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A unified Casson–Lin invariant for the real forms of SL(2)

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We introduce a unified framework for counting representations of knot groups into SU_2 and $\mathrm{SL}_2\mathbb{R}$. For a knot K in the 3-sphere, Lin and others showed that a Casson-style count of SU_2 representations with fixed meridional holonomy recovers the signature function of K. For knots whose complement contains no closed essential surface, we show there is an analogous count for $\mathrm{SL}_2\mathbb{R}$ representations. We then prove the $\mathrm{SL}_2\mathbb{R}$ count is determined by the SU_2 count and a single integer $h(K)$, allowing us to show the existence of various $\mathrm{SL}_2\mathbb{R}$ representations using only elementary topological hypotheses.

Combined with the translation extension locus of Culler and Dunfield, we use this to prove left-orderability of many 3-manifold groups obtained by cyclic branched covers and Dehn fillings on broad classes of knots. We give further applications to the existence of real parabolic representations, including a generalization of the Riley conjecture (proved by Gordon) to alternating knots. These invariants exhibit some intriguing patterns that deserve explanation, and we include many open questions.

The close connection between SU_2 and $\mathrm{SL}_2\mathbb{R}$ comes from viewing their representations as the real points of the appropriate $\mathrm{SL}_2\mathbb{C}$ character variety. While such real loci are typically highly singular at the reducible characters that are common to both SU_2 and $\mathrm{SL}_2\mathbb{R}$, in the relevant situations we show how to resolve these real algebraic sets into smooth manifolds. We construct these resolutions using the geometric transition $S^2 \to \mathbb{E}^2 \to \mathbb{H}^2$, studied from the perspective of projective geometry, and they allow us to pass between Casson–Lin counts of SU_2 and $\mathrm{SL}_2\mathbb{R}$ representations unimpeded.

57K10, 57K31; 20F60, 53A20, 57K18, 57R58

1 Introduction

In analogy with Casson's invariant for homology 3-spheres, Lin [1992] introduced an invariant of knots in S^3. In essence, Lin's invariant is a signed count of (conjugacy classes of) irreducible representations $\rho\colon \pi_1(S^3 - K) \to \mathrm{SU}_2$ where $\operatorname{tr}\rho(\mu) = 0$ for a meridian μ of K. This was generalized in [Herald 1997b; Heusener and Kroll 1998] to the *Casson–Lin invariant* $h^c_{\mathrm{SU}_2}(K) \in \mathbb{Z}$, which counts irreducible representations $\rho\colon \pi_1(S^3 - K) \to \mathrm{SU}_2$ with $\operatorname{tr}\rho(\mu) = c$ for a fixed value of $c \in [-2, 2]$. Here, one needs to exclude those c corresponding to roots of the Alexander polynomial $\Delta_K(t)$ on the unit circle, specifically avoiding $D_K := \{a + 1/a \mid a \in \mathbb{C} \text{ and } \Delta_K(a^2) = 0\}$.

Here, we introduce a similar invariant $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$, which counts representations $\rho\colon \pi_1(S^3 - K) \to \mathrm{SL}_2\mathbb{R}$ with $\operatorname{tr}\rho(\mu) = c$. The fact that $\mathrm{SL}_2\mathbb{R}$ is noncompact introduces difficulties that we sidestep, at least for this introduction, by requiring that K be *small*, that is, $S^3 - K$ contains no closed essential surface; in particular, this implies K is prime. Following the standard approach, we show that:

1.1 Theorem *If K is a small knot in S^3, there is an integer-valued invariant $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ for each $c \in [-2, 2] \setminus D_K$. If $h^c_{\mathrm{SL}_2\mathbb{R}}(K) \neq 0$, then there is an irreducible representation $\rho\colon \pi_1(S^3 - K) \to \mathrm{SL}_2\mathbb{R}$ with $\operatorname{tr}\rho(\mu) = c$.*

Fixing the knot K, we can view $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ and $h^c_{\mathrm{SU}_2}(K)$ as functions of c which are constant on the components of $[-2,2] \setminus D_K$. The central goal of this paper is to show that these two functions are related:

1.2 Theorem *If K is a small knot in S^3, there is an integer $h(K)$ such that $h(K) = h^c_{\mathrm{SU}_2}(K) + h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ for all $c \in [-2,2] \setminus D_K$.*

The SU_2 Casson–Lin invariant is determined by the Levine–Tristram signature function $\sigma_K\colon S^1 \to \mathbb{Z}$ [Herald 1997b; Heusener and Kroll 1998], specifically

(1.3) $$h^c_{\mathrm{SU}_2}(K) = -\tfrac{1}{2}\sigma_K(\omega^2), \quad \text{where } c = \omega + \bar{\omega} \text{ is in } [-2,2] \setminus D_K.$$

Hence, by Theorem 1.2, our new $\mathrm{SL}_2\mathbb{R}$ Casson–Lin invariant is determined by the signature function and the single integer $h(K)$. As we outline in Section 1.4, the connection of $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ to the classical signature function is actually key to its usefulness. Specifically, it will allow us to prove the existence of $\mathrm{SL}_2\mathbb{R}$ representations from elementary topological hypotheses.

For many knots, the invariant $h(K)$ is determined by the Trotter–Murasugi signature $\sigma(K) := \sigma_K(-1)$; specifically, for alternating knots and Montesinos knots we show $h(K) = -\frac{1}{2}\sigma(K)$ in Corollary 16.12 and Proposition 17.2. In contrast, for the positive torus knot $K = T(p,q)$, we have $h(k) = g(K) = \frac{1}{2}(p-1)(q-1)$ by Corollary 17.9, which is greater than $-\frac{1}{2}\sigma_K$ unless $(p,q) = (2, 2n+1)$, $(3,4)$ or $(3,5)$. Many similar examples can be found by considering knots which have lens space surgeries.

1.4 Applications

Our principal motivation for studying $h(K)$ and $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ is to prove the existence of irreducible representations to $\mathrm{SL}_2\mathbb{R}$ from $\pi_1(M_K)$, where $M_K = S^3 \setminus \nu(K)$ is the knot exterior, as well as the fundamental groups of manifolds constructed from K via branched coverings or Dehn surgery. Similar questions for representations to SU_2 have been extensively studied in the context of instanton Floer homology, for example in [Akbulut and McCarthy 1990; Collin and Steer 1999; Kronheimer and Mrowka 2004; Baldwin and Sivek 2023].

In recent years, interest in representations to $\mathrm{SL}_2\mathbb{R}$ has been raised by the L-space conjecture of Boyer, Gordon and Watson [Boyer et al. 2013], which predicts that a prime 3-manifold Y is a Heegaard Floer L-space if and only if $\pi_1(Y)$ is not left-orderable. By Boyer, Rolfsen and Wiest [Boyer et al. 2005], the existence of certain representations $\pi_1(Y) \to \mathrm{SL}_2\mathbb{R}$ provides one of our most effective criteria for proving that $\pi_1(Y)$ is left-orderable; see the overview in [Culler and Dunfield 2018, Section 1.5].

A representation $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ is called *elliptic*, *parabolic* or *hyperbolic* according to whether $\rho(\mu)$ is an elliptic, parabolic or hyperbolic element of $\mathrm{SL}_2\mathbb{R}$. Our first application is a criterion for the existence of elliptic representations analogous to results of Herald and Heusener–Kroll for SU_2:

1.5 Corollary *If K is a small knot with $\sigma_K(\omega)$ nonconstant, then $\pi_1(M_K)$ admits an irreducible elliptic representation to $\mathrm{SL}_2\mathbb{R}$.*

The point here is that if $\sigma_K(\omega)$ is nonconstant, so is $h^c_{\mathrm{SU}_2}(K)$ by (1.3), and then also $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ by Theorem 1.2; hence, $h^c_{\mathrm{SL}_2\mathbb{R}}(K) \neq 0$ for some open set of c, giving the needed representation by Theorem 1.1. The complement of the figure-eight knot has no such representations, so the condition that $\sigma_K(\omega)$ is nonconstant in Corollary 1.5 is necessary. For context, recall for nontrivial K there is always an irreducible SU_2 representation, by the deep results of [Kronheimer and Mrowka 2004]. Reid has asked if the same is true for $\mathrm{SL}_2\mathbb{R}$, and Corollary 1.5 is perhaps the strongest general result in that direction.

Parabolic representations of $\pi_1(M_K)$ are of particular interest. Since $h^{\pm 2}_{\mathrm{SU}_2}(K) = 0$ by Theorem 12.22, the invariant $h(K)$ can be interpreted as a signed count of parabolic $\mathrm{SL}_2\mathbb{R}$ representations of $\pi_1(M_K)$; see Corollary 14.3. Combined with Corollary 16.12 and Proposition 17.2, we have:

1.6 Theorem *If K is a small knot with $\sigma(K) \neq 0$ that is either alternating or Montesinos, then $\pi_1(M_K)$ admits an irreducible parabolic representation to $\mathrm{SL}_2\mathbb{R}$. There are at least $|\sigma(K)|$ conjugacy classes of such representations when counted with Casson–Lin multiplicities.*

When K is a 2-bridge knot, the fact that K should have at least $\sigma(K)$ conjugacy classes of parabolic representations, without any hypothesis on multiplicities, was conjectured by Riley [1972] and recently proved by Gordon [2017]. Thus Theorem 1.6 can be naturally viewed as an extension of Riley's conjecture to alternating and Montesinos knots. In Section 17.14 we give a new proof of Gordon's result and suggest a refinement in the form of Conjecture 17.16.

Our most satisfactory results for left-orderability apply to cyclic branched covers. If $\Sigma_n(K)$ is the n-fold cyclic branched cover of K, we prove:

1.7 Theorem *If K is a small knot with $\sigma_K(\omega)$ nonconstant, then $\pi_1(\Sigma_n(K))$ is left-orderable for all sufficiently large n.*

Here, an explicit lower bound on n can be computed easily from the roots of $\Delta_K(t)$; see Theorem 16.7 and also Remark 16.9. For small knots, this answers a question raised by Boileau, Boyer and Gordon [Boileau et al. 2019], who proved a similar theorem, to the effect that large cyclic branched covers of quasipositive knots are not L-spaces.

For Dehn surgery, our results are more complicated to state, but can be described succinctly when K is a 2-bridge knot. If $\alpha = \frac{p}{q} \in \mathbb{Q}$, we write $M_K(\alpha)$ for the result of Dehn surgery with slope $p\mu + q\lambda$ on K. Then Theorem 16.22 gives:

1.8 Theorem *If K is a 2-bridge knot with $\sigma(K) \neq 0$, then $\pi_1(M_K(\alpha))$ is left-orderable either for all $\alpha \in (-\infty, 1)$ or for all $\alpha \in (-1, \infty)$.*

In the special case of double-twist knots, see the prior work of [Khoi et al. 2021; Gao 2022b]; for a family of 2-bridge knots not covered by Theorem 16.22, see [Le 2021].

If K is a 2-bridge knot other than the unknot or the trefoil, every nontrivial surgery on K results in a non-L-space, and thus should give a manifold whose fundamental group is left-orderable. Hence, Theorem 1.8 has a less satisfactory conclusion with respect to the L-space conjecture than Theorem 1.7, which can be used to settle it for all but finitely many manifolds in its family. This discrepancy primarily represents a failure of the technique of using elliptic $\mathrm{SL}_2\mathbb{R}$ representations to prove left-orderability, rather than a gap in our understanding of $\mathrm{SL}_2\mathbb{R}$ representations for these knots. If K is a 2-bridge knot with $\sigma(K) > 0$, we expect that the interval of left-orderable filling slopes can be improved to $(-\infty, \sigma(K)-1)$ using elliptic $\mathrm{SL}_2\mathbb{R}$ representations, but no further.

A similar phenomenon is evident for knots with lens space surgeries. By [Ozsváth and Szabó 2005a], if K has a positive lens space surgery, the set of L-space filling slopes of K is the interval $[2g(K)-1, \infty]$. In Section 16.24, and specifically Theorem 16.27, we use elliptic $\mathrm{SL}_2\mathbb{R}$ representations to prove that for many such knots, the set of left-orderable filling slopes contains an interval of the form $(-\infty, k)$, where k is a positive integer strictly less than $2g(K)-1$. The L-space conjecture predicts that the fillings in the interval $[k, 2g(K)-1)$ should be left-orderable as well, but it seems unlikely that we will be able to construct left-orderings there using representations to $\mathrm{SL}_2\mathbb{R}$.

Our applications to left-orderability are detailed in Section 16. We make heavy use of the translation extension locus of [Culler and Dunfield 2018], which organizes representations of $\pi_1(S^3 \setminus K)$ to the universal covering group $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of $\mathrm{SL}_2\mathbb{R}$. Indeed, that paper motivated much of our study here. Our key advance over [loc. cit.] is that $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ allows us to give concrete lower bounds on the sizes of the intervals of left-orderable branched covers/Dehn fillings, whereas in [loc. cit.] only the existence of a nonempty open interval is proved. In complete generality, for Dehn surgery such intervals can be quite small — see Figure 24 — meaning one cannot hope to improve on eg [ibid., Theorem 1.2] using only $\mathrm{SL}_2\mathbb{R}$ techniques without additional hypotheses.

Using this perspective, in Section 14.15 we refine the count $h(K) = h^2_{\mathrm{SL}_2\mathbb{R}}(K)$ of parabolic $\mathrm{SL}_2\mathbb{R}$ representations into the *extended Lin invariant* $\tilde{h}(K)$ in $\mathbb{Z}[t^{\pm 1}]$ by taking into account the "longitudinal heights" of lifted representations $\pi_1(S^3 - K) \to \widetilde{\mathrm{SL}_2\mathbb{R}}$. This invariant exhibits some interesting formal similarities to the Seiberg–Witten invariant of the knot complement. We suspect these should be explained by [Haydys 2022], relating solutions to the 2-spinor Seiberg–Witten equations to representations into $\mathrm{SL}_2\mathbb{R}$. We show in Theorem 16.21 that if $\tilde{h}(K)$ has positive degree, then one gets left-orderability for large ranges of Dehn fillings.

1.9 The character variety and the translation extension locus

We now outline the geometrical interpretations of $h^c_{\mathrm{SU}_2}(K)$ and $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$, which are best understood in terms of character varieties of representations to SU_2 and $\mathrm{SL}_2\mathbb{R}$; throughout, refer to Section 3 for precise definitions and technical background.

Recall that the SU_2 character variety of M_K is

$$X_{\mathrm{SU}_2}(M_K) := \mathrm{Hom}(\pi_1(M_K), \mathrm{SU}_2)/\sim,$$

where two representations ρ and ρ' are equivalent if they define the same character, ie $\operatorname{tr}\rho(x) = \operatorname{tr}\rho'(x)$ for all $x \in \pi_1(M_K)$; for SU_2, this is the same as saying ρ and ρ' are conjugate. Restriction gives a natural

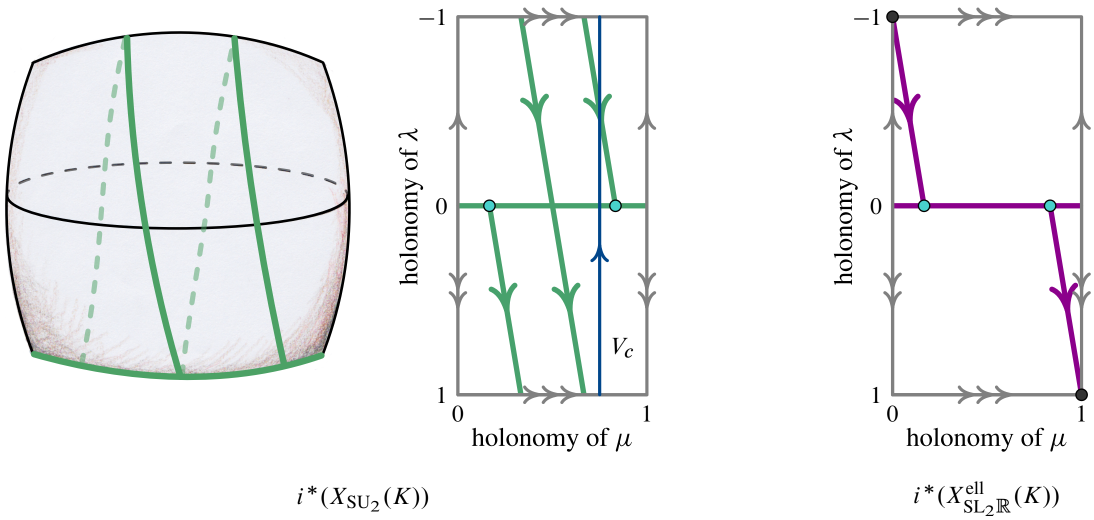


Figure 1: Left: the pillowcase orbifold $X_{\mathrm{SU}_2}(\partial M_K)$ containing the image of $X_{\mathrm{SU}_2}(M_K)$ under the restriction map, where K is the positive trefoil knot. Details are given in Section 14.4, but the horizontal and vertical coordinates on the pillowcase are the holonomies of μ and λ, where λ is the Seifert longitude. The light blue dots correspond to the roots of $\Delta_K(t)$ on the unit circle. Right: the corresponding picture for $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(M_K)$.

map

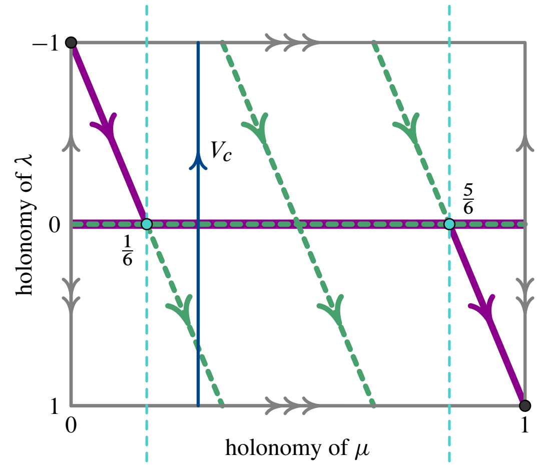
$$i^*\colon X_{\mathrm{SU}_2}(M_K) \to X_{\mathrm{SU}_2}(\partial M_K) \cong X_{\mathrm{SU}_2}(T^2).$$

The character variety $X_{\mathrm{SU}_2}(\partial M_K)$ is the pillowcase orbifold shown in Figure 1, left. The character variety $X_{\mathrm{SU}_2}(M_K)$ can be divided into the part $X^{\mathrm{red}}_{\mathrm{SU}_2}(M_K)$ coming from reducible representations, whose image is always the arc at the bottom of the pillowcase, and the part $X^{\mathrm{irr}}_{\mathrm{SU}_2}(M_K)$ coming from irreducible representations, which has expected dimension 1. If $X^{\mathrm{irr}}_{\mathrm{SU}_2}(M_K)$ is transversally cut out, it carries a natural orientation, and $h^c_{\mathrm{SU}_2}(K)$ can be interpreted [Heusener and Kroll 1998] as the intersection number between $i^*(X^{\mathrm{irr}}_{\mathrm{SU}_2}(M_K))$ and the vertical curve

$$V_c = \{[\rho] \mid \rho\colon \pi_1(\partial M_K) \to \mathrm{SU}_2 \text{ and } \operatorname{tr}\rho(\mu) = c\};$$

see Figure 1, left, where $h^c_{\mathrm{SU}_2}(K) = 1$ for the V_c shown. This orientation can also be understood in terms of Reidemeister torsion of cohomology with certain local coefficients [Dubois 2006].

The situation for $\mathrm{SL}_2\mathbb{R}$ is similar in many respects, but there are a few key differences. The part of $X_{\mathrm{SL}_2\mathbb{R}}(T^2)$ where $\pi_1(T^2)$ acts by elliptic elements is again a pillowcase, but now $X_{\mathrm{SL}_2\mathbb{R}}(T^2)$ also contains four noncompact hyperbolic components which intersect the elliptic component at the orbifold points of the pillowcase (see Section 14.4 and especially Figure 6). In this paper, we focus on the part $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(M_K)$ of $X_{\mathrm{SL}_2\mathbb{R}}(M_K)$ coming from representations whose restriction to $\pi_1(\partial M_K)$ is elliptic. This gives a very similar picture to the SU_2 case; see Figure 1, right. Provided $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(M_K)$ is transversally cut out, we will again be able to view $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ as the intersection number of $i^*(X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(M_K))$ with V_c.



σ_K	0	−2	0
$h^c_{\mathrm{SU}_2}$	0	1	0
$h^c_{\mathrm{SL}_2\mathbb{R}}$	1	0	1

Figure 2: For the positive trefoil knot K, this figure shows $i^*(X_{\mathrm{SU}_2}(M_K))$ in green (dotted lines) and $i^*(X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(M_K))$ in purple (thick solid lines) on a single pillowcase; compare Figure 1. Excluding the locus of reducible characters, which is common to both, they come together only at the points corresponding to roots of Δ_K on the unit circle. With respect to the orientations indicated, one has $h^c_{\mathrm{SU}_2}(K) = \langle i^*(X^{\mathrm{irr}}_{\mathrm{SU}_2}(M_K)), V_c\rangle$ and similarly for $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$. Hence, $h(K) = 1$ for this knot.

1.10 Sketch of the definition

As reinterpreted in [Heusener 2003], Lin's construction begins by taking a bridge diagram of K with n maxima. Splitting along the middle S^2, we can decompose $M_K = H_1 \cup_{S_{2n}} H_2$, where S_{2n} denotes the sphere with $2n$ punctures, and H_1 and H_2 are handlebodies (see Figure 5). If μ_i is a meridional loop about the i^{th} puncture, we define $X^c_{\mathrm{SU}_2}(S_{2n})$ to be the variety of characters where all μ_i have trace c. While $X^c_{\mathrm{SU}_2}(S_{2n})$ is singular at the reducible characters (a finite set), the irreducible characters $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_{2n})$ form a smooth open stratum of dimension $4n-6$. Each $L_i = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(H_i)$ is a half-dimensional subvariety of $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_{2n})$. For $c \notin D_K$, the intersection of L_1 and L_2 is compact; the invariant $h^c_{\mathrm{SU}_2}(K)$ is then defined to be the intersection number of L_1 and L_2 in $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_{2n})$. (One must of course also show that $h^c_{\mathrm{SU}_2}(K)$ does not depend on the particular bridge diagram.)

Turning to $\mathrm{SL}_2\mathbb{R}$, the difficulty of repeating this construction is that $\mathrm{SL}_2\mathbb{R}$ is not compact. However, by requiring that M_K be small (or more generally *real representation small*; see Section 3.13), the only possible noncompactness of $L_1 \cap L_2$ in the smooth stratum $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_{2n})$ is the kind already present for SU_2, namely intersections running out to some *reducible* character of $\pi_1(M_K)$. This is prevented by requiring $c \notin D_K$, just as in the SU_2 case, allowing us to define $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ for such c and so prove Theorem 1.1.

1.11 Motivation for Theorem 1.2

To motivate Theorem 1.2, recall from [Frohman and Klassen 1991] that if $c \in D_K$ corresponds to a simple root of $\Delta_K(t)$ on the unit circle, then one gets a 1-parameter family of representations $\rho_t : (-1, 1) \to \mathrm{SL}_2\mathbb{C}$ where ρ_t is an irreducible SU_2 representation for $t > 0$ and an irreducible $\mathrm{SL}_2\mathbb{R}$ representation for $t < 0$, and ρ_0 is a reducible representation with image in $\mathrm{SU}_2 \cap \mathrm{SL}_2\mathbb{R} = S^1$ with $\mathrm{tr}(\rho_0(\mu)) = c$. For simplicity, assume the parameter t actually corresponds to the trace of μ, say $\mathrm{tr}(\rho_t(\mu)) = c + t$. Then ρ_t will contribute to $h^{c+t}_{\mathrm{SU}_2}(K)$ for $t > 0$ and to $h^{c+t}_{\mathrm{SL}_2\mathbb{R}}(K)$ for $t < 0$; part of Theorem 1.2 is that these contributions are the same and hence $h^{c+t}_{\mathrm{SU}_2}(K) + h^{c+t}_{\mathrm{SL}_2\mathbb{R}}(K)$ is unchanged as t crosses 0. Figure 2 visualizes this for the trefoil by putting the SU_2 and $\mathrm{SL}_2\mathbb{R}$ pictures from Figure 6 together on the same pillowcase.

