Let $\mathrm{Aut}_1 \Lambda V$ be the kernel of this retraction. Every $\phi \in \mathrm{Aut}_1 \Lambda V$ fixes x, y, z and u , and is determined by the value $\phi(w) - w \in \Lambda^2 V^3 \subset (\Lambda V)^6$. It can be seen that there is an isomorphism

$$\mathrm{Aut}_1 \Lambda V \cong \mathrm{Hom}(V^6, \Lambda^2 V^3)$$

of groups with a $\mathrm{GL}(V)$ -action which sends $\phi \in \mathrm{Aut}_1 \Lambda V$ to the homomorphism $w \mapsto \phi(w) - w$. $\square$

The group $\mathrm{Aut} \Lambda$ of cdga automorphisms of Λ is the isotropy subgroup of the differential d inside $\mathrm{Aut} \Lambda V$. The subgroup $\mathrm{Aut}_1 \Lambda V$ of the preceding lemma clearly commutes with d , so it suffices to determine $\mathrm{Aut} \Lambda \cap \mathrm{GL}(V)$.

Lemma 5.46 *Let $W = \langle y, z \rangle \subset V^3$ be the $\mathbb{Q}$ -linear span of y and z . An automorphism $\phi \in \mathrm{GL}(V)$ commutes with d if and only if W is ϕ -stable, and $\phi(u) = (\det \phi|_W) \cdot u$.*

Proof Note that the line $\Lambda^2 W$ is precisely the span of yz , and also the image of the differential $d: (\Lambda V)^5 \hookrightarrow (\Lambda V)^6$. Hence ϕ commutes with d if and only if it carries $\Lambda^2 W$ to itself and it acts on $\Lambda^2 W$ and $(\Lambda V)^5$ by the same scalar. But $\phi(\Lambda^2 W) = \Lambda^2 \phi(W)$ is equal to $\Lambda^2 W$ if and only if $\phi(W) = W$. If this is the case, then ϕ scales $\Lambda^2 W$ by $\det \phi|_W$. $\square$

Proof of Proposition 5.42 The unipotent radical of $\mathrm{Aut} \Lambda V$, which we identified in Lemma 5.45, is contained in $\mathrm{Aut} \Lambda$, and hence in its unipotent radical. Thus the maximal reductive quotient of $\mathrm{Aut} \Lambda$ agrees with that of $\mathrm{Aut} \Lambda \cap \mathrm{GL}(V)$. From Lemma 5.46 it is easily seen that the unipotent radical of

$\text{Aut } \Lambda \cap \text{GL}(V)$ consists of the automorphisms that fix all generators except x and that send x to x plus an element of W , and the maximal reductive quotient is as described in the statement of the proposition.

To determine $\Gamma(X)$, recall that X is the homotopy fiber of the 6th integral cohomology class

$$K(\mathbb{Z}, 3)^3 \times K(\mathbb{Z}, 6) \rightarrow K(\mathbb{Z}, 6)$$

given by the cup product of the fundamental classes of two of the $K(\mathbb{Z}, 3)$ factors.

The integral cohomology ring of X agrees up to dimension 6 with

$$\Lambda_{\mathbb{Z}}(\bar{x}, \bar{y}, \bar{z}, \bar{w})/(\bar{y}\bar{z}, 2\bar{x}^2, 2\bar{y}^2, 2\bar{z}^2),$$

where the generators $\bar{x}, \bar{y}$ and $\bar{z}$ live in degree 3, $\bar{w}$ lives in degree 6, and each generator maps to the corresponding element of $H^*(\Lambda) \cong H^*(X; \mathbb{Q})$ under the change-of-coefficient homomorphism. Explicitly, $H^3(X) \cong \mathbb{Z}^3$ is spanned by $\bar{x}, \bar{y}, \bar{z}$, and $H^6(X) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^3$ is spanned by $\bar{x}\bar{y}, \bar{x}\bar{z}, \bar{w}$ and $\bar{x}^2, \bar{y}^2, \bar{z}^2$.

Thus the homotopy class of an endomorphism $f: X \rightarrow X$ is uniquely determined by the quadruple $(f^*\bar{x}, f^*\bar{y}, f^*\bar{z}, f^*\bar{w}) \in H^3(X)^3 \times H^6(X)$, and a quadruple $(\bar{x}', \bar{y}', \bar{z}', \bar{w}')$ is induced by an endomorphism if $\bar{y}' \smile \bar{z}' = 0$.

Thus an element $(\lambda, A, \mu) \in \text{GL}_1 \times \text{GL}_2 \times \text{GL}_1 \cong R(X)$ can be realized by a homotopy automorphism of X if and only if $\lambda, \mu \in \mathbb{Z}$, the matrix A has integer entries, and each row of A contains at least one even number (since $(a\bar{y} + b\bar{z})(c\bar{y} + d\bar{z}) = ac\bar{y}^2 + bd\bar{z}^2 \in H^6(X)$ is nonzero if ac or bd are odd). $\square$

Proof of Proposition 5.44 We begin by describing in some detail the structure of the dg Lie algebra $\mathfrak{g}(X)$. In positive degrees, $\mathfrak{g}(X)$ is just the full dg Lie algebra of derivations of the free cdga ΛV , and it is easy to list the basis in each degree.

In degree 0, $\mathfrak{g}(X)$ is the Lie algebra of the unipotent radical of $\text{Aut } \Lambda$. This radical sits in a (split) short exact sequence

$$1 \rightarrow \text{Hom}(V^6, \Lambda^2 V^3) \rightarrow \text{Aut}_u \Lambda \rightarrow \text{Hom}(\langle x \rangle, W) \rightarrow 1,$$

where the left-hand term is the unipotent radical of $\text{Aut } \Lambda V$ and the right-hand term is the unipotent radical of $\text{Aut } \Lambda \cap \text{GL}(V)$. From this, a basis for $\mathfrak{g}_0(X)$ can be read off. We list bases in all degrees in Table 1.

The differential δ is given by

$$u \frac{\partial}{\partial w} \mapsto yz \frac{\partial}{\partial w}, \quad \frac{\partial}{\partial y} \mapsto -z \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial z} \mapsto y \frac{\partial}{\partial u};$$

it vanishes on all the other generators.

Note that

$$\left[y \frac{\partial}{\partial x}, x \frac{\partial}{\partial u} \right] = y \frac{\partial}{\partial u} = \delta \left(\frac{\partial}{\partial z} \right)$$

while the other two pairs bracket to zero. Thus the Massey product contains

$$\left[\frac{\partial}{\partial z}, z \frac{\partial}{\partial w} \right] = \frac{\partial}{\partial w}.$$

degree	basis
0	$y \frac{\partial}{\partial x}, z \frac{\partial}{\partial x}, xy \frac{\partial}{\partial w}, xz \frac{\partial}{\partial w}, yz \frac{\partial}{\partial w}$
1	$u \frac{\partial}{\partial w}$
2	$x \frac{\partial}{\partial u}, y \frac{\partial}{\partial u}, z \frac{\partial}{\partial u}$
3	$x \frac{\partial}{\partial w}, y \frac{\partial}{\partial w}, z \frac{\partial}{\partial w}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$
5	$\frac{\partial}{\partial u}$
6	$\frac{\partial}{\partial w}$

Table 1

The indeterminacy is equal to the subset

$$\left[y \frac{\partial}{\partial x}, H_6(\mathfrak{g}(X)) \right] + \left[x \frac{\partial}{\partial u}, H_4(\mathfrak{g}(X)) \right] + \left[z \frac{\partial}{\partial w}, H_3(\mathfrak{g}(X)) \right]$$

of $H_6(\mathfrak{g}(X))$. This is easily seen to only contain 0. Thus the Massey product is indeed nontrivial as $\partial/\partial w$ is a nontrivial cycle, obstructing formality of $\mathfrak{g}(X)$. $\square$

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Resonant forms at zero for dissipative Anosov flows

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We study resonant differential forms at zero for transitive Anosov flows on 3-manifolds. We pay particular attention to the dissipative case, that is, Anosov flows that do not preserve an absolutely continuous measure. Such flows have two distinguished Sinai–Ruelle–Bowen 3-forms, $\Omega_{\text{SRB}}^{\pm}$, and the cohomology classes $[\iota_X \Omega_{\text{SRB}}^{\pm}]$ (where X is the infinitesimal generator of the flow) play a key role in the determination of the space of resonant 1-forms. When both classes vanish we associate to the flow a *helicity* that naturally extends the classical notion associated with null-homologous volume-preserving flows. We provide a general theory that includes horocyclic invariance of resonant 1-forms and SRB-measures as well as the local geometry of the maps $X \mapsto [\iota_X \Omega_{\text{SRB}}^{\pm}]$ near a null-homologous volume-preserving flow. Next, we study several relevant classes of examples. Among these are thermostats associated with holomorphic quadratic differentials, giving rise to quasi-Fuchsian flows as introduced by Ghys (1992). For these flows we compute explicitly all resonant 1-forms at zero, we show that $[\iota_X \Omega_{\text{SRB}}^{\pm}] = 0$ and give an explicit formula for the helicity. In addition we show that a generic time change of a quasi-Fuchsian flow is semisimple and thus the order of vanishing of the Ruelle zeta function at zero is $-\chi(M)$, the same as in the geodesic flow case. In contrast, we show that if (M, g) is a closed surface of negative curvature, the Gaussian thermostat driven by a (small) harmonic 1-form has a Ruelle zeta function whose order of vanishing at zero is $-\chi(M) - 1$.

35P05, 37C30, 37D40, 53C65, 58C40

To the memory of Will Merry.

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1 Introduction

If X is a vector field on a closed manifold $\mathcal{M}$, then typically the L^2 spectrum of X is not discrete but consists of essential spectrum. However, when X is *Anosov* (or *uniformly hyperbolic*), there is a hidden discrete *resonance* spectrum obtained by the action of X on some spaces of *anisotropic regularity*. This spectrum, and the associated *resonant states* (eigenfunctions of X) encode many significant properties of the flow, for instance one can read off whether X is ergodic or mixing (with respect to the Sinai–Ruelle–Bowen measure), or even exponentially mixing. Furthermore, the analogously defined space of resonant states Res_0^k of the Lie derivative $\mathcal{L}_X$ (acting on k -forms) at zero carries rich topological data, for instance it is related to the Betti numbers of $\mathcal{M}$ or to dynamical invariants of the flow generated by X , and is related to the periodic orbit spectrum (the set of periods of periodic orbits) through the *Ruelle zeta function*. In this paper, we are interested in this spectrum of $\mathcal{L}_X$ at zero.

In general, the dependence of the resonant space spectrum of $\mathcal{L}_X$ at zero on various topological or geometrical invariants of $\mathcal{M}$ and X is now understood to be very subtle. The goal of this paper is to cast some light on the more difficult case when X does not preserve a smooth measure and to illustrate the general picture with important classes of examples.

1.1 Geometric multiplicities of resonant forms

Assume $\mathcal{M}$ is a closed smooth 3-manifold, equipped with a smooth vector field X generating an *Anosov* flow φ_t . This means that there is a continuous and flow invariant splitting of the tangent space

$$T\mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u$$

into flow, stable and unstable directions, respectively, such that for some constants $C, \nu > 0$ and an arbitrary metric $|\bullet|$ on $\mathcal{M}$, and all $x \in \mathcal{M}$,

$$(1-1) \quad |d\varphi_t(x) \cdot v| \leq C e^{-\nu|t|} \cdot |v| \quad \text{for} \quad \begin{cases} t \geq 0, & v \in E_s(x), \\ t \leq 0, & v \in E_u(x). \end{cases}$$

It is well-known that the geodesic flow on the unit tangent bundle $\mathcal{M} = SM$ of a Riemannian manifold (M, g) with negative sectional curvature is Anosov [1967]. Write $E_{u/s}^* := (\mathbb{R}X \oplus E_{u/s})^\perp \subset T^*\mathcal{M}$, where $\bullet^\perp$ is the annihilator of $\bullet$.

Denote by Ω^k the bundle of differential k -forms and by $\Omega_0^k = \Omega^k \cap \ker \iota_X \subset \Omega^k$ the bundle of k -forms in the kernel of the contraction with the vector field. Write $\mathcal{D}'(\mathcal{M}; \Omega^k)$ for the space of distributional sections of Ω^k . Given $\ell \in \mathbb{Z}_{\geq 1}$, introduce the spaces of *generalised resonant k -forms at zero* by

$$\begin{aligned} \text{Res}^{k,\ell} &:= \{u \in \mathcal{D}'(\mathcal{M}; \Omega^k) \mid \mathcal{L}_X^\ell u = 0, \text{WF}(u) \subset E_u^*\}, & \text{Res}^{k,\infty} &:= \bigcup_{\ell \geq 1} \text{Res}^{k,\ell}, \\ \text{Res}_0^{k,\ell} &:= \text{Res}^{k,\ell} \cap \ker \iota_X, & \text{Res}_0^{k,\infty} &:= \text{Res}^{k,\infty} \cap \ker \iota_X, \end{aligned}$$

where $\text{WF}(u) \subset T^*M \setminus 0$ denotes the *wavefront set* of a distribution (see [Hörmander 2003, Chapter 8]), and $\mathcal{L}_X = \iota_X d + d\iota_X$ is the Lie derivative. These spaces are finite-dimensional: this is a nontrivial fact that follows from the construction of *anisotropic Sobolev spaces* tailored to the flow, on which $\mathcal{L}_X$ acts as a Fredholm operator; see [Faure and Sjöstrand 2011] or [Dyatlov and Zworski 2016].

Write $m_{k,0}^\infty := \dim \text{Res}_0^{k,\infty}$ for the *algebraic multiplicity*. When $\ell = 1$, denote by $\text{Res}_0^k := \text{Res}_0^{k,1}$ and $\text{Res}^k := \text{Res}^{k,1}$ the spaces of *resonant k -forms at zero*, and write $m_{k,0} := \dim \text{Res}_0^k$ and $m_k := \dim \text{Res}^k$ for the *geometric multiplicities*. Say that the action of $\mathcal{L}_X$ on Ω_0^k (resp. Ω^k) is *semisimple* (at zero) if $\text{Res}_0^k = \text{Res}_0^{k,\infty}$ (resp. $\text{Res}^k = \text{Res}^{k,\infty}$).

The *Ruelle zeta function*

$$(1-2) \quad \zeta_R(s) := \prod_{\gamma} (1 - e^{-sT_\gamma}), \quad \text{for } \text{Re}(s) \gg 1,$$

is a converging product for $\text{Re}(s)$ large enough and admits a meromorphic continuation to $s \in \mathbb{C}$ by the work of Giulietti, Liverani and Pollicott [Giulietti et al. 2013] and Dyatlov and Zworski [2016]. Here the product is over all primitive closed orbits of the flow. Define the *order of vanishing of the zeta function at zero* to be the unique integer $m_R(0)$ such that $s^{-m_R(0)}\zeta_R(s)$ is holomorphic and nonzero at $s = 0$. According to [Dyatlov and Zworski 2017, Section 3], we have

$$(1-3) \quad m_R(0) = m_{1,0}^\infty - m_{0,0}^\infty - m_{2,0}^\infty.$$

It is known that if X is a *transitive* Anosov vector field (ie it has a dense orbit), then the action of $\mathcal{L}_X$ on Ω^0 and Ω_0^2 is semisimple and $m_{0,0} = m_{2,0} = 1$; see Proposition 3.1 below. Under this assumption, $m_{1,0}^\infty$ is hence the only remaining unknown in (1-3).

We now introduce some important dynamical invariants of the X . Denote by Ω_{SRB}^+ the Sinai–Ruelle–Bowen (SRB) measure of X , that is, the unique invariant probability measure such that $\text{WF}(\Omega_{\text{SRB}}^+) \subset E_u^*$; similarly introduce the SRB measure Ω_{SRB}^- for the flow $-X$; see Section 2.6 for more details. We may equivalently regard $\Omega_{\text{SRB}}^\pm$ as (distributional) 3-forms, normalised so that they have integral 1. Since $\omega^\pm := \iota_X \Omega_{\text{SRB}}^\pm$ are closed, the 2-forms $\omega^\pm$ define de Rham cohomology classes in $H^2(\mathcal{M})$ that are Poincaré dual to the classical *winding cycles* of the SRB measures (also known as *asymptotic cycles* [Schwartzman 1957]). If both winding cycles vanish, ie $[\omega^+]_{H^2(\mathcal{M})} = [\omega^-]_{H^2(\mathcal{M})} = 0$, by Lemma 2.4 below we may write $\omega^\pm = d\tau^\pm$ for some $\tau^\pm \in \mathcal{D}'(\mathcal{M}; \Omega^1)$ (resp. $\tau^- \in \mathcal{D}'(\mathcal{M}; \Omega^1)$) with $\text{WF}(\tau^+) \subset E_u^*$ (resp. $\text{WF}(\tau^+) \subset E_u^*$) and it is possible to define the *helicity* $\mathcal{H}(X)$ by

$$(1-4) \quad \mathcal{H}(X) := \int_{\mathcal{M}} \tau^+(X) \Omega_{\text{SRB}}^- = \int_{\mathcal{M}} \tau^-(X) \Omega_{\text{SRB}}^+,$$

thanks to the wavefront set conditions. This quantity is independent of any choices (see Section 3) and it agrees with the well-known concept of helicity in the volume-preserving case. The following statement generalises [Cekić and Paternain 2020, Theorem 1.2].

Theorem 1.1 Assume X generates a transitive Anosov flow on $\mathcal{M}$. The **geometric** multiplicity for the action of $\mathcal{L}_X$ on Ω_0^1 and Ω^1 is determined as a function of $[\omega^\pm]_{H^2(\mathcal{M})}$ and $\mathcal{H}(X)$ by the following table:

cases	$[\omega^+] \neq 0$ $[\omega^-] \neq 0$	$[\omega^+] \neq 0$ $[\omega^-] = 0$	$[\omega^+] = 0$ $[\omega^-] \neq 0$	$[\omega^+] = [\omega^-] = 0$ $\mathcal{H}(X) \neq 0$	$[\omega^+] = [\omega^-] = 0$ $\mathcal{H}(X) = 0$
$d(\text{Res}_0^1)$	0	0	$\mathbb{C}\omega^+$	0	$\mathbb{C}\omega^+$
$m_{1,0} = \dim \text{Res}_0^1$	$b_1(\mathcal{M}) - 1$	$b_1(\mathcal{M})$	$b_1(\mathcal{M})$	$b_1(\mathcal{M})$	$b_1(\mathcal{M}) + 1$
$d(\text{Res}^1)$	0	0	$\mathbb{C}\omega^+$	$\mathbb{C}\omega^+$	$\mathbb{C}\omega^+$
$m_1 = \dim \text{Res}^1$	$b_1(\mathcal{M})$	$b_1(\mathcal{M})$	$b_1(\mathcal{M}) + 1$	$b_1(\mathcal{M}) + 1$	$b_1(\mathcal{M}) + 1$

Moreover, the map $\text{Res}^1 \cap \ker d \rightarrow H^1(\mathcal{M})$ given by $u \mapsto [u]$ is an isomorphism.

Note that there are two more cases compared to [Cekić and Paternain 2020, Theorem 1.2], due to the fact that $\Omega_{\text{SRB}}^+ \neq \Omega_{\text{SRB}}^-$ if the flow does not preserve a smooth volume; these are the second and third columns (the contact case belongs to the fourth column). We would like to point out that, to the best of our knowledge, there are no known examples of Anosov flows with zero helicity. Observe that Theorem 1.1 is really a statement about the 1-dimensional oriented foliation determined by X , as the resonant spaces Res_0^1 and the conditions defining the various cases are all independent of time changes of X . The theorem shows that $\dim \text{Res}^1$ is also invariant under time changes. However, it should be noted that in principle some of the resonant forms in Res^1 could be altered under time changes (in the first, third, and fourth cases) but not their key properties. We explain this in more detail in Remark 3.9.

Finally, we remark that Theorem 1.1 should have an analogue in the *nontransitive* case (examples of such Anosov flows were constructed in [Franks and Williams 1980]): indeed, then there are finitely many SRB measures in bijection with the kernel of $\mathcal{L}_X$ on Res^3 by [Butterley and Liverani 2007, Theorem 1] or [Bonthonneau et al. 2024, Theorem 3], and in principle a similar analysis applies.

Given that $[\omega^\pm]$ and $\mathcal{H}(X)$ feature prominently in Theorem 1.1 we next give some additional insights into their properties.

1.2 Helicity of the SRB measures

The helicity is traditionally defined in the context of null-homologous volume-preserving flows [Arnold and Khesin 1998]. One of our ancillary objectives is to explain that in the Anosov case, this quantity can be defined in dissipative situations using the SRB measures and that it captures similar features as in the volume-preserving case. For the quantity to be well-defined we require both cohomology classes $[\omega^\pm] = 0$ as explained above. Equation (1-4) defines $\mathcal{H}(X)$ taking advantage of the wave front set conditions and hence as a distributional pairing and in this form it is well-suited for our proof of Theorem 1.1. However, a natural question immediately arises: is it possible to express $\mathcal{H}(X)$ using *linking forms* as in [Contreras 1995] or [Kotschick and Vogel 2003]? This would interpret $\mathcal{H}(X)$ introduced by (1-4) as an averaged linking number.

We show that this is indeed the case. Let $K \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}; \pi_1^* \Omega^1 \otimes \pi_2^* \Omega^1)$ be the Schwartz kernel of the Green operator G of a fixed background Riemannian metric g ; by definition, if Δ_g denotes the Hodge Laplacian on 1-forms, G is given by 0 and Δ_g^{-1} on $\ker \Delta_g$ and its L^2 complement, respectively. Here π_1 and π_2 denote projections onto the first and second factors of $\mathcal{M} \times \mathcal{M}$, respectively. The *linking form* $L \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}; \pi_1^* \Omega^1 \otimes \pi_2^* \Omega^1)$ is defined as the double form

$$L(x, y) := \star_y d_y K(x, y),$$

where $\star$ denotes the Hodge star operator. It satisfies the property that when integrated over two knots in $\mathcal{M}$ it gives the *linking number* of the two knots; see [Kotschick and Vogel 2003, Proposition 1]. Set $\Lambda(x, y) := L(x, y)(X(x), X(y)) \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$; it is not hard to see that Λ is smooth outside the diagonal $\Delta(\mathcal{M})$ and at the diagonal it has a singularity of type $d(x, y)^{-1}$ (where $d(x, y)$ is the distance between x and y). Also, as explained in Section 9, the wavefront set calculus implies that $\Lambda(x, y) \Omega_{\text{SRB}}^+(x) \times \Omega_{\text{SRB}}^-(y)$ is well defined as a distribution. We have:

Theorem 1.2 *The function $\Lambda|_{\mathcal{M} \times \mathcal{M} \setminus \Delta(\mathcal{M})}$ is integrable with respect to $\Omega_{\text{SRB}}^+ \times \Omega_{\text{SRB}}^-$. The following formula holds:*

$$\begin{aligned} \mathcal{H}(X) &= \int_{(x,y) \in \mathcal{M} \times \mathcal{M}} \Lambda(x, y) \Omega_{\text{SRB}}^+(x) \times \Omega_{\text{SRB}}^-(y) \\ &= \lim_{R \rightarrow 0^+} \int_{(x,y) \in \mathcal{M} \times \mathcal{M} \setminus B_R} \Lambda(x, y) \Omega_{\text{SRB}}^+(x) \times \Omega_{\text{SRB}}^-(y), \end{aligned}$$

where in the last line $(B_R)_{R>0}$ is any nested family of neighbourhoods of $\Delta(\mathcal{M}) \subset \mathcal{M} \times \mathcal{M}$ such that $\bigcap_{R>0} B_R = \Delta(\mathcal{M})$, and the integral is interpreted classically.

We note that the first equality in the theorem is a direct consequence of the wavefront set calculus (somewhat similar to (1-4)) and of using the Green operator G . Indeed, we have

$$\mathcal{H}(X) = \int_{\mathcal{M}} \tau^+ \wedge d\tau^- = \int_{\mathcal{M}} G d^* d\tau^+ \wedge d\tau^- = \int_{(x,y) \in \mathcal{M} \times \mathcal{M}} \Lambda(x, y) \Omega_{\text{SRB}}^+(x) \times \Omega_{\text{SRB}}^-(y),$$

where in the second equality we used that τ^+ is equal to $G d^* d\tau^+$ up to closed terms (see (9-10) below), and in the third one that $\star_y L(x, y)$ is the Schwartz kernel of $G d^*$; see (9-7) below. The second equality of the theorem is its main component and we emphasise that its proof requires some care due to the singular nature of the SRB measures. Moreover, using recent results of Coles and Sharp [2023] it is also possible by means of Theorem 1.2 to express the helicity in terms of linkings of closed orbits; see Proposition 9.3. Finally we observe that the expression

$$\mathcal{H}(X) = \int_{\mathcal{M}} \tau^+ \wedge \omega^- = \int_{\mathcal{M}} \tau^- \wedge \omega^+$$

allows us also to directly interpret the helicity as a linking between the two SRB measures when both are homologically trivial (ie when $[\omega^\pm]_{H^2(\mathcal{M})} = 0$).

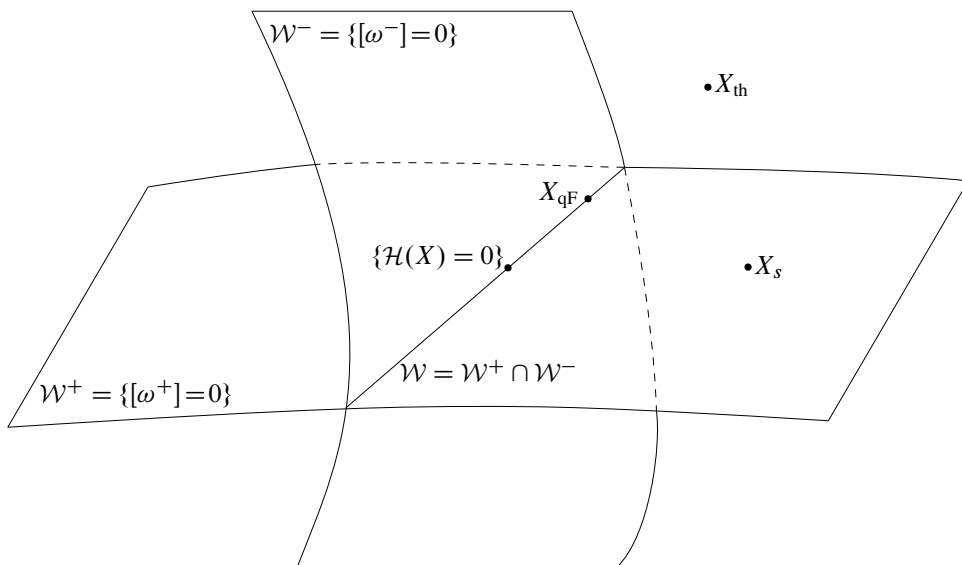


Figure 1: Local geometry of vector fields according to the value of the winding cycle and the helicity. Flows $X_s \in \mathcal{W}^+$, but $X_s \notin \mathcal{W}^-$, are constructed in [Proposition 6.2](#), quasi-Fuchsian flows $X_{\text{qF}} \in \mathcal{W}$ are studied in [Section 8](#), and thermostat flows satisfying $X_{\text{th}} \notin \mathcal{W}^+ \cup \mathcal{W}^-$ are provided by [Theorem 1.6](#).

1.3 Perturbation theory

Given the relevance of the classes $[\omega^\pm]$ it is natural to try to understand in more detail the structure of those Anosov vector fields X for which $[\omega^\pm(X)] = 0$ at least for X close to X_0 , where X_0 preserves a volume form Ω and is null-homologous (ie $[\omega^\pm(X_0)] = 0$). For what follows we fix such a vector field X_0 and we work in a fixed C^N -neighbourhood $\mathcal{U}$ of X_0 for N sufficiently large. Set

$$\mathcal{W}^\pm := \{X \in C^N(\mathcal{M}, T\mathcal{M}) \mid [\omega^\pm(X)] = 0\} \cap \mathcal{U} \quad \text{and} \quad \mathcal{W} := \mathcal{W}^+ \cap \mathcal{W}^-.$$

We shall show that $\mathcal{W}^\pm$ are C^1 Banach submanifolds near X_0 of codimension $b_1(\mathcal{M})$, the first Betti number of $\mathcal{M}$. The intersection $\mathcal{W}^+ \cap \mathcal{W}^-$ is transversal, that is, it is a C^1 Banach submanifold of codimension $2b_1(\mathcal{M})$. For the proof, see [Lemma 5.5](#) below. These results can be complemented with results on helicity: assuming additionally $\mathcal{H}(X_0) = 0$, the set $\{X \mid \mathcal{H}(X) = 0\} \subset \mathcal{W}$ is locally a C^1 Banach submanifold of codimension 1; see [Proposition 9.7](#) and [Figure 1](#). We reiterate that we do not know if such an X_0 exists.

1.4 Horocyclic invariance of resonant forms

Resonant states and forms often exhibit an additional kind of invariance. This is most evident on constant negatively curved surfaces, where a resonant state u at $s \in \mathbb{C}$ satisfying $(X + s)u = 0$ (and $\text{WF}(u) \subset E_u^*$) also satisfies $(X + s + 1)U_-u = 0$, where U_- is the smooth vector field spanning the unstable foliation E_u .

Since there are no resonances in the right half-plane, we conclude that $U_-^k u = 0$ for k large enough (this is studied in detail in [Dyatlov et al. 2015]).

When we change the constant curvature setting to an arbitrary Anosov flow on $\mathcal{M}$, the vector field U_- spanning E_u becomes only Hölder regular in general and the definition of the derivative $U_- u$ is not immediately clear. However, when X is a contact vector field and $|\operatorname{Re}(s)|$ is sufficiently small, u and U_- have enough regularity for $U_- u$ to exist as was observed by Guillarmou and Faure [2018] who showed $U_- u = 0$ in this situation. This observation was later used to show the existence of the first *spectral band* for resonances by Guillarmou and the first author [Cekić and Guillarmou 2021] (see Remark 4.7 for a further discussion). Let us also mention that [Tsujii 2018] and [Faure and Tsujii 2024] introduce a more abstract notion of ‘horocycle operators’.

In this paper, for *arbitrary* transitive Anosov flows in dimension 3, we are able to prove a version of horocyclic invariance for closed resonant 1-forms $u \in \operatorname{Res}_0^1$ (see Lemmas 4.2 and 4.4), including the fact that they are zero on the weak unstable bundle $\mathbb{R}X \oplus E_u$. The idea is to work with the regularity of the *weak unstable bundle*, which is $C^{1+\alpha}$ -regular for some $\alpha > 0$. Similar results were proved in the constant negative curvature setting; see [Cekić et al. 2022] and [Küster and Weich 2020]. We note that in contrast to [Guillarmou and Faure 2018], where the horocyclic invariance holds near the imaginary axis, the resonant forms at zero are “deep inside” the resonance spectrum (and as is well known, the leading resonance is given by the topological entropy). There are a couple of reasons making this possible:

- (1) contracting with vector fields costs less derivatives than differentiation, and
- (2) $d \operatorname{Res}_0^1$ is at most 1-dimensional, so elements Res_0^1 can be shown to have slightly more regularity than expected.

In addition, we are able to show that the SRB measures also satisfy a form of horocyclic invariance under the hypothesis that one weak bundle is smooth; see Lemma 4.13.

1.5 Semisimplicity

Having sorted out the resonant forms at zero, the only remaining obstacle to compute the order of vanishing at zero of the Ruelle zeta function using geometric multiplicities is semisimplicity for $\mathcal{L}_X$ acting on Ω_0^1 . As explained in [Cekić and Paternain 2020], this is a subtle issue that could in principle depend on the parametrisation of the flow. For example, [Cekić and Paternain 2020, Theorem 1.4] shows that there are time changes of geodesic flows of hyperbolic surfaces that are not semisimple. Here we propose a conjecture in this direction.

Conjecture 1.3 *Let X be a transitive Anosov vector field on a closed 3-manifold $\mathcal{M}$. There exists an open and dense set $\mathcal{O} \subset C^\infty(\mathcal{M}; \mathbb{R}_{>0})$ such that for $f \in \mathcal{O}$, the action of $\mathcal{L}_{fX}$ on Ω_0^1 is semisimple.*

A couple of remarks are in order. The conjecture is equivalent to showing that there is just *one* time change that is semisimple. This is easily seen using a pairing between resonant and *coresonant spaces* (resonant spaces for $-X$ are decorated with an additional subscript $*$) and noticing that semisimplicity is

characterised in terms of the nondegeneracy of this pairing; see [Lemma 2.1](#). Moreover, using the pairing the conjecture is implied by a positive answer to the following intriguing question:

Question Given $u \in \text{Res}_0^1$ and $u_* \in \text{Res}_{0*}^1$ with $u \wedge u_* = 0$, is it true that $u = 0$ or $u_* = 0$?

[Lemma 8.16](#) provides a positive answer to this question for an interesting class of flows, namely those Anosov flows with smooth weak bundles; see [Theorem 1.5](#) below. Note that semisimplicity is known in a limited number of circumstances, eg for Reeb Anosov flows by [\[Dyatlov and Zworski 2017\]](#) and for perturbations by [\[Cekić and Paternain 2020\]](#).

1.6 Examples

Let us illustrate [Theorem 1.1](#) with some specific examples.

1.6.1 Suspensions Let us first consider Anosov flows that are topologically orbit equivalent to the suspension of a linear hyperbolic toral automorphism $A \in \text{SL}(2, \mathbb{Z})$. The suspension of A lives on a solvable manifold $\mathcal{M}_A$ with $b_1(\mathcal{M}_A) = 1$. We have the following:

Corollary 1.4 *Let X be any Anosov vector field on $\mathcal{M}_A$. Then $[\omega^\pm] \neq 0$, $\text{Res}_0^1 = \text{Res}_{0*}^1 = \{0\}$ and $\zeta_R(s)$ has a pole of order 2 at $s = 0$, that is, $m_R(0) = -2$.*

Proof By [\[Plante 1981, Theorem B\]](#), any Anosov flow on $\mathcal{M}_A$ is topologically orbit equivalent to the suspension of A . Hence, it follows that the trivial homology class does not contain closed orbits of X and thus X is not homologically full. By [\[Sharp 1993, Theorem 1\]](#) we must have $[\omega^\pm] \neq 0$. Since $b_1(\mathcal{M}_A) = 1$, [Theorem 1.1](#) gives that $\text{Res}_0^1 = \text{Res}_{0*}^1 = \{0\}$ and therefore $\mathcal{L}_X$ is trivially semisimple on Ω_0^1 . Using (1-3) we deduce that $\zeta_R(s)$ has a pole of order 2 at $s = 0$. $\square$

1.6.2 Quasi-Fuchsian flows and the coupled vortex equations An interesting class of Anosov flows arises from the data given by a closed Riemann surface and a holomorphic differential of degree m on the surface, as we now explain.

Let (M, g) be a closed oriented Riemannian surface and denote by K_g its Gauss curvature. Introduce the unit sphere bundle as

$$SM := \{(x, v) \in TM \mid |v|_g = 1\},$$

which carries a natural smooth measure Ω given locally by the product of the volume form on M and the Lebesgue measure in the spherical fibres. Denote by X the geodesic vector field and by V the generator of (oriented) rotations in the fibres of SM . Define the canonical bundle $\mathcal{K} := (T_{\mathbb{C}}^*M)^{1,0}$ to be the holomorphic part of the complexified cotangent bundle $T_{\mathbb{C}}^*M$. For $m \in \mathbb{Z}_{\geq 0}$, there is a natural map

$$\pi_m^* : C^\infty(M; \mathcal{K}^{\otimes m}) \rightarrow C^\infty(SM), \quad \pi_m^* T(x, v) := T_x(v, v, \dots, v).$$

If $A \in C^\infty(M; \mathcal{K}^{\otimes m})$, we may set $\lambda = \text{Im}(\pi_m^* A)$ and we will be interested in the flow generated by the vector field $F := X + \lambda V$. Say that the pair (g, A) satisfies the *coupled vortex equations* if

$$(1-5) \quad \bar{\partial} A = 0, \quad K_g = -1 + (m-1)|A|_g^2.$$

If (1-5) are satisfied, [Mettler and Paternain 2019, Theorem 5.1] shows that F is Anosov, and by [loc. cit., Remark 5.3] for any A as above there is a unique metric in the conformal class $[g]$ that solves (1-5). Hence, this class of flows is parametrised by the data $([g], A)$. Moreover, by [loc. cit., Theorem 5.5], F preserves an absolutely continuous measure if and only if $A = 0$, showing that when $A \neq 0$ we are in a genuinely dissipative scenario.

The case $m = 2$ is special and in some sense these flows occupy a distinguished place among dissipative Anosov flows, much in the same way as geodesic flows of hyperbolic surfaces are special among contact/volume-preserving flows. The main reason they are special is that they have both weak bundles of class C^∞ , but the SRB measures are singular as long as $A \neq 0$ and hence they are not algebraic. Ghys [1992, Théorème B] introduced and studied this type of flow. His construction produces a smooth orientable Anosov foliation $\phi_{[g_1], [g_2]}$ on the bundle of positive half-lines tangent to the surface M . Here, g_i for $i = 1, 2$ are metrics of constant curvature -1 and $[g_i]$ are the corresponding points in Teichmüller space $\mathcal{T}(M)$. Any flow parametrising $\phi_{[g_1], [g_2]}$ has smooth weak stable foliation C^∞ -conjugate to the weak stable foliation of the constant curvature metric g_1 and smooth weak unstable foliation C^∞ -conjugate to the weak unstable foliation of the constant curvature metric g_2 . Moreover, a volume form is preserved if and only if $[g_1] = [g_2]$. Ghys named these flows *quasi-Fuchsian flows*, because his construction is analogous to the construction of quasi-Fuchsian groups obtained by coupling two Fuchsian groups. Theorem 4.6 in [Ghys 1993] ensures that for $m = 2$, the flow of F is C^∞ -orbit equivalent to a quasi-Fuchsian flow, so we shall call such an F also a quasi-Fuchsian vector field. As we noted above these are parametrised by $T^*\mathcal{T}(M)$ and in forthcoming work [Cekić and Paternain 2025] we will show that they exhaust all possible $\phi_{[g_1], [g_2]}$. Working with F instead of $\phi_{[g_1], [g_2]}$ has many advantages for our purposes as it gives us access to the vertical Fourier analysis introduced by Guillemin and Kazhdan [1980], which in turn will allow us to compute all resonant 1-forms at zero and the helicity of the SRB measures. As far as we are aware this is the first explicit helicity calculation in a dissipative context.

Theorem 1.5 *Let F be a quasi-Fuchsian vector field. Then $m_{1,0} = b_1(M)$, $[\omega^\pm] = 0$, and the helicity is*

$$\mathcal{H}(F) = \frac{1 + \frac{1}{2}e^+(F)}{2\pi \operatorname{vol}_g(M) + \int_{SM} a^2 \Omega}.$$

The function a is the unique Hölder solution to the equation

$$Fa + (1 + \tfrac{1}{2}V\lambda)a = -\lambda,$$

and $e^+(F)$ is the entropy production of the SRB measure, which is given explicitly by

$$e^+(F) = - \int_{SM} V\lambda \Omega_{\text{SRB}}^+.$$

Moreover, for an open and dense set of $f \in C^\infty(SM; \mathbb{R}_{>0})$, the action of $\mathcal{L}_{fF}$ on Ω_0^1 is semisimple and so for the time-changed vector field fF , we have $m_{\mathbb{R}}(0) = -\chi(M)$.

Curiously, we do not know if the action of $\mathcal{L}_F$ on Ω_0^1 is semisimple, except for A small.

1.6.3 Flows with $[\omega^+] = 0$ and $[\omega^-] \neq 0$ The next interesting class of examples that arises in relation to [Theorem 1.1](#) are flows in the second and third columns of the table. In fact, if X is an example with $[\omega^+] = 0$ and $[\omega^-] \neq 0$, then $-X$ is an example with $[\omega^+] \neq 0$ and $[\omega^-] = 0$. They have the special feature that Res_0^1 contains nonclosed 1-forms, something unseen in the volume-preserving case. That such examples must exist can now be easily inferred from the perturbation picture described in [Section 1.3](#), however we prefer to go one step further and give explicit smooth examples with the additional feature that the weak stable bundle is C^∞ . This is carried out in [Proposition 6.2](#). Incidentally, for these examples we show that semisimplicity for $\mathcal{L}_X$ acting on Ω^1 holds but fails for $\mathcal{L}_{-X}$. Also semisimplicity holds for $\mathcal{L}_{\pm X}$ acting on Ω_0^1 .

1.6.4 Homologically full flows with $[\omega^\pm] \neq 0$ The case of suspensions studied in [Corollary 1.4](#) certainly provides examples of flows with $[\omega^\pm] \neq 0$. However, here we provide such examples which in addition are *homologically full*, ie every homology class contains a closed orbit. The examples live on the unit tangent bundle of a surface with negative curvature (and hence they are topologically orbit equivalent to a geodesic flow) and are in fact *Gaussian thermostats* or *W-flows* as introduced in [\[Wojtkowski 2000\]](#) (they are reparametrisations of the geodesics of a Weyl connection).

Let (M, g) be a closed oriented surface of negative Gaussian curvature and let ρ be a smooth closed 1-form. Consider $\lambda := \pi_1^* \rho \in C^\infty(SM)$ defined by lifting the 1-form ρ via $\pi_1^* \rho(x, v) = \rho_x(v)$, and $F := X + \lambda V$. By [\[Wojtkowski 2000, Theorem 5.2\]](#) the flow of F is Anosov. Moreover, by [\[Dairbekov and Paternain 2007\]](#), the flow preserves an absolutely continuous measure if and only if $\rho = 0$. Observe that these flows are reversible, ie the flip $(x, v) \mapsto (x, -v)$ conjugates the flow of F with the flow of $-F$.

Theorem 1.6 *Let $F = X + \pi_1^* \rho V$ be the generator of a Gaussian thermostat with ρ **harmonic** and nonzero. Then $[\omega^+] = -[\omega^-] \neq 0$ and thus $m_{1,0} = b_1(M) - 1$. Moreover, for ρ small enough, the action of $\mathcal{L}_F$ on Ω_0^1 is semisimple and the order of vanishing of the Ruelle zeta function of F is $m_R(0) = -\chi(M) - 1$.*

A Gaussian thermostat as in [Theorem 1.6](#) may also be seen as the geodesic flow of the unique affine connection compatible with g and with torsion $\mathcal{T}(Y, Z) = -\star \rho(Y)Z + \star \rho(Z)Y$ [\[Przytycki and Wojtkowski 2008\]](#). Interestingly enough, we will also show that it is possible to produce flows with similar features when ρ is *exact* and nonzero; see [Proposition 5.9](#).

1.7 Outline of the paper

- In [Section 2](#) we recall some preliminary notions from microlocal analysis and resonances for Anosov flows.
- In [Section 3](#) we study general 3-dimensional Anosov flows that do not necessarily preserve a smooth measure and we determine the dimension of the resonant spaces of Lie derivatives at zero in terms of the helicity and the winding cycles of the associated SRB measures, proving [Theorem 1.1](#).

- In [Section 4](#) we show horocyclic invariance of resonant states and in [Section 5](#) we study the local manifold structure of $\mathcal{W}^\pm$.
- [Section 6](#) discusses examples of Anosov flows with smooth weak stable bundle, $[\omega^+] = 0$ and $[\omega^-] \neq 0$.
- [Section 7](#) introduces thermostats and provides a proof of [Theorem 1.6](#).
- [Section 8](#) is devoted to quasi-Fuchsian flows arising from the coupled vortex equations determined by a pair $([g], A)$, where A is a quadratic holomorphic differential. [Theorem 1.5](#) is proved here.
- [Section 9](#) establishes the relation between the helicity and the linking form given by [Theorem 1.2](#).
- Finally, there are two appendices: [Appendix A](#) deduces some standard properties of the ladder operators $\eta_\pm$, while [Appendix B](#) studies the behaviour of the geodesic vector field under rescaling.

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2 Preliminaries

Let X generate an Anosov flow φ_t on a closed manifold $\mathcal{M}$. Denote by Ω^k the vector bundle of differential k -forms, and by $\Omega_0^k = \Omega^k \cap \ker \iota_X$ the bundle of k -forms that are in the kernel of the contraction ι_X with the vector field X .

Let $\mathcal{E}$ be a vector bundle over $\mathcal{M}$. Denote by $\mathcal{D}'(\mathcal{M}; \mathcal{E})$ the space of distributional sections of $\mathcal{E}$. Given a closed conic set $\Gamma \subset T^*M$, write

$$\mathcal{D}'_\Gamma(\mathcal{M}; \mathcal{E}) = \{u \in \mathcal{D}'(\mathcal{M}; \mathcal{E}) \mid \text{WF}(u) \subset \Gamma\}.$$

For $m \in \mathbb{R}$, denote the space of *pseudodifferential operators* of order m acting on sections of $\mathcal{E}$ over $\mathcal{M}$ by $\Psi^m(\mathcal{M}; \mathcal{E})$; when this action is on functions, write simply $\Psi^m(\mathcal{M})$. For background on distribution theory and the wavefront set, and pseudodifferential operators, see [[Hörmander 2003](#), Chapters II and VIII], and [[Hörmander 2007](#), Chapter XVIII] or [[Dyatlov and Zworski 2019](#), Appendix E], respectively.

Let us note that the treatment of the Pollicott–Ruelle resonances as eigenvalues of the flow generator in anisotropic Banach or Hilbert spaces has been developed in the past twenty years by Baladi [[2005](#)], Baladi and Tsujii [[2007](#)], Blank, Keller and Liverani [[Blank et al. 2002](#)], Gouëzel and Liverani [[2006](#)], and Liverani [[2004](#)]. In this paper (see [Section 2.1](#) below) we take the viewpoint of microlocal methods for dynamical resonances, introduced by Faure and Sjöstrand [[2011](#)] and Dyatlov and Zworski [[2016](#)].

2.1 The resolvent and Pollicott–Ruelle resonances

For $s \in \mathbb{C}$ with sufficiently large real part, we may define the resolvent $R_k^+(s) = (\mathcal{L}_X + s)^{-1}$ acting on $L^2(\mathcal{M}; \Omega^k)$ by the expression

$$(2-1) \quad (\mathcal{L}_X + s)^{-1} = \int_0^\infty e^{-st} \varphi_{-t}^* dt : L^2(\mathcal{M}; \Omega^k) \rightarrow L^2(\mathcal{M}; \Omega^k),$$

where φ_{-t}^* denotes the pullback operation by φ_{-t} . It has been shown by Faure and Sjöstrand [2011] and Dyatlov and Zworski [2016] that $R_k^+(s)$ admits a meromorphic extension to the entire complex plane as a map $R_k^+(s) : C^\infty(\mathcal{M}; \Omega^k) \rightarrow \mathcal{D}'(\mathcal{M}; \Omega^k)$. At a pole $s_0 \in \mathbb{C}$, the resolvent admits a Laurent expansion (see [Dyatlov and Zworski 2016])

$$(2-2) \quad R_k^+(s) = R_k^{+,H}(s; s_0) + \sum_{i=1}^{J_k(s_0)} \frac{(-(\mathcal{L}_X + s_0))^{i-1} \Pi_k^+(s_0)}{(s - s_0)^i}.$$

Here $\Pi_k^+(s_0)$ is a finite-rank operator that extends to a map $\Pi_k^+(s_0) : \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^k) \rightarrow \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^k)$ and satisfies $\Pi_k^+(s_0)^2 = \Pi_k^+(s_0)$. Further, $J_k(s_0) \geq 1$ is the least integer for which $(\mathcal{L}_X + s_0)^{J_k(s_0)} \Pi_k^+(s_0) = 0$ and $R_k^{+,H}(s; s_0)$ is a holomorphic function defined near $s = s_0$. The map $R_k^{+,H}(s; s_0)$ also extends to a map $R_k^{+,H}(s; s_0) : \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^k) \rightarrow \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^k)$ by [Dyatlov and Zworski 2016]. We call the poles of $R_k^+(s)$ *resonances*.

Given $s_0 \in \mathbb{C}$ and $\ell \in \mathbb{Z}_{\geq 1}$, we introduce (similarly as in the introduction)

$$\text{Res}^{k,\ell}(s_0) = \{u \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^k) \mid (\mathcal{L}_X + s_0)^\ell u = 0\}.$$

Write $\text{Res}^k(s_0) := \text{Res}^{k,1}(s_0)$ and $\text{Res}^{k,\infty}(s_0) := \bigcup_{\ell \geq 1} \text{Res}^{k,\ell}(s_0)$, and call the elements of $\text{Res}^k(s_0)$ and $\text{Res}^{k,\infty}(s_0)$ *resonant states* and *generalised resonant states*, respectively. It can be checked that $s_0 \in \mathbb{C}$ is a pole of $R_k^+(s)$ if and only if $\text{Res}^k(s_0)$ is nontrivial, and that $\text{ran } \Pi_k^+(s_0) = \text{Res}^{k,\infty}(s_0)$. By the residue theorem, we have

$$(2-3) \quad \Pi_k^+ = \frac{1}{2\pi i} \oint_{s_0} R_k^+(s) ds,$$

where $\oint_{s_0}$ denotes integration along a small contour around $s = s_0$. Say that *semisimplicity* holds for the action of $\mathcal{L}_X$ on Ω^k at s_0 if $\text{Res}^{k,\infty}(s_0) = \text{Res}^k(s_0)$.

Similarly, the resolvent for the backwards flow $R_k^-(s) = (-\mathcal{L}_X + s)^{-1}$ is well-defined on $L^2(\mathcal{M}; \Omega^k)$ for all $s \in \mathbb{C}$ with large real part. It also admits a meromorphic extension as a map $R_k^-(s) : C^\infty(\mathcal{M}; \Omega^k) \rightarrow \mathcal{D}'(\mathcal{M}; \Omega^k)$. For $s_0 \in \mathbb{C}$, call the resonant states of $-\mathcal{L}_X + s_0$ *coresonant states* and decorate the analogous spaces with an additional subscript $*$: $\text{Res}_*^{k,\ell}(s_0)$, $\text{Res}_*^{k,\infty}(s_0)$, $\text{Res}_*^k(s_0)$. The projector at $s = s_0$ is denoted by $\Pi_k^-(s_0)$.

One may define analogous objects for the action of $\mathcal{L}_X$ on the bundle Ω_0^k . In that case, denote the resulting objects with an additional zero subscript:

$$\text{Res}_{0(*)}^{k,\ell}(s_0), \quad \text{Res}_{0(*)}^{k,\infty}(s_0), \quad \text{Res}_{0(*)}^k(s_0), \quad \Pi_{k,0}(s_0)^\pm.$$

For $s_0 = 0$, in order to simplify the notation we omit the 0 in parentheses, denoting the resulting objects as

$$\text{Res}_{0(*)}^{k,\ell}, \quad \text{Res}_{0(*)}^{k,\infty}, \quad \text{Res}_{0(*)}^k, \quad \Pi_k^\pm, \quad \Pi_{k,0}^\pm.$$

Since Π_k^+ is given by the contour integral (2-3) (and similarly for Π_k^-), and the Lie derivative $\mathcal{L}_X$ commutes with the exterior differential d and the contraction ι_X , we have, on the domain of $\Pi_k^\pm$,

$$(2-4) \quad d\Pi_k^\pm = \Pi_{k+1}^\pm d \quad \text{and} \quad \Pi_{k-1}^\pm \iota_X = \iota_X \Pi_k^\pm.$$

2.2 The pairings

Let $\alpha \in C^\infty(\mathcal{M}; \Omega^1)$ be an arbitrary 1-form such that $\alpha(X) = 1$. Introduce the following nondegenerate bilinear pairing $\langle\langle \bullet, \bullet \rangle\rangle$ (similar to the contact case studied in [Cekić et al. 2022]):

$$\langle\langle u, u_* \rangle\rangle := \int_{\mathcal{M}} \alpha \wedge u \wedge u_* \quad \text{for } u \in C^\infty(\mathcal{M}; \Omega_0^k), \quad u_* \in C^\infty(\mathcal{M}; \Omega_0^{2-k}).$$

Note that the pairing does not depend on the choice of α , and that by the wavefront set calculus it extends to $\mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega_0^k) \times \mathcal{D}'_{E_s^*}(\mathcal{M}; \Omega_0^{2-k})$.

If $A: C^\infty(\mathcal{M}; \Omega_0^k) \rightarrow \mathcal{D}'(\mathcal{M}; \Omega_0^k)$ is a continuous operator, let $A^T: C^\infty(\mathcal{M}; \Omega_0^{2-k}) \rightarrow \mathcal{D}'(\mathcal{M}; \Omega_0^{2-k})$ be its transpose with respect to $\langle\langle \bullet, \bullet \rangle\rangle$. In particular, observe it holds that

$$(2-5) \quad (\mathcal{L}_X + s)^T = -\mathcal{L}_X + s \quad \text{for } s \in \mathbb{C}.$$

It follows that, using (2-3) and meromorphic continuation from $s \in \mathbb{C}$ with large real part,

$$(2-6) \quad (\Pi_{k,0}^+)^T = \Pi_{2-k,0}^-.$$

More precisely, from (2-5), for large $\text{Re}(s)$ we have $\langle\langle (\mathcal{L}_X + s)^{-1} u, u_* \rangle\rangle = \langle\langle u, (-\mathcal{L}_X + s)^{-1} u_* \rangle\rangle$, and by meromorphic continuation for all $s \in \mathbb{C}$. Using (2-3) proves the claim. Moreover we can relate semisimplicity for $\mathcal{L}_X$ on Ω_0^k with the pairing $\langle\langle \bullet, \bullet \rangle\rangle$:

Lemma 2.1 *The Lie derivative $\mathcal{L}_X$ acting on Ω_0^1 is semisimple at $s = 0$ if and only if $\langle\langle \bullet, \bullet \rangle\rangle$ is nondegenerate on $\text{Res}_0^1 \times \text{Res}_{0*}^1$.*

Proof Assume first that the pairing is nondegenerate and $\mathcal{L}_X^2 u = 0$ for some $u \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega_0^1)$. Set $v := \mathcal{L}_X u$. Then $v \in \text{Res}_0^1$ and for any $u_* \in \text{Res}_{0*}^1$ we have

$$\langle\langle v, u_* \rangle\rangle = \langle\langle \mathcal{L}_X u, u_* \rangle\rangle = -\langle\langle u, \mathcal{L}_X u_* \rangle\rangle = 0.$$

By nondegeneracy $\mathcal{L}_X u = v = 0$ and the semisimplicity condition is verified.

For the other direction, assume that the semisimplicity condition holds. First we check that semisimplicity holds also for coresonant states. Indeed, since by assumption $\mathcal{L}_X \Pi_{1,0}^+ = \Pi_{1,0}^+ \mathcal{L}_X \equiv 0$, transposing we get $\mathcal{L}_X \Pi_{1,0}^- \equiv 0$, which proves the claim.

Next, assume $u \in \text{Res}_0^1$ satisfies $\langle\langle u, u_* \rangle\rangle = 0$ for all $u_* \in \text{Res}_{0*}^1$. Take any $\varphi \in C^\infty(\mathcal{M}; \Omega_0^1)$ and compute

$$0 = \langle\langle u, \Pi_{1,0}^- \varphi \rangle\rangle = \langle\langle u, \varphi \rangle\rangle,$$

where in the second equality we used (2-6) and $\Pi_{1,0}^+ u = u$. By the nondegeneracy of $\langle\langle \bullet, \bullet \rangle\rangle$, we conclude that $u \equiv 0$, proving the result. $\square$

Another nondegenerate pairing on $C^\infty(\mathcal{M}; \Omega^k) \times C^\infty(\mathcal{M}; \Omega^{3-k})$, with an extension to $\mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^k) \times \mathcal{D}'_{E_s^*}(\mathcal{M}; \Omega^{3-k})$, is given by

$$\langle u, u_* \rangle := \int_{\mathcal{M}} u \wedge u_*.$$

Note that similarly to $\langle\langle \bullet, \bullet \rangle\rangle$, with respect to this pairing the transpose satisfies

$$(2-7) \quad (\Pi_k^+)^T = \Pi_{3-k}^-.$$

In fact, we have:

Proposition 2.2 *The pairing $\langle \bullet, \bullet \rangle$ is nondegenerate on $\text{Res}^{k,\infty} \times \text{Res}_*^{3-k,\infty}$. Moreover, the action of $\mathcal{L}_X$ on Ω^k is semisimple if and only if the pairing $\langle \bullet, \bullet \rangle$ is nondegenerate on $\text{Res}^k \times \text{Res}_*^{3-k}$.*

Proof It suffices to consider some $u \in \text{Res}^{k,\infty}$ such that

$$\langle u, u_* \rangle = 0 \quad \text{for all } u_* \in \text{Res}_*^{k,\infty},$$

and show $u = 0$. In particular, for any $\varphi \in C^\infty(\mathcal{M}; \Omega^{3-k})$ it holds that

$$0 = \langle u, \Pi_{3-k}^- \varphi \rangle = \langle u, \varphi \rangle,$$

where we used (2-7) and $\Pi_k^+ u = u$ in the second equality. Therefore $u = 0$, by the fact that the pairing on $C^\infty(\mathcal{M}; \Omega^k) \times C^\infty(\mathcal{M}; \Omega^{3-k})$ is nondegenerate.

The second claim is proved analogously to Lemma 2.1, using (2-7). $\square$

2.3 Mapping properties of the resolvent

For $s \in \mathbb{R}$, denote by $C_*^s(\mathcal{M})$ the Hölder–Zygmund space of regularity s on $\mathcal{M}$. See [Bonthonneau and Lefeuvre 2023b, Section 2] or [Taylor 1997, Section 1.8] for a definition and background. In particular, a pseudodifferential operator of order m is a bounded map $C_*^s(\mathcal{M}) \rightarrow C_*^{s-m}(\mathcal{M})$. For s a nonnegative integer, these spaces agree with the usual Hölder spaces, and satisfy $C^k(\mathcal{M}) \subsetneq C_*^k(\mathcal{M})$ for $k \in \mathbb{Z}_{\geq 0}$. Let us write $C_*^{s-}(\mathcal{M}) := \cap_{t < s} C_*^t(\mathcal{M})$. We need the following statement about the mapping properties of the resolvent on Hölder–Zygmund spaces:

Proposition 2.3 *Let X generate an Anosov flow. Then:*

- (1) *Let $r \in C^\alpha(\mathcal{M})$ for some $\alpha > 0$ be such that $\liminf_{t \rightarrow \infty} \inf_{x \in \mathcal{M}} (1/t) \int_0^t \varphi_{-p}^* r(x) dp > \nu$ for some $\nu > 0$. Then there exist $\delta, \delta_1 > 0$ such that $(X + r + s)^{-1}$ exists and is holomorphic as a map $C^0(\mathcal{M}) \rightarrow C^0(\mathcal{M})$ and $C^\delta(\mathcal{M}) \rightarrow C^\delta(\mathcal{M})$ in the region $\text{Re}(s) > -\delta_1 > -\nu$.*

- (2) There exists $C > 0$, such that for any $\alpha > 0$ the resolvent $R_0^+(s) = (X + s)^{-1} : C_*^\alpha(\mathcal{M}) \rightarrow C_*^{-\alpha}(\mathcal{M})$ is meromorphic for $\operatorname{Re}(s) > -C\alpha$. In particular, the holomorphic part of the resolvent at zero $R_0^{+,H} : C^\infty(\mathcal{M}) \rightarrow \mathcal{D}'(\mathcal{M})$ extends as an operator $C_*^\alpha(\mathcal{M}) \rightarrow C_*^{-\alpha}(\mathcal{M})$ for all $\alpha > 0$.

Proof For item (1), it suffices to use the resolvent formula

$$(2-8) \quad (X + r + s)^{-1} = \int_0^\infty e^{-st} e^{-\int_0^t \varphi_{-p}^* r \, dp} \varphi_{-t}^* \, dt.$$

The conclusion on the mapping properties as a map $C^0(\mathcal{M}) \rightarrow C^0(\mathcal{M})$ follows directly from the formula, and the case of $C^\delta(\mathcal{M}) \rightarrow C^\delta(\mathcal{M})$ follows for small $\delta > 0$ by using interpolation between $C^0(\mathcal{M})$ and $C^1(\mathcal{M})$.

Item (2) follows from the recent result [Bonthonneau and Lefeuvre 2023a, Theorem 4.2], which was proved in the setting of the unit tangent bundle of a cusp manifold, but the proof carries over to our setting. Let us sketch the proof for completeness. Pick $A \in \Psi^0(\mathcal{M})$ such that $A = \mathbb{1}$ on a conical neighbourhood $\mathcal{C}_u^0$ of E_u^* and $\operatorname{WF}(A)$ is contained in a larger conical neighbourhood $\mathcal{C}_u^1$ which does not intersect $E_s^* \oplus E_0^*$ (by definition, E_0^* is the annihilator of $E_u \oplus E_s$). For $\alpha > 0$ introduce the anisotropic norm

$$\|u\|_\star := \|Au\|_{C_*^{-\alpha}} + \|(\mathbb{1} - A)u\|_{C_*^\alpha}.$$

Choose $T > 0$ such that for all $t \geq T$ we have $\Phi_t(\mathcal{C}_u^1) \subset \mathcal{C}_u^0$, where $\Phi_t(x, \xi) = (\varphi_t x, \xi \circ (d\varphi_t(x))^{-1})$ is the Hamiltonian lift on φ_t to $T^*\mathcal{M}$ (this choice is possible by the Anosov condition). Finally, as in [Bonthonneau and Lefeuvre 2023a, equation (4.7)], introduce the norm

$$\|u\|_{C^\alpha} := \int_0^T \|\varphi_{-t}^* u\|_\star \, dt,$$

which defines a Banach space $\mathbf{C}^\alpha(\mathcal{M})$. It is straightforward to show the continuous inclusions

$$C_*^\alpha(\mathcal{M}) \subset \mathbf{C}^\alpha(\mathcal{M}) \subset C_*^{-\alpha}(\mathcal{M}),$$

so our conclusion will follow if we can show that $R_0^+(s)$ extends as a family of meromorphic operators acting on $\mathbf{C}^\alpha(\mathcal{M})$ in the required region. As in [loc. cit., Lemma 4.3], it is possible to show there is a constant $C_0 > 0$ such that the propagator φ_{-t}^* satisfies the bound

$$(2-9) \quad \|\varphi_{-t}^* u\|_{C^\alpha} \leq C_0 e^{C_0 t \alpha} \|u\|_{C^\alpha}, \quad \text{where } \alpha > 0.$$

Next, by the radial sink/source estimates in Hölder–Zygmund spaces proved in [loc. cit.], following the proof of [loc. cit., Proposition 4.3] and using (2-9), it is possible to show that there exists a $C > 0$ such that for $\operatorname{Re}(s) > -C\alpha$,

$$\|u\|_{C^\alpha} \leq C(\|(X + s)u\|_{C^\alpha} + \|Ku\|_{C^\alpha}) \quad \text{for } u \in C^\infty(\mathcal{M}),$$

where K is a smoothing operator. From this estimate, it follows that $X + s$ is a semi-Fredholm operator (it has finite-dimensional kernel and closed range) with domain $\mathcal{D}(X) = \{u \in \mathbf{C}^\alpha(\mathcal{M}) \mid Xu \in \mathbf{C}^\alpha(\mathcal{M})\}$, and hence $R_0^+(s) : \mathbf{C}^\alpha(\mathcal{M}) \rightarrow \mathbf{C}^\alpha(\mathcal{M})$ admits a meromorphic continuation by the analytic Fredholm theorem, completing the proof. $\square$

2.4 Geometry of surfaces

For details about the content of this section, see [Guillemin and Kazhdan 1980; Merry and Paternain 2011; Singer and Thorpe 1976]. Let (M, g) be an oriented closed surface and let X be the geodesic vector field on the unit sphere bundle SM . Denote by V the generator of the fibrewise rotation action and by H the horizontal vector field. Let K_g be the Gauss curvature of (M, g) . Then

$$(2-10) \quad [H, V] = X, \quad [V, X] = H, \quad [X, H] = KV.$$

Dually to the global frame $\{X, H, V\}$, define a global frame of 1-forms $\{\alpha, \beta, \psi\}$. Set $\Omega := \alpha \wedge \psi \wedge \beta = \alpha \wedge d\alpha$ to be the canonical volume form on SM . For $k \in \mathbb{Z}$ and $x \in M$, introduce

$$\mathfrak{V}_k(x) = \{f \in C^\infty(S_x M) \mid Vf = ikf\}.$$

It is straightforward to see that $\mathfrak{V}_k \rightarrow M$ has the structure of a smooth line bundle; a smooth section of this bundle may be seen as a smooth function on SM . In fact if $\mathcal{K} := (T_{\mathbb{C}}^* M)^{1,0}$ is the associated *canonical* bundle and $\mathcal{K}^{-1} := (T_{\mathbb{C}}^* M)^{0,1}$, then for any $k \in \mathbb{Z}$, $\mathfrak{V}_k$ may be identified with the tensor power $\mathcal{K}^{\otimes k}$. For any $k \in \mathbb{Z}_{\geq 0}$ and $x \in M$, there are natural maps

$$\pi_k^*: (\otimes_S^k T_{\mathbb{C}}^* M)_x \rightarrow C^\infty(S_x M), \quad \text{with } \pi_k^* T(v) = T(v, \dots, v),$$

where $\otimes_S^k T_{\mathbb{C}}^* M$ denotes the bundle of symmetric tensors of degree k . Denote by $\otimes_S^k T_{\mathbb{C}}^* M|_{0-\text{tr}}$ the subbundle of *trace-free* symmetric tensors, where a symmetric k -tensor T at x is trace-free if for an orthonormal basis $(e_i)_{i=1}^2$ of $T_x M$, $T(e_1, e_1, \dots) + T(e_2, e_2, \dots) = 0$. It is straightforward to see that there are vector bundle isomorphisms

$$\begin{aligned} \otimes_S^k T_{\mathbb{C}}^* M|_{0-\text{tr}} &= \mathcal{K}^{\otimes k} \oplus \mathcal{K}^{-\otimes k} \quad \text{for all } k \in \mathbb{Z}_{>0}, \\ \pi_{|k|}^*: \mathcal{K}^{\otimes k} &\xrightarrow{\cong} \mathfrak{V}_k \quad \text{for all } k \in \mathbb{Z}. \end{aligned}$$

For $k \in \mathbb{Z}$, set $H_k := C^\infty(M; \mathfrak{V}_k) \subset C^\infty(SM)$. Then $\pi_{|k|}^*$ identifies smooth sections of $\mathcal{K}^k$ with H_k .

By decomposing into eigenstates of V , it is straightforward to see that

$$L^2(SM) = \bigoplus_{k \in \mathbb{Z}} L^2(M; \mathfrak{V}_k),$$

and for $f \in L^2(SM)$, let us write $f = \sum_{k \in \mathbb{Z}} f_k$ for the corresponding decomposition. We refer to f_k as the *Fourier mode of degree k* of f . Define the *degree* of f to be the maximal $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that $f_k \neq 0$ or $f_{-k} \neq 0$. Introduce the *raising/lowering* operators $\eta_\pm$:

$$\eta_+ = \frac{X - iH}{2} \quad \text{and} \quad \eta_- = \frac{X + iH}{2}.$$

By (2-10) it follows that

$$[\eta_\pm, V] = \mp i \eta_\pm,$$

and therefore $\eta_\pm: H_k \rightarrow H_{k \pm 1}$. Another consequence of (2-10) is the commutation relation

$$(2-11) \quad [\eta_+, \eta_-] = \frac{1}{2} i K_g V.$$

The Hodge star operator $\star: \Omega^1 \xrightarrow{\cong} \Omega^1$ is given by (oriented) rotation by $\pi/2$, and $\star: \Omega^0 \xrightarrow{\cong} \Omega^2$ by $\star(\mathbf{1}) = d\text{vol}_g$, the volume form of g , where $\mathbf{1}$ is the constant function equal to 1 everywhere. It can be checked that the codifferential on 1-forms is given by $d^* = -\star d\star$. Moreover, if X_- is defined on $H_{-1} \oplus H_1$ by $X_-(f_{-1} + f_1) = \eta_+ f_{-1} + \eta_- f_1$, by [Proposition A.3](#) we have

$$(2-12) \quad X_- \pi_1^* \gamma = -\frac{1}{2} \pi_0^* d^* \gamma \quad \text{for all } \gamma \in C^\infty(M; \Omega^1).$$

Finally, it is standard and follows from the expressions for X and H in local isothermal coordinates (see [\(A-1\)](#)) that

$$(2-13) \quad X \pi_0^* f = \pi_1^*(df) \quad \text{and} \quad H \pi_0^* f = -\pi_1^*(\star df) \quad \text{for all } f \in C^\infty(M),$$

and also similarly that

$$V \pi_1^* \gamma = -\pi_1^*(\star \gamma) \quad \text{and} \quad \iota_H \pi^*(\star \gamma) = \pi_1^* \gamma \quad \text{for all } \gamma \in C^\infty(M; \Omega^1).$$

2.5 Decomposition of differential forms

In this section we give a standard result about decomposition of differential forms, similar to [\[Dyatlov and Zworski 2017, Lemma 2.1\]](#). Denote by d^* the codifferential on $(\mathcal{M}, g)$, where g is an arbitrary Riemannian metric on $\mathcal{M}$. Denote by $\Delta_k := d^*d + dd^*$ the *Hodge Laplacian* on Ω^k .

Lemma 2.4 *Let $\Gamma \subset T^*\mathcal{M} \setminus 0$ be a closed conic set. Let $u \in \mathcal{D}'_\Gamma(\mathcal{M}; \Omega^k) \cap C_*^s(\mathcal{M}; \Omega^k)$ for some $s \in \mathbb{R}$. Then there exist $v \in \mathcal{D}'_\Gamma(\mathcal{M}; \Omega^{k-1}) \cap C_*^{s+1}(\mathcal{M}; \Omega^{k-1})$, $w \in \mathcal{D}'_\Gamma(\mathcal{M}; \Omega^{k+1}) \cap C_*^{s+2}(\mathcal{M}; \Omega^{k+1})$, and $\theta \in C^\infty(\mathcal{M}; \Omega^k)$, with $d^*v = 0$, $dw = 0$, such that*

$$u = dv + d^*w + \theta.$$

Moreover, if $du = 0$ then we may assume $w = 0$, thus we have $d\theta = 0$, and the cohomology class $[\theta] \in H^k(\mathcal{M})$ is well-defined.

Proof Let $Q_k \in \Psi^{-2}(\mathcal{M}; \Omega^k)$ be an elliptic parametrix of Δ_k , ie such that

$$Q_k \Delta_k - \text{Id}, \quad \text{with } \Delta_k Q_k - \text{Id} \in \Psi^{-\infty}(\mathcal{M}; \Omega^k).$$

Setting $\theta := u - \Delta_k Q_k u$, $v := d^* Q_k u$ and $w := d Q_k u$ concludes the proof of the first claim.

For the second claim, notice that using $[\Delta_k, d] = 0$ and the properties of the parametrix,

$$d Q_k \equiv Q_k \Delta_k d Q_k \equiv Q_k d \Delta_k Q_k \equiv Q_k d \quad \text{mod } \Psi^{-\infty}(\mathcal{M}).$$

Thus, using the proof of the first claim, we may assume that $w = 0$. If we write $u = dv_i + \theta_i$ for some v_i, θ_i as above for $i = 1, 2$, then $\theta_1 - \theta_2 = d(v_2 - v_1) \in C^\infty(\mathcal{M}; \Omega^k)$. Using $d^*(v_2 - v_1) = 0$, elliptic regularity implies $v_2 - v_1$ is smooth and thus $[\theta]$ is well-defined, finishing the proof. $\square$

2.6 SRB measures and entropy production

A probability measure Ω_{SRB}^+ on $\mathcal{M}$ is called a Sinai–Ruelle–Bowen (SRB) measure if there is a probability volume form Ω on $\mathcal{M}$ such that (see [Young 2002])

$$(2-14) \quad \Omega_{\text{SRB}}^+ = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi_{-t}^* \Omega \, dt$$

in the weak limit sense (ie when paired with any smooth function). It is known that for *transitive* Anosov flows (recall that by definition transitive flows have a dense orbit), there exists a unique SRB measure; see for instance [Butterley and Liverani 2007]. This measure may be seen as a resonant state for the Lie derivative acting on differential forms of top degree. For a microlocal point of view, see [Bonthonneau et al. 2025, Theorem 3], who characterise SRB measures as invariant measures whose wavefront set lies in E_u^* . In this paper, we will use the latter description. Similarly, there is a unique SRB measure for the flow $-X$ that we denote by Ω_{SRB}^- (it has wavefront set in E_s^*).

Next we examine the transformation law for SRB measures under a (positive) time-change $f \in C^\infty(\mathcal{M})$. Assume X is transitive with the unique SRB measure Ω_{SRB}^+ . Then the unique SRB measure $\Omega_{\text{SRB}}^+(fX)$ of fX is given by the formula

$$\Omega_{\text{SRB}}^+(fX) = \frac{f^{-1} \Omega_{\text{SRB}}^+}{\int_{\mathcal{M}} f^{-1} \Omega_{\text{SRB}}^+}.$$

Given an arbitrary smooth volume form Ω on $\mathcal{M}$, the *entropy production* of the SRB measure is given by

$$e^+(X) := - \int_{\mathcal{M}} \text{div}_\Omega(X) \Omega_{\text{SRB}}^+.$$

This quantity does not depend on the chosen volume form Ω , since given a smooth function $f > 0$, $\text{div}_{f\Omega}(X) - \text{div}_\Omega(X) = X(\log f)$. Note that

$$e^+(-X) = \int_{\mathcal{M}} \text{div}_\Omega(X) \Omega_{\text{SRB}}^- =: e^-(X).$$

An important property of entropy production was proved by Ruelle [1996, Theorem 1.2], who showed

$$e^+(X) \geq 0,$$

with equality if and only if X preserves a smooth measure. Later in the article we shall see several interesting classes of Anosov vector fields with $e^+(X) > 0$.

3 Dissipative Anosov flows, general case

In this section we discuss resonant forms at zero for an arbitrary topologically transitive Anosov flow X on an oriented 3-manifold $\mathcal{M}$ equipped with a Riemannian metric g and a smooth volume form Ω that integrates to one.

As a first step, we analyse the spaces Res_0^k for $k = 0, 2$, and Res^3 . The classification for $k = 1$ is relegated to Theorem 1.1.

Proposition 3.1 We have:

- (1) $\dim \operatorname{Res}^{0,\infty} = \dim \operatorname{Res}^{3,\infty} = 1$. The space $\operatorname{Res}^0 = \operatorname{Res}^{0,\infty}$ is spanned by constant functions, and Res^3 is spanned by the SRB measure Ω_{SRB}^+ .
- (2) $\dim \operatorname{Res}_0^{2,\infty} = 1$. The space Res_0^2 is spanned by $\iota_X \Omega_{\text{SRB}}^+$.

Proof The proof is a straightforward consequence of well-known facts. By [Bonthonneau et al. 2024, Theorem 3] (or [Butterley and Liverani 2007, Theorem 1]), the space Res^3 for a transitive flow is spanned by the SRB measure Ω_{SRB}^+ . Moreover, semisimplicity holds: if $\mathcal{L}_X^2 \tilde{\Omega} = 0$ for some $\tilde{\Omega} \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^3)$, then $\mathcal{L}_X \tilde{\Omega} = c \Omega_{\text{SRB}}^+$ for some $c \in \mathbb{C}$. Stokes' theorem implies that $c = 0$, showing semisimplicity for the action of $\mathcal{L}_X$ on Ω^3 and the semisimplicity for the action of X on Ω^0 (functions) follows by Proposition 2.2. By the latter fact it follows that $\operatorname{Res}^{0,\infty}$ is spanned by constant functions.

Next, if $\mathcal{L}_X u = 0$ for some $u \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega_0^2)$, then $du \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^3)$ and so $du = c \Omega_{\text{SRB}}^+$ for some $c \in \mathbb{C}$. By Stokes' theorem we get $c = 0$ and thus $du = 0$. Write $u = h \cdot \iota_X \Omega$ for some $h \in \mathcal{D}'_{E_u^*}(\mathcal{M})$. This implies $\mathcal{L}_X(h\Omega) = 0$ and so $h\Omega \in \operatorname{Res}^3$. This gives that Res_0^2 is spanned by $\iota_X \Omega_{\text{SRB}}^+$. Finally, to show that $\dim \operatorname{Res}_0^{2,\infty} = 1$, consider $u \in \mathcal{D}'_{E_u^*}(\mathcal{M}, \Omega_0^2)$ with $\mathcal{L}_X^2 u = 0$ and set $v = \mathcal{L}_X u$. We wish to show that $v = 0$. Since $v \in \operatorname{Res}_0^2$, there is a constant c such that $v = c \iota_X \Omega_{\text{SRB}}^+$. Thus

$$c = c \int_{\mathcal{M}} \Omega_{\text{SRB}}^+ = c \int_{\mathcal{M}} \alpha \wedge \iota_X \Omega_{\text{SRB}}^+ = \int_{\mathcal{M}} \alpha \wedge v = \int_{\mathcal{M}} \alpha \wedge \iota_X du = \int_{\mathcal{M}} du = 0,$$

and therefore $v = 0$ as desired. (Alternatively, by the version of Lemma 2.1 for $\operatorname{Res}_0^2 \times \operatorname{Res}_0^{0,*}$, semisimplicity follows from the semisimplicity of $-\mathcal{L}_X$ on Ω^0 established in the previous paragraph.) $\square$

3.1 Characterisation of Res_0^1

Let us follow a similar strategy as in [Cekić and Paternain 2020, Section 4]. By Proposition 3.1, the two SRB measures are given by

$$\Omega_{\text{SRB}}^{\pm} = \Pi_3^{\pm}(\Omega).$$

Recall that here Π_k^+ and Π_k^- are the projections onto $\operatorname{Res}^{k,\infty}$ and $\operatorname{Res}_*^{k,\infty}$, respectively, introduced in Section 2.1. Next, introduce the notation

$$\omega := \iota_X \Omega \quad \text{and} \quad \omega^{\pm} := \iota_X \Omega_{\text{SRB}}^{\pm}.$$

Define the winding cycle W^+ (resp. W^-) by setting, for any $u \in \mathcal{D}'_{E_s^*}(\mathcal{M}; \Omega^1)$ (resp. $u \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^1)$),

$$W^{\pm}(u) = \int_{\mathcal{M}} u(X) \Omega_{\text{SRB}}^{\pm}.$$

When we additionally impose that $du = 0$, the cohomology class $[u]_{H^1(\mathcal{M})}$ of u is well-defined by Lemma 2.4, $W^{\pm}(u)$ only depend on $[u]_{H^1(\mathcal{M})}$ since $\Omega_{\text{SRB}}^{\pm}$ are flow invariant, and so in particular $W^{\pm}$ descend to $H^1(\mathcal{M})$. In what follows, for simplicity we will often drop the index under the cohomology class. Note that $W^{\pm} \equiv 0$ (on $\ker d$) if and only if $[\omega^{\pm}] = 0$, by Lemma 2.4 and Poincaré duality.

Say that the SRB measure Ω_{SRB}^+ (resp. Ω_{SRB}^-) is *exact* or *null-homologous* if $[\omega^+] = 0$ (resp. $[\omega^-] = 0$). Equivalently, by Lemma 2.4 this means that $\omega^+ = d\tau^+$ (resp. $\omega^- = d\tau^-$) for some $\tau^+ \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^1)$ (resp. $\tau^- \in \mathcal{D}'_{E_s^*}(\mathcal{M}; \Omega^1)$). Still equivalently, applying the projector Π_2^+ (resp. Π_2^-) and using (2-4), we may assume that $\tau^+ \in \text{Res}^{1,\infty}$ (resp. $\tau^- \in \text{Res}_*^{1,\infty}$). Note that $\iota_X \tau^+ \in \text{Res}^{0,\infty}$ (resp. $\iota_X \tau^- \in \text{Res}_*^{0,\infty}$), and so by Proposition 3.1 we know $\iota_X \tau^+ = c^+$ (resp. $\iota_X \tau^- = c^-$) for some constants $c^\pm$, which implies $\tau^+ \in \text{Res}^1$ (resp. $\tau^- \in \text{Res}_*^1$). It may be assumed that $c^\pm \in \mathbb{R}$.

Let us introduce the *helicity* $\mathcal{H}(X)$ with respect to the SRB measures. Assume that *both* $\Omega_{\text{SRB}}^\pm$ are null-homologous, that is, $[\omega^+] = [\omega^-] = 0$. Introduce

$$\mathcal{H}(X) := c^+ = \int_{\mathcal{M}} \tau^+(X) \Omega_{\text{SRB}}^- = \int_{\mathcal{M}} \tau^+ \wedge d\tau^- = \int_{\mathcal{M}} \tau^- \wedge d\tau^+ = \int_{\mathcal{M}} \tau^-(X) \Omega_{\text{SRB}}^+ = c^-,$$

where we used Stokes' theorem in the fourth equality. It also follows from Stokes' theorem that $\mathcal{H}(X)$ is independent of the choices of both primitives $\tau^\pm$ of $\omega^\pm$, so the helicity $\mathcal{H}(X) \in \mathbb{R}$ is a well-defined quantity.

Proposition 3.2 *Let $f \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ and assume $\int_{\mathcal{M}} f \Omega_{\text{SRB}}^- = 0$. Then there exists $u \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ such that $Xu = f$.*

Proof By Proposition 3.1 and (2-2), near zero we may write

$$R_0^+(s) = R_0^{+,H}(s) + \frac{\Pi_0^+}{s}.$$

Therefore, by applying $X + s$ to this equation we obtain close to zero

$$(3-1) \quad (X + s)R_0^{+,H}(s) + \Pi_0^+ = \text{Id}.$$

Introduce $u := R_0^{+,H} f$, which lies in $\mathcal{D}'_{E_u^*}(\mathcal{M}; \mathcal{E})$ by the mapping properties of $R_0^{+,H}$. Then, assuming $\Pi_0^+ f = 0$, we have by (3-1) evaluated at $s = 0$,

$$f = f - \Pi_0^+ f = XR_0^{+,H} f = Xu.$$

Now we prove that $\Pi_0^+ f = 0$. For this, use that by (2-7) we have $(\Pi_0^+)^T = \Pi_3^-$ and

$$\Pi_0^+ f = \langle \Pi_0^+ f, \Omega_{\text{SRB}}^- \rangle = \langle f, \Pi_3^-(\Omega_{\text{SRB}}^-) \rangle = \langle f, \Omega_{\text{SRB}}^- \rangle = 0,$$

where in the first equality we used that $\Pi_0^+ f$ is constant and that Ω_{SRB}^- integrates to 1, and in the third equality that $\Pi_3^- \Omega_{\text{SRB}}^- = \Omega_{\text{SRB}}^-$ by definition, thus completing the proof. $\square$

We proceed with an auxiliary lemma.

Lemma 3.3 *There exists $\tilde{\tau} \in \text{Res}^1$ with $\iota_X \tilde{\tau} = 1$ and $d\tilde{\tau} = 0$, if and only if $[\omega^-] \neq 0$.*

Proof Assume first that such $\tilde{\tau}$ exists. Then $W^-(\tilde{\tau}) = 1$ and so $[\omega^-] \neq 0$ (recall that $W^- \equiv 0$ if and only if $[\omega^-] = 0$, by Lemma 2.4 and Poincaré duality), proving one of the implications. To see the other direction, assume $[\omega^-] \neq 0$. There is a closed, smooth 1-form η such that $W^-(\eta) = 1$. By Proposition 3.2

this implies we can solve

$$XF = 1 - \iota_X \eta, \quad \text{with } F \in \mathcal{D}'_{E_u^*}(\mathcal{M}).$$

Set $\tilde{\tau} = dF + \eta$. Then $\tilde{\tau}$ satisfies all of the assumptions of the lemma, completing the proof. $\square$

The next lemma determines $d(\text{Res}_0^1)$ as a function of $[\omega^\pm]$ and $\mathcal{H}(X)$.

Lemma 3.4 *There is a linear map $T: \text{Res}_0^1 \rightarrow \mathbb{C}$ such that $du = T(u)\omega^+$, where $u \in \text{Res}_0^1$. The map T satisfies the following:*

- (1) *If $[\omega^+] \neq 0$, or $[\omega^+] = [\omega^-] = 0$ and $\mathcal{H}(X) \neq 0$, then T is trivial.*
- (2) *If $[\omega^+] = 0$, and either $[\omega^-] = 0$ with $\mathcal{H}(X) = 0$, or $[\omega^-] \neq 0$, then T is surjective.*

Proof Let $u \in \text{Res}_0^1$. Then $du \in \text{Res}_0^2$, so $du = c\omega^+$ for some $c \in \mathbb{C}$ by Proposition 3.1. Setting $T(u) := c$ defines a linear map such that $du = T(u)\omega^+$. It clearly follows that $T \equiv 0$ if $[\omega^+] \neq 0$.

If $[\omega^+] = 0$, then $\omega^+ = d\tau^+$ for some $\tau^+ \in \text{Res}^1$ with $\iota_X \tau^+ = c^+$, where $c^+ \in \mathbb{C}$, and we have

$$d(u - T(u)\tau^+) = 0.$$

Now note that, using $\iota_X u = 0$, and the invariance of Ω_{SRB}^+ ,

$$(3-2) \quad -T(u)c^+ = W^-(u - T(u)\tau^+).$$

Assume first that $[\omega^-] = 0$; then also $W^- \equiv 0$. If $c^+ = \mathcal{H}(X) \neq 0$, (3-2) implies $T \equiv 0$; if $c^+ = \mathcal{H}(X) = 0$, then $\tau^+ \in \text{Res}_0^1$ and $d\tau^+ = \omega^+ \neq 0$, so $T \neq 0$.

Next, assume that $[\omega^-] \neq 0$. By Lemma 3.3, there is $\tilde{\tau} \in \text{Res}^1$ such that $d\tilde{\tau} = 0$ and $\iota_X \tilde{\tau} = 1$. Therefore $\tau^+ - c^+ \tilde{\tau} \in \text{Res}_0^1$ and $d(\tau^+ - c^+ \tilde{\tau}) = \omega^+ \neq 0$, so $T \neq 0$. This completes the proof. $\square$

Next, we compute the dimension of the space of closed elements of Res_0^1 .

Lemma 3.5 *There is an injection*

$$(3-3) \quad S: \ker T \hookrightarrow H^1(\mathcal{M}).$$

The injection can be described as follows: let $u \in \ker T$. Then there exists $F \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ such that

$$(3-4) \quad u - dF \in C^\infty(\mathcal{M}; \Omega^1)$$

and also $d(u - dF) = 0$. The injection map is

$$(3-5) \quad S: \ker T \rightarrow H^1(\mathcal{M}), \quad u \mapsto [u - dF].$$

An element $[\eta] \in H^1(\mathcal{M})$ is in the image of S if and only if

$$W^-(\eta) = \int_{\mathcal{M}} \eta(X) \Omega_{\text{SRB}}^- = 0.$$

Finally, we have

- (1) $\dim S(\ker T) = b_1(\mathcal{M})$ if $[\omega^-] = 0$,
- (2) $\dim S(\ker T) = b_1(\mathcal{M}) - 1$ if $[\omega^-] \neq 0$.

Proof Let $u \in \ker T$, so $du = 0$. By Lemma 2.4 there is an $F \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ such that $u - dF \in C^\infty(\mathcal{M}; \Omega^1)$ is closed. We claim that the class $S(u) = [u - dF] \in H^1(\mathcal{M})$ is independent of our choice of F . Let $G \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ be arbitrary such that $u - dG$ is smooth and closed. Then $d(F - G) \in C^\infty(\mathcal{M}; \Omega^1)$, so by Lemma 2.4, $F - G$ is smooth and thus $[u - dF] = [u - dG]$.

For injectivity, let us assume that $u - dF$ is exact. Moreover, without loss of generality we may assume $u = dF$. Then $\iota_X u = 0$ implies $XF = 0$, so by Proposition 3.1 we have that F is constant, so $u = 0$.

If $[\eta]$ is in the image of S , then $\eta = u - dF$ for some $F \in \mathcal{D}'_{E_u^*}(\mathcal{M})$. Contracting with X , we see that $\eta(X) = -XF$, so integrating gives

$$W^-(\eta) = \int_{\mathcal{M}} \eta(X) \Omega_{\text{SRB}}^- = 0.$$

Conversely, if the last integral is zero, Proposition 3.2 gives an $F \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ such that $\eta(X) = -XF$, so $u := \eta + dF \in \ker T$ and $S(u) = [\eta]$.

Finally, observe $S(\ker T) = \ker W^-|_{H^1(\mathcal{M})}$. Since $W^- \equiv 0$ if and only if $[\omega^-] = 0$, the conclusions follow. $\square$

Let us now put everything together and compute Res_0^1 in terms of $[\omega^\pm]$ and $\mathcal{H}(X)$.

Proof of Theorem 1.1 for Res_0^1 This is a direct consequence of the rank-nullity theorem and Lemmas 3.4 and 3.5, since we may write

$$\dim \text{Res}_0^1 = \dim S(\ker T) + \dim \text{ran } T. \quad \square$$

3.2 Characterisation of Res^1

In this subsection we compute the dimension of Res^1 and we complete the proof of Theorem 1.1. To this end we start with a simple lemma.

Lemma 3.6 *An element u is in Res^1 if and only if $\iota_X du = 0$ and $\iota_X u$ is constant. If there exists an element $\sigma \in \text{Res}^1$ with $\iota_X \sigma = 1$, then*

$$\text{Res}^1 = \text{Res}_0^1 \oplus \mathbb{R}\sigma.$$

If no such element exists, then $\text{Res}^1 = \text{Res}_0^1$.

Proof An element $u \in \text{Res}^1$ if and only if $\mathcal{L}_X u = \iota_X du + d\iota_X u = 0$. From this it follows that $\iota_X d\iota_X u = 0$ and thus $\iota_X u \in \text{Res}^0$, hence $\iota_X u$ is constant by Proposition 3.1.

Suppose there exists $\sigma \in \text{Res}^1$ such that $\iota_X \sigma = 1$. Given any $u \in \text{Res}^1$, we have $u - (\iota_X u)\sigma \in \text{Res}_0^1$ and thus $\text{Res}^1 = \text{Res}_0^1 \oplus \mathbb{R}\sigma$. If no such σ exists, then given any $u \in \text{Res}^1$ we must have $\iota_X u = 0$ and $u \in \text{Res}_0^1$. $\square$

We are now ready to prove the following proposition, which completes the proof of Theorem 1.1.

Proposition 3.7 Assume X generates a transitive Anosov flow on $\mathcal{M}$. The resonant 1-forms for $\mathcal{L}_X$ on Ω^1 at zero are determined in terms of $[\omega^\pm]_{H^2(\mathcal{M})}$, $\mathcal{H}(X)$, and Res_0^1 by the following table:

cases	$[\omega^+] \neq 0$ $[\omega^-] \neq 0$	$[\omega^+] \neq 0$ $[\omega^-] = 0$	$[\omega^+] = 0$ $[\omega^-] \neq 0$	$[\omega^+] = [\omega^-] = 0$ $\mathcal{H}(X) \neq 0$	$[\omega^+] = [\omega^-] = 0$ $\mathcal{H}(X) = 0$
Res^1	$\text{Res}_0^1 \oplus \mathbb{R}\sigma$	Res_0^1	$\text{Res}_0^1 \oplus \mathbb{R}\sigma$	$\text{Res}_0^1 \oplus \mathbb{R}\sigma$	Res_0^1
$d(\text{Res}^1)$	0	0	$\mathbb{C}\omega^+$	$\mathbb{C}\omega^+$	$\mathbb{C}\omega^+$
$\dim \text{Res}^1$	$b_1(\mathcal{M})$	$b_1(\mathcal{M})$	$b_1(\mathcal{M}) + 1$	$b_1(\mathcal{M}) + 1$	$b_1(\mathcal{M}) + 1$

In the table, the form $\sigma \in \text{Res}^1$ satisfies $\iota_X \sigma = 1$ as in Lemma 3.6. Moreover, the map

$$\text{Res}^1 \cap \ker d \rightarrow H^1(\mathcal{M}), \quad u \mapsto [u],$$

is an isomorphism.

Proof Consider cases as follows, which combined give a proof of the various cases in the table.

- (1) ($[\omega^-] \neq 0$) By Lemma 3.6 it suffices to show that there is a form σ with $d\sigma = 0$ and $\iota_X \sigma = 1$. This form is given by Lemma 3.3.
- (2) ($[\omega^\pm] = 0$ and $\mathcal{H}(X) \neq 0$) There is a primitive $\tau^+ \in \text{Res}^1$ with $d\tau^+ = \omega^+$ and $\iota_X \tau^+ = \mathcal{H}(X)$. Then $\sigma := \tau^+ / \mathcal{H}(X)$ satisfies the conditions of Lemma 3.6, completing the proof.
- (3) ($[\omega^+] \neq 0$ and $[\omega^-] = 0$) Suppose there is $\sigma \in \text{Res}^1$ with $\iota_X d\sigma = 0$ and $\iota_X \sigma = 1$. Since $d\sigma \in \text{Res}_0^2$ and $[\omega^+] \neq 0$ we must have $d\sigma = 0$ by Proposition 3.1. But by Lemma 3.3 this implies $[\omega^-] \neq 0$ and hence such σ cannot exist. By Lemma 3.6 it follows that $\text{Res}^1 = \text{Res}_0^1$ as claimed.
- (4) ($[\omega^\pm] = 0$ and $\mathcal{H}(X) = 0$) Once again we need to show that there is no $\sigma \in \text{Res}^1$ with $\iota_X d\sigma = 0$ and $\iota_X \sigma = 1$. Since $[\omega^-] = 0$ we must have $d\sigma = c\omega^+$, where $c \neq 0$ by Lemma 3.3. Since $c^{-1}\sigma(X) \neq 0$, this would give nonzero helicity and thus a contradiction.

Finally, the map $\text{Res}^1 \cap \ker d \rightarrow H^1(\mathcal{M})$, $u \mapsto [u]$, is clearly injective: if $u = df$ for $f \in \mathcal{D}'_{E_u^*}(\mathcal{M})$, then $Xf = c$. Integration against Ω_{SRB}^- gives $c = 0$ and thus f must be constant by Proposition 3.1. A glance at the table in the proposition shows that the map must be an isomorphism. $\square$

Remark 3.8 An interesting asymmetry arises in the second and third columns of the table in the previous proposition. Consider the case when we have a flow with $[\omega^+] = 0$ and $[\omega^-] \neq 0$. Then $\dim \text{Res}^1 = b_1(\mathcal{M}) + 1$ but $\dim \text{Res}_*^1 = b_1(\mathcal{M})$ (by applying the proposition to $-X$). Note that by virtue of Theorem 1.1, we always have $\dim \text{Res}_0^1 = \dim \text{Res}_{0*}^1$. Also note that thanks to the pairing between Res_0^1 and Res_{0*}^1 and Lemma 2.1, semisimplicity for $\mathcal{L}_X$ and $\mathcal{L}_{-X}$ acting on Ω_0^1 are equivalent. This is no longer the case for the actions of $\mathcal{L}_{\pm X}$ on Ω^1 as we shall see below.

Remark 3.9 As we mentioned before, $\dim \text{Res}^1$ is invariant under time changes. In the cases given by the first and third columns of the table there is $\sigma \in \text{Res}^1$ with $d\sigma = 0$ and $\iota_X \sigma = 1$. After a time change this particular form may no longer be in Res^1 , however, there is always a form of that type. A similar

remark applies to the fourth column, where there is always a form $\sigma \in \text{Res}^1$ such that $d\sigma = c\omega^+$ and $\iota_X \sigma = 1$ with $c \neq 0$. For example, if we begin with a Reeb Anosov vector field, after performing a time change most likely it will no longer be a Reeb vector field (as the bundle $E_u \oplus E_s$ stops being smooth); nevertheless it remains “weakly Reeb” in the sense that there still exists a form in Res^1 playing the same role as the contact form at least from the point of view of resonant states.

3.2.1 Semisimplicity on Ω^1 Finally, we discuss semisimplicity for the action of the Lie derivative on Ω^1 .

Proposition 3.10 *Assume there is $\tau^+ \in \text{Res}_0^1$ such that $d\tau^+ = \omega^+$ (this corresponds to item (2) of Lemma 3.4). Then the semisimplicity for the action of $\mathcal{L}_X$ on Ω^1 fails.*

Proof Let $\alpha^- \in \text{Res}_{*}^{1,\infty}$ such that $\iota_X \alpha^- = 1$. Such α^- exists since we may take any $\alpha \in C^\infty(\mathcal{M}; \Omega^1)$ with $\iota_X \alpha = 1$ and set $\alpha^- := \Pi_1^- \alpha$ (here we use (2-4)). Then

$$1 = \int_{\mathcal{M}} \alpha^- \wedge d\tau^+ = \int_{\mathcal{M}} d\alpha^- \wedge \tau^+ = \int_{\mathcal{M}} \alpha \wedge \iota_X d\alpha^- \wedge \tau^+ = \langle \iota_X d\alpha^-, \tau^+ \rangle,$$

where in the first equality we used $\iota_X \alpha^- = 1$ (so that $\alpha^- \wedge d\tau^+ = \Omega_{\text{SRB}}^+$), we used Stokes’ theorem in the second one, while in the third equality we used $\iota_X \tau^+ = 0$ and $\iota_X \alpha = 1$. Therefore, $\mathcal{L}_X \alpha^- = \iota_X d\alpha^- \neq 0$ and so $\alpha^- \notin \text{Res}_*^1$, and so by definition semisimplicity for the action of $\mathcal{L}_X$ on Ω^1 fails. $\square$

Remark 3.11 The described failure of semisimplicity for $\mathcal{L}_X$ acting on Ω^1 persists under time changes since the property that $d(\text{Res}_0^1) \neq 0$ is invariant under time changes, and that the same property is open in the set $\mathcal{W}^+ \setminus \mathcal{W}^-$ (defined in Section 5). Moreover, semisimplicity for the action of $\mathcal{L}_X$ on Ω_0^1 and Ω^1 can hold and fail, respectively; see Proposition 6.2, item (4).

3.3 Characterisation of Res^2

Here we complete the study of resonant forms at zero and compute the dimension of Res^2 .

Lemma 3.12 *A form u is in Res^2 if and only if $du = 0$ and $d\iota_X u = 0$.*

Proof By definition, being in Res^2 means that $\mathcal{L}_X u = 0$ and since $\mathcal{L}_X$ commutes with d and ι_X we must have $du \in \text{Res}^3$ and $\iota_X u \in \text{Res}_0^1$. By Proposition 3.1 there is a constant c such that $du = c\Omega_{\text{SRB}}^+$, and integrating we deduce that $c = 0$, and the lemma follows. $\square$

Lemma 3.13 *The map $\iota_X : \text{Res}^2 \rightarrow \text{Res}_0^1 \cap \ker d$ is surjective with right inverse $v \mapsto \Pi_2^+(\alpha \wedge v)$, where α is any smooth 1-form such that $\iota_X \alpha = 1$. Moreover, $\ker \iota_X = \mathbb{C} \omega^+$.*

Proof The map is well-defined by Lemma 3.12. Since ι_X commutes with Π_1^+ and Π_2^+ (see (2-4)), we have for $v \in \text{Res}_0^1$ that

$$\iota_X \Pi_2^+(\alpha \wedge v) = \Pi_1^+(\iota_X(\alpha \wedge v)) = \Pi_1^+(v) = v.$$

If in addition $dv = 0$, then $\Pi_2^+(\alpha \wedge v)$ is closed since $d\Pi_2^+(\alpha \wedge v) = \Pi_3^+(d\alpha \wedge v) = c \Omega_{\text{SRB}}^+$ for some constant $c \in \mathbb{C}$ by [Proposition 3.1](#), and $c = 0$ by integration by parts, so $\Pi_2^+(\alpha \wedge v) \in \text{Res}^2$.

The claim about the kernel of ι_X follows from item (2) in [Proposition 3.1](#). $\square$

[Lemma 3.13](#) shows that there exists a surjective linear map $C : \text{Res}^2 \rightarrow \mathbb{C}$ such that any $u \in \text{Res}^2$ may be written as

$$(3-6) \quad u = \Pi_2^+(\alpha \wedge v) + C(u)\omega^+, \quad v = \iota_X u.$$

From [Lemma 3.13](#) and [Theorem 1.1](#) applied to $\pm X$, we derive right away the following corollary.

Corollary 3.14 *Assume X generates a transitive Anosov flow on $\mathcal{M}$. The resonant 2-forms for the action of $\mathcal{L}_X$ on Ω^1 at zero satisfy*

$$\dim \text{Res}^2 = \dim \text{Res}_*^1.$$

Using (3-6) we may relate the pairings as follows. Given $u \in \text{Res}^2$ and $u_* \in \text{Res}_*^1$, we have

$$(3-7) \quad \langle u, u_* \rangle = \langle \Pi_2^+(\alpha \wedge v) + C(u)\omega^+, u_* \rangle = \langle \iota_X u, u_* \rangle + C(u) W^+(u_*),$$

where we used (2-7) and $\Pi_1^- u_* = u_*$ in the second equality. In the case $[\omega^+] = 0$ and $[\omega^-] \neq 0$, by [Proposition 3.7](#) (applied to $-X$) we know that $\text{Res}_*^1 = \text{Res}_{0*}^1$ and $d(\text{Res}_{0*}^1) = \{0\}$, hence $W^+(u_*) = 0$ for all $u_* \in \text{Res}_*^1$. Thus $u = \omega^+$ satisfies $\langle u, u_* \rangle = 0$ for all $u_* \in \text{Res}_*^1$, showing that semisimplicity fails for the action of $\mathcal{L}_X$ on Ω^2 , by [Proposition 2.2](#). But we expect semisimplicity to hold for the pairing $\langle \bullet, \bullet \rangle$ at least under small perturbations as in [Section 6](#).

In the case $[\omega^\pm] = 0$ and $\mathcal{H}(X) = 0$, we see also that $\text{Res}_*^1 = \text{Res}_{0*}^1$. There is however a nonclosed element of Res_{0*}^1 , but we still get $W^+(u_*) = 0$ for all $u_* \in \text{Res}_*^1$. Therefore $u = \omega^+$ satisfies $\langle u, u_* \rangle = 0$ for all $u_* \in \text{Res}_*^1$ and again, semisimplicity fails for the action of $\mathcal{L}_X$ on Ω^2 by [Proposition 2.2](#).

Remark 3.15 The pairings $\langle \bullet, \bullet \rangle$ and $\langle \bullet, \bullet \rangle$ are between Res_0^1 and Res_{0*}^1 , and Res^2 and Res_*^1 , respectively, and in both cases dimensions of the corresponding resonant spaces agree (this would be obvious under semisimplicity). We are not aware at the moment of an alternative proof of these facts that does not go through [Theorem 1.1](#).

The two conclusions following (3-7) about nonsemisimplicity follow directly from [Propositions 2.2](#) and [3.10](#), however our aim above was to relate the two pairings explicitly by (3-7).

4 Horocyclic invariance

According to Faure and Guillarmou [\[2018\]](#), the Ruelle resonant states for contact Anosov flows (in dimension 3) close to the imaginary axis exhibit horocyclic invariance. In a suitable sense, here we extend this idea to resonant 1-forms at zero of *transitive Anosov flows*.

Let X generate an arbitrary transitive Anosov flow on a closed orientable 3-manifold $\mathcal{M}$. By [Hasselblatt 1994, Corollary 1.8], it is known that the weak unstable/stable bundles $\mathbb{R}X \oplus E_{u/s}$ are $C^{1+\alpha}$ -Hölder regular for some $\alpha > 0$ (depending on the flow). In particular, if the $E_{u/s}$ are orientable, there are vector fields $Y_{u/s} \in C^{1+\alpha}(\mathcal{M}; T\mathcal{M})$ such that $\mathbb{R}X \oplus E_{u/s}$ are spanned by $\{X, Y_{u/s}\}$ at every point. Moreover, by [Hasselblatt 1994], there are C^β regular vector fields $U^{u/s}$ for some $\beta > 0$, differentiable along the flow, spanning $E_{u/s}$ at every point, as well as $r^{u/s} \in C^\beta(\mathcal{M})$ such that

$$(4-1) \quad [X, U^{u/s}] = -r^{u/s} U^{u/s}.$$

Let $\omega, \omega_{u/s} \in C^{1+\alpha}(\mathcal{M}; \Omega^1)$ be defined as

$$(4-2) \quad \omega_{u/s}(Y^{u/s}) = \omega_{u/s}(X) = 0, \quad \omega_{u/s}(Y^{s/u}) = 1, \quad \omega(Y^u) = \omega(Y^s) = 0, \quad \omega(X) = 1.$$

Let H and V be smooth vector fields approximating U^u and U^s , respectively, in the sense that

$$\|H - U^u\|_{C^0} + \|V - U^s\|_{C^0} < \varepsilon \quad \text{and} \quad \|[X, H] - [X, U^u]\|_{C^0} + \|[X, V] - [X, U^s]\|_{C^0} < \varepsilon$$

for a certain sufficiently small $\varepsilon > 0$, to be specified later. Indeed, such an approximation exists thanks to standard arguments; see eg [Dyatlov and Zworski 2019, Lemma E.45] in the related setting of the scale of Sobolev spaces. After possibly time-changing $U^{u/s}$ by some nowhere zero $b_{u/s} \in C^\beta(\mathcal{M})$ such that $\|b_{u/s} - 1\|_{C^0(\mathcal{M})} + \|Xb_{u/s}\|_{C^0(\mathcal{M})} < C\varepsilon$ for some uniform $C > 0$, we may assume that

$$(4-3) \quad U^u = H + r_V V - a_u X \quad \text{and} \quad U^s = r_H H + V - a_s X$$

for some $r_{H/V}, a_{u/s} \in C^\beta(\mathcal{M})$. Further, since $Y^{u/s} = x_{u/s} X + u_{u/s} U^{u/s}$ for some $x_{u/s}, u_{u/s} \in C^\beta(\mathcal{M})$ (where $u_{u/s} \neq 0$ pointwise), using (4-3) we immediately get that

$$x_{u/s} - u_{u/s} a_{u/s}, \quad u_{u/s}, \quad u_u r_V, \quad u_s r_H \in C^{1+\alpha}(\mathcal{M}).$$

In particular $u_{u/s} \in C^{1+\alpha}(\mathcal{M})$ implies

$$r_{H/V} = \frac{u_{s/u} r_{H/V}}{u_{s/u}} \in C^{1+\alpha}(\mathcal{M}).$$

After subtracting the X part in (4-3), we may redefine $Y^{u/s}$ to

$$(4-4) \quad Y^u := H + r_V V \quad \text{and} \quad Y^s := r_H H + V,$$

where $r_{H/V} \in C^{1+\alpha}(\mathcal{M})$, and $\{X, Y^{u/s}\}$ still spans the weak bundle $\mathbb{R}X \oplus E_{u/s}$ pointwise. In fact, if $\varepsilon > 0$ is small enough, by [Guillarmou and de Poyferré 2022, Theorem 2], the $r_{H/V}$ additionally satisfy a wavefront set condition

$$(4-5) \quad r_V \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \cap C^{1+\alpha}(\mathcal{M}) \quad \text{and} \quad r_H \in \mathcal{D}'_{E_s^*}(\mathcal{M}) \cap C^{1+\alpha}(\mathcal{M}).$$

It follows from (4-4), (4-3) and (4-1) that

$$(4-6) \quad Y^{u/s} = a_{u/s} X + U^{u/s} \quad \text{and} \quad [X, Y^{u/s}] = (X + r^{u/s})a_{u/s} X - r^{u/s} Y^{u/s}.$$

Integrating (4-1) the function r^u satisfies, for all $t \in \mathbb{R}$ and $z \in \mathcal{M}$,

$$(4-7) \quad d\varphi_{-t}(z)U^u(z) = e^{-\int_0^t r^u \circ \varphi_{-s}(z) ds} U^u(\varphi_{-t}z),$$

which implies in particular that (by definition of Anosov flows (1-1))

$$0 < \nu_{\min} := \lim_{T \rightarrow \infty} \inf_{z \in \mathcal{M}} \frac{1}{T} \int_0^T r^u \circ \varphi_{-s}(z) ds$$

is the minimal expansion rate.

In fact, more is true about the regularity of $r^{u/s}$:

Lemma 4.1 $Xr^{u/s}, r^{u/s}, Xr_{H/V}$ and $(X + r^{u/s})a_{u/s}$ all belong to $C^{1+\alpha}(\mathcal{M}) \cap \mathcal{D}'_{E_u^*}(\mathcal{M})$. Moreover, the $r_{H/V}$ satisfy a **Riccati type equation**.

Proof In the following, for a vector field Z , denote by $Z_{X/H/V}$ the component of Z in the $X/H/V$ direction, respectively. To simplify the notation we consider the unstable quantities only, the case of stable ones follows similarly. Write the commutator formula (4-6) in two ways, using (4-4):

$$\begin{aligned} [X, Y^u] &= ([X, H]_X + r_V[X, V]_X)X + ([X, H]_H + r_V[X, V]_H)H + ([X, H]_V + (X + [X, V]_V)r_V)V \\ &= (X + r^u)a_u X - r^u H - r^u r_V V. \end{aligned}$$

By equating the coefficients next to the vector fields X, H, V , we get using (4-5):

$$\begin{aligned} (X + r^u)a_u &= [X, H]_X + r_V[X, V]_X \in C^{1+\alpha}(\mathcal{M}) \cap \mathcal{D}'_{E_u^*}(\mathcal{M}), \\ -r^u &= [X, H]_H + r_V[X, V]_H \in C^{1+\alpha}(\mathcal{M}) \cap \mathcal{D}'_{E_u^*}(\mathcal{M}), \\ -r^u r_V &= [X, H]_V + (X + [X, V]_V)r_V. \end{aligned}$$

It follows that the left-hand side of the last equation is also in $C^{1+\alpha}(\mathcal{M})$ and so Xr_V is in $C^{1+\alpha}(\mathcal{M})$, and thus also by the second equation $Xr^u \in C^{1+\alpha}(\mathcal{M})$. This proves the first part of the claim, while the second part follows by substituting the second equation into the third one. $\square$

4.1 Horocyclic invariance of resonant 1-forms

Next we show that (closed) elements $u \in \text{Res}_0^1$ vanish on the weak unstable bundle, generalising the analogous claims for geodesic flows in constant negative curvature [Cekić et al. 2022; Küster and Weich 2020]. In particular, we will show that $u \in C_*^{-1-}(\mathcal{M}; \Omega^1)$ and since $\mathbb{R}X \oplus E_u$ is $C_*^{1+\alpha}$ -regular, the contraction is well-defined. We remark that it suffices to check that $\iota_Y u = 0$ ($\iota_X u = 0$ by definition), and then $\iota_Y u = 0$ for any $C_*^{1+\alpha}$ section Y of $\mathbb{R}X \oplus E_u$.

Lemma 4.2 (horocyclic invariance, I) *Suppose that X generates a transitive Anosov flow on a closed 3-manifold $\mathcal{M}$ such that E_u is orientable. Then:*

- (1) *If $u \in \text{Res}_0^1 \cap \ker d$, then $u \in C_*^{-1-}(\mathcal{M}; \Omega^1)$ and $\iota_Y u = 0$;*
- (2) *Assume X preserves a smooth probability volume form Ω . Then **any** $u \in \text{Res}_0^1$ is such that $u \in C_*^{-1-}(\mathcal{M}; \Omega^1)$ and $\iota_Y u = 0$.*

Proof Observe firstly that $U^u = Y^u - a_u X$ (see (4-6)) extends continuously as a differential operator to some $C_*^{-\varepsilon}(\mathcal{M}) \cap \mathcal{D}(X)$, where $\mathcal{D}(X)$ denotes the domain of X , ie all $u \in C_*^{-\varepsilon}(\mathcal{M})$ such that Xu is in $C_*^{-\varepsilon}(\mathcal{M})$, for any ε with $\min(\alpha, \beta) > \varepsilon > 0$.

We deal first with the case when $du = 0$. By Lemma 2.4 and Proposition 3.2, write $u = \theta + d\varphi$, where $\theta \in C^\infty(\mathcal{M}; \Omega^1)$, $d\theta = 0$, $X\varphi = -\iota_X \theta$, and $\varphi = -R_0^{+,H}(\iota_X \theta)$ is obtained by applying (holomorphic part of) the resolvent at zero. By Proposition 2.3 this implies $\varphi \in C_*^{0-}(\mathcal{M}) \cap \mathcal{D}'_{E_u^*}(\mathcal{M})$ and so $u \in C_*^{-1-}(\mathcal{M}; \Omega^1)$. Since $\Pi_0^+ R_0^{+,H} = 0$ (see Proposition 3.2), we have $\Pi_0^+ \varphi = 0$, or equivalently $\int_{\mathcal{M}} \varphi \Omega_{\text{SRB}}^- = 0$. Clearly $\varphi \in \mathcal{D}(X)$ and we claim that

$$U^u \varphi \in C^0(\mathcal{M}),$$

and in fact $U^u \varphi$ is Hölder regular for some positive exponent. To see this, it suffices to show

$$(4-8) \quad U^u \varphi = (X + r^u)^{-1} U^u X \varphi,$$

since $X\varphi = -\iota_X \theta \in C^\infty(\mathcal{M})$, $U^u X \varphi \in C^\beta(\mathcal{M})$, and $(X + r^u)^{-1}$ is a continuous map $C^0(\mathcal{M}) \rightarrow C^0(\mathcal{M})$ by Proposition 2.3.

Observe that for $\text{Re}(s) > 0$, $f, \phi \in C^\infty(\mathcal{M})$, we have in the sense of distributions (using the formula (2-8) from Proposition 2.3):

$$\begin{aligned} (4-9) \quad & \int_{\mathcal{M}} (X + r^u + s)^{-1} U^u f(x) \phi(x) d\text{vol}(x) \\ &= \int_{\mathcal{M}} \phi(x) d\text{vol}(x) \int_0^\infty e^{-\int_0^t (r^u \circ \varphi_{-p} + s) dp} U^u f(\varphi_{-t} x) dt \\ &= \int_{\mathcal{M}} \phi(x) d\text{vol}(x) \int_0^\infty e^{-st} df(\varphi_{-t} x) (d\varphi_{-t}(x) U^u(x)) dt \\ &= \int_{\mathcal{M}} \phi(x) d\text{vol}(x) \int_0^\infty e^{-st} U^u(\varphi_{-t}^* f)(x) dt \\ &= \lim_{N \rightarrow \infty} \int_{\mathcal{M}} U^u (X + s)^{-1} (\text{Id} - e^{-sN} \varphi_{-N}^*) f(x) \phi(x) d\text{vol}(x) \\ &= \int_{\mathcal{M}} U^u (X + s)^{-1} f(x) \phi(x) d\text{vol}(x), \end{aligned}$$

where $d\text{vol}$ is an arbitrary smooth volume form on $\mathcal{M}$. In the second line, we used (4-7), in the fourth we rewrote the integral as the resolvent and used that $\int_0^N e^{-st} U^u(\varphi_{-t}^* f) dt$ converges in $C^0(\mathcal{M})$, and in the last line we used that $U^u (X + s)^{-1} : C^0(\mathcal{M}) \rightarrow \mathcal{D}'(\mathcal{M})$ is continuous to take the limit as $N \rightarrow \infty$.

To show (4-8), observe that for $\text{Re}(s) > 0$,

$$(4-10) \quad (X + r^u + s)^{-1} U^u X \varphi = U^u (X + s)^{-1} X \varphi = U^u \varphi - s U^u (X + s)^{-1} \varphi,$$

where the first equality holds by (4-9). Now for $\text{Re}(s) > 0$ and s close to zero, $(X + s)^{-1} \varphi = R_0^{+,H}(s) \varphi$ since $\Pi_0^+ \varphi = 0$, and recalling that U^u extends continuously as a differential operator to $C_*^{-\varepsilon}(\mathcal{M}) \cap \mathcal{D}(X)$

(since $U^u = Y^u - a_u X$), and using $R_0^{+,H}(s)\varphi \in C_*^{-\varepsilon}(\mathcal{M})$ is uniformly bounded (by [Proposition 2.3](#)), taking $s \rightarrow 0$ proves the claim.

Now going back to $X\varphi = -\iota_X \theta$, we may apply U^u to get, using $d\theta = 0$ and (4-1),

$$U^u X\varphi = -U^u \iota_X \theta = d\theta(X, U^u) - X\iota_{U^u} \theta + \iota_{[X, U^u]} \theta = -(X + r^u) \iota_{U^u} \theta.$$

Since $\iota_{U^u} \theta$ and $U^u X\varphi$ belong to $C^0(\mathcal{M})$, applying $(X + r^u)^{-1}$ we conclude that

$$\iota_{U^u} \theta = -(X + r^u)^{-1} U^u X\varphi = -U^u \varphi,$$

where in the last equality we used (4-8). Using $\iota_{a_u} X\theta = -a_u X\varphi$, we conclude $0 = \iota_{Y^u} \theta + Y^u \varphi = \iota_{Y^u} u$, completing the proof in the case $du = 0$.

By [Proposition 3.1](#), since $du \in \text{Res}_0^2$ it remains to deal with the case $du = \iota_X \Omega_{\text{SRB}}^+$. By assumption, $\Omega_{\text{SRB}}^+ = \Omega_{\text{SRB}}^- = \Omega$ since Ω is a smooth and invariant, so in particular $du \in C^\infty(\mathcal{M}; \Omega_0^2)$. By [Lemma 2.4](#), there exist $\varphi \in \mathcal{D}'_{E_u^*}(\mathcal{M})$, a closed $\psi \in \mathcal{D}'_{E_u^*}(\mathcal{M}; \Omega^2)$, and a smooth $\theta \in C^\infty(\mathcal{M}; \Omega^1)$ such that

$$(4-11) \quad u = d\varphi + d^* \psi + \theta.$$

Since $d\psi = 0$, it follows that $dd^* \psi = \Delta_2 \psi$ (recall that Δ_2 is the Hodge Laplacian on Ω^2 of some Riemannian metric). Observe that

$$(4-12) \quad du = \Delta_2 \psi + d\theta \quad \text{for } du \in C^\infty(\mathcal{M}; \Omega^2),$$

and by elliptic regularity (equivalently, by applying the operator Q_2 from [Lemma 2.4](#)), we get that ψ is in $C^\infty(\mathcal{M}; \Omega^2)$. Set $\eta := d^* \psi + \theta \in C^\infty(\mathcal{M}; \Omega^1)$.

Using $\iota_X u = 0$, we get $X\varphi = -\iota_X \eta$. In particular, we may apply $R_0^{+,H}$ and assume that

$$\varphi = -R_0^{+,H} \iota_X \eta \in C_*^{0-}(\mathcal{M}) \cap \mathcal{D}'_{E_u^*}(\mathcal{M}),$$

by [Proposition 2.3](#). Observe that (4-9) still holds, and arguing as before we conclude that we may take the limit $s \rightarrow 0$ in (4-10), so (4-8) holds in this setting as well. Then we compute

$$U^u X\varphi = -U^u \iota_X \eta = d\eta(X, U^u) - X\iota_{U^u} \eta + \iota_{[X, U^u]} \eta = -(X + r^u) \iota_{U^u} \eta,$$

where in the first equality we used (4-11) and in the third equality we used that $\iota_X d\eta = \iota_X du = 0$ and (4-1). Since $\iota_{U^u} \eta$ and $U^u X\varphi$ are Hölder regular, applying $(X + r^u)^{-1}$ to the previous equation, using $X\varphi = -\iota_X \eta$ and (4-8), we get

$$U^u \varphi + \iota_{U^u} \eta = \iota_{Y^u} u = 0. \quad \square$$

Remark 4.3 The only unclear case in [Lemma 4.2](#) is when $du = \iota_X \Omega_{\text{SRB}}^+$ for some $u \in \text{Res}_0^1$ and X does not preserve a smooth measure. Let us follow the notation of the second part of [Lemma 4.2](#). Then $du \in (C^0(\mathcal{M}))'$ only and by Sobolev embeddings and elliptic regularity we eventually get that $\psi \in W^{2-\varepsilon, 3/(3-\varepsilon)-}(\mathcal{M}) \cap \mathcal{D}'_{E_u^*}(\mathcal{M})$ for all $0 < \varepsilon < 1$. Then the regularity of $\varphi = -R_0^{+,H} \iota_X \eta$ is not entirely clear: given these estimates, it belongs only to $C_*^{-2-}(\mathcal{M})$, which is not enough to define $Y^u \varphi$.

Next we give a lemma that relates Res_0^1 to a resonant state of $X + Q$ with an additional horocyclic invariance, where Q is some singular potential, generalising the case of geodesic flows in constant negative curvature [Dyatlov et al. 2015; Guillarmou et al. 2018; Cekić et al. 2022].

Lemma 4.4 (horocyclic invariance, II) *Let $u \in \text{Res}_0^1 \cap \ker d$. Then $c := \iota_V u \in C_*^{-1-}(\mathcal{M}) \cap \mathcal{D}'_{E_u^*}(\mathcal{M})$ satisfies*

$$\begin{aligned}(X + [X, V]_H \cdot r_V - [X, V]_V)c &= 0, \\ (H + [H, V]_H \cdot r_V - [H, V]_V)c + V(r_V c) &= 0.\end{aligned}$$

Similarly, any $u_* \in \text{Res}_{0*}^1 \cap \ker d$ satisfies, if $c_* := \iota_H u_*$,

$$\begin{aligned}(X + [X, H]_V \cdot r_H - [X, H]_H)c_* &= 0, \\ H(r_H c_*) + (V - [H, V]_V \cdot r_H + [H, V]_H)c_* &= 0.\end{aligned}$$

Proof Firstly, since $\{X, H, V\}$ form a smooth global frame, there is a smooth global dual coframe $\{\alpha, \beta, \psi\}$. By Lemma 4.2, $\iota_Y u = 0$. If $u = a\alpha + b\beta + c\psi$ for some $a, b, c \in \mathcal{D}_{E_u^*}(\mathcal{M}) \cap C_*^{-1-}(\mathcal{M})$, it follows that

$$\iota_X u = 0 \implies a = 0 \quad \text{and} \quad \iota_Y u = 0 \implies b + cr_V = 0.$$

A similar argument applies for any $u_* \in \text{Res}_{0*}^1 \cap \ker d$, so there is $\iota_H u_* =: c_* \in \mathcal{D}_{E_s^*}(\mathcal{M}) \cap C_*^{-1-}(\mathcal{M})$ such that

$$(4-13) \quad u = c(\psi - r_V \beta) \quad \text{and} \quad u_* = c_*(\beta - r_H \psi).$$

Now $du = 0$ implies

$$(4-14) \quad 0 = du(X, V) = X\iota_V u - \iota_{[X, V]}u = Xc - c([X, V]_V - r_V[X, V]_H),$$

where in the second equality we used $\iota_X u = 0$, and in the third one we applied (4-13). This gives the first equation for u , and the second one follows similarly:

$$(4-15) \quad 0 = du(H, V) = H\iota_V u - V\iota_H u - \iota_{[H, V]}u = Hc + V(r_V c) + c([H, V]_H \cdot r_V - [H, V]_V),$$

where we used (4-13) in the third equality. The case of $u_* \in \text{Res}_{0*}^1$ follows analogously. $\square$

We emphasise that the preceding result was not known even in the case of geodesic flows in variable negative curvature. Let us record this fact separately:

Proposition 4.5 *Let (M, g) be an Anosov surface, $\mathcal{M} = SM$ the unit sphere bundle, and X the geodesic vector field. Let $\{X, H, V\}$ be the orthonormal frame constructed in Section 2.4. Then every $u \in \text{Res}_0^1$ satisfies, where $c := \iota_V u$,*

$$(X - r^u)c = 0 \quad \text{and} \quad Hc + V(r^u c) = 0,$$

where r^u is the unique function such that $H + r^u V \in E_u$.

Proof Note firstly that $d(\text{Res}_0^1) = 0$ in this case, as follows from Theorem 1.1. Next, we observe that in this frame $r_V = r^u$. Indeed, this is well-known and follows for instance from Lemma 7.1 below for $\lambda = 0$

(ie in this case we may take $U^u = Y^u = H + r^u V$). Then the claim follows directly from [Lemma 4.4](#) and (2-10). $\square$

Remark 4.6 In [Lemma 4.4](#), the subtlety lies in the divergence type expression $V(r_V c)$, which cannot be expanded since the products $V(r_V) \cdot c$ and $r_V \cdot Vc$ typically do not make sense. However, as soon as the regularity of r_V is $C_*^{2+}(\mathcal{M})$ (ie when the weak unstable bundle is $C_*^{2+}(\mathcal{M})$ -regular), these products make sense and we have a classical interpretation of horocyclic invariance.

Remark 4.7 Using similar techniques one can easily prove horocyclic invariance for resonant states with resonances with real part sufficiently close to zero for arbitrary Anosov flows in dimension three. We believe this may be useful for studying the exponential speed of mixing of 3-dimensional Anosov flows as in [\[Cekić and Guillarmou 2021\]](#). We leave this for discussion elsewhere.

