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We consider a closed Riemannian manifold M of negative curvature and dimension at least 3, with marked length spectrum sufficiently close (multiplicatively) to that of a locally symmetric space N. Using the methods of Hamenstädt (1999), we show that the volumes of M and N are approximately equal. We then show that the smooth map $F: M \to N$ constructed by Besson, Courtois and Gallot (1996) is a diffeomorphism with derivative bounds close to 1 and which depend on the ratio of the two marked length spectrum functions. Thus, we refine the results of Hamenstädt and Besson, Courtois and Gallot, which show M and N are isometric if their marked length spectra are equal. We also prove a similar result for closed negatively curved surfaces using the methods of Otal (1990) and Pugh (1987).

37D40, 53C22, 53C24, 53C35; 37C27, 37D20

1 Introduction

The marked length spectrum $\mathcal{L}_g$ of a closed Riemannian manifold (M, g) of negative curvature is a function on the free homotopy classes of closed curves in M which assigns to each class the length of its unique geodesic representative. Since the set of free homotopy classes of closed curves in M can be identified with conjugacy classes in the fundamental group Γ of M, we will write $\mathcal{L}_g(\gamma)$ for the length of the unique geodesic representative of the conjugacy class of $\gamma \in \Gamma$ with respect to the metric g.

To what extent the function $\mathcal{L}_g$ determines the metric g is a question that has long been of interest. Notably, Burns and Katok [11, Conjecture 3.1] conjectured that the marked length spectrum completely determines the metric, that is, if g and g_0 are negatively curved metrics on M satisfying $\mathcal{L}_g = \mathcal{L}_{g_0}$ then g and g_0 are isometric. This marked length spectrum rigidity is known to be true in certain cases, due to Otal [43] and Croke [16] (independently) in dimension 2, Hamenstädt [32] in dimension at least 3 when one of the manifolds is assumed be locally symmetric (also using work of Besson, Courtois and Gallot [4]), and Guillarmou and Lefeuvre [29] if the metrics are assumed to be sufficiently close in a suitable C^k topology; see Guillarmou and Lefeuvre [29] and Guillarmou, Knieper and Lefeuvre [28]. There are other partial results, including generalizations beyond negative curvature; see the introduction to [29] for a more extensive history of the problem.

Still, even in the cases where rigidity does hold, there is more to be understood about the extent to which the marked length spectrum determines the metric. Namely, if two homotopy-equivalent manifolds have marked length spectra which are not equal but are close, is there some sense in which the metrics are close to being isometric?

In the case of hyperbolic surfaces, this question was answered by Thurston [51]. More specifically, if (M, g) and (N, g_0) are both surfaces of constant negative curvature, then the best possible Lipschitz constant for a map $F\colon M \to N$ in the same homotopy class as f is precisely

$$\sup_{\gamma\in\Gamma} \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)}.$$

In the general case of higher dimensions and variable negative curvature, the microlocal techniques of [29; 28] provide explicit estimates (in a suitable C^k norm) for how close the metrics are in terms of the ratio $\mathcal{L}_{g_0}/\mathcal{L}_g$, or more precisely the geodesic stretch; in fact, their results hold more generally for nonpositively curved metrics with Anosov geodesic flow. However, these estimates require g and g_0 to be sufficiently close metrics (in some C^k topology) on the same manifold.

In this paper, we provide new answers to this question in two cases: first, consider pairs of negatively curved metrics on a closed surface. To state our result precisely, let (M, g_0) be a closed surface of curvature between $-b^2$ and $-a^2$ for some $0 < a \le b$. Fix constants $0 < a \le b$, and $\alpha, R > 0$. Let $\mathcal{U}^{1,\alpha}_{g_0}(a, b, R)$ denote the set of all C^2 Riemannian metrics g on M with curvatures between $-b^2$ and $-a^2$ such that $\|g - g_0\|_{C^{1,\alpha}(M)} \le R$. We show pairs of such spaces become more isometric as their marked length spectra get closer to one another, refining the main result in Otal [43].

Theorem A *Suppose that (M, g_0) and $\mathcal{U}^{1,\alpha}_{g_0}(a, b, R)$ are as above. Fix $L > 1$. Then there exists $\varepsilon = \varepsilon(L, a, b, R, \alpha) > 0$ small enough so that for any pair $(M, g), (M, h) \in \mathcal{U}^{1,\alpha}_{g_0}(a, b, R)$ satisfying*

$$1 - \varepsilon \le \frac{\mathcal{L}_g}{\mathcal{L}_h} \le 1 + \varepsilon, \tag{1-1}$$

there exists an L-Lipschitz map $f\colon (M, g) \to (M, h)$.

Second, consider the case where (N, g_0) is a negatively curved locally symmetric space of dimension at least 3. Here, we quantify how close g and g_0 are to being isometric by estimating the derivative of a map $F\colon M \to N$ in terms of ε. This is considerably stronger than Theorem A, since we are able to determine how the Lipschitz constant depends on ε. This refines the rigidity result in [32, Corollary to Theorem A], which corresponds to the case $\varepsilon = 0$ in the theorem below.

In (1-1), the marking relating the lengths of the two metrics is the identity map. In our next theorem, we consider a more general setup consisting of a homotopy equivalence $f\colon (M, g) \to (N, g_0)$ (where M

and N need not be diffeomorphic a priori). Note that since M and N are $K(\pi,1)$ spaces, the existence of such a homotopy equivalence is equivalent to the existence of an isomorphism between the fundamental groups of M and N; more precisely, the homotopy class of f is determined by the isomorphism. We will assume the marked length spectra $\mathcal{L}_g$ and $\mathcal{L}_{g_0}$ are close in the following sense:

Hypothesis 1.1 Fix $\varepsilon > 0$. Suppose $f\colon (M,g) \to (N,g_0)$ is a homotopy equivalence of closed negatively curved manifolds and let Γ denote the fundamental group of M. Let f_* denote the map on fundamental groups induced by f. Assume that

$$1-\varepsilon \le \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)} \le 1+\varepsilon \quad \text{for all } \gamma \in \Gamma. \tag{1-2}$$

To obtain uniform estimates, we will also assume that (M,g) and (N,g_0) satisfy the following topological and geometric constraints.

Hypothesis 1.2 Fix $i_0, D > 0$ and $0 < a < b$. Suppose (M,g) and (N,g_0) are a pair of homotopy-equivalent closed Riemannian manifolds of dimension n at least 3, diameter at most D, injectivity radius at least i_0, and sectional curvatures contained in the interval $[-b^2,-a^2]$.

To state our result precisely, recall that the *Anosov splitting* of the geodesic flow on the unit tangent bundle T^1M refers to the flow-invariant decomposition of TT^1M into the stable, unstable and flow directions; see the introduction to [32]. The stable and unstable bundles are in general C^{α_0} Hölder continuous for some $0 < \alpha_0 \le 1$, which we will refer to as the *stable Hölder exponent* of M. If M is quarter-pinched, that is, $a^2/b^2 \in \left[\frac{1}{4},1\right]$, then the Anosov splitting is C^1; see Hirsch and Pugh [36]. Otherwise, we can take $\alpha_0 = 2a/b$; see Hasselblatt [33, Corollary 1.7].

Theorem B *Let (M,g) and (N,g_0) be a pair of homotopy-equivalent closed negatively curved Riemannian manifolds of dimension n at least 3 and fundamental group Γ. Suppose M and N satisfy the bounded geometry hypothesis (Hypothesis 1.2) for some $i_0, D > 0$ and $0 < a < b$. Suppose further that (N,g_0) is locally symmetric and that the curvature tensor $\mathcal{R}$ of M satisfies $\|\nabla\mathcal{R}\| \le R$ for some constant $R > 0$. Let $\alpha_0 = \alpha_0(a,b) = \min(2a/b,1)$ denote the above-defined stable Hölder exponent of (M,g).*

Then there exists $\varepsilon_0 = \varepsilon_0(a,b,R)$ such that for any $\varepsilon \in (0,\varepsilon_0]$ the following holds: Suppose there is a homotopy equivalence $f\colon M \to N$ so that the marked length spectra of M and N satisfy Hypothesis 1.1 for ε as above. Then, there is a smooth diffeomorphism $F\colon M \to N$, homotopic to f, such that for all $v \in TM$ we have

$$(1-C\varepsilon^{\alpha})\|v\|_g \le \|dF(v)\|_{g_0} \le (1+C\varepsilon^{\alpha})\|v\|_g \tag{1-3}$$

for some $C > 0$ depending only on $n, \Gamma, D, i_0, a, b, R$ and for any

$$\alpha < (1-\varepsilon)\frac{a^2}{b}\frac{\alpha_0^2}{16n}.$$

Remark 1.3 As in the previous theorem, we only assume that the marked length spectra of our metrics are close; we do not assume that the metrics themselves are close in any C^k topology, in contrast to [29; 28]. However, the hypotheses of Theorem B hold in particular if g is also a metric on N satisfying $\|g-g_0\|_{C^3(N)} \le R_0$, where R_0 is any fixed, but not necessarily small, constant. Moreover, the estimates for $\|dF(v)\|$ in (1-3) can in principle be computed explicitly in terms of ε, the C^3 distance $\|g-g_0\|_{C^3(N)}$, and the diameter, injectivity radius and sectional curvature bounds of the locally symmetric space (N, g_0).

Hamenstädt [32, Theorem A] proves that two negatively curved manifolds with the same marked length spectrum have the same volume provided that one of the manifolds has geodesic flow with C^1 Anosov splitting (instead of only Hölder); this condition holds in particular for locally symmetric spaces, as well as for quarter-pinched manifolds more generally; see Hirsch and Pugh [36].

Thus, if M and N satisfy the assumptions of Theorem B for $\varepsilon = 0$, they must have the same volume. Then, since the marked length spectrum determines the topological entropy of the geodesic flow, the fact that the two manifolds are isometric follows from the celebrated entropy rigidity theorem of Besson, Courtois and Gallot [3; 4].

To prove Theorem B, we start by proving an analogue of [32, Theorem A] under the assumption that the marked length spectra satisfy equation (1-2), ie we estimate the ratio $\mathrm{Vol}(M)/\mathrm{Vol}(N)$ in terms of ε.

Theorem C *Let (M, g) and (N, g_0) be a pair of homotopy-equivalent closed Riemannian manifolds as in Hypothesis 1.2. Suppose further that the geodesic flow on T^1N has C^1 Anosov splitting, and that the curvature tensor $\mathcal{R}$ of (M, g) satisfies $\|\nabla\mathcal{R}\| \le R$ for some $R > 0$. Let $\alpha_0 = \min(2a/b, 1)$ denote the stable Hölder exponent of M.*

Then there exists $\varepsilon_0 = \varepsilon_0(a, b, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the following holds: Suppose there is $f\colon M \to N$ such that the marked length spectra of M and N satisfy Hypothesis 1.1 for ε as above. Then

$$(1 - C\varepsilon^{\alpha})\,\mathrm{Vol}(M) \le \mathrm{Vol}(N) \le (1 + C\varepsilon^{\alpha})\,\mathrm{Vol}(M)$$

for some constant $C = C(n, \Gamma, D, i_0, a, b, R) > 0$ and any $\alpha < (1-\varepsilon)(a^2/b)\alpha_0^2$.

Remark 1.4 If N is locally symmetric, then $\mathrm{Vol}(N) \le (1+\varepsilon)^n\,\mathrm{Vol}(M)$ follows from Lemma 3.1 and the proof of the main theorem in [4]. (See Remark 3.6 for more details.) However, the lower bound for $\mathrm{Vol}(N)/\mathrm{Vol}(M)$ in Theorem C is also crucial for the proof of Theorem B.

Remark 1.5 If $\dim M = \dim N = 2$, then $(1-\varepsilon)^2\,\mathrm{Vol}(N) \le \mathrm{Vol}(M) \le (1+\varepsilon)^2\,\mathrm{Vol}(M)$ follows from work of Croke and Dairbekov [17, Theorem 1.1].

Outline We prove Theorem C in Section 2, which occupies the majority of this paper. As in [32], we study the Liouville measure on the unit tangent bundle of M, which is locally a product of the Liouville current on the space of geodesics and the Lebesgue measure on orbits of the geodesic flow [37, Theorem 2.1]. In Sections 2.1 and 2.2, we generalize results in Otal [43] and Hamenstädt [32], respectively, to "coarsely" control the Liouville currents of M and N; more precisely, we show the ratios of these measures is close to 1 (in terms of ε) on sets whose size is small, but fixed, in terms of ε. We also need to carefully control the time component of the Liouville measure, since when $\mathcal{L}_g$ does not coincide with $\mathcal{L}_{g_0}$, the corresponding geodesic flows are not conjugate. In Section 2.3, we use Hypothesis 1.1 to control the speed of the time changes by combining the constructions in Bourdon [7] and Gromov [27] (Theorem 2.38).

In Section 3, we find explicit estimates for the derivative of the natural map of Besson, Courtois and Gallot [4] when the two metrics have approximately equal volumes and entropies (Theorem 3.7). By Theorem C, the hypotheses of Theorem 3.7 hold in particular when the length functions are related as in Hypothesis 1.1. This proves the main theorem (Theorem B).

Finally, we prove Theorem A in Section 4. We prove the result by contradiction, obtaining a convergent sequence of counterexamples. We then use results in Pugh [47] together with the methods in Otal [43] to show the limits are isometric.

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2 Volume estimate

This section is devoted to proving Theorem C. The main tool is the *Liouville measure*, whose construction and properties we now briefly recall.

Liouville measure In order to determine the volume of (M, g), it suffices to determine the volume of the unit tangent bundle T^1M with respect to the *Liouville measure* μ^M. This measure is the volume induced by the Sasaki metric on M; locally it is the product of the Riemannian volume associated to the metric g on the base M and Lebesgue measure on the spherical fibers. Thus, estimating the ratio of the volumes of M and N is equivalent to estimating the ratio of the Liouville volumes of their respective unit tangent bundles.

To do so, Hamenstädt uses the following description of the Liouville measure, which is more compatible with the geodesic flow. Let ω be the 1-form on T^1M obtained by pulling back the canonical 1-form on T^*M to TM via the identification induced by the Riemannian metric and then restricting to T^1M. Then ω and $d\omega$ are both flow-invariant, ω is dual to the generating vector field for the geodesic flow, and ω is a contact form, meaning $\omega \wedge (d\omega)^{n-1}$ is a volume form on T^1M. The associated measure on T^1M coincides with the *Liouville measure*, up to a dimensional constant; see eg [10, Lemma 1.7] for a proof.

From the above description of the Liouville measure, it is apparent that it is geodesic-flow–invariant; moreover, it locally decomposes as a product of ω in the flow direction and $d\omega^{n-1}$ on transversals to the flow. In fact, $d\omega^{n-1}$ can be viewed as a measure on the *space of geodesics of* $\widetilde{M}$. This is defined to be the quotient of $T^1\widetilde{M}$ by the action of the geodesic flow, and can also be identified with the set $\partial^2\widetilde{M}$ of pairs of distinct points in the boundary. The associated measure is called the *Liouville current*, which we will denote by λ^M. (This is a special case of a more general correspondence between geodesic-flow–invariant measures on T^1M and $\pi_1(M)$-invariant measures on $\partial^2\widetilde{M}$; see [37, Theorem 2.1].)

Outline of the proof of Theorem C We will repeatedly use the following standard construction, part of which can be found in [4, Section 4]:

Construction 2.1 Let $f\colon M \to N$ be a homotopy equivalence of negatively curved manifolds. Let $\partial\widetilde{M}$ denote the visual boundary of $\widetilde{M}$. We can construct a map $\overline{f}\colon \partial\widetilde{M} \to \partial\widetilde{N}$ such that for all $\gamma \in \Gamma$ and all $\xi \in \partial\widetilde{M}$ we have $\overline{f}(\gamma.\xi) = (f_*\gamma).\overline{f}(\xi)$. Indeed, the homotopy equivalence $f\colon M \to N$ can be lifted to a Γ-equivariant map $\tilde{f}\colon \widetilde{M} \to \widetilde{N}$ such that $\tilde{f}$ is additionally a quasi-isometry (details in [4, Section 4]). Hence $\tilde{f}$ induces a Γ-equivariant map $\overline{f}$ between the boundaries $\partial\widetilde{M}$ and $\partial\widetilde{N}$.

Now recall the space of geodesics of $\widetilde{M}$ is the quotient of $T^1\widetilde{M}$ obtained by identifying any two unit tangent vectors on the same orbit of the geodesic flow. This space can be identified with the set $\partial^2\widetilde{M}$ of pairs of distinct points in $\partial\widetilde{M}$ by associating the equivalence class of the unit tangent vector v with the pair $(\pi(v), \pi(-v))$ of its forward and backward endpoints. Thus, the product $\overline{f} \times \overline{f}$ gives a map between the spaces of geodesics of $\widetilde{M}$ and $\widetilde{N}$. For notational simplicity, we will write this map as $\overline{f}\colon \partial^2\widetilde{M} \to \partial^2\widetilde{N}$.

Note that the case $\varepsilon = 0$ of Theorem C is Theorem A in [32]. We follow the same overall approach, which we now describe. If M and N have the same marked length spectrum, it is well-known that the geodesic flows ϕ^t on T^1M and ψ^t on T^1N are *conjugate*, which means there is a homeomorphism $\mathcal{F}\colon T^1M \to T^1N$ which satisfies $\mathcal{F}(\phi^t v) = \psi^t\mathcal{F}(v)$ for all $v \in T^1M$ and all $t \in \mathbb{R}$ (see [30; 7]; this also follows by combining the construction in [27] with Livsic's theorem, as outlined in [38, Section 2.2]). Moreover, $\mathcal{F}$ is compatible with the boundary map in Construction 2.1 in the sense that $\pi \circ \mathcal{F} = \overline{f} \circ \pi$.

Hamenstädt's proof of [32, Theorem A] consists of showing that $\mathcal{F}_*\mu^M = \mu^N$. Since $\mathcal{F}$ sends geodesics to geodesics in a *time-preserving* manner, it suffices to show the map $\overline{f}\colon \partial^2\widetilde{M} \to \partial^2\widetilde{N}$ preserves the Liouville current, ie $\overline{f}_*\lambda^M = \lambda^N$. The key tool used to relate equality of the marked length spectra to equality of the Liouville currents is the *cross-ratio*. (See Section 2.1 for the definition of the cross-ratio and how it relates to the marked length spectrum, and Section 2.2 for an explanation of how it relates to the Liouville current.)

In order to relate the Liouville currents in the case $\varepsilon > 0$, we start by generalizing [44, Theorem 2.2], ie we determine how perturbing the marked length spectrum as in (1-2) affects the cross-ratio (Proposition 2.4); to do so, we also use many techniques from [7]. Using Proposition 2.4, we refine the arguments in [32] to find certain families of "cubes" $K \subset \partial^2 \widetilde{M}$ so that the ratio of the measures $\lambda^M(K)$ and $\lambda^N(\overline{f}(K))$ is close to 1 in terms of ε (Proposition 2.37). We emphasize that we can only estimate this ratio for cubes whose diameter is *bounded below in terms of* ε. Indeed, the measures λ^M and $\overline{f}_* \lambda^N$ are not in general absolutely continuous (unless they are the same up to a multiplicative constant), so it is not possible to compare the volumes of arbitrarily small cubes in our case.

We then augment these cubes in the flow direction to obtain $(2n+1)$-dimensional cubes K' in $T^1 M$. Although the flows ϕ^t and ψ^t are in general not conjugate when $\varepsilon > 0$, so long as M and N are negatively curved and have isomorphic fundamental groups, their geodesic flows are *orbit-equivalent* [27]. This means there exists a homeomorphism $\mathcal{F}\colon T^1 M \to T^1 N$, and a function (cocycle) $a(t, v)$ such that

(2-1)
$$\mathcal{F}(\phi^t v) = \psi^{a(t,v)} \mathcal{F}(v)$$

for all $t \in \mathbb{R}$ and $v \in T^1 M$. (We also have $\pi \circ \mathcal{F} = \overline{f} \circ \pi$ as before.) In Theorem 2.38, we show that (1-2) implies the time change $a(t, v)$ is close to t in terms of ε. This allows us to estimate the ratios $\mu^M(K')/\mu^N(\mathcal{F}(K'))$. To complete the proof of the Theorem C, we estimate the error in approximating $T^1 M$ from above and below by these cubes of small, but fixed, size (Proposition 2.52). This requires a refined understanding of the shape, and not just the size, of the cubes K.

2.1 The cross-ratio

We now define the cross-ratio associated to any negatively curved Riemannian manifold (M, g). This concept will not only play a key role in our above-mentioned Liouville current comparison (Proposition 2.37), but also in the construction of our "almost conjugacy" (Theorem 2.38), where we generalize the construction in [7].

Let $\widetilde{M}$ denote the universal cover of M and let $\partial \widetilde{M}$ be the visual boundary of $\widetilde{M}$. Let $\pi : T^1 \widetilde{M} \to \partial \widetilde{M}$ denote the map which sends v to the forward boundary point of the geodesic determined by v. Let $\partial^4 \widetilde{M}$ denote pairwise distinct quadruples of points in $\partial \widetilde{M}$.

Definition 2.2 (Otal [44, Lemma 2.1]) Let $(\xi, \xi', \eta, \eta') \in\in \partial^4 \widetilde{M}$. Let $a_i, b_i, c_i, d_i \in \widetilde{M}$ be sequences converging to ξ, ξ', η, η', respectively. Define

(2-2)
$$[\xi, \xi', \eta, \eta'] = \lim_{i \to \infty} d(a_i, c_i) + d(b_i, d_i) - d(a_i, d_i) - d(b_i, c_i),$$

where d is the Riemannian distance function. By [44, Lemma 2.1], this limit exists and is independent of the chosen sequences a_i, b_i, c_i, d_i. We call $[\cdot, \cdot, \cdot, \cdot]$ the *cross-ratio*.

Theorem 2.2 in [44] shows that the cross-ratio is completely determined by the marked length spectrum, and the argument is not specific to dimension 2. The proof shows in particular that given $\xi, \xi', \eta, \eta' \in \partial^4 \widetilde{M}$ and $\delta > 0$, there exist free homotopy classes $\gamma_1, \dots, \gamma_4$ such that

(2-3)
$$|[\xi, \xi', \eta, \eta'] - (\mathcal{L}_g(\gamma_1) + \mathcal{L}_g(\gamma_2) - \mathcal{L}_g(\gamma_3) - \mathcal{L}_g(\gamma_4))| < \delta.$$

Hence if g and g_0 are such that $\mathscr{L}_g(\gamma) = \mathscr{L}_{g_0}(f(\gamma))$ for all free homotopy classes γ, it follows that $[\xi, \xi', \eta, \eta'] = [\overline{f}(\xi), \overline{f}(\xi'), \overline{f}(\eta), \overline{f}(\eta')]$ for all $\xi, \xi', \eta, \eta' \in \partial^4 \widetilde{M}$.

In this section we investigate what happens if instead $\mathscr{L}_g$ and $\mathscr{L}_{g_0} \circ f$ are ε-close. We start with the following straightforward observation.

Lemma 2.3 *Suppose that*

$$1 - \varepsilon \le \frac{\mathscr{L}_{g_0}(f(\gamma))}{\mathscr{L}_g(\gamma)} \le 1 + \varepsilon$$

for all $\gamma \in \Gamma$. Fix $L > 0$ and suppose $\gamma_1, \dots, \gamma_4$ are such that $\mathscr{L}_g(\gamma_i) \le L$ for $i = 1, \dots, 4$. Then

$$1 - 4L\varepsilon \le \frac{\mathscr{L}_{g_0}(f(\gamma_1)) + \mathscr{L}_{g_0}(f(\gamma_2)) - \mathscr{L}_{g_0}(f(\gamma_3)) - \mathscr{L}_{g_0}(f(\gamma_4))}{\mathscr{L}_g(\gamma_1) + \mathscr{L}_g(\gamma_2) - \mathscr{L}_g(\gamma_3) - \mathscr{L}_g(\gamma_4)} \le 1 + 4L\varepsilon.$$

Consequently, the above ratio is close to 1 when L is not too large in terms of ε, say $L = \varepsilon^{-1/2}$ for concreteness. This suggests that the quantity $[\xi, \xi', \eta, \eta']$ is multiplicatively close to $[\overline{f}(\xi), \overline{f}(\xi'), \overline{f}(\eta), \overline{f}(\eta')]$ whenever these two quantities can be approximated as in (2-3) by lengths of closed geodesics γ_i with $\mathscr{L}_g(\gamma_i) \le L$. To make this precise, we first explicitly determine how large the $\mathscr{L}_g(\gamma_i)$ need to be in terms of δ so that (2-3) holds; see Proposition 2.6. In other words, the size of $L = \varepsilon^{-1/2}$ limits the accuracy $\delta = \delta(\varepsilon)$ of the approximation achievable in (2-3).

As such, we prove that $[\xi, \xi', \eta, \eta']$ is multiplicatively close to $[\overline{f}(\xi), \overline{f}(\xi'), \overline{f}(\eta), \overline{f}(\eta')]$ for quadruples of points (ξ, ξ', η, η') such that $[\xi, \xi', \eta, \eta']$ is bounded away from zero, ie $[\xi, \xi', \eta, \eta'] \ge \delta(\varepsilon)$. In addition, for our proof of Theorem C, we only need this multiplicative closeness of cross-ratios when the distances between the four geodesics $[\xi, \eta']$, $[\xi', \eta]$, $[\xi, \eta]$, $[\xi', \eta']$ are not too large. More precisely, we show the following.

Proposition 2.4 *Let (M, g) and (N, g_0) be as in Hypotheses 1.1 and 1.2 for some $\varepsilon > 0$, i_0, $D > 0$ and $0 < a < b$. Let $\overline{f}\colon \partial\widetilde{M} \to \partial\widetilde{N}$ be as in Construction 2.1.*

Then there exist constants $r = r(a)$, $\delta_0 = \delta_0(\varepsilon, n, \Gamma, D, i_0, a, b)$ and $\varepsilon' = \varepsilon'(\delta_0, \varepsilon)$ such that the following holds: For any $(\xi, \xi', \eta, \eta') \in \partial^4 \widetilde{M}$ such that the pairwise distances between the geodesics $[\xi, \eta']$, $[\xi', \eta]$, $[\xi, \eta]$, $[\xi', \eta']$ are all less than r, and such that $[\xi, \xi', \eta, \eta'] \ge \delta_0$, we have

$$(1 - \varepsilon')[\xi, \xi', \eta, \eta'] \le [\overline{f}(\xi), \overline{f}(\xi'), \overline{f}(\eta), \overline{f}(\eta')] \le (1 + \varepsilon')[\xi, \xi', \eta, \eta'].$$

More specifically, we can take $\delta_0 = C\exp(-a\alpha\varepsilon^{-1/2}) \le C\varepsilon^{a\alpha}$ and $\varepsilon' = C\delta_0 + 4\sqrt{\varepsilon} \le 2C\delta_0$ for some constant $C = C(n, \Gamma, D, i_0, a, b)$ and any $\alpha < (1-\varepsilon)a/b$.

Remark 2.5 It is natural for the above ratio of cross-ratios to *not* remain bounded as $[\xi, \xi', \eta, \eta']$ approaches zero. For instance, when $n = 2$, Proposition 2.4 together with (4-2) means that the Liouville currents of M and N are mutually singular, which is indeed the case by [43] unless M and N are isometric.

In order to prove Proposition 2.4, we start by investigating the relation between L and δ in (2-3).