More generally, Theorem 1.2 can be viewed as a statement about representations of $\pi_1(M_K)$ transitioning from SU_2 to $\mathrm{SL}_2\mathbb{R}$ and vice versa as the trace of μ varies; any such transition must happen at a reducible representation corresponding to $c \in D_K$ (see eg Section 3.15). In particular, the starting point for Theorem 1.2 is the observation that

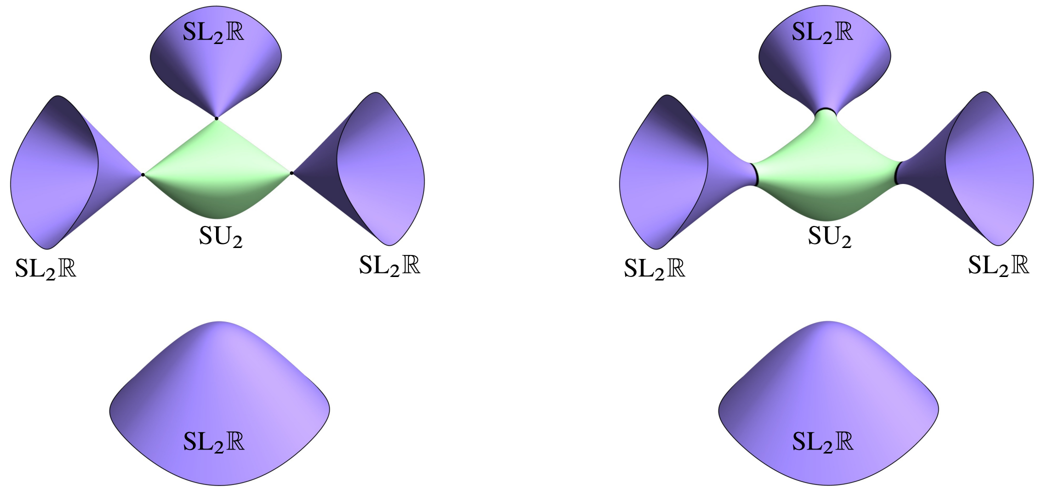
$$X^c(S_{2n}) := X^c_{\mathrm{SU}_2}(S_{2n}) \cup X^c_{\mathrm{SL}_2\mathbb{R}}(S_{2n})$$

is precisely the real points of the $\mathrm{SL}_2\mathbb{C}$ character variety $X^c_{\mathrm{SL}_2\mathbb{C}}(S_{2n})$. Moreover, the intersection of $X^c_{\mathrm{SU}_2}(S_{2n})$ and $X^c_{\mathrm{SL}_2\mathbb{R}}(S_{2n})$ is the set of reducible characters, which is finite. The key difficulty is that the reducible characters are highly singular points of $X^c(S_{2n})$, as shown in Figure 3, left, for the case $n = 2$. Thus it is very unclear how to "track" the relevant intersection numbers used to define $h^c_{\mathrm{SU}_2}(K)$ and $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ as one moves through the reducible locus.

To prove Theorem 1.2, we will resolve $X^c(S_{2n})$ to produce a smooth manifold $\mathscr{X}^c(S_{2n})$ containing half-dimensional smooth submanifolds $\mathscr{L}_1$ and $\mathscr{L}_2$ which are resolutions of the $X^c_{\mathrm{SU}_2}(H_i) \cup X^c_{\mathrm{SL}_2\mathbb{R}}(H_i)$. (See Figure 3, right, for a picture of $\mathscr{X}^c(S_{2n})$ when $n = 2$.) For any $c \in [-2, 2]$, we then define an invariant $h^c(K)$ as the intersection number between $\mathscr{L}_1$ and $\mathscr{L}_2$. When $c \notin D_K$, the intersection $\mathscr{L}_1 \cap \mathscr{L}_2$ will naturally subdivide into SU_2 and $\mathrm{SL}_2\mathbb{R}$ parts, showing $h^c(K) = h^c_{\mathrm{SU}_2}(K) + h^c_{\mathrm{SL}_2\mathbb{R}}(K)$. We then show there is a manifold $\mathscr{X}(S_{2n})$ with a submersion $\mathrm{tr}\colon \mathscr{X}(S_{2n}) \to [-2, 2]$ such that $\mathrm{tr}^{-1}(c) = \mathscr{X}^c(S_{2n})$. Moreover, $\mathscr{X}(S_{2n})$ will have submanifolds $\mathscr{L}_i$ such that $\mathscr{L}_i \cap \mathscr{X}^c(S_{2n}) = \mathscr{L}^c_i$. Then continuity of intersection numbers, in the form of Theorem 2.10, ensures that $h^c(K)$ is independent of c. This will prove the existence of $h(K)$, at least for a fixed bridge diagram.

1.12 Resolutions of real points of character varieties

Next, we discuss the general setting for building $\mathscr{X}^c(S_{2n})$, which we call a *resolution* of $X^c(S_{2n})$. For ease of exposition, we first discuss the corresponding resolution in the case of the free group $F_n = \langle s_1, \dots, s_n \rangle$ and restrict to $c \in (-2, 2)$. Let $X^c_{\mathrm{SL}_2\mathbb{C}}(F_n)$ be the character variety of $\mathrm{SL}_2\mathbb{C}$ representations ρ of F_n where all $\rho(s_i)$ have trace c. Its real points $X^c(F_n)$ are again the union of $X^c_{\mathrm{SU}_2}(F_n)$ and $X^c_{\mathrm{SL}_2\mathbb{R}}(F_n)$, which meet in exactly $X^{c,\mathrm{red}}(F_n)$. (Here, $X^c(F_n)$ is a real algebraic set with $X^c_{\mathrm{SU}_2}(F_n)$ and $X^c_{\mathrm{SL}_2\mathbb{R}}(F_n)$ real semialgebraic subsets.) It turns out both $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ and $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$ are smooth manifolds of dimension



$X^c(S_4) = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_4) \cup X^{c,\mathrm{red}}(S_4) \cup X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_4)$ $\qquad$ $\mathscr{X}^c(S_4) = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_4) \cup \mathscr{X}^{c,\mathrm{red}}(S_4) \cup X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_4)$

Figure 3: Left: $X^c(S_4)$ for $c = 2\cos\frac{2\pi}{5}$, drawn using the equations from [Benedetto and Goldman 1999]. Topologically, $X_{\mathrm{SU}_2}(S_4)$ is a 2-sphere, whereas $X_{\mathrm{SL}_2\mathbb{R}}(S_4)$ has four distinct components, each of which is a plane. The intersection $X^c_{\mathrm{SU}_2}(S_4) \cap X^c_{\mathrm{SL}_2\mathbb{R}}(S_4) = X^{\mathrm{red}}(S_4)$ is just three points, indicated by small dots. The bottom component of $X_{\mathrm{SL}_2\mathbb{R}}(S_4)$ corresponds to the Teichmüller space of hyperbolic structures on the orbifold with underlying space S^2 and four points labeled $\mathbb{Z}/5$. Right: the resolution $\mathscr{X}^c(S_n)$ where each of the three points in $X^{\mathrm{red}}(S_4)$ has been replaced by a circle.

$2n-3$, whereas $X^{c,\mathrm{red}}(F_n)$ is just 2^{n-1} points. Our resolution of $X^c(F_n)$ from Theorem 8.1 has the following properties, many of which can be visualized by comparing the left and right parts of Figure 3:

(1) The resolution is a smooth manifold $\mathscr{X}^c(F_n)$ equipped with a smooth surjection $\pi : \mathscr{X}^c(F_n) \to X(F_n)$. We take $\mathscr{X}^{c,\mathrm{red}}(F_n)$ and $\mathscr{X}^{c,\mathrm{irr}}(F_n)$ to be the preimages under π of $X^{c,\mathrm{red}}(F_n)$ and $X^{c,\mathrm{irr}}(F_n)$, respectively.

(2) The subset $\mathscr{X}^{c,\mathrm{red}}(F_n)$ is a smooth submanifold of $\mathscr{X}^c(F_n)$ of (real) codimension 1 and $\mathscr{X}^{c,\mathrm{irr}}(F_n)$ is an open submanifold.

(3) The map π restricts to a diffeomorphism between the open sets $\mathscr{X}^{c,\mathrm{irr}}(F_n)$ and $X^{c,\mathrm{irr}}(F_n)$. Each connected component of $\mathscr{X}^{c,\mathrm{irr}}(F_n)$ maps diffeomorphically under π to a connected component of either $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ or $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$.

(4) The subset $\mathscr{X}^{c,\mathrm{red}}(F_n)$ can be naturally identified with the "character variety" of representations to $U_0 = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & \bar{a} \end{smallmatrix} \right) \mid a, b \in \mathbb{C} \text{ with } |a| = 1 \right\}$ that are not conjugate to a representation with diagonal image.

(5) For an automorphism σ of F_n coming from the braid group, consider the induced automorphism σ^* of $X^c(F_n)$. There is a unique diffeomorphism $\tilde{\sigma}^*$ of $\mathscr{X}^c(F_n)$ which is compatible with σ^* in that sense that $\sigma^* \circ \pi = \pi \circ \tilde{\sigma}^*$.

In (4), we are replacing each character $\chi_0 \in X^{\mathrm{red}}(F_n)$ with something made out of reducible $\mathrm{SL}_2\mathbb{C}$ representations with that character. However, we should point out that U_0 is not conjugate into either

SU_2 or $\mathrm{SL}_2\mathbb{R}$. Indeed, any representation ρ from F_n into SU_2 or $\mathrm{SL}_2\mathbb{R}$ with character χ_0 is will in fact be diagonalizable in $\mathrm{SL}_2\mathbb{C}$. Still, the appearance of U_0 here is very natural from another vantage point: back in the setting of a knot K, de Rham showed the roots of Δ_K on the unit circle characterize the c where there are representations $\rho\colon \pi_1(M_K) \to U_0$ with $\operatorname{tr} \rho(\mu) = c$ that are not diagonalizable.

1.13 Geometric transition via projective geometry

The work of Frohman and Klassen [1991] mentioned in Section 1.11 starts from the fact that, after modding out by $\pm I$, the groups SU_2, U_0 and $\mathrm{SL}_2\mathbb{R}$ are the orientation-preserving isometry groups of the round sphere S^2, the Euclidean plane $\mathbb{E}^2$ and the hyperbolic plane $\mathbb{H}^2$. Their construction of the arc of characters, half in $X_{\mathrm{SU}_2}(K)$ and half in $X_{\mathrm{SL}_2\mathbb{R}}(K)$, uses a 1-parameter family of metrics on the plane that go from positive curvature to zero curvature to negative curvature. When the root of Δ_K is simple, they show how a nondiagonalizable representation $\rho\colon \pi_1(M_K) \to \mathrm{Isom}^+(\mathbb{E}^2)$ can be deformed into isometries of these nearby metrics.

To build the resolution $\mathscr{X}^c(F_n)$, we follow the lead of [loc. cit.] and consider the geometric transition $S^2 \to \mathbb{E}^2 \to \mathbb{H}^2$, which we study via projective geometry using the perspective of [Cooper et al. 2018, Section 2.1]. Because we are restricting to representations of F_n where the generators go to elliptic elements of a fixed trace, a relevant representation into SU_2, U_0, or $\mathrm{SL}_2\mathbb{R}$ is largely determined by the fixed points of the generators in their action on S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$. This leads us to study configuration spaces of n points in each of these three geometries in Section 5, where the points are taken modulo isometry (for S^2 and $\mathbb{H}^2$) or similarity (for $\mathbb{E}^2$). Here, we require that not all n points are the same, and denote the resulting configuration spaces by $\mathscr{C}_n(S^2)$, $\mathscr{C}_n(\mathbb{E}^2)$ and $\mathscr{C}_n(\mathbb{H}^2)$, which have dimensions $2n-3$, $2n-4$ and $2n-3$, respectively.

We now sketch a natural way of combining these configuration spaces together into a single smooth manifold. To begin, we consider $\mathscr{C}_n(\mathbb{H}^2)$, which has two ends: one where all the points coalesce and the other where their diameter goes to infinity. We focus on the former end and leave the other alone. Given an element in $\mathscr{C}_n(\mathbb{H}^2)$, we can rescale the metric on $\mathbb{H}^2$ so that the diameter of the set of points is exactly 1 in the new metric. The closer the original points are in $\mathbb{H}^2$, the flatter the rescaled metric is. When the original points are very close, the new space is nearly isometric to $\mathbb{E}^2$ on the scale of the points in the final configuration. This makes it plausible that we should compactify this end of $\mathscr{C}_n(\mathbb{H}^2)$ by adding a copy of $\mathscr{C}_n(\mathbb{E}^2)$ at infinity to produce a manifold with boundary. Looking now at the end of $\mathscr{C}_n(S^2)$ where all the points come together, the exact same story applies to suggest that we should also compactify this end by adding a copy of $\mathscr{C}_n(\mathbb{E}^2)$. We could then glue our two compactifications together to get a nice manifold structure on $\mathscr{C}_n(\mathbb{H}^2) \cup \mathscr{C}_n(\mathbb{E}^2) \cup \mathscr{C}_n(S^2)$.

To resolve $\mathscr{X}^c(F_n)$, the correct object to look at involves "signed points" to account for the rotation directions of the generators at their fixed points. We assemble the relevant configuration spaces into a smooth manifold $\mathscr{Y}$ in Theorem 5.9, and then use it to build $\mathscr{X}^c(F_n)$ in Theorem 8.1. The generalizations to $X(S_n)$ and to allowing c to vary are too involved for this introduction but are tackled in Sections 9 and 10.

1.14 Open problems

Our work here raises many questions, both general and specific. General problems include:

(1) Can the definition of $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ be extended to all knots, not just those which are real representation small? Naively, could one simply ignore any noncompact components of the intersection? Alternatively, could the sheaf-theoretic perspective of [Abouzaid and Manolescu 2020] in the case of $\mathrm{SL}_2\mathbb{C}$ be adapted to our real-algebraic setting? Given its connection to the signature, one expects that $h(K_1 \# K_2) = h(K_1) + h(K_2)$. However, the connected sum of nontrivial knots is typically not real representation small, meaning that we cannot make sense of $h(K_1 \# K_2)$. More generally, it would be nice to be able to define and compute h for satellite knots.

(2) Can we count hyperbolic representations with fixed trace and so extend the definition of $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ to all $c \in \mathbb{R} \setminus D_K$? Assuming this can be done, can $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ jump as we move through $c \in D_K$ for $|c| > 2$? If it does not, we would get interesting consequences about ideal points of $X_{\mathrm{SL}_2\mathbb{R}}(K)$ in terms of $h(K)$. A weak version of such an argument is provided by Corollary 15.13, which counts the number of such points modulo 2.

(3) Can our theory be extended to knots in other closed 3-manifolds? One approach would be to use bridge diagrams in Heegaard splittings and (g, n) knots. Here, the bridge presentations we use are $(0, n)$ knots and the doubly pointed Heegaard diagrams from [Ozsváth and Szabó 2004] are the $(g, 1)$ case. The first step would be to understand how to resolve the real points of the relevant character variety of a surface of genus g with $2n$ punctures. Our approach here uses that representations of $\pi_1(S_{2n})$ have image generated by elliptic elements, which is no longer the case in higher genus. This more general perspective might also help us understand the behavior of h for Berge knots and other $(1, 1)$ knots.

(4) Is it possible to give a more intrinsic definition of $h^c_{\mathrm{SL}_2\mathbb{R}}$ by working in the space of all $\mathrm{SL}_2\mathbb{R}$ connections, as in Herald's treatment [1997b] of the SU_2 Casson–Lin invariant? If this could be done, it might provide a way to extend h to knots in general 3-manifolds. What would a proof of Theorem 1.2 look like in this context?

(5) The translation extension locus of [Culler and Dunfield 2018] shares some interesting similarities with the moduli space of solutions to the Seiberg–Witten equations on the knot complement equipped with a cylindrical end. In this setting, the extended Lin invariant $\tilde{h}(K)$ is analogous to the Seiberg–Witten invariant. Haydys [2022] relates solutions to the 2-spinor Seiberg–Witten equations on a manifold Y to the space of $\mathrm{SL}_2\mathbb{R}$ connections on Y. Can these similarities be explained by this? Conjectures 17.11 and 17.16, which concern $\tilde{h}(K)$ for Berge knots and 2-bridge knots, were originally formulated with this analogy in mind.

(6) The proof of Theorem 1.8 relies on the fact that if K is 2-bridge, then every parabolic $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ lifts to $\tilde{\rho}\colon \pi_1(M_K) \to \widetilde{\mathrm{SL}_2\mathbb{R}}$ where $\rho(\lambda)$ has nonzero translation number. Are there other interesting classes of knots for which this statement holds? This would enable one to prove results similar to

Theorem 16.22 for such knots. It is conceivable that alternating or Montesinos knots have this property; see Remark 16.23.

Among the specific problems, we point out Conjectures 17.11 and 17.16 which describe the expected structure of the enhanced Lin invariant for Berge knots and 2-bridge knots. Other specific problems include those of Section 9.26, Remark 15.9, Question 15.14, Remark 16.23, Remark 16.28 and Question 17.12.

Plan of the paper

The first half of the paper is devoted to the definition of h and the proof of Theorem 1.2. Sections 2–4 contain background material. Section 2 gives our conventions for smooth manifolds and orientations, and collects some basic results about intersection numbers that are needed for the main theorems. Section 3 gives background on representations into $\mathrm{SL}_2\mathbb{C}$ and its subgroups SU_2 and $\mathrm{SL}_2\mathbb{R}$, their associated character varieties, and various standard or easy lemmas. Section 4 discusses representations into $\widetilde{\mathrm{SL}_2\mathbb{R}}$ and shows there is a "character variety" in this context (Theorem 4.3).

Sections 5–7 discuss configurations of points in S^2, $\mathbb{E}^2$ and $\mathbb{H}^2$, and how they fit together to form a space $\mathscr{Y}$ in Theorem 5.9. Section 8 then uses the space $\mathscr{Y}$ to build the resolution of $X^c(F_n, S)$ for $c \in (-2, 2)$ in Theorem 8.1. Section 9 builds the corresponding resolution for the punctured sphere S_{2n} in Theorem 9.2. Section 10 contains Theorems 10.1 and 10.2, which build the "parametrized" resolutions where c is allowed to vary. Section 11 studies orientations on what has been constructed so far. Section 12 shows the functoriality of $\mathscr{X}^c$ with respect to automorphisms coming from the braid group, and uses this to define $h^c_{\mathrm{SU}_2}$, $h^c_{\mathrm{SL}_2\mathbb{R}}$, and h^c for the plat closure of a braid. For a fixed braid, Theorem 12.21 shows that h^c is independent of c and Theorem 12.22 proves that $h = h^c_{\mathrm{SU}_2} + h^c_{\mathrm{SL}_2\mathbb{R}}$. Section 13 shows that $h^c_{\mathrm{SU}_2}$, $h^c_{\mathrm{SL}_2\mathbb{R}}$ and h^c do not depend on choice of plat diagram, completing the proofs of Theorems 1.1 and 1.2.

The second half of the paper is focused on the applications outlined in Section 1.4. Section 14 refines what h says about parabolic representations and begins to study how it relates to $\widetilde{\mathrm{SL}_2\mathbb{R}}$ representations and the translation extension locus of [Culler and Dunfield 2018]; this leads to Definition 14.17 for the extended Lin invariant $\tilde{h}(K)$. Section 15 establishes further properties of $h(K)$ and $\tilde{h}(K)$, including their behavior under mirroring. Therein, Section 15.3 pins down the connection between our version of $h^c_{\mathrm{SU}_2}(K)$ and that in [Heusener 2003], formally proving (1.3) as Corollary 15.5. Section 16 computes $h(K)$ for alternating knots and gives the applications to left-orderability. Finally, Section 17 computes h for Montesinos knots and torus knots, and gives some conjectures about the extended Lin invariant for Berge knots and 2-bridge knots.

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2 Background on orientations and smooth topology

2.1 Manifolds

Our terminology for smooth manifolds is as follows. A *smooth embedding* $S \to M$ of smooth manifolds S and M is a smooth immersion that is a homeomorphism onto its image. The image of a smooth embedding is called a *smooth submanifold*. When S has boundary, we insist ∂S is contained in ∂M with S transverse to ∂M; we refer to the situation when ∂S is contained in the interior of M as an *embedded submanifold with boundary*. A smooth submanifold S is *closed* when it is a closed subset of M; this is equivalent to the inclusion $S \to M$ being a proper map.

We will need the following basic fact starting in Section 9:

2.2 Lemma *Suppose $\pi\colon M \to N$ is a surjective submersion of smooth manifolds without boundary. If $S \subset M$ is a closed submanifold that is a union of fibers of π, then $\pi(S)$ is a closed submanifold of N of the same codimension as S.*

Proof By [Lee 2013, Proposition 4.28], the map π is a topological quotient map, and hence $\pi(S)$ is closed, and so it will be a closed submanifold if it is a submanifold at all. Set $m = \dim M$ and $n = \dim N$, and let k be the codimension of S. To see that $\pi(S)$ is an embedded submanifold of the claimed codimension, we apply the local slice criterion of [ibid., Theorem 5.8]: it is enough to show that given $s_0 \in S$, we can find a chart U on N containing $\pi(s_0)$ such that $\pi(S) \cap U$ is a local slice of the form $\mathbb{R}^{n-k} \times (0, \dots, 0)$ in $U \cong \mathbb{R}^n$.

Choose local charts $V \cong \mathbb{R}^m$ on M and $U \cong \mathbb{R}^n$ on N so that $s_0 = 0$ in V, the image $\pi(V)$ is U, and the map $\pi\colon V \to U$ is projection onto the first n coordinates. Let $S' = S \cap V$. Since $\pi(V) = U$ and S is a union of fibers of π, we have $\pi(S) \cap U = \pi(S')$. As S has codimension k, we can shrink our charts so that additionally there is a submersion $f\colon \mathbb{R}^m \to \mathbb{R}^k$ with $S' = f^{-1}(0)$. Since S' contains $0 \times \mathbb{R}^{m-n}$, the last $m-n$ columns of $D_0 f$ are 0, and hence the initial n columns of $D_0 f$ have rank k. Thus, after zooming in if necessary, the restriction of f to $\mathbb{R}^n \times (0, \dots, 0)$ is also a submersion. In particular, $H = S' \cap (\mathbb{R}^n \times (0, \dots, 0))$ is a smooth embedded submanifold of $\mathbb{R}^n \times (0, \dots, 0)$, and, as π is coordinate projection, $S' = H \times \mathbb{R}^{m-n}$. As π is a local diffeomorphism onto U when restricted to $\mathbb{R}^n \times (0, \dots, 0)$, we have that $\pi(S')$ is contained in a local slice near $\pi(s_0)$, as needed. □

2.3 Orientations

A short exact sequence of oriented vector spaces

$$0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$$

is *compatible* when, given a positively oriented basis $\langle a_1, \ldots, a_n \rangle$ of A and a positively oriented basis $\langle \pi(b_1), \ldots, \pi(b_m) \rangle$ of C, the basis $\langle b_1, \ldots, b_m, i(a_1), \ldots, i(a_n) \rangle$ for B is also positively oriented. For such a sequence, orientations on any two of A, B and C determine a unique orientation on the third that makes the sequence compatible. With this convention, the standard orientation on the direct sum $A \oplus B$ is the one compatible with the sequence $0 \to B \to A \oplus B \to A \to 0$.

When $E \xrightarrow{\pi} B$ is a smooth submersion of oriented manifolds, the fibers $\pi^{-1}(b)$ can be *compatibly oriented* by the requirement that for all $e \in E$ the sequence

$$0 \to T_e F_e \xrightarrow{di} T_e E \xrightarrow{d\pi} T_b B \to 0$$

be compatibly oriented, where $b = \pi(e)$ and $F_e = \pi^{-1}(b)$. In the special case of a smooth fiber bundle of oriented manifolds $F \to E \xrightarrow{\pi} B$ with structure group $\mathrm{Diff}^+(F)$, orientations on any two of F, E and B determine a unique orientation on the third that makes the orientations compatible with respect to π. Finally, if $M \subset M'$ is an embedding of smooth oriented manifolds, the normal bundle $\nu_{M'/M}$ is oriented by the requirement that the short exact sequence

$$0 \to TM \to TM' \to \nu_{M'/M} \to 0$$

be compatibly oriented.

We orient the boundary of a manifold via the "outwards normal first" convention of [Lee 2013, Chapter 15]. Thus, if $M = \{x \in \mathbb{R}^n \mid x_1 \le 0\}$ and $\langle e_1, e_2, \ldots, e_n \rangle$ gives the preferred orientation of M, then an oriented basis for ∂M is $\langle e_2, \ldots, e_n \rangle$.

2.4 Remark Our orientation conventions for both short exact sequences and ∂M differ from [Heusener 2003]: we use (base) $\oplus$ (fiber) and "outwards normal first" and whereas he uses (fiber) $\oplus$ (base) and "inwards normal last".

The rest of this section is not used until Section 12 or later, so you will want to skip ahead to Section 3 at first reading. If V is an oriented vector space, we write $-V$ for the vector space with the opposite orientation.