Remark 4.8 [Proposition 4.5](#) extends easily to arbitrary Anosov thermostats $F = X + \lambda V$ as introduced in [Section 7](#): if $u \in \text{Res}_0^1 \cap \ker d$ and $c = \iota_V u$, then

$$(F - r^u + V(\lambda))c = 0 \quad \text{and} \quad Hc + \lambda c + V(r^u c) = 0.$$

The form u can be written as $u = c(-r^u \beta - \lambda \alpha + \psi)$.

4.2 Horocyclic invariance of SRB measures

Next, we show that SRB measures exhibit some horocyclic invariance. Let us fix a smooth volume form Ω on $\mathcal{M}$, and denote $\lambda := -(X + r^u)a_u$ (Hölder regular by [Lemma 4.1](#)), and recall that by (4-6),

$$(4-16) \quad [X, Y^u] = -\lambda X - r^u Y^u.$$

Lemma 4.9 (horocyclic invariance, III) *Let $W \in C^\infty(\mathcal{M})$. Let α_W be the Hölder continuous solution of*

$$(4-17) \quad (X + r^u)\alpha_W = (Y^u - \lambda)W.$$

Then the following commutation relation holds:

$$[X + W, Y^u + \alpha_W] = -\lambda(X + W) - r^u(Y^u + \alpha_W).$$

As a consequence, if the SRB measure is given by $\Omega_{\text{SRB}}^+ = f\Omega$ for some $f \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ satisfying $(X + \text{div}_\Omega X)f = 0$, then we have

$$(4-18) \quad (X + \text{div}_\Omega X + r^u)(Y^u + \alpha_{\text{div}_\Omega X})f = 0.$$

Proof The first result is a straightforward computation:

$$\begin{aligned} [X + W, Y^u + \alpha_W] &= -\lambda X - r^u Y^u + X(\alpha_W) - Y^u(W) \\ &= -\lambda(X + W) - r^u(Y^u + \alpha_W) + \underbrace{\lambda W + r^u \alpha_W + X(\alpha_W) - Y^u(W)}_{=0, \text{ by (4-17)}}, \end{aligned}$$

where we use (4-16) in the first equality.

For the conclusion about SRB measures, set $W := \text{div}_\Omega X$. Since $Y^u \in C^{1+\alpha}(\mathcal{M}; T\mathcal{M})$, and α_W is Hölder regular, while $f\Omega$ is a measure, we may apply the commutation above to f to directly obtain (4-18). $\square$

Remark 4.10 The operator $X + \operatorname{div}_\Omega X + r^u$ is invertible (eg on $L^1(\mathcal{M}, \Omega)$), however the space to which $(Y^u + \alpha_{\operatorname{div}_\Omega X})f$ a priori belongs is very irregular. However, it is expected that (4-18) implies $(Y^u + \alpha_{\operatorname{div}_\Omega X})f = 0$, and we will indeed see this is the case in the next section under a regularity assumption on the weak unstable bundle.

4.3 Case of one smooth weak bundle

In this section we assume that the weak unstable vector bundle $\mathbb{R}X \oplus E_u$ is smooth, or equivalently, that the vector field Y^u is smooth. Indeed, this is clear by the construction of Y^u at the top of Section 4; the fact that Y^u is somewhat noncanonical does not play any role and what we care about is its regularity and the fact that it is pointwise linearly independent of X . It follows that $r_V \in C^\infty(\mathcal{M})$ and by inspection of the proof of Lemma 4.1, we have $r^u \in C^\infty(\mathcal{M})$. Then the proof of Lemma 4.2 simplifies, and we get horocyclic invariance for resonant 1-forms u that are not closed, ie we have:

Lemma 4.11 Assume $Y^u \in C^\infty(\mathcal{M}; T\mathcal{M})$. Then:

- (1) Let $u \in \operatorname{Res}^1$ such that $\iota_X u = D \in \mathbb{C}$. Then $\iota_{Y^u} u = Da_u$.
- (2) Let $u \in \operatorname{Res}_0^1$. Then there is a constant $D \in \mathbb{C}$ such that for $c := \iota_V u$,

$$(4-19) \quad \begin{aligned} (X + [X, V]_H \cdot r_V - [X, V]_V)c &= 0, \\ (Y^u + [H, V]_H \cdot r_V + V(r_V) - [H, V]_V)c &= -\frac{1}{2}D \Omega_{\text{SRB}}^+(X, H, V). \end{aligned}$$

Denote by $\mathcal{S}$ the set of distributional solutions of (4-19) for $D \in \mathbb{C}$. Then the map $P : \operatorname{Res}_0^1 \rightarrow \mathcal{S}$, $u \mapsto \iota_V u$, is an isomorphism.

Proof For the first claim, it can be checked that in fact the proof of the second part of Lemma 4.2 carries over in this setting, that is, the summary of Remark 4.3 applies. However, let us give a slightly different and more direct argument.

The idea is to express the contractions in two ways: via meromorphic extension of the resolvent (see Section 2.1) and via the resolvent integral. Let us for simplicity first consider the case $D = 0$. Via the resolvent integral, we consider for large $\operatorname{Re}(s)$ and $\gamma \in C^\infty(\mathcal{M}; \Omega_0^1)$ the expression, for some $x \in \mathcal{M}$,

$$(4-20) \quad \begin{aligned} \iota_{Y^u}(\mathcal{L}_X + s)^{-1}\gamma(x) &= \int_0^\infty e^{-st} \varphi_{-t}^* \gamma(x)(Y^u(x)) dt \\ &= \int_0^\infty e^{-st} \gamma(\varphi_{-t}x)(d\varphi_{-t}(x)Y^u(x)) dt \\ &\leq C \|\gamma\|_{C^0} \|U^u\|_{C^0} \int_0^\infty e^{-t(\operatorname{Re}(s) + v_{\min} - \varepsilon)} dt < \infty, \end{aligned}$$

where $\varepsilon > 0$, $C = C(\varepsilon) > 0$, we recall by (4-6) that $Y^u = a_u X + U^u$, and by the Anosov property, there exists $C' = C'(\varepsilon) > 0$ such that

$$\|d\varphi_{-t}(y)v\| \leq C' e^{-(v_{\min} - \varepsilon)t} \|v\| \quad \text{for all } y \in \mathcal{M} \text{ and all } v \in E_u(y).$$

Therefore

$$s \mapsto \iota_{Y^u}(\mathcal{L}_X + s)^{-1}\gamma \in C^0(\mathcal{M})$$

is a holomorphic map in the region $\operatorname{Re}(s) > -\nu_{\min}$.

On the other hand, by [Section 2.1](#) we know that $R_1^+(s) = (\mathcal{L}_X + s)^{-1} : C^\infty(\mathcal{M}; \Omega^1) \rightarrow \mathcal{D}'(\mathcal{M}; \Omega^1)$ admits a meromorphic extension to $s \in \mathbb{C}$. Since Y^u is smooth, it follows that $s \mapsto \iota_{Y^u}(\mathcal{L}_X + s)^{-1}\gamma \in \mathcal{D}'(\mathcal{M})$ is meromorphic. Since the two extensions agree for $\operatorname{Re}(s) \gg 1$, it follows that they are identical and holomorphic for $\operatorname{Re}(s) > -\nu_{\min}$. If $u \in \operatorname{Res}_0^1$, we have $u = \Pi_1^+\gamma$ for some $\gamma \in C^\infty(\mathcal{M}; \Omega_0^1)$, and since $\Pi_1^+ = (1/(2\pi i)) \oint_0 (\mathcal{L}_X + s)^{-1} ds$, we conclude that $\iota_{Y^u}u = (1/(2\pi i)) \oint_0 \iota_{Y^u}(\mathcal{L}_X + s)^{-1}\gamma ds = 0$, which concludes the proof of the first part.

For the case $D \neq 0$, it suffices to observe that upon replacing ι_{Y^u} with ι_{U^u} in (4-20) and taking $\gamma \in C^\infty(\mathcal{M}; \Omega^1)$, we similarly get that $s \mapsto \iota_{U^u}(\mathcal{L}_X + s)^{-1}\gamma \in C^0(\mathcal{M})$ is holomorphic for $\operatorname{Re}(s) > -\nu_{\min}$. Next, expressing

$$\iota_{U^u}(\mathcal{L}_X + s)^{-1}\gamma = \iota_{Y^u}(\mathcal{L}_X + s)^{-1}\gamma - a_u(X + s)^{-1}\iota_X\gamma,$$

we conclude, using also [Proposition 2.3](#) for the second term, that $s \mapsto \iota_{U^u}(\mathcal{L}_X + s)^{-1}\gamma$ is meromorphic (as a composition) for $\operatorname{Re}(s) > -\varepsilon$ for some $\varepsilon > 0$ small enough. Then the conclusion follows similarly to before, since $u = \Pi_1^+\gamma$ for some $\gamma \in C^\infty(\mathcal{M}; \Omega^1)$.

To derive (4-19), note firstly that by [Proposition 3.1](#) we may write $du = -\frac{1}{2}D\iota_X\Omega_{\text{SRB}}^+$ for some $D \in \mathbb{C}$. The first equation of (4-19) follows from $\iota_X du = 0$, item (1), and the computation carried out in (4-14). The other equation is a consequence of the computation in (4-15); note that the divergence term $V(r_V c)$ can now be expanded and the leading term becomes $Y^u c$.

Finally, P is clearly injective by item (1), ie $\iota_V u = 0$ implies $u = 0$. In the other direction, if c solves (4-19) we deduce that $\operatorname{WF}(c) \subset E_u^*$, where we also use that $\operatorname{WF}(\Omega_{\text{SRB}}^+) \subset E_u^*$. Moreover, we have $P(c(\psi - r_V \beta)) = c$ and it is straightforward to check that $c(\psi - r_V \beta) \in \operatorname{Res}_0^1$ by using the computations (4-14) and (4-15). $\square$

Remark 4.12 The method of [Lemma 4.11](#), item (1), applies in general to $u \in \operatorname{Res}_0^1$ under the assumption that the resolvent is a meromorphic map $(\mathcal{L}_X + s)^{-1} : C^\infty(\mathcal{M}; \Omega^1) \rightarrow C_*^{-1-}(\mathcal{M}; \Omega^1)$ for s close to zero. In that case one may define the composition $\iota_{Y^u}(\mathcal{L}_X + s)^{-1}$ and the proof carries over.

Lemma 4.13 If $\Omega_{\text{SRB}}^+ = f\Omega$ for some $f \in \mathcal{D}'_{E_u^*}(\mathcal{M})$, then

$$(Y^u + \alpha_{\operatorname{div}_\Omega X})f = 0.$$

Proof Observe firstly by [Lemma 4.1](#) that $\lambda = -(X + r^u)a_u$ satisfies $\operatorname{WF}(\lambda) \subset E_u^*$. Using that r^u and Y^u are smooth, it follows that the solution $\alpha_{\operatorname{div}_\Omega X}$ of (4-17) with $W = \operatorname{div}_\Omega X$ satisfies $\operatorname{WF}(\alpha_{\operatorname{div}_\Omega X}) \subset E_u^*$. Therefore $(Y^u + \alpha_{\operatorname{div}_\Omega X})f$ has wavefront set in E_u^* , so it is a resonant state (at zero) of the operator $X + \operatorname{div}_\Omega X + r^u$ by [Lemma 4.9](#). However, $s \mapsto (X + \operatorname{div}_\Omega X + r^u + s)^{-1} : L^1(\mathcal{M}, \Omega) \rightarrow L^1(\mathcal{M}, \Omega)$ is holomorphic for $\operatorname{Re}(s) > -\nu_{\min}$ and so this resonant state is equal to the zero function. $\square$

5 Perturbation theory

Let X_0 be a fixed smooth Anosov vector field generating a transitive Anosov flow. In this section we study the local manifold structure of flows with vanishing winding cycles using the implicit function theorem.

5.1 Anisotropic Sobolev spaces

We will use the perturbation theory developed by Bonthonneau [2020], for which we need to introduce anisotropic Sobolev spaces. Recall that these are given, for any $k \in \mathbb{N}_0$, by

$$\mathcal{H}_{rG,t}(\mathcal{M}; \Omega^k) := e^{-r \operatorname{Op}(G)} H^t(\mathcal{M}; \Omega^k) \quad \text{for } r \geq 0 \text{ and } t \in \mathbb{R}.$$

Here Op is a quantisation procedure on $\mathcal{M}$, $G(x, \xi) = m(x, \xi) \log(1 + |\xi|)$ is a logarithmically growing symbol on $T^*\mathcal{M}$, and $m(x, \xi)$ satisfies certain conditions with respect to any vector field X close to X_0 in $C^1(\mathcal{M}; T\mathcal{M})$ norm; these conditions are explicitly stated in [Bonthonneau 2020, equation (4)]. We note that the main theorem of [loc. cit.] features only $\mathcal{H}_{rG,0}(\mathcal{M}; \Omega^k)$; however, it is observed at the end of [loc. cit., Section 2] that the methods carry over to $\mathcal{H}_{rG,t}(\mathcal{M}; \Omega^k)$ (with a different *threshold*).

When $t = 0$ we will use the notation $\mathcal{H}_{rG}(\mathcal{M}; \Omega^k) := \mathcal{H}_{rG,0}(\mathcal{M}; \Omega^k)$, and when clear from context, we will simply write $\mathcal{H}_{rG,t}$ for $\mathcal{H}_{rG,t}(\mathcal{M}; \Omega^k)$ and $\mathcal{H}_{rG}$ for $\mathcal{H}_{rG}(\mathcal{M}; \Omega^k)$. Denote the domain of X (note that the domain *does* depend on X) by

$$\mathcal{D}_{rG,t}^X(\mathcal{M}; \Omega^k) := \{u \in \mathcal{H}_{rG,t}(\mathcal{M}; \Omega^k) \mid \mathcal{L}_X u \in \mathcal{H}_{rG,t}(\mathcal{M}; \Omega^k)\}.$$

For $N > 1$ and $\eta > 0$ write

$$\mathcal{U}_{N,\eta} := \{X \in C^N(\mathcal{M}; T\mathcal{M}) \mid \|X - X_0\|_{C^N} < \eta\},$$

$$\mathcal{R}_{N,\eta} := \{(X, s) \in \mathcal{U}_{N,\eta} \times \mathbb{C} \mid s \text{ is a resonance of } \mathcal{L}_X \text{ on } \Omega^k, \operatorname{Re}(s) > -1\}.$$

Here η will always be chosen small enough so that the flows in $\mathcal{U}_{N,\eta}$ are Anosov; this may be done by [Fisher and Hasselblatt 2019, Corollary 5.1.11].

We summarise the contents of [Bonthonneau 2020, Theorem 1 and Corollary 2] in the following lemma (see also the end of [Bonthonneau 2020, Section 2]):

Lemma 5.1 *There exist $N_0 > 0$, $\eta > 0$ and $C_0 \in \mathbb{R}$ such that the following holds. For any $N > N_0$, for all $X \in \mathcal{U}_{N,\eta}$, and all $t \in \mathbb{R}$ such that $r > C_0 + |t|$,*

$$\mathcal{L}_X + s: \mathcal{D}_{rG,t}^X(\mathcal{M}; \Omega^k) \rightarrow \mathcal{H}_{rG,t}(\mathcal{M}; \Omega^k) \quad \text{for } \operatorname{Re}(s) > -1$$

is a Fredholm operator, and its inverse, when s is not a resonance, is given by $(\mathcal{L}_X + s)^{-1}$. Moreover, the set $\mathcal{R}_{N,\eta} \subset \mathcal{U}_{N,\eta} \times \mathbb{C}$ is closed and the resolvent $(\mathcal{L}_X + s)^{-1}$ is bounded locally uniformly away from $\mathcal{R}_{N,\eta}$.

We remark that Bonthonneau works in infinite regularity, but the microlocal methods from this reference only require the control of a finite number of derivatives and thus carry over.

5.2 Derivative of the winding cycle

Now specialise to $\dim \mathcal{M} = 3$ and fix a smooth probability volume form Ω on $\mathcal{M}$. Set, for some large N to be specified later,

$$(5-1) \quad \mathcal{W}^\pm := \{X \in C^N(\mathcal{M}; T\mathcal{M}) \mid [\omega^\pm(X)]_{H^2(\mathcal{M})} = 0\} \cap \mathcal{U}_{N,\eta} \quad \text{and} \quad \mathcal{W} := \mathcal{W}^+ \cap \mathcal{W}^-,$$

where $\omega^\pm(X)$ denotes the winding cycles of X . Our goal is to show eventually in [Section 5.3](#) that $\mathcal{W}^\pm$ are locally transversal Banach manifolds near certain flows. Of course, for this we will assume that $X_0 \in \mathcal{W}^\pm$, however there is no need to do so in this subsection since we will only study the dependence of $\omega^\pm(X)$ on X .

Recall that $\omega^\pm(X) = \iota_X \Pi_3^{+,X} \Omega$, where for any k , $\Pi_k^{+,X}$ is the spectral projector at zero of $\mathcal{L}_X$ on k -forms. Here, we use that $\Pi_3^{+,X} \Omega =: \Omega_{\text{SRB}}^{+,X}$ is the SRB measure of X ; it is a probability measure since we assumed that Ω is a probability measure. For the remainder of this section, in order to simplify the notation we drop the superscript H in the holomorphic part at zero of the resolvent $R_k^{+,H,X}$ of $\mathcal{L}_X$ acting on Ω^k and write $R_k^{+,X}$ instead. When clear from context we will also drop the vector field X from the notation.

Let us first study the X dependence in $\Pi_3^{+,X}$.

Lemma 5.2 *There exist $N_0 > 0$, $\eta > 0$ and $C_0 \in \mathbb{R}$, such that for some $r > C_0 + 3$ and $N > N_0$, the map*

$$X \mapsto \Pi_3^{+,X} : \mathcal{H}_{rG}(\mathcal{M}; \Omega^3) \rightarrow \mathcal{H}_{rG,-3}(\mathcal{M}; \Omega^3) \quad \text{for } X \in \mathcal{U}_{N,\eta}$$

is C^1 -regular. Moreover, for N_0 large enough, the map

$$X \mapsto \iota_X \Pi_3^{+,X} : \mathcal{H}_{rG}(\mathcal{M}; \Omega^3) \rightarrow \mathcal{H}_{rG,-3}(\mathcal{M}; \Omega^2) \quad \text{for } X \in \mathcal{U}_{N,\eta}$$

is C^1 -regular. Finally, for any $(X, Y) \in \mathcal{U}_{N,\eta} \times C^N(\mathcal{M}; T\mathcal{M})$ and $N > N_0$,

$$D[\omega^+(X)](Y) = \Pi_2^{+,X} \iota_Y \Omega_{\text{SRB}}^{+,X} + dR_1^{+,X} \iota_X \iota_Y \Omega_{\text{SRB}}^{+,X}.$$

Here, we remark that $D[\omega^+(X)](Y)$ denotes the derivative of ω^+ at X in the direction of Y .

Proof Let γ be a small smooth contour around zero, such that $\mathcal{L}_{X_0}$ acting on Ω^3 has no nonzero resonances on a neighbourhood of the closed domain that γ bounds. We recall here that since X_0 is transitive, the rank of the spectral projector Π_3^{+,X_0} is equal to 1. By [Lemma 5.1](#), there is a neighbourhood V of γ such that for some small enough $\eta > 0$, for any $X \in \mathcal{U}_{N,\eta}$, $\mathcal{L}_X$ has no resonances in V and $(\mathcal{L}_X + s)^{-1}$ is uniformly bounded as a map on $\mathcal{H}_{rG}$. Therefore we may define

$$(5-2) \quad \tilde{\Pi}_3^{+,X} := \frac{1}{2\pi i} \oint_\gamma (\mathcal{L}_X + s)^{-1} ds \quad \text{for } X \in \mathcal{U}_{N,\eta}.$$

By [\[Cekić and Paternain 2020, Lemma 6.1\]](#), the rank of $\tilde{\Pi}_3^{+,X}$ is constant for $X \in \mathcal{U}_{N,\eta}$ for small enough $\eta > 0$; therefore $\tilde{\Pi}_3^{+,X} = \Pi_3^{+,X}$.

Note that by the resolvent identity, for any $X, Y \in \mathcal{U}_{N,\eta}$ and $s \in V$,

$$(5-3) \quad (\mathcal{L}_X + s)^{-1} - (\mathcal{L}_Y + s)^{-1} = (\mathcal{L}_X + s)^{-1} (\mathcal{L}_Y - \mathcal{L}_X) (\mathcal{L}_Y + s)^{-1}.$$

By Lemma 5.1, $(\mathcal{L}_Y + s)^{-1}$ is uniformly bounded for $s \in V$ as a map on $\mathcal{H}_{rG}$ and $(\mathcal{L}_X + s)^{-1}$ is uniformly bounded as map on $\mathcal{H}_{rG,-1}$, if $r > C_0 + 1$. We would like to estimate $\mathcal{L}_X - \mathcal{L}_Y = \mathcal{L}_{X-Y}$ as a map $\mathcal{H}_{rG} \rightarrow \mathcal{H}_{rG,-1}$. Equivalently, we would like to estimate the norm of the operator

$$P = (1 + \Delta_g)^{-\frac{1}{2}} e^{r \operatorname{Op}(G)} \mathcal{L}_{X-Y} e^{-r \operatorname{Op}(G)} : L^2(\mathcal{M}; \Omega^k) \rightarrow L^2(\mathcal{M}; \Omega^k).$$

By the composition formula for pseudodifferential operators and the Calderón–Vaillancourt theorem (see [Zworski 2012, Theorems 4.14 and 4.23] for these statements in the Euclidean setting), it follows that the norm $\|P\|_{L^2 \rightarrow L^2}$ depends on a finite number of derivatives of the symbols of the operators in the composition. Therefore if we take N large enough,

$$\|\mathcal{L}_X - \mathcal{L}_Y\|_{\mathcal{H}_{rG} \rightarrow \mathcal{H}_{rG,-1}} = \|P\|_{L^2 \rightarrow L^2} \leq C \|X - Y\|_{C^N}$$

for some $C > 0$ (depending also on r). This estimate and (5-3) finally show that the map

$$X \mapsto (\mathcal{L}_X + s)^{-1} : \mathcal{H}_{rG} \rightarrow \mathcal{H}_{rG,-1} \quad \text{for } X \in \mathcal{U}_{N,\eta}$$

is Lipschitz continuous for $s \in V$. Therefore, taking the limits we get

$$D[(\mathcal{L}_X + s)^{-1}](Y) = -(\mathcal{L}_X + s)^{-1} \mathcal{L}_Y (\mathcal{L}_X + s)^{-1} \quad \text{for } (X, Y) \in \mathcal{U}_{N,\eta} \times C^N(\mathcal{M}; T\mathcal{M})$$

as maps $\mathcal{H}_{rG} \rightarrow \mathcal{H}_{rG,-2}$. In fact, by the above discussion (for instance we need to use the bound $\|\mathcal{L}_{X-Y}\|_{\mathcal{H}_{rG,-1}, \mathcal{H}_{rG,-2}} \leq C \|X - Y\|_{C^N}$ for some $C > 0$), the right-hand side is only continuous as a map $\mathcal{H}_{rG} \rightarrow \mathcal{H}_{rG,-3}$, since for each of the operators we lose one derivative. We conclude that

$$X \mapsto (\mathcal{L}_X + s)^{-1} : \mathcal{H}_{rG} \rightarrow \mathcal{H}_{rG,-3} \quad \text{for } X \in \mathcal{U}_{N,\eta} \text{ and } s \in V$$

is C^1 -regular. Here and throughout we ask that $r > C_0 + 3$.

Since $\Pi_3^{+,X} = (1/(2\pi i)) \oint_{\gamma} (\mathcal{L}_X + s)^{-1} ds$, the first claim follows. The second claim follows by the C^1 properties of the multiplication $C^N(\mathcal{M}) \times \mathcal{H}_{rG,-3}(\mathcal{M}) \rightarrow \mathcal{H}_{rG,-3}(\mathcal{M})$ for large enough N .

Next, we compute the first derivative of the SRB measure:

$$(5-4) \quad \begin{aligned} D[\Omega_{\text{SRB}}^{+,X}](Y) &= -\frac{1}{2\pi i} \oint_{\gamma} (\mathcal{L}_X + s)^{-1} \mathcal{L}_Y (\mathcal{L}_X + s)^{-1} \Omega \\ &= -(R_3^{+,X} \mathcal{L}_Y \Pi_3^{+,X} + \Pi_3^{+,X} \mathcal{L}_Y R_3^{+,X}) \Omega = -dR_2^{+,X} \iota_Y \Omega_{\text{SRB}}^{+,X}, \end{aligned}$$

where in the second equality we used the residue theorem and the expansion (2-2), and in the last equality we used $\mathcal{L}_Y = d\iota_Y + \iota_Y d$, as well as $\Pi_3^{+,X} d = 0$, $dR_3^{+,X} = 0$, $d\Pi_3^{+,X} = 0$, and $dR_2^{+,X} = R_3^{+,X} d$. The final formula now easily follows:

$$D[\iota_X \Omega_{\text{SRB}}^{+,X}](Y) = \iota_Y \Omega_{\text{SRB}}^{+,X} - \iota_X dR_2^{+,X} \iota_Y \Omega_{\text{SRB}}^{+,X} = \Pi_2^{+,X} \iota_Y \Omega_{\text{SRB}}^{+,X} + dR_1^{+,X} \iota_X \iota_Y \Omega_{\text{SRB}}^{+,X},$$

where in the second equality we used $\mathcal{L}_X = \iota_X d + d\iota_X$, $\mathcal{L}_X R_2^{+,X} = \text{Id} - \Pi_2^{+,X}$ (which follows from (2-2)) and $\iota_X R_2^{+,X} = R_1^{+,X} \iota_X$. This completes the proof. $\square$

Remark 5.3 In the dynamical systems literature, the regularity of the SRB measure with respect to perturbations has been intensively studied, and statements similar to [Lemma 5.2](#) but in a different functional setting have been obtained; see [\[Ruelle 2008, Theorem A\]](#), [\[Butterley and Liverani 2007, Theorem 2\]](#) and [\[Gouëzel and Liverani 2006, Theorem 2.7\]](#). The point of this lemma is to provide an alternative microlocal proof of the regularity statement, and to compute the first variation of the winding cycle in this setting. We also remark that in the dynamical systems literature the formula (5-4) for the derivative of the SRB measure is sometimes known as the *linear response* formula.

5.3 Banach manifolds $\mathcal{W}^\pm$

We are now in shape to study the local Banach manifold structure of the spaces $\mathcal{W}^\pm$. Assume that X_0 preserves a smooth probability volume Ω , so that $\Omega_{\text{SRB}}^+ = \Omega_{\text{SRB}}^- = \Omega$, and assume that $[\omega^+] = [\omega^-] = 0$.

We first prove an auxiliary result:

Lemma 5.4 For $Y \in C^\infty(\mathcal{M}; T\mathcal{M})$, define $\mathcal{Z}^\pm(Y) := \Pi_2^\pm \iota_Y \Omega$. Then the following map is surjective, unless X_0 is a contact flow:

$$\mathcal{Z}: C^\infty(\mathcal{M}; T\mathcal{M}) \rightarrow \text{Res}^{2,\infty} \times \text{Res}_*^{2,\infty}, \quad Y \mapsto (\mathcal{Z}^+(Y), \mathcal{Z}^-(Y)).$$

If X_0 is contact, then $\text{ran}(\mathcal{Z})$ is of codimension 1, and the projection of $\text{ran}(\mathcal{Z})$ to $H^2(\mathcal{M}) \times H^2(\mathcal{M})$ is surjective. In either case, $\mathcal{Z}^+$ and $\mathcal{Z}^-$ are surjective onto $\text{Res}^{2,\infty}$ and $\text{Res}_*^{2,\infty}$, respectively, which further project surjectively to $H^2(\mathcal{M})$.

Proof Assume for the sake of contradiction that the map $\mathcal{Z}$ is not surjective. Then, by [Proposition 2.2](#), there is nonzero $(u_*, u) \in \text{Res}_*^{1,\infty} \times \text{Res}^{1,\infty}$ such that

$$\int_{\mathcal{M}} u_* \wedge \Pi_2^+ \iota_Y \Omega + \int_{\mathcal{M}} u \wedge \Pi_2^- \iota_Y \Omega = 0 \quad \text{for all } Y \in C^\infty(\mathcal{M}; T\mathcal{M}).$$

It follows by (2-7) that

$$(5-5) \quad \int_{\mathcal{M}} \iota_Y (u + u_*) \Omega = 0 \quad \text{for all } Y \in C^\infty(\mathcal{M}; T\mathcal{M}),$$

which translates into $u + u_* = 0$. By the wavefront set condition, it follows that $u \in C^\infty(\mathcal{M}; \Omega^1)$. By using the contraction/expansion properties, we get that $u|_{E_u \oplus E_s} = 0$, and since $\iota_{X_0} u \in \text{Res}^0$, we have $\iota_{X_0} u$ is a constant. By [\[Hurder and Katok 1990, Theorem 2.3\]](#) it follows that the flow is either a suspension of an Anosov diffeomorphism on a torus contradicting $[\omega^+] = 0$, or u is a contact form. This completes the proof of the first claim.

For the second claim, assume that X_0 is contact with contact form $\alpha \in C^\infty(\mathcal{M}; \Omega^1)$, satisfying $\iota_{X_0} \alpha = 1$ and $\mathcal{L}_{X_0} \alpha = 0$; then we may assume (up to rescaling) that $\Omega = \alpha \wedge d\alpha$. Since $\iota_{X_0} \text{Res}^{2,\infty} \subset \text{Res}_0^1$ (here we use that $\mathcal{L}_{X_0}$ is semisimple on Ω_0^1 as shown by [\[Dyatlov and Zworski 2017, Proposition 3.1\(3\)\]](#)), and

every $u \in \text{Res}^{2,\infty}$ decomposes invariantly as $u = \alpha \wedge \iota_{X_0} u + (u - \alpha \wedge \iota_{X_0} u)$, we conclude that

$$(5-6) \quad \text{Res}^{2,\infty} = \alpha \wedge \text{Res}_0^1 \oplus \text{Res}_0^2 = \alpha \wedge \text{Res}_0^1 \oplus \mathbb{C} d\alpha,$$

where the semisimplicity of $\mathcal{L}_{X_0}$ on Ω_0^2 was proved in [Proposition 3.1](#); hence in particular $\text{Res}^2 = \text{Res}^{2,\infty}$ and similarly $\text{Res}_*^2 = \text{Res}_*^{2,\infty}$. Since $(\Pi_2^\pm)^T = \Pi_1^\mp$ by (2-7) and $\Pi_1^\pm \alpha = \alpha$, we have

$$\int_{\mathcal{M}} \alpha \wedge \Pi_2^+ \iota_Y \Omega = \int_{\mathcal{M}} \alpha \wedge \iota_Y \Omega = \int_{\mathcal{M}} \alpha \wedge \Pi_2^- \iota_Y \Omega,$$

and so we get that

$$\text{ran}(\mathcal{Z}) \subset \left\{ (u, u_*) \in \text{Res}^2 \oplus \text{Res}_*^2 \mid \int_{\mathcal{M}} \alpha \wedge (u - u_*) = 0 \right\}.$$

We claim that the equality holds. We have $\mathcal{Z}(X_0) = (d\alpha, d\alpha)$ and so using (5-6) it suffices to show that

$$C^\infty(\mathcal{M}; T\mathcal{M}) \rightarrow \text{Res}_0^1 \times \text{Res}_{0*}^1, \quad Y \mapsto (\iota_{X_0} \mathcal{Z}^+(Y), \iota_{X_0} \mathcal{Z}^-(Y)) = -(\Pi_1^+ \iota_Y d\alpha, \Pi_1^- \iota_Y d\alpha),$$

is surjective. Here in the equality we used that $\Omega = \alpha \wedge d\alpha$ and $\iota_{X_0} \mathcal{Z}^\pm(Y) = \Pi_1^\pm \iota_{X_0} \iota_Y \Omega = -\Pi_1^\pm \iota_Y d\alpha$. Again, if not, by [Lemma 2.1](#) there exist $u_* \in \text{Res}_{0*}^1$ and $u \in \text{Res}_0^1$ (at least one of which is nonzero) such that for all Y ,

$$0 = \int_{\mathcal{M}} \alpha \wedge (u_* \wedge \Pi_1^+ \iota_Y d\alpha + u \wedge \Pi_1^- \iota_Y d\alpha) = \int_{\mathcal{M}} \alpha \wedge (u_* + u) \wedge \iota_Y d\alpha = \int_{\mathcal{M}} \iota_Y (u + u_*) \Omega,$$

where we used (2-6) in the second equality, and $(u + u_*) \wedge d\alpha = 0$ in the third equality. This implies $u + u_* = 0$, so $u \in C^\infty(\mathcal{M}; \Omega_0^1)$, which by the same argument as above translates to $u = 0$ (and so also $u_* = 0$). This contradicts the assumption and proves the claim about $\mathcal{Z}$.

That the projection onto $H^2(\mathcal{M}) \times H^2(\mathcal{M})$ is surjective now follows from the observation that for a smooth closed 2-form η

$$\Pi_2^+ \eta = (\text{Id} - R_2^+ \mathcal{L}_{X_0}) \eta = \eta - dR_1^+ \iota_{X_0} \eta,$$

so Π_2^+ restricts to an injection on $H^2(\mathcal{M})$ to $\text{Res}^{2,\infty}$, and similarly for Π_2^- . Since $d\alpha$ is exact, we conclude that $\text{ran}(\mathcal{Z})$ contains $\alpha \wedge \text{Res}_0^1 \times \alpha \wedge \text{Res}_{0*}^1$, which projects onto $H^2(\mathcal{M}) \times H^2(\mathcal{M})$, completing the proof.

The final claim about the surjectivity of $\mathcal{Z}^\pm$ follows easily from the method in the first paragraph, and the fact that $\text{Res}^{2,\infty}$ and $\text{Res}_*^{2,\infty}$ surject onto $H^2(\mathcal{M})$. $\square$

Next, we apply the preceding lemma to show the Banach manifold structure:

Lemma 5.5 *Assume X_0 preserves a smooth volume and that the winding cycles are zero, ie $[\omega^+(X_0)] = [\omega^-(X_0)] = 0$. Then $\mathcal{W}^\pm$ are C^1 Banach submanifolds near X_0 of codimension $b_1(\mathcal{M})$. The intersection $\mathcal{W}^+ \cap \mathcal{W}^-$ is transversal, that is, it is a C^1 Banach submanifold of codimension $2b_1(\mathcal{M})$.*

Proof Let us first show the claims about $\mathcal{W}^\pm$. Let $(e_i)_{i=1}^{b_1(\mathcal{M})}$ be a basis of $H^1(\mathcal{M}; \mathbb{R})$, and for $X \in \mathcal{U}_{N,\eta}$, define the maps

$$(5-7) \quad \mathcal{F}^\pm(X) = \left(\int_{\mathcal{M}} e_1 \wedge \iota_X \Pi_3^{\pm, X} \Omega, \dots, \int_{\mathcal{M}} e_{b_1(\mathcal{M})} \wedge \iota_X \Pi_3^{\pm, X} \Omega \right) \in \mathbb{R}^{b_1(\mathcal{M})}.$$

In suitable topologies given by [Lemma 5.2](#), the maps $\mathcal{F}^\pm$ are C^1 , and we may compute their derivative along a C^N -regular vector field Y as

$$(5-8) \quad D_{X_0} \mathcal{F}^\pm(Y) = \left(\int_{\mathcal{M}} e_1 \wedge \Pi_2^+ \iota_Y \Omega, \dots, \int_{\mathcal{M}} e_{b_1(\mathcal{M})} \wedge \Pi_2^+ \iota_Y \Omega \right).$$

We claim that $D_{X_0} \mathcal{F}^\pm$ are surjective. For that, it suffices to observe by [Lemma 5.4](#) that $\text{Res}^{2,\infty}$ projects surjectively onto $H^2(\mathcal{M})$ and $Y \mapsto \Pi_2^+ \iota_Y \Omega$ is surjective onto $\text{Res}^{2,\infty}$. Therefore, Poincaré duality implies the initial claim. An application of the implicit function theorem [[Lang 1999](#), Theorem 5.9] shows that $\mathcal{W}^\pm$ are locally C^1 Banach manifolds of codimension $b_1(\mathcal{M})$, as required.

Next, consider the map $\mathcal{F}(X) := (\mathcal{F}^+(X), \mathcal{F}^-(X)) \in \mathbb{R}^{2b_1(\mathcal{M})}$. As in the paragraph above, we may compute the derivative of $\mathcal{F}$ at X_0 , and conclude by [Lemma 5.4](#) that $D_{X_0} \mathcal{F}$ is surjective onto $\mathbb{R}^{2b_1(\mathcal{M})}$ in both contact and noncontact cases. The surjectivity is then deduced by Poincaré duality, and the claim follows by applying the implicit function theorem. $\square$

Remark 5.6 It is expected that [Lemma 5.5](#) is valid in the C^∞ -regularity setting. For that, we would need to show that the assumptions of the Nash–Moser implicit function theorem are satisfied (see [[Hamilton 1982](#), Chapter 4]), that is, that the involved mappings, and the right inverse for the derivative map, are *tame in a neighbourhood* of X_0 . Since this would produce further technical difficulties, we refrained from this exploration.

Remark 5.7 Throughout this section we assumed that X_0 preserves a smooth volume Ω . The reason is that otherwise, we would not be able to show the surjectivity of the derivative: for instance following the proof of [Lemma 5.4](#) to show the surjectivity of $\mathcal{Z}^+$, we would reach the point where $u_* \wedge \iota_{X_0} \Omega_{\text{SRB}}^+ = 0$ for some $u_* \in \text{Res}_*^{1,\infty}$. The problem is to show that $u_* = 0$, which is very similar to the issue of semisimplicity raised in [Section 7](#) and [Lemma 8.16](#).

5.4 Resonant 1-forms for nearly hyperbolic metrics

Resonant 1-forms for hyperbolic metrics are special in the sense that for them it is possible to show that they are in one-to-one correspondence with holomorphic/antiholomorphic 1-forms on the surface by means of a pushforward map to Fourier modes of degree ± 1 given by

$$u \mapsto (\iota_V u)_1 + (\iota_V u)_{-1} \quad \text{for } u \in \text{Res}_0^1.$$

(See for example [Lemma 8.7](#) for the case $A = 0$.) It is natural to ask if there is a similar correspondence for an arbitrary negatively curved metric. In this subsection we show that this is *not* the case. More precisely, the main result of this section is [Proposition 5.8](#), where we construct conformal perturbations

for which this correspondence is invalid. As a byproduct we derive [Proposition 5.9](#) to construct *specific* Gaussian thermostats that have both winding cycles nonzero.

We begin by deriving a relation between resonant 1-forms and Fourier modes. Let (M, g) be a negatively curved surface and let X be the geodesic vector field on the unit sphere bundle $\mathcal{M} = SM$. Let $\{X, H, V\}$ be the global frame introduced in [Section 2.4](#). For $\theta \in H^1(M)$, a real-valued 1-form, we set

$$(5-9) \quad u := \Pi_1^+ \pi^* \theta = \pi^* \theta + d\varphi \in \text{Res}_0^1, \quad \varphi := -R_0^+ \iota_X \pi^* \theta, \quad c := V\varphi \in C_*^{-1-}(\mathcal{M}),$$

where the regularity claim follows from [Lemma 4.2](#). It follows from [Proposition 4.5](#) that $u = c(-r^u \beta + \psi)$, where r^u solves a Riccati equation and $Y^u = H + r^u V$, and

$$(X - r^u)c = 0, \quad Hc + V(r^u c) = 0.$$

Reducing to Fourier modes, this translates to, for each $k \in \mathbb{Z}$ (see [Lemma 8.4](#) for the case of the hyperbolic metric)

$$(5-10) \quad 2\eta_- c_{k+1} - (k+1)(r^u c)_k = 0 \quad \text{and} \quad 2\eta_+ c_{k-1} + (k-1)(r^u c)_k = 0.$$

Note that the Fourier modes c_k and $(r^u c)_k$ are smooth thanks to the wavefront set condition and the fact that $V \notin \mathbb{R}X \oplus E_u$. Indeed, by (4-5) we have that $r^u \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \cap C_*^{1+\alpha}(\mathcal{M})$, and from (5-9) it follows that $c \in \mathcal{D}'_{E_u^*}(\mathcal{M}) \cap C_*^{-1-}(\mathcal{M})$. Therefore, the product $r^u c$ is well-defined, and by the usual wavefront set analysis moreover we may obtain that $\text{WF}(r^u c) \subset E_u^*$; see for example [[Hörmander 2003](#), Theorem 8.2.10]. Let us first assume that $k = 0$. On smooth functions f , the zeroth Fourier mode is (up to constant) equal to $f_0(x) = \int_{S_x M} f(x, v) dv$, where dv denotes the volume form on $S_x M$, ie f_0 is the pushforward of f . The wavefront set calculus, see [[Melrose 2003](#), Proposition 4.19], then shows that indeed c_0 and $(r^u c)_0$ are smooth, proving the claim. The case of nonzero k is similar.

Restricting the first and second equations in (5-10) to $k = -1$ and $k = 1$, respectively, we get $\eta_- c_0 = 0$ and $\eta_+ c_0 = 0$, which implies that c_0 is constant. Since $c = V\varphi$, integrating by parts implies that $c_0 = 0$. If g is the hyperbolic metric we have $r^u \equiv 1$ and so $\eta_- c_1 = 0$. However, we will see in this section that $\eta_- c_1 \neq 0$ in general, by perturbation theory.

Proposition 5.8 *Let g_0 be a hyperbolic metric on M and fix a closed nonexact (real) 1-form θ on M . For an open and dense set of $h \in C^\infty(M)$, there exists an $\varepsilon > 0$ such that for $0 < |s| < \varepsilon$ and for the metric $g_s := e^{-2sh} g_0$, the Fourier mode c_1 of c defined by (5-9) is not holomorphic, ie $\eta_- c_1 \neq 0$.*

Proof By (5-10) we have $\eta_- c_1 = \frac{1}{2}(r^u c)_0 = \frac{1}{2}(Xc)_0$, so it suffices to consider the latter quantity. Denote by SM_s the unit sphere bundle of g_s , by $\ell_s: SM_0 \rightarrow SM_s$ given by $\ell_s(x, v) = (x, e^{sh} v)$ the natural rescaling map, and by X_s and V_s the geodesic and vertical vector fields on SM_s , respectively; also denote by $r^u(s)$ the solution to the Riccati equation of g_s . Then by the third line of [Lemma B.1](#) (and (2-13)),

$$(5-11) \quad Z_s := \ell_s^* X_s = e^{s\pi_0^* h} (X - sVX\pi_0^* h \cdot V), \quad \ell_s^* V_s = V.$$

If $\pi(s)$ is the footpoint projection $SM_s \rightarrow M$, denote $\varphi(s) := -R_0^{+,X_s} \iota_{X_s} \pi(s)^* \theta$ so that $\varphi(0) = \varphi$ and if we set $c(s) := V_s \varphi(s)$, then

$$(5-12) \quad d(s) := \ell_s^* c(s) = -VR_0^{+,Z_s} \iota_{Z_s} \pi^* \theta.$$

In suitable topologies determined by [Lemma 5.2](#), we compute the derivative at zero of (5-12) (in various expressions, the dot denotes taking the derivative at zero):

$$(5-13) \quad \dot{d} = V(R_0^+ \dot{Z} \Pi_0^+ + \Pi_0^+ \dot{Z} R_0^+) \iota_X \pi^* \theta - VR_0^+ \iota_{\dot{Z}} \pi^* \theta = -VR_0^+ (\pi_0^* h \cdot \iota_X \pi^* \theta),$$

where in the first equality we used the formula for the derivative of the resolvent derived in the proof of [Lemma 5.2](#) and in the second one we used $V \Pi_0^+ = 0$, $\Pi_0^+ \iota_X \pi^* \theta = 0$, as well as (5-11) and $\iota_V \pi^* \theta = 0$. Therefore from (5-12)

$$\begin{aligned} (Z_s \dot{d}(s))_0 &= ((\pi_0^* h \cdot X - VX \pi_0^* h \cdot V) c)_0 - (XVR_0^+ (\pi_0^* h \cdot \iota_X \pi^* \theta))_0 \\ &= -(X \pi_0^* h \cdot c)_0 + (HR_0^+ (\pi_0^* h \cdot \iota_X \theta))_0 = (A + B)h, \end{aligned}$$

where in the first equality we used (5-13), in the second equality we used that $c_0 = 0$, $V^2 X \pi_0^* h = -X \pi_0^* h$ and (2-10); and in the last equality we used $X = \eta_+ + \eta_-$ and introduced the operators

$$A(\bullet) := -2 \operatorname{Re}(c_{-1} \eta_+ \pi_0^*(\bullet)), \quad B(\bullet) := (HR_0^+ (\pi_0^*(\bullet) \cdot \iota_X \theta))_0,$$

acting on $C^\infty(M)$. The operator B is actually pseudodifferential of order zero, as follows from [\[Cekić and Lefeuvre 2025, Proposition 4.1\]](#); note that there we deal with $R_0^+ + R_0^-$ instead of R_0^+ , but the same proof applies. The other operator A is a differential operator of degree 1. Since it has no degree-zero terms, it is clearly nonzero since c_{-1} is nonzero (in fact it is zero only at a finite number of points since c_{-1} is antiholomorphic). Therefore $A + B \in \Psi^1(M)$ is a nonzero pseudodifferential operator of order 1 and hence it is nonzero on an open and dense set of $C^\infty(M)$.

We conclude that the first derivative of $(Z_s d(s))_0$ at $s = 0$ is nonzero for an open and dense set of $h \in C^\infty(M)$. Since (where we identify the notation for the Fourier modes on SM_s and SM_0)

$$\ell_s^*(r^u(s)c(s))_0 = \frac{1}{2} \ell_s^*(X_s c(s))_0 = \frac{1}{2} (Z_s d(s))_0,$$

it follows that for an open and dense set of $h \in C^\infty(M)$, there exists an $\varepsilon > 0$ small enough such that for $0 < |s| < \varepsilon$, $(r^u(s)c(s))_0 \neq 0$, ie $(c(s))_1$ is *not* holomorphic, which completes the proof. $\square$

We conclude this section with an application to the winding cycles of thermostats:

Proposition 5.9 *Let (M, g_0) be a hyperbolic surface. For an open and dense set of $h \in C^\infty(M)$, there exist $f \in C^\infty(M)$ and $\varepsilon, \delta > 0$ such that for any metric $g = e^{-2sh} g_0$ with $\varepsilon/2 < |s| < \varepsilon$, any vector field $F_t := X_g + tX_g \pi_0^* f \cdot V_g$ with $0 < |t| < \delta$ generates an Anosov flow which has both nonzero winding cycles.*

Proof We adopt the notation of [Proposition 5.8](#), which shows that for any fixed nonexact closed (real) 1-form θ , there is an open and dense set of $h \in C^\infty(M)$, and $\varepsilon = \varepsilon(h) > 0$, such that $(r^u(s)c(s))_0 \neq 0$ for $0 < |s| < \varepsilon$. Write Ω_s for the canonical volume form on SM_s .

By [Lemma 5.2](#), the first variation at $t = 0$ of the winding cycle $[\omega^-(F_t)]$ in the direction of θ is

$$\begin{aligned} \int_{SM_s} \pi^* \theta \wedge \Pi_2^{-, X_s} (X_s \pi_0^* f \cdot \iota_{V_s} \Omega_s) &= \int_{SM_s} (\pi^* \theta + d\varphi(s)) \wedge X_s \pi_0^* f \cdot \iota_{V_s} \Omega_s \\ &= \int_{SM_s} c(s) X_s \pi_0^* f \cdot \Omega_s = - \int_{SM_s} (X_s c(s))_0 \cdot \pi_0^* f \cdot \Omega_s. \end{aligned}$$

By the proof of [Proposition 5.8](#), $(X_s c(s))_0 \neq 0$, so there exists an $f \in C^\infty(M)$ such that the last integral is nonzero, which shows $[\omega^-(F_t)] \neq 0$ for $\varepsilon/2 < s < \varepsilon$ and $0 < |t| < \delta$. A similar argument applies to $[\omega^+(F_t)]$, or alternatively, this follows from the reversibility of the flow, completing the proof. $\square$

6 Anosov flows with one smooth weak foliation

In this section we look at a particular subclass of Anosov flows on closed 3-manifolds, namely those that have one weak foliation of class C^∞ . To be definite, let us assume that this is the stable weak foliation. In this context it is known that the flow is topologically orbit equivalent to the suspension of a toral automorphism on $\mathbb{T}^2$ or, up to finite covers, to the geodesic flow of a hyperbolic surface [[Ghys 1993](#), Theorem 4.7]. We are mostly interested in flows that are not suspensions (since we already know what happens in that case) and we shall ignore finite covers for the sake of simplicity. Under these assumptions we may as well assume that $\mathcal{M}$ is the unit tangent bundle of a closed oriented hyperbolic surface (M, g) .

Let X denote an Anosov vector field with C^∞ weak stable bundle on SM , and let X_0 denote the geodesic vector field of a hyperbolic metric g . By [[Ghys 1993](#), Theorem 5.3], there is a diffeomorphism $f: SM \rightarrow SM$ that conjugates the weak foliation of X with the weak foliation of X_0 . This implies that there are $a, b \in C^\infty(SM)$ such that

$$f_*(X) = aX_0 + b(H - V)$$

since $\{X_0, H - V\}$ is a basis for the weak stable bundle of X_0 (recall that the global frame $\{X_0, H, V\}$ on SM was introduced in [Section 2.4](#)). Without loss of generality we may remove the diffeomorphism f from the notation and assume that we have an Anosov vector field X on SM of the form

$$(6-1) \quad X = aX_0 + b(H - V)$$

for some $a, b \in C^\infty(SM)$. This implies that the bundle spanned by $\{X_0, H - V\}$ is invariant under the flow of X and thus it must be one of the weak bundles. By switching the signs of a and b if necessary, we may suppose it is the weak stable bundle. It follows that

$$(6-2) \quad E_{s,X}^* = E_{s,X_0}^*.$$

Lemma 6.1 We have $\text{Res}_{0*}^1(X_0) \subset \text{Res}_{0*}^1(X)$ and $[\omega^+] = 0$, where $\omega^+ = \iota_X \Omega_{\text{SRB}}^+$.

Proof Take $u_* \in \text{Res}_{0*}^1(X_0)$. We know that $du_* = 0$ (for instance, by Lemma 8.5 below for $A = 0$) and $u_*(H - V) = 0$ by horocyclic invariance, Lemma 4.2. Since $u_*(X_0) = 0$ it follows that $u_*(X) = 0$ and thus $u_* \in \text{Res}_{0*}^1(X)$ since (6-2) holds. We also know that the map $\text{Res}_{0*}^1(X_0) \rightarrow H^1(SM)$ given by $u_* \mapsto [u_*]$ is an isomorphism. In other words, given any smooth closed 1-form η on SM there is $f \in \mathcal{D}'_{E_s^*}(SM)$ such that $\eta + df \in \text{Res}_{0*}^1(X_0)$. Thus $\eta(X) + df(X) = 0$ and

$$\int_{SM} \eta(X) \Omega_{\text{SRB}}^+ = - \int df(X) \Omega_{\text{SRB}}^+ = 0,$$

showing $[\omega^+] = 0$, as desired. $\square$

Next, consider the standard volume form Ω in SM . Since X_0, H and V preserve Ω , we have

$$\text{div}_\Omega X = X_0(a) + (H - V)(b).$$

Let $\lambda \in C^\infty(SM; \mathbb{R})$ be given by $\lambda = \lambda_{-m} + \lambda_m \in H_{-m} \oplus H_m$, where $\eta_{-m} = 0$ and $m \geq 1$, and $\lambda_{-m} = \bar{\lambda}_m$. By Lemma A.2 we know that this implies that $X_0 V \lambda = m H \lambda$. For a and b we choose

$$a = 1 + \frac{V(\lambda)}{m}, \quad b = -\lambda.$$

This gives

$$(6-3) \quad \text{div}_\Omega X = V \lambda.$$

The case $m = 1$ corresponds to $\lambda = \pi_1^* \theta$ (and hence $V \lambda = -\pi_1^* \star \theta$), where θ is a real harmonic 1-form; see the proof of Proposition A.4. For this case we will prove:

Proposition 6.2 Let (M, g) be a closed oriented hyperbolic surface and let θ be a nonzero harmonic 1-form. For $s \in \mathbb{R}$, consider the vector field

$$X_s = (1 - s\pi_1^* \star \theta)X_0 - s\pi_1^* \theta(H - V).$$

Then for all $s \neq 0$ sufficiently small, the Anosov vector field X_s has the following properties:

- (1) The weak stable bundle is C^∞ , but the weak unstable bundle is only of class $C^{1+\alpha}$ for some $\alpha > 0$.
- (2) $[\omega^+] = 0$ and $[\omega^-] \neq 0$.
- (3) There is $u \in \text{Res}_0^1(X_s)$ that is not closed, but $\text{Res}_{0*}^1(X_s) = \text{Res}_{0*}^1(X_0)$, and all its elements are closed.
- (4) Semisimplicity for the actions of $\mathcal{L}_{X_s}$ and $\mathcal{L}_{-X_s}$ on Ω^1 hold and fail, respectively. Semisimplicity for the action of $\mathcal{L}_{\pm X_s}$ on $\Omega^1 \cap \ker \iota_{X_s}$ holds.

Proof Note that

$$-s\pi^* \star \theta(X_s) = sV\lambda(1 + sV\lambda) + s^2\lambda^2 = sV\lambda + s^2[(V\lambda)^2 + \lambda^2],$$

where in the second equality we used that $\pi^*\theta(H) = \pi_1^*\theta$, which is a straightforward computation which follows from (A-1). Thus

$$(6-4) \quad -s \int_{SM} \pi^* \star \theta(X) \Omega_{\text{SRB}}^-(s) = s \int_{SM} V\lambda \Omega_{\text{SRB}}^-(s) + s^2 \int_{SM} ((V\lambda)^2 + \lambda^2) \Omega_{\text{SRB}}^-(s).$$

Using (6-3) we see that

$$e^-(s) = s \int_{SM} V\lambda \Omega_{\text{SRB}}^-(s)$$

is the entropy production of $\Omega_{\text{SRB}}^-(s)$. We know that $e^-(s) \geq 0$, with equality if and only if $sV(\lambda)$ is a coboundary for X_s . Indeed, recall that by [Ruelle 1996, Theorem 1.2], the entropy production vanishes if and only if the flow of X_s preserves a smooth measure. On the other hand, the flow of X_s preserves a smooth measure if and only if its divergence $sV(\lambda)$ is a coboundary. We claim that $sV(\lambda)$ cannot be coboundary for $s \neq 0$. If it were, the integral of $V(\lambda)$ along every closed orbit of X_s would be zero. Since X_s is topologically conjugate to X_0 and every homology class contains a closed orbit of X_0 , this would imply that $[\star\theta]$ pairs to zero with each homology class, which is absurd since $[\star\theta] \neq 0$. This gives $e^-(s) > 0$ for $s \neq 0$, and going back to (6-4) we deduce that $[\omega^-(X_s)] \neq 0$ for $s \neq 0$.

The flow of X_s has a smooth weak foliation by construction; hence Lemma 6.1 gives $[\omega^+] = 0$, and this shows item (2). To complete the proof of item (1), it suffices to note that if the weak unstable bundle were also C^∞ , then Lemma 6.1 applied to $-X_s$ would give that $[\omega^-] = 0$, which contradicts $[\omega^-] \neq 0$.

Next, item (3) follows from Theorem 1.1 and item (2).

Finally, by item (3), there is a $u \in \text{Res}_0^1(X_s)$ that is nonclosed, so semisimplicity for the action of $\mathcal{L}_{-X_s}$ on Ω^1 fails by Proposition 3.10. For the action of $\mathcal{L}_{X_s}$ on Ω^1 , we notice that by combining item (2) and Proposition 3.7, we have $\dim \text{Res}^1(X_s) = b_1(M) + 1 = \dim \text{Res}^1(X_0)$. Therefore by the upper-semicontinuity of $\dim \text{Res}^{1,\infty}$ (see [Cekić and Paternain 2020, Lemma 6.2]), we conclude that the required semisimplicity holds (note that $\mathcal{L}_{X_0}$ is semisimple).

The claim about the action of $\mathcal{L}_{X_s}$ on $\Omega^1 \cap \ker \iota_{X_s}$ follows from the fact that the action of $\mathcal{L}_{X_s}$ on Ω^1 is semisimple, and the analogous claim about $\mathcal{L}_{-X_s}$ then follows directly by Lemma 2.1. $\square$

Remark 6.3 Anosov flows with weak bundles as in item (1) of the previous proposition are always dissipative, ie they do not preserve a smooth volume. Indeed if a smooth volume form is being preserved, then both weak bundles would be smooth by [Hurder and Katok 1990, Corollary 3.5].

7 Thermostats

In this section we focus our attention on a particular kind of flow.

Let (M, g) be a closed oriented Riemannian surface and let $\lambda \in C^\infty(SM; \mathbb{R})$. We will be interested in the flow of the vector field $F = X + \lambda V$. The integral curves of F have the form $(\gamma, \dot{\gamma})$, where $\gamma: \mathbb{R} \rightarrow M$ solves the ODE

$$\ddot{\gamma} = \lambda(\gamma, \dot{\gamma})J\dot{\gamma},$$

where the acceleration is computed using the Levi-Civita connection of g and $J : TM \rightarrow TM$ is rotation by $\pi/2$ according to the orientation of the surface. We call these vector fields *thermostats* (sometimes they are referred to as “ λ -geodesic flows”). The flow of F models the motion of a particle under the influence of a force that is orthogonal to the velocity and with magnitude λ . For suitable choices of g and λ the vector field F will be a dissipative Anosov flow.

In [Dairbekov and Paternain 2007] it was proved that when F is Anosov, $E_{u/s}$ are transversal to the vertical direction and so there are $r^{u/s} \in C^{1+\alpha}(\mathcal{M})$ (as the weak stable/unstable bundles $\mathbb{R}F \oplus E_{u/s}$ are $C^{1+\alpha}$, see Section 4) functions such that $Y^{u/s} := H + r^{u/s}V \in \mathbb{R}F \oplus E_{u/s}$. In fact, they satisfy the Riccati equations (see [Dairbekov and Paternain 2007, Lemma 4.3] or [Merry and Paternain 2011, Proposition 8.19]):

$$(7-1) \quad Fr^{u/s} + (r^{u/s})^2 + K - H\lambda + \lambda^2 - V\lambda \cdot r^{u/s} = 0.$$

We may therefore compute the following commutators:

Lemma 7.1 (commutator stable/unstable) $[F, Y^{u/s}] = -\lambda F - r^{u/s}Y^{u/s}$.

Proof This is a straightforward computation, which we carry out just for r^u (it is the same for r^s):

$$\begin{aligned} [X + \lambda V, H + r^u V] &= [X, H] + Xr^u \cdot V + r^u[X, V] - H\lambda \cdot V - \lambda[H, V] + \lambda V r^u \cdot V - r^u V \lambda \cdot V \\ &= KV + Xr^u \cdot V - r^u H - H\lambda \cdot V - \lambda X + (\lambda V r^u - r^u V \lambda)V \\ &= -\lambda F + (\lambda^2 + K + Fr^u - H\lambda - r^u V \lambda)V - r^u H \\ &= -\lambda F - r^u(H + r^u V) \\ &= -\lambda F - r^u Y^u, \end{aligned}$$

where we used (2-10) in the second equality and (7-1) in the last one. This completes the proof. $\square$

Write $\Omega_{\text{SRB}}^+ = f\Omega$ for the SRB measures of F , where Ω is the canonical volume form on SM (see Section 2.4) and where $f \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ satisfies

$$(7-2) \quad 0 = (F + \text{div}_\Omega F)f = (F + V\lambda)f = Xf + V(\lambda f).$$

In what follows we will identify Ω_{SRB}^+ with f . Note that all the Fourier modes f_k of f are smooth thanks to the wavefront set condition and the fact that $V \notin \mathbb{R}F \oplus E_{u/s}$; see the paragraph after (5-10) where an argument is given. From (7-2) it follows that $(Xf)_0 = 0$ and so $\eta_- f_1 + \eta_+ f_{-1} = 0$. Let θ be the real-valued 1-form defined by the relation

$$(7-3) \quad \pi_1^* \theta = f_1 + f_{-1}.$$

Therefore $X_- \pi_1^* \theta = 0$ and by (2-12) we obtain

$$d \star \theta = 0,$$

ie $\star \theta$ defines a cohomology class in $H^1(M)$. Recall that $\omega^\pm = \iota_F \Omega_{\text{SRB}}^\pm$.

Lemma 7.2 We have $[\star\theta]_{H^1(M)} = 0$ if and only if $[\omega^+] = 0$, ie the winding cycle of Ω_{SRB}^+ vanishes.

Proof Observe that ω^+ is exact if and only if (by Poincaré duality) for every smooth closed real-valued 1-form β on SM we have

$$0 = \int_{SM} \beta \wedge \iota_F \Omega_{\text{SRB}}^+ = \int_{SM} f\beta(F) \Omega.$$

Since the pullback $\pi^*: H^1(M) \rightarrow H^1(SM)$ is an isomorphism (this is well-known, see eg [Merry and Paternain 2011, Corollary 8.10]), the previous condition is equivalent to

$$0 = \int_{SM} f\beta(X) \Omega \quad \text{for all } \beta \in C^\infty(M; \Omega^1) \cap \ker d.$$

Since $\beta(X) = \pi_1^* \beta$, this is equivalent to

$$0 = \int_{SM} (f_1\beta_{-1} + f_{-1}\beta_1) \Omega = \langle \pi_1^* \theta, \pi_1^* \beta \rangle_{L^2(\mathcal{M})} = \pi \langle \theta, \beta \rangle_{L^2(M)} = \pi \int_M \beta \wedge \star \theta.$$

By Poincaré duality, we conclude that the last equality is equivalent to $[\star\theta]_{H^1(M)} = 0$. $\square$

A case of particular interest arises when λ is an odd function in the velocity variable as this results in a *reversible* vector field F . Let $\mathcal{J}(x, v) = (x, -v)$ denote the flip map. It satisfies (see eg [Cekić and Paternain 2020, Proposition 5.1]) $\mathcal{J}^* X = -X$, $\mathcal{J}^* V = V$ and $\mathcal{J}^* \lambda = -\lambda$, and therefore $\mathcal{J}^* F = -F$. It follows that $\mathcal{J}^*$ sends $E_{u/s}$ to $E_{s/u}$ and $E_{u/s}^*$ to $E_{s/u}^*$. Moreover, the following holds.

Lemma 7.3 We have $\Omega_{\text{SRB}}^+ = \mathcal{J}^* \Omega_{\text{SRB}}^-$ and the winding cycles satisfy

$$[\omega^+] = -[\omega^-].$$

Moreover, $\mathcal{J}^* \text{Res}_{0*}^1 = \text{Res}_{0*}^1$.

Proof Since $\mathcal{J}^* E_u^* = E_s^*$ and $\mathcal{L}_F \mathcal{J}^* = -\mathcal{J}^* \mathcal{L}_F$, we get $\mathcal{J}^* \text{Res}_{0*}^k = \text{Res}_{0*}^k$ for any $k = 0, 1, 2, 3$. Since $\mathcal{J}$ is orientation-preserving, the claim about SRB measures follows from Proposition 3.1. For the other claim, it follows from $\mathcal{J}^* \omega^- = -\omega^+$ and the fact that $\mathcal{J}$ is homotopic to the identity (via rotations). $\square$

In particular, this lemma applies to *Gaussian* thermostats, that is, when $\lambda = \pi_1^* \rho$, where ρ is a smooth 1-form on M . Recall that if the curvature of g is negative and ρ is closed, the vector field F is Anosov; see [Mettler and Paternain 2019, Remark 4.4].

7.1 Coupled vortex equations

Let (M, g) be a compact oriented surface. Recall that an interesting class of thermostats is obtained when λ is generated by a holomorphic differential A of degree $m \geq 2$, that is, a section of $\mathcal{K}^{\otimes m}$, where $\mathcal{K} = (T_{\mathbb{C}}^* M)^{1,0}$ is the holomorphic part of $T_{\mathbb{C}}^* M$, and it satisfies the *coupled vortex equations* (see Appendix A for the notation)

$$(7-4) \quad \bar{\partial} A = 0, \quad K_g = -1 + (m-1)|A|_g^2.$$

It is known that when (7-4) holds, the flow of F for

$$(7-5) \quad \lambda := \operatorname{Im}(\pi_m^* A)$$

is Anosov; see [Mettler and Paternain 2019, Theorem 5.1]. By (A-5) and (A-4) it holds that

$$(7-6) \quad 4|\lambda_m|^2 = |A|_g^2 \quad \text{and} \quad \lambda_m = \frac{\pi_m^* A}{2i}.$$

Moreover, when $m \geq 2$, $\bar{\partial}A = 0$ is equivalent, by Lemma A.2, to

$$(7-7) \quad \eta_- \lambda_m = \eta_+ \lambda_{-m} = 0.$$

We finally note that by [Mettler and Paternain 2019, Lemma 5.2], we have

$$(7-8) \quad -1 \leq K_g < 0.$$

7.2 Proof of Theorem 1.6

In this case $\lambda = \pi_1^* \rho$, where ρ is harmonic and nonzero. Let us compute the entropy production:

$$0 \leq e^+ = - \int_{SM} \operatorname{div}_\Omega(F) \Omega_{\text{SRB}}^+ = - \int_{SM} V \lambda \cdot f \Omega = \int_{SM} \pi_1^*(\star \rho) \pi_1^* \theta \Omega = \pi \int_M \star \rho \wedge \star \theta,$$

where θ was defined by (7-3) and we used that $V \pi_1^* = -\pi_1^* \star$ on 1-forms. Note that the first inequality follows from Section 2.6. If $[\star \theta]_{H^1(M)} = 0$ we have $e^+ = 0$ and by [Dairbekov and Paternain 2007, Theorem A] we obtain that $[\star \rho]_{H^1(M)} = 0$, a contradiction. By Lemma 7.2 we conclude that the winding cycle $[\omega^+] \neq 0$. The claim that $[\omega^+] = -[\omega^-]$ is a consequence of Lemma 7.3. The claim that $m_{1,0} = b_1(M) - 1$ follows from the classification in Theorem 1.1.

The remaining claims are consequences of the general theory of perturbations and (1-3) as we now outline, following [Cekić and Paternain 2020, Section 7]. In fact, for $0 \neq \rho \in \mathcal{H}^1(M)$ small enough, we will show that $\mathcal{L}_{F_\rho}$, where $F_\rho := X + \pi_1^* \rho V$, has a splitting nonzero resonance when acting on 1-forms. This implies that the action of $\mathcal{L}_{F_\rho}$ on $\Omega^1 \cap \ker \iota_{F_\rho}$ is semisimple, by [Cekić and Paternain 2020, Lemma 6.2] and the fact that $\dim \operatorname{Res}^1(F_\rho) = b_1(M)$ (as follows from the previous paragraph and Theorem 1.1).

For ρ sufficiently small, by Lemma 5.1 there exists an anisotropic Sobolev space $\mathcal{H}_{rG,t} := \mathcal{H}_{rG,t}(SM; \Omega^1)$, for $t \in \mathbb{R}$, such that by a statement analogous to Lemma 5.2, the map

$$\mathcal{T}: \mathcal{H}^1(M) \rightarrow \mathcal{H}_{rG,-3}, \quad \rho \mapsto \mathcal{L}_{F_\rho} \tilde{\Pi}_1^+(\rho) \alpha,$$

is C^1 -regular, where $\tilde{\Pi}_1^+(\rho)$ is the projector onto resonant 1-forms of F_ρ near zero, defined analogously to (5-2), and α is the contact form of X . We compute for $\rho \in \mathcal{H}^1(M)$ that

$$(7-9) \quad D_0 \mathcal{T}(\rho) = \partial_s|_{s=0} \tilde{\Pi}_1^+(s\rho) \mathcal{L}_{F_{s\rho}} \alpha = \partial_s|_{s=0} \tilde{\Pi}_1^+(s\rho) s \mathcal{L}_{\pi_1^* \rho V} \alpha = \Pi_1^+(\lambda \beta),$$

where in the first equality we used that $\mathcal{L}_{F_{s\rho}}$ and $\tilde{\Pi}_1^+(s\rho)$ commute, in the second one we used $\mathcal{L}_X \alpha = 0$, and for the last one we recall that $\lambda := \pi_1^* \rho$. We claim that $D_0 \mathcal{T}$ is injective. Indeed, let $u := \Pi_1^+(\lambda \beta)$

for $\rho \neq 0$. Observe that $\Pi_1^+(V\lambda \cdot \alpha) = 0$, for instance by noting that $\iota_X \Pi_1^+(V\lambda \cdot \alpha) = 0$, and using that the pairing $\langle\langle \bullet, \bullet \rangle\rangle$ is nondegenerate by [Lemma 2.1](#) and [\[Dyatlov and Zworski 2017, Proposition 3.1\(3\)\]](#). Thus, as $\pi^*(\star\rho) = -V\lambda \cdot \alpha + \lambda\beta$ (since $\pi_1^*(\star\rho) = -V\lambda$ and $\iota_H \pi^*(\star\rho) = \lambda$), we get $u = \Pi_1^+(\pi^*(\star\rho)) \neq 0$ (recall that $\Pi_1^+ : H^1(M) \rightarrow \text{Res}_0^1$ is an isomorphism). Compute the quantity

$$(7-10) \quad W := \int_{SM} u(\lambda V) \Omega = \int_{SM} \alpha \wedge u \wedge \lambda\beta = \langle\langle u, \lambda\beta \rangle\rangle = \langle\langle u, \Pi_1^-(\lambda\beta) \rangle\rangle = \langle\langle u, \mathcal{J}^*u \rangle\rangle,$$

where in the last line we used $\Pi_1^+u = u$ and (2-6), and $\mathcal{J}^*\Pi_1^+ = \Pi_1^-\mathcal{J}^*$. By [\[Cekić and Paternain 2020, Lemma 7.2\]](#), $W > 0$ since $u \neq 0$, proving the claim about injectivity of $D_0\mathcal{T}$. Therefore, by Taylor expansion $\mathcal{T}$ is injective close to the zero 1-form (using that the unit ball in $\mathcal{H}^1(M)$ is compact). Thus for $\rho \neq 0$ small, we have that $\mathcal{T}(\rho) \neq 0$, and also, using that $\dim \text{Res}^1(F_\rho) = b_1(M)$ (see previous paragraphs) and [\[Cekić and Paternain 2020, Lemma 6.2\]](#) (the rank of $\tilde{\Pi}_1^+$ is locally constant), either $\mathcal{T}(\rho)$ is a splitting resonant state corresponding to a nonzero resonance, in which case $\mathcal{L}_{F_\rho}\mathcal{T}(\rho) \neq 0$, or we lose semisimplicity at zero for the action of $\mathcal{L}_{F_\rho}$ on Ω^1 , in which case $\mathcal{L}_{F_\rho}\mathcal{T}(\rho) = 0$ (ie there is a Jordan block of size 2). In what follows, we show that there is a splitting resonance.

Next, we compute the derivatives of $\mathcal{L}_{F_\rho}\mathcal{T}(\rho)$, which is C^4 -regular as a map $\mathcal{H}^1(M) \rightarrow \mathcal{H}_{rG, -t}$ for ρ small enough and for some t large enough (similarly to the case above). In fact, we have

$$D_0[\mathcal{L}_{F_\rho}\mathcal{T}](\rho) = \partial_s|_{s=0} \mathcal{L}_{F_{s\rho}}\mathcal{T}(s\rho) = \mathcal{L}_{\pi_1^*\rho V} \underbrace{\mathcal{T}(0)}_{=0} + \mathcal{L}_X \Pi_1^+(\pi_1^*\rho \cdot \beta) = 0,$$

where in second equality we used (7-9) and $\mathcal{T}(0) = 0$, and in the third one that $\mathcal{L}_X \Pi_1^+ = 0$. The second and third derivatives are computed as in [\[Cekić and Paternain 2020, Section 7.2\]](#). Indeed, we have

$$D_0^2 \mathcal{L}_{F_\rho} \mathcal{T}(\rho) = \partial_s^2|_{s=0} \mathcal{L}_{F_{s\rho}} \mathcal{T}(s\rho) = 2\partial_s|_{s=0} \mathcal{L}_{F_{s\rho}} \underbrace{\tilde{\Pi}_1^+(s\rho)(\pi_1^*\rho \cdot \beta)}_{=:u_s} = 2\dot{z}u = 0$$

at $s = 0$, where u_s satisfies $\mathcal{L}_{F_{s\rho}}u_s = z_s u_s$ for some $z_s \in \mathbb{R}$ and similarly to [\[Cekić and Paternain 2020, equation \(7.9\)\]](#), the map $s \mapsto z_s$ is C^2 -regular. Therefore the last equality follows from the first linearisation in [\[Cekić and Paternain 2020, Section 7.2\]](#), which shows $\dot{z} = 0$. For the third derivative, we similarly have

$$D_0^3 \mathcal{L}_{F_\rho} \mathcal{T}(\rho) = \partial_s^3|_{s=0} \mathcal{L}_{F_{s\rho}} \tilde{\Pi}_1^+(s\rho)(\pi_1^*\rho \cdot \beta) = 3\partial_s^2|_{s=0} (z_s u_s) = 3\ddot{z}u,$$

where in the last line we used that $z = \dot{z} = 0$ at $s = 0$. By the same argument as in the second linearisation of [\[Cekić and Paternain 2020, Section 7.2\]](#), and using that $W > 0$ by (7-10), we conclude that $\ddot{z} < 0$, and since $u \neq 0$ by the discussion above this proves that the third derivative is nonzero when $\rho \neq 0$.

By means of a Taylor expansion, it follows that the map $\mathcal{H}^1(M) \rightarrow \mathcal{H}_{rG, -t}$, $\rho \mapsto \mathcal{L}_{F_\rho}\mathcal{T}(\rho)$, satisfies the property that $\mathcal{L}_{F_\rho}\mathcal{T}(\rho) \neq 0$ for $\rho \neq 0$ small enough, and so a resonance splits (in fact, as the proof shows, necessarily to the right half-plane), proving the semisimplicity claim and completing the proof. $\square$

8 Quadratic holomorphic differentials

In this section we consider the coupled vortex equations (7-4) for $m = 2$ and we prove [Theorem 1.5](#). These thermostats are *quasi-Fuchsian* flows as defined by Ghys [1992]. Paternain [2007] conjectured that all quasi-Fuchsian flows arise this way. In this case, the weak stable/unstable bundles are *smooth* and moreover, by [Mettler and Paternain 2019, Section 6.1],

$$(8-1) \quad r^u = 1 + \frac{1}{2}V\lambda \quad \text{and} \quad r^s = -1 + \frac{1}{2}V\lambda.$$

(The case $m = 3$ corresponds to Hilbert geodesic flows and the cases $m \geq 4$ are largely unstudied; see [Mettler and Paternain 2019]. The case $m = 2$ is the only one for which it is possible to write down explicitly the solutions $r^{u/s}$ to the Riccati equation.)

Recall that $\lambda = \lambda_{-2} + \lambda_2$ is a holomorphic differential of degree two, that is, $\eta_- \lambda_2 = \eta_+ \lambda_{-2} = 0$; $\overline{\lambda_2} = \lambda_{-2}$ since λ is real-valued. Also, recall that $Y^{u/s} = H + r^{u/s}V$ (see [Section 7](#)), and that we write $\Omega_{\text{SRB}}^+ = f\Omega$ for some $f \in \mathcal{D}'_{E_u^*}(\mathcal{M})$ and $\Omega = \alpha \wedge \psi \wedge \beta$.

Let us first restate the horocyclic invariance result of [Lemma 4.11](#) in this setting.

Lemma 8.1 (horocyclic invariance for holomorphic differentials) *Let $u \in \text{Res}_0^1$. Then there is a constant $c \in \mathbb{C}$ such that $du = \frac{1}{2}cf\iota_F\Omega$ and $h := \iota_{Y^s}u$ satisfies*

$$(8-2) \quad (F + r^s)h = 0, \quad (Y^u - \lambda)h = cf.$$

Denote by $\mathcal{S}$ the set of distributional solutions of (8-2) for $c \in \mathbb{C}$. Then the map $\text{Res}_0^1 \rightarrow \mathcal{S}$ given by $u \mapsto \iota_{Y^s}u$ is an isomorphism.

Proof By [Lemma 4.11](#) we have that $\iota_{Y^u}u = 0$. Next, similarly to [Lemma 4.4](#), the first equation is derived from

$$0 = du(F, Y^s) = Fh - \iota_{[F, Y^s]}u = (F + r^s)h,$$

where in the second equality we used $\iota_F u = 0$, and in the third one we used [Lemma 7.1](#). For the second equation, we have

$$(8-3) \quad \frac{c}{2}f\Omega(F, Y^u, Y^s) = du(Y^u, Y^s) = Y^u h - \iota_{[Y^u, Y^s]}u,$$

where in the last equality we used $\iota_{Y^u}u = 0$. Then we have the following computation:

$$\begin{aligned} (8-4) \quad [Y^u, Y^s] &= [H + r^u V, H + r^s V] \\ &= Hr^s \cdot V + r^s[H, V] - Hr^u \cdot V - r^u[H, V] + r^u V r^s \cdot V - r^s V r^u \cdot V \\ &= H(r^s - r^u) \cdot V + (r^s - r^u) \cdot X + (r^u V r^s - r^s V r^u) V \\ &= -2X + \left(\left(1 + \frac{V\lambda}{2} \right) \frac{V^2\lambda}{2} - \left(-1 + \frac{V\lambda}{2} \right) \frac{V^2\lambda}{2} \right) V \\ &= -2X - 4\lambda V = -2F - \lambda(Y^u - Y^s), \end{aligned}$$

where we used (2-10) in the second line, that $r^s - r^u = -2$ and (8-1) in the third line, and $V^2\lambda = -4\lambda$ and $Y^u - Y^s = 2V$ in the final line. Also, we compute

$$\Omega(F, Y^u, Y^s) = \alpha \wedge \psi \wedge \beta(X + \lambda V, H + r^u V, H + r^s V) = -r^s + r^u = 2,$$

which combined with (8-4) and (8-3) shows (8-2) and concludes the proof of the first claim.

For the final claim, the proof is straightforward and analogous to the proof of the final part of Lemma 4.11 and we omit it. $\square$

8.1 Recurrence relations

We will use the ladder operators $\eta_{\pm}$ to derive from Lemma 8.1 the recurrence relations for the solutions h of the system in (8-2).

Lemma 8.2 *The PDE system (8-2) is satisfied if and only if, for every $k \in \mathbb{Z}$,*

$$(8-5) \quad \begin{aligned} 2\eta_- h_{k+1} - (k+1)h_k + 2\lambda_{-2}h_{k+2}i(k+1) &= icf_k, \\ 2\eta_+ h_{k-1} + (k-1)h_k + 2\lambda_2h_{k-2}i(k-1) &= -icf_k. \end{aligned}$$

Proof To derive these equations, we first rewrite the system (8-2) more explicitly:

$$Xh + \lambda Vh + \left(-1 + \frac{1}{2}V\lambda\right)h = 0, \quad Hh + \left(1 + \frac{1}{2}V\lambda\right)Vh - \lambda h = cf.$$

Multiplying the second equation by i , and adding and subtracting from the first one, we get

$$\begin{aligned} 2\eta_- h + \left(\lambda + i + \frac{1}{2}iV\lambda\right)Vh + \left(-1 + \frac{1}{2}V\lambda - i\lambda\right)h &= icf, \\ 2\eta_+ h + \left(\lambda - i - \frac{1}{2}iV\lambda\right)Vh + \left(-1 + \frac{1}{2}V\lambda + i\lambda\right)h &= -icf. \end{aligned}$$

Now writing $V\lambda/2 = i\lambda_2 - i\lambda_{-2}$ and $\lambda = \lambda_{-2} + \lambda_2$, we rewrite these equations as

$$\begin{aligned} 2\eta_- h + (i + 2\lambda_{-2})Vh + (-1 - 2i\lambda_{-2})h &= icf, \\ 2\eta_+ h + (-i + 2\lambda_2)Vh + (-1 + 2i\lambda_2)h &= -icf. \end{aligned}$$

Now rewriting these equalities with terms of fixed degree grouped, we get

$$\begin{aligned} 2\eta_- h + iVh - h + 2\lambda_{-2}(Vh - ih) &= icf, \\ 2\eta_+ h - iVh - h + 2\lambda_2(Vh + ih) &= -icf. \end{aligned}$$

The equations in (8-5) readily follow by identifying the degree- k components. $\square$

We start with the cases $k = \pm 1$ in (8-5).

Lemma 8.3 *Restricting to $k = \pm 1$ in (8-5), we get*

$$(8-6) \quad 2\eta_- h_0 = icf_{-1}, \quad 2\eta_+ h_0 = -icf_1.$$

This system of equations has a solution h_0 if and only if $c = 0$ or the winding cycle $[\iota_F \Omega_{\text{SRB}}^+]_{H^2(\mathcal{M})}$ vanishes. These solutions, if they exist, are unique up to adding a constant.

Proof Note that by conjugating the first equation in (8-6), and adding and subtracting from the second one, we get equations for $\operatorname{Re}(h_0)$ and $\operatorname{Im}(h_0)$, and c is replaced with $\operatorname{Re}(c)$ and $\operatorname{Im}(c)$, respectively (note that f is real-valued, so $\bar{f}_1 = f_{-1}$). This argument allows us to assume that h_0 and c are real-valued to begin with.

Since η_- is elliptic acting on $H_0 \cong C^\infty(M)$ and mapping to $H_{-1} \cong C^\infty(M, \mathcal{K}^{-1})$ (by Lemma A.2), by Fredholm theory and since $(\eta_+)^* = -\eta_-$, if $c \neq 0$ the first equation has a solution if and only if

$$(8-7) \quad \int_{\mathcal{M}} f_{-1} g_1 \Omega = 0 \quad \text{for all } g_1 \in \ker \eta_-|_{H_1}.$$

Introducing $g_{-1} := \bar{g}_1$ for some $g_1 \in \ker \eta_-|_{H_1}$, we may write $\pi_1^* \gamma = g_1 + g_{-1}$ for some real-valued 1-form γ on the base. Therefore, using (2-12) γ is coclosed, that is, $d \star \gamma = 0$, and similarly $d\gamma = 0$ by using $V\pi_1^* = -\pi_1^* \star$. (In fact, the condition $\eta_- g_1 = 0$ is equivalent to the fact that γ is a real-valued harmonic 1-form on the base.)

Recall that $\pi_1^* \theta = f_1 + f_{-1}$ (where θ was defined in (7-3)), so the condition (8-7) is equivalent to

$$\int_{\mathcal{M}} (f_1 + f_{-1})(g_1 + g_{-1}) \Omega = \int_{\mathcal{M}} (f_1 + f_{-1})(g_1 - g_{-1}) \Omega = 0 \quad \text{for all } g_1 \in \ker \eta_-|_{H_1},$$

which in turn is simply equivalent, using $g_1 - g_{-1} = -iV(g_1 + g_{-1}) = i\pi_1^*(\star \gamma)$, to

$$\int_{\mathcal{M}} \pi_1^* \theta \cdot \pi_1^* \gamma \Omega = \int_{\mathcal{M}} \pi_1^* \theta \cdot \pi_1^*(\star \gamma) \Omega = 0 \quad \text{for all } \gamma \in \mathcal{H}^1(M),$$

where $\mathcal{H}^1(M)$ denotes the set of harmonic 1-forms. Since $\star \gamma \in \mathcal{H}^1(M)$ if and only if $\gamma \in \mathcal{H}^1(M)$, the second condition is redundant, and it is equivalent to having

$$\int_M \star \theta \wedge \gamma = 0 \quad \text{for all } \gamma \in \mathcal{H}^1(M).$$

By Hodge decomposition, this is equivalent to $[\star \theta]_{H^1(M)} = 0$, so we conclude by Lemma 7.2.

If $c = 0$, set $h_0 = \pi_0^* h_{00}$ for some $h_{00} \in C^\infty(M)$. By (2-13), $\eta_+ h_0 = \eta_- h_0 = 0$ implies that $dh_{00} = 0$, and so h_{00} and h_0 are constant functions. Similarly, this argument proves uniqueness of solution to (8-6) up to constants, completing the proof. $\square$

We now try to figure out how h_1 depends on h_0 . To do this we have:

Lemma 8.4 *The system (8-5) is equivalent to the set of equations*

$$(8-8) \quad \begin{aligned} 2(\eta_- - 2i\lambda_{-2}\eta_+)h_{k+1} + (k+1)\underbrace{(-1 + 4\lambda_2\lambda_{-2})}_{=K_g}h_k &= c(if_k - 2\lambda_{-2}f_{k+2}), \\ 2(2i\lambda_2\eta_- + \eta_+)h_{k+1} + (k+1)\underbrace{(-4\lambda_2\lambda_{-2} + 1)}_{=-K_g}h_{k+2} &= c(-2\lambda_2f_k - if_{k+2}), \end{aligned}$$

valid for every $k \in \mathbb{Z}$.

Proof We use the system (8-5), plugging k in the first equation and $k + 2$ in the second one, to obtain

$$2\eta_- h_{k+1} - (k+1)h_k + 2\lambda_- h_{k+2} i(k+1) = i c f_k,$$

$$2\eta_+ h_{k+1} + (k+1)h_{k+2} + 2\lambda_2 h_k i(k+1) = -i c f_{k+2}.$$

The first equation of (8-8) follows by multiplying the second equation by $2i\lambda_-$ and subtracting from the first one; the second one follows from multiplying the first one by $2i\lambda_2$ and adding to the second one.

That the two systems are equivalent can be seen as follows. Denote the first and the second equation of (8-5) by $B = 0$ and $C = 0$, respectively. Then (8-8) takes the form

$$B - 2i\lambda_- C = 0, \quad 2i\lambda_2 B + C = 0.$$

From here it is easy to see that $B = C = 0$ is equivalent to this system of equations since

$$1 - 4\lambda_2 \lambda_- = 1 - |A|^2 = -K_g > 0,$$

where we use (7-6) in the first equality, (7-4) in the second, and (7-8) for the final inequality. $\square$

Note that the leading operators arising in (8-8) are conjugate to one another, that is,

$$(8-9) \quad \mu_- := \eta_- - 2i\lambda_- \eta_+, \quad \mu_+ := \eta_+ + 2i\lambda_2 \eta_-, \quad \mu_+^* = -\mu_-.$$

Here we also use (7-7), so that $\lambda_2 \eta_- = \eta_- \lambda_2$, $\lambda_- \eta_+ = \eta_+ \lambda_-$, and $\eta_+^* = -\eta_-$.

Lemma 8.5 *There are no solutions of (8-8) with $c \neq 0$. In particular, we have $d(\text{Res}_0^1) = 0$, ie all resonant 1-forms in the kernel of ι_F are closed.*

Proof Using the notation of Lemma 8.1, it suffices to show that for $u \in \text{Res}_0^1$ with $du = \frac{1}{2} c \iota_F \Omega$, we have $c = 0$. By Lemma 8.4, it is equivalent to showing that the system (8-8) implies that $c = 0$. Applying the first equation in (8-8) for $k = 0$ and the second one for $k = -2$, we get

$$2\mu_- h_1 + K_g h_0 = c(i f_0 - 2\lambda_- f_2),$$

$$2\mu_+ h_{-1} + K_g h_0 = c(-2\lambda_2 f_{-2} - i f_0).$$

Subtracting the two equations, we obtain

$$2(\mu_- h_1 - \mu_+ h_{-1}) = 2ic(f_0 - i(\lambda_2 f_{-2} - \lambda_- f_2)) = 2ic(f_0 - \frac{1}{2}(V\lambda \cdot f)_0).$$

Integrating over SM and using that $\mu_\pm^* = -\mu_\mp$, we get that the left-hand side is zero, so

$$(8-10) \quad 0 = 2ic \int_{\mathcal{M}} (f_0 - \frac{1}{2}(V\lambda \cdot f)_0) \Omega.$$

We have $\int_{\mathcal{M}} f_0 \Omega = \int_{\mathcal{M}} \Omega_{\text{SRB}}^+ = 1$. On the other hand, we have

$$-\int_{\mathcal{M}} (V\lambda \cdot f)_0 \Omega = -\int_{\mathcal{M}} V\lambda \cdot f \Omega = -\int_{\mathcal{M}} \text{div}_\Omega(F) \Omega_{\text{SRB}}^+ = e^+(F),$$

where $\text{div}_\Omega F = V\lambda$ is the divergence of F and we recall the entropy production $e^+(F)$ was introduced in Section 2.6, and that it satisfies $e^+(F) \geq 0$. We conclude that the integral on the right-hand side of (8-10) is strictly positive, and so $c = 0$, completing the proof. $\square$

Next, we show that the zero-order Fourier mode of h always vanishes.

Lemma 8.6 Assume h is real-valued and satisfies (8-8). Then $h_0 = 0$.

Proof By Lemma 8.5, we know $c = 0$ in (8-8) and (8-6). In fact, from Lemma 8.3 it follows that h_0 is constant. Integrating the first equation of (8-8) for $k = 0$, and using $\mu_+^* = -\mu_-$, we get

$$0 = -2 \int_{\mathcal{M}} \mu_- h_1 \Omega = h_0 \cdot \int_{\mathcal{M}} K_g \Omega.$$

By (7-8) we have $K_g < 0$, which implies $h_0 = 0$. □

Now we will construct h_k by hand solving (8-8) iteratively. For this we need to compute $\ker \mu_-$ on H_1 , because by Lemma 8.6, $\mu_- h_1 = 0$ is the initial equation. This and other needed properties of the operators $\mu_{\pm}$ are proved in Section 8.2 below.

Lemma 8.7 The map $\mathcal{J}: \mathcal{S} \rightarrow \ker \mu_+|_{H_{-1}} \oplus \ker \mu_-|_{H_1}$ given by $h \mapsto (h_{-1}, h_1)$ is an isomorphism on the set $\mathcal{S}$ of solutions of the system (8-8). Moreover, $\dim \text{Res}_0^1 = b_1(M)$.

Proof Recall first that by Lemma 8.5, for any solution $0 \neq h \in \mathcal{S}$ we have $c = 0$; also $h_0 = 0$ by Lemma 8.6. The map $\mathcal{J}$ is injective: if $h_1 = h_{-1} = 0$, then by applying first and second equations in (8-8), we get $h_k = 0$ for $k < 0$, and $h_k = 0$ for $k > 0$, respectively (here we use that $K_g < 0$ by (7-8)). Therefore $h \equiv 0$.

To show surjectivity, let $h_1 \in \ker \mu_-|_{H_1}$. Then set $h_k = 0$ for $k \leq 0$ and observe that equations in (8-8) take the form (plugging $k - 2$ in the second one and k in the first one), using¹ $K_g = -1 + 4|\lambda_2|^2$:

$$(8-11) \quad 2\mu_+ h_{k-1} = (k-1)K_g h_k,$$

$$(8-12) \quad 2\mu_- h_{k+1} = -(k+1)K_g h_k.$$

Next, set $h_2, h_3, \dots$ to be defined inductively using (8-11) for $k = 2, 3, \dots$ (note that they are all smooth, since h_1 is). Therefore (8-11) is now satisfied for all k , by definition of h_k for $k \leq 0$. Also, (8-12) is satisfied for $k \leq -1$ trivially, and for $k = 0$ by definition of h_1 . Therefore, we are left to check that (8-12) holds for $k \geq 1$.

We prove this by induction on k . If (8-12) holds for $k \leq \ell$ (starting with $\ell = 0$), then we have

$$\begin{aligned} 2\mu_- h_{\ell+2} &= 2\mu_- \left(\frac{2\mu_+ h_{\ell+1}}{K_g(\ell+1)} \right) = \frac{4}{\ell+1} \mu_- \left(\frac{1}{K_g} \mu_+ h_{\ell+1} \right) \\ &= \frac{4}{\ell+1} \left(\frac{i}{2} K_g V h_{\ell+1} + \mu_+ \left(\frac{1}{K_g} \mu_- h_{\ell+1} \right) \right) \\ &= -2K_g h_{\ell+1} - 2\mu_+ h_{\ell} \\ &= -2K_g h_{\ell+1} - \ell K_g h_{\ell+1} = -(\ell+2)K_g h_{\ell+1}. \end{aligned}$$

¹This agrees with [Guillarmou et al. 2018, equations (3.12) and (3.13)] in the constant curvature case, ie $A \equiv 0$.

Here we used (8-11) for $k = \ell + 2$ in the first line, Lemma 8.10 in the second line, the fact that $Vh_{\ell+1} = i(\ell + 1)h_{\ell+1}$ and (8-12) for $k = \ell$ in the third line, and finally (8-11) for $k = \ell + 1$ in the last line. This proves that (8-12) holds for $k = \ell + 1$, which completes the proof of the induction.

It is left to show that $h := \sum_{k=1}^{\infty} h_k$ converges in the distributional sense, and for that it suffices to prove $\|h_k\|_{L^2(SM)} = \mathcal{O}(|k|^N)$ as $k \rightarrow \infty$ for some $N > 0$, since the Fourier modes of an arbitrary $\varphi \in C^\infty(SM)$ decay faster than any polynomial. In what follows, norms and inner products will be in $L^2(SM)$. Notice that for $k \geq 2$,

$$\begin{aligned} \|h_k\|^2 &= \frac{2}{k-1} \langle h_k, \frac{1}{K_g} \mu_+ h_{k-1} \rangle = -\frac{2}{k-1} \left\langle \mu_- \left(\frac{1}{K_g} h_k \right), h_{k-1} \right\rangle \\ &= -\frac{2}{k-1} \left(\left\langle \frac{1}{K_g} \mu_- h_k, h_{k-1} \right\rangle + \left\langle \mu_- \left(\frac{1}{K_g} \right) h_k, h_{k-1} \right\rangle \right) \\ &= \frac{1}{k-1} \left(k \|h_{k-1}\|^2 - 2 \left\langle \mu_- \left(\frac{1}{K_g} \right) h_k, h_{k-1} \right\rangle \right). \end{aligned}$$

Here we used (8-11) and $\mu_+^* = -\mu_-$ in the first line, and (8-12) (plugging in $k-1$) in the third line. Therefore, by the Cauchy–Schwarz and AM–GM inequalities,

$$\|h_k\|^2 \leq \frac{k}{k-1} \|h_{k-1}\|^2 + \frac{1}{k-1} \left(\|h_k\|^2 + \left\| \mu_- \left(\frac{1}{K_g} \right) \right\|_{L^\infty}^2 \|h_{k-1}\|^2 \right),$$

which gives, after setting $Z := \lceil \|\mu_-(1/K_g)\|_{L^\infty}^2 \rceil \in \mathbb{N}$,

$$\|h_k\|^2 \leq \frac{k+Z}{k-2} \|h_{k-1}\|^2.$$

Iterating the last inequality, we obtain

$$\|h_k\|^2 \leq \frac{k+Z}{k-2} \cdot \frac{k-1+Z}{k-3} \cdots \frac{3+Z}{2} \cdot \|h_2\|^2.$$

It follows that $\|h_k\|^2 = \mathcal{O}(|k|^{Z+2})$, which proves the claim and shows $\mathcal{J}(h) = (0, h_1)$.

If $h_{-1} \in \ker \mu_+|_{H_{-1}}$, then $\overline{h_{-1}} \in \ker \mu_-|_{H_1}$, so the construction above gives an $\bar{h}$ with $\mathcal{J}\bar{h} = (0, \overline{h_{-1}})$, implying $\mathcal{J}h = (h_{-1}, 0)$. Thus $\mathcal{J}$ is surjective, completing the proof.

The final claim now follows from Lemma 8.11 below. □

Finally, we may show that the winding cycle of Ω_{SRB}^+ is trivial.

Lemma 8.8 *The winding cycle of Ω_{SRB}^+ vanishes, ie $[\iota_F \Omega_{\text{SRB}}^+]_{H^2(\mathcal{M})} = 0$. Also, the helicity $\mathcal{H}(F)$ is nonzero.*

Proof By Lemmas 8.5 and 8.7, we know $d(\text{Res}_0^1) = 0$ and $\dim \text{Res}_0^1 = b_1(M)$, respectively; in fact, by analogous arguments for coresonant states, ie arguing for the flow $-F$, the same lemmas

imply $d(\text{Res}_{0*}^1) = 0$ and $\dim \text{Res}_{0*}^1 = b_1(M)$. Therefore, the map $T_*: \text{Res}_{0*}^1 \rightarrow \mathbb{C}$ constructed (now for coresonances) in [Lemma 3.4](#) is trivial and the map $S_*: \text{Res}_{0*}^1 \rightarrow H^1(\mathcal{M})$ constructed (again, for coresonances) in [Lemma 3.5](#) is an isomorphism. By the last part of [Lemma 3.5](#) we conclude that the winding cycle of Ω_{SRB}^+ vanishes.

The final conclusion now follows from the classification in [Theorem 1.1](#). $\square$

So far, we have established the claims in [Theorem 1.5](#) that assert that $m_{1,0} = b_1(M)$ and $[\omega^\pm] = 0$, modulo properties of the operators $\mu_\pm$ to be proved next.

8.2 Properties of $\mu_\pm$

Recall that $\mu_\pm$ were defined in (8-9). In this section we compute the principal symbols of $\mu_\pm$, the size of their kernels, and also obtain the analogue of the formula (2-11) for the operators $\mu_\pm$. According to [Section 2.4](#), we will freely identify $\mu_\pm$ with operators acting on sections of tensor powers $\mathcal{K}^{\otimes m}$ for $m \in \mathbb{Z}$.

Lemma 8.9 *For any $m \in \mathbb{Z}$, the operators $\mu_\pm$ are elliptic.*

Proof As $\mu_+^* = -\mu_-$, it suffices to consider μ_- only; it also suffices to consider the case $m \geq 0$. In fact, for each $(x, \xi) \in T^*M$ the principal symbol $\sigma(\mu_-)(x, \xi): \mathcal{K}^{\otimes m}(x) \rightarrow \mathcal{K}^{\otimes(m-1)}(x)$ is a linear map and hence may be identified with an element of $\bar{\mathcal{K}}(x)$ (the dual bundle). Consider local isothermal coordinates on $U \subset M$, that is, such that $g|_U = e^{2\psi}(dx^2 + dy^2)$ for some locally defined ψ . Fix an arbitrary $(z_0, \xi) \in T^*U$ and take $\chi, S \in C^\infty(M)$ such that $S(z_0) = 0$ and $dS(z_0) = \xi$, and $\text{supp}(\chi) \subset U$ with $\chi = 1$ near z_0 . Using [Lemma A.2](#) (here we use the fact that the principal symbol of a pseudodifferential operator is recovered by oscillatory testing, see eg [\[Zworski 2012, Theorem 4.19\]](#) in the related Euclidean case and when S is linear, and see also [\[Hörmander 1965, Definition 2.1\]](#))

$$\begin{aligned} \sigma(\mu_-)(z_0, \xi)(dz^m) &= \lim_{h \rightarrow 0} h \mu_-(\chi e^{iS/h} dz^m)(z_0) \\ &= \lim_{h \rightarrow 0} h \left(\frac{\partial(e^{iS/h})}{\partial \bar{z}} e^{-2\psi} \cdot dz^{m-1} + \bar{A}_0 d\bar{z}^2 \otimes e^{2m\psi} \frac{\partial(e^{-2m\psi} e^{iS/h})}{\partial z} dz^{m+1} \right)(z_0) \\ &= i \left(\frac{\partial S}{\partial \bar{z}} + \bar{A}_0 e^{-2\psi} \frac{\partial S}{\partial z} \right)(z_0) \cdot e^{-2\psi(z_0)} dz^{m-1}, \end{aligned}$$

where in the second line we wrote $A = A_0 dz^2$, so that by (7-5), we have

$$\lambda_2 = \frac{\pi_2^* A}{2i} \quad \text{and} \quad \lambda_{-2} = -\frac{\pi_2^*(\bar{A}_0 d\bar{z}^2)}{2i},$$

and in the final line, we also used

$$\pi_2^*(d\bar{z} \otimes dz)(z_0, v) = d\bar{z}(v) \cdot dz(v) = v_x^2 + v_y^2 = e^{-2\psi}.$$

Using the identification of the symbol with an element of $\bar{\mathcal{K}}(z_0)$, we conclude that

$$\sigma(\mu_-)(z_0, \xi) = d\bar{z} \times i\xi \left(\frac{\partial}{\partial \bar{z}} + \bar{A}_0(z_0)e^{-2\psi(z_0)} \frac{\partial}{\partial z} \right).$$

Recall that $|A| = |A_0||dz|^2 = |A_0|e^{-2\psi}$ and that by (7-4) and (7-8) we have $|A|^2 = 1 + K_g < 1$. Therefore $|A_0|e^{-2\psi} < 1$ and writing $|A_0(z_0)| = r$ and $A_0(z_0) = re^{i\Upsilon}$ for some $\Upsilon \in \mathbb{R}$, the relation $\sigma(\mu_-)(z_0, \xi) = 0$ is equivalent to

$$\begin{aligned} 0 &= \xi_x + re^{-2\psi(z_0)}(\xi_x \cos \Upsilon - \xi_y \sin \Upsilon), \\ 0 &= \xi_y + re^{-2\psi(z_0)}(-\xi_x \sin \Upsilon - \xi_y \cos \Upsilon). \end{aligned}$$

If $\cdot$ denotes Euclidean inner product in $\mathbb{R}^2$, this implies

$$\begin{aligned} \xi_x^2 + \xi_y^2 &= (re^{-2\psi(z_0)})^2 (|(\xi_x, \xi_y) \cdot (\cos \Upsilon, -\sin \Upsilon)|^2 + |(\xi_x, \xi_y) \cdot (\sin \Upsilon, \cos \Upsilon)|^2) \\ &= (re^{-2\psi(z_0)})^2 (\xi_x^2 + \xi_y^2). \end{aligned}$$

Since $re^{-2\psi(z_0)} < 1$, this is equivalent to $\xi = 0$, showing that μ_- is elliptic. $\square$

Now we compute the commutator $[\mu_+, \mu_-]$:

Lemma 8.10 We have

$$(8-13) \quad [\mu_+, \mu_-] = -\frac{iK_g^2 V}{2} + \frac{\mu_+ K_g}{K_g} \cdot \mu_- - \frac{\mu_- K_g}{K_g} \cdot \mu_+.$$

In fact, this is equivalent to the formula

$$\left[\frac{\mu_-}{K_g}, \frac{\mu_+}{K_g} \right] = \frac{i}{2} V,$$

or once again, equivalently,

$$\mu_- \left(\frac{1}{K_g} \mu_+ \right) = \frac{i}{2} K_g V + \mu_+ \left(\frac{1}{K_g} \mu_- \right).$$

Proof Using the definition (8-9), $[\mu_+, \mu_-]$ is equal to

$$\begin{aligned} [\eta_+ + 2i\lambda_2\eta_-, \eta_- - 2i\lambda_{-2}\eta_+] &= [\eta_+, \eta_-] - 2i\eta_+\lambda_{-2} \cdot \eta_+ - 2i\eta_-\lambda_2 \cdot \eta_- \\ &\quad + 4(\lambda_2\eta_-(\lambda_{-2}) \cdot \eta_+ + |\lambda_2|^2\eta_-\eta_+ - \lambda_{-2}\eta_+(\lambda_2) \cdot \eta_- - |\lambda_2|^2\eta_+\eta_-) \\ &= (1 - 4|\lambda_2|^2)[\eta_+, \eta_-] + 4(\eta_- (|\lambda_2|^2) \cdot \eta_+ - \eta_+ (|\lambda_2|^2) \cdot \eta_-) \\ &= -K_g \cdot \frac{i}{2} K_g V + \eta_- K_g \cdot \eta_+ - \eta_+ K_g \cdot \eta_-, \end{aligned}$$

where in the second equality we used $\eta_-\lambda_2 = \eta_+\lambda_{-2} = 0$, and in the last equality the commutator (2-11), as well as (7-4). Coming back to the definition (8-9) of $\mu_{\pm}$, we express $\eta_{\pm}$ in terms of $\mu_{\pm}$:

$$(8-14) \quad \eta_- = -\frac{1}{K_g}(2i\lambda_{-2}\mu_+ + \mu_-), \quad \eta_+ = \frac{1}{K_g}(2i\lambda_2\mu_- - \mu_+).$$

Substituting (8-14) into the previous equality, $[\mu_+, \mu_-] + \frac{1}{2}iK_g^2V$ is equal to

$$\begin{aligned} & -\frac{1}{K_g^2}(2i\lambda_{-2}\mu_+ + \mu_-)K_g \cdot (2i\lambda_2\mu_- - \mu_+) + \frac{1}{K_g^2}(2i\lambda_2\mu_- - \mu_+)K_g \cdot (2i\lambda_{-2}\mu_+ + \mu_-) \\ &= \frac{1}{K_g^2}((2i\lambda_{-2}\mu_+ + \mu_-)K_g + (-4|\lambda_2|^2\mu_- - 2i\lambda_{-2}\mu_+K_g)) \cdot \mu_+ \\ & \quad + \frac{1}{K_g^2}((2i\lambda_2\mu_- - \mu_+)K_g + (4|\lambda_2|^2\mu_+ - 2i\lambda_2\mu_-K_g)) \cdot \mu_- \\ &= -\frac{\mu_-K_g}{K_g} \cdot \mu_+ + \frac{\mu_+K_g}{K_g} \cdot \mu_-, \end{aligned}$$

where we used that $K_g = -1 + 4|\lambda_2|^2$ in the last line. This proves the first formula.

For the second formula, compute

$$\begin{aligned} \left[\frac{\mu_-}{K_g}, \frac{\mu_+}{K_g} \right] &= \frac{1}{K_g}\mu_- \left(\frac{\mu_+}{K_g} \right) - \frac{1}{K_g}\mu_+ \left(\frac{\mu_-}{K_g} \right) \\ &= -\frac{1}{K_g^2}[\mu_+, \mu_-] + \frac{1}{K_g^3}(\mu_+K_g \cdot \mu_- - \mu_-K_g \cdot \mu_+) = \frac{iV}{2}, \end{aligned}$$

where we used (8-13) in the last equality. The final formula is a straightforward restatement of this one. $\square$

Finally, we are able to compute the index of $\mu_{\pm}$ explicitly:

Lemma 8.11 We have $\ker \mu_+|_{H_m} = \{0\}$ for any $m > 0$. Also, $\ker \mu_+|_{H_0}$ is spanned by constant functions. Moreover, for any $m \geq 0$, the analytical index satisfies $\text{ind}(\mu_+|_{H_m}) = \text{ind}(\eta_+|_{H_m})$.

In particular, $\dim \ker \mu_-|_{H_1} = \frac{1}{2}b_1(M)$.

Proof Assume $\mu_+f = 0$, where $f \in H_m$ for some $m \geq 0$. Using the final identity of Lemma 8.10, multiplying with $\bar{f}$ and integrating over $\mathcal{M}$, we get

$$\begin{aligned} 0 &= \frac{i}{2} \int_{\mathcal{M}} K_g V f \cdot \bar{f} \Omega + \int_{\mathcal{M}} \mu_+ \left(\frac{1}{K_g} \mu_- f \right) \cdot \bar{f} \Omega \\ &= -\frac{m}{2} \int_{\mathcal{M}} K_g |f|^2 \Omega - \int_{\mathcal{M}} \frac{1}{K_g} |\mu_- f|^2 \Omega \geq 0, \end{aligned}$$

since by (7-8), $K_g < 0$, and we used $\mu_+^* = -\mu_-$. If $m > 0$, this shows $f = 0$ and completes the proof of the first part of the statement.

For the second part, if $m = 0$, from the previous paragraph we get that $\mu_-f = 0$. By (8-14), and from the assumption $\mu_+f = 0$, we conclude that $\eta_+f = \eta_-f = 0$. This implies in particular that $Xf = 0$, which by (2-13) implies $df = 0$, which in turn gives that f is a constant.

Finally, let $P_t := \eta_- - t \cdot 2i\lambda_{-2}\eta_+$ be a continuous deformation of operators for $t \in [0, 1]$. By inspecting the proof of Lemma 8.9 the operators P_t are indeed elliptic in this region, when acting on sections of $\mathcal{K}^{\otimes m}$ (or equivalently, on H_m) for any $m \geq 0$. By topological invariance of the Fredholm index we

conclude that $\mu_- = P_1$ and $\eta_- = P_0$ have identical indices, ie $\text{ind}(\mu_-|_{H_m}) = \text{ind}(\eta_-|_{H_m})$. In particular, for $m = 1$ we get that, using $\mu_+^* = -\mu_-$,

$$\dim \ker \mu_-|_{H_1} - \dim \ker \mu_+|_{H_0} = \text{ind}(\mu_-|_{H_1}) = \text{ind}(\eta_-|_{H_1}) = \frac{1}{2}b_1(M) - 1,$$

where in the last equality we used [Proposition A.4](#). Since $\dim \ker \mu_+|_{H_0} = 1$, the final claim follows. $\square$

8.3 Horocyclic invariance of the SRB measure

In this section we derive equations for the horocyclic invariance of the SRB measure. Let $a := a_u$ in $\mathcal{D}'_{E_u^*}(\mathcal{M})$ be the Hölder regular solution of

$$(8-15) \quad (F + r^u)a = -\lambda.$$

Note that this a is such that $U^u = Y^u - aF$ (the notation coming from [Section 4](#)):

$$[F, U^u] = [F, Y^u] - Fa \cdot F = -r^u Y^u - (Fa + \lambda) \cdot F = -r^u(Y^u - aF) = -r^u U^u.$$

Using the results of [Section 4](#), we have:

Lemma 8.12 *If $\Omega_{\text{SRB}}^+ = f\Omega$ is the SRB measure, it holds that*

$$(Y^u - 2\lambda + 2a)f = 0.$$

Note that af is well-defined as $f\Omega$ is an actual *measure*.

Proof According to the notation of [Lemma 4.9](#) and by [Lemma 4.13](#), it suffices to show that

$$\alpha_{\text{div}_\Omega} F = 2a - 2\lambda.$$

Since $\text{div}_\Omega F = V\lambda$, for this it suffices to show the identity

$$B := (F + r^u)(2a - 2\lambda) - (Y^u - \lambda)V\lambda = 0.$$

In fact, by the definition (8-15) of a we compute

$$\begin{aligned} B &= -2\lambda - 2(X + \lambda V + (1 + \frac{1}{2}V\lambda))\lambda - (H + (1 + \frac{1}{2}V\lambda)V)V\lambda + \lambda V\lambda \\ &= -2\lambda - 2X\lambda - 2\lambda V\lambda - 2\lambda - \lambda V\lambda - HV\lambda + 4\lambda + 2\lambda V\lambda + \lambda V\lambda = 0, \end{aligned}$$

where we used that $(2X + HV)\lambda = 0$ (which follows from [Lemma A.2](#)) and $V^2\lambda = -4\lambda$. $\square$

[Lemma 4.9](#) implies that f satisfies the system

$$(8-16) \quad (F + V\lambda)f = 0, \quad (Y^u - 2\lambda + 2a)f = 0.$$

Under the assumption that f is a measure, or more generally, that it belongs to some Sobolev space with a small negative exponent (so that af makes sense), this system implies that $f\Omega$ is really the SRB measure.

Remark 8.13 One might hope to explicitly solve (8-16) similarly to how we did in [Lemma 8.7](#) for elements of Res_0^1 and the system (8-2). However, the issue here is that a has *infinite Fourier content*, that is, its degree is infinite — otherwise, it would be smooth, and so the bundle E_u would be smooth, which

generically does not happen [Paternain 2007]. Therefore, the recurrence relations stemming from (8-16) would now involve *all* Fourier modes of f , which complicates the situation significantly compared to (8-8).

Remark 8.14 Since $\lambda_{\pm 2}$ vanish at finitely many points due to holomorphicity of A (exactly at $4g - 4$ of points, the degree of the holomorphic bundle $\mathcal{K}^{\otimes 2}$), we see that the solution to $(F + V\lambda)f = 0$ is uniquely determined by f_0, f_1, f_2 (the $f_3, \dots$ are determined by dividing by $\lambda_{\pm 2}$ outside zeros of A and by continuity on the zeros).

8.4 Generic semisimplicity

In this section we tackle the question of semisimplicity for quasi-Fuchsian vector fields. We will need the following claim, proved in [Cekić et al. 2022, Lemma 4.8].

Lemma 8.15 (linear algebra lemma) *Let $V \subset \mathbb{M}_{n \times n}$ be a (real or complex) linear subspace of the set of $n \times n$ complex matrices such that for each $u, v \in \mathbb{C}^n \setminus \{0\}$, there exists $B \in V$ such that $(Bu, v) \neq 0$. (Here $(\bullet, \bullet)$ denotes the canonical inner product on $\mathbb{C}^n$.) Then V contains a dense set of invertible matrices.*

Next, we need to show that the product of certain distributions equals zero if and only if one of the distributions itself is zero.

Lemma 8.16 (product lemma) *Let $P_{\pm} = Y + Q^{\pm}$ and $P_{1,2} = Y_{1,2} + Q_{1,2}$, be first-order differential operators with C^{∞} coefficients, where $Y, Y_1, Y_2 \in C^{\infty}(\mathcal{M}; T\mathcal{M})$ form a smooth global frame and $Q^{\pm}, Q_{1,2} \in C^{\infty}(\mathcal{M})$. Assume $u_{\pm} \in \mathcal{D}'(\mathcal{M})$ satisfy $P_{\pm}u_{\pm} = P_1u_+ = P_2u_- = 0$. Then the product u_+u_- is well-defined, and*

$$u_+u_- = 0 \implies \text{supp}(u_+)^c \cup \text{supp}(u_-)^c = \mathcal{M}.$$

Proof The basic idea of the proof is to reduce the product of distributions to a tensor product in suitable coordinates. Since $\text{WF}(u_+) \subset (\mathbb{R}Y \oplus \mathbb{R}Y_1)^{\perp} \subset T^*\mathcal{M}$ and $\text{WF}(u_-) \subset (\mathbb{R}Y \oplus \mathbb{R}Y_2)^{\perp}$ have zero intersection, where we recall that $\bullet^{\perp}$ denotes the annihilator of $\bullet$, the distributional product is well-defined by [Hörmander 2003, Theorem 8.2.10]. Let $x \in \mathcal{M}$ and pick flow-box coordinates $(x_1, x_2, x_3) \in [-\varepsilon, \varepsilon]^3$ for some $\varepsilon > 0$ near $x = (0, 0, 0)$ such that $Y = \partial_{x_1}$ and so

$$(8-17) \quad (\partial_{x_1} + Q_{\pm})u_{\pm} = 0.$$

By (8-17) we have $\text{WF}(u_{\pm}) \subset \mathbb{R}dx_2 \oplus \mathbb{R}dx_3$, so since the conormal bundle of the slice $S_c := \{x_1 = c\}$ for each $c \in [-\varepsilon, \varepsilon]$ is $\mathbb{R}dx_1$, we may restrict $u_{\pm}$ to S_c by the wavefront set calculus; see [Hörmander 2003, Corollary 8.2.7]. On each S_c , using (8-17) we may write the equations $P_1u_+ = 0$ and $P_2u_- = 0$ as

$$\underbrace{(a_+ \partial_{x_2} + b_+ \partial_{x_3} + Q'_1)}_{=: Z_+} u_+ = 0 \quad \text{and} \quad \underbrace{(a_- \partial_{x_2} + b_- \partial_{x_3} + Q'_2)}_{=: Z_-} u_- = 0,$$

where $a_{\pm}, b_{\pm}, Q'_{1,2}$ depend on (c, x_2, x_3) , and $Z_{\pm}$ are pointwise linearly independent in TS_c . Since the restriction of the product to S_c is the product of restrictions (well-defined similarly to the above since

$Z_{\pm}$ are linearly independent), and the invariance (8-17) is valid, it suffices to show $u_+|_{S_0}$ or $u_-|_{S_0} = 0$ near $(0, 0)$.

By using suitable smooth integrating factors, without loss of generality assume that $Q'_1 = Q'_2 = 0$, so that $Z_{\pm}u_{\pm} = 0$ on S_0 . This reduces the problem to a statement in 2 dimensions. We may put Z_+ into flow-box coordinates $(y_1, y_2) \in (-\delta, \delta)^2$ for some $\delta > 0$, defined on S_0 such that $x = (0, 0)$, $Z_+ = \partial_{y_1}$, and (possibly after a time-change) $Z_- = d\partial_{y_1} + \partial_{y_2}$, where $d \in C^{\infty}((-\delta, \delta)^2)$, so that

$$\partial_{y_1}u_+ = 0 \quad \text{and} \quad (d\partial_{y_1} + \partial_{y_2})u_- = 0.$$

Consider the flow α_t of Z_- , and the time $t = t(y_1, y_2) \in \mathbb{R}$, defined on a subdomain $(-\delta_1, \delta_1)^2$ for some $0 < \delta_1 < \delta$, such that $\text{pr}_2 \circ \alpha_{-t}(y_1, y_2) = 0$, where for $i = 1, 2$, pr_i denotes the projection onto the coordinate y_i . Consider the map

$$G: (-\delta_1, \delta_1)^2 \rightarrow (-\delta, \delta)^2, \quad (y_1, y_2) \mapsto (\text{pr}_1 \circ \alpha_{-t}(y_1, y_2), y_2),$$

and note that after taking $\delta_1 > 0$ small enough, G is a diffeomorphism onto its image by the inverse mapping theorem (note that $DG(0, 0) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ on $\{y_2 = 0\}$ since $G(y_1, 0) = (y_1, 0)$). Denote by (G_1, G_2) the components of G . By the wavefront set calculus, define $v_+ := u_+|_{\{0\} \times (-\delta, \delta)}$ and $v_- := u_-|_{(-\delta, \delta) \times \{0\}}$. Since pullback of distributions under submersions is well-defined, see [Hörmander 2003, Theorem 6.1.2], the invariance $Z_{\pm}u_{\pm} = 0$ translates to $G_2^*v_+ = u_+$ and $G_1^*v_- = u_-$.

Let $\delta_2 > 0$ be such that $[-\delta_2, \delta_2]^2 \subset G((-\delta_1, \delta_1)^2)$ and $\delta_2 < \delta_1$. Let $\psi_1, \psi_2 \in C_0^{\infty}((-\delta_2, \delta_2))$ be arbitrary. Then

$$\begin{aligned} & \int_{G((-\delta_1, \delta_1)^2)} v_-(z_1)v_+(z_2)\psi_1(z_1)\psi_2(z_2) dz_1 dz_2 \\ &= \int_{(-\delta_1, \delta_1)^2} u_-(y_1, y_2)u_+(y_1, y_2)(G_1^*\psi_1)(y_1, y_2)\psi_2(y_2)J(y_1, y_2) dy_1 dy_2 = 0, \end{aligned}$$

since $u_+u_- = 0$, and where J is the Jacobian of G . Since ψ_1 and ψ_2 were arbitrary we conclude that either $v_+|_{(-\delta_2, \delta_2)}$ or $v_-|_{(-\delta_2, \delta_2)}$ have to vanish and so either $u_+ = G_2^*v_+$ or $u_- = G_1^*v_-$ vanish in a neighbourhood of x , completing the proof. $\square$

Finally, we formulate the main result of this section: for a generic time-change, semisimplicity is valid for Res_0^1 as claimed in Theorem 1.5.

Lemma 8.17 *For an open and dense set of $a \in C^{\infty}(\mathcal{M}; \mathbb{R}_{>0})$, the Lie derivative action $\mathcal{L}_{aF}$ on Ω_0^1 is semisimple.*

Proof For $a \in C^{\infty}(\mathcal{M}; \mathbb{R})$, define the pairing

$$A_a(u, u_*) = \int_{\mathcal{M}} a\alpha \wedge u \wedge u_* \quad \text{for } (u, u_*) \in \text{Res}_0^1 \times \text{Res}_{0*}^1.$$

Further identifying $\text{Res}_{0(*)}^1$ with $\mathbb{C}^{b_1(M)}$ (according to [Lemma 8.7](#)), we may identify $A_{\mathbf{a}}$ with an $n \times n$ complex matrix in $\mathbb{M}_{n \times n}$. Introduce the finite-dimensional real-linear vector space

$$V := \{A_{\mathbf{a}} \in \mathbb{M}_{n \times n} \mid \mathbf{a} \in C^\infty(\mathcal{M}; \mathbb{R})\}.$$

For $\mathbf{a} \in C^\infty(\mathcal{M}; \mathbb{R}_{>0})$, denote by $\langle\langle \bullet, \bullet \rangle\rangle_{\mathbf{a}} = \langle\langle (1/\mathbf{a}) \bullet, \bullet \rangle\rangle$ the pairing associated to $\mathbf{a}F$ introduced in [Section 2.2](#). Since the weak stable/unstable bundles of $\mathbf{a}F$ and F agree by [[de la Llave et al. 1986](#), Lemma 2.1], the $\text{Res}_{0(*)}^1$ spaces with respect to $\mathbf{a}F$ and F agree. Thus $\langle\langle \bullet, \bullet \rangle\rangle_{\mathbf{a}}$ may be identified with the matrix $A_{1/\mathbf{a}}$, and by [Lemma 2.1](#) to prove the present lemma it suffices to show that for an open and dense set of $\mathbf{a} \in C^\infty(\mathcal{M}; \mathbb{R}_{>0})$, the matrix $A_{1/\mathbf{a}}$ is invertible. Openness is immediate and it suffices to show density.

We first show that V contains a dense set of invertible matrices; again, since invertibility is an open condition, this set is also open. Let $0 \neq u \in \text{Res}_0^1$ and $0 \neq u_* \in \text{Res}_{0*}^1$. Then

$$A_{\mathbf{a}}(u, u_*) = \int_{\mathcal{M}} \mathbf{a} h h_* \alpha \wedge \omega_u \wedge \omega_s,$$

where $h = \iota_{Y^s} u$, $h_* = \iota_{Y^u} u_*$ and $\omega_{u/s} \in C^\infty(\mathcal{M}; T^*\mathcal{M})$ were defined in (4-2); we also used [Lemma 4.2](#) and its analogue for coresonant states. If $A_{\mathbf{a}}(u, u_*) = 0$ for all $\mathbf{a} \in C^\infty(\mathcal{M}; \mathbb{R})$, then $h h_* \equiv 0$. By Lemmas [8.1](#) and [8.5](#), $(F + r^s)h = (Y^u - \lambda)h = 0$, and similarly $(-F - r^u)h_* = (Y^s - \lambda)h_* = 0$ and so [Lemma 8.16](#) applies to give that $\text{supp}(h)^c \cup \text{supp}(h_*)^c = \mathcal{M}$. However, [[Weich 2017](#), Theorem 1] shows that h and h_* (being resonant states for $F + r^s$ and $-F - r^u$, respectively) either vanish everywhere or have full support, which implies that either $h \equiv 0$ or $h_* \equiv 0$, contradiction. Therefore, by [Lemma 8.15](#), the claim about density follows.

Observe that for an arbitrary open set $U \subset C^\infty(\mathcal{M}; \mathbb{R})$, we have $\{A_{\mathbf{a}} \in \mathbb{M}_{n \times n} \mid \mathbf{a} \in U\} \subset V$ is open, and so by density of invertible elements in V , we conclude that for a dense set of $\mathbf{a} \in C^\infty(\mathcal{M}; \mathbb{R})$, the matrix $A_{\mathbf{a}}$ is invertible. Since $C^\infty(\mathcal{M}; \mathbb{R}_{>0}) \subset C^\infty(\mathcal{M}; \mathbb{R})$ is also open, the main claim follows. $\square$

Remark 8.18 In fact, in order to establish generic semisimplicity under time changes as in [Lemma 8.17](#) it is equivalent to showing semisimplicity for only *one* time change. This follows from basic linear algebra.

8.5 Computation of the helicity

By [Lemma 8.8](#) we know that the winding cycles of $F = X + \lambda V$ are zero; therefore there exists a primitive $\tau^+ \in \text{Res}^1$ such that

$$(8-18) \quad d\tau^+ = \iota_F \Omega_{\text{SRB}}^+, \quad \iota_F \tau^+ = \mathcal{H}(F) =: B.$$

Recall that a was defined in (8-15) and that it satisfies $Y^u = U^u + aF$; also recall $\Omega_{\text{SRB}}^+ = f\Omega$ for some $f \in \mathcal{D}'_{E_u^*}(\mathcal{M})$, where Ω is the canonical measure on $\mathcal{M} = SM$. Write $f^+ := f$ in this section. We first prove an auxiliary horocyclic invariance result similar to [Lemma 8.1](#), but valid for τ^+ .

Lemma 8.19 Let $c := \iota_V \tau^+$. Then

$$(8-19) \quad (F + r^s)c = -aB, \quad (Y^u - \lambda)c = -f^+ + B(1 + Va).$$

Moreover, the following identity holds:

$$(8-20) \quad -(Y^u - \lambda)r^s = (F + r^u)\lambda.$$

Proof By the first part of [Lemma 4.11](#) we know that τ^+ satisfies

$$\iota_H \tau^+ + r^u c = \iota_{Y^u} \tau^+ = aB.$$

Using this relation, as well as $[F, V] = -(H + V\lambda \cdot V)$, the first equation of (8-19) follows from expanding $d\tau^+(F, V) = 0$ similarly to [Lemma 4.4](#). The second equation similarly follows by expanding the equation $d\tau^+(H, V) = -f^+$, as in [Lemma 4.4](#).