Proposition 2.6 *There exist constants $C = C(a, b, \operatorname{diam}(M))$ and $r = r(a)$ such that the following holds: For any $\delta > 0$ and for any $(\xi, \xi', \eta, \eta') \in \partial^4 \widetilde{M}$ such that the pairwise distances between the geodesics $[\xi, \eta'], [\xi', \eta], [\xi, \eta], [\xi', \eta']$ are all less than r, there exist free homotopy classes γ_i for $i = 1, 2, 3, 4$, with $\mathcal{L}_g(\gamma_i) \le L := (1/a)\log(C\delta^{-1})$ such that*

$$|[\xi, \xi', \eta, \eta'] - (\mathcal{L}_g(\gamma_1) + \mathcal{L}_g(\gamma_2) - \mathcal{L}_g(\gamma_3) - \mathcal{L}_g(\gamma_4))| < \delta.$$

For this we will use many ideas from our paper [12], which pertains to controlling the marked length spectrum only using information about "short geodesics" of length less than some fixed L. Let W^{si} for $i = \mathrm{s}, \mathrm{u}$ denote the strong stable and strong unstable foliations for the geodesic flow ϕ^t on $T^1 M$. Recall that their leaves have the following geometric description; see [1]. Let $v \in T^1 \widetilde{M}$. Denote by $p \in \widetilde{M}$ the footpoint of v and by $\xi \in \partial \widetilde{M}$ the forward projection of v to the boundary at infinity. Let $B_{\xi,p}$ denote the associated Busemann function on $\widetilde{M}$ and let $H_{\xi,p}$ denote the horosphere given by the zero set of $B_{\xi,p}$. Then the lift of the leaf $W^{\mathrm{ss}}(v)$ to $T^1 \widetilde{M}$ is given by $\{-\operatorname{grad} B_{\xi,p}(q) \mid q \in H_{p,\xi}\}$. Analogously, the lift of $W^{\mathrm{su}}(v)$ is given by $\{\operatorname{grad} B_{\eta,p}(q) \mid q \in H_{\eta,p}\}$, where η denotes the forward projection of $-v$ to $\partial \widetilde{M}$.

For $v \in T^1 \widetilde{M}$ and $R > 0$, let $W^{si}_R(v) = W^{si}(v) \cap B(v, R)$, where $B(v, R)$ denotes a ball of radius R centered at v with respect to the Sasaki metric on $T^1 \widetilde{M}$. (See, for instance, [13, Exercise 3.2] for the definition of the Sasaki metric on TM induced by the Riemannian metric g on M.)

Lemma 2.7 *Fix $D > 0$ and let $u_1, u_2 \in T^1 \widetilde{M}$ with $d(u_1, u_2) \le D$. Suppose, in addition, that u_1 and $-u_2$ do not lie on the same oriented geodesic. Then there exists some constant C, depending only on a, b, D, together with a time $\sigma \in [0, D]$, such that*

$$W^{\mathrm{ss}}_C(u_1) \cap W^{\mathrm{su}}_C(\phi^\sigma u_2) \neq \varnothing.$$

Remark 2.8 If one is not concerned about which geometric quantities the above constant C depends on, then the statement follows from the standard fact that Anosov flows have local product structure; see, for instance, [20, Proposition 6.2.2].

Proof By assumption, we have either $\pi(u_1) \neq \pi(-u_2)$ or $\pi(-u_1) \neq \pi(u_2)$. Without loss of generality, we will assume that $\pi(u_1) \neq \pi(-u_2)$. Then there exists a point u, often denoted by $[u_1, u_2]$, together with a time $\sigma = \sigma(u_1, u_2)$, such that $u = W^{\mathrm{ss}}(u_1) \cap \phi^\sigma W^{\mathrm{su}}(u_2)$. To see this geometrically, let p_1 and p_2 denote the footpoints of u_1 and u_2, respectively. Let $\xi = \pi(u_1)$ and $\eta = \pi(-u_2)$. Let $p_1' \in H_{p_1,\xi}$ and $p_2' \in H_{p_2,\eta}$ be points on the geodesic determined by ξ and η. Let $u_1' = -\operatorname{grad} B_{\xi,p_1}(p_1')$ and $u_2' = \operatorname{grad} B_{\eta,p_2}(p_2')$. Let $\sigma = \sigma(u_1, u_2)$ be such that $\phi^\sigma u_2' = u_1'$. Then $u_1' = W^{\mathrm{ss}}(u_1) \cap \phi^\sigma W^{\mathrm{su}}(u_2)$.

By the above construction, $|\sigma| = d(p_1', p_2') \le D$; see also [12, Lemma 3.1]. By [12, Proposition 3.3] we see that $d(p_1, p_1')$ and $d(\phi^\sigma p_2, \phi^\sigma p_2')$ are both bounded above by $C' d(u_1, u_2)$, where C' is a constant depending only on a, b. Letting $C = 2C'D$ completes the proof. □

Lemma 2.9 *Let (M, g) be a closed Riemannian manifold with sectional curvatures in the interval $[-b^2, -a^2]$ and diameter bounded above by D. Given $\delta > 0$, there exists a constant $C = C(a, b, D)$ such that the following holds: Let $S = (1/a)\log(C/\delta)$. Then for every $v \in T^1M$ and every $T_0 > 0$, there exists $w \in T^1M$, together with a time $T \in [0, S]$, satisfying*

(1) $d(\phi^t v, \phi^t w) < \delta$ *for all* $t \in [-T_0, T_0]$,

(2) $\phi^{2(T_0+T)} w = w$.

Remark 2.10 Since the geodesic flow of (M, g) satisfies the specification property [38, Section 18.3], it follows that given $\delta > 0$, there is an S depending on δ and (M, g) satisfying the conclusion of the above lemma. We show more specifically that there is a *uniform* choice of S which works for all (M, g) satisfying the hypotheses of the above lemma.

Proof Fix $\delta' > 0$ (which will later be chosen based on δ). Set $v_1 = \phi^{-T_0} v$ and $v_2 = \phi^{T_0} v$. Let $\tilde{v}_1$ be a lift of v_1 to $T^1\widetilde{M}$. Then of course $\phi^{2T_0}\tilde{v}_1$ is a lift of v_2, but there is moreover a deck transformation $\gamma \in \Gamma$ such that $\tilde{v}_2 := \gamma\phi^{2T_0}\tilde{v}_1$ is distance less than D away from $\tilde{v}_1$. For $j = 1, 2$, let $P(\tilde{v}_j, \delta') = \bigcup_{v' \in W^{\mathrm{ss}}_{\delta'}(\tilde{v}_j)} W^{\mathrm{su}}_{\delta'}(v') \subset T^1\widetilde{M}$ be a local transversal to the geodesic flow ϕ^t. Let $R(\tilde{v}_j, \delta') = \bigcup_{t \in (-\delta'/2, \delta'/2)} \phi^t P(\tilde{v}_j, \delta) \subset T^1\widetilde{M}$ be a local flow box.

Let $R(v_j, \delta')$ denote the projection of $R(\tilde{v}_j, \delta')$ to T^1M. We claim there is a constant S', depending only on δ', a, b, D, such that for any $R(v_1, \delta')$ and $R(v_2, \delta')$ as above, there are $T_i \in [0, S']$ so that $\phi^{-T_1} R(v_1, \delta') \cap \phi^{T_2} R(v_2, \delta') \neq \varnothing$. To see this, let C denote the constant in Lemma 2.7. By the hyperbolicity of the geodesic flow, there exists $T_1 > 0$ sufficiently large such that $\phi^{-T_1+D} W^{\mathrm{ss}}_{\delta'}(\tilde{v}_1) \supset W^{\mathrm{ss}}_C(\phi^{-T_1+D}\tilde{v}_1)$. In fact, by the remark after [34, Proposition 4.1], for any $T > 0$ and $\tilde{v}_1' \in W^{\mathrm{ss}}(\tilde{v}_1)$ we have $d(\phi^{-T}\tilde{v}_1, \phi^{-T}\tilde{v}_1') \geq ce^{aT} d(\tilde{v}_1, \tilde{v}_1')$, where $c = c(a)$ is a constant depending only on a. As such, T_1 can be taken smaller than $S = (1/a)\log(C'/\delta')$ for some $C' = C'(a, b, D)$. Similarly, there exists $0 < T_2 \leq S$ such that $\phi^{T_2 - D} W^{\mathrm{su}}_{\delta'}(v_2) \supset W^{\mathrm{su}}_C(\phi^{T_2-D} v_2)$.

By Lemma 2.7 applied to $u_1 = \phi^{-T_1} v_1$ and $u_2 = \phi^{T_2-D} v_2$, we have that there exists $\sigma \in [-D, D]$ such that $\phi^{-T_1} W^{\mathrm{ss}}_{\delta'}(v_1) \cap \phi^{T_2+\sigma-D} W^{\mathrm{su}}_{\delta'}(v_2) \neq \varnothing$. Note that $T_2' := T_2 + \sigma - D \leq T_2 \leq S$. By construction of the $R(v_i, \delta')$, it follows that $\phi^{-T_1} R(v_1, \delta') \cap \phi^{T_2'} R(v_2, \delta') \neq \varnothing$ as claimed.

Now let $u \in \phi^{-T_1} R(v_1, \delta') \cap \phi^{T_2'} R(v_2, \delta')$. This means $d(\phi^{T_1} u, v_1), d(\phi^{-T_2'} u, v_2) < \delta'$, which by the shadowing property for Anosov flows [38, Theorem 18.1.6] in turn implies that

$$\{\phi^t v\}_{t \in [-T_0, T_0]} \cup \{\phi^t(-u)\}_{t \in [-T_2', T_1]}$$

is $\eta = \eta(\delta', g)$-shadowed by a periodic orbit $\{\phi^t w\}_{t \in [0, S_0]}$ for some S_0 satisfying $|2(T_0+S) - S_0| < \eta$. An argument similar to [12, Lemma 3.13] shows that for this particular flow, we can take $\eta = \eta(\delta', g) = C'\delta'$ for some constant C' depending only on a and b. Finally, choosing δ' small enough so that $C\delta' < \delta$ completes the proof. □

Given $\xi, \xi', \eta, \eta' \in \partial^4 \widetilde{M}$, let $q_1, q_2 \in \widetilde{M}$ be the unique points on the geodesics through $[\xi, \eta']$ and $[\xi', \eta]$, respectively, such that the geodesic segment $[q_1, q_2]$ is perpendicular to both $[\xi, \eta']$ and $[\xi', \eta]$. Let v_1 be the unit tangent vector based at q_1 such that $\pi(v_1) = \xi$ and $\pi(-v_1) = \eta$. Let v_2 be the unit tangent vector based at q_2 such that $\pi(v_2) = \xi'$ and $\pi(-v_2) = \eta'$. We then have the following:

Lemma 2.11 *Let v_1 and v_2 be as above. Then there exists small enough $r = r(a, b)$ so that the following holds: Fix $\delta \in (0, r)$. Then for any $(\xi, \xi', \eta, \eta') \in \partial^4 \widetilde{M}$ such that $\delta \le [\xi, \xi', \eta, \eta']$ and such that the distances between the geodesics $[\xi, \eta']$, $[\xi', \eta]$, $[\xi, \eta]$, $[\xi', \eta']$ are all less than r, there exists $T \le (2/a)\log(16(a\delta)^{-1})$ such that the stable horospheres associated to $W^{ss}(\phi^T v_1)$ and $W^{ss}(\phi^T v_2)$ are disjoint.*

For the proof, we will use a formula for the cross-ratio in terms of the *Gromov product*, following Bourdon [7]. Recall that for $\xi, \xi' \in \partial \widetilde{M}$ and $p \in \widetilde{M}$, the Gromov product of ξ and ξ' with respect to p is given by

$$(\xi \mid \xi')_p := \lim_{(x,x') \to (\xi,\xi')} \tfrac{1}{2}(d(p,x) + d(p,x') - d(x,x')). \tag{2-4}$$

This quantity is also equal to half the length of the portion of the bi-infinite geodesic determined by ξ and ξ' which lies in the intersection of the horoballs bounded by the zero sets of the Busemann functions $B_{\xi,p}$ and $B_{\xi',p}$; see [6, Section 2.4].

For $\xi, \xi' \in \partial \widetilde{M}$, let $d_p(\xi, \xi') = e^{-(\xi|\xi')_p}$ be the Bourdon–Gromov distance. By [7, Proposition 1.1(d)], the cross-ratio can be written as

$$e^{\frac{1}{2}[\xi,\xi',\eta,\eta']} = \frac{d_p(\xi,\eta)\, d_p(\xi',\eta')}{d_p(\xi,\eta')\, d_p(\xi',\eta)} \tag{2-5}$$

for any $p \in \widetilde{M}$.

We will also use a characterization of $d_p(\xi, \xi')$ in terms of comparison angles. Recall that, by hypothesis, the sectional curvatures of (M, g) are bounded above by $-a^2$. Consider now the rescaling of g which has sectional curvatures bounded above by -1; the associated distance function is then written as $a\, d_g$ with respect to the original distance function d_g. (This normalization is so that we are working with a CAT(-1) space; see the introduction of [7] for the definition.) Now given $p, x, x' \in \widetilde{M}$, consider the associated comparison triangle in the hyperbolic space of constant curvature -1, and let $\theta_p(x, x')$ denote the associated comparison angle. Then, for $\xi, \xi' \in \partial \widetilde{M}$, let

$$\theta_p(\xi, \xi') := \lim_{(x,x') \to (\xi,\xi')} \theta_p(x, x').$$

By [6, Section 2.5], we then have

$$d_p(\xi, \xi')^a = \sin\bigl(\tfrac{1}{2}\theta_p(\xi, \xi')\bigr). \tag{2-6}$$

We will also use the key fact that $\theta_p(\cdot, \cdot)$ is a metric on $\partial \widetilde{M}$ for all $p \in \widetilde{M}$. (This is where the CAT(-1) hypothesis, and hence the above preliminary rescaling of the metric, is used; see [7, Proposition 1.2(b)] and [6, Lemma 2.5.4].)

Finally, for the proof of Lemma 2.11, we also require the following.

Lemma 2.12 *Suppose (M, g) has sectional curvatures in the interval $[-b^2, -a^2]$. Let $u_0 \in T^1_{p_0}\widetilde{M}$ and let $B\colon \widetilde{M} \to \mathbb{R}$ denote the associated Busemann function. Given $\delta > 0$, there exists small enough $r = r(\delta, a, b)$ so that the following holds: For any p on the horosphere $\{B = 0\}$, let p' denote its orthogonal projection onto the hypersurface $\exp_p(\operatorname{grad} B^{\perp})$. Then $d(p_0, p') \le r$ ensures $d(p_0, p) \le \delta$.*

Proof Let $s_0 > 0$. For any $v \in T^1_{p_0}\widetilde{M}$ which is orthogonal to $u_0 = \operatorname{grad} B(p_0)$, we define a number $r(s_0, v)$ as follows. Let $\gamma(s)$ denote the unit-speed geodesic such that $\gamma(0) = p_0$ and $\gamma'(0) = v$. Let x denote the point on $\{B = 0\}$ obtained by flowing the point $\gamma(s_0)$ along the vector field $-\operatorname{grad} B$. Let x' denote the orthogonal projection of x onto $\exp_{p_0} \operatorname{grad} B^{\perp}$. Define $r(s_0, v) = d(p_0, x')$.

Now suppose $s_0 \le 1/(2b)$. We claim that $r(s_0, v) \ge \frac{3}{4}s_0$ for all $v \in T^1_{p_0}\widetilde{M} \cap u^{\perp}$. Indeed, we have $r(s_0, v) = d(p_0, x') \ge s_0 - d(x', \gamma(s_0)) \ge s_0 - d(x, \gamma(s_0))$. By [12, Lemma 3.4], we have $d(x, \gamma(s_0)) \le \frac{1}{2}bs_0^2$. This gives $r(s_0, v) \ge s_0\bigl(1 - \frac{1}{2}bs_0\bigr) \ge \frac{3}{4}s_0$ as desired.

Now let p' be a point in the hypersurface $\exp_{p_0}(\operatorname{grad} B^{\perp})$ such that $d(p_0, p_1) \le r \le r_0 := \frac{1}{2}(1/(2b))$. Let ν be the unit normal vector of this hypersurface at p', and let p be the point on $\{B = 0\}$ obtained by intersecting $\{B = 0\}$ with the geodesic $\eta(t)$ determined by $\eta(0) = p'$ and $\eta'(0) = p$. Then p and p' are as in the statement of the lemma. By the previous paragraph, our choice of r_0 guarantees that the geodesic determined by the points $p \in \widetilde{M}$ and $\pi(u_0) \in \partial\widetilde{M}$ does in fact intersect $\exp_{p_0}(\operatorname{grad} B^{\perp})$ at some point y, and moreover, we have $s_0 := d(y, p_0) \le \frac{4}{3}r$. By [34, Proposition 4.7], we then have that there is a constant $C = C(a)$ such that $d(p_0, p) \le C(a)s_0 \le \frac{4}{3}C(a)r$. This completes the proof. □

Proof of Lemma 2.11 If the horospheres associated to $W^{ss}(v_1)$ and $W^{ss}(v_2)$ are disjoint, then taking $T = 0$ proves the claim; if not, let q be a point lying on both horospheres, which minimizes the distance to the segment $[q_1, q_2]$.

Recall from above that the Gromov product $(\xi \mid \xi')_q$ is equal to half of the length of the segment of the geodesic $[\xi, \xi']$ which lies in the intersection of the horoballs determined by $W^{ss}(v_1)$ and $W^{ss}(v_2)$. This means precisely that flowing v_1 and v_2 for time $T = 2(\xi \mid \xi')_q$ guarantees disjointness of the desired horospheres; thus, we need to bound $(\xi \mid \xi')_q$ from above in terms of δ.

Since $[\xi, \xi', \eta, \eta'] \ge \delta$, equation (2-5) gives

$$1 + \tfrac{1}{2}\delta \le e^{\frac{1}{2}[\xi,\xi',\eta,\eta']} = \frac{d_q(\xi, \eta)\, d_q(\xi', \eta')}{d_q(\xi, \eta')\, d_q(\xi', \eta)}.$$

Hence one of $d_q(\xi', \eta)/d_q(\xi, \eta)$ or $d_q(\xi, \eta')/d_q(\xi', \eta')$ is bounded above by $\bigl(1 + \frac{1}{2}\delta\bigr)^{-1/2} \le 1 - \frac{1}{4}\delta$. We will assume $d_q(\xi', \eta)/d_q(\xi, \eta) \le 1 - \frac{1}{4}\delta$; the other case is similar. This means

$$\sin\bigl(\tfrac{1}{2}\theta_q(\xi', \eta)\bigr) \le \bigl(1 - \tfrac{1}{4}\delta\bigr)^a \sin\bigl(\tfrac{1}{2}\theta_q(\xi, \eta)\bigr). \tag{2-7}$$

Next, we claim there is a small enough choice of $r = r(a,b)$ so that $\sin\bigl(\frac{1}{2}\theta_q(\xi',\eta)\bigr) = d_q(\xi',\eta)^a \geq \frac{1}{2}$. Since q_2 is on the geodesic determined by ξ' and η, we have $d_{q_2}(\xi',\eta) = 1$. By [7, Proposition 1.1(b)], we then have $d_q(\xi',\eta) = \exp\bigl(\frac{1}{2}B_{\xi'}(q_2,q) + \frac{1}{2}B_\eta(q_2,q)\bigr) \geq \exp(-d(q,q_2))$. Now let δ_0 be sufficiently small so that $d(q,q_2) < \delta_0$ implies $\exp(-d(q,q_2)) \geq \frac{1}{2}$. Applying Lemma 2.12 with $p_0 = q_2$ and $q = p$ shows that $d(q_2,q)$ can be made less than δ_0 provided the distance between q_2 and the orthogonal projection of q onto $[q_1,q_2]$, which is bounded above by $d(q_1,q_2) \leq r$, is sufficiently small in terms of δ_0, a and b.

Now let $x = \frac{1}{2}\theta_q(\xi',\eta)$ and $y = \frac{1}{2}\theta_q(\xi,\eta)$. The previous paragraph, together with (2-7), shows $\sin(y) \geq \frac{1}{2}$. Setting $4\varepsilon = 1 - \bigl(1 - \frac{1}{4}\delta\bigr)^a$ and using (2-7), we have

$$\begin{aligned}
\sin x &\leq \bigl(1 - \tfrac{1}{4}\delta\bigr)^a \sin y = (1-4\varepsilon)\sin y \\
&\leq (1-\varepsilon^2)\sin y - 2\varepsilon \sin y \\
&\leq \cos\varepsilon \sin y - \varepsilon && \bigl(\text{since } \sin y \geq \tfrac{1}{2}\bigr) \\
&\leq \cos\varepsilon \sin y - \sin\varepsilon\cos y \\
&= \sin(y-\varepsilon).
\end{aligned}$$

Since $x, y \in \bigl[0, \frac{\pi}{2}\bigr]$ and $\theta \mapsto \sin\theta$ is increasing on this interval, we conclude $\varepsilon \leq y - x$. By the triangle inequality for θ_q, we have

$$\tfrac{1}{2}\theta_q(\xi,\xi') \geq \tfrac{1}{2}(\theta_q(\xi,\eta) - \theta_q(\xi',\eta)) = y - x \geq \varepsilon \geq \tfrac{1}{8}a\delta.$$

As such,

$$e^{-a(\xi|\xi')_q} = d_q(\xi,\xi')^a = \sin\bigl(\tfrac{1}{2}\theta_q(\xi,\xi')\bigr) \geq \tfrac{1}{4}\theta_q(\xi,\xi') \geq \tfrac{1}{16}a\delta.$$

This gives $(\xi \mid \xi')_p \leq (1/a)\log(16(a\delta)^{-1})$. □

Proof of Proposition 2.6 If $[\xi,\xi',\eta,\eta'] < \delta$, taking all γ_i to be the trivial free homotopy class shows the statement of the proposition. As such, we will now assume that $\delta \leq [\xi,\xi',\eta,\eta']$. By Lemma 2.11, there exist $T^+, T^- \leq (1/a)\log(C/\delta)$ (for some universal constant C) such that the stable horospheres associated to $W^{\mathrm{ss}}(\phi^{T^+}v_1)$ and $W^{\mathrm{ss}}(\phi^{T^+}v_2)$ are disjoint, and the unstable horospheres associated to $W^{\mathrm{su}}(\phi^{T^-}v_1)$ and $W^{\mathrm{su}}(\phi^{T^-}v_2)$ are disjoint. Denote these horospheres by H_ξ, $H_{\xi'}$, H_η, $H_{\eta'}$, respectively.

Now fix $\delta' > 0$ (which will later be specified in terms of δ). We can increase T^+ more if necessary so that the intersection points of the geodesics $[\xi,\eta]$ and $[\xi,\eta']$ with H_ξ are of distance less than $\frac{1}{8}\delta'$ apart; similarly, we further increase T^+ so that the analogous property holds for the horosphere $H_{\xi'}$. We also increase T^- so that the analogous property holds for H_η and $H_{\eta'}$. By the remark after Proposition 4.1 in [34], we can do all this while keeping $T^\pm$ bounded above by $(1/a)\log(c(a)d(q_1,q_2)/\delta')$, for some constant $c = c(a)$ depending only on a.

For simplicity we will first complete the proof in the case where all the above four horospheres H_ξ, $H_{\xi'}$, H_η, $H_{\eta'}$ are pairwise disjoint. Let $l_{\xi,\eta'}$ denote the length of the segment of the geodesic determined by ξ and η' which lies in the complement of the above four horospheres. Define $l_{\xi',\eta}, l_{\xi',\eta'}, l_{\xi,\eta}$ analogously. In this case, the proof of [44, Lemma 2.1] shows that

$$[\xi,\xi',\eta,\eta'] = l_{\xi,\eta} + l_{\xi',\eta'} - l_{\xi,\eta'} - l_{\xi',\eta}. \tag{2-8}$$

Using Lemma 2.9, there is a constant $C = C(a,b,\operatorname{diam}(M))$ such that we can increase $T^{\pm}$ by at most $(1/a)\log(C/\delta')$ so that the lengths $l_{\xi,\eta'}$ and $l_{\xi',\eta}$ are each within δ' of lengths of closed geodesics. By the Anosov closing lemma (see [12, Lemma 3.13]), together with the previous paragraph, we have that $l_{\xi,\eta}$ and $l_{\xi',\eta'}$ are within $C\delta'$ of lengths of closed geodesics, for some $C = C(a,b)$. Choosing δ' so that $2\delta' + 2C\delta' < \frac{1}{2}\delta$ completes the proof in this case.

Now suppose instead that the above four horospheres are not disjoint. Without loss of generality, assume H_ξ intersects another one of $H_{\xi'}, H_\eta, H_{\eta'}$ nontrivially. By Lemma 2.11, H_ξ does not overlap with $H_{\xi'}$. If H_ξ intersects H_η or $H_{\eta'}$, then the overlap between the two horoballs contains a segment of the geodesic $[\xi,\eta]$ or $[\xi,\eta']$, respectively. In this case, the proof of [44, Lemma 2.1] then shows that the formula (2-8) holds after changing the sign in front of $l_{\xi,\eta}$ or $l_{\xi,\eta'}$, respectively. So the same argument as in the previous paragraph completes the proof. □

The next step of our argument is to obtain an analogue of (2-3) for the marked length spectrum $\mathcal{L}_{g_0} \circ f_*$. To do so, we use an orbit equivalence of geodesic flows, as in (2-1). In Theorem 2.38(2) in Section 2.3 below, we show that the orbit equivalence constructed in [27] is Hölder continuous with controlled Hölder constant and exponent. That is, for all $v, w \in T^1M$, we have

$$d(\mathcal{F}(v), \mathcal{F}(w)) \le C d(v,w)^\alpha, \tag{2-9}$$

where C is a constant depending only on n, Γ, a, b, i_0, D and α is any positive number strictly less than $(1-\varepsilon)a/b$.

Proposition 2.13 *Let $\delta > 0$. Suppose that $(\xi,\xi',\eta,\eta') \in \partial^4 \widetilde{M}$ and γ_i for $i = 1,2,3,4$ are such that*

$$|[\xi,\xi',\eta,\eta'] - (\mathcal{L}_g(\gamma_1) + \mathcal{L}_g(\gamma_2) - \mathcal{L}_g(\gamma_3) - \mathcal{L}_g(\gamma_4))| < \delta.$$

Let $\delta' = 100\, C\delta^\alpha$, where C and α are as in (2-9). Then

$$|[f(\xi), f(\xi'), f(\eta), f(\eta')] - (\mathcal{L}_g(f\gamma_1) + \mathcal{L}_g(f\gamma_2) - \mathcal{L}_g(f\gamma_3) - \mathcal{L}_g(f\gamma_4))| < \delta'.$$

Proof Let $H_\xi, H_{\xi'}, H_\eta, H_{\eta'}$ be as in the proof of Proposition 2.6. For simplicity, we will complete the proof in the case where $H_\xi, H_{\xi'}, H_\eta, H_{\eta'}$ are all disjoint (if not, the sign-change argument at the end of the proof of Proposition 2.6 can be used similarly here). For $i = 1,\dots,4$, let $v_i \in T^1\widetilde{M}$ be such that the terms on the right-hand side of (2-8) are of the form $\{\phi^t v_i\}_{t\in[-T_i^-, T_i^+]}$ for some times $T_i^{\pm} > 0$. Recall that, by construction, $d(\phi^{T_1^+} v_1, \phi^{T_3^+} v_3) < \frac{1}{8}\delta$. As in the proof of Proposition 2.6, we can increase the $T_i^{\pm}$ if necessary so that there are unit tangent vectors w_i satisfying $d(\phi^t v_i, \phi^t w_i) < \delta$ for all $t \in [-T_i^-, T_i^+]$, and such that the segments $\{\phi^t w_i\}_{t\in[-T_i^-, T_i^+]}$ project to periodic orbits in T^1M.