2.5 Lemma *Suppose the following diagram of oriented finite-dimensional vector spaces commutes*:

$$\begin{array}{ccccc} V_{11} & \longrightarrow & V_{12} & \longrightarrow & V_{13} \\ \downarrow & & \downarrow & & \downarrow \\ V_{21} & \longrightarrow & V_{22} & \longrightarrow & V_{23} \\ \downarrow & & \downarrow & & \downarrow \\ V_{31} & \longrightarrow & V_{32} & \longrightarrow & V_{33} \end{array}$$

If all of the rows and the leftmost two columns are short exact, then the rightmost column is short exact as well. If additionally they are compatibly oriented, the rightmost column is also compatibly oriented if and only if $(\dim V_{13})(\dim V_{31})$ *is even.*

Proof The rightmost column is short exact by the (3×3)-lemma, so it remains to puzzle over the orientations. With our conventions, the middle term can be written as an oriented direct sum in two ways:

$$\begin{aligned} V_{22} &= V_{23} \oplus V_{21} = V_{23} \oplus (V_{31} \oplus V_{11}) \\ &= V_{32} \oplus V_{12} = (V_{33} \oplus V_{31}) \oplus (V_{13} \oplus V_{11}) \\ &= (-1)^{(\dim V_{13})(\dim V_{31})} V_{33} \oplus V_{13} \oplus V_{31} \oplus V_{11}. \end{aligned}$$

Canceling the $V_{31} \oplus V_{11}$ from the ends of the first and last lines gives

$$V_{23} = (-1)^{(\dim V_{13})(\dim V_{31})} V_{33} \oplus V_{13},$$

proving the lemma. □

2.6 Corollary *Suppose* $M \subset M'$ *and* $X \subset X'$ *are inclusions of smooth oriented manifolds and that* $f\colon M' \to X'$ *is a smooth submersion that restricts to a smooth submersion on* M*. Let* $N' = f^{-1}(X)$ *and* $N = N' \cap M$*. The submersions* $f\colon M' \to X'$ *and* $f\colon M \to X'$ *determine orientations on* N' *and* N*, respectively, and we have* $(\nu_{M'/M})|_N \cong (-1)^e \nu_{N'/N}$*, where* $e = (\operatorname{codim}_{N'} N)(\operatorname{codim}_{X'} X)$*.*

Proof We have a commuting diagram of vector bundles on N

$$\begin{array}{ccccc} TN & \longrightarrow & TN' & \longrightarrow & \nu_{N'/N} \\ \downarrow & & \downarrow & & \downarrow \\ TM & \longrightarrow & TM' & \longrightarrow & \nu_{M'/M} \\ \downarrow & & \downarrow & & \downarrow \\ \nu_{X'/X} & \longrightarrow & \nu_{X'/X} & \longrightarrow & 0 \end{array}$$

to which Lemma 2.5 applies, proving our claim. □

A very similar argument can be used to prove:

2.7 Corollary *Suppose that* $Y \subset Y'$ *is an embedding of smooth oriented manifolds. Suppose* G *is an oriented connected Lie group acting freely and properly on* Y'*, where this action leaves* Y *invariant. Then* $X = Y/G$ *and* $X' = Y'/G$ *inherit orientations from the orientations on* G*,* Y*, and* Y'*. The action of* G *on* Y *extends to an action on* $\nu_{Y'/Y}$*, giving* $\nu_{X'/X} \cong (\nu_{Y'/Y})/G$ *as oriented vector bundles.*

2.8 Intersection numbers

The invariants of Theorems 1.1 and 1.2 are defined as intersection numbers of certain pairs of submanifolds. In this final subsection, we record the basic facts we will use starting in Section 12; they are well known, but we could not find a good reference.

2.9 Theorem *Let M be a smooth oriented $2n$-manifold without boundary, possibly noncompact. Suppose A and B are oriented closed submanifolds of M of dimension n. If $A \cap B$ is compact, then there exists a compactly supported ambient isotopy of A to A' such that A' and B intersect transversely. The algebraic intersection number of A' and B is independent of the choice of such isotopy, and will be denoted by $\langle A, B\rangle_M$.*

Proof First, we construct the claimed ambient isotopy. Let $f\colon A \to M$ be the inclusion map. As f is proper, we can choose a compact n-dimensional submanifold $V \subset A$ with boundary containing $f^{-1}(B)$ in its interior. Then f is trivially transverse to B on the closed set $A \setminus \operatorname{int}(V)$. By [Wall 2016, Proposition 4.5.7], there is an arbitrarily small perturbation of f to a smooth f' transverse to B where $f = f'$ outside $\operatorname{int}(V)$. By [ibid., Proposition 4.4.4], we can arrange that $f'|_V$ is also an embedding that is isotopic to $f|_V$ via an isotopy that is constant on ∂V. (Here, you have to go back to the proof of [ibid., Corollary 2.2.5] to see that the isotopy is constant where f and f' agree.) Applying [ibid., Theorem 2.4.2] to the isotopy from f to f', we get a compactly supported ambient isotopy of f to f', as required.

Because our isotopies are compactly supported, the proof that $\langle A, B\rangle_M$ is well defined is essentially the same as when M is compact. Specifically, consider an isotopy of embeddings $F\colon A \times I \to M$ where F_0 and F_1 are transverse to B that is constant outside a compact set $K \subset A$ containing $F^{-1}(B)$. We can use [ibid., Proposition 4.5.7] to perturb F to a map F' that agrees with F on both $A \times \{0,1\}$ and $A \setminus K$ and is transverse to B. Then $(F')^{-1}(B)$ is a closed 1-dimensional submanifold of $A \times I$ contained in the compact set $K \times I$. Therefore $(F')^{-1}(B)$ is a finite union of oriented circles and arcs with endpoints in $A \times \{0,1\}$, and the usual argument from eg [Guillemin and Pollack 1974, Chapter 3] shows that $\langle F_0(A), B\rangle_M = \langle F_1(A), B\rangle_M$, as needed. □

2.10 Theorem *Let M be a smooth oriented $(2n+1)$-manifold with boundary, possibly noncompact, with $\pi\colon M \to I$ a submersion where $\partial M = \pi^{-1}(\partial I)$. For $t \in I$, let M_t be the closed submanifold $\pi^{-1}(t)$. Suppose that A and B are closed oriented submanifolds of M of dimension $n+1$, that both $\pi|_A$ and $\pi|_B$ are submersions, and that $A \cap B$ is compact. Consider the n-dimensional submanifolds $A_t = A \cap M_t$ and $B_t = B \cap M_t$. Then the intersection number $\langle A_t, B_t\rangle_{M_t}$ is independent of $t \in I$.*

Proof It suffices to show that $\langle A_0, B_0\rangle_{M_0} = \langle A_1, B_1\rangle_{M_1}$ where $I = [0,1]$, since any case can be reduced to this one by replacing M with $\pi^{-1}([a,b])$ and reparametrizing $[a,b]$ by $[0,1]$.

Applying Theorem 2.9 to A_0 and B_0 in M_0, we get a compactly supported isotopy of M_0 that takes A_0 to an $A_0' \subset M_0$ that is transverse to B_0. We also have an analogous isotopy of M_1 taking A_1 to some A_1' transverse to B_1. Since $\partial M = M_0 \cup M_1$, we can use the collar structure near ∂M to extend these isotopies to a compactly supported isotopy of M that takes A to some A' such that A_0' and A_1' are as previously constructed. As embedded submanifolds, both A' and B are transverse to ∂M and hence the inclusion map $f'\colon A' \to M$ is transverse to B near $\partial A' = A'_0 \cup A'_1$. Now f' is proper and $A' \cap B$ is compact, so we can choose an open set $U \subset A'$ with compact closure containing $(f')^{-1}(B)$. As f' is transverse to B on $\partial A' \cup (A' \setminus U)$, by [Wall 2016, Proposition 4.5.7] we can perturb f' to f'' which is transverse to B without changing the values on $\partial A' \cup (A' \setminus U)$. Thus $(f'')^{-1}(B)$ is a closed submanifold of A' of codimension n which is contained inside the compact set $\overline{U}$, that is, a finite union of circles and arcs with endpoints in $\partial A'$. This 1-manifold comes with an orientation which we can use in the standard way to conclude that $\langle A'_0, B_0\rangle_{M_0} = \langle A'_1, B_1\rangle_{M_1}$, as needed. □

In the setting of Theorem 2.9, suppose that, in addition to being compact, the intersection $A \cap B$ has only finitely many connected components. Now, connected components are always closed, and since $A \cap B$ has only finitely many such, each connected component Z of $A \cap B$ is also open in $A \cap B$. Thus we can find an open $U \subset M$ with $U \cap A \cap B = Z$. In this situation, there is a local intersection number $\langle A, B\rangle|_Z := \langle A \cap U, B \cap U\rangle_U$ that is independent of the choice of U. Note that

(2.11)
$$\langle A, B\rangle = \sum_Z \langle A, B\rangle|_Z,$$

where the sum runs over the connected components Z of $A \cap B$.

Now suppose that we are in the situation of Theorem 2.10, where moreover $A \cap B$ has finitely many connected components. If W is a connected component of $A \cap B$, pick an open neighborhood U of it with $U \cap A \cap B = W$. We then have a local intersection number for each $t \in I$ given by $\langle A_t, B_t\rangle|_W := \langle A_t \cap U, B_t \cap U\rangle_{U_t}$, which is independent of U. Applying Theorem 2.10 to $(U, A \cap U, B \cap U)$ and using (2.11) gives:

2.12 Corollary *The intersection number $\langle A_t, B_t\rangle|_W$ is independent of t. If each $A_t \cap B_t$ has finitely many connected components, then, for all t,*

$$\langle A_t, B_t\rangle|_W = \sum_{Z \subset W} \langle A_t, B_t\rangle|_Z,$$

where the sum is over the connected components of $A_t \cap B_t$ contained in W.

Finally, recall that smooth submanifolds X and Y of M intersect *cleanly* when $X \cap Y$ is a smooth submanifold with $T_x(X \cap Y) = T_x X \cap T_x Y$ for all $x \in X \cap Y$. This is a weaker notion than intersecting transversely; for example, all pairs of affine subspaces in $\mathbb{R}^n$ intersect cleanly.

2.13 Lemma *Let M be a smooth oriented submanifold of M' with $A', B' \subset M'$ smooth oriented submanifolds, each of which intersects cleanly with M. Suppose $A' \cap B' \subset M$ and let $A = A' \cap M$, $B = B' \cap M$. Assume further that there are oriented bundles V_A, V_B on M such that $\nu_{A'/A} \cong V_A|_A$, $\nu_{B'/B} \cong V_B|_B$ and $\nu_{M'/M} = V_A \oplus V_B$ as oriented vector bundles. If Z is a compact component of $A \cap B = A' \cap B'$ which is an open subset of $A \cap B$, then*

$$(\langle A', B'\rangle_{M'})|_Z = (-1)^{(\dim A)(\dim V_B)} (\langle A, B\rangle_M)|_Z.$$

Proof If A and B are transverse at x as submanifolds of M, then A' and B' are transverse at x as submanifolds of M', and we are just comparing the oriented vector space

$$T_x A' \oplus T_x B' = (V_A|_x \oplus T_x A) \oplus (V_B|_x \oplus T_x B)$$

with $\nu_{M'/M} \oplus T_x M = (V_A|_x \oplus V_B|_x) \oplus (T_x A \oplus T_x B)$.

In general, the assumption that A' and B' intersect cleanly with M means that, after restricting to a tubular neighborhood of M, we can assume that $M' = \nu_{M'/M}$, $A' = V_A|_A$ and $B' = V_B|_B$. Suppose f_t is an isotopy of A such that $f_1(A)$ is transverse to B. By choosing a connection on V_A, we can extend f_t to an isotopy F_t of A' with the property that $F_t(V_A|_x) = V_A|_{f_t(x)}$. Then $F_1(A')$ is transverse to B' and $F_1(A') \cap B' = f_1(A) \cap B$, so the statement follows from the transverse case. □

3 Background on character varieties

In this section, we establish notation for the character varieties we will consider and recall some of their basic properties; for further background, see [Culler and Shalen 1983, Section 1.4]. Beyond their mention in the introduction, the contents of this section are not used until Section 8. Let Γ be a finitely generated group. The representation variety $R_{\mathbb{C}}(\Gamma)$ is $\mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C})$ viewed as an affine algebraic set over $\mathbb{C}$. The group $\mathrm{SL}_2\mathbb{C}$ acts on $R_{\mathbb{C}}(\Gamma)$ by conjugation, and the character variety $X_{\mathbb{C}}(\Gamma)$ is the geometric invariant theory quotient of $R_{\mathbb{C}}(\Gamma)$ by this action. For each $g \in \Gamma$, there is a regular function $\mathrm{tr}_g \colon X_{\mathbb{C}}(\Gamma) \to \mathbb{C}$ defined by $[\rho] \mapsto \mathrm{tr}(\rho(g))$. The character variety $X_{\mathbb{C}}(\Gamma)$ is also an affine algebraic set over $\mathbb{C}$; concretely, there is a finite set of $g_i \in \Gamma$ such that the tr_{g_i} give global coordinates for $X_{\mathbb{C}}(\Gamma)$.

We will be interested in representations where a distinguished finite subset $S = \{s_1, \dots, s_n\}$ of elements of Γ share a common conjugacy class in $\mathrm{SL}_2\mathbb{C}$. Here, a typical pair (Γ, S) to keep in mind is when $\Gamma = \langle s_1, \dots, s_n \rangle$ is the free group generated by S.

Define the $\mathrm{SL}_2\mathbb{C}$ *representation variety of the pair* (Γ, S) to be

$$R_{\mathbb{C}}(\Gamma, S) := \{\rho \in R_{\mathbb{C}}(\Gamma) \mid \text{all } \rho(s_i) \text{ and } \rho(s_j) \text{ are conjugate in } \mathrm{SL}_2\mathbb{C}\}.$$

Despite the name, $R_{\mathbb{C}}(\Gamma, S)$ is rarely an affine algebraic set, but rather is constructible, a term which we now recall. If W and V are affine algebraic sets (ie are Zariski closed in $\mathbb{C}^m$), their difference $W - V$ is a *quasiaffine algebraic set*. A *constructible set* is then a finite union of quasiaffine sets. For context, recall that the image of an affine algebraic set under a polynomial map is generally only constructible, but the image of a constructible set under such a map is still constructible. We define $X_{\mathbb{C}}(\Gamma, S)$ to be the image of $R_{\mathbb{C}}(\Gamma, S) \subset R_{\mathbb{C}}(\Gamma, \varnothing) = R_{\mathbb{C}}(\Gamma)$ under the natural map $\tau \colon R_{\mathbb{C}}(\Gamma) \to X_{\mathbb{C}}(\Gamma)$. Hence, $X_{\mathbb{C}}(\Gamma, S)$ is a constructible subset of the locus $\{\mathrm{tr}_{s_i} = \mathrm{tr}_{s_j}\}$ of $X_{\mathbb{C}}(\Gamma)$.

A representation $\rho \in R_{\mathbb{C}}(\Gamma)$ is *reducible* when $\rho(\Gamma)$ leaves invariant a line in $\mathbb{C}^2$; otherwise ρ is *irreducible*. The corresponding subsets of $R_{\mathbb{C}}(\Gamma, S)$ are denoted by $R^{\mathrm{red}}_{\mathbb{C}}(\Gamma, S)$ and $R^{\mathrm{irr}}_{\mathbb{C}}(\Gamma, S)$, respectively. These two sets have disjoint images under τ and are denoted by $X^{\mathrm{red}}_{\mathbb{C}}(\Gamma, S)$ and $X^{\mathrm{irr}}_{\mathbb{C}}(\Gamma, S)$. The subsets $R^{\mathrm{red}}_{\mathbb{C}}(\Gamma, S)$ and $X^{\mathrm{red}}_{\mathbb{C}}(\Gamma, S)$ are Zariski closed in $R_{\mathbb{C}}(\Gamma, S)$ and $X_{\mathbb{C}}(\Gamma, S)$, respectively, and hence $R^{\mathrm{irr}}_{\mathbb{C}}(\Gamma, S)$ and $X^{\mathrm{irr}}_{\mathbb{C}}(\Gamma, S)$ are Zariski open. For χ in $X^{\mathrm{irr}}_{\mathbb{C}}(\Gamma, S)$, all representations in $\tau^{-1}(\chi)$ are conjugate, whereas this need not be the case for χ in $X^{\mathrm{red}}_{\mathbb{C}}(\Gamma, S)$.

We will need the following general fact about character varieties:

3.1 Lemma *If $\varphi\colon \Gamma_1 \to \Gamma_2$ is an epimorphism, then the induced map $\varphi^*\colon X_{\mathbb{C}}(\Gamma_2) \to X_{\mathbb{C}}(\Gamma_1)$ given by $[\rho] \mapsto [\rho \circ \varphi]$ is proper in the classical topology.*

Proof First note that, for any Γ, a closed subset $C \subset X_{\mathbb{C}}(\Gamma)$ is compact if and only if $\mathrm{tr}_g(C)$ is bounded for all $g \in \Gamma$; this is because the functions tr_g give coordinates on $X_{\mathbb{C}}(\Gamma)$. So suppose $K \subset X_{\mathbb{C}}(\Gamma_1)$ is compact and set $L = (\varphi^*)^{-1}(K)$ in $X_{\mathbb{C}}(\Gamma_2)$. Given $g_2 \in \Gamma_2$, choose $g_1 \in \Gamma_1$ with $g_2 = \varphi(g_1)$. Then $\mathrm{tr}_{g_2}(L) = \mathrm{tr}_{g_1}(K)$ and hence the former is bounded as the latter is. Thus L is compact and φ^* is proper. □

3.2 Real characters

Our focus throughout this paper is on the real characters of Γ, that is, the subset $X(\Gamma, S)$ of $X_{\mathbb{C}}(\Gamma, S)$ where $\mathrm{tr}_g(\chi) \in \mathbb{R}$ for all $g \in \Gamma$. The algebraic set $X_{\mathbb{C}}(\Gamma, \varnothing)$ is defined over $\mathbb{Q}$ [Morgan and Shalen 1984, Section III.1], and the same is true for $X_{\mathbb{C}}(\Gamma, S)$; hence, complex conjugation preserves $X_{\mathbb{C}}(\Gamma, S)$ with fixed-point set the real locus $X(\Gamma, S)$. In particular, the subset $X(\Gamma, S)$ is a *real semialgebraic set*, ie it is defined by a system of real polynomial equations and inequalities. Note that $\tau^{-1}(X(\Gamma, S))$ is usually considerably larger than the real locus of $R_{\mathbb{C}}(\Gamma, S)$. However, we will show that the points in $X(\Gamma, S)$ all come from representations into the subgroups SU_2 and $\mathrm{SL}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{C}$. For any subgroup H of $\mathrm{SL}_2\mathbb{C}$, let $R_H(\Gamma, S)$ be the subset of $R_{\mathbb{C}}(\Gamma, S)$ consisting of ρ where $\rho(\Gamma) \subset H$, and let $X_H(\Gamma, S)$ be the image of $R_H(\Gamma, S)$ under τ.

3.3 Proposition *The set $X(\Gamma, S)$ is $X_{\mathrm{SU}_2}(\Gamma, S) \cup X_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S)$. Moreover, for*

$$D = \mathrm{SU}_2 \cap \mathrm{SL}_2\mathbb{R} = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right\} \cong S^1,$$

we have $X_{\mathrm{SU}_2}(\Gamma, S) \cap X_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S) = X^{\mathrm{red}}(\Gamma, S) = X_D(\Gamma, S)$. Consequently, $X^{\mathrm{irr}}(\Gamma, S)$ is the disjoint union of $X^{\mathrm{irr}}_{\mathrm{SU}_2}(\Gamma, S)$ and $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S)$.

Proof By [ibid., Proposition III.1.1], $X(\Gamma, \varnothing) = X_{\mathrm{SU}_2}(\Gamma, \varnothing) \cup X_{\mathrm{SL}_2\mathbb{R}}(\Gamma, \varnothing)$. Moreover, by [Culler and Dunfield 2018, Lemma 2.10], $X_{\mathrm{SU}_2}(\Gamma, \varnothing) \cap X_{\mathrm{SL}_2\mathbb{R}}(\Gamma, \varnothing) = X^{\mathrm{red}}(\Gamma, \varnothing) = X_D(\Gamma, \varnothing)$. To transfer these facts to $X(\Gamma, S)$, it suffices to show

$$X_H(\Gamma, S) = X_H(\Gamma, \varnothing) \cap X(\Gamma, S) \quad \text{for } H \text{ each of } D, \mathrm{SU}_2 \text{ and } \mathrm{SL}_2\mathbb{R}.$$

As the definitions immediately give $X_H(\Gamma, S) \subset X_H(\Gamma, \varnothing) \cap X(\Gamma, S)$, we need only show the other containment.

Given $\chi \in X_D(\Gamma, \varnothing) \cap X(\Gamma, S)$, choose $\rho \in R_D(\Gamma, \varnothing)$ with $\tau(\rho) = \chi$. Now, on $X(\Gamma, S)$ we have $\mathrm{tr}_{s_i} = \mathrm{tr}_{s_j}$ for all $s_i, s_j \in S$. Moreover, elements of D are conjugate in $\mathrm{SL}_2\mathbb{C}$ if and only if they have the same trace. Hence, ρ must be in $R_D(\Gamma, S)$ and hence χ is also in $X_D(\Gamma, S)$, as needed.

The identical argument works for SU_2, but not $\mathrm{SL}_2\mathbb{R}$ as here there are two conjugacy classes of elements with trace 2 and -2. However, since $X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, \varnothing) = X_D(\Gamma, \varnothing)$, we need only consider $\chi \in X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, \varnothing)$, where every fiber of τ in $R_{\mathbb{C}}(\Gamma, \varnothing)$ consists of conjugate representations. In particular, if $\chi \in X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, \varnothing) \cap X(\Gamma, S)$ comes from $\rho \in R^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, \varnothing)$, it follows that $\rho \in R^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S)$ since some ρ' with the same character is in $R_{\mathbb{C}}(\Gamma, S)$. □

Both $X_{\mathrm{SU}_2}(\Gamma, S)$ and $X_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S)$ are real semialgebraic sets since they are images of the real semialgebraic subsets $R_{\mathrm{SU}_2}(\Gamma, S)$ and $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S)$ of $R_{\mathbb{C}}(\Gamma, S)$ under the polynomial map τ that is defined over $\mathbb{Q}$. Note that while $X_{\mathrm{SU}_2}(\Gamma, \varnothing)$ can be identified with the topological quotient $R_{\mathrm{SU}_2}(\Gamma, \varnothing)/\mathrm{SU}_2$, the analogous statement for $\mathrm{SL}_2\mathbb{R}$ is false. This is for two reasons: first, because there are nonconjugate elements of $\mathrm{SL}_2\mathbb{R}$ with the same trace; and second, because the normalizer of $\mathrm{SL}_2\mathbb{R}$ in $\mathrm{SL}_2\mathbb{C}$ is larger than just $\mathrm{SL}_2\mathbb{R}$. To be precise, the normalizer is the full stabilizer of the copy of $\mathbb{H}^2 \subset \mathbb{H}^3$ that is preserved by $\mathrm{SL}_2\mathbb{R}$, and hence is a double cover of the disconnected group $\mathrm{Isom}(\mathbb{H}^2)$. We use $\mathrm{SL}_2^{\pm}(\mathbb{R})$ to denote the normalizer of $\mathrm{SL}_2\mathbb{R}$ in $\mathrm{SL}_2\mathbb{C}$.

Note also that

$$R_{\mathrm{SU}_2}(\Gamma, S) = \{\rho \in R_{\mathrm{SU}_2}(\Gamma) \mid \text{all } \rho(s_i) \text{ and } \rho(s_j) \text{ are conjugate in } \mathrm{SU}_2\}$$

and

$$R_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S) = \{\rho \in R_{\mathrm{SL}_2\mathbb{R}}(\Gamma) \mid \text{all } \rho(s_i) \text{ and } \rho(s_j) \text{ are conjugate in } \mathrm{SL}_2^{\pm}(\mathbb{R})\}.$$

Recalling that, for χ in $X^{\mathrm{irr}}_{\mathbb{C}}(\Gamma, S)$, all representations in $\tau^{-1}(\chi) \subset R_{\mathbb{C}}(\Gamma, S)$ are conjugate, it is easy to show:

3.4 Lemma *As topological spaces, $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S)$ is the quotient of $R^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma, S)$ by the conjugation action of $\mathrm{SL}_2^{\pm}(\mathbb{R})$.*

3.5 The free group

We now establish some basic facts about the character variety of the free group. We start by considering $X_{\mathbb{C}}(F_n, \varnothing)$ for the free group F_n on generators $S = \{s_1, \ldots, s_n\}$. By [Heusener and Porti 2004, Proposition 5.8], for $n > 2$, the smooth locus of $X_{\mathbb{C}}(F_n, \varnothing)$ is precisely $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, \varnothing)$; for $n \leq 2$, the entirety of $X_{\mathbb{C}}(F_n, \varnothing)$ is smooth. We define

$$X^*_{\mathbb{C}}(F_n) = \{[\rho] \in X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{C}}(F_n, \varnothing) \mid \rho(s_i) \neq \pm I \text{ for all } i\},$$

which is a Zariski-open subset of $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, \varnothing)$ as one has

$$X^*_{\mathbb{C}}(F_n) = \{\chi \in X^{\mathrm{irr}}_{\mathbb{C}}(F_n, \varnothing) \mid \text{for all } i \text{ there exists } j \text{ with } \chi([s_i, s_j]) \neq 2\}.$$

Since $\chi \in X^{\mathrm{red}}_{\mathbb{C}}(F_n, \varnothing)$ satisfies $\chi([s_i, s_j]) = 2$ for all i, j, we have

$$X^*_{\mathbb{C}}(F_n) = X_{\mathbb{C}}(F_n, \varnothing) \setminus \bigcup_{i=1}^{n} \{\chi \in X_{\mathbb{C}}(F_n, \varnothing) \mid \chi([s_i, s_j]) = 2 \text{ for all } j\}.$$

In particular, $X^*_{\mathbb{C}}(F_n)$ is a quasiaffine algebraic set.