The final equation (8-20) follows from the following computation:

$$\begin{aligned} -Y^u r^s + \lambda(r^s - r^u) - F\lambda &= -(H + (1 + \tfrac{1}{2}V\lambda)V)(-1 + \tfrac{1}{2}V\lambda) - 2\lambda - (X + \lambda V)\lambda \\ &= -\tfrac{1}{2}HV\lambda + 2\lambda + \lambda V\lambda - 2\lambda - X\lambda - \lambda V\lambda \\ &= -(\tfrac{1}{2}HV + X)\lambda = -2(\eta_{-}\lambda_2 + \eta_{+}\lambda_{-2}) = 0, \end{aligned}$$

which completes the proof. $\square$

We are now in shape to compute B in terms of a and the entropy production (defined in [Section 2.6](#)). We will denote by $\text{vol}_\bullet(M)$ the volume of M equipped with the Riemannian metric $\bullet$.

Lemma 8.20 The following identity holds:

$$(8-21) \quad \mathcal{H}(F) = \frac{1 + \frac{1}{2}e^+(F)}{2\pi \text{vol}_g(M) + \int_{\mathcal{M}} a^2 \Omega}.$$

Next, assume $A \neq 0$ and set $A(s) = sA$ for $s \in \mathbb{R}$. Consider the Riemannian metric g_s such that $(g_s, A(s))$ solves the coupled vortex equations (7-4), and the associated thermostat vector field F_s . Then

$$\mathcal{H}(F_s) = \mathcal{O}\left(\frac{1}{s}\right) \quad \text{as } s \rightarrow \infty.$$

Proof Multiplying the first equation of (8-19) by λ and integrating with respect to Ω , we get (we omit the volume form for simplicity)

$$\begin{aligned} -B \int_{\mathcal{M}} a\lambda &= \int_{\mathcal{M}} (F + r^s)c \cdot \lambda = - \int_{\mathcal{M}} c(F + r^u)\lambda \\ &= \int_{\mathcal{M}} c(Y^u - \lambda)r^s = - \int_{\mathcal{M}} (Y^u - \lambda)c \cdot r^s \\ &= - \int_{\mathcal{M}} (-f^+ + B(1 + Va))r^s \\ &= -(1 + \tfrac{1}{2}e^+(F)) + B \cdot 2\pi \text{vol}_g(M) - 2B \int_{\mathcal{M}} a\lambda. \end{aligned}$$

In the second line we integrated by parts and used (8-1), in the third line we used (8-20), in the fourth line we integrated by parts and used (8-1) again, in the fifth line we used the second equation of (8-19), in the sixth line we used that $\operatorname{div}_\Omega F = V\lambda$, equation (8-1), the definition of entropy production, and integration by parts. To prove the first claim, it now suffices to observe that

$$-\int_{\mathcal{M}} a\lambda = \int_{\mathcal{M}} (F + r^u)a \cdot a = -\frac{1}{2} \int_{\mathcal{M}} a^2(-2r^u + V\lambda) = \int_{\mathcal{M}} a^2,$$

where in the first equality we used (8-15), in the second one we integrated by parts, and in the final one we used (8-1).

Next, consider the family of solutions $g_s := e^{2u(s)}g_0$ to the coupled vortex equations (7-4) corresponding to $A(s)$, and giving $\lambda(s) := s \operatorname{Im}(\pi_{2,s}^*A) \in C^\infty(SM_s)$, where $\pi_{2,s}^*$ is the pullback on 2-tensors to the unit sphere bundle SM_s of g_s , and V_s is the vertical vector field on SM_s . Then the curvature of g_s being negative (see (7-8)) translates to

$$(8-22) \quad 0 > K_{g_s} = -1 + s^2 e^{-4u(s)} |A|_{g_0}^2 \iff e^{2u(s)} > s |A|_{g_0}.$$

The volume form scales as $d\operatorname{vol}_{g_s} = e^{2u(s)} d\operatorname{vol}_{g_0}$ and so by (8-22) we get

$$(8-23) \quad \operatorname{vol}_{g_s}(M) = \int_M d\operatorname{vol}_{g_s} = \int_M e^{2u(s)} d\operatorname{vol}_{g_0} \geq s \int_M |A|_{g_0} d\operatorname{vol}_{g_0}.$$

Observe next that the entropy production is bounded: indeed, pointwise we have

$$|V_s \lambda(s)| = s |\operatorname{Im}(2i \pi_{2,s}^* A)| \leq 2s |\pi_{2,s}^* A| = 2s |A|_{g_s} = 2s e^{-2u(s)} |A|_{g_0} \leq 2,$$

where in the second equality we used (A-5), and in the final estimate we used (8-22). This shows that the entropy production is bounded by

$$(8-24) \quad |e^+(F_s)| \leq \int_{SM_s} |V_s \lambda(s)| \Omega_{\text{SRB}}^+(F_s) \leq 2.$$

Finally, it follows from the formula (8-21), as well as the estimates (8-23) and (8-24), that

$$\mathcal{H}(F_s) \leq \frac{2}{2\pi \operatorname{vol}_g(M)} \leq \frac{1}{s\pi \int_M |A|_{g_0} d\operatorname{vol}_{g_0}},$$

which completes the proof. $\square$

Theorem 1.5 now follows directly from Lemmas 8.7, 8.8, 8.17 and 8.20.

9 Helicity and linking

In this final section we give an interpretation of helicity as an averaged quantity with respect to the SRB measures, based on the wavefront set calculus, and the works of Coles and Sharp [2023] and Kotschick and Vogel [2003]. We also give an interpretation of $\mathcal{H}(X)$ as an asymptotic weighted averaged sum

over closed orbits. Throughout, we will assume that $[\omega^+] = [\omega^-] = 0$ and so there are distributional 1-forms $\tau^\pm$ such that

$$(9-1) \quad \iota_X \Omega_{\text{SRB}}^\pm = d\tau^\pm = \omega^\pm, \quad \text{where } \tau^+ \in \text{Res}^1, \tau^- \in \text{Res}_*^1.$$

Throughout the section we will assume that $\dim \mathcal{M} = 3$.

9.1 The linking form

In [Kotschick and Vogel 2003, Section 2], the linking from is constructed for any pair (N_1, N_2) of null-homologous submanifolds whose sum of dimensions plus one is equal to the dimension of the ambient space, and the linking number is expressed as its double integral over $N_1 \times N_2$. We proceed in a similar fashion here.

Fix a Riemannian metric g on $\mathcal{M}$ and write

$$(9-2) \quad \Omega_{\text{SRB}}^\pm = f^\pm d\text{vol}_g, \quad \text{where } f^+ \in \mathcal{D}'_{E_u^*}(\mathcal{M}), f^- \in \mathcal{D}'_{E_s^*}(\mathcal{M}).$$

Denote by $\mathcal{H}^i$ the space of harmonic i -forms on $(\mathcal{M}, g)$ and by $(\mathcal{H}^i)^\perp$ its L^2 orthogonal complement; let $\mathcal{P}$ denote the orthogonal projection onto $\mathcal{H}^i$. Note that the Hodge Laplacian $\Delta = \Delta_i: (\mathcal{H}^i)^\perp \rightarrow (\mathcal{H}^i)^\perp$ is an isomorphism, so we may introduce $G \in \Psi^{-2}(\mathcal{M})$ by asking that $G = 0$ on $\mathcal{H}^i$ and $G = \Delta^{-1}$ on $(\mathcal{H}^i)^\perp$. Then, by definition,

$$(9-3) \quad G\Delta = \Delta G = \text{Id} - \mathcal{P}, \quad \mathcal{P}G = 0.$$

From $G\Delta = \Delta G$ and the fact that $[\Delta, d] = 0$, $[\Delta, d^*] = 0$ and $[\Delta, \star] = 0$, it follows that

$$(9-4) \quad [G, d] = 0, \quad [G, d^*] = 0, \quad [G, \star] = 0.$$

Denote by $K \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}; \text{pr}_1^* \Omega^i \otimes \text{pr}_2^* \Omega^i)$ the Schwartz kernel of G ; here pr_1 and pr_2 denote projections onto the first and second factors of $\mathcal{M} \times \mathcal{M}$, respectively. Since $G \in \Psi^{-2}(\mathcal{M}; \Omega^i)$, it follows that K is smooth outside of $\Delta(\mathcal{M})$, and by [Neri 1970, Theorem 1.5], that $d(x, y)|K(x, y)|$ is bounded on $\mathcal{M} \times \mathcal{M} \setminus \Delta(\mathcal{M})$ (this uses $\dim \mathcal{M} = 3$), where $d(x, y)$ is the Riemannian distance function and $\Delta(\mathcal{M}) \subset \mathcal{M} \times \mathcal{M}$ is the diagonal.

Then for any $\alpha \in C^\infty(\mathcal{M}; \Omega^i)$,

$$(9-5) \quad G\alpha(x) = \int_{y \in \mathcal{M}} \langle \alpha(y), K(x, y) \rangle_g d\text{vol}_g(y) = \int_{y \in \mathcal{M}} \alpha(y) \wedge \star_y K(x, y),$$

where $\langle \bullet, \bullet \rangle_g$ is the natural inner product on the fibres of Ω^i . Specialising to $i = 1$, the *linking form* $L \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}; \text{pr}_1^* \Omega^1 \otimes \text{pr}_2^* \Omega^1)$ is defined as

$$L(x, y) := \star_y d_y K(x, y).$$

It satisfies the property that when integrated over two knots in $\mathcal{M}$ it gives the *linking number* of the two knots, see [Kotschick and Vogel 2003, Proposition 1] (or [Vogel 2003, Section 3]), ie

$$(9-6) \quad \text{lk}(K_1, K_2) = \int_{K_1} \int_{K_2} L,$$

where $\text{lk}(\bullet, \bullet)$ denotes the linking number (taking values in the rationals $\mathbb{Q}$), and K_1 and K_2 are two submanifolds of dimension 1, both of which have a trivial Poincaré dual over $\mathbb{R}$. Then [Kotschick and Vogel 2003, Proposition 1] shows that $\text{lk}(K_1, K_2)$ does not depend on the choice of g and agrees with the definition through differential forms — also, it takes values in the rationals $\mathbb{Q}$, see the discussion in [Kotschick and Vogel 2003, Section 2]; if we assume furthermore that the Poincaré duals of K_1 and K_2 are trivial over $\mathbb{Z}$, then $\text{lk}(K_1, K_2) \in \mathbb{Z}$. In fact, in this paper, we define $\text{lk}(K_1, K_2)$ as in (9-6), where K_1 and K_2 are *any* two submanifolds of dimension 1. The difference then is that $\text{lk}(K_1, K_2)$ *does depend* on the choice of the metric g (through L), as opposed to the case when K_1 and K_2 are assumed to have trivial Poincaré duals. The same definition appears in [Coles and Sharp 2023, Section 7.2].

Note that $\star_y L(x, y)$ is actually the Schwartz kernel of the operator Gd^* , since for any $\alpha \in C^\infty(\mathcal{M}; \Omega^2)$, by (9-5) we have

$$(9-7) \quad Gd^*\alpha(x) = \int_{y \in \mathcal{M}} d^*\alpha(y) \wedge \star_y K(x, y) = \int_{y \in \mathcal{M}} \alpha(y) \wedge L(x, y).$$

It follows that $d(x, y)^2 |L(x, y)|$ is bounded on $\mathcal{M} \times \mathcal{M} \setminus \Delta(\mathcal{M})$. Set

$$(9-8) \quad \Lambda(x, y) := L(x, y)(X(x), X(y)) \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}).$$

This distribution satisfies the following important properties:

Lemma 9.1 *The distribution $\Lambda(x, y)$ is the Schwartz kernel of the operator $P := \iota_X Gd^* \iota_X \star$. Moreover, $P \in \Psi^{-2}(\mathcal{M})$ and thus there is a constant $C_\Lambda > 0$ such that*

$$d(x, y) |\Lambda(x, y)| \leq C_\Lambda \quad \text{for } (x, y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta(\mathcal{M}).$$

Finally, $\Lambda(x, y) = \Lambda(y, x)$ and $f^+(x)\Lambda(x, y)f^-(y) \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ is well-defined as a distribution.

We remark that the bound on $\Lambda(x, y)$ was shown [Coles and Sharp 2023, Lemma 9.2] by a computation in local coordinates, whereas here we employ a global approach.

Proof For any $f \in C^\infty(\mathcal{M})$ we compute

$$Pf(x) = \int_{y \in \mathcal{M}} f(y) \iota_{X(y)} d\text{vol}_g(y) \wedge \iota_{X(x)} L(x, y) = \int_{y \in \mathcal{M}} f(y) d\text{vol}_g(y) \cdot \overbrace{\iota_{X(y)} \iota_{X(x)} L(x, y)}^{=\Lambda(x, y)}.$$

Here we used (9-7) in the first equality and the anticommuting property of the contraction in the second; this completes the proof of the first claim.

For the second claim, simply write

$$P = [\iota_X, G]d^*\iota_X \star + \overbrace{G(\iota_X d^* + d^*\iota_X)}^{=:Q} \iota_X \star \in \Psi^{-2}(\mathcal{M}),$$

since the commutators $[\iota_X, G]$ and Q are pseudodifferential operators of degrees -3 and 0 , respectively. Indeed, the principal symbol of G , given by $\sigma(G)(x, \xi) = |\xi|_g^{-2}(x, \xi) \times \text{Id}_{\Omega^1}$, is diagonal, and $\sigma(\iota_X)(x, \xi) = \iota_{X(x)}$ and $\sigma(d^*)(x, \xi) = \iota_{\xi^\#}$, where $\xi^\#$ is obtained by applying the musical isomorphism to ξ . Since $\iota_{X(x)}\iota_{\xi^\#} = -\iota_{\xi^\#}\iota_{X(x)}$, this shows that Q has degree 0 , which completes the proof.

Next, the symmetry $\Lambda(x, y) = \Lambda(y, x)$ is equivalent to P being formally self-adjoint, which follows from integration by parts, by using that G is self-adjoint, and applying (9-4).

For the final claim it suffices to observe that

$$(9-9) \quad \text{WF}(\Lambda) \subset \{(x, x, \xi, -\xi) \mid x \in \mathcal{M}, \xi \in T_x^* \mathcal{M}\}$$

since Λ is a kernel of a pseudodifferential operator, and that f^+ and f^- have disjoint wavefront sets, hence the wavefront set calculus applies and the product $f^+(x)\Lambda(x, y)f^-(y)$ is well-defined. $\square$

We proceed with the proof of [Theorem 1.2](#), which is the main result of this section.

Proof of Theorem 1.2 Observe that for any $\alpha \in \mathcal{D}'(\mathcal{M}; \Omega^1)$ it holds that

$$(9-10) \quad Gd^*\alpha = G\Delta\alpha - Gdd^*\alpha = \alpha - \mathcal{P}\alpha - dh,$$

where $h = Gd^*\alpha$, and we used (9-3) and (9-4) in the second equality. Now we may compute

$$(9-11) \quad \begin{aligned} \mathcal{H}(X) &= \int_{\mathcal{M}} \tau^- \wedge d\tau^+ = \int_{\mathcal{M}} Gd^*d\tau^- \wedge d\tau^+ \\ &= \int_{\mathcal{M}} \iota_X Gd^*\iota_X \star f^- \cdot f^+ d\text{vol}_g = \int_{\mathcal{M}} P f^- \cdot f^+ d\text{vol}_g \\ &= \int_{(x,y) \in \mathcal{M} \times \mathcal{M}} f^+(x)\Lambda(x, y)f^-(y) d\text{vol}_g(y) \times d\text{vol}_g(x). \end{aligned}$$

Here in the second equality we used (9-10) for $\alpha = \tau^-$ and integration by parts, in the third equality we used (9-1) and (9-2), in the fourth we used the definition of P , and in the fifth we used that $f^+\Lambda f^-$ is a well-defined distribution by [Lemma 9.1](#).

Finally, to interpret the integral in the fifth line as a limit of classical integrals, consider the regularisations $\Lambda_\varepsilon := E_\varepsilon \Lambda \in C^\infty(\mathcal{M} \times \mathcal{M})$ (where the mollifiers E_ε are introduced in [Section 9.2](#) below). Then we have the chain of limits

$$\begin{aligned} \mathcal{H}(X) &= \lim_{\varepsilon \rightarrow 0} \int_{(x,y) \in \mathcal{M} \times \mathcal{M}} \Lambda_\varepsilon(x, y) \Omega_{\text{SRB}}^+(x) \times \Omega_{\text{SRB}}^-(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{(x,y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta(\mathcal{M})} \Lambda_\varepsilon(x, y) \Omega_{\text{SRB}}^+(x) \times \Omega_{\text{SRB}}^-(y) \\ &= \int_{(x,y) \in \mathcal{M} \times \mathcal{M} \setminus \Delta(\mathcal{M})} \Lambda(x, y) \Omega_{\text{SRB}}^+(x) \times \Omega_{\text{SRB}}^-(y). \end{aligned}$$

Here in the first equality we used the second part of [Lemma 9.2](#) to obtain $\Lambda_\varepsilon \rightarrow \Lambda$ in $\mathcal{D}'_\Gamma(\mathcal{M} \times \mathcal{M})$, where Γ is given by the right-hand side of (9-9), and we used the sequential continuity of multiplication under the wavefront set condition; see [\[Hörmander 2003, Chapter 8\]](#). Next, in the second line we used [\[Coles and Sharp 2023, Lemma 9.7\]](#), which guarantees that the $\Omega_{\text{SRB}}^+ \times \Omega_{\text{SRB}}^-$ measure of $\Delta(\mathcal{M}) \subset \mathcal{M} \times \mathcal{M}$ is zero. In the final line we used the first part of [Lemma 9.2](#) for $S = \Delta(\mathcal{M})$ and $t = 1$, which says that Λ_ε is dominated by the function $Cd(x, y)^{-1} \sim Cd((x, y), \Delta(\mathcal{M}))^{-1}$ for some uniform $C > 0$ (recall by [Lemma 9.1](#) that Λ has the singularity $d(x, y)^{-1}$ at the diagonal and is smooth outside of it), which is integrable with respect to $\Omega_{\text{SRB}}^+ \times \Omega_{\text{SRB}}^-$ by [\[Coles and Sharp 2023, Lemma 9.3\(ii\)\]](#) as well as [Lemma 9.6](#), where the latter lemma is applied to Hölder continuous functions $\psi = -r^u$ and $\psi = r^s$ which give the SRB measures as equilibrium measures — recall that $r^{u/s}$ were introduced in (4-1), see also [Section 9.3](#). This also shows the first part of the lemma and moreover, applying the dominated convergence theorem completes the proof. $\square$

9.2 Regularisation of the kernel

We now discuss the auxiliary mollification result used in the proof of [Theorem 1.2](#). Given a metric g_N on a manifold N of dimension $n = \dim N$, introduce the family of mollifiers defined for $u \in \mathcal{D}'(N)$ and $\varepsilon > 0$

$$(9-12) \quad E_\varepsilon u(x) := \frac{1}{F_\varepsilon(x)} \int_N \chi\left(\frac{d(x, y)}{\varepsilon}\right) u(y) d\text{vol}_g(y) \quad \text{for } x \in N.$$

Here $\chi \in C^\infty(\mathbb{R}_{\geq 0}; [0, 1])$ is a nonincreasing cut-off function supported in $[0, 1]$ with values in $[0, 1]$ such that $\chi = 1$ close to zero; also $F_\varepsilon \in C^\infty(N)$ is chosen such that $E_\varepsilon \mathbf{1} \equiv \mathbf{1}$ and for some $C > 1$,

$$(9-13) \quad C^{-1}\varepsilon^n \leq F_\varepsilon(x) \leq C\varepsilon^n \quad \text{for all } \varepsilon > 0 \text{ and all } x \in N.$$

By [\[Dyatlov and Zworski 2016, equation \(2.18\)\]](#) we know that

$$(9-14) \quad E_\varepsilon \in \Psi^{-\infty}(N), \quad E_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \text{Id} \quad \text{in } \Psi^{0+}(N),$$

where the latter limit is understood in the Fréchet topologies of $\Psi^k(N)$ for every $k > 0$. We prove the following approximation result:

Lemma 9.2 *Let $S \subset N$ be a smooth submanifold of dimension s . Denote by $d(\bullet, S)$ the distance function to S . Then for any $t \in (0, n - s)$, there exists $C_t > 0$ such that*

$$[E_\varepsilon d(\bullet, S)^{-t}](x) \leq C_t d(x, S)^{-t} \quad \text{for all } \varepsilon > 0 \text{ and all } x \notin S.$$

Moreover, if $u \in \mathcal{D}'_\Gamma(N)$ for some closed conic set Γ , we have

$$E_\varepsilon u \xrightarrow{\varepsilon \rightarrow 0} u \quad \text{in } \mathcal{D}'_\Gamma(N).$$

The sequential topology in the space $\mathcal{D}'_\Gamma(N)$ is introduced in [\[Hörmander 2003, Chapter 8\]](#).

Proof We first show the second claim. For that, consider an arbitrary $\varphi \in C^\infty(N)$. By (9-14), we have ${}^tE_\varepsilon\varphi \rightarrow \varphi$ in $C^\infty(N)$ as $\varepsilon \rightarrow 0$, where the superscript ^{t} denotes the transposed operator. It follows that

$$\langle E_\varepsilon u, \varphi \rangle = \langle u, {}^tE_\varepsilon \varphi \rangle \xrightarrow{\varepsilon \rightarrow 0} \langle u, \varphi \rangle,$$

which means that $E_\varepsilon u \xrightarrow{\varepsilon \rightarrow 0} u$ in $\mathcal{D}'(N)$. To show convergence in $\mathcal{D}'_\Gamma(N)$, it suffices to show that for an arbitrary $A \in \Psi^0(N)$ with $\text{WF}(A) \cap \Gamma = \emptyset$, $AE_\varepsilon u$ is uniformly (in $\varepsilon > 0$) bounded in $C^\infty(N)$; see [Hörmander 2003, Definition 8.2.2]. Let $B \in \Psi^0(N)$ be such that $\text{WF}(B) \cap \text{WF}(A) = \emptyset$ and $\text{WF}(\text{Id} - B)$ is contained in the complement of an open conic neighbourhood of Γ . Then

$$AE_\varepsilon u = AE_\varepsilon Bu + AE_\varepsilon(\text{Id} - B)u,$$

where by construction and (9-14) we have $AE_\varepsilon B \in \Psi^{-\infty}(N)$ with uniformly bounded seminorms, $(\text{Id} - B)u \in C^\infty(N)$ (as $\text{WF}(u) \subset \Gamma$) and since again by (9-14) we have $AE_\varepsilon \in \Psi^{0+}(N)$ with uniformly bounded seminorms. The claim immediately follows.

For the first claim, from the definition of E_ε and by the triangle inequality, it follows that

$$[E_\varepsilon d(\cdot, S)^{-t}](x) \leq (d(x, S) - \varepsilon)^{-t}, \quad d(x, S) > \varepsilon,$$

and so we immediately get, setting $u_\varepsilon := E_\varepsilon d(\cdot, S)^{-t}$,

$$(9-15) \quad u_\varepsilon(x) \leq 2^t d(x, S)^{-t}, \quad d(x, S) \geq 2\varepsilon.$$

Next, consider the injectivity radius $\iota_N > 0$ of the metric g_N . Identify an ε_0 -neighbourhood of S with $T := N_{\leq \varepsilon_0} S = \{(x, \xi) \in NS \mid x \in S, |\xi| \leq \varepsilon_0\}$ via the parametrisation by normal geodesics to S , where NS denotes the normal bundle to S . By taking $\varepsilon_0 \in (0, \iota_N)$ small enough, we may assume there exists $C_0 > 1$ such that for any $(x, \xi), (y, \eta) \in T$ with distance at most ε_0 apart, we have

$$(9-16) \quad C_0^{-1} d((x, \xi), (y, \eta)) \leq d(x, y) + |\xi - P_{y \rightarrow x} \eta| \leq C_0 d((x, \xi), (y, \eta)),$$

where $P_{y \rightarrow x}: T_y N \rightarrow T_x N$ is the parallel transport with respect to the Levi-Civita connection along the unique short geodesic from y to x . For any $(x, \xi) \in T \setminus S$ with $d((x, \xi), S) \leq 2\varepsilon$ (and $\varepsilon > 0$ small enough) and for any $r \in (0, 1)$, we make the following computation:

$$\begin{aligned} u_\varepsilon(x, r\xi) &= \frac{1}{F_\varepsilon(x, r\xi)} \int_{y \in T_x S, |y| \leq C_1 \varepsilon} \int_{\eta \in N_x S, |\eta| \leq C_1 \varepsilon} \chi\left(\frac{d((x, r\xi), (y, \eta))}{\varepsilon}\right) |\eta|^{-t} J_x(y, \eta) dy d\eta \\ &\leq C \varepsilon^{-n} \int_{y \in T_x S, |y| \leq C_1 \varepsilon} \int_{\eta \in N_x S, |\eta| \leq C_1 \varepsilon} \chi\left(\frac{C_0^{-1} \left(\frac{1}{r}|y| + \left|\xi - \frac{1}{r}\eta\right|\right)}{\varepsilon/r}\right) |\eta|^{-t} J_x(y, \eta) dy d\eta \\ &= C \varepsilon^{-n} r^{n-t} \int_{\substack{y' \in T_x S \\ |y'| \leq C_1 \varepsilon/r}} \int_{\substack{\eta' \in N_x S \\ |\eta'| \leq C_1 \varepsilon/r}} \chi\left(\frac{C_0^{-1} (|y'| + |\xi - \eta'|)}{\varepsilon/r}\right) |\eta'|^{-t} J_x(ry', r\eta') dy' d\eta' \\ &\leq C \varepsilon^{-n} r^{n-t} \|J_x\|_\infty (2C_1)^s \left(\frac{\varepsilon}{r}\right)^s \int_{\eta' \in N_x S, |\eta'| \leq C_1 \varepsilon/r} |\eta'|^{-t} d\eta' \end{aligned}$$

$$\begin{aligned}
&\leq \text{vol}(\mathbb{S}^{n-s-1}) C \varepsilon^{-n} r^{n-t} \|J_x\|_\infty (2C_1)^s \left(\frac{\varepsilon}{r}\right)^s \int_0^{C_1 \varepsilon/r} \rho^{-t+n-s-1} d\rho \\
&\leq \frac{\text{vol}(\mathbb{S}^{n-s-1})}{-t+n-s} C \varepsilon^{-n} r^{n-t} \|J_x\|_\infty 2^s (C_1)^{-t+n} \left(\frac{\varepsilon}{r}\right)^{-t+n} \\
&\leq C'(n, s, t) \varepsilon^{-t} \leq C' 2^t d((x, \xi), S)^{-t},
\end{aligned}$$

where in the first line we identified y with points in a geodesic ball of radius $C_1 \varepsilon$ centred at $x \in S$, for a suitable uniform constant $C_1 > 0$, and η with vectors in the fibre $N_x S$ via parallel transport $P_{y \rightarrow x}$; $J_x(y, \eta)$ is the Jacobian of the volume form $d\text{vol}_{g_N}$ in the coordinates (y, η) (uniformly bounded from above). In the second line we used (9-13), (9-16) and the monotonicity of χ , while in the third one we changed the coordinates by $\eta' = \eta/r$ and $y' = y/r$. In the fourth line we estimated χ by 1 from above, integrated in y' and bounded from above the volume of the unit ball in $\mathbb{R}^s$ by 2^s ; in the fifth we used the polar coordinate system in $N_x S$; throughout, $\text{vol}(\mathbb{S}^k)$ denotes the volume of the unit sphere $\mathbb{S}^k \subset \mathbb{R}^{k+1}$. In the sixth line, we used the assumption that $d(\bullet, S)^{-t}$ is integrable (ie $t < n - s$). In the seventh line, we introduced the constant $C' = C'(n, s, t) > 0$. In the final estimate we used the assumption that $d((x, \xi), S) \leq 2\varepsilon$. (The role of $r \in (0, 1)$ is to keep track of the natural scaling in this estimate.)

Since $(x, \xi) \in T \setminus S$ with $d((x, \xi), S) \leq 2\varepsilon$ (with $\varepsilon > 0$ small enough), as well as $r \in (0, 1)$ were arbitrary, combining the preceding estimate with (9-15) completes the proof. $\square$

9.3 Relation with linking of closed orbits

Here we indicate how the results of [Coles and Sharp 2023] can be directly used in conjunction with Theorem 1.2 to give another formula for the helicity in terms of linkings of closed orbits, generalising [Coles and Sharp 2023, Theorem 1.1]. We first very briefly introduce some notation regarding the thermodynamic formalism that will be used only in this section — we refer the reader to [Coles and Sharp 2023, Section 3], [Katok and Hasselblatt 1995] and [Merry and Paternain 2011, Chapter 10] for more details.

Denote by $\mathcal{PM}(X)$ the set of flow-invariant Borel probability measures on $\mathcal{M}$, and given $\nu \in \mathcal{PM}(X)$ denote by $h(\nu)$ the *measure-theoretic entropy* of the time 1 map, φ_1 . Given a Hölder continuous function ψ , denote by $P(\psi)$ the *pressure* of ψ , ie

$$P(\psi) = \sup \left\{ h(\nu) + \int_{\mathcal{M}} \psi d\nu \mid \nu \in \mathcal{PM}(X) \right\}.$$

There exists a unique $\mu_\psi \in \mathcal{PM}(X)$ that attains the supremum above, called the *equilibrium state*. It is well-known that the SRB measures are equilibrium states, more precisely we have (see [Merry and Paternain 2011, Chapter 10.3])

$$\Omega_{\text{SRB}}^+ = \mu_{-r^u}, \quad \Omega_{\text{SRB}}^- = \mu_{r^s},$$

where we recall that $r^{u/s}$ were defined in (4-1).

For $T > 0$, denote by $\mathcal{P}_T$ the set of closed orbits of φ_t with period in $(T - 1, T]$; denote by $\mathcal{P}_T(0) \subset \mathcal{P}_T$ the subset of closed orbits which are trivial in homology $H_1(\mathcal{M}; \mathbb{R})$. Given $\gamma \in \mathcal{P}_T$, set μ_γ to be the probability Dirac measure on γ , that is, if T_γ is the period of γ , then

$$\mu_\gamma(f) = \frac{1}{T_\gamma} \int_\gamma f \quad \text{for } f \in C^0(\mathcal{M}).$$

Given a Hölder function ψ , introduce the weighted orbital probability measures [Coles and Sharp 2023, Section 9]

$$(9-17) \quad \mu_{\psi,T} := \frac{\sum_{\gamma \in \mathcal{P}_T} e^{\int_\gamma \psi} \mu_\gamma}{\sum_{\gamma \in \mathcal{P}_T} e^{\int_\gamma \psi}}, \quad \mu_{\psi,T}^0 := \frac{\sum_{\gamma \in \mathcal{P}_T(0)} e^{\int_\gamma \psi} \mu_\gamma}{\sum_{\gamma \in \mathcal{P}_T(0)} e^{\int_\gamma \psi}}.$$

These measures are used to define the weighted sums

$$(9-18) \quad \mathcal{L}(T) := \int_{\mathcal{M} \times \mathcal{M}} \Lambda d\mu_{-r^u,T}^0 \times d\mu_{r^s,T+1} = \frac{\sum_{\gamma \in \mathcal{P}_T(0), \gamma' \in \mathcal{P}_{T+1}} \frac{\text{lk}(\gamma, \gamma')}{T_\gamma T_{\gamma'}} e^{-\int_\gamma r^u + \int_{\gamma'} r^s}}{\sum_{\gamma \in \mathcal{P}_T(0), \gamma' \in \mathcal{P}_{T+1}} e^{-\int_\gamma r^u + \int_{\gamma'} r^s}},$$

where we recall Λ was introduced in (9-8) (it is smooth outside of the diagonal), and in the second equality we used the definition (9-17) of weighted orbital measures and the defining property of the linking form L from (9-6).

Proposition 9.3 *The following formula holds:*

$$\mathcal{H}(X) = \lim_{T \rightarrow \infty} \mathcal{L}(T).$$

Proof By [Coles and Sharp 2023, Theorem 4.1], we get that $\mu_{r^s,T} \rightarrow \Omega_{\text{SRB}}^-$ in the weak limit sense and moreover, by [loc. cit., Theorem 6.7] that $\mu_{-r^u,T}^0 \rightarrow \mu_{-r^u + f_{[\tau]}}$, where $[\tau] = [\tau](-r^u) \in H^1(\mathcal{M}; \mathbb{R})$ is a cohomology class depending on $-r^u$ and $f_{[\tau]}(x) = \tau(X(x))$. More precisely, $[\tau]$ is defined as the minimizer of the function $\beta: H^1(\mathcal{M}; \mathbb{R}) \rightarrow \mathbb{R}$ defined by $\beta([\varpi]) := P(-r^u + f_{[\varpi]})$ (as the notation suggests, $\mu_{-r^u + f_{[\tau]}}$ and $P(-r^u + f_{[\varpi]})$ do not depend on the choice of the primitives τ and ϖ):

$$\beta([\tau]) = \inf_{[\varpi] \in H^1(\mathcal{M}; \mathbb{R})} P(-r^u + f_{[\varpi]}).$$

That this infimum is uniquely attained follows from [loc. cit., Proposition 5.1] by noting that X is *homologically full*, ie that every integral class in $H_1(\mathcal{M}; \mathbb{Z})$ is represented by a closed orbit, which in turn follows from the assumption that the winding cycle $[\omega^+]$ of Ω_{SRB}^+ is zero and [loc. cit., Proposition 3.1]. From [loc. cit., Proposition 5.1] it also follows that the function $[\varpi] \mapsto \beta([\varpi])$ is strictly convex and by [Lalley 1987; Sharp 1992] that it is real-analytic, and that its derivative at zero in the direction of $[\varpi] \in H^1(\mathcal{M}; \mathbb{R})$ is

$$D\beta(0)([\varpi]) = \int_{\mathcal{M}} f_{[\varpi]} \Omega_{\text{SRB}}^+ = \int_{\mathcal{M}} \varpi(X) \Omega_{\text{SRB}}^+ = 0,$$

as $[\omega^+] = 0$. By uniqueness of the infimum it follows that $[\tau] = 0$ and so $\mu_{-r^u,T}^0 \rightarrow \Omega_{\text{SRB}}^+$ weakly.

Next, for $R > 0$ set $B_R := \{(x, y) \in \mathcal{M} \times \mathcal{M} \mid d(x, y) < R\} \supset \Delta(\mathcal{M})$. By [Coles and Sharp 2023, Lemma 9.9] and as a consequence of Fubini's theorem as in [loc. cit., proof of Lemma 9.9], there exist $Q, \alpha > 0$ such that for all $R, T > 0$ (note here that the supports of $\mu_{-r^u, T}^0$ and $\mu_{r^s, T+1}$ do not intersect so Λ is smooth in the domain of the following integral),

$$(9-19) \quad \int_{B_R} |\Lambda| d\mu_{-r^u, T}^0 \times d\mu_{r^s, T+1} \leq QR^\alpha.$$

Take an arbitrary $\delta > 0$. By (9-19) and Theorem 1.2 we may take $R > 0$ small enough such that for all $T > 0$,

$$\int_{B_R} |\Lambda| d\mu_{-r^u, T}^0 \times d\mu_{r^s, T+1} < \delta \quad \text{and} \quad \left| \int_{B_R} \Lambda \Omega_{\text{SRB}}^+ \times \Omega_{\text{SRB}}^- \right| < \delta,$$

respectively. Next, by the preceding paragraph for $T > 0$ large enough we have

$$\left| \int_{\mathcal{M} \times \mathcal{M} \setminus B_R} \Lambda d\mu_{-r^u, T}^0 \times d\mu_{r^s, T+1} - \int_{\mathcal{M} \times \mathcal{M} \setminus B_R} \Lambda \Omega_{\text{SRB}}^+ \times \Omega_{\text{SRB}}^- \right| < \delta.$$

Combining the two previous inequalities and by the triangle inequality, for $R > 0$ small enough and $T > 0$ large enough, we get

$$\left| \int_{\mathcal{M} \times \mathcal{M}} \Lambda d\mu_{-r^u, T}^0 \times d\mu_{r^s, T+1} - \int_{\mathcal{M} \times \mathcal{M}} \Lambda \Omega_{\text{SRB}}^+ \times \Omega_{\text{SRB}}^- \right| < 3\delta.$$

Since $\delta > 0$ was arbitrary, using Theorem 1.2 concludes the proof. $\square$

Remark 9.4 As we have already mentioned there are no known examples of Anosov flows with zero helicity. The formulas in [Coles and Sharp 2023, Theorem 1.1] and Proposition 9.3 give another point of view to this problem, however since the linkings of closed orbits can be both positive and negative in theory there could be cancellations in (9-18) that in the limit give zero.

Remark 9.5 In a recent article, Dang and Rivière [2024, Theorem 1.2] show that the linking of two distinct closed geodesics (which are homologically trivial) in the unit sphere bundle of a negatively curved surface can be expressed as the value at zero of a certain Poincaré series involving lengths of orthogeodesics.

Remark 9.6 Marty [2023] proves the existence of a unique continuous symmetric bilinear form on the space of null-homologous Borel invariant probability measures of a transitive Anosov flow such that it extends the linking number between two null-homologous closed orbits. It is an interesting question to decide if $\mathcal{H}(X)$ agrees with Marty's linking between the SRB measures (when they are both null-homologous).

9.4 First variation of the helicity

Here we compute the first variation formula for helicity and derive some consequences about the set of Anosov flows with zero helicity. We will use the notation and the perturbation theory derived in Section 5.

We will assume (9-1) for the vector field X_0 . As follows from the second equality of (9-11), we have for $X \in \mathcal{W}$ near X_0 ,

$$(9-20) \quad \mathcal{H}(X) = \int_{\mathcal{M}} G d^* \iota_X \Omega_{\text{SRB}}^- \wedge \iota_X \Omega_{\text{SRB}}^+.$$

For technical reasons, we will now assume that X_0 preserves a smooth volume Ω . Recall that the spaces $\mathcal{W}^\pm$ and $\mathcal{W} = \mathcal{W}^+ \cap \mathcal{W}^-$ defined in (5-1), are locally C^1 Banach submanifolds of codimensions $b_1(\mathcal{M})$ and $2b_1(\mathcal{M})$, respectively, by Lemma 5.5. We prove:

Proposition 9.7 *Assume X_0 preserves a smooth volume Ω such that $[\iota_{X_0}\Omega] = 0$ and $\mathcal{H}(X_0) = 0$. Then, the set $\{X \in \mathcal{W} \mid \mathcal{H}(X) = 0\} \subset \mathcal{W}$ is locally a C^1 Banach submanifold of codimension 1.*

Proof We first show that the map $\mathcal{W} \rightarrow \mathbb{R}$, $X \mapsto \mathcal{H}(X)$, is C^1 -regular in suitable topologies (for $X \in C^N(\mathcal{M}; T\mathcal{M})$ for N large enough). Coming back to Lemma 5.1, it follows from [Bonthonneau 2020, Section 2] that the function $m(x, \xi)$ and the analogous function $m'(x, \xi)$ for the flow $-X_0$ defined on $T^*\mathcal{M}$ may be chosen with values in $[-\frac{1}{2}, 1]$, such that m is equal to $-\frac{1}{2}$ in a conic neighbourhood of E_u^* and equal to 1 outside of slightly larger conic neighbourhood of E_u^* , and m' is chosen to satisfy the analogous property with respect to E_s^* . If $G'(x, \xi) \sim m'(x, \xi) \log(1 + |\xi|)$ is a logarithmically growing symbol on $T^*\mathcal{M}$, for r large enough, it follows that the distributional wedge product is well-defined as a map $\mathcal{H}_{rG, -3}(\mathcal{M}; \Omega^1) \times \mathcal{H}_{rG', -3}(\mathcal{M}; \Omega^2) \rightarrow \mathcal{D}'(\mathcal{M}; \Omega^3)$. More precisely, for this wedge product to make sense, by using the definition of anisotropic spaces, as well as integration by parts it suffices to have (for simplicity, this is viewed on functions)

$$(1 + \Delta_g)^{\frac{3}{2}} e^{-r \text{Op}(G)} e^{-r \text{Op}(G')} (1 + \Delta_g)^{\frac{3}{2}} \in \Psi^0(\mathcal{M}).$$

In turn, by the composition rule for pseudodifferential operators it suffices to have $r(m + m') - 6 \geq 0$. By construction we have $m + m' \geq \frac{1}{2}$, so this is satisfied for large enough r . Thus, by noting that the pseudodifferential operator Gd^* of degree -1 maps $\mathcal{H}_{rG, -3}(\mathcal{M}; \Omega^2)$ to $\mathcal{H}_{rG, -3}(\mathcal{M}; \Omega^1)$, the required regularity follows from the expression (9-20) by applying Lemma 5.2.

Next, we compute the first derivative $D_{X_0}\mathcal{H}$. By the final formula of Lemma 5.2 we get for Y in $C^N(\mathcal{M}; T\mathcal{M})$ that

$$(9-21) \quad D_{X_0}\mathcal{H}(Y) = \int_{\mathcal{M}} G d^* \Pi_2^+ \iota_Y \Omega_{\text{SRB}}^+ \wedge \iota_{X_0} \Omega_{\text{SRB}}^- + \int_{\mathcal{M}} G d^* \iota_{X_0} \Omega_{\text{SRB}}^+ \wedge \Pi_2^- \iota_Y \Omega_{\text{SRB}}^-.$$

Here we used (9-10) for $\alpha = \iota_{X_0} R_2^{\pm, H} \iota_Y \Omega_{\text{SRB}}^\pm$ and the fact that $\iota_{X_0}\alpha = 0$ to get rid of the second term in the last formula in Lemma 5.2.

As X_0 has zero helicity, it is not a contact flow. By Lemma 5.4, there exists $Y \in C^\infty(\mathcal{M}; T\mathcal{M}) \cap T_{X_0}\mathcal{W}$ such that $\Pi_2^+ \iota_Y \Omega = d\alpha^+$ and $\Pi_2^- \iota_Y \Omega = 0$, where $\Pi_1^+ \beta =: \alpha^+ \in \text{Res}^{1, \infty}$ is defined for some β in $C^\infty(\mathcal{M}; \Omega^1)$ with $\iota_{X_0}\beta = 1$ (the analogous construction for $\text{Res}_*^{1, \infty}$ was carried out in Proposition 3.10)

and so α^+ satisfies $\iota_{X_0}\alpha^+ = 1$. Then the first variation formula (9-21) above gives, using also (9-10),

$$D_{X_0}\mathcal{H}(Y) = \int_{\mathcal{M}} \alpha^+ \wedge d\tau^- + 0 = 1,$$

since $\alpha^+ \wedge d\tau^- = \Omega_{\text{SRB}}^-$, which shows that the derivative $D_{X_0}\mathcal{H}$ is surjective onto $\mathbb{R}$ and the main claim follows from the implicit function theorem. $\square$

Remark 9.8 If we restrict to $\mathcal{V}_\Omega = \{X \in C^N(\mathcal{M}; T\mathcal{M}) \mid \mathcal{L}_X \Omega = 0\}$, ie flows preserving the fixed smooth volume form Ω , then it is straightforward (even much simpler than the dissipative case, since there is no need for anisotropic spaces) to see that

- (1) the manifolds $\mathcal{W} := \mathcal{W}^+ \cap \mathcal{V}_\Omega = \mathcal{W}^- \cap \mathcal{V}_\Omega \subset \mathcal{V}_\Omega$ are of codimension $b_1(\mathcal{M})$ (follows from (5-7) and (5-8), and the implicit function theorem), and
- (2) the first derivative of the helicity at X_0 is nonzero, so $\{X \in \mathcal{W} \mid \mathcal{H}(X) = 0\} \subset \mathcal{W}$ is locally a C^1 -regular Banach manifold (follows from (9-20) by noting that $\Omega_{\text{SRB}}^+ = \Omega_{\text{SRB}}^- = \Omega$ is fixed).

Appendix A Revision of elementary facts about $\eta_\pm$

Here we recall some properties of the operators $\eta_\pm$ introduced in Section 2.4; we will follow the notation introduced in that section. Let (x, y) denote local isothermal coordinates on $U \subset M$, so $g|_U = e^{2\psi}(dx^2 + dy^2)$. Then on $SM|_U$ we have another set of coordinates,

$$(x, y, \theta) \mapsto (x, y, e^{-\psi} \cos \theta, e^{-\psi} \sin \theta) \in SM|_U \quad \text{for } (x, y, \theta) \in U \times \mathbb{S}^1.$$

In these coordinates, Merry and Paternain [2011, page 36] show that

$$\begin{aligned} X(x, y, \theta) &= e^{-\psi} (\cos \theta \cdot \partial_x + \sin \theta \cdot \partial_y + (-\partial_x \psi \cdot \sin \theta + \partial_y \psi \cdot \cos \theta) \partial_\theta), \\ H(x, y, \theta) &= e^{-\psi} (-\sin \theta \cdot \partial_x + \cos \theta \cdot \partial_y - (\partial_x \psi \cdot \cos \theta + \partial_y \psi \cdot \sin \theta) \partial_\theta), \end{aligned} \tag{A-1}$$

so that we compute $\eta_\pm = \frac{1}{2}(X \mp iH)$, $\partial/\partial\bar{z} = \frac{1}{2}(\partial_x + i\partial_y)$, $\partial/\partial z = \frac{1}{2}(\partial_x - i\partial_y)$:

$$\eta_-(x, y, \theta) = e^{-\psi} e^{-i\theta} \left(\frac{\partial}{\partial\bar{z}} - i \frac{\partial\psi}{\partial\bar{z}} \cdot \partial_\theta \right), \quad \eta_+(x, y, \theta) = e^{-\psi} e^{i\theta} \left(\frac{\partial}{\partial z} + i \frac{\partial\psi}{\partial z} \cdot \partial_\theta \right). \tag{A-2}$$

Now let A be a section of $\mathcal{K}^{\otimes m}$, where $\mathcal{K}$ are the holomorphic 1-forms (spanned locally by dz). Then the pullback π^*A is a section of $\pi^*\mathcal{K}^{\otimes m} \subset \otimes^m T^*(SM)$ (here the pullback is in the sense of tensor pullback and we recall $\pi: SM \rightarrow M$ is the projection), and in fact we claim:

Lemma A.1 *The bundle $\pi^*\mathcal{K} \subset T^*(SM)$ is spanned by $\alpha + i\beta$.*

Proof To see this, denote by J the complex structure on the surface (M, g) , and note that by definition, for any $u \in T_{(x,v)}SM$ we have

$$\begin{aligned} \alpha(x, v)(u) &= g_x(v, d\pi(x, v)u) = \pi^*(g_x(v, \bullet))(u), \\ \beta(x, v)(u) &= g_x(Jv, d\pi(x, v)u) = \pi^*(g_x(Jv, \bullet))(u), \end{aligned}$$

so α and β indeed define sections of $\pi^*\mathcal{K}$, and we get

$$(\alpha + i\beta)(x, v)(u) = g_x(v + iJ(x)v, d\pi(x, v)u).$$

It is easy to see, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, that

$$g_x(v + iJv, \partial_x) = e^{2\psi}(v_x - iv_y), \quad g_x(v + iJv, \partial_y) = e^{2\psi}(v_y + iv_x) = ie^{2\psi}(v_x - iv_y),$$

so we conclude that we get

$$g_x(v + iJv, \bullet) = e^{2\psi}(v_x - iv_y) dz,$$

and finally, using that $e^{2\psi}(v_x^2 + v_y^2) = 1$,

$$(A-3) \quad (\alpha + i\beta)(x, v) = \frac{\pi^* dz}{dz(v)},$$

which proves the claim. $\square$

Therefore if $A = fdz^m$ in U , we may write, using (A-3),

$$\pi^* A = \pi_0^* f \cdot (\pi^* dz)^m = \pi_0^* f \cdot (dz(v))^m \cdot (\alpha + i\beta)^m = \overbrace{\pi_0^* f \cdot e^{-m\psi} e^{im\theta}}^{\tilde{a}:=} \cdot (\alpha + i\beta)^m,$$

where the function $\tilde{a}$ defined by the local expression extends globally to SM (since α and β are global).

Since $\pi_m^*(fdz^m)(z, v) = f(z) \cdot (dz(v))^m = \tilde{a}(z, v)$, it follows that $\tilde{a} = \pi_m^* A$, so

$$\pi^* A = \pi_m^* A \cdot (\alpha + i\beta)^m.$$

In [Mettler and Paternain 2019], the authors write $\tilde{a} = (Va/m) + ia = 2ia_m$, where $a = a_{-m} + a_m$ is real-valued (ie $a_{-m} = \bar{a}_m$), from which it follows that

$$(A-4) \quad a_m = \frac{\pi_m^* A}{2i}.$$

Therefore we obtain the relation

$$\lambda = a = a_m + \bar{a}_m = 2\operatorname{Re}(a_m) = 2\operatorname{Re}\left(\frac{\pi_m^* A}{2i}\right) = -\operatorname{Re}(i\pi_m^* A) = \operatorname{Im}(\pi_m^* A)$$

between $\lambda := a$ and A . Now using the convention that $|dz|^2 = \frac{1}{2}(|dx|^2 + |dy|^2) = e^{-2\psi}$, we get

$$|A(z)|_g^2 = |f(z)|^2 \cdot |dz|^{2m} = |f(z)|^2 \cdot e^{-2m\psi},$$

and using $dz(v) = e^{-\psi} e^{i\theta}$,

$$4|a_m(z, v)|^2 = |\pi_m^* A(z, v)|^2 = |f(z)|^2 \cdot |dz(v)|^{2m} = |f(z)|^2 e^{-2m\psi}.$$

We conclude with the relation between the norms:

$$(A-5) \quad 4|a_m(z, v)|^2 = |f(z)|^2 \cdot e^{-2m\psi} = |A|_g^2.$$

Next, we compute explicitly the expressions in local coordinates for the operators $\eta_{\pm}$ acting on tensor powers of the canonical bundle $\mathcal{K}$.

Lemma A.2 The following identity holds for $m \geq 0$:

$$\eta_- \pi_m^*(f dz^m) = \pi_{m-1}^* \bar{\partial}(f dz^m).$$

Equivalently, we have

$$\begin{aligned} \eta_- \pi_m^*(f dz^m) &= \pi_0^* \left(\frac{\partial f}{\partial \bar{z}} \right) e^{-2\psi} \cdot \pi_{m-1}^*(dz^{m-1}), \\ \eta_+ \pi_m^*(f dz^m) &= e^{2m\psi} \frac{\partial(f e^{-2m\psi})}{dz} \cdot \pi_{m+1}^*(dz^{m+1}). \end{aligned}$$

Therefore $\bar{\partial}A = 0$ if and only if $\eta_- \pi_m^* A = 0$. If $m \geq 1$, then $\bar{\partial}A = 0$ if and only if $XVa = mHa$, or alternatively if and only if $HVa + mXa = 0$.

Here

$$\bar{\partial}(f dz^m) = \frac{\partial f}{\partial \bar{z}} \cdot d\bar{z} \otimes dz^m$$

is well-defined since dz is a holomorphic section of the holomorphic vector bundle $\mathcal{K}$.

Proof Using (A-2), and $dz(v) = e^{-\psi} e^{i\theta}$, compute

$$\begin{aligned} \eta_- \pi_m^*(f dz^m) &= \eta_-(\pi_0^* f e^{-m\pi_0^* \psi} e^{im\theta}) = \eta_-(\pi_0^* f) \cdot \pi_m^*(dz^m) + \pi_0^* f \cdot \eta_-(e^{-m\pi_0^* \psi} e^{im\theta}) \\ &= \pi_0^* \left(\frac{\partial f}{\partial \bar{z}} \right) \cdot \underbrace{e^{-(m+1)\pi_0^* \psi} e^{i(m-1)\theta}}_{= e^{-2\pi_0^* \psi} \pi_{m-1}^*(dz^{m-1})} + \pi_0^* f \cdot \eta_-(e^{-m\pi_0^* \psi} e^{im\theta}). \end{aligned}$$

We claim that $\eta_-(e^{-m\psi} e^{im\theta}) = 0$. Indeed, we compute using (A-2) that

$$\eta_-(e^{-m\psi} e^{im\theta}) = e^{-(m+1)\psi} e^{i(m-1)\theta} \cdot \left(-m \frac{\partial \psi}{\partial \bar{z}} - i \frac{\partial \psi}{\partial \bar{z}} \cdot im \right) = 0.$$

Next, compute the right-hand side

$$\begin{aligned} \pi_{m-1}^* \bar{\partial}(f dz^m) &= \pi_0^* \left(\frac{\partial f}{\partial \bar{z}} \right) \cdot \pi_1^*(d\bar{z}) \cdot \pi_1^*(dz)^m = \pi_0^* \left(\frac{\partial f}{\partial \bar{z}} \right) \cdot e^{-(m+1)\psi} \cdot e^{i(m-1)\theta} \\ &= \pi_0^* \left(\frac{\partial f}{\partial \bar{z}} \right) \cdot e^{-2\psi} \pi_{m-1}^*(dz^{m-1}), \end{aligned}$$

which completes the proof of the first identity. The other identity follows analogously:

$$\begin{aligned} \eta_+ \pi_m^*(f dz^m) &= \eta_+(\pi_0^* f) \cdot \pi_m^*(dz^m) + \pi_0^* f \cdot \eta_+(e^{-m\psi} e^{im\theta}) \\ &= e^{-(m+1)\psi} e^{i(m+1)\theta} \pi_0^* \left(\frac{\partial f}{\partial z} \right) + \pi_0^* f \cdot e^{-\psi} e^{i\theta} e^{-m\psi} e^{im\theta} \left(-m \frac{\partial \psi}{\partial z} + i \frac{\partial \psi}{\partial z} \times im \right) \\ &= e^{-(m+1)\psi} e^{i(m+1)\theta} \left(\pi_0^* \left(\frac{\partial f}{\partial z} \right) - 2m\pi_0^* f \times \frac{\partial \psi}{\partial z} \right) \\ &= \pi_{m+1}^*(dz^{m+1}) \cdot e^{2m\psi} \frac{\partial(f e^{-2m\psi})}{\partial z}. \end{aligned}$$

For the final conclusion, observe

$$\begin{aligned} XVa - mHa = 0 &\iff X(ia_m - ia_{-m}) - H(a_m + a_{-m}) = 0 \\ &\iff (X + iH)a_m - (X - iH)a_{-m} = 0, \end{aligned}$$

which holds if and only if $\eta_- a_m = 0$, which by (A-4) completes the proof. The other equivalence is similarly obtained. $\square$

Recall that X_- on $H_{-1} \oplus H_1$ is defined by $X_-(f_{-1} + f_1) = \eta_+ f_{-1} + \eta_- f_1$. Then:

Proposition A.3 *It holds that*

$$X_- \pi_1^* \gamma = -\frac{1}{2} \pi_0^* d^* \gamma \quad \text{for all } \gamma \in C^\infty(SM; \Omega^1).$$

Proof It suffices to consider the operator η_- and in local isothermal coordinates, $\gamma = f dz$. By Lemma A.2,

$$\eta_- \pi_1^*(f dz) = e^{-2\psi} \pi_0^* \left(\frac{\partial f}{\partial \bar{z}} \right).$$

On the other hand, we compute

$$\begin{aligned} \pi_0^* d^*(f dz) &= -\pi_0^*(\star d \star f dz) = i \pi_0^*(\star df \wedge dz) \\ &= i \pi_0^* \left(\star \left(\frac{\partial f}{\partial \bar{z}} \right) 2i dx \wedge dy \right) = -2e^{-2\psi} \pi_0^* \left(\frac{\partial f}{\partial \bar{z}} \right). \end{aligned}$$

Here we used that in isothermal coordinates $\star dz = -i dz$, $d\bar{z} \wedge dz = 2i dx \wedge dy$, and that the volume form is equal to $e^{2\psi} dx \wedge dy$. $\square$

Proposition A.4 *We have*

$$\dim \ker(\eta_-|_{H_0}) = \dim \ker(\eta_+|_{H_0}) = 1.$$

Moreover, it holds that

$$\dim \ker(\eta_-|_{H_1}) = \dim \ker(\eta_+|_{H_{-1}}) = \frac{1}{2} b_1(M).$$

Proof For the first claim, by (2-13) note that $f \in \ker(\eta_-|_{H_0})$ if and only if $df = i \star df$. It follows that $\|df\|_{L^2(M)} = 0$ and so f is constant. For the other case note that $\ker(\eta_+|_{H_0})$ is obtained by conjugation from $\ker(\eta_-|_{H_0})$.

Next, we claim that the map

$$(A-6) \quad \mathcal{H}^1(M) \rightarrow \ker \eta_+|_{H_{-1}} \oplus \ker \eta_-|_{H_1}, \quad \gamma \mapsto (f_{-1}, f_1),$$

is well-defined and an isomorphism, where $\pi_1^* \gamma = f_{-1} + f_1$ is the splitting of $\pi_1^* \gamma$ into Fourier modes. Since complex conjugation provides an isomorphism between $\ker \eta_+|_{H_{-1}}$ and $\ker \eta_-|_{H_1}$, the claim will follow.

To see (A-6), by Proposition A.3 we have

$$(A-7) \quad \begin{aligned} -\frac{1}{2}\pi_0^*(d^*\gamma) &= X-\pi_1^*\gamma = \eta_+f_{-1} + \eta_-f_1, \\ -\frac{1}{2}\pi_0^*(d^*\star\gamma) &= X-\pi_1^*\star\gamma = -X-V\pi_1^*\gamma = -i(-\eta_+f_1 + \eta_-f_1), \end{aligned}$$

where in the second line we also use $V\pi_1^* = -\pi_1^*\star$. Therefore, γ is both closed and coclosed if and only if $\eta_-f_1 = \eta_+f_{-1} = 0$. This shows that the map (A-6) is well-defined and moreover an isomorphism, completing the proof. $\square$

Appendix B Conformal rescaling of the geodesic vector field

In this appendix we study the behaviour of the geodesic vector field under a conformal rescaling of a Riemannian metric on a surface. Let $g_1 = e^{-2f}g$ be a conformal scaling of the metric of the surface (M, g) , for some $f \in C^\infty(M)$. Consider the scaling diffeomorphism

$$\ell_1: SM \rightarrow SM_1, \quad (x, v) \mapsto \left(x, \frac{v}{\|v\|_{g_1}}\right) = (x, e^f v),$$

where SM_1 denotes the unit sphere bundle of (M, g_1) . Let X_1 be the geodesic vector field on SM_1 and denote by π_1 the footpoint projection $SM_1 \rightarrow M$. Moreover, denote by $\alpha_1, \beta_1, \psi_1$ the global frame constructed in Section 2.4 on SM_1 . Then:

Lemma B.1 For $(x, v) \in SM$, we have

$$X_f := \ell_1^*X_1 = e^f X + \pi_1^*(\star d(e^f))V, \quad \ell_1^*V_1 = V.$$

In particular, X_f is a time-change by e^f of the thermostat flow $X - V(\pi_1^*(df))V$ (as $\pi_1^*\star = -V\pi_1^*$).

Proof Let us compute $\ell_1^*\psi_1, \ell_1^*\alpha_1, \ell_1^*\beta_1$ in terms of ψ, α, β . We start with α_1 :

$$\ell_1^*\alpha_1(x, v)(\xi) = \alpha_1(x, e^f v)(d\ell_1(x, v)\xi) = \underbrace{\langle d\pi_1 \circ d\ell_1(x, v)\xi, e^f v \rangle_{g_1}}_{=d\pi} = e^{-f}\alpha(x, v)(\xi).$$

Then we have, if $\beta(x, v)(\xi) = \langle d\pi(x, v)\xi, iv \rangle_g$,

$$\ell_1^*\beta_1(x, v)(\xi) = \beta_1(x, e^f v)(d\ell_1(x, v)\xi) = \langle d\pi_1 \circ d\ell_1(x, v)\xi, ie^f v \rangle_{g_1} = e^{-f}\beta(x, v)(\xi).$$

Recall that the 1-form ψ is defined as

$$\psi(x, v)(\xi) = \langle \mathcal{K}(x, v)\xi, iv \rangle_g,$$

where $\mathcal{K}$ is the connection map, defined by

$$\mathcal{K}(x, v)(\xi) = \frac{DZ}{dt}(0) \in T_x M,$$

where $c: (-\varepsilon, \varepsilon) \rightarrow TM$ is any curve satisfying $c(0) = (x, v)$, $\dot{c}(0) = \xi$ and $c(t) = (\gamma(t), Z(t))$, and D/dt is the covariant derivative along γ with respect to the Levi-Civita connection ∇ of (M, g) . Let

∇^1 be the Levi-Civita connection of (M, g_1) . Then by Koszul's formula we have, for arbitrary vector fields Y and T (see [Lee 2018, Proposition 7.29]),

$$\nabla_Y^1 T = \nabla_Y T - Yf \cdot T - Tf \cdot Y + g(Y, T)\nabla f.$$

It follows that

$$\frac{D^1 Z(t)}{dt} = \nabla_{\dot{\gamma}}^1 Z = \nabla_{\dot{\gamma}} Z - df(\dot{\gamma}) \cdot Z - df(Z) \cdot \dot{\gamma} + \langle \dot{\gamma}, Z \rangle \cdot \nabla f.$$

Recall that $Z(0) = v$ and $\dot{\gamma}(0) = d\pi(x, v)\xi$, and observe that $\ell_1 \circ c(t) = (\gamma(t), e^f Z(t))$ is a curve adapted to $(x, e^f v)$ and $d\ell_1(x, v)\xi$. Therefore

$$\begin{aligned} \ell_1^* \psi_1(x, v)(\xi) &= \langle \mathcal{K}_1(x, e^f v)(d\ell_1(x, v)\xi), i e^f v \rangle_{g_1} = e^{-f} \left\langle \frac{D^1(e^f Z)}{dt} \Big|_{t=0}, i v \right\rangle_g = \left\langle \frac{D^1 Z}{dt} \Big|_{t=0}, i v \right\rangle_g \\ &= \left\langle \frac{DZ}{dt} \Big|_{t=0}, i v \right\rangle_g - 0 - df(x, v) \cdot \langle d\pi(x, v)\xi, i v \rangle_g + \langle d\pi(x, v)\xi, v \rangle_g \cdot \langle \nabla f(x), i v \rangle_g \\ &= \psi(x, v)(\xi) - df(x, v) \cdot \beta(x, v)(\xi) + df(x, i v) \cdot \alpha(x, v)(\xi), \end{aligned}$$

where we used that $Z(0) = v \perp i v$ in the third and fourth equalities. Since $df(x, i v) = -\star df(x, v)$, as both i and $\star$ are rotations by $\pi/2$ counterclockwise, we conclude that

$$\ell_1^* \psi_1 = \psi - \pi_1^*(df)\beta - \pi_1^*(\star df)\alpha.$$

Now we use that

$$\ell_1^* \alpha_1(\ell_1^* X_1) = e^{-f} \alpha(\ell_1^* X_1) = 1, \quad \ell_1^* \beta_1(\ell_1^* X_1) = \ell_1^* \psi_1(\ell_1^* X_1) = 0,$$

to conclude that writing $\ell_1^* X_1 = aX + bH + cV$, by the above expressions,

$$a = e^f, \quad b = 0, \quad c = \pi_1^*(\star df)e^f = \pi_1^*(\star de^f),$$

which completes the proof of the first claim. The formula for $\ell_1^* V_1$ readily follows as well, completing the proof. $\square$

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Quantisation of derived Poisson structures

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We prove that every 0-shifted Poisson structure on a derived Artin n -stack admits a curved A_∞ deformation quantisation whenever the stack has perfect cotangent complex; in particular, this applies to LCI schemes, where it gives a DQ algebroid quantisation. Whereas the Kontsevich–Tamarkin approach to quantisation for smooth varieties hinges on invariance of the Hochschild complex under affine transformations, we instead exploit the observation that the Hochschild complex carries an anti-involution, and that such anti-involutive deformations of the complex of polyvectors are essentially unique. We also establish analogous statements for deformation quantisations in C^∞ and analytic settings.

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Introduction

A deformation quantisation of a Poisson structure on a geometric object Y is a noncommutative deformation, parametrised by power series in $\hbar$, of the functions $\mathcal{O}_Y$ on Y , such that the classical limit $\hbar \rightarrow 0$ recovers the Poisson structure. Classically, this means looking at associative deformations $\star_\hbar$ of the multiplication on $\mathcal{O}_Y$, with the Poisson bracket then given by $\{a, b\} := \lim_{\hbar \rightarrow 0} (a \star_\hbar b - b \star_\hbar a)/\hbar$. In algebro-geometric settings, the local deformation quantisations tend not to glue strictly, leading to the notion of DQ algebroid quantisations as deformations of the algebra regarded as a linear category on one object, so the cocycle condition only holds up to an inner automorphism satisfying further coherence relations. In derived geometry, $\mathcal{O}_Y$ is homologically enriched so quantisations have to be formulated in terms of more exotic algebraic structures.

By [Kontsevich 2003], every Poisson manifold admits a deformation quantisation. For smooth algebraic varieties in characteristic 0, Kontsevich [2001] and Yekutieli [2005] proved an analogous statement, showing that all Poisson structures admit DQ algebroid quantisations. Via local choices of connections, the question reduced to constructing quantisations of affine space. These could then be handled as in [Tamarkin 1999; Kontsevich 1999; Yekutieli 2015; Van den Bergh 2007]: formality of the E_2 operad

associates to the Hochschild complex a deformation of the P_2 -algebra of multiderivations, and invariance under affine transformations ensures that it is the unique deformation. The same argument extends to graded manifolds [Cattaneo and Felder 2007], but requires the differential to be trivial.¹

We now consider generalisations of this question to singular schemes, and more generally to derived stacks in characteristic 0, considering quantisations of 0-shifted Poisson structures in the sense of [Pridham 2017; Calaque et al. 2017] and their analytic and $\mathcal{C}^\infty$ analogues [Pridham 2020a; 2020b]. For positively shifted structures, the analogous question is a formality, following from the equivalence $E_{n+1} \simeq P_{n+1}$ of operads. Quantisations for nondegenerate 0-shifted Poisson structures were established in [Pridham 2018], and we now consider degenerate quantisations as well, addressing the remaining unsolved case of [Toën 2014, Conjecture 5.3],² long regarded as the hardest.³

The construction of nondegenerate quantisations in [Pridham 2018; 2022] only relied on the fact that the Hochschild complex is an anti-involutive deformation of the complex of multiderivations. Our strategy here is closer to [Tamarkin 1999; Kontsevich 1999] in that we establish an equivalence between the two complexes. As in [Pridham 2018; 2022], the key observation is still that the Hochschild complex of a differential graded-commutative algebra (CDGA) carries an anti-involution corresponding to the endofunctor on deformations sending an algebra to its opposite. Via a formality quasi-isomorphism for the E_2 operad corresponding to an even associator, the Hochschild complex becomes an anti-involutive deformation of the P_2 -algebra of multiderivations.

We show (Corollary 1.19) that such deformations are essentially unique whenever the complexes of polyvectors and of multiderivations are quasi-isomorphic. This condition is satisfied when the CDGA has perfect cotangent complex, and so gives the following:

Theorem 2.21 *Let A be a CDGA, $\mathcal{C}^\infty$ -DGA or DGA with entire functional calculus. Whenever A has perfect cotangent complex in the relevant theory, any choice of even 1-associator yields a natural quasi-isomorphism*

$$D_{\oplus}^{\text{poly}}(A)_{[-1]} \simeq \bigoplus_{p \geq 0} \mathbf{R}\underline{\text{Hom}}_A(\mathbf{L}\Omega_A^p, A)_{[p-1]}$$

of differential graded Lie algebras (DGLAs) between the relevant complex of polydifferential operators and the algebra of derived polyvectors, compatible with canonical filtrations and involutions. In the CDGA case, we can replace $D_{\oplus}^{\text{poly}}(A)$ with the cohomological Hochschild complex $\text{CC}_{R, \oplus}(A)$.

¹Shortly after this manuscript first appeared, [Liao et al. 2018] established a Kontsevich–Duflo formality quasi-isomorphism for finite-dimensional dg $\mathcal{C}^\infty$ -manifolds, indirectly induced from formality for graded manifolds. A similar argument should apply to derived affine schemes with perfect cotangent complexes.

²Although the statement for derived DM stacks was specifically claimed in [Toën 2014, Theorem 5.4], it is not even stated in the reference provided, whose proof is only relevant to strictly positive shifts, and it lacks the constraints on the cotangent complex needed to rule out known affine counterexamples.

³See for instance the survey [Pantev and Vezzosi 2018, Section 3], but beware that deformation quantisations for negative shifts, starting with the BV quantisations of [Pridham 2019], take a far more subtle form than proposed there, and that the declared aim of [Pantev and Vezzosi 2018, Section 2] is somewhat moot given the uncited [Pridham 2017].

The following underived consequence is a special case of [Corollary 2.27](#) (applied to a cofibrant replacement and taking H_0 of the output). It was previously known only for nonsingular spaces.

Corollary *Take the ring A of functions on an affine scheme, affine C^∞ -space, or affine analytic space, with only local complete intersection (LCI) singularities, together with a Poisson structure $\{-, -\}$ on A . There then exist associative deformations $\star_{\hbar}$ of the product on $A[[\hbar]]$ satisfying*

$$\{a, b\} \equiv (a \star_{\hbar} b - b \star_{\hbar} a)/\hbar \pmod{\hbar}.$$

The coefficients of $\star_{\hbar}$ are given by differential operators, and the deformation can be chosen to be anti-involutive in the sense that $b \star_{\hbar} a = a \star_{-\hbar} b$.

[Theorem 2.21](#) enjoys sufficient functoriality to yield the following global consequence (a special case of [Corollary 2.31](#), which also considers derived objects and gives a complete parametrisation):

Corollary *Given an algebraic, C^∞ , or analytic space or Deligne–Mumford stack $\mathfrak{X}$ with only LCI singularities, every Poisson structure on the sheaf $\mathcal{O}_{\mathfrak{X}}$ of functions admits DQ algebroid deformations $\mathcal{A}_{\hbar}$ of $\mathcal{O}_{\mathfrak{X}}[[\hbar]]$. These can be chosen to be self-dual in the sense that $\mathcal{A}_{-\hbar} \simeq \mathcal{A}_{\hbar}^{\text{opp}}$.*

Deformation quantisations of 1-shifted coisotropic structures also follow as an immediate consequence ([Corollary 2.36](#)) of [Theorem 2.21](#). A 1-shifted coisotropic structure on a morphism $\mathfrak{X} \rightarrow \mathfrak{Y}$ is a notion corresponding to a 1-shifted Poisson structure ϖ on $\mathfrak{Y}$ acting on a 0-shifted Poisson structure π on $\mathfrak{X}$. The deformation quantisations established in [Corollary 2.36](#) when $\mathfrak{X}$ has perfect cotangent complex consist of an almost commutative E_2 -algebra deformation of $(\mathcal{O}_{\mathfrak{Y}}, \varpi)$ acting appropriately on a deformation quantisation of $(\mathcal{O}_{\mathfrak{X}}, \pi)$.

In [Section 3](#), we extend these results to derived Artin n -stacks. This follows by essentially the same argument, but is much more technically complicated because of the subtleties in formulating polyvectors and Hochschild complexes for Artin stacks. Locally, these are defined as Tate (ie sum–product) realisations Tôt of double complexes arising from formally étale charts by stacky thickenings of derived affine schemes (Lie algebroids, broadly speaking). Those total complexes do not satisfy the conditions of [Corollary 1.19](#), so we introduce an intermediate category through which Tôt factorises, and in which the P_2 -algebra of polyvectors has no nontrivial involutive deformations.

For stacky thickenings of derived affine schemes, this leads to an equivalence ([Theorem 3.35](#)) between polyvectors and polydifferential operators or the Hochschild complex, generalising [Theorem 2.21](#) above. Via a form of étale functoriality, that yields the following corollary, where E_1 quantisations of $\mathfrak{X}$ are almost commutative curved A_∞ -algebra deformations of the rings of functions on formally étale stacky derived affine charts.

Corollary 3.38 *Given a derived Artin n -stack $\mathfrak{X}$ (algebraic, C^∞ , or analytic) with perfect cotangent complex, any even associator yields a map from the space of 0-shifted Poisson structures on $\mathfrak{X}$ to the space of self-dual E_1 quantisations of $\mathfrak{X}$.*

These quantisations give rise to curved A_∞ deformations of the dg category of perfect $\mathcal{O}_{\mathfrak{X}}$ -complexes, $\hbar$ -semilinearly anti-involutive with respect to the dg endofunctor $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(-, \mathcal{O}_{\mathfrak{X}})$, whenever $\mathfrak{X}$ is strongly quasiconvex (ie quasiconvex, quasiseparated ...).

Any 1-shifted Lagrangian structure on a morphism gives rise to a 1-shifted coisotropic structure by [Melani and Safronov 2018, Theorem 4.22]. Examples of these from [Calaque 2015] include quotient stacks $[Y/G]$ when Y is equipped with a Hamiltonian or quasi-Hamiltonian structure (Lagrangian over $[\mathfrak{g}^*/G]$ and $[G/G^{\text{ad}}]$, respectively), and the derived moduli stack $\text{Perf}(\Sigma)$ of perfect complexes on a del Pezzo surface Σ (Lagrangian over $\text{Perf}(E)$ for an elliptic curve $E \subset \Sigma$). A 1-shifted coisotropic structure on a morphism gives a 0-shifted Poisson structure on its source, to which Corollary 3.38 can then be applied. We can however say more, with Corollary 3.41 showing that, when the source has perfect cotangent complex, 1-shifted coisotropic structures admit deformation quantisations in the form of an E_2 quantisation of the target acting on an E_1 quantisation of the source.

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Notation and terminology We write CDGAs (commutative differential graded algebras) and DGAs (differential graded associative algebras) as chain complexes (homological grading), and denote the differential on a chain complex by δ .

The graded vector space underlying a chain (resp. cochain) complex V is denoted by $V_\#$ (resp. $V^\#$). Since we often have to work with chain and cochain structures separately, we denote shifts by subscripts and superscripts, respectively, so $(V_{[i]})_n := V_{i+n}$ and $(V^{[i]})^n := V^{i+n}$.

Given a DGAA A , and A -modules M and N in chain complexes, we write $\underline{\text{Hom}}_A(M, N)$ for the chain complex given by

$$\underline{\text{Hom}}_A(M, N)_i = \text{Hom}_{A_\#}(M_\#, N_{\#[i]}),$$

with differential $\delta f = \delta_N \circ f \pm f \circ \delta_M$, where $V_\#$ denotes the graded vector space underlying a chain complex V .

When we need to compare chain and cochain complexes, we make use of the equivalence u from chain complexes to cochain complexes given by $(uV)^i := V_{-i}$, and refer to this as rewriting the chain complex as a cochain complex (or vice versa). On suspensions, this has the effect that $u(V_{[n]}) = (uV)^{[-n]}$.

1 Involutively filtered deformations of Poisson algebras

We will assume that all filtrations are increasing and exhaustive, unless stated otherwise. An action of the algebraic group $\mathbb{G}_m$ on a vector space V is equivalent to a decomposition $V = \bigoplus_i \mathcal{W}_i V$; given a $\mathbb{G}_m$ -equivariant vector space V , we then define a weight filtration W on V by setting $W_n V := \bigoplus_{i \leq n} \mathcal{W}_i V$. For $\mathbb{G}_m$ -equivariant complexes U and V , we write $\mathcal{W}_i \underline{\text{Hom}}(U, V)$ for the complex $\prod_j \underline{\text{Hom}}(\mathcal{W}_j U, \mathcal{W}_{i+j} V)$

of homomorphisms of weight i , with similar conventions for complexes of derivations etc; beware that the inclusion $\bigoplus_i \mathcal{W}_i \underline{\mathrm{Hom}}(U, V) \rightarrow \underline{\mathrm{Hom}}(U, V)$ is not surjective in general.

1.1 Involutively filtered deformations of $\mathcal{P}$ -algebras

We now adapt an idea developed in [Pridham 2016, Section 14.3.2], using Rees constructions to interpret the problem of recovering a filtered algebra from its associated graded algebra as a deformation problem.

Definition 1.1 We say that a vector space V is involutively filtered if it is equipped with a filtration W and an involution e which preserves W and acts on $\mathrm{gr}_i^W V$ as multiplication by $(-1)^i$.

We say that a chain complex $V_\bullet$ is quasi-involutively filtered if it is equipped with a filtration W by subcomplexes and an involution e which preserves W and acts on $H_*(\mathrm{gr}_i^W V)$ as multiplication by $(-1)^i$.

Observe that if V is involutively filtered, then the involution gives an eigenspace decomposition $V = V^{e=1} \oplus V^{e=-1}$. Because $(\mathrm{gr}_{i-1}^W V)^{e=(-1)^i} = 0$, we have $W_{2j+1} V^{e=1} = W_{2j} V^{e=1}$ and $W_{2j} V^{e=-1} = W_{2j-1} V^{e=-1}$. If $V_\bullet$ is quasi-involutively filtered, we similarly have that $W_{2j} V^{e=1} \subset W_{2j+1} V^{e=1}$ and $W_{2j-1} V^{e=-1} \subset W_{2j} V^{e=-1}$ are quasi-isomorphic subcomplexes.

Definition 1.2 Given a quasi-involutively filtered chain complex $(V_\bullet, W, e)$, define the involutively filtered chain complex $(V_\bullet, W^e, e)$ by setting

$$W_n^e V_\bullet := (W_n V_\bullet)^{e=(-1)^n} \oplus (W_{n-1} V_\bullet)^{e=(-1)^{n-1}}.$$

Lemma 1.3 After localisation at filtered quasi-isomorphisms, the inclusion functor from involutively filtered chain complexes to quasi-involutively filtered chain complexes gives an equivalence of ∞ -categories.

Proof We can identify the involutively filtered objects as those $(V_\bullet, W, e)$ for which the natural transformation $\varepsilon_V: (V_\bullet, W^e, e) \rightarrow (V_\bullet, W, e)$ is an isomorphism, so the functor $(V_\bullet, W, e) \mapsto (V_\bullet, W^e, e)$ is right adjoint to the inclusion functor, with ε the counit of the adjunction.

It remains to show that ε_V is a filtered quasi-isomorphism for all quasi-involutively filtered chain complexes. Observe that $W_{n-1} V_\bullet$ is contained in $W_n^e V_\bullet$, with quotient isomorphic to $\mathrm{gr}_n^W V_\bullet^{e=(-1)^n}$, which implies that the inclusion $W_n^e V_\bullet \hookrightarrow W_n V_\bullet$ is indeed a quasi-isomorphism by quasi-involutivity of W . $\square$

Definition 1.4 Define the $\mathbb{G}_m$ -equivariant algebra $\mathbb{Q}[\hbar]$ by setting $\hbar$ to have weight 1 for the $\mathbb{G}_m$ -action (and implicitly homological degree 0).

Definition 1.5 For a quasi-involutively filtered chain complex $(V, W_i V, e)$ over $\mathbb{Q}$, we define

$$\mathrm{Rees}_W^e(V) := \bigoplus_n \hbar^n \{v \in W_n V : e(v) = (-1)^n v\}.$$

This is a $\mathbb{G}_m$ -equivariant chain complex of $\mathbb{Q}[\hbar^2]$ -modules, where the $\mathbb{G}_m$ -action is induced by the action on $\hbar$.

Lemma 1.6 The involutive Rees functor of [Definition 1.5](#) gives an equivalence between the category of involutively filtered $\mathbb{Q}$ -vector spaces and the category of flat $\mathbb{G}_m$ -equivariant $\mathbb{Q}[\hbar^2]$ -modules.

For quasi-involutively filtered complexes (V, W, e) , it also satisfies $\text{Rees}_W^e(V) \cong \text{Rees}_W^e(V)$, and so factorises the ∞ -equivalence of [Lemma 1.3](#).

Proof We adapt the equivalence between exhaustively filtered vector spaces and flat $\mathbb{G}_m$ -equivariant $\mathbb{Q}[\hbar]$ -modules from for instance [\[Pridham 2016, Lemma 2.1\]](#).

The functor Rees^e has a left adjoint, which sends a $\mathbb{G}_m$ -equivariant $\mathbb{Q}[\hbar^2]$ -module M in complexes to the complex $M/(\hbar^2 - 1)$ equipped with involution $-1 \in \mathbb{G}_m$ and filtration

$$W_n(M/(\hbar^2 - 1)) := \left(\bigoplus_{i \leq n} \mathcal{W}_i M \right) / (\hbar^2 - 1) \left(\bigoplus_{i \leq n-2} \mathcal{W}_i M \right) \cong \mathcal{W}_n M \oplus \mathcal{W}_{n-1} M;$$

flatness of M ensures that the map $W_n(M/(\hbar^2 - 1)) \rightarrow M/(\hbar^2 - 1)$ is indeed an inclusion.

Evaluation at $\hbar = 1$ gives a morphism $\text{Rees}_W^e(V)/(\hbar^2 - 1) \rightarrow V$, the counit of the adjunction. When V is involutively filtered, this map is a filtered isomorphism. Even when V is not involutively filtered, it follows easily that $W_n(\text{Rees}_W^e(V)/(\hbar^2 - 1)) \cong W_n^e V$. Thus the involutive Rees functor factorises the ∞ -equivalence of [Lemma 1.3](#), and the counit of the adjunction is a natural isomorphism when restricted to involutively filtered objects.

Meanwhile, for any $\mathbb{G}_m$ -equivariant $\mathbb{Q}[\hbar^2]$ -module M we have

$$\mathcal{W}_n \text{Rees}_W^e(M/(\hbar^2 - 1)) \cong (\mathcal{W}_n \oplus \mathcal{W}_{n-1} M)^{e=(-1)^n} = \mathcal{W}_n M,$$

so the unit of the adjunction is a natural isomorphism. □

Remark 1.7 Although we are assuming that filtrations are exhaustive, beware that we are not assuming they are Hausdorff. For instance, the $\mathbb{G}_m$ -equivariant $\mathbb{Q}[\hbar^2]$ -module $V[\hbar^2, \hbar^{-2}]$ corresponds under [Lemma 1.6](#) to the vector space V with trivial involution and filtration $W_i V = V$ for all i .

However, in applications we only ever work with filtrations W on algebras satisfying the stronger condition $W_{-1} V = 0$, and our filtrations on operads will always be complete.

For the remainder of this subsection, we fix a $\mathbb{G}_m$ -equivariant operad $\mathcal{P}$ in chain complexes over $\mathbb{Q}$.

Definition 1.8 The category of quasi-involutively filtered $(\mathcal{P}, W, e)$ -algebras consists of quasi-involutively filtered chain complexes $(V, W_i V, e)$ of $\mathbb{Q}$ -vector spaces equipped with a $\mathcal{P}$ -algebra structure which is compatible with the involution $-1 \in \mathbb{G}_m$ on $\mathcal{P}$ and compatible with the filtration in the sense that the structure maps $\mathcal{P}(n) \otimes V^{\otimes n} \rightarrow V$ restrict to maps

$$\mathcal{W}_i \mathcal{P}(n) \otimes W_r(V^{\otimes n}) \rightarrow W_{r+i} V$$

for all r and i , where $W_r(V^{\otimes n}) := \sum_{r_1 + \dots + r_n = r} (W_{r_1} V) \otimes \dots \otimes (W_{r_n} V)$.

The involutive Rees functor is clearly lax monoidal, ie $\text{Rees}_W^e(V) \otimes_{\mathbb{Q}[\hbar^2]} \text{Rees}_W^e(V) \rightarrow \text{Rees}_W^e(U \otimes_{\mathbb{Q}} V)$, as is its left adjoint $M \mapsto M/(\hbar^2 - 1)$, so [Lemma 1.6](#) has the following immediate consequence:

Lemma 1.9 *The functor of [Definition 1.5](#) gives an equivalence of ∞ -categories from the category of quasi-involutively filtered $(\mathcal{P}, W, e)$ -algebras localised at filtered quasi-isomorphisms to the ∞ -category of $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebras in chain complexes of flat $\mathbb{Q}[\hbar^2]$ -modules, localised at quasi-isomorphisms.*

Note that there is a projective model structure on $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebras in chain complexes, for which fibrations are surjections and weak equivalences are quasi-isomorphisms. Existence of this cofibrantly generated model structure follows from [\[Hirschhorn 2003, Theorem 11.3.2\]](#) applied to the forgetful functor to $\mathbb{G}_m$ -equivariant chain complexes of $\mathbb{Q}$ -vector spaces.

Definition 1.10 Given a $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebra A , we write $\underline{\text{Der}}_{\mathcal{P}, \mathbb{G}_m}(A, M)$ for the complex of $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -derivations from A to M . We also write $\mathbf{R}\underline{\text{Der}}_{\mathcal{P}, \mathbb{G}_m}(A, M)$ for the complex of derived derivations, given as in [\[Quillen 1970\]](#) by $\underline{\text{Der}}_{\mathcal{P}, \mathbb{G}_m}(\tilde{A}, M)$ for any cofibrant replacement $\tilde{A}$ of A in the projective model structure.

Here, M is a Beck A -module, meaning that $A \oplus M$ is a $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebra for which the projection map $A \oplus M \rightarrow A$ and the addition map $(A \oplus M) \times_A (A \oplus M) \rightarrow A \oplus M$ are both $\mathcal{P}$ -algebra homomorphisms. Explicitly,

$$\underline{\text{Der}}_{\mathcal{P}, \mathbb{G}_m}(A, M)_n := \text{Hom}(A, A \oplus \text{cone}(M)_{[n+1]}) \times_{\text{Hom}(A, A)} \{\text{id}\},$$

where Homs on the right-hand side are in the category of $\mathcal{P}$ -algebras in $\mathbb{G}_m$ -equivariant chain complexes. The differential $\underline{\text{Der}}_{\mathcal{P}, \mathbb{G}_m}(A, M)_n \rightarrow \underline{\text{Der}}_{\mathcal{P}, \mathbb{G}_m}(A, M)_{n-1}$ is then induced by the obvious map $\text{cone}(M)_{[n+1]} \rightarrow \text{cone}(M)_{[n]}$.

Proposition 1.11 *Take quasi-involutively filtered $(\mathcal{P}, W, e)$ -algebras A and B such that the filtration on B is complete (satisfied in particular if $W_{-1}B = 0$), together with a morphism $f : \text{gr}^W A \rightarrow \text{gr}^W B$ of the associated $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebras.*

If

$$\text{H}_i \mathbf{R}\underline{\text{Der}}_{\mathcal{P}, \mathbb{G}_m}(\text{gr}^W A, \hbar^{2(n+1)} \text{gr}^W B) \cong 0$$

for all $n \geq 0$ and all $i \geq -1$, then the homotopy fibre over f of the associated graded functor

$$\text{gr}^W : \mathbf{R}\text{map}_{(\mathcal{P}, W, e)}(A, B) \rightarrow \mathbf{R}\text{map}_{\mathcal{P}, \mathbb{G}_m}(\text{gr}^W A, \text{gr}^W B)$$

on mapping spaces in the respective ∞ -categories is contractible.

Proof By [Lemma 1.9](#), $\mathbf{R}\text{map}_{(\mathcal{P}, W, e)}(A, B) \simeq \mathbf{R}\text{map}_{\mathbb{Q}[\hbar^2] \otimes \mathcal{P}, \mathbb{G}_m}(\text{Rees}_W^e A, \text{Rees}_W^e B)$. Since the filtration on B is complete, $\text{Rees}_W^e B \cong \varprojlim_n (\text{Rees}_W^e B)/\hbar^{2n}$, the limit being taken in the $\mathbb{G}_m$ -equivariant category

(ie separately in each weight). Since the maps $(\operatorname{Rees}_W^e B)/\hbar^{2n+2} \rightarrow (\operatorname{Rees}_W^e B)/\hbar^{2n}$ are all surjective, they are fibrations in the projective model structure, and thus the limit is a homotopy limit, so

$$\mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e A, \operatorname{Rees}_W^e B) \simeq \varprojlim_n \mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e A, (\operatorname{Rees}_W^e B)/\hbar^{2n}),$$

where we write $\mathcal{P}[\hbar^2]$ for the operad $\mathbb{Q}[\hbar^2] \otimes \mathcal{P}$

Since the natural map $(\operatorname{Rees}_W^e A)/\hbar^2 \rightarrow \operatorname{gr}^W A$ is a quasi-isomorphism, and similarly for B , we have

$$\mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e A, (\operatorname{Rees}_W^e B)/\hbar^2) \simeq \mathbf{Rmap}_{\mathcal{P}, \mathbb{G}_m}(\operatorname{gr}^W A, \operatorname{gr}^W B),$$

so it suffices to show that the maps

$$\mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e A, (\operatorname{Rees}_W^e B)/\hbar^{2n+2}) \rightarrow \mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e A, (\operatorname{Rees}_W^e B)/\hbar^{2n})$$

are all equivalences.

We now invoke a standard obstruction theory argument. Since $\hbar^{2n}\mathbb{Q} \subset \mathbb{Q}[\hbar^2]/\hbar^{2n+2}$ is an ideal, there is an obvious CDGA structure on the chain complex $C' := \operatorname{cone}(\hbar^{2n}\mathbb{Q} \rightarrow \mathbb{Q}[\hbar^2]/\hbar^{2n+2})$, with the quotient map $C' \rightarrow \mathbb{Q}[\hbar^2]/\hbar^{2n}$ being a quasi-isomorphism. Moreover, since $(\hbar^{2n}) \cdot (\hbar^2) = 0$ in $\mathbb{Q}[\hbar^2]/\hbar^{2n+2}$, the natural surjection $C' \rightarrow \operatorname{cone}(\hbar^{2n}\mathbb{Q} \xrightarrow{0} \mathbb{Q})$ is also a CDGA map, with

$$\mathbb{Q}[\hbar^2]/\hbar^{2n+2} \cong C' \times_{\operatorname{cone}(\hbar^{2n}\mathbb{Q} \xrightarrow{0} \mathbb{Q})} \mathbb{Q}.$$

Tensoring with $\operatorname{Rees}_W^e(B)$ over $\mathbb{Q}[\hbar^2]$, this gives rise to a quasi-isomorphism

$$C := \operatorname{cone}(\hbar^{2n} : \operatorname{Rees}_W^e(B)/\hbar^2 \rightarrow \operatorname{Rees}_W^e(B)/\hbar^{2n+2}) \xrightarrow{\alpha} \operatorname{Rees}_W^e(B)/\hbar^{2n}$$

of $\mathbb{G}_m$ -equivariant $\mathcal{P}[\hbar^2]$ -algebras, together with an isomorphism

$$\operatorname{Rees}_W^e(B)/\hbar^{2n+2} \cong C \times_{\hbar^{2n}(\operatorname{Rees}_W^e(B)/\hbar^2)_{[-1]} \oplus \operatorname{Rees}_W^e(B)/\hbar^2} \operatorname{Rees}_W^e(B)/\hbar^2.$$

That fibre product is a homotopy fibre product because α is surjective, so we have a homotopy pullback square

$$\begin{array}{ccc} \mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e(A), \operatorname{Rees}_W^e(B)/\hbar^{2n+2}) & \longrightarrow & \mathbf{Rmap}_{\mathcal{P}, \mathbb{G}_m}(\operatorname{gr}^W A, \operatorname{gr}^W B) \\ \downarrow & & \downarrow \\ \mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e(A), \operatorname{Rees}_W^e(B)/\hbar^{2n}) & \longrightarrow & \mathbf{Rmap}_{\mathcal{P}, \mathbb{G}_m}(\operatorname{gr}^W A, \operatorname{gr}^W B \oplus \hbar^{2n} \operatorname{gr}^W B_{[-1]}) \end{array}$$

Taking homotopy fibres over $f \in \mathbf{Rmap}_{\mathcal{P}, \mathbb{G}_m}(\operatorname{gr}^W A, \operatorname{gr}^W B)$ gives a homotopy fibre sequence

$$\begin{aligned} \mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e(A), \operatorname{Rees}_W^e(B)/\hbar^{2n+2})_f &\rightarrow \mathbf{Rmap}_{\mathcal{P}[\hbar^2], \mathbb{G}_m}(\operatorname{Rees}_W^e(A), \operatorname{Rees}_W^e(B)/\hbar^{2n})_f \\ &\rightarrow \mathbf{Rmap}_{\mathcal{P}, \mathbb{G}_m}(\operatorname{gr}^W A, \operatorname{gr}^W B \oplus \hbar^{2n} \operatorname{gr}^W B_{[-1]})_f, \end{aligned}$$

but

$$\pi_j \mathbf{Rmap}_{\mathcal{P}, \mathbb{G}_m}(\operatorname{gr}^W A, \operatorname{gr}^W B \oplus \hbar^{2n} \operatorname{gr}^W B_{[-1]})_f \cong H_{j-1} \mathbf{RDer}_{\mathcal{P}, \mathbb{G}_m}(\operatorname{gr}^W A, \hbar^{2n} \operatorname{gr}^W B),$$

which is 0 by hypothesis for all $j > 0$. Thus the base of the fibration is contractible, which gives the desired equivalence. $\square$

Corollary 1.12 *If A is a quasi-involutively filtered $(\mathcal{P}, W, e)$ -algebra such that the filtration is complete and $H_i \mathbf{RDer}_{\mathcal{P}, \mathbb{G}_m}(\mathrm{gr}^W A, \hbar^{2n} \mathrm{gr}^W A) \cong 0$ for all $n > 0$ and all $i \geq -1$, then A is quasi-isomorphic to $\mathrm{gr}^W A$ as a $(\mathcal{P}, W, e)$ -algebra.*

Proof Proposition 1.11 implies that the ∞ -category of quasi-involutively filtered $(\mathcal{P}, W, e)$ -algebras B with fixed quasi-isomorphism $\mathrm{gr}^W B \simeq \mathrm{gr}^W A$ is contractible (by taking f to be the identity map on $\mathrm{gr}^W A$). Since A and $\mathrm{gr}^W A$ both lie in this category, it follows that they are quasi-isomorphic. $\square$

Corollary 1.13 *The associated graded functor from the ∞ -category of quasi-involutively filtered $(\mathcal{P}, W, e)$ -algebras to the category of $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebras becomes an equivalence when restricted to objects A satisfying the conditions*

- (1) $\mathrm{gr}_i^W A \simeq 0$ for all $i < 0$, and
- (2) $\mathbf{RDer}_{\mathcal{P}, \mathbb{G}_m}(\mathrm{gr}^W A, M) \simeq 0$ for all Beck $\mathrm{gr}^W A$ -modules M with $\mathcal{W}_i M \simeq 0$ for all $i < 2$.

The quasi-inverse functor is given by sending a $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebra to the underlying filtered algebra.

In particular, note that these conditions are satisfied if $\mathrm{gr}^W A$ has a cofibrant replacement generated in weights ≤ 1 .

Proof Given A satisfying condition (2), B satisfying condition (1), and a morphism $f: \mathrm{gr}^W A \rightarrow \mathrm{gr}^W B$, observe that the Beck A -module $\hbar^{2(n+1)} \mathrm{gr}^W B$ is acyclic in weights below $2n + 2$, so for all $n \geq 0$ we have

$$\mathbf{RDer}_{\mathcal{P}, \mathbb{G}_m}(\mathrm{gr}^W A, \hbar^{2(n+1)} \mathrm{gr}^W B) \simeq 0$$

by hypothesis on A . Thus Proposition 1.11 ensures that the restricted functor is full and faithful. The functor sending a grading to its underlying increasing filtration is clearly a right inverse to gr^W , and hence a quasi-inverse. $\square$

Remark 1.14 If we take a quasi-involutively filtered dg operad $\mathcal{Q}$ with an involutive equivalence $\mathrm{gr}^W \mathcal{Q} \simeq \mathcal{P}$, then the conditions of Proposition 1.11 give the same conclusion when A and B are quasi-involutively filtered $(\mathcal{Q}, W, e)$ -algebras, with the same proof, because the associated graded pieces are the same. Since $\mathrm{gr}^W \mathcal{Q}$ -algebras are not then canonically $\mathcal{Q}$ -algebras, the analogous statement to Corollary 1.13 is just that the associated graded functor from quasi-involutively filtered $(\mathcal{Q}, W, e)$ -algebras to the category of $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebras is full and faithful when restricted to objects satisfying those conditions.

Essential surjectivity also follows if we assume $H_{<0} \mathcal{P} = 0$ and restrict to $\mathcal{Q}$ -algebras A satisfying $H_{<0}(\mathrm{gr}^W A) = 0$, because if the condition $H_{-2} \mathbf{RDer}_{\mathcal{P}, \mathbb{G}_m}(B, \hbar^{2(n+1)} B) = 0$ holds for all $n \geq 0$, then there must exist a quasi-involutively filtered $(\mathcal{Q}, W, e)$ -algebra A with $\mathrm{gr}^W A \simeq B$. This follows by the following deformation argument similar to [Kontsevich and Soibelman 2000; Hinich 2004]. By taking a cofibrant model for B concentrated in nonnegative degrees and lifting generators, we have a quasifree

Rees $_W^e(\mathcal{Q})$ -algebra $\tilde{B}$ in graded vector spaces, on which we wish to define a closed derivation $\tilde{\delta}$ lifting δ ; the quasifreeness and concentration conditions will ensure the dg algebra $(\tilde{B}, \tilde{\delta})$ is cofibrant. We construct $\tilde{\delta}$ on generators inductively in $\hbar^2$, with the potential obstruction to lifting from $\tilde{B}/\hbar^{2n}$ to $\tilde{B}/\hbar^{2(n+1)}$ being the class $[\tilde{\delta} \circ \tilde{\delta}]$ in the space of derivations above.

1.2 Almost commutative Poisson algebras

We now consider nonunital P_k -algebras (ie $(k-1)$ -shifted Poisson algebras); these are nonunital CDGAs equipped with a Lie bracket of chain degree $k-1$ acting as a biderivation. They are governed by an operad P_k which can be written as $\text{Com} \circ (s^{1-k} \text{Lie})$ via a distributive law (cf [Loday and Vallette 2012, Section 8.6]), for the operads Com and Lie governing nonunital commutative algebras and Lie algebras, where the shift $s\mathcal{P}$ of a dg operad $\mathcal{P}$ is given by $(s\mathcal{P})(n) := \mathcal{P}(n)_{[n-1]}$.

Definition 1.15 The $\mathbb{G}_m$ -equivariant dg operad P_k^{ac} is the dg operad $\text{Com} \circ s^{1-k} \hbar^{-1} \text{Lie}$, where $(\hbar^j \mathcal{P})(i) := \hbar^{j(i-1)} \mathcal{P}(i)$ for any operad $\mathcal{P}$, and as in Definition 1.4, $\hbar$ has degree 0 and weight 1 for the $\mathbb{G}_m$ -action.

Define a quasi-involutive almost commutative P_k -algebra over a CDGA R to be an $(R \otimes P_k^{\text{ac}}, W, e)$ -algebra A in quasi-involutively filtered chain complexes. Here $R \otimes \mathcal{P}$ is the operad $(R \otimes \mathcal{P})(n) := R \otimes \mathcal{P}(n)$ with operad structure coming from the distributive law (as in [loc. cit., Section 8.6]) $\mathcal{P}(n) \otimes R^{\otimes n} \rightarrow R \otimes \mathcal{P}(n)$ given by the multiplication on R .

Thus a $\mathbb{G}_m$ -equivariant P_k^{ac} -algebra is a P_k -algebra equipped with a $\mathbb{G}_m$ -action for which multiplication has weight 0 and the Lie bracket has weight -1 . A quasi-involutive almost commutative P_k -algebra over R is a P_k -algebra in R -modules, equipped with an increasing filtration W satisfying $W_i \cdot W_j \subset W_{i+j}$ and $[W_i, W_j] \subset W_{i+j-1}$, together with an involution $*$ preserving the filtration, satisfying $(a \cdot b)^* = a^* \cdot b^*$ and $[a, b]^* = -[a^*, b^*]$, and acting as $(-1)^i$ on $H_* \text{gr}_i^W$.

Note that the chain degree of a nonzero element of the operad P_k^{ac} is always $1-k$ times its weight.

Lemma 1.16 The Koszul dual of the $\mathbb{G}_m$ -equivariant dg operad P_k^{ac} is given by $s^{k-1} \hbar P_k^{\text{ac}}$.

Proof The proof of [Loday and Vallette 2012, 8.6.5] adapts to this shifted and graded setting (cf [Fresse 2017, Appendix C]) to give

$$\begin{aligned} (P_k^{\text{ac}})^! &= (\text{Com} \circ s^{1-k} \hbar^{-1} \text{Lie})^! = (s^{1-k} \hbar^{-1} \text{Lie})^! \circ \text{Com}^! \cong (s^{k-1} \hbar \text{Com}) \circ \text{Lie} \\ &= s^{k-1} \hbar (\text{Com} \circ (s^{1-k} \hbar^{-1} \text{Lie})) = s^{k-1} \hbar P_k^{\text{ac}}. \end{aligned} \quad \square$$

We now fix a CDGA R over $\mathbb{Q}$. There is a model structure on the category of $\mathbb{G}_m$ -equivariant R -chain complexes (the projective model structure), in which fibrations are surjections.

Before we state the next proposition, recall that the cotangent complex $\mathbf{L}\Omega^1$ for CDGAs is defined as the left-derived functor of Kähler differentials Ω^1 . For a morphism $f: A \rightarrow B$ of CDGAs, that means $\mathbf{L}\Omega_{B/A}^1$

(defined up to B -linear quasi-isomorphism) is given by the B -module $\Omega_{\tilde{B}/A}^1 \otimes_{\tilde{B}} B$ for any factorisation $A \rightarrow \tilde{B} \rightarrow B$ of f as a cofibration followed by a weak equivalence.

For a morphism $A \rightarrow B$ of nonunital R -CDGAs, we then write $\mathbf{L}\Omega_{B/A}^1 := \mathbf{L}\Omega_{(R \oplus B)/(R \oplus A)}^1$, using the equivalence between B -modules and modules for the unital ring $R \oplus B$. Note that this agrees with the definition above for morphisms of unital CDGAs because then there are isomorphisms $R \oplus A \cong R \times A$ and $R \oplus B \cong R \times B$ sending (r, x) to $(r, r + x)$, with $\mathbf{L}\Omega_{(R \times B)/(R \times A)}^1 \simeq \mathbf{L}\Omega_{B/A}^1$ since cotangent complexes satisfy étale descent and $\mathbf{L}\Omega_{R/R}^1 \simeq 0$.

The following lemma runs on similar lines to [Melani 2016]:

Lemma 1.17 *Given a $\mathbb{G}_m$ -equivariant P_k^{ac} -algebra A in R -chain complexes which is cofibrant as a $\mathbb{G}_m$ -equivariant R -CDGA (unital or nonunital), together with a Beck A -module M , there is a model for $\mathbf{R}\underline{\text{Der}}_{P_k^{\text{ac}} \otimes R, \mathbb{G}_m}(A, M)$ which has a complete decreasing filtration $F^1 \supset F^2 \supset \dots$, with associated graded complexes quasi-isomorphic to*

$$\underline{\text{Hom}}_A(\text{CoS}_A^P((\Omega_{A/R}^1)[-k]), \hbar^{p-1}M)_{[-k]}^{\mathbb{G}_m} \quad \text{or} \quad \underline{\text{Hom}}_{R \oplus A}(\text{CoS}_{R \oplus A}^P((\Omega_{A/R}^1)[-k]), \hbar^{p-1}M)_{[-k]}^{\mathbb{G}_m}$$

in the unital and nonunital cases, respectively, where Ω^1 denotes the complex of Kähler differentials of the underlying CDGA and $\text{CoS}_A^P(N) := \text{CoSymm}_A^P(N) = (N^{\otimes_A P})^{\Sigma_P}$, the cosymmetric powers.

Proof Adapting the formulae of [Loday and Vallette 2012, Section 11.2] to the R -linear setting, the Koszul duality map $\alpha: (s(P_k^{\text{ac}})^1)^\vee \rightarrow P_k^{\text{ac}}$ gives rise to a cobar–bar adjunction $\Omega_\alpha \dashv B_\alpha$ between P_k^{ac} -algebras in R -chain complexes and $s(P_k^{\text{ac}})^1$ -coalgebras in R -chain complexes. Moreover, $\Omega_\alpha B_\alpha A$ is a cofibrant resolution of A [loc. cit., Theorem 11.4.7], so

$$\mathbf{R}\underline{\text{Der}}_{P_k^{\text{ac}} \otimes R, \mathbb{G}_m}(A, M) \simeq \underline{\text{Der}}_{P_k^{\text{ac}} \otimes R, \mathbb{G}_m}(\Omega_\alpha B_\alpha A, M).$$

Since $\Omega_\alpha C$ is freely generated by $C_{[1]}$, the latter complex can be rewritten as

$$(\underline{\text{Hom}}_R(B_\alpha A, M)_{[-1]}^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -]),$$

where ω_A encodes the data defining the P_k^{ac} -algebra structure on A and the bracket is defined via the convolution product of [loc. cit., 6.4.4] (which is of weight 0 for the $\mathbb{G}_m$ -action).

Substituting for B_α , we can rewrite this complex as

$$\begin{aligned} & \left(\prod_n (\hbar^{n-1} P_k^{\text{ac}}(n)_{[k(n-1)]} \otimes^{\Sigma_n} \underline{\text{Hom}}_R(A^{\otimes n}, M))_{[-1]}^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -] \right) \\ & \cong (\underline{\text{Hom}}_R(\text{CoSymm}^+(\hbar^{-1} \text{CoLie}(A_{[-1]})_{[1-k]})_{[k]} \hbar, M)_{[-1]}^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -]). \end{aligned}$$

Since the differential on $B_\alpha A$ is nonincreasing on cosymmetric powers, it follows that this dual complex has a complete decreasing filtration F , with F^p given by terms involving CoSymm^i for $i \geq p$. The associated graded pieces are then given by

$$\text{gr}_F^p \cong (\underline{\text{Hom}}_R(\text{CoSymm}^p(\hbar^{-1} \text{CoLie}(A_{[-1]})_{[1-k]})_{[k]} \hbar, M)_{[-1]}^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -]).$$

In order to proceed further, we use the Koszul duality map $\beta: (s \operatorname{Lie})^\vee \rightarrow \operatorname{Com}$, which as in [loc. cit., Section 11] gives us a cofibrant resolution $A' := \Omega_{\operatorname{Com} B_{\operatorname{Com}} A} A$ of the nonunital CDGA underlying A , on generators $(B_{\operatorname{Com}} A)_{[1]}$. Repeating the argument above for this instance of Koszul duality gives

$$\begin{aligned} & (\underline{\operatorname{Hom}}_R(\operatorname{CoSymm}^p(\hbar^{-1} \operatorname{CoLie}(A_{[-1]})_{[1-k]})_{[k]} \hbar, M)^{\mathbb{G}_m}, \delta + [\omega_A \circ \alpha, -]) \\ & \cong \underline{\operatorname{Hom}}_{R \oplus A'}(\operatorname{CoS}_{R \oplus A'}^p((\hbar^{-1} \Omega_{A'/R}^1)_{[-k]}) \hbar, M)_{[-k]}^{\mathbb{G}_m} \simeq \underline{\operatorname{Hom}}_{R \oplus A}(\operatorname{CoS}_{R \oplus A}^p((\hbar^{-1} \Omega_{A/R}^1)_{[-k]}) \hbar, M)_{[-k]}^{\mathbb{G}_m}, \end{aligned}$$

where the final line follows because A is cofibrant and $A' \rightarrow A$ a quasi-isomorphism. When A is unital, the first factor of $R \times A \cong R \oplus A$ acts trivially on $\Omega_{A/R}^1$ and on the A -module M , so we may replace $R \oplus A$ with A throughout. $\square$

Proposition 1.18 *If B is a nonnegatively weighted $\mathbb{G}_m$ -equivariant P_k^{ac} -algebra over a CDGA R for which the map $(\mathcal{W}_1 \mathbf{L} \Omega_{B/\mathcal{W}_0 B}^1) \otimes_{(R \oplus \mathcal{W}_0 B)}^{\mathbf{L}} (R \oplus B) \rightarrow \mathbf{L} \Omega_{B/\mathcal{W}_0 B}^1$ is a quasi-isomorphism, then*

$$\mathbf{R} \underline{\operatorname{Der}}_{P_k^{\operatorname{ac}} \otimes R, \mathbb{G}_m} (B, M) \simeq 0$$

for all Beck B -modules M with $\mathcal{W}_i M \simeq 0$ for all $i < 2$.

Proof Without loss of generality, we may assume that B is cofibrant. For the unital algebra $B' := R \oplus B$, with R in weight 0, we have $\Omega_{B'}^1 \cong \Omega_B^1$ and $\Omega_{\mathcal{W}_0 B'}^1 \cong \Omega_{\mathcal{W}_0 B}^1$, and hence an exact triangle

$$\Omega_{\mathcal{W}_0 B}^1 \otimes_{\mathcal{W}_0 B'} B' \rightarrow \Omega_B^1 \rightarrow \Omega_{B/\mathcal{W}_0 B}^1 \rightarrow \Omega_{\mathcal{W}_0 B}^1 \otimes_{\mathcal{W}_0 B'} B'_{[-1]},$$

which by hypothesis simplifies to

$$\Omega_{\mathcal{W}_0 B}^1 \otimes_{\mathcal{W}_0 B'} B' \rightarrow \Omega_B^1 \rightarrow (\mathcal{W}_1 \Omega_{B/\mathcal{W}_0 B}^1) \otimes_{\mathcal{W}_0 B'} B' \rightarrow \Omega_{\mathcal{W}_0 B}^1 \otimes_{\mathcal{W}_0 B'} B'_{[-1]}.$$

Since Ω_B^1 is quasi-isomorphic to a B -module (equivalently, B' -module) freely generated by terms of weights 0 and 1, we thus have that $\mathbf{R} \underline{\operatorname{Hom}}_{B'}(\operatorname{CoS}_{B'}^p((\Omega_{B/R}^1)_{[-k]}), N)^{\mathbb{G}_m}$ is acyclic for any Beck module N with weights $> p$, or more generally with $\mathcal{W}_i N$ acyclic for $i \leq p$. Setting $N := \hbar^{p-1} M$, the statement now follows from the convergent spectral sequence

$$\operatorname{Ext}_{B'}^j(\operatorname{CoS}_{B'}^p((\Omega_{B/R}^1)_{[-k]}), \hbar^{p-1} M)^{\mathbb{G}_m} \Rightarrow H_{k-j} \mathbf{R} \underline{\operatorname{Der}}_{P_k^{\operatorname{ac}} \otimes R, \mathbb{G}_m} (B, M)$$

arising from the complete filtration of Lemma 1.17. $\square$

Corollary 1.19 *The associated graded functor from the ∞ -category of quasi-involutive almost commutative P_k -algebras to $\mathbb{G}_m$ -equivariant P_k^{ac} -algebras becomes an equivalence when restricted to those objects B with $W_{-1} B = 0$ for which the map*

$$(\operatorname{gr}_1^W \mathbf{L} \Omega_{B/W_0 B}^1) \otimes_{(R \oplus W_0 B)}^{\mathbf{L}} (R \oplus \operatorname{gr}^W B) \rightarrow \mathbf{L} \Omega_{\operatorname{gr}^W B/W_0 B}^1$$

of commutative cotangent complexes is a quasi-isomorphism.

The quasi-inverse functor is given by sending a $\mathbb{G}_m$ -equivariant P_k^{ac} -algebra to the underlying filtered algebra.

Proof Substitute Proposition 1.18 into Corollary 1.13. $\square$

Remark 1.20 For noninvolutive deformations, the analogue of [Corollary 1.19](#) does not hold. The noninvolutive analogue of [Corollary 1.13](#) involves homology of $\mathbf{R}\underline{\mathrm{Der}}_{\mathcal{P}, \mathbb{G}_m}(\mathrm{gr}^W B, M)$ for M concentrated in weights ≥ 1 , giving a first term which is seldom acyclic. This is similar to phenomena arising in [\[Pridham 2018; 2019; 2022\]](#), where the only obstruction to quantisation is first-order, and can be eliminated by restricting to involutive quantisations.

Remark 1.21 Following [Remark 1.14](#), if we take an involutively filtered dg operad $(\mathcal{P}, W)$ with an involutive equivalence $\mathrm{gr}^W \mathcal{P} \simeq P_k^{\mathrm{ac}}$, then the conditions of [Corollary 1.19](#) also ensure (via the argument of [\[Kontsevich and Soibelman 2000; Hinich 2004\]](#)) that for B a nonnegatively weighted $\mathbb{G}_m$ -equivariant P_k^{ac} -algebra concentrated in nonnegative homological degrees, the space of quasi-involutive almost commutative derived $\mathcal{P}$ -algebras (B', W) with $\mathrm{gr}^W B' \simeq B$ is contractible.

In particular, when $k = 1$ we can take $\mathcal{P}$ to be the BD_1 operad, given by the PBW filtration on the associative operad as in [\[Calaque et al. 2017, Section 3.5.1\]](#) (named by analogy with the Beilinson–Drinfeld (BD or BD_0) algebras of [\[Costello and Gwilliam 2017, Section 2.2\]](#)). The argument above then gives an essentially unique filtered associative dg algebra (B', W) equipped with a filtered involution $(B')^{\mathrm{opp}} \cong B'$ and an equivalence $\mathrm{gr}^W B' \simeq B$. When B is an algebra of polyvectors, B' will thus be given by the ring of differential operators $\mathcal{D}(\omega^{1/2})$ (in the sense of [\[Ginzburg 1998, Section 2.2\]](#)) on a square root $\omega^{1/2}$ of the dualising bundle whenever this exists, and B' gives a ring of twisted differential operators generalising $\mathcal{D}(\omega^{1/2})$ even when the dualising complex is not a line bundle, or has no square root.

2 Quantisations on derived Deligne–Mumford stacks

2.1 Hochschild complexes

Definition 2.1 For a cofibrant CDGA A over R , we define the filtered chain complex

$$(\mathrm{CC}_{R, \oplus}(A)_\bullet, \tau^{\mathrm{HH}})$$

to be the direct sum total complex (written as a chain complex) of the double complex $\underline{\mathrm{CC}}^\bullet_R(A)$ given by

$$\underline{\mathrm{CC}}^n_R(A) = \underline{\mathrm{Hom}}_R(A^{\otimes_R n}, A),$$

with Hochschild differential $b: \underline{\mathrm{CC}}^{n-1} \rightarrow \underline{\mathrm{CC}}^n$ given by

$$\begin{aligned} (bf)(a_1, \dots, a_n) \\ = a_1 f(a_2, \dots, a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_n) + (-1)^n f(a_1, \dots, a_{n-1}) a_n. \end{aligned}$$

The filtration τ^{HH} on $\underline{\mathrm{CC}}^\bullet_R(A)$ and $\mathrm{CC}_{R, \oplus}(A)_\bullet$ is given by good truncation in the Hochschild direction, so $\tau_p^{\mathrm{HH}} \mathrm{CC}_{R, \oplus}(A)_\bullet \subset \mathrm{CC}_{R, \oplus}(A)_\bullet$ is the subspace

$$\prod_{i=0}^{p-1} \underline{\mathrm{Hom}}(A^{\otimes_R i}, A)_{[i]} \times \ker(b: \underline{\mathrm{Hom}}(A^{\otimes_R p}, A) \rightarrow \underline{\mathrm{Hom}}(A^{\otimes_R (p+1)}, A))_{[p]}.$$

Remarks 2.2 Beware that $\mathrm{CC}_{R,\oplus}(A)_\bullet$ is usually a proper subcomplex of the cohomological Hochschild complex $\mathrm{CC}_{R,\Pi}(A)_\bullet$ of A , which is defined by taking the product, rather than sum, total complex. Since $\mathrm{CC}_{R,\oplus}(A)_\bullet = \bigcup_p \tau_p^{\mathrm{HH}} \mathrm{CC}_{R,\Pi}(A)_\bullet$, our constructions will be consistent with those of [Pridham 2018].

There is a filtration γ from [Pridham 2022, Definition 1.18] which is quasi-isomorphic to τ^{HH} under the cofibrancy hypothesis of Definition 2.1, but which also behaves well whenever A is cofibrant as an R -module (rather than as an R -CDGA) and gives an involutive, rather than quasi-involutive, filtration analogous to Lemma 2.8. For simplicity, we will just use τ^{HH} .

Recall that a brace algebra B is a chain complex equipped with a cup product in the form of a chain map

$$B \otimes B \rightrightarrows B,$$

and braces in the form of maps

$$\{-\}\{-, \dots, -\}_r: B \otimes B^{\otimes r} \rightarrow B_{[r]}$$

satisfying the conditions of [Voronov 2000, Section 3.2] (where brace algebras are called homotopy G -algebras) with respect to the differential. The commutator of the brace $\{-\}\{-\}_1$ is a Lie bracket, so for any brace algebra B , there is a natural DGLA structure on $B_{[-1]}$.

There is a natural brace algebra structure on $\mathrm{CC}_{R,\oplus}(A)$ by [Voronov 2000, Theorem 3.1].

The following are taken from [Pridham 2022, Section 1.2.1]:

Definition 2.3 Given a brace algebra B , define the opposite brace algebra B^{opp} to have the same elements as B , but multiplication $b^{\mathrm{opp}} \smile c^{\mathrm{opp}} := (-1)^{\deg b \deg c} (c \smile b)^{\mathrm{opp}}$ and brace operations given by the multiplication $(B B^{\mathrm{opp}}) \otimes (B B^{\mathrm{opp}}) \rightarrow B B^{\mathrm{opp}}$ on the bar construction induced by the isomorphism $(B B^{\mathrm{opp}}) \cong (B B)^{\mathrm{opp}}$, the opposite coalgebra. Explicitly,

$$\{b^{\mathrm{opp}}\}\{c_1^{\mathrm{opp}}, \dots, c_m^{\mathrm{opp}}\} := \pm \{b\}\{c_m, \dots, c_1\}^{\mathrm{opp}},$$

where $\pm = (-1)^{m(m+1)/2 + (\deg b - m)(\sum_i \deg c_i - m) + \sum_{i < j} \deg c_i \deg c_j}$.

Observe that when a filtered brace algebra B is almost commutative, then so is B^{opp} .

Definition 2.4 Define a filtration τ on the brace operad Br of [Voronov 2000] by good truncation aritywise, so $(\tau_{\geq p} \mathrm{Br})(n) := \tau_{\geq p} \mathrm{Br}(n)$.

We refer to (brace, τ)-algebras in filtered complexes as almost commutative brace algebras. We define a quasi-involutive almost commutative brace algebra to be an almost commutative brace algebra (B, F) equipped with an involution $(B, F) \cong (B^{\mathrm{opp}}, F)$ of (brace, τ)-algebras which acts on $H_*(\mathrm{gr}_i^F B)$ as multiplication by $(-1)^i$.

Lemma 2.5 The filtered Hochschild complex $(\mathrm{CC}_{R,\oplus}(A)_\bullet, \tau^{\mathrm{HH}})$ naturally has the structure of an almost commutative brace algebra.

Proof The double complex $\underline{\mathrm{CC}}_R^\bullet(A)$ is a brace algebra in the cochain direction, as a consequence of [Voronov 2000, Theorem 3.1]. Compatibility of tensor products with good truncation then makes $(\underline{\mathrm{CC}}_R^\bullet(A), \tau^{\mathrm{HH}})$ a (Br, τ) -algebra in double complexes, and passing to total complexes gives the required result. $\square$

Definition 2.6 Define $\mathrm{Pol}(A/R, 0)$ to be the $\mathbb{G}_m$ -equivariant P_2^{ac} -algebra $\bigoplus_{p \geq 0} \underline{\mathrm{Hom}}_A(\Omega_{A/R}^p, A)_{[p]}$ with the obvious graded-commutative multiplication and the Schouten–Nijenhuis Lie bracket, given by interpreting $\underline{\mathrm{Hom}}_A(\Omega_{A/R}^p, A)$ as the complex of antisymmetric p -derivations $A^p \rightarrow A$.

Lemma 2.7 Let A be a cofibrant R -CDGA. Then the HKR isomorphism gives a quasi-isomorphism $\mathrm{HKR}: \mathrm{gr}^{\tau^{\mathrm{HH}}} \mathrm{CC}_{R, \oplus}(A)_\bullet \rightarrow \mathrm{Pol}(A/R, 0)$ of $\mathbb{G}_m$ -equivariant $\mathrm{gr}^\tau \mathrm{Br}$ -algebras, where the $\mathrm{gr}^\tau \mathrm{Br}$ -algebra structure on $\mathrm{Pol}(A/R, 0)$ comes from the quasi-isomorphism $\mathrm{gr}^\tau \mathrm{Br} \rightarrow H_* \mathrm{Br} \cong P_2$ of dg operads from [McClure and Smith 2002].

Proof By turning the structural differential δ on A off and on again, the HKR isomorphism gives us a quasi-isomorphism $(\mathrm{HH}_R^p(A_\#), \delta) \simeq \underline{\mathrm{Hom}}_A(\Omega_A^p, A)$, since A is cofibrant.⁴ Thus $\mathrm{gr}_p^{\tau^{\mathrm{HH}}} \underline{\mathrm{CC}}_R^\bullet(A)$ maps quasi-isomorphically to $\underline{\mathrm{Hom}}_A(\Omega_A^p, A)^{[-p]}$.

Now, the action of $\mathrm{gr}^\tau \mathrm{Br}$ on $\mathrm{HH}_R^*(A_\#)$ factors through its quotient $H_* \mathrm{Br} \cong P_2$. This operad is generated by the commutative product and Lie bracket, and a simple check shows that the cup product and Gerstenhaber bracket on Hochschild cohomology act in the prescribed fashion under the HKR isomorphism. $\square$

Lemma 2.8 Given a cofibrant CDGA A over R , there is an anti-involution

$$-i : \mathrm{CC}_{R, \oplus}(A)^{\mathrm{opp}} \rightarrow \mathrm{CC}_{R, \oplus}(A)$$

given by

$$-i(f)(a_1, \dots, a_m) = (-1)^{\sum_{i < j} \deg a_i \deg a_j} (-1)^{m(m+1)/2} f(a_m, \dots, a_1),$$

making $(\mathrm{CC}_{R, \oplus}(A), \tau^{\mathrm{HH}})$ into a quasi-involutive almost commutative brace algebra.

Proof The (brace, τ) -algebra structure is given by Lemma 2.5. The involution is given in [Braun 2014, Section 2.1], and comes from the antipode on the cofree tensor coalgebra $\bigoplus (A_{[-1]})^{\otimes n}$ regarded as the universal enveloping coalgebra of a cofree graded Lie coalgebra. Quasi-involutivity is then a simple calculation as in [Pridham 2018, Lemma 1.15], and compatibility with the brace structure follows as in [loc. cit., Remark 2.22]. $\square$

2.2 Polydifferential operators

There is a variant of the Hochschild complex defined using polydifferential operators instead of R -linear maps. In the algebraic setting it is quasi-isomorphic to the Hochschild complex, and so leads to the same theory, but in smooth and analytic settings it gives a much better behaved object.

⁴In fact, it suffices for A to be quasismooth in the sense of [Kontsevich 1994] (not to be confused with the clashing sense of [Toën 2009]) or even ind-quasismooth, ie for A_0 to be ind-smooth and the graded algebra $A_\#$ to be freely, or even projectively, generated over it. The cofibrant hypothesis on A can be accordingly relaxed throughout this paper.

The following adapts [Carchedi and Roytenberg 2012] as in [Pridham 2020a; 2020b]:

Definition 2.9 A $\mathcal{C}^\infty$ -DGA (over a base $R = \mathbb{R}$), resp. an EFC-DGA (over a base $R = K$, a complete valued field), is an R -CDGA for which

- A_0 is equipped with a $\mathcal{C}^\infty$ -ring structure (resp. entire functional calculus) enhancing its commutative R -algebra structure, and
- the derivation $\delta: A_0 \rightarrow A_{-1}$ is a $\mathcal{C}^\infty$ -derivation (resp. EFC-derivation).

A morphism $A \rightarrow B$ of $\mathcal{C}^\infty$ -DGAs (resp. EFC-DGAs) is then a morphism of R -CDGAs preserving the $\mathcal{C}^\infty$ (resp. EFC) structure.

A $\mathcal{C}^\infty$ (resp. EFC) structure on A_0 is a product-preserving set-valued functor $\mathbb{R}^n \mapsto (A_0)^n$ (resp. $K^n \mapsto (A_0)^n$) on the category with objects $\{\mathbb{R}^n\}_{n \geq 0}$ (resp. $\{K^n\}_{n \geq 0}$) and morphisms consisting of $\mathcal{C}^\infty$ (resp. K -analytic) maps. Explicitly, the set A_0 is equipped, for every $\mathcal{C}^\infty$ (resp. analytic) function f in n variables with an operation $\Phi_f: (A_0)^n \rightarrow A_0$. The condition on $\delta: A_0 \rightarrow A_{-1}$ then says that the map $(\text{id}, \delta): A_0 \rightarrow A_0 \times A_{-1}$ is a $\mathcal{C}^\infty$ (resp. EFC) morphism.