Now let $l_{f(\xi),f(\eta)} = d(\mathscr{F}(\phi^{T_1^+} v_1), \mathscr{F}(\phi^{-T_1^-} v_1))$ and define $l_{f(\xi),f(\eta')}, l_{f(\xi'),f(\eta)}, l_{f(\xi),f(\eta')}$ analogously. The previous paragraph, combined with (2-9), gives

$$|[\bar{f}(\xi), \bar{f}(\xi'), \bar{f}(\eta), \bar{f}(\eta')] - (l_{f(\xi),f(\eta)} + l_{f(\xi'),f(\eta')} - l_{f(\xi'),f(\eta)} - l_{f(\xi),f(\eta')})| < 8\delta',$$

where $\delta' = \bar{C}\left(\frac{1}{8}\delta\right)^\alpha$. Next, note that $d(\mathscr{F}(\phi^t v_i), \mathscr{F}(\phi^t w_i)) < \delta'$ for all $t \in [-T_i, T_i]$. This means

$$|d(\mathscr{F}(\phi^{T_i} v_i), \mathscr{F}(\phi^{-T_i} v_i)) - d(\mathscr{F}(\phi^{T_i} w_i), \mathscr{F}(\phi^{-T_i} w_i))| < 2\delta'$$

for each i. Finally, note that if γ_i denotes the free homotopy class associated to the periodic orbit determined by w_i, then $2T_i = \mathscr{L}_g(\gamma_i)$, and moreover, $\mathscr{F}(w_i)$ is tangent to a periodic geodesic of length $\mathscr{L}_{g_0}(f(\gamma_i))$. This shows $d(\mathscr{F}(\phi^{T_i} w_i), \mathscr{F}(\phi^{-T_i} w_i)) = \mathscr{L}_{g_0}(f(\gamma_i))$, which completes the proof. □

Proof of Proposition 2.4 Given $\varepsilon > 0$, set $L = \varepsilon^{-1/2}$. Let C be the constant from Proposition 2.6. Set $\delta = Ce^{-a\varepsilon^{-1/2}}$. Then L and δ are related as in the statement of Proposition 2.6. Set $\delta_0 = 3\delta^{\alpha/2}$, where α is as in (2-9). Take $(\xi, \xi', \eta, \eta') \in \partial^4 \widetilde{M}$ satisfying the hypothesis of Proposition 2.6, and suppose further that $[\xi, \xi', \eta, \eta'] \geq \delta_0$. Now set

$$x = [\xi, \xi', \eta, \eta'], \qquad y = \mathscr{L}_g(\gamma_1) + \mathscr{L}_g(\gamma_2) - \mathscr{L}_g(\gamma_3) - \mathscr{L}_g(\gamma_4),$$
$$x' = [f(\xi), f(\xi'), f(\eta), f(\eta')], \quad y' = \mathscr{L}_{g_0}(f(\gamma_1)) + \mathscr{L}_{g_0}(f(\gamma_2)) - \mathscr{L}_{g_0}(f(\gamma_3)) - \mathscr{L}_{g_0}(f(\gamma_4)).$$

Note that we are assuming $x \geq 3\delta^{\alpha/2}$. We also have

(1) $|x - y| < \delta$ by Proposition 2.6,

(2) $|x' - y'| < \delta' = C\delta^\alpha$ by Proposition 2.13,

(3) $1 - 4\sqrt{\varepsilon} \leq y'/y \leq 1 + 4\sqrt{\varepsilon}$ by Lemma 2.3.

We want to show the ratio x'/x is close to 1. By (1) and (2), we have

$$\frac{y' - \delta'}{y + \delta} \leq \frac{x'}{x} \leq \frac{y' + \delta'}{y - \delta},$$

which rearranges to

$$-\frac{1}{y + \delta}\left(\delta' + \frac{\delta y'}{y}\right) \leq \frac{x'}{x} - \frac{y'}{y} \leq \frac{1}{y - \delta}\left(\delta' + \frac{\delta y'}{y}\right).$$

Since $x \geq 3\delta^{\alpha/2}$, we have $y \geq 2\delta^{\alpha/2}$. For $\delta < 1$, we then get $y - \delta \geq \delta^{\alpha/2}$. Hence $(y \pm \delta)^{-1} \leq \delta^{-\alpha/2}$, which gives

$$\left|\frac{x'}{x} - \frac{y'}{y}\right| \leq \delta^{-\alpha/2}(C\delta^\alpha + 2\delta) \leq C'\delta^{\alpha/2}.$$

Using (3), we see that $1 - \varepsilon' \leq x'/x \leq 1 + \varepsilon'$ for $\varepsilon' = C\delta_0 + 4\sqrt{\varepsilon}$. □

2.2 The Liouville current

In this section, we use our findings on the cross-ratio from Proposition 2.4 to relate the Liouville currents λ^M and λ^N. We first explain the connection between the cross-ratio and the temporal function. (The temporal function was also used in the proof of Lemma 2.7).

Definition 2.14 (see, for instance, [20, Proposition 6.2.2]) Given w_1 and w_2 such that $\pi(-w_1) \neq \pi(w_2)$, there is a unique time $\sigma = \sigma(w_1, w_2)$, which we call the *temporal function*, such that

$$W^{\mathrm{su}}(w_1) \cap W^{\mathrm{ss}}(\phi^\sigma w_2) \neq \varnothing;$$

moreover, the above intersection consists of a single point denoted by $[w_1, w_2]$, and called the *Bowen bracket* of w_1 and w_2.

We then have (see, for instance, [31, page 495])

$$[\pi(w_1), \pi(w_2), \pi(-w_1), \pi(-w_2)] = \sigma(w_1, w_2) + \sigma(w_2, w_1). \tag{2-10}$$

Now, we relate the temporal function to $d\omega$ as in [30, Lemma 1]; see also [40, Appendix B]. Let $v \in T^1 M$ and suppose $w_1 \in W^{\mathrm{su}}(v)$ and $w_2 \in W^{\mathrm{ss}}(v)$. Consider a C^1 surface Σ bounded by five arcs (each contained in either a stable leaf, an unstable leaf or a flow line) connecting the points $v, w_1, [w_1, w_2], w_2, \phi^\sigma w_2$. By Stokes' theorem, together with the fact that the stable and unstable bundles are in the kernel of ω, we obtain

$$\int_\Sigma d\omega = \int_{\partial\Sigma} \omega = \sigma(w_1, w_2) = [\pi(w_1), \pi(w_2), \pi(-w_1), \pi(-w_2)], \tag{2-11}$$

where in the last equality, we used that $\sigma(w_2, w_1) = 0$, since $v = W^{\mathrm{ss}}(w_2) \cap W^{\mathrm{su}}(w_1)$.

Remark 2.15 In the special case where the Bowen bracket is C^1 (ie if the Anosov splitting is C^1) then we can construct Σ as in the proof of [30, Lemma 1] as follows: let $s \mapsto c_1(s)$ be a curve in $W^{\mathrm{su}}(v)$ joining v and w_1, and let $t \mapsto c_2(t)$ be a curve in $W^{\mathrm{ss}}(v)$ joining v and w_2; then we can take Σ to be the image of $(s,t) \mapsto [c_1(s), c_2(t)]$.

If $\mathcal{F}$ conjugates the geodesic flows of M and N, then $\sigma(w_1, w_2) = \sigma(\mathcal{F}(w_1), \mathcal{F}(w_2))$, since $\mathcal{F}$ preserves the strong stable and unstable foliations. If, moreover, $\mathcal{F}$ is C^1 (which is the case if, for instance, the Anosov splittings of M and N are both C^1, see [30]), and Σ is the surface in the above remark, then $\mathcal{F}(\Sigma)$ is a C^1 surface of the same form. This shows $\mathcal{F}$ preserves the symplectic form $d\omega$ and hence the Liouville volume form.

In general, the marked length spectrum allows us to relate $\sigma(w_1, w_2)$ and $\sigma(\mathcal{F}(w_1), \mathcal{F}(w_2))$ by way of the cross-ratio (Proposition 2.4 and (2-10)). However, it is less clear how $\int_\Sigma d\omega_M$ and $\int_{\mathcal{F}(\Sigma)} d\omega_N$ are related when $\mathcal{F}$ is neither C^1 nor time-preserving. In [32, Corollary 3.12], Hamenstädt estimates the

Liouville current of embedded balls in $\partial^2 \tilde{N}$ which satisfy a *quasisymplecticity* property [32, page 123], which essentially means that equality approximately holds in (2-11). (The argument relies on the Bowen bracket of N being C^1.)

In [32, page 123], Hamenstädt constructs particular examples of quasisymplectic balls in $\partial^2 \tilde{M}$, whose Liouville current can be independently estimated without relying on the Bowen bracket. Equality of marked length spectra, and hence cross-ratios, means that the images of these balls by $\bar{f}$ have controlled Liouville current. All the above-mentioned estimates/approximations go to zero as the size of the balls goes to zero. Finally, Hamenstädt approximates $T^1 M$ and $T^1 N$ with arbitrary small such balls from above and below to estimate their total Liouville measures; see the measures $\mathcal{S}$ and $\mathcal{P}$ defined, respectively, on pages 124 and 131 of [32].

In the $\varepsilon > 0$ case, we follow the same overall approach, but, in light of Proposition 2.4, we can only compare balls of size bounded away from zero. Thus, we need to make the various estimates and approximations referred to above more explicit. In Section 2.2.1, we study the particular examples of quasisymplectic balls from [32, page 123]. In Section 2.2.2, we study the images of these balls by the map $\bar{f} : \partial^2 \tilde{M} \to \partial^2 \tilde{N}$, and use quasisymplecticity to estimate their Liouville current.

2.2.1 E_v-cubes

In this subsection, we analyze Hamenstädt's construction of transversals to the geodesic flow on $T^1 M$ with controlled Liouville current [32, page 123]. We start with some notation and preliminaries on various subbundles of $T(T^1 \tilde{M})$. (See, for instance, [31, page 495].)

Let $P : T^1 \tilde{M} \to \tilde{M}$ denote the footpoint map. The *vertical bundle* $\mathcal{V}$ is the kernel of the differential $dP : TT^1 \tilde{M} \to T\tilde{M}$. For each v, the elements of $\mathcal{V}(v)$ are tangent to a curve in $T^1_p \tilde{M}$ (for $p = P(v)$); the tangent vectors of such curves are perpendicular to v. Let $\kappa : TT^1 \tilde{M} \to \tilde{M}$ denote the connector map, which is defined as follows. Given $X \in T_v T^1 \tilde{M}$, let $v(t)$ be a curve in $T^1 \tilde{M}$ which is tangent to X. Then $\kappa(X)$ is by definition the covariant derivative $\nabla_{(Pv)'(0)} v(0)$. These maps give an identification of $T_v T^1 \tilde{M}$ with $T^1_p \tilde{M} \oplus T^1_p \tilde{M}$, where $V \mapsto (dP(V), \kappa(V))$. The *Sasaki metric* on $T^1 \tilde{M}$ is given by

$$\langle V, W \rangle_v = \langle dP(V), dP(W) \rangle + \langle \kappa(V), \kappa(W) \rangle,$$

where the inner products on the right-hand side are with respect to the metric g.

If X_0 denotes the geodesic spray, ie the vector field on $T^1 \tilde{M}$ generating the geodesic flow, then $\kappa(X_0) = 0$. The *horizontal bundle* is defined to be the orthogonal complement of $\mathbb{R} X_0 \oplus \mathcal{V}$ with respect to the Sasaki metric; as such, the horizontal directions and $\mathbb{R} X_0$ span the kernel of κ.

If ω denotes the canonical contact form on $T^1 \tilde{M}$ introduced above, then its exterior derivative has the form

$$d\omega(V, W) = \langle \kappa(V), dP(W) \rangle - \langle \kappa(W), dP(V) \rangle \tag{2-12}$$

for all $V, W \in TT^1 \tilde{M}$; see, for instance, [10, Proposition 1.4(ii)].

It will also be important for us to understand how the (strong) stable and unstable subbundles TW^{ss} and TW^{su} relate to the horizontal and vertical subbundles. For this, recall from Section 2.1 that curves $v(t) \in W^{\mathrm{ss}}(v)$ are such that the footpoint curve $Pv(t)$ is a curve in the horosphere determined by $P(v) \in \widetilde{M}$ and $\pi(v) \in \partial\widetilde{M}$, and that $v(t)$ is the inward normal to the horosphere at the point $Pv(t)$. As such, for $V \in T_v W^{\mathrm{ss}}$, let w denote its horizontal component $dP(V)$ (which is tangent to $P(W^{\mathrm{ss}}(v))$). To determine the vertical component $\kappa(V)$, let $U(v)$ denote the second fundamental form of the horosphere $PW^{\mathrm{ss}}(v)$. For fixed $\xi \in \partial\widetilde{M}$, let Z^ξ denote the vector field on $\widetilde{M}$ so that $\pi(Z^\xi(q)) = \xi$ for all $q \in \widetilde{M}$. Then Z^ξ is a C^1 vector field, and moreover, its covariant derivative is given by

$$\nabla_Y Z^\xi = -U(Z^\xi)(Y) \tag{2-13}$$

for all $Y \in T\widetilde{M}$ [34, Proposition 3.1]. In particular, $\kappa(V)$ is given by $-U(v)w$ for $w = dP(v)$ as above. Similarly, if $W \in T_v W^{\mathrm{su}}$ and $w = dP(W)$, then $\kappa(W) = U(-v)w$.

Finally, we recall that $U(v)$ depends Hölder continuously on v. Using the discussion above, this follows from the fact that the subbundles TW^{ss}, TW^{su} are Hölder continuous. More precisely, from the proof of [1, Appendix, Proposition 4.4], it follows that the modulus of continuity of both the stable and unstable subbundles depends only on the expansion and contraction constants of the flow, together with the norm of the derivative of the geodesic flow. These quantities depend on the sectional curvature bounds of (M, g) and the variation of the sectional curvatures, respectively. Moreover, as mentioned in the introduction, the Hölder exponent can be controlled in terms of a and b alone; more precisely, the exponent is given by $\alpha_0 = \min(2a/b, 1)$; see [33, Corollary 1.7]. As such, we have the following:

Lemma 2.16 *Let (M, g) be a Riemannian manifold whose sectional curvatures are contained in the interval $[-b^2, -a^2]$, and suppose the curvature tensor $\mathcal{R}$ satisfies $\|\nabla\mathcal{R}\| \le R$ for some $R > 0$. Then there exist positive constants $C = C(a, b, R)$ and $\alpha_0 = \min(2a/b, 1)$ such that for all $v_1, v_2 \in T^1M$, we have*

$$\|U(v_1) - U(v_2)\| \le C d(v_1, v_2)^{\alpha_0}.$$

Hypothesis 2.17 For the remainder of this section, we will assume the curvature tensors of M and N both satisfy $\|\nabla\mathcal{R}\| \le R$.

We now begin to define *E_v-cubes*. Fix $v \in T^1\widetilde{M}$ and write $p = P(v)$. Let $X_1, \dots X_{n-1}$ be an orthonormal basis for the fiber of the vertical subbundle $\mathcal{V}_v \subset T_v T^1 M$. Via this basis we will identify $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ with $X := \sum_i x_i X_i \in \mathcal{V}_v$. Write $v_i = \kappa(X_i) \subset T_p^1\widetilde{M}$, and note that all v_i are perpendicular to v, and hence tangent to the stable horosphere determined by $W^{\mathrm{ss}}(v)$. Let $Y_i \in T_v W^{\mathrm{ss}}$ be such that $dP(Y_i) = v_i$ and $\kappa(Y_i) = -U(v)v_i$ for each $i = 1, \dots, n-1$. Via this basis, we can identify $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ with $Y := \sum_i y_i Y_i \in T_v W^{\mathrm{ss}}$. We also note that

$$d\omega(X_i, Y_j) = \delta_{ij}. \tag{2-14}$$

Let g^v denote the restriction of the Sasaki metric to $\mathcal{V}_v$; in other words, g^v is the round metric on the unit tangent sphere $T^1_p\widetilde{M}$. Let $\exp^v\colon \mathcal{V}_v \to T^1_p\widetilde{M}$ denote the exponential map with respect to this metric. Let g^{ss} be the metric dual to g^v via $d\omega$; in other words, g^{ss} is the horospherical metric induced by g, obtained from identifying $W^{ss}(v)$ with the horosphere $P(W^{ss}(v))$. Let $\exp^{ss}\colon T_v W^{ss} \to W^{ss}(v)$ denote the exponential map with respect to g^{ss}.

Definition 2.18 [32, page 123] Let $v \in T^1\widetilde{M}$. Given $(x, y) \in \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$, let (X, Y) be the associated point in $\mathcal{V}_v \oplus T_v W^{ss}$ as above. Let $E_v(x, y)$ be the unit tangent vector whose endpoints at infinity are given by $\pi(E_v(x, y)) = \pi(\exp^v(X))$ and $\pi(-E_v(x, y)) = \pi(-\exp^{ss}(Y))$, and such that, in addition, the footpoint $P(E_v(x, y))$ lies in the stable horosphere $P(W^{ss}(v))$. Let $Q(r) \subset \mathbb{R}^{n-1}$ denote the cube $[-r, r]^{n-1}$. We call $E_v(Q(r) \times Q(r)) \subset T^1\widetilde{M}$ an E_v *r-cube* or simply an *E_v-cube*.

The goal of this subsection is to prove the following.

Proposition 2.19 *Let $\alpha_0 = \min(2a/b, 1)$ be the stable exponent of (M, g). There exists a constant $C > 0$, depending only on n, a, b, R, such that*

$$1 - Cr^{\alpha_0} \le \frac{\lambda^M\big(E_v(Q(r) \times Q(r))\big)}{\mathrm{vol}_{\mathbb{R}^{n-1}}(Q(r) \times Q(r))} \le 1 + Cr^{\alpha_0}.$$

Remark 2.20 In [32, page 123, (i)], a similar statement is claimed (for balls instead of cubes), ie it is stated (without proof) that the above ratio is between $1 \pm \eta(r)$ for some function $\eta(r)$ such that $\eta(r) \to 0$ as $r \to 0$. We prove a refined version of this statement: in addition to proving that the above ratio is close to 1, we give a more precise description of the function $\eta(r)$, which is crucial for our purposes.

To prove Proposition 2.19, we first construct smooth coordinates on the space of geodesics $\partial^2\widetilde{M}$. Let H denote the horosphere $P(W^{ss}(v))$. Fix $r > 0$ small and let $\mathcal{H}_r \subset \partial^2\widetilde{M}$ be the set of geodesics $\gamma(t)$ intersecting H such that

(1) the point of intersection $\gamma(t_0) \in H$ satisfies $d_{ss}(\gamma(t_0), p) \le r$,

(2) the angle $\gamma'(t_0)$ makes with the normal vector $\nu(\gamma(t_0))$ to H at the point $\gamma(t_0)$ is at most r.

Given $(s_1, \dots, s_{n-1}) \in B(r)$, the image of the map $(s_1, \dots, s_{n-1}) \mapsto P\big(\exp^{ss}\big(\sum_i s_i Y_i\big)\big)$ consists of points on H whose horospherical distance to the point $p := P(v)$ is at most r. These coordinates give rise to vector fields $\partial/\partial s_i$ tangent to H in a horospherical neighborhood of p. Now given $\theta \in [0, 2\pi]$, together with a unit vector w such that $Pw \in H$, let $R_i(\theta)w$ denote the rotation of w by the angle θ in the 2-dimensional subspace of $T^1_{Pw}\widetilde{M}$ spanned by the normal vector ν and the tangent vector $\partial/\partial s_i$. Finally, set

$$\Phi\colon B(r) \times B(r) \to T^1\widetilde{M}, \quad (\theta_1, \dots, \theta_{n-1}, s_1, \dots, s_{n-1}) \mapsto R_1(\theta_1) \dots R_{n-1}(\theta_{n-1}) \exp^{ss}\bigg(\sum_i s_i Y_i\bigg).$$

We now determine how the volume form $d\omega^{n-1}$ looks in these coordinates.

Lemma 2.21 *Write* $\theta = (\theta_1, \dots, \theta_{n-1})$ *and* $s = (s_1, \dots, s_{n-1})$ *for short. Then*

$$\Phi^*(d\omega^{n-1}) = f(\theta, s)\, d\theta_1 \wedge \dots \wedge d\theta_{n-1} \wedge ds_1 \wedge \dots \wedge ds_{n-1},$$

where

$$1 - C\theta \le \frac{f(\theta, s)}{n-1} \le 1 + C\theta,$$

and C *is a constant depending only on the sectional curvature bounds* a *and* b.

Proof Write

$$\Phi^* d\omega = \sum_{i \neq j} a_{ij}\, d\theta_i \wedge d\theta_j + \sum_{i,j} b_{ij}\, ds_i \wedge d\theta_j + \sum_{i \neq j} c_{ij}\, ds_i \wedge ds_j.$$

It follows from (2-12) that $a_{ij} = 0$ for all i, j. Next,

$$b_{ij} = d\omega(\partial/\partial s_i, \partial/\partial\theta_j) = \langle \kappa(\partial/\partial\theta_i), dP(\partial/\partial s_j)\rangle = \delta_{ij} \cos\theta_i.$$

Finally,

$$c_{ij} = d\omega(\partial/\partial s_i, \partial/\partial s_j).$$

When $\theta = 0$, we have that $\partial/\partial s_i$ is tangent to a curve of normal vectors to H; in other words, $\partial/\partial s_i \in TW^{\mathrm{ss}}$. Since TW^{ss} is a Lagrangian subbundle for $d\omega$, it follows that $c_{ij} = 0$ when $\theta = 0$. This shows that $f(0,0) = 1$.

If $\theta \neq 0$, we can write $\partial/\partial s_i = \cos\|\theta\| V + \sin\|\theta\| W_i$, where V is tangent to the normal vector field along the curve $s \mapsto P(\exp^{\mathrm{ss}}(sY_i))$, and W_i is tangent to a vector field along the same horospherical geodesic which remains tangent to the given horosphere. We then have

$$c_{ij} = \cos\|\theta\| \sin\|\theta\| d\omega(V, W_j) + \cos\|\theta\| \sin\|\theta\| d\omega(V, W_i) + \sin\|\theta\| \sin\|\theta\| d\omega(W_i, W_j). \tag{2-15}$$

Now note that

$$\kappa(V_i) = -U((s,0))\left(\frac{d}{ds} P \exp^{\mathrm{ss}}(sY_i)\right).$$

Next, since W_i is parallel in the horospherical metric, its covariant derivative along $s \mapsto P(\exp^{\mathrm{ss}}(sY_i))$ only has a component in the normal direction of the horosphere. Using this, we obtain

$$\kappa(W_i) = \frac{D}{ds} W_i = \left\langle \frac{D}{ds} W_i, \nu \right\rangle \nu = -\left\langle W_i, \frac{D}{ds}\nu \right\rangle \nu = -\langle W_i, U(s,0)\rangle \nu. \tag{2-16}$$

Since the mean curvature of horospheres in (M, g) can be bounded in terms of the sectional curvature bounds a and b (see [9, Corollary 4.2]), by (2-16) we see that there exists a constant $C = C(a,b)$ such that $|d\omega(V_i, W_j)|, |d\omega(W_i, W_j)| \le C$. Noting that $|\sin\theta| \le |\theta|$ completes the proof. □

Remark 2.22 The coordinates Φ used above are reminiscent of Otal's construction [43] in the $n = 2$ case; see Lemma 4.6 below. The difference is that Otal uses geodesics, whereas we use horocycles, as totally geodesic hypersurfaces are extremely rare when $n > 2$.

In order to understand how E_v-cubes look in the Φ coordinates, we study images of "vertical slices" $\Phi^{-1} \circ E_v(\{x_0\} \times Q(r))$. We proceed as follows:

(1) Show that $\Phi^{-1} \circ E_v(\{x_0\} \times Q(r))$ has bounded s-coordinates (Lemma 2.29).

(2) Show that $\Phi^{-1} \circ E_v(\{x_0\} \times Q(r))$ has nearly constant θ coordinates (Lemma 2.30).

(3) Show that the projection of the above set to the s-coordinate is roughly the shape of a parallelogram $L(Q(r))$, where $L\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is a linear map with $\det L = 1$ (Lemma 2.31).

To prove Proposition 2.19, we then combine steps (2) and (3) with Lemma 2.21.

Fix $y \in Q(r)$. Let $X = x_0/\|x_0\|$ and $Y = y/\|y\|$. Let $\tau = \|y\|/\|x_0\|$. Consider the curve $c(t) = E_v(t\tau X_0, tY)$ for $t \in [0, \|y\|]$. To prove Lemma 2.29, we seek to understand the horospherical distance between the footpoints of v and $c(t)$.

Recall that $\pi\colon T^1\widetilde{M} \to \partial\widetilde{M}$ denotes the forward projection of a unit tangent vector to the boundary at infinity. For $q \in \widetilde{M}$, let π_q denote the restriction of π to $T^1_q\widetilde{M}$. We will also consider the restriction of π to $W^{\mathrm{su}}(w)$; we denote this restriction by $\pi|_{W^{\mathrm{su}}(w)}$. (While we will use several ideas from [31] below, our above notation for restrictions of π differs from Hamenstädt's.)

Remark 2.23 Compositions of the form $\pi_p^{-1} \circ \pi_q$ will frequently appear — these are also known as the *holonomy maps of the weak stable foliation* between vertical transversals $T^1_q\widetilde{M}$ and $T^1_p\widetilde{M}$; similarly, the weak stable holonomy between unstable transversals will appear. (See also [1, Appendix, Definition 3.3].)

Using the above notation, we can express $c(t)$ as a composition of holonomies:

$$c(t) = \phi^{\bar{t}(t)}\pi|^{-1}_{W^{\mathrm{su}}(\exp^{\mathrm{ss}}(tY))} \circ \pi\big(\pi^{-1}_{P(\exp^{\mathrm{ss}}(tY))} \circ \pi(\exp^v(t\tau X_0))\big). \tag{2-17}$$

In other words, the point $c(\|y\|)$ can be obtained from v by the following procedure:

(1) Move v to a vector based at $P(\exp^{\mathrm{ss}}(y))$ and pointing toward $\exp^v x_0$ along the curve $\zeta(t) = \pi^{-1}_{P(\exp^{\mathrm{ss}}(tY))} \circ \pi(\exp^v(t\tau X_0))$.

(2) Consider now the vertical curve in $T^1_{P(\exp^{\mathrm{ss}} y)}\widetilde{M}$ connecting $\exp^{\mathrm{ss}}(y)$ and $\zeta(\|y\|)$; call it $\bar{\zeta}(t)$. Then $\psi(t) = \pi|^{-1}_{W^{\mathrm{su}}(\exp^{\mathrm{ss}}(ty))} \circ \pi(\bar{\zeta}(t))$ is a curve in $W^{\mathrm{su}}(\exp^{\mathrm{ss}}(y))$ with the same forward endpoints in $\partial\widetilde{M}$ as in $\bar{\zeta}(t)$.

(3) Flow each unit tangent vector of $\psi(t)$ until the basepoint intersects the stable horosphere $PW^{\mathrm{ss}}(v)$.

Before we bound the horospherical displacement resulting from each of the above steps, we prove a key preliminary lemma, which shows the stable holonomies $\pi_p \circ \pi_q^{-1}$ are almost shape-preserving when $d(p,q)$ is small. In general, it is known these maps are Hölder continuous, but not Lipschitz continuous. However, these holonomies are known to be absolutely continuous, and moreover, the Jacobians are close to 1 if transversals $T^1_p\widetilde{M}$ and $T^1_q\widetilde{M}$ are close to one another (this follows by a similar argument to the proof of [1, Appendix, Theorem 5.1], which concerns the holonomy of the *strong* stable foliation);

in other words, such weak stable holonomy maps are almost volume-preserving. In the lemma below, we show the stronger property that these holonomies are almost shape-preserving when applied to disks whose radius is comparable to $d(p,q)$.