Consider the map $\mathrm{tr}_S\colon X^*_{\mathbb{C}}(F_n) \to \mathbb{C}^n$ given by $\mathrm{tr}_S(\chi) = (\chi(s_1), \dots, \chi(s_n))$. For $c \in \mathbb{C}$, we take $\boldsymbol{c} = (c, \dots, c) \in \mathbb{C}^n$ and define

$$X^{c,\mathrm{irr}}_{\mathbb{C}}(F_n, S) := \mathrm{tr}_S^{-1}(\boldsymbol{c}) \subset X^*_{\mathbb{C}}(F_n).$$

Observe that $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S) = \bigcup_{c \in \mathbb{C}} X^{c,\mathrm{irr}}_{\mathbb{C}}(F_n, S)$.

3.6 Lemma *The sets $X^{c,\mathrm{irr}}_{\mathbb{C}}(F_n, S)$ and $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S)$ are quasiaffine and smooth. The map*

$$\mathrm{tr}\colon X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S) \to \mathbb{C}, \quad \chi \mapsto \chi(s_1),$$

is a submersion, with $\mathrm{tr}^{-1}(c) = X^{c,\mathrm{irr}}_{\mathbb{C}}(F_n, S)$.

Proof For clarity, we denote the map $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S) \to \mathbb{C}$ in the statement as tr_{s_1} to distinguish it from $\mathrm{tr}_S\colon X^*_{\mathbb{C}}(F_n) \to \mathbb{C}^n$ and also $\mathrm{tr}\colon \mathrm{SL}_2\mathbb{C} \to \mathbb{C}$. Set

$$R^*_{\mathbb{C}}(F_n) = \{\rho\colon F_n \to \mathrm{SL}_2\mathbb{C} \mid \rho(s_i) \neq \pm I \text{ for all } i\} \cong (\mathrm{SL}_2\mathbb{C} \setminus \{\pm I\})^n,$$

so $\tau(R^*_{\mathbb{C}}(F_n)) = X^*_{\mathbb{C}}(F_n)$. The only critical points of the map $\mathrm{tr}\colon \mathrm{SL}_2\mathbb{C} \to \mathbb{C}$ are $\pm I$, so $\mathrm{tr}_S \circ \tau\colon R^*_{\mathbb{C}}(F_n) \to \mathbb{C}^n$ is a surjective submersion. It follows that tr_S is a surjective submersion. Now $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S) = \mathrm{tr}_S^{-1}(\Delta)$, where $\Delta = \{\boldsymbol{c} \mid c \in \mathbb{C}\} \subset \mathbb{C}^n$, so $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S)$ is a smooth quasiaffine algebraic set. The restriction $\mathrm{tr}_S\colon X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S) \to \Delta$ is also a surjective submersion whose composition with any coordinate projection $\Delta \to \mathbb{C}$ is $\mathrm{tr}_{s_1}\colon X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S) \to \mathbb{C}$; hence, the latter map is a surjective submersion and $X^{c,\mathrm{irr}}_{\mathbb{C}}(F_n, S) = \mathrm{tr}_{s_1}^{-1}(c)$ is also smooth and quasiaffine. □

For $c \in \mathbb{R}$, we define $X^{c,\mathrm{irr}}(F_n, S)$ and $X^{\mathrm{irr}}(F_n, S)$ to be the real loci of $X^{c,\mathrm{irr}}_{\mathbb{C}}(F_n, S)$ and $X^{\mathrm{irr}}_{\mathbb{C}}(F_n, S)$, respectively. Since they are the real loci of smooth algebraic sets, we see that:

3.7 Corollary *The sets $X^{c,\mathrm{irr}}(F_n, S)$ and $X^{\mathrm{irr}}(F_n, S)$ are smooth.*

Using Proposition 3.3, we have

$$X^{c,\mathrm{irr}}(F_n, S) = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n, S) \amalg X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S) \quad \text{and} \quad X^{\mathrm{irr}}(F_n, S) = \bigcup_{c \in \mathbb{R}} X^{c,\mathrm{irr}}(F_n, S),$$

as well as the following consequence of Lemma 3.6:

3.8 Corollary *The map* $\mathrm{tr}\colon X^{\mathrm{irr}}(F_n, S) \to \mathbb{R}$ *is a submersion.*

We define $R^{c,\mathrm{irr}}_{\mathbb{C}}(F_n,S)$ inside $R^{\mathrm{irr}}_{\mathbb{C}}(F_n,S)$ to be the preimage of $X^{c,\mathrm{irr}}_{\mathbb{C}}(F_n,S)$ under τ and use analogous notation for the other target groups.

3.9 Lemma *The maps $\tau\colon R^{c,\mathrm{irr}}(F_n,S)\to X^{c,\mathrm{irr}}(F_n,S)$ and $\tau\colon R^{\mathrm{irr}}(F_n,S)\to X^{\mathrm{irr}}(F_n,S)$ are submersions.*

Proof As $X^{\mathrm{irr}}(F_n,S)$ is the disjoint union of $X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n,S)$ and $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ by Proposition 3.3, this breaks up into two cases, corresponding to SU_2 and $\mathrm{SL}_2\mathbb{R}$. We do $\mathrm{SL}_2\mathbb{R}$; the other case is identical.

By [Culler and Dunfield 2018, Lemma 2.11], for any group Γ the map $\tau\colon R_{\mathrm{SL}_2\mathbb{R}}(\Gamma,\varnothing)\to X_{\mathrm{SL}_2\mathbb{R}}(\Gamma,\varnothing)$ has the weak-path lifting property: given $f\colon I\to X_{\mathrm{SL}_2\mathbb{R}}(\Gamma,\varnothing)$ there exists $\tilde f\colon I\to X_{\mathrm{SL}_2\mathbb{R}}(\Gamma,\varnothing)$ with $f=\tau\circ\tilde f$. By [loc. cit.], if $f(0)$ is in $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma,\varnothing)$, then $\tilde f(0)$ can be required to be any representation in $\tau^{-1}(f(0))$. To show $\tau\colon R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)\to X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ is a submersion, consider $\rho\in R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ and v any tangent vector to $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ at $\tau(\rho)$. Choose a smooth path $f\colon I\to X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ with $f(0)=\tau(\rho)$ and $f'(0)=v$. Let $\tilde f\colon I\to R_{\mathrm{SL}_2\mathbb{R}}(F_n,\varnothing)$ be a lift with $\tilde f(0)=\rho$. As $R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ is the preimage under τ of $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$, it follows that $\tilde f(I)$ is contained in $R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ and so $\tilde f'(0)$ is a tangent vector to $R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ that maps to v under $D\tau$, proving τ is a submersion. The proof that $\tau\colon R^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)\to X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ is a submersion is identical. □

3.10 Lemma *When c is not in $(-2,2)$, the set $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n,S)$ is empty. For $c=\pm2$, every $[\rho]\in X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ has $\rho(s_i)\neq\pm I$ for all i. The same holds for any pair (Γ,S) where Γ is generated by conjugates of elements of S.*

Proof The first claim is immediate for $|c|>2$ since SU_2 has no elements with that trace. For $|c|=2$, if $[\rho]\in X^{c}_{\mathrm{SU}_2}(F_n,S)$, the only possibilities for $\rho(s_i)$ are I or $-I$ depending on the sign of c, which means the image of ρ is contained in $\{\pm I\}$ and so $[\rho]$ is reducible; hence, $X^{\pm2,\mathrm{irr}}_{\mathrm{SU}_2}(F_n,S)$ is empty. Finally, suppose $c=\pm2$ and $[\rho]\in X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ with $\rho\in R^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$. As all $\rho(s_i)$ are conjugate by definition, if any $\rho(s_i)=\pm I$ then all are, which is impossible as ρ is irreducible. □

We will also need the following standard lemma:

3.11 Lemma *If an irreducible $\rho\colon\Gamma\to\mathrm{PSL}_2\mathbb{R}$ is centralized by $A\in\mathrm{PSL}_2\mathbb{R}$, then $A=I$.*

Proof We provide the proof to caution that the conclusion does not hold when we allow A to be in the larger group $\mathrm{PSL}_2\mathbb{C}$. Following [Heusener and Porti 2004, Section 3], a representation $\hat\rho\colon\Gamma\to\mathrm{PSL}_2\mathbb{C}$ is Ad-reducible if it leaves invariant either a point in $P^1(\mathbb{C})$ *or* a geodesic in $\mathbb{H}^3$. Moreover, ρ is Ad-irreducible exactly when its stabilizer under the conjugation action of $\mathrm{PSL}_2\mathbb{C}$ is trivial by [ibid., Proposition 3.16(ii)]. So if $\rho\colon\Gamma\to\mathrm{PSL}_2\mathbb{R}$ is Ad-irreducible, then the lemma holds. If instead $\rho\colon\Gamma\to\mathrm{PSL}_2\mathbb{R}$ is irreducible but Ad-reducible, then it is a metabelian representation into the stabilizer of some geodesic L in $\mathbb{H}^3$, which is a copy of $\mathbb{C}^\times \rtimes (\mathbb{Z}/2)$. Since any finite representation into $\mathrm{PSL}_2\mathbb{R}$ is reducible, $\rho(\Gamma)$ contains elements that translate along L. This forces L to be in $\mathbb{H}^2$. As ρ is irreducible, $\rho(\Gamma)$ also contains elliptic elements of order two that interchange the two endpoints of L. Thus there is a unique nontrivial centralizer of $\rho(\Gamma)$, namely the elliptic of order two with fixed-point set L. As that element is not in $\mathrm{PSL}_2\mathbb{R}$, we are done. □

3.12 Reducibles

Suppose $\rho \in R^{\mathrm{red}}_{\mathbb{C}}(F_n, S)$, so $\rho(F_n)$ preserves a line in $\mathbb{C}^2$. Conjugating by an element of $\mathrm{SL}_2\mathbb{C}$, we may assume this line is spanned by e_1. Hence, the elements of $\rho(\Gamma)$ are upper triangular matrices, say

$$\rho(s_j) = \begin{pmatrix} \lambda_j & * \\ 0 & \lambda_j^{-1} \end{pmatrix}.$$

Since ρ has the same character as an abelian representation, namely the one which sends s_j to $\begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j^{-1} \end{pmatrix}$, we see $X^{\mathrm{red}}_{\mathbb{C}}(F_n, S)$ is the image of the subset $R^{\mathrm{ab}}_{\mathbb{C}}(F_n, S)$ of abelian representations.

Now suppose that $[\rho]$ is a real character, and moreover one in $X^c(F_n, S)$, so for each generator we have $\operatorname{tr}\rho(s_j) = \lambda_j + \lambda_j^{-1} = c \in \mathbb{R}$. If we further assume that $c \in (-2, 2)$, then $\lambda_j \in S^1$. Letting $\alpha = \cos^{-1}\frac{c}{2}$, we can write $\lambda_j = e^{i\epsilon_j\alpha}$, where $\epsilon_j = \pm 1$. Hence, $[\rho]$ is determined by the vector $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$. It is easy to see that the representations determined by ϵ and ϵ' have the same character if and only if $\epsilon = \pm\epsilon'$. Hence, the set $X^{c,\mathrm{red}}(F_n, S)$ is in bijection with $\{\pm 1\}^n/\pm\mathbb{1}$, where $\mathbb{1} = (1, \dots, 1)$. We denote the reducible character associated to $[\epsilon] \in \{\pm 1\}^n/\pm\mathbb{1}$ by $\chi^c_{[\epsilon]}$, or just $\chi_{[\epsilon]}$ when c is clear from context.

3.13 Knot complements

Suppose M is the exterior of a knot K in S^3, and $\mu \in \pi_1(\partial M)$ is a meridian. We will use $X(K)$ to denote $X(M, \{\mu\})$; as $S = \{\mu\}$ has only one element, $X(M, \{\mu\}) = X(M)$, and so the $X(K)$ notation is just encoding that $\operatorname{tr}\colon X(K) \to \mathbb{R}$ is tr_μ. Similarly, we define $X^c(K) := X^c(M, \{\mu\})$. The knot K is *small* when M does not contain a closed essential surface. Our main results will apply to small knots as well as to the following much larger class. Specifically, we say K is *real representation small* if the preimage of $[-2, 2]$ under $\operatorname{tr}\colon X(K) \to \mathbb{R}$ is compact. As the nomenclature suggests:

3.14 Lemma *If K is small then it is real representation small. Moreover, for each $c \in \mathbb{R}$, the set $X^c(K) = \operatorname{tr}^{-1}(c)$ is finite.*

We prove this using the Culler–Shalen machinery.

Proof We need to prove $E := \operatorname{tr}_\mu^{-1}([-2, 2])$ in $X(M)$ is compact. Consider the complex affine algebraic set $X_{\mathbb{C}}(M)$. By [Cooper et al. 1994, Section 2.4], as M is small, every irreducible component of $X_{\mathbb{C}}(M)$ has complex dimension 1. As there are finitely many such components C, it suffices to prove each $E \cap C$ is compact. Following [Culler and Shalen 1983, Section 1.3], let $\widetilde{C}$ be a smooth projective curve with a birational isomorphism $\iota\colon \widetilde{C} \to C$. In particular, $\widetilde{C}$ is a compact Riemann surface. The points where ι is regular (ie defined) will be denoted by $\widetilde{C}_{\mathrm{reg}}$; the finitely many points in $\widetilde{C} \setminus \widetilde{C}_{\mathrm{reg}}$ are called *ideal points*. The trace map $\mathrm{tr}_\mu\colon C \to \mathbb{C}$ induces a *regular* map $\mathrm{tr}_\mu\colon \widetilde{C} \to P^1(\mathbb{C})$.

We next claim $\widetilde{E} := \iota^{-1}(E \cap C)$ is closed in $\widetilde{C}$; throughout this proof, open and closed refer to the classical rather than Zariski topology. We know E is closed in $X(M)$ which is closed in $X_{\mathbb{C}}(M)$; thus, $E \cap C$ is closed in C. Hence, $\widetilde{E}$ is closed in $\widetilde{C}_{\mathrm{reg}}$, so it suffices to show that no ideal point x is a limit point of $\widetilde{E}$. If x were such an ideal point, then $\mathrm{tr}_\mu(x) \in [-2, 2]$ by continuity of $\mathrm{tr}_\mu\colon \widetilde{C} \to P^1(\mathbb{C})$. By [Culler et al. 1987, Chapter 1], there is an essential surface S associated to x, and, as $\mathrm{tr}_\mu(x) \neq \infty$, it follows that S is either closed or has boundary some number of parallel copies of μ. In the latter case, [ibid., Theorem 2.0.3] implies that M also contains a closed essential surface. In either case, this contradicts that M is small, so $\widetilde{E}$ is closed in $\widetilde{C}$. As $\widetilde{C}$ is compact, $\widetilde{E}$ is compact and, by definition, it is contained in $\widetilde{C}_{\mathrm{reg}}$. As $\iota(\widetilde{C}_{\mathrm{reg}})$ is all of C, $E \cap C = \iota(\widetilde{E})$ is the continuous image of a compact set and hence compact, as desired. So K is real representation small.

The final claim that each $X^c(K)$ is finite follows provided each $\mathrm{tr}_\mu\colon \widetilde{C} \to P^1(\mathbb{C})$ is nonconstant. This is the case since otherwise we would again have an ideal point where $\mathrm{tr}_\mu(x) \neq \infty$, violating smallness. □

3.15 Reducible representations and the Alexander polynomial

As noted in Section 3.12, the subset $X^{\mathrm{red}}_{\mathbb{C}}(K)$ of reducible characters is the same as the characters of diagonal representations. As the latter factor through $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, we can parametrize $X^{\mathrm{red}}_{\mathbb{C}}(K)$ by $\mathrm{tr}_\mu(\chi) \in \mathbb{C}$; we will use χ^c to denote the reducible character with $\mathrm{tr}_\mu = c$.

For the Alexander polynomial $\Delta_K(t)$ of K, define

$$D_K := \{a + 1/a \mid a \in \mathbb{C} \text{ with } \Delta_K(a^2) = 0\}.$$

Note that $\Delta_K(\pm 1)$ is always an odd integer, so D_K is finite with 0 and ± 2 not in D_K. By de Rahm, the set D_K is exactly the $c \in \mathbb{C}$ where there is a reducible representation ρ to $\mathrm{SL}_2\mathbb{C}$ with *nonabelian image* and character χ^c; see [Cooper et al. 1994, Section 6] or [Burde et al. 2014, Proposition 14.6]. Any reducible character that is the limit of irreducible ones must come from D_K:

3.16 Lemma [Cooper et al. 1994, Proposition 6.2] *If a reducible character χ^c is in the Zariski closure of $X^{\mathrm{irr}}_{\mathbb{C}}(K)$, then $c \in D_K$.*

4 Representations to the universal covering group

Next, we discuss the basics of representations into the Lie group $\widetilde{\mathrm{SL}_2\mathbb{R}}$, the universal cover of $\mathrm{SL}_2\mathbb{R}$; throughout, see [Culler and Dunfield 2018, Section 3] for background and the basic facts that we use, noting that $\widetilde{\mathrm{SL}_2\mathbb{R}} = \widetilde{\mathrm{PSL}_2\mathbb{R}}$. This material will not be used until Section 14.12.

The Lie group $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is the universal central extension of $\mathrm{SL}_2\mathbb{R}$ by $\pi_1(\mathrm{SL}_2\mathbb{R}) \cong \mathbb{Z}$,

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{SL}_2\mathbb{R}} \to \mathrm{SL}_2\mathbb{R} \to 1. \tag{4.1}$$

Note the $\mathbb{Z}$ in (4.1) is actually index two in the center $\langle c\rangle \cong \mathbb{Z}$ in $\widetilde{\mathrm{SL}_2\mathbb{R}}$, with $c \mapsto -I$ under the map $\widetilde{\mathrm{SL}_2\mathbb{R}} \to \mathrm{SL}_2\mathbb{R}$ and $\widetilde{\mathrm{SL}_2\mathbb{R}}/\langle c\rangle = \mathrm{PSL}_2\mathbb{R}$.

For a finitely generated group Γ, the obstruction to lifting $\rho\colon \Gamma \to \mathrm{SL}_2\mathbb{R}$ to a representation $\widetilde{\rho}\colon \Gamma \to \widetilde{\mathrm{SL}_2\mathbb{R}}$ is the Euler class $e(\rho) \in H^2(\Gamma;\mathbb{Z})$. When $e(\rho) = 0$, the cohomology group $H^1(\Gamma;\mathbb{Z})$ acts freely on the set of lifts: viewing $\alpha \in H^1(\Gamma;\mathbb{Z})$ as a homomorphism from Γ to the $\mathbb{Z}$ in (4.1), we set $(\alpha\cdot\widetilde{\rho})(g) = \alpha(g)\widetilde{\rho}(g)$ for $g \in \Gamma$, which is another representation as $\alpha(\Gamma)$ is central in $\widetilde{\mathrm{SL}_2\mathbb{R}}$. Consider the set of liftable $\mathrm{SL}_2\mathbb{R}$ representations

$$R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)_0 = \{\rho \in R_{\mathrm{SL}_2\mathbb{R}}(\Gamma) \mid e(\rho) = 0\},$$

which is a union of connected components of $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ by [ibid., Section 3.3]. The group $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is not linear and so has a real analytic rather than real algebraic structure. Still, $R_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$ makes sense as a real analytic variety. The cohomology group $H^1(\Gamma;\mathbb{Z})$ acts freely and properly discontinuously on $R_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$ with quotient $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)_0$; in particular, $R_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma) \to R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)_0$ is a covering map.

Before introducing the analogue of the character variety for $\widetilde{\mathrm{SL}_2\mathbb{R}}$, we take a brief detour to study $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma) = R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)/\!\!/\!\sim$, where $/\!\!/$ denotes the geometric invariant theory quotient and $\sim$ is the equivalence relation generated by conjugation by elements of $\mathrm{SL}_2\mathbb{R}$. Equivalently, $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ is the polystable quotient of $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ under the action of $\mathrm{SL}_2\mathbb{R}$ by conjugation, as discussed in Section 4.4 below. This differs from $X_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$, which is $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)/\!\!/\mathrm{SL}_2^{\pm}(\mathbb{R})$, as discussed in Section 3.2. In particular, the conjugation action of $\left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$ on $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ descends to a $\mathbb{Z}/2$ action that quotients $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ down to $X_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$; thus the natural map $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma) \to X_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ is a (possibly branched) 2-fold cover. As with $X_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$, the space $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ is a real semialgebraic set [Richardson and Slodowy 1990, Theorem 7.6], and thus Hausdorff and locally contractible by [Basu et al. 2006, Theorem 5.43]. The following will be needed in Section 14:

4.2 Lemma *The map $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma) \to X_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ can be branched only at reducible characters. In particular, each $\chi \in X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ has two distinct preimages in $\widetilde{X}^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$.*

Proof Lest you think the claim obvious, we point out that it fails for $\mathrm{PSL}_2\mathbb{R}$ (consider an infinite dihedral group in $\mathrm{PSL}_2\mathbb{R}$ generated by two distinct elements of order 2). Suppose $\rho \in R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ represents χ. The claim is equivalent to showing that the $\mathrm{SL}_2^{\pm}(\mathbb{R})$-orbit of ρ has two connected components (each of which is then a distinct $\mathrm{SL}_2\mathbb{R}$-orbit, giving two distinct points in $\widetilde{X}^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$). Since the stabilizer of an irreducible representation in $\mathrm{SL}_2\mathbb{C}$ is just $\{\pm I\}$ (eg use the setup of the proof of [Culler and Shalen 1983, Lemma 1.5.1]), we see $\mathrm{SL}_2^{\pm}(\mathbb{R})\cdot\rho$ is disconnected by the orbit–stabilizer theorem. □

We will define $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$ to be the polystable quotient of $R_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$ under the action of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ by conjugation; see Section 4.4 and especially Corollary 4.7 for details. The main result of this section tells us that while $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is not an algebraic group, it still has a very reasonable "character variety" in the form of $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$:

4.3 Theorem *The image $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)_0$ of $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)_0$ in $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ is a union of connected components. The map $R_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma) \to R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)_0$ induces a regular covering map $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma) \to \widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)_0$, where the covering group is $H^1(\Gamma;\mathbb{Z})$.*

We point out that the final conclusion of this theorem does not hold for the central extension $\mathbb{Z}/2 \to \mathrm{SL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}$; the map $X_{\mathrm{SL}_2\mathbb{C}}(\Gamma) \to X_{\mathrm{PSL}_2\mathbb{C}}(\Gamma)$ can have branching, at irreducible representations no less [Heusener and Porti 2004, Section 4]. The proof of Theorem 4.3 is unconnected with the rest of this paper, and so the trusting reader can skip ahead to Section 5.

4.4 Polystable quotients

Following [Richardson and Slodowy 1990, Section 7.2], consider the following construction. Let H be a locally compact group acting on a locally compact space V. This action is *polystable* when for all v in V there is a unique closed H-orbit in the closure of $H \cdot v$. For a polystable action, let V^* be those points in V whose H-orbits are closed, which is called the set of *polystable points*. Now define the set $V/\!/H$ to be V^*/H. Let $\pi\colon V \to V/\!/H$ send v to $[x]$, where $H \cdot x$ is the unique closed orbit in the closure of $H \cdot v$. Finally, give $V/\!/H$ the quotient topology induced by π; this space is called the *polystable quotient*. As the notation suggests, the space $V/\!/H$ is also the geometric invariant theory quotient in many situations, including when H is a reductive real or complex linear algebraic group acting on an affine variety [Richardson and Slodowy 1990]. In particular, this gives an alternative perspective on the construction of $R_{\mathbb{C}}(\Gamma) \to X_{\mathbb{C}}(\Gamma)$. Since the polystable quotient is purely a topological construction, we can try to apply it to the conjugation action of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ on $R_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$ where there is no geometric invariant theory quotient to be had; we succeed in doing so in Corollary 4.7.