As in [Carchedi and Roytenberg 2012, Theorem 6.10], the corresponding categories of algebras carry cofibrantly generated model structures obtained by adjunction from the model category of chain complexes, so weak equivalences are quasi-isomorphisms and fibrations are surjections. For most applications (though not Section 2.4.3), it suffices to know that examples of cofibrant⁵ $\mathcal{C}^\infty$ -DGAs and EFC-DGAs A are given by CDGAs $A = A_{\geq 0}$ which are freely generated as graded-commutative A_0 -algebras, with A_0 respectively a ring of smooth functions on $\mathbb{R}^n$ or a ring of analytic functions on K^n . Morphisms are CDGA morphisms with additional restrictions in degree 0 corresponding to $\mathcal{C}^\infty$ (resp. holomorphic) morphisms between manifolds.

Both $\mathcal{C}^\infty$ -DGAs and EFC-DGAs similarly have notions of derivations which are more restrictive than those of the underlying CDGAs, with resulting modules of differential forms; for rings of functions on manifolds, these correspond respectively to smooth and analytic forms on the manifold. This leads to a notion of differential operators, which have the expected form in terms of coordinates as in [Pridham 2020b, Definition 3.1]; for intrinsic definitions, see [loc. cit., Remark 3.2].

Definition 2.10 For a cofibrant R -CDGA (resp. $\mathcal{C}^\infty$ -DGA, resp. EFC-DGA) A , we define the filtered chain complex

$$(D_{\oplus}^{\text{poly}}(A)_{\bullet}, \tau^{\text{HH}})$$

of polydifferential operators as follows. First let

$$\underline{D}^{\text{poly}, n}(A) := \mathcal{D}\text{iff}(A^{\coprod n}, A),$$

⁵Henceforth cofibrancy is only used to give the correct cotangent complexes, so the condition on A can be relaxed by asking only that A_0 consist respectively of functions on a smooth manifold or holomorphic functions on a Stein manifold, and that $A = A_{\geq 0}$ be projectively generated as a graded-commutative A_0 -algebra.

the complex of R -linear algebraic (resp. $\mathcal{C}^\infty$, resp. EFC) differential operators from the n -fold coproduct $A^{\amalg n}$ (ie $A^{\otimes n}$ in the algebraic case) to the $A^{\amalg n}$ -module A . This carries a Hochschild differential $b: \underline{D}^{\text{poly}, n-1} \rightarrow \underline{D}^{\text{poly}, n}$ given by the same formula as in [Definition 2.1](#), leading to a double complex $\underline{D}^{\text{poly}, \bullet}(A)$. We then define $D_{\oplus}^{\text{poly}}(A)_{\bullet}$ to be the direct sum total complex of $\underline{D}^{\text{poly}, \bullet}(A)$.

The filtration τ^{HH} on $\underline{D}^{\text{poly}, \bullet}(A)$ and $D_{\oplus}^{\text{poly}}(A)_{\bullet}$ is given by good truncation in the Hochschild direction.

The subcomplex $D_{\oplus}^{\text{poly}}(A)_{\bullet} \subset \text{CC}_{R, \oplus}(A)$ is clearly closed under the brace operations, where R is taken to be $\mathbb{R}$ (resp. K) in the $\mathcal{C}^\infty$ (resp. EFC) setting, so $D_{\oplus}^{\text{poly}}(A)_{\bullet}$ is a brace algebra over R .

Definition 2.11 Given a $\mathcal{C}^\infty$ -DGA A (resp. EFC-DGA A over K) define $\text{Pol}(A, 0)$ to be the $\mathbb{G}_m$ -equivariant P_2^{ac} -algebra of $\mathcal{C}^\infty$ (resp. EFC) polyvectors of A , defined by replacing $\Omega_{A/R}^1$ in [Definition 2.6](#) with the complexes $\Omega_{A, \mathcal{C}^\infty}^1$ and $\Omega_{A/K, \text{EFC}}^1$ of $\mathcal{C}^\infty$ and EFC differentials, respectively.

Lemma 2.12 For any cofibrant R -CDGA (resp. $\mathcal{C}^\infty$ -DGA, resp. EFC-DGA) A , we have a natural quasi-isomorphism HKR: $\text{gr}^{\tau^{\text{HH}}} D_{\oplus}^{\text{poly}}(A)_{\bullet} \rightarrow \text{Pol}(A/R, 0)$ of $\mathbb{G}_m$ -equivariant gr^{τ} Br-algebras.

Proof Observe that for the order filtration F on differential operators, we have $\text{gr}_j^F \underline{D}^{\text{poly}, n}(A) \cong \underline{\text{Hom}}_A(\text{Sym}_A^j((\Omega_A^1)^{\oplus n}), A)$. Thus $\text{gr}_j^F \underline{D}^{\text{poly}, n}(A)$ is the A -linear dual of the free A -algebra generated by $(\Omega_A^1)^{\oplus n}$, and the Hochschild differential on these copies of Ω_A^1 is easily seen to act as the total differential of the simplicial nerve of Ω_A^1 . Thus the Eilenberg–Zilber theorem gives quasi-isomorphisms

$$\text{gr}_p^F D_{\oplus}^{\text{poly}}(A)_{\bullet} \rightarrow \underline{\text{Hom}}_A(\text{Sym}_A^p((\Omega_A^1)_{[-1]}), A) \cong \underline{\text{Hom}}_A(\Omega_A^p, A)_{[p]}.$$

These are compatible with τ^{HH} , so the restriction to $D_{\oplus}^{\text{poly}}(A)_{\bullet}$ of the HKR map is a quasi-isomorphism. The rest of the proof follows as in [Lemma 2.7](#). $\square$

Lemma 2.13 Given a cofibrant R -CDGA (resp. $\mathcal{C}^\infty$ -DGA, resp. EFC-DGA) A , there is an anti-involution

$$-i: D_{\oplus}^{\text{poly}}(A)_{\bullet}^{\text{opp}} \rightarrow D_{\oplus}^{\text{poly}}(A)_{\bullet}.$$

making $(D_{\oplus}^{\text{poly}}(A)_{\bullet}^{\text{opp}}, \tau^{\text{HH}})$ into a quasi-involutive almost commutative brace algebra.

Proof Most of the properties follow immediately from [Lemma 2.8](#), since $D_{\oplus}^{\text{poly}}(A)_{\bullet} \subset \text{CC}_{R, \oplus}(A)$ is a brace subalgebra. It only remains to show that the involution acts as $(-1)^j$ on $H_* \text{gr}_j^{\tau^{\text{HH}}} D_{\oplus}^{\text{poly}}(A)_{\bullet}^{\text{opp}}$. By [Lemma 2.12](#), these groups are $H_{*+j} \underline{\text{Hom}}_A(\Omega_A^j, A)$, on which the induced involution acts as $(-1)^j$ by [\[Pridham 2018, Lemma 1.15\]](#), giving quasi-involutivity. $\square$

2.3 Involutions from the Grothendieck–Teichmüller group

In order to exploit the structure provided by the anti-involutions on brace algebras, we need a formality isomorphism for the brace operad which intertwines the functors on brace algebras and Gerstenhaber

algebras sending algebras to their opposites. Drinfeld associators which are even are known to provide equivalences for Lie bialgebras satisfying an analogous property [Enriquez and Halbout 2010], and we now deduce similar results for brace algebras.

Heuristically, the idea is that the involution opp_{Br} of the brace operad which sends a brace algebra to its opposite corresponds under Dunn additivity to the involution $\text{id} * \text{opp}_{E_1}$ of $E_2 \simeq E_1 * E_1$ which sends E_1 -algebras in the second factor to their opposites. The group space of homotopy automorphisms of E_2 is homotopy equivalent to the orthogonal group $O(2, \mathbb{R})$ acting on discs in the obvious way, with both this involution and Drinfeld's involution corresponding to reflections, and thus being homotopic. Since the interaction of Dunn additivity with solutions of the Deligne conjecture is not fully set out, we will establish equivalence by a less direct argument.

By [Drinfeld 1990; Bar-Natan 1998] as interpreted in [Fresse 2017], the Grothendieck–Teichmüller group $\text{GT}(\mathbb{Q})$, defined as certain automorphisms of the pro-unipotent completion of the free group F_2 on two generators, acts naturally on a model for the operad $\mathbf{C}_\bullet(E_2, \mathbb{Q})$ of chains of E_2 as an operad in cocommutative dg coalgebras (equivalently, as a dg Hopf operad). GT is a pro-algebraic group with reductive quotient $\mathbb{G}_m$.

Definition 2.14 As in [Bar-Natan 1998, Section 4.1], define $P \in \text{GT}(\mathbb{Q})$ to be the automorphism sending each generator of F_2 to its inverse. This lies over $-1 \in \mathbb{G}_m$, and by [Drinfeld 1990, Proposition 4.1] is the unique nontrivial object-preserving automorphism of the operad (in groupoids) of parenthesised braids.

Lemma 2.15 *The brace operad Br equipped with its involution opp_{Br} is naturally quasi-isomorphic as a C_2 -equivariant $\mathbb{Q}$ -linear dg operad to the operad $\mathbf{C}_\bullet(E_2, \mathbb{Q})$ of chains of the little discs operad equipped with its involution $P \in \text{GT}$.*

Proof By [McClure and Smith 2002, Theorem 1.1], the brace operad Br is naturally $\mathbb{Q}$ -linearly quasi-isomorphic to $\mathbf{C}_\bullet(E_2, \mathbb{Q})$. As discussed in [Kaufmann 2007, Sections 3.2 and 4.5], there are models E'_2 for E_2 for which the involution opp_{Br} is induced by an involution t . In other words, there exist a topological operad E'_2 , a quasi-isomorphism $\mathbf{C}_\bullet(E'_2, \mathbb{Q}) \rightarrow \text{Br}$ intertwining t and opp_{Br} , and a zigzag of weak equivalences between E'_2 and E_2 .

Now, by [Horel 2017, Theorem 8.5], the group space $\text{Aut}^h(E_2)$ of homotopy automorphisms of E_2 is weakly equivalent to $O(2, \mathbb{R}) \simeq \{\pm 1\} \ltimes B\mathbb{Z}$, with the nontrivial class in $\pi_0 \text{Aut}^h(E_2)$ being $[P]$. We know that t cannot be homotopic to the identity because opp_{Br} acts nontrivially on $\text{gr}^\tau \text{Br} \simeq P_2$. Moreover, since the second cohomology group of C_2 with coefficients in the nontrivial representation on $\mathbb{Z}$ is zero, it follows that the homotopy fibre of the map $\mathbf{Rmap}_{\text{TopGp}}(C_2, \text{Aut}^h(E_2)) \rightarrow \text{Hom}_{\text{Gp}}(C_2, \pi_0 \text{Aut}^h(E_2))$ over the nontrivial homomorphism $C_2 \rightarrow \{\pm 1\}$ is connected (and in fact equivalent to $B\mathbb{Z}$, corresponding to the space of reflections inside $O(2, \mathbb{R})$). Thus after identification via the quasi-isomorphism $\mathbf{C}_\bullet(E_2, \mathbb{Q}) \simeq \text{Br}$ above, P and t must induce homotopic involutions $C_2 \rightarrow \text{Aut}^h(E_2)$. $\square$

Definition 2.16 Denote the pro-unipotent radical of the pro-algebraic group GT (defined over $\mathbb{Q}$) by GT^1 . Write $\mathrm{Levi}_{\mathrm{GT}}$ for the set of Levi decompositions $\mathrm{GT} \cong \mathbb{G}_m \ltimes \mathrm{GT}^1$; equivalently, this is the set of sections of the natural map $\mathrm{GT} \rightarrow \mathbb{G}_m$. We then define $\mathrm{Levi}_{\mathrm{GT}}^P \subset \mathrm{Levi}_{\mathrm{GT}}$ to be the set of sections w satisfying $w(-1) = P$.

Taking base change to arbitrary commutative $\mathbb{Q}$ -algebras A , we can extend this definition to give sets $\mathrm{Levi}_{\mathrm{GT}}(A)$ and $\mathrm{Levi}_{\mathrm{GT}}^P(A)$ of decompositions of the pro-algebraic group $\mathrm{GT} \times \mathrm{Spec} A$ over A .

Every Levi decomposition $\mathrm{GT} \cong \mathrm{GT}^1 \rtimes \mathbb{G}_m$ of GT automatically equips GT^1 with a grading. Since the natural $\mathbb{G}_m$ -action on the abelianisation of GT^1 has weight 1, each such grading splits the lower central series filtration on GT^1 , giving an isomorphism between GT^1 and the graded Grothendieck–Teichmüller group GRT^1 . Thus $\mathrm{Levi}_{\mathrm{GT}}$ is naturally isomorphic to the space of Drinfeld 1-associators. The subset $\mathrm{Levi}_{\mathrm{GT}}^P \subset \mathrm{Levi}_{\mathrm{GT}}$ then corresponds to 1-associators which are even in the sense of [Bar-Natan 1998, Section 4.1].

Lemma 2.17 *The functor $\mathrm{Levi}_{\mathrm{GT}}^P$ is an affine scheme over $\mathbb{Q}$ equipped with the structure of a trivial torsor for the subgroup scheme $(\mathrm{GT}^1)^P \subset \mathrm{GT}^1$ given by the centraliser of P .*

Proof We expand the argument from [Pridham 2018, Remark 2.22]. By the general theory [Hochschild and Mostow 1969] of pro-algebraic groups in characteristic 0, the set $\mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ is nonempty, and for all commutative $\mathbb{Q}$ -algebras A , the group $\mathrm{GT}^1(A)$ acts transitively on $\mathrm{Levi}_{\mathrm{GT}}(A)$ via the adjoint action. Because the graded quotients of the lower central series of GT^1 have nonzero weight for the adjoint $\mathbb{G}_m$ -action, the centralisers of this action are trivial and $\mathrm{Levi}_{\mathrm{GT}}(A)$ is a torsor for $\mathrm{GT}^1(A)$.

Now, choose any Levi decomposition $w_0 \in \mathrm{Levi}_{\mathrm{GT}}(\mathbb{Q})$ and let $w_0(-1) = Pu$ for $u \in \mathrm{GT}^1(\mathbb{Q})$. Since P and $w_0(-1)$ are both of order 2, we have $u = \mathrm{ad}_P(u^{-1})$. Writing $u = \exp(v)$ for v in the pro-nilpotent Lie algebra $\mathfrak{gt}^1$, and setting $u^{1/2} := \exp(\frac{1}{2}v)$, we have $u^{1/2} = \mathrm{ad}_P(u^{-1/2})$, giving $w := \mathrm{ad}_{u^{1/2}} \circ w_0 \in \mathrm{Levi}_{\mathrm{GT}}^P(\mathbb{Q})$. Thus $\mathrm{Levi}_{\mathrm{GT}}^P$ is nonempty, so $\mathrm{Levi}_{\mathrm{GT}}^P \subset \mathrm{Levi}_{\mathrm{GT}}$ is a torsor for the subgroup $(\mathrm{GT}^1)^P \subset \mathrm{GT}^1$ fixing P under the adjoint action. $\square$

Proposition 2.18 *Every 1-associator $w \in \mathrm{Levi}_{\mathrm{GT}}$ induces a zigzag θ_w of filtered quasi-isomorphisms between (Br, τ) and (P_2, τ) . If the 1-associator is even (ie $w \in \mathrm{Levi}_{\mathrm{GT}}^P$), then the quasi-isomorphisms are C_2 -equivariant, preserving the involutions given by $-1 \in \mathbb{G}_m$ on P_2^{ac} and by Definition 2.3 on Br .*

The quasi-isomorphism θ_w is compatible with the natural maps $s^{-1} \mathrm{Lie} \rightarrow \mathcal{P}_2$ and $s^{-1} \mathrm{Lie} \rightarrow \mathrm{Br}$ from the shifted Lie operad $s^{-1} \mathrm{Lie}$.

Proof As in Lemma 2.15, we have a zigzag of C_2 -equivariant quasi-isomorphisms between $(\mathrm{Br}, \mathrm{opp}_{\mathrm{Br}})$ and $(C_\bullet(B \mathrm{PaB}, \mathbb{Q}), P)$, the dg operad of chains on the operad of parenthesised braids, equipped with their respective involutions.

As explained succinctly in [Petersen 2014], formality of the operad $C_\bullet(B \text{ PaB}, \mathbb{Q})$ is a consequence of its GT-action and the observation that the Grothendieck–Teichmüller group GT is a pro-unipotent extension of $\mathbb{G}_m$. The pro-unipotent radical $GT^1 = \ker(GT \rightarrow \mathbb{G}_m)$ acts trivially on homology $H_*(B \text{ PaB}, \mathbb{Q}) \cong P_2$, inducing a $\mathbb{G}_m$ -action on P_2 .

Since the $\mathbb{G}_m$ -action has weight 0 on $H_0 \text{ Br}(2)$ and weight -1 on $H_1 \text{ Br}(2)$, and since these generate the whole operad, it follows that the $\mathbb{G}_m$ -action on $H_j \text{ Br}(k)$ has weight $-j$ for all j and k . Thus we have a $\mathbb{G}_m$ -equivariant isomorphism

$$H_*(B \text{ PaB}, \mathbb{Q}) \cong P_2^{\text{ac}}$$

with the $\mathbb{G}_m$ -equivariant operad $P_2^{\text{ac}} = \text{Com} \circ s^{-1} \hbar^{-1} \text{ Lie}$ of Definition 1.15. The element $P \in GT(\mathbb{Q})$ lies over $-1 \in \mathbb{G}_m(\mathbb{Q})$, so the involution P on Br induces the action of $-1 \in \mathbb{G}_m$ on P_2^{ac} under the isomorphism above.

Any Levi decomposition $w: \mathbb{G}_m \rightarrow GT$ gives a $\mathbb{G}_m$ -action on $C_\bullet(B \text{ PaB}, \mathbb{Q})$, ie a weight decomposition. As in [Petersen 2014], the weight decomposition associated to w thus induces a zigzag of $\mathbb{G}_m$ -equivariant quasi-isomorphisms between $C_\bullet(B \text{ PaB}, \mathbb{Q})$ and P_2^{ac} . The involution $-1 \in \mathbb{G}_m$ of the latter then necessarily corresponds to the involution $w(-1) \in GT$ of the former.

These quasi-isomorphisms of operads combine to give θ_w , and the quasi-isomorphisms respect the involutions when $w(-1) = P$. The good truncation filtration τ of Definition 2.4 is defined similarly on any dg operad in place of Br, and in particular there is a good truncation on the quasi-isomorphic $\mathbb{Q}$ -linear dg operads $C_\bullet(B \text{ PaB}, \mathbb{Q})$ and $C_\bullet(E_2, \mathbb{Q})$ of chains⁶ on the topological operad E_2 . Any chain map automatically preserves the good truncation filtration, so θ_w is a filtered quasi-isomorphism.

Finally, the natural morphism from the Lie operad to Br is given in each arity by inclusion of the top weight term for the decreasing filtration, ie

$$\{\text{Br}(n)\}_n \leftarrow \{\tau_{\geq n-1} \text{Br}(n)\}_n \xrightarrow{\sim} \{H_{n-1} \text{Br}(n)_{[1-n]}\}_n \cong \{\text{Lie}(n)_{[1-n]}\}_n = \{s^{-1} \text{Lie}(n)\}_n.$$

The same construction applied to each dg operad in the zigzag ensures that our map $(P_2, \tau) \rightarrow (\text{Br}, \tau)$ in the homotopy category of filtered dg operads (or C_2 -equivariant filtered dg operads for w even) respects the natural maps from the shifted Lie operad on each side. $\square$

Definition 2.19 Given a Levi decomposition $w \in \text{Levi}_{GT}(\mathbb{Q})$ of the pro-algebraic group GT over $\mathbb{Q}$, we denote by p_w the ∞ -functor from almost commutative brace algebras to almost commutative P_2 -algebras coming from the map $\theta_w: (P_2, \tau) \rightarrow (\text{Br}, \tau)$. This preserves the underlying filtered L_∞ -algebras up to equivalence.

When $w \in \text{Levi}_{GT}^P(\mathbb{Q})$, this induces an ∞ -functor from quasi-involutive almost commutative brace algebras to quasi-involutive almost commutative P_2 -algebras, which we also denote by p_w .

⁶The latter is presumably what is meant by the filtration given by the Postnikov tower in [Calaque et al. 2017, Section 3.5.1], since the Postnikov tower itself consists of quotients rather than subspaces, with kernels given by good truncation.

Explicitly, [Proposition 2.18](#) gives us a filtered quasi-isomorphism $\tilde{\theta}_w: \tilde{P}_2^{\text{ac}} \rightarrow (\text{Br}, \tau)$ from any $\mathbb{G}_m$ -equivariant cofibrant replacement $\tilde{P}_2^{\text{ac}}$ of $\tilde{P}_2$, and $\tilde{\theta}_w$ is involutive when w is even. This automatically gives any involutive almost commutative brace algebra the structure of an involutive almost commutative $\tilde{P}_2^{\text{ac}}$ -algebra, and we can then obtain p_w by applying the derived left adjoint of the forgetful functor from $\text{Rees}_W^{-1 \in \mathbb{G}_m}(P_2^{\text{ac}})$ -algebras to $\text{Rees}_W^{-1 \in \mathbb{G}_m}(\tilde{P}_2^{\text{ac}})$ -algebras induced by the morphism $\tilde{P}_2^{\text{ac}} \rightarrow P_2^{\text{ac}}$.

2.4 Existence of deformation quantisations

2.4.1 Formality

Lemma 2.20 Take a cofibrant R -CDGA (resp. $\mathcal{C}^\infty$ -DGA, resp. K -EFC-DGA) A with perfect cotangent complex $\Omega_{A/R}^1$ (resp. $\Omega_{A,\mathcal{C}^\infty}^1$, resp. $\Omega_{A/K,\text{EFC}}^1$). For the nonnegatively weighted $\mathbb{G}_m$ -equivariant P_2^{ac} -algebra $\text{Pol}(A, 0)$ of polyvectors (of [Definition 2.6](#), resp. [Definition 2.11](#)), the map

$$(\mathcal{W}_1 \mathbf{L}\Omega_{\text{Pol}(A,0)/A}^1) \otimes_A^{\mathbf{L}} \text{Pol}(A, 0) \rightarrow \mathbf{L}\Omega_{\text{Pol}(A,0)/A}^1$$

of commutative cotangent complexes is a quasi-isomorphism.

Proof Write $\Omega_{A,\square}^p$ for $\Omega_{A/R}^p$ (resp. $\Omega_{A,\mathcal{C}^\infty}^p$, resp. $\Omega_{A/K,\text{EFC}}^p$). Since $\Omega_{A,\square}^1$ is perfect, we have a $\mathbb{G}_m$ -equivariant quasi-isomorphism

$$\bigoplus_{p \geq 0} \mathbf{L}\text{Sym}_A^p(\text{Hom}_A(\Omega_{A,\square}^1, A)_{[1]}) \rightarrow \bigoplus_{p \geq 0} \text{Hom}_A(\Omega_{A,\square}^p, A)_{[p]}, \quad \mathbf{L}\text{Sym}_A(\mathcal{W}_1 \text{Pol}(A, 0)) \rightarrow \text{Pol}(A, 0)$$

of CDGAs, from which the statement follows immediately. $\square$

Theorem 2.21 Given a cofibrant R -CDGA (resp. $\mathcal{C}^\infty$ -DGA or K -EFC-DGA) A with perfect cotangent complex $\Omega_{A/R}^1$ (resp. $\Omega_{A,\mathcal{C}^\infty}^1$ or $\Omega_{A/K,\text{EFC}}^1$), the quasi-involutively filtered DGLA underlying the complex of polydifferential operators $D_{\oplus}^{\text{poly}}(A)_{[-1]}$ with the increasing filtration τ^{HH} of [Definition 2.10](#) is filtered quasi-isomorphic to the graded DGLA $\text{Pol}(A, 0)_{[-1]}$ from [Definition 2.6](#) (resp. [Definition 2.11](#)).

This quasi-isomorphism depends only on a choice of even 1-associator $w \in \text{Levi}_{\text{GT}}^P$, and is natural with respect to all morphisms $(D_{\oplus}^{\text{poly}}(A), \tau^{\text{HH}}) \rightarrow (D_{\oplus}^{\text{poly}}(A'), \tau^{\text{HH}})$ in the ∞ -category of quasi-involutive almost commutative brace algebras.

When A is an R -CDGA with perfect cotangent complex, the same statements hold for the Hochschild complex $\text{CC}_{R,\oplus}(A)$ in place of $D_{\oplus}^{\text{poly}}(A)$.

Proof [Lemma 2.13](#) shows that $(D_{\oplus}^{\text{poly}}(A), \tau^{\text{HH}})$ is a quasi-involutive almost commutative brace algebra. We have a quasi-isomorphism $\text{HKR}: \text{gr}^{\tau^{\text{HH}}} D_{\oplus}^{\text{poly}}(A) \xrightarrow{\sim} \text{Pol}(A, 0)$ of graded gr^τ Br-algebras by [Lemma 2.12](#).

[Lemmas 2.8](#) and [2.7](#) give the corresponding statements for $\text{CC}_{R,\oplus}(A)$ in the algebraic setting; since $D_{\oplus}^{\text{poly}}(A) \rightarrow \text{CC}_{R,\oplus}(A)$ is thus a filtered quasi-isomorphism, it suffices to prove the statements for polydifferential operators in place of the Hochschild complex.

Applying the ∞ -functor p_w of [Definition 2.19](#) for some even associator $w \in \text{Levi}_{\text{GT}}^P(\mathbb{Q})$ (or even a point in the space $\text{Levi}_{\text{GT}}^P(R)$) gives a quasi-involutive almost commutative P_2 -algebra $p_w D_{\oplus}^{\text{poly}}(A)$ together with a zigzag of quasi-isomorphisms of $\mathbb{G}_m$ -equivariant P_2^{ac} -algebras between its associated graded algebra and $\text{Pol}(A, 0)$.

By [Lemma 2.20](#), $\mathbf{L}\Omega_{\text{Pol}(A,0)/A}^1$ satisfies the conditions of [Corollary 1.19](#), giving an essentially unique equivalence of involutive almost commutative P_2 -algebras $\alpha_{w,A}: p_w D_{\oplus}^{\text{poly}}(A) \simeq p_w \text{gr}^{\tau^{\text{HH}}} D_{\oplus}^{\text{poly}}(A)$, natural with respect to quasi-involutive almost commutative brace algebra morphisms of $D_{\oplus}^{\text{poly}}(A)$. Composition with the HKR quasi-isomorphism then gives us the required equivalence

$$\text{HKR} \circ \alpha_{w,A}: p_w D_{\oplus}^{\text{poly}}(A) \simeq \text{Pol}(A, 0). \quad \square$$

Remark 2.22 When applied to polynomial rings in the algebraic setting and to $C^\infty(\mathbb{R}^n)$ in the smooth setting, the statement of [Theorem 2.21](#) recovers [[Kontsevich 1999](#), Theorem 4; [Tamarkin 1999](#), Section 3]. For more general smooth varieties it recovers [[Van den Bergh 2007](#), Theorem 1.1]. The preliminary steps are the same, but the arguments for eliminating the potential first-order deformation are very different, as we consider anti-involutive deformations while Tamarkin and Kontsevich looked at invariance under affine transformations, which do not exist even locally for our more general rings.

Complete intersection singularities have perfect cotangent complexes, so [Theorem 2.21](#) also promotes Kontsevich's quasi-isomorphism [[2007](#), Proposition 1] to an L_∞ quasi-isomorphism.

2.4.2 Affine quantisations

Definition 2.23 Given a differential graded Lie algebra (DGLA) L with homological grading, define the Maurer–Cartan set by

$$\text{MC}(L) := \{\omega \in L_{-1} \mid \delta\omega + \frac{1}{2}[\omega, \omega] = 0 \in L_{-2}\}.$$

Following [[Hinich 2001](#)], define the Maurer–Cartan space $\underline{\text{MC}}(L)$ (a simplicial set) of a nilpotent DGLA L by

$$\underline{\text{MC}}(L)_n := \text{MC}(L \otimes_{\mathbb{Q}} \Omega(\Delta^n)_\bullet),$$

where

$$\Omega(\Delta^n)_\bullet = \mathbb{Q}[t_0, t_1, \dots, t_n, \delta t_0, \delta t_1, \dots, \delta t_n] / \left(\sum t_i - 1, \sum \delta t_i \right)$$

is the commutative dg algebra of de Rham polynomial forms on the n -simplex, with the t_i of degree 0 and δt_i of chain degree -1 , so $\Omega(\Delta^n)_{-m} = \Omega^m(\Delta^n)$, the space of m -forms.

Given an inverse system $L = \{L_\alpha\}_\alpha$ of nilpotent DGLAs, define

$$\text{MC}(L) := \varprojlim_\alpha \text{MC}(L_\alpha), \quad \underline{\text{MC}}(L) := \varprojlim_\alpha \underline{\text{MC}}(L_\alpha).$$

Note that $\text{MC}(L) = \text{MC}(\varprojlim_\alpha L_\alpha)$, but $\underline{\text{MC}}(L) \neq \underline{\text{MC}}(\varprojlim_\alpha L_\alpha)$.

Definition 2.24 Given a cofibrant R -CDGA A (resp. $\mathcal{C}^\infty$ -DGA or EFC-DGA), as in [Pridham 2017, Definition 1.1] we complete the P_2 -algebra $\text{Pol}(A, 0)$ of Definition 2.6 (resp. Definition 2.11) to give a P_2 -algebra

$$\widehat{\text{Pol}}(A, 0) := \prod_{p \geq 0} \mathcal{W}_p \text{Pol}(A, 0)$$

of polyvectors. This is $\prod_{p \geq 0} \underline{\text{Hom}}_A(\Omega_{A/R}^p, A)_{[p]}$ in the algebraic setting and corresponding expressions using $\mathcal{C}^\infty$ or EFC differentials in the other settings.

This is not graded, but does have a decreasing filtration

$$F^i \widehat{\text{Pol}}(A, 0) := \prod_{p \geq i} \mathcal{W}_p \text{Pol}(A, 0).$$

The space $\mathcal{P}(A, 0)$ of 0-shifted Poisson structures on A is then defined in [Pridham 2017, Definition 1.5] to be

$$\underline{\text{MC}}(F^2 \widehat{\text{Pol}}(A, n)_{[-1]});$$

this is equivalent to the space of P_1 -algebras with underlying CDGA quasi-isomorphic to A .

Definition 2.25 Given a cofibrant R -CDGA, $\mathcal{C}^\infty$ -DGA or K -EFC-DGA A , adapting [Pridham 2018, Definition 1.16] as in [Pridham 2020a; 2020b], we define the DGLA $Q \widehat{\text{Pol}}(A, 0)_{[-1]}$ of quantised polyvectors by setting

$$Q \widehat{\text{Pol}}(A, 0) := \prod_{p \geq 0} \tau_p^{\text{HH}} D_{\oplus}^{\text{poly}}(A)_{\bullet} \hbar^{p-1};$$

observe that because the Gerstenhaber bracket satisfies $[\tau_p^{\text{HH}}, \tau_q^{\text{HH}}] \subset \tau_{p+q-1}^{\text{HH}}$, this is indeed a DGLA.

Define a decreasing filtration $\tilde{F}$ on $Q \widehat{\text{Pol}}(A, 0)$ by the subcomplexes

$$\tilde{F}^i Q \widehat{\text{Pol}}(A, 0) := \prod_{j \geq i} \tau_j^{\text{HH}} D_{\oplus}^{\text{poly}}(A) \hbar^{j-1}.$$

This filtration is complete and Hausdorff, with $[\tilde{F}^i, \tilde{F}^j] \subset \tilde{F}^{i+j-1}$, and in particular this makes $\tilde{F}^2 Q \widehat{\text{Pol}}(A, 0)_{[-1]}$ into a pro-nilpotent filtered DGLA.

The space $Q\mathcal{P}(A, 0)$ of 0-shifted quantisations of A is then defined to be (adapting [Pridham 2018, Definition 1.23])

$$\underline{\text{MC}}(\tilde{F}^2 Q \widehat{\text{Pol}}(A, n)_{[-1]}).$$

The subspace $Q\mathcal{P}(A, 0)^{\text{sd}} \subset Q\mathcal{P}(A, 0)$ of self-dual quantisations then consists of fixed points for the involution $(-)^*$ given by $\Delta^*(\hbar) := i(\Delta)(-\hbar)$, for the involution i of Lemma 2.8.

Remark 2.26 Since for an R -CDGA A , the inclusion $D_{\oplus}^{\text{poly}}(A)_{\bullet} \subset \text{CC}_{R, \oplus}(A)_{\bullet}$ is a filtered quasi-isomorphism, replacing polyvectors with the Hochschild complex gives an equivalent construction in the algebraic setting. Similarly the filtration γ from [Pridham 2022, Definition 1.18] is quasi-isomorphic to τ^{HH} , so the definitions are also equivalent to those of [loc. cit.].

As in [loc. cit., Remark 2.15], $\mathcal{QP}(A, 0)$ is thus equivalent to the space of curved BD_1 -algebras (almost commutative $\hbar$ -adic associative algebras) deforming A ; when A lies in nonnegative chain degrees no curvature is possible on objects, but curvature still leads to additional higher morphisms coming from inner automorphisms. The objects in the $\mathcal{C}^\infty$ and EFC settings admit a similar interpretation, but with additional constraints from the restriction to polydifferential operators.

As in [loc. cit., Remark 2.15], $\mathcal{QP}(A, 0)^{\text{sd}}$ consists of curved BD_1 -algebras $\tilde{A}$ deforming A and equipped with an anti-involution $\tilde{A}^{\text{opp}} \cong \tilde{A}$ which is semilinear under the transformation $\hbar \mapsto -\hbar$.

Corollary 2.27 *Given a cofibrant R -CDGA (resp. $\mathcal{C}^\infty$ -DGA or K -EFC-DGA) A with perfect cotangent complex $\Omega_{A/R}^1$ (resp. $\Omega_{A,\mathcal{C}^\infty}^1$ or $\Omega_{A/K,\text{EFC}}^1$), the space $\mathcal{QP}(A, 0)$ of 0-shifted quantisations of A is equivalent to the Maurer–Cartan space*

$$\begin{aligned} \underline{\text{MC}}\left(\left(A_{[-1]}\hbar \times \underline{\text{Hom}}_A(\Omega_{A,\square}^1, A)\hbar \times \prod_{p \geq 2} \underline{\text{Hom}}_A(\Omega_{A,\square}^p, A)_{[p-1]}\hbar^{p-1}\right)[[\hbar]]\right) \\ \cong \underline{\text{MC}}(F^2 \widehat{\text{Pol}}(A, 0)_{[-1]} \times \hbar F^1 \widehat{\text{Pol}}(A, 0)_{[-1]} \times \hbar^2 \widehat{\text{Pol}}(A, 0)_{[-1]}[[\hbar]]), \end{aligned}$$

where the DGLA structure comes from the Schouten–Nijenhuis bracket, and $\Omega_{A,\square}^p$ is $\Omega_{A/R}^p$ (resp. $\Omega_{A,\mathcal{C}^\infty}^p$ or $\Omega_{A/K,\text{EFC}}^p$).

The subspace $\mathcal{QP}(A, 0)^{\text{sd}} \subset \mathcal{QP}(A, 0)$ of self-dual quantisations is then equivalent to the Maurer–Cartan space

$$\begin{aligned} \underline{\text{MC}}\left(\left(A_{[-1]}\hbar \times \underline{\text{Hom}}_A(\Omega_{A,\square}^1, A)\hbar^2 \times \prod_{p \geq 2} \underline{\text{Hom}}_A(\Omega_{A,\square}^p, A)_{[p-1]}\hbar^{p-1}\right)[[\hbar^2]]\right) \\ \cong \underline{\text{MC}}(F^2 \widehat{\text{Pol}}(A, 0)_{[-1]} \times \hbar^2 \widehat{\text{Pol}}(A, 0)_{[-1]}[[\hbar^2]]). \end{aligned}$$

In particular, there exist self-dual associative quantisations for every Poisson structure on A .

Proof The quasi-isomorphism of Theorem 2.21 gives us a quasi-isomorphism

$$\tilde{F}^2 \mathcal{Q} \widehat{\text{Pol}}(A, 0)_{[-1]} \simeq \prod_{p \geq 2, i \leq p} \mathcal{W}_i \text{Pol}(A, 0)_{[-1]} \hbar^{p-1} = \prod_{i \geq 0} \hbar^{\max(i-1, 1)} \mathcal{W}_i \text{Pol}(A, 0)_{[-1]}[[\hbar]]$$

of pro-nilpotent filtered DGLAs, and hence an equivalence of the respective Maurer–Cartan spaces. We then use the isomorphism

$$F^2 \widehat{\text{Pol}}(A, 0) \times \hbar F^1 \widehat{\text{Pol}}(A, 0) \times \hbar^2 \widehat{\text{Pol}}(A, 0)[[\hbar]] \xrightarrow{\cong} \prod_i \hbar^{\max(i-1, 1)} \mathcal{W}_i \text{Pol}(A, 0)[[\hbar]]$$

given by multiplying $\mathcal{W}_i \text{Pol}(A, 0)$ by $\hbar^{i-1}$; this gives a DGLA morphism because the Schouten–Nijenhuis bracket satisfies $[\mathcal{W}_i, \mathcal{W}_j] \subset \mathcal{W}_{i+j-1}$.

Now, the involution $*$ acts on $H_*(\text{gr}_i^{\tau^{\text{HH}}} \text{CC}_R(A))\hbar^j$ and on $\mathcal{W}_i \text{Pol}(A, 0)\hbar^j$ as $(-1)^{i+j-1}$. Under the isomorphism above given by division by $\hbar^{i-1}$, this corresponds to the involution $\hbar \mapsto -\hbar$ of $F^2 \widehat{\text{Pol}}(A, 0) \times \hbar F^1 \widehat{\text{Pol}}(A, 0) \times \hbar^2 \widehat{\text{Pol}}(A, 0)[[\hbar]]$, so the fixed points are $F^2 \widehat{\text{Pol}}(A, 0) \times \hbar^2 \widehat{\text{Pol}}(A, 0)[[\hbar^2]]$.

Existence of quantisations then follows by inclusion of the first term in $F^2 \widehat{\mathcal{P}\mathcal{O}\mathcal{L}}(A, 0) \times \hbar^2 \widehat{\mathcal{P}\mathcal{O}\mathcal{L}}(\mathcal{O}, 0) \llbracket \hbar^2 \rrbracket$, giving a morphism $\mathcal{P}(A, 0) \rightarrow Q\mathcal{P}(A, 0)^{\text{sd}} \subset Q\mathcal{P}(A, 0)$. $\square$

Remark 2.28 In principle, the proof of [Corollary 2.27](#) could be used to construct deformation quantisations. It would be similar to explicit application of [\[Tamarkin 1999\]](#) to smooth varieties, for which the author is unaware of any examples. After choosing an even associator, the first step would be to describe the resulting strong homotopy P_2 -algebra operations on Hochschild cochains as combinations of brace operations. One would then have to solve iteratively for the essentially unique involutive filtered equivalence between that strong homotopy P_2 -algebra and the algebra of polyvectors, with the resulting L_∞ equivalence generating quantisations from Poisson structures.

2.4.3 Étale functoriality revisited Functoriality for Hochschild complexes and polyvectors is subtle, but exists with respect to homotopy étale morphisms as in [\[Pridham 2018, Section 3.1; 2017, Section 2.1.2\]](#). We now revisit and generalise the results there.

Definition 2.29 Given a category $\mathcal{C}$, let $\int BC$ be the Grothendieck construction of the nerve of $\mathcal{C}$. Explicitly, objects of $\int BC$ are pairs $(m, A: [m] \rightarrow \mathcal{C}) = (m, A(0) \rightarrow A(1) \rightarrow \cdots \rightarrow A(m))$ for $m \geq 0$, and a morphism $u: (m, A: [m] \rightarrow \mathcal{C}) \rightarrow (n, B: [n] \rightarrow \mathcal{C})$ is a morphism $u: \mathbf{n} \rightarrow \mathbf{m}$ in the simplex category such that $B = A \circ u$.

Lemma 2.30 The category $\mathcal{C}$ is equivalent to the simplicial localisation of $\int BC$ at the class $\mathcal{W}_0$ of morphisms $u: (m, A: [m] \rightarrow \mathcal{C}) \rightarrow (n, A \circ u: [n] \rightarrow \mathcal{C})$ with $u(0) = 0 \in \mathbf{m}$.

Proof The functor $\rho: \int BC \rightarrow \mathcal{C}$ is given by $\rho(m, A) = A(0)$ on objects and $\rho(u) = A(\gamma): A(0) \rightarrow A(u(0))$ on morphisms $u: (m, A: [m] \rightarrow \mathcal{C}) \rightarrow (n, A \circ u: [n] \rightarrow \mathcal{C})$, for the unique morphism $\gamma: 0 \rightarrow u(0)$ in $\mathbf{m}$.

In order to show that this induces an equivalence $L^{\mathcal{W}_0}(\int BC) \simeq \mathcal{C}$ of simplicial categories, by [\[Dwyer and Kan 1987, Theorem 2.2\]](#) it suffices to show that the functor $\rho^*: (s\text{Set})^{\mathcal{C}} \rightarrow (s\text{Set})^{\int BC, \mathcal{W}_0}$ from the model category of simplicial set-valued functors on $\mathcal{C}$ to the model category of $\mathcal{W}_0$ -restricted $\int BC$ -diagrams is a right Quillen equivalence.

Now, giving a simplicial set-valued functor F on $\int BC$ is equivalent to giving a bisimplicial set $\int F$ over the nerve BC , with $(\int F)_n := \coprod_{x \in B_m \mathcal{C}} F(x)$. Thus the category $(s\text{Set})^{\int BC, \mathcal{W}_0}$ is equivalent to $ss\text{Set} \downarrow BC$. Moreover, F sends morphisms in $\mathcal{W}_0$ to weak equivalences if and only if the maps $F(n, A(0) \rightarrow \cdots \rightarrow A(n)) \rightarrow F(0, A(0))$ are weak equivalences, which amounts to saying that the maps

$$\left(\int F \right)_n \xrightarrow{(\partial_1)^n} \left(\int F \right)_0 \times_{B_0 \mathcal{C}, (\partial_1)^n} B_n \mathcal{C}$$

are weak equivalences. When $\int F$ is Reedy fibrant, this is equivalent to saying that $\int F$ is a left fibration over BC in the sense of [\[Boavida de Brito 2018\]](#).

The equivalence $\int^{-1}: s\text{Set} \downarrow BC \rightarrow (s\text{Set})^{\int BC, \mathcal{W}_0}$ of categories is a right Quillen functor for the left fibration model structure on $s\text{Set} \downarrow BC$, and the observation above implies that it is a Quillen equivalence. Composing this with the right Quillen equivalence of [loc. cit., Theorem A] (in its variant for left fibrations) then gives the required result. $\square$

If $\mathcal{C}$ is the category of R -CDGAs, EFC-DGAs, or $\mathcal{C}^\infty$ -DGAs, then for any diagram $D: I \rightarrow \mathcal{C}$, we can define $D_{\oplus}^{\text{poly}}(D)$ and $\text{Pol}(D, 0)$ by substituting $\underline{\text{Hom}}_B(M, N)$ with the equaliser of the obvious diagram

$$\prod_{i \in I} \underline{\text{Hom}}_{B(i)}(M(i), N(i)) \Rightarrow \prod_{f: i \rightarrow j \text{ in } I} \underline{\text{Hom}}_{B(i)}(M(i), f_* N(j))$$

throughout the corresponding definitions for algebras. These constructions behave well for diagrams $D: [m] \rightarrow \mathcal{C}_{c, \twoheadrightarrow}$ to the subcategory $\mathcal{C}_{c, \twoheadrightarrow} \subset \mathcal{C}$ of cofibrant objects and surjective morphisms. Since a morphism $u: [n] \rightarrow [m]$ naturally induces a morphism $u^\#: D_{\oplus}^{\text{poly}}(D) \rightarrow D_{\oplus}^{\text{poly}}(u^* D)$ by restriction, the constructions $D \mapsto (D_{\oplus}^{\text{poly}}(D), \tau^{\text{HH}})$ and $D \mapsto \text{Pol}(D, 0)$ define functors on the Grothendieck construction $\int BC_{c, \twoheadrightarrow}$.

Since all objects of $\mathcal{C}$ are fibrant, the functor $\mathcal{C}_{c, \twoheadrightarrow} \rightarrow \mathcal{C}$ induces a weak equivalence of simplicial categories on localisation at weak equivalences. We are unaware of a reference for this fact, but [Dwyer and Kan 1980, Proposition 5.2] gives a closely related statement with a very similar proof. Since the category admits a right homotopy calculus of fractions, the key to the argument is that for any morphism $f: A \rightarrow B$ in $\mathcal{C}_c$, factorising the graph of f leads to a weakly equivalent span $(A \xleftarrow{\sim} P_f A \rightarrow B)$ in $\mathcal{C}_{c, \twoheadrightarrow}$ with the first map a trivial fibration. The homotopy equivalence is given by the span $(A \xleftarrow{\sim} P_f A \rightarrow P_{\text{id}_B} B)$ (which lies in $\mathcal{C}_{c, \twoheadrightarrow}$ whenever f is a fibration) and its canonical levelwise trivial fibrations to the span above and to the span $(A = A \rightarrow B)$.

When $u(0) = 0$ and the morphisms in D are all homotopy étale, the morphism $u^\#$ is moreover a filtered quasi-isomorphism, so the restriction of $D_{\oplus}^{\text{poly}}$ to the subcategory $\int B(\mathcal{C}_{c, \twoheadrightarrow}, \text{ét})$ of homotopy étale morphisms sends morphisms in $\mathcal{W}_0$ to filtered quasi-isomorphisms. It then follows from Lemma 2.30 and quasi-isomorphism invariance that $(D_{\oplus}^{\text{poly}}, \tau^{\text{HH}})$ and $\text{Pol}(-, 0)$ define ∞ -functors on the ∞ -category $\mathbf{LC}^{\text{ét}}$ of R -CDGAs, EFC-DGAs, or $\mathcal{C}^\infty$ -DGAs, and homotopy étale morphisms; the HKR quasi-isomorphism is then a natural transformation between them. Taking Maurer–Cartan elements as in Definition 2.25 then gives an ∞ -functor $Q\mathcal{P}(-, 0)$ from $\mathbf{LC}^{\text{ét}}$ to the ∞ -category of simplicial sets.

2.4.4 Global quantisations Incorporating the homotopy étale functoriality of Section 2.4.3 into the functoriality of Theorem 2.21 immediately leads to the following generalisation of Corollary 2.27 on taking derived global sections on the étale site:

Corollary 2.31 *Given a derived DM n -stack $\mathfrak{X}$ over R with perfect cotangent complex $\mathbf{L}\Omega_{\mathfrak{X}/R}^1$, the space $Q\mathcal{P}(\mathfrak{X}, 0) := \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, Q\mathcal{P}(\mathcal{O}, 0))$ of 0-shifted quantisations of $\mathfrak{X}$ in [Pridham 2018, Definitions 1.23*

and 3.9] and its subspace $Q\mathcal{P}(\mathfrak{X}, 0)^{\text{sd}}$ of self-dual (or involutive) quantisations in [loc. cit., Definition 1.33] are respectively equivalent to the Maurer–Cartan spaces

$$\begin{aligned} & \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \underline{\text{MC}}(F^2 \widehat{\text{Pol}}(\mathcal{O}, 0)_{[-1]} \times \hbar F^1 \widehat{\text{Pol}}(\mathcal{O}, 0)_{[-1]} \times \hbar^2 \widehat{\text{Pol}}(\mathcal{O}, 0)_{[-1]}[[\hbar]]), \\ & \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \underline{\text{MC}}(F^2 \widehat{\text{Pol}}(\mathcal{O}, 0)_{[-1]} \times \hbar^2 \widehat{\text{Pol}}(\mathcal{O}, 0)_{[-1]}[[\hbar^2]]). \end{aligned}$$

In particular, every Poisson structure $\pi \in \mathcal{P}(\mathfrak{X}, 0) = \mathbf{R}\Gamma(\mathfrak{X}_{\text{ét}}, \underline{\text{MC}}(F^2 \widehat{\text{Pol}}(\mathcal{O}, 0)_{[-1]}))$ admits self-dual quantisations in the form of almost commutative curved A_∞ -deformations $\mathcal{A}_\hbar$ of $\mathcal{O}_\mathfrak{X}$ with $\mathcal{A}_{-\hbar} \simeq \mathcal{A}_\hbar^{\text{opp}}$.

The analogous statements for derived $\mathcal{C}^\infty$ and derived analytic DM n -stacks (in the sense of [Pridham 2020a]) with perfect cotangent complexes also hold.

Remarks 2.32 The hypotheses of Corollary 2.31 are satisfied by any derived Deligne–Mumford stack locally of finite presentation over the CDGA R . When $R = H_0 R$, this includes underived schemes X which are local complete intersections over R , in which case the cotangent complex $\mathbf{L}\Omega_{X/R}^1$ is concentrated in homological degrees $[0, 1]$. For such underived schemes, a quantisation in the sense of the corollary reduces to the usual notion, namely a DQ algebroid deformation of $\mathcal{O}_X$ over $R[[\hbar]]$. For more details, see [Pridham 2022, Remarks 2.15, 4.10, and 4.17].

In analytic settings, Corollary 2.31 similarly gives DQ algebroid quantisations for local complete intersections. Since $\mathcal{C}^\infty$ spaces tend to embed in affine space under mild finiteness hypotheses, Corollary 2.27 also gives strict quantisations for LCI $\mathcal{C}^\infty$ spaces.

Our hypothesis that $\mathfrak{X}$ have perfect cotangent complex cannot be removed, since Mathieu’s example [1997] gives a nonquantisable Poisson structure on a non-LCI scheme.

Remark 2.33 The space $Q\mathcal{P}^+(\mathfrak{X}, 0)$ of strict deformation quantisations, that is, quantisations by BD_1 -algebras (associative almost commutative algebras) rather than algebroids, is given by replacing $Q\widehat{\text{Pol}}(\mathcal{O}_\mathfrak{X}, 0)$ with the kernel $\tilde{F}^2 Q\widehat{\text{Pol}}^+(\mathcal{O}_\mathfrak{X}, 0)$ of the natural map $\tilde{F}^2 Q\widehat{\text{Pol}}(\mathcal{O}_\mathfrak{X}, 0) \rightarrow \hbar \mathcal{O}_\mathfrak{X}[[\hbar]]$, and similarly by $\tilde{F}^2 Q\widehat{\text{Pol}}^+(\mathcal{O}_\mathfrak{X}, 0)^{\text{sd}} := \ker(\tilde{F}^2 Q\widehat{\text{Pol}}(\mathcal{O}_\mathfrak{X}, 0) \rightarrow \hbar \mathcal{O}_\mathfrak{X}[[\hbar^2]])$ for $Q\mathcal{P}^+(\mathfrak{X}, 0)^{\text{sd}}$. These maps are not in general compatible with the formality quasi-isomorphism of Corollary 2.31, but they are pro-nilpotent surjections of DGLAs, and so lead to homotopy fibre sequences

$$\begin{aligned} Q\mathcal{P}^+(\mathfrak{X}, 0) & \rightarrow Q\mathcal{P}(\mathfrak{X}, 0) \rightarrow \mathbf{R}\Gamma(\mathfrak{X}, \hbar \mathcal{O}_\mathfrak{X}[[\hbar]])_{[-2]}, \\ Q\mathcal{P}^+(\mathfrak{X}, 0)^{\text{sd}} & \rightarrow Q\mathcal{P}(\mathfrak{X}, 0)^{\text{sd}} \rightarrow \mathbf{R}\Gamma(\mathfrak{X}, \hbar \mathcal{O}_\mathfrak{X}[[\hbar^2]])_{[-2]}. \end{aligned}$$

In particular, this means that if $H^2(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) = 0$ then every (self-dual) quantisation in Corollary 2.31 comes from a strict (self-dual) quantisation.

2.4.5 Quantisation of 1-shifted coisotropic structures In [Melani and Safronov 2018, Section 5.3], existence of quantisations for n -shifted coisotropic structures is established as a direct consequence of the formality of the E_n operad, for $n > 1$. Theorem 2.21 allows us to establish the corresponding result for 1-shifted coisotropic structures.

Before defining quantised coisotropic structures, we need a few preliminaries. We will use the term A_∞ -brace algebra to refer to algebras B for the brace operad Br of [Willwacher 2016, Section 3.4], corresponding to the restricted class of B_∞ -algebras from [Getzler and Jones 1994, Section 5.2]; this enlarges the class of brace algebras from Section 2.1 by having an A_∞ -algebra structure rather than just an associative cup product, and can be encoded as a bialgebra structure on the dg coalgebra $\text{B}_{\text{Ass}} B$ given by the bar construction, with restrictions.

Definition 2.34 Given a 0-shifted quantisation $\Delta \in \text{MC}(\tilde{F}^2 Q \widehat{\text{Pol}}(A, 0)[1])$, define the centre of (A, Δ) by

$$T_\Delta Q \widehat{\text{Pol}}(A, 0) := \prod_{p \geq 0} F_p \text{CC}^\bullet_R(A) \hbar^p,$$

with derivation $\delta \pm b + [\Delta, -]$ (necessarily square-zero by the Maurer–Cartan conditions). The A_∞ operations $\smile + \{\Delta\}\{\}_2$ and $\{\Delta\}\{\}_i$ for $i \geq 3$, and unchanged brace operations $\{\}\{\}_j$ for $j \geq 1$, make this an A_∞ brace algebra. Explicitly, this structure corresponds to twisting the differential on the bialgebra $\text{B}_{\text{Ass}} T_0 Q \widehat{\text{Pol}}(A, 0)$ by the commutator of Δ with respect to the product induced by the brace operations.

Similarly, given a 0-shifted Poisson structure $\pi \in \text{MC}(F^2 \widehat{\text{Pol}}(A, 0))$, define the centre of (A, π) by

$$T_\pi \widehat{\text{Pol}}(A, 0) := (\widehat{\text{Pol}}(A, 0), \delta_{\widehat{\text{Pol}}} + [\pi, -]),$$

with the same commutative product and shifted Lie bracket as $\widehat{\text{Pol}}(A, 0)$ making this a P_2 -algebra.

We now take the model for BD_2 given by the completed Rees construction of the good truncation filtration on the A_∞ brace operad, so a BD_2 -algebra is an almost commutative A_∞ brace algebra in complete $R[[\hbar]]$ -modules equipped with the $\hbar$ -adic filtration.

The following is slightly weaker than the $n = 1$ case of [Melani and Safronov 2018, Definition 5.13] since we cannot require the quantisation of $\mathfrak{X}$ to be strict.

Definition 2.35 A 1-shifted quantised coisotropic structure on a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of (algebraic) derived DM n -stacks consists of

- (1) a quantised 0-shifted Poisson structure Δ on $\mathfrak{X}$,
- (2) a quantised 1-shifted Poisson structure on $\mathfrak{Y}$ in the form of a BD_2 -algebra $\tilde{\mathcal{O}}_{\mathfrak{Y}}$ deforming the CDGA $\mathcal{O}_{\mathfrak{Y}}$ in the sense that it is equipped with a CDGA quasi-isomorphism $\tilde{\mathcal{O}}_{\mathfrak{Y}} \otimes_{R[[\hbar]]}^{\mathbf{L}} R \rightarrow \mathcal{O}_{\mathfrak{Y}}$, and
- (3) a BD_2 -algebra morphism

$$F: f^{-1} \tilde{\mathcal{O}}_{\mathfrak{Y}} \rightarrow T_\Delta Q \widehat{\text{Pol}}(\mathcal{O}_{\mathfrak{X}}, 0)$$

of étale hypersheaves on $\mathfrak{X}$, where the target is defined using the functoriality of Section 2.4.3.

We say that this quantisation is self-dual if Δ is self-dual, the BD_2 -algebra $\tilde{\mathcal{O}}_{\mathfrak{Y}}$ is given a quasi-involutive structure, and F intertwines the involutions.

Note that on reducing modulo $\hbar$ and applying the HKR quasi-isomorphism, this gives a 0-shifted Poisson structure π on $\mathfrak{X}$, a strong homotopy P_2 -algebra structure ϖ on the CDGA $\mathcal{O}_{\mathfrak{Y}}$ (ie a 1-shifted Poisson structure), and a zigzag $f^{-1}\mathcal{O}_{\mathfrak{Y}} \xleftarrow{\sim} f^{-1}\tilde{\mathcal{O}}_{\mathfrak{Y}}/\hbar \rightarrow T_{\pi}\widehat{\mathrm{Pol}}(\mathcal{O}_{\mathfrak{X}}, 0)$ of strong homotopy P_2 -algebra morphisms, with the first a quasi-isomorphism. This last is equivalent to giving a morphism $\rho: f^{-1}\mathcal{O}_{\mathfrak{Y}} \rightarrow T_{\pi}\widehat{\mathrm{Pol}}(\mathcal{O}_{\mathfrak{X}}, 0)$ in the ∞ -category of P_2 -algebras, so (π, ϖ, ρ) are precisely the data of a 1-shifted coisotropic structure on f in the sense of [Melani and Safronov 2018].

Corollary 2.36 *Given a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of (algebraic) derived DM n -stacks such that $\mathfrak{X}$ has perfect cotangent complex, every 1-shifted coisotropic structure on f admits self-dual quantisations in the sense of Definition 2.35.*

Proof The twisting in Definition 2.35 commutes with formality, by [Dolgushev and Willwacher 2015, Theorem 1.1]. Alternatively, we can interpret (F, Δ) as a curved brace morphism, with any formality equivalence for the unital brace operad as a Hopf operad compatibly inducing formality for the curved brace operad by Koszul duality.

Applying the ∞ -equivalence p_w for any even associator w , and then composing with the filtered quasi-isomorphism $\mathrm{HKR} \circ \alpha_{w,B}$ from the proof of Theorem 2.21, we thus see that a quantisation is equivalent to the data of Theorem 2.21 together with a formal deformation $p_w\tilde{\mathcal{O}}_{\mathfrak{Y}}$ of $\mathcal{O}_{\mathfrak{Y}}$ over $R[[\hbar]]$ equipped with an $R[[\hbar]]$ -linear 1-shifted Poisson structure, and a morphism

$$f^{-1}p_w\tilde{\mathcal{O}}_{\mathfrak{Y}} \rightarrow \left(\prod_{p \geq 0} \mathrm{L}\Lambda_{\mathcal{O}_{\mathfrak{X}}}^p(T_{\mathfrak{X}/R})_{[p]}\hbar^p[[\hbar]], \delta + \{\alpha_w\Delta, -\} \right)$$

of strong homotopy P_2 -algebras over $R[[\hbar]]$. In the notation of [Melani and Safronov 2018], this is a formal $P_{[2,1]}[[\hbar]]$ -algebra deformation of a $P_{[2,1]}$ -algebra (ie 1-shifted coisotropic) structure on f , except that we allow our $P_{[2,1]}[[\hbar]]$ -algebra to incorporate $\hbar^2$ -curvature in the underlying $P_1[[\hbar]]$ -algebra deformation of $\mathcal{O}_{\mathfrak{X}}$. When the quantisation is self-dual, the deformation is invariant under the involution $\hbar \mapsto -\hbar$, or equivalently induced by a deformation over $R[[\hbar^2]]$.

In particular, every $P_{[2,1]}$ -algebra structure $\mathcal{P}$ on $(\mathcal{O}_{\mathfrak{Y}}, f^{-1}\mathcal{O}_{\mathfrak{Y}} \rightarrow \mathcal{O}_{\mathfrak{X}})$ admits a self-dual quantisation corresponding under the equivalence above to the trivial $P_{[2,1]}[[\hbar]]$ -algebra deformation $\mathcal{P}[[\hbar]]$ of $\mathcal{P}$. $\square$

Remark 2.37 (analytic and $\mathcal{C}^\infty$ analogues) For structures defined as in Definition 2.35, the proof of Corollary 2.36 is still valid for analytic and $\mathcal{C}^\infty$ derived DM n -stacks. However, such structures are too weak to quantise the correct notion of coisotropic structures in those cases, since the strong homotopy P_2 -algebra $\tilde{\mathcal{O}}_{\mathfrak{Y}}/\hbar$ and strong homotopy P_2 -algebra morphism $F/\hbar$ should be defined in terms of EFC or $\mathcal{C}^\infty$ multiderivations.

Instead of constructing BD_2 -algebras and morphisms in the category of complexes over $R[[\hbar]]$, the solution is to work in the pretriangulated dg category generated by $\mathcal{O}_{\mathfrak{Y}}[[\hbar]]$ -modules and differential operators. Although this dg category is not monoidal, we can still define algebras by endowing it with the dg multicategory (ie coloured dg operad) structure in which multioperations $(M_1, \dots, M_r) \rightarrow N$

are given by (EFC or $\mathcal{C}^\infty$) polydifferential operators. Because an EFC or $\mathcal{C}^\infty$ polydifferential operator which is an algebraic derivation is automatically an EFC or $\mathcal{C}^\infty$ multiderivation, P_k -algebras in that dg multicategory do correspond to $(k-1)$ -shifted Poisson structures.

Working in that dg multicategory and recasting [Definition 2.35](#) accordingly, the proof of [Corollary 2.36](#) then gives genuine deformation quantisations of 1-shifted coisotropic structures in analytic and $\mathcal{C}^\infty$ settings.

3 Quantisations on derived Artin stacks

When applied to (derived) Artin stacks, the definition of Poisson structures in [\[Pridham 2017, Section 3\]](#) or [\[Calaque et al. 2017\]](#) and their quantisations in [\[Pridham 2018, Definitions 1.23 and 3.9\]](#) is more subtle than for derived DM stacks, since polyvectors and the Hochschild complex are not functorial with respect to smooth morphisms.

This is resolved in [\[Pridham 2017, Section 3.1\]](#) by observing that the formal completion of a derived Artin stack $\mathfrak{X}$ along an affine atlas $f: U \rightarrow \mathfrak{X}$ with f smooth can be recovered from a commutative algebra in double complexes.

These lead to sufficiently functorial constructions of polyvectors and Hochschild complexes, but only after passing to sum–product total complexes Tôt (Tate realisations in the terminology of [\[Calaque et al. 2017\]](#)). Generalising quantisation results to derived Artin stacks is thus far from straightforward, and to establish them we introduce an intermediate category $\mathcal{U}_{P_k^{\text{ac}}}[\hbar^2]$ in [Section 3.3](#) which delicately balances the requirements of functoriality and deformations.

Remark 3.1 While the results of [Section 1](#) easily adapt to P_k^{ac} -algebras in double complexes, the P_{n+2} -algebra of n -shifted polyvectors only satisfies the analogue of [Proposition 1.18](#) before applying Tôt . In terms of our notation below, $\mathcal{P}ol(-, n)$ —roughly corresponding to $\text{Pol}^{\text{int}}(-, n)$ in [\[loc. cit.\]](#)— is not sufficiently functorial, while the algebra $\text{Tôt}_{\mathbb{G}_m} \mathcal{P}ol(-, n)$ —roughly corresponding to $\text{Pol}^t(-, n)$ in [\[loc. cit.\]](#)— cannot simply be described as derived symmetric powers of the tangent complex, so we cannot constrain its deformations.

3.1 Double complexes and stacky Hochschild complexes

Definition 3.2 A stacky CDGA is a chain cochain complex (ie a double complex)

$$A_\bullet = (A_\bullet^0 \xrightarrow{\partial} A_\bullet^1 \xrightarrow{\partial} A_\bullet^2 \xrightarrow{\partial} \cdots)$$

equipped with a commutative product $A \otimes A \rightarrow A$ and unit $\mathbb{Q} \rightarrow A$. Given a chain CDGA R , a stacky CDGA over R is then a morphism $R \rightarrow A$ of stacky CDGAs, where we silently regard R as a stacky CDGA concentrated in cochain degree 0.

A stacky $\mathcal{C}^\infty$ -DGA (resp. EFC-DGA over K) is a stacky CDGA over $\mathbb{R}$ (resp. K) $A_\bullet$ equipped with a $\mathcal{C}^\infty$ -DGA (resp. EFC-DGA) structure on $A_\bullet^0$ and such that $\partial: A_0^0 \rightarrow A_0^1$ is a $\mathcal{C}^\infty$ -derivation (resp. EFC-derivation) in the sense of [\[Pridham 2020a, Definition 2.9\]](#).

As explained in [Pridham 2017, Remark 3.32], these correspond to the “graded mixed” CDGAs of [Calaque et al. 2017] (but beware that the latter are something of a misnomer, not having mixed differentials). The structure in the chain direction encodes derived information, while the cochain direction encodes stacky structure.

For general derived Artin n -stacks, these formal completions are constructed in [Pridham 2017, Section 3.1] by forming affine hypercovers as in [Pridham 2013], and then applying the functor D^* (left adjoint to denormalisation) to obtain a stacky CDGA. When $\mathfrak{X}$ is a derived Artin 1-stack, the formal completion of an affine atlas $U \rightarrow \mathfrak{X}$ is simply given by the relative de Rham complex

$$O(U) \xrightarrow{\partial} \Omega_{U/\mathfrak{X}}^1 \xrightarrow{\partial} \Omega_{U/\mathfrak{X}}^2 \xrightarrow{\partial} \cdots,$$

which arises by applying the functor D^* to the Čech nerve of U over $\mathfrak{X}$.

Lemma 3.3 *There is a cofibrantly generated model structure on the category of cochain chain complexes in which fibrations are surjections and weak equivalences are levelwise quasi-isomorphisms in the chain direction. For any chain operad $\mathcal{P}$, this induces a cofibrantly generated model structure on $\mathcal{P}$ -algebras in cochain chain complexes, in which fibrations and weak equivalences are those of the underlying cochain chain complexes.*

Similarly, for any $\mathbb{G}_m$ -equivariant chain operad $\mathcal{P}$, there is a cofibrantly generated model structure on the category of $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebras in cochain chain complexes with the same cofibrations, fibrations, and weak equivalences.

Proof This follows as in the proof of [Pridham 2017, Lemma 3.4]. Each generating (trivial) cofibration in the non- $\mathbb{G}_m$ -equivariant setting gives rise to a $\mathbb{Z}$ -indexed family of generating (trivial) cofibrations in the $\mathbb{G}_m$ -equivariant setting, corresponding to choices of weights for the generators. $\square$

Note that since $\mathcal{P}$ is assumed to be a chain operad, the operations for the $\mathcal{P}$ -algebras A in Lemma 3.3 are maps

$$\mathcal{P}(n)_i \otimes A_{j_1}^{m_1} \otimes \cdots \otimes A_{j_n}^{m_n} \rightarrow A_{i+(j_1+\cdots+j_n)}^{m_1+\cdots+m_n}.$$

We refer to such $\mathcal{P}$ -algebras in cochain chain complexes as stacky $\mathcal{P}$ -algebras; beware that this conflicts slightly with the terminology of Definition 3.2 because we allow stacky $\mathcal{P}$ -algebras to have terms of negative cochain degree, while our stacky CDGAs are concentrated in nonnegative cochain degrees.

We also have a more subtle variant:

Lemma 3.4 *Given a chain CDGA R , there is a cofibrantly generated model structure on the category of $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebras A over R of nonnegative weights (ie $\mathcal{W}_i A = 0$ for all $i < 0$), in which fibrations are surjections and weak equivalences are levelwise quasi-isomorphisms.*

Moreover, the forgetful functor from this category to the category of $\mathbb{G}_m$ -equivariant stacky R -CDGAs preserves cofibrant objects.

Proof We apply [Hirschhorn 2003, Theorem 11.3.2] to the forgetful functor mapping to nonnegatively weighted $\mathbb{G}_m$ -equivariant double complexes of $\mathbb{Q}$ -vector spaces. The only nontrivial condition to check for the first statement is that the left adjoint F sends trivial cofibrations to levelwise quasi-isomorphisms. For the second statement it suffices to show that for the class C of cofibrations of double complexes, pushouts of morphisms in FC over FC -cells (ie transfinite iterated pushouts of morphisms in FC) are cofibrations of $\mathbb{G}_m$ -equivariant stacky R -CDGAs.

Since the forgetful functor factors through the category of nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky $s^{1-k}\hbar^{-1}$ Lie-algebras (by forgetting the commutative R -algebra structure but keeping the Lie bracket), the left adjoint F factors as $(R \otimes_{\mathbb{Q}} \text{Sym}_{\mathbb{Q}}) \circ L_+$, where L_+ is the left adjoint to the forgetful functor from that intermediate category. For the Lie bracket of chain degree $k-1$ and $\mathbb{G}_m$ -weight -1 , the functor L_+ sends a $\mathbb{G}_m$ -equivariant double complex V to the quotient $L(V)/(\mathcal{W}_{<0}L(V))$ of the free graded Lie algebra $L(V)$ (with Koszul signs) by the Lie ideal generated by terms of negative weight.

Now, a cofibration of double complexes is an injective map $U \hookrightarrow V$ for which the quotient V/U is acyclic in the cochain direction (ie $H^i(V_j/U_j) = 0$ for all i and j); it is trivial if it is moreover acyclic in the chain direction. On V/U there is thus a contracting cochain homotopy $h: (V/U)_{\#}^{\#} \rightarrow V/U_{\#}^{\#[-1]}$ such that the graded commutator $[\partial, h]$ is the identity, and if the cofibration is trivial, there is also a contracting chain homotopy $h': (V/U)_{\#}^{\#} \rightarrow (V/U)_{\#[1]}^{\#}$ such that $[\delta, h']$ is the identity. If we give V an increasing filtration with $\text{Fil}_{-1}V = 0$, $\text{Fil}_0V = U$, and $\text{Fil}_1V = V$, then the Lie bracket induces an exhaustive increasing filtration Fil on $L_+(V)$, with $\text{gr}^{\text{Fil}} L_+(V) \cong L_+(U \oplus (V/U))$.

We can then define a derivation

$$H: \text{gr}^{\text{Fil}} L_+(V)_{\#}^{\#} \rightarrow \text{gr}^{\text{Fil}} L_+(V)_{\#}^{\#[-1]} \quad (\text{resp. } H': \text{gr}^{\text{Fil}} L_+(V)_{\#}^{\#} \rightarrow \text{gr}^{\text{Fil}} L_+(V)_{\#[1]}^{\#}),$$

given on generators by 0 on U and h (resp. h') on V/U . It follows that the commutator $[\partial, H]$ (resp. $[\delta, H]$) is a derivation acting as 0 on generators U and as the identity on generators V/U , so it must act as multiplication by p on $\text{gr}_p^F L_+(V)$.

The maps $\text{Fil}_p L_+V \rightarrow \text{Fil}_{p+1} L_+V$ are thus cofibrations (resp. trivial cofibrations) of double complexes for all $p \geq 0$, since $h/(p+1)$ and $H/(p+1)$ provide the necessary homotopies on the quotient, so $L_+U \rightarrow L_+V$ is a cofibration (resp. trivial cofibration) of double complexes. Passing to symmetric powers over R then gives that $F(U) \rightarrow F(V)$ is a cofibration (resp. trivial cofibration) of $\mathbb{G}_m$ -equivariant CDGAs, proving the first statement.

For the second statement, observe that cochain acyclicity of the quotient allows us to split any cofibration of double complexes if we turn off the chain differential. Any FC -cell A thus has the form $(F(U_{\#}), \delta)$ for some differential δ , and a pushout B of A along a map in FC takes the form $(F(U_{\#} \oplus V_{\#}), \delta)$ with $\delta(U \oplus V) \subset A$ and V acyclic in the cochain direction. Taking the increasing filtration Fil of $L_+(U \oplus V)$ by powers of V , the quotients $\text{gr}_p^{\text{Fil}} L_+(U_{\#} \oplus V_{\#})$ are cochain acyclic with $\delta(\text{Fil}_{p+1}) \subset \text{Sym}(\text{Fil}_p)$, so

taking symmetric powers gives a sequence of cofibrations $\mathrm{Symm}(\mathrm{Fil}_p) \rightarrow \mathrm{Symm}(\mathrm{Fil}_{p+1})$ of stacky sub-CDGAs of B ; composing them all implies that the morphism $A \rightarrow B$ is a cofibration of $\mathbb{G}_m$ -equivariant stacky CDGAs. $\square$

Hochschild complexes and multiderivations on stacky CDGAs are then defined in terms of the semi-infinite total complex:

Definition 3.5 The subcomplex $\mathrm{T\hat{o}t} V \subset \mathrm{Tot}^\Pi V$ is given by

$$(\mathrm{T\hat{o}t} V)_m := \left(\bigoplus_{i < 0} V_{i+m}^i \right) \oplus \left(\prod_{i \geq 0} V_{i+m}^i \right)$$

with differential $\partial \pm \delta$. This functor has the properties that it maps levelwise quasi-isomorphisms in the chain direction to quasi-isomorphisms, and that it is lax monoidal.

Applied to the internal Hom functor $\mathcal{H}om$, this construction gives chain complexes

$$\underline{\mathrm{Hom}}_R(M, N) := \mathrm{T\hat{o}t} \mathcal{H}om_R(M, N)$$

for cochain complexes M and N of R -modules in chain complexes, and hence a dg enhancement of the monoidal category of R -cochain chain complexes.

Definition 3.6 For a stacky DGAA A (ie an associative algebra in double complexes) over a chain CDGA R , we define the internal cohomological Hochschild complex $\mathcal{C}C_{R,\oplus}(A)$ by replacing $\underline{\mathrm{Hom}}$ with $\mathcal{H}om$ in [Definition 2.1](#) (as in [\[Pridham 2022, Definition 2.7\]](#)) to give a chain cochain complex. Note that for the conventions we have chosen, this means that the Hochschild differential is acting in the chain direction.

We similarly define $\mathcal{D}_\oplus^{\mathrm{poly}}(A) \subset \mathcal{C}C_{R,\oplus}(A)$ to be the double subcomplex of polydifferential operators (including the $\mathcal{C}^\infty$ setting with $R = \mathbb{R}$ and the EFC setting with $R = K$, a complete valued field), defined exactly as in [Definition 2.10](#), but with an additional grading coming from the cochain grading on A .

The filtration τ^{HH} on $\mathcal{C}C_{R,\oplus}(A)$ and $\mathcal{D}_\oplus^{\mathrm{poly}}(A)$ is then given by good truncation in the Hochschild direction.

Definition 3.7 For A a cofibrant stacky R -CDGA (resp. stacky $\mathcal{C}^\infty$ -DGA, resp. stacky EFC-DGA), define the $\mathbb{G}_m$ -equivariant stacky P_{n+2}^{ac} -algebra $\mathcal{P}ol(A, n)$ by

$$\mathcal{P}ol(A, n) := \bigoplus_{p \geq 0} \mathcal{H}om_A(\mathrm{CoS}_A^p((\Omega_{A,\square}^1)_{[-n-1]}), A),$$

where p is the weight for the $\mathbb{G}_m$ -action, the commutative multiplication and Lie bracket are defined in the usual way for polyvectors, and $\Omega_{A,\square}^p$ is $\Omega_{A/R}^p$ (resp. $\Omega_{A,\mathcal{C}^\infty}^p$, resp. $\Omega_{A/K,\mathrm{EFC}}^p$).

In particular, note that $\mathcal{P}ol(A, 0) = \bigoplus_{p \geq 0} \mathcal{H}om_A(\Omega_{A,\square}^p, A)_{[p]}$.

3.2 Étale functoriality and $\underline{\text{Hom}}$ -equivalences

We now consider conditions for a morphism $f: P \rightarrow Q$ of A -modules to be a $\underline{\text{Hom}}_A$ -homotopy equivalence, ie for the homology class $[f] \in H_0 \underline{\text{Hom}}_A(P, Q)$ to have an inverse in $H_0 \underline{\text{Hom}}_A(Q, P)$. Writing $\sigma^{\leq n} M := M / \sigma^{>n} M$ for the brutal cotruncation in the cochain direction, we have:

Lemma 3.8 *If A is a stacky CDGA concentrated in nonnegative cochain degrees, and P and Q are cofibrant A -modules in double complexes, with the chain complexes $(P \otimes_A A^0)^i$ and $(Q \otimes_A A^0)^i$ zero for all $i < r$ and acyclic for all $i > s$, then a morphism $f: P \rightarrow Q$ is a $\underline{\text{Hom}}_A$ -homotopy equivalence whenever the map $\text{Tot } \sigma^{\leq s}(P \otimes_A A^0) \rightarrow \text{Tot } \sigma^{\leq s}(Q \otimes_A A^0)$ is a quasi-isomorphism. Under this condition, the morphism $f^*: \mathcal{H}om_A(Q, A) \rightarrow \mathcal{H}om_A(P, A)$ is also a $\underline{\text{Hom}}_A$ -homotopy equivalence.*

Proof Since P is cofibrant, brutal truncation $\{\sigma^{\geq p} M\}_p$ of any A -module M in cochain chain complexes M induces surjections $\mathcal{H}om_A(P, \sigma^{\geq i} M) \rightarrow \mathcal{H}om_A(P, \sigma^{\geq i-1} M)$ with kernels $\mathcal{H}om_A(P, M^i)^{[-i]}$. The second hypothesis on P implies that $(\mathcal{H}om_A(P, M^i)^{[-i]})^m$ is acyclic for $i > m + s$, thus giving a quasi-isomorphism

$$\mathcal{H}om_A(P, M)^m \cong \varprojlim_n \mathcal{H}om_A(P, \sigma^{\leq n} M)^m \xrightarrow{\sim} \mathcal{H}om_A(P, \sigma^{\leq m+s} M)^m.$$

Moreover, if $M^i = 0$ for all $i < t$, then $\mathcal{H}om_A(P, \sigma^{\leq m+s} M)^m = 0$ for all $m < t - s$, so $\mathcal{H}om_A(P, M)^m \simeq 0$ for all $m < t - s$. Applying Tot then gives

$$\underline{\text{Hom}}(P, M) \simeq \text{Tot}^\Pi \sigma^{\geq t-s} \mathcal{H}om(P, M).$$

Now, we have quasi-isomorphisms $\text{Tot } \sigma^{\leq s'}(P \otimes_A A^0) \rightarrow \text{Tot } \sigma^{\leq s'}(Q \otimes_A A^0)$ for all $s' \geq s$, giving quasi-isomorphisms $\text{Tot}^\Pi \sigma^{\geq -s'} \mathcal{H}om_A(Q, M^i) \rightarrow \text{Tot}^\Pi \sigma^{\geq -s'} \mathcal{H}om_A(P, M^i)$ for all i . Writing M as the limit of the tower $\cdots \rightarrow \sigma^{\leq t+1} M \rightarrow \sigma^{\leq t} M = (M^t)^{[-t]}$ and setting $s' = s + i - t$, these give a quasi-isomorphism

$$\text{Tot}^\Pi \sigma^{\geq t-s} \mathcal{H}om_A(Q, M) \rightarrow \text{Tot}^\Pi \sigma^{\geq t-s} \mathcal{H}om_A(P, M).$$

For all M bounded below in the cochain direction, we therefore have quasi-isomorphisms

$$f^*: \underline{\text{Hom}}_A(Q, M) \rightarrow \underline{\text{Hom}}_A(P, M).$$

The first hypothesis on P implies that $P \cong \sigma^{\geq r} P$, so it is bounded below. We may therefore take $M = P$, giving a class $[g] \in H_0 \underline{\text{Hom}}_A(Q, P)$ with $f^*[g] = [\text{id}]$. Thus $[g]$ is inverse to $[f] \in H_0 \underline{\text{Hom}}_A(P, Q)$, so f is indeed a $\underline{\text{Hom}}_A$ -homotopy equivalence.

Finally, observe that the contravariant functor $\mathcal{H}om_A(-, A)$ on A -modules is a $\mathcal{H}om_A$ -enriched functor, having natural maps

$$\mathcal{H}om_A(M, N) \rightarrow \mathcal{H}om_A(\mathcal{H}om_A(N, A), \mathcal{H}om_A(M, A))$$

for all A -modules M and N in chain cochain complexes, compatible with composition. Applying Tôt then gives maps

$$\underline{\mathrm{Hom}}_A(M, N) \rightarrow \underline{\mathrm{Hom}}_A(\mathrm{Hom}_A(N, A), \mathrm{Hom}_A(M, A))$$

compatible with composition, so $[f]$ and $[g]$ give rise to mutually inverse elements of

$$\mathrm{H}_0 \underline{\mathrm{Hom}}_A(\mathrm{Hom}_A(Q, A), \mathrm{Hom}_A(P, A)) \quad \text{and} \quad \mathrm{H}_0 \underline{\mathrm{Hom}}_A(\mathrm{Hom}_A(P, A), \mathrm{Hom}_A(Q, A)). \quad \square$$

If D denotes an $[m]$ -diagram $(A(0) \xrightarrow{f_1} A(1) \xrightarrow{f_2} \dots \xrightarrow{f_m} A(m))$ of cofibrant stacky CDGAs with all f_i surjective, then adapting [Pridham 2018, Section 3.1] (cf Section 2.4.3), gives a stacky involutive almost commutative brace algebra $\mathcal{CC}_{R,\oplus}(D)$ (resp. $\mathcal{D}_{\oplus}^{\mathrm{poly}}(D)$) with restriction maps

$$\mathcal{CC}_{R,\oplus}(D) \rightarrow \mathcal{CC}_{R,\oplus}(u^* D) \quad \text{and} \quad \mathcal{D}_{R,\oplus}^{\mathrm{poly}}(D) \rightarrow \mathcal{D}_{R,\oplus}^{\mathrm{poly}}(u^* D)$$

for all maps $u: [m'] \rightarrow [m]$. For $\mathcal{C}$ the relevant category of stacky CDGAs, EFC-DGAs, or $\mathcal{C}^\infty$ -DGAs, these combine to give a functor on the Grothendieck construction $\int B\mathcal{C}_{c,\rightarrow}$ of the nerve (cf Section 2.4.3).

Similarly, setting

$$\mathcal{P}ol(A, B; n) := \bigoplus_{p \geq 0} \mathrm{Hom}_A(\mathrm{CoS}_A^p((\Omega_{A,\square}^1)_{[-n-1]}), B),$$

where $\Omega_{A,\square}^1$ is $\Omega_{A/R}^1$ (resp. $\Omega_{A,\mathcal{C}^\infty}^1$, resp. $\Omega_{A/K,\mathrm{EFC}}^1$) and $\mathcal{P}ol(D, n)$ is

$$\mathcal{P}ol(A(0), n) \times_{\mathcal{P}ol(A(0), A(1); n)} \mathcal{P}ol(A(1), n) \times_{\mathcal{P}ol(A(1), A(2); n)} \dots \times_{\mathcal{P}ol(A(m-1), A(m); n)} \mathcal{P}ol(A(m), n)$$

gives a stacky P_{n+2}^{ac} -algebra with restriction maps

$$\mathcal{P}ol(D, n) \rightarrow \mathcal{P}ol(u^* D, n)$$

for all maps $u: [m'] \rightarrow [m]$.

As in [Pridham 2018, Definition 1.7], we say that a map $A \rightarrow B$ of stacky CDGAs is homotopy formally étale if the map

$$\{\mathrm{Tot} \sigma^{\leq q}(\mathbf{L}\Omega_A^1 \otimes_A^{\mathbf{L}} B^0)\}_q \rightarrow \{\mathrm{Tot} \sigma^{\leq q}(\mathbf{L}\Omega_B^1 \otimes_B^{\mathbf{L}} B^0)\}_q$$

is a pro-quasi-isomorphism, where $\sigma^{\leq q}$ denotes the brutal cotruncation. When the maps f_i are all homotopy formally étale, the map $\mathcal{P}ol(A(0) \xrightarrow{f_1} A(1) \xrightarrow{f_2} \dots \xrightarrow{f_n} A(n), n) \rightarrow \mathcal{P}ol(A(0), n)$ induced by the inclusion $[0] \rightarrow [n]$ becomes a quasi-isomorphism on applying Tôt, as shown in [Pridham 2018, Section 3.1]. However, it will not usually be a levelwise filtered quasi-isomorphism. Crucially for us, a slightly stronger property than Tôt-quasi-isomorphism holds, as we will see in Lemma 3.10 below.

Definition 3.9 Given a stacky CDGA A and A -modules P and Q in cochain chain complexes, we define $\mathbf{R}\underline{\mathrm{Hom}}_A(P, Q)$ to be any of the quasi-isomorphic complexes $\underline{\mathrm{Hom}}_A(\tilde{P}, Q)$ given by replacing P with a cofibrant replacement $\tilde{P} \rightarrow P$ in the model structure of Lemma 3.3; all choices are quasi-isomorphic because all objects are fibrant, so quasi-isomorphisms between cofibrant objects are homotopy equivalences in the strict model structure, inducing quasi-isomorphisms on Hom and hence on $\underline{\mathrm{Hom}}$.

We say that a map $f: P \rightarrow Q$ is an $\mathbf{RH}\hat{\mathbf{om}}_A$ -homotopy equivalence if the induced homology class $[f] \in H_0 \mathbf{RH}\hat{\mathbf{om}}_A(P, Q)$ has an inverse in $H_0 \mathbf{RH}\hat{\mathbf{om}}_A(Q, P)$.

Lemma 3.10 Take a diagram $D = (A(0) \xrightarrow{f_1} A(1) \xrightarrow{f_2} \dots \xrightarrow{f_m} A(m))$ of cofibrant stacky R -CDGAs (resp. stacky EFC-DGAs, resp. stacky $\mathcal{C}^\infty$ -DGAs) concentrated in nonnegative cochain degrees with all f_i surjective, such that

- (1) there exists $s \geq 0$ for which the chain complexes $(\Omega_{A(i), \square}^1 \otimes_{A(i)} A(i)^0)^r$ are acyclic for all $r > s$, and
- (2) the maps f_i are all homotopy formally étale in the sense that the maps

$$\mathrm{Tot} \sigma^{\leq s}(\Omega_{A(i-1), \square}^1 \otimes_{A(i-1)} A(i)^0) \rightarrow \mathrm{Tot} \sigma^{\leq s}(\Omega_{A(i), \square}^1 \otimes_{A(i)} A(i)^0)$$

are quasi-isomorphisms of $A(i)^0$ -modules.

Then the natural morphisms

$$\mathcal{W}_i \mathcal{P}ol(D, n) \rightarrow \mathcal{W}_i \mathcal{P}ol(A(0), n)$$

are all $\mathbf{RH}\hat{\mathbf{om}}_{A(0)}$ -homotopy equivalences.

Proof Since each f_i is surjective and taking symmetric invariants is an exact functor, the question reduces to showing that for $A = A(i-1)$ and $B = A(i)$, the maps

$$\mathcal{H}om_B((\Omega_{B, \square}^1)^{\otimes B^p}, B) \rightarrow \mathcal{H}om_A((\Omega_{A, \square}^1)^{\otimes A^p}, B) \cong \mathcal{H}om_B((\Omega_{A, \square}^1)^{\otimes A^p} \otimes_A B, B)$$

are $\mathbf{RH}\hat{\mathbf{om}}_{A(0)}$ -homotopy equivalences for all $i > 0$.

The boundedness hypotheses ensure that the chain complexes $((\Omega_{A, \square}^1)^{\otimes p} \otimes_A A^0)^i$ and $((\Omega_{B, \square}^1)^{\otimes p} \otimes_B B^0)^i$ are acyclic for $i > sp$. Combined with the homotopy formally étale hypothesis, this ensures that the maps $(\Omega_{A, \square}^1)^{\otimes p} \otimes_A B \rightarrow (\Omega_{B, \square}^1)^{\otimes p}$ satisfy the conditions of Lemma 3.8, so are $\hat{\mathbf{H}}\mathbf{om}_B$ -homotopy equivalences, and hence $\mathbf{RH}\hat{\mathbf{om}}_{A(0)}$ -homotopy equivalences a fortiori. $\square$

Lemma 3.11 If $A \rightarrow B$ is a morphism of cofibrant $\mathbb{G}_m$ -equivariant stacky CDGAs with nonnegative $\mathbb{G}_m$ -weights such that $\mathcal{W}_0 A \rightarrow \mathcal{W}_0 B$ is a levelwise quasi-isomorphism and the morphisms $\mathcal{W}_i A \rightarrow \mathcal{W}_i B$ are $\hat{\mathbf{H}}\mathbf{om}_{\mathcal{W}_0 A}$ -homotopy equivalences for all $i \geq 0$, then the morphisms

$$\mathcal{W}_i(\Omega_A^1 \otimes_A \mathcal{W}_0 A) \rightarrow \mathcal{W}_i(\Omega_B^1 \otimes_B \mathcal{W}_0 B)$$

are $\hat{\mathbf{H}}\mathbf{om}_{\mathcal{W}_0 A}$ -homotopy equivalences for all i .

Proof Since $\mathcal{W}_0 A \rightarrow \mathcal{W}_0 B$ is a levelwise quasi-isomorphism, so too is $\Omega_{\mathcal{W}_0 A}^1 \rightarrow \Omega_{\mathcal{W}_0 B}^1$. Using the exact sequence $0 \rightarrow \Omega_{\mathcal{W}_0 A}^1 \otimes_{\mathcal{W}_0 A} A \rightarrow \Omega_A^1 \rightarrow \Omega_{A/\mathcal{W}_0 A}^1 \rightarrow 0$, we can thus replace Ω_A^1 and Ω_B^1 with $\Omega_{A/\mathcal{W}_0 A}^1$ and $\Omega_{B/\mathcal{W}_0 B}^1$ to give an equivalent statement.

Because A is cofibrant and $\mathcal{W}_{>0} A$ is the augmentation ideal of the morphism $A \rightarrow \mathcal{W}_0 A$, we have $\Omega_{A/\mathcal{W}_0 A}^1 \otimes_A \mathcal{W}_0 A \cong (\mathcal{W}_{>0} A)/((\mathcal{W}_{>0} A) \cdot (\mathcal{W}_{>0} A))$, so the Koszul resolution (equivalently, the commutative bar construction as in the proof of Lemma 1.17) gives a canonical $\mathcal{W}_0 A$ -module resolution of $\Omega_{A/\mathcal{W}_0 A}^1 \otimes_A \mathcal{W}_0 A$ equipped with an increasing filtration whose graded pieces are $\mathrm{CoLie}_n(\mathcal{W}_{>0} A_{[-1]})[1]$.

In weight i , this becomes a finite filtration only involving terms with $n \leq i$, so it suffices to observe that the maps $\mathcal{W}_j \operatorname{CoLie}_{\mathcal{W}_0 A}^n(\mathcal{W}_{>0} A_{[-1]}) \rightarrow \mathcal{W}_j \operatorname{CoLie}_{\mathcal{W}_0 B}^n(\mathcal{W}_{>0} B_{[-1]})$ are all $\underline{\operatorname{Hom}}_{\mathcal{W}_0 A}$ -homotopy equivalences because they are finite tensor expressions in the terms $\mathcal{W}_k A$ and $\mathcal{W}_k B$. $\square$

3.3 Poisson L_∞ -morphisms in the Tate category

For this section, we fix a chain CDGA R over $\mathbb{Q}$ to act as our base ring.

We now introduce a variant of Definition 3.5 incorporating nonnegatively weighted $\mathbb{G}_m$ -actions.

Definition 3.12 Define the nonnegatively weighted Tate dg category $\mathcal{T}_{R,\operatorname{dg}}^+$ as follows. Objects are $\mathbb{G}_m$ -equivariant R -modules in chain cochain complexes for which the $\mathbb{G}_m$ -weights are nonnegative. Morphisms are given by the complexes

$$\mathcal{T}_{R,\operatorname{dg}}^+(M, N) := \prod_{i \geq 0} \underline{\operatorname{Hom}}_R(\mathcal{W}_i M, \mathcal{W}_i N)$$

with the obvious composition rule.

The nonnegatively weighted Tate category $\mathcal{T}_R^+$ is defined to have the same objects, but with morphisms $Z_0 \mathcal{T}_{R,\operatorname{dg}}^+(M, N)$.

Note that the tensor product defines a bifunctor on these categories, since $\hat{\operatorname{T}}\hat{\operatorname{t}}$ commutes with finite direct sums, so

$$\mathcal{T}_{R,\operatorname{dg}}^+(M \otimes_R M', P) = \prod_{i \geq 0, j \geq 0} \underline{\operatorname{Hom}}_R(\mathcal{W}_i M \otimes_R \mathcal{W}_j M', \mathcal{W}_{i+j} P),$$

meaning that the maps

$$\underline{\operatorname{Hom}}_R(\mathcal{W}_i M, \mathcal{W}_i N) \otimes_R \underline{\operatorname{Hom}}_R(\mathcal{W}_j M', \mathcal{W}_j N') \rightarrow \underline{\operatorname{Hom}}_R(\mathcal{W}_i M \otimes_R \mathcal{W}_j M', \mathcal{W}_{i+j}(N \otimes N'))$$

coming from lax monoidality of $\hat{\operatorname{T}}\hat{\operatorname{t}}$ induce natural maps

$$\mathcal{T}_{R,\operatorname{dg}}^+(M, N) \otimes_R \mathcal{T}_{R,\operatorname{dg}}^+(M', N') \rightarrow \mathcal{T}_{R,\operatorname{dg}}^+(M \otimes_R M', N \otimes_R N').$$

Note that our hypothesis that the $\mathbb{G}_m$ -weights be nonnegative is essential for this to hold, since otherwise the expression $\mathcal{W}_n(M \otimes_R M') = \bigoplus_{i+j=n} \mathcal{W}_i M \otimes_R \mathcal{W}_j M'$ would not be finite.

Also observe that $\mathcal{T}_{R,\operatorname{dg}}^+$ has a dg-subcategory consisting of $\mathbb{G}_m$ -equivariant R -linear morphisms of double complexes, which we refer to as strict morphisms, given by $\prod_{i \geq 0} Z^0 \operatorname{Hom}_R(\mathcal{W}_i M, \mathcal{W}_i N) \subset \mathcal{T}_{R,\operatorname{dg}}^+(M, N)$.

Definition 3.13 Given a stacky CDGA A equipped with a $\mathbb{G}_m$ -action of nonnegative weights, we define $\mathcal{T}_{A,\operatorname{dg}}^+$ to be the dg category whose objects are $\mathbb{G}_m$ -equivariant A -modules in chain cochain complexes for which the $\mathbb{G}_m$ -weights are nonnegative and whose complexes $\mathcal{T}_{A,\operatorname{dg}}^+(M, N)$ of morphisms are given by the natural equaliser

$$\mathcal{T}_{R,\operatorname{dg}}^+(M, N) \rightrightarrows \mathcal{T}_{R,\operatorname{dg}}^+(A \otimes_R M, N)$$

coming from the observations above.

The category $\mathcal{T}_A^+$ is then defined to have the same objects, but with morphisms $Z_0 \mathcal{T}_{A,\operatorname{dg}}^+(M, N)$.

Since the bifunctor $\underline{\mathrm{Hom}}_R$ respects finite colimits in the first argument, for any morphism $A \rightarrow B$ of nonnegatively weighted $\mathbb{G}_m$ -equivariant CDGAs, the forgetful functor $\mathcal{T}_{B,\mathrm{dg}}^+ \rightarrow \mathcal{T}_{A,\mathrm{dg}}^+$ then has a left adjoint given by $M \mapsto M \otimes_A B$.

Since $\mathcal{T}_R^+$ does not contain arbitrary coproducts, and in particular since infinite direct sums are not coproducts there, we cannot directly adapt [Section 1.2](#). Instead, in order to express our obstructions in terms of polyvectors we have to enlarge the class of morphisms as follows.

For now, fix a nonnegatively weighted $\mathbb{G}_m$ -equivariant chain CDGA S , which in applications will be either $R[\hbar^2]$ or $R[\hbar^2]/\hbar^{2m}$.

Definition 3.14 Take a symmetric monoidal pretriangulated dg category $\mathcal{C}$ equipped with a tensor functor from the dg category of $\mathbb{G}_m$ -representations, and take P_k^{ac} -algebras A and B in $\mathcal{C}$; write $\hbar^p: \mathcal{C} \rightarrow \mathcal{C}$ for the dg functor given by tensoring with the $\mathbb{G}_m$ -representation of weight p .

We then define Poisson L_∞ -morphisms from A to B to consist of sequences $f = (f_1, f_2, \dots)$, with $f_p \in \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{Sym}^p(A_{[1-k]}), \hbar^{p-1}B)_{p-k}$, such that

- (1) f is an L_∞ -morphism (ie satisfies the formulae of [\[Loday and Vallette 2012, Proposition 10.2.13\]](#)), and
- (2) f satisfies the multiplicative property

$$f_{n+1}(ab, x_1, \dots, x_n) = \sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}_{p,q}}} \pm f_{p+1}(a, x_{\sigma(1)}, \dots, x_{\sigma(p)}) f_{q+1}(b, x_{\sigma(p+1)}, \dots, x_{\sigma(n)}),$$

where $\pm$ denotes the appropriate Koszul sign (see [Remark 3.15](#)) and $\mathrm{Sh}_{p,q}$ the set of (p, q) -shuffle permutations.

When A and B are weighted $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebras over S , we write $\mathrm{Tate}_{P_k^{\mathrm{ac}}, S, L_\infty}(A, B)$ for the set of Poisson L_∞ -morphisms from A to B in $\mathcal{T}_{S,\mathrm{dg}}^+$.

Again, observe that the category from [Lemma 3.4](#) arises as a subcategory of $\mathrm{Tate}_{P_k^{\mathrm{ac}}, R, L_\infty}(A, B)$, consisting of morphisms f with $f_1 \in Z_0 Z^0 \mathcal{H}om_R(A, B) \subset \mathcal{T}_{R,\mathrm{dg}}^+(A, B)$ and $f_p = 0$ for all $p > 1$; we refer to these as strict morphisms.

Remark 3.15 Adapting [\[Voronov 2018, Theorem 1\]](#), a conceptual interpretation [Definition 3.14\(2\)](#) is that the tangent map of the Maurer–Cartan map $\omega \mapsto \sum_n f_n(\omega, \dots, \omega)/n!$ is universally multiplicative. Explicitly, for any pro-nilpotent CDGA C and any element $\omega \in (A \otimes C)_{-1}$ (not necessarily satisfying the Maurer–Cartan equation), condition (2) implies that the C -linear map

$$\sum_{n \geq 0} f_{n+1}(-, \omega, \dots, \omega)/n!: A \otimes C \rightarrow B \otimes C$$

is multiplicative. Condition (2) can be explicitly recovered from this property by letting C contain free independent variables $t_1, \dots, t_n$ with $\deg(t_i) = -\deg(x_i) - 1$, then setting $\omega = \sum x_i t_i$.

Lemma 3.16 *Poisson L_∞ -morphisms are closed under composition of L_∞ -morphisms.*

Proof This can be checked by direct substitution or as in [Voronov 2018, Theorems 3 and 6].

Alternatively, we can form bar constructions with respect to the P_k and Lie operads (cf Lemma 1.17) in the ind-category $\text{ind}(\mathcal{C})$, which does have infinite coproducts. An L_∞ -morphism f corresponds to a cocommutative coalgebra map $B_{\text{Lie}}(f): B_{S^{1-k}\hbar^{-1}\text{Lie}}A \rightarrow B_{S^{1-k}\hbar^{-1}\text{Lie}}B$ in the ind-category, with f being given by restriction to cogenerators B . Since $B_{S^{1-k}\hbar^{-1}\text{Lie}}A$ is a quotient of the P_k^{ac} -coalgebra $B_{P_k^{\text{ac}}}A$, the same procedure defines a map $B_{P_k^{\text{ac}}}(f): B_{P_k^{\text{ac}}}A \rightarrow B_{P_k^{\text{ac}}}B$ of $\mathbb{G}_m$ -equivariant graded P_k^{ac} -coalgebras (ie a degree-0 morphism in the dg category but not necessarily closed). The conditions of Definition 3.14 are then precisely the evaluation on cogenerators B of the condition that $B_{P_k^{\text{ac}}}(f)$ be closed under the differential.

Thus a Poisson L_∞ -morphism from A to B in $\mathcal{C}$ corresponds to a P_k^{ac} -coalgebra map $B_{P_k^{\text{ac}}}A \rightarrow B_{P_k^{\text{ac}}}B$ in $\text{ind}(\mathcal{C})$ whose corestriction to the natural coalgebra quotient $B_{S^{1-k}\hbar^{-1}\text{Lie}}B$ factors through the corresponding quotient $B_{S^{1-k}\hbar^{-1}\text{Lie}}A$ of $B_{P_k^{\text{ac}}}A$. This property is manifestly closed under composition. $\square$

Definition 3.17 Given a $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebra A over S and a Beck A -module M in the strict category of $\mathbb{G}_m$ -equivariant S -modules in double complexes, define the chain complex $\mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}},S,L_\infty}(A, M)$ to consist in degree n of $\mathbb{G}_m$ -equivariant Poisson L_∞ -derivations $A \rightarrow \text{cone}(M)_{[n]}$ in $\mathcal{T}_S^+$, with differential $\delta: \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}},S,L_\infty}(A, M)_n \rightarrow \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}},S,L_\infty}(A, M)_{n-1}$ induced by the obvious map $\text{cone}(M)_{[n]} \rightarrow \text{cone}(M)_{[n-1]}$.

The following acts as our substitute for Lemma 1.17 in the stacky setting:

Lemma 3.18 *Given a nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebra A over R which is cofibrant as a $\mathbb{G}_m$ -equivariant R -CDGA, together with a Beck A -module M , there is a complete decreasing filtration $F^1 \supset F^2 \supset \dots$ on the complex $\mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}},R,L_\infty}(A, M)$ with associated graded complexes*

$$\mathcal{T}_A^+(\text{Symm}_A^P((\Omega_{A/R}^1)_{[-k]}), \hbar^{p-1}M)_{[-k]},$$

where Ω^1 denotes the double complex of Kähler differentials of the underlying stacky CDGA.

Proof Unwinding Definition 3.14, a $\mathbb{G}_m$ -equivariant Poisson L_∞ -derivation in $\mathcal{T}_R^+$ is an L_∞ -derivation θ for which the elements $\theta_p \in \mathcal{T}_{R,\text{dg}}^+(\text{Symm}^P(A_{[1-k]}), \hbar^{p-1}M)_{p-k}$ satisfy

$$\theta_p(ab, y_2, \dots, y_p) = a\theta_p(b, y_2, \dots, y_p) \pm \theta_p(a, y_2, \dots, y_p)b,$$

which, since the operations are symmetric, says precisely that they are multiderivations.

Thus if we forget differentials, we have an isomorphism of graded R -modules

$$\mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}},R,L_\infty}(A, M)_\# \cong \prod_{p \geq 1} \mathcal{T}_{A,\text{dg}}^+(\text{Symm}_A^P((\Omega_{A/R}^1)_{[-k]}), \hbar^{p-1}M)_{[-k]},$$

and we can then set F^j to be the product of the terms with $p \geq j$. Since $\mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, R, L_\infty}(A, M)$ is a subcomplex of the complex of L_∞ -derivations, the p^{th} component $(\delta\theta)_p$ is an expression in terms of θ_j for $j \leq p$. If $\theta_p = 0$ for all $p < j$, this implies that $(\delta\theta)_p = 0$ for all $p < j$, so $\delta(F^j) \subset F^j$ and we have a filtration by subcomplexes. Moreover, $(\delta\theta)_j = \delta(\theta_j)$ under those conditions, so $\delta\theta \in \delta(\theta_j) + F^{j+1}$, giving the required description of the associated graded complex. $\square$

Definition 3.19 Say that a morphism $A \rightarrow B$ of stacky $\mathcal{P}$ -algebras is an abelian extension if it is surjective and if whenever a $\mathcal{P}$ -algebra operation on A has more than one input in $\ker(A \rightarrow B)$, the output is zero. Say that a morphism $A \rightarrow B$ is a nilpotent extension if it is a finite composition of abelian extensions.

Note that the condition implies that $\ker(A \rightarrow B)$ is naturally a Beck A -module, and that this structure is induced by a Beck B -module structure. It also gives us a stacky $\mathcal{P}$ -algebra isomorphism $A \times_B A \cong A \times_B (B \oplus \ker(A \rightarrow B))$.

The following proposition is the technical key to this section:

Proposition 3.20 Take strict morphisms $B' \xrightarrow{g} B \xrightarrow{h} \bar{B} \xleftarrow{f} A$ of nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebras over S such that g is an abelian extension with kernel I , h is a nilpotent extension, and the natural Beck B -module structure on I is induced by a Beck $\bar{B}$ -module structure.

We then have a short exact sequence

$$0 \rightarrow Z_0 \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, S, L_\infty}(A, f_*I) \rightarrow \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, B')_f \xrightarrow{g_*} \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, B)_f$$

of groups and sets, where $(-)_f$ denotes the fibre over $f \in \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, \bar{B})$.

Moreover, if A is cofibrant as a stacky CDGA in the model structure of [Lemma 3.3](#), then the short exact sequence extends to a further term $o_g: \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, B)_f \rightarrow H_{-1} \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, S, L_\infty}(A, f_*I)$, so $o_g(\phi) = 0$ if and only if ϕ lies in the image of g_* .

Proof Given two elements $\phi, \psi \in \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, B')_f$ with $g \circ \phi = g \circ \psi$, consider the element $\phi - \psi \in \prod_{p \geq 1} \mathcal{T}_{S, \text{dg}}^+(\text{Symm}_S^p(A_{[1-k]}), \hbar^{p-1}B)_{p-k}$. Since g is an abelian extension with I a Beck $\bar{B}$ -module, we have an isomorphism

$$B' \times_{\bar{B}} (\bar{B} \oplus I) \xrightarrow{\cong} B' \times_B B', \quad (b, (\bar{b}, x)) \mapsto (b, b + x)$$

of P_k^{ac} -algebras. Noting that the inclusion functor from the strict dg category of double complexes to $\mathcal{T}_{S, \text{dg}}^+$ preserves finite limits, it follows that $(f, \phi - \psi): A \rightarrow \bar{B} \oplus I$ defines a Poisson L_∞ -morphism of $\mathbb{G}_m$ -equivariant P_k^{ac} -algebras $\mathcal{T}_S^+$. Expanding this out in terms of [Definition 3.14](#), this says precisely that $\phi - \psi \in Z_0 \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, S, L_\infty}(A, f_*I)$, which gives the first statement.

For the second statement, we start with an intermediate lemma.

Lemma When A is cofibrant as a stacky CDGA, every element $\rho \in \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, B)_f$ admits a lift $\tilde{\rho}$ in $\prod_{p \geq 1} \mathcal{T}_{S, \text{dg}}^+(\text{Symm}_S^p(A_{[1-k]}), \hbar^{p-1}B')_{p-k}$ satisfying the multiplicativity property of [Definition 3.14](#).

Proof of lemma Since this lifting property is preserved on passage to a retract, we may assume that A is freely generated as a commutative algebra by a trigraded (ie $\mathbb{G}_m$ -equivariant bigraded) $\mathbb{Q}$ -vector space $V = \bigoplus_i \mathcal{W}_i V_{\#}^{\#}$; forgetting the differentials gives $A_{\#}^{\#} \cong \bigoplus_n S \otimes_{\mathbb{Q}} \text{Sym}_{\mathbb{Q}}^n(V_{\#}^{\#})$.

If we let α be the restriction of ρ to V , we can choose a lift $\tilde{\alpha} \in \prod_{p \geq 1} \mathcal{T}_{\mathbb{Q}}^+(\text{Sym}_{\mathbb{Q}}^p(V_{[1-k]}), \hbar^{p-1} B')_{p-k}$, since $B' \rightarrow B$ is surjective. Repeated application of the multiplicativity property then gives us elements $\tilde{\rho}_p$ of $\text{Tot}^{\Pi}(\mathcal{H}om_S(\text{Sym}_S^p(A_{[1-k]}), \hbar^{p-1} B'))_{p-k}$ agreeing with $\tilde{\alpha}_p$ on generators $V \subset A$, but we need to check that each term lies in $\mathcal{T}_{\text{dg}}^+ \subset \text{Tot}^{\Pi} \mathcal{H}om$. The iterated multiplicativity property gives, for $r := \sum_{j=1}^p r_j$ and $L = r + 1 - p$,

$$\tilde{\rho}_p(\text{Symm}^{r_1} V \otimes \cdots \otimes \text{Symm}^{r_p} V) \subset \sum_{\sum_{l=1}^L m_l = r} \tilde{\alpha}_{m_1}(V, \dots, V) \cdots \tilde{\alpha}_{m_L}(V, \dots, V),$$

which in particular means that $m_l \leq p$ for all l , so $\tilde{\rho}_p$ depends only on $\{\tilde{\alpha}_m\}_{m \leq p}$.

Writing $\tilde{\alpha}_{p,i}$ for the component of $\tilde{\alpha}$ in $\underline{\text{Hom}}_{\mathbb{Q}}(\mathcal{W}_i \text{Sym}_{\mathbb{Q}}^p(V_{[1-k]}), \mathcal{W}_{i-p+1} B')_{p-k}$, and $\tilde{\rho}_{p,i}$ for the restriction of $\tilde{\alpha}_p$ to $\mathcal{W}_i$, we moreover have that $\tilde{\rho}_{p,i}$ depends only on the finite set $\{\tilde{\alpha}_{m,j}\}_{m \leq p, j \leq i}$. Since the set is finite, by definition of $\underline{\text{Hom}}$ there exists an integer N such that $\tilde{\alpha}_{m,j}$ lies in cochain degrees $[-N, \infty)$ for all $m \leq p$ and $j \leq i$.

We can decompose $\tilde{\alpha}_{m,j}$ as $\tilde{\alpha}_{m,j}^+ + \tilde{\alpha}_{m,j}^-$ into terms of nonnegative and negative cochain degrees. Since f is a strict morphism, the image of $\tilde{\alpha}_{m,j}^-$ must lie in the kernel J' of the nilpotent surjection $B' \rightarrow \bar{B}$. If n is the index of nilpotence of J' , then at most $n - 1$ of the terms $\tilde{\alpha}_{m,j}^-$ can feature in a product to give a nonzero contribution. Thus the cochain degree of $\tilde{\rho}_{p,i}$ is bounded below by $-(n - 1)N$ (crucially independent of r), so $\tilde{\rho}_{p,i} \in \underline{\text{Hom}}_S(\mathcal{W}_i \text{Sym}_S^p(A_{[1-k]}), \mathcal{W}_{i+1-p} B')_{p-k}$, proving the lemma. $\square$

To complete the proof of the proposition, we can now proceed by a standard obstruction argument.

Given $\rho \in \text{Tate}_{P_k^{\text{ac}}, S, L_{\infty}}(A, B)_f$, we choose a lift $\tilde{\rho}$ as in the lemma. This will be an L_{∞} morphism if and only if the induced coalgebra morphism $\text{B}_{\text{Lie}}(\tilde{\rho}) : \text{B}_{S^{1-k} \text{ Lie}} A \rightarrow \text{B}_{S^{1-k} \text{ Lie}} B'$, as in the proof of [Lemma 3.16](#), commutes with $\partial \pm \delta$. The resulting commutator $\kappa(\tilde{\rho}) \in \prod_p \mathcal{T}_{S, \text{dg}}^+(\text{Sym}_S^p(A_{[1-k]}), \hbar^{p-1} B')_{p-k-1}$ is thus the obstruction to $\tilde{\rho}$ being an L_{∞} morphism. Since its projection $\kappa(\rho)$ to B is 0, it follows that $\kappa(\tilde{\rho})$ is an L_{∞} derivation from A to I . Because $B' \rightarrow B$ is an abelian extension, it moreover follows that $\kappa(\tilde{\rho})$ is a Poisson L_{∞} derivation and that the Poisson L_{∞} A -module structure on I is that induced from $\rho : A \rightarrow B$, so

$$\kappa(\tilde{\rho}) \in Z_{-1} \underline{\mathbb{D}\hat{\text{er}}}_{P_k^{\text{ac}}, S, L_{\infty}}(A, f_* I).$$

Any other lift of ρ is of the form $\tilde{\rho} + \theta$ for $\theta \in \underline{\mathbb{D}\hat{\text{er}}}_{P_k^{\text{ac}}, S, L_{\infty}}(A, f_* I)_0$, since $B' \rightarrow B$ is an abelian extension. Then $\kappa(\tilde{\rho} + \theta) = \kappa(\tilde{\rho}) + [\partial \pm \delta, \theta]$, the abelian property killing all other terms, so there exists a lift of ρ to $\text{Tate}_{P_k^{\text{ac}}, S, L_{\infty}}(A, B')$ if and only if $\kappa(\tilde{\rho})$ is a boundary, ie whenever $o_g(\rho) := [\kappa(\tilde{\rho})]$ is 0 in $H_{-1} \underline{\mathbb{D}\hat{\text{er}}}_{P_k^{\text{ac}}, S, L_{\infty}}(A, f_* I)$. $\square$

Remark 3.21 The cofibrancy condition in [Proposition 3.20](#) is stronger than the proof uses, since it suffices for A to be a retract of a quasifree object. That is an instance of a general phenomenon that deformation theory works most naturally with derived categories of the second kind.

Definition 3.22 Define the category $\mathcal{U}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}$ (resp. $\mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$) as follows. Objects are nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky $P_k^{\text{ac}} \otimes_{\mathbb{Q}} R[\hbar^2]/\hbar^{2m}$ -algebras (resp. $P_k^{\text{ac}} \otimes_{\mathbb{Q}} R[\hbar^2]$ -algebras) which are levelwise flat over $R[\hbar^2]/\hbar^{2m}$ (resp. $R[\hbar^2]$).

Morphisms from A to B then consist of those elements of

$$\text{Tate}_{P_k^{\text{ac}}, R[\hbar^2]/\hbar^{2m}, L_{\infty}}(A, B) \quad (\text{resp. } \text{Tate}_{P_k^{\text{ac}}, R[\hbar^2], L_{\infty}}(A, B))$$

which are strict morphisms modulo $\hbar^2$, ie morphisms of $\mathbb{G}_m$ -equivariant $P_k^{\text{ac}} \otimes_{\mathbb{Q}} R$ -algebras in the strict category of double complexes.

We then say that a morphism $A \rightarrow B$ is a weak equivalence if the induced (strict) morphism $A/\hbar^2 \rightarrow B/\hbar^2$ is a levelwise quasi-isomorphism, ie induces isomorphisms $H_i(A^j/\hbar^2) \rightarrow H_i(B^j/\hbar^2)$ for all i and j .

In particular, note that $\mathcal{U}_{P_k^{\text{ac}}, R} = \mathcal{U}_{P_k^{\text{ac}}[\hbar^2]/\hbar^2, R}$ is the strict category of nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky $(P_k^{\text{ac}} \otimes_{\mathbb{Q}} R)$ -algebras, ie the category with morphisms of cochain degree 0 and with no higher L_{∞} terms.

Also observe that since $\mathcal{W}_i(\hbar^{2m} B) = 0$ for all $i < 2m$, we have

$$\text{H}\hat{\text{om}}_{R[\hbar^2]}(\mathcal{W}_i A, \mathcal{W}_i B) = \varprojlim_m \text{H}\hat{\text{om}}_{R[\hbar^2]}(\mathcal{W}_i A, \mathcal{W}_i(B/\hbar^{2m})),$$

so substitution in [Definition 3.12](#) gives $\mathcal{T}_{R[\hbar^2], \text{dg}}^+(A, B) \cong \varprojlim_m \mathcal{T}_{R[\hbar^2]/\hbar^{2m}, \text{dg}}^+(A/\hbar^{2m}, B/\hbar^{2m})$ and thus

$$\mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}(A, B) \cong \varprojlim_m \mathcal{U}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}(A/\hbar^{2m}, B/\hbar^{2m}).$$

Definition 3.23 Letting $\mathcal{U}$ be either of the categories $\mathcal{U}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}$ or $\mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$, and taking objects $A, B \in \mathcal{U}$, define the simplicial set $\underline{\mathcal{U}}(A, B)$ to be given by

$$n \mapsto \mathcal{U}(A, B \otimes \Omega(\Delta^n)_{\bullet}),$$

for the CDGA $\Omega(\Delta^n)_{\bullet}$ of polynomial de Rham forms as in [Definition 2.23](#).

Corollary 3.24 Given $A, B \in \mathcal{U}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m+2}, R}$, with A cofibrant in the model structure on $\mathbb{G}_m$ -equivariant $(P_k^{\text{ac}} \otimes_{\mathbb{Q}} R[\hbar^2]/\hbar^{2m+2})$ -algebras from [Lemma 3.3](#), the map

$$\theta: \underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m+2}, R}(A, B) \rightarrow \underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}(A/\hbar^{2m}, B/\hbar^{2m})$$

is a Kan fibration of simplicial sets.

Moreover, for each element $f \in \underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}(A/\hbar^{2m}, B/\hbar^{2m})$, there is a functorial obstruction $o(f) \in H_{-1} \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, R, L_{\infty}}(A/\hbar^2, f_* \hbar^{2m}(B/\hbar^2))$ which vanishes if and only if the fibre $\theta^{-1}(f)$ is nonempty. If $o(f)$ vanishes, then $\pi_i(\theta^{-1}(f))$ is naturally a torsor for $H_i \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, \mathcal{T}_R^+, L_{\infty}}(A/\hbar^2, f_* \hbar^{2m}(B/\hbar^2))$.

Proof These are fairly standard consequences of [Proposition 3.20](#). Since finite limits behave well in $\text{Tate}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R, L_{\infty}}$, for any finite simplicial set K we have

$$\text{Hom}_{\text{sSet}}(K, \underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}(A, B)) \cong \mathcal{U}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}(A, B \otimes \Omega(K)_{\bullet}),$$

where $\Omega^n(K) := \text{Hom}_{\text{sSet}}(K, \Omega^n(\Delta^{\bullet}))$.

Letting $S := R[\hbar^2]/\hbar^{2m+2}$, the relative horn-filling condition for θ then amounts to surjectivity of

$$\begin{aligned} \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, B \otimes \Omega(\Delta^n)_\bullet)_F \\ \rightarrow \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, (B \otimes \Omega(\Delta^n)_\bullet/\hbar^{2m}) \times_{B \otimes \Omega(\Lambda^{n,k})_\bullet/\hbar^{2m}} B \otimes \Omega(\Lambda^{n,k})_\bullet)_F, \end{aligned}$$

for all $F \in \mathcal{U}_{P_k^{\text{ac}}, R}(A/\hbar^2, B/\hbar^2)_n \subset \text{Tate}_{P_k^{\text{ac}}, S, L_\infty}(A, B \otimes \Omega(\Delta^n)_\bullet/\hbar^2)$, where the subscript denotes the fibre over F .

Now, the map

$$B \otimes \Omega(\Delta^n)_\bullet \rightarrow (B \otimes \Omega(\Delta^n)_\bullet/\hbar^{2m}) \times_{B \otimes \Omega(\Lambda^{n,k})_\bullet/\hbar^{2m}} B \otimes \Omega(\Lambda^{n,k})_\bullet$$

is an abelian extension with kernel $\hbar^{2m}(B/\hbar^2) \otimes \ker(\Omega(\Delta^n)_\bullet \rightarrow \Omega(\Lambda^{n,k})_\bullet)$. The latter complex is acyclic, so the obstruction of [Proposition 3.20](#) vanishes, giving the required surjectivity for θ to be a Kan fibration.

The obstruction $o(f)$ is then given by applying [Proposition 3.20](#) to the abelian extension $B \rightarrow B/\hbar^{2m}$ of $(P_k^{\text{ac}} \otimes R[\hbar^2]/\hbar^{2m+2})$ -algebras. Then applying [Proposition 3.20](#) to the extensions $B \otimes \Omega(\Delta^n)_\bullet \rightarrow B \otimes \Omega(\Delta^n)_\bullet/\hbar^{2m}$ and writing $V(n) := \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, R, L_\infty}(A/\hbar^2, f_*\hbar^{2m}(B/\hbar^2) \otimes \Omega(\Delta^n)_\bullet)$ gives us a faithful transitive action of $Z_0 V(n)$ on $\theta^{-1}(f)_n$ for all n , and hence of $\pi_i Z_0 V(-)$ on $\pi_i \theta^{-1}(f)$. Since A is cofibrant, the simplicial chain complex $V(-)$ is a Reedy fibrant replacement of the chain complex $V(0)$, so the simplicial abelian group $Z_0 V(-)$ has homotopy groups $\pi_i Z_0 V(-)$ isomorphic to $H_i V(0)$, as required. $\square$

Corollary 3.25 For $\mathcal{U} := \mathcal{U}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}$ (resp. $\mathcal{U} := \mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$), with $A, B \in \mathcal{U}$ such that A is cofibrant in the model structure on $\mathbb{G}_m$ -equivariant stacky $(P_k^{\text{ac}} \otimes R[\hbar^2]/\hbar^{2m})$ -algebras (resp. $(P_k^{\text{ac}} \otimes R[\hbar^2])$ -algebras) from [Lemma 3.4](#), the simplicial set $\underline{\mathcal{U}}(A, B)$ is a model for the derived mapping space

$$\mathbf{Rmap}_{\mathcal{U}}(A, B)$$

of morphisms from A to B in the simplicial localisation $L^{\mathcal{W}}\mathcal{U}$ of $\mathcal{U}$ at weak equivalences.

Proof We first show by induction on m that the functor $\underline{\mathcal{U}}(-, -)$ preserves weak equivalences in both factors, provided we restrict to cofibrant $P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}$ -algebras in the first factor. For $m = 1$, this follows from [\[Hovey 1999, Section 5\]](#), since it is a right function complex with respect to the model structure of [Lemma 3.4](#). For the natural maps

$$\underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m+2}, R}(A, B) \rightarrow \underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2m}, R}(A/\hbar^{2m}, B/\hbar^{2m}),$$

the description of [Corollary 3.24](#) implies that the homotopy fibres are invariant under weak equivalences in A and B , which gives the inductive step. The case $\mathcal{U} := \mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$ then follows by taking the limit over m ; since all maps are fibrations, the limit is a homotopy limit.

In the model category of [Lemma 3.4](#), there exists a Reedy cofibrant cosimplicial frame $\tilde{A}^\bullet$ of A [\[Hovey 1999, Section 5.4\]](#), leading to a simplicial set $n \mapsto \mathcal{U}(\tilde{A}^n, B)$. The invariance above, together with an identical argument that $\mathcal{U}(\tilde{A}^\bullet, -)$ preserves weak equivalences, then gives us weak equivalences

$$\mathcal{U}(\tilde{A}^\bullet, B) \rightarrow \text{diag } \underline{\mathcal{U}}(\tilde{A}^\bullet, B) \leftarrow \underline{\mathcal{U}}(A, B)$$

of simplicial sets. In the model category of restricted diagrams from [Toën and Vezzosi 2005, Section 2.3.2], the natural morphisms $\mathcal{U}(A, -) \rightarrow \mathcal{U}(\tilde{A}^n, -)$ are all weak equivalences, since the maps $\tilde{A}^n \rightarrow A$ are weak equivalences in $\mathcal{U}$. Because $\mathcal{U}(\tilde{A}^\bullet, B)$ is a model for $\operatorname{holim}_{n \in \Delta^{\text{opp}}} \mathcal{U}(\tilde{A}^n, -)$, the natural map $\mathcal{U}(A, -) \rightarrow \mathcal{U}(\tilde{A}^\bullet, -)$ is also a weak equivalence in the category of restricted diagrams. The result now follows by [Dwyer and Kan 1987], as interpreted in [Toën and Vezzosi 2005, Theorem 2.3.5], since we have shown that $\mathcal{U}(\tilde{A}^\bullet, -) \simeq \underline{\mathcal{U}}(A, -)$ preserves weak equivalences. $\square$

3.4 Uniqueness results for deformations

3.4.1 Uniqueness for stacky $P_k^{\text{ac}}[\hbar^2]$ -algebras

Corollary 3.26 Take $A, B \in \mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$ such that A is cofibrant in the model structure on $\mathbb{G}_m$ -equivariant stacky $(P_k^{\text{ac}} \otimes_{\mathbb{Q}} R[\hbar^2])$ -algebras from Lemma 3.4. If

$$H_i \mathbb{D}\hat{\text{er}}_{P_k^{\text{ac}}, R, L_\infty}(A/\hbar^2 A, \hbar^{2n} f_*(B/\hbar^2 B)) \cong 0$$

for all $i \geq -1$ and $n \geq 1$, and all morphisms $f: A/\hbar^2 A \rightarrow B/\hbar^2 B$ in $\mathcal{U}_{P_k^{\text{ac}}, R}$, then the natural map

$$\underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2], R}(A, B) \rightarrow \underline{\mathcal{U}}_{P_k^{\text{ac}}, R}(A/\hbar^2 A, B/\hbar^2 B)$$

is a trivial fibration.

Proof Under these hypotheses, Corollary 3.24 gives fibrations

$$\underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2n+2}, R}(A/\hbar^{2n+2}, B/\hbar^{2n+2}) \rightarrow \underline{\mathcal{U}}_{P_k^{\text{ac}}[\hbar^2]/\hbar^{2n}, R}(A/\hbar^{2n}, B/\hbar^{2n})$$

for all $n \geq 1$, and shows that their fibres are contractible, making them trivial fibrations. The result then follows by taking the limit over all n . $\square$

Definition 3.27 The lax monoidal dg functor $\text{Tôt}_{\mathbb{G}_m}$ from $\mathbb{G}_m$ -equivariant double complexes to $\mathbb{G}_m$ -equivariant chain complexes is given by $\text{Tôt}_{\mathbb{G}_m} A := \bigoplus_i \text{Tôt } \mathcal{W}_i A$.