Lemma 2.24 *Let $p(t)=P(\exp^{\mathrm{ss}}(tY))$ for t near 0. Let $\zeta(t)=\pi_{p(t)}^{-1}\circ\pi_p(\exp^v(tX))$. Let $W(t)$ be a unit-norm vector field along $p(t)$ which is either normal to the horosphere $P(W^{\mathrm{ss}}(v))$, or parallel relative to the horospherical metric induced by g. Then*

$$\langle\zeta(t)-\exp^{\mathrm{ss}}(tY),W(t)\rangle=t\langle X,W(0)\rangle+e(t),$$

where $\|e(t)\|/t\le Ct^{\alpha_0}$ for some constant $C=C(a,b,R)$.

Proof While the holonomy $\pi_{p(t)}^{-1}\circ\pi_p$ is not in general differentiable for $t\neq0$, the key observation is that $\zeta(t)$ is differentiable at $t=0$.

Let $\xi_t=\pi(\exp^v(tX))$ and let Z^{ξ_t} as in (2-13). Then $\zeta(t)=Z^{\xi_t}(p(t))$ and $\exp^{\mathrm{ss}}(tY)=Z^{\xi_0}(p(t))$. We start by finding a naive a priori upper bound for $\|Z^{\xi_t}(p(t))-Z^{\xi_0}(p(t))\|$ by considering a third point $Z^{\xi^t}(p(0))$. By definition of ξ_t, the distance between $Z^{\xi^t}(p(0))$ and $Z^{\xi_0}(p(0))$ is $t\|X\|$. Next, note that the vectors $Z^{\xi^t}(p(0))$ and $Z^{\xi^t}(p(t))$ are on the same weak stable leaf, and their footpoints are distance less than t apart. Let t_0 be the time such that $\phi^{t_0}Z^{\xi^t}(p(0))$ and $Z^{\xi_t}q(t)$ are on the same strong stable leaf. The distance between the footpoints of these two vectors is still at most t, and [12, Lemma 3.9] implies their Sasaki distance is bounded above by $(1+b)t$. Finally, the triangle inequality gives

$$\|Z^{\xi_t}(p(t))-Z^{\xi_0}(p(t))\|\le(b+3)t. \tag{2-18}$$

Next, we claim $\|\nabla_{p'(t)}W(t)\|$ is bounded above by a constant $C(a,b)$ depending only on the sectional curvature bounds of the metric g. This follows from similar arguments to (2-16).

Now fix $t>0$ small, and for $s\in(-t,t)$, define

$$\begin{aligned}f_1(s)&=\langle\pi_{p(s)}^{-1}\circ\pi(\exp^v tX),W(s)\rangle=\langle Z^{\xi_t}(p(s)),W(s)\rangle,\\ f_2(s)&=\langle\pi_p^{-1}\circ\pi(\exp^v(sX)),W(0)\rangle=\langle\exp^v(sX),W(0)\rangle,\\ f_3(s)&=\langle\pi_{p(s)}^{-1}\circ\pi(v),W(s)\rangle=\langle Z^{\xi_0}(p(s)),W(s)\rangle.\end{aligned}$$

Observe that by the mean value theorem there exist $t_i\in[0,t]$ for $i=1,2,3$ such that

$$\begin{aligned}\langle\zeta(t)-\exp^{\mathrm{ss}}(tY),W(t)\rangle&=\langle Z^{\xi_t}(p(t))-Z^{\xi_0}(p(t)),W(t)\rangle\\&=(f_1(t)-f_1(0))+(f_2(t)-f_2(0))-(f_3(t)-f_3(0))\\&=t\big(f_1'(t_1)+f_2'(t_2)+f_3'(t_3)\big).\end{aligned}$$

We have

$$f_2'(t_2)=\frac{d}{ds}\Big|_{s=t_2}\langle\exp^v(sX),W(0)\rangle=\langle X+e(t_2),V(0)\rangle,$$

where $|e(t)| \le Ct$ for some universal constant (depending on the geometry of the round sphere). Since $Z^{\xi_0}(p(s))$ is the normal to the stable horosphere determined by v, we have $f_3'(s) = 0$ for all s. Using (2-13), we compute

$$0 = f_3'(s) = \big\langle U\big(Z^{\xi_0}(p(s))\big), W(s)\big\rangle + \langle Z^{\xi_0}(p(s)), \nabla_{p'(s)} W(s)\rangle.$$

Finally,

$$\begin{aligned}|f_1'(t_1)| &= \big|\big\langle U\big(Z^{\xi_t}(p(t_1))\big), W(t_1)\big\rangle + \langle Z^{\xi_t}(p(t_1)), \nabla_{p'(t_1)} W(t_1)\rangle\big| \\ &= \big|\big\langle U\big(Z^{\xi_t}(p(t_1))\big) - U\big(Z^{\xi_0}(p(t_1))\big), W(t_1)\big\rangle + \langle Z^{\xi_t}(p(t_1)) - Z^{\xi_0}(p(t_1)), \nabla_{p'(t_1)} W(t_1)\rangle\big| \\ &\le C\|Z^{\xi_t}(p(t)) - Z^{\xi_0}(p(t))\|^{\alpha_0} + \|\nabla_{p'(t_1)} W(t_1)\| \|Z^{\xi_t}(p(t)) - Z^{\xi_0}(p(t))\|,\end{aligned}$$

where we used Lemma 2.16 in the last line. By (2-18) and the bound for $\|\nabla_{p'(t)} W(t)\|$, we see that $|f_1'(t_1)| \le Ct^{\alpha_0}$ for some $C = C(a,b,R)$. □

Lemma 2.25 *Let $\phi(t)$ be a smooth curve in $W^{su}(v)$ with $\phi(0) = v$ and so that, in addition, the footpoint curve $P\phi(t)$ is a geodesic in the horosphere $P(W^{su}(v))$. Then consider the curve $\zeta(t) = \pi_p^{-1} \circ \pi(\phi(t))$ in $T_p^1 \widetilde{M}$. We claim that $\zeta(t)$ is differentiable at* 0 *with initial tangent vector $(U(v) + U(-v))(P\phi)'(0)$ and, moreover,*

$$\zeta(t) - v = t(U(v) + U(-v))(P\phi)'(0) + e(t),$$

where $\|e(t)\|/t \le Ct^{\alpha_0}$, for some constant $C > 0$ depending only on a, b, R.

Remark 2.26 The fact that $\zeta'(0)$ exists and has the claimed form is the content of [31, Lemma 2.1]; however, for the purposes of this paper, we require a more precise understanding of the error term.

Proof Let $Z_t = Z^{\pi(\phi(t))}$ as in (2-13); let Y be any parallel vector field along the footpoint curve $P\phi$. By the proof of [31, Lemma 2.1], we then have

$$\frac{1}{t}\langle Z_t(p) - v, Y\rangle = \frac{1}{t}\int_0^t \langle U(Z_t)((P\phi)'(s), Y\rangle\, ds + \frac{\langle \phi(t), Y\rangle - \langle \phi(0), Y\rangle}{t}.$$

As $t \to 0$, the first term limits to $\langle U(v)(P\phi)'(0)), Y\rangle$, the value of the integrand at $s = 0$; the $o(t)$ error term depends on the variation of the integrand on the interval $[0,t]$. The proof of [12, Lemma 3.9] shows that $d((P\phi)'(s), (P\phi)'(0)) \le Cs$ for some constant that depends only on a and b. By Lemma 2.16, the variation of U depends only on a, b, R. So the error term is bounded above by $t(Ct^{\alpha})$, where C and α are constants depending only on a, b, R.

For the next term, we first note that, by the mean value theorem, there exists $s_t \in [0,t]$ such that

$$\begin{aligned}\frac{\langle \phi(t), Y\rangle - \langle \phi(0), Y\rangle}{t} &= \frac{d}{ds}\Big|_{s=s_t} \langle \phi(s), Y\rangle \\ &= \Big\langle \frac{D}{ds}\Big|_{s=s_t} \phi(s), Y\Big\rangle \qquad \text{(since } Y \text{ is parallel along } P\phi(t)\text{)} \\ &= \langle U(-\phi(s_t))((P\phi)'(s_t)), Y\rangle.\end{aligned}$$

As $t \to 0$ the above limits to $\langle U(-v)(P\Phi'(0)), Y\rangle$, its value at $t=0$. Analysis analogous to that in the above paragraph shows the error term has the desired form. □

Corollary 2.27 *Let $\zeta(t) = \exp^v(tX)$. Let $\psi(t) = \pi|^{-1}_{W^{\mathrm{su}}(v)} \circ \pi(\zeta(t))$. Let* exp *denote the exponential map at $p = Pv$ of the horosphere $PW^{\mathrm{su}}(v)$. Then the horospherical distance between $P\psi(t)$ and $\exp\bigl(t(U(v)+U(-v))^{-1}X\bigr)$ is bounded above by $t(Ct^{\alpha_0})$ for some $C = C(a,b,R)$.*

Proof By [31, Corollary 2.2], it is known that $(P\circ\psi)'(0)$ exists and is equal to $(U(v)+U(-v))^{-1}(X)$; we need to understand the error term. Let $Y(t) \in T_pPW^{\mathrm{su}}(v)$ be such that $\exp(tY(t)) = P\psi(t)$. Let $\phi_t(s) = \exp(sY(t))$. Let $\overline{\zeta}_t(s) = \pi_p^{-1}\circ\pi\bigl(\nu(\phi_t(s))\bigr)$. By the previous lemma,

$$\overline{\zeta}_t(s) - v = s(U(v)+U(-v))Y(t) + e(s),$$

which, after setting $s=t$, rearranges to

$$tY(t) = (U(v)+U(-v))^{-1}(\zeta(t)-v) - (U(v)+U(-v))^{-1}e(t).$$

Noting that $\zeta(t) - v = tX + o(t)$ (where the error term depends on the geometry of the round sphere), we see that

$$tP\psi'(0) = (U(v)+U(-v))^{-1}tX = tY(t) + e(t) \tag{2-19}$$

for some $e(t)$ such that $\|e(t)\|/t \le Ct^{\alpha_0}$. After applying exp to both sides, for any $|t| \le 1$ we get that the error is controlled by a and b. This completes the proof. □

Lemma 2.28 *Let $v \in T^1\widetilde{M}$ and let $u \in W^{\mathrm{su}}(v)$. Let $l \in \mathbb{R}^{>0}$ such that $P\phi^l u \in PW^{\mathrm{ss}}(v)$. Then the following hold.*

(1) *There exist constants c and C, depending only on a and b, such that*
$$c \le \frac{d_{\mathrm{su}}(Pv, Pu)}{d_{\mathrm{ss}}(Pv, P\phi^l v)} \le C.$$

(2) *There exists a constant $C = C(a,b,R)$ so that the time l satisfies*
$$l \le C d_{\mathrm{su}}(v,u)^2.$$

Proof Let $\gamma(s)$ be the geodesic perpendicular to v such that $\gamma(s)$ intersects the geodesic $P\phi^l u$. Let $\eta(s)$ be the unit-speed geodesic with $\eta(0) = Pv$ and $\eta(s_0) = P\phi^l u$. By [34, Theorem 4.6], we have that the ratio $d_{\mathrm{ss}}(Pv, P\phi^l u)/s_0$ is bounded between two constants c_1 and c_2, depending only on a and b.

Next, we estimate the angle θ between $\gamma'(0)$ and $\eta'(0)$. Let $V(s)$ denote the parallel transport of v along $\eta(s)$. Note that θ is also the angle at $P\phi^l u$ between $V(s_0)$ and the normal vector ν of the stable horosphere $PW^{\mathrm{ss}}(u)$. This allows us to estimate θ as follows:

$$\cos\theta = \langle V(s_0), \nu(s_0)\rangle = \langle V(0), \nu(0)\rangle + \int_0^{s_0} \langle -U(\nu(s))(\eta'(s)), V(s)\rangle\, ds. \tag{2-20}$$

Now we can use this, together with [34, Proposition 4.7], to see that the ratio $s_0/d_{\mathrm{su}}(v,u)$ is between two constants c_1' and c_2', depending only on a and b. From this, (1) follows.

To prove (2), let B denote the Busemann function whose zero set is the unstable horosphere $PW^{\mathrm{ss}}(v)$. Then $l = B(\eta(s_0))$. By [34] (see also [12, Section 3]), we have $f(\eta(s)) \le s\sin(\theta) + Cs^2$, where C depends only on a and b. Moreover, from (2-20), we have that $\sin(\theta) \le C's_0$ for $C' = C(a,b,R)$. Thus, $f(\eta(s)) \le C''s^2$, which proves (2). □

For the statements of the following lemmas, let $P_\theta, P_s\colon \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ denote the projections $(\theta,s) \mapsto \theta$ and $(\theta,s) \mapsto s$, respectively.

Lemma 2.29 *Fix $x_0 \in B(r) \subset \mathbb{R}^{n-1}$ and consider $E_v(\{x_0\} \times B(r))$. Then there exists a constant $C = C(a,b,R)$, together with $S \le \|x_0\| + Cr$, so that the footpoints satisfy*

$$P_s\big(\Phi^{-1} \circ E_v(\{x_0\} \times B(r))\big) \subset P_s\big(\Phi^{-1} \circ E_v(\{0\} \times B(S))\big).$$

Proof Combine (2-17) with Lemma 2.24, Corollary 2.27 and Lemma 2.28. □

Lemma 2.30 *Fix $x_0 \in B(r) \subset \mathbb{R}^{n-1}$ and consider again $E_v(\{x_0\} \times B(r))$. Then there exists a positive constant $C = C(a,b,R)$, together with $\eta(r) \le rCr^{\alpha_0}$ such that*

$$P_\theta\big(\Phi^{-1} \circ E_v(\{x_0\} \times B(r))\big) \in B_{x_0}(\eta).$$

Proof Let $y \in B(r)$ and consider the curve $c(t) = E_v(t\tau X_0, tY)$ from the proof of the discussion above Remark 2.23. By the previous lemma, the footpoint of $c(\|y\|)$ coincides with the footpoint of $\exp^{\mathrm{ss}}(sY)$ for some $s \le \|x_0\| + C\|y\|$. As such, we can write $c(\|y\|) = \pi^{-1}_{P(\exp^{\mathrm{ss}}(sY))}(\pi(\exp^v x_0))$. By Lemma 2.24, we see that

$$c(\|y\|) - \exp^{\mathrm{ss}}(sY) = \|x_0\| + e(s)$$

for $e(s)$ satisfying $\|e(s)\|/s \le Cs^{\alpha_0}$ for some $C = C(a,b,R)$. □

Lemma 2.31 *Let $A_x = P_s \circ \Phi^{-1} \circ E_v(\{x\} \times Q(r))$. Then there exists $C = C(a,b,R)$, together with a linear map $L\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ with $\|L\| \le C$ such that*

$$Q_{Lx}((1 - Cr^{\alpha_0})r) \subset A_x \subset Q_{Lx}((1 + Cr^{\alpha_0})r).$$

Proof Throughout this proof, we will use $d\omega$ to identify the tangent spaces $\mathcal{V}_v$ and T_vW^{ss} as follows; see also (2-14) above for more details. Let $X_1, \dots, X_{n-1} \in \mathcal{V}_v$ be an orthonormal basis, and let $Y_1, \dots, Y_{n-1}$ be the basis of T_vW^{ss} such that $d\omega(X_i, Y_j) = \delta_{ij}$.

Let $Q(r)$ denote the cube $[-r,r]^{n-1} \subset \mathbb{R}^{n-1} \cong T_vW^{\mathrm{ss}}$ and let $H = -\exp^{\mathrm{ss}}(Q(r)) \subset W^{\mathrm{su}}(-v)$. Now consider $O = \pi_p^{-1} \circ \pi(H)$. Let $E(r) = (U(v) + U(-v))^{-1}Q(r) \subset T_vW^{\mathrm{ss}}$. By Lemma 2.25 , viewing $E(r) \subset \mathcal{V}_v$, we have

$$\exp^v((1 - Cr^\alpha)E(r)) \subset O \subset \exp^v((1 + Cr^\alpha)E(r)).$$

Now let $w = \exp^v x$ and consider $W^{ss}(w)$. Let $w_0 \in W^{ss}(w)$ be such that $\pi(-w_0) = \pi(-v)$. By Corollary 2.27, the unstable distance between w_0 and $\exp^{ss}\big((U(v)+U(-v))^{-1}(\exp^v(x))\big)$ is bounded above by $\|x\|C\|x\|^{\alpha_0}$ for some $C = C(a,b,R)$. We will see that the linear map $L\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ in the statement of the lemma is given by $(U(v)+U(-v))^{-1}$ viewed as a map from $\mathscr{V}_v$ with respect to the basis $\{X_i\}_{i=1}^m$ to $T_v W^{ss}$ with respect to the basis $\{Y_i\}_{i=1}^m$.

Let q denote the footpoint of w_0. Let Y_j' for $j = 1,\dots,n-1$ be the basis for $T_{w_0}W^{ss}$ obtained by parallel transporting the Y_i along the horospherical geodesic from p to q, with respect to the induced horospherical metric on $PW^{ss}(v)$. Let $X_i' \in \mathscr{V}_{w_0}$ for $i = 1,\dots,n-1$ be the basis dual to the Y_i with respect to $d\omega$. Now let $O' = \pi_q^{-1} \circ \pi(O)$. By Lemma 2.24 and the above, we have

$$\exp^v_{w_0}\big((1-C'r^{\alpha_0})(E(r))\big) \subset O \subset \exp^v_{w_0}\big((1+C'r^{\alpha_0})(E(r))\big).$$

Next, let $H' = \pi^{-1}_{W^{su}(-w_0)} \circ \pi(O)$. By Corollary 2.27, we have

$$\exp^{ss}\big((1-C'r^{\alpha_0})(U(w)+U(-w))E(r)\big) \subset H \subset \exp^{ss}\big((1+C'r^{\alpha_0})(U(w)+U(-w))E(r)\big).$$

By Lemma 2.16, we have $(1-Cr^{\alpha_0})Q(r) \subset (U(w)+U(-w))E(r) \subset (1+Cr^{\alpha_0})Q(r)$, which means $\exp^{ss}_{w_0}((1-Cr^{\alpha_0})Q(r)) \subset H' \subset \exp^{ss}_{w_0}((1+Cr^{\alpha_0})Q(r))$.

Finally, to obtain $E_v(\{x_0\}\times Q(r))$ from H', we must simply flow each unit tangent vector in H' until its footpoint reaches the stable horosphere $PW^{ss}(v)$. Let B denote the Busemann function determined by v. Given any $q' \in H'$, let $\gamma(s)$ denote the geodesic joining p and $q' = \gamma(s_0)$. Then the amount of time we must flow q' to reach $PW^{ss}(v)$ is at most $B(\gamma(s_0))$. We have $B(\gamma(s_0)) = s_0 \sin(\|x\|) + o(s_0^2)$, and it follows from [34, Lemma 4.3] that the error term depends only on s_0 and b, and we have that $s_0 \le Cr^2$ for some $C = C(a,b,R)$. Let u_0 be the vector with basepoint on $PW^{ss}(v)$ on the same flow line as w_0. We have shown that the distance between u_0 and Lx is bounded above by $\|x\|C\|x\|^{\alpha_0}$ for some $C = C(a,b,R)$.

Moreover, by the Anosov property of the geodesic flow (see eg [34, Proposition 4.1]), we see that flowing points of H' for time l distorts distances by a factor contained in $[e^{-bl}, e^{bl}]$. Combining with our above bound for $B(\gamma(s_0))$ completes the proof. □

Proof of Proposition 2.19 By Lemma 2.21, it suffices to estimate the ratio between the Euclidean volume of $\mathfrak{Q}_r := \Phi^{-1} \circ E_v(Q(r)\times Q(r))$ and that of $Q(r)\times Q(r)$. First, we note that by Lemma 2.30, images of vertical slices $\Phi^{-1}\circ E_v(\{x\}\times Q(r))$ are nearly vertical slices in (θ,s) coordinates; more precisely, there exists $\eta = \eta(r) \le rCr^{\alpha_0}$ such that

$$\Phi^{-1}\circ E_v(\{x\}\times Q(r)) \subset Q_x(\eta)\times A_x,$$

where $Q_x(\eta)$ denotes the translated cube $x + [-\eta,\eta]^{n-1}$. This suggests that $\mathfrak{Q}_r$ is approximately the shape of a parallelogram. More precisely, let L be the linear transformation from Lemma 2.31. Let

$\bar{L}\colon \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ be the shear transformation given by $(\theta, s) \mapsto (\theta, s + L(\theta))$. Since $\det(\bar{L}) = 1$, it suffices to show that there exist $r_1 = (1 - Cr^{\alpha_0})r$ and $r_2 = (1 + Cr^{\alpha_0})r$ such that

$$\bar{L}(Q(r_1) \times Q(r_1)) \subset \mathcal{Q}_r \subset \bar{L}(Q(r_2) \times Q(r_2)). \tag{2-21}$$

We start by noting that it follows from Lemma 2.30 that $Q(r_1) \subset P_\theta(\mathcal{Q}_r) \subset Q(r_2)$, where $r_1 = (1-Cr^{\alpha_0})r$ and $r_2 = (1 + Cr^{\alpha_0})r$, and we can additionally increase C to ensure that $r_1 < r - \eta$ and $r_2 > r + \eta$. We then have

$$\bigcup_{x \in Q(r_1)} \{x\} \times A_x \subset \Phi^{-1} \circ E_v(Q(r) \times Q(r)) \subset \bigcup_{x \in Q(r_2)} \{x\} \times A_x.$$

By the previous lemma, $Q_{Lx}(r_1) \subset A_x \subset Q_{Lx}(r_2)$. The claim now follows from the fact that

$$\bar{L}(Q(r) \times Q(r)) = \bigcup_{x \in Q(r)} \{x\} \times Q_{Lx}(r). \qquad \square$$

2.2.2 Quasisymplecticity

While Proposition 2.19 allows us to determine the Liouville currents of E_v-cubes in $\partial^2 \widetilde{M}$, it is not clear that images of E_v-cubes by the boundary map $\bar{f}\colon \partial^2 \widetilde{M} \to \partial^2 \widetilde{N}$ are still E_v-cubes. However, the boundary map does preserve a coarser property of these cubes, defined in terms of the cross-ratio, which Hamenstädt calls *quasisymplecticity*. It turns out that for manifolds whose geodesic flow has C^1 Anosov splitting, this quasisymplectic property of an embedded cube is enough to estimate its Liouville current [32, Lemma 3.11].

Definition 2.32 [32, page 123] Fix $\eta > 0$. Let $B(r) \subset \mathbb{R}^n$ be the ball of radius r centered at the origin, and let $\phi_0(x, y)$ denote the dot product of $x, y \in \mathbb{R}^n$. Let $\beta_1, \beta_2\colon B(r) \to \partial \widetilde{M}$ be continuous embeddings such that

$$|[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)] - \phi_0(x, y)| \le \eta r^2$$

for all $x, y \in B(r)$. We say the image $\beta_1(B(r)) \times \beta_2(B(r)) \subset \partial \widetilde{M} \times \partial \widetilde{M} \setminus \Delta$ is a $(1+\eta)$-*quasisymplectic r-ball*.

It follows from [31, Corollary 2.10], together with the short computation in Lemma 2.33 below, that for any $\eta > 0$, there exists a sufficiently small r such that $\pi\big(E_v(B(r) \times B(r))\big)$ is $(1+\eta)$-quasisymplectic. The proof relies on the connection between the cross-ratio and the *temporal function*, which was discussed in (2-10) above.

Hamenstädt shows that if $c(t)$ is a curve in $T^1 M$ with $c(0) = v$ and $c'(0) = \lambda X_0 + Y + Z$ with $Y \in T_v W^{\mathrm{ss}}$ and $Z \in T_v W^{\mathrm{su}}$, then we have the following Taylor expansion at $t = 0$:

$$\sigma(v, c(t)) + \sigma(c(t), v) = t^2 d\omega(Z, Y) + o(t^2). \tag{2-22}$$

(This is also done in [40, Lemma B.7].) Next, we note the following.

Lemma 2.33 *For $(X,Y)\in\mathscr{V}_v\oplus T_vW^{\mathrm{ss}}$, write $c(t)=E_v(tX,tY)$; see Definition 2.18. Write $c'(0)=\lambda X_0+Y'+Z$ for $Y'\in T_vW^{\mathrm{ss}}$ and $Z\in T_vW^{\mathrm{su}}$. Then $Y=Y'$ and*

$$d\omega(X,Y)=d\omega(Z,Y).$$

Proof By construction, $\pi^{-1}_{W^{\mathrm{su}}(v)}(c(t))=-\exp^{\mathrm{ss}}(tY)$. So $Y=(\pi^{-1}_{W^{\mathrm{su}}(v)}\circ c)'(0)$, which is equal to Y' by [31, Corollary 2.2]. Similarly, we have $\pi_p^{-1}(c(t))=\exp^v(tX)$. Hence $X=(\pi_p^{-1}\circ c)'(0)$. By [31, Lemma 2.1], we then have

$$\kappa(X)=U(v)\,dPc'(0)+\kappa(c'(0))=U(v)\,dP(Y)+U(v)\,dP(Z)+U(-v)\,dP(Z)-U(v)\,dP(Y).$$

Next, since $dP(X)=0$, the formula (2-12) gives,

$$\begin{aligned}d\omega(X,Y)&=\langle\kappa(X),dP(Y)\rangle\\&=\langle U(v)\,dPc'(0)+\kappa c'(0),dP(Y)\rangle\\&=\langle(U(v)+U(-v))\,dP(Z),dP(Y)\rangle+\langle(U(v)+U(-v))\,dP(Y),dP(Y)\rangle\\&=d\omega(Z,Y)+d\omega(Y,Y),\end{aligned}$$

where the last line follows from [31, Lemma 2.7]. □

For $c(t)$ as in the previous lemma, we have that the left-hand side of (2-22) equals

$$[\pi(v),\pi(\exp^v(tX)),\pi(-v),\pi(-\exp^{\mathrm{ss}}(tY))].$$

By (2-14), we have $\phi_0(X,Y)=d\omega(X,Y)$. Thus, given any $\eta>0$, there exists a sufficiently small r such that E_v r-balls are $(1+\eta)$-quasisymplectic. For the purposes of this paper, we need to show that η depends only on r and our other fixed geometric data:

Proposition 2.34 *There exists a constant $C=C(a,b,R)$ such that for any $v\in T^1\widetilde{M}$ and any $r>0$, we have that $E_v(B(r)\times B(r))$ is $(1+Cr^{\alpha_0})$-quasisymplectic.*

Proof We need to more explicitly understand the $o(t^2)$ error term in (2-22) for the particular curve $c(t)=E_v(tX,tY)$. First, let $\phi(t)=\pi|^{-1}_{W^{\mathrm{su}}(v)}c(t)$. As in the proof of [31, Lemma 2.6], we write $\sigma(v,c(t))=f_1(t)+f_2(t)$, where

$$f_1(t)=B_{\phi(t)}(Pc(t))-B_{\phi(t)}(Pc(0))\quad\text{and}\quad f_2(t)=B_{\phi(t)}(P\phi(0))-B_{\phi(t)}(P\phi(t)).$$

We begin by Taylor expanding $f_1(t)$. We will focus on the $o(t^2)$ error term, since the second-degree Taylor polynomial is computed in [31, Lemma 2.6].