For ease of notation, set $G = \mathrm{SL}_2\mathbb{R}$, $\widetilde{G} = \widetilde{\mathrm{SL}_2\mathbb{R}}$ and $\overline{G} = \mathrm{PSL}_2\mathbb{R} = G/Z(G) = \widetilde{G}/Z(\widetilde{G})$. We are interested in the conjugation action of $\overline{G}$ on $R_G(\Gamma)$ and $R_{\widetilde{G}}(\Gamma)$. There are five types of orbits $\overline{G} \cdot \rho$ for $\rho \in R_G(\Gamma)$:

1. When the image of ρ is in $Z(G)$, we have $\overline{G} \cdot \rho = \{\rho\}$, which is closed.
2. When ρ is irreducible, the orbit $\overline{G} \cdot \rho$ is closed and homeomorphic to $\overline{G}$ itself (compare Lemma 3.11).
3. When ρ is a reducible and has a unique fixed point in $\mathbb{H}^2$, the orbit $\overline{G} \cdot \rho$ is closed and homeomorphic to $\mathbb{H}^2$.
4. When ρ is reducible and has exactly two fixed points in $P^1(\mathbb{R})$, it is conjugate to a representation into $\left\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times\right\}$. The orbit $\overline{G} \cdot \rho$ is again closed and homeomorphic to $S^1 \times \mathbb{R}$.
5. When ρ is reducible and has a unique fixed point in $P^1(\mathbb{R})$, the orbit $\overline{G} \cdot \rho$ is not closed. There is a unique orbit of type (4) in the closure of $\overline{G} \cdot \rho$: if ρ is upper triangular, that orbit is that of its "diagonal part". Indeed, if ρ is upper triangular, define ρ_t for $t \in [0,1]$ by
$$\rho_t(g) = \begin{pmatrix} a & bt^2 \\ 0 & a^{-1} \end{pmatrix} \quad \text{when } \rho(g) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$
This is a smooth path in $R_G(\Gamma)$ where ρ_0 is diagonal and ρ_t is conjugate to ρ for $t > 0$ by $\left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & t \end{smallmatrix}\right)$.

For an action of a group H on a space V, for $v \in V$ define

$$[v]_H = \{w \in V \mid \text{the closures of } H \cdot v \text{ and } H \cdot w \text{ intersect}\}.$$

When H acts polystably on V with induced map $\pi\colon V \to V/\!\!/H$, we have $[v]_H = \pi^{-1}(\pi(v))$ for each v in V. The basic facts about the action of $\overline{G}$ on $R_G(\Gamma)$ and its quotient are then:

4.5 Lemma *The action of $\overline{G}$ on $R_G(\Gamma)$ is polystable. The fibers of $\tau\colon R_G(\Gamma) \to \widetilde{X}_G(\Gamma) = R_G(\Gamma)/\!\!/\overline{G}$ are closed and path connected. Finally, $\tau^{-1}(\tau(\rho)) = [\rho]_{\overline{G}}$ for each $\rho \in R_G(\Gamma)$.*

Proof The first claim can be deduced with some effort from the concrete description of the orbits, but it is also a general fact about actions of real reductive groups [ibid., Theorem 7.3.1]. As noted above, $\widetilde{X}_G(\Gamma)$ is Hausdorff, so points are closed and hence so are their preimages under the continuous map τ. That the fibers are path connected follows from the discussion in (5). Finally, as previously mentioned, the last statement holds for any polystable action. □

We turn now to studying $\pi\colon R_{\widetilde{G}}(\Gamma) \to R_G(\Gamma)_0$, which we recall is a regular cover with deck group $H^1(\Gamma;\mathbb{Z})$.

4.6 Lemma *For each $\rho \in R_G(\Gamma)_0$, there is an open set U containing $[\rho]_{\overline{G}}$ where the following holds: For every lift $\widetilde{\rho} \in R_{\widetilde{G}}(\Gamma)$ of ρ, there is an open $\widetilde{U} \subset \pi^{-1}(U)$ containing $[\widetilde{\rho}]_{\overline{G}}$ where $\pi|_{\widetilde{U}}$ is a homeomorphism onto U. Moreover, $[\widetilde{\rho}]_{\overline{G}}$ is closed in $R_{\widetilde{G}}(\Gamma)$ and π restricts to a homeomorphism $[\widetilde{\rho}]_{\overline{G}} \to [\rho]_{\overline{G}}$.*

Proof Consider the continuous translation number function $\operatorname{trans}\colon \widetilde{G} \to \mathbb{R}$ discussed in [Culler and Dunfield 2018, Section 3.1], which is a conjugacy invariant. Let $s_1, \dots, s_d$ in Γ generate $H_1(\Gamma;\mathbb{Z})/(\text{torsion})$. Define $T\colon R_{\widetilde{G}}(\Gamma) \to \mathbb{R}^d$ by $T(\rho)_i = \operatorname{trans}(\rho(s_i))$, which is continuous and constant on $\overline{G}$-orbits. Let $\alpha \in H^1(\Gamma;\mathbb{Z})$ act on $x \in \mathbb{R}^d$ by $(\alpha \cdot x)_i = x_i + 2\alpha(s_i)$. If c is the preferred generator of $Z(\widetilde{G})$, one has $\operatorname{trans}(\tilde{g}c^k) = \operatorname{trans}(\tilde{g}) + k$; from this it follows that T is $H^1(\Gamma;\mathbb{Z})$-equivariant (recall here that the $\mathbb{Z}$ in (4.1) is $\langle c^2\rangle$, not $\langle c\rangle$).

For now, fix any $\widetilde{\rho}$ in $\pi^{-1}(\rho)$. Let $\widetilde{V}$ be the open box in $\mathbb{R}^d$ about $T(\widetilde{\rho})$ that is the product of intervals $\bigl(t_i - \frac{1}{3}, t_i + \frac{1}{3}\bigr)$, where $t_i = T(\widetilde{\rho})_i$. Let $\widetilde{C}$ be the closed box which is the product of intervals $\bigl[t_i - \frac{1}{6}, t_i + \frac{1}{6}\bigr]$. Consider $\widetilde{U} = T^{-1}(\widetilde{V})$ and $\widetilde{D} = T^{-1}(\widetilde{C})$, which are open and closed, respectively, with $\widetilde{\rho} \in \widetilde{D} \subset \widetilde{U}$; moreover, both $\widetilde{D}$ and $\widetilde{U}$ are invariant under the $\overline{G}$ action. As $T(\widetilde{\rho}) = T(\overline{G} \cdot \widetilde{\rho}) = T([\widetilde{\rho}]_{\overline{G}})$, both $[\widetilde{\rho}]_{\overline{G}}$ and its closure are contained in $\widetilde{D}$.

For all nonzero $\alpha \in H^1(\Gamma;\mathbb{Z})$, $\widetilde{V} \cap \alpha \cdot \widetilde{V}$ is empty since α moves some coordinate of $\mathbb{R}^d$ by at least two. As T is equivariant, $\widetilde{U} \cap \alpha \cdot \widetilde{U}$ is empty. This implies $\pi|_{\widetilde{U}}$ is injective. As $\widetilde{U}$ is open and π is an open map, $\pi|_{\widetilde{U}}$ is a homeomorphism onto the open set $U = \pi(\widetilde{U})$. To show $\pi(\widetilde{D})$ is closed in $R_G(\Gamma)$, we note that its complement is the image under the open map π of the open set $T^{-1}\bigl(\mathbb{R}^d \setminus \coprod\{\alpha \cdot \widetilde{C} \mid \alpha \in H^1(\Gamma;\mathbb{Z})\}\bigr)$.

As π is $\overline{G}$-equivariant, the subsets $\pi(\widetilde{D})$ and U are also invariant under the $\overline{G}$ action. So $\overline{G}\cdot\rho$ and indeed its closure is contained in $\pi(\widetilde{D})$. If ρ_1 is any representation where the closure of $\overline{G}\cdot\rho_1$ meets the closure of $\overline{G}\cdot\rho$, then $\overline{G}\cdot\rho_1$ must meet U and hence be contained in it. Thus $[\rho]_{\overline{G}}$ is contained in U, and of course $\pi([\widetilde{\rho}]_{\overline{G}})$ is contained in $[\rho]_{\overline{G}}$; in fact, since $\pi|_{\widetilde{U}}$ is a $\overline{G}$-equivariant homeomorphism, they are equal. So $[\widetilde{\rho}]_{\overline{G}}$ is closed in $\widetilde{U}$ since $[\rho]_{\overline{G}}$ is closed in $R_G(\Gamma)_0$ by Lemma 4.5. Since it is also a subset of $\widetilde{D}$ which is closed in $R_{\widetilde{G}}(\Gamma)$, it follows that $[\widetilde{\rho}]_{\overline{G}}$ is closed in $R_{\widetilde{G}}(\Gamma)$, as needed.

This establishes that U has the properties claimed in the statement of the lemma for the particular lift $\widetilde{\rho}$ that we fixed near the start. The $H^1(\Gamma;\mathbb{Z})$ action on $R_{\widetilde{G}}(\Gamma)$ shows that the same U also works for any other lift $\widetilde{\rho}'$, completing the proof. □

It follows from Lemma 4.6 that π maps the closure of $\overline{G}\cdot\widetilde{\rho}$ to the closure of $\overline{G}\cdot\rho$ homeomorphically and G-equivariantly. Since the action of $\overline{G}$ on $R_G(\Gamma)$ is polystable, the action of $\overline{G}$ on $R_{\widetilde{G}}(\Gamma)$ is polystable as well. Hence, we can define $\widetilde{X}_{\widetilde{G}}(\Gamma) = R_{\widetilde{G}}(\Gamma)/\!\!/\overline{G}$.

4.7 Corollary *The fibers of* $R_{\widetilde{G}}(\Gamma)\to\widetilde{X}_{\widetilde{G}}(\Gamma)$ *are closed and the map* $\pi\colon R_{\widetilde{G}}(\Gamma)\to R_G(\Gamma)$ *induces a continuous map* $\overline{\pi}\colon \widetilde{X}_{\widetilde{G}}(\Gamma)\to\widetilde{X}_G(\Gamma)$ *that makes the obvious diagram commute. Similarly, for each* $\gamma\in\Gamma$, *the function* $\mathrm{trans}_\gamma\colon R_{\widetilde{G}}(\Gamma)\to\mathbb{R}$ *given by* $\mathrm{trans}_\gamma(\rho) := \mathrm{trans}(\rho(\gamma))$ *descends to a continuous map* $\mathrm{trans}_\gamma\colon \widetilde{X}_{\widetilde{G}}(\Gamma)\to\mathbb{R}$.

Proof The first claim is immediate from Lemma 4.6, since each $[\widetilde{\rho}]_{\overline{G}}$ is closed. For the second claim, we need to check that each fiber of $R_{\widetilde{G}}(\Gamma)\to\widetilde{X}_{\widetilde{G}}(\Gamma)$ has constant image under the composition $R_{\widetilde{G}}(\Gamma)\to R_G(\Gamma)\to\widetilde{X}_G(\Gamma)$. This also follows from Lemma 4.6, since $\pi([\widetilde{\rho}]_{\overline{G}}) = [\rho]_{\overline{G}}$ and $\widetilde{X}_G(\Gamma) = R_G(\Gamma)/\!\!/\overline{G}$. Finally, the last claim follows since trans_γ is constant on each fiber $[\widetilde{\rho}]_{\overline{G}}$. □

Consider the maximal compact subgroup $\overline{K} = \mathrm{PSO}_2$ in $\overline{G}$. From [Richardson and Slodowy 1990], there is a closed subset $\mathcal{M}$ of $R_G(\Gamma)$ such that:

(1) The closed $\overline{G}$-orbits in $R_G(\Gamma)$ are exactly those that meet $\mathcal{M}$.

(2) For $\rho\in\mathcal{M}$, the intersection of $\overline{G}\cdot\rho$ with $\mathcal{M}$ is $\overline{K}\cdot\rho$.

(3) There is a continuous $\overline{K}$-equivariant deformation retraction $\varphi\colon R_G(\Gamma)\times[0,1]\to R_G(\Gamma)$ of $R_G(\Gamma)$ onto $\mathcal{M}$ that is *along the orbits of* $\overline{G}$; that is, for all $\rho\in R_G(\Gamma)$, we have $\{\varphi(\rho,t)\mid t\in[0,1)\}\subset\overline{G}\cdot\rho$ and $\varphi(\rho,1)$ is in the closure of $\overline{G}\cdot\rho$.

(4) The map $\mathcal{M}/\overline{K}\to R_G(\Gamma)/\!\!/\overline{G}$ induced by the inclusion $\mathcal{M}\to R_G(\Gamma)$ is a homeomorphism. (That this map is a continuous bijection follows from (1) and (2).)

We call a closed $\mathcal{M}$ satisfying (1)–(4) a *Kempf–Ness* subset [1979]. Their usefulness lies in that (4) allows us to understand the topology of $R_G(\Gamma)/\!\!/\overline{G}$ in terms of an ordinary topological quotient by a *compact* group. To prove Theorem 4.3, we will need:

4.8 Theorem *Let* $\mathcal{M}_0 = \mathcal{M}\cap R_G(\Gamma)_0$ *and set* $\widetilde{\mathcal{M}} = \pi^{-1}(\mathcal{M}_0)$ *inside* $R_{\widetilde{G}}(\Gamma)$. *Then* $\widetilde{\mathcal{M}}$ *is a Kempf–Ness subset for the action of* $\overline{G}$ *on* $R_{\widetilde{G}}(\Gamma)$.

Proof We check the properties in turn, with ρ denoting a representation in $R_G(\Gamma)_0$ and $\widetilde{\rho}$ one of its preimages in $R_{\widetilde{G}}(\Gamma)$. Throughout, we use Lemma 4.6 without further comment.

For (1), to start we know that $[\rho]_{\overline{G}}$ and $[\widetilde{\rho}]_{\overline{G}}$ are closed in $R_G(\Gamma)$ and $R_{\widetilde{G}}(\Gamma)$, respectively, and that the restriction $\pi\colon [\widetilde{\rho}]_{\overline{G}} \to [\rho]_{\overline{G}}$ is a homeomorphism. Thus $\overline{G}\cdot\widetilde{\rho}$ is closed if and only if $\overline{G}\cdot\rho$ is closed, and of course $\overline{G}\cdot\widetilde{\rho}$ meets $\widetilde{\mathcal{M}}$ if and only if $\overline{G}\cdot\rho$ meets $\mathcal{M}_0$. Hence, property (1) for $R_G(\Gamma)$ gives the same for $R_{\widetilde{G}}(\Gamma)$. Property (2) follows since π restricts to a $\overline{G}$-equivariant bijection from $\overline{G}\cdot\widetilde{\rho}$ to $\overline{G}\cdot\rho$.

For (3), recall $R_G(\Gamma)$ is a real semialgebraic set and hence locally contractible; in particular, every connected component is path connected. Also, $R_G(\Gamma)_0$ is a union of connected components of $R_G(\Gamma)$. If φ is the deformation retraction for $R_G(\Gamma)$, consider

$$\begin{array}{ccc} R_{\widetilde{G}}(\Gamma)\times[0,1] & \overset{\widetilde{\varphi}}{\dashrightarrow} & R_{\widetilde{G}}(\Gamma) \\ \Big\downarrow{\scriptstyle \pi\times\mathrm{id}} & & \Big\downarrow{\scriptstyle \pi} \\ R_G(\Gamma)_0\times[0,1] & \xrightarrow{\ \varphi\ } & R_G(\Gamma)_0 \end{array}$$

Here the vertical maps are covering maps and φ is a homotopy equivalence. Now $R_{\widetilde{G}}(\Gamma)\times[0,1]$ is locally path connected since $R_G(\Gamma)\times[0,1]$ is, so basic covering space theory gives us a lift $\widetilde{\varphi}$ of $\varphi\circ(\pi\times\mathrm{id})$ whose restriction $\widetilde{\varphi}_0$ is the identity map on $R_{\widetilde{G}}(\Gamma)$. Since π is bijective on each $\overline{G}$-orbit, the claims that $\widetilde{\varphi}$ is $\overline{K}$-equivariant and is along the orbits of $\overline{G}$ follow from the corresponding properties of φ.

To see that φ is a deformation retract onto $\widetilde{\mathcal{M}}$, suppose $\widetilde{\rho}\in\widetilde{\mathcal{M}}$. Note the path $\widetilde{\varphi}(\widetilde{\rho},t)$ for $t\in[0,1]$ is a lift of $\varphi(\rho,t)$ which is the constant path at ρ. Since $\pi^{-1}(\rho)$ is discrete, $\widetilde{\varphi}(\widetilde{\rho},t)$ must be the constant path at $\widetilde{\rho}$. In particular, each φ_t restricts to the identity on $\widetilde{\mathcal{M}}$. As $\widetilde{\varphi}_1(R_{\widetilde{G}}(\Gamma))$ is contained in $\pi^{-1}(\mathrm{im}(\varphi_1))=\widetilde{\mathcal{M}}$, we have shown $\widetilde{\varphi}$ satisfies (3).

For (4), we use the argument from [Richardson and Slodowy 1990, Section 9.6]. By (1) and (2), the inclusion $\widetilde{\mathcal{M}}\to R_{\widetilde{G}}(\Gamma)$ induces a continuous bijection $\widetilde{\mathcal{M}}/\overline{K}\to R_{\widetilde{G}}(\Gamma)/\!\!/\overline{G}$. If $\pi_{\overline{K}}$ is the quotient map $\widetilde{\mathcal{M}}\to\widetilde{\mathcal{M}}/\overline{K}$, we claim the composite $\pi_{\overline{K}}\circ\widetilde{\varphi}_1\colon R_{\widetilde{G}}(\Gamma)\to\widetilde{\mathcal{M}}/\overline{K}$ is constant on each fiber of π, ie on each $[\widetilde{\rho}]_{\overline{G}}$. To see this, note that since $\widetilde{\varphi}$ is along $\overline{G}$-orbits, $\widetilde{\varphi}_1(\widetilde{\rho})\in[\widetilde{\rho}]_{\overline{G}}$, so, if $\widetilde{\rho}\in\widetilde{\mathcal{M}}$, we have $\varphi_1^{-1}(\overline{K}\cdot\widetilde{\rho})=[\widetilde{\rho}]_{\overline{G}}$. In particular, $\pi_{\overline{K}}\circ\widetilde{\varphi}_1$ descends to the right-hand map in

$$\widetilde{\mathcal{M}}/\overline{K}\to R_{\widetilde{G}}(\Gamma)/\!\!/\overline{G}\to\widetilde{\mathcal{M}}/\overline{K}.$$

The definitions give that the composition above is the identity on $\widetilde{\mathcal{M}}/\overline{K}$. Both maps are continuous, and the leftmost one is a bijection, so both are homeomorphisms. □

Proof of Theorem 4.3 First, we show that $\widetilde{X}_G(\Gamma)_0$ is a union of connected components. By Lemma 4.5, the fibers of $\tau\colon R_G(\Gamma)\to\widetilde{X}_G(\Gamma)$ are connected. Since $R_G(\Gamma)_0$ is a union of components, this implies that $\tau^{-1}(\widetilde{X}_G(\Gamma)_0)=R_G(\Gamma)_0$, so, like $R_G(\Gamma)_0$, the subset $\widetilde{X}_G(\Gamma)_0$ is both open and closed, ie a union of components.

To show $\widetilde{X}_{\widetilde{G}}(\Gamma) \to \widetilde{X}_G(\Gamma)_0$ is a covering map, consider Kempf–Ness subsets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ as in Theorem 4.8 and set $\mathcal{M}_0 = \mathcal{M} \cap R_G(\Gamma)_0$. We have the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}/\overline{K} & \longrightarrow & \widetilde{X}_{\widetilde{G}}(\Gamma) \\ \downarrow & & \downarrow \\ \mathcal{M}_0/\overline{K} & \longrightarrow & \widetilde{X}_G(\Gamma)_0 \end{array}$$

where the horizontal maps are homeomorphisms and vertical ones are induced by $\pi\colon R_{\widetilde{G}}(\Gamma) \to R_G(\Gamma)_0$. Moreover, the top homeomorphism is equivariant with respect to the $H^1(\Gamma;\mathbb{Z})$ action. Now $\pi\colon \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}_0$ is a $\overline{K}$-equivariant covering map with covering group $H^1(\Gamma;\mathbb{Z})$. As $\overline{K}$ is compact, this implies that $\widetilde{\mathcal{M}}/\overline{K} \to \mathcal{M}_0/\overline{K}$ is also a $H^1(\Gamma;\mathbb{Z})$ cover, which proves the theorem since the diagram commutes. □

5 Geometric point configurations

Having dispensed with the background, we move to the main body of the paper. Our first task is to construct a resolution of the character variety $X^c(F_n, S)$, in the sense of Section 1.12. We do this in Sections 5–8. The goal of this section is to state Theorem 5.9, which describes a smooth manifold $\mathcal{Y}$ built out of configurations of $n > 1$ points in all three 2-dimensional geometries: $\mathbb{E}^2$, S^2 and $\mathbb{H}^2$. We prove this theorem in Section 7. In Section 8, we show that $\mathcal{Y}$ is the desired resolution of $X^c(F_n, S)$.

5.1 Conventions and models

Throughout this section, the integer $n > 1$. We fix for the whole paper the following preferred models for the three 2-dimensional geometries $\mathbb{E}^2$, S^2 and $\mathbb{H}^2$: First, we define the Euclidean plane $\mathbb{E}^2$ as $\mathbb{R}^2$ with the Riemannian metric associated to the standard quadratic form $x^2 + y^2$ oriented so that e_1, e_2 is a positive basis. Second, the sphere S^2 is the points in $\mathbb{R}^3$ where $x^2 + y^2 + z^2 = 1$ with the Riemannian metric determined by restricting the quadratic form $x^2 + y^2 + z^2$ to the tangent bundle TS^2; we orient S^2 so that e_1, e_2 is a positive basis for $T_{e_3}S^2$. Finally, the hyperbolic plane $\mathbb{H}^2$ is the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -1$ with the Riemannian metric coming from $x^2 + y^2 - z^2$, where e_1, e_2 is a positive basis for $T_{e_3}\mathbb{H}^2$.

5.2 The Euclidean case

Consider the action of the orientation-preserving isometry group $\mathrm{Isom}^+(\mathbb{E}^2)$ on ordered n-tuples of points in $\mathbb{E}^2$, that is, the diagonal action of $\mathrm{Isom}^+(\mathbb{E}^2)$ on $(\mathbb{E}^2)^n$. This action is free except when all the points are the same, in which case the stabilizer is S^1. Restricting to $\mathscr{E}_n = (\mathbb{E}^2)^n \setminus \Delta$, where $\Delta = \{(p, p, \dots, p) \mid p \in \mathbb{E}^2\}$, we have a free action of $\mathrm{Isom}^+(\mathbb{E}^2)$. As this action is also proper, the quotient $\mathscr{E}_n/\mathrm{Isom}^+(\mathbb{E}^2)$ is a smooth manifold. The manifold $\mathscr{E}_n/\mathrm{Isom}^+(\mathbb{E}^2)$ is noncompact with two ends: one corresponding to all the points coalescing and the other to the diameter of the set of points going to infinity.

Let $\mathrm{Sim}^+(\mathbb{E}^2)$ denote the group of orientation-preserving similarities of $\mathbb{E}^2$, which is generated by $\mathrm{Isom}^+(\mathbb{E}^2)$ and dilations about any given point. While the action of $\mathrm{Sim}^+(\mathbb{E}^2)$ on $\mathbb{E}^2$ is not proper (point stabilizers are noncompact), its action on $\mathscr{E}_n$ is proper, roughly because the amount of dilation is detected in the change in the distance between any pair of distinct points. So we get a manifold

$$\mathscr{C}_n(\mathbb{E}^2) = \mathscr{E}_n/\mathrm{Sim}^+(\mathbb{E}^2),$$

whose topology is as follows:

5.3 Lemma *The manifold $\mathscr{C}_n(\mathbb{E}^2)$ is diffeomorphic to $P^{n-2}(\mathbb{C})$.*

Proof We will typically denote a point in $\mathscr{E}_n$ by $v = (v_1, v_2, \dots, v_n)$, where each $v_i \in \mathbb{E}^2$ is regarded as a vector in $\mathbb{R}^2$. Using translations and dilations, we see that the subset

$$\mathscr{A} = \left\{ v \in \mathscr{E}_n \;\middle|\; v_1 = 0 \text{ and } \sum_{i=2}^{n} |v_i|^2 = 1 \right\}$$

meets every $\mathrm{Sim}^+(\mathbb{E}^2)$-orbit, and indeed that $\mathscr{C}_n(\mathbb{E}^2)$ is the quotient of $\mathscr{A}$ under the S^1 action of rotations about 0. Now $\mathscr{A}$ is just $S^{2n-3} \subset (\mathbb{R}^2)^{n-1} = \mathbb{C}^{n-1}$, and so the quotient under the circle action is exactly $P^{n-2}(\mathbb{C})$, as claimed. □

5.4 Signed points

We will also need to work with n ordered points where each one has an associated sign in $\{\pm 1\}$. The set of such configurations is $\mathscr{E}_n^{\pm} = \mathscr{E}_n \times \{\pm 1\}^n$, where, if $(v, \epsilon) \in \mathscr{E}_n^{\pm}$, then ϵ_i is the sign associated to the point v_i. We define an action of the full group of similarities $\mathrm{Sim}(\mathbb{E}^2)$ on $\mathscr{E}_n^{\pm}$ so that an orientation-preserving element leaves the signs unchanged, but an orientation-reversing element multiplies them all by -1. The action of $\mathrm{Sim}(\mathbb{E}^2)$ on $\mathscr{E}_n^{\pm}$ is again free and proper, and we set

$$\mathscr{C}_n^{\pm}(\mathbb{E}^2) = \mathscr{E}_n^{\pm}/\mathrm{Sim}(\mathbb{E}^2)$$

Note that the signs are important to make this action free: any v in $\mathscr{E}$ where all v_i lie on a common line L is fixed by reflection in L.