Lemma 3.28 There is an ∞ -functor $\text{Tôt}_{\mathbb{G}_m}$ from the ∞ -category of nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky $P_k^{\text{ac}} \otimes_{\mathbb{Q}} R[\hbar^2]$ -algebras to the ∞ -category of $\mathbb{G}_m$ -equivariant $(P_k^{\text{ac}} \otimes_{\mathbb{Q}} R[\hbar^2])$ -algebras, and this extends to an ∞ -functor on $L^{\mathcal{V}} \mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$.

Proof Since Tôt is lax monoidal, $\text{Tôt}_{\mathbb{G}_m}$ defines a functor from $\mathbb{G}_m$ -equivariant stacky $\mathcal{P}$ -algebras to $\mathbb{G}_m$ -equivariant $\mathcal{P}$ -algebras for all chain operads $\mathcal{P}$, and this automatically yields an ∞ -functor because it preserves weak equivalences, sending levelwise quasi-isomorphisms to quasi-isomorphisms.

Similarly, $\text{Tôt}_{\mathbb{G}_m}$ defines a functor from $\mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$ to the category of $\mathbb{G}_m$ -equivariant Poisson L_∞ -morphisms between $\mathbb{G}_m$ -equivariant $(P_k^{\text{ac}} \otimes_{\mathbb{Q}} R[\hbar^2])$ -algebras. As in the proof of Lemma 3.16, these are a fortiori $R[\hbar^2]$ -linear $(P_k)_\infty$ -morphisms, ie ∞ -morphisms in the sense of [Loday and Vallette 2012, Section 10.2.2].

Composing with the $R[\hbar^2]$ -linear analogue of the rectification functor $\Omega_{P_k^{\text{ac}}} \mathbf{B}_{P_k^{\text{ac}}}$ of Theorem 11.4.7 in [loc. cit.] thus gives us a functor $F := \Omega_{P_k^{\text{ac}}} \mathbf{B}_{P_k^{\text{ac}}} \text{Tôt}_{\mathbb{G}_m}$ on $\mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$, with the restriction of F to the

category of strict morphisms admitting a natural quasi-isomorphism from $\mathrm{T\hat{ot}}_{\mathbb{G}_m}$. In order to show that F induces the required ∞ -functor, it suffices to show that it preserves weak equivalences. By the $R[\hbar^2]$ -linear analogue of [loc. cit., Proposition 11.4.11], that amounts to showing that the functor $\mathrm{T\hat{ot}}_{\mathbb{G}_m}$ sends weak equivalences to ∞ -quasi-isomorphisms (ie $(P_k)_\infty$ -morphisms whose first component is a quasi-isomorphism).

If $f: A \rightarrow B$ is a weak equivalence in $\mathcal{U}$, then $f_1: A/\hbar^2 \rightarrow B/\hbar^2$ is a levelwise quasi-isomorphism, so $f_1: \mathrm{T\hat{ot}} \mathcal{W}_i(A/\hbar^2) \rightarrow \mathrm{T\hat{ot}} \mathcal{W}_i(B/\hbar^2)$ is a quasi-isomorphism. Now, the flatness hypotheses on objects of $\mathcal{U}_{P_k^{\mathrm{ac}}, R}[\hbar^2]$ ensure that the maps $\hbar^{2k}: (A/\hbar^2) \rightarrow (\hbar^{2k} A/\hbar^{2k+2} A)$ are isomorphisms and similarly for B , so by induction $f_1: \mathrm{T\hat{ot}} \mathcal{W}_i(A/\hbar^{2k}) \rightarrow \mathrm{T\hat{ot}} \mathcal{W}_i(B/\hbar^{2k})$ is a quasi-isomorphism for all k . Taking $k > \frac{1}{2}i$ then shows that the map $f_1: \mathrm{T\hat{ot}} \mathcal{W}_i A \rightarrow \mathrm{T\hat{ot}} \mathcal{W}_i B$ is a quasi-isomorphism for all i . $\square$

Definition 3.29 Given a nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebra A over R and a Beck A -module M in the strict category of $\mathbb{G}_m$ -equivariant double complexes, define $\mathbf{R}\mathbb{D}\hat{\mathrm{er}}_{P_k^{\mathrm{ac}}, R, L_\infty}(A, M) := \mathbb{D}\hat{\mathrm{er}}_{P_k^{\mathrm{ac}}, R, L_\infty}(\tilde{A}, M)$ for any cofibrant replacement $\tilde{A}$ of A as a $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebra; note that by Lemma 3.18 this is well defined up to quasi-isomorphism because Lemma 3.4 implies that $\tilde{A}$ is cofibrant as a stacky CDGA.

Theorem 3.30 For $\bar{A} := (A/\hbar^2 A)$, the functors

$$A \mapsto \bigoplus_i \mathrm{T\hat{ot}} \mathcal{W}_i A \quad \text{and} \quad A \mapsto \bigoplus_i \mathrm{T\hat{ot}} \mathcal{W}_i (\bar{A}[\hbar^2])$$

from the ∞ -category of nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky $(P_k^{\mathrm{ac}} \otimes_{\mathbb{Q}} R[\hbar^2])$ -algebras to the ∞ -category of $\mathbb{G}_m$ -equivariant $(P_k^{\mathrm{ac}} \otimes_{\mathbb{Q}} R[\hbar^2])$ -algebras become naturally equivalent when restricted to objects A which are flat over $R[\hbar^2]$ and satisfy

($\dagger$) $\mathbf{R}\mathbb{D}\hat{\mathrm{er}}_{P_k^{\mathrm{ac}}, R, L_\infty}(\bar{A}, M) \simeq 0$ for all Beck $\bar{A}$ -modules M of pure $\mathbb{G}_m$ -weight ≥ 2 .

Proof We adapt the proof of Corollary 1.13. As a preliminary, note that for all $P, N \in \mathcal{T}_A^+$, we have $\mathcal{T}_A^+(P, N) \cong \varprojlim_r \mathcal{T}_A^+(P, N/\mathcal{W}_{>r} N)$ essentially by construction. Since $\bar{A}$ is cofibrant, we can therefore deduce by a filtration argument that $\mathbf{R}\mathbb{D}\hat{\mathrm{er}}_{P_k^{\mathrm{ac}}, R, L_\infty}(\bar{A}, N) \simeq 0$ whenever $\mathbf{R}\mathbb{D}\hat{\mathrm{er}}_{P_k^{\mathrm{ac}}, R, L_\infty}(\bar{A}, \mathcal{W}_r N) \simeq 0$ for all r , which ($\dagger$) above ensures for all Beck $\bar{A}$ -modules N with $\mathcal{W}_i N = 0$ for all $i < 2$.

In particular, whenever A satisfies ($\dagger$) and B is nonnegatively weighted, $\mathbf{R}\mathbb{D}\hat{\mathrm{er}}_{P_k^{\mathrm{ac}}, R, L_\infty}(\bar{A}, \hbar^{2i} \bar{B}) \simeq 0$ for all $i \geq 1$, so Corollary 3.26 implies that the map $\underline{\mathcal{U}}_{P_k^{\mathrm{ac}}, R}[\hbar^2](A, B) \rightarrow \underline{\mathcal{U}}_{P_k^{\mathrm{ac}}, R}(A/\hbar^2 A, B/\hbar^2 B)$ is a weak equivalence. By Corollary 3.25, this means that the ∞ -functor $A \mapsto A/\hbar^2 A$ from $L^{\mathcal{W}}\mathcal{U}_{P_k^{\mathrm{ac}}, R}[\hbar^2]$ to $L^{\mathcal{W}}\mathcal{U}_{P_k^{\mathrm{ac}}, R}$ becomes full and faithful when restricted to objects satisfying ($\dagger$) above. Since the functor $C \mapsto C[\hbar^2]$ is right inverse to reduction mod $\hbar^2$, it follows that the identity functor is naturally equivalent to the functor $A \mapsto (A/\hbar^2 A)[\hbar^2]$ on this ∞ -subcategory $(L^{\mathcal{W}}\mathcal{U}_{P_k^{\mathrm{ac}}, R}[\hbar^2])^\%$.

Since the ∞ -functor $\mathrm{T\hat{ot}}_{\mathbb{G}_m}$ factors through $(L^{\mathcal{W}}\mathcal{U}_{P_k^{\mathrm{ac}}, R}[\hbar^2])$ by Lemma 3.28, composition gives us the required equivalence $\mathrm{T\hat{ot}}_{\mathbb{G}_m} A \simeq \mathrm{T\hat{ot}}_{\mathbb{G}_m}((A/\hbar^2 A)[\hbar^2])$, natural in objects A satisfying ($\dagger$). $\square$

3.4.2 Uniqueness for involutive almost commutative stacky P_k -algebras Adapting Definitions 1.1 and 1.15, we have:

Definition 3.31 A cochain chain complex $V_\bullet^\bullet$ is quasi-involutively filtered if it is equipped with a filtration W by double subcomplexes and an involution e which preserves W and acts on $H_*(\mathrm{gr}_i^W V^j)$ as multiplication by $(-1)^i$ for all j .

Define a quasi-involutive almost commutative stacky P_k -algebra over R to be an $(R \otimes P_k^{\mathrm{ac}}, W, e)$ -algebra A in quasi-involutively filtered cochain chain complexes.

We have the following immediate analogue of Lemma 1.9:

Lemma 3.32 The Rees functor of Definition 1.5 gives an equivalence of ∞ -categories from the category of quasi-involutive almost commutative stacky P_k -algebras over a chain CDGA R , localised at filtered levelwise quasi-isomorphisms, to the ∞ -category of $\mathbb{G}_m$ -equivariant $(R \otimes_{\mathbb{Q}} P_k^{\mathrm{ac}})$ -algebras in cochain chain complexes of flat $\mathbb{Q}[\hbar^2]$ -modules, localised at levelwise quasi-isomorphisms.

Applying Lemmas 1.9 and 3.32 and taking functorial cofibrant replacement, Theorem 3.30 immediately gives the following:

Corollary 3.33 The functors $\mathrm{T\hat{o}t}$ and $\mathrm{T\hat{o}t}^W$ from the ∞ -category of quasi-involutive almost commutative stacky P_k -algebras in double complexes (localised at filtered levelwise quasi-isomorphisms) to the ∞ -category of quasi-involutive almost commutative P_k -algebras become naturally equivalent when restricted to objects A satisfying the conditions

- (1) $W_{-1}A^j = 0$ for all j , and
- (2) $\mathbf{R}\hat{\mathbf{D}}\hat{\mathbf{e}}\mathbf{r}_{P_k^{\mathrm{ac}}, \mathcal{T}_R^+, L_\infty}(\mathrm{gr}^W A, M) \simeq 0$ for all Beck W_0A -modules M of pure $\mathbb{G}_m$ -weight ≥ 2 .

The following analogue of Proposition 1.18 is now an immediate consequence of Lemmas 3.18 and 3.11:

Proposition 3.34 If B is a nonnegatively weighted $\mathbb{G}_m$ -equivariant stacky P_k^{ac} -algebra over a CDGA R for which the map $(\mathcal{W}_1 \mathbf{L}\Omega_{B/\mathcal{W}_0 B}^1) \otimes_{\mathcal{W}_0 B}^{\mathbf{L}} B \rightarrow \mathbf{L}\Omega_{B/\mathcal{W}_0 B}^1$ of commutative cotangent complexes is a quasi-isomorphism, then

$$\mathbf{R}\hat{\mathbf{D}}\hat{\mathbf{e}}\mathbf{r}_{P_k^{\mathrm{ac}}, \mathcal{T}_R^+, L_\infty}(B, M) \simeq 0$$

for all Beck $\mathcal{W}_0 B$ -modules M pure of $\mathbb{G}_m$ -weight ≥ 2 .

Moreover, if A is another such algebra, equipped with a morphism $A \rightarrow B$ such that $\mathcal{W}_0 A \rightarrow \mathcal{W}_0 B$ is a levelwise quasi-isomorphism, and if the morphisms $\mathcal{W}_i A \rightarrow \mathcal{W}_i B$ are $\mathbf{R}\hat{\mathbf{H}}\hat{\mathbf{o}}\mathbf{m}_{\mathcal{W}_0 A}$ -homotopy equivalences for all $i \geq 0$, then we also have $\mathbf{R}\hat{\mathbf{D}}\hat{\mathbf{e}}\mathbf{r}_{P_k^{\mathrm{ac}}, \mathcal{T}_R^+, L_\infty}(B, M) \simeq 0$.

3.5 Quantisations on derived Artin stacks

We are now in a position to generalise Theorem 2.21 to stacky CDGAs, and hence Corollary 2.31 to derived Artin n -stacks.

3.5.1 Quantisation of 0-shifted Poisson structures By [Pridham 2018, Lemma 1.14], based on [Voronov 2000, Section 3], $\mathcal{CC}_R(A)$ is equipped with a stacky brace algebra structure. Since A is commutative, Lemma 2.8 moreover adapts to make $(\mathcal{CC}_{R,\oplus}(A), \tau^{\text{HH}})$ (resp. $\mathcal{D}_{\oplus}^{\text{poly}}(A)$) into a quasi-involutive stacky almost commutative brace algebra.

For $w \in \text{Levi}_{\text{GT}}^P(\mathbb{Q})$, Definition 2.19 gives an equivalence p_w between stacky quasi-involutive almost commutative brace algebras and stacky quasi-involutive almost commutative P_2 -algebras by considering the respective algebras in the dg category of cochain complexes of R -modules in chain complexes. However, the proof of Theorem 2.21 does not immediately adapt to this setting, because functoriality for stacky Hochschild complexes and stacky polyvectors is much more subtle, which is why we have had to involve $\widehat{\text{Hom}}$.

Writing $\Omega_{A,\square}^1$ for the cotangent module associated to the relevant theory (commutative, $\mathcal{C}^\infty$, or EFC) as in Corollary 2.27, we have:

Theorem 3.35 *Take a cofibrant stacky R -CDGA (resp. stacky $\mathcal{C}^\infty$ -DGA or stacky K -EFC-DGA) A with:*

($\ddagger$) *the cotangent complex $(\Omega_{A,\square}^1)^\#$ is perfect as an $A^\#$ -module.*

Then the quasi-involutively filtered DGLA underlying the complex of polydifferential operators

$$(\text{Tôt } \mathcal{D}_{\oplus}^{\text{poly}}(A)_{[-1]}, \tau^{\text{HH}})$$

is filtered quasi-isomorphic to the graded DGLA

$$\text{Pol}_R(A, 0)_{[-1]} := \bigoplus_{p \geq 0} \widehat{\text{Hom}}_A(\Omega_{A,\square}^p, A)_{[p-1]}$$

of derived polyvectors on A , where the Lie algebra structure is given by the Schouten–Nijenhuis bracket.

This quasi-isomorphism depends only on a choice of even 1-associator $w \in \text{Levi}_{\text{GT}}^P$, and is natural with respect to homotopy étale functoriality induced by [Pridham 2017, Section 3.4.2; 2018, Section 3.1] and its $\mathcal{C}^\infty$ and EFC analogues.

When A is a cofibrant stacky R -CDGA satisfying ($\ddagger$), the same statements hold for the Hochschild complex $\mathcal{CC}_{R,\oplus}(A)$ in place of $\mathcal{D}_{\oplus}^{\text{poly}}(A)$.

Proof We adapt the proof of Theorem 2.21. As explained in Section 3.1, the cofibrant hypothesis ensures that Lemma 2.8 adapts to show that $(\mathcal{D}_{\oplus}^{\text{poly}}(A), \tau^{\text{HH}})$ is a quasi-involutive almost commutative stacky brace algebra, with a levelwise graded quasi-isomorphism HKR: $\text{gr}^{\tau^{\text{HH}}} \mathcal{D}_{\oplus}^{\text{poly}}(A) \xrightarrow{\sim} \text{Pol}(A, 0)$. Since $\mathcal{D}_{\oplus}^{\text{poly}}(A) \rightarrow \mathcal{CC}_{R,\oplus}(A)$ is a filtered levelwise quasi-isomorphism in the stacky R -CDGA setting, it suffices to focus on $\mathcal{D}_{\oplus}^{\text{poly}}(A)$ in all settings.

For any even associator w , the ∞ -functor p_w of Definition 2.19 then gives an involutive almost commutative stacky P_2 -algebra $p_w(\mathcal{D}_{R,\oplus}^{\text{poly}}, \tau^{\text{HH}})$, with its associated graded algebra having a zigzag of levelwise quasi-isomorphisms of $\mathbb{G}_m$ -equivariant stacky P_2^{ac} -algebras to $\text{Pol}(A, 0)$.

Since $(\Omega_{A,\square}^1)^\#$ is assumed to be perfect, [Lemma 2.20](#) adapts verbatim to give a levelwise quasi-isomorphism $\mathbf{L}\Omega_{\mathcal{P}ol(A,0)/A}^1 \simeq \mathcal{P}ol(A,0) \otimes_A^{\mathbf{L}} \mathcal{W}_1 \mathcal{P}ol(A,0)$ of A -modules. Thus $\mathcal{P}ol(A,0)$ satisfies the conditions of [Proposition 3.34](#).

By base change, we know that $\Omega_{A,\square}^1 \otimes_A A^0$ must be a perfect A^0 -module in double complexes, so all but finitely many of the A^0 -modules $(\Omega_{A,\square}^1 \otimes_A A^0)^r$ must be acyclic. Thus if we take a diagram $D = (A(0) \rightarrow A(1) \rightarrow \cdots \rightarrow A(m))$ of homotopy formally étale surjections between $m+1$ such stacky CDGAs, the hypotheses of [Lemma 3.10](#) are satisfied, so the maps $\mathcal{W}_i \mathcal{P}ol(D,0) \rightarrow \mathcal{W}_i \mathcal{P}ol(A(0),0)$ are $\mathbf{R}\widehat{\mathbf{H}\mathbf{om}}_{A(0)}$ -homotopy equivalences.

Thus $\mathcal{P}ol(D,0)$ also satisfies the conditions of [Proposition 3.34](#), so $p_w(\mathcal{D}_{R,\oplus}^{\text{poly}}(D), \tau^{\text{HH}})$ satisfies the conditions of [Corollary 3.33](#), giving a zigzag of filtered involutive quasi-isomorphisms

$$\alpha_{w,D}: p_w \bigcup_i \widehat{\mathbf{T}\mathbf{ot}}(\tau_i^{\text{HH}} \mathcal{D}_{\oplus}^{\text{poly}}(D)) \simeq \bigoplus_i \widehat{\mathbf{T}\mathbf{ot}} \mathcal{W}_i \mathcal{P}ol(D,0),$$

natural with respect to all morphisms $(\mathcal{D}_{\oplus}^{\text{poly}}(D), \tau^{\text{HH}}) \rightarrow (\mathcal{D}_{\oplus}^{\text{poly}}(D'), \tau^{\text{HH}})$ in the ∞ -category of quasi-involutive almost commutative stacky brace algebras, for all such diagrams D and D' .

If we write $\mathcal{C}$ for the category of stacky R -CDGAs (resp. stacky $\mathcal{C}^\infty$ -DGAs or stacky K -EFC-DGAs), with $\mathcal{C}_{c,\rightarrow}$ the subcategory of cofibrant objects and surjective morphisms, then as in [Section 3.2](#) we have functors $D \mapsto (\mathcal{D}_{\oplus}^{\text{poly}}(D), \tau^{\text{HH}})$ and $D \mapsto \mathcal{P}ol(D,0)$ on the Grothendieck construction $\int B\mathcal{C}_{c,\rightarrow}$ of the nerve. If we further restrict to the full subcategory $\mathcal{C}_{c,\rightarrow}^{\text{lfp}} \subset \mathcal{C}_{c,\rightarrow}$ on objects satisfying $(\ddagger)$, then the natural transformation α_w above gives a natural equivalence between the respective functors from $\int B\mathcal{C}_{c,\rightarrow}^{\text{lfp}}$ to the category of quasi-involutive almost commutative P_2 -algebras.

Consider the ∞ -category $\mathbf{L}\mathcal{C}^{\text{lfp},\acute{\text{e}}\text{t}}$ given by the localisation at levelwise quasi-isomorphisms of the category of homotopy formally étale morphisms between stacky R -CDGAs (resp. stacky $\mathcal{C}^\infty$ -DGAs or stacky K -EFC-DGAs) satisfying $(\ddagger)$. By [Lemma 2.30](#), $\mathbf{L}\mathcal{C}^{\text{lfp},\acute{\text{e}}\text{t}}$ arises as a simplicial localisation of $\int B(\mathcal{C}_{c,\rightarrow}^{\text{lfp},\acute{\text{e}}\text{t}})$, for the subcategory $\mathcal{C}_{c,\rightarrow}^{\text{lfp},\acute{\text{e}}\text{t}} \subset \mathcal{C}_{c,\rightarrow}^{\text{lfp}}$ of homotopy formally étale morphisms. Since the maps $\widehat{\mathbf{T}\mathbf{ot}} \mathcal{W}_i \mathcal{P}ol(D,0) \rightarrow \widehat{\mathbf{T}\mathbf{ot}} \mathcal{W}_i \mathcal{P}ol(A(0),0)$ above are quasi-isomorphisms, our totalised functors descend to that localisation, so we also have an equivalence

$$\alpha_w: \bigcup_i \widehat{\mathbf{T}\mathbf{ot}}(\tau_i^{\text{HH}} \mathcal{D}_{\oplus}^{\text{poly}}(-)) \simeq \bigoplus_i \widehat{\mathbf{T}\mathbf{ot}} \mathcal{W}_i \mathcal{P}ol(-,0)$$

of ∞ -functors from $\mathbf{L}\mathcal{C}^{\text{lfp},\acute{\text{e}}\text{t}}$ to the ∞ -category of quasi-involutive almost commutative P_2 -algebras. $\square$

Definition 3.36 Given a cofibrant stacky R -CDGA, stacky $\mathcal{C}^\infty$ -DGA, or stacky K -EFC-DGA A , adapting [[Pridham 2018](#), Definition 1.16] as in [[Pridham 2020a](#); [2020b](#)], we define the filtered DGLA $(Q \widehat{\mathbf{Pol}}(A,0)_{[-1]}, \widetilde{F})$ of quantised polyvectors by setting

$$\widetilde{F}^i Q \widehat{\mathbf{Pol}}(A,0) := \prod_{p \geq i} \widehat{\mathbf{T}\mathbf{ot}} \tau_p^{\text{HH}} \mathcal{D}_{\oplus}^{\text{poly}}(A) \hbar^{p-1};$$

observe that the Gerstenhaber bracket satisfies $[\tau_p^{\text{HH}}, \tau_q^{\text{HH}}] \subset \tau_{p+q-1}^{\text{HH}}$, so $[\tilde{F}^i, \tilde{F}^j] \subset \tilde{F}^{i+j-1}$, making $\tilde{F}^2 Q \widehat{\text{Pol}}(A, 0)_{[-1]}$ into a pro-nilpotent filtered DGLA.

The space $Q\mathcal{P}(A, 0)$ of 0-shifted quantisations of A is then defined to be (adapting [Pridham 2018, Definition 1.23])

$$\varprojlim_i \underline{\text{MC}}(\tilde{F}^2 Q \widehat{\text{Pol}}(A, 0)_{[-1]} / \tilde{F}^i).$$

The subspace $Q\mathcal{P}(A, 0)^{\text{sd}} \subset Q\mathcal{P}(A, 0)$ of self-dual quantisations then consists of fixed points for the involution $(-)^*$ given by $\Delta^*(\hbar) := i(\Delta)(-\hbar)$, for the involution i of Lemma 2.8.

These definitions all extend to diagrams $(A(0) \rightarrow \cdots \rightarrow A(m))$ in place of A , giving homotopy étale functoriality as in the proof of Theorem 3.35.

For a strongly quasicompact derived Artin n -stack $\mathfrak{X}$, the space $Q\mathcal{P}(\mathfrak{X}, 0)$ and its variants are defined in [Pridham 2018, Section 3.1] by first taking an Artin $(n+2)$ -hypergroupoid resolution $X_\bullet$ of $\mathfrak{X}$, then applying the left adjoint D^* of the denormalisation functor D , forming a cosimplicial stacky CDGA $j \mapsto D^*O(X^{\Delta^j})$ with homotopy formally étale structure morphisms. This can be thought of as giving a formally étale simplicial resolution of $\mathfrak{X}$ by derived Lie algebroids. We then set

$$\mathcal{P}(\mathfrak{X}, 0) := \text{holim}_{j \in \Delta} \mathcal{P}(D^*O(X^{\Delta^j}), 0), \quad Q\mathcal{P}(\mathfrak{X}, 0) := \text{holim}_{j \in \Delta} Q\mathcal{P}(D^*O(X^{\Delta^j}), 0);$$

this is shown to be well defined because the constructions are invariant under smooth hypercovers. Similar constructions work verbatim for EFC and $\mathcal{C}^\infty$ analogues.

These constructions extend beyond the strongly quasicompact setting, either by allowing the hypercover to involve disjoint unions of derived affine schemes, or as in [Pridham 2024] by working directly with a functor on $D_*\mathfrak{X}$ on stacky CDGAs which admits formally étale affine hypercovers. Indeed, analogously to [Pridham 2020c, Section 2.3.4], this approach allows the definitions to extend to any homogeneous (also known as infinitesimally cohesive on one factor) derived stack with a bounded below cotangent complex.

In the following lemma note that, as with any Fermat theory, modules over dg $\mathcal{C}^\infty$ -rings and dg EFC-rings are just modules over the underlying CDGAs.

Lemma 3.37 *If Y is a derived Artin m -hypergroupoid, then the stacky CDGA $A := D^*O(Y)$ satisfies $(\Omega_{A, \square}^1 \otimes_A A^0)^r \simeq 0$ for all $r > m$. If the associated derived Artin m -stack $\mathfrak{Y} := Y^\sharp$ has perfect cotangent complex, then $(\Omega_A^1)^\sharp$ is perfect as an $A^\sharp$ -module.*

Proof First, note that the double complex $\Omega_{A, \square}^1 \otimes_A A^0$ is just the Dold–Kan normalisation N_c of the cosimplicial chain complex $\Omega_{Y, \square}^1 \otimes_{O(Y)} A^0$, since both constructions have the same right adjoint. The term N_c^r in cochain degree r is thus isomorphic to the cokernel $\Omega_{Y_r/M_{\wedge^{r,0}}Y}^1 \otimes_{O(Y_r)} O(Y_0)$ of the $(r, 0)^{\text{th}}$ partial matching map, which by hypothesis is a trivial cofibration in degrees $r > m$, and so $(\Omega_{A, \square}^1 \otimes_A A^0)^r \simeq 0$.

The map $\mathcal{H}om_A(\Omega_A^1, N)^0 \rightarrow \mathcal{H}om_A(\Omega_A^1, \sigma^{\leq m} N)^0$ to the brutal cotruncation is thus a quasi-isomorphism for all A^0 -modules N in double complexes. That quasi-isomorphism extends to all A -modules N in double complexes which are concentrated in nonnegative cochain degrees, since $N \cong \varprojlim_r \sigma^{\geq r} N$ with the quotients being A^0 -modules.

All the partial matching maps are smooth, so the argument of the first paragraph also shows that the chain complexes $(\Omega_{A, \square}^1 \otimes_A A^0)^r$ are projective A^0 -modules in chain complexes, and in particular perfect complexes, for all $r > 0$. Since $Y_0 \rightarrow \mathfrak{Y}$ is an Artin m -atlas and $\mathfrak{Y}$ has perfect cotangent complex, so does Y_0 , meaning that the A^0 -module $\Omega_{A^0, \square}^1 = (\Omega_{A, \square}^1 \otimes_A A^0)^0$ is also a perfect complex. Since the vanishing result above implies that $(\Omega_A^1 \otimes_A A^0)^\# \simeq \bigoplus_{r=0}^m ((\Omega_{A, \square}^1 \otimes_A A^0)^r)^{[-r]}$, we deduce that it is perfect as an A^0 -module.

The functor $\mathcal{H}om_A(\Omega_A^1, -)^0$ thus commutes with filtered homotopy colimits of A^0 -modules in double complexes, and hence (via quotients of the brutal truncation filtration) with filtered colimits of A -modules concentrated in cochain degrees $[0, m]$. Since cotruncation commutes with filtered colimits, it follows from the cotruncation property above that $\mathcal{H}om_A(\Omega_A^1, -)^0$ commutes with all filtered homotopy colimits of A -modules, which is equivalent to saying that $(\Omega_A^1)^\#$ is perfect as an $A^\#$ -module. $\square$

Corollary 3.38 *Given a derived Artin n -stack $\mathfrak{X}$ over R with perfect cotangent complex, any even associator $w \in \text{Levi}_{\text{GT}}^P$ gives rise to a map*

$$\mathcal{P}(\mathfrak{X}, 0) \rightarrow Q\mathcal{P}^{\text{sd}}(\mathfrak{X}, 0)$$

from the space of 0-shifted Poisson structures on $\mathfrak{X}$ to the space of self-dual E_1 quantisations of $\mathfrak{X}$ in the sense of [Pridham 2018, Definitions 1.23 and 3.9]. When $\mathfrak{X}$ is strongly quasicompact, these quantisations give rise to curved A_∞ deformations $(\text{per}_{\text{dg}}(\mathfrak{X})[[\hbar]], \{m^{(i)}\}_{i \geq 0})$ of the dg category $\text{per}_{\text{dg}}(\mathfrak{X})$ of perfect $\mathcal{O}_{\mathfrak{X}}$ -complexes, $\hbar$ -semilinearly anti-involutive with respect to the dg endofunctor $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(-, \mathcal{O}_{\mathfrak{X}})$ on $\text{per}_{\text{dg}}(\mathfrak{X})$.

The analogous statements for derived $\mathcal{C}^\infty$ and derived analytic Artin n -stacks (in the sense of [Pridham 2020a]) with perfect cotangent complexes also hold.

Proof The first statement and its $\mathcal{C}^\infty$ and analytic analogues follow immediately by substituting Theorem 3.35 in the definitions above, via Lemma 3.37, and passing to homotopy limits.

By [Pridham 2018, Proposition 3.11], E_1 quantisations of $\mathfrak{X}$ give rise to curved A_∞ deformations of $\text{per}_{\text{dg}}(\mathfrak{X})$, as a consequence of the corresponding statement [loc. cit., Proposition 3.11] for stacky CDGAs. The latter follows by establishing that the natural restriction map $\text{CC}_R(\text{Per}(A)) \rightarrow \text{CC}_R(A)$ is a filtered quasi-isomorphism, for a bi-dg category $\text{Per}(A)$ of perfect modules associated to A [loc. cit., Definition 1.24]. The identity $A \cong A^{\text{opp}}$ extends to the dualisation functor $(-)^{\vee}$ on $\text{Per}(A)$, which is a homotopy involution in the sense that it extends to an action of a simplicial resolution of the group C_2 . The restriction map above is then equivariant for this homotopy C_2 -action, and so induces equivalences on homotopy fixed points of the respective spaces of deformations.

Explicitly, for strongly self-dual A -modules P_j , the self-dual condition on a curved A_∞ deformation is that the following diagram commutes for each of the A_∞ operations $m^{(i)}(\hbar)$:

$$\begin{array}{ccc} \mathrm{Hom}(P_0, P_1) \otimes \cdots \otimes \mathrm{Hom}(P_{i-1}, P_i) & \xrightarrow{m_{P_0, \dots, P_i}^{(i)}(-\hbar)} & \mathrm{Hom}(P_0, P_i)[[\hbar]] \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Hom}(P_i^\vee, P_{i-1}^\vee) \otimes \cdots \otimes \mathrm{Hom}(P_1^\vee, P_0^\vee) & \xrightarrow{\pm m_{P_i^\vee, \dots, P_0^\vee}^{(i)}} & \mathrm{Hom}(P_i^\vee, P_0^\vee)[[\hbar]] \end{array} \quad \square$$

Remark 3.39 The hypotheses of [Corollary 3.38](#) are satisfied by any derived Artin stack locally of finite presentation over the CDGA R . When $R = H_0 R$, this includes those underived Artin n -stacks $\mathfrak{X}$ which admit Artin n -atlases by schemes which are local complete intersections over R , in which case the cotangent complex $\mathbf{L}\Omega_{\mathfrak{X}/R}^1$ is concentrated in homological degrees $[-n, 1]$.

3.5.2 Quantisation of 1-shifted coisotropic structures In [\[Melani and Safronov 2018, Definition 5.14\]](#), shifted coisotropic structures and their quantisations are defined for stacky CDGAs, and hence derived Artin stacks, in terms of $P_{[n+1, n]}$ -algebras and $BD_{[n+1, n]}$ -algebras in the unweighted Tate dg category $\mathcal{T}_{R, \mathrm{dg}}^0 \subset \mathcal{T}_{R, \mathrm{dg}}^+$, the full dg subcategory of objects concentrated in weight 0. Unwinding our definition of a 0-shifted quantisation gives a curved strong homotopy BD_1 -algebra in $\mathcal{T}_{R, \mathrm{dg}}^0$.

Since we have to allow for curvature, we need a more general definition than [\[loc. cit.\]](#), similar to [Definition 2.35](#). We then encounter the subtlety that $\mathcal{T}_{R, \mathrm{dg}}^0$ does not have infinite limits, so we work with the pro-category $\mathrm{pro}(\mathcal{T}_{R, \mathrm{dg}}^0)$ in order to accommodate the algebra of quantised polyvectors. The monoidal functor $\mathcal{T}_{R, \mathrm{dg}}^+ \rightarrow \mathrm{pro}(\mathcal{T}_{R, \mathrm{dg}}^0)$ sending V to the pro-object $\{\prod_{0 \leq i \leq j} \mathcal{W}_j V\}_j$ allows us to make the following definition.

Definition 3.40 A 1-shifted quantised coisotropic structure on a morphism $f: A \rightarrow B$ of stacky CDGAs consists of

- (1) a quantised 0-shifted Poisson structure Δ on B ,
- (2) a quantised 1-shifted Poisson structure on A in the form of a system of flat strong homotopy $BD_2/\hbar^k$ -algebras $\tilde{A}/\hbar^k$ in $\mathcal{T}_{R[\hbar]/\hbar^k, \mathrm{dg}}^0$, equipped with a weak equivalence $\tilde{A}/\hbar \rightarrow A$ of commutative algebras in $\mathcal{T}_{R, \mathrm{dg}}^0$, and
- (3) a strong homotopy morphism of BD_2 -algebras in $\mathrm{pro}(\mathcal{T}_{R, \mathrm{dg}}^0)$:

$$F: \tilde{A} \rightarrow T_\Delta \widehat{\mathcal{Q}} \mathrm{Pol}(B, 0) := \left\{ \left(\prod_{0 \leq i \leq j} \tau_j^{\mathrm{HH}} D_\oplus^{\mathrm{poly}}(B) \hbar^j, \delta + \{\Delta, -\}, \smile + \{\Delta\}\}_2, \{\Delta\}\}_{i \geq 3} \right) \right\}_j.$$

Define a 1-shifted quantised coisotropic structure on a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of derived Artin n -stacks to consist of compatible 1-shifted quantised coisotropic structures on all morphisms $f: A \rightarrow B$ of stacky CDGAs with A and B homotopy formally étale over $\mathfrak{Y}$ and $\mathfrak{X}$, respectively, using the homotopy formally étale functoriality of [Section 3.2](#).

We say that this quantisation is self-dual if Δ is self-dual, the BD_2 -algebras $\tilde{A}$ are given compatible quasi-involutive structures, and F intertwines the involutions.

Corollary 3.26, and hence the proof of **Theorem 3.30**, gives equivalences in $L^{\mathcal{W}}\mathcal{U}_{P_k^{\text{ac}}[\hbar^2], R}$, and hence a fortiori as $P_k^{\text{ac}}[\hbar^2]$ -algebras in $\mathcal{T}_{R, \text{dg}}^+$. Substituting in these results, the proof of **Corollary 2.36** adapts immediately to give:

Corollary 3.41 *Given a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ of derived Artin n -stacks such that $\mathfrak{X}$ has perfect cotangent complex, every 1-shifted coisotropic structure on f admits self-dual quantisations in the sense of **Definition 3.40**.*

For analytic and $\mathcal{C}^\infty$ analogues, the considerations of **Remark 2.37** apply equally to derived Artin stacks, so the proof of **Corollary 3.41** immediately gives abstract BD_2 -algebra quantisations in those settings, which we can strengthen to genuine quantisations of the analytic or $\mathcal{C}^\infty$ coisotropic structures by replacing $\mathcal{T}_{\text{dg}}^0$ with the multicategory of polydifferential operators.

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Volume growth of Funk geometry and the flags of polytopes

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We consider the Holmes–Thompson volume of balls in the Funk geometry on the interior of a convex domain. We conjecture that for a fixed radius, this volume is minimized when the domain is a simplex and the ball is centered at the barycenter, or in the centrally symmetric case, when the domain is a Hanner polytope. This interpolates between Mahler’s conjecture and Kalai’s flag conjecture. We verify this conjecture for unconditional domains.

For polytopal Funk geometries, we study the asymptotics of the volume of balls of large radius, and compute the two highest-order terms. The highest depends only on the combinatorics, namely on the number of flags. The second highest depends also on the geometry, and thus serves as a geometric analogue of the centro-affine area for polytopes.

We then show that for any polytope, the second highest coefficient is minimized by a unique choice of center point, extending the notion of Santaló point. Finally, we show that, in dimension two, this coefficient, with respect to the minimal center point, is uniquely maximized by affine images of the regular polygon.

[51K99](#), [51M10](#), [52A38](#), [52A40](#), [52B05](#)

1 Introduction

We are interested in the volume of metric balls in the Funk geometry — a Finsler geometry with a nonreversible metric defined on the interior of a convex body, closely related to the Hilbert metric. The forward metric balls in this geometry take a particularly simple form: they are merely scaled versions of the body itself. We use the Holmes–Thompson definition of volume.

Our interest stems in part from a connection with a longstanding conjecture in convex geometry. As the radius R of the ball tends to zero, the volume is asymptotic to R^n times the *Mahler volume* (volume product) of the body, with the polar body taken relative to the center of the ball. The Mahler conjecture states that the Mahler volume is minimized when the body is a simplex, or, if the body is required to be centrally symmetric, when it is a Hanner polytope. For a recent survey of the problem, see Fradelizi, Meyer and Zvavitch [8].

We recall that a Hanner (or Hansen–Lima) polytope is any polytope that can be constructed from line segments by taking Cartesian products and polar bodies. One may ask if the same bodies minimize the volume for all $R > 0$.

Conjecture 1.1 *Let K be a centrally symmetric convex body in $\mathbb{R}^n$, and let $R > 0$. Then, the volume of the (forward) ball in the Funk geometry of radius R centered at the origin is minimized when the body is a Hanner polytope. Explicitly,*

$$\mathrm{Vol}_K(B_K(R)) \geq \mathrm{Vol}_H(B_H(R)) = \frac{2^n (\log(2e^R - 1))^n}{n! \omega_n},$$

where H is any Hanner polytope of dimension n , and ω_n the volume of the n -dimensional Euclidean ball of unit radius.

If K is not assumed to be centrally symmetric, then the centered n -dimensional simplex minimizes $\mathrm{Vol}_K(B_K(R))$.

Our first theorem shows that as the radius becomes large, the asymptotic behavior of the volume of the ball is determined by the number of flags of the polytope. Let P be a polytope of dimension n endowed with its Funk geometry, and denote again by $\mathrm{Vol}_P(B_P(R))$ the Holmes–Thompson volume of the ball of radius R about the origin, which we assume to be contained in the interior of P . Also, denote by $\mathrm{Flags}(P)$ the set of flags of P .

Theorem 1.2 *For any polytope P of dimension n ,*

$$\lim_{R \rightarrow \infty} \frac{\omega_n}{R^n} \mathrm{Vol}_P(B_P(R)) = \frac{|\mathrm{Flags}(P)|}{(n!)^2}.$$

That the asymptotic volume on the left-hand side is proportional to the number of flags can be established by adapting the proof of Vernicos and Walsh [24] from Hilbert geometry, where the same statement holds with a different constant. The proof there makes use of the invariance of the volume under collineations (invertible projective transformations), which holds for any reasonable definition of volume in the Hilbert case. For the Funk metric, however, invariance of the volume under collineations is a special property of the Holmes–Thompson definition, as observed by Faifman [6]. The explicit constant $(n!)^{-2}$ can then be determined, for example, by computing the case of Hanner polytopes, which is done in Section 4.

We will however take a different approach, and deduce this theorem from a comprehensive computation of the first two terms of the asymptotic expansion of $\mathrm{Vol}_P(B_P(R))$, as explained in Theorem 1.5. Let us mention here that volume growth behavior in metric geometry similar to Theorem 1.2 has been observed by Pansu [16] on the universal cover of nilmanifolds, and by Burago and Ivanov [4] in the special case of tori, allowing characterization of flat ones (see also [24] for an analogous rigidity result in Hilbert geometry by the last two authors). However, an explicit second-order term, which we obtain in the present paper, is usually far more complicated, and may not even exist.

In light of [Theorem 1.2](#), our conjecture would imply that, out of all centrally symmetric polytopes of a given dimension, Hanner polytopes have the fewest flags. This statement is known as the *flag conjecture* and has been ascribed in [\[23, Chapter 22\]](#) to Kalai. It appears as a special case of Conjecture C of Kalai [\[10\]](#), although this conjecture was shown to be false in general in Sanyal, Werner and Ziegler [\[20\]](#).

We show that [Conjecture 1.1](#) is true for *unconditional bodies*, and describe all equality cases under that assumption. Recall that an unconditional body is one that is symmetric through reflections in the coordinate hyperplanes.

Theorem 1.3 *Let $K \subset \mathbb{R}^n$ be an unconditional convex body, and let H be any Hanner polytope of the same dimension. Then,*

- $\text{Vol}_K(B_K(R)) \geq \text{Vol}_H(B_H(R))$ for all $R > 0$;
- equality holds if and only if K is a Hanner polytope.

Letting R tend to infinity and invoking [Theorem 1.2](#), we then deduce the following.

Corollary 1.4 *For an unconditional n -dimensional convex polytope P , and a Hanner polytope H of the same dimension, we have*

$$|\text{Flags}(P)| \geq |\text{Flags}(H)| = 2^n n!.$$

While we lose track of the equality cases when passing to the limit, we do obtain necessary conditions for equality as a corollary of [Theorem 1.5](#) — see [Corollary 1.6](#) below. We remark that recently, the related 3^d conjecture of Kalai concerning the minimal total face number of a symmetric polytope was proved in the unconditional case by Chambers and Portnoy [\[5\]](#), with equality cases established by Sanyal and Winter [\[21\]](#).

Since the highest-order term in the asymptotics of the volume depends only on the combinatorics of the polytope and not on its geometry, it is interesting to look also at the second-highest order term. Let us stress once again that this second-order term is usually not easy to obtain, for instance for the closely related Hilbert metric, a second-order term is not known. In the present setting of Funk geometry, the second-order term is a new affine invariant of convex polytopes.

Recall that polytopes have the following property: if F and F' are two faces of dimension $i - 1$ and $i + 1$, respectively, and $F \subset F'$, then there are exactly two faces G of dimension i such that $F \subset G \subset F'$. So, for each $i \in \{0, \dots, n - 1\}$ and flag $f \in \text{Flags}(P)$, there is a unique flag, which we denote $r_i f$, that differs from f only by having a different face of dimension i . The group $\langle r_0, \dots, r_{n-1} \rangle$ generated by these maps is known as the *monodromy group* or *connection group* of the polytope P . To express the second term in the asymptotics of the volume, we will need a particular element of this group, the *complete flip* $r := r_{n-1} \circ \dots \circ r_0$.

Theorem 1.5 *The volume of the ball of radius R in the Funk geometry on a polytope P satisfies*

$$\omega_n \text{Vol}_P(B_P(R)) = c_0(P)R^n + c_1(P)R^{n-1} + o(R^{n-1}),$$

where

$$c_0(P) = \frac{|\text{Flags}(P)|}{(n!)^2}$$

and

$$(1) \quad c_1(P) = \frac{n}{(n!)^2} \sum_{f \in \text{Flags}(P)} \log(1 - \langle (rf)_{n-1}, f_0 \rangle).$$

Here, we are identifying the facet $(rf)_{n-1}$ with the associated vertex of the dual polytope.

Let $c_1(P, x)$ denote the corresponding coefficient with x taking the place of the origin as the center of the ball. Recall from Berck, Bernig and Vernicos [2] and Faifman [6] that for a sufficiently regular convex body $K \subset \mathbb{R}^n$, one has

$$\omega_n \text{Vol}_P(B_K(R)) \sim \frac{1}{n-1} 2^{-\frac{1}{2}(n-1)} \Omega_n(K) e^{\frac{1}{2}(n-1)R},$$

where $\Omega_n(K)$ is the centro-affine surface area of K . Thus $c_1(P)$ can be seen as a geometric analogue for polytopes of the centro-affine surface area. Both $c_1(P, x)$ and the centro-affine surface area can be seen as versions of the Margulis function [13] in the respective settings of Funk geometry.

Corollary 1.6 *For an unconditional n -dimensional convex polytope P , and a Hanner polytope H of the same dimension, assume*

$$|\text{Flags}(P)| = |\text{Flags}(H)|.$$

Then for any flag $f \in \text{Flags}(P)$, we have $-f_0 \in (rf)_{n-1}$.

We remark that all centrally symmetric 2-level polytopes satisfy this condition.

We next consider the dependence of the volume of the metric ball on the choice of its center. Vernicos and Walsh [25] show that, for any $R > 0$, the volume $\text{Vol}_P(B_P(R))$ is minimized by a unique choice of center, denoted by $s_R(P)$, generalizing the notion of the Santaló point of a convex body. Here we study the behavior of this point as R tends to infinity.

Theorem 1.7 *Let P be a polytope. Then, $x \mapsto c_1(P, x)$ is proper and strictly convex, and finite in the interior of P , and hence attains its minimum at a unique point $s_\infty(P)$ in the interior.*

This point can be characterized as follows: $s_\infty(P) = 0$ if and only if the center of mass of the vertices of P° lies at the origin, where each vertex is assigned a weight equal to the number of flags of P° that contain it.

Moreover, we have

$$s_\infty(P) = \lim_{R \rightarrow \infty} s_R(P).$$

The characterization of $s_\infty(P)$ in this theorem is analogous to that of the classical Santaló point, namely that the Santaló point is at the origin if and only if the center of mass of the polar body is at the origin.

In dimension two, the second term (1) of [Theorem 1.5](#) reduces to

$$(2) \quad c_1(P) := \frac{1}{2} \sum_{i,j:i \sim j} \log(1 - \langle e_i, v_j \rangle),$$

where the v_j are the vertices of the polygon, the e_i are the edges, which correspond to the vertices of the dual polytope, and $i \sim j$ means that $\langle e_i, v_j \rangle \neq 1$ and $\langle e_i, v_{j+k} \rangle = 1$ for some $k \in \{-1, 1\}$. We consider how the functional c_1 varies as the polygon varies, with the number of vertices being fixed. It is of course $\mathrm{GL}_2(\mathbb{R})$ -invariant. We show that every stationary point of c_1 is the image under an element of $\mathrm{GL}_2(\mathbb{R})$ of the regular polygon.

Theorem 1.8 *Consider the space $\mathcal{P}_2^m$ of convex polygons with $m \geq 3$ vertices containing the origin in their interior. If the functional $c_1: \mathcal{P}_2^m \rightarrow \mathbb{R}$ is stationary at P , then P is a linear image of the regular polygon D^m with m vertices, centered at the origin. Moreover, among all $P \in \mathcal{P}_2^m$ with $s_\infty(P)$ at the origin, $c_1(P)$ is uniquely maximized by linear images of D^m .*

This should be compared with Meyer and Reisner [\[15\]](#) (see also Alexander, Fradelizi and Zvavitch [\[1\]](#)), where the regular polygon is shown to uniquely maximize the volume product among all polygons with a fixed number of vertices. One could ask if the regular polygon similarly maximizes $\mathrm{Vol}_P(B_P(R))$ for general $0 < R < \infty$.

1.1 Plan of the paper

After some preliminaries, we prove [Theorem 1.3](#) in [Section 4](#). In [Section 5](#), we decompose the polytope and its polar into flag simplices, and show that the volume of a ball can be expressed as the sum of a certain integral over all pairs consisting of a flag simplex of the polytope and a flag simplex of the polar. We then calculate the two highest-order terms contributed by each of these pairings. In [Section 6](#), we combine this information to prove [Theorem 1.5](#). In [Section 7](#), we prove [Theorem 1.7](#) concerning the convergence of the Funk–Santaló point, and in [Section 8](#) we study two-dimensional Funk geometries, proving [Theorem 1.8](#) and developing an exact expression in dimension two for the growth rate of the volume at any finite radius. We establish a recursive formula for the growth of the volume in the case of the simplex in [Section 9](#). We conclude in [Section 10](#) with some further conjectures, questions and thoughts.

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2 Preliminaries

2.1 Funk geometry

Let K be an open bounded convex body in $\mathbb{R}^n$. Consider a point $p \in K$ and a vector $\vec{v} \in \mathbb{R}^n$. Since K is convex, there exists a unique real number $t > 0$ such that $p + t\vec{v}$ is on the boundary ∂K of K . We define the *Funk weak norm* of $\vec{v}$ at p to be

$$F_K(p, \vec{v}) := \frac{1}{t}.$$

This has all the properties of a norm, except that it is not necessarily homogeneous, only positively homogeneous, and so the associated unit ball is not necessarily symmetric. Some call this the *tautological Finsler structure* associated to K , because at each point p of K the unit ball of the norm F_K is just the convex set K itself centered at p .

Given any two points p and q in K , we can look at all piecewise C^1 paths $\gamma: [0, 1] \rightarrow K$ from p to q , and compute their *Funk length*

$$l_F(\gamma) := \int_0^1 F_K(\gamma(t), \dot{\gamma}(t)) dt.$$

If we denote by b the intersection of the half line $[p, q)$ with ∂K , then the infimum of these lengths over all paths from p to q is the *Funk weak distance* and is equal to

$$d_F(p, q) := \log \frac{pb}{qb}.$$

Here, pb/qb is to be understood as the real number λ such that $\vec{pb} = \lambda \vec{qb}$. The quantity d_F is a weak distance in the sense that it is nonnegative for any two distinct points, it satisfies the triangle inequality, and $d_F(x, x) = 0$ for every point x . However, it is not symmetric. For simplicity, we will drop the adjective “weak” from now on.

The forward Funk metric ball centered at $p \in K$ of radius R is

$$B_K(p, R) := \{z \in K \mid d_K(p, z) \leq R\}.$$

One easily finds that

$$B_K(p, R) = p + (1 - e^{-R})(K - p),$$

that is, the image of K by the dilation of ratio $(1 - e^{-R})$ with respect to p . We shall denote by λ the number $(1 - e^{-R})$ in what follows to simplify the exposition.

The Holmes–Thompson, sometimes also called symplectic, volume in Funk geometry can be defined as follows. Given a Lebesgue measure μ on $\mathbb{R}^n$, there exists a unique dual Lebesgue measure μ^* on the dual of $\mathbb{R}^n$ such that, for any basis $e = (e_1, \dots, e_n)$ and its dual one $e^* = (e_1^*, \dots, e_n^*)$, we get

$$\mu^*(e_1^* \wedge \dots \wedge e_n^*) \cdot \mu(e_1 \wedge \dots \wedge e_n) = 1.$$

Denote by K^p the polar body of K with respect to $p \in K$. Then, for any Borel set U in K , its Holmes–Thompson measure is given by

$$\text{Vol}_K(U) := \int_U \frac{\mu^*(K^p)}{\omega_n} d\mu(p),$$

where ω_n is the volume of the n -dimensional Euclidean unit ball.

Typically in computations, once coordinates $p = (p_1, \dots, p_n)$ are fixed, one takes $d\mu(p) = d\text{Leb}_n(p) = dp_1 \cdots dp_n$, where $dp_i := p_i^*$, hence

$$\text{Vol}_K(U) = \omega_n^{-1} \int_U |K^p| dp \quad \text{with } |K^p| = \int_{K^p} dp_1^* \cdots dp_n^*.$$

2.2 Faces and flags

A *face* of a convex polytope P is the intersection of the polytope with a closed halfspace such that no point of the interior of the polytope lies on the boundary of the halfspace. Note that the polytope itself is a face, as is the empty set. The dimension of a face is the dimension of the affine subspace it generates. As usual, $(n-1)$ -dimensional faces are called *facets*, 0-dimensional ones *vertices* and 1-dimensional ones *edges*.

The set of faces of a polytope is a partially ordered set with the inclusion relation. In fact, it forms a lattice, called the *face lattice*.

A *flag* of P is a sequence $f = (f_0, f_1, \dots, f_n)$ of faces such that, for all $i \in \{0, \dots, n\}$, the face f_i is an i -dimensional face, and, for $i \in \{0, \dots, n-1\}$, we have $f_i \subset f_{i+1}$. We denote the set of flags of P by $\text{Flags}(P)$.

Polytopes have the following so-called *diamond property*: if F and F' are two faces of dimensions $i-1$ and $i+1$, respectively, and $F \subset F'$, then there are exactly two faces G of dimension i such that $F \subset G \subset F'$. For each $i \in \{0, \dots, n-1\}$, define the map $r_i: \text{Flags}(P) \rightarrow \text{Flags}(P)$ in the following way. For each flag f of P , and $j \in \{-1, \dots, n\}$, let

$$(r_i f)_j := \begin{cases} G' & \text{if } j = i, \\ f_j & \text{otherwise,} \end{cases}$$

where G' is the unique face of dimension i distinct from f_i satisfying $f_{i-1} \subset G' \subset f_{i+1}$.

The two flags $r_i f$ and f are said to be *i -adjacent*. The group $\langle r_0, \dots, r_{n-1} \rangle$ generated by these maps is known as the *monodromy group* or *connection group* of the polytope P .

2.3 Taking the polar with respect to different points

We have the following relationship between the polar of a convex body at the origin and its polar at another point.

Lemma 2.1 *Let K be a convex body, and x a point in $\text{int } K$. Then,*

$$K^x := (K - x)^\circ = \left\{ \frac{w}{1 - \langle w, x \rangle} \mid w \in K^\circ \right\}.$$

3 Properties of the Holmes–Thompson volume in Funk geometry

While the Funk geometry is only an affine invariant, many of its aspects are more naturally considered within projective geometry, as was observed in [6]. In particular, the associated Holmes–Thompson volume exhibits two projective invariance properties. Firstly, it is invariant under collineations.

Proposition 3.1 *If $g: \mathbb{RP}^n \rightarrow \mathbb{RP}^n$ is a collineation, and $K \subset \mathbb{R}^n$ is a convex body in an affine chart such that $gK \subset \mathbb{R}^n$, then $\text{Vol}_K(U) = \text{Vol}_{gK}(gU)$ for any Borel subset $U \subset K$.*

Secondly, the Holmes–Thompson volume in Funk geometry is invariant under projective polarity [6, Proposition 7.2]. This can be stated as follows.

Proposition 3.2 *If K and L are convex bodies in $\mathbb{R}^n$, with $K \subset \text{int}(L)$, and $p \in \text{int}(K)$, then $\text{Vol}_L(K) = \text{Vol}_{K^p}(L^p)$.*

We now establish a multiplicative property of the volume in Funk geometry.

Lemma 3.3 *If $K \subset \mathbb{R}^k$ and $L \subset \mathbb{R}^l$ are convex bodies, then*

$$(3) \quad (k+l)! \omega_{k+l} \text{Vol}_{K \times L}(B_{K \times L}((a, b), R)) = (k! \omega_k \text{Vol}_K(B_K(a, R))) (l! \omega_l \text{Vol}_L(B_L(b, R))).$$

Proof For convex bodies $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^l$, we define the convex body

$$A \times_1 B = \{(tx, (1-t)y) \mid x \in A, y \in B, 0 \leq t \leq 1\} \subset \mathbb{R}^{k+l}.$$

Assuming $0 \in A$ and $0 \in B$, it has Lebesgue volume

$$|A \times_1 B| = \frac{k!l!}{(k+l)!} |A| |B|.$$

Write $\lambda = 1 - e^{-R}$, and assume for simplicity that $a = b = 0$. It holds, for all (x, y) in the interior of $K \times L$, that $(K \times L)^{(x,y)} = K^x \times_1 L^y$, and so

$$\begin{aligned} \omega_{k+l} \text{Vol}_{K \times L}(B_{K \times L}((a, b), R)) &= \int_{\lambda(K \times L)} |(K \times L)^{(x,y)}| \, dx \, dy = \int_{\lambda K \times \lambda L} |K^x \times_1 L^y| \, dx \, dy \\ &= \frac{k!l!}{(k+l)!} \int_{\lambda K} |K^x| \, dx \int_{\lambda L} |L^y| \, dy, \end{aligned}$$

concluding the proof. $\square$

We can now calculate the volume of Funk balls in Hanner polytopes.

Lemma 3.4 *Let H be a Hanner polytope in $\mathbb{R}^n$ centered at the origin, and let $R > 0$ and $\lambda = 1 - e^{-R}$. Then,*

$$\text{Vol}_H(B_H(R)) = \frac{2^n}{n! \omega_n} \left(\log \frac{1+\lambda}{1-\lambda} \right)^n = \frac{2^n}{n! \omega_n} (\log(2e^R - 1))^n.$$

Proof An elementary calculation shows that, for $Q = [-1, 1] \subset \mathbb{R}$, we have

$$\omega_1 \operatorname{Vol}_Q(B_Q(0, R)) = 2 \log \frac{1 + \lambda}{1 - \lambda}.$$

The conclusion follows on applying [Proposition 3.2](#) and [Lemma 3.3](#), since any Hanner polytope can be constructed from intervals by taking products and polars. $\square$

4 The Mahler inequality for unconditional bodies

It is convenient to extend the Funk Holmes–Thompson volume to functions, as was done in [\[6\]](#). For a convex function $\phi: V \rightarrow \mathbb{R} \cup \{+\infty\}$ defined on $V = \mathbb{R}^n$, and $R > 0$, set

$$\tilde{M}_\lambda(e^{-\phi}) = \frac{\lambda^n}{n!} \int_{V \times V^*} e^{-\phi(x) - \mathcal{L}\phi(\xi) + \lambda \langle x, \xi \rangle} dx d\xi,$$

where $\lambda = 1 - e^{-R}$, and $\mathcal{L}$ is the *Legendre transform* given by

$$\mathcal{L}\phi(\xi) = \sup_{x \in V} (\langle x, \xi \rangle - \phi(x)).$$

Clearly $\tilde{M}_\lambda(e^{-\phi}) = \tilde{M}_\lambda(e^{-\mathcal{L}\phi})$. Note also that $\mathbb{1}_K^\infty := -\log \mathbb{1}_K$ is a convex function when K is convex.

The following was verified in [\[6, Lemma 8.5\]](#).

Lemma 4.1 *For a convex body K and $0 < \lambda < 1$, one has*

$$\omega_n \operatorname{Vol}_K(\lambda K) = \tilde{M}_\lambda(\mathbb{1}_K) = \tilde{M}_\lambda(e^{-h_K}),$$

where h_K is the support function of K .

Proof The first equality is [\[6, Lemma 8.5\]](#). Since $\mathcal{L}(\mathbb{1}_K^\infty) = h_K$, we get the second one. $\square$

Theorem 4.2 *Let $K \subset \mathbb{R}^n$ be an unconditional convex body, and $H \subset \mathbb{R}^n$ a Hanner polytope centered at the origin. Then, for any $0 < R < \infty$, we have*

$$\operatorname{Vol}_K(B_K(R)) \geq \operatorname{Vol}_H(B_H(R)).$$

Equality is uniquely attained by Hanner polytopes.

Proof Consider an unconditional convex function $\phi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$.

Unconditional here means again that it is invariant under reflections in the coordinate hyperplanes. Define, for each $j \geq 0$,

$$I_{2j}(e^{-\phi}) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \langle x, \xi \rangle^{2j} e^{-\phi(x) - \mathcal{L}\phi(\xi)} dx d\xi.$$

Due to the unconditionality of ϕ , we have

$$\begin{aligned} I_{2j}(e^{-\phi}) &= \sum_{|I|=j} \binom{2j}{2I} \int_{\mathbb{R}^n} x^{2I} e^{-\phi(x)} dx \int_{\mathbb{R}^n} \xi^{2I} e^{-\mathcal{L}\phi(\xi)} d\xi \\ &= 4^n \sum_{|I|=j} \binom{2j}{2I} \int_{\mathbb{R}_+^n} x^{2I} e^{-\phi(x)} dx \int_{\mathbb{R}_+^n} \xi^{2I} e^{-\mathcal{L}\phi(\xi)} d\xi, \end{aligned}$$

where the sum is over all n -tuples $I = (i_1, \dots, i_n)$ of nonnegative integers with $|I| := \sum_{k=1}^n i_k = j$.

We are using here the notation $x^{2I} := \prod_{k=1}^n x_k^{2i_k}$ and

$$\binom{2j}{2I} := \frac{(2j)!}{(2i_1)! \cdots (2i_n)!}.$$

By the inequality of Fradelizi and Meyer [7, Theorem 6], we have

$$(4) \quad \int_{\mathbb{R}_+^n} x^{2I} e^{-\phi(x)} dx \int_{\mathbb{R}_+^n} \xi^{2I} e^{-\mathcal{L}\phi(\xi)} d\xi \geq \prod_{k=1}^n \frac{\Gamma(2i_k + 2)}{(2i_k + 1)^2},$$

where Γ denotes Euler's gamma function. Equality, for any given j and I , occurs (nonuniquely) when $\phi(x) = \|x\|_{\ell_1^n} := \sum |x_k|$, the support function of the cube $[-1, 1]^n$.

Denoting $\lambda = 1 - e^{-R}$, we have $B_K(R) = \lambda K$. Since $K = -K$, we have by Lemma 4.1,

$$\begin{aligned} \omega_n \text{Vol}_K(\lambda K) &= \tilde{M}_\lambda(\mathbb{1}_K) = \frac{\lambda^n}{n!} \int_{K \times \mathbb{R}^n} e^{-h_K(\xi) + \lambda \langle x, \xi \rangle} dx d\xi \\ &= \frac{\lambda^n}{n!} \int_{K \times \mathbb{R}^n} e^{-h_K(\xi)} \cosh(\lambda \langle x, \xi \rangle) dx d\xi \\ &= \frac{1}{n!} \sum_{j=0}^{\infty} \frac{1}{(2j)!} I_{2j}(e^{-h_K}) \lambda^{n+2j}. \end{aligned}$$

The last equality here is obtained by expanding the hyperbolic cosine as a power series, and using that $I_{2j}(\exp(-\mathbb{1}_K^\infty)) = I_{2j}(\exp(-h_K))$.

Thus all summands in the series for $\text{Vol}_K(\lambda K)$ are simultaneously minimized when h_K is the support function of the cube. That is, among all unconditional convex bodies in $\mathbb{R}^n$, the cube minimizes $\text{Vol}_K(\lambda K)$. Applying Lemma 3.4, we see that the minimum is also attained by any Hanner polytope.

For the equality cases, it follows from the proof that equality must hold in equation (4) for every I , in particular when $I = (0, \dots, 0)$, so that

$$\int_{\mathbb{R}_+^n} e^{-\mathbb{1}_K^\infty(x)} dx \int_{\mathbb{R}_+^n} e^{-h_K(\xi)} d\xi = 1,$$

that is,

$$\frac{n!}{4^n} |K| |K^\circ| = 1.$$

The minimizers of the volume product among unconditional convex bodies were shown to coincide with the Hanner polytopes by Meyer [14] and Reisner [17]. This concludes the proof. $\square$

5 A decomposition of the volume

In this section and the next, we determine the two highest-order terms of growth of the balls in the Funk geometry on a polytope P . Our strategy will be to decompose both P and its polar into flag simplices, and to consider separately each pairing of a flag simplex with a dual flag simplex.

To this end, for each face F of P , we make a choice of point $p(F)$ in the relative interior of F . When F is a vertex, this choice of course can only be the vertex itself. When $F = P$, we choose $p(P)$ to be the origin. Our choice of points leads to a flag decomposition of P : to each flag f of P is associated a flag simplex, which is the convex hull of the points $p(f_0), \dots, p(f_n)$. The interiors of these simplices are pairwise disjoint, and their union equals P .

Similarly, for each face G of P° , we make a choice of point $q(G)$ in the relative interior of G . Again, this leads to a flag decomposition of P° .

For each flag f of P , denote by $\Delta^f := \text{conv}\{p(f_0), \dots, p(f_n)\}$ the associated flag simplex. We also define, for each flag f and $\tau \in (0, 1)$, the truncated simplex $\Delta_\tau^f := (1 - \tau)\Delta^f$. Recall here that $p(P) = 0$. For each dual flag $g \in \text{Flags}(P^\circ)$ and $i \in \{0, \dots, n-1\}$, let $x_i^g(\cdot) := 1 - \langle q(g_i), \cdot \rangle$. The functions x_i^g provide a set of coordinates on $\mathbb{R}^n$, for any given dual flag g . We denote by Leb_n the n -dimensional Lebesgue measure.

Lemma 5.1 *The Holmes–Thompson volume of the ball $B_P(0, R)$ in the Funk geometry is given, for $\tau := \exp(-R)$, by*

$$\text{Vol}(B_P(0, R)) = \frac{1}{n! \omega_n} \sum_{f \in \text{Flags}(P)} \sum_{g \in \text{Flags}(P^\circ)} \int_{\Delta_\tau^f} \frac{dx_0^g \cdots dx_{n-1}^g}{x_0^g \cdots x_{n-1}^g}.$$

Proof The volume of the ball is

$$\text{Vol}(B_P(0, R)) = \frac{1}{\omega_n} \int_{B_P(0, R)} |P^y| \, d\text{Leb}_n(y),$$

where P^y denotes the polar of P with respect to the point y . Observe that $B_P(0, R)$ is the union of the simplices Δ_τ^f , and that the interiors of these simplices do not intersect. We conclude that

$$(5) \quad \text{Vol}(B_P(0, R)) = \frac{1}{\omega_n} \sum_{f \in \text{Flags}(P)} \int_{\Delta_\tau^f} |P^y| \, d\text{Leb}_n(y).$$

The n -dimensional Lebesgue volume of a simplex with vertices $v_0, \dots, v_{n-1}, 0$ is $|v_0 \wedge \cdots \wedge v_{n-1}|/n!$. So, we have

$$|P^\circ| = \frac{1}{n!} \sum_{g \in \text{Flags}(P^\circ)} |q(g_0) \wedge \cdots \wedge q(g_{n-1})|.$$

As we vary y over $\text{int } P$, the shape of the polar P^y changes according to [Lemma 2.1](#). Note that the combinatorics stay the same. In fact, the change of shape is always by a projective map. For each face G of P° and point y in $\text{int } P$, let

$$q^y(G) := \frac{q(G)}{1 - \langle q(G), y \rangle}.$$

Then, the $q^y(G)$ define a flag decomposition of P^y . So, we may write

$$\begin{aligned} (6) \quad |P^y| &= \frac{1}{n!} \sum_{g \in \text{Flags}(P^\circ)} |q^y(g_0) \wedge \cdots \wedge q^y(g_{n-1})| \\ &= \frac{1}{n!} \sum_{g \in \text{Flags}(P^\circ)} \frac{|q(g_0) \wedge \cdots \wedge q(g_{n-1})|}{(1 - \langle q(g_0), y \rangle) \cdots (1 - \langle q(g_{n-1}), y \rangle)}. \end{aligned}$$

Changing to the coordinates x_i^g changes the measure in the following way:

$$dx_0^g \cdots dx_{n-1}^g = |q(g_0) \wedge \cdots \wedge q(g_{n-1})| d\text{Leb}_n.$$

Combining this with [\(5\)](#) and [\(6\)](#) gives the result. □

So we see that the Funk volume can be computed by calculating an integral of a relatively simple function over a truncated flag simplex with respect to a coordinate system coming from the dual flag simplex. This is done for every combination of flag simplex and dual flag simplex, and the results are added.

We will see that the highest-order term as R tends to infinity is determined by the flag simplices combined with their duals; the second-highest order term is determined by these pairs as well as pairs where the flag simplex and the dual flag simplex are nearest neighbors, in the sense that the associated flags differ by exactly one face; in the following sections we calculate the contributions of the relevant pairs.

Let us fix notation that will be used throughout this section. Recall that $f(x) = O(x)$ means that $f(x)/x$ is bounded, whereas $f(x) = o(x)$ means that $f(x)/x$ converges to zero as x tends to infinity.

Recall that we have the following elementary integral: for $a \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$(7) \quad \int \frac{(\log x + a)^k}{x} dx = \frac{1}{k+1} (\log x + a)^{k+1} + C.$$

For each $k \in \mathbb{N}$, define the function $F^k: \mathbb{R}_{>0}^n \rightarrow \mathbb{R}$ by

$$F^k(x) := \frac{(\log x_n - \log \tau)^k}{x_1 \cdots x_n}.$$

Here $\mathbb{R}_{>0}$ is the set of positive real numbers. We use the convention that $\mathbb{N}$, the set of natural numbers, contains 0.

5.1 Combining a flag simplex and its own dual

We start off by considering the case where the flag simplex and the dual flag simplex are associated to the same flag. These pairs will be seen to contribute to both the R^n and R^{n-1} terms.

Let

$$(8) \quad \Delta := \text{conv}\{v_0, \dots, v_n\}$$

be an n -dimensional simplex with vertices

$$v_0 := (0, 0, \dots, 0), \quad v_1 := (v_{1,1}, 0, \dots, 0), \quad \dots, \quad v_n := (v_{n1}, v_{n2}, \dots, v_{nn}).$$

All nonzero coordinates are assumed positive.

Denote by Δ_τ the set $\Delta \setminus (\mathbb{R}^{n-1} \times [0, \tau))$, in other words, the subset of Δ where the last coordinate is greater than or equal to τ .

We are interested in the growth of the integral of F^k over Δ_τ , as τ decreases. It is actually the case $k = 0$ that we will use later, but it is important for the inductive proof to consider more general k .

We will need the following two lemmas. The first shows that, by scaling the coordinates, we may fit the simplex Δ into the “ordered simplex”.

Lemma 5.2 *Let Δ be a simplex of the form (8). Then, there exist positive real numbers a_i such that*

$$0 \leq a_n x_n \leq \dots \leq a_1 x_1 \leq 1 \quad \text{for all } x \in \Delta.$$

Proof First, choose $a_1 > 0$ so that $a_1 v_{i1} \leq 1$ for all i . Then, choose $a_2 > 0$ so that $a_2 v_{i2} \leq a_1 v_{i1}$ for all i . Continue in this way until all the a_i are chosen. It is clear that any convex combination of the v_i then satisfies the required relations. $\square$

The next lemma shows that, when we restrict our attention to a region where one of the coordinates is bounded below, we get a bound on the growth rate of the integral of F^k .

Lemma 5.3 *Let $U(R)$ be the integral of F^k over the intersection of Δ_τ with the region $\{x_j \geq \beta\}$, where $j \in \{1, \dots, n-1\}$ and $\beta > 0$, with $\tau = \exp(-R)$. Then, $U(R) = O(R^{n+k-j})$.*

Proof Using Lemma 5.2, we see that the domain of integration is contained in

$$[\alpha, \gamma]^j \times [\delta\tau, \gamma]^{n-j-1} \times [\tau, \gamma],$$

with δ, α and γ positive real numbers. The integrand can be integrated easily over this set with the help of (7), and thus we get the following bound:

$$U(R) \leq (\log \gamma - \log \alpha)^j (\log \gamma - \log \delta + R)^{n-j-1} \frac{(\log \gamma + R)^{k+1}}{k+1}. \quad \square$$

Lemma 5.4 Let Δ be of the form (8), and let Δ_τ be defined as above. For each $R > 0$ and $k \in \mathbb{N}$, write

$$V(R) := \frac{1}{k!} \int_{\Delta_\tau} F^k \, d\text{Leb}_n,$$

where $\tau = \exp(-R)$. Then, for all $k \in \mathbb{N}$,

$$V(R) = \frac{R^{n+k}}{(n+k)!} + \frac{R^{n+k-1}}{(n+k-1)!} \left(\sum_{i=1}^n \log v_{i,i} - \sum_{i=1}^{n-1} \log v_{i+1,i} \right) + o(R^{n+k-1}).$$

Proof We use induction on the dimension n . When $n = 1$, the integral can be evaluated exactly using (7), and we get that

$$V(R) = \frac{(\log v_{1,1} + R)^{k+1}}{(k+1)!}.$$

So, we see that the statement of the lemma is true in this case.

Assume that $n \geq 2$ and that the statement is true when the dimension is $n-1$. Choose $\alpha > 0$ such that $\alpha < v_{i,1}$ for all $1 \leq i \leq n$. Decompose the integral into two parts, where $x_1 \geq \alpha$, and where $x_1 < \alpha$. Consider the simplices $\Delta' = \Delta \cap \{x_1 = \alpha\}$ and $\Delta'_\tau = \Delta_\tau \cap \{x_1 = \alpha\}$. The vertices of Δ' are

$$v'_1 := (\alpha, 0, \dots, 0), \quad \dots, \quad v'_n := \left(\alpha, \alpha \frac{v_{n,2}}{v_{n,1}}, \dots, \alpha \frac{v_{n,n}}{v_{n,1}} \right).$$

Let $\hat{\Delta}'$ and $\hat{\Delta}'_\tau$ be the projections of Δ' and Δ'_τ , respectively, to the last $n-1$ coordinates.

Consider the part of Δ_τ with $x_1 \leq \alpha$. This part can be written

$$\left\{ (x_1, x_2, \dots, x_n) \mid 0 \leq x_1 \leq \alpha, \left(\frac{x_2 \alpha}{x_1}, \dots, \frac{x_n \alpha}{x_1} \right) \in \hat{\Delta}', x_n \geq \tau \right\}.$$

We introduce new coordinates $y_j := x_j \alpha / x_1$ for $j \in \{2, \dots, n\}$. In the coordinates $x_1, y_2, \dots, y_n$, the above set takes the form

$$\Delta_\tau \cap \{x_1 \leq \alpha\} = \left\{ \frac{\tau \alpha}{y_n} \leq x_1 \leq \alpha, y \in \hat{\Delta}'_\tau \right\}.$$

Thus, we can write the contribution to $V(R)$ of this piece as

$$\begin{aligned} & \frac{1}{k!} \int_{\hat{\Delta}'_\tau} \int_{\tau \alpha / y_n}^{\alpha} \left(\frac{x_1}{\alpha} \right)^{n-1} F^k \left(x_1, \frac{x_1}{\alpha} y_2, \dots, \frac{x_1}{\alpha} y_n \right) dx_1 \, d\text{Leb}_{n-1} \\ &= \frac{1}{k!} \int_{\hat{\Delta}'_\tau} \int_{\tau \alpha / y_n}^{\alpha} \frac{(\log y_n + \log x_1 - \log \alpha - \log \tau)^k}{x_1 y_2 \cdots y_n} dx_1 \, d\text{Leb}_{n-1}. \end{aligned}$$

We do the inner integral using (7), and then use the induction hypothesis:

$$\begin{aligned}
& \frac{1}{(k+1)!} \int_{\tilde{\Delta}'_\tau} \frac{(\log y_n + \log x_1 - \log \alpha - \log \tau)^{k+1}}{y_2 \cdots y_n} \Big|_{\tau\alpha/y_n}^\alpha d\text{Leb}_{n-1} \\
&= \frac{1}{(k+1)!} \int_{\tilde{\Delta}'_\tau} \frac{(\log y_n - \log \tau)^{k+1}}{y_2 \cdots y_n} d\text{Leb}_{n-1} \\
&= \frac{R^{n+k}}{(n+k)!} + \frac{R^{n+k-1}}{(n+k-1)!} \left(\sum_{i,j \geq 2: i=j} \log v'_{ij} - \sum_{i,j: i=j+1} \log v'_{ij} \right) + o(R^{n+k-1}).
\end{aligned}$$

The sum of the logarithm terms within the parentheses can be re-expressed as

$$\log \frac{\alpha}{v_{2,1}} + \sum_{i,j \geq 2: i=j} \log v_{ij} - \sum_{i,j \geq 2: i=j+1} \log v_{ij}.$$

Here we have used that $v'_{ij} = \alpha v_{ij} / v_{i1}$ for all i and j with $i \geq 2$.

Now we turn our attention to the part of Δ_τ with $x_1 \geq \alpha$. Let $\hat{\Delta}$ be the projection of Δ onto the last $n-1$ coordinates. Choose small $\epsilon > 0$. By [Lemma 5.3](#), we may neglect any region outside of the set

$$K_\beta := \{x \in \mathbb{R}^n \mid x_j \leq \beta \text{ for } 2 \leq j \leq n-1\},$$

where β is an arbitrary positive number. The definition of Δ in [\(8\)](#) implies that, by taking β small enough, we can ensure that

$$P_1 \subset K_\beta \cap \Delta \cap \{x \in \mathbb{R}^n \mid \alpha \leq x_1\} \subset P_2,$$

where

$$P_1 := K_\beta \cap ([\alpha, v_{1,1} - \epsilon] \times \hat{\Delta}) \quad \text{and} \quad P_2 := K_\beta \cap ([\alpha, v_{1,1} + \epsilon] \times \hat{\Delta}).$$

Denote by P_1^τ , P_2^τ and $\hat{\Delta}_\tau$ the intersections of the set $\{x \in \mathbb{R}^n \mid x_n \geq \tau\}$ with, respectively, P_1 , P_2 and $\hat{\Delta}$.

Let $K'_\beta := \{(x_j)_{j=2}^n \in \mathbb{R}^{n-1} \mid x_j \leq \beta \text{ for } 2 \leq j \leq n-1\}$ be the projection of K_β onto the last $n-1$ coordinates.

Using the induction hypothesis together with [Lemma 5.3](#),

$$\begin{aligned}
\frac{1}{k!} \int_{P_1^\tau} F^k(x) d\text{Leb}_n &= \left(\int_{[\alpha, v_{1,1} - \epsilon]} \frac{1}{x_1} dx_1 \right) \left(\frac{1}{k!} \int_{K'_\beta \cap \hat{\Delta}_\tau} \frac{(\log x_n - \log \tau)^k}{x_2 \cdots x_n} d\text{Leb}_{n-1} \right) \\
&= \left(\log(v_{1,1} - \epsilon) - \log \alpha \right) \left(\frac{R^{n+k-1}}{(n+k-1)!} + O(R^{n+k-2}) \right).
\end{aligned}$$

When the integral is over P_2^τ instead of P_1^τ , a similar equation holds with the sign of ϵ changed. Since ϵ is arbitrarily small, we conclude that

$$\frac{1}{k!} \int_{K_\beta \cap \Delta_\tau \cap \{x \in \mathbb{R}^n \mid \alpha \leq x_1\}} F^k(x) d\text{Leb}_n = (\log v_{1,1} - \log \alpha) \frac{R^{n+k-1}}{(n+k-1)!} + o(R^{n+k-1}).$$

Combining this equation with the previous calculation, we get the result. $\square$

The log terms in the statement of [Lemma 5.4](#) follow the following pattern, shown here for $n = 3$:

$$\begin{pmatrix} 0 & 0 & 0 \\ + & 0 & 0 \\ - & + & 0 \\ \cdot & - & + \end{pmatrix}.$$

The plus and minus signs represent the sign of the log term in the expression; a dot means that the term is not present, and 0 stands for a null coordinate. Note that when the vertices of Δ are of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ a & b & 0 \\ a & b & c \end{pmatrix},$$

the proof of the [Lemma 5.4](#) can be adapted to give an exact formula for $V(R)$:

$$V(R) = \frac{(R + \log c)^{n+k}}{(n+k)!}.$$

5.2 Combining a flag simplex with the dual of a neighbor

In this section, we take Δ to be of the same form as in equation (8), except with a single additional positive entry on the upper diagonal. That is, the entry $v_{i,i+1}$ is now positive for some $i \in \{0, \dots, n-1\}$. As before, let Δ_τ be the set $\Delta \setminus (\mathbb{R}^{n-1} \times [0, \tau))$ and, for each $R > 0$ and $k \in \mathbb{N}$, write

$$(9) \quad V_\Delta(R) := \frac{1}{k!} \int_{\Delta_\tau} F^k(x) \, d\text{Leb}_n,$$

where $\tau = \exp(-R)$. Sometimes we write simply $V(R)$ when the simplex is clear from context.

Lemma 5.5 *If the extra positive entry is $v_{0,1}$, then, for all $k \in \mathbb{N}$,*

$$(10) \quad V(R) = \frac{R^{n+k-1}}{(n+k-1)!} \cdot \left| \log \frac{v_{0,1}}{v_{1,1}} \right| + o(R^{n+k-1}).$$

If, on the other hand, the extra positive entry is $v_{i,i+1}$, with $i \in \{1, \dots, n-1\}$, then

$$(11) \quad V(R) = \frac{R^{n+k-1}}{(n+k-1)!} \cdot \left| \log \left(\frac{v_{i,i}}{v_{i,i+1}} \frac{v_{i+1,i+1}}{v_{i+1,i}} \right) \right| + o(R^{n+k-1}).$$

Proof Again, we use induction on the dimension n . When $n = 1$, we can calculate the integral exactly using (7), and get that, for sufficiently small τ ,

$$V(R) = \frac{1}{(k+1)!} |(\log v_{0,1} + R)^{k+1} - (\log v_{1,1} + R)^{k+1}|.$$

This implies the statement of the lemma, in the case where $n = 1$.

Now, assume that $n \geq 2$ and that the statement is true when the dimension is $n-1$. There are two possibilities to consider.

Case 1 ($v_{0,1} > 0$) For simplicity, we assume that $v_{0,1} > v_{1,1}$; the proof is similar when the inequality is in the other direction. The simplex Δ can be expressed as the difference (up to a set of measure zero) of the two simplices

$$\tilde{\Delta} := \text{conv}\{0, v_0, v_2, \dots, v_n\} \quad \text{and} \quad \tilde{\Delta}' := \text{conv}\{0, v_1, v_2, \dots, v_n\}.$$

Both simplices are of the form assumed in [Lemma 5.4](#). Applying this lemma to each of them and subtracting the results gives [\(10\)](#).

Case 2 ($v_{i,i+1} > 0$ for some $i \geq 1$) As in the proof of [Lemma 5.4](#), choose $\alpha > 0$ such that α is less than each of the first coordinates of the nonzero vertices of Δ .

Let Δ' be the intersection of the hyperplane $\{x_1 = \alpha\}$ with the simplex Δ . Observe that the vertices of Δ' are exactly the vertices of Δ (apart from the origin) rescaled so as to have the first coordinate equal to α . Denote these vertices by v'_i , where $i \in \{1, \dots, n\}$. Also, let Δ'_τ be the intersection of $\{x_1 = \alpha\}$ with the set $\Delta_\tau := \Delta \setminus \{x_n < \tau\}$, which is not necessarily a simplex.

Look at the part Δ_{α}^- of Δ_τ on the same side of the hyperplane as the origin. Just as in the proof of [Lemma 5.4](#), we do a change of coordinates and perform a one-dimensional integration to get that the contribution of this part is

$$\frac{1}{k!} \int_{\Delta_{\alpha}^-} F^k(x) \, d\text{Leb}_n = \frac{1}{(k+1)!} \int_{\Delta'_\tau} \frac{(\log x_n - \log \tau)^{k+1}}{x_2 \cdots x_n} \, d\text{Leb}_{n-1}.$$

Now we use the induction hypothesis. If $v'_{1,2} > 0$, we get that the contribution is

$$\frac{R^{n+k-1}}{(n+k-1)!} \cdot \left| \log \frac{v'_{1,2}}{v'_{2,2}} \right| + o(R^{n+k-1}).$$

Otherwise, we get

$$\frac{R^{n+k-1}}{(n+k-1)!} \cdot \left| \log \left(\frac{v'_{i,i}}{v'_{i,i+1}} \frac{v'_{i+1,i+1}}{v'_{i+1,i}} \right) \right| + o(R^{n+k-1}),$$

where the extra positive entry is $v'_{i,i+1}$. Recalling that $v'_{i,j} = \alpha v_{i,j} / v_{i,1}$ for each i and j , we get in both cases that the right-hand side of [\(11\)](#) is a lower bound on $V(R)$.

Now we want to show that the same expression is also an upper bound. Choose $\beta > 0$ such that β is greater than each of the first coordinates of the nonzero vertices of Δ . Let Δ'' be the projection of Δ onto the hyperplane $\{x_1 = \beta\}$ along rays passing through the origin. In other words, Δ'' is the intersection of that hyperplane with the cone over Δ' . Observe that the vertices of Δ'' are exactly the points $(\beta/\alpha)v_i$ with $i \in \{1, \dots, n\}$. Another important thing to note is that $\text{conv}(\{0\} \cup \Delta'')$ contains Δ . So we get an upper bound on $V(R)$ by calculating the integral in [\(9\)](#) with Δ_τ replaced by $\text{conv}(\{0\} \cup \Delta'') \cap \{x_n \geq \tau\}$. This calculation is formally the same as the one we have just done, with α replaced by β , and it has the same answer. $\square$

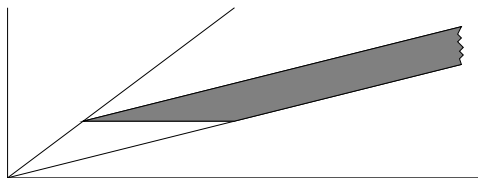


Figure 1: The integral of $1/xy$ over the infinite gray region is finite.

Let $\Phi_\tau: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the map $\Phi_\tau(x) := (1-\tau)x + \tau v_n$ that shrinks by a factor $1-\tau$ towards the point v_n . Write $\tilde{\Delta}_\tau := \Phi_\tau(\Delta)$. For each $R > 0$ and $k \in \mathbb{N}$, write

$$\tilde{V}_\Delta(R) := \frac{1}{k!} \int_{\tilde{\Delta}_\tau} F^k(x) \, d\text{Leb}_n,$$

where $\tau = \exp(-R)$. Again we omit the Δ when the simplex is clear from context. Observe that if the n^{th} coordinate of v_n is 1, and the n^{th} coordinate of every other v_i is zero, then $\Delta_\tau = \tilde{\Delta}_\tau$ and so $V_\Delta(R) = \tilde{V}_\Delta(R)$ for all $R > 0$.

The case when the extra positive entry is the coordinate $v_{n-1,n}$ presents a special difficulty, because in this case the region we must integrate over to compute the Funk volume is not Δ_τ but instead $\tilde{\Delta}_\tau$. We show however that, for our purposes, the difference between the two integrals is unimportant because it is of order R^{n-2} . We first consider the two-dimensional case.

Lemma 5.6 *Let m_1, m_2 and τ be positive real numbers, with $m_1 < m_2$. Define the region*

$$(12) \quad A_\tau := \left\{ (x, y) \in \mathbb{R}^2 \mid \tau \leq y, m_1 x \leq y \text{ and } m_1 \left(x - \frac{\tau}{m_2} \right) \geq y - \tau \right\}.$$

Then, the integral over A_τ of $1/xy$ against the Lebesgue measure is finite. Moreover, it depends on m_1 and m_2 , but not on τ .

The region A_τ is illustrated in Figure 1.

Proof Observe that the vertical distance between the two parallel lines bounding A_τ is $h := \tau(1 - m_1/m_2)$. Since $y \geq m_1 x$ for $(x, y) \in A_\tau$, and the integrand is decreasing in y , we have the following bound:

$$\int_{A_\tau} \frac{1}{xy} \, dy \, dx \leq \int_{\tau/m_2}^{\infty} \int_{m_1 x}^{m_1 x + h} \frac{1}{m_1 x^2} \, dy \, dx = \frac{-h}{m_1 x} \Big|_{\tau/m_2}^{\infty} = \frac{m_2}{m_1} - 1.$$

That the integral is independent of τ follows from the invariance of the integrand under the map $(x, y) \mapsto (\alpha x, \alpha y)$ mapping A_τ to $A_{\tau/\alpha}$, for any $\alpha > 0$. $\square$

We now consider the n -dimensional case. Note that $\tilde{\Delta}_\tau \subset \Delta_\tau$ for all τ .

Lemma 5.7 *Assume that Δ is of the form (8), except that the entry $v_{n-1,n}$ is positive. Also, assume that $v_n = (1, \dots, 1)$. Then,*

$$V_\Delta^*(R) := \int_{\Delta_\tau \setminus \tilde{\Delta}_\tau} \frac{d\text{Leb}_n}{x_1 \cdots x_n} = O(R^{n-2}).$$

Proof We treat the case where $v_{n-1,n-1} \geq v_{n-1,n}$. The other case can be handled similarly.

Let $x \in \Delta_\tau \setminus \tilde{\Delta}_\tau$. Since $x \in \Delta_\tau$ we have

$$x_n \geq \tau \quad \text{and} \quad \frac{x_n}{v_{n-1,n}} \geq \frac{x_{n-1}}{v_{n-1,n-1}}.$$

Since $x \in \Delta \setminus \tilde{\Delta}_\tau$, we have

$$\frac{x_n - \tau}{v_{n-1,n}} \leq \frac{x_{n-1} - \tau}{v_{n-1,n-1}}.$$

Let $W \subset \mathbb{R}^2$ be the set of pairs (x_{n-1}, x_n) such that these three inequalities hold.

Project Δ to the coordinate hyperplane $x_{n-1} = 0$ to obtain the simplex $\bar{\Delta}$. We may then apply [Lemma 5.2](#) to $\bar{\Delta}$ and find positive constants $a_1, \dots, a_{n-2}, a_n$ such that, for all $x \in \Delta$, we have

$$a_n x_n \leq a_{n-2} x_{n-2} \leq \dots \leq a_1 x_1 \leq 1.$$

In particular, for all $x \in \Delta_\tau$, we have

$$\tau \leq \frac{a_{n-2}}{a_n} x_{n-2} \leq \frac{a_{n-3}}{a_n} x_{n-3} \leq \dots \leq \frac{a_1 x_1}{a_n} \leq \frac{1}{a_n}.$$

Let $\Delta'_\tau \subset \mathbb{R}^{n-2}$ be the set of points $(x_1, \dots, x_{n-2})$ satisfying this inequality. Observe that this set is a simplex.

Since $\Delta_\tau \setminus \tilde{\Delta}_\tau$ is a subset of $\Delta'_\tau \times W$, we have

$$V_\Delta^*(R) \leq \int_{\Delta'_\tau} \frac{d\text{Leb}_{n-2}}{x_1 \cdots x_{n-2}} \cdot \int_W \frac{d\text{Leb}_2}{x_{n-1} x_n}.$$

The first factor grows like $O(R^{n-2})$ according to [Lemma 5.4](#), and the second is bounded, by [Lemma 5.6](#). $\square$

5.3 Combining a flag simplex with the dual of a nonneighbor

Let the n -dimensional vectors v_i for $i \in \{0, \dots, n\}$ be such that $v_{ij} > 0$ for all $j \leq i$. Here j indexes the coordinates and ranges from 1 to n . Assume that $\Delta := \text{conv}\{v_0, \dots, v_n\}$ is a nondegenerate n -dimensional simplex.

On $\mathbb{R}^n$ we use $\leq$ to denote the coordinatewise partial order, that is, $x \leq y$ if $x_j \leq y_j$ for all $j \in \{1, \dots, n\}$. If $A \subset \mathbb{R}^n$, we write $|A|$ for its Lebesgue volume.

Lemma 5.8 *Let Δ_1 and Δ_2 be nondegenerate simplices in $\mathbb{R}_+^n$ with vertices u_i for $i \in \{0, \dots, n\}$, and v_i for $i \in \{0, \dots, n\}$, respectively. Let $R > 0$, and take $k = 0$. If $u_i \leq v_i$ for all $i \in \{0, \dots, n\}$, then*

$$\frac{\tilde{V}_{\Delta_1}(R)}{|\Delta_1|} \geq \frac{\tilde{V}_{\Delta_2}(R)}{|\Delta_2|}.$$

Proof There exists an affine transformation T taking u_i to v_i for all i . Each $x \in \Delta_1$ can be expressed as a convex combination of the vertices $\{u_i\}$, and so Tx is a convex combination of the $\{v_i\}$ with the same coefficients. We conclude that $Tx \geq x$ for all $x \in \Delta_1$. Noting that F^0 is monotone decreasing and that T takes $\tilde{\Delta}_1^\tau := \Phi_\tau(\Delta_1)$ to $\tilde{\Delta}_2^\tau := \Phi_\tau(\Delta_2)$, we get

$$\frac{|\tilde{\Delta}_1^\tau|}{|\tilde{\Delta}_2^\tau|} \int_{\tilde{\Delta}_2^\tau} F^0(x) \, d\text{Leb}_n = \int_{\tilde{\Delta}_1^\tau} F^0(Tx) \, d\text{Leb}_n \leq \int_{\tilde{\Delta}_1^\tau} F^0(x) \, d\text{Leb}_n.$$

It just remains to observe that Φ^τ shrinks Lebesgue volume by the constant factor $(1 - \tau)^n$. $\square$

Lemma 5.9 Let Δ be a nondegenerate simplex in $\mathbb{R}_+^n$ with vertices u_i for $i \in \{0, \dots, n\}$, such that $u_{i,j} > 0$ when $i \geq j$. Assume that there are two distinct vertices v_{p-1} and v_{q-1} of Δ , with $p, q \in \{1, \dots, n\}$, such that $v_{p-1,p}$ and $v_{q-1,q}$ are positive. Take $k = 0$. Then,

$$\tilde{V}_\Delta(R) = o(R^{n-1}).$$

Proof Let $\epsilon := \min\{v_{ij} \mid v_{ij} > 0\}$ be the smallest strictly positive coordinate amongst the vertices of Δ . By Lemma 5.8, it is enough to show that the result is true for the simplex obtained from Δ by replacing the lower triangular coordinates, that is, v_{ij} with $i \geq j$, by ϵ , the two diagonal coordinates $v_{p-1,p}$ and $v_{q-1,q}$ by $\epsilon/2$, and all the other coordinates by zero. We verify that this simplex is nondegenerate by subtracting v_0 from each of the other vectors, and calculating the determinant of the resulting matrix, that is, $\det(v_{ij} - v_{0j})_{1 \leq i, j \leq n}$. This determinant equals $\epsilon^n/4$, as can be easily verified by separately considering the cases when $0 \in \{q-1, p-1\}$ and $0 \notin \{q-1, p-1\}$.

Thus we assume that Δ is of this new form. Now scale Δ so that all coordinates of all vertices lie in $\{0, \frac{1}{2}, 1\}$. Notice that $\tilde{V}_\Delta$ does not change, as the integrand is invariant under diagonal linear maps.

Since we now have $v_{nn} = 1$, the inclusion $\tilde{\Delta}_\tau \subset \Delta_\tau$ holds for each $\tau \in (0, 1)$, and so

$$(13) \quad \tilde{V}_\Delta(R) \leq V_\Delta(R) \quad \text{for all } R > 0.$$

By switching labels if necessary, we can assume that $q > p$. Let v' be the vector having the same coordinates as v_q , except for the q^{th} coordinate, which we set equal to 0. So,

$$v' = (1, \dots, 1, 0, 0, \dots, 0), \quad v_{q-1} = (1, \dots, 1, \frac{1}{2}, 0, \dots, 0), \quad v_q = (1, \dots, 1, 1, 0, \dots, 0).$$

Observe that the simplex Δ can be expressed as the difference (up to a set of measure zero) of the two simplices

$$\Delta_1 := \text{conv}\{v_0, \dots, v_{q-2}, v', v_{q-1}, v_{q+1}, \dots, v_n\},$$

$$\Delta_2 := \text{conv}\{v_0, \dots, v_{q-2}, v', v_q, v_{q+1}, \dots, v_n\}.$$

Here Δ_1 is the simplex obtained from Δ by replacing the vertex v_q with the vertex v' , and Δ_2 is obtained similarly by replacing v_{q-1} with v' . Both of these simplices are of the form assumed in Lemma 5.5. We apply this lemma to each of them, with $k = 0$, and see that the term of order $n - 1$ is the same in each case. Subtracting, we get that $V_\Delta(R) = o(R^{n-1})$. That the same holds for $\tilde{V}_\Delta(R)$ follows from (13). $\square$

6 The contribution of a flag simplex

Recall that a flag decomposition of a polytope involves, for each face F of the polytope, a choice of point $p(F)$ in the relative interior of F . The polytope P is itself a face, and we choose $p(P)$ to be the origin, which we have assumed to be in the interior of P . We also take a flag decomposition of the dual polytope, and so for each face F we also have a dual point $q(F)$ in the relative interior of the dual face of F . To simplify notation, we write

$$L(F, G) := \log(1 - \langle q(F), p(G) \rangle)$$

for faces F and G of the polytope P . Observe that $L(F, G)$ is finite as long as G is not a subset of F . For a face F of P , by F° we mean the face of P° that is dual to F .

We will make use of the following basic fact from linear algebra: the space of linear antisymmetric 2-forms on V , which can be thought of as the space $(\wedge^2 V)^*$ of linear functionals on the exterior square of V , is naturally isomorphic to the exterior square $\wedge^2(V^*)$ of the dual. The corresponding pairing $\wedge^2 V^* \times \wedge^2 V \rightarrow \mathbb{R}$ is given on simple 2-vectors by

$$\langle v_1 \wedge v_2, u_1 \wedge u_2 \rangle = \langle v_1, u_1 \rangle \langle v_2, u_2 \rangle - \langle v_1, u_2 \rangle \langle v_2, u_1 \rangle.$$

Lemma 6.1 Fix $1 \leq i \leq n-2$. Let f be a flag of P , and $g := r_i f$. For $j \in \{i-1, i, i+1\}$ choose points $p(f_j)$ in the relative interior of f_j , and $q(g_j)$ in the relative interior of g_j° . Write

$$\begin{aligned} u_j &:= p(f_j) - p(f_{j-1}) \quad \text{for } j \in \{i, i+1\}, \\ v'_j &:= q(g_j) - q(g_{j+1}) \quad \text{for } j \in \{i, i-1\}. \end{aligned}$$

Then $\langle v'_i \wedge v'_{i-1}, u_i \wedge u_{i+1} \rangle \geq 0$.

Proof Let E be the 2-dimensional affine plane through $q(g_j)$, where $j \in \{i-1, i, i+1\}$. It lies in the affine span of $g_{i-1}^\circ = f_{i-1}^\circ$, and since it contains $q(f_{i+1}^\circ)$, it intersects the face f_i° along an interval I . This interval cannot lie inside $f_{i+1}^\circ = g_{i+1}^\circ$, for in this case E would lie inside the affine span of g_i° , and so would not contain $q(f_{i-1}^\circ)$. Therefore, I passes through the relative interior of f_i° . Choose any point $q(f_i)$ in the interior of I , and define $v_i := q(f_i) - q(g_{i+1})$ and $v_{i-1} := q(g_{i-1}) - q(f_i)$. Since $\{v_i, v_{i-1}\}$ and $\{v'_i, v'_{i-1}\}$ are bases of the linear space of E having opposite orientations, $v_i \wedge v_{i-1} = c v'_i \wedge v'_{i-1}$ for some $c < 0$. Thus it remains to check that $\langle v_i \wedge v_{i-1}, u_i \wedge u_{i+1} \rangle \leq 0$.

Note that, since $\langle v_i, u_i \rangle = 0$, we have

$$\langle v_i \wedge v_{i-1}, u_i \wedge u_{i+1} \rangle = -\langle v_{i-1}, u_i \rangle \langle v_i, u_{i+1} \rangle.$$

Now

$$\langle v_{i-1}, u_i \rangle = \langle q(f_{i-1}) - q(f_i), p(f_i) - p(f_{i-1}) \rangle = \langle q(f_{i-1}), p(f_i) \rangle - 1 \leq 0,$$

and similarly $\langle v_i, u_{i+1} \rangle \leq 0$. This concludes the proof. $\square$

Lemma 6.2 *The volume of the Funk outward ball of radius R inside a polytope P satisfies*

$$(14) \quad \omega_n \text{Vol}_P(B_P(R)) = c_0(P)R^n + c_1(P)R^{n-1} + o(R^{n-1}),$$

where $c_0(P) = |\text{Flags}(P)|/(n!)^2$ and

$$(15) \quad c_1(P) = \frac{1}{n!(n-1)!} \sum_{f \in \text{Flags}(P)} \left(\sum_{i=0}^{n-1} L((r_i f)_i, f_i) - \sum_{i=0}^{n-1} L((r_i f)_i, f_{i+1}) \right).$$

Proof Lemma 5.1 tells us that to find the contribution of a flag f to the volume, we must consider it paired with each of the dual flag simplices. We use the results of the previous section to determine the contribution of each of these pairings. This contribution depends on the degree of proximity of the dual flag to f .

Indeed, Lemma 5.9 shows that only the dual of f and its nearest neighbors are important, when paired with f , while all other dual flags only contribute to the $o(R^{n-1})$ error term. Recall that a nearest neighbor of a flag is one that differs in exactly one face. Moreover, by Lemma 5.5, only the pairing of the flag with its own dual contributes to the R^n term. Together, those observations confirm the asymptotic formula (14), and it remains to compute the values of $c_0(P)$ and $c_1(P)$.

We apply Lemma 5.4 to the simplex having vertices v_i for $0 \leq i \leq n$, given by

$$v_{ij} = x_{j-1}^f(p(f_i)) = 1 - \langle q(f_{j-1}), p(f_i) \rangle \quad \text{for } 1 \leq j \leq n,$$

taking $k = 0$. The contribution of f , paired with its dual, to the leading term is $(1/n!)R^n$. Together with Lemma 5.1, the value for $c_0(P)$ follows. The contribution to $c_1(P)$ of f paired with its own dual is

$$\sum_{i=0}^{n-1} L(f_i, f_{i+1}) - \sum_{i=0}^{n-2} L(f_i, f_{i+2}).$$

When we consider a nearest neighbor having a different face of dimension i , all that changes is the $(i+1)^{\text{th}}$ coordinate of each vertex. Using Lemmas 5.5 and 5.7, we get that the contribution to c_1 of f paired with this neighboring dual flag simplex is, up to sign,

$$L((r_i f)_i, f_i) + L(f_{i-1}, f_{i+1}) - L(f_{i-1}, f_i) - L((r_i f)_i, f_{i+1})$$

when $i \in \{1, \dots, n-1\}$, while for $i = 0$ we get

$$L((r_0 f)_0, f_0) - L((r_0 f)_0, f_1).$$

We will now confirm that we have chosen the sign correctly, namely that these expressions are positive in all cases. For $i = 0$, this is straightforward to check, so we assume $i \geq 1$.

Write $f' := r_i f$. We must check that

$$(1 - \langle q(f'_i), p(f_i) \rangle)(1 - \langle q(f_{i-1}), p(f_{i+1}) \rangle) \geq (1 - \langle q(f_{i-1}), p(f_i) \rangle)(1 - \langle q(f'_i), p(f_{i+1}) \rangle).$$

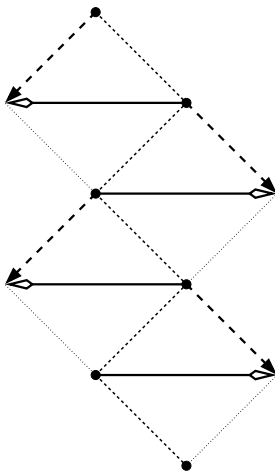


Figure 2: The contribution of a single flag simplex.

First consider the case $i = n - 1$. Then $p(f_{i+1}) = 0$, and it remains to check that

$$\langle q(f_{i-1}), p(f_i) \rangle \geq \langle q(f'_i), p(f_i) \rangle \iff \langle q(f_{i-1}) - q(f'_i), p(f_i) \rangle \geq 0,$$

which holds since $q(f_{i-1}) - q(f'_i) = c(q(f_i) - q(f_{i-1}))$ for some $c > 0$.

For $1 \leq i \leq n - 2$, set $u_j = p(f_j) - p(f_{j-1})$ for $j \in \{i, i + 1\}$, and $v'_j = q(f'_j) - q(f'_{j+1})$ for $j \in \{i - 1, i\}$.

One computes

$$\begin{aligned} \langle q(f'_i), p(f_i) \rangle &= 1 + \langle v'_i, u_i \rangle, \\ \langle q(f_{i-1}), p(f_{i+1}) \rangle &= 1 + \langle v'_i, u_i \rangle + \langle v'_i, u_{i+1} \rangle + \langle v'_{i-1}, u_i \rangle + \langle v'_{i-1}, u_{i+1} \rangle, \\ \langle q(f_{i-1}), p(f_i) \rangle &= 1 + \langle v'_i, u_i \rangle + \langle v'_{i-1}, u_i \rangle, \\ \langle q(f'_i), p(f_{i+1}) \rangle &= 1 + \langle v'_i, u_i \rangle + \langle v'_i, u_{i+1} \rangle. \end{aligned}$$

Then the desired inequality is easily seen to be equivalent to

$$\langle v'_i, u_i \rangle \langle v'_{i-1}, u_{i+1} \rangle \geq \langle v'_{i-1}, u_i \rangle \langle v'_i, u_{i+1} \rangle,$$

which holds by [Lemma 6.1](#).

Adding up all these contributions, recalling that $L(f_{n-1}, f_n) = 0$, we get the stated formula for $c_1(P)$. $\square$

This is illustrated in [Figure 2](#). The dotted line represents the flag, connecting the full polytope face at the top to the empty set at the bottom. The arrows represent the terms of the above expression. An arrow starting at a face F and ending at face G represents the term $\pm \log(1 - \langle q(G), p(F) \rangle)$, with a positive sign if the arrow is full, and a negative sign if the arrow is dashed.

It will be convenient to express the sum in [\(15\)](#) in a different way.

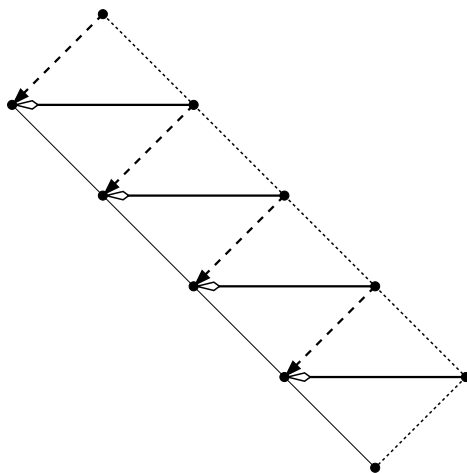


Figure 3: The contribution of a flag simplex after the sum is rearranged.

Lemma 6.3 Assume we are given a polytope P with a flag decomposition, and a flag decomposition of its dual. Then, the sum over the flags in (15) is equal to

$$\sum_{f \in \text{Flags}(P)} \left(\sum_{i=0}^{n-1} L((rf)_i, f_i) - \sum_{i=0}^{n-1} L((rf)_i, f_{i+1}) \right).$$

Proof Fix i , and observe that $r_{i-1} \cdots r_0$ is a permutation of the flags. If g and f are two flags such that $g = r_{i-1} \cdots r_0 f$, then

$$g_i = f_i, \quad g_{i+1} = f_{i+1} \quad \text{and} \quad (r_i g)_i = (rf)_i.$$

Therefore,

$$\begin{aligned} \sum_{g \in \text{Flags}(P)} L((r_i g)_i, g_i) &= \sum_{f \in \text{Flags}(P)} L((rf)_i, f_i), \\ \sum_{g \in \text{Flags}(P)} L((r_i g)_i, g_{i+1}) &= \sum_{f \in \text{Flags}(P)} L((rf)_i, f_{i+1}). \end{aligned}$$

Summing over all i gives the result. □

These terms are illustrated in Figure 3 (for a single flag f) using the same convention as before.

In the following lemma, we are using the convention that faces are closed. We use $\text{aff } F$ to denote the affine hull of a set F , and by $X - Y$ we mean the Minkowski sum of X and $-Y$.

Lemma 6.4 Let F and G be faces of P such that $F \subset G$ and $\dim G = \dim F + 1$. If $p, p' \in G$ and $q, q' \in F^\circ$, then

$$(1 - \langle q, p \rangle)(1 - \langle q', p' \rangle) = (1 - \langle q, p' \rangle)(1 - \langle q', p \rangle).$$

Proof Because the dimensions of F and G differ by exactly 1, we can write $p = s + \alpha v$ and $p' = s' + \alpha' v$, with $s, s' \in \text{aff } F$ and $\alpha, \alpha' \in \mathbb{R}$, and where v is some element of $(\text{aff } G \setminus \text{aff } F) - \text{aff } F$. Likewise, we can write $q = r + \beta u$ and $q' = r' + \beta' u$, with $r, r' \in \text{aff } G^\circ$ and $\beta, \beta' \in \mathbb{R}$, and where u is some element of $(\text{aff } F^\circ \setminus \text{aff } G^\circ) - \text{aff } G^\circ$.

Note that $\langle r, s \rangle = 1$ and $\langle r, v \rangle = \langle u, s \rangle = 0$. We calculate that $1 - \langle q, p \rangle = -\alpha\beta\langle u, v \rangle$, and the other factors in (16) can be calculated similarly. The conclusion then follows. $\square$

We are now ready to prove [Theorem 1.5](#).

Proof of Theorem 1.5 Let f be a flag of P . For each $i \in \{0, \dots, n-2\}$, we apply [Lemma 6.4](#) with $F := (rf)_i$ and $G := f_{i+1}$. Observe that $q((rf)_{i+1})$ and $q((rf)_i)$ are in F° , and that $p(f_{i+1})$ and $p(f_0)$ are in G . From the lemma,

$$L((rf)_{i+1}, f_0) + L((rf)_i, f_{i+1}) = L((rf)_{i+1}, f_{i+1}) + L((rf)_i, f_0).$$

It is clear that no proper face of the flag f is a subset of any proper face of rf . Hence, all four of these terms are finite. Summing over $i \in \{0, \dots, n-2\}$, and using that $L((rf)_{n-1}, f_n) = 0$ (this is because $p(P)$ is the origin), we get

$$L((rf)_{n-1}, f_0) + \sum_{i=0}^{n-1} L((rf)_i, f_{i+1}) = \sum_{i=0}^{n-1} L((rf)_i, f_i).$$

Summing over all flags, and comparing with [Lemmas 6.2](#) and [6.3](#), we get the result. $\square$

Proof of Corollary 1.6 Let Q be a cube of the same dimension as P . Both P and Q have $2^n n!$ flags. Using [Theorem 1.5](#) we may write

$$\omega_n \text{Vol}_P(B_P(R)) = \frac{2^n}{n!} R^n + \frac{n}{(n!)^2} \sum_{f \in \text{Flags}(P)} \log(1 - \langle (rf)_{n-1}, f_0 \rangle) R^{n-1} + o(R^{n-1}).$$

We have by [Theorem 1.3](#) that

$$\text{Vol}_P(B_P(R)) \geq \text{Vol}_Q(B_Q(R)) \quad \text{for all } R > 0.$$

As the leading term in the asymptotics for the volume is the same for Q and P , it must hold that

$$(17) \quad \sum_{f \in \text{Flags}(P)} \log(1 - \langle (rf)_{n-1}, f_0 \rangle) \geq \sum_{f \in \text{Flags}(Q)} \log(1 - \langle (rf)_{n-1}, f_0 \rangle).$$

For each $f \in \text{Flags}(Q)$, one easily verifies that $\langle (rf)_{n-1}, f_0 \rangle = -1$. Thus the right-hand side of (17) equals $2^n n! \log 2$.

For each $f \in \text{Flags}(P)$, it holds by the central symmetry of P that $\langle (rf)_{n-1}, f_0 \rangle \geq -1$, and so none of the summands on the left-hand side of (17) exceed $\log 2$. As there are $2^n n!$ summands, it follows that each summand must equal $\log 2$, that is, $\langle (rf)_{n-1}, f_0 \rangle = -1$ for all $f \in \text{Flags}(P)$. Thus $\langle (rf)_{n-1}, -f_0 \rangle = 1$, or equivalently, $-f_0 \in (rf)_{n-1}$ for all $f \in \text{Flags}(P)$, concluding the proof. $\square$

7 The limit of the Funk–Santaló point for polytopal Funk geometries as the radius tends to infinity

Santaló [19] studied the dependence of the Mahler volume of a convex body on the point in the body through which the polar is taken. He showed that there is a unique point in the interior of the body where this quantity is minimized. This point is now known as the *Santaló point* of the body.

This notion is generalized in [25], where, instead of the Mahler volume, the Holmes–Thompson volume of metric balls of the Funk geometry was considered. The following theorem appears there.

Theorem 7.1 [25] *Let K be a convex body, and let $R > 0$. Then, the function $f_{K,R}: K \rightarrow \mathbb{R}$, defined by*

$$f_{K,R}(x) := \text{Vol}_K(B_K(x, R)),$$

is proper and strictly convex, and hence attains its infimum at a unique point $s_R(K)$ in the interior of K .

Proper means that the function converges to infinity as the boundary of K is approached. We call the point $s_R(K)$ the *Funk–Santaló point* of the convex body K , for radius R .

It is shown in [25] that the Funk–Santaló point of a convex body converges to the Santaló point as the radius tends to zero. Here, we study the behavior of the Funk–Santaló point as the radius becomes large in the case of polytopes. We will see that this behavior is governed by the second-highest order term of the volume growth at infinity.

Recall that if a sequence f_j of convex functions converges pointwise on a dense subset D of an open convex set C in $\mathbb{R}^n$, then the limit function f exists on C , and the convergence is locally uniform on C ; see [18, Theorem 10.8].

The following lemma concerning the convergence of infima of convex functions is established in [25].

Lemma 7.2 *Let f_j be a sequence of lower semicontinuous convex functions defined on a convex body K in a finite-dimensional vector space, taking values in $\mathbb{R} \cup \{\infty\}$. Assume that f_j converges pointwise on $\text{int } K$ to a lower semicontinuous convex function f that is finite on $\text{int } K$. Then, $\inf f_j$ converges to $\inf f$. Moreover, if for each j , the function f_j attains its infimum at x_j , and this sequence of points has a limit point x , then f attains its infimum at x .*

Proof of Theorem 1.7 By Theorem 1.5 and Lemma 2.1, we have

$$\begin{aligned} (18) \quad c_1(P, x) &= \frac{1}{n!(n-1)!} \sum_{f \in \text{Flags}(P)} \log \left(1 - \left\langle \frac{(rf)_{n-1}}{1 - \langle rf, x \rangle}, f_0 - x \right\rangle \right) \\ &= C - \frac{1}{n!(n-1)!} \sum_{F \in F_{n-1}(P)} |\text{Flags}(F)| \log(1 - \langle q(F), x \rangle), \end{aligned}$$

where C is a constant that does not depend on x , $F_{n-1}(P)$ denotes the set of facets of P , and $q(F)$ is the vertex of P° corresponding to the facet F .

Notice that each nonconstant term is minus the logarithm of an affine function, and hence convex. In fact, the level sets of any one of these terms are parallel hyperplanes, and the term is strictly convex along line segments that do not lie in one of these hyperplanes. Since the vertices of P° , that is, the set of dual vectors $q(F)$ with $F \in F_{n-1}(P)$, affinely generate the whole dual space, these points separate $\mathbb{R}^n$, and so there is no nondegenerate line segment lying in a level set of every one of the terms. We conclude that $c_1(P, x)$ is strictly convex.

The function $c_1(P, x)$ is clearly continuous, and finite in $\text{int } P$. If x is on the boundary of P , then there is at least one facet F for which $x \in F$, or equivalently $\langle q(F), x \rangle = 1$. Since none of the terms can be $-\infty$, we deduce that $c_1(P, x) = \infty$. This establishes that $c_1(P, x)$ is proper.

It follows immediately that the infimum of $c_1(P, x)$ is attained at a unique point, which we denote by $s_\infty(P)$.

As $c_1(P, x)$ is smooth in x , $s_\infty(P)$ must be its unique stationary point. Thus

$$\sum_{F \in F_{n-1}(P)} |\text{Flags}(F)| \frac{1}{1 - \langle q(F), s_\infty(P) \rangle} q(F) = 0,$$

and assuming $s_\infty(P) = 0$ we obtain the asserted characterization of $s_\infty(P)$.

For each $R > 0$, define a function of $x \in \text{int } P$ by

$$f_{P,R}(x) := \frac{1}{R^{n-1}} \left(\omega_n \text{Vol}(B_P(x, R)) - \frac{1}{(n!)^2} |\text{Flags}(P)| R^n \right).$$

By [Theorem 7.1](#), this function is a proper convex function for all $R > 0$, and its infimum is attained at $s_R(P)$. By [Theorem 1.5](#) it holds that

$$\lim_{R \rightarrow \infty} f_{P,R}(x) = c_1(P, x) \quad \text{for all } x \in \text{int } P.$$

Thus the conditions of [Lemma 7.2](#) are satisfied, and it follows upon applying that lemma that $s_R(P)$ converges to $s_\infty(P)$ as $R \rightarrow \infty$. $\square$

8 Stationary points of the second-highest-order-term functional in 2D

In this section, we prove [Theorem 1.8](#), that is, that, in dimension two, the only stationary points of the second-highest order term $c_1(P)$ in the expansion of the volume of balls are the regular polytopes, up to linear transformations. We show moreover that they uniquely maximize $c_1(P, s_\infty(P))$.

The proof will rely on the following lemma concerning polygons where c_1 remains stationary under certain perturbations of a single vertex. Recall that $\mathcal{P}_2^m$ denotes the space of convex polygons in $\mathbb{R}^2$ with m vertices that contain the origin in their interior.

Lemma 8.1 Let v_1, v_2, v_3 be consecutive vertices of a polygon $K \in \mathcal{P}_2^m$. Denote by e_2, e_3 and e_4 the consecutive vertices of K° such that

$$\langle e_2, v_1 \rangle = \langle e_2, v_2 \rangle = 1 = \langle e_3, v_2 \rangle = \langle e_3, v_3 \rangle = \langle e_4, v_3 \rangle.$$

For small $t \in \mathbb{R}$, let K^t be the polygon obtained from K by moving v_2 by an amount $t(v_2 - v_1)$. If t is small enough, then K^t is a convex polygon. Assume that

$$\left. \frac{d}{dt} c_1(K^t) \right|_{t=0} = 0.$$

Then, there is an element γ of $\mathrm{GL}_2(\mathbb{R})$ such that $\gamma(v_1) = e_4$, $\gamma(v_2) = e_3$ and $\gamma(v_3) = e_2$.

Proof As t varies, so do v_2 and e_3 . So there are four terms in the expression for $c_1(K^t)$ that change with t . More precisely, by equation (2),

$$2c_1(K^t) = \log(1 - \langle e_1, v_2(t) \rangle) + \log(1 - \langle e_4, v_2(t) \rangle) + \log(1 - \langle e_3(t), v_1 \rangle) + \log(1 - \langle e_3(t), v_4 \rangle) + C,$$

where C is a constant independent of t . Here, v_4 and e_1 are the vertices of K and K° , respectively, such that v_2, v_3 and v_4 are consecutive, and e_1, e_2 and e_3 are consecutive. So

$$2 \left. \frac{d}{dt} c_1(K^t) \right|_{t=0} = -\frac{\langle e_1, v_2 - v_1 \rangle}{1 - \langle e_1, v_2 \rangle} - \frac{\langle e_4, v_2 - v_1 \rangle}{1 - \langle e_4, v_2 \rangle} - \frac{\langle \dot{e}_3, v_1 \rangle}{1 - \langle e_3, v_1 \rangle} - \frac{\langle \dot{e}_3, v_4 \rangle}{1 - \langle e_3, v_4 \rangle}.$$

Here, $\dot{e}_3$ denotes the derivative of e_3 with respect to t at $t = 0$.

To simplify calculations, we choose (e_3, e_2) as a coordinate basis for $\mathbb{R}^2$. With respect to this basis

$$v_1 = (x, 1), \quad v_2 = (1, 1) \quad \text{and} \quad v_3 = (1, y),$$

with x and y in $(-\infty, 1)$. Moreover, with respect to the dual basis,

$$e_2 = (0, 1), \quad e_3 = (1, 0) \quad \text{and} \quad e_4 = (w, z),$$

where w and z satisfy

$$(19) \quad w + yz = 1.$$

Since $\langle e_4, v_3 - v_2 \rangle > 0$, we have $z < 0$. A short calculation shows that

$$\dot{e}_3 = \frac{1-x}{1-y}(y, -1).$$

Denote the coordinates of v_4 by $v_4 = (a, b)$. Since $\langle e_4, v_3 \rangle = \langle e_4, v_4 \rangle = 1$, we have $w + yz = aw + bz = 1$, and hence

$$\frac{b-y}{1-a} = \frac{w}{z}.$$

Using these coordinates, and that $\langle e_1, v_1 \rangle = 1$, we get

$$2 \left. \frac{d}{dt} c_1(K^t) \right|_{t=0} = 1 - \frac{(1-x)w}{1-w-z} + \frac{1-xy}{1-y} - \frac{1-x}{1-y} \cdot \frac{ay-b}{1-a}.$$

Observing that

$$-\frac{ay-b}{1-a} = \frac{y(1-a) + (b-y)}{1-a} = y + \frac{w}{z},$$

we have

$$\begin{aligned} 2 \frac{d}{dt} c_1(K^t) \Big|_{t=0} &= 1 + \frac{(1-x)w}{(1-y)z} + \frac{1-xy}{1-y} + \frac{1-x}{1-y} \left(y + \frac{w}{z} \right) \\ &= \frac{2}{1-y} \left((1-x) \frac{w}{z} + 1 - xy \right). \end{aligned}$$

By assumption, this expression is zero, so we obtain a linear relation between w and z , which combined with (19) gives us a pair of equations having the unique solution

$$w = \frac{1-xy}{1-y} \quad \text{and} \quad z = -\frac{1-x}{1-y}.$$

The linear map defined by the matrix

$$\gamma := \frac{1}{1-y} \begin{pmatrix} -y & 1 \\ 1 & -1 \end{pmatrix}$$

satisfies $\gamma(v_1) = e_4$, $\gamma(v_2) = e_3$ and $\gamma(v_3) = e_2$. □

We will also need to know that the Funk–Santaló point depends smoothly on P .

Lemma 8.2 *The map $s_\infty: \mathcal{P}_2^m \rightarrow \mathbb{R}^2$ is C^∞ -smooth.*

Proof As s_∞ commutes with translations, it suffices to consider the open subset $U \subset \mathcal{P}_2^m$ consisting of polygons containing 0 in their interior. For $P \in U$, let e_i for $1 \leq i \leq m$ be a cyclical ordering of the vertices of P° . By Theorem 1.7 and equation (18), $s_\infty(P)$ is the unique minimum of the following strictly convex smooth function defined on the interior of P :

$$f_P(z) = -\sum_{i=1}^m \log(1 - \langle e_i, z \rangle).$$

It follows that $s_\infty(P)$ is the unique solution of the equation $H(P, \cdot) = 0$, where

$$H(P, z) := \nabla f_P(z) = \sum_{i=1}^m \frac{1}{1 - \langle e_i, z \rangle} e_i.$$

Let us verify that $\partial H / \partial z$ has rank 2, so that by the implicit function theorem $z = s_\infty(P)$ depends smoothly on P .

We write $z = (z_1, z_2)$ and $e_i = (e_i^1, e_i^2)$, for each i . Observe that the Jacobian is

$$\frac{\partial H_j}{\partial z_k} = \sum_{i=1}^m \frac{e_i^j e_i^k}{(1 - \langle e_i, z \rangle)^2} \quad \text{for } j \in \{1, 2\}.$$

For clarity we write $x(e_i), y(e_i)$ for the coordinates of e_i . So, the determinant is given by

$$\begin{aligned} \det \frac{\partial H}{\partial z} &= \sum_{i=1}^m \frac{x(e_i)^2}{(1 - \langle e_i, z \rangle)^2} \sum_{j=1}^m \frac{y(e_j)^2}{(1 - \langle e_j, z \rangle)^2} - \left(\sum_{i=1}^m \frac{x(e_i)y(e_i)}{(1 - \langle e_i, z \rangle)^2} \right)^2 \\ &= \sum_{1 \leq i < j \leq m} \frac{(x(e_i)y(e_j) - x(e_j)y(e_i))^2}{(1 - \langle e_i, z \rangle)^2(1 - \langle e_j, z \rangle)^2}. \end{aligned}$$

If $\partial H/\partial z$ were not of rank 2, then we would have $x(e_i)y(e_j) = x(e_j)y(e_i)$ for all i and j , that is, all the points e_i would lie on a line, which is impossible. $\square$

Lemma 8.3 *The function $f(P) = c_1(P, s_\infty(P))$ has a global maximum on $\mathcal{P}_2^m$ for each $m \geq 3$.*

Proof Let X_m denote the set of m -sided convex polygons $P \subset \mathbb{R}^2$ in John's position. Recall that P is said to be in John's position if the John ellipse of P — the unique maximal volume ellipse it contains — is the unit Euclidean ball centered at the origin.

Since $f(P)$ is invariant under affine transformations of P , and any $P \in \mathcal{P}_2^m$ has an affine image in John's position, it suffices to show that $f(P)$ attains a global maximum on X_m .