As in the proof of the Taylor expansion of the $f_2(t)$ term in [31, Lemma 2.6], we start by letting $X(t)\in T^1_p\widetilde{M}$ so that $\exp(tX(t))=Pc(t)$, where exp denotes the exponential map in the stable horosphere $PW^{\mathrm{ss}}(v)$. Now let $\psi_t(s)=\exp(sX(t))$. By the same argument as in the proof of Corollary 2.27 (see, in particular, equation (2-19)), we have, for any $\tau\in(-t,t)$, that $d(Pc'(\tau),\psi_t'(\tau))\le Ct^{\alpha}$ for some positive constants $C=C(a,b,R)$ and $\alpha=\alpha(a,b,R)$.

Since $\psi_t(t) = Pc(t)$, the fundamental theorem of calculus gives

$$f_1(t) = \int_0^t \langle Z_t, \psi_t'(s)\rangle \, ds,$$

where Z_t is the vector field on $\widetilde{M}$ such that $\pi(Z_t(x)) = \pi(\phi(t))$ for all $x \in \widetilde{M}$. We can then write

$$\langle Z_t, \psi_t'(s)\rangle = \langle Z_t, \psi_t'(0)\rangle + \int_0^s \Big(\langle U(Z_t)\psi_t'(\tau), \psi_t'(\tau)\rangle - \Big\langle Z_t, \frac{D}{d\tau}\psi_t'(\tau)\Big\rangle \Big) \, d\tau.$$

For the first term, by Lemma 2.24 combined with the discussion above, we deduce that $\langle Z_t, \psi_t'(0)\rangle \to \langle Z_0, Pc'(0)\rangle$, with Hölder error term of the form Ct^{α_0} for some $C = C(a,b,R)$. For the second term, we obtain a similar error term using Lemma 2.16. For the third term, we simplify, as in (2-16), and obtain

$$\frac{D}{d\tau}(\psi_t)'(\tau) = \big\langle U\big(Z_0(\psi_t(\tau))\big), \psi_t'(\tau)\big\rangle Z_0(\psi_t(\tau)).$$

Hence, we see that the above integral converges to the value of the integrand at $\tau = 0$, and moreover, the error term is again of the form Cs^{α_0}.

The argument for the expansion of $f_2(t)$ is very similar, and we omit it. □

In light of Proposition 2.4, the map $\bar{f} \colon \partial^2 \widetilde{M} \to \partial^2 \widetilde{N}$ preserves the quasisymplectic property to the following extent.

Lemma 2.35 *Let (M,g) and (N,g_0) as in the hypotheses of Theorem C. Let $\delta_0 = \delta_0(\varepsilon)$ and $\varepsilon' = \varepsilon'(\varepsilon)$ as in Proposition 2.4. Let $r > \delta_0$ and let $A(\delta_0, r)$ denote the annulus $B(r) \setminus B(\delta_0)$. Suppose that $\mathcal{B}$ is a $(1+\eta)$-quasisymplectic r-ball. Let $\bar{\eta} = (1+\eta)\varepsilon' + \eta$. Then $\bar{f}(\mathcal{B})$ satisfies the $(1+\bar{\eta})$-quasisymplectic condition for all $x, y \in A(\delta_0, r)$.*

In particular, if $r = 2\delta_0$, then we can take $\bar{\eta} \le C\varepsilon^{\alpha}$ for some C depending only on n, Γ, D, i_0, a, b, R and any $\alpha < \alpha_0(1-\varepsilon)a^2/b$.

Proof If $\mathcal{B}$ is a $(1+\eta)$-quasisymplectic r-ball, there are maps $\beta_i \colon B(r) \to \partial\widetilde{M}$ for $i = 1, 2$ with $\mathcal{B} = \beta_1(B(r)) \times \beta_2(B(r))$ such that

$$|[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M - \phi_0(x,y)| \le \eta r^2. \tag{2-23}$$

Using the triangle inequality, Proposition 2.4 and (2-23) gives

$$\begin{aligned}
&|[\bar{f}\circ\beta_1(x), \bar{f}\circ\beta_1(0), \bar{f}\circ\beta_2(y), \bar{f}\circ\beta_2(0)]_N - \phi_0(x,y)| \\
&\qquad \le \varepsilon'[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M + |[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M - \phi_0(x,y)| \\
&\qquad \le \varepsilon'[\beta_1(x), \beta_1(0), \beta_2(y), \beta_2(0)]_M + \eta r^2 \\
&\qquad \le \varepsilon'(1+\eta) r^2 + \eta r^2,
\end{aligned}$$

which shows that $\bar{\eta} = (1+\eta)\varepsilon' + \eta$. Noting that $\eta = Cr^{\alpha_0}$ (Proposition 2.34), together with the fact that ε' and δ_0 are both of the form $C\varepsilon^{A^{-1}a^2/b}$ (Proposition 2.4) completes the proof. □

Proposition 2.36 *Let (M,g) and (N,g_0) be as in the hypotheses of Theorem C. Let $r=2\delta_0=2\delta_0(\varepsilon)$ be as above. Then, for any $v\in T^1\widetilde{M}$, we have*

$$1-C\varepsilon^\alpha \le \frac{\lambda^N\big(\bar{f}\circ E_v(Q(r)\times Q(r))\big)}{\mathrm{vol}_{\mathbb{R}^{n-1}}(Q(r)\times Q(r))}\le 1+C\varepsilon^\alpha,$$

for some C depending only on n, Γ, D, i_0, a, b, R and any $\alpha<\alpha_0(1-\varepsilon)a^2/b$.

Proof Since N has C^1 Anosov splitting, the Bowen bracket $[w_1,w_2]:=W^{\mathrm{su}}(w_1)\cap W^{\mathrm{ss}}(w_2)$ is a C^1 function of w_1 and w_2; see Definition 2.14. We can thus use it to define C^1 coordinates Ψ on the space of geodesics as follows. Let $Y_1,\dots,Y_{n-1}$ be a basis of T_vW^{ss} such that the $dP(Y_i)$ are orthonormal with respect to the metric g. Now let $Z_1,\dots,Z_{n-1}$ be the dual basis of T_vW^{su} with respect to $d\omega$ at v, ie such that $d\omega_v(Y_i,Z_j)=\delta_{ij}$. Using these bases, we identify each of T_vW^{ss} and T_vW^{su} with a copy of $\mathbb{R}^{n-1}$, ie

$$T_vW^{\mathrm{ss}}\longleftrightarrow\mathbb{R}^{n-1},\quad Y=\sum y_iY_i\rightleftarrows(y_1,\dots,y_{n-1})=y,$$

and analogously for T_vW^{su}. Then we define

$$\Psi\colon\mathbb{R}^{n-1}\oplus\mathbb{R}^{n-1}\to T^1\widetilde{M},\quad (y,z)\mapsto[\exp^{\mathrm{ss}}(Y),\exp^{\mathrm{su}}(Z)].$$

Since $d\omega_v(Y_i,Z_j)=\delta_{ij}$, the pullback $\rho=\Psi^*d\omega$ agrees with the standard symplectic form ρ_0 on $\mathbb{R}^{n-1}\oplus\mathbb{R}^{n-1}$ at the origin, and by continuity, ρ is close to ρ_0 on small balls. We now make the latter statement more precise. Let $Y\in T_vW^{\mathrm{ss}}$ and $Z\in T_vW^{\mathrm{su}}$, let $\Sigma_{Y,Z}$ be the parametrized surface given by $(s,t)\mapsto\Psi(sy,tz)$ for $(s,t)\in[0,1]^2$. As in [32], define

$$\phi(y,z)=\int_{\Sigma_{y,z}}d\omega=\int_{\Psi^{-1}(\Sigma_{y,z})}\rho.$$

Since Ψ is C^1, we can apply Stokes' theorem to the leftmost integral above to obtain

$$\phi(y,z)=\int_{\partial\Sigma_{y,z}}\omega=\sigma(\exp^{\mathrm{ss}}(Y),\exp^{\mathrm{su}}(Z)).$$

Using the Taylor expansion for the temporal function σ in (2-22), together with the explicit estimate of the error term given by the proof of Proposition 2.34, we obtain

$$|\phi(y,z)-d\omega_v(Y,Z)|\le Cr(\|y\|\|z\|)$$

for some $C=C(a,b,R)$. (We have also used that the stable exponent $\alpha_0=1$ since N has C^1 Anosov splitting.) But $d\omega(Y,Z)=\sum_i y_iz_i=\int_{\Psi^{-1}(\Sigma_{y,z})}\rho_0$, where ρ_0 is the standard symplectic form on $\mathbb{R}^{n-1}\oplus\mathbb{R}^{n-1}$. This gives

$$|\phi(y,z)-\phi_0(y,z)|\le Cr\|y\|\|z\|. \tag{2-24}$$

(This is a quantitative version of property (4) in [32, page 130].)

Now let $\mathcal{F}\colon T^1M\to T^1N$ as in (2-1) and (2-9) be an orbit equivalence, and let

$$\beta=(\beta_1,\beta_2)\colon\mathbb{R}^{n-1}\oplus\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}\oplus\mathbb{R}^{n-1}$$

be given by $\Psi^{-1}\circ\mathcal{F}\circ E_v$. By [32, Lemma 3.9], we have

$$\begin{aligned}\phi(\beta_1 x,\beta_2 y) &= \phi(\Psi^{-1}\circ\mathcal{F}\circ E_v(x,y))\\ &= \big[\bar f(\pi(v)),\ \bar f(\pi(\exp^v tX)),\ \bar f(\pi(-v)),\ \bar f\big(\pi(-\exp^{ss}(tY))\big)\big].\end{aligned}$$

By the previous lemma, β restricted to $A(\delta_0,r)\times A(\delta_0,r)$ satisfies the $(1+\bar\eta(r))$-quasisymplectic condition. Using this, together with (2-24), an argument identical to that of [32, Lemma 3.11] shows there is a constant $k=k(n)$ such that β restricted to $A(\delta_0,r)\times A(\delta_0,r)$ satisfies the $(1+k(n)\bar\eta)$-quasisymplectic condition relative to the standard symplectic form ρ_0. The argument of [32, Proposition 3.4], restricted to $A(\delta_0,r)\times A(\delta_0,r)$, still shows there is a constant $k'=k'(n)$ such that β_1 and β_2 are $k'(n)\bar\eta$-close to a dual pair of linear maps L and L^{-t} on this domain. Since $\partial Q(r)\times\partial Q(r)\subset A(\delta_0,r)\times A(\delta_0,r)$, the volume of $\beta(Q(r)\times Q(r))$ with respect to ρ_0 is contained in the interval $\mathrm{vol}_{\mathbb{R}^{n-1}}(Q(r))^2[1-k(n)\bar\eta,1+k(n)\bar\eta]$. From (2-24), it follows that the ratio of $\mathrm{vol}_{\rho_0}\big(\beta(Q(r)\times Q(r))\big)$ and $\mathrm{vol}_{\rho}\big(\beta(Q(r)\times Q(r))\big)$ is contained in the interval $[1-Cr^\alpha,1+Cr^\alpha]$ for C depending only on n,a,b,R and $\alpha=\alpha_0A^{-1}a^2/b$. □

Combining Propositions 2.19 and 2.36 we obtain:

Proposition 2.37 *Let (M,g), (N,g_0) and $\alpha_0=\alpha_0(a,b)$ be as in the hypotheses of Theorem C. Let $r=2\delta_0=2\delta_0(\varepsilon)$ be as in Proposition 2.36. Then, for any $v\in T^1\widetilde M$, we have*

$$1-C\varepsilon^\alpha\le\frac{\lambda^M\big(E_v(Q(r)\times Q(r))\big)}{\lambda^N\big(\bar f\big(E_v(Q(r)\times Q(r))\big)\big)}\le 1+C\varepsilon^\alpha,$$

$\bar\eta\le C\varepsilon^\alpha$ for some C depending only on n, Γ, D, i_0, a, b, R and any $\alpha<\alpha_0(1-\varepsilon)a^2/b$.

2.3 A controlled orbit equivalence

Let ϕ^t denote the geodesic flow of M and let ψ^t denote the geodesic flow of N. If the marked length spectra of M and N are equal, then the flows ϕ^t and ψ^t are *conjugate* [30; 7], ie there is a homeomorphism $\mathcal{F}\colon T^1M\to T^1N$ such that

$$\mathcal{F}(\phi^t v)=\psi^t\mathcal{F}(v)$$

for all $t\in\mathbb{R}$, $v\in T^1M$. If, in addition to this, M and N have the same Liouville current, then T^1M and T^1N have the same Liouville measure, so $\mathrm{Vol}(M)=\mathrm{Vol}(N)$.

If the lengths of closed geodesics of M and N are instead ε-close as in (1-2), the geodesic flows may not be conjugate. However, so long as M and N are negatively curved and have isomorphic fundamental groups, their geodesic flows are *orbit-equivalent* [27]. This means there exists a homeomorphism $\mathcal{F}\colon T^1M\to T^1N$, and a function (cocycle) $a(t,v)$ such that

$$\mathcal{F}(\phi^t v)=\psi^{a(t,v)}\mathcal{F}(v) \tag{2-25}$$

for all $t\in\mathbb{R}$ and $v\in T^1M$.

In this section, we will use the assumption of approximately equal lengths (1-2) to show there is an orbit equivalence where the time-change $a(t, v)$ is close to t. In fact, we can more generally control the speed of the time change in terms of the ratio $\mathcal{L}_{g_0} \circ f_*/\mathcal{L}_g$ without necessarily assuming this ratio is close to 1. Moreover, we obtain an explicit formula for the Hölder exponent, which was used in the proof of Proposition 2.4.

Theorem 2.38 *Suppose (M, g) and (N, g_0) have diameter at most D and sectional curvatures in the interval $[-b^2, -a^2]$. Suppose there is $A \geq 1$ such that $f\colon (M, g) \to (N, g_0)$ and $h\colon (N, g_0) \to (M, g)$ are A-Lipschitz homotopy equivalences, with $f \circ h$ homotopic to the identity. Let $K_1, K_2 > 0$ such that for all $\gamma \in \Gamma$, we have*

$$K_1 \leq \frac{\mathcal{L}_{g_0}(f_*\gamma)}{\mathcal{L}_g(\gamma)} \leq K_2.$$

Then for any $\delta > 0$, there is an orbit equivalence $\mathcal{F}\colon T^1M \to T^1N$ which is C^1 along orbits, transversally Hölder continuous, and such that the following hold:

1. *Equation (2-25) holds with $(K_1 - \delta)t \leq a(t, v) \leq (K_2 + \delta)t$ for all $t \in \mathbb{R}$ and all $v \in T^1M$.*
2. *Let $\alpha = (K_1 - \delta)a/b$. Then there exists a constant $C = C(a, b, D, A)$ so that for every $v, w \in T^1M$ we have*
$$d(\mathcal{F}(v), \mathcal{F}(w)) \leq C d(v, w)^\alpha.$$

Remark 2.39 If M and N have dimension $n \geq 3$, then the constant A can be controlled in terms of the dimension n, the fundamental group Γ, the sectional curvature bound b, the upper bound D on the diameters, and a lower bound i_0 on the injectivity radii. See Proposition 2.41. As such, for $n \geq 3$, the constant C in part (2) of the above theorem depends only on n, Γ, a, b, D, i_0.

Remark 2.40 When $K_1 = 1 - \varepsilon$ and $K_2 = 1 + \varepsilon$, a similar result to (1) follows using the more general approximate Livsic theorems in [23] and [39], but our direct method yields a better estimate in this case. See [12, Remark 2.10] for more details. Moreover, in [12, Proposition 2.4] we showed a weaker form of (2) with $\alpha = A^{-1}a/b$.

Our construction of the map $\mathcal{F}$ follows the method of Gromov [27] and also uses ideas of Bourdon [7]. Both constructions rely heavily on the boundary homeomorphism $\overline{f}\colon \partial\widetilde{M} \to \partial\widetilde{N}$ induced by the homotopy equivalence $f\colon M \to N$; see Construction 2.1. As such, it is useful to first obtain control of the quasi-isometry constants of the lift $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$.

2.3.1 A controlled quasi-isometry In this section, we show that the lift $\widetilde{f}$ of the initial homotopy equivalence $f\colon M \to N$ in the setup of Theorems B and C can be modified within its homotopy class so that it lifts to a quasi-isometry $\widetilde{M} \to \widetilde{N}$ with controlled constants.

Proposition 2.41 *Let (M, g) and (N, g_0) be a pair of homotopy-equivalent closed Riemannian manifolds of dimension $n \geq 3$, with fundamental group Γ, diameter at most D, injectivity radius at least i_0, and* *sectional curvatures contained in the interval* $[-b^2, 0)$. *Then there exist constants* A *and* B, *depending only on* n, Γ, D, i_0, b, *such that any* Γ*-equivariant homotopy equivalence* $f' \colon \widetilde{M} \to \widetilde{N}$ *is homotopic to a* Γ*-equivariant map* $f \colon \widetilde{M} \to \widetilde{N}$ *satisfying*

$$A^{-1} d_g(x, y) - B \le d_{g_0}(f(x), f(y)) \le A\, d_g(x, y) \tag{2-26}$$

for all $x, y \in \widetilde{M}$.

As mentioned in Construction 2.1, it is well known that any homotopy equivalence $f' \colon M \to N$ inducing the starting isomorphism f_* of fundamental groups lifts to a quasi-isometry $\widetilde{M} \to \widetilde{N}$; see, for instance, [2, page 86]. As in [2, Proof of Lemma C.1], we note that since $f' \colon M \to N$ is a continuous map, it is homotopic to a smooth map; see, for instance, [5, Proposition 17.8]. Thus, since M is compact, we can assume without loss of generality there exists a constant A such that $f' \colon M \to N$ (as well as some homotopy inverse $h \colon N \to M$ of f') is A-Lipschitz, which gives an upper bound as in (2-26). From here, it is straightforward to see that the lower bound in (2-26) holds for some $B = B(A, D)$; see [12, Lemma 5.1]. Hence, to prove Proposition 2.41, it suffices to show the following.

Proposition 2.42 *Let* (M, g) *and* (N, g_0) *be as in the hypotheses of the previous proposition. Let* $f \colon (M, g) \to (N, g_0)$ *be a* C^1 *map. Then there exists a constant* $A = A(n, \Gamma, i_0, D, b)$ *such that* f *is homotopic to an* A*-Lipschitz map.*

Note the above proposition is false in dimension 2. Indeed, given $f \colon (M, g) \to (N, g_0)$, let $\mathrm{Lip}(f)_{g,g_0}$ denote its Lipschitz constant, ie the supremum over all $x \ne y \in M$ of the quantity $d_{g_0}(f(x), f(y))/d_g(x, y)$. Now let (M, g) be a hyperbolic surface and let $f \colon M \to M$ be the identity map. Since the mapping class group of M is not compact, there is a sequence $h_n \in \mathrm{Diff}(M)$ such that $\mathrm{Lip}(h_n)_{g,g} \to \infty$. As such, $\mathrm{Lip}(f)_{g,h_n^* g} \to \infty$ even though g and $h_n^* g$ are isometric, and hence have identical diameter, injectivity radius, and sectional curvature bounds. In dimensions 3 or more, on the other hand, we have the following finiteness of the mapping class group which is an immediate consequence of [26, Theorem 5.4 A].

Lemma 2.43 *Let* (M, g) *be a closed negatively curved manifold of dimension at least* 3. *Let* $\mathrm{Diff}(M)$ *denote the group of smooth self-diffeomorphisms of* M, *and let* $\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$ *denote the subgroup consisting of diffeomorphisms which are homotopic to the identity. Then the quotient group* $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ *is finite.*

Remark 2.44 In higher dimensions, the subgroup $\mathrm{Diff}_0(M)$ above may be larger than the subgroup $\mathrm{Diff}^0(M)$ of diffeomorphisms *isotopic* to the identity, equivalently, the connected component of the identity in $\mathrm{Diff}(M)$. The term "mapping class group" is also used to mean $\pi_0(\mathrm{Diff}(M)) = \mathrm{Diff}(M)/\mathrm{Diff}^0(M)$ elsewhere in the literature. The size of the latter group is a more delicate matter than what is needed for our purposes, and in fact, it follows from work of Farrell and Jones [18, Corollary 10.16 and equation (10.28)] that $\pi_0(\mathrm{Diff}(M))$ is infinite when $n \ge 11$.

Proof It is well known that $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ is a subgroup of the outer automorphism group $\mathrm{Out}(\pi_1(M))$. (See, for instance, [19, Remark 8].) Indeed, let $\mathrm{HE}(M)$ denote the group of self-homotopy equivalences of M and let $\mathrm{HE}_0(M)$ denote the subgroup of all such which are homotopic to the identity. Then $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ is a subgroup of $\mathrm{HE}(M)/\mathrm{HE}_0(M)$, and the latter is isomorphic to $\mathrm{Out}(\pi_1(M))$ since M is a $K(\pi, 1)$ space.

Since M is negatively curved of dimension at least 3, by [26, Theorem 5.4 A], $\mathrm{Out}(\pi_1(M))$ is finite. This completes the proof. □

Proof of Proposition 2.42 Given a C^1 map $f\colon (M, g) \to (N, g_0)$, write

$$\|df\|_{g,g_0} = \sup_{0\neq v\in TM} \frac{\|df_p(v)\|_{g_0}}{\|v\|_g}.$$

We will show that there is a constant $A = A(n, \Gamma, i_0, D, b)$ such that, after possibly changing f in its homotopy class, we have $\|df\|_{g,g_0} \le A$ for all g, g_0 as in the statement of the proposition. Suppose for contradiction that for each $n \in \mathbb{N}$ there exist metrics g_n and $(g_0)_n$ with injectivity radius at least i_0, diameter at most D and sectional curvatures in the interval $[-b^2, 0)$ such that for any C^1 map f_n homotopic to f, we have $\|df_n\|_{g_n,(g_0)_n} \ge n$.

By Gromov's systolic inequality [25, 0.1.A], there exists $v_0 = v_0(i_0, n)$ so that $\mathrm{Vol}_{g_n}(M)$ and $\mathrm{Vol}_{(g_0)_n}(N)$ are both bounded below by v_0 for all n. Hence, the sequences g_n and $(g_0)_n$ satisfy the bounded geometry hypotheses of the main theorem of [24]. This means that, after passing to subsequences, there exist $\phi_n \in \mathrm{Diff}(M)$, $(\phi_0)_n \in \mathrm{Diff}(N)$ and $C^{1,\beta}$ (for any $0 < \beta < 1$) Riemannian metrics g_∞ and $(g_0)_\infty$ on M and N, respectively, such that $\phi_n^* g_n \to g_\infty$ in $C^{1,\beta}(M)$ and $(\phi_0)_n^*(g_0)_n \to (g_0)_\infty$ in $C^{1,\beta}(N)$.

Since $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ is finite by Lemma 2.43, after passing to a further subsequence, we have that there exists a single $h \in \mathrm{Diff}(M)$ such that for all n, there exists $\psi_n \in \mathrm{Diff}_0(M)$ so that $\phi_n = h \circ \psi_n$. Analogously, there is $h_0 \in \mathrm{Diff}(N)$ and $(\psi_0)_n \in \mathrm{Diff}_0(N)$ such that $(\phi_0)_n = h_0 \circ (\psi_0)_n$. This means $\psi_n^* g_n \to (h^{-1})^* g_\infty$ and $(\psi_0)_n^*(g_0)_n \to ((h_0)^{-1})^*(g_0)_\infty$ in the $C^{1,\beta}$, and hence C^0, topologies. Hence

$$\|d((\psi_0)_n \circ f \circ \psi_n^{-1})\|_{g_n,(g_0)_n} = \|df\|_{\psi_n^* g_n,(\psi_0)_n^*(g_0)_n} \to \|df\|_{h_*^{-1} g_\infty, h_*^{-1} g_\infty}.$$

The right-hand side is finite because f is C^1 and M is compact. This contradicts the initial construction of the sequences g_n and $(g_0)_n$. □

2.3.2 Combining the constructions of Bourdon and Gromov

We now briefly recall the construction of Gromov [27]. The first step is to use f to construct a preliminary continuous map $\mathcal{F}_0\colon T^1M \to T^1N$ which sends geodesics to (reparametrized) geodesics, but not necessarily injectively. If p denotes the footpoint of v, then $\mathcal{F}_0(v)$ is defined to be the orthogonal projection of the point $f(p)$ onto the bi-infinite geodesic determined by $\big(f(\pi(-v)), f(\pi(v))\big) \in \partial^2\widetilde{N}$. Since $\mathcal{F}_0$ sends geodesics to geodesics, there exists a function (cocycle) $b\colon T^1M \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{F}_0(\phi^t v) = \psi^{b(t,v)}\mathcal{F}_0(v). \tag{2-27}$$

Later, we will explain Gromov's key averaging trick used to obtain an injective orbit equivalence $\mathcal{F}$ from $\mathcal{F}_0$; see (2-30) below.

If $\mathcal{L}_g = \mathcal{L}_{g_0} \circ f_*$, one can use the Livsic theorem to upgrade the orbit equivalence $\mathcal{F}$ to a time-preserving conjugacy; see [38, Chapter 2.2]. Alternatively, in [7], Bourdon directly shows that if $\bar{f}\colon \partial\tilde{M} \to \partial\tilde{N}$ is cross-ratio–preserving (which holds when the marked length spectra of M and N coincide), then there exists a conjugacy of geodesic flows $\mathcal{F}\colon T^1M \to T^1N$. We leverage some ideas from Bourdon's construction to relate the function $b(t,v)$ in (2-27) above to the cross-ratio, and then use an argument of Otal to in turn relate $b(t,v)$ to the ratio $\mathcal{L}_{g_0} \circ f_*/\mathcal{L}_g$.

We now recall Bourdon's construction [7, Section 1.4]. Let $\partial^3\tilde{M}$ denote the set of triples (ξ,ξ',η) of points in $\partial\tilde{M}$ which are pairwise distinct. The first step of Bourdon's construction is to use the *Gromov product* $(\xi\,|\,\xi')_p$ (defined in (2-4) above) to construct a fibration $\Pi\colon \partial^3\tilde{M} \to T^1\tilde{M}$ as follows: Given $(\xi,\xi',\eta) \in \partial\tilde{M}$, let $p(\xi,\xi',\eta)$ denote the unique point on the geodesic determined by ξ and ξ' such that $(\xi\,|\,\eta)_p = (\xi'\,|\,\eta)_p$. Let $\Pi(\xi,\xi',\eta)$ denote the tangent vector to $[\xi',\xi]$ at the point $p(\xi,\xi',\eta)$. Since the action of Γ on $\tilde{M} \cup \partial\tilde{M}$ preserves the Gromov product, the map Π is Γ-equivariant.

A calculation (using the formula (2-5)) shows that

$$d(p(\xi,\xi',\eta), p(\xi,\xi',\eta')) = [\xi,\xi',\eta,\eta']. \tag{2-28}$$

See [7, Section 1.3]. In particular, this shows that if $v = \Pi(\xi,\xi',\eta)$, then the fiber $\Pi^{-1}(v)$ consists of all triples (ξ,ξ',η') where η' satisfies the equation $[\xi,\xi',\eta,\eta'] = 0$. Hence, if $\bar{f}\colon \partial\tilde{M} \to \partial\tilde{N}$ preserves the cross-ratio, it preserves the fibers of Π. As such, the natural homeomorphism $\bar{f}^3\colon \partial^3\tilde{M} \to \partial\tilde{N}$ induced by $\bar{f}$ factors through the projections Π to a homeomorphism $T^1\tilde{M} \to T^1\tilde{N}$. Since $\bar{f}$ and Π are Γ-equivariant, the above homeomorphism further factors through to a homeomorphism $\mathcal{F}\colon T^1M \to T^1N$. By construction, $\mathcal{F}$ sends ϕ^t-orbits to ψ^t-orbits, and is thus of the form in (2-1). From (2-28) it is immediate that $a(t,v) = t$ for all $t \in \mathbb{R}$ and $v \in T^1\tilde{M}$.