There is an action of $\{\pm 1\}^n$ on $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$ by changing the signs of the points; more formally, the action of $\{\pm 1\}^n$ on $\mathscr{E}_n^{\pm}$ by multiplication on the second factor commutes with the action of $\mathrm{Sim}(\mathbb{E}^2)$, and so descends to an action on $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$. Using this action, you can show that $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$ is the disjoint union of 2^{n-1} copies of $\mathscr{C}_n(\mathbb{E}^2)$; a subtlety here is that while $(-1, \dots, -1)$ does not permute the connected components of $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$, it does act nontrivially on each of them when $n \geq 3$.

5.5 The hyperbolic case

For the hyperbolic plane $\mathbb{H}^2$, our focus will be on

$$\mathscr{C}_n(\mathbb{H}^2) = \mathscr{H}_n/\mathrm{Isom}^+(\mathbb{H}^2), \quad \text{where } \mathscr{H}_n = (\mathbb{H}^2)^n \setminus \Delta,$$

which, as in the Euclidean case, is a smooth manifold with two ends. We will show below that $\mathscr{C}_n(\mathbb{H}^2)$ is diffeomorphic to $\mathscr{C}_n(\mathbb{E}^2)\times\mathbb{R}\cong P^{n-2}(\mathbb{C})\times\mathbb{R}$, but for now just note that $\dim\mathscr{C}_n(\mathbb{H}^2)=2n-3$. There is also a signed variant

$$\mathscr{C}_n^{\pm}(\mathbb{H}^2)=\mathscr{H}_n\times\{\pm1\}^n/\mathrm{Isom}(\mathbb{H}^2)$$

defined as above, which again has an action of $\{\pm1\}^n$ and consists of 2^{n-1} copies of $\mathscr{C}_n(\mathbb{H}^2)$.

5.6 The spherical case

Turning now to the round sphere S^2, we must remove a little more to get a free action since a pair of antipodal points has nontrivial stabilizer. Specifically, we have a free action of $\mathrm{Isom}^+(S^2)$ on

$$\mathscr{S}_n=\{v\in(S^2)^n\mid v_i\notin\{-v_j,v_j\}\text{ for at least one }i,j\},$$

which is automatically a proper action since $\mathrm{Isom}^+(S^2)\cong\mathrm{SO}_3$ is compact. Hence,

$$\mathscr{C}_n(S^2)=\mathscr{S}_n/\mathrm{Isom}^+(S^2)$$

is a smooth manifold of dimension $2n-3$. This time, it has 2^{n-1} ends, corresponding to the different ways the points can coalesce to a pair of antipodal points; concretely, each v_i for $i>1$ can approach either v_1 or $-v_1$. We will show in Theorem 7.4 that each end is diffeomorphic to $\mathscr{C}_n(\mathbb{E}^2)\times\mathbb{R}$.

The action of $\{\pm1\}^n$ on $\mathscr{S}_n$ by coordinatewise multiplication descends to an action on $\mathscr{C}_n(S^2)$. This action is faithful but not free; for example, when $n=2$ the entire group fixes $[(e_1,e_2)]$. We will not use this directly, but the quotient orbifold $\mathscr{C}_n(S^2)/\{\pm1\}^n$ is $\mathscr{C}_n(P^2(\mathbb{R}))$, that is, configurations of n points in $P^2(\mathbb{R})$, not all the same, modulo $\mathrm{Isom}(P^2(\mathbb{R}))\cong\mathrm{PO}_3\cong\mathrm{SO}_3$.

5.7 Orientations

We will eventually use these configuration spaces to construct an invariant via an algebraic intersection number, so we next give them preferred orientations. These will be derived from the orientations of $\mathbb{E}^2$, $\mathbb{H}^2$ and S^2 specified in Section 5.1, and we use freely the orientation conventions of Section 2.3. To start, we need to orient $\mathrm{Sim}(\mathbb{E}^2)$, $\mathrm{Isom}(\mathbb{H}^2)$ and $\mathrm{Isom}(S^2)$. We orient the tangent bundle of an oriented surface X so that, if (x_1,x_2) are oriented coordinates on X, then $\langle x_1,x_2,\partial/\partial x_1,\partial/\partial x_2\rangle$ are oriented coordinates on TX. We orient the unit tangent bundle UTX by viewing it as the boundary of the submanifold $\{v\in TX\mid |v|\le1\}$; equivalently, if $\langle v_1,v_2\rangle$ is an oriented orthonormal basis of T_pX, then we orient $T_{(p,v_1)}(UTX)=T_pX\oplus T_{v_1}S^1$ by $\langle v_1,v_2,v_3\rangle$ where $v_3=\frac{d}{dt}\big|_{t=0}\cos(t)v_1+\sin(t)v_2$.

If we fix a tangent vector $v_0\in T\mathbb{E}^2$, then the map $\mathrm{Sim}^+(\mathbb{E}^2)\to T\mathbb{E}^2$ given by $g\mapsto g\cdot v_0$ is a diffeomorphism, and we orient $\mathrm{Sim}^+(\mathbb{E}^2)$ so that this map is orientation-preserving. We then orient all of $\mathrm{Sim}(\mathbb{E}^2)$ by insisting that the orientation be left-invariant. We orient $\mathrm{Isom}(\mathbb{H}^2)$ by the corresponding identification of $\mathrm{Isom}^+(\mathbb{H}^2)$ with $UT\mathbb{H}^2$ and orient $\mathrm{Isom}(S^2)$ analogously.

For the unsigned configuration spaces, we first give $\mathscr{E}_n$ the product orientation, and then orient $\mathscr{C}_n(\mathbb{E}^2)$ by requiring that the fiber bundle $\mathrm{Sim}^+(\mathbb{E}^2) \to \mathscr{E}_n \to \mathscr{C}_n(\mathbb{E}^2)$ be compatibly oriented. Analogously, we orient $\mathscr{C}_n(\mathbb{H}^2)$ and $\mathscr{C}_n(S^2)$ via $\mathrm{Isom}^+(\mathbb{H}^2) \to \mathscr{H}_n \to \mathscr{C}_n(\mathbb{H}^2)$ and $\mathrm{Isom}^+(S^2) \to \mathscr{S}_n \to \mathscr{C}_n(S^2)$, respectively.

The signed case is handled as follows, focusing on $\mathbb{E}^2$ for ease of notation. Orient $\mathbb{E}^2 \times \{\pm 1\}$ so that $\mathbb{E}^2 \times \{+1\}$ has the preferred orientation of $\mathbb{E}^2$ and $\mathbb{E}^2 \times \{-1\}$ has its reverse. Consider the homomorphism $\mathrm{sign}\colon \mathrm{Sim}(\mathbb{E}^2) \to \{\pm 1\}$ that is 1 on orientation-preserving similarities and -1 on orientation-reversing ones. We make $g \in \mathrm{Sim}(\mathbb{E}^2)$ act on $\mathbb{E}^2 \times \{\pm 1\}$ by $g \cdot (v, s) = (g(v), \mathrm{sign}(g) s)$ and note that this action is always orientation-preserving. Now orient $\mathscr{E}_n^{\pm} = \mathscr{E}_n \times \{\pm 1\}^n$ via the product orientation on $(\mathbb{E}^2 \times \{\pm 1\})^n$. The action of $\mathrm{Sim}(\mathbb{E}^2)$ on $\mathscr{E}_n^{\pm}$ preserves this orientation, and the orientation induced on $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$ from $\mathrm{Sim}(\mathbb{E}^2) \to \mathscr{E}_n^{\pm} \to \mathscr{C}_n^{\pm}(\mathbb{E}^2)$ will be our preferred one there.

Each of $\mathscr{C}_n^{\pm}(\mathbb{H}^2)$, $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$ and $\mathscr{C}_n(S^2)$ is acted on by $\{\pm 1\}^n$, and it will be important to understand how these actions interact with our preferred orientations. In all cases, an $\epsilon \in \{\pm 1\}^n$ preserves orientation if the product of the ϵ_i is $+1$ and reverses it otherwise.

5.8 Putting the geometries together

We now sketch a natural way of combining these configuration spaces for $\mathbb{H}^2$, $\mathbb{E}^2$ and S^2 together into a single smooth manifold. To begin, we consider the end of $\mathscr{C}_n(\mathbb{H}^2)$ where all the points coalesce. Given an element in $\mathscr{C}_n(\mathbb{H}^2)$, we can rescale the metric on $\mathbb{H}^2$ so that the diameter of the set of points is exactly 1 in the new metric. The closer the original points are in $\mathbb{H}^2$, the flatter the rescaled metric is. When the original points are very close, the new space is nearly isometric to $\mathbb{E}^2$ on the scale of the points in the final configuration. This makes it plausible that we should compactify this end of $\mathscr{C}_n(\mathbb{H}^2)$ by adding a copy of $\mathscr{C}_n(\mathbb{E}^2)$ at infinity to produce a manifold with boundary. Looking now at the particular end of $\mathscr{C}_n(S^2)$ where all the points come together, the exact same story applies to suggest that we should also compactify this end by adding a copy of $\mathscr{C}_n(\mathbb{E}^2)$. We could then glue our two compactifications together to get a nice manifold structure on $\mathscr{C}_n(\mathbb{H}^2) \cup \mathscr{C}_n(\mathbb{E}^2) \cup \mathscr{C}_n(S^2)$.

As discussed, the manifold $\mathscr{C}_n(S^2)$ has 2^{n-1} ends, and we can bring the others into our picture as follows. Regard an element of $\mathscr{C}_n(S^2)$ as n pairs of antipodal points where one point in each pair is labeled $+1$ and the other -1; the action of $\{\pm 1\}^n$ on $\mathscr{C}_n(S^2)$ now simply swaps the labels as appropriate. An end of $\mathscr{C}_n(S^2)$ can now be labeled by an ϵ in $\{\pm 1\}^n$ corresponding to the pattern of signs on the points that are coalescing; in fact, each end has two such labels, ϵ and $-\epsilon$, as there are two clusters of points we can focus on. The points in these two clusters have opposite signs and their positions differ by the antipodal map, which is orientation-reversing. Thus, to compactify all the ends of $\mathscr{C}_n(S^2)$, we should consider *signed* points in $\mathbb{E}^2$, up to the equivalence used to define $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$. Hence, we should be able to make $\mathscr{C}_n^{\pm}(\mathbb{E}^2) \cup \mathscr{C}_n(S^2)$ into a compact manifold with boundary to which we can attach a copy of $\mathscr{C}_n^{\pm}(\mathbb{H}^2)$.

That this whole sketch can be made precise is the content of:

5.9 Theorem *There is a smooth structure on $\mathscr{C}_n^{\pm}(\mathbb{H}^2) \cup \mathscr{C}_n^{\pm}(\mathbb{E}^2) \cup \mathscr{C}_n(S^2)$ making it into an oriented manifold $\mathscr{Y}$ of dimension $2n-3$ that is compatible with the previously defined smooth structures on $\mathscr{C}_n^{\pm}(\mathbb{H}^2)$, $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$ and $\mathscr{C}_n(S^2)$; the orientation of $\mathscr{Y}$ agrees with that of $\mathscr{C}_n(S^2)$ but is opposite of that on $\mathscr{C}_n^{\pm}(\mathbb{H}^2)$. The action of $\{\pm 1\}^n$ on the set $\mathscr{Y}$ is in fact an action by diffeomorphisms. The subset $\mathscr{C}_n^{\pm}(\mathbb{H}^2) \cup \mathscr{C}_n^{\pm}(\mathbb{E}^2)$ is a closed submanifold with boundary equal to $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$, and it is diffeomorphic to 2^{n-1} copies of $P^{n-2}(\mathbb{C}) \times [0,\infty)$.*

One consequence of Theorem 5.9 is that $\mathscr{Y}$ is diffeomorphic to $\mathscr{C}_n(S^2)$ itself. We will prove Theorem 5.9 over the next two sections.

6 Groups and quadratic forms

Our framework for building the smooth structure on $\mathscr{Y}$ will be the geometric transition $\mathbb{H}^2 \to \mathbb{E}^2 \to S^2$, studied via projective geometry from the perspective of [Cooper et al. 2018, Section 2.1]. To describe this transition, we use a family of quadratic forms and their automorphism groups.

6.1 The quadratic forms B_t

On $\mathbb{R}^3$ with coordinates (x, y, z), consider the family of quadratic forms

$$B_t = t(x^2 + y^2) + z^2, \quad \text{where } t \in \mathbb{R}.$$

We will use $B_t(\cdot\,,\cdot)$ for the associated bilinear forms. Let $\overline{Q}_t$ be the quadric surface where $B_t = 1$, which is an ellipsoid when $t > 0$, two planes when $t = 0$, and a hyperboloid of two sheets when $t < 0$. We define Q_t to be the subset of $\overline{Q}_t$ where $z > 0$.

The group $\mathbb{R}_{>0}$ acts on $\mathbb{R}^3$ by dilating the x- and y-coordinates:

$$s \cdot (x, y, z) = (sx, sy, z). \tag{6.2}$$

Note this action satisfies $B_t(s \cdot v) = B_{s^2 t}(v)$ for $v \in \mathbb{R}^3$ and hence $v \mapsto s \cdot v$ gives an isometry $(\mathbb{R}^3, B_t) \to (\mathbb{R}^3, B_{t/s^2})$. Thus $s \cdot \overline{Q}_t = \overline{Q}_{t/s^2}$. Consequently, each $(\overline{Q}_t, B_t)$ with $t > 0$ is isometric to $S^2 = (B_1, \overline{Q}_1)$ and each $(\overline{Q}_t, B_t)$ with $t < 0$ is isometric to $(\overline{Q}_{-1}, B_{-1})$, which is two copies of $\mathbb{H}^2$ with the negative of its metric from Section 5.1. We give the two planes making up $\overline{Q}_0$ the Euclidean metric associated to the quadratic form $x^2 + y^2$. We will always orient $\overline{Q}_t$ by the convention that u_1, u_2 is a positive basis for $T_v \overline{Q}_t$ if and only if v, u_1, u_2 is a positive basis for $\mathbb{R}^3$; in particular, e_1, e_2 is a positive basis for $T_{e_3} \overline{Q}_t$ and $e_1, -e_2$ is a positive basis for $T_{-e_3} \overline{Q}_t$.

6.3 The groups G_t

For $t \neq 0$, let $G_t = \mathrm{SO}(B_t)$ be the subgroup of $\mathrm{SL}_3\mathbb{R}$ that preserves B_t. For $t = 0$, we define G_0 to be the subgroup of $\mathrm{SL}_3\mathbb{R}$ that preserves B_0 and acts on $\overline{Q}_0$ by an isometry; concretely,

$$G_0 = \left\{ \begin{pmatrix} A & b \\ 0 & \det A \end{pmatrix} \in \mathrm{SL}_3\mathbb{R} \;\middle|\; A \in O(2) \right\}. \tag{6.4}$$

From our definition of the orientation on $\overline{Q}_t$, since $G_t \leq \mathrm{SL}_3\mathbb{R}$, we see that G_t acts on $\overline{Q}_t$ preserving orientation. For $t > 0$, the group $G_t \cong G_1 \cong \mathrm{Isom}^+(S^2)$ is connected. However, for $t \leq 0$, the group G_t has two connected components, corresponding to whether an element preserves or interchanges the two components of $\overline{Q}_t$.

If $t < 0$, the identity component of G_t is

$$\mathrm{Isom}^+(Q_t) \cong \mathrm{Isom}^+(Q_{-1}) = \mathrm{Isom}^+(\mathbb{H}^2).$$

We can regard the full isometry group $\mathrm{Isom}(Q_t)$ as the subgroup of the full orthogonal group $O(B_t)$ that preserves Q_t. There is an isomorphism $\mathrm{Isom}(Q_t) \to G_t$ given by $g \mapsto (\det g)g$, so $G_t \cong G_{-1} \cong \mathrm{Isom}(\mathbb{H}^2)$. Note that the action of G_{-1} on $\overline{Q}_{-1}$ corresponds to the action of $\mathrm{Isom}(\mathbb{H}^2)$ on $\mathbb{H}^2 \times \{\pm 1\}$ from Section 5.5, with Q_t playing the role of $\mathbb{H}^2 \times \{1\}$. A similar argument shows that $G_0 \cong \mathrm{Isom}(\mathbb{E}^2)$.

When $t = 0$, the groups $\mathbb{R}_{>0}$ and G_0 both act on $\overline{Q}_0$; their actions can be combined to give the action of a larger group $\widehat{G}_0 \cong \mathrm{Sim}(\mathbb{E}^2)$. More precisely, define $\widehat{G}_0$ to be the subgroup of $\mathrm{SL}_3\mathbb{R}$ generated by G_0 and matrices of the form $\left(\begin{smallmatrix} sI & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $s > 0$. The action of $\widehat{G}_0$ preserves $\overline{Q}_0$. Restriction to $\overline{Q}_0$ defines a homomorphism from $\widehat{G}_0$ to $\mathrm{Sim}(\mathbb{E}^2)$, which you can check is an isomorphism.

6.5 The groups U_t

The groups G_t are closely related to certain subgroups of $\mathrm{SL}_2\mathbb{C}$ which we now describe. For $t \in \mathbb{R}$, let

$$U_t = \left\{ \begin{pmatrix} a & b \\ -t\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C} \text{ with } |a|^2 + t|b|^2 = 1 \right\}.$$

For $t \neq 0$, the group U_t is the subgroup of $\mathrm{SL}_2\mathbb{C}$ that preserves the Hermitian form given by $J_t = \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$, as can be seen by solving $A^* J_t A = J_t$ in the form $A^* J_t = J_t A^{-1}$. In particular, $U_1 = \mathrm{SU}_2$ and $U_{-1} = \mathrm{SU}_{1,1} \cong \mathrm{SL}_2\mathbb{R}$. More generally, if $T_s = \left(\begin{smallmatrix} s & 0 \\ 0 & 1 \end{smallmatrix}\right)$, you can easily check that $T_s U_t T_s^{-1} = U_{t/s^2}$, so $U_t \cong U_1 = \mathrm{SU}_2$ for $t > 0$ and $U_t \cong U_{-1} = \mathrm{SU}_{1,1}$ for $t < 0$.

Let U_t' be the normalizer of U_t in $\mathrm{SL}_2\mathbb{C}$. When $t \neq 0$, we define

$$\gamma_t = \begin{pmatrix} 0 & t^{-1/2} \\ -t^{1/2} & 0 \end{pmatrix},$$

which is in U_t', with conjugation by γ_t inducing the automorphism of U_t that is complex conjugation of the matrix entries. For $t > 0$, we have $U_t' = U_t$, and, for $t = 0$, the group U_0' consists of upper triangular matrices with determinant 1. For $t < 0$, the subgroup U_t' is conjugate to the subgroup $\mathrm{SL}_2^{\pm}(\mathbb{R})$ from Section 3.2; it is generated by U_t and γ_t.

The Lie algebra $\mathfrak{u}_t$ of U_t is easy to compute: when $t \neq 0$, the relation $A^* J_t = J_t A^{-1}$ turns into $X^* J_t = -J_t X$; combining this with the trace 0 condition defining $\mathfrak{sl}_2\mathbb{C}$ gives

(6.6) $$\mathfrak{u}_t = \left\{ i \begin{pmatrix} z & -x - iy \\ -t(x - iy) & -z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

It is easy to check that this formula for $\mathfrak{u}_t$ is also valid when $t = 0$. Let $\varphi_t : \mathbb{R}^3 \to \mathfrak{u}_t$ be given by

$$\varphi_t(x, y, z) = i \begin{pmatrix} z & -x - iy \\ -t(x - iy) & -z \end{pmatrix}.$$

From now on, we identify $\mathfrak{u}_t$ with $\mathbb{R}^3$ via φ_t, for example getting a linear action of G_t on $\mathfrak{u}_t$ by requiring $g \cdot \varphi_t(v) = \varphi_t(g \cdot v)$. With this identification, a routine calculation shows that the Killing form on $\mathfrak{u}_t$ is a multiple of B_t, specifically $-8B_t$. As any Lie algebra automorphism in $\mathrm{Aut}(\mathfrak{u}_t)$ preserves the Killing form, we see $\mathrm{Aut}(\mathfrak{u}_t) \le O(B_t)$.

As U_t' normalizes U_t, we get a homomorphism $\psi_t : U_t' \to \mathrm{Aut}(\mathfrak{u}_t)$ by the adjoint action $\psi_t(A) = \mathrm{Ad}_A$. A key relationship between the groups G_t and U_t is:

6.7 Lemma *For $t \neq 0$, the action of G_t on $\mathfrak{u}_t$ gives $G_t = \mathrm{Aut}(\mathfrak{u}_t)$ as subgroups of $\mathrm{GL}(\mathfrak{u}_t)$. Moreover, $\mathrm{Aut}(\mathfrak{u}_t) = \mathrm{Aut}(U_t) = \psi_t(U_t') \cong U_t'/\pm I$.*

Proof First, note $\mathfrak{u}_t$ carries a natural orientation determined by the ordered basis $v, w, [v, w]$, where v and w are any linearly independent elements of $\mathfrak{u}_t$. Now $\mathrm{Aut}(\mathfrak{u}_t)$ preserves this orientation and the Killing form, giving $\mathrm{Aut}(\mathfrak{u}_t) \le \mathrm{SO}(B_t) = G_t$.

Now U_t' acts on U_t by conjugation, giving $\widetilde{\psi}_t : U_t' \to \mathrm{Aut}(U_t)$ whose kernel is $\pm I$. The group U_t is connected, so the derivative at I gives an injective homomorphism $\mathrm{Aut}(U_t) \to \mathrm{Aut}(\mathfrak{u}_t)$. Note that the ψ_t defined previously is the composition of $\widetilde{\psi}_t$ and $\mathrm{Aut}(U_t) \to \mathrm{Aut}(\mathfrak{u}_t)$, so the kernel of ψ_t is also $\pm I$.

So $\psi_t(U_t) \cong U_t/\{\pm I\}$ is a connected Lie subgroup of dimension 3 inside G_t, which forces $\psi_t(U_t)$ to be the identity subgroup of G_t. If $t > 0$, then $U_t' = U_t$ and G_t are all connected, so $\psi_t(U_t') = G_t$. If $t < 0$, recall that, for $u \in \mathfrak{u}_t$, we have $\mathrm{Ad}_{\gamma_t}(u) = \bar{u}$, implying

$$\psi_t(\gamma_t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{is in } G_t,$$

and again $\psi_t(U_t') = G_t$. All the claims of the lemma now follow since $\psi_t(U_t') \le \mathrm{Aut}(U_t) \le \mathrm{Aut}(\mathfrak{u}_t) \le G_t$ must all actually be equalities. □

The analogue of Lemma 6.7 when $t = 0$ is:

6.8 Lemma *As subgroups of $\mathrm{GL}(\mathfrak{u}_0)$, we have $\mathrm{Aut}(\mathfrak{u}_0) = \widehat{G}_0$. Moreover, $\mathrm{Aut}(\mathfrak{u}_0) = \mathrm{Aut}(U_0)$. Finally, $\psi_0(U_0') \cong U_0'/\pm I$ and $\psi_0(U_0) \cong U_0/\pm I$ are the identity components of $\widehat{G}_0$ and G_0, respectively.*

Proof Write $\mathfrak{u}_0 = L \oplus \langle h \rangle$, where $L = \left\{ \left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right) \mid \alpha \in \mathbb{C} \right\}$, which we view as a 1-dimensional complex vector space, and $h = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. We have $[v, w] = 0$ if $v, w \in L$, and $[h, v] = 2iv$ for $v \in L$. Observe that $g \in \mathrm{Aut}(\mathfrak{u}_0)$ must preserve L, and so is represented by a matrix of the form $\left(\begin{smallmatrix} A & * \\ 0 & a \end{smallmatrix}\right)$. In order for g to respect the bracket, we must have $[ah, A(v)] = A(2iv)$, or equivalently $iaA(v) = A(iv)$. As a self-map of L, we know A is $\mathbb{R}$-linear, so applying this relation twice we see that $a^2 = 1$. When $a = 1$, the map A is $\mathbb{C}$-linear; when $a = -1$, it is conjugate linear. In either case, $A = s \cdot \overline{A}$, where $s \in \mathbb{R}_{>0}$ and $\overline{A} \in O(2)$. Hence, $\mathrm{Aut}(\mathfrak{u}_0) \subset \widehat{G}_0$.