Let B denote the Euclidean unit disc around the origin. It follows from John's theorem that each polygon $P \in X_m$ is contained in $2B$. Also, as P contains B , it holds that the polar body P° lies inside B . We conclude that for all $v \in P$ and $e \in P^\circ$, one has $|\langle e, v \rangle| \leq 2$. It follows that each summand in the expression in (2) for $c_1(P) = c_1(P, 0)$ is bounded from above by $\frac{1}{2} \log 3$.

Let $v_1, \dots, v_m$ be the consecutive vertices of P , with cyclical index, and let $e_1, \dots, e_m$ be the vertices of P° such that $\langle e_i, v_i \rangle = \langle e_i, v_{i-1} \rangle = 1$ for all i .

By Blaschke's selection theorem, X_m is precompact within the space $\mathcal{H}(\mathbb{R}^2)$ of convex bodies equipped with the Hausdorff distance. Its closure $\bar{X}_m$ in $\mathcal{H}(\mathbb{R}^2)$ is $X_m \cup \partial X_m$, where $\partial X_m := X_{m-1} \cup \dots \cup X_3$ is the geometric boundary of X_m .

Suppose now that $P \in X_m$ approaches the boundary ∂X_m . This implies that there is some i for which either $|v_i - v_{i+1}|$ or $|e_i - e_{i+1}|$ tends to zero. If the former is the case, then

$$1 - \langle e_i, v_{i+1} \rangle = \langle e_i, v_i - v_{i+1} \rangle$$

tends to zero. Similarly, in the latter case, $1 - \langle e_{i+1}, v_{i-1} \rangle$ tends to zero.

We conclude that at least one of the summands in (2) for $c_1(P)$ tends to $-\infty$. Since we have seen that each of the summands is bounded above, we conclude that $c_1(P)$ tends to $-\infty$, as P approaches the boundary. It follows that $f(P)$ also tends to the same limit as $P \rightarrow \partial X_m$, since

$$f(P) = c_1(P, s_\infty(P)) \leq c_1(P, 0) = c_1(P),$$

by the definition of $s_\infty(P)$.

Using this, and that the function $f(P)$ is continuous on X_m , we deduce that it attains a global maximum in X_m , and thus in $\mathcal{P}^m$, concluding the proof. $\square$

Proof of Theorem 1.8 Let the function c_1 be stationary at a polygon $K \in \mathcal{P}_2^m$. Let $v_1, \dots, v_m$ be the consecutive vertices of K with cyclical index, and $e_1, \dots, e_m$ the consecutive vertices of K° as in Lemma 8.1, so that $\langle e_i, v_i \rangle = \langle e_i, v_{i-1} \rangle = 1$ for all i . By that lemma, there is an element γ of $\mathrm{GL}_2(\mathbb{R})$ such that the triple of vertices (v_1, v_2, v_3) is mapped to the triple (e_4, e_3, e_2) of dual vertices. We apply the same lemma again, this time to K° , to get that there exists another element γ' of $\mathrm{GL}_2(\mathbb{R})$ such that the triple (e_2, e_3, e_4) is mapped to (v_4, v_3, v_2) . Composing these maps, we get an element γ'' of $\mathrm{GL}_2(\mathbb{R})$ taking the triple of points (v_1, v_2, v_3) to (v_2, v_3, v_4) . Similarly, we have an element of $\mathrm{GL}_2(\mathbb{R})$ taking the triple (v_2, v_3, v_4) to (v_3, v_4, v_5) . Note that this is actually the same map, because it agrees with γ'' on v_2 and v_3 . Proceeding inductively, we get that γ'' takes v_j to v_{j+1} for all j . It follows that $(\gamma'')^m$ is the identity map. The iterates $(\gamma'')^k$ for $k \in \{0, \dots, m-1\}$ form a finite subgroup of $\mathrm{GL}_2(\mathbb{R})$, and are therefore contained in some maximal compact subgroup conjugate to the orthogonal group $\mathrm{O}_2(\mathbb{R})$. Thus, we may write $\gamma'' = ghg^{-1}$ for some $g \in \mathrm{GL}_2(\mathbb{R})$ and $h \in \mathrm{O}_2(\mathbb{R})$. So, $K = gD_m$, where D_m is the regular polygon $D_m := \mathrm{conv}\{h^k(g^{-1}v_1) \mid k \in \{0, \dots, m-1\}\}$.

It remains to show that $P = D_m$ maximizes $c_1(P, s_\infty(P))$ among all $P \in \mathcal{P}_2^m$ with $s_\infty(P)$ at the origin, uniquely up to linear transformations.

By Lemma 8.2, $s_\infty: \mathcal{P}_2^m \rightarrow \mathbb{R}^2$ is smooth. As it commutes with translations, it is everywhere submersive. It follows that $Y_m := \{P \in \mathcal{P}_2^m \mid s_\infty(P) = 0\}$ is a smooth submanifold of $\mathcal{P}_2^m$.

By Lemma 8.3, the function $c_1(P, s_\infty(P))$ has a global maximum on $\mathcal{P}_2^m$. In fact, since it is affinely invariant, the maximum, which we denote by P_0 , can be taken to be in Y_m . Note that on Y_m , we have $c_1(P, s_\infty(P)) = c_1(P)$.

We can write the tangent space at P_0 as

$$T_{P_0}\mathcal{P}_2^m = T_{P_0}Y_m \oplus T_{P_0}(GP_0),$$

where $G = \mathbb{R}^2$ is the group of translations of $\mathbb{R}^2$.

It follows from the definition of s_∞ that P_0 is a local minimum of $c_1(P)$ in the directions of $T_{P_0}(GP_0)$, while by the above it is a local maximum in the directions of $T_{P_0}Y_m$. Thus, P_0 is a stationary point of $c_1(P)$ in $\mathcal{P}_2^m$. By the first part of the theorem, P_0 coincides with D_m up to a linear transformation. The assertion of the theorem readily follows. $\square$

Next we consider the derivative of $V_K(\lambda) := \omega_n \mathrm{Vol}_P(B_K(R))$, where the radius R is related to λ by $R = -\log(1 - \lambda)$. In general, this derivative can be written as an integral over the boundary of the ball. For $K = P \in \mathcal{P}_2^m$, this integral can be computed explicitly, as follows.

Let $v_1, v_2, \dots, v_m$ be the consecutive vertices of P with cyclical index, and $e_1, \dots, e_m$ the consecutive vertices of P° , so that $\langle e_i, v_i \rangle = \langle e_i, v_{i-1} \rangle = 1$ for all i . For each i and j , write $G_{ij} := 1 - \lambda \langle e_j, v_j \rangle$, understood as a function of λ .

Proposition 8.4 *It holds that*

$$\begin{aligned} \frac{dV_K(\lambda)}{d\lambda} &= \sum_{i,j} \lambda \cdot \left| \log \frac{G_{i,j} G_{i+1,j+1}}{G_{i,j+1} G_{i+1,j}} \right| \cdot \frac{|\langle e_i, v_j \rangle \langle e_{i+1}, v_{j+1} \rangle - \langle e_i, v_{j+1} \rangle \langle e_{i+1}, v_j \rangle|}{|G_{i,j} G_{i+1,j+1} - G_{i,j+1} G_{i+1,j}|} \\ &= \sum_{i,j} \left| \log \frac{G_{i,j} G_{i+1,j+1}}{G_{i,j+1} G_{i+1,j}} \right| \cdot \left| \lambda - \frac{\langle e_i, v_j \rangle + \langle e_{i+1}, v_{j+1} \rangle - \langle e_i, v_{j+1} \rangle - \langle e_{i+1}, v_j \rangle}{\langle e_i, v_j \rangle \langle e_{i+1}, v_{j+1} \rangle - \langle e_i, v_{j+1} \rangle \langle e_{i+1}, v_j \rangle} \right|^{-1}. \end{aligned}$$

Proof Observe that the proof of [Lemma 5.1](#) can be repeated to show that

$$\omega_n \text{Vol}(B_P(0, R)) = \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \int_{\lambda \Delta(0, v_i, v_{i+1})} \frac{dy_j dy_{j+1}}{y_j y_{j+1}},$$

where $y_j = 1 - \langle e_j, \cdot \rangle$ and $\Delta(0, v_i, v_{i+1})$ is the triangle with the listed vertices. We then parametrize $p = s(1-t)v_i + stv_{i+1}$, with $0 \leq s \leq \lambda$ and $0 \leq t \leq 1$, and use the fundamental theorem of calculus to deduce the stated formula. $\square$

Let us finish this section by pointing out that for the regular polygon D_m we have a very simple formula for the second-order term:

$$c_1(D_m) = 2m \log \left(2 \sin \left(\frac{\pi}{m} \right) \right) = 2m \log \left(\frac{l}{r} \right),$$

where l is the length of an edge, and r is the radius of the circumscribed circle.

9 Simplices

Theorem 9.1 *Consider the Funk geometry of the simplex $\Delta_n \subset \mathbb{R}^n$. The Holmes–Thompson volume of the forward ball $\Delta_n(R)$ of radius R centered at the barycenter is $1/\omega_n$ times*

$$(20) \quad V_{\Delta_n}(R) = \frac{n+1}{n!} \int_{\Delta_n(R)} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}.$$

We also have a recursive formula for $n \geq 2$,

$$(21) \quad V'_{\Delta_n}(R) = \frac{n+1}{n} \frac{1}{1 - \frac{e^{-R}}{n+1}} V_{\Delta_{n-1}} \left(R + \log \left(1 + \frac{1}{n} - \frac{1}{n} e^{-R} \right) \right).$$

For all $n \geq 1$ it holds for $R \rightarrow \infty$ that

$$(22) \quad V_{\Delta_n}(R) = \frac{(n+1)!}{(n!)^2} R^n (1 + n \log(n+1) R^{-1} + o(R^{-1})),$$

while for $R \rightarrow 0$ we have

$$(23) \quad V_{\Delta_n}(R) = \frac{(n+1)^{n+1}}{(n!)^2} (R^n - \frac{1}{2} n R^{n+1} + o(R^{n+1})).$$

Proof Consider the standard n -dimensional simplex Δ_n , with vertices $C_0 = 0, C_1 = e_1, \dots, C_n = e_n$, where e_i is the standard basis. We will consider a point $C = \sum_{i=0}^n \alpha_i C_i \in \Delta_n$, with $\alpha_i \geq 0$ and $\sum \alpha_i = 1$, whose barycentric coordinates are therefore $(\alpha_0, \dots, \alpha_n)$. Then in standard Euclidean coordinates,

$$C = (\alpha_1, \dots, \alpha_n).$$

Notice that $\alpha_0 = 0$ corresponds to the face F_0 belonging to the hyperplane $\sum_{i=1}^n x_i = 1$, and for $1 \leq i \leq n$, $\alpha_i = 0$ corresponds to the face F_i belonging to the hyperplane $x_i = 0$. It follows that the vectors

$$V_1 = (-\alpha_1, 0, \dots, 0), \quad V_2 = (0, -\alpha_2, 0, \dots, 0), \quad \dots, \quad V_n = (0, \dots, 0, -\alpha_n),$$

are unit vectors with respect to the Funk metric at C . The same is true for the vector

$$V_0 \left(\frac{1 - \sum_{i=1}^n \alpha_i}{n}, \dots, \frac{1 - \sum_{i=1}^n \alpha_i}{n} \right).$$

Hence for $i = 1, \dots, n$, the hyperplanes $X_i = -\alpha_i$ are the faces of the tangent unit ball, and so is the hyperplane $\sum_{i=1}^n X_i = 1 - \sum_{i=1}^n \alpha_i = \alpha_0$. Then the dual simplex can be seen as the convex hull of the following vectors in the dual space:

$$W_1 = \left(\frac{-1}{\alpha_1}, 0, \dots, 0 \right), \quad \dots, \quad W_n = \left(0, \dots, \frac{-1}{\alpha_n} \right) \quad \text{and} \quad W_0 = \frac{1}{\alpha_0} (1, \dots, 1).$$

The volume of this simplex is the sum of the volumes of the simplices defined by 0 and any n of this points. For instance, the volume defined by 0, $W_1, \dots, W_n$ is exactly

$$\frac{1}{n!} \frac{1}{\alpha_1 \cdots \alpha_n},$$

hence by adding all these terms we get the Holmes–Thompson density in the form

$$(24) \quad \beta_{\Delta_n}^*(A) = \frac{1}{n!} \sum_{i=0}^n \frac{1}{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_n} = \frac{1}{n!} \frac{1}{\alpha_0 \cdots \alpha_n},$$

where the $\hat{}$ means that the term is removed). Recall that the Funk ball $\Delta_n(R)$ is the image of Δ_n under the dilation with ratio $(1 - e^{-R})$ centered at the barycenter $(1/(n+1))(1, \dots, 1)$. Then

$$V_{\Delta_n}(R) = \frac{1}{n!} \sum_{i=0}^n \int_{\Delta_n(R)} \frac{1}{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_n} dx.$$

If for $i = 1, \dots, n$ we use the change of variable

$$F_i(x) = \left(x_1, \dots, x_{i-1}, 1 - \sum_{k=1}^n x_k, x_{i+1}, \dots, x_n \right),$$

we get that

$$\int_{\Delta_n(R)} \frac{1}{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_n} dx = \int_{\Delta_n(R)} \frac{1}{\alpha_1 \cdots \alpha_n} dx,$$

which allows us to get the formula

$$(25) \quad V_{\Delta_n}(R) = \frac{n+1}{n!} \int_{\Delta_n(R)} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$$

for the volume of the forward ball of radius R centered at the barycenter of the simplex in euclidean coordinates.

Denote by

$$A_0 = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1} \right)$$

the barycenter of Δ_n . Then for $1 \leq k \leq n$,

$$A_k(R) = A_0 + (1 - e^{-R}) \left(\frac{1}{n-k} \sum_{j=1}^{n-k} e_j - A_0 \right)$$

are the vertices of a simplex $S_n(R)$ in the barycentric subdivision of $\Delta_n(R)$.

Explicitly,

$$(26) \quad A_1(R) = \left(\frac{1}{n} - \frac{e^{-R}}{n(n+1)}, \dots, \frac{1}{n} - \frac{e^{-R}}{n(n+1)}, \frac{e^{-R}}{n+1} \right),$$

$\vdots$

$$(27) \quad A_k(R) = \left(\underbrace{\frac{1}{n-k+1} - \frac{k}{(n+1)(n-k+1)}}_{n-k \text{ times}} e^{-R}, \underbrace{\frac{e^{-R}}{n+1}}_{k \text{ times}} \right),$$

$\vdots$

$$A_n(R) = \left(\frac{e^{-R}}{n+1}, \dots, \frac{e^{-R}}{n+1} \right).$$

Therefore, because of the symmetries involved, using the right equality in (24) with $X = x_1 + \cdots + x_n$, we have

$$(28) \quad \int_{\Delta_n(R)} \frac{dx_1 \cdots dx_n}{(1-X)x_1 \cdots x_n} = (n+1)! \int_{S_n(R)} \frac{dx_1 \cdots dx_n}{(1-X)x_1 \cdots x_n}.$$

Let us focus on

$$(29) \quad W_n(R) = \int_{S_n(R)} \frac{dx_1 \cdots dx_n}{(1-x_1 - \cdots - x_n)x_1 \cdots x_n},$$

so that

$$V_{\Delta_n}(R) = (n+1)W_n(R).$$

We can rewrite $W_n(R)$ using Fubini's theorem, and writing $s = x_n$,

$$W_n(R) = \int_{e^{-R}/(n+1)}^{1/(n+1)} \frac{ds}{s} \int_{S_n(-\log(s(n+1))) \cap \{x_n=s\}} \frac{dx_1 \cdots dx_{n-1}}{((1-s) - x_1 - \cdots - x_{n-1})x_1 \cdots x_{n-1}}.$$

Now let us proceed with the change of variable $t_i = x_i/(1-s)$ for $i = 1, \dots, n-1$ in the inner integral. This gives

$$(30) \quad W_n(R) = \int_{e^{-R}/(n+1)}^{1/(n+1)} \frac{ds}{s(1-s)} \int_{S_{n-1}(-\log(sn/(1-s)))} \frac{dt_1 \cdots dt_{n-1}}{(1-t_1 - \cdots - x_{n-1}) \cdot t_1 \cdots t_{n-1}} \\ = \int_{e^{-R}/(n+1)}^{1/(n+1)} \frac{ds}{s(1-s)} W_{n-1}\left(-\log \frac{s \cdot n}{1-s}\right).$$

Hence it holds that

$$W'_n(R) = \frac{1}{1 - \frac{e^{-R}}{n+1}} W_{n-1}\left(-\log \frac{n}{(n+1)e^R - 1}\right),$$

from which we get the following differential equation by multiplying by $(n+1)$:

$$(31) \quad V'_{\Delta_n}(R) = \frac{n+1}{n} \frac{1}{1 - \frac{e^{-R}}{n+1}} V_{\Delta_{n-1}}\left(R + \log\left(1 + \frac{1}{n} - \frac{1}{n}e^{-R}\right)\right),$$

as claimed.

Next consider the asymptotic behavior as $R \rightarrow 0$. The one-dimensional case follows from [Lemma 3.4](#), yielding

$$V_{\Delta_1}(R) = 2 \log(2e^R - 1) = 2R + 2 \log 2 + 2 \log(1 - \frac{1}{2}e^{-R}) = 4R - 2R^2 + o(R^2).$$

Next we proceed by induction, utilizing the recursive formula. Writing

$$V_{\Delta_n}(R) = a_n R^n + b_n R^{n+1} + o(R^{n+1}),$$

and taking into account that

$$R + \log\left(1 + \frac{1}{n+1} - \frac{1}{n+1}e^{-R}\right) = \frac{n+2}{n+1}R \left(1 - \frac{R}{2(n+1)} + o(R)\right),$$

we get that

$$(32) \quad W_{\Delta_n}(R) := V_{\Delta_n}\left(R + \log\left(1 + \frac{1}{n+1} - \frac{1}{n+1}e^{-R}\right)\right) \\ = \left(\frac{n+2}{n+1}\right)^n a_n R^n + \left(\frac{n+2}{n+1}\right)^n \left(\frac{n+2}{n+1} b_n - a_n \frac{n}{2(n+1)}\right) R^{n+1} + o(R^{n+1}).$$

Now equation (31) implies

$$V'_{\Delta_{n+1}}(R) = \left(\frac{n+2}{n+1}\right)^2 \left(1 - \frac{R}{n+1} + o(R)\right) W_{\Delta_n}(R),$$

to obtain

$$V'_{\Delta_{n+1}}(R) = \left(\frac{n+2}{n+1}\right)^{n+2} a_n R^n \left(1 - \frac{R}{n+1} + o(R)\right) \times \left(1 + \frac{1}{a_n} \left(\frac{n+2}{n+1} b_n - a_n \frac{n}{2(n+1)}\right) R + o(R)\right),$$

which finally gives us

$$(33) \quad V'_{\Delta_{n+1}}(R) = \left(\frac{n+2}{n+1}\right)^{n+2} a_n R^n \left(1 + \frac{n+2}{a_n(n+1)}(b_n - \tfrac{1}{2}a_n)R + o(R)\right).$$

After integration, we get

$$a_{n+1} = \left(\frac{n+2}{n+1}\right)^{n+2} \frac{1}{n+1} a_n \quad \text{and} \quad b_{n+1} = \left(\frac{n+2}{n+1}\right)^{n+2} \frac{1}{n+1} (b_n - \tfrac{1}{2}a_n).$$

Taking into account that $a_1 = 4$ and $b_1 = -2$, we easily deduce by induction that for all $n \geq 2$,

$$a_n = \frac{(n+1)^{n+1}}{(n!)^2} \quad \text{and} \quad b_n = -\frac{n}{2} \frac{(n+1)^{n+1}}{(n!)^2},$$

as stated.

Finally, the asymptotics as $R \rightarrow \infty$ follow at once from [Theorem 1.5](#), or they can be deduced from the recursive formula as in the $R \rightarrow 0$ regime. $\square$

10 Reflections and questions

10.1 Flags of polytopes and comparison of volume growth

Another conjecture related to flags and Mahler volume, and which may be related to Funk volume, is:

Conjecture 10.1 *Let P be a centrally symmetric polytope in dimension n . Then,*

$$|\text{Flags}(P)| \geq \frac{(n!)^2}{2^n} |P| |P^\circ|.$$

Kalai [\[11\]](#) talks about this conjecture in a YouTube video, around 32 minutes in. He ascribes it to conversations with Freij, Henze, Schmitt and Ziegler, about a decade ago, based on computer experimentation. We might be so bold as to make the following conjecture.

Conjecture 10.2 *Let Ω be a centrally symmetric convex body, and let H be a Hanner polytope of the same dimension. Then, for any r and R positive real numbers with $r < R$,*

$$\frac{\text{Vol}_\Omega(B_\Omega(R))}{\text{Vol}_\Omega(B_\Omega(r))} \geq \frac{\text{Vol}_H(B_H(R))}{\text{Vol}_H(B_H(r))}.$$

This is a strengthening of [Conjecture 10.1](#). Indeed, we recover that conjecture from [Conjecture 10.2](#) by letting r approach zero, and R approach infinity.

Conjecture 10.2 reminds us of the Bishop–Gromov theorem in Riemannian geometry, but with the inequality reversed. The inequality in Funk geometry directly corresponding to the Bishop–Gromov theorem would be

$$\frac{\text{Vol}_\Omega(B_\Omega(R))}{\text{Vol}_\Omega(B_\Omega(r))} \leq \frac{\text{Vol}_E(B_E(R))}{\text{Vol}_E(B_E(r))},$$

where E is a centered ellipsoid. This inequality is false: considering small radii, it would imply the inequality

$$\int_{\Omega \times \Omega^\circ} \langle x, \xi \rangle^2 \, dx \, d\xi \leq \frac{n}{(n+2)^2} |\Omega| |\Omega^\circ|,$$

which was shown to be false for large n by Klartag [12, Proposition 1.5], even for unconditional bodies.

10.2 Some further questions and remarks

The paper [9] studies Hansen polytopes of split graphs. This class contains polytopes that are only slightly worse than the Hanner polytopes with respect to the Mahler conjecture, the flag conjecture, and also Kalai’s 3^d conjecture concerning the minimal number of faces of a centrally symmetric polytope. It would be interesting to see how closely these polytopes come to minimizing the volume of Funk balls.

In [22], it was shown that the number of flags of a polytope determines the asymptotics of the volume of its floating body. This phenomenon was later studied in greater generality in [3]. Since for smooth bodies the asymptotics is governed by the affine surface area, the authors consider the flag number to be an analogue of the affine surface area for polytopes. One could ask if it is possible to obtain a refinement that depends on the geometry of the polytope, similarly to how we have the second-highest order term for the Funk volume.

It was conjectured in [6] that, if one considers the volume of a ball in a Funk geometry centered about the centroid of the body, then this quantity is maximized when the body is an ellipsoid. This is an upper bound counterpart to [Conjecture 1.1](#). It likewise interpolates between two inequalities, this time that of Blaschke–Santaló when the radius goes to zero, and the centro-affine isoperimetric inequality when the radius tends to infinity; see [25]. This upper bound conjecture was established [6] in the case of unconditional bodies.

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Invariant prime ideals in equivariant Lazard rings

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Let A be an abelian compact Lie group. We compute the spectrum of invariant prime ideals of the A -equivariant Lazard ring, or equivalently the spectrum of points of the moduli stack of A -equivariant formal groups. We further show that this spectrum is homeomorphic to the Balmer spectrum of compact A -spectra, with the comparison map induced by equivariant complex bordism homology.

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1 Introduction

Let us pose the question: what algebraic input do we need to develop equivariant versions of chromatic homotopy theory?

Chromatic homotopy theory studies stable homotopy theory through the lens of formal groups, building on Quillen's identification [1969; 1971] of the complex bordism ring $\pi_*\text{MU}$ with the Lazard ring. Around the same time, tom Dieck [1970] introduced for every compact Lie group A an equivariant analog of MU , the homotopical A -equivariant complex bordism MU_A . Letting A be abelian, Cole, Greenlees and Kriz [Cole et al. 2000] many years later found the correct notion of an A -equivariant formal group law. Recently, the first author generalized work of Hanke and Wiemeler [2018] and showed that $\pi_*^A\text{MU}_A$ is indeed the universal ring for A -equivariant formal group laws, thus establishing an equivariant analog of Quillen's theorem for the equivariant Lazard ring L_A .

Many structural features of stable homotopy theory can be explained through the chromatic perspective. The central notion of chromatic homotopy theory is that of *height*. Honda classified formal groups over a separably closed field of characteristic p in terms of the height $0 \leq n \leq \infty$. Thus, the points of the moduli stack of formal groups $\mathcal{M}_{\text{FG}}$ correspond to pairs (p, n) with $n = 0$ if and only if $p = 0$. Hopkins and Smith [1998] showed that the same classification pertains to thick subcategories of finite spectra: Given a finite spectrum X , its MU-homology MU_*X defines a coherent sheaf over $\mathcal{M}_{\text{FG}}$. Taking the support of MU_*X in the Zariski spectrum $|\mathcal{M}_{\text{FG}}|$ of points, we obtain a support theory on compact spectra. The thick subcategory theorem states that this support theory is the universal one. In other words, the induced map $|\mathcal{M}_{\text{FG}}| \rightarrow \text{Spec}(\text{Sp}^c)$ to the Balmer spectrum [2005; 2010] of compact spectra is a homeomorphism.

We show the following equivariant generalization (a more precise statement of which we give as Theorem 1.6):

Theorem 1.1 *Let A be an abelian compact Lie group. Then the spectrum of points of the moduli stack $\mathcal{M}_{\text{FG}}^A$ of A -equivariant formal groups is homeomorphic to the Balmer spectrum of compact A -spectra, with the comparison map induced by a support theory based on complex bordism homology $(\text{MU}_A)_*$.*

This establishes MU_A and the theory of equivariant formal groups as fundamental tools for building equivariant versions of chromatic homotopy theory.

For abelian groups as above, the Balmer spectrum of finite A -spectra has been computed completely by Balmer and Sanders [2017] (the case $A = C_p$), Barthel, Hausmann, Naumann, Nikolaus, Noel and Stapleton [Barthel et al. 2019] (the finite abelian case) and Barthel, Greenlees and Hausmann [Barthel et al. 2020] (the general abelian case). In a surprising turn of history, it was the algebraic counterpart which had not been computed before. As a set, both $|\mathcal{M}_{\text{FG}}^A|$ and $\text{Spec}(\text{Sp}_A^c)$ decompose as the disjoint union of one copy of $|\mathcal{M}_{\text{FG}}| \cong \text{Spec}(\text{Sp}^c)$ for every closed subgroup of A . Thus, the correct notion of height of an A -equivariant formal group F over a field of characteristic p consists of a pair: a height n of a nonequivariant formal group and a closed subgroup $B \subset A$ such that F is induced along the zig-zag $A \rightarrow A/B \leftarrow \{1\}$.

The more subtle information lies in the topology of the spectrum, which encodes on the algebraic level how heights can deform and on the homotopical level the chromatic interdependence of the various geometric fixed points $\Phi^B X$ of a compact A -spectrum X .

We will detail our results below in the language of invariant prime ideals. Crucially, we exhibit equivariant lifts v_n of the classical v_n and show that in many cases they provide a sequence of generators of invariant prime ideals. The nonequivariant v_n play an important role in many of the highlights of chromatic homotopy theory, like the greek-letter construction [Ravenel 1986], the construction of the Morava K-theories or the periodicity theorem [Hopkins and Smith 1998], and we hope that our equivariant lifts open the prospect of generalizing these to the equivariant context.

1.1 Invariant prime ideals and statement of results

As indicated above, the main theorem can also be stated in terms of invariant prime ideals of the equivariant Lazard ring L_A , as we now explain. Similarly to the nonequivariant case, L_A is the ground ring of a flat Hopf algebroid (L_A, S_A) , classifying A -equivariant formal group laws and their strict isomorphisms. By [Hausmann 2022, Theorem 5.52] and the discussion thereafter, this is isomorphic to the graded Hopf algebroid $(\pi_*^A \mathrm{MU}_A, \pi_*^A \mathrm{MU}_A \otimes \mathrm{MU}_A)$ of cooperations. The associated stack is the moduli stack of A -equivariant formal groups. Hence, the category of graded (L_A, S_A) -comodules is equivalent to the category of quasicoherent sheaves over $\mathcal{M}_{\mathrm{FG}}^A$.

Recall that an ideal I of L_A is called *invariant* (in the sense of Hopf algebroids) if it is a subcomodule, ie if $\eta_L(I)S_A = \eta_R(I)S_A$ for the left and right unit $\eta_L, \eta_R: L_A \rightarrow S_A$. Every invariant *prime* ideal $\mathfrak{p}$ gives rise to a point of the moduli stack of prime ideals via the quotient field of $L_A/\mathfrak{p}$. This defines a map from the set of invariant prime ideals $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ to $|\mathcal{M}_{\mathrm{FG}}^A|$, which we show to be a homeomorphism in Theorem 4.7.

For the nonequivariant Lazard ring, Morava, as well as Landweber [1973, Theorem 2.7], showed that the invariant prime ideals are precisely the ideals $I_{p,n} = (v_0, \dots, v_{n-1})$ for a prime p and $n \in \mathbb{N} \cup \{\infty\}$ (with $I_{p,0}$ being the 0-ideal for all p).

To describe the invariant prime ideals in the equivariant case we recall that L_A contains universal Euler classes e_V for all characters $V \in A^*$, and that equivariant Lazard rings are contravariantly functorial in continuous group homomorphisms. In particular, all equivariant Lazard rings are algebras over the nonequivariant Lazard ring.

Then, given a nonequivariant invariant prime ideal $I_{p,n}$ and a closed subgroup B of A , we obtain an invariant prime ideal $I_{B,p,n}^A \subseteq L_A$ as the kernel of the composite

$$L_A \xrightarrow{\mathrm{res}_B^A} L_B \rightarrow \Phi^B L \rightarrow \Phi^B L \otimes_L L/I_{p,n}.$$

Here, $\Phi^B L$ is defined as the localization of L_B away from all the Euler classes of nontrivial characters for B . The ring $\Phi^B L$ is an algebraic version of geometric fixed points and indeed agrees with the coefficient ring of the B -geometric fixed points of MU_B .

Theorem 1.2 (Theorem 4.7) *For every abelian compact Lie group A , the map*

$$\mathrm{Sub}(A) \times \mathrm{Spec}^{\mathrm{inv}}(L) \rightarrow \mathrm{Spec}^{\mathrm{inv}}(L_A), \quad (B, I_{p,n}) \mapsto I_{B,p,n}^A,$$

is a bijection.

Here, $\mathrm{Sub}(A)$ is the set of all closed subgroups of A . Hence, as for the Balmer spectrum, the invariant prime ideals of L_A decompose as a set as one copy of $\mathrm{Spec}^{\mathrm{inv}}(L)$ for every closed subgroup of A . And similarly to the Balmer spectrum, the main work then lies in understanding the Zariski topology,

in particular in determining the containments between invariant prime ideals associated to different subgroups.

We obtain the following:

Theorem 1.3 (Theorem 5.1) *There is an inclusion $I_{B,p,n}^A \subseteq I_{B',q,n'}^A$ if and only if*

- (1) B' is a subgroup of B ,
- (2) $p = q$ or $n = 0$ (in which case $I_{B,p,0}^A = I_{B,q,0}^A$), the components $\pi_0(B/B')$ are a p -group and $n' \geq n + \text{rank}_p(\pi_0(B/B'))$.

Comparing with [Barthel et al. 2020] we see that these correspond precisely to the inclusions in the Balmer spectrum, but with roles reversed: There is an inclusion $I_{B,p,n}^A \subseteq I_{B',q,n'}^A$ if and only if there is an inclusion $P_{B',q,n'}^A \subseteq P_{B,p,n}^A$. Here, $P_{B,p,n}^A = \{X \in \text{Sp}_A^c \mid K(n)_*(\Phi^B X) = 0\}$ are the thick subcategories of Sp_A^c with $K(n)$ being Morava K-theory at the prime p .

To show that $I_{B,p,n}^A$ indeed includes into $I_{B',q,n'}^A$ when conditions (1) and (2) are satisfied, one can reduce to the case $A = \mathbf{T}$ the circle group where it is straightforward to describe explicit generators for the invariant prime ideals. The main step in ruling out further inclusions is the construction of equivariant refinements $\bar{v}_{n-1} \in L_{C_p^n}$ of the elements $v_{n-1} \in L$ which exhibit maximal height shifts (Definition 5.21, Proposition 5.24). Roughly speaking, $\bar{v}_{n-1}$ is of height -1 at the top group C_p^n (ie it lies in the ideal $I_{C_p^n,p,0}^{C_p^n}$) while it is of height $n-1$ at the trivial group (ie it lies in the ideal $I_{\{1\},p,n}^{C_p^n}$ but not in $I_{\{1\},p,n-1}^{C_p^n}$). This is the algebraic analog of the existence of finite C_p^n -spectra of underlying type n whose C_p^n -geometric fixed points are rationally nontrivial as in [Barthel et al. 2019, Section 4] and [Kuhn and Lloyd 2024, Section 7]. More precisely, $\bar{v}_{n-1}$ is canonically defined only modulo a certain smaller ideal (analogously to v_{n-1} only being defined uniquely up to the ideal I_{n-1} and up to a unit). More details are given in Section 5.3.

We further show that — at least over elementary abelian p -groups — the elements $\bar{v}_i$ give rise to generators of the invariant prime ideals:

Theorem 1.4 (Theorem 6.1) *For all primes p and $n \in \mathbb{N}$ the elements*

$$p_1^* \bar{v}_0, p_2^* \bar{v}_1, \dots, p_{n-1}^* \bar{v}_{n-2}, \bar{v}_{n-1}$$

generate the ideal $I_{C_p^n,p,0}^{C_p^n}$. Here, $p_i: C_p^n \rightarrow C_p^i$ denotes the projection to the first i coordinates.

Suitable restrictions of the $\bar{v}_n$ then form generators for the ideals $I_{C_p^n,p,m}^{C_p^n}$ at higher height m ; see Section 6. We emphasize that in contrast to the nonequivariant situation, the sequence of the $p_i^* \bar{v}_{i-1}$ is *not* a regular sequence. In fact, since $I_{C_p^n,p,0}^{C_p^n}$ consists precisely of the Euler-class-power torsion, it does not contain a nonzero divisor and hence cannot be generated by a regular sequence (unless $n = 0$). The torsion in the ring $L_{C_p^n}$ is closely linked to the torsion in the group of characters $(C_p^n)^*$. Hence one might hope that

$I_{B,n}^A$ is generated by a regular sequence whenever A is a torus, and indeed that is the case in all the cases we understand; see [Remark 6.7](#).

Finally, to describe the Zariski topology we need one additional ingredient. When A is infinite, the set of closed subgroups $\text{Sub}(A)$ carries a nontrivial metric topology, turning it into a totally disconnected compact Hausdorff space. Together with the inclusions between the invariant prime ideals, this topology determines the Zariski topology on $\text{Spec}^{\text{inv}}(L_A)$.

Theorem 1.5 ([Theorem 7.5](#)) *The Zariski topology on $\text{Spec}^{\text{inv}}(L_A)$ has as basis the closed subsets C which are*

- (i) *closed under upward inclusions (ie if $I_{B',q,n'}^A \in C$ and $I_{B',q,n'}^A \subseteq I_{B,p,n}^A$, then $I_{B,p,n}^A \in C$), and*
- (ii) *are locally constant on $\text{Sub}(A)$ in the sense that every $B \in \text{Sub}(A)$ has a neighborhood U such that, for every n and p , either $I_{B',p,n}^A \in C$ for all $B' \in U$ or $I_{B',p,n}^A \notin C$ for all $B' \in U$.*

Comparing with [\[Barthel et al. 2020\]](#), we see that this description precisely matches the computation of the topology on the Balmer spectrum, with $I_{B,p,n}^A$ replaced by $P_{B,p,n}^A$. Hence the assignment

$$I_{B,p,n}^A \mapsto P_{B,p,n}^A$$

yields a homeomorphism from $\text{Spec}^{\text{inv}}(L_A)$ (and hence $|\mathcal{M}_{\text{FG}}^A|$) to $\text{Spec}(\text{Sp}_A^c)$. In the last section we explain that this comparison map can be obtained less ad hoc via MU_A -homology:

Theorem 1.6 ([Section 8.1](#)) *Let X be a compact A -spectrum and $I_{B,p,n}^A$ an invariant prime ideal. Then the localization $(\text{MU}_A)_{I_{B,p,n}^A} \wedge X$ is nontrivial if and only if the B -geometric fixed points $\Phi^B X$ are of type $\leq n$ at p , ie if and only if $P_{B,p,n}^A$ is in the Balmer support of X .*

This shows that

$$X \mapsto \text{supp}((\text{MU}_A)_* X) \subseteq \text{Spec}^{\text{inv}}(L_A) \cong |\mathcal{M}_{\text{FG}}^A|$$

defines a universal support theory on compact A -spectra and thus a homeomorphism $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$. Here, $(\text{MU}_A)_* X$ is the Mackey functor recording $(\text{MU}_B)_* \text{res}_B^A X$ for all closed subgroups B of A , and $\text{supp}((\text{MU}_A)_* X)$ is defined as the set of invariant prime ideals at which the localization of this Mackey functor is nontrivial.

Knowing that $X \mapsto \text{supp}((\text{MU}_A)_* X)$ is a (not necessarily universal) support theory with a characterization in terms of geometric fixed points as above already yields a continuous bijection $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$. This reproves one half of the main theorems of [\[Barthel et al. 2019\]](#) and [\[Barthel et al. 2020\]](#). This half has been dubbed the chromatic Smith fixed point theorem in [\[Kuhn 2021\]](#) and [\[Balderrama and Kuhn 2024\]](#), where other proofs are given. To establish that our support theory is universal (and hence that $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$ is also open), we need additional topological input from [\[Barthel et al. 2019; 2020\]](#); see also [\[Kuhn and Lloyd 2024\]](#).

We show in [Proposition 8.11](#) that the support theory from [Theorem 1.6](#) can alternatively be built by viewing $(\underline{\mathrm{MU}}_A)_* X$ as defining quasicoherent sheaves on $\mathcal{M}_{\mathrm{FG}}^B$ for every closed $B \subseteq A$; the union of the supports of these sheaves agrees with $\mathrm{supp}((\underline{\mathrm{MU}}_A)_* X)$ under the homeomorphism $\mathrm{Spec}^{\mathrm{inv}}(L_A) \cong |\mathcal{M}_{\mathrm{FG}}^A|$. To show this, we establish in [Proposition 8.8](#) how the adjunction between Sp^A and $\mathrm{Sp}^{A/B}$ defined by geometric fixed points and pullback corresponds on the algebraic level to pullback and pushforward along the open immersion $\mathcal{M}_{\mathrm{FG}}^{A/B} \subseteq \mathcal{M}_{\mathrm{FG}}^A$ from [Proposition 3.12](#).

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2 Equivariant formal groups

The aim of this section is to recall some basic definitions and properties about equivariant formal groups and equivariant formal group laws from [\[Cole et al. 2000; Greenlees 2001; Strickland 2011; Hausmann 2022\]](#). To make our paper more self-contained, we also replicate some of the proofs in our language, and we provide some small extensions of known results. Our treatment of equivariant formal groups is far from exhaustive, and especially [\[Strickland 2011\]](#) contains a wealth of results we do not touch upon.

2.1 Basic definitions

In this subsection, we will recall the notions of an equivariant formal group and an equivariant formal group law over a commutative ring k . The definition of an equivariant formal group law is due to Cole, Greenlees and Kriz [\[Cole et al. 2000\]](#), and our definition of an equivariant formal group will be a variant of that of Strickland [\[2011\]](#).

For us, a *formal k -algebra* is a complete linearly topologized commutative k -algebra with a countable system of open ideals generating the topology. We define a functor F from formal k -algebras to the pro-category of commutative k -algebras, sending a formal k -algebra R to the pro-system (R/I) with $I \subseteq R$ running over all open ideals. The essential image is contained in the full subcategory $\mathrm{Pro}^{\mathrm{cont}, \mathrm{surj}}(\mathrm{CRing})$ on those pro-objects indexed on a countable directed set with surjective transition maps and actually equals this subcategory by the discussion after Definition 5.4 in [\[Yasuda 2009\]](#). On this subcategory, taking the limit of the pro-system defines a right adjoint to F , which is also a one-sided inverse. Thus, F is fully faithful and defines an equivalence onto $\mathrm{Pro}^{\mathrm{cont}, \mathrm{surj}}(\mathrm{CRing})$.

For us, the category of *formal k -schemes* is the opposite of that of formal k -algebras. Using the above, it can be viewed as the category of ind-objects in affine k -schemes, indexed by a countable directed set with closed immersions as transition maps. We will sometimes use the notation $\mathrm{Spec} R$ or $\mathrm{Spf} R$ for the formal k -scheme associated to a formal k -algebra R , and $\mathbb{O}_X$ for the formal k -algebra associated to a formal k -scheme X . The product on affine k -schemes induces one on formal k -schemes, and this corresponds to the completed tensor product on formal k -algebras.

We set $S = \mathrm{Spec} k$. Given a countable set M , we view $S \times M$ as a formal scheme, namely as the colimit over all $S \times N$ with $N \subseteq M$ finite. This corresponds to giving k^M the product topology. If we just write M , we will apply this construction to $\mathrm{Spec} \mathbb{Z}$ instead of S .

For a compact Lie group A , we will denote by $A^* = \mathrm{Hom}(A, T)$ its Pontryagin dual, which is always a discrete group. We use ϵ for the unit element in A^* .

Definition 2.1 Given a compact abelian Lie group A , an *A -equivariant formal group over k* consists of a commutative group object X in formal k -schemes together with a group homomorphism $\varphi: S \times A^* \rightarrow X$ of formal k -schemes satisfying the following two conditions:

- (1) For the composite $\varphi_\epsilon: S \xrightarrow{\mathrm{id}_S \times \epsilon} S \times A^* \xrightarrow{\varphi} X$, the augmentation ideal $I_\epsilon = \ker(R \rightarrow k)$ of the induced map is fpqc-locally on k a free R -module of rank 1, where $R = \mathbb{O}_X$.
- (2) The topology on R is generated by products of the ideals $I_V = \ker(R \xrightarrow{\varphi_V^*} k)$ for $V \in A^*$ and $\varphi_V: S \xrightarrow{\mathrm{id}_S \times V} S \times A^* \xrightarrow{\varphi} X$.

Remark 2.2 In the above definition, one can easily replace $\mathrm{Spec} k$ by an arbitrary quasicompact scheme S , with formal S -schemes being a suitable subcategory of the ind-category of Aff_S , the category of schemes affine over S .

Our definition differs in two aspects from that put forward in [Strickland 2011, Definition 2.15]. First, Strickland restricts to finite A^* . Second, Strickland asks $\ker(\mathbb{O}_X \rightarrow k)$ to be free of rank 1 instead of locally free [loc. cit., Proposition 2.10]. We changed it so that our definition satisfies descent. We could have asked, equivalently, that the augmentation ideal is Zariski locally on k a free R -module of rank 1 because line bundles satisfy fpqc-descent.

Remark 2.3 If we leave out the second condition in Definition 2.1, we get a notion that we call an *A -equivariant group*. The category of A -equivariant formal groups embeds into that of A -equivariant groups and this inclusion has a right adjoint, called *completion*. Concretely, this replaces R in the notation in Definition 2.1 by the formal k -algebra $\lim_{V_1, \dots, V_n \in A^*} R/(I_{V_1} \cdots I_{V_n})$. We will only use A -equivariant groups to complete them to A -equivariant formal groups.

If we fix a trivialization of the augmentation ideal I_ϵ in Definition 2.1, we can spell out the definition in more algebraic terms and obtain the notion of an equivariant formal group law.

Definition 2.4 An A -equivariant formal group law over k is a quadruple

$$(R, \Delta, \theta, y(\epsilon))$$

of a formal k -algebra R , a continuous comultiplication $\Delta: R \rightarrow R \hat{\otimes} R$, a map of k -algebras $\theta: R \rightarrow k^{A^*}$ and an orientation $y(\epsilon) \in R$, such that

- (i) the comultiplication is a map of k -algebras which is cocommutative, coassociative and counital for the augmentation $\theta_\epsilon: R \rightarrow k$,
- (ii) the map θ is compatible with the coproduct, and the topology on R is generated by finite products of the kernels of the component functions $\theta_V: R \rightarrow k$ for $V \in A^*$, and
- (iii) the element $y(\epsilon)$ is regular and generates the kernel of θ_ϵ .

Remark 2.5 If we want to remember the base of an equivariant formal group law, we sometimes also write it as a quintuple $(k, R, \Delta, \theta, y(\epsilon))$.

Lemma 2.6 An A -equivariant formal group $(\text{Spec } R, \varphi)$ over k together with an R -linear isomorphism $I_\epsilon \cong R$ of the augmentation ideal is equivalent datum to an A -equivariant formal group **law** over k .

Proof The maps Δ and θ are induced by the multiplication on $\text{Spec } R$ and φ , respectively. The element $y(\epsilon)$ corresponds to the trivialization of $I_\epsilon = \ker(R \rightarrow k)$. $\square$

Given any equivariant formal group $G = (\varphi: A^* \rightarrow X)$ over $S = \text{Spec } k$, we obtain for every $V \in A^*$ a morphism $\varphi_V: S \xrightarrow{\text{id}_S \times V} S \times A^* \xrightarrow{\varphi} X$. If $R = \mathbb{O}_X$, this corresponds to the morphism $\theta_V: R \rightarrow k$. Moreover, φ composed with left multiplication defines an A^* -action on X ; for every $V \in A^*$ this gives a map $l_V: R \rightarrow R$. In terms of the data of an equivariant formal group law $F = (k, R, \Delta, \theta, y(\epsilon))$, this can explicitly be written as

$$l_V: R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{\theta_V \hat{\otimes} \text{id}_R} R.$$

Given $V \in A^*$, we set

$$y(V) = l_V(y(\epsilon)) \in R,$$

which generates the kernel of θ_{V-1} . If A is trivial, we have $R \cong k[[y]]$. We want to describe an analog for general A . A *complete flag* for A is a sequence of characters $f = V_1, V_2, \dots \in (A^*)^{\mathbb{N}}$ such that every character appears infinitely often. Given such a flag and $n \in \mathbb{N}$, we set $W_n = V_n \oplus V_{n-1} \oplus \dots \oplus V_1$ and

$$y(W_n) = y(V_n)y(V_{n-1}) \cdots y(V_1).$$

Then every element x of R can be written uniquely as

$$(2.7) \quad x = \sum_{n \in \mathbb{N}} a_n^f y(W_n)$$

for coefficients $a_n^f \in k$ [Cole et al. 2000, Lemma 13.2]. Hence, as a k -module, R is isomorphic to a countable infinite product of copies of k .

We refer to [Cole et al. 2000] and [Hausmann 2022] for more information about equivariant formal group laws.

2.2 Lazard rings

Our aim in this subsection is to recall the definition of the universal ring for equivariant formal group laws and to clarify its universal property. Let us begin by considering a very strict form of morphisms of equivariant formal group laws.

Definition 2.8 A *morphism* between A -equivariant formal group laws $(k_1, R_1, \Delta_1, \theta_1, y(\varepsilon)_1)$ and $(k_2, R_2, \Delta_2, \theta_2, y(\varepsilon)_2)$ is a pair of maps $f: k_1 \rightarrow k_2$ and $g: R_1 \rightarrow R_2$ which are compatible with both the comultiplications Δ and the augmentations θ , and which send $y(\varepsilon)_1$ to $y(\varepsilon)_2$.

This leads to a category $A\text{-FGL}$ of A -equivariant formal group laws. In [Cole et al. 2000, Section 14] it is shown that this category has an initial object F^{uni} , the ground ring of which is called the *A -equivariant Lazard ring* and denoted by L_A . In fact, the category of A -equivariant formal group laws is equivalent to the category of commutative rings under L_A . To discuss this, note first that the forgetful functor

$$A\text{-FGL} \rightarrow \text{CRing}, \quad (k, R, \Delta, \theta, y(\varepsilon)) \mapsto k,$$

into the category of commutative rings is cofibered in groupoids. Concretely this boils down to the following two observations:

- Every morphism of A -equivariant formal group laws whose first component $f: k_1 \rightarrow k_2$ is the identity map is an isomorphism. One observes indeed that the diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\quad} & R_2 \\ & \searrow \cong & \swarrow \cong \\ & \prod_{\mathbb{N}} k_1 = \prod_{\mathbb{N}} k_2 & \end{array}$$

obtained from equation (2.7) commutes. We call such an isomorphism living over the identity a *very strict isomorphism* between A -equivariant formal group laws, in order to distinguish from other kinds of isomorphisms.

- Given a morphism $f: k_1 \rightarrow k_2$ and an A -equivariant formal group law F over k_1 , one can define a pushforward f_*F over k_2 with the usual universal property. Its underlying k_2 -algebra is given by a completion of $R \otimes_{k_1} k_2$, where R is the underlying k_1 -algebra of F ; see [Greenlees 2001, Section 2.E], [Hausmann 2022, Section 2.3].

Note that the only automorphism of an A -equivariant formal group law over k which is also a very strict isomorphism is the identity map. Hence, given two A -equivariant formal group laws over k , there either exists a unique very strict isomorphism between them or none at all. For this reason it is usually harmless to identify two very strictly isomorphic A -equivariant formal group laws, and we will often do so.

Now, given a map $f: L_A \rightarrow k$ there is an induced A -equivariant formal group law f_*F^{uni} over k obtained by pushing forward the universal A -equivariant formal group law. Given an A -equivariant formal group law F over k , we can apply this to the first component $f: L_A \rightarrow k$ of the unique map of A -equivariant formal group laws $F^{\text{uni}} \rightarrow F$. The resulting morphism $f_*F^{\text{uni}} \rightarrow F$ is necessarily a very strict isomorphism. So, as claimed above, we obtain:

Corollary 2.9 *The functor*

$$\text{CAlg}_{L_A} \rightarrow A\text{-FGL}, \quad (f: L_A \rightarrow k) \mapsto f_*F^{\text{uni}},$$

from commutative L_A -algebras to A -equivariant formal group laws is an equivalence of categories. An inverse is given by sending an A -equivariant formal group law $F = (k, R, \Delta, \theta, y(\varepsilon))$ to the first component $f: L_A \rightarrow k$ of the unique morphism $F^{\text{uni}} \rightarrow F$.

Remark 2.10 The above proof is an instance of a general characterization of initial objects X in categories cofibered in groupoids $\mathcal{C} \xrightarrow{F} \mathcal{D}$, namely that pushforward defines an equivalence of $\mathcal{D}_{F(X)/-}$ with $\mathcal{C}$.

Remark 2.11 Nonequivariantly the k -algebra R is often fixed to be the power series ring $k[[y(\varepsilon)]]$ rather than a ring only isomorphic to it. With this convention, the category of formal group laws is *isomorphic* (not merely equivalent) to the category of commutative rings under L . In other words, every very strict isomorphism of formal group laws is the identity. Equivariantly one needs to be a little more careful: The statement “ A -equivariant formal group laws are represented by the A -equivariant Lazard ring” is only true up to this notion of very strict isomorphism.

2.3 Global functoriality

In this subsection, we will discuss both a covariant and a contravariant functoriality of the category of A -equivariant formal groups in A .

Definition 2.12 Let $\alpha: B \rightarrow A$ be a group homomorphism and let $G = (B^* \rightarrow X)$ be a B -equivariant formal group. We define the *corestriction* α_*G to be the A -equivariant formal group which is the completion of the A -equivariant group $(A^* \xrightarrow{\alpha^*} B^* \rightarrow X)$.

Proposition 2.13 *For every injective group homomorphism $\alpha: B \rightarrow A$, the functor α_* from B -equivariant formal groups to A -equivariant formal groups is fully faithful. The essential image consists of those A -equivariant formal groups where the homomorphism from A^* factors through B^* .*

Proof For every B -equivariant formal group G , by construction α_*G is the same group object as G in formal schemes with the structure morphism $A^* \rightarrow B^* \rightarrow G$ (since $A^* \rightarrow B^*$ is surjective, no completion is necessary). This implies full faithfulness. $\square$

Upon choosing coordinates, [Definition 2.12](#) corresponds to the construction in Section 2.4 of [\[Hausmann 2022\]](#): given a B -equivariant formal group law $F = (k, R, \Delta, \theta, y(\varepsilon))$ and a group homomorphism $\alpha: B \rightarrow A$, there is an induced A -equivariant formal group law $\alpha_* F$ over the same ring k , given by completing R at products of the ideals $I_V = \ker(R \xrightarrow{\theta_V} k)$ for those $V \in B^*$ which are in the image of $\alpha^*: A^* \rightarrow B^*$. This defines a functor α_* from B -equivariant formal group laws to A -equivariant formal group laws, which induces a map $\alpha^*: L_A \rightarrow L_B$ on Lazard rings. Hence we obtain a functor

$$L: (\text{abelian compact Lie groups})^{\text{op}} \rightarrow \text{commutative rings},$$

which we call the *global Lazard ring*. As shown in [\[Hausmann 2022\]](#), L_A is isomorphic to $\pi_*^A \text{MU}_A$ and our map α^* corresponds to the restriction map on that level, explaining our terminology.

Remark 2.14 By Pontryagin duality, the opposite category of abelian compact Lie groups is equivalent to the category of finitely generated abelian groups. Therefore, everything in this paper could alternatively be phrased in terms of finitely generated abelian groups rather than abelian compact Lie groups, and the more algebraically minded reader might prefer to do so.

In addition to this covariant functoriality, there is also a contravariant functoriality.

Definition 2.15 Let $\alpha: A \rightarrow C$ be a surjective group homomorphism and let $G = (C^* \rightarrow X)$ be a C -equivariant group. We define the *coinduction* $\alpha^* G$ to be the A -equivariant group $(A^* \rightarrow X \times_{C^*} A^*)$, where the target denotes the quotient of $X \times A^*$ by the antidiagonal C^* -action.

Lemma 2.16 For $\alpha: A \rightarrow C$ a surjective group homomorphism and G a C -equivariant formal group, $\alpha^* G$ is an A -equivariant formal group, ie needs no additional completion.

Proof If $G = (C^* \rightarrow \text{Spec } R)$ is a C -equivariant formal group, then

$$\text{Spec } R \times_{C^*} A^* \cong \text{Spec } \text{Map}_{C^*}(A^*, R).$$

Any choice of coset representatives defines an isomorphism $\text{Map}_{C^*}(A^*, R) \cong \prod_{A^*/C^*} R$ of formal k -algebras, with respect to the product topology on the latter. As products of complete rings are complete, the claim follows. $\square$

Proposition 2.17 Let $\alpha: A \rightarrow C$ be a surjective group homomorphism. As functors between C -equivariant formal groups and A -equivariant formal groups, α^* is the left adjoint of α_* .

Proof For an A -equivariant group $G = (A^* \rightarrow X)$, define $\tilde{\alpha}_* G$ as $C^* \rightarrow A^* \rightarrow X$. Then α^* and $\tilde{\alpha}_*$ are adjoints between C -equivariant groups and A -equivariant groups in the sense of [Remark 2.3](#). Since completion is a right adjoint, the result follows from the previous lemma. $\square$

2.4 Euler classes

Given an A -equivariant formal group law $F = (R, \Delta, \theta, y(\varepsilon))$ over k , we can define Euler classes in k . Recall that for $V \in A^*$, we set $y(V) = l_V(y(\varepsilon)) \in R$. The corresponding Euler class is

$$e_V = \theta_\varepsilon(y(V)) = \theta_V(y(\varepsilon)) \in k.$$

In terms of the associated equivariant formal group $G = (\varphi: A^* \rightarrow X)$, we have

$$S \times_{\varphi_\varepsilon, X, \varphi_V} S \cong \operatorname{Spec}(k \otimes_{\theta_\varepsilon, R, \theta_V} k) \cong \operatorname{Spec}(k/e_V)$$

with $S = \operatorname{Spec} k$ and φ_V being the composite $S \cong S \times \{V\} \subseteq S \times A^* \xrightarrow{\varphi} X$. This implies:

Lemma 2.18 *For a given A -equivariant formal group law with notation as above:*

- (1) *The Euler class e_V is invertible if and only if $S \times_{\varphi_\varepsilon, X, \varphi_V} S = \emptyset$.*
- (2) *The Euler class e_V is zero if and only if $\varphi_V = \varphi_\varepsilon$.*

Thus, the vanishing or invertibility of Euler classes does not depend on chosen coordinates. This allows us to generalize these concepts to A -equivariant formal groups in the following way:

Definition 2.19 For an A -equivariant formal group $G = (\varphi: A^* \rightarrow X)$, we say that the *Euler class e_V is invertible* if $S \times_{\varphi_\varepsilon, X, \varphi_V} S = \emptyset$ and that *e_V is zero* if $\varphi_V = \varphi_\varepsilon$.

Informally, e_V is invertible if and only if the images of $S \times \{\varepsilon\}$ and $S \times \{V\}$ in X are disjoint.

Example 2.20 Let $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[x^{\pm 1}]$ be the multiplicative group over $S = \operatorname{Spec} \mathbb{Z}$. We choose the group homomorphism $\varphi: C_2 = (C_2)^* \rightarrow \mathbb{G}_m$ picking the units $\{\pm 1\}$ in $\mathbb{Z}$. This defines the structure of a C_2 -equivariant group, and its completion is a C_2 -equivariant formal group we call $\hat{\mathbb{G}}_m^{C_2}$. Let $V \in (C_2)^*$ be the unique nontrivial character. One computes

$$\operatorname{Spec} \mathbb{Z} \times_{\varphi_\varepsilon, \hat{\mathbb{G}}_m^{C_2}, \varphi_V} \operatorname{Spec} \mathbb{Z} \cong \operatorname{Spec} \mathbb{Z} \times_{\varphi_\varepsilon, \mathbb{G}_m, \varphi_V} \operatorname{Spec} \mathbb{Z} \cong \operatorname{Spec} \mathbb{Z}/2.$$

Thus, $e_V = \pm 2$, depending on the choice of coordinate. See also [Strickland 2011, Section 7] and [Greenlees 2001, Section 7] for more information on this and related examples. Note in particular that our example is the pushforward of the true C_2 -equivariant multiplicative formal group (given by a completion of $\operatorname{Spec} \mathbb{Z}[(C_2 \times T)^*]$) along the map $\mathbb{Z}[C_2^*] \rightarrow \mathbb{Z}$ classifying ± 1 .

We will use the following lemma several times, taken from [Hausmann 2022, Corollary 2.8] and the explanation thereafter:

Lemma 2.21 *Let $B \subseteq A$ be a subgroup. Then the restriction map $L_A \rightarrow L_B$ is surjective, with kernel I_A^B generated by the Euler classes e_V where V runs over a generating set of $\ker(A^* \rightarrow B^*)$.*

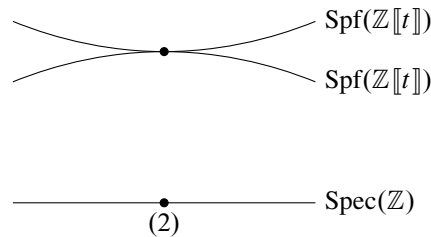


Figure 1: A schematic picture of $\hat{\mathbb{G}}_m^{C_2}$ from [Example 2.20](#).

Similarly, the following holds on the level of equivariant formal groups.

Proposition 2.22 *Let G be an A -equivariant formal group.*

- (1) *Let $\alpha: A \rightarrow C$ be a surjective group homomorphism. Assume that for $V \notin \mathrm{im}(C^* \rightarrow A^*)$, the Euler class e_V is invertible. Then $\alpha^* \alpha_* G \rightarrow G$ is an isomorphism.*
- (2) *Let $\alpha: B \rightarrow A$ be an injective group homomorphism. Assume that for $V \in \ker(A^* \rightarrow B^*)$, we have $e_V = 0$. Then G is in the essential image of α_* .*

Proof Let $G = (\varphi: A^* \rightarrow X)$. Fixing a coordinate $y(\epsilon)$ Zariski-locally and choosing a complete flag, X is defined by the directed system $(\mathrm{Spec} R/(y(V_1) \cdots y(V_n)))_n$ and each of these terms has underlying space $\bigcup_{i=1}^n \mathrm{im}(\varphi_{V_i}) \subseteq \mathrm{Spec} R$.

In the first item, [Lemma 2.18](#) implies that $\mathrm{im}(\varphi_V)$ and $\mathrm{im}(\varphi_W)$ intersect each other in $\mathrm{Spec} \mathbb{O}_X$ only if $[V] = [W] \in A^*/C^*$. Thus, the underlying space of $(\mathrm{Spec} R/(y(V_1) \cdots y(V_n)))_n$ decomposes into closed subspaces $\bigsqcup_{v \in A^*/C^*} \bigcup_{V_i \in v} \mathrm{im}(\varphi_{V_i})$, of which only finitely many are nonempty. This induces decompositions of the schemes $\mathrm{Spec} R/(y(V_1) \cdots y(V_n))$ and we obtain thus an A^* -equivariant isomorphism $G \cong \bigsqcup_{A^*/C^*} \alpha_* G$ on the level of formal schemes.

By construction, $\alpha^* \alpha_* G$ decomposes in the same way. On the unit copy, the map $\alpha^* \alpha_* G \rightarrow G$ is an isomorphism since $\alpha_* \alpha^* \alpha_* G \rightarrow \alpha_* G$ is one by [Proposition 2.17](#). For the other copies, this follows by the A^* -equivariance of the map $\alpha^* \alpha_* G \rightarrow G$.

For the second item: by definition, the structure morphism $\varphi: A^* \rightarrow X$ of G factors as $A^* \rightarrow B^* \xrightarrow{\varphi'} X$. The result follows from [Proposition 2.13](#). $\square$

The second part is also true in the setting of equivariant formal group laws, as shown in [[Hausmann 2022](#), Lemma 2.7]. The analog of the first part becomes more complicated as a coordinate on $\tilde{G}$ does not determine a canonical coordinate on $\alpha^* \tilde{G}$ and hence α^* does not define a functor from C -equivariant formal group laws to A -equivariant formal group laws without additional choices; we will discuss this further in [Proposition 2.25](#).

Corollary 2.23 *Let $\alpha: A \rightarrow B$ be a surjective group homomorphism. Then α^* is a fully faithful embedding from the category of B -equivariant formal groups to that of A -equivariant formal groups. The image consists of those A -equivariant formal groups such that e_V is invertible for V not in the image of $B^* \rightarrow A^*$.*

Proof If G is a B -equivariant formal group, then α^*G has the property that e_V is invertible for V not in the image of $B^* \rightarrow A^*$ by construction. By the preceding proposition, invertibility of these Euler classes characterizes the image of α^* . Moreover, α^* is fully faithful since $G \rightarrow \alpha_*\alpha^*G$ is an isomorphism by construction. $\square$

The following proposition essentially provides a classification of equivariant formal groups over fields. The same statement already appears in [Strickland 2011, Corollary 8.3].

Proposition 2.24 *Let k be a field and G be an A -equivariant formal group over k . Denote by A/B the Pontryagin dual of the subgroup $\{V \in A^* \mid e_V = 0\} \subseteq A^*$. Denote further by $A \xrightarrow{q} A/B \xleftarrow{i} \{1\}$ the obvious morphisms. Then $G \cong q^*i_*p_*G$, where p_*G is the **nonequivariant** formal group defined by $p: A \rightarrow \{1\}$.*

Proof Assume $e_V = 0$, ie $\varphi_\epsilon = \varphi_V$. By the A^* -action, this implies $\varphi_W = \varphi_{VW}$ for every $W \in A^*$. Setting $W = V^{-1}$, this implies $e_{V^{-1}} = 0$. Moreover, if $\varphi_\epsilon = \varphi_W$, this implies $\varphi_\epsilon = \varphi_{VW}$. Thus, $\{V \in A^* \mid e_V = 0\} \subseteq A^*$ is indeed a subgroup.

Since k is a field, $e_V = 0$ if and only if e_V is not invertible. By Proposition 2.22, we thus have $G \cong q^*i_*\Gamma$ for some nonequivariant formal group Γ over k . One computes $\Gamma \cong p_*q^*i_*\Gamma \cong p_*G$. $\square$

For any A , let $\Phi^A L = L_A[e_V^{-1}]$ be the localization of L_A away from all Euler classes e_V for $V \neq \epsilon$. Our results above let us compute $\Phi^A L$ quite explicitly; cf [Greenlees 2001, Corollary 6.4] and [Hausmann 2022, Proposition 2.11].

Proposition 2.25 *There is a ring isomorphism of the form*

$$\Phi^A L \cong L[(b_0^V)^{\pm 1}, b_i^V \mid i > 0, V \in A^* - \{\epsilon\}],$$

where the b_i^V are independent polynomial variables.

Proof The ring $\Phi^A L$ classifies A -equivariant formal group laws $F = (R, \Delta, \theta, y(\epsilon))$ such that e_V is invertible for all $V \neq \epsilon$. By Proposition 2.22, $R \cong \text{map}(A^*, \hat{R})$, where $\hat{R}$ is the completion of R at the augmentation ideal. The structure of F determines the structure of a nonequivariant formal group law p_*F on $\hat{R}$, where $p: A \rightarrow \{1\}$ is the projection; in particular, we obtain an isomorphism $\hat{R} \cong k[[y]]$, where y is the image of $y(\epsilon)$. Conversely, Δ and θ are determined by p_*F . In particular, θ_ϵ is the composite $\text{map}(A^*, \hat{R}) \xrightarrow{\text{ev}_\epsilon} \hat{R} \rightarrow k$. Thus we see that F is determined by p_*F , plus a choice of $y(\epsilon)$ mapping to y under ev_ϵ . Such $y(\epsilon)$ are exactly those elements $(y^V) \in \text{map}(A^*, k[[y]])$ such that $y^\epsilon = y$ and $y^V = b_0^V + b_1^V y + \dots$ with $b_0^V \in k^\times$. This gives the result. $\square$

Remark 2.26 The elements $b_i^V \in \Phi^A L$ in the previous proposition already come from elements $\gamma_i^V \in L_A$, which are uniquely determined by the property that

$$e_{\epsilon \otimes \tau} = \gamma_0^V + \gamma_1^V e_{V^{-1} \otimes \tau} + \gamma_2^V (e_{V^{-1} \otimes \tau})^2 + \cdots + \gamma_n^V (e_{V^{-1} \otimes \tau})^n \in L_{A \times T} / (e_{V^{-1} \otimes \tau})^{n+1}$$

for all $n \in \mathbb{N}$. Here, $\tau \in T^*$ denotes the tautological character for the circle group T , and we view $L_{A \times T}$ as an algebra over L_A via restriction along the projection $A \times T \rightarrow A$. In particular, γ_0^V equals the Euler class e_V . The elements γ_i^V are natural in the sense that $\alpha^* \gamma_i^V = \gamma_i^{\alpha^* V}$ for every group homomorphism $\alpha: B \rightarrow A$. We refer to [Hausmann 2022, Section 2.7] for more details on this construction.

2.5 The relationship between Lazard rings at different groups and their completions

While not needed for our *classification* of invariant prime ideals, it will be necessary for our study of *containments* between invariant prime ideals to have a deeper look upon how Lazard rings at different groups relate. These properties are all based on the identification of the global Lazard ring with equivariant complex bordism in [Hausmann 2022].

Proposition 2.27 [Hausmann 2022, Proposition 5.50, Corollary 5.33, Lemma 5.28] (1) For every A and every nontorsion character $V \in A^*$, the sequence

$$0 \rightarrow L_A \xrightarrow{e_V} L_A \xrightarrow{\text{res}_{\ker(V)}^A} L_{\ker(V)} \rightarrow 0$$

is exact. In particular, all Euler classes $e_V \in L_A$ for nontorsion characters V are nonzero divisors.

(2) For every A , the complete L_A -Hopf algebra R of the universal A -equivariant formal group law is canonically isomorphic to the completion

$$\lim_{n \in \mathbb{N}, V_1, \dots, V_n \in A^*} (L_{A \times T}) / I_{V_1} \cdots I_{V_n},$$

where I_{V_j} is the kernel of the restriction map $(\text{id}, V_j)^*: L_{A \times T} \rightarrow L_A$. More generally, $R^{\widehat{\otimes} n}$ is a completion of $L_{A \times T^n}$. Under this identification,

- (a) the comultiplication $R \rightarrow R \widehat{\otimes} R$ and the augmentations $\theta_V: R \rightarrow L_A$ are induced by the maps $(\text{id}_A, m)^*: L_{A \times T} \rightarrow L_{A \times T \times T}$ and $(\text{id}_A, V)^*: L_{A \times T} \rightarrow L_A$ on completion, and
- (b) the elements $y(V)$ are the image of $e_{V \otimes \tau} \in L_{A \times T}$ under the completion map, where $\tau \in T^*$ is the tautological character.

The special case of (2) for A the trivial group is particularly important: Completing L_{T^n} at the kernel I of the augmentation $L_{T^n} \rightarrow L$ yields a power series ring on n generators $L[[y_1, y_2, \dots, y_n]]$, where y_i is the Euler class of the i^{th} projection $T^n \rightarrow T$. Moreover, the isomorphism

$$(L_{T^n})_I^{\widehat{}} \cong L[[y_1, y_2, \dots, y_n]]$$

is natural in T^n , where

- the functoriality of $(L_{T^n})_I^{\widehat{}}$ is induced by the global functoriality of L_{T^n} , and
- the functoriality of $L[[y_1, y_2, \dots, y_n]]$ is through the universal formal group law over L .

For example, for any $n \in \mathbb{N}$ the square

$$\begin{array}{ccc} L_{\mathbf{T}} & \xrightarrow{c} & L[[y]] \\ [n]^* \downarrow & & \downarrow [n]_F \\ L_{\mathbf{T}} & \xrightarrow{c} & L[[y]] \end{array}$$

commutes, where $[n]: \mathbf{T} \rightarrow \mathbf{T}$ is the n^{th} power map, and $[n]_F: L[[y]] \rightarrow L[[y]]$ sends y to the n -series $[n]_F(y)$ with respect to the universal formal group law. This implies that the Euler class $e_{\tau^n} \in L_{\mathbf{T}}$ for the n^{th} power map on $\mathbf{T}$ is sent to the n -series $[n]_F(y)$ under the completion map. Similar statements hold for the collection of $L_{A \times \mathbf{T}^n}$ for fixed A and varying $\mathbf{T}^n$.

We further record, where again $\tau: \mathbf{T} \rightarrow \mathbf{T}$ denotes the identity character:

Corollary 2.28 *Let $x \in L_{\mathbf{T}}$ be an element whose image in $L[[y]]$ is of the form $\lambda y^k + \text{higher-order terms}$. Then x is uniquely divisible by e_{τ}^k , and the quotient x/e_{τ}^k restricts to λ at the trivial group.*

Proof First we note that $e_{\tau} \in L_{\mathbf{T}}$ is a regular element by Proposition 2.27. Hence division by e_{τ} is always unique if possible. By induction on k the corollary then follows from the following facts, for an element $z \in L_{\mathbf{T}}$ and its image $c(z) \in L_{\mathbf{T}}$:

- (1) z is divisible by e_{τ} if and only if $\text{res}_1^{\mathbf{T}}(z) = 0$.
- (2) The leading coefficient of $c(z)$ is equal to $\text{res}_1^{\mathbf{T}}(z)$.
- (3) If $\text{res}_1^{\mathbf{T}}(z) = 0$ and hence z is divisible by e_{τ} , then $c(z/e_{\tau}) = c(z)/y$. □

The global functoriality makes every equivariant Lazard ring L_A into an algebra over the nonequivariant Lazard ring L such that all restriction maps $\alpha^*: L_A \rightarrow L_B$ are L -algebra maps. The following property is shown in [Hausmann 2022, Corollary 5.50] as a consequence of work of Comezaña [1996].

Proposition 2.29 *For every abelian compact Lie group A , the equivariant Lazard ring L_A is free as a module over L .*

Corollary 2.30 *The exact sequences of L -modules in Proposition 2.27(1) are split exact. In particular, they remain exact after applying any additive functor.*

The following special case is of particular importance to us:

Corollary 2.31 *Let $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $I_n \subseteq L$ be the ideal generated by $(v_0, \dots, v_{n-1})$. Then for every A and every nontorsion character $V \in A^*$, the sequence*

$$0 \rightarrow L_A/(I_n \cdot L_A) \xrightarrow{e_V} L_A/(I_n \cdot L_A) \xrightarrow{\text{res}_{\ker(V)}^A} L_{\ker(V)}/(I_n \cdot L_{\ker(V)}) \rightarrow 0$$

is exact. In particular, the Euler classes $e_V \in L_A$ for nontorsion characters $V \in A^*$ remain nonzero divisors in L_A/I_n . In the terminology of [Hausmann 2022], the assignment

$$A \mapsto L_A/(I_n \cdot L_A)$$

(together with the image of e_{τ} under $L_{\mathbf{T}} \rightarrow L_{\mathbf{T}}/(I_n \cdot L_{\mathbf{T}})$) is a regular global group law.

3 The Lazard Hopf algebroid and its associated stack

3.1 Strict isomorphisms and the Lazard Hopf algebroid

In this subsection, we will introduce one of our main objects of study, the Hopf algebroid (L_A, S_A) for equivariant formal group laws. There is a hierarchy of notions of isomorphisms between (equivariant) formal group laws, namely

- isomorphisms, which do not need to respect the coordinate and are thus really isomorphisms between the underlying (equivariant) formal groups;
- strict isomorphisms, which respect the coordinate up to quadratic terms;
- very strict isomorphisms as in [Section 2.2](#), which respect the coordinate strictly.

Already classically, strict isomorphisms are especially relevant since the Hopf algebroid modeled on them gives $(\mathrm{MU}_*, \mathrm{MU}_* \mathrm{MU})$.

Definition 3.1 A *strict isomorphism* between two A -equivariant formal group laws

$$(k, R_1, \Delta_1, \theta_1, y(\varepsilon)_1) \quad \text{and} \quad (k, R_2, \Delta_2, \theta_2, y(\varepsilon)_2)$$

over the same ground ring k is a k -linear isomorphism

$$\varphi: R_1 \xrightarrow{\cong} R_2$$

of Hopf algebras over k^{A^*} such that $y(\varepsilon)_1$ is sent to $y(\varepsilon)_2$ modulo I_ε^2 , where I_ε is the augmentation ideal in R_2 . Explicitly, this means that $(\varphi \otimes \varphi) \circ \Delta_1 = \Delta_2 \circ \varphi$, $\theta_2 \circ \varphi = \theta_1$ and $\varphi(y(\varepsilon)_1) = x \cdot y(\varepsilon)_2$ for some unit $x \in R_2$ which augments to $1 \in k$.

By definition, strict isomorphisms need not preserve the coordinate, hence they are generally not morphisms of formal group laws in the sense of [Definition 2.8](#). On the other hand, every very strict isomorphism is both a strict isomorphism and an isomorphism in the category of A -equivariant formal group laws.

Let SI be the category of strict isomorphisms of A -equivariant formal group laws. More precisely, its objects are quadruples (k, F^1, F^2, φ) consisting of a commutative ring k , two A -equivariant formal group laws F^1 and F^2 over k and a strict isomorphism φ between them. Morphisms between two such quadruples $(k_1, F_1^1, F_1^2, \varphi_1)$ and $(k_2, F_2^1, F_2^2, \varphi_2)$ are given by a pair of morphisms $f_1: F_1^1 \rightarrow F_2^1$ and $f_2: F_1^2 \rightarrow F_2^2$ in the sense of [Definition 2.8](#), with the same underlying map $k_1 \rightarrow k_2$, such that

$$\begin{array}{ccc} R_1^1 & \xrightarrow{\varphi_1} & R_1^2 \\ f_1 \downarrow & & \downarrow f_2 \\ R_2^1 & \xrightarrow{\varphi_2} & R_2^2 \end{array}$$

commutes.

Proposition 3.2 *The category SI has an initial object, whose underlying ring S_A is a localization of an infinite polynomial ring $L_A[a_1^f, a_2^f, \dots]$ over the Lazard ring L_A .*

By Remark 2.10, we can equivalently say that there is an object $(S_A, F^{\text{uni}}, \tilde{F}^{\text{uni}}, \varphi) \in \text{SI}$ such that the functor

$$\text{CAlg}_{S_A} \rightarrow \text{SI}, \quad (f: S_A \rightarrow k) \mapsto (k, f_* F^{\text{uni}}, f_* \tilde{F}^{\text{uni}}, f_* \varphi),$$

is an equivalence of categories. Here, we use that SI is again cofibered in groupoids over commutative rings.

Before we prove the proposition, it will be good to review two general results about the maps $\theta_V: R \rightarrow k^{A^*} \rightarrow k$ for an A -equivariant formal group law $(k, R, \Delta, \theta, y(\varepsilon))$. Recall that after choosing a complete flag f we can write every element $x \in R$ uniquely as

$$x = \sum_{n \in \mathbb{N}} a_n^f y(W_n).$$

Given $V \in A^*$, we have

$$\theta_V(x) = \sum_{n \in \mathbb{N}} a_n^f \theta_V(y(W_n)) = \sum_{n \in \mathbb{N}} a_n^f e_{V W_n},$$

where $e_{V W_n}$ is defined as the product $e_{V \cdot V_n} e_{V \cdot V_{n-1}} \cdots e_{V \cdot V_1}$. This is a finite sum, since $e_{V W_n} = 0$ if there exists some $i \leq n$ where $V_i = V^{-1}$. We obtain:

Lemma 3.3 *The augmentation θ_V is a linear combination of a_n^f whose coefficients are products of Euler classes.*

Moreover, we have the following:

Lemma 3.4 *Let $F = (k, R, \Delta, \theta, y(\varepsilon))$ be an A -equivariant formal group law. Then an element $x \in R$ is a unit if and only if $\theta_V(x)$ is a unit in k for all $V \in A^*$.*

Proof See [Hausmann 2022, Lemma 2.3]. □

Proof of Proposition 3.2 For every object $(k, F^1, F^2, \varphi) \in \text{SI}$ we can define a new A -equivariant formal group law $\tilde{F}^2$ for which the components k, R, Δ and θ agree with those of F^1 , but $\tilde{y}(\varepsilon)_2$ is defined as $\varphi^{-1}(y(\varepsilon)_2)$, the preimage of the coordinate of F^2 under φ . Then we obtain a new object $(k, F^1, \tilde{F}^2, \text{id}_{R_1})$, which is isomorphic in SI to (k, F^1, F^2, φ) via the commutative square

$$\begin{array}{ccc} F^1 & \xrightarrow{\text{id}} & \tilde{F}^2 \\ \text{id} \downarrow & & \downarrow \varphi \\ F^1 & \xrightarrow{\varphi} & F^2 \end{array}$$

Note that the two vertical maps are in fact very strict isomorphisms of A -equivariant formal group laws. In summary, every object of SI is isomorphic to one of the form (k, F_1, F_2, id) , where F_1 and F_2 are given by the same k^{A^*} -augmented k -Hopf algebra and the strict isomorphism is the identity. This is the

same data as a single A -equivariant formal group law F together with a second choice of coordinate $y(\varepsilon)_2 = x \cdot y(\varepsilon)_1$ for some unit $x \in R$ which augments to 1. Up to very strict isomorphism we can further assume that F is the pushforward along a map $L_A \rightarrow k$. We claim that the functor sending an A -equivariant formal group law to the set of all units $x \in R$ augmenting to 1 is representable by an L_A -algebra S_A . Indeed, a presentation for S_A is given by

$$(3.5) \quad S_A = L_A[a_1^f, a_2^f, \dots][P_V(1, a_1^f, a_2^f, \dots)^{-1} \mid V \in A^* - \{\varepsilon\}],$$

where f is a complete flag starting with ε and P_V is the linear combination expressing θ_V in terms of the coefficients with respect to f ; see Lemmas 3.3 and 3.4. Here we use that the elements augmenting to 1 are precisely the elements of the form $1 + \sum_{n \in \mathbb{N}^+} a_n^f y(W_n)$, with $y(W_n)$ as in the end of Section 2.1.

To conclude, let F^{uni} be the pushforward of the universal equivariant formal group law along $L_A \rightarrow S_A$, and let $\tilde{F}^{\text{uni}}$ be the same formal group with the coordinate replaced with $(1 + \sum_{n \in \mathbb{N}^+} a_n^f y(W_n)) \cdot y(\varepsilon)$. The functor

$$\text{CAlg}_{S_A} \rightarrow \text{SI}, \quad (f: S_A \rightarrow k) \mapsto (k, f_* F^{\text{uni}}, f_* \tilde{F}^{\text{uni}}, \text{id}),$$

is fully faithful by Section 2.2, and we have demonstrated it to be essentially surjective. Thus, it is an equivalence of categories, as was to be shown. $\square$

Remark 3.6 The same proof shows that the category of all (not necessarily strict) isomorphisms also has an initial object, whose underlying ring is $S_A[(a_0^f)^{\pm 1}]$. In fact, the only difference in the proof is that the equation for the unit x is now of the form $\sum_{n \in \mathbb{N}} a_n^f y(W_n)$ so that the presentation for the analog of S_A becomes

$$L_A[a_0^f, a_1^f, a_2^f, \dots][(a_0^f)^{-1}, P_V(a_0^f, a_1^f, a_2^f, \dots)^{-1} \mid V \in A^* - \{\varepsilon\}].$$

Since $P_V(a_0^f, a_1^f, a_2^f, \dots) = a_0^f P_V(1, a_1^f/a_0^f, a_2^f/a_0^f, \dots)$, this ring is indeed isomorphic to

$$L_A\left[(a_0^f)^{\pm 1}, \frac{a_1^f}{a_0^f}, \frac{a_2^f}{a_0^f}, \dots\right]\left[P_V\left(1, \frac{a_1^f}{a_0^f}, \frac{a_2^f}{a_0^f}, \dots\right)^{-1} \mid V \in A^* - \{\varepsilon\}\right] \cong S_A[(a_0^f)^{\pm 1}].$$

There are functors

$$s, t: \text{SI} \rightarrow A\text{-FGL}$$

sending a strict isomorphism to its source and target, respectively, as well as an “identity” functor

$$A\text{-FGL} \rightarrow \text{SI},$$

an “inverse” functor

$$i: \text{SI} \rightarrow \text{SI}$$

and a “composition”

$$c: \text{SI} \times \text{SI} \rightarrow \text{SI},$$

which restrict to the full subcategory of those objects of SI where the isomorphism φ is given by the identity. By representability we obtain analogous source and target maps $s, t: L_A \rightarrow S_A$, an identity map $S_A \rightarrow L_A$, an inverse map $i: S_A \rightarrow S_A$ and a composition map $c: S_A \rightarrow S_A \otimes_{L_A} S_A$.

Corollary 3.7 *The pair (L_A, S_A) together with the above structure defines a Hopf algebroid ie a cogroupoid object in commutative rings. The associated functor*

$$\mathbf{CRing} \rightarrow \mathbf{Groupoids}$$

is equivalent to the one sending a commutative ring k to the groupoid of A -equivariant formal group laws over k and strict isomorphisms between them.

Remark 3.8 The Hopf algebroid (L_A, S_A) has a natural grading. This can be constructed in three equivalent ways:

- Hausmann [2022, Corollary 5.6] shows that the global Lazard ring admits a unique grading that defines a graded global group law. By the same arguments, the “universal global group law with a strict n -tuple of coordinates” $L^{(n)}$ [loc. cit., Section 5.8] also carries a unique grading for every n such that the coordinates have degree -2 . The source–target maps $s, t: L \rightarrow L^{(2)}$, the identity map $L^{(2)} \rightarrow L$, the inverse map $i: L^{(2)} \rightarrow L^{(2)}$ and the composition map $c: L^{(2)} \rightarrow L^{(3)}$ all preserve this grading, since they are defined through their effect on the respective coordinates. Evaluating these maps at a group A yields the Hopf algebroids (L_A, S_A) , which hence inherit a grading compatible with all restriction and inflation maps.
- The groupoid-valued functor represented by (L_A, S_A) admits a $\mathbb{G}_m$ -action from multiplying the coordinate of the equivariant formal group law by a unit u and acting on strict isomorphisms by multiplying a_n^f by u^n . This corresponds to an even grading on (L_A, S_A) , putting eg a_n^f in degree $2n$.
- By its interpretation as representing the groupoid of A -equivariant formal group laws and isomorphisms between them (see Remark 3.6), $(L_A, S_A[(a_0^f)^{-1}])$ obtains the structure of a Hopf algebroid. An element s in S_A is of degree $2n$ if applying the composition map $S_A[(a_0^f)^{-1}] \rightarrow S_A[(a_0^f)^{-1}] \otimes_{L_A} S_A[(a_0^f)^{-1}]$ to s gives $(a_0^f)^n c(s)$, where $c: S_A \rightarrow S_A \otimes_{L_A} S_A$ is the composition map of (L_A, S_A) .

One can show that this is the same grading coming from the isomorphism

$$(L_A, S_A) \cong (\pi_*^A \mathbf{MU}_A, \pi_*^A \mathbf{MU}_A \wedge \mathbf{MU}_A)$$

from [loc. cit., Theorem E].

Given a strict isomorphism $\varphi: F_1 \cong F_2$ of B -equivariant formal group laws and a group homomorphism $\alpha: B \rightarrow A$, we obtain an induced strict isomorphism $\alpha_* \varphi: \alpha_* F_1 \cong \alpha_* F_2$ by completion. This assignment is compatible with composition of strict isomorphisms. Therefore, the functor L from Section 2.3 extends to a functor

$$L: (\text{abelian compact Lie groups})^{\text{op}} \rightarrow \text{Hopf algebroids},$$

which we call the *global Lazard Hopf algebroid*.

3.2 The moduli stack of equivariant formal groups

Throughout this subsection we let A be an abelian compact Lie group. Our goal is to discuss the stack $\mathcal{M}_{\text{FG}}^A$ of equivariant formal groups and its relationship to the Lazard Hopf algebroid. Our stacks will always be stacks for the fpqc-topology on commutative rings. More precisely, we will consider stacks as the full sub-2-category of the 2-category of (pseudo)functors from commutative rings to groupoids on those (pseudo)functors satisfying fpqc-descent on objects and morphisms (the usual stack conditions).¹

Let us define $\mathcal{M}_{\text{FG}}^A$, leaving the verification that it is indeed an fpqc-stack to [Proposition 3.10](#).

Definition 3.9 Let $\mathcal{M}_{\text{FG}}^A$ be the (pseudo)functor sending a commutative ring to the groupoid of A -equivariant formal groups over it (with morphisms arbitrary isomorphisms between them).

We have discussed above that the Hopf algebroid (L_A, S_A) represents the functor sending a commutative ring to the groupoid of A -equivariant formal group laws and strict isomorphisms between them. As discussed in [Remark 3.8](#), (L_A, S_A) is naturally a graded Hopf algebroid. On the other hand, as we have discussed in [Remark 3.6](#), the ungraded Hopf algebroid $(L_A, S_A[a_0^{\pm 1}])$ classifying A -equivariant formal group laws and all isomorphisms between them. It is easy to see that this is precisely the ungraded Hopf algebroid associated to the graded Hopf algebroid (L_A, S_A) in the sense of [\[Meier and Ozornova 2020, Section 4.1\]](#).² We will follow [\[loc. cit., Definition 4.1\]](#) by defining the stack associated to a graded Hopf algebroid as the fpqc-stackification of the groupoid-valued functor corepresented by its associated ungraded Hopf algebroid. In particular, the stack associated to the graded Hopf algebroid (L_A, S_A) is the same as the stack associated to the ungraded Hopf algebroid $(L_A, S_A[a_0^{\pm 1}])$. Equivalently, it is the quotient of the stack associated to (L_A, S_A) by the $\mathbb{G}_m$ -action induced by the grading.

Proposition 3.10 (1) The groupoid-valued presheaf $\mathcal{M}_{\text{FG}}^A$ from [Definition 3.9](#) is an fpqc-stack.

(2) Sending an A -equivariant formal group law to its underlying A -equivariant formal group defines an equivalence from the stack associated to the graded Hopf algebroid (L_A, S_A) to $\mathcal{M}_{\text{FG}}^A$.

Proof Let us first show that (1) implies (2): denote the stack associated to the graded Hopf algebroid (L_A, S_A) by $\mathcal{X}_A$; equivalently, this is the stack associated to the ungraded Hopf algebroid $(L_A, S_A[a_0^{\pm 1}])$. Since the augmentation ideal of every A -equivariant formal group (X, φ) is by definition fpqc-locally trivial, (X, φ) comes after fpqc-base change from an A -equivariant formal group law (see [Lemma 2.6](#)). Thus, $\mathcal{X}_A \rightarrow \mathcal{M}_{\text{FG}}^A$ is essentially surjective as a functor of stacks. Moreover, it is fully faithful since

¹We write (pseudo)functors since stacks are typically not strict functors into the 2-category of groupoids. All 2-morphisms in our setting are invertible; thus we can also decide to work in the setting of $(\infty, 1)$ -categories and look at all functors from commutative rings to the $(\infty, 1)$ -category of groupoids. In this setting, the “pseudoness” is implicit; which is why we set “pseudo” in brackets.

²Meier and Ozornova [\[2020\]](#) use an algebraic grading convention, while we use a topological one; thus one has to double all degrees.

isomorphisms between A -equivariant formal group laws are precisely isomorphisms of the underlying A -equivariant formal groups. Thus, it remains to show (1), ie that $\mathcal{M}_{\text{FG}}^A$ satisfies fpqc-descent on morphisms and objects.

Given two formal k -schemes X and Y , the functor

$$k\text{-algebras} \rightarrow \text{Set}, \quad K \mapsto \text{Hom}_K(X \times_{\text{Spec } k} \text{Spec } K, Y \times_{\text{Spec } k} \text{Spec } K),$$

is an fpqc-sheaf. Indeed, if we view X and Y as ind-objects (X_i) and (Y_j) , we can rewrite this Hom as $\lim_i \text{colim}_j \text{Hom}_K(X_i \times_{\text{Spec } k} \text{Spec } K, Y_j \times_{\text{Spec } k} \text{Spec } K)$ and equalizers commute with filtered colimits in sets. This easily implies that $\mathcal{M}_{\text{FG}}^A$ satisfies fpqc-descent on morphisms.

Let $\text{Fin}(A^*)$ denote the directed set of finite multi subsets³ of A^* , ordered by inclusion. Sending an A -equivariant formal group (X, φ) over k to the system $(\text{Spec } \mathbb{O}_X/\mathcal{I})$, where $\mathcal{I}$ runs over all finite products of the $\ker(\mathbb{O}_X \xrightarrow{\phi(V)^*} k)$ for $V \in A^*$, defines a functor from A -equivariant formal groups over k to the subcategory $\mathcal{C}(k)$ of $\text{Fun}(\text{Fin}(A^*), \text{Aff}_k)$ where all transition maps are closed immersions. The associated object in $\text{Ind}(\text{Aff}_k)$ defines the underlying formal scheme of X . Moreover, the functors $\mathcal{M}_{\text{FG}}^A(k) \rightarrow \mathcal{C}(k)$ assemble into a natural transformation $\mathcal{M}_{\text{FG}}^A \rightarrow \mathcal{C}$ of (pseudo)functors from commutative rings to groupoids.

Given a descent datum D for A -equivariant formal groups for an fpqc-cover $k \rightarrow K$, we thus obtain a descent diagram for $\mathcal{C}$ along $k \rightarrow K$. This in turn defines a $\text{Fin}(A^*)$ -diagram of descent data for Aff_k along $k \rightarrow K$. As Aff and closed immersions satisfy fpqc-descent, we obtain a $\text{Fin}(A^*)$ -diagram in Aff_k such that all transition maps are closed immersions. This defines a formal k -scheme X . Denoting the functor

$$(\text{formal schemes over } k) \rightarrow (\text{descent data for formal schemes along } k \rightarrow K)$$

by $\mathcal{D}$, one checks that $\mathcal{D}(X)$ is the underlying descent datum of formal schemes for D . Moreover, $\mathcal{D}$ preserves products. By descent for morphisms between formal k -schemes, X obtains a group structure and also a group homomorphism $\varphi: S \times A^* \rightarrow X$. Conditions (1) and (2) for an A -equivariant formal group are fulfilled by construction. $\square$

Remark 3.11 As every equivariant formal group comes Zariski-locally from an equivariant formal group law, in our case only a Zariski-stackification was necessary to pass from $(L_A, S_A[u^{\pm 1}])$ to $\mathcal{M}_{\text{FG}}^A$.

Proposition 3.12 Let $B \subseteq A$ be a subgroup of an abelian group A . Denote by $\alpha: B \rightarrow A$ the inclusion and by $q: A \rightarrow A/B$ the projection.

- (i) The functor q^* induces an open immersion $\mathcal{M}_{\text{FG}}^{A/B} \rightarrow \mathcal{M}_{\text{FG}}^A$, whose image is the common nonvanishing locus of the Euler classes e_V for all $V \notin \text{im}((A/B)^* \rightarrow A^*)$.
- (ii) The functor α_* induces a closed immersion $\mathcal{M}_{\text{FG}}^B \rightarrow \mathcal{M}_{\text{FG}}^A$, inducing an equivalence of $\mathcal{M}_{\text{FG}}^B$ to the common vanishing locus of the e_V for all $V \in \ker(A^* \rightarrow B^*)$.

³A *multiset* is a version of a set where elements can occur more than once. Formally speaking, a *multi subset* of a set S is a function $S \rightarrow \mathbb{Z}_{\geq 0}$, indicating which element occurs how often. It is *finite* if the function has finite support. A multi subset F of S is *contained* in another multi subset G of S if and only if $F(s) \leq G(s)$ for all $s \in S$.



Figure 2: Decompositions of $\mathcal{M}_{\text{FG}}^{C_4}$ into open and closed substacks, using lighter shading for open and darker shading for closed. Every ellipse stands for the image of an immersion $\mathcal{M}_{\text{FG}} \rightarrow \mathcal{M}_{\text{FG}}^{C_4}$, as in Remark 3.14.

Proof The first part is a reformulation of Corollary 2.23. The immersion is open since it is open after pullback to $\text{Spec } L_A$.

For the second, note that by Proposition 2.22 every A -equivariant formal group such that $e_V = 0$ for all $V \in \ker(A^* \rightarrow B^*)$ is of the form $\alpha_* G$ for G a B -equivariant formal group. Moreover, by construction, this vanishing of Euler classes is true for all A -equivariant formal groups of the form $\alpha_* G$ and thus characterizes the image of α_* . The substack given by this image is closed since it is closed after pullback to $\text{Spec } L_A$. Moreover, α_* is fully faithful by Proposition 2.13. $\square$

Remark 3.13 With notation as in the preceding proposition, we have $H \cong q_* q^* H$ for every A/B -equivariant formal group H . Thus, the inclusion of substacks $\mathcal{M}_{\text{FG}}^{A/B} \rightarrow \mathcal{M}_{\text{FG}}^A$ in the preceding proposition allows for a retraction (up to isomorphism) in the 2-category of stacks.

Remark 3.14 Let A be a compact abelian Lie group. Proposition 3.12 allows us to define for every subgroup $B \subseteq A$ an immersion $\mathcal{M}_{\text{FG}} \rightarrow \mathcal{M}_{\text{FG}}^A$ as the composition of the open immersion $\mathcal{M}_{\text{FG}}^{B/B} \rightarrow \mathcal{M}_{\text{FG}}^B$ and the closed immersion $\mathcal{M}_{\text{FG}}^B \rightarrow \mathcal{M}_{\text{FG}}^A$. Proposition 4.4 allows us to view the collection of all such immersions as a stratification of $\mathcal{M}_{\text{FG}}^A$.

Example 3.15 Proposition 3.12 gives closed immersions of $\mathcal{M}_{\text{FG}}$ and $\mathcal{M}_{\text{FG}}^{C_2}$ into $\mathcal{M}_{\text{FG}}^{C_4}$. The first is the common vanishing locus of all Euler classes (which equals the vanishing locus of the Euler class of one of the two generators of $(C_4)^*$, see Proposition 2.24). Its complement is the open substack given by the nonvanishing locus of the Euler class of the generators and hence equivalent to $\mathcal{M}_{\text{FG}}^{C_4/C_2}$. The second, ie the closed immersion of $\mathcal{M}_{\text{FG}}^{C_2}$, equals the vanishing locus of $e_{[2]}$ for $[2]: C_4 \xrightarrow{[2]} C_4 \hookrightarrow T$. Its complement is an open substack equivalent to $\mathcal{M}_{\text{FG}}^{C_4/C_4}$, the nonvanishing locus of all Euler classes of nontrivial characters.

For a general group A the situation is more complicated, and we cannot expect that the complement of the closed substack $\mathcal{M}_{\text{FG}}^B$ in $\mathcal{M}_{\text{FG}}^A$ can be expressed as a single open substack $\mathcal{M}_{\text{FG}}^{A/C}$ in general and vice versa. In general, the complement of $\mathcal{M}_{\text{FG}}^B$ in $\mathcal{M}_{\text{FG}}^A$ can be written as the union of the open substacks $\mathcal{M}_{\text{FG}}^{A/C}$ where C runs over the minimal subgroups of A not contained in B . We have indicated the situation for $A = C_2 \times C_2$ in Figure 3.

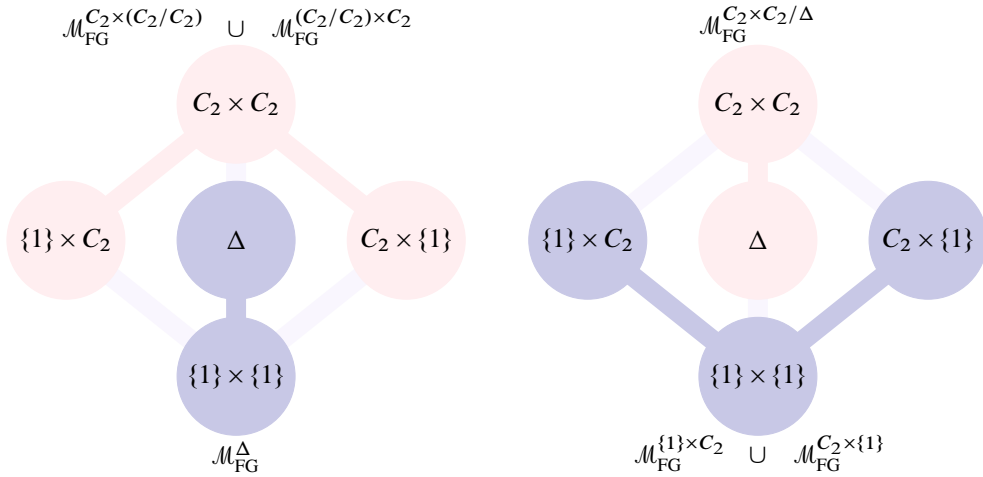


Figure 3: Decompositions of $\mathcal{M}_{\text{FG}}^{C_2 \times C_2}$ into open and closed substacks, Δ being the diagonal subgroup. Every circle stands for the image of an immersion $\mathcal{M}_{\text{FG}} \rightarrow \mathcal{M}_{\text{FG}}^{C_2 \times C_2}$, as in Remark 3.14.

4 Points of the moduli stack of equivariant formal groups and invariant prime ideals

The goal of this section is to classify the points of $\mathcal{M}_{\text{FG}}^A$ and the invariant prime ideals of (L_A, S_A) . Although the latter could be done without the former, we feel that both questions are of the same importance and the stack point of view makes some issues more transparent.

4.1 The space associated to a stack

As mentioned above, given a graded flat Hopf algebroid (A, Γ) , we can associate an ungraded Hopf algebroid $(A, \Gamma[u^{\pm 1}])$ to it; see eg [Meier and Ozornova 2020, Section 4.1].⁴ The category of comodules over the latter is equivalent to that of graded comodules over (A, Γ) . The stack $\mathcal{X}$ associated to (A, Γ) is by definition the stack associated to $(A, \Gamma[u^{\pm 1}])$, ie the fpqc-stackification of the presheaf of groupoids represented by $(A, \Gamma[u^{\pm 1}])$ on the category of all schemes. We denote the resulting morphism $\text{Spec } A \rightarrow \mathcal{X}$ by π .

Definition 4.1 Let (A, Γ) be a Hopf algebroid with units η_L and η_R . An ideal $I \subseteq A$ is called *invariant* if $\eta_L(I)\Gamma = \eta_R(I)\Gamma$. If (A, Γ) is graded, we will assume that I is also graded, ie generated by homogeneous elements.

It is easy to check that invariant ideals in a graded Hopf algebroid (A, Γ) correspond exactly to graded subcomodules of A and thus to ideal sheaves on $\mathcal{X}$. Here, we use that $\pi^*: \text{QCoh}(\mathcal{X}) \rightarrow \text{QCoh}(\text{Spec } A) \simeq \text{Mod}_A$

⁴Standard conventions force us to use A as part of the notation of a general Hopf algebroid, while A stands for a compact abelian Lie group in most of this article. We trust that this does not cause confusion.

refines to an equivalence from quasicoherent $\mathbb{O}_{\mathcal{X}}$ -modules to graded (A, Γ) -comodules [Meier and Ozornova 2020, Proposition 4.3].

Following [Laumon and Moret-Bailly 2000, Section 5] and [Stacks 2005–, Tag 04XL] in the case of Artin stacks, we can associate a topological space to $\mathcal{X}$. To that purpose, we define a morphism $\mathcal{U} \rightarrow \mathcal{X}$ of stacks to be an *open immersion* if the pullback $\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec} A \rightarrow \mathrm{Spec} A$ is an open immersion.

Definition 4.2 For $\mathcal{X}$ as above, define the underlying set of $|\mathcal{X}|$ to consist of equivalence classes of morphisms $x: \mathrm{Spec} K \rightarrow \mathcal{X}$ for K a field; the equivalence relation is generated by isomorphisms and $x \sim (\mathrm{Spec} L \rightarrow \mathrm{Spec} K \xrightarrow{x} \mathcal{X})$, where L is a field extension of K . We call a subset of $|\mathcal{X}|$ *open* if it is the image of $|\mathcal{U}| \rightarrow |\mathcal{X}|$ for an open immersion $\mathcal{U} \rightarrow \mathcal{X}$.

Equivalently, we can characterize the opens as the images of those opens in $\mathrm{Spec} A$ that are invariant, ie have the same preimage along both the left and right unit $\mathrm{Spec} \Gamma[u^{\pm 1}] \rightarrow \mathrm{Spec} A$. Indeed: since morphisms of stacks satisfy descent along the fpqc-cover $\mathrm{Spec} A \rightarrow \mathcal{X}$ and $\mathrm{Spec} A \times_{\mathcal{X}} \mathrm{Spec} A \simeq \mathrm{Spec} \Gamma[u^{\pm 1}]$, an open immersion $\mathcal{U} \rightarrow \mathcal{X}$ corresponds to an open immersion $\mathcal{V} \rightarrow \mathrm{Spec} A$ with an isomorphism $\mathcal{V} \times_{\mathrm{Spec} A} \mathrm{Spec} \Gamma[u^{\pm 1}] \cong \mathrm{Spec} \Gamma[u^{\pm 1}] \times_{\mathrm{Spec} A} \mathcal{V}$ over $\mathrm{Spec} \Gamma[u^{\pm 1}]$ satisfying a cocycle condition. But the category of open immersions into some X with isomorphisms over X between them is equivalent to the discrete category of open subsets of X , yielding the equivalence we claimed. Since invariant opens in $\mathrm{Spec} A$ form a topology, we deduce that the opens in $|\mathcal{X}|$ form a topology. One further checks that the map induced by any morphism of stacks is continuous.

Our definition of the topological space associated to a stack coincides with that of Laumon and Moret-Bailly [2000] and [Stacks 2005–] (who do it for Artin stacks) in the intersection of their domains, eg when $\mathcal{X}$ is an affine scheme, where we get the usual topology.

Proposition 4.3 Let (A, Γ) be a graded Hopf algebroid with associated stack $\mathcal{X}$. Then:

- (1) For every invariant prime ideal $I \subseteq A$, the image of $V(I) \subseteq |\mathrm{Spec} A|$ is closed in $|\mathcal{X}|$ and $\eta_I = |\pi|(\eta)$ for η the generic point of $V(I)$ (ie the point in $|\mathrm{Spec} A|$ corresponding to I) is generic, that is, $\{\overline{\eta_I}\} = |\pi|(V(I))$.
- (2) Mapping I to η_I defines an injection from the set of invariant prime ideals $\mathrm{Spec}^{\mathrm{inv}}(A)$ in A to $|\mathcal{X}|$. Equipping $\mathrm{Spec}^{\mathrm{inv}}(A)$ with the subspace topology from $\mathrm{Spec} A$, this map is continuous; if it is a bijection, it is a homeomorphism.

Proof For the first point, observe first that for every invariant ideal I , the set $V(I)$ is invariant and hence the complement of $V(I)$ defines an invariant open. Thus the image in $|\mathcal{X}|$ is open. Moreover, $|\pi|^{-1}(|\pi|(V(I))) = V(I)$. Thus $|\pi|(V(I))$ is the complement of $|\pi|(|\mathrm{Spec} A| \setminus V(I))$ and thus closed.

Assume now that I is an invariant prime ideal. Let η_I be the image of the generic point η of $V(I)$. Then $\{\overline{\eta_I}\} \subseteq |\pi|(V(I)) = |\pi|\{\overline{\eta}\} \subseteq \{\overline{\eta_I}\}$ and hence $\{\overline{\eta_I}\} = |\pi|(V(I))$.

For the injectivity of $\mathrm{Spec}^{\mathrm{inv}}(A) \rightarrow |\mathcal{X}|$, let $I, J \subseteq A$ be two invariant prime ideals with $\eta_I = \eta_J$. By the first point, this implies that $|\pi|(V(I)) = |\pi|(V(J))$ and hence $V(I) = V(J)$. Thus, $I = J$.

The continuity of $\mathrm{Spec}^{\mathrm{inv}}(A) \rightarrow |\mathcal{X}|$ follows from that of $\mathrm{Spec} A \rightarrow |\mathcal{X}|$. An arbitrary closed set of $\mathrm{Spec}^{\mathrm{inv}}(A)$ is of the form $V(I) \cap \mathrm{Spec}^{\mathrm{inv}}(A)$ for some ideal $I \subseteq A$. Set $I' = \bigcap_{I \subseteq J} J$, where the $J \subseteq A$ run over all invariant ideals containing I . Then $V(I) \cap \mathrm{Spec}^{\mathrm{inv}}(A) = V(J) \cap \mathrm{Spec}^{\mathrm{inv}}(A)$. If $|\pi|: \mathrm{Spec}^{\mathrm{inv}}(A) \rightarrow |\mathcal{X}|$ is a bijection, $|\pi|^{-1}(|\pi|(V(J))) = V(J)$ implies $|\pi|(V(J) \cap \mathrm{Spec}^{\mathrm{inv}}(A)) = |\pi|(V(J))$, and this is closed by the first part. $\square$

We warn the reader that in general, the preimage of an irreducible closed subset of $|\mathcal{X}|$ won't be irreducible in $|\mathrm{Spec} A|$ and thus does not correspond to an invariant prime ideal.

We recall that $|\mathcal{M}_{\mathrm{FG}}|$ has been computed by Honda: its points are classified by a pair (p, n) , where $p \geq 0$ is the characteristic of the field and $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the height of the formal group, with $n = 0$ if and only if $p = 0$. More precisely, Honda defined for each prime field of characteristic p the Honda formal group law of height n , and showed that any formal group law over a separably closed field is isomorphic to the pushforward of a Honda formal group law. As any field embeds into a separably closed field, the classification of points of $\mathcal{M}_{\mathrm{FG}}$ follows. We will often identify points of $\mathcal{M}_{\mathrm{FG}}$ with pairs (p, n) .

Proposition 4.4 *Let A be a compact abelian Lie group. We obtain a bijection of sets*

$$|\mathcal{M}_{\mathrm{FG}}^A| \rightarrow |\mathcal{M}_{\mathrm{FG}}| \times \mathrm{Sub}(A),$$

with $\mathrm{Sub}(A)$ being the set of closed subgroups of A , as follows: For a given A -equivariant formal group G over a field, the associated point in $|\mathcal{M}_{\mathrm{FG}}|$ is defined by the pushforward of G along $p: A \rightarrow \{e\}$, and the associated subgroup of A is Pontryagin dual to $A^*/\{V \in A^* \mid e_V = 0\}$.

Proof Given a closed subgroup B of A and a nonequivariant formal group Γ over a field k , we obtain an A -equivariant formal group via $q^*i_*\Gamma$, where $A \xrightarrow{q} A/B \xleftarrow{i} \{1\}$. By Proposition 2.24, every A -equivariant formal group G over a field k is isomorphic to one of this form. Here, the projection $A \rightarrow A/B$ is Pontryagin dual to the subgroup inclusion $\{V \in A^* \mid e_V = 0\} \subseteq A^*$, and thus B is uniquely defined by G . Moreover, necessarily $\Gamma \cong p_*G$. This implies that $|\mathcal{M}_{\mathrm{FG}}^A| \rightarrow |\mathcal{M}_{\mathrm{FG}}| \times \mathrm{Sub}(A)$ is surjective and that two A -equivariant formal groups G_1 and G_2 defined over the same field k are isomorphic if and only if they have the same associated closed subgroup and their completions p_*G_1 and p_*G_2 are isomorphic as nonequivariant formal groups.

To show that $|\mathcal{M}_{\mathrm{FG}}^A| \rightarrow |\mathcal{M}_{\mathrm{FG}}| \times \mathrm{Sub}(A)$ is injective, assume that two points

$$G_i: \mathrm{Spec} K_i \rightarrow \mathcal{M}_{\mathrm{FG}}^A$$

for $i = 1, 2$ define the same element of $|\mathcal{M}_{\mathrm{FG}}| \times \mathrm{Sub}(A)$ for separable closed fields K_i . By the above we know that $G_1 \cong q^*i_*p_*G_1$ and $G_2 \cong q^*i_*p_*G_2$, where i and q are defined with respect to the

same subgroup B of A . Moreover, the heights of p_*G_1 and p_*G_2 and the characteristics of K_1 and K_2 agree by assumption. Let Γ be a formal group law of the same height as the p_*G_i , but defined over the common prime field K of the K_i . Both p_*G_i are isomorphic to the pushforward of Γ to the K_i . Thus, the A -equivariant formal group $q^*i_*\Gamma$ defines the same element in $|\mathcal{M}_{\text{FG}}^A|$ as the $G_i \cong q^*i_*p_*G_i$. Hence G_1 and G_2 define the same element of $|\mathcal{M}_{\text{FG}}^A|$. $\square$

Remark 4.5 By Proposition 3.12, there is an open immersion $\mathcal{M}_{\text{FG}}^{A/B} \rightarrow \mathcal{M}_{\text{FG}}^A$ for any closed subgroup $B \subseteq A$ and hence $|\mathcal{M}_{\text{FG}}^{A/B}|$ is homeomorphic to an open subset of $|\mathcal{M}_{\text{FG}}^A|$. Given $C \subseteq A/B$, the point corresponding to (C, p, n) is $(q^{-1}(C), p, n)$ for $q: A \rightarrow A/B$ the projection. Indeed, for an A/B -equivariant formal group G over a field corresponding to (C, p, n) , we have

$$\{V \in A^* \mid e_V = 0\} = q^*(\{V \in (A/B)^* \mid e_V = 0\}) = q^*((A/B)/C)^*.$$

The Pontryagin dual of $A^*/q^*((A/B)/C)^*$ is precisely $q^{-1}(C)$ since $A/q^{-1}(C) \cong (A/B)/C$.

4.2 Invariant prime ideals of (L_A, S_A)

Our goal in this section is to classify the invariant prime ideals of the Hopf algebroid (L_A, S_A) . For that purpose let us introduce and recall some notation.

As before, let $\Phi^B L = L_B[e_V^{-1}]$ be the localization of L_B away from all Euler classes e_V for $V \neq \epsilon$ and let $\Phi^B S = S_B \otimes_{L_B} \Phi^B L$. We denote by v_n the coefficient of x^{p^n} in the p -series $[p]_{F^{\text{uni}}}(x)$ of the universal formal group law F^{uni} over L . We further denote by $I_{p,n}$ the ideal $(v_0 = p, v_1, v_2, \dots, v_{n-1}) \subseteq L$, ie the unique invariant prime ideal at height n containing p . We also include the case of $I_{p,0} = 0$. In this section, we will denote this ideal by $I_{0,0}$ to uniformize notation with respect to the residue characteristic.⁵

Construction 4.6 For every triple (B, p, n) of a closed subgroup B of A , a prime p and $n \in \overline{\mathbb{N}}$ we define an invariant ideal $I_{B,p,n}^A \subseteq L_A$ as the preimage of $\Phi^B L \cdot I_{p,n} \subseteq \Phi^B L$ along the composite map of Hopf algebroids

$$(L_A, S_A) \rightarrow (L_B, S_B) \rightarrow (\Phi^B L, \Phi^B S).$$

The $\Phi^B L \cdot I_{p,n}$ are indeed prime ideals since by Proposition 2.25 the quotient ring $\Phi^B L / \Phi^B L \cdot I_{p,n}$ is of the form $(L/I_{p,n})[(b_0^V)^{\pm 1}, b_i^V \mid i > 0, V \in A^* - \{\epsilon\}]$ and hence an integral domain. Thus, the $I_{B,p,n}^A$ are prime as well.

To simplify notation, we will from now on often write $\Phi^B L / I_{p,n}$ instead of $\Phi^B L / \Phi^B L \cdot I_{p,n}$ and likewise in similar situations.

⁵Our choice of v_n is different from the more commonly chosen Araki generators v'_n , which satisfy $[p](x) = \sum_n^{F^{\text{uni}}} v'_n x^{p^n}$; see for instance [Ravenel 1986, A.2.2.4]. Thus, modulo $I'_{p,n} = (v'_0 = p, v'_1, \dots, v'_{n-1})$, the p -series is of the form $v'_n x^{p^n} + \dots$. Therefore, $v_n \equiv v'_n \pmod{I'_{p,n}}$. Inductively, we see that $I_{p,n} = I'_{p,n}$.

Theorem 4.7 *The assignment*

$$\mathrm{Sub}(A) \times |\mathcal{M}_{\mathrm{FG}}| \rightarrow \mathrm{Spec}^{\mathrm{inv}}(L_A), \quad (B, p, n) \mapsto I_{B,p,n}^A,$$

is a bijection. In other words, the ideals $I_{B,p,n}^A$ are pairwise different and constitute all the invariant prime ideals in L_A .

The map $\mathrm{Spec}^{\mathrm{inv}}(L_A) \rightarrow |\mathcal{M}_{\mathrm{FG}}^A|$ from [Proposition 4.3](#) is a homeomorphism.

Proof By [Proposition 4.3](#), we know that $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ injects into $|\mathcal{M}_{\mathrm{FG}}^A|$ and the latter we computed to be $\mathrm{Sub}(A) \times |\mathcal{M}_{\mathrm{FG}}|$ as a set. Thus, it suffices to show that the element of $\mathrm{Sub}(A) \times |\mathcal{M}_{\mathrm{FG}}|$ associated to $I_{B,p,n}^A \in \mathrm{Spec}^{\mathrm{inv}}(L_A)$ is precisely $(B, (p, n))$, where p is a prime number if $n > 0$ and 0 if $n = 0$.

To spell this out concretely, let k be the field of fractions of $L_A/I_{B,p,n}^A$. We denote by F the pushed forward A -equivariant formal group law over k and by G the corresponding A -equivariant formal group. Since $\mathrm{Spec} k \rightarrow \mathrm{Spec} L_A$ hits the point corresponding to the prime ideal $I_{B,p,n}^A$, the corresponding point in $|\mathcal{M}_{\mathrm{FG}}^A|$ is represented by $\mathrm{Spec} k \xrightarrow{G} \mathcal{M}_{\mathrm{FG}}^A$. By the classification in [Proposition 4.4](#), we need to show three things:

- (1) the set of $V \in A^*$ such that $e_V = 0$ in k is precisely $\ker(A^* \rightarrow B^*)$,
- (2) k has characteristic p (which is clear), and
- (3) the pushforward p_*G along $A \xrightarrow{p} \{1\}$ has height n .

For the first, recall from [Lemma 2.21](#) that $I_B^A = \ker(\mathrm{res}: L_A \rightarrow L_B)$ is generated by the Euler classes e_V for all $V \in \ker(A^* \rightarrow B^*)$. These Euler classes must vanish in k since $L_A \rightarrow k$ factors through L_A/I_B^A . If V is not in $\ker(A^* \rightarrow B^*)$, then $e_V \neq 0$ in L_B and hence also in $\Phi^B L/I_{p,n}$ (as else $\Phi^B L/I_{p,n} = 0$). Since $L_A/I_{B,p,n}^A$ injects into $\Phi^B L/I_{p,n}$, the Euler class e_V is actually nonzero in $L_A/I_{B,p,n}^A$ and hence invertible in k . This shows the first point.

The pushforward p_*G is classified by the composite

$$g: L \rightarrow L_A \rightarrow L_A/I_{B,p,n}^A \rightarrow k.$$

The ideal $I_{p,n} \cdot L_A$ maps to 0 in $\Phi^B L/I_{p,n}$ and is hence contained in $I_{B,p,n}^A$. Therefore, g factors through $L/I_{p,n}$ and the height of p_*G is at least n . If $n = \infty$, we are done. If $n < \infty$, it remains to show that v_n is nonzero in $L_A/I_{B,p,n}^A$ and hence in k . By definition of $I_{B,p,n}^A$, the map $L_A \rightarrow \Phi^B L/I_{p,n}$ factors over $L_A/I_{B,p,n}^A$. By [Proposition 2.25](#), $\Phi^B L/I_{p,n}$ is an integral domain of the form $(L/I_{p,n})[(b_0^V)^{\pm 1}, b_i^V]$. In particular, v_n is nontrivial in $\Phi^B L/I_{p,n}$ and hence in $L_A/I_{B,p,n}^A$. This shows that p_*G is of height n as desired, which finishes the proof. $\square$

Unraveling the definition, we obtain the following description of the ideal $I_{B,p,n}^A$, where I_B^A still denotes the kernel of the restriction map $L_B \rightarrow L_A$.

Lemma 4.8 *An element $x \in L_A$ lies in $I_{B,p,n}^A$ if and only if there exists an A -representation W with $W^B = 0$, such that*

$$x \cdot e_W \in (I_B^A, I_{p,n}).$$

Proof By definition, $I_{B,p,n}^A$ is the kernel of the composition

$$L_A \xrightarrow{\text{res}_B^A} L_B \rightarrow L_B/L_B \cdot I_{p,n} \rightarrow \Phi^B L/\Phi^B L \cdot I_{p,n}.$$

The composition $L_A \rightarrow L_B \rightarrow L_B/I_{p,n}$ is surjective with kernel $(I_B^A, I_{p,n})$, and the map $L_B/I_{p,n} \rightarrow \Phi^B L/I_{p,n}$ inverts all Euler classes $e_{\bar{W}}$ for B -representations $\bar{W}$ with $\bar{W}^B = 0$. The product of Euler classes is an Euler class again. Hence, if $x \in L_A$ is contained in $I_{B,p,n}^A$, its image $\bar{x}$ in $L_B/I_{p,n}$ must be annihilated by such an Euler class $e_{\bar{W}}$. We can extend $\bar{W}$ to an A -representation W and find that $x \cdot e_W$ is contained in $(I_B^A, I_{p,n})$, as desired.

For the opposite direction, if we assume that $x \cdot e_W \in (I_B^A, I_{p,n})$ for some A -representation W with $W^B = 0$, then $\bar{x} \cdot e_{\text{res}_B^A W} = 0$ in $L_B/I_{p,n}$. Since $\text{res}_B^A W$ has trivial B -fixed points, $e_{\text{res}_B^A W}$ becomes invertible in $\Phi^B L/I_{p,n}$ and hence $\bar{x}$ is taken to 0 there. Therefore, x is contained in $I_{B,p,n}^A$. $\square$

When B is a torus, the ideal $I_{B,p,n}^A$ is easy to describe explicitly:

Corollary 4.9 *If B is a torus, then $I_{B,p,n}^A = (I_B^A, I_{p,n})$.*

Proof When B is a torus, every nontrivial character is nontorsion and thus the map

$$L_B/I_{p,n} \rightarrow \Phi^B L/I_{p,n}$$

is injective (Corollary 2.31). Hence, $I_{B,p,n}^A$ is equal to the kernel of

$$L_A \rightarrow L_A/I_{p,n} \rightarrow L_B/I_{p,n},$$

which is generated by I_B^A and $I_{p,n}$. $\square$

We note the following useful corollary:

Corollary 4.10 *Let A be a torus, $I_{p,n} \subseteq L$ an invariant prime ideal with $n < \infty$, and consider the augmentation ideal*

$$I = \ker(L_A/I_{p,n} \rightarrow L/I_{p,n}),$$

ie the ideal generated by all the Euler classes. Then the intersection $J = \bigcap_{k \in \mathbb{N}} I^k \subseteq L_A/I_{p,n}$ equals the 0-ideal.

For example, this shows that no element in the T -equivariant Lazard ring L_T is infinitely often divisible by the Euler class e .

Proof Since I is an invariant ideal of $L_A/I_{p,n}$, so are all its powers and the intersection thereof. Moreover, J equals the kernel of the completion map (see [Section 2.5](#))

$$L_A/I_{p,n} \rightarrow L/I_{p,n}[[y_1, \dots, y_r]],$$

where r is the rank of A and the y_i are the images of the Euler classes ranging through a basis of A^* . Since $L/I_{p,n}[[y_1, \dots, y_r]]$ is an integral domain, J must be prime and hence an invariant prime ideal.

Therefore J must be of the form $I_{B,p,m}^A$ (or rather its image under the projection $L_A \rightarrow L_A/I_{p,n}$) for some subgroup B and height $m \geq n$. Note that v_n is not contained in I , hence in particular not in J . This means that we must have $m = n$. Moreover, given a character $V = V_1^{\otimes k_1} \otimes \dots \otimes V_r^{\otimes k_r} \in A^*$ expressed in the chosen basis $V_1, \dots, V_r$ above, the image of e_V under the completion map is given by

$$F([k_1]_F(y_1), \dots, [k_r]_F(y_r)) \in L/I_{p,n}[[y_1, \dots, y_r]],$$

where F is the universal formal group law pushed forward to $L/I_{p,n}$. Since we assumed n to be a finite height, the $[k]$ -series of F is nontrivial whenever k is nonzero. It follows that for any nontrivial V the image of e_V is nontrivial in $L/I_{p,n}[[y_1, \dots, y_r]]$. Hence J contains no Euler class e_V and we must have $B = A$, ie

$$J = I_{A,p,n}^A = 0 \subseteq L_A/I_{p,n},$$

as desired. □

5 Inclusions between invariant prime ideals

Nonequivariantly the ideals $I_{p,n} \subseteq L$ form ascending towers

$$(0) = I_{p,0} \subseteq I_{p,1} \subseteq I_{p,2} \subseteq \dots,$$

essentially by definition. Except for the overlap at (0) , there are no inclusions between the towers for different primes p . For invariant prime ideals in the equivariant Lazard ring L_A we saw that we have one tower

$$I_{B,p,0}^A \subseteq I_{B,p,1}^A \subseteq I_{B,p,2}^A \subseteq \dots$$

for every pair of a closed subgroup B and prime p . Again, there will be no interplay between the towers associated to different primes (except for the overlap at height 0). However, there are additional inclusions connecting the towers for different subgroups B, B' at the same prime p . This relationship between the heights at different subgroups is one of the essential properties of equivariant formal groups. It is closely related to the blue-shift phenomenon in stable homotopy theory. We say more about this in [Section 8](#).

To see that there is no inclusion between towers associated with different primes, note that $p = v_0 \in I_{B,p,n}^A$ whenever $n \geq 1$. It is easy to see that p maps nontrivially under $L_A \rightarrow \Phi^{B'} L/I_{q,n'}$ for $q \neq p$ (since the target is free over $L/I_{q,n}$ by [Proposition 2.25](#)). Hence there cannot be an inclusion $I_{B,p,n}^A \subseteq I_{B',q,n'}^A$ for $n \geq 1$ and $p \neq q$. Moreover, if $n = 0$ we have $I_{B,p,0}^A = I_{B,q,0}^A$. Hence we can reduce to studying containments between invariant prime ideals associated to the same prime p .

New convention For this reason and to simplify notation, from now on and for the rest of the paper we implicitly localize at a fixed prime p . That is, we consider the p -localized Lazard ring L_A and denote its invariant prime ideals simply by $I_{B,n}^A$, omitting the chosen prime p . We further sometimes abbreviate $I_{A,n}^A$ to $I_{A,n}$.

Hence our goal is to understand for which pairs of subgroups B, B' and natural numbers n, n' there is an inclusion

$$I_{B,n}^A \subseteq I_{B',n'}^A.$$

We will show the following:

Theorem 5.1 *There is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$ if and only if the following conditions are satisfied:*

- (1) B' is a subgroup of B and $\pi_0(B/B')$ is a p -group.
- (2) We have $n' \geq n + \text{rank}_p(\pi_0(B/B'))$.