To obtain quantitative control of the Bourdon construction, we start by noting that the quasi-invariance of distance functions established in Proposition 2.41 (together with the Morse lemma [8, Theorem III.H.1.7]) implies a quasi-invariance of the Gromov product; see, for instance, [22, Chapter 5, Proposition 15]. This in turn implies the following bi-Hölder equivalence of Bourdon–Gromov distances (defined above (2-5)): there exist constants C_1, C_2, depending only on a, A, D, such that

$$C_1 d_p(\xi,\eta)^A \le d_{f(p)}(\bar{f}(\xi), \bar{f}(\eta)) \le C_2 d_p(\xi,\eta)^{1/A} \tag{2-29}$$

for every $p \in \tilde{M}$ and every $\xi,\eta \in \partial\tilde{M}$. Here, and throughout this section, A is as in (2-26), and can be further controlled in terms of geometric data as stated in Proposition 2.42.

In the following lemma, we show that the images of the two maps $\mathcal{F}_0 \circ \Pi_M$ and $\Pi_N \circ \bar{f}^3 \circ \Pi_M$ from $\partial^3\tilde{M} \to T^1\tilde{N}$ are a controlled distance apart.

Lemma 2.45 *Let $v = \Pi(\xi_1, \xi_2, \eta)$. Then there exists a constant $C = C(a, A, D)$ such that for any $(\xi_1, \xi_2, \eta) \in T^1 M$, we have $d\big(\mathscr{F}_0(\Pi(\xi_1, \xi_2, \eta)), \Pi(\bar{f}(\xi_1), \bar{f}(\xi_2), \bar{f}(\eta))\big) \le C$.*

Remark 2.46 Since the maps in the statement of the lemma are all Γ-equivariant, the existence of some such bound C follows from compactness of $T^1 M$, but, as usual, we need to take greater care determining which geometric quantities this constant depends on.

Proof Let p denote the footpoint of v. By the Morse lemma (see [8, Theorem III.H.1.7]), there is a constant $C = C(A, D, a)$ such that the distance between $f(p)$ and $\mathscr{F}_0(v)$ is at most C.

Next, by definition of Π we have $d_p(\xi_1, \eta) = d_p(\xi_2, \eta)$. The fact that p lies on the geodesic (ξ_1, ξ_2), together with the triangle inequality, gives

$$1 = d_p(\xi_1, \xi_2) \le d_p(\xi_1, \eta) + d_p(\xi_2, \eta) = 2d_p(\xi_i, \eta).$$

Hence $\frac{1}{2} \le d_p(\xi_1, \eta) = d_p(\xi_2, \eta) \le 1$. By (2-29), there are constants C_1, C_2, depending only on a, A, D, such that $C_1 \le d_{f(p)}(\bar{f}(\xi_i), \bar{f}(\eta)) \le C_2$ for $i = 1, 2$. If q is the footpoint of $\mathscr{F}_0(v)$ then the claim in the previous paragraph, together with [45, equation (2)], gives $C_1' \le d_q(\bar{f}(\xi_i), \bar{f}(\eta)) \le C_2'$, for C_i' depending only on a, A, D.

Now let $q(t)$ be a unit-speed parametrization of the geodesic $(\bar{f}(\xi_1), \bar{f}(\xi_2))$ so that $q(0) = q$. Recall that $\Pi(\bar{f}(\xi_1), \bar{f}(\xi_2), \bar{f}(\eta))$ is the point $q(t_0) \in \tilde{N}$ such that $d_{q(t_0)}(\bar{f}(\xi_1), \bar{f}(\eta)) = d_{q(t_0)}(\bar{f}(\xi_2), \bar{f}(\eta))$. To prove the lemma, we thus need to show $t_0 \le C = C(A, D, a)$. To this end, we use the change of basepoint formula of the Gromov product [7, 1.1(b)] to rewrite the previous equality as

$$\frac{d_q(\bar{f}(\xi_1), \bar{f}(\eta))}{d_q(\bar{f}(\xi_2), \bar{f}(\eta))} = \exp\big(\tfrac{1}{2}\big(B_{\bar{f}(\xi_2)}(q, q(t_0)) - B_{\bar{f}(\xi_1)}(q, q(t_0))\big)\big) = e^{t_0}.$$

Since the left-hand side is bounded between C_1'/C_2' and C_2'/C_1', this completes the proof. □

Proposition 2.47 *Suppose $\mathscr{L}_g(\gamma) \ge K\mathscr{L}_{g_0}(f(\gamma))$ for all $\gamma \in \Gamma$. Then there exists $C = C(a, b, A, D)$ such that for all $\xi, \xi', \eta, \eta' \in \partial\widetilde{M}$ we have*

$$[\xi, \xi', \eta, \eta'] \ge K[\bar{f}(\xi), \bar{f}(\xi'), \bar{f}(\eta), \bar{f}(\eta')] + KC.$$

Remark 2.48 The fact that there exists some C such that the above holds is proved by Otal in [44, Proposition 4.2]. We follow his proof, but we replace all instances of orthogonal projection of η onto (ξ, ξ') with Bourdon's projection $\Pi(\xi, \xi', \eta)$, which simplifies the proof and allows for better quantitative control. For instance, the role of [44, Lemma 4.1] in Otal's proof is played by [7, Proposition 1.3] below.

Proof Given ξ, ξ', η, η', let p and p' denote the footpoints of $v = \Pi(\xi, \xi', \eta)$ and $v' = \Pi(\xi, \xi', \eta')$, respectively. By [7, Proposition 1.3], we have $d(p, p') = [\xi, \xi', \eta, \eta']$. Let $\delta_0 > 0$ which will remain fixed throughout this proof. Then by Lemma 2.9, there is a constant $S = S(\delta, a)$ and times $t_1, t_2 \in [0, S]$ such that the geodesic-flow–line joining $\phi^{-t_1}v$ and $\phi^{t_2}v$ is δ_0-close to the lift of some closed geodesic γ. This means

$$\mathscr{L}_g(\gamma) \le d(p,p') + 2(S+\delta_0) = [\xi,\xi',\eta,\eta'] + 2(S+\delta_0).$$

By [12, Proposition 2.4], $\mathscr{F}_0$ is uniformly continuous with modulus of continuity depending explicitly on A, D, a, b. This means there exists some constant $S' = S'(S,a,A,D)$ such that $\mathscr{L}_{g_0}(f(\gamma)) \le d(\mathscr{F}_0(v), \mathscr{F}_0(v')) + S'_0$. Finally, let C be the constant from Lemma 2.45. This lemma gives

$$d(\mathscr{F}_0(v), \mathscr{F}_0(v')) \le 2C + d\big(\Pi(\bar{f}(\xi), \bar{f}(\xi'), \bar{f}(\eta)), \Pi(\bar{f}(\xi), \bar{f}(\xi'), \bar{f}(\eta'))\big).$$

Since $d\big(\Pi(\bar{f}(\xi), \bar{f}(\xi'), \bar{f}(\eta)), \Pi(\bar{f}(\xi), \bar{f}(\xi'), \bar{f}(\eta'))\big) = [\bar{f}(\xi), \bar{f}(\xi'), \bar{f}(\eta), \bar{f}(\eta')]$, this completes the proof. □

Lemma 2.49 *Let $b(t,v)$ as in (2-27); let (M,g) and (N,g_0) as in the hypotheses of Theorem 2.38. Then there is a constant $C = C(A,D,a,b)$ such that*

$$K_1(t-C) \le b(t,v) \le K_2(t+C)$$

for all $v \in T^1M$ and all $t \in \mathbb{R}$.

Proof Let v and $\phi^t v$ be given by $\Pi(\xi,\xi_2,\eta_1)$ and $\Pi(\xi_1,\xi_2,\eta_2)$. Then we have $t = [\xi_1,\xi_2,\eta_1,\eta_2]$. In Lemma 2.45 we showed there exists a constant C such that for any $v = \Pi(\xi_1,\xi_2,\eta_1)$, we have $d\big(\mathscr{F}_0(v), \Pi(\bar{f}(\xi_1), \bar{f}(\xi_2), \bar{f}(\eta_1))\big) \le C$. Hence

$$\begin{aligned}
b(t,v) &= d(\mathscr{F}_0(v), \mathscr{F}_0(\phi^t v)) \\
&\le 2C + d\big(\Pi(\bar{f}(\xi_1), \bar{f}(\xi_2), \bar{f}(\eta_1)), \Pi(\bar{f}(\xi_1), \bar{f}(\xi_2), \bar{f}(\eta_2))\big) && \text{(by Lemma 2.45)} \\
&= 2C + [\bar{f}(\xi_1), \bar{f}(\xi_2), \bar{f}(\eta_1), \bar{f}(\eta_2)] && \text{(by [7, Proposition 1.3])} \\
&\le 2C + K_2[\xi_1,\xi_2,\eta_1,\eta_2] + C' && \text{(by Proposition 2.47)} \\
&= 2C + (1+\varepsilon)t + C'.
\end{aligned}$$

The proof of the lower bound for $b(t,v)$ is analogous. □

We now use an averaging trick of Gromov to turn $\mathscr{F}_0$ into an orbit equivalence $\mathscr{F}$ satisfying the conclusions of Theorem 2.38. Let C be the constant from the previous lemma. Given $\delta > 0$, choose $l > 0$ so that $K_i C/l < \delta$ for $i = 1,2$. For $t \ge 0$ and $v \in T^1\widetilde{M}$, consider the average

$$a_l(t,v) := \frac{1}{l}\int_t^{t+l} b(s,v)\,ds. \tag{2-30}$$

Then set $\mathscr{F}(v) = \psi^{a_l(0,v)}\mathscr{F}_0(v)$.

Proof of Theorem 2.38 (1) By the fundamental theorem of calculus,

$$\frac{d}{dt}a_l(t,v) = \frac{b(t+l,v) - b(t,v)}{l} = \frac{b(l,\phi^t v)}{l}.$$

Hence, a_l is injective so long as the right-hand side above is never zero. In our case, Lemma 2.49 shows the much stronger result that

$$\frac{d}{dt}a_l(t,v) \in [K_1-\delta, K_2+\delta].$$

By [12, Proposition 5.4], letting $\mathscr{F}(v) = \psi^{a_l(0,v)}\mathscr{F}_0(v)$ completes the proof of (1).

(2) We can replace the conclusion of [12, Lemma 5.5] with Lemma 2.49 above. Since the constant C in Lemma 2.49 can be explicitly controlled in terms of A, D, a, b, the same arguments as in the remainder of [12] go through verbatim to show $\mathscr{F}$ has the desired Hölder exponent α and Hölder constant C, which completes the proof of (2). □

2.4 Proof of the volume estimate

Now we will explicitly relate the Liouville current λ on $\partial^2\widetilde{M}$ and the Liouville measure μ on $T^1\widetilde{M}$. Let X denote the vector field on T^1M which generates the geodesic flow. For every $v \in T^1M$, we can choose local coordinates $(t, x_1, \dots, x_m)$ near v so that $\partial/\partial t = X$. Then $(0, x_1, \dots, x_m)$ defines a local smooth hypersurface $K_0 \subset T^1M$ which is transverse to X. Let $K = \pi(K_0) \subset \partial^2\widetilde{M}$. Then $\int_{K_0}(d\omega)^{n-1} = \lambda(K)$.

For $T > 0$ define $K_T = \{\phi^t v \mid v \in K_0,\ t \in [-\tfrac{1}{2}T, \tfrac{1}{2}T]\}$. If T is sufficiently small, then with respect to our choice of local coordinates, we have $K_T = \{(t, x_1, \dots, x_m) \mid -\tfrac{1}{2}T \le t \le \tfrac{1}{2}T\}$ and $\omega = dt$. We thus obtain

$$\mu(K_T) = \int_{K_T} \omega \wedge (d\omega)^{n-1} = T\int_{K_0}(d\omega)^{n-1} = T\lambda(K).$$

Now let $K_0 = K(v,r)_0 := E_v(Q(r) \times Q(r)) \subset T^1M$. Then by Proposition 2.37, we have

$$(1-C\varepsilon^\alpha)\lambda^M(K_0) \le \lambda^N(\bar{f}(K_0)) \le (1+C\varepsilon^\alpha)\lambda^M(K_0).$$

Now let $\mathscr{F}$ be as in Theorem 2.38. The discussion above then implies

$$(1-C\varepsilon^\alpha)(1-2\varepsilon)\mu^M(K_r) \le \lambda^N(\mathscr{F}(K_r)) \le (1+C\varepsilon^\alpha)(1+2\varepsilon)\mu^M(K_r). \tag{2-31}$$

To complete the proof of Theorem C, we will estimate the volumes of T^1M and T^1N by approximating them with flow boxes K_r as above. Since we cannot make r arbitrarily small ($r = C\varepsilon^\alpha$, and ε is fixed), we proceed to quantify the error of such an approximation.

Lemma 2.50 *Let $\delta_0 = \delta_0(\varepsilon)$ be as in Proposition 2.4, let $r = 2\delta_0$, and let ε be sufficiently small that the error term $e(r)$ in Lemma 2.24 satisfies $\|e(r)\|/r < \tfrac{1}{2}$. Let $K(v,r)_0 = E_v(Q(r) \times Q(r))$. Then there is a constant c, depending only on b, such that for any $v \in T^1M$, we have that $K(v,r)_r$ contains the Sasaki ball $B(v,cr)$.*

Proof By construction, $K(v,r)_0 \supset W_r^{\mathrm{ss}}(v) := B(v,r) \cap W^{\mathrm{ss}}(v)$. By Lemma 2.24 and our choice of ε, for each $w \in W_r^{\mathrm{ss}}(v)$, there is a neighborhood of size $\tfrac{1}{2}r$ of the vertical leaf $\mathscr{V}(w)$ contained in K_0^v. In other words, $K_0^v \supset \bigcup_{w \in W_r^{\mathrm{ss}}(v)} \mathscr{V}_{r/2}(w)$.

Set $c = (2(1+b))^{-1}$. First we consider $u \in B(v, cr)$ such that $Pu \in PW^{\mathrm{ss}}(v)$. Let $\overline{u}$ be the vector with the same footpoint as u such that $\overline{u} \in W^{\mathrm{ss}}(v)$. Then $d(u, \overline{u}) \le cr$. Moreover, by [12, Lemma 3.9], it follows that $\overline{u} \in W_r^{\mathrm{ss}}(v)$.

Now suppose that $u \in B(v, cr)$. Let u_0 be the element in $\{\phi^t u\}_{t\in\mathbb{R}}$ which is closest to v. Then the distances between the footpoints of u, u_0 and v are all at most cr. Let $u_1 = \phi^t u_0$ be such that the footpoint of u_1 is on $PW^{\mathrm{ss}}(v)$. By [12, Lemma 3.9], we see that $t_1 \le Cr^2$. Hence $u_1 \in B(v, cr)$ and the argument in the previous paragraph shows $u_1 \in K(v, r)_0$, which completes the proof. □

Lemma 2.51 *Let $r > 0$ be sufficiently small, as in the previous lemma. Then there exists a constant $C = C(n, b)$ and an integer $m \le \mathrm{vol}(T^1 M) C r^{-(2n+1)}$, together with $v_1, \dots, v_m \in T^1 M$, such that $T^1 M = \bigcup_{i=1}^m K(v_i, r)_r$.*

Proof We will argue similarly to the proof of [12, Lemma 2.1]. Let $v_1, \dots, v_m$ be a maximal cr-separated subset of $T^1 M$ with respect to the Sasaki metric. Then the $B(v_i, cr)$ cover $T^1 M$, and by the previous lemma, so do the $K_r^{v_i}$. To estimate m, we note that the smaller balls $B\big(v_i, \tfrac{1}{2}cr\big)$ are disjoint, and hence

$$m \inf_{v\in T^1 M} \mathrm{vol}\big(B\big(v, \tfrac{1}{2}cr\big)\big) \le \mathrm{vol}(T^1 M).$$

By [12, Lemma 4.1], we have $\mathrm{vol}\big(B\big(v, \tfrac{1}{2}cr\big)\big) \ge c'\big(\tfrac{1}{2}r\big)^{n+1}$ for some $c' = c'(c, n)$. This means that $m \le C\,\mathrm{vol}(T^1 M) r^{-(2n+1)}$, which completes the proof. □

Proposition 2.52 *Let $r > 0$ be as in the previous lemma. There exists a constant C, depending only on n, a, b, R, such that the following hold:*

(1) *There exist $v_1, \dots, v_m$ such that the flow boxes $K^i := K(v_i, r)_r$ are disjoint and*

$$\sum_{i=1}^m \mu^M(K^i) \ge (1 - Cr)\mu^M(T^1 M).$$

(2) *There exist $v_1, \dots, v_m$ such that the flow boxes $K^i := K(v_i, r)_r$ cover $T^1 M$ and*

$$\sum_{i=1}^m \mu^M(K^i) \le (1 - Cr)\mu^M(T^1 M).$$

Proof First we choose r_0' sufficiently small so that each $K(v, r_0')_{r_0'}$ has diameter less than the injectivity radius of $T^1 M$ with respect to the Sasaki metric. Indeed, by Lemmas 2.29 and 2.30, we have $\mathrm{diam}(K(v, r_0')_{r_0'}) \le Cr_0'$ for some $C = C(a, b, R)$.

Set $n' = 2n + 1 = \dim(T^1 M)$. By the previous lemma, we need $m \le C\,\mathrm{vol}(T^1 M) r_0'^{-n'}$ sets of the form $K(v_i, r_0')_{r_0'}$ to cover $T^1 M$. By the proof of Proposition 2.19, each $K(v_i, r_0')_{r_0'}$ is contained in a "parallelogram" we will refer to as $P(v_i, r_0)$, where $r_0 = (1 + Cr_0'^{\alpha_0})r_0'$. Next, choose $k \in \mathbb{N}$ so that $r_0/k < r$. Subdivide each $P(v_i, r_0)$ into $k^{n'}$ "small parallelograms" of the form $P(v_i^j, r_0/k)$. Now delete all small "boundary" parallelograms, ie all $P(v_i^j, r_0/k)$ which are adjacent to the boundary of $P(v_i, r_0)$. Call this deleted region $B \subset T^1 M$. Using that $m \leq C \operatorname{vol}(T^1 M) r_0^{-n'}$, we see that

$$\mu(B) \leq \frac{k^{n'} - (k-2)^{n'}}{k^{n'}} r_0^{n'} C \operatorname{vol}(T^1 M) r_0^{-n'} \leq \frac{C' r_0^{-1} r_0}{k} \operatorname{vol}(T^1 M) \leq C'' r \operatorname{vol}(T^1 M),$$

where C'' depends only n, a, b, R, i_M.

We claim that $T^1 M \setminus B$ can be disjointly covered by a subset of the remaining "small" parallelograms. Suppose an interior parallelogram $P(v_1^j, r_0/k)$ intersects a parallelogram $P(v_2^l, r_0/k)$. Since $P(v_1^j, r_0/k)$ is not a boundary parallelogram, we have that $P(v_2^l, r_0/k)$ is entirely covered by (interior and boundary) parallelograms of the form $P(v_1^{j'}, r_0/k)$. As such, deleting $P(v_2^l, r_0/k)$ does not expose any part of $T^1 M \setminus B$. Continuing to delete overlapping cubes in this manner proves the claim. Finally, to prove (1), we use the proof of Proposition 2.19 to find sets of the form $K(v_i^j, r)_r$ inside each of the $P(v_i^j, r)_r$ such that $\mu^M(K(v_i^j, r)_r / P(v_i^j, r)_r) \leq 1 - C r^{\alpha_0}$ for some $C = C(n, a, b, R)$.

By the above deletion argument, we can also cover $T^1 M$ with small parallelograms such that any overlapping parallelograms intersect the boundary region B. This proves (2). □

Proof of Theorem C Let $K^1, \dots, K^m$ be disjoint flow boxes satisfying part (1) of the Proposition 2.52. Let $\mathcal{F}$ as in Theorem 2.38. Since $\mathcal{F}$ is injective, the $\mathcal{F}(K^i)$ are disjoint as well, which, using (2-31), gives

$$\mu^N(T^1 N) \geq \mu^N\Big(\bigcup_i \mathcal{F}(K^i)\Big) = \sum_i \mu^N(\mathcal{F}(K^i)) \geq (1-\varepsilon') \sum_i \mu^M(K^i) \geq (1-C\delta_0)(1-\varepsilon')\mu^M(T^1 M),$$

where ε' is of the form $C\varepsilon^\alpha$ for $\alpha < \alpha_0^2(1-\varepsilon)a^2/b$, as desired.

Next, we take $K^1, \dots, K^m$ as in (2) of the previous lemma. Then, surjectivity of $\mathcal{F}$ gives

$$\mu^N(T^1 N) = \mu^N\Big(\bigcup_i \mathcal{F}(K^i)\Big) \leq \sum_i \mu^N(\mathcal{F}(c_i)) \leq (1+\varepsilon') \sum_i \mu^M(K^i) \leq (1+\varepsilon')(1+C\delta_0)\mu^M(T^1 M),$$

which completes the proof. □

3 Estimates for the BCG map

If $\mathcal{L}_g = \mathcal{L}_{g_0}$, then it follows from [32, Theorem A] that $\operatorname{Vol}(M, g) = \operatorname{Vol}(N, g_0)$. Since $\mathcal{L}_g$ determines the topological entropy of the geodesic flow, the entropy rigidity theorem of Besson, Courtois and Gallot [4] states there is an isometry $F \colon M \to N$.

In the case where $1 - \varepsilon \leq \mathcal{L}_{g_0}/\mathcal{L}_g \leq 1 + \varepsilon$ (equation (1-2)), Theorem C states that the volumes of M and N satisfy $1 - C\varepsilon^\alpha \leq \operatorname{Vol}(N)/\operatorname{Vol}(M) \leq 1 + C\varepsilon^\alpha$, where C and α are positive constants depending only on $n, \Gamma, a, b, i_0, D, R$. Moreover, the entropies are related as follows.

Lemma 3.1 *Let h denote the topological entropy of the geodesic flow. Then with the above marked length spectrum assumptions*, *we have*

$$\frac{1}{1+\varepsilon}h(g) \le h(g_0) \le \frac{1}{1-\varepsilon}h(g). \tag{3-1}$$

Proof This follows from the following description of the topological entropy in terms of periodic orbits due to Margulis [41]:

$$h(g) = \lim_{t\to\infty} \frac{1}{t}\log P_g(t), \tag{3-2}$$

where $P_g(t) = \#\{\gamma \mid l_g(\gamma) \le t\}$. □

We use the results of Theorem C and Lemma 3.1 to modify the proof in [4] that there is an isometry $F\colon M \to N$. More specifically, we use the same construction for the map F as in [4] and show that the matrix of dF_p with respect to suitable orthonormal bases is close to the identity matrix.

3.1 Construction of the BCG map

From now on, we will assume N is a locally symmetric space. This means $\widetilde{N}$ is either a real, complex or quaternionic hyperbolic space or the Cayley hyperbolic space of real dimension 16; let $d = 1, 2, 4$ or 8, respectively.

We now normalize g_0 so that the sectional curvatures are all -1 in the case $d = 1$ and contained in the interval $[-4, -1]$ otherwise. The requisite scaling factor depends only on $-a^2$ and $-b^2$, the original sectional curvature bounds for g_0. We also rescale g by the same factor so that (1-2) still holds. For notational convenience, we will continue to refer to the sectional curvature bounds of the renormalized (M, g) as $-b^2$ and $-a^2$. We also note that since $\dim N \ge 3$, Mostow rigidity [42] implies (N, g_0) is determined up to isometry by its fundamental group Γ. Thus, from now on, any constants arising from the geometry of N, such as the diameter and the injectivity radius, can be thought of as depending only on Γ.

We first recall the construction of the map $F\colon M \to N$ in [4]. We then summarize the proof that F is an isometry in the case of equal entropies and volumes, before explaining how to modify it for approximately equal entropies and volumes.

Given $p \in M$, let μ_p be the Patterson–Sullivan measure on $\partial\widetilde{M}$. Let $\bar{f}\colon \partial\widetilde{M} \to \partial\widetilde{N}$ as before; see Construction 2.1. Define $F(p) = \operatorname{bar}(f_*\mu_p)$, where bar denotes the barycenter map; see [4] for more details. We call F the *BCG map*. By the definition of the barycenter, the BCG map has the implicit description

$$\int_{\partial\widetilde{N}} dB_{F(p),\xi}(\,\cdot\,)\, d(\bar{f}_*\mu_p)(\xi) = 0, \tag{3-3}$$

where $\xi \in \partial\widetilde{N}$ and $B_{F(p),\xi}$ is the Busemann function on $(\widetilde{N}, g_0)$. By the implicit function theorem, the BCG map F is C^1 (actually, C^2 since Busemann functions on $\widetilde{M}$ are C^2 [1, Proposition IV.3.2]), and its derivative dF_p satisfies, for all $v \in T_pM$ and $u \in T_{F(p)}N$ (see [4, (5.2)]),

$$\int_{\partial \widetilde{N}} \text{Hess}\, B^N_{F(p),\xi}(dF_p(v),u)\, d(\bar{f}_*\mu_p)(\xi) = h(g) \int_{\partial \widetilde{N}} dB^N_{F(p),\xi}(u)\, dB^M_{p,\bar{f}^{-1}(\xi)}(v)\, d(\bar{f}_*\mu_p)(\xi). \tag{3-4}$$

In light of this, it is natural to define the following quadratic forms K and H:

$$\langle K_{F(p)}u,u\rangle := \int_{\partial \widetilde{N}} (\text{Hess}\, B_{F(p),\xi})(u)\, d(\bar{f}_*\mu_p)(\xi), \tag{3-5}$$

$$\langle H_{F(p)}u,u\rangle := \int_{\partial \widetilde{N}} (dB_{F(p),\xi}(u))^2\, d(\bar{f}_*\mu_p)(\xi), \tag{3-6}$$

where $\langle\cdot,\cdot\rangle$ denotes the Riemannian inner product coming from g_0; see [4, page 636].

Without any assumptions about the volumes or entropies, the following three inequalities hold; see [49] for the Cayley case.

Lemma 3.2 [4, Lemma 5.4] $$|\text{Jac}\, F(p)| \le \frac{h^n(g)}{n^{n/2}} \frac{\det(H)^{1/2}}{\det(K)}.$$

Lemma 3.3 [3, Lemma B3] *Let $n \ge 3$ and let H and K be the $n \times n$ positive definite symmetric matrices coming from the operators in (3-5) and (3-6), respectively. Then*

$$\frac{\det H}{\det(K)^2} \le \frac{(n-1)^{2n(n-1)/(n+d-2)}}{(n+d-2)^{2n}} \frac{\det(H)^{(n-d)/(n+d-2)}}{\det(I-H)^{2(n-1)/(n+d-2)}},$$

with equality if and only if $H = (1/n)I$.

Lemma 3.4 [3, Lemma B4] *Let H be an $n \times n$ positive definite symmetric matrix with trace 1, where $n \ge 3$. Let $1 < \alpha \le n-1$. Then*

$$\frac{\det H}{\det(I-H)^\alpha} \le \left(\frac{n^\alpha}{n(n-1)^\alpha}\right)^n.$$

Moreover, equality holds if and only if $H = (1/n)I$.