For the converse, observe the adjoint action gives a homomorphism $\widetilde{\psi}_0 \colon U_0' \to \mathrm{Aut}(U_0)$ whose kernel is $\pm I$. As before, $\mathrm{Aut}(U_0) \to \mathrm{Aut}(\mathfrak{u}_0)$ is injective as U_0 is connected, and so ψ_0 also has kernel $\pm I$. Hence, $\psi_0(U_0') = U_0'/\{\pm I\}$ is a connected Lie group of dimension 4 inside $\widehat{G}_0$. As $\dim \widehat{G}_0 = 4$, this means $\psi_0(U_0') = U_0'/\{\pm I\}$ is the identify component of $\widehat{G}_0$. The elements of $\mathrm{Aut}(U_0)$ and $\mathrm{Aut}(\mathfrak{u}_0)$ induced by entrywise complex conjugation correspond to

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{in } \widehat{G}_0.$$

As $\widehat{G}_0$ is generated by its identity component and C, we have $\mathrm{Aut}(U_0) = \mathrm{Aut}(\mathfrak{u}_0) = \widehat{G}_0$, as desired. □

6.9 Further properties of G_t and U_t

Combining Lemmas 6.7 and 6.8, we will view G_t as a subgroup of $\mathrm{Aut}(U_t)$, and hence G_t acts on U_t. Indeed, for $t \neq 0$, the homomorphism $\psi_t \colon U_t' \to G_t$ is surjective, so concretely the action of G_t on U_t is given by $\psi_t(A) \cdot B = ABA^{-1}$. For all t, the element

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

acts on $\mathfrak{u}_t$ as complex conjugation, and $C = \psi_t(\gamma_t)$ for $t \neq 0$. The action also satisfies

$$\exp(g \cdot u) = g \cdot \exp u \quad \text{and} \quad \mathrm{tr}(g \cdot B) = \mathrm{tr}\, B,$$

which may be easily checked for $g \in \mathrm{im}\, \psi_t(U_t')$ or $g = C$, and so hold for all $g \in G_t$.

Recall that $T_s U_t T_s^{-1} = U_{t/s^2}$, where $T_s = \left(\begin{smallmatrix} s & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The adjoint action of T_s on $\mathfrak{sl}_2\mathbb{C}$ thus takes $\mathfrak{u}_t$ to $\mathfrak{u}_{t/s^2}$; under our identification of the latter two with $\mathbb{R}^3$, this is precisely the dilation action defined in (6.2). In particular, it carries $\overline{Q}_t \subset \mathfrak{u}_t$ to $\overline{Q}_{t/s^2} \subset \mathfrak{u}_{t/s^2}$ and B_t to B_{t/s^2}. Moreover, the diagram

$$\begin{array}{ccc} \overline{Q}_t & \xrightarrow{\mathrm{Ad}_{T_s}} & \overline{Q}_{t/s^2} \\ {\scriptstyle \psi_t(g)}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{t/s^2}(T_s g T_s^{-1})} \\ \overline{Q}_t & \xrightarrow{\mathrm{Ad}_{T_s}} & \overline{Q}_{t/s^2} \end{array}$$

commutes for each $g \in U_t$, since both compositions send $v \in \mathfrak{u}_t$ to $T_s g v g^{-1} T_s^{-1}$.

6.10 Remark We will always orient $\mathfrak{u}_t$ by taking (x, y, z) from (6.6) to be oriented coordinates. This is consistent with our earlier conventions in the following sense: if we orient G_t as isometries of $\overline{Q}_t$ using the convention of Section 5.7, then the homomorphism $U_t \to G_t$ is orientation-preserving. We leave the detailed check of this to you, but note that as the orientations of $\overline{Q}_t$, G_t and U_t all vary continuously in t, it suffices to check this for $t = 0$.

6.11 A $\mathbb{C}^\times$ action

The action of $\mathbb{R}_{>0}$ on $\mathbb{R}^3$ extends to an action of $\mathbb{C}^\times$ as follows. Writing $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, where (x, y, z) is identified with $(x + iy, z)$, we define $u \cdot (w, z) = (uw, z)$ for $u \in \mathbb{C}^\times$. An easy calculation shows that

$$\varphi_{t/|u|^2}(u \cdot v) = T_u \varphi_t(v) T_u^{-1}, \quad \text{where } T_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.12}$$

The $\mathbb{C}^\times$ action combines the actions of $\mathbb{R}_{>0}$ and the adjoint action of the subgroup of U_t consisting of diagonal matrices, in the sense that if $\iota\colon S^1 \to U_t$ is given by

$$\iota(u) = \begin{pmatrix} \sqrt{u} & 0 \\ 0 & \sqrt{u}^{-1} \end{pmatrix},$$

then $\varphi_t(u \cdot v) = \iota(u) \cdot \varphi_t(v)$.

6.13 The exponential map

Suppose $A \in U_t$, and let $c = \operatorname{tr} A$. A quick look at the definitions of U_t and $\mathfrak{u}_t$ shows that we can write $A = \frac{1}{2} c I + u$, where $u \in \mathfrak{u}_t$. Moreover, if $g \in G_t$, we have $g \cdot A = \frac{1}{2} c I + g \cdot u$, where G_t acts on U_t in the right-hand side of the equality and acts on $\mathfrak{u}_t$ in the left-hand side; this is easily checked when $g \in \operatorname{im} \psi_t$ and when $g = C$, and so holds for all elements of G_t.

6.14 Lemma *If $u \in \overline{Q}_t \subset \mathfrak{u}_t$, then $\exp(\alpha u) = (\cos\alpha) I + (\sin\alpha) u$. Geometrically, the element $\exp(\alpha u)$ acts on $\overline{Q}_t$ as the isometry in G_t that fixes u and rotates about it anticlockwise by angle 2α with respect to the orientation of $\overline{Q}_t$ fixed in Section 6.1.*

Proof We first prove the claims for $h = (0, 0, 1) \in \overline{Q}_t$, which corresponds to $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in \mathfrak{u}_t$ under φ_t. First, we have

$$\exp(\alpha h) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} = (\cos\alpha) I + (\sin\alpha) h,$$

as claimed. A straightforward calculation shows

$$\psi_t(\exp(\alpha h)) = \begin{pmatrix} \cos(2\alpha) & -\sin(2\alpha) & 0 \\ \sin(2\alpha) & \cos(2\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which verifies the geometric claim.

For a general $u \in \overline{Q}_t$, choose $g \in G_t$ with $g \cdot h = u$. Then

$$\exp(\alpha u) = \exp(g \cdot \alpha h) = g \cdot \exp(\alpha h) = (\cos\alpha) I + (\sin\alpha) g \cdot h = (\cos\alpha) I + (\sin\alpha) u.$$

We also have $\psi_t(\exp(\alpha u)) = \psi_t(g \cdot \exp(\alpha h)) = g\psi_t(\exp(\alpha h))g^{-1}$ and, as noted in Section 6.3, the action of G_t on $\overline{Q}_t$ preserves orientation. Therefore, $\exp(\alpha u)$ acts as a 2α-anticlockwise rotation about u, as desired. □

6.15 Corollary *If $c \in (-2,2)$ and $c = 2\cos\alpha$, the map $r : \overline{Q}_t \to \operatorname{tr}^{-1}(c) \cap U_t$ given by $r(u) = \exp(\alpha u)$ is a diffeomorphism.*

Proof It is evident from Lemma 6.14 that $\operatorname{tr}(\exp(\alpha u)) = 2\cos\alpha$, so $\operatorname{im} r \subset \operatorname{tr}^{-1}(c)$. The hypothesis $c \in (-2,2)$ implies that $\sin\alpha \neq 0$, so the same lemma also shows that r and its derivative are injective. If $\operatorname{tr} A = c$, we can write $A = (\cos\alpha)I + (\sin\alpha)u$ for some $u \in \mathfrak{u}_t$; the condition that $\det A = 1$ implies that $u \in \overline{Q}_t$, so r is surjective. □

7 Joining configuration spaces

In this section, we construct the smooth structure on $\mathscr{C}_n(S^2) \cup \mathscr{C}_n^{\pm}(\mathbb{E}^2) \cup \mathscr{C}_n^{\pm}(\mathbb{H}^2)$ promised in Section 5.8. Consider the set

$$\overline{\mathscr{R}}_n = \bigl\{(t, v_1, \dots, v_n) \in \mathbb{R} \times (\mathbb{R}^3)^n \mid v_i \in \overline{Q}_t \text{ for all } i;\ v_i \notin \{v_j, -v_j\} \text{ for some } i, j \text{ if } n > 1\bigr\}$$

By calculating the Jacobian matrix of its n defining equations, you can check $\overline{\mathscr{R}}_n$ is a smooth $(2n+1)$-dimensional submanifold of $\mathbb{R} \times (\mathbb{R}^3)^n \cong \mathbb{R}^{3n+1}$. A typical element of $\overline{\mathscr{R}}_n$ will be denoted by v with constituent vectors v_i and t value denoted by $t(v)$. Given $w \in \mathbb{R}^3$ which does not lie on the z-axis, there is a unique t such that $w \in Q_t$. It follows that for $n > 1$, any $v \in \overline{\mathscr{R}}_n$ is uniquely determined by the vectors $(v_1, \dots, v_n)$.

Recall that $\mathbb{R}_{>0}$ acts on $\mathbb{R}^3$ by dilating the x- and y-coordinates: $s \cdot (x, y, z) = (sx, sy, z)$. Since $s \cdot \overline{Q}_t = \overline{Q}_{t/s^2}$, the group $\mathbb{R}_{>0}$ acts on $\overline{\mathscr{R}}_n$ by

$$s \cdot (t, v_1, \dots, v_n) = (t/s^2, s \cdot v_1, \dots, s \cdot v_n).$$

For $t \in \mathbb{R}$, we set $\overline{\mathscr{R}}_n^t = \{v \in \overline{\mathscr{R}}_n \mid t(v) = t\}$. The group G_t acts on the slice $\overline{\mathscr{R}}_n^t$ via $g \cdot (t, v_1, \dots, v_n) = (t, g \cdot v_1, \dots, g \cdot v_n)$. The action of $\mathbb{R}_{>0}$ intertwines these actions, in the sense that, if $g \in G_t$ and we define $(s \cdot g)(v) = s \cdot (g \cdot (s^{-1} \cdot v))$ for $v \in \overline{Q}_{t/s^2}$, then $s \cdot g \in G_{t/s^2}$.

We define the space $\mathscr{Y}$ to be the quotient of $\overline{\mathscr{R}}_n$ by the simultaneous actions of $\mathbb{R}_{>0}$ and all the G_t, where $\mathscr{Y}$ has the quotient topology. We let $\mathscr{Y}_-$, $\mathscr{Y}_0$ and $\mathscr{Y}_+$ be the images of $\overline{\mathscr{R}}_n^{<0}$, $\overline{\mathscr{R}}_n^{0}$ and $\overline{\mathscr{R}}_n^{>0}$ in $\mathscr{Y}$. The main result of this section is:

7.1 Theorem *The space $\mathscr{Y}$ can be given the structure of a smooth $(2n-3)$-dimensional manifold such that the quotient map $\sigma : \overline{\mathscr{R}}_n \to \mathscr{Y}$ is a smooth submersion. The subsets $\mathscr{Y}_-$ and $\mathscr{Y}_+$ are open submanifolds of $\mathscr{Y}$, while $\mathscr{Y}_0$ is a closed codimension-1 submanifold. They are diffeomorphic to $\mathscr{C}_n^{\pm}(\mathbb{H}^2)$, $\mathscr{C}_n(S^2)$ and $\mathscr{C}_n^{\pm}(\mathbb{E}^2)$, respectively. Finally, the action of $\{\pm 1\}^n$ on $\mathscr{Y}$ is by diffeomorphisms.*

We will need the following technical tool in our eventual proof of Theorem 7.1:

7.2 Lemma *Suppose $(t, v) \in \overline{\mathcal{R}}_1$ with $v = (x, y, z)$ and $z > 0$. Let L be the unique shortest geodesic segment in Q_t joining v to $e_3 = (0, 0, 1)$. Then*

$$\begin{pmatrix} 1-tx^2/(1+z) & -txy/(1+z) & x \\ -txy/(1+z) & 1-ty^2/(1+z) & y \\ -tx & -ty & z \end{pmatrix}$$

is an element of G_t taking e_3 to v. When $t \leq 0$, it is a translation whose axis contains L; when $t > 0$, it is a rotation with angle $\theta < \pi$ whose invariant equator contains L. Finally, the inverse of $W(t, v)$ is $W(t, (-x, -y, z))$.

Proof When $t = 0$, we have $z = 1$ and the formula for $W(t, v)$ greatly simplifies, making it clear that $W(t, v)$ is the element of G_0 that translates e_3 to v. So we henceforth assume $t \neq 0$. A tedious but straightforward calculation using that $Q_t(v) = 1$ shows that the columns of $W(t, v)$ have the same B_t inner products as the standard basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$. Hence, $W(t, v)$ preserves B_t, and moreover it is in G_t as we can connect $W(t, v)$ to the identity matrix by moving v to e_3 along L. Now $W(t, v)$ takes e_3 to v, and we can see the claimed geometric description by noting that $W(t, v)$ fixes $(-y, x, 0)$, which is a basis for the B_t-orthogonal complement to the plane spanned by $\{e_3, v\}$. Noting that e_3, v and $(-x, -y, z)$ all lie on the geodesic containing L, the fact that $W(t, v)$ and $W(t, (-x, -y, z))$ are inverses follows from their geometric characterizations. □

7.3 A local model

Rather than tackle Theorem 7.1 all at once, we start with a more specialized statement that exhibits all the key issues. Given $0 < \delta < \frac{\pi}{2}$, let $\mathcal{V}_n = \{v \in \overline{\mathcal{R}}_n \mid \text{all } v_i \in Q_t\}$, and consider

$$\mathcal{R}_n = \{v \in \mathcal{V}_n \mid B_t(v_1, v_i) > \cos\delta \text{ for all } i\}.$$

Note that $\mathcal{R}_n$ is an open subset of $\overline{\mathcal{R}}_n$. When $t \leq 0$, the condition $B_t(v_1, v_i) > \cos\delta$ is automatically satisfied: applying an element of G_t, we can assume that $v_1 = e_3$ and then $B_t(v_1, v_i)$ is just the z-coordinate of $v_i \in Q_t$, which is at least 1. When $t > 0$, the condition on $B_t(v_1, v_i)$ is equivalent to the geometric distance in Q_t between v_1 and v_i being less than δ. Since the dilation action of $\mathbb{R}_{>0}$ on $\overline{\mathcal{R}}_n$ preserves the B_t, the set $\mathcal{R}_n$ is invariant under this action.

Let $\mathcal{Y}^\delta = \sigma(\mathcal{R}_n)$, where $\sigma\colon \overline{\mathcal{R}}_n \to \mathcal{Y}$ is the quotient map, and take $\mathcal{Y}^\delta_-$, $\mathcal{Y}^\delta_0$ and $\mathcal{Y}^\delta_+$ to be its intersection with the subsets $\mathcal{Y}_-$, $\mathcal{Y}_0$ and $\mathcal{Y}_+$ of $\mathcal{Y}$. We will show:

7.4 Theorem *For any $n > 1$ and $0 < \delta < \frac{\pi}{2}$, there is a smooth structure on $\mathcal{Y}^\delta$ making it diffeomorphic to $P^{n-2}(\mathbb{C}) \times \mathbb{R}$. With respect to this smooth structure, the projection $\sigma\colon \mathcal{R}_n \to \mathcal{Y}^\delta$ is a submersion. The subsets $\mathcal{Y}^\delta_-$, $\mathcal{Y}^\delta_0$ and $\mathcal{Y}^\delta_+$ are identified with $P^{n-2}(\mathbb{C}) \times (-\infty, 0)$, $P^{n-2}(\mathbb{C}) \times \{0\}$ and $P^{n-2}(\mathbb{C}) \times (0, \infty)$, respectively.*

The smooth structure on $\mathcal{Y}^\delta$ is uniquely characterized by the requirement that σ be a submersion; this is a consequence of the following basic fact that we will use repeatedly: if $\pi\colon M \to \overline{M}$ is a surjective submersion and $f\colon M \to N$ is any smooth map that is constant on each fiber of π, then the induced map $\bar{f}\colon \overline{M} \to N$ is smooth [Lee 2013, Theorem 4.30].

To prove Theorem 7.4, we focus on a subset $\mathcal{R}_n''$ of $\mathcal{R}_n$ which still surjects $\mathcal{Y}^\delta$ but where the fibers $\mathcal{R}_n'' \xrightarrow{\sigma} \mathcal{Y}^\delta$ are simple enough that we can push the smooth structure forward. To this end, consider the (degenerate) quadratic form H on $\mathbb{R}^3$ given by x^2+y^2 and define

$$\mathcal{R}_n' = \{v \in \mathcal{R}_n \mid v_1 = e_3\} \quad \text{and} \quad \mathcal{R}_n'' = \left\{ v \in \mathcal{R}_n' \;\middle|\; \sum_{i=2}^n H(v_i) = 1 \right\}.$$

Unrolling all the definitions, if we set $v_i = (x_i, y_i, z_i)$, then equivalently

$$\textbf{(7.5)}\quad \mathcal{R}_n'' = \left\{ (t, v_1, \dots, v_n) \in \mathbb{R}^{3n+1} \;\middle|\; t(x_i^2+y_i^2)+z_i^2 = 1,\ z_i > \cos\delta \text{ for all } i;\ v_1 = e_3;\ \sum_{i=2}^n x_i^2+y_i^2 = 1 \right\}.$$

It is straightforward to check that $\mathcal{R}_n'$ and $\mathcal{R}_n''$ are smooth submanifolds of $\mathcal{R}_n$.

7.6 Lemma *There are submersions $\mathcal{R}_n \xrightarrow{p_1} \mathcal{R}_n'$ and $\mathcal{R}_n' \xrightarrow{p_2} \mathcal{R}_n''$ that commute with $\mathcal{R}_n \xrightarrow{\sigma} \mathcal{Y}^\delta$.*

Proof Let us start with $p_2\colon \mathcal{R}_n' \to \mathcal{R}_n''$, which we define via the $\mathbb{R}_{>0}$ action by

$$v \mapsto \frac{1}{\sqrt{\sum H(v_i)}} \cdot v.$$

The sets $s \cdot \mathcal{R}_n''$ as s varies are a family of local sections to p_2 which pass through every point of $\mathcal{R}_n'$, showing that p_2 is a submersion.

Our map $p_1\colon \mathcal{R}_n \to \mathcal{R}_n'$ is defined using Lemma 7.2 to move v_1 to e_3 in a consistent way: given $v \in \mathcal{R}_n$ with $v_i = (x_i, y_i, z_i)$, define

$$p_1(v) = W(t(v), v_1)^{-1} \cdot v = W(t(v), (-x_1, -y_1, z_1)) \cdot v,$$

which is a smooth function by the formula for W in Lemma 7.2. To see this is a submersion at $w \in \mathcal{R}_n$, pick a path $\gamma\colon (t(w)-\eta, t(w)+\eta) \to \mathcal{V}_1$ for some $\eta > 0$ with $t(\gamma(s)) = s$ for all s and $\gamma(t(w)) = w_1$. Then the map $\beta_w\colon \mathcal{R}_n' \to \mathcal{R}_n$ given by

$$\beta_w(v) = W\big(t(v), \gamma(t(v))\big) \cdot v$$

gives a local section to p_1 whose image contains w. □

Note that both $\mathcal{R}_n'$ and $\mathcal{R}_n''$ are invariant under the S^1 action of rotations about the z-axis. This S^1 is a subgroup of G_t for all t, giving a continuous map

$$j\colon \mathcal{R}_n''/S^1 \to \mathcal{Y}^\delta \quad \text{via the composition} \quad \mathcal{R}_n'' \xrightarrow{p_3} \mathcal{R}_n''/S^1 \to \sigma(\mathcal{R}_n'') \hookrightarrow \mathcal{Y}^\delta,$$

where p_3 is the topological quotient map $\mathcal{R}_n'' \to \mathcal{R}_n''/S^1$.

7.7 Lemma *The map j is a homeomorphism.*

Proof This S^1 can be viewed as the subgroup of G_t that stabilizes e_3, so any $g \in G_t \setminus S^1$ moves $\{v \in \mathcal{R}'_n \mid t(v) = t\}$ to something disjoint from $\mathcal{R}'_n$. Additionally, any $s \neq 1$ in $\mathbb{R}_{>0}$ takes $\mathcal{R}''_n$ completely off itself since $H(s \cdot v) = s^2 H(v)$. It follows that j is injective. Lemma 7.6 implies that j is surjective, so it remains to check that j^{-1} is continuous.

Given $U \subset \mathcal{R}''_n / S^1$ open, we must check that $j(U)$ is open in $\mathcal{Y}^\delta$, ie that $\sigma^{-1}(j(U))$ is open in $\mathcal{R}_n$. This too follows from Lemma 7.6, since $\sigma^{-1}(j(U)) = p_1^{-1}\big(p_2^{-1}(p_3^{-1}(U))\big)$, and p_1, p_2 and p_3 are all continuous. □

Proof of Theorem 7.4 The S^1 action on $\mathcal{R}''_n$ is free and proper, so $\mathcal{R}''_n / S^1 = \mathcal{Y}^\delta$ inherits a smooth structure from $\mathcal{R}''_n$, where we have identified $\mathcal{R}''_n / S^1$ with $\mathcal{Y}^\delta$ via j from Lemma 7.7. In addition, $\sigma \colon \mathcal{R}_n \to \mathcal{Y}^\delta$ is a submersion, since $\sigma = p_3 \circ p_2 \circ p_1$ and each of the individual factors is a submersion.

It remains to identify the topology of $\mathcal{Y}^\delta$ and its various pieces. Consider the smooth map $\kappa \colon \mathcal{R}''_n \to \mathbb{R} \times S^{2n-3}$ defined by $\kappa(v) = (t(v), x_2, y_2, \dots, x_n, y_n)$, where as usual $v_i = (x_i, y_i, z_i)$. From the description of $\mathcal{R}''_n$ in (7.5), we see κ is injective with image the open set

(7.8)
$$\mathcal{T}_n = \left\{ (t, x_2, y_2, \dots, x_n, y_n) \in \mathbb{R} \times S^{2n-3} \;\middle|\; t < \frac{\sin^2(\delta)}{\max_i (x_i^2 + y_i^2)} \right\}$$

and moreover that κ has a smooth inverse on $\mathcal{T}_n$. Therefore, the map κ is a diffeomorphism onto $\mathcal{T}_n$. There is a natural action of S^1 on $\mathbb{R} \times S^{2n-3}$ by viewing the latter as a subset of $\mathbb{R} \times \mathbb{C}^{n-1}$ and acting diagonally on the $\mathbb{C}$ factors. As this action is compatible with the S^1 action on $\mathcal{R}''_n$, we have an embedding of $\mathcal{Y}^\delta = \mathcal{R}''_n / S^1$ into $\mathbb{R} \times P^{n-2}(\mathbb{C})$. Since $\mathcal{T}_n$ is simply the region lying below the graph of a piecewise smooth function $S^{2n-3} \to \mathbb{R}$, we see that $\mathcal{Y}^\delta$ is indeed diffeomorphic to $\mathbb{R} \times P^{n-2}(\mathbb{C})$. The remaining conclusions of the theorem are now easily checked. □

Proof of Theorem 7.1 For $0 < \delta < \frac{\pi}{2}$, let $\mathcal{R}^*_n = \{v \in \overline{\mathcal{R}}_n \mid B_t(v_1, v_i) > \cos\delta \text{ for all } i\}$, which contains $\mathcal{R}_n$ as an open subset. A $v \in \overline{\mathcal{R}}_n$ with $t(v) > 0$ is in $\mathcal{R}^*_n$ if and only if the geometric distance in $Q_{t(v)}$ from v_1 to v_i is less than δ for all i. When $t(v) \le 0$, a $v \in \overline{\mathcal{R}}_n$ is in $\mathcal{R}^*_n$ exactly when the v_i all belong to the same component of $\overline{Q}_{t(v)}$. As $\mathcal{R}^*_n$ is an open subset of $\overline{\mathcal{R}}_n$ which is invariant under the actions of $\mathbb{R}_{>0}$ and the G_t, $\sigma(\mathcal{R}^*_n)$ is an open subset of $\mathcal{Y}$. Note that $\sigma(\mathcal{R}_n) \subset \sigma(\mathcal{R}^*_n)$; we claim that these two sets are equal. Indeed, given $v \in \mathcal{R}^*_n$, we can find $g \in G_{t(v)}$ with $g \cdot v_1 = e_3$. Since $\delta < \frac{\pi}{2}$, the element $g \cdot v$ is in $\mathcal{R}_n$, and so $\sigma(\mathcal{R}^*_n) \subset \sigma(\mathcal{R}_n)$. Hence, $\sigma(\mathcal{R}^*_n) = \sigma(\mathcal{R}_n) = \mathcal{Y}^\delta$ is open in $\mathcal{Y}$.