Hence, for example there are inclusions $I_{\mathbb{T},n}^T \subseteq I_{1,n}^T$ and $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$, but $I_{C_p^k,n}^{C_p^k}$ is not contained in $I_{1,n+k-1}^{C_p^k}$. In fact the theorem can be formally reduced to checking those three special cases, as we will see below. The theorem says, in particular, that given a chain of inclusions $B' \subseteq B \subseteq A$, the question of whether $I_{B,n}^A$ is contained in $I_{B',n'}^A$ does not depend on the ambient group A , but only on B, B', n and n' .

Theorem 5.1 can be interpreted as a statement about the heights of geometric fixed points of localizations of L_A , in the following way. Recall from [Hovey and Strickland 2005, Definition 4.1] that the *height* $\text{ht}(R)$ of an L -algebra R is the maximal n such that $R/(I_n \cdot R) \neq 0$; equivalently, it is the minimal number n such that $I_{n+1} \cdot R = R$. If there is no such n , the height is understood to be infinite. If $R = 0$, the height is -1 . Denoting by $\Phi^A((L_A)_{I_{B,n}^A})$ the localization of L_A inverting all Euler classes and all elements outside of $I_{B,n}^A$, we have the following:

Corollary 5.2 *Let $B \subseteq A$ be a closed subgroup, and let $n \in \mathbb{N}$. If $\pi_0(A/B)$ is not a p -group, or if $\pi_0(A/B)$ is a p -group but $\text{rank}_p(\pi_0(A/B)) > n$, then the geometric fixed points $\Phi^A((L_A)_{I_{B,n}^A})$ are trivial. Otherwise, their height is given by*

$$\text{ht}(\Phi^A((L_A)_{I_{B,n}^A})) = n - \text{rank}_p(\pi_0(A/B)).$$

Proof We have $I_m \cdot \Phi^A((L_A)_{I_{B,n}^A}) = \Phi^A((L_A)_{I_{B,n}^A})$ if and only if there exists an element of L_A not contained in $I_{B,n}^A$ which is mapped to $I_m \cdot \Phi^A L_A$ under the geometric fixed point map. Since $I_{A,m}$ is defined precisely as the preimage of $I_m \cdot \Phi^A L_A$, this in turn is equivalent to $I_{A,m}^A$ not being contained in $I_{B,n}^A$.

By **Theorem 5.1**, if $\pi_0(A/B)$ is not a p -group or if $\pi_0(A/B)$ is a p -group but $\text{rank}_p(\pi_0(A/B)) > n$, then $I_{A,0}^A$ is not contained in $I_{B,n}^A$. Since $I_0 = (0)$, this implies that the geometric fixed points $\Phi^A((L_A)_{I_{B,n}^A}) = I_0 \cdot \Phi^A((L_A)_{I_{B,n}^A})$ are trivial.

If $\pi_0(A/B)$ is a p -group and $r = \text{rank}_p(\pi_0(A/B)) \leq n$, then the theorem tells us that $I_{A,n-r}^A \subseteq I_{B,n}^A$ and $I_{A,n-r+1}^A \not\subseteq I_{B,n}^A$. Hence we have $I_{n-r} \cdot \Phi^A((L_A)_{I_{B,n}^A}) \neq \Phi^A((L_A)_{I_{B,n}^A})$ and $I_{n-r+1}^A \cdot \Phi^A((L_A)_{I_{B,n}^A}) = \Phi^A((L_A)_{I_{B,n}^A})$, as claimed. $\square$

Remark 5.3 The techniques of this paper can be used to compute the height of geometric fixed points for many complex oriented theories. We give one example of this in [Proposition 8.6](#) below.

Remark 5.4 There is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$ if and only if in $|\mathcal{M}_{FG,(p)}^A|$, the point corresponding to $I_{B',n'}^A$ lies in the closure of the point corresponding to $I_{B,n}^A$. Thus, [Theorem 5.1](#) can be interpreted as a result about the topology of $|\mathcal{M}_{FG,(p)}^A|$.

The proof of [Theorem 5.1](#) takes up the remainder of this section.

5.1 Formal reduction to the p -toral case

Our first step is the following:

Lemma 5.5 *If there is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$, then B' is a subgroup of B and $n' \geq n$.*

Proof When B' is not a subgroup of B we can choose a character $V \in A^*$ which is trivial when restricted to B but nontrivial when restricted to B' . Its Euler class e_V then restricts to 0 in L_B , in particular it is contained in $I_{B,n}^A$ for all n . On the other hand, its restriction to B' becomes an invertible element in the nontrivial ring $\Phi^{B'} L/I_{n'}$ and is hence nontrivial. Therefore e_V is not an element of $I_{B',n'}^A$. It follows that $I_{B,n}^A$ cannot be contained in $I_{B',n'}^A$.

If $n' < n$, then $v_{n'}$ is contained in I_n but not in $I_{n'}$. This implies that $v_{n'}$ (now thought of as an element of L_A) is contained in $I_{B,n}^A$ but not in $I_{B',n'}^A$, since $\Phi^{B'} L/I_{n'}$ is a nontrivial free module over $L/I_{n'}$ by [Proposition 2.25](#). Hence, again, $I_{B,n}^A$ cannot be contained in $I_{B',n'}^A$. $\square$

Lemma 5.6 *Let $B' \subseteq B$ be an inclusion of subgroups of A and $n, n' \in \mathbb{N}$. Then there is an inclusion*

$$I_{B,n}^A \subseteq I_{B',n'}^A$$

if and only if there is an inclusion

$$I_{B,n}^B \subseteq I_{B',n'}^B$$

if and only if there is an inclusion

$$I_{B/B',n}^{B/B'} \subseteq I_{B'/B',n'}^{B/B'}.$$

Proof The first two statements are equivalent since the restriction map $\text{res}_B^A: L_A \rightarrow L_B$ identifies L_B with a quotient of L_A , and the ideals $I_{B,n}^A$ and $I_{B',n'}^A$ are the preimages of the ideals $I_{B,n}^B$ and $I_{B',n'}^B$ under the quotient projection.

Phrased differently, the projection $L_A \rightarrow L_B$ induces a closed embedding of the stack of B -equivariant formal groups into the stack of A -equivariant formal groups. On spectra, the image consists precisely of these $I_{B'',n}^A$ with $B'' \subseteq B$. This implies the desired equivalence by [Remark 5.4](#).

For the second equivalence, recall the open embedding $|\mathcal{M}_{\text{FG}}^{B/B'}| \subseteq |\mathcal{M}_{\text{FG}}^B|$ from [Remark 4.5](#), sending the point $(B''/B', n)$ to (B'', n) for every $B' \subseteq B'' \subseteq B$. Since the closure relation among points in a subspace can be detected in the subspace, [Remark 5.4](#) gives the result. $\square$

Taken together, the previous two lemmas allow us to reduce to the case $B = A$ and $B' = 1$ and understand under what conditions there is an inclusion

$$I_{A,n}^A \subseteq I_{1,n'},$$

or in other words whether the restriction map $L_A \rightarrow L$ maps $I_{A,n}^A$ into the ideal $I_{n'}$.

Our next goal is to show that we can further reduce to the case where $\pi_0 A$ is a p -group. For this we choose a prime q and consider the Euler class $e_{\tau^q} \in L_{\mathbf{T}}$, ie the pullback of $e_{\tau} \in L_{\mathbf{T}}$ along the q^{th} power map $[q]: \mathbf{T} \rightarrow \mathbf{T}$. The Euler class e_{τ^q} restricts to 0 at the trivial group and is hence uniquely divisible by e_{τ} . We set $\mathbf{v}_0^{(q)} \in L_{\mathbf{T}}$ to be the unique element satisfying $e_{\tau^q} = \mathbf{v}_0^{(q)} \cdot e_{\tau}$ (the reason for this choice of notation will become clear in [Section 5.2](#)).

Under the completion map $L_{\mathbf{T}} \rightarrow L[[e_{\tau}]]$, the Euler class e_{τ^q} is sent to the q -series $[q]_F(e_{\tau})$ of the universal formal group law. Hence, $\mathbf{v}_0^{(q)}$ is sent to the quotient $[q]_F(e_{\tau})/e_{\tau}$, whose leading coefficient equals q . Since the restriction of any element in $L_{\mathbf{T}}$ to the trivial group equals the leading coefficient of its image in $L[[e_{\tau}]]$, we see that $\text{res}_1^{\mathbf{T}} \mathbf{v}_0^{(q)} = q \in L$. We further set $\bar{\mathbf{v}}_0^{(q)} \in L_{C_q}$ to be the restriction of $\mathbf{v}_0^{(q)}$, and find that it satisfies

$$\bar{\mathbf{v}}_0^{(q)} \cdot e_{\bar{\tau}} = 0, \quad \text{res}_1^{C_q} \bar{\mathbf{v}}_0^{(q)} = q.$$

Here, $\bar{\tau}$ denotes the restriction of the tautological character $\tau \in \mathbf{T}^*$ to C_q . We are now ready to show:

Lemma 5.7 *If $\pi_0 A$ is not a p -group, then there is no inclusion of the form $I_{A,n}^A \subseteq I_{1,n'}$.*

Proof If $\pi_0 A$ is not a p -group we can choose a surjection $f: A \rightarrow C_q$ with $q \neq p$ a prime. Then the element $f^* \bar{\mathbf{v}}_0^{(q)} \in L_A$ satisfies the equation $f^* \bar{\mathbf{v}}_0^{(q)} \cdot e_{f^* \bar{\tau}} = 0$. Since f is surjective, the character $f^* \bar{\tau} \in A^*$ is nontrivial. Hence, $f^* \bar{\mathbf{v}}_0^{(q)}$ is an element of $I_{A,0}^A$ and hence also of $I_{A,n}^A$. Its restriction to the trivial group equals that of $\bar{\mathbf{v}}_0^{(q)}$, which is q and hence a unit in the (p -localized) ring $L/I_{n'}$. In other words, $f^* \bar{\mathbf{v}}_0^{(q)}$ is not an element of $I_{1,n'}^A$. Hence $I_{A,0}^A$ does not include into $I_{1,n'}^A$. $\square$

Combined with [Lemmas 5.5](#) and [5.6](#) we obtain:

Corollary 5.8 *If there is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$, then B' is a subgroup of B , $n' \geq n$ and the quotient B/B' is p -toral, ie a product of a p -group and a torus.*

5.2 Proof of inclusions

The next goal is to prove the “if” part of [Theorem 5.1](#), ie to show that if the conditions on B , B' , n and n' stated there are satisfied, we do have an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$. We start with the easiest case:

Lemma 5.9 *Let $B' \subseteq B$ be an inclusion of subgroups of A such that B/B' is a torus. Then there is an inclusion $I_{B,n}^A \subseteq I_{B',n}^A$ for all n .*

Proof By [Lemma 5.6](#) we can reduce to $B = A$ and $B' = 1$ the trivial group. Hence A is a torus. [Corollary 4.9](#) then implies that $I_{1,n}^A$ is equal to the ideal generated by I_n and the augmentation ideal I_1^A , whereas $I_{A,n}^A$ is generated by I_n only. Clearly the latter is contained in the former. $\square$

We now turn to showing that there are inclusions $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$. We will show that $I_{C_{p^k},n}^{C_{p^k}}$ is generated by I_n plus one additional element which reduces to v_n under the restriction map $L_{C_{p^k}}/I_n \rightarrow L/I_n$. We start with the case $k = 1$ and recall again that the Euler class e_{τ^p} is sent to the p -series $[p]_F(e_{\tau})$ under the completion map $L_{\mathbf{T}} \rightarrow L[[e_{\tau}]]$. Modulo I_n , this p -series is of the form $v_n e_{\tau}^{p^n} + \text{higher-order terms}$. Hence [Corollary 2.28](#) implies that there exists a unique element $\psi_p^{(n)} \in L_{\mathbf{T}}/I_n$ such that $e_{\tau^p} = \psi_p^{(n)} e_{\tau}^{p^n}$. This element satisfies $\text{res}_1^{\mathbf{T}}(\psi_p^{(n)}) = v_n$, ie it is the coefficient of x^{p^n} in the p -series of the universal formal group law.

We then set

$$\psi_{p^k}^{(n)} = [p^{k-1}]^*(\psi_p^{(n)}) \in L_{\mathbf{T}}/I_n$$

for all $k \geq 2$, where $[p^{k-1}]$ is the multiplication-by- p^{k-1} map on the circle. By functoriality, $\psi_{p^k}^{(n)}$ also restricts to v_n at the trivial group. In fact, it already restricts to v_n at $L_{C_{p^{k-1}}}/I_n$. This is because $C_{p^{k-1}}$ is the kernel of $[p^{k-1}]$ and hence the restriction map factors through the trivial group. Applying $[p^{k-1}]^*$ to the defining equation for $\psi_p^{(n)}$, we also obtain $e_{p^k} = \psi_{p^k}^{(n)} e_{p^{k-1}}^{p^n}$; here and in the following, we will often abbreviate e_{τ^m} to e_m .

Proposition 5.10 *For every $k \geq 1$ and $n \in \mathbb{N}$ the element $\psi_{p^k}^{(n)}$ generates the kernel of*

$$\phi_{C_{p^k}}^{\mathbf{T}} : L_{\mathbf{T}}/I_n \rightarrow L_{C_{p^k}}/I_n \rightarrow \Phi^{C_{p^k}} L/I_n.$$

Hence, $I_{C_{p^k},n}^{\mathbf{T}}$ is generated by the regular sequence $v_0, v_1, \dots, v_{n-1}, \psi_{p^k}^{(n)}$.

Corollary 5.11 *The ideal $I_{C_{p^k},n}^{C_{p^k}}$ is generated by I_n and the restriction $\bar{\psi}_{p^k}^{(n)}$ of $\psi_{p^k}^{(n)}$ to $L_{C_{p^k}}/I_n$.*

Proof of Proposition 5.10 Let $x \in L_{\mathbf{T}}/I_n$ be an element mapping to 0 in $\Phi^{C_{p^k}} L/I_n$, ie an element in the image of $I_{C_{p^k},n}^{\mathbf{T}}$ under the reduction map $L_{\mathbf{T}} \rightarrow L_{\mathbf{T}}/I_n$. By [Lemma 4.8](#) we know that we have an equation of the form

$$(5.12) \quad x \cdot e_1^{b_1} \cdots e_{p^{k-1}}^{b_{p^{k-1}}} = y \cdot e_{p^k} = y' \cdot \psi_{p^k}^{(n)}$$

for some $b_i \in \mathbb{N}$, $y \in L_{\mathbf{T}}/I_n$ and $y' = y \cdot (e_{p^k}/\psi_{p^k}^{(n)})$ since $I_{C_{p^k}}^{\mathbf{T}} = (e_{p^k})$ by [Lemma 2.21](#).

If l is coprime to p , then $e_{p^l l}$ and e_{p^l} become multiples of one another modulo e_{p^k} : indeed, the corresponding characters in $(C_{p^k})^*$ generate the same subgroup and thus [Lemma 2.21](#) implies that $e_{p^l l}$ and e_{p^l} generate the same ideal in $L_{C_{p^k}}$. It follows that equation (5.12) gives rise to an equation

$$x \cdot e_1^{a_0} \cdot e_p^{a_1} \cdots e_{p^{k-1}}^{a_{k-1}} = y'' \cdot \psi_{p^k}^{(n)},$$

with only Euler classes for powers of p appearing on the left-hand side. We claim that y'' must be divisible by the entire product $e_1^{a_0} \cdot e_p^{a_1} \cdots e_{p^{k-1}}^{a_{k-1}}$. Since L_T/I_n is an integral domain, this implies that x is divisible by $\psi_{p^k}^{(n)}$, as desired. To see the claim, recall that $\psi_{p^k}^{(n)}$ restricts to v_n in $L_{C_{p^l}}/I_n$ for all $l < k$. Hence if $a_l > 0$ we have that

$$0 = \text{res}_{C_{p^l}}^T(y'' \cdot \psi_m^{(n)}) = \text{res}_{C_{p^l}}^T(y'') \cdot v_n \in L_{C_{p^l}}/I_n.$$

Since v_n is a regular element in $L_{C_{p^l}}/I_n$ by [Proposition 2.29](#), this implies that $\text{res}_{C_{p^l}}^T(y'') = 0$ and hence y'' is (uniquely) divisible by e_{p^l} . This argument can be iterated by replacing y'' by y''/e_{p^l} , and the statement follows. $\square$

Remark 5.13 One can show that more generally there exist elements $\psi_m^{(n)} \in L_T/I_n$ for all $m \in \mathbb{N}$ uniquely determined by the equations

$$(5.14) \quad e_m = \prod_{t|m} (\psi_t^{(n)})^{p^{v_p(m/t) \cdot n}},$$

where $v_p(-)$ denotes the p -adic valuation of a natural number. The element $\psi_m^{(n)}$ generates the kernel of

$$\phi_{C_m}^T: L_T/I_n \rightarrow L_{C_m}/I_n \rightarrow \Phi^{C_m} L/I_n,$$

and its restriction to the trivial group is given by

$$\text{res}_1^T(\psi_m^{(n)}) = \begin{cases} 0 & \text{if } m = 1, \\ v_n & \text{if } m = p^l \text{ and } l > 0, \\ q & \text{if } m = q^l \text{ with } q \neq p \text{ prime and } l > 0, \\ 1 & \text{otherwise.} \end{cases}$$

We note also that every $\psi_m^{(n)}$ is prime, since the geometric fixed points $\Phi^{C_m} L/I_n$ are integral domains. The elements $\psi_m^{(0)}$ were previously considered in [[Hausmann 2022](#), Proposition 5.46], denoted by ψ_m there.

Remark 5.15 The proof of [Proposition 5.10](#) applies in a more general context. Let X be a global group law in the sense of [[Hausmann 2022](#), Definition 5.1]. As the global Lazard ring L is the initial global group law, there is a unique map $L \rightarrow X$. Assume that the map $L = L(1) \rightarrow X(1)$ sends I_n to 0, and that for every $l = 0, \dots, k$ the Euler classes e_{p^l} in $X(T)$ are regular elements and v_n is a regular element in $X(C_{p^l}) = X(T)/e_{p^l}$ (for example this is the case if v_n is regular in $X(T)$ and e_{p^l} remains a regular element modulo v_n). Then the image of $\psi_{p^k}^{(n)}$ in $X(T)$ generates the kernel of the composition

$$\Phi_{C_{p^k}}^T: X(T) \rightarrow X(C_{p^k}) \rightarrow \Phi^{C_{p^k}} X.$$

For example, this applies to the coefficients of many Borel-equivariant complex oriented spectra, which can be used to compute their blue-shift numbers. We make use of this in [Proposition 8.6](#).

Corollary 5.16 *We have an inclusion $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$.*

Proof By [Corollary 5.11](#), $I_{C_{p^k},n}^{C_{p^k}}$ is generated by I_n and $\bar{\psi}_{p^k}^{(n)}$. Since I_n is clearly contained in $I_{1,n+1}^{C_{p^k}}$ (even in $I_{1,n}^{C_{p^k}}$), we can reduce modulo I_n and need to show that $\bar{\psi}_{p^k}^{(n)}$ is taken to 0 under the composition

$$L_{C_{p^k}}/I_n \xrightarrow{\text{res}_1^T} L/I_n \rightarrow L/I_{n+1}.$$

But this is clear, since $\bar{\psi}_{p^k}^{(n)}$ restricts to v_n at the trivial group and v_n lies inside I_{n+1} . □

We now have all the ingredients to prove the “if” direction in [Theorem 5.1](#):

Corollary 5.17 *Let $B' \subseteq B$ be an inclusion of subgroups of A such that B/B' is p -toral, and $n', n \in \mathbb{N}$ such that $n' \geq n + \text{rank}_p(\pi_0(B/B'))$. Then there is an inclusion $I_{B,n}^A \subseteq I_{B',n'}^A$.*

Proof By [Lemma 5.6](#) we can assume that $A = B$ and that $B' = 1$ is the trivial subgroup. Hence, A is a p -toral group. Let A^0 denote the path component of the identity. We have $I_{A^0,n'}^A \subseteq I_{1,n'}^A$ by [Lemma 5.9](#). Hence it suffices to show that $I_{A,n}^A$ is contained in $I_{A^0,n'}^A$. For this, making use of [Lemma 5.6](#) once more, we can assume that $A^0 = 1$ and hence A is a finite abelian p -group. We write $m = \text{rank}_p(A)$ and choose a filtration of subgroups

$$1 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m = A$$

such that every subquotient A_i/A_{i-1} is a cyclic p -group. By [Corollary 5.16](#) and [Lemma 5.6](#) we see that

$$I_{A,n}^A \subseteq I_{A_{m-1},n+1}^A \subseteq I_{A_{m-2},n+2}^A \subseteq \cdots \subseteq I_{1,n+m}^A \subseteq I_{1,n'}^A.$$

The final inclusion follows from the assumption $n' \geq n + \text{rank}_p(\pi_0(B/B')) = n + m$. This finishes the proof. □

5.3 Proof of noninclusions

For the “only if” direction we need to rule out further inclusions between prime ideals by constructing elements whose restrictions exhibit a large “height shift”. We will abbreviate $I_{A,n}^A$ to $I_{A,n}$ in the following, for various groups A . The goal is for every $n \in \mathbb{N}$ to construct an element $x \in L_{C_p^{n+1}}$ which lies in the ideal $I_{C_p^{n+1},0}$ and whose restriction to the trivial group lies outside of $I_n \subseteq L$.

It turns out to be more natural to define such an element modulo a subideal of $I_{C_p^{n+1},0}$, namely the ideal generated by the inflation of the ideal $I_{C_p^n,0}$ along the projection $p_{C_p^n}: C_p^{n+1} \cong C_p^n \times C_p \rightarrow C_p^n$. That is, we construct an element

$$\bar{v}_n \in I_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0}.$$

By [Corollary 5.17](#) the restriction map $L_{C_{p^n}} \rightarrow L$ takes $I_{C_p^n,0}$ into I_n . Therefore, the restriction map $L_{C_p^{n+1}} \rightarrow L$ takes $p^*I_{C_p^n,0}$ into I_n and we obtain an induced restriction map

$$L_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0} \rightarrow L / I_n.$$

We will see that, under this restriction, $\bar{v}_n$ is sent to v_n . Later in [Section 6](#) we will show that $\bar{v}_n$ in fact forms a generator of the quotient $I_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0}$ and that suitable inflations and restrictions of these elements generate all the invariant prime ideals at elementary abelian p -groups.

We now turn to the construction of the element $\bar{v}_n$. We set $A = C_p^n$ and first define an element v_n in the ring $L_{A \times T} / p_A^* I_{A,0}$, whose restriction to $L_{A \times C_p} / p_A^* I_{A,0}$ then yields $\bar{v}_n$.

For every character $V \in A^*$, [Proposition 2.27](#) yields a short exact sequence

$$0 \rightarrow L_{A \times T} \xrightarrow{e_{V \otimes \tau}} L_{A \times T} \xrightarrow{(\text{id}, V^{-1})^*} L_A \rightarrow 0,$$

where we use that $(\text{id}, V^{-1}): A \rightarrow A \times T$ identifies A with the kernel of $(V \otimes \text{id}): A \times T \rightarrow T$. The inflation map $L_A \rightarrow L_{A \times T}$ provides an L_A -linear splitting of this exact sequence if we view $L_{A \times T}$ as an L_A -module via the same inflation map. Thus, we obtain a short exact sequence

$$(5.18) \quad 0 \rightarrow L_{A \times T} / p_A^* I_{A,0} \xrightarrow{e_{V \otimes \tau}} L_{A \times T} / p_A^* I_{A,0} \xrightarrow{(\text{id}, V^{-1})^*} L_A / I_{A,0} \rightarrow 0$$

of $L_A / I_{A,0}$ -modules.

Remark 5.19 We will see below in [Section 6](#) that the ideal in $L_{A \times T}$ generated by $p_A^* I_{A,0}$ equals the invariant prime ideal $I_{A \times T,0}$. In particular, $L_{A \times T} / p_A^* I_{A,0}$ is again an integral domain. At this point it is only clear that $p_A^* I_{A,0}$ is contained in $I_{A \times T,0}$.

We now consider the Euler class $e_{\epsilon \otimes \tau^p} \in L_{A \times T} / I_{A,0}$. We have $(\text{id}, V^{-1})^*(e_{\epsilon \otimes \tau^p}) = e_{V^{-p}} = 0$, since A is an elementary abelian p -group and hence every character is p -torsion. By exactness, it follows that $e_{\epsilon \otimes \tau^p}$ is divisible by $e_{V \otimes \tau}$ for all $V \in A^*$. We want to define $\bar{v}_n$ as the quotient of $e_{\epsilon \otimes \tau^p}$ by the product over all the $e_{V \otimes \tau}$. For this we first need to check that the different $e_{V \otimes \tau}$ are coprime. To understand this, we consider the following:

Lemma 5.20 *Let $x, y \in L_{A \times T} / p_A^* I_{A,0}$, and assume that y restricts to a nonzero element under the map $L_{A \times T} / p_A^* I_{A,0} \xrightarrow{(\text{id}, V^{-1})^*} L_A / I_{A,0}$. Then x is divisible by $e_{V \otimes \tau}$ if and only if $x \cdot y$ is divisible by $e_{V \otimes \tau}$.*

Proof By exactness of (5.18), x is divisible by $e_{V \otimes \tau}$ if and only if $(\text{id}, V^{-1})^* x = 0$ in $L_A / I_{A,0}$. Since $(\text{id}, V^{-1})^*(y)$ is nonzero by assumption and $L_A / I_{A,0}$ is an integral domain, this is the case if and only if

$$(\text{id}, V^{-1})^*(x \cdot y) = (\text{id}, V^{-1})^* x \cdot (\text{id}, V^{-1})^* y = 0,$$

which in turn is equivalent to $x \cdot y$ being divisible by $e_{V \otimes \tau}$. □

Given a character $V_2 \neq V$, the restriction of $e_{V_2 \otimes \tau}$ along $(\text{id}, V^{-1})^*$ equals the nontrivial Euler class $e_{V_2 V^{-1}} \in L_A/I_{A,0}$. Hence the lemma applies to y being any product of Euler classes of the form $e_{V' \otimes \tau}$ with $V' \neq V$. We find that in the quotient $L_{A \times T}/p_A^* I_{A,0}$, the Euler class $e_{\epsilon \otimes \tau^p}$ is uniquely divisible by the product $\prod_{V \in A^*} e_{V \otimes \tau}$. To summarize:

Definition 5.21 Let $n \in \mathbb{N}$ and $A = C_p^n$. We define $v_n \in L_{A \times T}/p_A^* I_{A,0}$ to be the unique element satisfying

$$e_{\epsilon \otimes \tau^p} = v_n \cdot \prod_{V \in A^*} e_{V \otimes \tau} \in L_{A \times T}/p_A^* I_{A,0},$$

and we set

$$\bar{v}_n = \text{res}_{A \times C_p}^{A \times T}(v_n) \in L_{A \times C_p}/p_A^* I_{A,0}.$$

Remark 5.22 The elements $v_0 \in L_T$ and $\bar{v}_0 \in L_{C_p}$ agree with the elements $v_0^{(p)}$ and $\bar{v}_0^{(p)}$ respectively from [Section 5.1](#).

Furthermore, the element $\psi_p^{(n)} \in L_T/I_n$ introduced before [Proposition 5.10](#) is equal to the restriction of $v_n \in L_{A \times T}/p^* I_{A,0}$ to L_T/I_n . (Note that the latter makes sense since we know from [Corollary 5.17](#) that the restriction of $I_{A,0} \subseteq L_{C_p^n}$ to the nonequivariant Lazard ring L lands in I_n .) Indeed, the defining relation for v_n above restricts to the equation

$$e_{\tau^p} = \text{res}_T^{A \times T}(v_n) \cdot \prod_{V \in A^*} \text{res}_T^{A \times T}(e_{V \otimes \tau}) = \text{res}_T^{A \times T}(v_n) \cdot e_{\tau}^{p^n} \in L_T/I_n,$$

since each $e_{V \otimes \tau}$ restricts to e_{τ} at the circle group. Since $\psi_p^{(n)}$ was defined as the unique element of L_T/I_n satisfying $e_{\tau^p} = \psi_p^{(n)} \cdot e_{\tau}^{p^n}$, the claim follows. Consequently, $\bar{\psi}_p^{(n)} \in L_{C_p}/I_n$ agrees with the restriction of $\bar{v}_n \in L_{A \times C_p}/p^* I_{A,0}$ to L_{C_p}/I_n .

Remark 5.23 [Lemma 5.20](#) also applies in the ring $L_{A \times T}$ itself (ie before quotienting by $p_A^* I_{A,0}$) if we demand that $(\text{id}, V)^* y$ is a regular element, rather than just being nonzero. These two conditions are equivalent in $L_A/I_{A,0}$ since it is an integral domain. However, the Euler classes are not regular elements in L_A , hence the lemma does not apply for y a product of the $e_{V' \otimes \tau}$. In fact $e_{\epsilon \otimes \tau^p}$ is not divisible by $\prod_{V \in A^*} e_{V \otimes \tau}$ before dividing out $p_A^* I_{A,0}$, even though it is divisible by each individual $e_{V \otimes \tau}$. One can see this by restricting from $A \times T$ to T : If $e_{\epsilon \otimes \tau^p}$ was divisible by the product of all the $e_{V \otimes \tau}$, this would imply that its restriction $e_{\tau^p} \in L_T$ was divisible by $e_{\tau}^{p^n}$, since every $e_{V \otimes \tau}$ restricts to e_{τ} . But $e_{\tau^p} \in L_T$ is divisible by e_{τ} precisely once, since e_{τ^p}/e_{τ} restricts to p at the trivial group; see [Section 5](#). It is for this reason that it is most natural to define v_n and $\bar{v}_n$ in this quotient. As we will see now, this matches nicely with the fact that v_n is most naturally defined in the quotient L/I_n .

Proposition 5.24 (1) The element $\bar{v}_n$ defines a class in the ideal $I_{A \times C_p,0}/p_A^* I_{A,0}$, ie it is sent to 0 under the map $L_{A \times C_p,0}/p_A^* I_{A,0} \rightarrow \Phi^{A \times C_p} L$.

(2) The restriction map

$$L_{A \times C_p, 0} / p_A^* I_{A, 0} \rightarrow L / I_n$$

takes $\bar{v}_n$ to v_n .

Proof (1) The equation

$$e_{\epsilon \otimes \tau^p} = v_n \cdot \prod_{V \in A^*} e_{V \otimes \tau}$$

reduces to the equation

$$0 = \bar{v}_n \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}$$

in $L_{A \times C_p} / p_A^* I_{A, 0}$, where $\bar{\tau}$ denotes the restriction of $\tau \in T^*$ to C_p . Note that each $(A \times C_p)$ -character of the form $V \otimes \bar{\tau}$ is nontrivial. Hence $\bar{v}_n$ forms Euler-power torsion and therefore maps to 0 in the geometric fixed points.

(2) By Remark 5.22, the intermediate restriction $\text{res}_T^{A \times T} v_n \in L_T / I_n$ equals the element $\psi_p^{(n)}$ constructed before Proposition 5.10. Since $\text{res}_1^T \psi_p^{(n)} = v_n$, this finishes the proof. $\square$

Corollary 5.25 If x_n is a preimage of $\bar{v}_n$ under the projection $L_{C_p^{n+1}} \rightarrow L_{C_p^{n+1}} / p_{C_p}^* I_{C_p^n, 0}$ and $B \subseteq C_p^{n+1}$ is a subgroup of rank $0 \leq m \leq n+1$, then

$$x_n \in I_{B, n+1-m}^{C_p^{n+1}} - I_{B, n-m}^{C_p^{n+1}}.$$

Proof By the previous proposition, x_n is an element of $I_{C_p^{n+1}, 0}^{C_p^{n+1}}$. As A/B has rank $(n+1-m)$, we know by Corollary 5.17 that x_n must lie in $I_{B, n+1-m}^{C_p^{n+1}}$.

If x_n were an element of $I_{B, n-m}^{C_p^{n+1}}$, then applying Corollary 5.17 to the inclusion of the trivial group into B shows that x_n is also an element of $I_{1, n}^{C_p^{n+1}}$. This contradicts the fact that, modulo I_n , we have $\text{res}_1^{C_p^{n+1}}(x_n) = \text{res}_1^{C_p^{n+1}}(\bar{v}_n) = v_n$. $\square$

Corollary 5.26 If $B' \subseteq B$ is a p -toral inclusion of subgroups of A (that is, B/B' is p -toral) and if $n' < n + \text{rank}_p(\pi_0(B/B'))$, then $I_{B, n}^A$ does not include into $I_{B', n'}^A$.

Proof By Lemma 5.6 we can reduce to the case $A = B$ and $B' = 1$. Let $r = \text{rank}_p(\pi_0(A))$, and $q: A \rightarrow C_p^r$ be a surjection. Let $x_{n+r-1} \in L_{C_p^{n+r}}$ as in Corollary 5.25; then the restriction $\text{res}_{C_p^r}^{C_p^{n+r}}(x_{n+r-1})$ is an element of $I_{C_p^r, n}^{C_p^r}$ but not an element of $I_{1, n+r-1}^{C_p^r}$. Therefore

$$x = q^*(\text{res}_{C_p^r}^{C_p^{n+r}}(x_{n+r-1}))$$

is an element of $I_{A, n}^A$ whose restriction to the trivial group is not contained in I_{n+r-1} . In other words, x is an element of $I_{A, n}^A$ but not an element of $I_{1, n+r-1}^A$. Since by assumption we have $n' \leq n+r-1$ and hence $I_{1, n'}^A \subseteq I_{1, n+r-1}^A$, this proves that $I_{A, n}^A$ does not include into $I_{1, n'}^A$. $\square$

Combined with Corollaries 5.8 and 5.17, this proves Theorem 5.1.

6 Generators for invariant prime ideals

In this section we show that over elementary abelian p -groups the elements $\bar{v}_n$ — together with the Euler classes — generate all invariant prime ideals under restriction and inflation maps. More precisely, we show the following theorem:

Theorem 6.1 (1) For every torus B and $n \in \mathbb{N}$, the ideal $I_{C_p^n \times B, 0} = I_{C_p^n \times B, 0}^{C_p^n \times B}$ is generated by the elements

$$p_1^*(\bar{v}_0), p_2^*(\bar{v}_1), \dots, p_{n-1}^*(\bar{v}_{n-2}), p_n^*(\bar{v}_{n-1}),$$

where $p_i: C_p^n \times B \rightarrow C_p^i$ is the projection to the first i factors.

(2) For every $m \in \mathbb{N}$ and every inclusion $i: C_p^n \rightarrow C_p^{n+m}$, the restriction map

$$(i \times B)^*: L_{C_p^{n+m} \times B} \rightarrow L_{C_p^n \times B}$$

maps $I_{C_p^{n+m} \times B, 0}$ surjectively onto $I_{C_p^n \times B, m}$.

Remark 6.2 Implicit in the statement of the theorem is that each $p_i^*(\bar{v}_{i-1})$ is well-defined modulo the ideal generated by $p_1^*(\bar{v}_0), \dots, p_{i-1}^*(\bar{v}_{i-2})$. By definition, $p_i^*(\bar{v}_{i-1})$ is an element of the quotient by the subideal generated by $p_{i-1}^* I_{C_p^{i-1}, 0}$. Applying the theorem to rank $n-1$ and $B=0$ we see that this ideal is indeed generated by $p_1^*(\bar{v}_0), \dots, p_{i-1}^*(\bar{v}_{i-2})$, so the sequence of elements makes sense. Hence, the theorem and sequence should be interpreted in an inductive manner.

Combining both parts it follows that $I_{C_p^n \times B, m}$ is generated by

$$(i \times B)^* p_1^*(\bar{v}_0), (i \times B)^* p_2^*(\bar{v}_1), \dots, (i \times B)^* p_{n+m-1}^*(\bar{v}_{n+m-2}), (i \times B)^* \bar{v}_{n+m-1},$$

where $i: C_p^n \rightarrow C_p^{n+m}$ is any inclusion. The choice of inclusion will generally affect the resulting generators. For example, setting $n=m=1$ and B the trivial group: If we choose $i_1: C_p \rightarrow C_p^2$ to be the inclusion into the first factor, the composite $p_1 \circ i_1$ becomes the identity. Hence we obtain that $I_{C_p, 1}$ is generated by the elements $\bar{v}_0$ and $i_1^*(\bar{v}_1)$. If we alternatively use the inclusion $i_2: C_p \rightarrow C_p^2$ into the second factor the composite $p_1 \circ i_2$ becomes the constant map, yielding the generators $v_0 = p$ and $i_2^*(\bar{v}_1)$ (ie the same ones as in [Corollary 5.11](#), as $\bar{\psi}_p^{(1)}$ equals $i_2^*(\bar{v}_1)$). Furthermore, it follows that generators for ideals of the form $I_{C_p^n \times B, m}^A$ with B a torus can be obtained as the union of Euler classes $(e_V)_{V \in \mathcal{B}}$ for a basis $\mathcal{B}$ of $\ker(A^* \rightarrow (C_p^n \times B)^*)$ together with a choice of generators for $I_{C_p^n \times B, m}^{C_p^n \times B}$.

We prove both parts of [Theorem 6.1](#) by induction on n , alongside the following statement:

(★) For every torus B and $m \in \mathbb{N}$ the ideal $I_{C_p^n \times B, m}$ is generated by $p_{C_p^n}^* I_{C_p^n, m}$.

For (1), the induction base case $n=0$ is the statement that $I_{B, 0} = I_{B, 0}^B$ is the 0-ideal for any torus B ([Corollary 4.9](#)); this also shows (★) for $n=0$. For (2) we need to see that the restriction $L_{C_p^m \times B} \rightarrow L_B$

maps $I_{C_p^m \times B, 0}$ surjectively onto $I_{B, m}$, which we know is generated by I_m by Corollary 4.9. For $i = 0, \dots, m-1$ we can consider the elements $\bar{v}_i \in I_{C_p^{i+1}, 0} / p_i^* I_{C_p^i, 0}$, which reduce to $v_i \in L / I_{i-1}$. It follows that the inflation of $\bar{v}_i$ to C_p^m via any choice of surjection $C_p^m \times B \rightarrow C_p^{i+1}$ gives an element of a quotient of $I_{C_p^m \times B, 0}$ which reduces to v_i in L_B / I_{i-1} . Since I_i / I_{i-1} is generated by v_i , the claim follows.

We now assume that Theorem 6.1 holds for an elementary abelian p -group A of rank n and show it also holds for $A \times C_p$. For any $m \in \mathbb{N}$ we consider the surjection

$$L_{A \times T \times B} / p_A^* I_{A, m} \rightarrow L_{A \times C_p \times B} / p_A^* I_{A, m},$$

with kernel generated by e_{τ^p} for τ the tautological T -character pulled back to $A \times T \times B$. We first claim that if $V \in A^*$ is nontrivial, then the Euler class $p_A^*(e_V)$ is a nonzero divisor in $L_{A \times C_p \times B} / p_A^* I_{A, m}$. To see this, we use that since $T \times B$ is a torus we can apply the induction hypothesis to $L_{A \times T \times B}$. In particular, we know that the statement $(\star)$ holds for $A \cong C_p^n$ and hence $p_A^* I_{A, m}$ generates the ideal $I_{A \times T \times B, m}$. Therefore, $L_{A \times T \times B} / p_A^* I_{A, m}$ is an integral domain. So we have to show that $p_A^*(e_V)$ still acts regularly modulo e_{τ^p} . Since both Euler classes are regular, this is equivalent to showing that e_{τ^p} is regular modulo $p_A^*(e_V)$. We have an isomorphism

$$L_{A \times T \times B} / (p_A^* I_{A, m}, p_A^*(e_V)) \cong L_{\ker(V) \times T \times B} / p_{\ker(V)}^* (\text{res}_{\ker(V)}^A I_{A, m}).$$

By the induction hypothesis for (2), we know that $I_{A, m}$ restricts onto $I_{\ker(V), m+1}$, hence the latter quotient identifies with $L_{\ker(V) \times T \times B} / p_{\ker(V)}^* I_{\ker(V), m+1}$. Again we know by the induction hypothesis that this quotient is an integral domain, and e_{τ^p} is clearly a nontrivial element. So the claim follows and we have shown that $p_A^* I_{A, m}$ generates the Euler power torsion in $L_{A \times C_p \times B}$ (at height m) for characters inflated up from A .

Hence to understand the full ideal $I_{A \times C_p \times B, m}$ it suffices to further divide by the Euler-power torsion for the remaining torsion characters in $(A \times C_p \times B)^*$ (there is no Euler-power torsion for nontorsion characters by Proposition 2.27). These torsion characters are of the form $V \otimes \bar{\tau}^k$, where $V \in A^*$, $\bar{\tau}$ is the restriction of $\tau \in T^*$ to C_p and $k \in \{1, \dots, p-1\}$. Furthermore we can assume that $k = 1$: Any $V \otimes \bar{\tau}^k$ has some power of the form $V' \otimes \bar{\tau}$ and hence $e_{V' \otimes \bar{\tau}}$ is a multiple of $e_{V \otimes \bar{\tau}^k}$. Thus, $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$ is generated by Euler-power torsion for characters of the form $V \otimes \bar{\tau}$.

Again it is beneficial to pass to the integral domain $L_{A \times T \times B} / p_A^* I_{A, m}$ to understand the Euler-power torsion for these characters. We have the following:

Lemma 6.3 *Let $A = C_p^n$ be an elementary abelian p -group, B a torus and $m \in \mathbb{N}$. Further, let $x \in L_{A \times T \times B} / p_A^* I_{A, m}$ be an element satisfying*

$$x \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}^{n_V} = y \cdot e_{\tau^p}$$

for some y and collection of natural numbers n_V . Then x lies in the ideal generated by

$$p_{A \times T}^* \text{res}_{A \times T}^{A \times C_p^m \times T} (v_{n+m}).$$

Proof By applying $p_{A \times C_p^m \times T}^*$ to the defining property of v_{n+m} (Definition 5.21) we obtain the equation

$$e_{\tau^p} = p_{A \times C_p^m \times T}^*(v_{n+m}) \cdot \prod_{V \in (A \times C_p^m)^*} e_{V \otimes \tau}$$

in $L_{A \times C_p^m \times T \times B} / p_{A \times C_p^m}^* I_{A \times C_p^m, 0}$. Restricting from $A \times C_p^m$ to $A = C_p^n$ yields

$$(6.4) \quad e_{\tau^p} = p_{A \times T}^* \operatorname{res}_{A \times T}^{A \times C_p^m \times T} (v_{n+m}) \cdot \prod_{V \in A^*} e_{V \otimes \tau}^{p^m}$$

in the quotient $L_{A \times T \times B, m} / p_A^* I_{A, m}$. This uses that every character of A extends to p^m different characters of $A \times C_p^m$, and that the restriction of $I_{A \times C_p^m, 0} \subseteq L_{A \times C_p^m}$ lands in the ideal $I_{A, m} \subseteq L_A$. For the rest of the proof we write z for the element $p_{A \times T}^* \operatorname{res}_{A \times T}^{A \times C_p^m \times T} (v_{n+m})$. With x as in the statement of the lemma, we hence obtain an equation of the form

$$(6.5) \quad x \cdot \prod_{V \in A^*} e_{V \otimes \tau}^{n_V} = z \cdot \prod_{V \in A^*} e_{V \otimes \tau}^{p^m} \cdot y,$$

and we need to show that x is a multiple of z . The Euler classes $e_{V \otimes \tau}$ fit into short exact sequences of the form

$$0 \rightarrow L_{A \times T \times B} / p_A^* I_{A, m} \xrightarrow{e_{V \otimes \tau}} L_{A \times T \times B} / p_A^* I_{A, m} \xrightarrow{((\operatorname{id}, V^{-1}) \times B)^*} L_{A \times B} / p_A^* I_{A, m} \rightarrow 0,$$

analogously to equation (5.18). The induction hypothesis implies that the quotient $L_{A \times B} / p_A^* I_{A, m}$ is an integral domain. We know that z restricts to $v_{n+m} \in L / I_{n+m}$ at the trivial group. In particular it must restrict nontrivially under each $((\operatorname{id}, V^{-1}) \times B)^*$. Hence the above short exact sequence implies: If an Euler class $e_{V \otimes \tau}$ divides an element of the form $z \cdot \alpha$, then $e_{V \otimes \tau}$ divides α . Applying this iteratively to equation (6.5) (and using that $L_{A \times T \times B} / p_A^* I_{A, m}$ is an integral domain as well by the induction hypothesis) we see that $\prod_{V \in A^*} e_{V \otimes \tau}^{n_V}$ must divide the term $\prod_{V \in A^*} e_{V \otimes \tau}^{p^m} \cdot y$. Dividing on both sides shows that x is a multiple of z , as desired. $\square$

Corollary 6.6 *The quotient*

$$I_{A \times C_p \times B, m} / p_A^* I_{A, m}$$

is generated by the element

$$\operatorname{res}_{A \times C_p \times B}^{A \times C_p^m \times C_p \times B} (p_{A \times C_p^m \times C_p}^* \bar{v}_{n+m}) = p_{A \times C_p}^* (\operatorname{res}_{A \times C_p}^{A \times C_p^m \times C_p} \bar{v}_{n+m}).$$

Proof We saw above that the quotient $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$ is generated by Euler-power torsion for characters of the form $V \otimes \bar{\tau}$. An element $\bar{x}$ of $L_{A \times C_p \times B} / p_A^* I_{A, m}$ is such a torsion element if and only if it is the reduction of an element $x \in L_{A \times T \times B} / p_A^* I_{A, m}$ satisfying the conditions of the lemma. Since the reduction of $p_{A \times T}^* \operatorname{res}_{A \times T}^{A \times C_p^m \times T} (v_{n+m})$ equals $p_{A \times C_p}^* \operatorname{res}_{A \times C_p}^{A \times C_p^m \times C_p} (\bar{v}_{n+m})$, it follows that $\bar{x}$ lies in the ideal generated by the latter.

As $p_{A \times C_p}^* \operatorname{res}_{A \times C_p}^{A \times C_p^m \times C_p} (\bar{v}_{n+m})$ is Euler-power torsion itself, it forms a generator of $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$. $\square$

Finishing the proof of Theorem 6.1 Setting $m = 0$ in the corollary shows that $\bar{v}_n$ generates the quotient $I_{A \times C_p \times B, 0} / p_A^* I_{A, 0}$. By the induction hypothesis, $I_{A, 0}$ is generated by $p_1^*(\bar{v}_0), \dots, p_{n-1}^*(\bar{v}_{n-2}), \bar{v}_{n-1}$. Combined this proves (1) for the group $A \times C_p$.

For (2) and general m , we first note that it suffices to show the statement for any choice of injection $i: A \times C_p \rightarrow C_p^{n+m+1}$ since any two only differ by postcomposition with an automorphism of C_p^{n+m+1} . We can hence pick the canonical inclusion $A \times C_p \rightarrow A \times C_p^m \times C_p$ avoiding C_p^m . By the induction hypothesis we know that $I_{A \times C_p^m, 0}$ surjects onto $I_{A, m}$. From the diagram

$$\begin{array}{ccc} I_{A \times C_p^m, 0} & \xrightarrow{\text{res}} & I_{A, m} \\ p_{A \times C_p^m}^* \downarrow & & \downarrow p_A^* \\ I_{A \times C_p^m \times C_p \times B, 0} & \xrightarrow{\text{res}} & I_{A \times C_p \times B, m} \end{array}$$

we see that $p_A^*(I_{A, m})$ is contained in the image of the lower-horizontal arrow. Furthermore, Corollary 6.6 implies that $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$ is generated by the restriction of an element of $I_{A \times C_p^m \times C_p \times B, m}$. Finally, we need to see that $(\star)$ holds true for $C_p^{n+1} \cong A \times C_p$. This is a direct consequence of Corollary 6.6, as by definition both $p_A^* I_{A, m}$ and $p_{A \times C_p}^*(\text{res}_{A \times C_p}^{A \times C_p^m \times C_p} \bar{v}_{n+m})$ lie in the image of $p_{A \times C_p}^* I_{A \times C_p, m}$. $\square$

Remark 6.7 Unlike the sequence $v_0, \dots, v_{n-1}$, the sequence

$$p_1^*(\bar{v}_0), p_2^*(\bar{v}_1), \dots, p_{n-1}^*(\bar{v}_{n-2}), \bar{v}_{n-1}$$

isn't regular. In fact, the ideal $I_{C_p^n, 0}$ generated by these elements is precisely that of Euler-power torsion.

This can be corrected by passing to a torus: The ideal $I_{C_p^n, 0}^{T^n}$ is generated by the sequence

$$p_1^*(v_0), p_2^*(v_1), \dots, p_{n-1}^*(v_{n-2}), v_{n-1}.$$

Here, each $p_{i+1}^*(v_i)$ is the element of

$$(6.8) \quad L_{T^n} / (p_1^*(v_0), p_2^*(v_1), \dots, p_i^*(v_{i-1})) \cong L_{C_p^i \times T^{n-i}} / I_{C_p^i \times T^{n-i}, 0}$$

obtained as the inflation of $v_i \in L_{C_p^i \times T} / I_{C_p^i \times T, 0}$ along the projection to the first coordinate of T^{n-i} . Since each successive quotient $L_{C_p^i \times T^{n-i}} / I_{C_p^i \times T^{n-i}, 0}$ is an integral domain, the regularity of the sequence is clear once we have demonstrated the isomorphism claimed in (6.8). Similarly one shows that $I_{C_p^n, m}^{T^n}$ is generated by a regular sequence of length $n + m$.

To establish the isomorphism, first note that $(e_{\tau^p}) \subseteq (p_{C_p^{i-1} \times T}^*(v_{i-1}))$ in the quotient ring

$$L_{C_p^{i-1} \times T^{n-i+1}} / p_{C_p^{i-1}}^* I_{C_p^{i-1}, 0}$$

by Lemma 6.3. Thus, Corollary 6.6 with $m = 0$ gives

$$L_{T^n} / (p_1^*(v_0), p_2^*(v_1), \dots, p_i^*(v_{i-1})) \cong L_{C_p^i \times T^{n-i}} / (p_1^*(\bar{v}_0), p_2^*(\bar{v}_1), \dots, p_i^*(\bar{v}_{i-1})),$$

and the claimed isomorphism becomes Theorem 6.1.

7 The Zariski topology on the spectrum of invariant prime ideals

The goal of this section is to describe the Zariski topology on $\mathrm{Spec}^{\mathrm{inv}}(L_A)$, or equivalently the topology on the space $|\mathcal{M}_{\mathrm{FG}}^A|$ (Theorem 4.7). By definition, the closed subsets of $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ are the subsets of the form

$$V(X) = \{I_{B,n}^A \in \mathrm{Spec}^{\mathrm{inv}}(L_A) \mid X \subseteq I_{B,n}^A\}$$

for some subset X of L_A . Hence we need to determine the collections of invariant prime ideals that arise as $V(X)$ for some X . We now fix a subset X . Since a containment $X \subseteq I_{B,n}^A$ automatically implies $X \subseteq I_{B,n'}^A$ for all $n' \geq n$, it suffices to understand, for every closed subgroup B of A , the smallest value of $n \in \mathbb{N} \cup \{\infty\}$ such that $X \subseteq I_{B,n}^A$. In other words we need to determine the function

$$\mathrm{ht}_X: \mathrm{Sub}(A) \rightarrow \overline{\mathbb{N}}_- = \{-1\} \cup \mathbb{N} \cup \{\infty\} \quad \text{defined by} \quad \mathrm{ht}_X(B) = \sup\{n \mid X \not\subseteq I_{B,n}^A\},$$

where we set $\sup(\emptyset) = -1$. We note that with this definition the height function of the image of $v_n \in L$ in L_A (thought of as a one-element set) is constantly n . This follows from the fact that $\Phi^B L$ is a free L -module by Proposition 2.25; cf. also the proof of Theorem 4.7. Moreover, we have:

Example 7.1 Let $x_n \in L_{C_p^{n+1}}$ be a lift of $\bar{v}_n \in L_{C_p^{n+1}}/p_{C_p^n}^* I_{C_p^n}$. Then Corollary 5.25 implies that $\mathrm{ht}_{x_n}(B) = n - \mathrm{rk}(B)$.

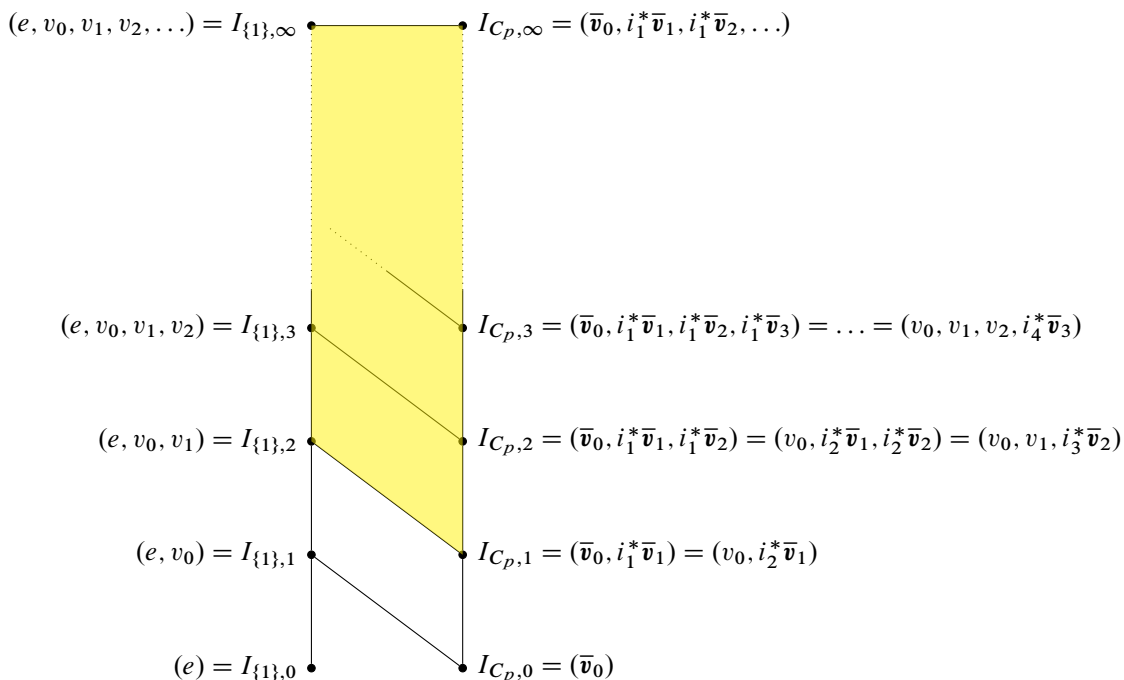


Figure 4: A picture of $\mathrm{Spec}^{\mathrm{inv}}(L_{C_p})$, localized at p , including different choices of generators, with $i_j: C_p \rightarrow C_p^k$ denoting the j^{th} canonical inclusion (generators arising from the further inclusions $C_p \rightarrow C_p^k$ are omitted). The shaded area depicts the closure of the point $I_{C_p, 1}$.

Our goal is to understand which functions $\text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ arise as such height functions. In the previous section we showed that there are inclusions between invariant prime ideals associated to different subgroups of A . These translate to conditions between the different values of ht_X : If $B' \subseteq B$ is an inclusion of subgroups of A such that B/B' is p -toral, and X is contained in $I_{B,n}^A$, then X is also contained in $I_{B',n+\text{rank}_p(\pi_0(B/B'))}^A$. In terms of the height function this translates to the inequality

$$\text{ht}_X(B') \leq \text{ht}_X(B) + \text{rank}_p(\pi_0(B/B')).$$

This leads us to the following definition:

Definition 7.2 A function $f: \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ is called *admissible* if it satisfies the inequality

$$f(B') \leq f(B) + \text{rank}_p(\pi_0(B/B'))$$

for every p -toral inclusion $B' \subseteq B$ of closed subgroups of A . Here, $B' \subseteq B$ is p -toral if B/B' is a product of a torus and a p -group.

By the above considerations, any height function ht_X is admissible. When the group A is finite, it turns out that the converse also holds: Any admissible function is realized by a height function ht_X . For positive dimensional A there is an additional condition on top of admissibility. To state this condition, we recall that choosing an invariant Riemannian metric d on A also equips the set of closed subgroups $\text{Sub}(A)$ with the Hausdorff metric, the underlying topology of which does not depend on the chosen metric on A . This turns $\text{Sub}(A)$ into a compact totally disconnected metric space, in which a sequence $(B_i)_{i \in \mathbb{N}}$ of closed subgroups converges to another closed subgroup $B \in \text{Sub}(A)$ if and only if almost all B_i are subgroups of B and for every element $b \in B$ the distance function $d(b, B_i)$ converges to zero; see [tom Dieck 1979, Section 5.6]. If $B_i \rightarrow B$, we have the following two implications about representations:

- (1) Let W be a (finite-dimensional) representation of B with $W^B = 0$. Thus, writing W as a sum of characters V_i , none of the V_i is trivial. For sufficiently large i , no $\ker(V_i)$ contains B_i (since $\ker(V_i) \subseteq B$ is a closed proper subgroup) and thus $W^{B_i} = 0$.
- (2) Let V and W be two characters of B such that $\text{res}_{B_i}^B V = \text{res}_{B_i}^B W$ for all sufficiently large i . If $V \neq W$, then $(V \cdot W^{-1})^B = 0$, in contradiction with the previous point. Thus $V = W$.

We have the following:

Proposition 7.3 For every **finite** subset $X \subseteq L_A$, the height function ht_X is a locally constant function on $\text{Sub}(A)$.

Proof We start by noting that

$$\text{ht}_X(B) = \max(\text{ht}_{\{x\}}(B) \mid x \in X).$$

The maximum of a finite number of locally constant functions is again locally constant. Hence it suffices to understand that $\text{ht}_{\{x\}}$ is locally constant for any element x of L_A .

Now let $(B_i)_{i \in \mathbb{N}}$ denote a sequence of subgroups of A converging to a subgroup B . We need to show that $\text{ht}_{\{x\}}(B_i) = \text{ht}_{\{x\}}(B)$ for almost all i . Without loss of generality we can assume that the B_i are subgroups of B . Replacing x by $\text{res}_B^A(x)$ if necessary we can further restrict to the case $A = B$.

We first assume that $x \in I_{A,n}^A$ for some n , and show that then also $x \in I_{B_i,n}^A$ for almost all B_i . If $x \in I_{A,n}^A$, there exists an A -representation W with $W^A = 0$, such that $e_W \cdot x$ lies in the ideal $L_A \cdot I_n$; see Lemma 4.8. For all i large enough we have $W^{B_i} = 0$, meaning that for these i the restriction $e_{\text{res}_{B_i}^A W}$ becomes invertible in $\Phi^{B_i} L$. It follows that $\text{res}_{B_i}^A x$ maps to the ideal generated by I_n in $\Phi^{B_i} L$, in other words x is contained in $I_{B_i,n}^A$.

For the other direction, we assume that x is not contained in $I_{A,n}^A$ for some n , and show that then also $x \notin I_{B_i,n}^A$ for almost all B_i . For this we recall from Remark 2.26 the construction of elements γ_j^V for every character $V \in G^*$ and $j \in \mathbb{N}$ satisfying the following three properties:

- (1) $\alpha^*(\gamma_j^V) = \gamma_j^{\alpha^*(V)}$ for every group homomorphism $\alpha: G' \rightarrow G$,
- (2) $\gamma_0^V = e_V$, and
- (3) The canonical map $L[e_V^{\pm 1}, \gamma_j^V \mid V \in G^* - \{\epsilon\}, j > 0] \rightarrow \Phi^G L$ is an isomorphism for all abelian compact Lie groups G .

Now, if x is not contained in $I_{A,n}^A$, it maps to a nontrivial element in

$$\Phi^A L / I_n = L / I_n [e_V^{\pm 1}, \gamma_j^V \mid V \in A^* - \{\epsilon\}, j > 0].$$

In other words, there exists an A -representation W with $W^A = 0$ and pairwise different nontrivial characters $V_1, \dots, V_k$ such that $e_W \cdot x \in L_A$ is a polynomial over L in the classes $\gamma_j^{V_l}$ for $l = 1, \dots, k$ and $j \geq 0$, not all of whose coefficients are contained in I_n . For all i large enough we have that

- (i) $W^{B_i} = 0$, and
- (ii) all the characters $V_1, \dots, V_k$ restrict to pairwise different and nontrivial characters of B_i .

It then follows that for these i the element $e_{\text{res}_{B_i}^A W} \cdot \text{res}_{B_i}^A x$ equals the corresponding polynomial in the classes $\gamma_j^{\text{res}_{B_i}^A V_l}$, implying that it maps to a nontrivial element in $\Phi^{B_i} L / I_n$. In other words, x is not contained in $I_{B_i,n}^A$ for i large enough. This finishes the proof. $\square$

Together with admissibility, this property characterizes the height functions of finite subsets of L_A :

Proposition 7.4 *Given a function $f: \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$, the following are equivalent:*

- (1) *There exists a finite subset $X \subseteq L_A$ such that $f = \text{ht}_X$.*
- (2) *The function f is admissible and locally constant.*

Proof We have already shown the implication 1. $\Rightarrow$ 2.

It remains to show that given a locally constant admissible function f , there exists a finite subset $X \subseteq L_A$ with $f = \text{ht}_X$. We start with the following claim: If f is admissible, then given any pair of subgroups $B, B' \subseteq A$, there exists an element $x_{B,B'} \in L_A$ such that $\text{ht}_{x_{B,B'}}(B) = f(B)$ and $\text{ht}_{x_{B,B'}}(B') \leq f(B')$.

To see this, we distinguish between three cases:

- (i) If B is not a subgroup of B' , we can choose a character $V \in A^*$ which restricts to the trivial character over B' but to a nontrivial character over B . Then $x_{B,B'} = e_V \cdot v_{f(B)}$ has the desired properties, since $\text{ht}_{e_V \cdot v_n}(B) = n$ and $\text{ht}_{e_V \cdot v_n}(B') = -1$. Here and in the following, we set $v_{-1} = 0$, $v_0 = p$ and $v_\infty = 1$.
- (ii) If B is a subgroup of B' with $\pi_0(B'/B)$ not a p -group, we choose a prime $q \neq p$ dividing the order of $\pi_0(B'/B)$ and a surjection $g: B' \rightarrow C_q$ containing B in the kernel. Then we set $y = v_{f(B)} \cdot g^*(\bar{v}_0^{(q)}) \in L_{B'}$, where $\bar{v}_0^{(q)} \in L_{C_q}$ is the element introduced in Section 5.1. Then $\bar{v}_0^{(q)}$ is an element of $I_{C_q,0}^{C_q}$ and its restriction to the trivial group is given by q and hence a unit. It follows that $\text{ht}_y(B') = -1$, since $\text{ht}_{g^*(\bar{v}_0^{(q)})}(B') = -1$. Moreover, $\text{ht}_y(B) = f(B)$, since $\text{res}_{B'}^B(y) = v_{f(B)} \cdot q$. Hence, we can set $x_{B,B'}$ to be any lift of y to an element of L_A .
- (iii) The remaining case is when B is a subgroup of B' with $\pi_0(B'/B)$ a p -group. Let r be the minimum of the p -rank of $\pi_0(B'/B)$ and the number $f(B) + 1$, and choose a surjection $g: B'/B \rightarrow C_p^r$. By Corollary 5.25 we know for $f(B) < \infty$ that there exist an element $x_{f(B)} \in L_{C_p^{f(B)+1}}$ such that $\text{ht}_{x_{f(B)}}(1) = f(B)$ and $\text{ht}_{x_{f(B)}}(C_p^{f(B)+1}) = -1$. (We set $x_{-1} = 0$.) We can choose an embedding $C_p^r \rightarrow C_p^{f(B)+1}$ and restrict $x_{f(B)}$ to an element $y \in L_{C_p^r}$. Then we have $\text{ht}_y(1) = f(B)$ and $\text{ht}_y(C_p^r) = f(B) - r$; see Corollary 5.25. If $f(B) = \infty$, choose $y = 1$. It follows that $\text{ht}_{g^*(y)}(B) = f(B)$ and $\text{ht}_{g^*(y)}(B') \leq f(B) - r \leq f(B')$, since f is admissible. Hence, $g^*(y)$ has the desired properties.

Now given any such pair (B, B') there exists an open neighborhood $U_{B'}$ of B' on which both $\text{ht}_{x_{B,B'}}$ and f are constant. The $U_{B'}$ for varying B' form an open cover of the compact space $\text{Sub}(A)$. Let $U_{B'_1}, \dots, U_{B'_k}$ be a finite subcover. We then set

$$x_B = x_{B,B'_1} \cdot x_{B,B'_2} \cdots x_{B,B'_k}$$

to be the product of the corresponding elements. For any closed subgroup B' we have

$$\text{ht}_{x_B}(B') = \min(\text{ht}_{x_{B,B'_j}}(B') \mid j = 1, \dots, k).$$

For $B' = B$ this gives $\text{ht}_x(B) = f(B)$, since $\text{ht}_{x_{B,B'_j}}(B) = f(B)$ for all j . Any B' is contained in $U_{B'_i}$ for some i , yielding

$$\text{ht}_{x_B}(B') \leq \text{ht}_{x_{B,B'_i}}(B') = \text{ht}_{x_{B,B'_i}}(B'_i) \leq f(B'_i) = f(B').$$

In summary, the height function ht_{x_B} is less than or equal to f everywhere and agrees with f at B itself. Once more we can apply that ht_{x_B} and f are locally constant to find that ht_{x_B} and f in fact agree on a neighborhood V_B of B . Letting B vary, this yields an open cover of $\text{Sub}(A)$, for which we can choose a finite subcover $V_{B_1}, \dots, V_{B_l}$. Hence, for every $B \in \text{Sub}(A)$, there exists some $i \in \{1, \dots, l\}$ such that $f(B) = \text{ht}_{x_{B_i}}$. Finally, we define X to be the set $\{x_{B_1}, \dots, x_{B_l}\}$. Since ht_X is given by the maximum of the functions ht_{B_i} , it follows that X has the desired property. $\square$

For the following theorem, we remind the reader of our convention that everything is implicitly localized at a fixed prime p .

Theorem 7.5 *The Zariski topology on $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ has as a basis the closed sets*

$$V_f = \{I_{B,n} \mid n > f(B)\}$$

for all locally constant, admissible functions $f : \mathrm{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$.

Proof A basis for the Zariski topology is given by the sets $V(X)$ for all finite subsets X of L_A . As we saw above, $V(X)$ is determined by its height function ht_X as

$$V(X) = \{I_{B,n} \mid n > \mathrm{ht}_X(B)\}.$$

By [Proposition 7.4](#), the functions $\mathrm{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ that occur as height functions of finite subsets are precisely the locally constant admissible functions, which finishes the proof. $\square$

8 Comparison with A -spectra

In this final section, we discuss the relationship of the algebraic results of the previous sections with the theory of A -spectra. Throughout, we will fix a prime p and implicitly localize at it.

8.1 The universal support theory via MU_A -homology

We begin by comparing our classification of invariant prime ideals with the Balmer spectrum of compact p -local A -spectra. We recall from [\[Balmer 2005\]](#) that a prime ideal $\mathfrak{p}$ of a tensor-triangulated category $\mathcal{C}$ is defined to be a thick tensor-ideal with the additional property that if $X \otimes Y$ is contained in $\mathfrak{p}$, then $X \in \mathfrak{p}$ or $Y \in \mathfrak{p}$. The set of all prime ideals assembles to a topological space $\mathrm{Spec}(\mathcal{C})$, the Balmer spectrum, with the topology generated by the closed sets $\mathrm{supp}(X) = \{\mathfrak{p} \mid X \notin \mathfrak{p}\}$ for all objects $X \in \mathcal{C}$.

Here, the support function $\mathrm{supp}(-)$, assigning a closed set of the Balmer spectrum to every object of $\mathcal{C}$, is the *universal support theory* in the sense of Balmer. This means that it is terminal among pairs (T, σ) of a topological space T and a function

$$\sigma : \mathrm{ob}(\mathcal{C}) \rightarrow \{\text{closed subsets of } T\}$$

satisfying

$$\sigma(0) = \emptyset, \quad \sigma(1) = T, \quad \sigma(X \oplus Y) = \sigma(X) \cup \sigma(Y), \quad \sigma(\Sigma X) = \sigma(X) \quad \text{and} \quad \sigma(X \otimes Y) = \sigma(X) \cap \sigma(Y)$$

for all $X, Y \in \mathrm{ob}(\mathcal{C})$, and $\sigma(X) \subseteq \sigma(Y) \cup \sigma(Z)$ whenever there exists a distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$. See [\[Balmer 2005\]](#) for more details.

In the case of compact p -local A -spectra Sp_A^c for an abelian compact Lie group A , the Balmer spectrum was computed in [Balmer and Sanders 2017; Barthel et al. 2019; 2020]. Given a closed subgroup B and $n \in \mathbb{N} \cup \{\infty\}$, one defines a prime ideal

$$P_{B,n}^A = \{X \in \mathrm{Sp}_A^c \mid K(n)_*(\Phi^B X) = 0\},$$

where $K(n)$ denotes the n^{th} Morava K -theory, and $\Phi^B X$ is the B -geometric fixed point spectrum of X , a compact p -local spectrum.

Theorem 8.1 [Balmer and Sanders 2017; Barthel et al. 2019; 2020] *The map*

$$\mathrm{Sub}(A) \times \overline{\mathbb{N}} \rightarrow \mathrm{Spec}(\mathrm{Sp}_A^c), \quad (B, n) \mapsto P_{B,n}^A,$$

defines a bijection. Moreover, the topology on $\mathrm{Spec}(\mathrm{Sp}_A^c)$ has a basis given by the closed sets defined by

$$\{P_{B,n}^A \mid n \geq f(B)\},$$

where f ranges through all admissible functions $\mathrm{Sub}(A) \rightarrow \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Here, “admissible” is meant in the sense of Definition 7.2. Hence, comparing to Theorem 7.5, we see that the assignment $I_{B,n}^A \mapsto P_{B,n}^A$ defines a homeomorphism from $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ to $\mathrm{Spec}(\mathrm{Sp}_A^c)$.⁶ The computation of $\mathrm{Spec}(\mathrm{Sp}_A^c)$ is also analogous to the one of $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ in the way that computing the underlying set is relatively straightforward (and is in fact known for all compact Lie groups), with most work going into understanding the topology.

Hence, we can view the universal support theory of Sp_A^c to take values in the invariant prime ideals of L_A . The goal of the remainder of this section is to construct this universal support theory more intrinsically using MU_A -homology and the structure of equivariant formal group laws described in this paper. The idea is the following: Given a compact p -local A -spectrum X , we can consider its equivariant complex bordism homology $(\underline{\mathrm{MU}}_A)_* X$. Here, underlining MU_A indicates that we take the A -Mackey functor valued homology of X , ie we record the collection of $(\mathrm{MU}_B)_*(\mathrm{res}_B^A X)$ for all closed subgroups B of A , together with restriction and transfer maps between them. We reiterate that we will always work at the fixed prime p and p -localize everything implicitly.

Since the coefficients $\pi_*^A \mathrm{MU}_A$ are isomorphic to the Lazard ring L_A and moreover the cooperations $\pi_*^A \mathrm{MU}_A \wedge \mathrm{MU}_A$ agree with S_A (see [Hausmann 2022, Theorem 5.52] and the discussion thereafter), the groups $(\mathrm{MU}_A)_* X$ form a graded A -Mackey functor in (L_A, S_A) -comodules. As such, we can take its support in the invariant prime ideals

$$\mathrm{supp}((\underline{\mathrm{MU}}_A)_* X) = \{I_{B,n}^A \mid ((\underline{\mathrm{MU}}_A)_* X)_{I_{B,n}^A} \neq 0\} \subseteq \mathrm{Spec}^{\mathrm{inv}}(L_A).$$

⁶The reader may have noticed that in this section our admissible functions f take values in $\overline{\mathbb{N}}$, while in the last section they took values in $\overline{\mathbb{N}}_-$. Likewise, we consider the condition $n \geq f(B)$ here, and considered $n > f(B)$ before. A shift by one shows that the topologies agree. The shift is caused by wanting v_n to have height n in the last section.

Remark 8.2 In general, $\text{supp}((\underline{\text{MU}}_A)_* X)$ is different from $\text{supp}((\text{MU}_A)_* X)$. Take for example $A = C_2$, $p = 2$ and $X = S^\sigma$, the circle with action given by reflection at a line. We have

$$\text{MU}_*(\text{res}_1^{C_2} S^\sigma)_{I_{1,0}} = \text{MU}_*(S^1)_{\mathbb{Q}} \neq 0.$$

The module $(\text{MU}_{C_2})_*(S^\sigma)_{I_{1,0}^{C_2}}$ is rational as well since $2 \notin I_{1,0}^{C_2}$. Rationally, $(\text{MU}_{C_2})_*(S^\sigma)$ splits into the coinvariants $(\text{MU}_{C_2})_*(S^\sigma)_{C_2} = 0$ and the geometric fixed points $(\text{MU}_{C_2})_*(S^\sigma)[e^{-1}]$, for e the Euler class of the unique nontrivial character. The element $\bar{v}_0$ from [Definition 5.21](#) becomes zero in the geometric fixed points, but restricts to 2 and is thus not in $I_{1,0}^{C_2}$; thus $(\text{MU}_{C_2})_*(S^\sigma)[e^{-1}]$ becomes zero as well after localization at $I_{1,0}^{C_2}$, and $(\text{MU}_{C_2})_*(S^\sigma)_{I_{1,0}^{C_2}} = 0$.

Remark 8.3 (transfer maps) The isomorphisms $L_A \cong \pi_*^A \text{MU}_A$ imply that on top of the contravariant restriction maps along group homomorphisms there also exist transfer maps $L_B \rightarrow L_A$ for inclusions $B \subseteq A$ of finite index. In other words, the collection of all equivariant Lazard rings L_A forms a “global Green functor” on the family of abelian compact Lie groups, in the sense of [\[Schwede 2018, Definition 5.1.3\]](#). Strickland [\[2011, Section 23\]](#) defines transfers for equivariant formal groups and shows that they agree with the topological transfers for complex oriented equivariant cohomology theories, in particular MU_A .

Using the methods from this paper, transfers on equivariant Lazard rings can be determined as follows: By Frobenius reciprocity, it suffices to compute $\text{tr}_B^A: L_B \rightarrow L_A$ on $1 \in L_B$ since the restriction is surjective. Inductively, we can further assume that $A/B \cong C_p$. Furthermore,

$$\text{tr}_B^A(1) = q^* \text{tr}_{\{1\}}^{C_p}(1) \quad \text{for } q: A \rightarrow A/B \cong C_p,$$

since transfers are compatible with inflation maps; see eg [\[Schwede 2018, Theorem 4.2.6 ff.\]](#). Hence it suffices to identify $\text{tr}_{\{1\}}^{C_p}(1)$. We claim that it equals $\bar{v}_0 \in L_{C_p}$. Indeed, any transfer maps to zero in the geometric fixed points and is thus an element of $I_{C_p,0}$. We know that $\bar{v}_0$ generates $I_{C_p,0}$. Hence, we can write $\text{tr}_{\{1\}}^{C_p}(1) = x \cdot \bar{v}_0$ for $x \in L_{C_p}$. Writing x as $x_0 + x' \cdot e$ for e a nontrivial Euler class and $x_0 \in L$, we get $\text{tr}_{\{1\}}^{C_p}(1) = x_0 \cdot \bar{v}_0$ since $e \cdot \bar{v}_0 = 0$ by the definition of $\bar{v}_0$. We obtain $x_0 \cdot p = \text{res}_{\{1\}}^{C_p} \text{tr}_{\{1\}}^{C_p}(1) = p$ since $\bar{v}_0$ restricts to p , and thus $x_0 = 1$ and $\text{tr}_{\{1\}}^{C_p}(1) = \bar{v}_0$.

We will now see that our notion of support is another model for the universal support theory on compact A -spectra.

We first note a major inconvenience: It is unclear whether $(\text{MU}_A)_* X$ is a finitely generated L_A -module, even for compact X . This is in contrast with the nonequivariant situation, where finite generation of $\text{MU}_* X$ for compact X follows from the fact that L is a polynomial ring and hence coherent. An analogous statement is unknown for equivariant Lazard rings. In particular, it is a priori unclear that $\text{supp}((\underline{\text{MU}}_A)_* X)$ is indeed a *closed* subset of $\text{Spec}^{\text{inv}}(L_A)$ and we have to prove this by hand; see [Proposition 8.5](#) below.

The following proposition gives the relationship between the MU_A -homology support theory described above and geometric fixed points.

Proposition 8.4 *Let B be a subgroup of A , $n \in \overline{\mathbb{N}}$ and X a compact A -spectrum. Then the following are equivalent:*

- (i) *The Mackey functor $((\underline{\mathrm{MU}}_A)_*(X))_{I_{B,n}^A}$ is trivial.*
- (ii) *The $(\mathrm{MU}_B)_*$ -module $((\mathrm{MU}_B)_*(\mathrm{res}_B^A X))_{I_{B,n}^B}$ is trivial.*
- (iii) *The B -geometric fixed points $\Phi^B(X)$ are of chromatic type $\geq n + 1$.*

We give two proofs of this proposition below, one using the results of [Balmer and Sanders 2017; Barthel et al. 2019; 2020] and one using the methods from this paper. The latter one is complicated by the fact that we don't know whether $(\mathrm{MU}_A)_*X$ is always finitely generated.

We have the following corollary:

Proposition 8.5 *Let X be a compact A -spectrum. Then its support $\mathrm{supp}((\underline{\mathrm{MU}}_A)_*X)$ is a closed subset of $\mathrm{Spec}^{\mathrm{inv}}(L_A)$.*

Moreover, the assignment

$$\mathrm{ob}(\mathrm{Sp}_A^c) \rightarrow \mathrm{Spec}^{\mathrm{inv}}(L_A), \quad X \mapsto \mathrm{supp}((\underline{\mathrm{MU}}_A)_*X),$$

is a support theory on compact A -spectra.

Proof We consider the type function

$$\mathrm{type}_X: \mathrm{Sub}(A) \rightarrow \overline{\mathbb{N}}, \quad X \mapsto \mathrm{type}(\Phi^B X),$$

which is locally constant by [Barthel et al. 2020, Proposition 4.3]. By Proposition 8.4, $I_{B,n}$ is an element of $\mathrm{supp}((\underline{\mathrm{MU}}_A)_*X)$ if and only if $\mathrm{type}_X(B) < n + 1$. Since the support $\mathrm{supp}((\underline{\mathrm{MU}}_A)_*X)$ is closed under inclusion, Theorem 5.1 implies that $I_{B',n} \in \mathrm{supp}((\underline{\mathrm{MU}}_A)_*X)$ if $I_{B,n-\mathrm{rk}_p(\pi_0(B/B'))} \in \mathrm{supp}((\underline{\mathrm{MU}}_A)_*X)$ for every p -toral inclusion $B' \subseteq B$. Thus type_X is admissible in the sense of Definition 7.2. Theorem 7.5 implies that $\mathrm{supp}((\underline{\mathrm{MU}}_A)_*X)$ is closed, as desired.

For the second part, all required properties of a support theory follow easily from exactness of localization $(-)_I_{B,n}^A$ except for the one on the interplay with smash products. This in turn follows from the third characterization in Proposition 8.4, since the type of a smash product of two compact spectra is the maximum of the two types. $\square$

By the universal property of the Balmer spectrum, we obtain a continuous map

$$\mathrm{Spec}^{\mathrm{inv}}(L_A) \rightarrow \mathrm{Spec}(\mathrm{Sp}_A^c).$$

Proposition 8.4 makes it clear that this map sends $I_{B,n}^A$ to $P_{B,n}^A$ and is hence bijective. Therefore we can conclude from the results of this paper that the topology on $\mathrm{Spec}(\mathrm{Sp}_A^c)$ is at least as coarse as the one on $\mathrm{Spec}^{\mathrm{inv}}(L_A)$. In other words, our proof of the existence of inclusions $I_{B,n} \subseteq I_{B',n'}$ gives another

proof of the analogous inclusion $P_{B',n'}^A \subseteq P_{B,n}^A$ on the topological side. The fact that there are no further topological inclusions requires additional arguments, namely the existence of compact A -spectra with “maximal type shifting behavior”; see [Barthel et al. 2019, Section 4] or [Kuhn and Lloyd 2024, Section 7]. Knowing this, we see that $X \mapsto \text{supp}((\underline{\text{MU}}_A)_* X)$ is a universal support theory on compact A -spectra.

It remains to give the proof of Proposition 8.4, which we will do in three steps:

(i) $\implies$ (ii) The L_A -action on $(\text{MU}_B)_*(X)$ factors through $\text{res}_B^A: L_A \rightarrow L_B$ and $I_{B,n}^A = (\text{res}_B^A)^{-1}(I_{B,n}^B)$, so it follows that we have an isomorphism $((\text{MU}_B)_*(X))_{I_{B,n}^A} \cong ((\text{MU}_B)_*(X))_{I_{B,n}^B}$. In particular, the vanishing of the entire Mackey-functor $(\underline{\text{MU}}_*(X))_{I_{B,n}^A}$ implies the vanishing at the subgroup B as a special case.

(ii) $\iff$ (iii) Note that $I_{B,n}^B$ contains none of the nontrivial Euler classes for B , hence the nontrivial Euler classes act invertibly on $((\text{MU}_B)_*(X))_{I_{B,n}^B}$. It follows that we have an isomorphism

$$((\text{MU}_B)_*(X))_{I_{B,n}^B} \cong ((\Phi^B \text{MU}_B)_* \Phi^B X)_{I_{B,n}^B} \cong \Phi^B ((\text{MU}_B)_{I_{B,n}^B})_* \Phi^B X.$$

Modulo I_n , the ring $\Phi^B ((\text{MU}_B)_{I_{B,n}^B})_*$ embeds into the field of fractions of $\Phi^B L/I_n$, which is nontrivial. Moreover, v_n is not contained in $I_{B,n}^B$ and hence is invertible in the localization. It follows that the localization $(\Phi^B \text{MU}_B)_{I_{B,n}^B}$ is an MU -algebra of height n , ie its vanishing detects compact spectra of type $\geq n+1$.

(iii) $\implies$ (i) Let X be a compact A -spectrum such that $\Phi^B(X)$ is of type $> n$. By the previous paragraph we know that $((\text{MU}_B)_*(X))_{I_{B,n}^A} = 0$, and we have to show that $((\text{MU}_{\tilde{B}})_*(X))_{I_{B,n}^A} = 0$ for all other closed subgroups $\tilde{B}$, too. We first assume that $\tilde{B}$ does not contain B . Then there exists a character $V \in A^*$ which restricts to the trivial character for $\tilde{B}$ but to a nontrivial character for B . It follows that e_V is not contained in $I_{B,n}^A$ and hence acts invertibly on $(\text{MU}_{\tilde{B}})_{I_{B,n}^A}$. On the other hand e_V restricts to 0 in $\text{MU}_{\tilde{B}}$, so it follows that $(\text{MU}_{\tilde{B}})_{I_{B,n}^A} = 0$ and in particular $((\text{MU}_{\tilde{B}})_*(X))_{I_{B,n}^A} = 0$, as desired. Hence we can assume that $\tilde{B}$ contains B as a subgroup. Since the statement then no longer depends on the ambient group, we can reduce to the case $\tilde{B} = A$.

Hence we need to show that $(\text{MU}_A)_*(X)_{I_{B,n}^A} = 0$. By induction on the pair (dimension, cardinality of $\pi_0(A/B)$) it follows that all the localizations at smaller intermediate groups $B \subseteq \tilde{B} \subseteq A$ vanish, and hence the homotopy groups of $(\text{MU}_A)_{I_{B,n}^A} \wedge X$ are concentrated at A . In particular this implies that the map $((\text{MU}_A)_{I_{B,n}^A})_* X \rightarrow ((\Phi^A \text{MU}_A)_{I_{B,n}^A})_* \Phi^A X$ is an isomorphism and all Euler classes e_V for nontrivial characters act invertibly on $((\text{MU}_A)_{I_{B,n}^A})_* X$. Now, if V restricts to the trivial character in B^* (such a V always exists since we can assume that B is a proper subgroup of A), its Euler class e_V lies in the maximal ideal $I_{B,n}^A$ of $((\text{MU}_A)_{I_{B,n}^A})_*$.

If we knew that $((\text{MU}_A)_{I_{B,n}^A})_* X$ is finitely generated over $(\text{MU}_A)_*$ we could apply Nakayama’s lemma to see directly that $((\text{MU}_A)_{I_{B,n}^A})_* X = 0$, as desired. Since we do not know this, we have to argue differently: By Corollary 5.2, we know that $\Phi^A((\text{MU}_A)_{I_{B,n}^A})_*$ is trivial if $\pi_0(A/B)$ is not a p -group, and is of height $n - \text{rank}_p(\pi_0(A/B))$ otherwise (where negative heights again mean that the theory is trivial). Hence,

what we want to show is that if $\Phi^B X$ is of type $\geq n+1$, then $\Phi^A X$ is of type $\geq n+1 - \text{rank}_p(\pi_0(A/B))$. Indeed, this implies that $((\text{MU}_A)_{I_{B,n}^A})_* X \cong ((\Phi^A \text{MU}_A)_{I_{B,n}^A})_* \Phi^A X$ is trivial.

This precise statement about $\Phi^A X$ is one of the main results of [Barthel et al. 2020], building on [Balmer and Sanders 2017] and [Barthel et al. 2019]. Hence, using these results, Proposition 8.4 follows. Alternatively, rather than importing we can reprove the above statement using the methods from this paper. Note that by induction on the rank of A/B and by replacing X by the compact A/B -spectrum $\Phi^B X$ it suffices to show two special cases:

- (1) If X is a compact C_{p^k} -spectrum of underlying type $\geq n+1$, then the type of $\Phi^{C_{p^k}} X$ is at least n .
- (2) If X is a compact T -spectrum of underlying type $\geq n+1$, then the type of $\Phi^T X$ is also $\geq n+1$.

For (1) it suffices to find a complex oriented theory E of height n (meaning here that E_* vanishes on a compact spectrum X if and only if X is of type $\geq n+1$) such that $\Phi^{C_{p^k}} \underline{E}$ is of height $\geq n-1$, where $\underline{E} = F(EG_+, E)$ denotes the Borel theory associated to E (see the proofs of [Barthel et al. 2019, Corollary 3.12] or [Barthel et al. 2020, Proposition 6.10] for details on this argument). In [Barthel et al. 2019] it was shown that Morava E -theory $E = E_n$ has this property, building on results of Hopkins, Kuhn and Ravenel [Hopkins et al. 2000] on the p -divisible group associated to E_n and extending earlier work of Greenlees, Hovey and Sadofsky. Using Remark 5.15 we obtain similar results more generally:

Proposition 8.6 *Let E be any complex oriented ring spectrum of height n which is Landweber exact over $\text{MU}/I_{n-1} = \text{MU}/(v_0, \dots, v_{n-2})$, ie I_{n-1} acts trivially on $\pi_* E$, $v_{n-1} \in \pi_* E$ is a regular element and v_n is a unit modulo v_{n-1} . Then $\Phi^{C_{p^k}} \underline{E}$ is of height $n-1$.*

Proof We apply Remark 5.15 and check that its assumptions are satisfied. We have $(\underline{E}^T)_* = E_*[[e]]$, with Euler classes e_n given by the n -series $[n]_F(e)$ for the formal group law associated to E . If n is a power of p , the leading term of this Euler class is a power of v_{n-1} , which we assumed to be regular. Modulo v_{n-1} , the leading term becomes v_n which is a unit since E/v_{n-1} is of height n . Hence by Remark 5.15 we find that the pushforward of the element $\psi_{p^k}^{(n-1)}$ to $E_*[[e]]$, ie the element $\sum_{i=0}^{\infty} a_i [p^{k-1}]_F(e)$ for $\sum_{i=0}^{\infty} a_i e^i = [p]_F(e)/e^{p^{n-1}}$, generates the kernel of the composite

$$E_*[[e]] \rightarrow E_*[[e]]/[p^k]_F(e) = (\underline{E}^{C_{p^k}})_* \rightarrow (\Phi^{C_{p^k}} \underline{E})_*.$$

The leading term of $\psi_{p^k}^{(n-1)}$ equals v_{n-1} , which is not a unit. Hence $\Phi^{C_{p^k}} \underline{E}$ is nontrivial and since I_{n-1} acts trivially it must be of height $\geq n-1$. Furthermore, reducing modulo v_{n-1} the leading coefficient of $\psi_{p^k}^{(n-1)}$ equals a power of v_n . Since v_n is a unit modulo v_{n-1} , it follows that so is $\psi_{p^k}^{(n-1)}$. Therefore $(\Phi^{C_{p^k}} \underline{E})_*/v_{n-1}$ is trivial as an algebra over $E_*[[e]]/(v_{n-1}, \psi_{p^k}^{(n-1)}) = 0$, and hence $\Phi^{C_{p^k}} \underline{E}$ is of height exactly $n-1$. $\square$

Example 8.7 Given any Landweber exact complex oriented ring spectrum E of height n , its quotient E/I_{n-1} satisfies the assumptions of the previous proposition. It follows that also in this case the height of $\Phi^{C_{p^k}} E$ equals $n-1$. For example this applies to Johnson–Wilson spectra, to $\text{MU}[v_n^{-1}]$ or to Morava E -theories.

Similarly, for (2) one needs to find a complex oriented theory E of height n such that $\Phi^T \underline{E}$ is also of height n . This is more elementary and satisfied by any p -local complex oriented theory E of height n . To see this, note again that the Euler classes e_n are given by the n -series $[n]_F(e) \in E_*[[e]]$. Writing $n = p^k \cdot m$ with m coprime to p , we find that $[n]_F(e)$ is a unit multiple of $[p^k]_F(e)$. Modulo I_n , for $k > 0$ the leading term of $[p^k]_F(e)$ is a power of v_n , which by assumption is a unit in E_*/I_n . It follows that, modulo I_n , the coefficients of $\Phi^T \underline{E}$ are given by $E_*/I_n((e))$, which is always nontrivial when E_*/I_n is.

8.2 Change of groups and the structure of $\mathcal{M}_{\text{FG}}^A$

For any A -spectrum X , the groups $\text{MU}_{2*}^A(X)$ come equipped with the structure of a graded (L_A, S_A) -comodule. Since the stack $\mathcal{M}_{\text{FG}}^A$ is the stack associated to this graded Hopf algebroid, we obtain an associated quasicoherent sheaf $\mathcal{F}^A(X)$ on $\mathcal{M}_{\text{FG}}^A$; see [Meier and Ozornova 2020, Proposition 4.3]. This is the 0th graded piece of a $\text{QCoh}(\mathcal{M}_{\text{FG}}^A)$ -valued homology theory on A -spectra, whose i^{th} piece $\mathcal{F}_i^A(X)$ is given by $\text{MU}_{2*+i}^A(X)$. We end this section with a closer look at how the structure of $\mathcal{M}_{\text{FG}}^A$ relates to this homology theory.

Recall (Proposition 3.12) that for a closed subgroup $B \subseteq A$, there is an open immersion $j: \mathcal{M}_{\text{FG}}^{A/B} \rightarrow \mathcal{M}_{\text{FG}}^A$ and a closed immersion $i: \mathcal{M}_{\text{FG}}^B \rightarrow \mathcal{M}_{\text{FG}}^A$. We obtain corresponding adjunctions

$$\text{QCoh}(\mathcal{M}_{\text{FG}}^A) \xrightleftharpoons[j_*]{j^*} \text{QCoh}(\mathcal{M}_{\text{FG}}^{A/B}) \quad \text{and} \quad \text{QCoh}(\mathcal{M}_{\text{FG}}^A) \xrightleftharpoons[i_*]{i^*} \text{QCoh}(\mathcal{M}_{\text{FG}}^B).$$

Believing that the structure of $\mathcal{M}_{\text{FG}}^A$ dictates the structure of the ∞ -category Sp_A of A -spectra, we expect a relation to the adjunctions

$$\text{Sp}_A \xrightleftharpoons[P_{A/B}^*]{\underline{\Phi}^B} \text{Sp}_{A/B} \quad \text{and} \quad \text{Sp}_A \xrightleftharpoons[\text{coind}_B^A]{\text{res}_B^A} \text{Sp}_B.$$

Here, denoting by $\mathcal{F}[B]$ the family of subgroups of A not containing B and by $q: A \rightarrow A/B$ the projection, we define $\underline{\Phi}^B(X) = (\tilde{E}\mathcal{F}[B] \wedge X)^B$ and $P_{A/B}^*(Y) = \tilde{E}\mathcal{F}[B] \wedge q^*X$. Note that the definition of $\underline{\Phi}^B$ is made so that its underlying spectrum is Φ^B and more generally $\Phi^{C/B} \underline{\Phi}^B \simeq \Phi^C$ for every $B \subseteq C \subseteq A$. For more details on the first adjunction, see [Lewis et al. 1986, Section II.9], [Hill 2012, Section 4.1] and [Meier et al. 2023, Section 2.2]. For the benefit of the reader, we give a brief sketch of its basic properties: The adjunction between q^* and $(-)^H$ induces maps

$$\text{id} \rightarrow \underline{\Phi}^B P_{A/B}^* \quad \text{and} \quad P_{A/B}^* \underline{\Phi}^B \rightarrow \tilde{E}\mathcal{F}[B] \wedge (-),$$

which can be checked to be equivalences on geometric fixed points. The inverses

$$\underline{\Phi}^B P_{A/B}^* \xrightarrow{\sim} \text{id} \quad \text{and} \quad \text{id} \rightarrow \tilde{E}\mathcal{F}[B] \wedge (-) \xrightarrow{\sim} P_{A/B}^* \underline{\Phi}^B$$

form the counit and unit of the adjunction. In particular, $P_{A/B}^*$ is fully faithful and its image agrees with that of $\tilde{E}\mathcal{F}[B] \wedge$. Since smashing with $\tilde{E}\mathcal{F}[H]$ is idempotent and hence symmetric monoidal, $P_{A/B}^*$ is symmetric monoidal as well and so is $\underline{\Phi}^B$.

Proposition 8.8 For every abelian compact Lie group A and closed subgroup $B \subseteq A$, the diagram

$$\begin{array}{ccc} \mathrm{Sp}_A & \xrightleftharpoons[\mathcal{P}_{A/B}^*]{\Phi^B} & \mathrm{Sp}_{A/B} \\ \downarrow \mathcal{F}^A & & \downarrow \mathcal{F}^{A/B} \\ \mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}^A) & \xrightleftharpoons[j_*]{j^*} & \mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}^{A/B}) \end{array}$$

commutes, ie there are natural isomorphisms $j^* \mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(\Phi^B X)$ for $X \in \mathrm{Sp}_A$ and $j_* \mathcal{F}^{A/B}(Y) \cong \mathcal{F}^A(\mathcal{P}_{A/B}^* Y)$ for $Y \in \mathrm{Sp}_{A/B}$.

Moreover, there is a natural isomorphism $i_* \mathcal{F}^B(Z) \cong \mathcal{F}^A(\mathrm{coind}_B^A Z)$ for $Z \in \mathrm{Sp}_B$.

No such isomorphism can be expected for i^* and res_B^A in general. One reason is that i^* is not flat, but even a spectral sequence relating i^* and res_B^A seems not to exist for A not a torus for reasons related to [Remark 8.2](#).

Before proving the proposition we need a lemma, in which we will use the Hopf algebroid $(\Phi^B L_A, \Phi^B S_A)$. Here $\Phi^B L_A$ is obtained from L_A by inverting e_V for all $V \notin \mathrm{im}((A/B)^* \rightarrow A^*)$, and the ring $\Phi^B S_A$ is defined as $\Phi^B L_A \otimes_{L_A} S_A \otimes_{L_A} \Phi^B L_A$. This classifies strict isomorphism between equivariant formal group laws where the relevant Euler classes are invertible on source and target. Since the invertibility of Euler classes only depends on the underlying equivariant formal group and not on the choice of coordinate, this simplifies to $S_A \otimes_{L_A} \Phi^B L_A$.

Lemma 8.9 For $Y \in \mathrm{Sp}_B$ and $q: A \rightarrow A/B$ the projection, there are natural isomorphisms

$$\Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_*(q^* Y) \xrightarrow{\cong} \Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_*(\mathcal{P}_{A/B}^* Y) \xrightarrow{\cong} (\Phi^B \mathrm{MU}_A)_*(Y)$$

and

$$\Phi^B L_A \otimes_{L_{A/B}} (\mathrm{MU}_{A/B})_*(Y) \xrightarrow{\cong} (\Phi^B \mathrm{MU}_A)_*(Y)$$

of graded $(\Phi^B L_A, \Phi^B S_A)$ -comodules, where the map $L_{A/B} \rightarrow \Phi^B L_A$ is defined as the composite $L_{A/B} \xrightarrow{q^*} L_A \rightarrow \Phi^B L_A$.

Proof A model for $\tilde{E}\mathcal{F}[B]$ is given by $S^{\infty W}$ for W the sum of all characters $V \in \mathcal{V}$, for $\mathcal{V}$ the set of characters V not restricting to 1 in B or, equivalently, $V \notin \mathrm{im}((A/B)^* \rightarrow A^*)$. Indeed, $W^C = 0$ for $C \subseteq A$ if and only if $B \subseteq C$, as this is equivalent to none of the $V \in \mathcal{V}$ restricting to 1 in C .

In other words, smashing with $\tilde{E}\mathcal{F}[B]$ is the same as inverting all the maps $S^0 \rightarrow S^V$ for $V \in \mathcal{V}$. For an MU_A -module, this is equivalent to inverting e_V for $V \in \mathcal{V}$.

We have

$$\begin{aligned} \Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_*(q^* Y) &\cong (\mathrm{MU}_A)_*(q^* Y)[e_V^{-1} : V \in \mathcal{V}] \cong \pi_*(\mathrm{MU}_A \wedge q^* Y \wedge \tilde{E}\mathcal{F}[B])^A \\ &\cong \pi_*(\Phi^B(\mathrm{MU}_A \wedge \mathcal{P}_{A/B}^* Y))^{A/B} \cong \pi_*(\Phi^B \mathrm{MU}_A \wedge Y)^{A/B} \\ &\cong (\Phi^B \mathrm{MU}_A)_*(Y). \end{aligned}$$

Replacing q^*Y by $P_{A/B}^*Y$ in the chain of isomorphisms above yields a similar chain of isomorphisms. All the isomorphisms are isomorphisms of comodules since the isomorphisms are natural in the MU_A -variable and we can plug into this variable the left and right unit $\mathrm{MU}_A \rightarrow \mathrm{MU}_A \wedge \mathrm{MU}_A$.

To construct the second isomorphism, note that as part of the global structure of equivariant MU , there is a ring map $q^*\mathrm{MU}_{A/B} \rightarrow \mathrm{MU}_A$; see [Linskens et al. 2025]. Applying $\Phi^B = (- \wedge \tilde{E}\mathcal{F}[H])^B$ yields a ring map $\mathrm{MU}_{A/B} \rightarrow \Phi^B \mathrm{MU}_A$. This induces a morphism

$$\Phi^B L_A \otimes_{L_{A/B}} (\mathrm{MU}_{A/B})^*(Y) \rightarrow (\Phi^B \mathrm{MU}_A)^*(Y).$$

It is enough to show that this is an isomorphism for finite Y and hence for $Y = (A/B)/(B'/B)_+ \cong A/B'_+$ for $B \subseteq B' \subseteq A$. In this case, the map becomes

$$(8.10) \quad \Phi^B L_A \otimes_{L_{A/B}} L_{B'/B} \rightarrow (\Phi^B \mathrm{MU}_A)^*(A/B'_+).$$

The natural map

$$L_A \otimes_{L_{A/B}} L_{B'/B} \rightarrow L_{B'}$$

is an isomorphism since $L_A \rightarrow L_{B'}$ is a surjection with kernel generated by the Euler classes e_V for those $V \in A^*$ restricting trivially to B' ; these are exactly the images of the Euler classes e_W for those $W \in (A/B)^*$ restricting trivially to B'/B .

Thus, (8.10) becomes

$$\Phi^B L_A \otimes_{L_A} \mathrm{MU}_A^*(A/B'_+) \cong \Phi^B L_A \otimes_{L_A} L_{B'} \rightarrow (\Phi^B \mathrm{MU}_A)^*(A/B'_+),$$

which is an isomorphism by the first part. Similar to the first part, all isomorphisms are isomorphisms of comodules again. $\square$

Proof of Proposition 8.8 We establish first the isomorphism $j^*\mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(\Phi^B X)$ for A -spectra X . Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec} L_A & \longleftarrow & \mathrm{Spec} \Phi^B L_A & \longrightarrow & \mathrm{Spec} L_{A/B} \\ \downarrow & \lrcorner & \searrow \varphi & & \swarrow \\ \mathcal{M}_{\mathrm{FG}}^A & \xleftarrow{j} & \mathcal{M}_{\mathrm{FG}}^{A/B} & & \end{array}$$

where the down-right arrow comes from applying q_* to an A -equivariant formal group classified by a morphism to $\mathrm{Spec} \Phi^B L_A$, and the right-pointing horizontal morphisms come from the composition $L_{A/B} \xrightarrow{q^*} L_A \rightarrow \Phi^B L_A$. The square is a pullback square by Proposition 3.12. Thus φ is faithfully flat and hence φ^* induces an equivalence of $\mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}}^{A/B})$ to graded $(\Phi^B L_A, \Phi^B S_A)$ -comodules.

The comodule corresponding to $j^*\mathcal{F}_X^A$ is $\Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_{2*} X$. As in (the proof of) the first isomorphism in Lemma 8.9, we observe that this is isomorphic to

$$\begin{aligned} \Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_{2*}(X \wedge \tilde{E}\mathcal{F}[B]) &\cong \Phi^B L_A \otimes_{L_A} (\mathrm{MU}_A)_{2*}(P_{A/B}^* \Phi^B X) \\ &\cong (\Phi^B \mathrm{MU}_A)_{2*}(\Phi^B X). \end{aligned}$$

By the second isomorphism in [Lemma 8.9](#), this is isomorphic to the comodule corresponding to $\mathcal{F}_{\underline{\Phi}^B X}^{A/B}$, ie to $\underline{\Phi}^B L_A \otimes_{L_{A/B}} (\text{MU}_{A/B})_{2*}(\underline{\Phi}^B X)$. This establishes the first claimed isomorphism of sheaves.

Since the counit $\underline{\Phi}^B P_{A/B}^* \rightarrow \text{id}_{\text{Sp}_{A/B}}$ is an equivalence, we can assume for the proof of $j_* \mathcal{F}^{A/B}(Y) \cong \mathcal{F}^A(P_{A/B}^* Y)$ for $Y \in \text{Sp}_{A/B}$ that $Y = \underline{\Phi}^B X$ for some $X \in \text{Sp}_A$, and we obtain from the first isomorphism in this case a natural isomorphism

$$j^* j_* \mathcal{F}^{A/B}(Y) \cong j^* j_* j^* \mathcal{F}^A(X) \cong j^* \mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(Y) \cong j^* \mathcal{F}^A(P_{A/B}^* Y).$$

The Euler classes for characters $V \notin \text{im}((A/B)^* \rightarrow A^*)$ act invertibly on $\mathcal{F}^A(P_{A/B}^* Y)$, and [Proposition 3.12](#) implies that $\mathcal{F}^A(P_{A/B}^* Y)$ is in the image of j_* . Since the counit

$$j^* j_* \rightarrow \text{id}_{\text{QCoh}(\mathcal{M}_{\text{FG}}^{A/B})}$$

is an isomorphism, the claimed isomorphism follows.

Lastly, we show $i_* \mathcal{F}^B(Z) \cong \mathcal{F}^A(\text{coind}_B^A Z)$ for $Z \in \text{Sp}_B$. This follows from the chain

$$\begin{aligned} (\text{MU}_A)_{2*}(\text{coind}_B^A Z) &\cong \pi_{2*}^A(\text{MU}_A \otimes \text{coind}_B^A Z) \cong \pi_{2*}^A(\text{coind}_B^A(\text{MU}_B \otimes Z)) \\ &\cong \pi_{2*}^B(\text{MU}_B \otimes Z) \cong \text{MU}_{2*}^B(Z) \end{aligned}$$

of isomorphisms of (L_A, S_A) -comodules. $\square$

We end this section by connecting the stacky point of view with support theories. For a compact A -spectrum X , we may consider the support $\text{supp } \mathcal{F}_i^A(X) \subseteq |\mathcal{M}_{\text{FG}}^A|$, corresponding to those $x: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}}^A$ such that $x^* \mathcal{F}_i^A(X)$ is nontrivial. As remarked above, in general the sheaves $\mathcal{F}_i^A(X)$ do not play well with restrictions. Thus, for an A -spectrum X , one should consider all $\mathcal{F}_i^B(\text{res}_B^A X) \in \text{QCoh}(\mathcal{M}_{\text{FG}}^B)$ for $B \subseteq A$ in conjunction. Using the closed embeddings $\mathcal{M}_{\text{FG}}^B \hookrightarrow \mathcal{M}_{\text{FG}}^A$, the correct notion of support of $\mathcal{F}_*^A(X)$ becomes thus $\text{supp } \mathcal{F}_*(X) = \bigcup_{B \subseteq A} \text{supp } \mathcal{F}_*^B(\text{res}_B^A X)$.

Proposition 8.11 *For any compact A -spectrum X , the subsets*

$$\text{supp } \mathcal{F}_*(X) \subseteq |\mathcal{M}_{\text{FG}}^A| \quad \text{and} \quad \text{supp}((\text{MU}_A)_* X) \subseteq \text{Spec}^{\text{inv}}(L_A)$$

correspond to each other under the bijection from [Theorem 4.7](#). Thus, $\text{supp } \mathcal{F}_*(-)$ is a universal support theory as well.

Proof Consider a point $[x: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}}^A] \in |\mathcal{M}_{\text{FG}}^A|$ corresponding to a pair (B, n) . Then by [Proposition 3.12](#) and [Remark 4.5](#), x factors as $\bar{x}: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}} = \mathcal{M}_{\text{FG}}^{B/B}$ composed with $\mathcal{M}_{\text{FG}}^{B/B} \xrightarrow{j} \mathcal{M}_{\text{FG}}^B \xrightarrow{i} \mathcal{M}_{\text{FG}}^A$. By [Proposition 8.8](#), the pullback functor j^* corresponds to $\underline{\Phi}^B$. Thus, $x \in \text{supp } \mathcal{F}_*^B(\text{res}_B^A X)$ if and only if $\bar{x}^* \mathcal{F}_*^{\{1\}}(\Phi^B X) \neq 0$, ie if $\Phi^B X$ has chromatic type at least n . We have seen in [Proposition 8.4](#), this happens if and only if $I_{B,n}^A \in \text{supp}((\text{MU}_A)_* X)$.

Now suppose $I_{B,n}^A \notin \text{supp}((\text{MU}_A)_* X)$ and let $B \subseteq \tilde{B} \subseteq A$. Thus $(\text{MU}_{\tilde{B}})_*(\text{res}_{\tilde{B}}^A(X))_{I_{B,n}^{\tilde{B}}}$ vanishes. Since $[x] \in |\mathcal{M}_{\text{FG}}^{\tilde{B}}|$ is the image of the prime ideal $I_{B,n}^{\tilde{B}} \in \text{Spec } L_{\tilde{B}} \cong \text{Spec}(\text{MU}_{\tilde{B}})_*$, we see by [Lemma 2.6](#) that $x: \text{Spec } k \rightarrow \mathcal{M}_{\text{FG}}^{\tilde{B}}$ factors over $\tilde{x}: \text{Spec } k \rightarrow \text{Spec}(L_{\tilde{B}})_{I_{B,n}^{\tilde{B}}}$. Thus, we can identify $x^* \mathcal{F}_*^{\tilde{B}}(X)$ with $\tilde{x}^*(\text{MU}_{\tilde{B}})_*(\text{res}_{\tilde{B}}^A(X))_{I_{B,n}^{\tilde{B}}} = 0$, and $x \notin \text{supp } \mathcal{F}_*(X)$. $\square$

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A Weyl’s law for singular Riemannian foliations with applications to invariant theory

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We prove a version of Weyl’s law for the basic spectrum of a closed singular Riemannian foliation $(M, \mathcal{F})$ with basic mean curvature. In the special case of $M = \mathbb{S}^n$, this gives an explicit formula for the volume of the leaf space $\mathbb{S}^n/\mathcal{F}$ in terms of the algebra of basic polynomials. In particular, $\text{Vol}(\mathbb{S}^n/\mathcal{F})$ is a rational multiple of $\text{Vol}(\mathbb{S}^m)$, where $m = \dim(\mathbb{S}^n/\mathcal{F})$.

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1 Introduction

Given a compact, connected n -dimensional Riemannian manifold M , the Laplacian operator Δ is a self-adjoint operator with a discrete spectrum $0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$. While the eigenvalues themselves strongly depend on the geometry of M , the celebrated Weyl’s theorem states that the *growth* of the eigenvalues only depends on the dimension and volume of M : in other words, letting $S(t) = \max\{k \mid \tilde{\lambda}_k < t\}$ denote the counting function for the eigenvalues of Δ , we have that

$$S(t) \sim \frac{\omega_n}{(2\pi)^n} \text{Vol}(M) t^{n/2},$$

where ω_n denotes the volume of the unit n -ball, and $f \sim g$ means that $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$. Weyl’s law was generalized by Brüning and Heintze [1978/79] and Donnelly [1978] to the context of compact manifolds equipped with an isometric action by a compact Lie group G ; in their version, the growth of the counting function for the G -invariant eigenfunctions of Δ depends on the volume and dimension of the orbit space M/G .

Singular Riemannian foliations $(M, \mathcal{F})$ are, roughly speaking, partitions of a Riemannian manifold M into connected submanifolds (called *leaves*) locally equidistant to one another; see Section 2.1.1 for a more detailed definition. They generalize the decomposition into orbits of isometric actions of connected groups, which we also refer to as *homogeneous* singular Riemannian foliations. Singular Riemannian foliations also appear as natural generalizations of isoparametric foliations in symmetric spaces, Riemannian submersions, and the dual foliation in noncompact manifolds with nonnegative curvature. Recently, the understanding of singular Riemannian foliations has seen rapid development, and despite providing a

proper generalization of group actions (that is, there exist infinitely many nonhomogeneous examples, even in spheres; see [Radeschi 2014]), it has been shown that they retain a lot of the rigid properties that isometric actions enjoy; see [Galaz-Garcia and Radeschi 2015; Ge and Radeschi 2015; Mendes and Radeschi 2019]. This is especially true when M is a round sphere $\mathbb{S}^n$, which appears as an “infinitesimal model” for general singular Riemannian foliations around a point; see [Lytchak and Radeschi 2018; Mendes and Radeschi 2020b].

Given a singular Riemannian foliation $(M, \mathcal{F})$ with closed leaves, one can define a *leaf space* $M/\mathcal{F}$ which is an analogue to the orbit space, and the algebra of *basic functions* defined as the functions which are constants along the leaves. If $\mathcal{F}$ has *basic mean curvature*, in the sense that the mean curvature vectors of the leaves of $\mathcal{F}$ project to a vector field in $M/\mathcal{F}$, then the Laplace operator restricts to a self-adjoint operator on the space of basic functions, with a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$; see Section 2.1.3. The first main result can then be described as Weyl’s law for closed singular Riemannian foliations:

Theorem A *Let M be a compact Riemannian manifold and $\mathcal{F}$ a singular Riemannian foliation of M with closed leaves and basic mean curvature. Let $N(t) = \max\{i \mid \lambda_i < t\}$ be the counting function for the basic spectrum, and $X = M/\mathcal{F}$ be the leaf space. Then*

$$N(t) \sim \frac{\text{vol}(X)\omega_m}{(2\pi)^m} t^{m/2} \quad \text{as } t \rightarrow \infty,$$

where $m = \dim X$ and $\omega_m = \text{vol } \mathbb{D}^m$.

The condition of basic mean curvature is always satisfied when $\mathcal{F}$ is homogeneous or when M is the Euclidean space, a round sphere [Alexandrino and Radeschi 2015], as well as complex and quaternionic projective spaces [Alexandrino and Radeschi 2016, page 2205].

In the homogeneous case, Theorem A follows as a special case of Corollary 3.5 of Brüning and Heintze [1978/79] and the Main Theorem of Donnelly [1978]. In the *regular case*, namely when all leaves have the same dimension, this result was proved by Richardson [2010] without the assumptions of closed leaves and basic mean curvature.

A surprising result we prove using Weyl’s law is the following:

Theorem B *If $(\mathbb{S}^n, \mathcal{F})$ is a closed singular Riemannian foliation (resp. $G \rightarrow O(n+1)$ is an orthogonal representation) with m -dimensional leaf space (resp. orbit space), then $\text{Vol}(\mathbb{S}^n/\mathcal{F})$ (resp. $\text{Vol}(\mathbb{S}^n/G)$) is a rational multiple of $\text{Vol}(\mathbb{S}^m)$.*

To the best of the authors’ knowledge, Theorem B is novel even in the homogeneous case, even though it is clearly true when G is finite. Since the quotients $\mathbb{S}^n/\mathcal{F}$ can be thought as singular spaces all of whose geodesics are closed (in the sense of quotient geodesics from [Mendes and Radeschi 2020a, Section 3.4]), Theorem B is similar in spirit to [Weinstein 1977, Theorem 3] which applies to *manifolds* all of whose geodesic are closed.

In the case of foliations, [Theorem B](#) follows immediately from [Theorem C](#) below, in which the rational number $\text{Vol}(\mathbb{S}^n/\mathcal{F})/\text{Vol}(\mathbb{S}^m)$ is computed explicitly in terms of the algebraic structure of $(\mathbb{S}^n, \mathcal{F})$. Recall that a singular Riemannian foliation $(\mathbb{S}^n, \mathcal{F})$ extends to an infinitesimal foliation $(\mathbb{R}^{n+1}, \mathcal{F})$, ie a singular Riemannian foliation with the origin as a leaf. In this case, it was shown in [\[Mendes and Radeschi 2020a\]](#) that the algebra A of basic polynomials is a so-called *Laplacian algebra* (ie a graded algebra preserved by the multiplication by $r^2 = \sum_i x_i^2$ and by the Laplacian $\Delta = \sum_i \partial^2/\partial x_i^2$) which completely determines $\mathcal{F}$. Furthermore, there is a dictionary between the geometric properties of an infinitesimal foliation and the algebraic properties of its Laplacian algebra A . The next main theorem adds a line to this dictionary:

Theorem C *Let $(V, \mathcal{F})$, with $\dim V = n + 1$, be an infinitesimal foliation with closed leaves, and let $A = \bigoplus_i A_i \subset \mathbb{R}[V]$ be the associated Laplacian algebra of basic polynomials. Denote its Hilbert series by $H(z) = \sum_i (\dim A_i) z^i$. Let $\mathbb{S}V$ denote the unit sphere in V , and m the dimension of the leaf space $X = \mathbb{S}V/\mathcal{F}$.*

Then:

- (a) *The Laurent series of $H(z)$ at $z = 1$ has the form*

$$H(z) = \frac{\text{Vol}(X)}{\text{Vol}(\mathbb{S}^m)} (1-z)^{-m-1} + (\dots)(1-z)^{-m} + \dots.$$

- (b) *A is a **free module over a free subalgebra**, ie $A = \bigoplus_{i=1}^{\ell} \mathbb{R}[f_0, \dots, f_m] g_i$ for some homogeneous $f_0, \dots, f_m, g_1, \dots, g_{\ell} \in A$, and*

$$\frac{\text{Vol}(X)}{\text{Vol}(\mathbb{S}^m)} = \frac{\ell}{\deg f_0 \cdots \deg f_m}.$$

The analogue of [Theorem C](#) for representations of disconnected groups also holds; see [Proposition 40](#).

Let us comment on the main aspects of the proofs. [Theorem A](#) follows a framework similar to the homogeneous version by Brüning and Heintze [\[1978/79\]](#). In their proof of Weyl's law, however, results from transformation groups and representation theory were used in a fundamental way in many places, which are not available in the foliated case. Some of these obstacles have been overcome using recent results in the literature of singular Riemannian foliations, most notably the averaging operator introduced by Lytchak and Radeschi [\[2018\]](#), and the foliated slice theorem of Mendes and Radeschi [\[2019\]](#). Others require new uniform estimates on the geometry of leaves (see [Section 3](#)), which culminate in the following result:

Theorem D *Given a compact connected Riemannian manifold M and a singular Riemannian foliation $\mathcal{F}$ with closed leaves, there exists a constant $C = C(M, \mathcal{F})$ such that, for every $p \in M$, and every $s > 0$, one has*

$$\int_{L_p} \exp\left(-\frac{d_M(p, q)^2}{s}\right) dq \leq C s^{\frac{1}{2} \dim L_p},$$

where L_p denotes the leaf through p .

The proof of [Theorem C](#) relies on the observation that the Hilbert series $H(z)$ for the Laplacian algebra A of basic polynomials of $(V, \mathcal{F})$ contains, via the theory of spherical harmonics, the same information as the basic spectrum for the Laplacian of $\mathbb{S}V$. Using well-known techniques from enumerative combinatorics, we relate the pole of $H(z)$ at $z = 1$ with the growth of the coefficients of $H(z)$, hence to the growth of basic eigenvalues of $(\mathbb{S}V, \mathcal{F})$, which is given by the foliated Weyl's law ([Theorem A](#)).

1.1 Remarks, open questions and generalizations

The results of [Theorem A](#) lead to possible generalizations. We list them here as questions for the interested reader.

Given a singular Riemannian foliation $(M, \mathcal{F})$, regardless of whether the mean curvature is basic or not, there always is a basic Laplacian $\Delta_b: L^2(M)^b \rightarrow L^2(M)^b$ defined as $\Delta_b = dd^* + d^*d$, where $d: L^2(M) \rightarrow L^2(\Omega^1(M))^b$ is the restriction of the differential to the space of basic functions (see [[Richardson 2010](#), Section 1]) and d^* is its adjoint. If the mean curvature of $\mathcal{F}$ is not basic, however, Δ_b does not coincide with the restriction of the usual Laplacian to the space $L^2(M)^b$.

Question 1 *Given a singular Riemannian foliation $(M, \mathcal{F})$ with closed leaves on a compact manifold M , denoting $N_b(t)$ the counting function for the eigenvalues of Δ_b , does the Weyl's law*

$$N_b(t) \sim \frac{\omega_m}{(2\pi)^m} \text{Vol}(M/\mathcal{F}) t^{m/2}, \quad \text{where } m = \dim M/\mathcal{F},$$

hold?

In the regular case (ie when all the leaves have the same dimension) this was proved by Richardson [[2010](#)].

Similarly, one could consider foliations with nonclosed leaves:

Question 2 *Given a singular Riemannian foliation $(M, \mathcal{F})$ with nonclosed leaves, does a Weyl's law still hold?*

Given a singular Riemannian foliation $(M, \mathcal{F})$ with nonclosed leaves, Molino's conjecture (proved by Alexandrino and Radeschi [[2017](#)]) states that replacing the leaves of $\mathcal{F}$ with their closures gives rise to a new singular Riemannian foliation $\bar{\mathcal{F}}$, with the same basic functions as $\mathcal{F}$. Therefore, a positive answer to [Question 1](#) would imply that, even in this case, the counting function N_b for the basic Laplacian would have asymptotics

$$N_b(t) \sim \frac{\omega_{\bar{m}}}{(2\pi)^{\bar{m}}} \text{Vol}(M/\bar{\mathcal{F}}) t^{\bar{m}/2}, \quad \text{where } \bar{m} = \dim M/\bar{\mathcal{F}}.$$

One may also consider a variant of [Question 2](#), where $\mathcal{F}$ is assumed to have basic mean curvature. It is unclear to the authors whether this implies that $\bar{\mathcal{F}}$ has basic mean curvature.

Regarding the Weyl's law, a natural question is whether more geometric information on a singular Riemannian foliation $(M, \mathcal{F})$ (with, say, closed leaves and basic mean curvature) can be obtained from finer information about the eigenvalues of the basic Laplacian — more specifically, from the asymptotics

of the Laplace transform $n(s) := s \int_0^\infty e^{-st} N(t) dt$ of the basic counting function $N(t)$, as $s \rightarrow 0$. The proof of [Theorem A](#) hinges on the estimate that

$$n(s) \sim s^{-m/2} \frac{\Gamma(m/2 + 1) \omega_m}{(2\pi)^m} \text{Vol}(M/\mathcal{F})$$

(see [Proposition 27](#)), but a more precise estimate could yield more profound connections with the geometry of the leaf space:

Question 3 *What are the next terms in the asymptotics for $n(s)$? Can they be described in terms of geometric quantities of $M/\mathcal{F}$?*

Estimates of this type were obtained by Richardson [\[2010\]](#) for regular Riemannian foliations. In the special case of closed leaves, he obtained (Theorem 3.5):

$$n(s) = \frac{1}{(4\pi s)^{m/2}} \left(\text{Vol}(M/\mathcal{F}) + \frac{\sqrt{\pi}}{2} \text{Vol}(\partial M/\mathcal{F}) s^{1/2} + C(M/\mathcal{F}) s + O(s^{3/2}) \right),$$

where $C(M/\mathcal{F})$ depends on the total curvature, as well as the codimension-2 strata of $M/\mathcal{F}$. Similar estimates were obtained by Brüning and Heintze [\[1984\]](#) in the homogeneous case.

In the special case of a round sphere $M = \mathbb{S}^n$, given the correspondence between singular Riemannian foliations $(\mathbb{S}(V), \mathcal{F})$ and Laplacian algebras $A \subseteq \mathbb{R}[V]$, one can ask whether the terms in the asymptotics of $n(s)$ correspond somehow to the terms in the Laurent expansion for the Hilbert series $H(t)$ of A , at $t = 1$:

Question 4 *Let A be the Laplacian algebra of a singular Riemannian foliation $(\mathbb{S}^n, \mathcal{F})$, and $H(z)$ be the Hilbert series of A . If the Laurent series of $H(z)$ at $z = 1$ is of the form $H(z) = (1 - z)^{-m-1} \sum_k a_k z^k$, what is (if any) the correspondence between the coefficients a_k and the terms in the asymptotics of $n(s)$?*

We end this section with:

Question 5 *In the context of [Theorem B](#), what rational numbers appear as*

$$\frac{\text{Vol}(\mathbb{S}^n/\mathcal{F})}{\text{Vol}(\mathbb{S}^m)} \quad \text{or} \quad \frac{\text{Vol}(\mathbb{S}^n/G)}{\text{Vol}(\mathbb{S}^m)}?$$

Are they all the reciprocal of a natural number?

1.2 Structure of the paper

The paper is structured as follows: in [Section 2](#), we recall the main definitions and important results that we will use throughout the paper. [Section 3](#) has a more geometric flavor and is devoted to the proof of [Theorem D](#). In [Section 4](#), which has a more analytic flavor, we prove [Theorem A](#). Finally, the more algebraic [Section 5](#) contains the proof of [Theorem C](#).

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2 Preliminaries

2.1 Singular Riemannian foliations

2.1.1 Basic definitions and facts The main objects of study in the present paper are *singular Riemannian foliations*, first defined by Molino [1988]. By definition, a singular Riemannian foliation is a partition $\mathcal{F}$ of a complete Riemannian manifold M into connected, injectively immersed smooth submanifolds (called *leaves*), which is generated by a set of smooth vertical fields, and such that every geodesic of M that meets one leaf orthogonally is orthogonal to every other leaf that it meets. The last condition is equivalent to the leaves being locally at a constant distance apart [Radeschi 2025, Proposition 2.3].

The main examples are fibers of a Riemannian submersion, leaves of a (regular) Riemannian foliation, the parallel and focal sets of an isoparametric submanifold of a sphere, and orbits of an isometric group action (sometimes called “homogeneous” singular Riemannian foliations).

We say $\mathcal{F}$ is *closed* if all the leaves are closed. In this case, and assuming M complete, the condition involving normal geodesics is equivalent to (global) equidistance of every pair of leaves [Radeschi 2025, Remark 2.5].

We say $\mathcal{F}$ is *infinitesimal* when M is isometric to some Euclidean space $\mathbb{R}^{n+1}$, and the origin $\{0\}$ is a leaf. In this case any sphere centered at the origin is a union of leaves (in other words, is “saturated” by $\mathcal{F}$) and, by Molino’s homothetic transformation lemma [1988, Lemma 6.2], $\mathcal{F}$ is determined by its restriction to any such sphere. In fact, the map $\mathcal{F} \mapsto \mathcal{F}|_{\mathbb{S}^n}$ is a bijection between infinitesimal singular Riemannian foliations of $\mathbb{R}^{n+1}$ and singular Riemannian foliations of $\mathbb{S}^n$.

Given a singular Riemannian foliation $(M, \mathcal{F})$, the set of leaves is called the *leaf space*, and is denoted by $M/\mathcal{F}$. When the leaves are closed, the leaf space $M/\mathcal{F}$ has a natural metric structure, with respect to which the natural projection map $\sigma: M \rightarrow M/\mathcal{F}$ is a “manifold submetry”. For more information on submetries and manifold submetries, see [Kapovitch and Lytchak 2022; Mendes and Radeschi 2020a].