Combining the above three inequalities (setting $\alpha = 2(n-1)/(n-d)$) together with the fact that $h(g_0) = n+d-2$, we obtain:

Lemma 3.5 [4, Proposition 5.2(i)] $$|\text{Jac}\, F(p)| \le \left(\frac{h(g)}{h(g_0)}\right)^n.$$

As in the proof of [4, Theorem 5.1], the above lemma relates the volumes of M and N as follows:

$$\text{Vol}(N,g_0) \le \int_M |F^* d\text{Vol}| = \int_M |(\text{Jac}\, F)\, d\text{Vol}| \le \left(\frac{h(g)}{h(g_0)}\right)^n \text{Vol}(M,g). \tag{3-7}$$

Remark 3.6 This, together with Lemma 3.1, improves one of the inequalities in Theorem C in the special case where N is a locally symmetric space.

With this setup in mind, the argument in [4] showing that F is an isometry consists of the following components:

(1) If the volumes and entropies are equal, then the inequalities in (3-7) are all equalities, which gives equality in Lemma 3.5.

(2) Thus, equality also holds in Lemmas 3.2 and 3.3, from which it follows that
$$H = \frac{1}{n}I \quad \text{and} \quad K = \frac{n+d-2}{n}I = \frac{h(g_0)}{n}I.$$
See [4, page 639].

(3) With H and K as above, the end of the proof of Proposition 5.2(ii) in [4] shows that
$$dF_p = \frac{h(g_0)}{h(g)}I,$$
which means F is an isometry in the case where the entropies are equal. This concludes the proof of Theorem 1 in [4].

Assuming instead that $1-\varepsilon \le \mathscr{L}_{g_0}/\mathscr{L}_g \le 1+\varepsilon$, the equalities of volumes and entropies are replaced with the conclusions of Theorem C and Lemma 3.1, respectively. As such, to prove our main theorem (Theorem B), it remains to show the following:

Theorem 3.7 *Let (M, g) be a closed Riemannian manifold of dimension at least 3 with fundamental group Γ, diameter at most D, injectivity radius at least i_M, and sectional curvatures contained in the interval $[-b^2, -a^2]$. Let (N, g_0) be a locally symmetric space with sectional curvatures contained in the interval $[-4, -1]$.*

Then there exists small enough $\varepsilon_0 = \varepsilon_0(n, i_M)$ so that whenever $\varepsilon_1, \varepsilon_2 \le \varepsilon_0$ and

$$(1-\varepsilon_1)\operatorname{vol}(M) \le \operatorname{vol}(N) \quad \text{and} \quad (1-\varepsilon_2)h(g_0) \le h(g) \le (1+\varepsilon_2)h(g_0), \tag{3-8}$$

there is a C^2 map $F\colon M \to N$ homotopic to f and constants
$$C = C(n, \Gamma, b, D) \quad \text{and} \quad \delta = 1 - \frac{1-\varepsilon_1}{1+\varepsilon_2}$$
such that for all $v \in TM$ we have
$$(1 - C\delta^{1/16n})\|v\|_g \le \|dF(v)\|_{g_0} \le (1 + C\delta^{1/16n})\|v\|_g.$$

Remark 3.8 Under the assumptions of Theorem B, we have $i_M \ge i_0$ where i_0 is a constant depending only on ε and Γ. Indeed, (1-2), together with the fact that the injectivity radius of M is half the length of the shortest closed geodesic in M (see [46, page 178]), gives $i_M \ge (1/(1+\varepsilon))i_N$, and i_N depends only on Γ by Mostow rigidity.

To obtain estimates for $\|dF_p\|$ in terms of δ, we proceed as in the above outline:

(1) We show equality almost holds in (3-7); that is, we find a lower bound for $\operatorname{Jac} F(p)$ of the form $(1-\beta)(h(g)/h(g_0))^n$ for suitable small $\beta > 0$ (Proposition 3.18).

(2) This implies the eigenvalues of H are all close to $1/n$ and the eigenvalues of K are all close to $h(g_0)/n$ (Proposition 3.21).

(3) With H and K as above, we mimic the proof of [4, Proposition 5.2(ii)] to obtain bounds for $\|dF_p\|$, which completes the proof of Theorem 3.7.

The main difficulty is step (1), where we cannot simply mimic the arguments in [4]. Indeed, with the hypotheses of Theorem 3.7, the inequalities in (3-7) become

$$\frac{1-\varepsilon_1}{1+\varepsilon_2}\left(\frac{h(g)}{h(g_0)}\right)^n \operatorname{Vol}(M) \leq \int_M |\operatorname{Jac} F| \leq \left(\frac{h(g)}{h(g_0)}\right)^n \operatorname{Vol}(M), \tag{3-9}$$

which does not give a lower bound for the integrand. In order to obtain a lower bound for $|\operatorname{Jac} F|$, we use the above lower bound for its integral together with a Lipschitz bound for the function $p \mapsto |\operatorname{Jac} F(p)|$ (Proposition 3.17). The fact that this function is Lipschitz is immediate from the fact that F is C^2; however, it is not clear a priori how the Lipschitz bound depends on (M,g).

3.2 Lipschitz bound for F

Recall the BCG map F is defined implicitly (see (3-3)), and its derivative dF_p satisfies the equation

$$\langle K dF_p(v), u\rangle = h(g)\int_{\partial\widetilde{N}} dB^N_{F(p),\xi}(u)\, dB^M_{p,\bar{f}^{-1}\xi}(v)\, d(\bar{f}_*\mu_p)(\xi).$$

(See (3-4) and (3-5).) In order to use this equation to find a Lipschitz bound for $\operatorname{Jac} F(p)$, we start by bounding the quadratic form K away from zero (Lemma 3.9). Recall

$$\langle K_{F(p)}u, u\rangle := \int_{\partial\widetilde{N}} (\operatorname{Hess} B_\xi)_{F(p)}(u)\, d(\bar{f}_*\mu_p)(\xi). \tag{3-10}$$

Note that K depends not only on the symmetric space (N, g_0), but also on (M, g), since μ_p is the Patterson–Sullivan measure on $\partial\widetilde{M}$ defined with respect to the metric g.

For the remainder of the paper, we will let $A = A(n, \Gamma, D, i_0, b)$ denote the constant from the conclusion of Proposition 2.41. That is, for all $p, q \in \widetilde{M}$, we have

$$A^{-1} d(p,q) - B \leq d(f(p), f(q)) \leq A\, d(p,q), \tag{3-11}$$

where B depends only on A and the diameter bound D.

Lemma 3.9 *There is a* $\kappa = \kappa(n, \Gamma, A, D) > 0$ *such that* $\langle K_{F(p)}u, u\rangle \geq \kappa$ *for all* $p \in \widetilde{M}$, $u \in T^1_{F(p)}\widetilde{N}$.

Proof First we examine the integrand in (3-10). Fix $p \in \widetilde{M}$ and $u \in T^1_{F(p)}\widetilde{N}$. Consider $(\operatorname{Hess} B_\xi)_{F(p)}(u)$. Let $v_{F(p),\xi}$ be the unit tangent vector based at $F(p)$ so that the geodesic with initial vector v has forward boundary point ξ, ie $v_{F(p),\xi}$ is the gradient of $B_{\xi,F(p)}$. Let θ_ξ denote the angle between $v_{F(p),\xi}$ and u.

Then we can write $u = (\cos\theta_\xi)v_{F(p),\xi} + (\sin\theta_\xi)w$ for some unit vector w perpendicular to $v_{F(p),\xi}$. Since $(\text{Hess } B_\xi)_{F(p)}(u) = \langle \nabla_u v_{F(p),\xi}, u\rangle$, we obtain $(\text{Hess } B_\xi)_{F(p)}(u) = \sin^2\theta_\xi (\text{Hess } B_\xi)_{F(p)}(w)$. Let R denote the curvature tensor of $(\widetilde{N}, \widetilde{g}_0)$. Using the formula (see [14, page 16])

$$(\text{Hess } B_\xi)_{F(p)}(\,\cdot\,) = \sqrt{-R(v_{F(p),\xi}, \cdot\,, v_{F(p),\xi}, \cdot\,)}, \tag{3-12}$$

together with the fact the sectional curvatures of $\widetilde{N}$ are at most -1, it follows that

$$(\text{Hess } B_\xi)_{F(p)}(u) \geq \sin^2\theta_\xi.$$

Hence, the integrand in the definition of $K_{F(p)}$ is 0 if and only if $\theta_\xi = 0, \pi$. This occurs precisely when $\xi = \pi(\pm u)$, where π is the projection of a unit tangent vector to its forward boundary point in $\partial\widetilde{N}$.

Now fix $T > 0$, whose size will be specified later. Let $q_{\pm T}$ denote the footpoints of $\phi^{\pm T}u$, respectively. Given $y \in \widetilde{N}$ and $\eta \in \partial\widetilde{N}$, let $c_{y,\eta}$ denote the unique g_0 geodesic through y and η. Let

$$\mathbb{O}_\pm = \{\eta \in \partial\widetilde{N} \mid c_{F(p),\eta} \cap B(q_{\pm T}, 1) \neq \varnothing\}.$$

Then (3-11) implies that

$$\bar{f}^{-1}(\mathbb{O}_\pm) \subset \{\xi \in \partial\widetilde{M} \mid c_{f^{-1}(F(p)),\xi} \cap B(f^{-1}(q_{\pm T}), R)\}$$

for some $R \leq A + B$. By the shadow lemma for Patterson–Sullivan measures (see eg [48, Lemma 1.3]), together with (3-11), we have

$$\begin{aligned}\mu_p(\bar{f}^{-1}(\mathbb{O}_\pm)) &\leq \exp\bigl(-h(g)\bigl(d_g\bigl(f^{-1}(F(p)), f^{-1}(q_{\pm T})\bigr) - 2R\bigr)\bigr)\\ &\leq \exp\bigl(-h(g)(A^{-1}T - 2(A+B))\bigr).\end{aligned}$$

Choose T sufficiently large so the above line is less than $\frac{1}{4}$. Then there is $\theta = \theta(g_0, T) = \theta(n, T)$ so that $\mathbb{O}_\pm$ consists of points $\pi(w)$ where $w \in T^1_{F(p)}\widetilde{N}$ makes angle less than θ with $\pm u$, respectively. This means the integrand in the definition of K is bounded below by $\sin\theta$ on the set $\partial\widetilde{N} \setminus (\mathbb{O}_+ \cup \mathbb{O}_-)$. Since we have shown this set has measure at least $\frac{1}{2}$, this completes the proof. □

This lower bound for κ allows us to find an a priori Lipschitz bound for F. While the fact that F is Lipschitz follows from the fact that F is C^2, it is not clear a priori which properties of (M, g) this Lipschitz constant depends on. In the end, this Lipschitz constant will turn out to be close to 1 in a way that depends only on $\varepsilon, n, \Gamma, \Lambda, A$ by Theorem B.

Lemma 3.10 *Let F be the BCG map. Then $\|dF_p\| \leq h(g)/\kappa$ for all $p \in \widetilde{M}$.*

Proof Using (3-4), we get the following inequality by applying Cauchy–Schwarz (see [4, (5.3)]) together with the fact that $\|dB(w)\| \leq \|w\|$ for any Busemann function:

$$\langle K_{F(p)}\, dF_p v, u\rangle \leq h(g)\|v\|\|u\|.$$

Now let $\|v\| = 1$ and let $u = dF_p(v)$. Then the above inequality and Lemma 3.9 give

$$\kappa\|dF_p v\|^2 \leq \langle K_{F(p)}\, dF_p(v), dF_p(v)\rangle \leq h(g)\|dF_p(v)\|.$$

Thus $\|dF_p(v)\| \leq h(g)/\kappa$, which completes the proof. □

3.3 Lipschitz constant for Jac $F(p)$

Let $p, q \in \widetilde{M}$ and let $c(t)$ be the unit-speed geodesic joining p and q such that $c(0) = p$. Let $P_{c(t)}$ denote parallel transport along the curve $c(t)$. For $i = 1, 2$, let $u_i \in T^1_{F(p)}\widetilde{N}$ and let $U_i(t) = P_{F(c(t))}u_i$.

We begin by finding a bound for the derivative of the function $t \mapsto \langle K_{F(c(t))}U_1(t), U_2(t)\rangle$ for $0 \le t \le T_0$.

This bound will depend only on ε_2, n, Γ, A, D and T_0.

Lemma 3.11 *Let $K^{\xi}_{F(p)}(u_1, u_2) = (\operatorname{Hess} B_{\xi})_{F(p)}(u_1, u_2)$. Let $U_i(t) = P_{F(c(t))}u_i$, as above. Then the function $t \mapsto K^{\xi}_{F(c(t))}(U_1(t), U_2(t))$ has derivative bounded by a constant depending only on ε_2, n, Γ, A and D.*

Proof Let $X = (d/dt)|_{t=0}F(c(t))$. Then it suffices to find a uniform bound for $\|X(K^{\xi}(U_1, U_2))\|$ on $\widetilde{N}$. Since the U_i are parallel along X, we have $X(K^{\xi}(U_1, U_2)) = \nabla K^{\xi}(U_1, U_2, X)$; see [13, Definition 4.5.7]. So $\|X(K^{\xi}(U_1, U_2))\| \le \|\nabla K^{\xi}\|\|U_1\|\|U_2\|\|X\|$. Since $\|X\| \le h(g)/\kappa$ by the previous lemma and $\|U_1\| = \|U_2\| = 1$, it remains to control $\|\nabla K^{\xi}\|$. We claim that this quantity is uniformly bounded on $\widetilde{N}$.

First note that if a is an isometry fixing ξ, then

$$K^{\xi}_x(v, w) = K^{\xi}_{a(x)}(a_*v, a_*w).$$

Now fix $x_0 \in \widetilde{N}$ and let $e_1, \dots, e_n \in T_{x_0}\widetilde{N}$ be an orthonormal frame. For any other $x \in \widetilde{N}$, there exists an isometry a taking x to x_0 fixing ξ (since $\widetilde{N}$ is a symmetric space). As such, we can extend the e_i to vector fields E_i on all of $\widetilde{N}$. Then the quantity

$$\nabla K^{\xi}(E_i, E_j, E_k) = E_k(K^{\xi}(E_i, E_j)) - K^{\xi}(\nabla_{E_k}E_i, E_j) - K^{\xi}(\nabla_{E_k}E_i, E_j)$$

is invariant by isometries a fixing ξ, and is thus constant on $\widetilde{N}$. This shows the desired claim that $\|\nabla K^{\xi}\|$ is uniformly bounded on $\widetilde{N}$. The bound depends only on the symmetric space $\widetilde{N}$ and hence only on the dimension n. □

Lemma 3.12 *Consider the function*

$$t \mapsto \langle K_{F(c(t))}U_1(t), U_2(t)\rangle \quad \text{for } 0 \le t \le T_0.$$

Its derivative is bounded by a constant depending only on ε, n, Γ, A, D and T_0.

Proof Note that $\bar{f}_*\mu_{c(t)}(\xi) = \exp[-h(g)B^M_{\bar{f}^{-1}(\xi)}(p, c(t))]\,\bar{f}_*\mu_p(\xi)$. Then

$$\langle K_{F(c(t))}U_1(t), U_2(t)\rangle = \int_{\partial\widetilde{N}} K^{\xi}(U_1(t), U_2(t))\exp[-h(g)B^M_{\bar{f}^{-1}(\xi)}(p, c(t))]\,\bar{f}_*\mu_p(\xi).$$

The first term in the integrand is bounded above as a consequence of (3-12), and this bound depends only on the dimension n. By the previous lemma, the derivative of this function is bounded by a constant depending only on ε_2, n, Γ, A, D. Since $|B_{\bar{f}^{-1}(\xi)}(p,c(t))| \le d(p,c(t)) \le T_0$, the second term is bounded by a constant depending only on n, ε and T_0. The same is true of its derivative, since Busemann functions have gradient 1. Hence the derivative of $\langle K_{F(c(t))}U_1(t), U_2(t)\rangle$ is bounded by a constant depending only on the desired parameters. □

Corollary 3.13 *The function $t \mapsto \det K_{F(c(t))}$ on the interval $0 \le t \le T_0$ is L_1-Lipschitz for some $L_1 = L_1(\varepsilon, n, \Gamma, A, D, T_0)$.*

Proof By Lemma 3.12, the entries of the matrix $K_{F(c(t))}$ (with respect to a g_0-orthonormal basis) vary in a Lipschitz way. Using (3-12), we see that for u_1 and u_2 unit vectors, the expression $\operatorname{Hess} B^N_{F(p),\xi}(u_1,u_2)$ is uniformly bounded above by some constant depending only on $(\widetilde{N}, \tilde{g}_0)$. Since the entries of the matrix $K_{F(c(t))}$ are Lipschitz and bounded, it follows that the determinant of this matrix is Lipschitz. □

Recall that (3-4) implies

$$\langle K_{F(p)}\, dF_p(v), u\rangle = h(g) \int_{\partial \widetilde{M}} dB^N_{F(p),\bar{f}(\xi)}(u)\, dB^M_{p,\xi}(v)\, d\mu_p(\xi).$$

This formula, together with the Lipschitz bound for $p \mapsto \det K_{F(p)}$ established in Corollary 3.13, will allow us to find a Lipschitz bound for $p \mapsto \det(dF_p) = \operatorname{Jac} F(p)$.

Lemma 3.14 *Let p, q and $c(t)$ be as above. Then the function*

$$t \mapsto dB^M_{c(t),\xi}(P_{c(t)}v)$$

is $\frac{1}{2}b$-Lipschitz for all $v \in T^1_p\widetilde{M}$.

Proof We have

$$\frac{d}{dt}\Big|_{t=0} dB_{c(t),\xi}(P_{c(t)}v) = \operatorname{Hess} B_{p,\xi}(c'(0), v) = \operatorname{Hess} B_{p,\xi}(c'(0)^T, v^T),$$

where $c'(0)^T$ and v^T are the components of $c'(0)$ and v in the direction tangent to the horosphere through p and ξ.

Using that $\operatorname{Hess} B_{p,\xi}$ is bilinear and positive definite on $\operatorname{grad} B^{\perp}_{p,\xi}$, we obtain

$$4 \operatorname{Hess} B_{p,\xi}(c'(0)^T, v^T) \le \operatorname{Hess} B_{p,\xi}(c'(0)^T + v^T, c'(0)^T + v^T).$$

Let $v' = c'(0)^T + v^T$ and note that $\|v'\| \le 2$. Let $\beta(s)$ be a curve in the horosphere such that $\beta'(0) = v'$. Consider the geodesic variation $j(s,t) = \exp_{\beta(s)}(t \operatorname{grad} B_{\beta(s),\xi})$ and let $J(t) = (d/ds)|_{s=0} j(s,t)$ be the associated Jacobi field. Then $J(0) = v'$ and $J'(0) = \nabla_{v'} \operatorname{grad} B_{p,\xi}$. This means

$$\operatorname{Hess} B_{p,\xi}(c'(0)^T + v^T, c'(0)^T + v^T) = \langle J'(0), J(0)\rangle.$$

Let $\chi = \frac{1}{2}\sqrt{a^2+b^2}$. According to [9, Corollary 4.2],

$$\langle J'(0), J(0)\rangle \le \langle J'(0) + \chi J(0), J(0)\rangle \le |J(0)|(\chi - a).$$

Since $|J(0)| = |v'| \le 2$, we get $4(d/dt)|_{t=0} dB_{c(t),\xi}(P_{c(t)}v) \le \langle J'(0), J(0)\rangle \le 2b$. □

Lemma 3.15 *The function* $t \mapsto dB^N_{F(c(t)),\xi}(P_{F(c(t))}u)$ *is* $(h(g)/\kappa + 1)$*-Lipschitz for all* $u \in T^1_{F(p)}\widetilde{N}$.

Proof We repeat the same proof as in the previous lemma, but replacing λ^2 and Λ^2 with 1 and 4, respectively. In this case, $\chi - \lambda < 1$. This gives

$$\frac{d}{dt}\bigg|_{t=0} dB^N_{F(c(t)),\xi}(P_{F(c(t))}u) = \operatorname{Hess} B^N_{F(p),\xi}\big(dF_p(c'(0)), u\big) < |dF_p(c'(0)) + u|.$$

Since $c'(0)$ has norm 1, the Lipschitz bound from Lemma 3.10 gives $|dF_p(c'(0)) + u| \le (h(g)/\kappa) + 1$, which completes the proof. □

Lemma 3.16 *The function* $t \mapsto \det K_{F(c(t))} \operatorname{Jac} F(c(t))$ *on the interval* $0 \le t \le T_0$ *is* L_2*-Lipschitz, where* L_2 *depends only on* ε, n, Γ, b, A, D *and* T_0.

Proof Consider the function

$$t \mapsto h(g) \int_{\partial \widetilde{M}} dB^N_{F(c(t)),f(\xi)}(P_{F(c(t))}u)\, dB^M_{c(t),\xi}(P_{c(t)}v) e^{-h(g)B_\xi(p,c(t))}\, d\mu_p(\xi).$$

The first two terms in the integrand are bounded by 1 in absolute value. The third term is bounded above by a constant depending only on ε, n and T_0 as in the proof of Lemma 3.12 and $h(g) \le (1+\varepsilon)h(g_0)$ by Lemma 3.1. Moreover, the three terms in the integrand are each Lipschitz — the first two by Lemmas 3.15 and 3.14, respectively, and the last one as in the proof of Lemma 3.12. Since the entries of the matrix $K_{F(c(t))}(dF_{c(t)})$ are bounded and Lipschitz, the determinant of this matrix is also Lipschitz. □

Proposition 3.17 *The function* $p \mapsto |\operatorname{Jac} F(p)|$ *is* L*-Lipschitz, where the constant* L *depends only on* ε_2, n, Γ, b, A *and* D.

Proof Since $K_{F(p)}$ is a symmetric matrix, it has an orthonormal basis of eigenvectors u_i. Moreover, $\langle K_{F(p)}u_i, u_i\rangle \ge \kappa\langle u_i, u_i\rangle$ by Lemma 3.9. It follows that $\det K_{F(p)} \ge \kappa^n$. Using this, we obtain

$$\begin{aligned}
\kappa^n|\operatorname{Jac} F(p) - \operatorname{Jac} F(q)| &\le |\det K_{F(p)} \operatorname{Jac} F(p) - \det K_{F(p)} \operatorname{Jac} F(q)| \\
&\le L_2 d(p,q) + |\operatorname{Jac} F(q)|\,|\det K_{F(p)} - \det K_{F(q)}| \quad \text{(Lemma 3.16)} \\
&\le L_2 d(p,q) + (1+\varepsilon_2)^n h(g_0) L_1 d(p,q),
\end{aligned}$$

where the last inequality follows from Corollary 3.13 and Lemmas 3.5 and 3.1. Moreover, Corollary 3.13 and Lemma 3.16 imply that L_1 and L_2 depend only on ε_2, n, Γ, b, A and D. Lemma 3.9 states that κ depends only on ε, n and Γ. □

3.4 Lower bound for $|\operatorname{Jac} F(p)|$

Now that we have a Lipschitz bound for $\operatorname{Jac} F(p)$, we can use (3-9) to show equality almost holds in the inequality $\operatorname{Jac} F(p) \le (h(g)/h(g_0))^n$ (Lemma 3.5). Note that Lemma 3.5 together with the following proposition both require $n \ge 3$.

Proposition 3.18 *Let (M, g), (N, g_0) and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \delta > 0$ be as in Theorem 3.7. Then there is sufficiently small $\varepsilon_0 = \varepsilon_0(n, \Gamma)$ and $C = C(n, \Gamma, b, A, D)$ so that*

$$(1 - C\delta^{1/2n})\left(\frac{h(g)}{h(g_0)}\right)^n \le |\mathrm{Jac}\, F(p)| \quad \text{for all } p \in \widetilde{M}.$$

We need two preliminary lemmas. Let ν denote the measure on M coming from the Riemannian volume.

Lemma 3.19 *Let $\phi\colon M \to \mathbb{R}$ be a ν-measurable function such that $\phi \ge 0$. Suppose the integral of ϕ satisfies $0 \le \int_M \phi \le \delta$. Let $B = \{x \in M \mid \phi > \omega\}$, where ω is some constant. Then $\nu(B) \le \delta/\omega$.*

Proof Note that $\omega\nu(B) \le \int_B \phi \le \int_X \phi \le \delta$, which gives the desired bound. □

Lemma 3.20 *Let i_M denote the injectivity radius of M and let $c(n)$ denote the volume of the unit ball in $\mathbb{R}^n$. Fix $\delta < c(n)(i_M)^n$. Let $B \subset M$ be an open set with $\nu(B) < \delta$. Then there is an $r = r(\delta)$ such that for any $p \in B$ there is a $q \in M \setminus B$ with $d(p, q) \le r$. Moreover, $r \le c(n)^{-1/n}\delta^{1/n}$.*

Proof Let $p \in B$. Let $q \in M \setminus B$ be the point such that $d(p, q) = \min_{x \in M \setminus B} d(p, x)$. Let $r = d(p, q)$. Then the open ball $B(p, r)$ is contained in the set B. We consider the cases $r \le i_M$ and $r > i_M$ separately.

In the case $r \le i_M$, we can apply Theorem 3.101(ii) in [21] to obtain the inequality $\mathrm{Vol}\, B(p, r) \ge c(n)r^n$, where $c(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Since $B(p, r) \subset B$, this gives $r^n \le \delta/c(n)$.

In the case $r > i_M$, we do not have the above volume estimate for the ball $B(p, r)$. However, since $B(p, i_M) \subset B(p, r) \subset B$, the same argument as in the first case gives a bound $(i_M)^n \le \delta/c(n)$. This is a contradiction for δ as small as in the statement of the lemma, so we must be in the first case. □

Proof of Proposition 3.18 Let $\phi(p) = (h(g)/h(g_0))^n - |\mathrm{Jac}\, F(p)| \ge 0$. Then, by Lemma 3.5 and (3-8), we have

$$0 \le \int_M \phi \le \delta\left(\frac{h(g)}{h(g_0)}\right)^n \mathrm{vol}(M).$$

Let

$$M_{\sqrt{\delta}} = \left\{|\mathrm{Jac}\, F(p)| \ge (1 - \sqrt{\delta})\left(\frac{h(g)}{h(g_0)}\right)^n\right\}.$$

Lemma 3.19 gives $\nu(M \setminus M_{\sqrt{\delta}}) \le \sqrt{\delta}\, \mathrm{vol}(M)$. By [4, Theorem 1.1(i)], together with $h(g) \le (1+\varepsilon_2)h(g_0)$, we have $\mathrm{vol}(M) \le (1+\varepsilon_2)C$ for $C = C(n, \Gamma)$. Then, if δ is sufficiently small (in terms of n and $\mathrm{inj}(M, g)$), we have

$$\nu(M \setminus M_{\sqrt{\delta}}) \le c(n)(\mathrm{inj}(M, g))^n, \tag{3-13}$$

so the hypotheses of Lemma 3.20 are satisfied. The lemma gives $r(\delta) \le c(n)^{-1/n}\delta^{1/2n}$ so that for all $p \in M \setminus M_\alpha$ there is $q \in M_{\sqrt{\delta}}$ satisfying $d(p,q) < r(\delta)$. Applying Proposition 3.17 with $T_0 = r(\delta_0)$, we then have

$$(1-\sqrt{\delta})\left(\frac{h(g)}{h(g_0)}\right)^n \le |\mathrm{Jac}\, F(q)| \le Lr(\delta) + |\mathrm{Jac}\, F(p)|$$

for some $L = L(n, \Gamma, b, A, D)$. Rearranging and using that $(1-\varepsilon_2)h(g_0) \le h(g)$ gives

$$\left(1-\sqrt{\delta}-(1-\varepsilon_2)^{-n}Lr(\delta)\right)\left(\frac{h(g)}{h(g_0)}\right)^n \le \mathrm{Jac}\,|F(p)|,$$

which completes the proof. □

3.5 Estimates for $\|dF_p\|$

Recall that $H_{F(p)}$ and $K_{F(p)}$ are symmetric bilinear forms on $T_{F(p)}\widetilde{N}$; see (3-5) and (3-6). We will use the lower bound we just established for Jac $F(p)$ in Proposition 3.18 to show that H and K are close to scalar matrices. This will then allow us to mimic the proof of [4, Proposition 5.2(ii)] to find bounds for the derivative of the BCG map that are close to 1. Note that $n \ge 3$ is assumed throughout this section.