A function defined on M is $\mathcal{F}$ -*basic* (or just *basic*, if $\mathcal{F}$ is understood from context) when it is constant on the leaves of $\mathcal{F}$. Equivalently, when it is the pullback by σ of a function defined on the “base” $M/\mathcal{F}$. We sometimes denote sets of basic functions with the superscript b , for example, $C^\infty(M)^b$ for the set of all smooth basic (real-valued) functions on M .

2.1.2 Invariant theory If $(V, \mathcal{F})$ is an infinitesimal closed singular Riemannian foliation, the algebra of all basic *polynomials*, denoted by $A = \mathcal{B}(\mathcal{F}) = \mathbb{R}[V]^b$, is large enough that it separates leaves, and moreover, it is a finitely generated algebra. This is known as the *algebraicity theorem* [Lytchak and Radeschi 2018]. When $(V, \mathcal{F})$ is homogeneous, that is, when the leaves are given as the orbits of an orthogonal representation of a compact group, this algebra A is called the “algebra of invariants”, and the algebraicity theorem reduces to classical facts in invariant theory.

2.1.3 Principal part Given a singular Riemannian foliation $(M, \mathcal{F})$ on a complete manifold M , there exists an open, dense, saturated, full-measure subset $M_0 \subset M$, called the *principal part*, such that $M_0/\mathcal{F} \subset M/\mathcal{F}$ is a smooth Riemannian manifold, and $\sigma|_{M_0}: M_0 \rightarrow M_0/\mathcal{F}$ is a Riemannian submersion; see [Lytchak and Radeschi 2018, page 3]. (To justify the name, note that when $\mathcal{F}$ is homogeneous, M_0 is the union of the principal orbits.) When M is compact and $\mathcal{F}$ is closed, the map $\sigma|_{M_0}: M_0 \rightarrow M_0/\mathcal{F}$ is proper, hence a locally trivial fiber bundle by Ehresmann’s lemma. In particular, the function $h: M_0/\mathcal{F} \rightarrow \mathbb{R}$ given by $h(x) = \text{vol}(\sigma^{-1}(x))$, and its pullback $h \circ \sigma|_{M_0}: M_0 \rightarrow \mathbb{R}$, given by $p \mapsto \text{vol}(L_p)$, are smooth.

We denote by $H(p) \in T_p M$ the mean curvature vector of the leaf L_p as a submanifold of M , for every $p \in M_0$. Note that H is a smooth vector field on M_0 . We say $\mathcal{F}$ has *basic mean curvature* when H is σ -related to a (smooth) vector field $\underline{H}$ on $M_0/\mathcal{F}$. By [Alexandrino and Radeschi 2015, Proposition 3.1], this is automatically the case when M is a round sphere, or M is Euclidean space.

2.1.4 Tubular neighborhoods and the slice theorem Let M be a closed Riemannian manifold, and $\mathcal{F}$ a closed singular Riemannian foliation. Fix a leaf L , and $\epsilon > 0$ be smaller than the normal injectivity radius of L . Denote by νL the normal bundle of L , by $\nu L^{\leq \epsilon}$ (resp. $\nu L^{< \epsilon}$) the set of all vectors in νL of length at most ϵ (resp. less than ϵ), by $B_\epsilon(L)$ the (open) tube of radius ϵ around L , that is, the set of all points at distance less than ϵ from L , and by $\exp: \nu L \rightarrow M$ the normal exponential map. Note that the restriction of $\exp$ to $\nu L^{< \epsilon}$ is a diffeomorphism onto $B_\epsilon(L)$. We denote by $\pi: B_\epsilon(L) \rightarrow L$ the nearest-point projection, and note that $\pi \circ \exp$ coincides with the natural projection $\nu L \rightarrow L$ on $\nu L^{< \epsilon}$. Note also that the restriction of π to any leaf $L' \subset B_\epsilon(L)$ is a submersion $L' \rightarrow L$. This follows from [Alexandrino et al. 2013, Theorem 2.2(b)], and see also [Mendes and Radeschi 2019, Proposition 13].

Let $p \in L$. The *disconnected slice foliation* (see [Mendes and Radeschi 2019, Section 3.3]) of $\mathcal{F}$ at p , denoted by $\mathcal{F}^p$, is a partition of $\nu_p L$ obtained by first intersecting the leaves of $\mathcal{F}$ with $\exp \nu_p L^{\leq \epsilon}$, then pulling back this decomposition by $\exp$, and finally extending it via homothetic transformations to all of $\nu_p L$. This partition $\mathcal{F}^p$ of the Euclidean space $\nu_p L$ is a “disconnected infinitesimal foliation” in the sense of [Mendes and Radeschi 2019, Definition 8], and its restriction to any sphere centered at the

origin is a “disconnected singular Riemannian foliation” in the sense of [Definition 15](#) below. By the slice theorem [\[Mendes and Radeschi 2019, Theorem A\]](#), the disconnected slice foliation $\mathcal{F}^q$ at any other $q \in L$ is equivalent to $\mathcal{F}^p$ in the sense that there exists an isometry $v_p L \rightarrow v_q L$ sending leaves onto leaves.

2.1.5 Averaging operator Given a singular Riemannian foliation $(M, \mathcal{F})$ with compact leaves, and an L^2 function $f: M \rightarrow \mathbb{R}$, we define its average $\text{Av}(f): M \rightarrow \mathbb{R}$ over $\mathcal{F}$ by

$$\text{Av}(f)(p) = \frac{1}{\text{vol}(L_p)} \int_{q \in L_p} f(q) d\text{vol}^{L_p}(q),$$

where $d\text{vol}^{L_p}$ denotes the Riemannian volume form of the compact Riemannian manifold L_p and $\text{vol}(L_p)$ its volume. We will frequently write dq instead of $d\text{vol}^{L_p}(q)$ to simplify notation. When $\mathcal{F}$ has basic mean curvature, the averaging operator Av takes smooth functions to smooth functions, and commutes with the Laplace operator on the principal part M_0 ; see [\[Lytschak and Radeschi 2018, Theorem 3.3 and Lemma 3.2\]](#), and see also [\[Park and Richardson 1996\]](#) for similar considerations in the case of regular Riemannian foliations.

2.2 Hörmander’s estimate

We briefly recall Hörmander’s estimate on spectral functions, which is one of the main analytical ingredients in the proof of Weyl’s law for singular Riemannian foliations.

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{H}}$ and let A be a self-adjoint, possibly unbounded linear operator on $\mathcal{H}$. Given any characteristic function $\chi_{\mathcal{U}}$ on a Borel measurable set $\mathcal{U}$ in $\mathbb{R}$, one can use Borel functional calculus to define an orthogonal projection $\chi_{\mathcal{U}}(A)$ on $\mathcal{H}$; see [\[Reed and Simon 1972, page 262\]](#).

Given any $\lambda \in \mathbb{R}$, define $E(\lambda) := \chi_{(-\infty, \lambda)}(A)$. The family of projections $\{E(\lambda)\}$ is called the *spectral resolution* of the self-adjoint linear operator A .

Example 6 Suppose that $A: \mathcal{H} \rightarrow \mathcal{H}$ is a positive, self-adjoint linear operator with discrete spectrum

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$$

and an orthonormal eigenbasis $\{u_i\}$. By the spectral theorem in projection-valued measure form [\[Reed and Simon 1972, Theorem VIII.6\]](#), the spectral resolution of A is simply the projection to the sum of eigenspaces whose corresponding eigenvalues are less than λ . In other words,

$$(1) \quad E(\lambda)(f) = \chi_{(-\infty, \lambda)}(A)(f) = \sum_{\lambda_i < \lambda} (f, u_i)_{\mathcal{H}} \cdot u_i.$$

To define spectral functions, let Ω be a smooth, m -dimensional manifold with a smooth density μ . Let $\hat{P}: L^2(\Omega, \mu) \rightarrow L^2(\Omega, \mu)$ be a formally positive self-adjoint extension of an elliptic operator P on $C^\infty(\Omega)$. Let $\{E(\lambda)\}$ be the spectral resolution of $\hat{P}$. It is a standard result [\[Hörmander 1955, Section 3.7\]](#)

that the ellipticity of P implies that for each t , there exists a unique kernel $\theta(x, y; t) \in C^\infty(\Omega \times \Omega)$ such that for any $f \in L^2(\Omega, \mu)$,

$$(2) \quad E(t)(f)(x) = \int_{\Omega} \theta(x, y; t) f(y) d\mu(y).$$

The kernels $\theta(x, y; t)$ are called *spectral functions*.

Hörmander proved the following asymptotics for the spectral functions:

Theorem 7 [Hörmander 1968] *Let p be the principal symbol of an elliptic differential operator P of order k and let $d\xi$ be the measure induced by $d\mu$ on each fiber of the cotangent bundle $T^*\Omega$. Define*

$$R(x, \lambda) := \lambda^{-m/k} \theta(x, x; \lambda) - (2\pi)^{-m} \int_{B_x} d\xi,$$

where $B_x = \{\xi \in T_x^*\Omega \mid p(\xi) < 1\}$. Then on every compact subset of Ω , $R(x, \lambda) = O(\lambda^{-1/m})$ uniformly as $\lambda \rightarrow \infty$.

2.3 Basic Laplacian

Let (M, g) be a compact Riemannian manifold, $\mathcal{F}$ a closed singular Riemannian foliation with basic mean curvature, and let $\Delta(f) := -\operatorname{div} \nabla(f)$ be the Laplace operator on functions.

Recall from Section 2.1.5 that the averaging operator Av commutes with the Laplace operator Δ on $C^\infty(M)$. Therefore, the Laplace operator Δ sends basic smooth functions to basic smooth functions. We then extend the restriction $\Delta|_{C^\infty(M)^b}$ of Δ on the space of basic functions, to obtain a self-adjoint linear operator $R: L^2(M)^b \rightarrow L^2(M)^b$ with domain $\mathcal{D}(R) = H^2(M)^b$.

Let M_0 be the principal part of M and let $X_0 := \sigma(M_0)$ be the manifold part of X such that $\sigma: M_0 \rightarrow X_0$ is a Riemannian submersion. Given any point $x \in X_0$, define $h(x)$ to be the volume of the leaf $L_{\sigma^{-1}(x)}$, and recall from Section 2.1.3 that h is a smooth function. By Fubini's theorem for Riemannian submersions, the pullback of σ is a Hilbert space isometry $\sigma^*: L^2(X_0, h) \rightarrow L^2(M_0)^b$, where $L^2(X_0, h)$ is the set of square-integrable functions on X_0 with respect to the measure $h(x) d\operatorname{vol}_{X_0}$.

Let σ_* be the inverse of σ^* . Define $T: L^2(X_0, h) \rightarrow L^2(X_0, h)$ to be

$$T := \sigma_* \circ R \circ \sigma^*,$$

with the domain $\mathcal{D}(T) = \sigma_*(\mathcal{D}(R))$. A simple calculation proves the following lemma that relates the operator T to the Laplace operator on X_0 :

Lemma 8 *For any $f \in C_c^\infty(X_0)$, we have*

$$T(f) = \Delta_{X_0} f + \underline{H} f,$$

where $\underline{H}$ is the projection of the mean curvature vector field H through σ .

By [Lemma 8](#), the operator T is a self-adjoint extension of an elliptic differential operator, whose principal symbol is the same as that of the Laplace operator on X_0 .

Remark 9 As M is compact, the Laplace operator has discrete spectrum:

$$0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$$

Since $T: L^2(X_0, h) \rightarrow L^2(X_0, h)$ is equivalent to $R: L^2(M)^b \rightarrow L^2(M)^b$, it must have a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots,$$

and we can define the counting function

$$N(t) = \max\{k \mid \lambda_k < t\}.$$

Furthermore, since the Laplace operator commutes with Av , there exists a smooth, orthonormal basis of eigenfunctions $\{\phi_k\}$ for $L^2(M)$ with a subsequence $\{\phi_{k_i}\}$ that generates the vector subspace $L^2(M)^b$.

Define $\varphi_i := \sigma_*(\phi_{k_i})$. Then $\{\varphi_i\}$ form a smooth orthonormal basis of T -eigenfunctions for $L^2(X_0, h)$.

Proposition 10 Let $\theta^T(x, y; t)$ and $\theta^\Delta(x, y; t)$ be the spectral function of T and Δ , respectively, and let $\{\phi_k\}$ and $\{\varphi_i\}$ be the smooth orthonormal bases in [Remark 9](#). Then for each $t \in \mathbb{R}$, $p, q \in M$ and $x, y \in X_0$ we have the following:

- (a) $\theta^T(x, y; t) = \sum_{\lambda_i < t} \varphi_i(x) \varphi_i(y)$ and $\theta^\Delta(p, q; t) = \sum_{\tilde{\lambda}_k < t} \phi_k(p) \phi_k(q)$.
- (b) For each pair of $p, q \in M$ and each $t \in \mathbb{R}$,

$$\theta^T(\sigma(p), \sigma(q); t) = (\text{Av} \otimes \text{Av}) \theta^\Delta(p, q; t).$$

Proof By (1), $\sum_{\lambda_i < t} \varphi_i(x) \varphi_i(y)$ and $\sum_{\tilde{\lambda}_k < t} \phi_k(p) \phi_k(q)$ are two smooth functions satisfying (2). Hence (a) follows from the uniqueness of spectral functions.

To prove (b), simply note that

$$\begin{aligned} \theta^T(\sigma(p), \sigma(q); t) &= \sum_{\lambda_i < t} \varphi_i(\sigma(p)) \varphi_i(\sigma(q)) = \sum_{\lambda_i < t} \sigma^* \varphi_i(p) \sigma^* \varphi_i(q) \\ &= \sum_{\tilde{\lambda}_k < t} \text{Av} \phi_k(p) \text{Av} \phi_k(q) = (\text{Av} \otimes \text{Av}) \theta^\Delta(p, q; t). \end{aligned} \quad \square$$

By applying [Theorem 7](#) to the spectral function of T , we have the following:

Proposition 11 Let $\theta^T(x, y; t)$ be the spectral function of the elliptic differential operator that generates T . Then for each $x \in X_0$,

$$\theta^T(x, x; t) \sim t^{m/2} \frac{\omega_m}{h(x)(2\pi)^m}$$

as $t \rightarrow \infty$, where ω_m is the volume of unit balls in $\mathbb{R}^m$.

2.4 Heat kernels

Let $\phi_i(x)$ be the orthonormal eigenbasis for Δ constructed in [Remark 9](#). A well-known result by Minakshisundaram and Pleijel [\[1949\]](#) (see also [\[Buser 2010, Theorem 7.2.18\]](#)) states that, for each $\sigma > \dim(M)$, the Dirichlet series $\sum_i \lambda_i^{-\sigma} \phi_i(p) \phi_i(q)$ converges uniformly and absolutely on $M \times M$. Combining with (a) in [Proposition 10](#), we see that the spectral function $\theta^\Delta(p, q; t)$ has uniform polynomial growth, in the sense that there exists positive constants C and N such that for all $p, q \in M$,

$$|\theta^\Delta(p, q; t)| < Ct^N.$$

Using [Proposition 10\(a\)](#) and (b), we see that $\theta^T(x, y; t)$ also has uniform polynomial growth. Hence, for any $s > 0$, $x, y \in X_0$, and $p, q \in M_0$, one can take the Laplace transform of θ^Δ and θ^T to define

$$(3) \quad \Gamma_s^T(x, y) := s \int_0^\infty e^{-st} \theta^T(x, y; t) dt \quad \text{and} \quad \Gamma_s^\Delta(p, q) := s \int_0^\infty e^{-st} \theta^\Delta(p, q; t) dt.$$

The functions $\Gamma_s^T(x, y)$ and $\Gamma_s^\Delta(p, q)$ are called the heat kernels and they are the Schwartz kernels for the operators e^{-sT} and $e^{-s\Delta}$ respectively; see [\[Brüning and Heintze 1978/79, page 181\]](#). Note that $\Gamma_s^\Delta(p, q)$ coincides with the fundamental solution to the heat equation on M . The parametrix construction for the fundamental solutions of heat equations, which was first introduced in [\[Minakshisundaram and Pleijel 1949\]](#), immediately implies the following estimate:

Proposition 12 *There exists positive constants C_1 and C_2 such that for any pair of points $p, q \in M^n$,*

$$|\Gamma_s^\Delta(p, q)| \leq C_1 s^{-n/2} e^{-C_2 d_M^2(p, q)/s}.$$

Proof The proposition directly follows from equation (45) on page 154 of [\[Chavel 1984\]](#). □

2.5 Tauberian and abelian theorems

Given two real-valued functions $f(t)$ and $g(t)$. We write $f(t) \sim g(t)$ as $t \rightarrow \infty$ (resp. $t \rightarrow 0$) if $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$ (resp. $\lim_{t \rightarrow 0} f(t)/g(t) = 1$). We state the Tauberian theorem and abelian theorem together below.

Theorem 13 [\[Feller 1971, Theorem 2 on page 445\]](#) *Let $F: [0, \infty) \rightarrow \mathbb{R}$ be a function with bounded variation such that the Laplace transform $f(s) = \int_0^\infty e^{-st} F(t) dt$ exists. Then for fixed $\alpha \in [0, \infty)$ and $C \in \mathbb{R}$, each of the relations*

$$(4) \quad sf(s) \sim \frac{C}{s^\alpha} \quad \text{as } s \rightarrow 0,$$

$$(5) \quad F(t) \sim \frac{C}{\Gamma(\alpha + 1)} t^\alpha \quad \text{as } t \rightarrow \infty,$$

implies the other.

Historically, the implication from (5) to (4) is called an abelian theorem, and the converse is called a Tauberian theorem [\[Feller 1971, page 445\]](#).

3 Geometric estimate

The main goal of this section is to prove [Theorem D](#) by induction on the dimension of M . For such an inductive argument to work, we will, in fact, prove a more general statement ([Theorem 18](#) below), that applies to *disconnected* singular Riemannian foliations. First, we need the following definition.

Definition 14 Given a closed singular Riemannian foliation $(M, \mathcal{F})$, an isometry of $M/\mathcal{F}$ is called *liftable* if it is induced by an isometry of M that permutes the leaves of $\mathcal{F}$.

Definition 15 Let $(M, \mathcal{F}_0)$ be a closed singular Riemannian foliation with canonical projection $\sigma_0: M \rightarrow M/\mathcal{F}_0$, and let Γ be a finite group acting on $M/\mathcal{F}_0$ by liftable isometries. The *disconnected singular Riemannian foliation* (with closed leaves) on M induced by $(\mathcal{F}_0, \Gamma)$ is the partition $\mathcal{F}$ of M into sets of the form $\sigma_0^{-1}(\Gamma \cdot x)$, for $x \in M/\mathcal{F}_0$, which are called the *leaves of $\mathcal{F}$* .

Given a singular Riemannian foliation with disconnected leaves as above, we observe that its leaf space is $M/\mathcal{F} = (M/\mathcal{F}_0)/\Gamma$, and the canonical projection $\sigma: M \rightarrow M/\mathcal{F}$ is the composition of $\sigma_0: M \rightarrow M/\mathcal{F}_0$ with the quotient map $M/\mathcal{F}_0 \rightarrow (M/\mathcal{F}_0)/\Gamma$.

Example 16 Given a compact disconnected Lie group G acting on a Riemannian manifold M by isometries, the partition into G -orbits is an example of disconnected singular Riemannian foliation. In this case $\mathcal{F}_0$ is the orbit decomposition into G_0 -orbits, where G_0 is the identity component of G , $\Gamma = G/G_0$ and the action of G/G_0 on M/G_0 is the natural one.

Remark 17

- While the isometries in Γ are required to be liftable, we do not assume that the entire action of Γ lifts to an action on M . This is in general not the case even in the example above.
- [Definition 15](#) is very similar to [[Mendes and Radeschi 2019](#), Definition 8], with the exceptions that the latter allows for $\mathcal{F}$ and $\mathcal{F}_0$ to have nonclosed leaves, and the total space is assumed to be Euclidean space.
- The key fact we will use below is that any slice foliations of any closed singular Riemannian foliation is a disconnected singular Riemannian foliation in the sense of [Definition 15](#); see [Section 2.1.4](#).

The following theorem generalizes [Theorem D](#):

Theorem 18 Given a compact connected Riemannian manifold M and a disconnected singular Riemannian foliation $\mathcal{F}$ with closed leaves, there exists a constant $C = C(M, \mathcal{F})$ such that, for every $p \in M$, and every $s > 0$, one has

$$(6) \quad \int_{L_p} \exp\left(-\frac{d_M(p, q)^2}{s}\right) dq \leq C s^{\frac{1}{2} \dim L_p},$$

where the integral is computed with respect to the Riemannian volume form of L_p .

3.1 Reductions

To make the proof of [Theorem 18](#) shorter, we prove a couple of reductions as separate lemmas, starting with the reduction to the case where the leaves are connected:

Lemma 19 *Let $M, \mathcal{F}$ be as in [Theorem 18](#), and $\mathcal{F}_0, \Gamma$ as in [Definition 15](#). Suppose the conclusion of [Theorem 18](#) holds for $(M, \mathcal{F}_0)$. Then it holds for $(M, \mathcal{F})$.*

Proof We are assuming the existence of $C = C(M, \mathcal{F}_0)$ such that, for all $p \in M$, and all $s > 0$,

$$(7) \quad \int_{L_p} \exp\left(-\frac{d_M(p, q)^2}{s}\right) dq \leq C s^{\frac{1}{2} \dim L_p},$$

where L_p denotes the $\mathcal{F}_0$ -leaf through p . We claim that, for $(M, \mathcal{F})$, we can take $C(M, \mathcal{F}) = C|\Gamma|2^{\dim M}$.

Indeed, let $s > 0$, $p \in M$, and let L denote the $\mathcal{F}$ -leaf through p . Then L is the disjoint union of $\mathcal{F}_0$ -leaves $L_1, \dots, L_k$, where $k \leq |\Gamma|$, and we may assume $p \in L_1$. Note that each L_j is the image of L_1 under an isometry of M , so, in particular, they all have the same dimension, which we may denote by $\dim L$. Let $p_1 = p$, and p_j be a point of L_j that is closest to p . Then, for any $q \in L_j$, we have, by the triangle inequality,

$$d_M(q, p) \geq \frac{d_M(q, p) + d_M(p, p_j)}{2} \geq \frac{d_M(q, p_j)}{2}.$$

Therefore

$$\int_L \exp\left(-\frac{d_M(p, q)^2}{s}\right) dq \leq \sum_{j=1}^k \int_{L_j} \exp\left(-\frac{d_M(p_j, q)^2}{4s}\right) dq \leq |\Gamma| C (4s)^{\frac{1}{2} \dim L},$$

where in the last inequality we have used (7) for each L_j , and $k \leq |\Gamma|$. Since $(4s)^{\frac{1}{2} \dim L} \leq 2^{\dim M} s^{\frac{1}{2} \dim L}$, this finishes the proof of the claim. $\square$

The next lemma corresponds to the case in [Theorem 18](#) where $\mathcal{F}$ has a single leaf:

Lemma 20 *Let M be a compact connected Riemannian manifold. Then, there exists a constant C such that, for every $p \in M$, and every $s > 0$, one has*

$$\int_M \exp\left(-\frac{d_M(p, q)^2}{s}\right) dq \leq C s^{\frac{1}{2} \dim M}.$$

Proof See [[Brüning and Heintze 1978/79](#), Lemma 4.5]. $\square$

Remark 21 (scaling-invariance) If [Theorem 18](#) holds for a given manifold (M, g) , then it holds for $(M, \lambda^2 g)$, for any $\lambda > 0$, with the same constant C .

3.2 Sasaki metrics

We will need to put a special metric on the normal bundle of a leaf L in order to prove [Lemma 24](#) below, so we collect here the definition and some basic properties.

Definition 22 Let L be a closed Riemannian manifold, and $\pi: E \rightarrow L$ be a vector bundle equipped with an inner product $\langle \cdot, \cdot \rangle_x$ on the fiber $E_x = \pi^{-1}(x)$ for each $x \in L$, and a metric connection ∇ . Parallel translation determines a choice of “horizontal space” $H_{(x,v)}$ at each $(x, v) \in E$ (where $x \in L$ and $v \in E_x$), and we define the *Sasaki metric* on E by declaring $H_{(x,v)}$ to be orthogonal to the “vertical space” $V_{(x,v)} = \ker d\pi_{(x,v)} = T_{(x,v)}E_x = E_x$, using the given inner product on $V_{(x,v)} = E_x$, and pulling back the given inner product on $T_x L$ to $H_{(x,v)}$.

The construction above is also described in [\[Besse 1987, Example 9.60\]](#).

Proposition 23 *In the notation of the definition above, endow E with the Sasaki metric. Then:*

- (a) $\pi: E \rightarrow L$ is a Riemannian submersion.
- (b) For each $x \in L$, the fiber E_x is flat and totally geodesic.
- (c) E is complete.
- (d) There exists $r > 0$ such that, for all $x \in L$, one has $d_E = d_{E_x}$ on the set $E_x^{\leq r} = \{v \in E_x \mid \|v\| \leq r\}$.

Proof (a) Follows immediately from the definition.

(b) For each $x \in L$, the Riemannian metric on E_x is flat by definition, being isometric to the Euclidean space $(E_x, \langle \cdot, \cdot \rangle_x)$. For the second claim, see [\[Besse 1987, Theorem 9.59 and Example 9.60\]](#).

(c) See [\[Besse 1987, Theorem 9.59 and Example 9.60\]](#).

(d) Identify L with the zero section in E , and note that, for every $r > 0$, the closed tube of radius r around L in E coincides with $\bigcup_{x \in L} E_x^{\leq r}$. Since E is complete, the injectivity radius function on E is continuous [\[Chavel 2006, Theorem III.2.1\]](#), and thus, by compactness of L , there exists $r > 0$ such that the injectivity radius is larger than $2r$ on $\bigcup_{x \in L} E_x^{\leq r}$.

Let $x \in L$, and $v, w \in E_x^{\leq r}$. Then $\|v - w\| \leq 2r$, and, in particular, $d_E((x, v), (x, w)) \leq 2r$. Thus there is a unique $\xi \in T_{(x,w)}E$ with $\|\xi\| \leq 2r$ which exponentiates to (x, v) . Since $v - w$ is another vector with the same property (because E_x is totally geodesic), we obtain $\xi = v - w$. Therefore $d_E((x, v), (x, w)) = \|\xi\| = \|v - w\| = d_{E_x}(v, w)$. $\square$

3.3 Uniform estimates

The next two technical lemmas will be used in the proof of [Theorem 18](#), by providing uniform information about the geometry of leaves around a fixed one. We start with an analogue of [\[Brüning and Heintze 1978/79, Lemma 4.7\]](#).

Lemma 24 Let L be a leaf of the closed singular Riemannian foliation $\mathcal{F}$ on the closed Riemannian manifold M . Assume L has codimension at least one. Denote by $\pi: B_\epsilon(L) \rightarrow L$ the closest-point projection, where ϵ is smaller than the normal injectivity radius of L . Let $E = \nu L$ denote the normal bundle of L in M , and $\exp$ the normal exponential map $\nu L \rightarrow M$. Then there exist $C_1 > 0$ and $r \in (0, \epsilon)$ such that, for every $y \in L$ and $p \in B_r(L)$, there exists $v = v(p, y) \in \nu_y L$ such that $\exp(v) \in L_p$ and, for all $q \in \pi^{-1}(y) \cap L_p$, one has

$$C_1 d_M^2(p, q) \geq \begin{cases} d_L^2(\pi(p), \pi(q)) + d_{\text{sph}}^2(v, \exp^{-1}(q)) & \text{if } \text{codim}(L) \geq 2, \\ d_L^2(\pi(p), \pi(q)) & \text{if } \text{codim}(L) = 1. \end{cases}$$

Here d_{sph} denotes the distance in the round sphere in $\nu_y L$ of radius equal to the distance between L and L_p (that is, the leaf through p).

Proof To simplify notation, denote by ψ the inverse of the diffeomorphism

$$\exp|_{E^{<\epsilon}}: E^{<\epsilon} \rightarrow B_\epsilon(L).$$

Fix a metric connection on E (eg the normal connection), and endow E with the associated Sasaki metric; see Section 3.2. Since ϵ is strictly smaller than the normal injectivity radius, there exists $K > 0$ such that, for all $\eta \leq \epsilon$, the map $\exp|_{E^{<\eta}}: E^{<\eta} \rightarrow B_\eta(L)$ is K -bi-Lipschitz.

Choose $r > 0$ that is smaller than the r given in Proposition 23(d), and such that $2r < \epsilon$.

We claim that, for all $p, q \in B_r(L)$,

$$d_M(p, q) \geq C d_E(\psi(p), \psi(q)),$$

where $C = (2r + \text{diam}(L))^{-1} K^{-1} r$. We consider two cases.

Case 1 ($d_M(p, q) \geq r$) Using the triangle inequality, we obtain

$$d_{B_r(L)}(p, q) \leq 2r + \text{diam}(L).$$

Thus

$$\begin{aligned} d_M(p, q) &\geq r \cdot \frac{2r + \text{diam}(L)}{2r + \text{diam}(L)} \geq \frac{r}{2r + \text{diam}(L)} d_{B_r(L)}(p, q) \\ &\geq \frac{r}{2r + \text{diam}(L)} \frac{1}{K} d_{E^{\leq r}}(\psi(p), \psi(q)) \\ &\geq \frac{r}{2r + \text{diam}(L)} \frac{1}{K} d_E(\psi(p), \psi(q)) = C d_E(\psi(p), \psi(q)). \end{aligned}$$

Case 2 ($d_M(p, q) < r$) Then $q \in B_r(p) \subset B_\epsilon(L)$, so that

$$\begin{aligned} d_M(p, q) &= d_{B_\epsilon(L)}(p, q) \geq K^{-1} d_{E^{\leq \epsilon}}(\psi(p), \psi(q)) \\ &\geq K^{-1} d_E(\psi(p), \psi(q)) \geq C d_E(\psi(p), \psi(q)), \end{aligned}$$

which finishes the proof of the claim.

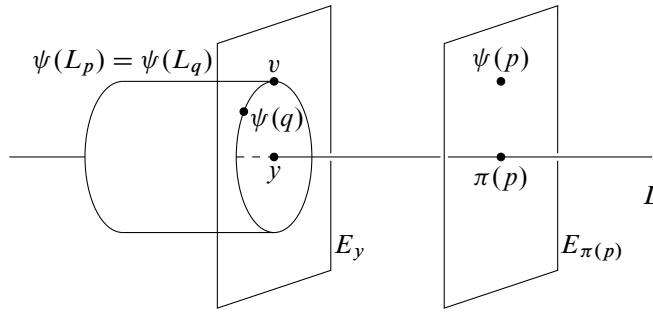


Figure 1

The claim above reduces the proof of the statement of the lemma to a similar statement, where $d_M(p, q)$ is replaced with $d_E(\psi(p), \psi(q))$.

If $\text{codim}(L) = 1$, the desired inequality follows from the fact (see [Proposition 23\(a\)](#)) that $\pi: E \rightarrow L$ is a Riemannian submersion. Assume $\text{codim}(L) \geq 2$.

Let $y \in L$, $p \in B_r(L)$. Choose a point v of $E_y \cap \psi(L_p)$ minimizing $d_E(v, \psi(p))$. Let $q \in \pi^{-1}(y) \cap L_p$, which is equivalent to $\psi(q) \in E_y \cap \psi(L_p)$. See [Figure 1](#).

By the triangle inequality,

$$d_E(\psi(q), v) \leq d_E(\psi(q), \psi(p)) + d_E(\psi(p), v).$$

By the choice of v ,

$$d_E(\psi(p), v) \leq d_E(\psi(q), \psi(p)).$$

Using [Proposition 23\(d\)](#) and some Euclidean geometry, we obtain

$$d_E(\psi(q), v) = d_{E_y}(\psi(q), v) \geq \frac{2}{\pi} d_{\text{sph}}(\psi(q), v).$$

Combining the last three inequalities, we obtain

$$(8) \quad d_E(\psi(q), \psi(p)) \geq \frac{1}{\pi} d_{\text{sph}}(\psi(q), v).$$

On the other hand, since $\pi: E \rightarrow L$ is a Riemannian submersion (see [Proposition 23\(a\)](#)), we have

$$d_E(\psi(q), \psi(p)) \geq d_L(y, \pi(p)).$$

Squaring this inequality and adding the square of (8), we obtain

$$2d_E^2(\psi(q), \psi(p)) \geq d_L^2(y, \pi(p)) + \frac{1}{\pi^2} d_{\text{sph}}^2(\psi(q), v),$$

from which the desired inequality follows (with $C_1 = 2\pi^2/C^2$). □

The closest-point projection to a leaf, when restricted to a nearby leaf, is known to be a submersion; see [Section 2.1.4](#). The next lemma states that it is, in fact, not too far from being a *Riemannian* submersion; see [\[Lytschak 2024, Proposition 1.3\]](#) for a more precise and general statement. It will be used to control the Jacobian appearing in a coarea formula inside the proof of [Theorem 18](#) below.

Lemma 25 *Let L be a leaf of the closed singular Riemannian foliation $\mathcal{F}$ on the closed Riemannian manifold M . Assume L has codimension at least one, and denote by $\pi: B_r(L) \rightarrow L$ the closest-point projection, where $r > 0$ is strictly smaller than the normal injectivity radius of L . For each $q \in B_r(L)$, let L_q denote the leaf through q , and $d\pi'_q: T_q L_q \rightarrow T_{\pi(q)} L$ the restriction of $d\pi_q$ to $T_q L_q$. Let $f(q)$ be the smallest eigenvalue of the self-adjoint linear map*

$$d\pi'_q \circ (d\pi'_q)^*: T_{\pi(q)} L \rightarrow T_{\pi(q)} L.$$

Then $C_2 = \inf_{q \in B_r(L)} f(q)$ is positive.

Proof Denote by R the normal injectivity radius of L . Recall from [Section 2.1.4](#) that, for every $q \in B_R(L)$, the map $\pi|_{L_q}: L_q \rightarrow L$ is a submersion; in particular $f(q) > 0$.

Assume, for a contradiction, that $C_2 = 0$. Then, there exists a sequence $q_i \in B_r(L)$ with $\lim f(q_i) = 0$. Choose $v_i \in T_{\pi(q_i)} L$ to be a unit eigenvector of $d\pi'_{q_i} \circ (d\pi'_{q_i})^*$ associated to the eigenvalue $f(q_i)$.

By compactness, we may assume, after passing to a subsequence, that q_i converges to $q \in \overline{B_r(L)} \subset B_R(L)$, and that v_i converge to $v \in T_{\pi(q)} L$.

Since $d\pi'_q$ is surjective, let $w \in T_q L_q$ such that $d\pi'_q(w) = v$. Since $\mathcal{F}$ is a singular Riemannian foliation, it is generated by smooth vector fields. In particular, there exist vectors $w_i \in T_{q_i} L_{q_i}$ converging to w , so that $\lim d\pi_{q_i}(w_i) = d\pi_q(w) = v$. Then

$$\lim \langle (d\pi'_{q_i})^* v_i, w_i \rangle = \lim \langle v_i, d\pi_{q_i} w_i \rangle = \langle v, v \rangle = 1.$$

On the other hand,

$$\lim \|(d\pi'_{q_i})^* v_i\|^2 = \lim \langle d\pi'_{q_i} \circ (d\pi'_{q_i})^* v_i, v_i \rangle = \lim f(q_i) = 0$$

and $\lim \|w_i\| = \|w\| < \infty$, so that, by the Cauchy–Schwarz inequality,

$$\lim \langle (d\pi'_{q_i})^* v_i, w_i \rangle = 0,$$

a contradiction. Therefore $C_2 > 0$. □

3.4 Proof of [Theorem 18](#)

Proof of [Theorem 18](#) Let $\mathcal{F}_0$ and Γ as in [Definition 15](#).

We use induction on $\dim M$. If $\dim M = 1$, then either $\mathcal{F} = \mathcal{F}_0$ has a single leaf, in which case we can apply [Lemma 20](#), or $\mathcal{F}_0$ is the trivial foliation by points, for which the conclusion is trivial, so we can apply [Lemma 19](#). Assume $\dim M > 1$.

By Lemma 19, we may assume that $\mathcal{F} = \mathcal{F}_0$, that is, $\mathcal{F}$ is a singular Riemannian foliation (with connected, closed leaves).

By Lemma 20, we may assume that M is not a single leaf. Since M is connected, that means every leaf has codimension at least one.

Since M is compact, it is enough to prove that, for every leaf L , there exists $r, C_L > 0$ such that (6) holds for all p in the tube $B_r(L)$ and all $s > 0$, with C_L in place of C . Indeed, M can be covered by finitely many such tubes, so one can take C to be the maximum of the corresponding constants C_L . Therefore, we fix a leaf L .

Take $r > 0$ and C_1 given by Lemma 24 (in particular r is smaller than the normal injectivity radius of L), and C_2 given in Lemma 25. Let $p \in B_r(L)$. The restriction of π to L_p is a submersion $L_p \rightarrow L$ (see Section 2.1.4), with Jacobian

$$\text{Jac}(q) = \sqrt{\det(d\pi'_q \circ (d\pi'_q)^*)},$$

which, by Lemma 25, satisfies $\text{Jac}(q) \geq C_2^{\dim L/2}$ for all $q \in L_p$. To simplify notation, let

$$g_s(q) = \exp\left(-\frac{d_M(p, q)^2}{s}\right).$$

Then

$$\int_{q \in L_p} g_s(q) d\text{vol}^{L_p}(q) \leq C_2^{-\frac{1}{2} \dim L} \int_{q \in L_p} g_s(q) \text{Jac}(q) d\text{vol}^{L_p}(q).$$

Applying the coarea formula [Chavel 2006, page 160, Exercise III.12(c)], the integral on the right becomes

$$(9) \quad \int_{y \in L} \left[\int_{q \in L_p \cap \pi^{-1}(y)} g_s(q) d\text{vol}^{L_p \cap \pi^{-1}(y)}(q) \right] d\text{vol}^L(y).$$

The next step is to use the inequality in Lemma 24, so we split in two cases: $\text{codim}(L) = 1$ and $\text{codim}(L) \geq 2$.

Case 1 ($\text{codim}(L) = 1$) By Lemma 24, $g_s(q) \leq \exp(-d_L^2(\pi(p), \pi(q))/C_1 s)$. The leaves L and L_p have the same dimension and, for each $y \in L$, the set $L_p \cap \pi^{-1}(y) \subseteq \{\exp_y x \mid x \in \nu_y L, \|x\| = d_M(y, p)\}$ consists of at most two points. Thus (9) is less than or equal to

$$2 \int_{y \in L} \exp\left(\frac{-d_L^2(\pi(p), y)}{C_1 s}\right) d\text{vol}^L(y),$$

which, by Lemma 20, is at most

$$2C_3(C_1 s)^{\frac{1}{2} \dim L} = 2C_3 C_1^{\frac{1}{2} \dim L_p} s^{\frac{1}{2} \dim L_p}$$

for some C_3 that depends only on L . Therefore,

$$\int_{L_p} \exp\left(-\frac{d_M(p, q)^2}{s}\right) dq \leq [C_2^{-\frac{1}{2} \dim L} 2C_3 C_1^{\frac{1}{2} \dim L_p}] s^{\frac{1}{2} \dim L_p},$$

finishing the proof of (6) in Case 1.

Case 2 ($\text{codim}(L) \geq 2$) By [Lemma 24](#),

$$g_s(q) \leq \exp\left(\frac{-d_L^2(\pi(p), \pi(q))}{C_1 s}\right) \exp\left(\frac{-d_{\text{sph}}^2(v, \exp^{-1}(q))}{C_1 s}\right),$$

where $v = v(p, y)$ depends only on p and y .

Thus (9) is at most

$$(10) \quad \int_{y \in L} e^{-d_L^2(\pi(p), y)/C_1 s} \left[\int_{q \in L_p \cap \pi^{-1}(y)} e^{-d_{\text{sph}}^2(v, \exp^{-1}(q))/C_1 s} d\text{vol}^{L_p \cap \pi^{-1}(y)}(q) \right] d\text{vol}^L(y).$$

Next we make a change of variables in the inner integral above, using the normal exponential map $\exp: v_y L \rightarrow \pi^{-1}(y)$. This is a diffeomorphism when restricted to the ball of radius r in $v_y L$, and, since L is compact, it is bi-Lipschitz with a constant depending only on L, r . Since $d\text{vol}^{L_p \cap \pi^{-1}(y)}$ is the Riemannian volume form of the Riemannian manifold $L_p \cap \pi^{-1}(y)$ endowed with the induced Riemannian structure from $\pi^{-1}(y)$, there exists C_4 , depending only on L and r , such that the inner integral in (10) is at most

$$(11) \quad C_4 \int_{z \in L_v} e^{-d_{\text{sph}}^2(v, z)/C_1 s} d\text{vol}^{L_v}(z).$$

Here $L_v = \exp^{-1}(L_p \cap \pi^{-1}(y))$ is the leaf through v of the (disconnected) slice foliation $\mathcal{F}^y$ of the Euclidean space $v_y L$; see [Section 2.1.4](#).

We apply the inductive hypothesis to the restriction of $\mathcal{F}^y$ to the unit sphere $\mathbb{S}(v_y L)$ and conclude that there exists C_5 such that, for all $w \in \mathbb{S}(v_y L)$,

$$(12) \quad \int_{z \in L_w} e^{-d_{\text{sph}}^2(w, z)/C_1 s} d\text{vol}^{L_w}(z) \leq C_5 (C_1 s)^{\frac{1}{2} \dim L_w}.$$

By [Remark 21](#), the same conclusion holds with w replaced with any $v \in v_y L$ (and with the same constant C_5), because homothetic transformations of $v_y L$ take leaves of $\mathcal{F}^y$ to leaves of $\mathcal{F}^y$; see the second paragraph after [\[Mendes and Radeschi 2019, Definition 8\]](#). In principle, the constant C_5 depends on $y \in L$, but in fact that it can be chosen independently of y , because the slice foliation does not depend on y ; see [Section 2.1.4](#).

Combining the inequality (9) $\leq$ (10) with the estimates (11) and (12), and noting that $\dim L_v + \dim L = \dim L_p$, we obtain

$$\int_{q \in L_p} g_s(q) d\text{vol}^{L_p}(q) \leq C_2^{-\frac{1}{2} \dim L} C_4 C_5 (C_1 s)^{\frac{1}{2} (\dim L_p - \dim L)} \int_{y \in L} e^{-d_L^2(\pi(p), y)/C_1 s} d\text{vol}^L(y).$$

Finally, by [Lemma 20](#), there exists C_3 (depending only on L) such that the integral on the right is at most $C_3 (C_1 s)^{\frac{1}{2} \dim L}$. Thus

$$\int_{q \in L_p} g_s(q) d\text{vol}^{L_p}(q) \leq (C_2^{-\frac{1}{2} \dim L} C_4 C_5 C_3 C_1^{\frac{1}{2} \dim L_p}) s^{\frac{1}{2} \dim L_p},$$

which concludes the proof of (6) in Case 2. $\square$

4 Weyl's law

Throughout this section, we will use the terminology introduced in [Section 2](#).

We follow the approach in [\[Brüning and Heintze 1978/79, Section 3\]](#) to prove Weyl's law ([Theorem A](#)). The proof is roughly organized as follows. First, we derive a trace formula for the operator e^{-sT} , where $T: L^2(X_0, h) \rightarrow L^2(X_0, h)$ is the operator defined in [Section 2.3](#). This formula relates the Laplace transform of the counting function $N(t)$ of T to an integral of its heat kernel. At each point in X_0 , one can apply Hörmander's estimate ([Theorem 7](#)) and the abelian theorem ([Theorem 13](#)) to obtain the asymptotics for the heat kernel. [Theorem D](#) allows us to integrate the asymptotics of the heat kernel. The asymptotics for $N(t)$ then follows from the Tauberian theorem and the trace formula.

4.1 A trace formula

The counting function $N(t)$ and Γ_s^T are related via the following trace formula for e^{-sT} :

Proposition 26 *The function $N(t)$ has polynomial growth as $t \rightarrow \infty$. Furthermore,*

$$(13) \quad s \int_0^\infty e^{-st} N(t) dt = \int_{X_0} \Gamma_s^T(x, x) h(x) dx,$$

where $\Gamma_s^T(x, y)$ is the heat kernel of T defined by [\(3\)](#).

Proof Let $S(t)$ be the number of eigenvalues of the Laplace operator Δ on M that are less than t . The standard Weyl's law for the Laplace operator on compact Riemannian manifolds implies that $S(t)$ must have polynomial growth as $t \rightarrow \infty$. Since the growth rate of $N(t)$ cannot exceed that of $S(t)$, $N(t)$ must have polynomial growth as $t \rightarrow \infty$.

To prove the integral formula, let $\{\varphi_i\}$ be the orthonormal basis for $L^2(X_0, h)$ in [Remark 9](#) such that $T(\varphi_i) = \lambda_i \varphi_i$. Then by [Proposition 10\(a\)](#),

$$(14) \quad N(t) = \sum_{\lambda_i < t} \|\varphi_i(x)\|_{L^2(X_0, h)}^2 = \int_{X_0} \theta^T(x, x, t) h(x) dx.$$

Hence,

$$(15) \quad s \int_0^\infty e^{-st} N(t) dt = s \int_0^\infty e^{-st} \int_{X_0} \theta^T(x, x, t) h(x) dx dt.$$

Since $\theta^T(x, x, t)$ has uniform polynomial growth (see [Section 2.4](#)) we may use Fubini's theorem to interchange the integral and obtain

$$(16) \quad s \int_0^\infty e^{-st} N(t) dt = \int_{X_0} \int_0^\infty s e^{-st} \theta^T(x, x, t) dt h(x) dx = \int_{X_0} \Gamma_s^T(x, x) h(x) dx,$$

which proves the integral formula. □

Proposition 11 and **Theorem 13** directly imply the following asymptotic estimate for the heat kernel:

Proposition 27 For each point $x \in X_0$, as $s \rightarrow 0$ we have

$$\Gamma_s^T(x, x) \sim s^{-m/2} \Gamma\left(\frac{m}{2} + 1\right) \frac{\omega_m}{h(x)(2\pi)^m}.$$

4.2 Uniform estimates for the heat kernel

Proposition 27 in the previous section can be restated by saying that the function $h(x)s^{m/2}\Gamma_s^T(x, x)$, as a function of s and x , converges pointwise to a constant as $s \rightarrow 0$. Such convergence however might not be uniform, as X_0 is not compact in general. Nevertheless, the main result in this section uses **Theorem D** to show that this function is uniformly bounded. This will allow us to use the Dominated Convergence Theorem which, together with the Tauberian Theorem, will give Weyl's law.

Theorem 28 Let M^n be a compact n -dimensional manifold and let $\mathcal{F}$ be a closed singular Riemannian foliation with basic mean curvature. Then there exists a constant C such that for all $x \in X_0$,

$$|s^{m/2}\Gamma_s^T(x, x)h(x)| < C,$$

where $m = \dim(X_0)$.

To prove **Theorem 28**, we first relate the heat kernel of Δ to that of T .

Lemma 29 Let Γ_s^T and Γ_s^Δ be the heat kernel of T and Δ constructed by (3). Then for each pair of points $p, q \in M_0$, we have

$$\Gamma_s^T(\sigma(p), \sigma(q)) = (\text{Av} \otimes \text{Av})\Gamma_s^\Delta(p, q).$$

Proof Let $\{\phi_k\}$ and $\{\varphi_i\}$ be the orthonormal basis of $L^2(M_0)$ and $L^2(X_0, h)$ constructed in **Remark 9** and let $\theta^T(x, y; t)$ and $\theta^\Delta(x, y; t)$ be the spectral functions of T and Δ respectively.

Since $\theta^T(x, y; t)$ has uniform polynomial growth, **Proposition 10** and Fubini's theorem imply

$$\begin{aligned} \Gamma_s^T(\sigma(p), \sigma(q)) &= s \int_0^\infty e^{-st} \theta^T(\sigma(p), \sigma(q); t) dt = s \int_0^\infty e^{-st} (\text{Av} \otimes \text{Av}) \theta^\Delta(p, q; t) dt \\ &= (\text{Av} \otimes \text{Av}) \Gamma_s^\Delta(p, q). \end{aligned} \quad \square$$

Proof of Theorem 28 Let $\mathcal{F}$ be a closed singular Riemannian foliation with basic mean curvature and $\dim(X_0) = m$. Let $x \in X_0$ and $p \in \sigma^{-1}(x)$. By **Lemma 29** and **Proposition 12**,

$$\begin{aligned} |s^{m/2}\Gamma_s^T(x, x)h(x)| &= \left| \frac{s^{m/2}}{\text{vol}(L_p)} \int_{L_p} \int_{L_p} \Gamma_s^\Delta(p', q') dp' dq' \right| \\ &\leq \sup_{q \in L_p} s^{m/2} \int_{L_p} |\Gamma_s^\Delta(p', q)| dp' \leq C_1 \sup_{q \in L_p} s^{(m-n)/2} \int_{L_p} e^{-C_2 d_M^2(p', q)/s} dp'. \end{aligned}$$

Since $\dim(L_p) = n - m$, the theorem follows from **Theorem D**. $\square$

4.3 Proof of Weyl's law

Proof of Theorem A Since X_0 has finite volume, Proposition 27, Theorem 28 and the dominated convergence theorem imply that

$$\int_{X_0} \Gamma_s^T(x, x) h(x) dx \sim s^{-m/2} \Gamma\left(\frac{m}{2} + 1\right) \frac{\omega_m}{(2\pi)^m} \text{vol}(X_0).$$

Theorem A then follows from Proposition 26 (trace formula) and Theorem 13 (the Tauberian theorem). $\square$

5 Foliations of the sphere and invariant theory

In this section we consider the special case of closed singular Riemannian foliations $(M, \mathcal{F})$ where M is a round sphere. The main goal is to relate the volume of the leaf space to algebraic invariants of the algebra of basic polynomials (see Section 2.1.2), culminating in the proof of Theorem C. This is an application of Weyl's law (Theorem A) and the theory of spherical harmonics.

5.1 The Cohen–Macaulay property and Hilbert series

Let V be an $(n+1)$ -dimensional real vector space with inner product, and let $\mathbb{R}[V]$ denote the $\mathbb{R}$ -algebra of real-valued polynomial functions on V . Fix an orthonormal basis for V , via which the algebra $\mathbb{R}[V]$ is isomorphic to $\mathbb{R}[x_0, \dots, x_n]$.

Following [Mendes and Radeschi 2020a, Definition 4]:

Definition 30 A subalgebra $A \subset \mathbb{R}[V]$ is called *Laplacian* if it contains $r^2 = \sum_i x_i^2$ and is preserved by the Laplace operator $\Delta = \sum_i \partial^2 / \partial x_i^2$.

The typical examples of Laplacian algebras are rings of invariants $A = \mathbb{R}[V]^G$, where G is a group acting on V by linear isometries; and the ring of basic polynomials $A = \mathcal{B}(\mathcal{F})$, where $\mathcal{F}$ is an infinitesimal singular Riemannian foliation of V . More generally, by Mendes and Radeschi [2020a; 2023], Laplacian algebras are in one-to-one correspondence with infinitesimal manifold submetries of V .

Definition 31 [Derksen and Kemper 2015, Definition 2.5.6] Let $A \subset \mathbb{R}[V]$ be a graded subalgebra. A subset $\{f_0, \dots, f_m\} \subset A$ of homogeneous elements is called a *homogeneous system of parameters* if they are algebraically independent, and A is a finitely generated module over the subalgebra $\mathbb{R}[f_0, \dots, f_m]$.

We note that finitely generated graded subalgebras $A \subset \mathbb{R}[V]$ always have a homogeneous system of parameters, by the Noether normalization lemma; see [Derksen and Kemper 2015, Corollary 2.5.8].

Definition 32 [Derksen and Kemper 2015, Proposition 2.6.3] Let $A \subset \mathbb{R}[V]$ be a finitely generated graded subalgebra. We say A is *Cohen–Macaulay* if, for every homogeneous system of parameters $f_0, \dots, f_m$, the algebra A is a *free* module over $\mathbb{R}[f_0, \dots, f_m]$.

We note that, in the above definition, the quantifier “for every” can be replaced with “there exists”, resulting in an equivalent definition; see [Derksen and Kemper 2015, Proposition 2.6.3].

Our first result is the observation that the celebrated Hochster–Roberts theorem [Derksen and Kemper 2015, Theorem 2.6.5] generalizes from rings of invariants to arbitrary Laplacian algebras:

Proposition 33 *Suppose that $A \subset \mathbb{R}[V]$ is a Laplacian algebra. Then A is graded, finitely generated, and Cohen–Macaulay.*

Proof By [Mendes and Radeschi 2020a, Corollary 22], the Laplacian algebra A is graded, and by [Mendes and Radeschi 2020a, Lemma 24], it is finitely generated. By [Derksen and Kemper 2015, Theorem 2.6.11], it suffices to show that, for every ideal $I \subset A$, one has $(I\mathbb{R}[V]) \cap A = I$.

By [Mendes and Radeschi 2020a, Theorem 23], there exists a Reynolds operator $\Pi: \mathbb{R}[V] \rightarrow A$. This means Π is a linear projection onto A , and $\Pi(fg) = f\Pi(g)$ for all $f \in A$ and $g \in \mathbb{R}[V]$. Let $I \subset A$ be an ideal. The inclusion $I \subset (I\mathbb{R}[V]) \cap A$ is clear. To prove the reverse inclusion, let $f \in (I\mathbb{R}[V]) \cap A$. Then $f = \sum_i g_i f_i$ for $f_i \in I$ and $g_i \in \mathbb{R}[V]$. Applying Π to this equation, and using the assumption that $f \in A$, so that $\Pi(f) = f$, we obtain $f = \sum_i f_i \Pi(g_i)$, which proves that $f \in I$, because I is an ideal of A , and $\Pi(g_i) \in A$. Therefore $(I\mathbb{R}[V]) \cap A = I$, and A is Cohen–Macaulay. $\square$

We note that the proof that $(I\mathbb{R}[V]) \cap A = I$ above is analogous to the proof of [Derksen and Kemper 2015, Lemma 2.6.10].

Definition 34 Given any graded algebra $A = \bigoplus_{k=0}^{\infty} A_k$, we define its Hilbert series by

$$H(z) = \sum_{k=0}^{\infty} \dim(A_k) z^k.$$

The next lemma is an application of Weyl’s law (Theorem A) together with the theory of spherical harmonics to relate the volume of the leaf space to the growth of coefficients of a Hilbert series.

Lemma 35 *Let $\mathcal{F}$ be a closed infinitesimal singular Riemannian foliation of V , and let $X = \mathbb{S}V/\mathcal{F}$ be the leaf space. Let $A \subset \mathbb{R}[V]$ be the associated Laplacian algebra of basic polynomials, and let $H(z)$ denote its Hilbert series. Then the coefficients of the power series*

$$(1+z)H(z) = \sum_{k=0}^{\infty} b_k z^k$$

satisfy

$$b_k \sim \frac{\text{vol}(X)\omega_m}{(2\pi)^m} k^m \quad \text{as } k \rightarrow \infty,$$

where $m = \dim X$ and $\omega_m = \text{vol } \mathbb{D}^m$.

Proof The singular Riemannian foliation $\mathcal{F}$ has basic mean curvature; see [Section 2.1.3](#). By [Theorem A](#), the counting function $N(t)$ for the basic spectrum satisfies

$$(17) \quad N(t) \sim \frac{\text{vol}(X)\omega_m}{(2\pi)^m} t^{m/2} \quad \text{as } t \rightarrow \infty.$$

By the theory of spherical harmonics (see eg [\[Howe and Tan 1992, Exercise 12\(e\), page 118\]](#)), the eigenvalues of the sphere $\mathbb{S}V$ are $\{k(k+n-1) \mid k = 0, 1, \dots\}$, with corresponding eigenspaces the restrictions to $\mathbb{S}V$ of the space $\mathcal{H}_k \subset \mathbb{R}[V]_k$ of all homogeneous harmonic polynomials of degree k . Moreover, one has the direct sum decomposition

$$\mathbb{R}[V]_k = \mathcal{H}_k \oplus r^2 \mathbb{R}[V]_{k-2} \quad \text{for all } k \geq 2.$$

Since A is Laplacian, we obtain the analogous decomposition

$$A_k = (A \cap \mathcal{H}_k) \oplus r^2 A_{k-2}$$

see [\[Mendes and Radeschi 2023, Lemma 4\(b\)\]](#). Note that the basic eigenvalues are $k(k+n-1)$, with multiplicity $\dim(A \cap \mathcal{H}_k)$. Denoting by $G(z)$ the corresponding generating function

$$G(z) = \sum_{k=0}^{\infty} \dim(A \cap \mathcal{H}_k) z^k,$$

we obtain the formula $G(z) = (1-z^2)H(z)$, and thus

$$(1+z+z^2+\dots)G(z) = (1+z)H(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Note that

$$b_k = \sum_{j=0}^k \dim(A \cap \mathcal{H}_j) = N(k(k+n-1)).$$

Using [\(17\)](#), we obtain the desired asymptotic. □

The next lemma is purely algebraic, and will be used in combination with [Lemma 35](#) in the proof of [Theorem C](#). Moreover, part (b) gives algebraic constraints on the Hilbert series of the algebra of basic polynomials of an infinitesimal singular Riemannian foliation. These constraints seem to be new, and may be of independent interest. We remind the reader that here, and throughout this article, $f(t) \sim g(t)$ as $t \rightarrow \infty$ means that $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$.

Lemma 36 *Let $A \subset \mathbb{R}[V]$ be a graded, finitely generated, Cohen–Macaulay algebra, and denote its Hilbert series by $H(z)$. Let $f_0, \dots, f_m \in A$ be a homogenous system of parameters (so $m+1$ is the Krull dimension of A). Assume that the coefficients of the power series*

$$(1+z)H(z) = \sum_{k=0}^{\infty} b_k z^k$$

satisfy

$$(18) \quad b_k \sim Ck^{m'} \quad \text{as } k \rightarrow \infty$$

for some $C \neq 0$ and $m' \in \mathbb{N}$.

Then:

- (a) $m = m'$.
- (b) $H(z)$ is a rational function whose poles are roots of unity, such that $z = 1$ is a pole of order $m + 1$, $z = -1$ is (possibly) a pole of order at most $m + 1$. All other poles have order at most m .
- (c) The Laurent series of $H(z)$ at $z = 1$ has the form

$$\frac{Cm!}{2}(1-z)^{-m-1} + (\cdots)(1-z)^{-m} + \cdots.$$

- (d) The constant C appearing in (18) is given by

$$C = \frac{2 \dim_{\mathbb{R}[f_0, \dots, f_m]} A}{m! \deg f_0 \cdots \deg f_m},$$

where $\dim_{\mathbb{R}[f_0, \dots, f_m]} A$ is the rank of A as a free, finitely generated module over $\mathbb{R}[f_0, \dots, f_m]$.

Proof Since the algebra A is Cohen–Macaulay, it is a finite free module over $\mathbb{R}[f_0, \dots, f_m]$. Let $g_1, \dots, g_\ell$ be a basis consisting of homogeneous elements of A , so that $\ell = \dim_{\mathbb{R}[f_0, \dots, f_m]} A$. Thus A has the presentation, sometimes called a “Hironaka decomposition”,

$$A = \mathbb{R}[f_0, \dots, f_m] g_1 \oplus \cdots \oplus \mathbb{R}[f_0, \dots, f_m] g_\ell,$$

which implies that the Hilbert series $H(z)$ has the form

$$(19) \quad H(z) = \frac{z^{\deg(g_1)} + \cdots + z^{\deg(g_\ell)}}{(1 - z^{\deg(f_0)}) \cdots (1 - z^{\deg(f_m)})}.$$

The first sentence of (b) follows immediately from this formula, that is: 1 is a pole of order $m + 1$ and all poles have order at most $m + 1$. Moreover, factoring $(1 - z)^{-m-1}$ out of the above expression, and plugging $z = 1$ in the remaining term, we obtain the Laurent series of $H(z)$ at $z = 1$, namely

$$\frac{Dm!}{2}(1-z)^{-m-1} + (\cdots)(1-z)^{-m} + \cdots, \quad \text{where } D = \frac{2 \dim_{\mathbb{R}[f_0, \dots, f_m]} A}{m! \deg f_0 \cdots \deg f_m}.$$

To prove (c) and (d), it remains to prove that $D = C$, where C is the constant that appears in the hypothesis (18) above.

In order to make use of (18), we consider the partial fraction decomposition of $(1 + z)H(z)$, and separate the terms associated to poles of maximum possible order, namely $m + 1$. These terms will dominate the behavior of b_k as $k \rightarrow \infty$.

More precisely, let us denote by $\Omega' \subset \mathbb{C}$ the (finite) set of all poles of $(1 + z)H(z)$, and writing $d = \gcd(\deg(f_0), \dots, \deg(f_m))$, let $\Omega \subset \mathbb{C}$ denote the set of all d^{th} roots of unity, that is, the “potential” poles of order $m + 1$. For each $\omega \in \Omega'$, denote by $d(\omega)$ the order of ω as a pole of $(1 + z)H(z)$. Note that every $\omega \in \Omega' \setminus \Omega$ is a root of unity with order $d(\omega) \leq m$.

The partial fraction decomposition of $(1+z)H(z)$ is

$$(20) \quad (1+z)H(z) = P(z) + \sum_{\omega \in \Omega} \alpha_{\omega} (\omega - z)^{-m-1} + \sum_{\omega \in \Omega'} \sum_{j=1}^{d_{\omega}} \beta_{j,\omega} (\omega - z)^{-j},$$

where $P(z)$ is the polynomial part of $(1+z)H(z)$ and $d_{\omega} = \min(d(\omega), m)$. Thus the first sum above corresponds to poles of order $m+1$, and part of our goal is to show that $\alpha_{\omega} = 0$ for all $\omega \in \Omega \setminus \{1\}$. Using the binomial series, we obtain

$$(21) \quad (\omega - z)^{-j} = \omega^{-j} (1 - (\omega^{-1}z))^{-j} = \omega^{-j} \sum_{k=0}^{\infty} \binom{k+j-1}{k} \omega^{-k} z^k.$$

Putting equations (20) and (21) together, the formula for $(1+z)H(z)$ becomes

$$P(z) + \sum_k \left(\sum_{\omega \in \Omega} \alpha_{\omega} \omega^{-k-m-1} \binom{k+m}{k} + \sum_{\omega \in \Omega'} \sum_{j=1}^{d_{\omega}} \beta_{j,\omega} \omega^{-k-j} \binom{k+j-1}{k} \right) z^k$$

For fixed j , the binomial $\binom{k+j-1}{k}$ is a polynomial in k with leading term $k^{j-1}/(j-1)!$ so we get

$$b_k \sim \left(\sum_{\omega \in \Omega} \frac{1}{m!} \alpha_{\omega} \omega^{-k-m-1} \right) k^m + (\text{lower-order terms}) \quad \text{as } k \rightarrow \infty.$$

We note that the sum multiplying k^m is periodic in k . If it vanished identically, that is,

$$\sum_{\omega \in \Omega} \frac{1}{m!} \alpha_{\omega} \omega^{-k-m-1} = 0 \quad \text{for all } k,$$

then, averaging over all k from 1 to d , we would get

$$0 = \frac{1}{d} \sum_{k=1}^d \sum_{\omega \in \Omega} \frac{1}{m!} \alpha_{\omega} \omega^{-k-m-1} = \frac{1}{d} \sum_{\omega \in \Omega} \frac{1}{m!} \alpha_{\omega} \omega^{-m-1} \left(\sum_{k=1}^d \omega^{-k} \right) = \frac{1}{m!} \alpha_1,$$

where the last equality follows from $\sum_{k=1}^d \omega^{-k} = 0$ for $\omega \in \Omega \setminus \{1\}$. Since $\alpha_1 \neq 0$ (because 1 is a pole of order $m+1$), we thus know that the asymptotic behavior of b_k as $k \rightarrow \infty$ is dominated by the term

$$\frac{1}{m!} \left(\sum_{\omega \in \Omega} \alpha_{\omega} \omega^{-m-1} \omega^{-k} \right) k^m \quad \text{as } k \rightarrow \infty.$$

Our assumption that

$$b_k \sim C k^{m'} \quad \text{as } k \rightarrow \infty$$

implies that $m = m'$ (proving (a)), and that the periodic function

$$\frac{1}{m!} \left(\sum_{\omega \in \Omega} \alpha_{\omega} \omega^{-m-1} \omega^{-k} \right)$$

is constant (hence equal to its average) and equal to C :

$$(22) \quad \frac{1}{m!} \alpha_1 = \frac{1}{m!} \sum_{\omega \in \Omega} \alpha_{\omega} \omega^{-m-1} \omega^{-k} = C \quad \text{for all } k.$$

Using the partial fraction decomposition (20), this implies $C = D$, which finishes the proof of (c) and (d). Similarly, fixing $\omega_0 \in \Omega \setminus \{1\}$, multiplying (22) with ω_0^k , and adding for $k = 1, \dots, d$ yields $\alpha_{\omega_0} = 0$, thus proving (b). $\square$

Remark 37 The arguments in the proof above are standard in enumerative combinatorics; see for example [Wilf 2006, Chapter 5] for a general reference, and [Israel 2012] for the specific inspiration.

Remark 38 The proof of Lemma 36 could be considerably shortened, in the case of Laplacian algebras, if we knew that a degree-2 generator (eg r^2) were part of a system of homogeneous parameters. We believe this is the case, but cannot prove it. Furthermore, we point out that the conclusions of Lemma 36 do not hold for general Cohen–Macaulay algebras. For example in $\mathbb{R}[t^3]$ a system of parameters is just $f_0 = t^3$ (hence $m = 0$) but in the series $(1+z)H(z) = \sum_k b_k t^k$, the limit b_k/k^m oscillates between 0 and 1 — thus $b_k \not\sim Ck^{m'}$ for any m' . The authors do not know, however, whether every Cohen–Macaulay algebra containing r^2 satisfies the conclusions of the lemma.

5.2 Proof of Theorem C and its representation-theoretic analogue

In this section we prove the following, slightly more precise version of Theorem C:

Theorem 39 Let $(V, \mathcal{F})$, with $\dim V = n + 1$, be an infinitesimal foliation with closed leaves. Let $A = \bigoplus_i A_i \subset \mathbb{R}[V]$ be the associated Laplacian algebra of basic polynomials, and let $H(z) = \sum_i (\dim A_i) z^i$ denote its Hilbert series. Denote by SV the unit sphere in V , and by m the dimension of the leaf space $X = SV/\mathcal{F}$.

Then:

- (a) The Laurent series of $H(z)$ at $z = 1$ has the form

$$H(z) = \frac{\text{Vol}(X)}{\text{Vol}(\mathbb{S}^m)} (1-z)^{-m-1} + (\dots)(1-z)^{-m} + \dots.$$

- (b) A is a Cohen–Macaulay algebra with Krull dimension $m + 1$, and, for every homogeneous system of parameters $f_0, \dots, f_m \in A$,

$$\frac{\text{Vol}(X)}{\text{Vol}(\mathbb{S}^m)} = \frac{\dim_{\mathbb{R}[f_0, \dots, f_m]} A}{\deg f_0 \cdots \deg f_m},$$

where $\dim_{\mathbb{R}[f_0, \dots, f_m]} A$ is the rank of A as a finite free module over its subalgebra $\mathbb{R}[f_0, \dots, f_m]$.

Proof By [Proposition 33](#), the algebra A is graded, finitely generated, and Cohen–Macaulay.

By [Lemma 35](#), the coefficients of the power series

$$(1+z)H(z) = \sum_{k=0}^{\infty} b_k z^k \quad \text{satisfy} \quad b_k \sim C k^m \quad \text{as } k \rightarrow \infty,$$

where $C = (\text{vol}(X)\omega_m)/(2\pi)^m$, $m = \dim X$ and $\omega_m = \text{vol } \mathbb{D}^m$.

By [Lemma 36](#), the Krull dimension of A is $m+1$, the leading term in the Laurent series of $H(z)$ at $z=1$ is $\frac{1}{2}Cm!(1-z)^{-m-1}$, and

$$C = \frac{2 \dim_{\mathbb{R}[f_0, \dots, f_m]} A}{m! \deg f_0 \cdots \deg f_m}.$$

Thus

$$\frac{\text{vol}(X)}{\text{vol}(\mathbb{S}^m)} = \frac{\dim_{\mathbb{R}[f_0, \dots, f_m]} A}{\deg f_0 \cdots \deg f_m} \cdot \frac{2(2\pi)^m}{m! \omega_m \text{vol}(\mathbb{S}^m)}.$$

Finally, it remains to show that $\text{vol}(\mathbb{S}^m)\omega_m = 2(2\pi)^m/m!$. This can be done with a straightforward induction argument using the recursive formulas $\omega_{i+1} = \text{vol}(\mathbb{S}^i)/(i+1)$ and $\text{vol}(\mathbb{S}^{i+1}) = 2\pi\omega_i$. $\square$

The analogous result to [Theorem C](#) for orthogonal representations is:

Proposition 40 *Let V be the Euclidean vector space of dimension $n+1$, and $G \rightarrow O(V)$ an orthogonal representation of the compact group G . Let $A = \mathbb{R}[V]^G \subset \mathbb{R}[V]$ be the associated algebra of invariant polynomials, and let $H(z)$ denote its Hilbert series. Denote by $\mathbb{S}V$ the unit sphere in V , and by m the dimension of the orbit space $X = \mathbb{S}V/G$.*

Then:

- (a) *The Laurent series of $H(z)$ at $z=1$ has the form*

$$H(z) = \frac{\text{Vol}(X)}{\text{Vol}(\mathbb{S}^m)}(1-z)^{-m-1} + (\cdots)(1-z)^{-m} + \cdots.$$

- (b) *A is a Cohen–Macaulay algebra with Krull dimension $m+1$, and, for every homogeneous system of parameters $f_0, \dots, f_m \in A$,*

$$\frac{\text{Vol}(X)}{\text{Vol}(\mathbb{S}^m)} = \frac{\dim_{\mathbb{R}[f_0, \dots, f_m]} A}{\deg f_0 \cdots \deg f_m},$$

where $\dim_{\mathbb{R}[f_0, \dots, f_m]} A$ is the rank of A as a finite free module over its subalgebra $\mathbb{R}[f_0, \dots, f_m]$.

Proof The proof is completely analogous to the proof of [Theorem 39](#), except that the analogue of [Lemma 35](#) for representations is proved using the main results of [[Donnelly 1978](#); [Brüning and Heintze 1978/79](#)] instead of [Theorem A](#). $\square$

Even though [Proposition 40](#) follows from [[Donnelly 1978](#); [Brüning and Heintze 1978/79](#)] using classical results, to the best of the authors' knowledge this fact has not been noticed for general (infinite) groups G .

5.3 Examples

In this section we give examples to illustrate [Theorem C](#) and its version for representations, [Proposition 40](#).

Example 41 Suppose G is a *finite* group, and $G \rightarrow O(V)$ is injective (ie faithful). Then the orbit space $X = \mathbb{S}V/G$ is obtained from a fundamental domain of the action by boundary identifications. In particular, $\dim X = \dim \mathbb{S}V$ and $\text{Vol}(X)/\text{Vol}(\mathbb{S}V) = |G|^{-1}$. Moreover, [Proposition 40\(a\)](#) follows immediately from Molien's formula; see [\[Neusel and Smith 2002, Proposition 3.1.4\]](#).

Specializing even more:

Example 42 Suppose $G \subset O(V)$ is a finite group generated by reflections. Then $A = \mathbb{R}[V]^G$ is free by the Chevalley–Shephard–Todd theorem [\[Neusel and Smith 2002, Theorem 7.1.4\]](#), so that [Proposition 40\(b\)](#) reduces to the well-known fact that

$$|G| = \deg f_0 \cdots \deg f_m$$

for an algebraically independent generating set $f_0, \dots, f_m$ of $\mathbb{R}[V]^G$; see [\[Neusel and Smith 2002, Proposition 3.1.5\]](#).

An example illustrating [Theorem C](#):

Example 43 (Clifford foliations) The assertions and definitions below can be found in [\[Radeschi 2014\]](#).

Let $V = \mathbb{R}^{2l}$ and $C = (P_0, \dots, P_m)$ be a Clifford system on V , which implies $m \leq l$. Let $\psi: V \rightarrow \mathbb{R}^{m+2}$ be the polynomial map given by

$$\psi(x) = (\psi_1, \dots, \psi_{m+2})(x) = (\|x\|^2, \langle P_0 x, x \rangle, \dots, \langle P_m x, x \rangle).$$

If $m \neq l-1$, the fibers of ψ form an infinitesimal singular Riemannian foliation $\mathcal{F}_C$ of V , called a *Clifford foliation*. The leaf space $X = \mathbb{S}V/\mathcal{F}_C$ is isometric to a hemisphere in $\mathbb{S}^{m+1}(\frac{1}{2})$ when $m \leq l-2$, and to $\mathbb{S}^m(\frac{1}{2})$ when $m = l$. Thus the quotient of volumes $\text{vol } X / \text{vol}(\mathbb{S}^{\dim X})$ appearing in [Theorem C](#) is 2^{-m-2} , respectively 2^{-m} .

On the other hand, $\psi_1, \dots, \psi_{m+2}$ actually generate the algebra A of basic polynomials of $\mathcal{F}_C$; see [\[Mendes and Radeschi 2020b, proof of Theorem C\]](#). When $m \leq l-2$, they are algebraically independent because the image of ψ has nonempty interior. Each ψ_i has degree two, so this confirms the conclusion of [Theorem C\(b\)](#).

When $m = l$, there is one relation between the generators, namely $\psi_1^2 = \psi_2^2 + \dots + \psi_{m+2}^2$. The polynomials $\psi_1, \dots, \psi_{m+1}$ form a homogeneous system of parameters, and A is a free module of rank 2 over $\mathbb{R}[\psi_1, \dots, \psi_{m+1}]$. Thus the right-hand side of the equation in [Theorem C\(b\)](#) is $2/2^{m+1} = 2^{-m}$, as expected.

Example 44 (Hopf fibrations) Special cases of the previous example (all with $m = l$) are the infinitesimal singular Riemannian foliations of $\mathbb{R}^4 = \mathbb{C}^2$, $\mathbb{R}^8 = \mathbb{H}^2$ and $\mathbb{R}^{16} = \mathbb{C}a^2$, whose restrictions to the unit sphere form the fibers of one of the Hopf fibrations $\mathbb{S}^3 \rightarrow \mathbb{S}^2(\frac{1}{2})$, $\mathbb{S}^7 \rightarrow \mathbb{S}^4(\frac{1}{2})$, and $\mathbb{S}^{15} \rightarrow \mathbb{S}^8(\frac{1}{2})$.

Concretely in the complex case, the infinitesimal singular Riemannian foliation of $\mathbb{R}^4 = \mathbb{C}^2$ is given by the orbits under the action of the unit complex numbers $a \in \mathbb{S}^1$ by $a(z, w) = (az, aw)$. The algebra A of invariants is generated by the degree-2 real polynomials $\psi_1 = |z|^2 + |w|^2$, $\psi_2 = |z|^2 - |w|^2$, $\psi_3 = 2 \operatorname{Re}(z\bar{w})$ and $\psi_4 = 2 \operatorname{Im}(z\bar{w})$. The only relation is $\psi_1^2 = \psi_2^2 + \psi_3^2 + \psi_4^2$. A possible choice of system of parameters is ψ_1, ψ_2, ψ_3 , for which $\dim_{\mathbb{R}[\psi_1, \psi_2, \psi_3]} A = 2$, thus illustrating part (b) of either [Theorem C](#) or [Proposition 40](#), because $2/2^3 = \operatorname{vol} \mathbb{S}^2(\frac{1}{2}) / \operatorname{vol} \mathbb{S}^2(1)$.

6 A note on manifold submetries

An even more general concept than closed singular Riemannian foliations is given by manifold submetries, which are maps $\sigma: M \rightarrow X$, from a smooth Riemannian manifold to a metric space, whose fibers are smooth and mutually equidistant. This generalizes closed singular Riemannian foliations in essentially two ways, first by not requiring the presence of smooth vector fields generating the tangent spaces of the fibers (although it is conjectured that such vector fields would exist anyway), and by allowing the fibers to be disconnected. The structure of submetries from Riemannian manifolds (and, in particular, manifold submetries) has been studied by Kapovitch and Lytchak [\[2022\]](#). In the particular case where M is the unit sphere in $\mathbb{R}^n$, it was proved by Mendes and Radeschi [\[2020a; 2023\]](#) that there is a one-to-one correspondence between manifold submetries from M and Laplacian algebras contained in $\mathbb{R}[x_1, \dots, x_n]$.

The statement of [Theorem A](#) still makes sense for manifold submetries, and the authors believe it still holds. However, the main obstacle to generalizing our proof to the manifold submetry case is the lack of a slice theorem; see [Section 2.1.4](#). More specifically, given $L = \sigma^{-1}(x)$, the differential $d_p\sigma: \mathbb{S}(v_p L) \rightarrow \Sigma_x X$ (where $\Sigma_x X$ is the space of directions of X at x) might, in principle, depend on the choice of $p \in L$.

The slice theorem was used in a fundamental way in the inductive step to prove the geometric estimate [\(12\)](#) in the proof of [Theorem D](#). In view of the discussion above, proving that the estimate [\(12\)](#) still holds essentially requires a stronger version of [Theorem D](#), applied to a manifold submetry $\sigma: M \rightarrow X$, where the constant C only depends on X , rather than the specific σ .

After this paper was finished, we were informed by A Lytchak of his new manuscript [\[Lytchak 2024\]](#), in which he obtains a number of results bounding the geometry of leaves in a manifold submetry $\sigma: M \rightarrow X$, which only depend on the geometry of M and X rather than on σ itself. In particular, he obtains a more general and accurate version of [Lemma 25](#); see Proposition 1.3 and Corollary 7.4 of [\[Lytchak 2024\]](#). The authors believe that the results in [\[Lytchak 2024\]](#), together with previous results on the structure of transnormal submetries in [\[Kapovitch and Lytchak 2022\]](#), could constitute important ingredients to generalizing [Theorem D](#) in the way mentioned above, and hence [Theorem A](#), to the manifold submetry case.

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The tautological ring of $\overline{\mathcal{M}}_{g,n}$ is rarely Gorenstein

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We prove that the tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ and $RH^*(\overline{\mathcal{M}}_{g,n})$ are not Gorenstein when $g \geq 2$ and $2g + n \geq 24$, extending results of Petersen and Tommasi in genus 2. The proof uses the intersection of tautological classes with nontautological bielliptic cycles. We conjecture the converse: the tautological rings should be Gorenstein when $g = 0, 1$ or $g \geq 2$ and $2g + n < 24$. The conjecture is known for $g = 0, 1$ by work of Keel and Petersen, and we prove several new cases of this conjecture for $RH^*(\overline{\mathcal{M}}_{g,n})$ when $g \geq 2$.

14C15, 14C17

1 Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space of stable curves of genus g with n markings. The tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ is the $\mathbb{Q}$ -subalgebra of the Chow ring of $\overline{\mathcal{M}}_{g,n}$ generated by the decorated strata classes, and its image in cohomology under the cycle class map is $RH^*(\overline{\mathcal{M}}_{g,n})$. Decorated strata classes are Chow classes corresponding to stable graphs of genus g with n legs, decorated by κ classes on the vertices and ψ classes on the legs and half-edges; see Graber and Pandharipande [12] and Pandharipande [21]. Finding a complete set of relations among these generators is a fundamental open problem in the intersection theory of $\overline{\mathcal{M}}_{g,n}$. Pixton [26] conjectured relations among the tautological generators—the *3-spin relations*, also known as the generalized Faber–Zagier relations or Pixton’s relations—that were shown to hold in cohomology by Pandharipande, Pixton and Zvonkine [22] and in Chow by Janda [15]. No other relations have been found, despite significant investigation; see Clader and Janda [7] and Janda [16].

Conjecture 1 (Pixton) *The 3-spin relations are complete for $R^*(\overline{\mathcal{M}}_{g,n})$ and $RH^*(\overline{\mathcal{M}}_{g,n})$.*

Before Pixton’s conjecture, it had been speculated that the tautological ring is Gorenstein (see Faber and Pandharipande [10], Hain and Looijenga [14] and Pandharipande [20]), extending Faber’s conjecture [9] on $R^*(\mathcal{M}_g)$. We say $R^*(\overline{\mathcal{M}}_{g,n})$ is Gorenstein if all the intersection pairings

$$R^i(\overline{\mathcal{M}}_{g,n}) \times R^{3g-3+n-i}(\overline{\mathcal{M}}_{g,n}) \rightarrow R^{3g-3+n}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$$

are perfect, and similarly for $RH^*(\overline{\mathcal{M}}_{g,n})$. If the Gorenstein property holds, a tautological class is zero if and only if it intersects trivially with every tautological class in complementary codimension, thereby providing an algorithm to determine all of the relations in the tautological ring. The Gorenstein property

holds in genus 0 and 1 (see Keel [17] and Petersen [23]), but Petersen and Tommasi [25] showed it fails in genus 2 when there are sufficiently many marked points. Further investigation by Petersen [24] shows that $R^*(\bar{\mathcal{M}}_{2,n})$ and $RH^*(\bar{\mathcal{M}}_{2,n})$ are not Gorenstein when $n \geq 20$. Our main result is that the tautological rings are Gorenstein in only finitely many cases when $g \geq 2$.

Theorem 2 *Neither $R^*(\bar{\mathcal{M}}_{g,n})$ nor $RH^*(\bar{\mathcal{M}}_{g,n})$ is Gorenstein if $g \geq 2$ and $2g + n \geq 24$.*

In fact, it suffices to prove Theorem 2 for cohomology, as a direct calculation [25, Corollary 2.5] shows that if $RH^*(\bar{\mathcal{M}}_{g,n})$ is not Gorenstein, then neither is $R^*(\bar{\mathcal{M}}_{g,n})$.

The central difficulty in Conjecture 1 is the study of classes in the Gorenstein kernel, classes that intersect trivially with every tautological class in complementary codimension. Conjecture 1 predicts the existence of nonzero classes in the Gorenstein kernel, and proving that such classes are nonzero thus requires the failure of the Gorenstein property. At the moment, there are no known cases where Conjecture 1 holds but the Gorenstein property fails. The proof of Theorem 2 uses a new method for detecting nonzero classes in the Gorenstein kernel: intersecting with nontautological classes. Recently, several ways of studying the cycle theory of $\bar{\mathcal{M}}_{g,n}$ beyond the tautological ring have appeared; see Schmitt and van Zelm [27], Lian [18] and Canning, Larson and Payne [4]. Lian as well as Schmitt and van Zelm propose adding Hurwitz cycles parametrizing covers of curves of positive genus to the calculus. Such cycles tend to be nontautological (see Graber and Pandharipande [12], van Zelm [28], Lian [19] and Arena, Canning, Clader, Haburcak, Li, Mok and Tamborini [1]), whereas the Hurwitz cycles parametrizing covers of genus-0 curves are tautological (see Faber and Pandharipande [11]).

Remark 3 Conjecture 1 and the Gorenstein property have natural analogues for moduli spaces of curves of compact type; see Pixton [26], Faber and Pandharipande [10] and Pandharipande [20]. In Canning, Larson and Schmitt [5], it is shown that if $g \geq 2$ and $2g + n \geq 12$, then neither $R^*(\mathcal{M}_{g,n}^{\text{ct}})$ nor $RH^*(\mathcal{M}_{g,n}^{\text{ct}})$ is Gorenstein. Moreover, [5] shows that the restriction of the 3-spin relations to $\mathcal{M}_6^{\text{ct}}$ are complete, providing the first known case where Pixton's conjecture holds but the tautological ring is not Gorenstein.

The proof of Theorem 2 is carried out in Sections 2, 3 and 4. In Section 2, using direct calculations in the tautological ring (Lemmas 11 and 12), we reduce to the cases when $2g + n = 24$. To prove these cases, we recall in Section 3 the result of Petersen and Tommasi [25] and Petersen [24] when $g = 2$: there exists a nonzero tautological class $\alpha \in RH^{24}(\bar{\mathcal{M}}_{2,20})$ such that $\alpha \cdot \beta = 0$ for all $\beta \in RH^{22}(\bar{\mathcal{M}}_{2,20})$. Let $\xi: \bar{\mathcal{M}}_{2,20} \rightarrow \bar{\mathcal{M}}_{g,n}$ be an iterated self-gluing map, which exists for $(g, n) = (2 + k, 20 - 2k)$ for $0 \leq k \leq 10$, where k is the number of gluings, ie $g \geq 2$ and $2g + n = 24$. We show that $\xi_*\alpha$ pairs trivially with every tautological class in complementary codimension. It thus suffices to show that $\xi_*\alpha \neq 0$, which we do in Section 4 by pulling back to $\bar{\mathcal{M}}_{2,20}$ and intersecting with a nontautological bielliptic cycle. The intersection computation is done using pullback formulas for Hurwitz cycles under ξ^* developed by Schmitt and van Zelm [27].

A key feature of the proof is that it does not require any explicit knowledge of $H^*(\overline{\mathcal{M}}_{g,n})$ when $(g, n) \neq (2, 20)$. Petersen and Tommasi's proof [25] that $\mathrm{RH}^*(\overline{\mathcal{M}}_{2,20})$ is not Gorenstein requires essentially complete knowledge of $H^*(\overline{\mathcal{M}}_{2,20})$, which is obtained through an argument with the Leray spectral sequence and the study of the cohomology of local systems on $\mathcal{A}_2$, where $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties of dimension g . It seems unlikely that this proof strategy will generalize to higher genus. In particular, when $g \geq 4$, $\mathcal{M}_g$ is not an open substack of $\mathcal{A}_g$, so drawing conclusions about the cohomology of $\mathcal{M}_g$ from the cohomology of $\mathcal{A}_g$ is much more difficult. We hope that the idea of reducing to a few pairs (g, n) where both g and n are relatively small may also be useful for studying Conjecture 1.

Remark 4 The proof of Theorem 2 provides a geometric reason for the failure of the Gorenstein property: the existence of the nontautological bielliptic cycle in $H^{22}(\overline{\mathcal{M}}_{2,20})$. The failure of the Gorenstein property for $\mathcal{M}_{g,n}^{\mathrm{ct}}$ does not require the existence of a nontautological cycle; see Canning, Larson and Schmitt [5]. Nevertheless, the failure of the Gorenstein property on $\mathcal{M}_g^{\mathrm{ct}}$ is closely related to the existence of a nontautological class on $\mathcal{A}_g$; see Canning, Oprea and Pandharipande [6].

We conjecture the converse of Theorem 2 should also hold.

Conjecture 5 The tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ and $\mathrm{RH}^*(\overline{\mathcal{M}}_{g,n})$ are Gorenstein if and only if $g = 0, 1$ or $g \geq 2$ and $2g + n < 24$.

Remark 6 If Conjecture 5 is true, the remaining open cases can in principle be checked computationally, using the Sage package `admcycles` [8] to compute pairing matrices. Because of the number of decorated strata classes, this computation is not currently feasible.

In cohomology, some cases of Conjecture 5 follow immediately from results in the literature. In particular, the $g = 0, 1, 2$ cases are known; see Keel [17], Petersen [23; 24] and Petersen and Tommasi [25]. For the $g \geq 3$ cases, one can use that the cohomology rings are generated by tautological classes for many cases when g and n are small; see Canning and Larson [2, Theorem 1.4]. It follows by Poincaré duality that the cohomological tautological ring is Gorenstein in these cases. Table 1 summarizes what is already known from [17; 23; 25; 24; 2].

g	0	1	2	3	4	5	6	7
$c(g)$	∞	∞	20	9	7	5	3	1
$24 - 2g$	24	22	20	18	16	14	12	10

Table 1: $\mathrm{RH}^*(\overline{\mathcal{M}}_{g,n})$ is Gorenstein for $n < c(g)$. When $g \geq 2$ and $n < 24 - 2g$, Conjecture 5 predicts $\mathrm{RH}^*(\overline{\mathcal{M}}_{g,n})$ is Gorenstein.

g	0	1	2	3	4	5	6	7	8
$d(g)$	∞	∞	20	12	10	8	6	4	1
$24 - 2g$	24	22	20	18	16	14	12	10	8

Table 2: The even cohomology of $\bar{\mathcal{M}}_{g,n}$ is tautological for $n < d(g)$. Hence, $\mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$ is Gorenstein for $n < d(g)$. When $g \geq 2$ and $n < 24 - 2g$, [Conjecture 5](#) predicts $\mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$ is Gorenstein.

Remark 7 When $g = 0, 1$, $R^*(\bar{\mathcal{M}}_{g,n}) = \mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$. We do not know if this equality holds when $g = 2$ and $n < c(g)$, but it is true for $g = 2$ and $n < 10$ by [\[2, Theorem 1.4\]](#). For $3 \leq g \leq 7$ and $n < c(g)$, $R^*(\bar{\mathcal{M}}_{g,n}) = \mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$ by [\[2, Theorem 1.4\]](#). Thus, $R^*(\bar{\mathcal{M}}_{g,n})$ is Gorenstein in these cases.

We can prove several more cases by studying just the even cohomology. Given $0 \leq g \leq 8$, we define $d(g)$ as in [Table 2](#).

Theorem 8 For $g \leq 8$ and $n < d(g)$, the even cohomology of $\bar{\mathcal{M}}_{g,n}$ is tautological. Therefore, $\mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$ is Gorenstein for $g \leq 8$ and $n < d(g)$.

We do not know the analogous result for $R^*(\bar{\mathcal{M}}_{g,n})$. In [Section 5](#), we prove [Theorem 8](#) using the stratification of $\bar{\mathcal{M}}_{g,n}$ by topological type and previously known results about the cohomology (both even and odd) of moduli spaces of curves of low genus; see Canning and Larson [\[2\]](#) and Canning, Larson and Payne [\[3; 4\]](#). When $g \geq 2$, [Conjecture 5](#) predicts that $\mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$ is Gorenstein for 110 pairs (g, n) . [Theorem 8](#) shows that $\mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$ is Gorenstein for 61 of those pairs.

When $\mathrm{RH}^*(\bar{\mathcal{M}}_{g,n})$ is known to have the Gorenstein property, the even cohomology algebra $H^{2*}(\bar{\mathcal{M}}_{g,n})$ is tautologically generated. It seems likely that the Gorenstein property is simply an accident, occurring only for small g and n because of the nonexistence of nontautological classes. Together with [Theorem 2](#), [Conjecture 9](#) below implies [Conjecture 5](#) in cohomology by Poincaré duality.

Conjecture 9 If $2g + n < 24$, $H^{2*}(\bar{\mathcal{M}}_{g,n})$ is generated by tautological classes.

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2 Reduction to the case $2g + n = 24$

We first reduce [Theorem 2](#) to the cases when $g \geq 2$ and $2g + n = 24$.

Lemma 10 *The self-gluing and forgetful pushforward maps in top degree*

$\mathrm{RH}^{6g-8+2n}(\overline{\mathcal{M}}_{g-1,n+2}) \rightarrow \mathrm{RH}^{6g-6+2n}(\overline{\mathcal{M}}_{g,n})$ and $\mathrm{RH}^{6g-4+2n}(\overline{\mathcal{M}}_{g,n+1}) \rightarrow \mathrm{RH}^{6g-6+2n}(\overline{\mathcal{M}}_{g,n})$
are isomorphisms.

Proof We have $\mathrm{H}^{6g-6+2n}(\overline{\mathcal{M}}_{g,n}) = \mathrm{RH}^{6g-6+2n}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$ for any g and n by [\[13\]](#). The result follows by Poincaré duality and the fact that zero cycles in homology push forward nontrivially for degree reasons. $\square$

Lemma 11 *If $\mathrm{RH}^*(\overline{\mathcal{M}}_{g,n})$ is not Gorenstein, then $\mathrm{RH}^*(\overline{\mathcal{M}}_{g,n+1})$ is not Gorenstein.*

Proof Let $\pi: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the map forgetting the last marked point. Then

$$\pi^*: \mathrm{RH}^*(\overline{\mathcal{M}}_{g,n}) \rightarrow \mathrm{RH}^*(\overline{\mathcal{M}}_{g,n+1})$$

is injective because for any $\alpha \in \mathrm{RH}^*(\overline{\mathcal{M}}_{g,n})$,

$$\frac{1}{2g-2+n} \pi_*(\pi^* \alpha \cdot \psi_{n+1}) = \alpha.$$

Let $\beta \in \mathrm{RH}^*(\overline{\mathcal{M}}_{g,n})$ be a nonzero class such that

$$\beta \cdot \gamma = 0$$

for all $\gamma \in \mathrm{RH}^*(\overline{\mathcal{M}}_{g,n})$ of complementary degree. Then $\pi^* \beta \neq 0$, and for all $\delta \in \mathrm{RH}^*(\overline{\mathcal{M}}_{g,n+1})$ of complementary degree

$$\pi_*(\pi^* \beta \cdot \delta) = \beta \cdot \pi_* \delta = 0.$$

Because π_* is injective in the top degree by [Lemma 10](#), $\mathrm{RH}^*(\overline{\mathcal{M}}_{g,n+1})$ is not Gorenstein. $\square$

Lemma 12 *If $\mathrm{RH}^*(\overline{\mathcal{M}}_g)$ is not Gorenstein, then $\mathrm{RH}^*(\overline{\mathcal{M}}_{g+1})$ is not Gorenstein.*

Proof Let $\beta \in \mathrm{RH}^k(\overline{\mathcal{M}}_g)$ be a nonzero class such that

$$\beta \cdot \gamma = 0$$

for all $\gamma \in \mathrm{RH}^{6g-6-k}(\overline{\mathcal{M}}_g)$. Set $\alpha = \pi^* \beta$, where $\pi: \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ is the forgetful map. Let

$$\varphi: \overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1} \rightarrow \overline{\mathcal{M}}_{g+1}$$

be the map attaching an elliptic tail. For $\zeta \in \mathrm{RH}^{6g-2-k}(\overline{\mathcal{M}}_{g+1})$, we have

$$\varphi_*(\alpha \otimes 1) \cdot \zeta = \varphi_*((\alpha \otimes 1) \cdot \varphi^* \zeta) \in \mathrm{RH}^{6g}(\overline{\mathcal{M}}_{g+1}) \cong \mathbb{Q}.$$

Consider the class $(\alpha \otimes 1) \cdot \varphi^* \zeta$. Because ζ is tautological,

$$\varphi^* \zeta \in (\mathrm{RH}^{6g-2-k}(\bar{\mathcal{M}}_{g,1}) \otimes \mathrm{RH}^0(\bar{\mathcal{M}}_{1,1})) \oplus (\mathrm{RH}^{6g-4-k}(\bar{\mathcal{M}}_{g,1}) \otimes \mathrm{RH}^2(\bar{\mathcal{M}}_{1,1}))$$

under the Künneth decomposition [12, Proposition 1]. The product of $\alpha \otimes 1$ with any class in the first summand is trivial for dimension reasons. The product of $\alpha \otimes 1$ with the second summand is zero by the argument in the proof of Lemma 11.

It suffices to show that $\varphi_*(\alpha \otimes 1) \neq 0$. By the self-intersection formula [12, equation 11],

$$\varphi^* \varphi_*(\alpha \otimes 1) = \alpha \cdot (\delta_{1,\emptyset} - \psi_1) \otimes 1 + \alpha \otimes (-\psi_1).$$

Here, $\delta_{1,\emptyset}$ is the boundary divisor on $\bar{\mathcal{M}}_{g,1}$ given by the gluing map $\bar{\mathcal{M}}_{1,1} \times \bar{\mathcal{M}}_{g-1,2} \rightarrow \bar{\mathcal{M}}_{g,1}$. We see that

$$\mathrm{pr}_{1*}(\varphi^* \varphi_*(\alpha \otimes 1)) = -\frac{1}{24}\alpha \neq 0,$$

where $\mathrm{pr}_1: \bar{\mathcal{M}}_{g,1} \times \bar{\mathcal{M}}_{1,1} \rightarrow \bar{\mathcal{M}}_{g,1}$ is projection onto the first factor. Thus, $\varphi_*(\alpha \otimes 1) \neq 0$. $\square$

Combining Lemmas 11 and 12, we see it suffices to prove Theorem 2 in the cases when $g \geq 2$ and $2g + n = 24$.

3 Admissible covers and cohomology in genus 2

We collect background information for the proof of Theorem 2 on moduli spaces of admissible double covers and the cohomology of $\bar{\mathcal{M}}_{2,n}$.

3.1 Admissible covers and nontautological cycles

We consider the stacks $\bar{\mathcal{H}}_{g,G,\xi}$ of admissible G -covers of genus g with monodromy data ξ , where G is a finite group, as in [27, Definitions 3.2 and 3.5]. See [27] for complete details. In this paper, we always take $G = \mathbb{Z}/2\mathbb{Z}$. In this special case, the monodromy data is simply a tuple $\xi = (a_1, \dots, a_b)$ of generators for the stabilizer group of points in the fibers of the source curve over b ordered points of the target curve, controlling the ramification behavior. The group law will be written additively, and so ξ has entries in $\{0, 1\}$. An entry $a_i = 1$ corresponds to marking a ramification point of the source curve over the i^{th} marked point of the target curve. An entry $a_j = 0$ corresponds to marking two points of the source pairwise switched by the covering involution over the j^{th} marked point of the target.

As a rule, we will mark every point of ramification when considering moduli of admissible double covers. That is, if we are given the genus h of the target curve, the monodromy data ξ will contain $2g - 4h + 2$ entries 1, the maximal number allowed by the Riemann–Hurwitz formula. Conversely, ξ and g determine h .

The source map

$$\phi: \bar{\mathcal{H}}_{g,G,\xi} \rightarrow \bar{\mathcal{M}}_{g,2g-4h+2+2n_2}$$

sends an admissible cover to its source curve together with all of the marked points determined by the monodromy data ξ ; see [27, Theorem 3.7]. When $\xi = (1^{2g-4h+2}, 0^{n_2})$, we denote the image of the source map by $\overline{\mathcal{B}}_{g \rightarrow h, 2g-4h+2, 2n_2}$. One can further compose with the forgetful map $\overline{\mathcal{M}}_{g, 2g-4h+2+2n_2} \rightarrow \overline{\mathcal{M}}_{g, n_1+2n_2}$, forgetting the first $2g-4h+2-n_1$ points. The image of $\overline{\mathcal{B}}_{g \rightarrow h, 2g-4h+2, 2n_2}$ under this map is denoted by $\overline{\mathcal{B}}_{g \rightarrow h, n_1, 2n_2}$. When $h = 1$, these loci play a fundamental role because of the following result of Graber and Pandharipande [12] (when $g = 2$ and $n_2 = 10$) and van Zelm [28].

Theorem 13 (Graber–Pandharipande, van Zelm) *The class $[\overline{\mathcal{B}}_{g \rightarrow 1, 0, 2n_2}] \in H^{22}(\overline{\mathcal{M}}_{g, 2n_2})$ is nontautological if $g \geq 2$ and $g + n_2 = 12$.*

Remark 14 The proof of Theorem 2 only uses Theorem 13 when $g = 2$. In fact, the $g > 2$ cases of Theorem 13 will follow from the proof of Theorem 2, using equation (3-1) below and the fact that tautological classes pull back to tautological classes. The argument is not the same as van Zelm’s proof of the $g \geq 3$ part of Theorem 13, although it is quite similar.

On the other hand, classes of hyperelliptic loci are tautological:

Theorem 15 (Faber and Pandharipande [11, Proposition 1]) *The class $[\overline{\mathcal{B}}_{g \rightarrow 0, n_1, 2n_2}]$ is tautological for any g, n_1, n_2 .*

3.2 Cohomology in genus 2

The cohomology of $\overline{\mathcal{M}}_{2,n}$ for $n \leq 20$ is completely understood by Petersen and Tommasi [25, Theorems A and B] and Petersen [24, Theorem 3.8].

Theorem 16 (Petersen–Tommasi, Petersen) *Let $k \in \mathbb{N}$ be even.*

- (1) *For $n < 20$, $\mathrm{RH}^k(\overline{\mathcal{M}}_{2,n}) = H^k(\overline{\mathcal{M}}_{2,n})$.*
- (2) *For $k \neq 22$, $\mathrm{RH}^k(\overline{\mathcal{M}}_{2,20}) = H^k(\overline{\mathcal{M}}_{2,20})$.*
- (3) *The image of the pushforward map $H^{k-2}(\widetilde{\partial}\overline{\mathcal{M}}_{2,20}) \rightarrow H^k(\overline{\mathcal{M}}_{2,20})$ lies in $\mathrm{RH}^k(\overline{\mathcal{M}}_{2,20})$, where $\widetilde{\partial}\overline{\mathcal{M}}_{2,20}$ is the normalization of the boundary.*

Theorem 16(2) combined with Theorem 13 implies that $\mathrm{RH}^*(\overline{\mathcal{M}}_{2,20})$ is not Gorenstein.

In contrast to the essentially complete result of Theorem 16, little is known about the structure of the cohomology of $\overline{\mathcal{M}}_{g, 2n_2}$ beyond Theorem 13 when $g \geq 3$ and $g + n_2 = 12$. The proof of Theorem 2 relies on pulling back to $\overline{\mathcal{M}}_{2,20}$ and leveraging the complete description of its cohomology, as follows.

By Poincaré duality and Theorem 16(2), the intersection pairing

$$H^{22}(\overline{\mathcal{M}}_{2,20}) \times \mathrm{RH}^{24}(\overline{\mathcal{M}}_{2,20}) \rightarrow \mathbb{Q}$$

is perfect. Hence, we can identify $\mathrm{RH}^{24}(\bar{\mathcal{M}}_{2,20})$ with the dual of $\mathrm{H}^{22}(\bar{\mathcal{M}}_{2,20})$. Pick $\alpha \in \mathrm{RH}^{24}(\bar{\mathcal{M}}_{2,20})$ such that $\alpha \cdot [\bar{\mathcal{B}}_{2 \rightarrow 1,0,20}] = 1$, but $\alpha \cdot \beta = 0$ for every tautological class $\beta \in \mathrm{RH}^{22}(\bar{\mathcal{M}}_{2,20})$. (The class α can be obtained by completing $[\bar{\mathcal{B}}_{2 \rightarrow 1,0,20}]$ together with a basis for the tautological subspace $\mathrm{RH}^{22}(\bar{\mathcal{M}}_{2,20})$ to a basis for $\mathrm{H}^{22}(\bar{\mathcal{M}}_{2,20})$ and choosing α to be the first basis vector in the dual basis.)

Consider the gluing map

$$\varphi_g: \bar{\mathcal{M}}_{2,20} \rightarrow \bar{\mathcal{M}}_{g,2n_2},$$

given by identifying the last $g - 2$ pairs of marked points on $\bar{\mathcal{M}}_{2,20}$. Note that $g + n_2 = 12$. Let $\gamma \in \mathrm{RH}^{24}(\bar{\mathcal{M}}_{g,2n_2})$. By the projection formula,

$$\varphi_{g*} \alpha \cdot \gamma = \varphi_{g*} (\alpha \cdot \varphi_g^* \gamma) = 0$$

because $\varphi_g^* \gamma$ is tautological. By iteratively applying [Lemma 10](#), φ_{g*} is an isomorphism in the top degree. Therefore, if $\varphi_{g*} \alpha \neq 0$, $\mathrm{RH}^*(\bar{\mathcal{M}}_{g,2n_2})$ is not Gorenstein. In fact, it suffices to show that $\varphi_{12*} \alpha \neq 0$, as φ_g factors φ_{12} . We will show that $\varphi_{12*} \alpha \neq 0$ by computing the intersection of $\varphi_{12*} \alpha$ with the bielliptic locus in genus 12. By the projection formula,

$$\varphi_{12*} \alpha \cdot [\bar{\mathcal{B}}_{12 \rightarrow 1,0,0}] = \varphi_{12*} (\alpha \cdot \varphi_{12}^* [\bar{\mathcal{B}}_{12 \rightarrow 1,0,0}]).$$

By [Lemma 10](#) applied iteratively, it suffices to show that $\alpha \cdot \varphi_{12}^* [\bar{\mathcal{B}}_{12 \rightarrow 1,0,0}] \neq 0$. We claim that

$$(3-1) \quad \varphi_{12}^* [\bar{\mathcal{B}}_{12 \rightarrow 1,0,0}] = c [\bar{\mathcal{B}}_{2 \rightarrow 1,0,20}] + T,$$

where $c \neq 0$ and $T \in \mathrm{RH}^{22}(\bar{\mathcal{M}}_{2,20})$. Assuming [\(3-1\)](#), we have

$$\alpha \cdot \varphi_{12}^* [\bar{\mathcal{B}}_{12 \rightarrow 1,0,0}] = \alpha \cdot c [\bar{\mathcal{B}}_{2 \rightarrow 1,0,20}] + \alpha \cdot T = c \neq 0.$$

To finish the proof of [Theorem 2](#), we thus only need to show that equation [\(3-1\)](#) holds.

Remark 17 The nontautological part of $\mathrm{H}^{22}(\bar{\mathcal{M}}_{2,20})$ spans a copy of the irreducible $\mathbb{S}_{20}$ representation associated to the partition (2^{10}) by [\[24, Remark 3.10\]](#), which makes equation [\(3-1\)](#) surprisingly simple.

4 Pulling back $\bar{\mathcal{B}}_{12 \rightarrow 1,0,0}$: proof of equation [\(3-1\)](#)

Schmitt and van Zelm [\[27\]](#) developed a method for pulling back admissible cover cycles to boundary strata $\bar{\mathcal{M}}_\Gamma$ of $\bar{\mathcal{M}}_{g,n}$ associated to a stable graph Γ . We apply their method to prove equation [\(3-1\)](#).

Let A be the stable graph with one vertex of genus 2 and 10 loop edges, with the half-edges labeled by $\{1, \dots, 20\}$ and edges given by $(1, 2), (3, 4), \dots, (19, 20)$. Then $\bar{\mathcal{M}}_A = \bar{\mathcal{M}}_{2,20}$, and the gluing morphism

$$\varphi_A: \bar{\mathcal{M}}_A \rightarrow \bar{\mathcal{M}}_{12}$$

is the map φ_{12} , defined in the previous section. Let B denote the graph obtained from A by adding 22 legs, corresponding to the ramification points of a double cover of a genus 1 curve by a genus 12 curve. The associated gluing map is denoted by

$$\varphi_B: \overline{\mathcal{M}}_B \rightarrow \overline{\mathcal{M}}_{12,22}.$$

Set $\xi = (1^{22})$. We have the composition

$$\overline{\mathcal{H}}_{12,G,\xi} \xrightarrow{\phi} \overline{\mathcal{M}}_{12,22} \xrightarrow{\pi} \overline{\mathcal{M}}_{12},$$

where the first map is the source curve map and the second map forgets the markings. The image in $\overline{\mathcal{M}}_{12}$ is by definition $\overline{\mathcal{B}}_{12 \rightarrow 1,0,0}$. We want to compute the pullback

$$\varphi_A^*[\overline{\mathcal{B}}_{12 \rightarrow 1,0,0}] = \varphi_A^* \pi_* \phi_* [\overline{\mathcal{H}}_{12,G,\xi}].$$

4.1 The fiber product and admissible G -graphs

The first step is to describe the fiber product of φ_B and ϕ , which Schmitt and van Zelm [27, Proposition 4.3] do combinatorially in terms of admissible G -graphs. We summarize what is needed from their work here. We continue to take $G = \mathbb{Z}/2\mathbb{Z}$, as it simplifies the discussion of the monodromy data.

Given an admissible cover in $\overline{\mathcal{H}}_{g,G,\xi}$, we take the dual graph Γ of the source curve C . The graph Γ inherits an action of G . In addition to the graph Γ with its G action, we record the stabilizer group of each half-edge and leg. This data allows us to remember the ramification behavior at the nodes and markings. The data of Γ with the G action and the stabilizer group of each half-edge and leg is called an admissible G graph and will be denoted by (Γ, G) . See [27, Section 3.5] for a completely combinatorial definition.

To each admissible G graph (Γ, G) , there is a space $\overline{\mathcal{H}}_{(\Gamma,G)}$ and a morphism

$$\varphi_{(\Gamma,G)}: \overline{\mathcal{H}}_{(\Gamma,G)} \rightarrow \overline{\mathcal{H}}_{g,G,\xi}$$

whose image is the closure of the locus of admissible covers with admissible G graph (Γ, G) , defined as follows. Let V be a set of vertices of Γ consisting of one vertex from each orbit of the G action. Let $G_v \subset G$ denote the stabilizer of $v \in V$, and let ξ_v be the vector of monodromy data associated to the half-edges and legs incident to v (recall that an admissible G graph includes the data of the stabilizer group of each half-edge and leg, allowing for the assignment of monodromy data). Then

$$\overline{\mathcal{H}}_{(\Gamma,G)} = \prod_{v \in V} \overline{\mathcal{H}}_{g(v),G_v,\xi_v},$$

and the morphism $\varphi_{(\Gamma,G)}: \overline{\mathcal{H}}_{(\Gamma,G)} \rightarrow \overline{\mathcal{H}}_{g,G,\xi}$ is obtained by gluing according to the graph Γ . That $\varphi_{(\Gamma,G)}$ is well-defined is established in [27, page 30]. There is also a natural morphism

$$\phi_{(\Gamma,G)}: \overline{\mathcal{H}}_{(\Gamma,G)} \rightarrow \overline{\mathcal{M}}_\Gamma$$

obtained by remembering the source curve and forgetting the G action such that the diagram

$$\begin{array}{ccc} \overline{\mathcal{H}}(\Gamma, G) & \xrightarrow{\varphi(\Gamma, G)} & \overline{\mathcal{H}}_{g, G, \xi} \\ \downarrow \phi(\Gamma, G) & & \downarrow \phi \\ \overline{\mathcal{M}}_{\Gamma} & \xrightarrow{\varphi_{\Gamma}} & \overline{\mathcal{M}}_{g, n} \end{array}$$

commutes, where φ_{Γ} is the gluing map associated to the stable graph Γ .

Recall that a morphism of stable graphs $f: \Gamma \rightarrow \Gamma'$ (also called a Γ' structure on Γ) induces a gluing morphism

$$\varphi_f: \overline{\mathcal{M}}_{\Gamma} \rightarrow \overline{\mathcal{M}}_{\Gamma'}.$$

If (Γ, G) is an admissible G graph, then a Γ' structure on (Γ, G) is simply a morphism of the underlying stable graphs $f: \Gamma \rightarrow \Gamma'$. Such a morphism includes the data of an injection $\beta: E(\Gamma') \hookrightarrow E(\Gamma)$ on the sets of edges. The Γ' structure $f: \Gamma \rightarrow \Gamma'$ is called generic if $G \cdot \text{Im}(\beta) = E(\Gamma)$. That is, every edge of Γ is in the G orbit of an edge of Γ' . Given a generic Γ' structure $f: \Gamma \rightarrow \Gamma'$, there is a morphism

$$\phi_f: \overline{\mathcal{H}}(\Gamma, G) \xrightarrow{\phi(\Gamma, G)} \overline{\mathcal{M}}_{\Gamma} \xrightarrow{\varphi_f} \overline{\mathcal{M}}_{\Gamma'}.$$

Let $\mathfrak{H}_{\Gamma'}$ denote the set of isomorphism classes of admissible G graphs Γ with generic Γ' structures $f: \Gamma \rightarrow \Gamma'$. Setting $\Gamma' = B$, we have the following specialization of [27, Proposition 4.3].

Proposition 18 *The diagram*

$$(4-1) \quad \begin{array}{ccc} \coprod_{(\Gamma, G, f) \in \mathfrak{H}_B} \overline{\mathcal{H}}(\Gamma, G) & \xrightarrow{\coprod \varphi(\Gamma, G)} & \overline{\mathcal{H}}_{12, G, \xi} \\ \downarrow \coprod \phi_f & & \downarrow \phi \\ \overline{\mathcal{M}}_B & \xrightarrow{\varphi_B} & \overline{\mathcal{M}}_{12, 22} \end{array}$$

is Cartesian.

4.2 Excess intersection and the pullback formula

The excess intersection class associated to the diagram (4-1) can be computed separately on each component. The excess bundle E_f is by definition the quotient of the normal bundles $E_f = \phi_f^* N_{\varphi_B} / N_{\varphi(\Gamma, G)}$. The top Chern class of E_f is computed in [27, Proposition 4.7]. Let $\beta: E(B) \rightarrow E(\Gamma)$ be the map on edges induced by the generic B structure (Γ, G, f) . Let N be a set of representatives for the orbit of the G action on $E(\Gamma)$ such that $N \subset \text{Im}(\beta)$. Then

$$c_{\text{top}}(E_f) = \prod_{(h, h') \in \text{Im}(\beta) \setminus N} (-\psi_h - \psi'_{h'}).$$

By definition, these ψ classes are pulled back from $\overline{\mathcal{M}}_B$ under ϕ_f , as noted in the discussion following [27, Theorem 4.9]. In fact, since they are associated to edges of B , they are pulled back from $\overline{\mathcal{M}}_A$ under the composition

$$\overline{\mathcal{H}}(\Gamma, G) \xrightarrow{\phi_f} \overline{\mathcal{M}}_B \xrightarrow{\pi_B} \overline{\mathcal{M}}_A,$$

where the map $\pi_B: \bar{\mathcal{M}}_B \rightarrow \bar{\mathcal{M}}_A$ simply forgets legs. Now consider the diagram

$$\begin{array}{ccc} \coprod_{(\Gamma, G, f) \in \mathfrak{H}_B} \bar{\mathcal{H}}_{(\Gamma, G)} & \xrightarrow{\coprod \varphi_{(\Gamma, G)}} & \bar{\mathcal{H}}_{12, G, \xi} \\ \downarrow \coprod \phi_f & & \downarrow \phi \\ \bar{\mathcal{M}}_B & \xrightarrow{\varphi_B} & \bar{\mathcal{M}}_{12, 22} \\ \downarrow \pi_B & & \downarrow \pi \\ \bar{\mathcal{M}}_A & \xrightarrow{\varphi_A} & \bar{\mathcal{M}}_{12} \end{array}$$

Specializing [27, Theorem 4.12] to the case at hand, we have

$$(4-2) \quad \varphi_A^*[\bar{\mathcal{B}}_{12 \rightarrow 1, 0, 0}] = \varphi_A^* \pi_* \phi_* [\bar{\mathcal{H}}_{12, G, \xi}] = \sum_{(\Gamma, G, f) \in \mathfrak{H}_B} \pi_{B*}(c_{\text{top}}(E_f) \cdot \phi_{f*}[\bar{\mathcal{H}}_{(\Gamma, G)}]).$$

4.3 Pullback computation

We will now establish equation (3-1) using equation (4-2). We examine each term in the summation on the right-hand side of equation (4-2):

$$\sum_{(\Gamma, G, f) \in \mathfrak{H}_B} \pi_{B*}(c_{\text{top}}(E_f) \cdot \phi_{f*}[\bar{\mathcal{H}}_{(\Gamma, G)}]).$$

We first study a large class of $(\Gamma, G, f) \in \mathfrak{H}_B$ that contribute only tautological classes.

Lemma 19 Suppose $(\Gamma, G, f) \in \mathfrak{H}_B$ satisfies any of the following properties:

- (1) $\bar{\mathcal{H}}_{(\Gamma, G)}$ is not of the expected codimension.
- (2) Γ has no vertices of genus 2.
- (3) G acts nontrivially on the vertices.
- (4) The quotient graph Γ/G has first Betti number 1.
- (5) Γ has loop edges.
- (6) G acts trivially on an edge.
- (7) Γ has vertices v_1, v_2 connected by $n \geq 1$ edges and $n \neq 2$.

Then $\pi_{B*}(c_{\text{top}}(E_f) \cdot \phi_{f*}[\bar{\mathcal{H}}_{(\Gamma, G)}])$ is tautological.

Proof Suppose (Γ, G, f) is such that (1) holds. Then the excess bundle E_f is of rank at least 1. Because $c_{\text{top}}(E_f)$ is pulled back from $\bar{\mathcal{M}}_A$, the term

$$\pi_{B*}(c_{\text{top}}(E_f) \cdot \phi_{f*}[\bar{\mathcal{H}}_{(\Gamma, G)}])$$

is a product of algebraic classes of lower codimension on $\bar{\mathcal{M}}_A = \bar{\mathcal{M}}_{2, 20}$ by the projection formula. Hence, by Theorem 16(2), $\pi_{B*}(c_{\text{top}}(E_f) \cdot \phi_{f*}[\bar{\mathcal{H}}_{(\Gamma, G)}])$ is tautological.

From now on, we suppose (Γ, G, f) does not satisfy property (1). Then there is no excess contribution, and we study only the terms of the form $\pi_{B*}\phi_{f*}[\overline{\mathcal{H}}_{(\Gamma, G)}]$. Now suppose (Γ, G, f) is such that (2) holds. First, assume that Γ has a vertex of genus 1. Then $\pi_{B*}\phi_{f*}[\overline{\mathcal{H}}_{(\Gamma, G)}]$ is supported on the boundary of $\overline{\mathcal{M}}_A$, and hence is tautological by [Theorem 16\(3\)](#). We argue similarly if all vertices of Γ are of genus 0.

From now on, we assume (Γ, G, f) does not satisfy property (2), and so has a unique vertex of genus 2. Suppose that (Γ, G, f) is such that G acts nontrivially on the vertices of Γ . We pick a pair v_1 and v_2 of genus 0 vertices interchanged by the G action. The G action fixes all of the legs because the monodromy data is $\xi = (1^{22})$. Therefore, there can be no legs attached to v_1 and v_2 . Under the pushforward π_{B*} forgetting the legs, the vertices v_1 and v_2 are not contracted, meaning $\pi_{B*}\phi_{f*}[\overline{\mathcal{H}}_{(\Gamma, G)}]$ is supported on the boundary of $\overline{\mathcal{M}}_A$, and is thus tautological.

From now on, we assume (Γ, G, f) does not satisfy property (3). Suppose that Γ/G has first Betti number 1. It follows that every vertex of the quotient graph is of genus 0, as the target curve has genus 1. Therefore, in the product decomposition

$$\overline{\mathcal{H}}_{(\Gamma, G)} = \prod_{v \in V} \overline{\mathcal{H}}_{g(v), G_v, \xi_v},$$

the term corresponding to the genus 2 vertex parametrizes admissible hyperelliptic covers. Such spaces contribute only tautological classes by [Theorem 15](#).

From now on, we assume (Γ, G, f) does not satisfy property (4). Suppose that Γ has a vertex with a loop edge. The quotient graph Γ/G must have a loop edge because G acts trivially on the vertices. This contradicts the assumption that (Γ, G, f) does not satisfy property (4).

From now on, we assume (Γ, G, f) does not satisfy property (5). Consider the case when Γ has vertices v_1 and v_2 connected by an edge e that is fixed under the G action. Because the B structure is generic, e must be an edge of B . But e is not a loop edge, whereas all the edges of B are. Therefore, there must exist another path in Γ connecting v_1 and v_2 , in which all constituent edges are contracted by the B structure $f: \Gamma \rightarrow B$. But then the quotient graph Γ/G must have nonzero first Betti number, as the images of v_1 and v_2 are connected by at least two edges, contradicting that (Γ, G, f) does not satisfy property (4).

From now on, we assume (Γ, G, f) does not satisfy property (6). Suppose that Γ has vertices v_1 and v_2 connected by $n \geq 1$ edges $e_1, \dots, e_n$. If n is odd, then because G acts trivially on the vertices, it must also fix an edge, violating the assumption that it does not satisfy property (6). Therefore, n is even. If $n \geq 4$, then the quotient graph Γ/G has first Betti number at least 1, violating the assumption that it does not satisfy property (4). $\square$

Proof of equation (3-1) We calculate using equation (4-2):

$$\varphi_A^*[\overline{\mathcal{B}}_{12 \rightarrow 1,0,0}] = \varphi_A^* \pi_* \phi_* [\overline{\mathcal{H}}_{12, G, \xi}] = \sum_{(\Gamma, G, f) \in \mathfrak{S}_B} \pi_{B*}(c_{\text{top}}(E_f) \cdot \phi_{f*}[\overline{\mathcal{H}}_{(\Gamma, G)}]).$$

We need only to consider the terms on the right-hand side satisfying none of the properties listed in [Lemma 19](#). By the negation of property (1), such terms are of the form

$$\pi_{B*}\phi_{f*}[\overline{\mathcal{H}}_{(\Gamma,G)}].$$

By the negation of properties (2) and (5), there is one genus 2 vertex, all other vertices are of genus 0, and there are no loops on any vertex. By the negation of property (4), the first Betti number of the quotient graph Γ/G is 0. Hence, the image of the unique vertex of genus 2 is a vertex of genus 1. By the negation of property (3), the vertices of genus 0 map bijectively to vertices of genus 0 of Γ/G . By Riemann–Hurwitz, we see that every vertex of Γ has two markings. By the negation of properties (6) and (7), any two vertices v_1 and v_2 of Γ are either not connected by edges or are connected by exactly two edges pairwise switched by the G action.

We consider the case where there is a genus 0 vertex v_1 not connected to the genus 2 vertex. If v_1 is connected to at least two other vertices, then the component corresponding to v_1 is not contracted under the map forgetting the marked points. It follows that $\pi_{B*}\phi_{f*}[\overline{\mathcal{H}}_{(\Gamma,G)}]$ is supported on the boundary, and is thus tautological. If v_1 is connected to only one other vertex, then under the map forgetting the marked points, v_1 is contracted. The resulting graph has a genus 0 vertex with a loop, and thus $\pi_{B*}\phi_{f*}[\overline{\mathcal{H}}_{(\Gamma,G)}]$ is supported on the boundary.

Finally, we consider the case when no two genus 0 vertices are connected by edges. Upon forgetting the marked points, all of the genus 0 components are contracted, and the cycle $\pi_{B*}\phi_{f*}[\overline{\mathcal{H}}_{(\Gamma,G)}]$ is a positive multiple of $[\overline{\mathcal{B}}_{2 \rightarrow 1,0,20}]$. $\square$

5 Proof of Theorem 8

We prove [Theorem 8](#) by induction on g and n . The proof idea is analogous to [\[2, Lemma 4.1\]](#), but extra care is needed for possible contributions from odd cohomology in the boundary. The statement holds for $g = 0$ by [\[17\]](#), $g = 1$ by [\[23\]](#), and $g = 2$ by [\[24\]](#). For any g, n and k , there is an exact sequence

$$(5-1) \quad H^{2k-2}(\partial\widetilde{\mathcal{M}}_{g,n}) \rightarrow H^{2k}(\overline{\mathcal{M}}_{g,n}) \rightarrow W_{2k}H^{2k}(\mathcal{M}_{g,n}) \rightarrow 0,$$

where $\partial\widetilde{\mathcal{M}}_{g,n}$ is the normalization of the boundary. For $3 \leq g \leq 8$, $n < d(g)$ and k arbitrary, we have that

$$W_{2k}H^{2k}(\mathcal{M}_{g,n}) = RH^{2k}(\mathcal{M}_{g,n})$$

by [\[2, Theorem 1.4\]](#) and [\[4, Theorem 1.10 and Lemma 4.3\]](#). Therefore, using the exact sequence (5-1), it suffices to study the boundary contribution.

Up to quotients by finite groups, the normalization of the boundary $\partial\widetilde{\mathcal{M}}_{g,n}$ is the disjoint union of spaces of the form $\overline{\mathcal{M}}_{g-1,n+2}$ and $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$, where $g_1 + g_2 = g$ and $n_1 + n_2 = n$. By induction on g and n , the even cohomology of $\overline{\mathcal{M}}_{g-1,n+2}$ and $\overline{\mathcal{M}}_{g_1,n_1+1}$ and $\overline{\mathcal{M}}_{g_2,n_2+1}$ are tautological. We thus only

g	0	1	2	3	4	5	6	7
$e(g)$	∞	11	10	9	7	5	3	1

Table 3: The odd cohomology of $\bar{\mathcal{M}}_{g,n}$ vanishes for $n < e(g)$.

need to consider the summands in the Künneth decomposition

$$H^{2k-2}(\bar{\mathcal{M}}_{g_1, n_1+1} \times \bar{\mathcal{M}}_{g_2, n_2+1}) \cong \bigoplus_{p+q=2k-2} H^p(\bar{\mathcal{M}}_{g_1, n_1+1}) \otimes H^q(\bar{\mathcal{M}}_{g_2, n_2+1})$$

when both p and q are odd. It follows from [2, Theorem 1.4] that for $n_1 + 1 < e(g_1)$ (resp. $n_2 + 1 < e(g_2)$), as defined in Table 3, all of the odd cohomology of $\bar{\mathcal{M}}_{g_1, n_1+1}$ (resp. $\bar{\mathcal{M}}_{g_2, n_2+1}$) vanishes. Therefore, for $3 \leq g \leq 8$ and $n < d(g)$, the terms

$$H^p(\bar{\mathcal{M}}_{g_1, n_1+1}) \otimes H^q(\bar{\mathcal{M}}_{g_2, n_2+1})$$

with p and q odd vanish identically, as either $H^p(\bar{\mathcal{M}}_{g_1, n_1+1})$ or $H^q(\bar{\mathcal{M}}_{g_2, n_2+1})$ is zero. $\square$

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