Proposition 3.21 *Let $F\colon \widetilde{M} \to \widetilde{N}$ be the BCG map and assume there is a constant $\beta > 0$ such that the Jacobian of F satisfies*

$$(1-\beta)\left(\frac{h(g)}{h(g_0)}\right)^n \le |\mathrm{Jac}\, F(p)|$$

for all $p \in \widetilde{M}$ (as in the conclusion of Proposition 3.18). Let $H_{F(p)}$ and $K_{F(p)}$ be the symmetric bilinear forms on $T_{F(p)}\widetilde{N}$ defined in (3-5) and (3-6). Then there is a constant $C = C(n,\Gamma)$ such that

$$(1-C\beta^{1/2})\frac{1}{n}\langle v,v\rangle \le \langle H_{F(p)}v,v\rangle \le (1+C\beta^{1/2})\frac{1}{n}\langle v,v\rangle,$$
$$(1-C\beta^{1/4})\frac{h(g_0)}{n}\langle v,v\rangle \le \langle K_{F(p)}v,v\rangle \le (1+C\beta^{1/4})\frac{h(g_0)}{n}\langle v,v\rangle$$

for all $p \in M$ and all $v \in T_{F(p)}\widetilde{N}$.

The lower bound on Jac $F(p)$ can be thought of as equality almost holding in Lemma 3.5. This lower bound, together with the inequalities in Lemmas 3.2 and 3.3, implies that equality almost holds in Lemma 3.4, that is,

$$\beta^{2(n+d-2)/(n-d)}\left(\frac{n^{\alpha-1}}{(n-1)^\alpha}\right)^n \le \frac{\det H}{\det(I-H)^\alpha} \le \left(\frac{n^{\alpha-1}}{(n-1)^\alpha}\right)^n, \quad \text{where } \alpha = \frac{2(n-1)}{n-d}. \tag{3-14}$$

In order to prove Proposition 3.21, we will first show that since $(1-\beta)$ is close to 1, the matrix H is almost $(1/n)I$. This proves the first part of the proposition.

Lemma 3.22 *Let H be a symmetric positive definite $n\times n$ matrix with trace 1 for $n\geq 3$. Let $1<\alpha\leq n-1$ and let*

$$m=\left(\frac{n^{\alpha-1}}{(n-1)^{\alpha}}\right)^{n}.$$

Suppose that

$$\frac{\det H}{\det(I-H)^{\alpha}}\geq(1-\beta)^{2(n+d-2)/(n-d)}m$$

for β as in Proposition 3.18. Let λ_i denote the eigenvalues of H. Then there is a constant C, depending only on n, such that

$$(1-C\sqrt{\beta})\frac{1}{n}\leq\lambda_i\leq(1+C\sqrt{\beta})\frac{1}{n}\quad\text{for } i=1,\dots,n.$$

Proof Since $1\leq d\leq 8$, there is a universal constant C so that $1\leq 2(n+d-2)/n-d\leq C$ for all $n\geq 3$; hence $(1-\beta)^{2(n+d-2)/(n-d)}\geq 1-\bar{C}\beta$ for some universal constant $\bar{C}$. It follows from [3, Proposition B.5] (see [49] for the Cayley case), Lemma 3.2 and Proposition 3.18 that there is a constant $B(n)>0$ such that

$$\sum_{i=1}^{n}\left(\lambda_i-\frac{1}{n}\right)^2\leq\frac{\bar{C}\beta}{B}.$$

This means $|\lambda_i-1/n|\leq C\sqrt{\beta}$ for some $C=C(n)$, which completes the proof. □

Next, we need an analogue of Lemma 3.22 for the arithmetic–geometric mean inequality.

Lemma 3.23 *Let L be a symmetric positive-definite $n\times n$ matrix with $c_1\leq\operatorname{trace}L\leq c_2$ for positive constants c_1,c_2 depending only on n and Γ. Suppose*

$$\det L\geq(1-\omega)\left(\frac{1}{n}\operatorname{trace}L\right)^{n}$$

for some $\omega>0$. Let $\mu_1,\dots,\mu_n$ denote the eigenvalues of L. Then there is a constant $C=C(n,\Gamma)$ such that

$$(1+C\sqrt{\omega})\frac{\operatorname{trace}L}{n}\leq\mu_i\leq(1-C\sqrt{\omega})\frac{\operatorname{trace}L}{n}\quad\text{for } i=1,\dots,n.$$

Proof We will use the approach of the proof of [3, Proposition B5]. Let $\phi(\mu_1,\dots,\mu_n)=\log(\mu_1\cdots\mu_n)$. Since ϕ is concave, there is a constant $B>0$ such that the inequality

$$\log(\mu_1\cdots\mu_n)\leq\log\left(\frac{\operatorname{trace}L}{n}\right)^{n}-B\sum_{i=1}^{n}\left(\mu_i-\frac{\operatorname{trace}L}{n}\right)^2$$

holds on the set of all $\mu_i\geq 0$ satisfying $\mu_1+\cdots+\mu_n=\operatorname{trace}L$. The constant B depends only on the function ϕ. In other words, it does not depend on any topological or geometric properties of the manifolds M and N other than the number $n=\dim M=\dim N$.

Since L is positive definite, we know that $0 < \mu_i < \operatorname{trace} L$ for all i. So there exists a $T = T(\varepsilon, n, \Gamma, \Lambda)$ such that $B\sum_{i=1}^n (\mu_i - \operatorname{trace} L/n)^2 \leq T$. Following the same steps as in the proof of [3, Proposition B.5], we then obtain

$$\sum_{i=1}^n \left(\mu_i - \frac{\operatorname{trace} L}{n}\right)^2 \leq \frac{\omega}{B\,(1-e^{-T})/T}.$$

Using the boundedness assumption $c_1 \leq \operatorname{trace}(L) \leq c_2$ completes the proof. □

Proof of Proposition 3.21 First, note that by [3, Proposition B1] and Lemma 3.22, there is a constant $C = C(n,\Gamma)$ so that $\det K \geq (1 - C\sqrt{\beta})(h(g_0)/n)^n$. So equality almost holds in the arithmetic–geometric mean inequality. By Lemma 3.23, the eigenvalues of K are between $(1 - C\beta^{1/4})h(g_0)/n$ and $(1 + C\beta^{1/4})h(g_0)/n$, as desired. □

Proof of Theorem 3.7 We closely follow the proof of [4, Proposition 5.2(ii)]. First note that it suffices to prove the claim for v a unit vector. Using the definitions of H and K together with the Cauchy–Schwarz inequality, we obtain

$$\langle K dF_p v, u\rangle \leq h(g)\langle Hu, u\rangle^{1/2}\left(\int_{\partial\widetilde{M}} (dB_{p,\xi}(v))^2\, d\mu_p(\xi)\right)^{1/2}.$$

(See [4, (5.3)].) Using the upper bound for H in Proposition 3.21, the above inequality implies

$$\langle K dF_p v, u\rangle \leq \sqrt{1 + C\beta^{1/2}}\,\frac{h(g)}{\sqrt{n}}\|u\|\left(\int_{\partial\widetilde{M}} (dB_{p,\xi}(v))^2 d\mu_p(\xi)\right)^{1/2}.$$

Now let $u = dF_p(v)/\|dF_p(v)\|$. Using the lower bound for K in Proposition 3.21 gives

$$\|dF_p\| \leq (1 + C'\beta^{1/4})\frac{h(g)}{h(g_0)}\sqrt{n}\left(\int_{\partial\widetilde{M}} (dB_{p,\xi}(v))^2 d\mu_p(\xi)\right)^{1/2}.$$

Now let $L = dF_p \circ dF_p^T$ and let v_i be an orthonormal basis for $T_p\widetilde{M}$. Then, since $\|dB_{p,\xi}\| \leq \|v\| = 1$, we get

$$\operatorname{trace} L = \sum_{i=1}^n \langle Lv_i, v_i\rangle = \sum_{i=1}^n \langle dF_p(v_i), dF_p(v_i)\rangle \leq \left((1 + C'\beta^{1/4})\frac{h(g)}{h(g_0)}\right)^2 n.$$

Combining this with Proposition 3.18 and the arithmetic–geometric mean inequality gives

$$(1-\beta)^2\left(\frac{h(g)}{h(g_0)}\right)^{2n} \leq |\operatorname{Jac} F(p)|^2 = \det L \leq \left(\frac{1}{n}\operatorname{trace} L\right)^n \leq \left((1 + C'\beta^{1/4})\frac{h(g)}{h(g_0)}\right)^{2n}. \tag{3-15}$$

Hence the hypotheses of Lemma 3.23 hold with $\omega = C''\beta^{1/4}$. Lemma 3.23 thus implies

$$(1 - C''\beta^{1/8})\frac{1}{n}\operatorname{trace} L\langle v, v\rangle \leq \langle Lv, v\rangle = \langle dF_p v, dF_p v\rangle \leq (1 + C''\beta^{1/8})\frac{1}{n}\operatorname{trace} L\langle v, v\rangle.$$

Finally, using (3-15) gives

$$(1-\beta)^{2/n}\left(\frac{h(g)}{h(g_0)}\right)^2 \leq \frac{\langle dF_p v, dF_p v\rangle}{\langle v, v\rangle} \leq (1 + C''\beta^{1/8})\left((1 + C'\beta^{1/4})\frac{h(g)}{h(g_0)}\right)^2,$$

which completes the proof. □

4 Surfaces

In this section, we prove a generalization of Otal [43] and Croke's [16] marked length spectrum rigidity result for negatively curved surfaces. Using the arguments of Otal [43], we show that pairs of negatively curved metrics on a surface become more isometric as the ratio of their marked length spectrum functions gets closer to 1. Aside from some background on the Liouville measure and Liouville current from Section 2, this section does not rely on earlier parts of this paper.

Let M be a closed surface of higher genus. Fix $0 < a \leq b$. Let g_0 be a C^2 Riemannian metric on M with sectional curvatures contained in the interval $[-b^2, -a^2]$. Fix $\alpha, R > 0$. Let $\mathcal{U}^{1,\alpha}_{g_0}(a, b, R)$ consist of all C^2 metrics g on M such that $\|g - g_0\|_{C^{1,\alpha}} \leq R$. In this section we will prove the following theorem about surfaces whose marked length spectra are close:

Theorem A *Let M, a, b, R, g_0 be as above. Fix $L > 1$. Then there exists $\varepsilon = \varepsilon(L, a, b, R, \alpha) > 0$ small enough so that for any pair $(M, g), (M, h) \in \mathcal{U}^{1,\alpha}_{g_0}(a, b, R)$ satisfying*

$$1 - \varepsilon \leq \frac{\mathcal{L}_g}{\mathcal{L}_h} \leq 1 + \varepsilon, \tag{4-1}$$

there exists an L-Lipschitz map $f : (M, g) \to (M, h)$.

By the Arzelà–Ascoli theorem, any sequence $(M, g_n) \in \mathcal{U}^{1,\alpha}_{g_0}(a, b, R)$ has a subsequence g_{n_k} which converges in the following sense: there is a Riemannian metric g_0 on M such that in local coordinates we have $g^{ij}_{n_k} \to g^{ij}_0$ in the $C^{1,\alpha}$ norm, and the limiting g^{ij}_0 have regularity $C^{1,\alpha}$. In particular, the C^0 convergence of the $g^{ij}_{n_k}$ implies the distance functions converge in the following sense:

Lemma 4.1 *Given any $A > 1$, there is a sufficiently large k so that for all $p, q \in M$ we have $A^{-1}\, d_{g_0}(p, q) \leq d_{g_{n_k}}(p, q) \leq A\, d_{g_0}(p, q)$.*

We will prove Theorem A by contradiction. Indeed, suppose the statement is false. Then for every $\varepsilon > 0$, there are $(M, g_\varepsilon), (M, h_\varepsilon) \in \mathcal{U}^{1,\alpha}_{g_0}(a, b, R)$ satisfying (4-1) so that there is no L-Lipschitz map $f : (M, g_\varepsilon) \to (M, h_\varepsilon)$. By the Arzelà–Ascoli theorem, there is a subsequence $\varepsilon_n \to 0$ such that $(M, g_{\varepsilon_n}) \to (M, g_0)$ and $(M, h_{\varepsilon_n}) \to (M, h_0)$ in the sense described above. From now on we will relabel g_{ε_n} as g_n and h_{ε_n} as h_n. To prove the main theorem, it suffices to prove the following statement:

Proposition 4.2 *Let (M, g_0) and (M, h_0) be the $C^{1,\alpha}$ limits of the counterexamples above. Then there is a map $f : M \to M$ such that for all $p, q \in M$ we have $d_{g_0}(p, q) = d_{h_0}(f(p), f(q))$.*

Proof of Theorem A Fix $L > 1$ and suppose the theorem is false. Let (M, g_n) and (M, h_n) be the convergent sequences of counterexamples defined above. Since $(M, g_n) \to (M, g_0)$, Lemma 4.1 gives large enough n so that $(\sqrt{L})^{-1}\, d_{g_0}(p,q) \le d_{g_n}(p,q) \le \sqrt{L}\, d_{g_0}(p,q)$ for all $p,q \in M$, and similarly for d_{h_n}. Then Proposition 4.2 gives

$$d_{g_n}(p,q) \le \sqrt{L} d_{h_0}(f(p), f(q)) \le L\, d_{h_n}(f(p), f(q)).$$

So $f\colon (M,g_n) \to (M,h_n)$ is an L-Lipschitz map, which is a contradiction. □

4.1 The marked length spectra of (M, g_0) and (M, h_0)

To prove Proposition 4.2, we will first show (M, g_0) and (M, h_0) have the same marked length spectrum. Then we will construct an isometry $f\colon (M, g_0) \to (M, h_0)$. We use the same main steps as in [43]; however, since g_0 and h_0 are only of $C^{1,\alpha}$ regularity, there are additional technicalities that arise when verifying the requisite properties of the Liouville measure and Liouville current in this context.

We first recall some additional properties of the limit (M, g_0). By a theorem of Pugh [47, Theorem 1], this limiting metric will have a Lipschitz geodesic flow, and the geodesics themselves are of $C^{1,1}$ regularity. Moreover, the exponential maps converge uniformly on compact sets [47, Lemma 2], which is equivalent to the following:

Lemma 4.3 *Let ϕ_n and ϕ_0 denote the geodesic flows on (T^1M, g_n) and (T^1M, g_0), respectively. Fix $T > 0$ and let $K \subset T^1M$ compact. Then $\phi_n^t v \to \phi_0^t v$ uniformly for $(t, v) \in [0, T] \times K$.*

In addition, the space (M, g_0) is CAT$(-a^2)$ because it is a suitable limit of such spaces [8, Theorem II.3.9]. Thus, even though the curvature tensor is not defined for the $C^{1,\alpha}$ metric g_0, this limiting space still exhibits many key properties of negatively curved manifolds. One such property, heavily used in Otal's proof of marked length spectrum rigidity [43], is the fact that the angle sum of a nondegenerate geodesic triangle is strictly less than π; see [13, Lemma 12.3.1(ii)]. This still holds for CAT$(-a^2)$ spaces, essentially by definition [8, Proposition II.1.7 4].

Moreover, we can define the marked length spectrum of (M, g_0) the same way as for negatively curved manifolds. The fact that there exists a geodesic representative for each homotopy class is a general application of the Arzelà–Ascoli theorem; see [8, Proposition I.3.16]. The proof that this geodesic representative is unique in the negatively curved case immediately generalizes to the CAT$(-a^2)$ case; see [13, Lemma 12.3.3].

We will now show that (M, g_0) and (M, h_0) have the same marked length spectrum. We start with a preliminary lemma.

Lemma 4.4 *Let $\langle\gamma\rangle$ be a free homotopy class. Let γ_0 and γ_n denote the geodesic representatives with respect to g_0 and g_n, respectively. Write $\gamma_0(t) = \phi_0^t v_0$ and $\gamma_n(t) = \phi_n^t v_n$. Then for all $0 \le t \le l_{g_0}(\gamma_0)$, we have $\phi_n^t v_n \to \phi_0^t v_0$ in T^1M as $n \to \infty$.*

Proof Let $T = l_{g_0}(\gamma_0)$. By Lemma 4.3, choose n large enough so that $d(\phi_n^t v_0, \phi_0^t v_0) < \varepsilon$ for all $t \in [0, T]$. In particular, $\phi_n^T v_0$ is close to $\phi_0^T v_0 = v_0$. The Anosov closing lemma applied to the geodesic flow on $(T^1 M, g_n)$ gives $\phi_n^t v_0$ is shadowed by a closed orbit. By construction, this closed orbit is close to γ_0 and is also homotopic to it, which completes the proof. □

Proposition 4.5 *The Riemannian surfaces* (M, g_0) *and* (M, h_0) *have the same marked length spectrum.*

Proof The previous lemma, together with Lemma 4.1, implies $l_{g_n}(\gamma_n) \to l_{g_0}(\gamma_0)$ as $n \to \infty$. Let $\widetilde{\gamma}_n$ be the geodesic representatives of $\langle \gamma \rangle$ with respect to the h_n metrics. Then we also have $l_{h_n}(\widetilde{\gamma}_n) \to l_{h_0}(\widetilde{\gamma}_0)$ as $n \to \infty$. Since $\mathcal{L}_{g_n}/\mathcal{L}_{h_n} \to 1$, we obtain $l_{g_0}(\gamma_0)/l_{h_0}(\widetilde{\gamma}_0) = 1$, which completes the proof. □

4.2 Liouville current

Now that we have two surfaces with the same marked length spectrum, we will follow the method of [43] to show they are isometric. Two key tools used in Otal's proof are the Liouville current and the Liouville measure (both defined at the beginning of Section 2). In this section and the next, we will construct analogous measures for the limit (M, g_0) and show they still satisfy the properties required for Otal's proof.

Recall that the Liouville current is a Γ-invariant measure on the space of geodesics of $\widetilde{M}$; see Section 2. Recall as well the following relation between the cross-ratio and Liouville current for surfaces. Let $\xi, \xi', \eta, \eta' \in \partial \widetilde{M}$ be four distinct points. Since $\partial \widetilde{M}$ is a circle, the pair of points (ξ, ξ') determines an interval in the boundary (after fixing an orientation). Let $(\xi, \xi') \times (\eta, \eta') \in \partial^2 \widetilde{M}$ denote the geodesics starting in the interval (ξ, ξ') and ending in the interval (η, η'). Then

$$\lambda((\xi, \xi') \times (\eta, \eta')) = \tfrac{1}{2}[\xi, \xi', \eta, \eta']. \tag{4-2}$$

(See [43, Proof of Theorem 2] and [35, Theorem 4.4].)

We can use the above equation to define the Liouville current λ_0 on (M, g_0). Let λ_n denote the Liouville current with respect to the smooth metric g_n. It is then clear from Proposition 2.4 that $\lambda_n(A) \to \lambda_0(A)$ for any Borel set $A \subset \partial^2 \widetilde{M}$.

We now recall a key property of the Liouville current used in Otal's proof. We begin by defining coordinates on the space of geodesics: Fix $v \in T^1 M$ and $T > 0$, and let $t \mapsto \eta(t)$ be the geodesic segment of length T with $\eta'(0) = v$. Let $\mathcal{G}_v^T$ denote the (bi-infinite) geodesics which intersect the geodesic segment η transversally. Let $b\colon [0, T] \times (0, \pi) \to T^1 M$ be the map defined by sending (t, θ) to the unit tangent vector with footpoint $\eta(t)$ obtained by rotating $\eta'(t)$ by angle θ. We can then identify each vector $b(t, \theta)$ with a unique geodesic in $\mathcal{G}_v^T$; see [43, page 155]. (This is similar, but not the same, as our construction in Lemma 2.21 above; see Remark 2.22.)

When g is a smooth Riemannian metric on M, the Liouville current with respect to the above coordinates is of the form $\frac{1}{2} \sin\theta \, d\theta \, dt$. The same proof works for the measure λ_0 defined in terms of the $C^{1,\alpha}$ Riemannian metric g_0. To see this, let $P\colon TM \to M$ denote the footpoint map. Since g is $C^{1,\alpha}$, it has a connection of C^α regularity, so we can define the connector map κ as in Section 2.2.1. For $v \in T^1M$ and V, W, define the C^α 2-form

$$\tau_v(V, W) = \langle dPW, \kappa_v V\rangle - \langle dPV, \kappa_v W\rangle.$$

In the case of a smooth Riemannian metric, the above formula is the coordinate expression for the symplectic form $d\omega$, as in (2-12). Since the g_{ij}^n and their derivatives converge to those of g_0, this means τ is the limit of the $d\omega^n$ for the metrics g_n. Since each $d\omega^n$ is invariant under the geodesic flow ϕ_n, Lemma 4.3 implies τ is invariant under the geodesic flow g_0. Therefore, we can think of τ as a C^α 2-form on the space of geodesics, which in turn gives rise to a measure.

Lemma 4.6 *Let $b\colon [0, T] \times (0, \pi) \to T^1M$ as above. Then $b^*\tau = \sin\theta\, d\theta\, dt$.*

Proof Fix (t, θ) and let $u = b(t, \theta)$. Let $\beta_1(t)$ denote the coordinate curve $t \mapsto b(t, \theta)$. This gives a parallel vector field along η making fixed angle θ with η'. Thus if V is the vector tangent to β_1 at u, we get $\kappa_v V = 0$ and $d\pi V = \eta'(t)$. This latter vector is obtained by rotating u by angle θ, which we will denote by $\theta \cdot u$.

Next, let $\beta_2(\theta)$ denote the coordinate curve $\theta \mapsto b(t, \theta)$ and let W be the vector tangent to β_2 at u. Since $\beta_2(\theta)$ is a curve in the fiber over $\eta(t)$, which means $d\pi(W) = 0$. This curve traces out a circle in the unit tangent space, and its tangent vector is thus perpendicular to the circle. This means that $\kappa_v(W) = \frac{\pi}{2} \cdot u$.

Hence

$$\tau_{b(t,\theta)}(W, V) = \left\langle \tfrac{\pi}{2} \cdot u, \theta \cdot u\right\rangle - \langle 0, 0\rangle = \sin\theta,$$

as claimed. □

We now claim the measure on the space of geodesics coming from the symplectic form $\frac{1}{2}\tau$ is equal to the Liouville current. Indeed, this follows from [43, Theorem 2]. To show that this theorem is still true for (M, g_0), it suffices to verify that the geodesic flow ϕ_0 satisfies the specification property; see the proof of Proposition 2.6, along with Lemma 2.9 and Remark 2.10. Since, as mentioned above, (M, g_0) is a CAT$(-a^2)$ space, the desired specification property is given by [15, Theorem 3.2].

Since (M, g_0) and (M, h_0) are CAT$(-a^2)$ spaces, as in Construction 2.1 we can define a correspondence of geodesics $\phi\colon (\partial^2\widetilde{M}, g_0) \to (\partial^2\widetilde{M}, h_0)$. The following fact is still true in this context [43, page 156].

Proposition 4.7 *Let $\mathcal{G}_v \subset \partial^2\widetilde{M}$ be a coordinate chart with coordinates (t, θ), and let $\phi(\mathcal{G}_v) = \mathcal{G}_{\phi(v)}$ have coordinates (t, θ'). Then ϕ takes the measure $\sin\theta\, d\theta\, dt$ to $\sin\theta'\, d\theta'\, dt'$.*

4.3 Liouville measure

Let μ_n denote the Liouville measure on T^1M with respect to the metric g_n on M. Let g_n^S denote the associated Sasaki metric on T^1M. Then μ_n is a constant multiple of the measure arising from the

Riemannian volume form of g_n^S. In local coordinates, the measure μ_n can be written in terms of the g_n^{ij} and their first derivatives. Since $g_n^{ij} \to g_0^{ij}$ in the $C^{1,\alpha}$ norm, we see that the measures μ_n converge to a measure μ_0, which is the Riemannian volume associated to the C^α Sasaki metric g_0^S. Hence, the measure μ_0 can be written locally as the product $dm \times d\theta$, where dm is the Riemannian volume on M coming from g_0, and $d\theta$ is Lebesgue measure on the circle $T_p^1 M$.

We now recall the average change in angle function $\Theta' \colon [0,\pi] \to [0,\pi]$ from [43, Section 2]. First Otal considers the function $\theta' \colon T^1 M \times [0,\pi] \to \mathbb{R}$ defined as follows. Given a unit tangent vector v and an angle θ, let $\theta \cdot v$ denote the vector obtained by rotating v by θ. Consider lifts of the geodesics determined by v and $\theta \cdot v$ passing through the same point in $\widetilde{M}$. The correspondence of geodesics ϕ (see above Proposition 4.7 and Construction 2.1) takes intersecting geodesics to intersecting geodesics (since $\dim M = 2$). Let $\theta'(\theta, v)$ denote the angle between the image geodesics in $(\widetilde{M}, h_0)$ at their point of intersection. Finally, let $\Theta'(\theta) = \int_{T^1 M} \theta'(\theta, v)\, d\mu_0(v)$.

The function Θ' satisfies symmetry and subadditivity properties [43, Proposition 6]. Indeed, the proof of [43, Proposition 6] uses the above local product structure of the Liouville measure along with the fact that in negative curvature, the angle sum of a nondegenerate geodesic triangle is strictly less than π. As mentioned before, this latter fact holds for CAT$(-a^2)$ spaces as well [8, Proposition II.1.7.4].

To deduce the third key property of Θ' (see [43, Proposition 7] for the exact statement), we require the following fact about μ_0, which holds by [50] in the original smooth case. Since ϕ_0 is a geodesic flow on a CAT(-1) space, it satisfies a sufficiently strong specification property such that the proof of [50] works verbatim in this context; see [15, Theorem 3.2, Lemma 4.5].

Proposition 4.8 *Let $f \colon T^1 M \to \mathbb{R}$ be a continuous function. Let $\varepsilon > 0$. Then there is a closed geodesic γ_0 such that*

$$\left| \int_{T^1 M} f\, d\mu_0 - \frac{1}{l_{g_0}(\gamma_0)} \int_{\gamma_0} f\, dt \right| < \varepsilon.$$

4.4 Constructing a distance-preserving map $f \colon (M, g_0) \to (M, h_0)$

Using Propositions 4.7 and 4.8, the proof of [43, Proposition 7] shows the hypotheses of [43, Lemma 8] are satisfied. Thus, the function Θ' defined at the beginning of Section 4.3 is the identity. From this, it follows that ϕ takes triples of geodesics intersecting in a single point to triples of geodesics intersecting in a single point; see the proof of [43, Theorem 1]. We then define $f \colon (M, g_0) \to (M, h_0)$ exactly as in [43]: given $p \in \widetilde{M}$, take any two geodesics through p. Then their images under ϕ must also intersect in a single point, which we call $\tilde{f}(p)$. Then $\tilde{f}$ is distance-preserving and Γ-equivariant by the same argument as in [43].

This proves Proposition 4.2, and hence Theorem A is proved.

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