The group $\{\pm 1\}^n$ acts on $\overline{\mathcal{R}}_n$, where the action is given by $(\epsilon \cdot v)_i = \epsilon_i v_i$. This action commutes with the actions of G_t and $\mathbb{R}_{>0}$. Let $\mathbb{1}$ be the identity element of $\{\pm 1\}^n$. The subgroup $\pm\mathbb{1}$ preserves $\overline{\mathcal{R}}^*_n$, and if $\epsilon \neq \pm\mathbb{1}$, then $\epsilon \cdot \mathcal{R}^*_n \cap \mathcal{R}^*_n = \varnothing$. For $\chi \in \{\pm 1\}^n / \pm\mathbb{1}$, let $\mathcal{R}^\chi_n = \chi \cdot \mathcal{R}^*_n$. The sets $\mathcal{R}^\chi_n$ are disjoint open subsets of $\overline{\mathcal{R}}_n$ which are invariant under the actions of $\mathbb{R}_{>0}$ and the G_t. Together with $\overline{\mathcal{R}}^{>0}_n$, they form an open cover of $\overline{\mathcal{R}}_n$.

Let $\mathcal{Y}^\chi = \sigma(\mathcal{R}_n^\chi)$, which is an open subset of $\mathcal{Y}$. By Theorem 7.4, $\mathcal{Y}^\chi$ can be given the structure of a smooth manifold diffeomorphic to $P^{n-2}(\mathbb{C}) \times \mathbb{R}$ for which the map $\sigma\colon \mathcal{R}_n^\chi \to \mathcal{Y}^\chi$ is a submersion.

The set $\overline{\mathcal{R}}_n^{>0}$ is invariant under the actions of $\mathbb{R}_{>0}$ and G_t, so $\mathcal{Y}_+ = \sigma(\overline{\mathcal{R}}_n^{>0})$ is an open subset of $\mathcal{Y}$. The set $\overline{\mathcal{R}}_n^1$ is a global slice for the action of $\mathbb{R}_{>0}$ on $\overline{\mathcal{R}}_n^{>0}$, so

$$\mathcal{Y}_+ \cong \overline{\mathcal{R}}_n^1/G_1 \cong \mathcal{S}_n/\mathrm{Isom}^+(S^2) = \mathcal{C}_n(S^2)$$

can be given the structure of a smooth manifold. The map $\sigma\colon \overline{\mathcal{R}}_n^{>0} \to \mathcal{C}_n(S^2)$ is the composition of the map $p_4\colon \overline{\mathcal{R}}_n^{>0} \to \overline{\mathcal{R}}_n^1$ given by $p_4(v) = \sqrt{t(v)}\cdot v$ with the projection $p_5\colon \overline{\mathcal{R}}_n^1 \to \mathcal{C}_n(S^2)$. Both p_4 and p_5 are submersions, so $\sigma\colon \overline{\mathcal{R}}_n^{>0} \to \mathcal{C}_n(S^2)$ is a submersion as well.

In summary, we have shown that the $\mathcal{Y}^\chi$, together with $\mathcal{Y}_+$, form an open cover of $\mathcal{Y}$, each of whose elements is a smooth manifold. To show that $\mathcal{Y}$ is a smooth manifold, we must show that the transition functions relating different sets in our cover are smooth and that $\mathcal{Y}$ is Hausdorff. For the first, consider the overlap $\mathcal{Y}_+^\chi = \mathcal{Y}^\chi \cap \mathcal{Y}_+$. Viewing $\mathcal{Y}_+^\chi$ as a subset of $\mathcal{Y}^\chi$ and identifying $\mathcal{Y}_+$ with $\mathcal{C}_n(S^2)$, we get a transition function $g_\chi\colon \mathcal{Y}_+^\chi \to \mathcal{C}_n(S^2)$. As g_χ is induced by the smooth map $\sigma\colon \overline{\mathcal{R}}_n^{>0} \to \mathcal{C}_n(S^2)$, which is constant on the fibers of the submersion $\sigma\colon \overline{\mathcal{R}}_n^{>0} \cap \mathcal{R}_n^* \to \mathcal{Y}_+^\chi$, we know g_χ is smooth by [Lee 2013, Theorem 4.30]. A similar argument shows g_χ^{-1} is smooth, making all transition functions smooth, as needed.

To show $\mathcal{Y}$ is Hausdorff, suppose x and y are distinct points of $\mathcal{Y}$. The open sets $\mathcal{Y}^\chi$ and $\mathcal{Y}_+$ forming the cover are themselves Hausdorff, so, if x and y both belong to the same open set U of the cover, they can be separated by sets U_x, U_y which are open in U and hence open in $\mathcal{Y}$. On the other hand, if $x \in \mathcal{Y}^{\chi_1}$ and $y \in \mathcal{Y}^{\chi_2}$ with $\chi_1 \neq \chi_2$, then x and y are separated by $\mathcal{Y}^{\chi_1}$ and $\mathcal{Y}^{\chi_2}$. The only remaining case is when $x \in \mathcal{Y}^\chi \setminus \mathcal{Y}_+$ and $y \in \mathcal{Y}_+ \setminus \mathcal{Y}^\chi$. Since $x \notin \mathcal{Y}_+$, we have $x \in \chi \cdot \mathcal{Y}^{\delta_0}$ for all $0 < \delta_0 < \frac{\pi}{2}$. Write $y = \sigma(v)$ for $v \in \overline{\mathcal{R}}_n^1$, and let $\delta_0 = \frac{1}{2}\max(d(v_i, v_j))$. Then x and y are separated by the open sets $U_x = \chi \cdot \mathcal{Y}_n^{\delta_0}$ and $U_y = \{\sigma(v) \mid t(v) > 0 \text{ and } \max(d(v_i, v_j)) > \delta_0\}$.

With this smooth structure on $\mathcal{Y}$, the group $\{\pm 1\}^n$ acts by diffeomorphisms since our open cover was constructed using that very action and we already knew $\{\pm 1\}^n$ acts smoothly on $\mathcal{Y}_+ \cong \mathcal{C}_n(S^2)$. Moreover, the map $\sigma\colon \overline{\mathcal{R}}_n \to \mathcal{Y}$ is a smooth submersion, since its restriction to each set in our open cover of $\overline{\mathcal{R}}_n$ is a smooth submersion. It remains to check the descriptions of $\mathcal{Y}_-$, $\mathcal{Y}_0$ and $\mathcal{Y}_+$ given in the statement. We have already shown that $\mathcal{Y}_+$ is a smooth submanifold of $\mathcal{Y}$ diffeomorphic to $\mathcal{C}_n(S^2)$. The same argument shows that $\mathcal{Y}_-$ is diffeomorphic to $\mathcal{C}_n^\pm(\mathbb{H}^2)$. Finally, Theorem 7.4 implies that $\mathcal{Y}_0$ is a closed submanifold of $\mathcal{Y}$. The subspace $\mathcal{Y}_0$ is the quotient of $\mathcal{R}_0$ by the simultaneous actions of G_0 and $\mathbb{R}_{>0}$. This is the quotient of

$$\{v \in \overline{Q}_0^n \mid v_i \neq \pm v_j \text{ for some } i, j\}$$

by the action of the group $\widehat{G}_0$, which is the space $\mathcal{E}_n^\pm/\mathrm{Sim}(\mathbb{E}^2) = \mathcal{C}_n^\pm(\mathbb{E}^2)$. So $\mathcal{Y}_0 \cong \mathcal{C}_n^\pm(\mathbb{E}^2)$, as needed. □

Applying Theorem 7.4 to the open cover $\mathcal{Y}_+ \cup \{\mathcal{Y}^\chi\}_{\chi \in \{\pm 1\}^n/\pm\mathbb{1}}$ of $\mathcal{Y}$ in the preceding proof of Theorem 7.1, we see that $\mathcal{Y}$ is diffeomorphic to its subset $\mathcal{Y}_+$ and hence:

7.9 Corollary *The manifold $\mathcal{Y}$ is diffeomorphic to $\mathcal{C}_n(S^2)$.*

As we oriented $\mathcal{C}_n(S^2)$ in Section 5.7, the corollary shows that $\mathcal{Y}$ is orientable. Moreover, with the orientation on $\mathcal{C}_n(\mathbb{H}^2)$ from Section 5.7:

7.10 Proposition *If we orient $\mathcal{Y}$ so that the induced orientation on $\mathcal{Y}_+$ matches that of $\mathcal{C}_n(S^2)$, then the induced orientation on $\mathcal{Y}_-$ is the reverse of that on $\mathcal{C}_n(\mathbb{H}^2)$.*

Proof We begin with the case $n = 2$. The maps $\mathcal{C}_2(\mathbb{H}^2) \to \mathbb{R}_{>0}$ and $\mathcal{C}_2(S^2) \to (0, \pi)$ that compute the distance between the two points are diffeomorphisms which, given our conventions in Section 5.7, are either both orientation-preserving or both orientation-reversing (it is the former, but we do not need this). Recall the set $\mathcal{T}_2$ from (7.8). Then we can identify $\mathcal{Y}^\delta$ with the slice $\{(t, 1, 0) \mid t < \sin^2(\delta)\}$ of the S^1 action on $\mathcal{T}_2$. For $t > 0$, increasing t increases the distance between the two points in $\mathcal{C}_2(S^2) \cap \mathcal{Y}^\delta$. However, for $t < 0$, increasing t decreases the distance on $\mathcal{C}_2(\mathbb{H}^2)$. Thus, in this case $\mathcal{C}_2(S^2)$ and $\mathcal{C}_2(\mathbb{H}^2)$ give opposite orientations to $\mathcal{Y}^\delta$.

For general n, the moral is that, for $t > 0$, we act by $\sqrt{t}$ to normalize a point of $\mathcal{R}''_n$ into $\mathcal{V}^1_n = \mathcal{S}_n$, whereas, for $t < 0$, we act by $\sqrt{-t}$ to normalize a point of $\mathcal{R}''_n$ into $\mathcal{V}^{-1}_n = \mathcal{H}_n$; as the orientations on $\mathcal{V}^1_n$ and $\mathcal{V}^{-1}_n$ match in a suitable sense, the orientation flip between $\sqrt{t}$ and $\sqrt{-t}$ means that $\mathcal{C}^\delta_n(S^2)$ and $\mathcal{C}_n(\mathbb{H}^2)$ give opposite orientations to $\mathcal{Y}^\delta$.

This proves the proposition for the subset of $\mathcal{C}^\pm_n(\mathbb{H}^2)$ where all points have the same sign. As discussed at the end of Section 5.7, the action of an element on $\{\pm 1\}^n$ on $\mathcal{C}_n(S^2)$ and $\mathcal{C}^\pm_n(\mathbb{H}^2)$ either preserves the orientation of both or reverses it for both. As the action of $\{\pm 1\}^n$ is transitive on the components of $\mathcal{C}^\pm_n(\mathbb{H}^2)$, this completes the proof. □

Finally, we can assemble all the pieces to give the result promised in Section 5:

Proof of Theorem 5.9 All but the last claim are covered by Theorem 7.1 and Proposition 7.10; the last claim follows from Theorem 7.4 and the $\{\pm 1\}^n$ action. □

8 The character variety of the free group

Fix $c \in (-2, 2)$ and consider the free group F_n generated by $S = \{s_1, \ldots, s_n\}$. Recall from Sections 3.5 and 3.12 that the character variety $X^c(F_n, S) = X^c_{\mathrm{SL}_2\mathbb{R}}(F_n, S) \cup X^c_{\mathrm{SU}_2}(F_n, S)$ is a smooth manifold away from the reducible locus, which consists of 2^{n-1} reducible characters $\chi^c_{[\epsilon]}$ labeled by $[\epsilon] \in \{\pm 1\}^n/\pm\mathbb{1}$. The main result of this section is that the space $\mathcal{Y}$ from Theorem 7.1 is a resolution of $X^c(F_n, S)$:

8.1 Theorem *For each* $c \in (-2, 2)$, *there is a proper smooth surjective map* $\pi_c\colon \mathcal{Y} \to X^c(F_n, S)$. *The function* π_c *maps* $\mathcal{Y}_- \cong \mathscr{C}_n^{\pm}(\mathbb{H}^2)$ *diffeomorphically onto* $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S)$ *and maps* $\mathcal{Y}_+ \cong \mathscr{C}_n(S^2)$ *diffeomorphically onto* $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n, S)$. *It maps the component of* $\mathcal{Y}_0 \cong \mathscr{C}_n^{\pm}(\mathbb{E}^2)$ *corresponding to* $[\epsilon] \in \{\pm 1\}^n / \pm\mathbb{1}$ *to the reducible character* $\chi^c_{[\epsilon]}$.

To understand the statement that π_c is smooth, note that while only the dense open set $X^{c,\mathrm{irr}}(F_n, S)$ is a smooth manifold for $n > 2$, the semialgebraic set $X^c(F_n, S)$ can be viewed as a subset of some $\mathbb{R}^m$ using suitable trace functions as coordinates. While the same space $\mathcal{Y}$ appears as the resolution for all $c \in (-2, 2)$, when we want to emphasize the value of c, we write $\mathscr{X}^c(F_n, S)$ for $\mathcal{Y}$. Furthermore, we set

$$\mathscr{X}^{c,\mathrm{irr}}(F_n, S) := \pi_c^{-1}(X^{c,\mathrm{irr}}(F_n, S)) \cong \mathscr{C}_n(S^2) \cup \mathscr{C}_n^{\pm}(\mathbb{H}^2) \tag{8.2}$$

and note that $\mathscr{X}^{c,\mathrm{irr}}(F_n, S)$ is dense in $\mathscr{X}(F_n, S)$ by the last sentence in Theorem 5.9. Similarly, we define $\mathscr{X}^{c,\mathrm{red}}(F_n, S) = \pi_c^{-1}(X^{c,\mathrm{red}}(F_n, S)) \cong \mathscr{C}_n^{\pm}(\mathbb{E}^2)$.

We will define π_c using a map π_c' from the space $\overline{\mathscr{R}}_n$ of Section 7, whose quotient is $\mathcal{Y}$, to the representation variety $R_{\mathbb{C}}(F_n, S)$. In doing so, we will make heavy use of the subgroups U_t of $\mathrm{SL}_2\mathbb{C}$ from Section 6.5. Recall the conjugacy class of U_t in $\mathrm{SL}_2\mathbb{C}$ depends only on whether t is positive, negative or zero. Thus, much of the distinction between the U_t is meaningless at the level of character varieties. However, this nicely varying family U_t will be crucial in writing down a smooth $\pi_c'\colon \overline{\mathscr{R}}_n \to R_{\mathbb{C}}(F_n, S)$ even though $U_1 = \mathrm{SU}_2$ and $U_{-1} \cong \mathrm{SL}_2\mathbb{R}$ are radically different subgroups of $\mathrm{SL}_2\mathbb{C}$.

Turning now to the reducible locus, the points of $\mathscr{X}^{c,\mathrm{red}}(F_n, S)$ can be understood in terms of representations to U_0 as follows. Recall from Sections 6.3 and 6.5 that $G_0 \cong \mathrm{Isom}(\mathbb{E}^2)$ and $U_0 \to G_0$ is a double cover onto the identity component of G_0, which is $\mathrm{Isom}^+(\mathbb{E}^2)$. Recall also that G_0 is contained in $\widehat{G}_0 \cong \mathrm{Sim}(\mathbb{E}^2)$, which is $\mathrm{Aut}(U_0)$ by Lemma 6.8. A representation ρ to U_0 is *fully reducible* when it is conjugate to one with diagonal image. Now set

$$R^c_{U_0}(F_n, S) = \{\rho\colon F_n \to U_0 \mid \text{all } \operatorname{tr}\rho(s_i) = c \text{ and } \rho \text{ not fully reducible}\}. \tag{8.3}$$

We define $X^c_{U_0}(F_n, S)$ to be the quotient $R^c_{U_0}(F_n, S)/\widehat{G}_0$. Despite the notation, the set $X^c_{U_0}(F_n, S)$ is not a true character variety: it turns out to be a manifold of dimension $2n-4$, whereas the representations in $R^c_{U_0}(F_n, S)$ have only the finitely many distinct characters $\chi^c_{[\epsilon]}$. We will show:

8.4 Theorem *The group* $\widehat{G}_0 \cong \mathrm{Aut}(U_0)$ *acts freely and properly on* $R^c_{U_0}(F_n, S)$, *so* $X^c_{U_0}(F_n, S)$ *has a natural smooth structure, and there is a diffeomorphism* $\widetilde{\pi}_c^{\mathrm{red}}\colon \mathscr{X}^{c,\mathrm{red}}(F_n, S) \to X^c_{U_0}(F_n, S)$. *Under this identification, the restriction* $\pi_c\colon \mathscr{X}^{c,\mathrm{red}}(F_n, S) \to X^{c,\mathrm{red}}(F_n, S)$ *corresponds to the map* $X^c_{U_0}(F_n, S) \to X^{c,\mathrm{red}}(F_n, S)$ *taking* $[\rho]$ *to its character.*

Thus we can think of Theorem 8.1 as saying we can resolve $X^c(F_n, S)$ by "blowing up" the finitely many reducible representations into the space of "characters" of U_0 representations that are not fully reducible.

8.5 Constructing π_c

Given $v \in \overline{Q}_t$ and $\alpha \in (0, \pi)$, we regard v as the element $\varphi_t(v)$ of the Lie algebra $\mathfrak{u}_t$ and define $A(\alpha, t, v) = \exp(\alpha\varphi_t(v)) \in U_t$; as per Lemma 6.14, the element $A(\alpha, t, v)$ acts on $\overline{Q}_t$ as a 2α–anticlockwise rotation about v with

$$A(\alpha, t, v) = (\cos\alpha)I + (\sin\alpha)\varphi_t(v). \tag{8.6}$$

Note that $\operatorname{tr}(A(\alpha, t, v)) = 2\cos\alpha$ and $A(\alpha, t, -v) = A(\alpha, t, v)^{-1}$. For $c \in (-2, 2)$, let $\alpha = \cos^{-1}\frac{c}{2}$. We define $\pi'_c : \overline{\mathcal{R}}_n \to R^c_{\mathbb{C}}(F_n, S)$ by sending $v \in \overline{\mathcal{R}}_n$ to the representation where

$$s_i \mapsto A(\alpha, t(v), v_i).$$

If $t(v) \neq 0$, the condition that $v_i \neq \pm v_j$ for some i, j implies that $\pi'_c(v)$ is irreducible. When $t(v) = 0$, that condition instead implies that $\pi'_c(v)$ is not fully reducible.

First, we study the image of $\pi'_c(\overline{\mathcal{R}}_n)$ under the projection $\tau : R^c_{\mathbb{C}}(F_n, S) \to X^c_{\mathbb{C}}(F_n, S)$:

8.7 Lemma *For each $t \neq 0$, the restriction of π'_c to $\overline{\mathcal{R}}^t_n$ gives a diffeomorphism to $R^{c,\mathrm{irr}}_{U_t}(F_n, S)$. Also, the restriction of π'_c to $\overline{\mathcal{R}}^0_n$ gives a diffeomorphism to $R^c_{U_0}(F_n, S)$ as defined in* (8.3). *Finally, the subset $\tau(\pi'_c(\overline{\mathcal{R}}_n))$ is equal to $X^c(F_n, S)$.*

Proof To start in on the initial claim where $t \neq 0$ is fixed, note that for $v \in \overline{\mathcal{R}}^t_n$ each $A(\alpha, t, v_i)$ is in U_t and so $\pi'_c(v) \in R^{c,\mathrm{irr}}_{U_t}(F_n, S)$. Next, we show $\pi'_c : \overline{\mathcal{R}}^t_n \to R^{c,\mathrm{irr}}_{U_t}(F_n, S)$ is onto. By Corollary 6.15, given $\rho \in R^{c,\mathrm{irr}}_{U_t}(F_n, S)$, there are unique $v_i \in \overline{Q}_t$ with $\rho(s_i) = A(\alpha, t, v_i)$. At least one v_i is not $\pm v_1$ as otherwise all $\rho(s_i)$ would be $\rho(s_1)^{\pm 1}$, violating the condition that ρ is irreducible. Hence, the associated v is in $\overline{\mathcal{R}}^t_n$, establishing that $\pi'_c(\overline{\mathcal{R}}^t_n) = R^{c,\mathrm{irr}}_{U_t}(F_n, S)$. By uniqueness of the v_i, in fact $\pi'_c : \overline{\mathcal{R}}^t_n \to R^{c,\mathrm{irr}}_{U_t}(F_n, S)$ is a bijection; Corollary 6.15 implies it is also a diffeomorphism since locally π'_c on $\overline{\mathcal{R}}^t_n$ is the restriction of the product of n copies of the map r in the corollary. This completes the proof of the first claim, and the second claim covering the case $t = 0$ follows by the identical argument with "irreducible" replaced with "not fully reducible".

Turning now to the final claim of the lemma, first recall that

$$X^c(F_n, S) = X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S) \cup X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n, S) \cup X^{c,\mathrm{red}}(F_n, S).$$

When $t < 0$, the subgroup $U_t \leq \mathrm{SL}_2\mathbb{C}$ is conjugate to $\mathrm{SL}_2\mathbb{R}$, so

$$\tau(\pi'_c(\mathcal{R}^t_n)) = \tau(R^{c,\mathrm{irr}}_{U_t}(F_n, S)) = \tau(R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S)) = X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S).$$

For $t > 0$, the subgroup U_t is conjugate to SU_2 and so $\tau(\pi'_c(\mathcal{R}^t_n)) = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n, S)$. It thus remains to show that $\tau(\pi'_c(\mathcal{R}^0_n)) = X^{c,\mathrm{red}}(F_n S)$.

In one direction, given $v \in \overline{\mathcal{R}}^0_n$, let $\epsilon_i \in \{\pm 1\}$ be the z-coordinate of v_i, and so

$$A(\alpha, 0, v_i) = \begin{pmatrix} e^{i\epsilon_i\alpha} & * \\ 0 & e^{-i\epsilon_i\alpha} \end{pmatrix}.$$

Hence, the character of $\pi_c(v)$ is the reducible character $\chi^c_{[\epsilon]}$ from Section 3.12. Conversely, given $\chi^c_{[\epsilon]} \in X^{c,\mathrm{red}}(F_n, S)$, take $v_1 = (1, 0, \epsilon_1)$ and all other $v_i = (0, 0, \epsilon_i)$ to get $v \in \overline{\mathcal{R}}^0_n$ with $\pi'_c(v) = \chi_{[\epsilon]}$. Combined, we have $\tau(\pi'_c(\mathcal{R}^0_n)) = X^{c,\mathrm{red}}(F_n S)$, completing the proof of the lemma. □

8.8 Lemma *There is a smooth map* $\pi_c\colon \mathcal{Y} \to X^c(F_n, S)$ *making the following diagram commute*:

(8.9)
$$\begin{array}{ccc} \overline{\mathcal{R}}_n & \xrightarrow{\pi'_c} & \mathrm{im}(\pi'_c) \\ \big\downarrow \sigma & & \big\downarrow \tau \\ \mathcal{Y} & \xrightarrow{\pi_c} & X^c(F_n, S) \end{array}$$

Proof To show that π_c exists as a function, we must check that $\tau \circ \pi'_c$ is invariant under the actions of G_t and $\mathbb{R}_{>0}$. First, recall from Section 6.5 that $G_t \subset \mathrm{Aut}(U_t)$ and that the action of G_t on U_t satisfies $\exp(g \cdot v) = g \cdot \exp v$ and so

(8.10)
$$A(\alpha, t, g \cdot v) = g \cdot A(\alpha, t, v).$$

Using this, we define an action of G_t on $R^c_{U_t}(F_n, S)$ by $(g \cdot \rho)(s) = g \cdot \rho(s)$, where $g \cdot \rho$ is also a representation of F_n as g acts as an element of $\mathrm{Aut}(U_t)$; with this action, π'_c is G_t-equivariant. Since $\mathrm{tr}(g \cdot B) = \mathrm{tr}\, B$ for all $B \in U_t$, we have $\tau(g \cdot \rho) = \tau(\rho)$. Equation (8.10) implies that $\tau(\pi'_c(g \cdot v)) = \tau(g \cdot \pi'_c(v)) = \tau(\pi'_c(v))$, and thus $\tau \circ \pi'_c$ is invariant under each G_t action, as desired.

Turning to the $\mathbb{R}_{>0}$ action, for the more general action of $u \in \mathbb{C}^\times$, equation (6.12) gives that

(8.11)
$$A(\alpha, t/|u|^2, u \cdot v) = T_u A(\alpha, t, v) T_u^{-1}, \quad \text{where } T_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

So, for $s \in \mathbb{R}_{>0}$, it follows that $\pi'_c(s \cdot v) = T_s \cdot \pi'_c(v)$, where T_s acts by conjugation, giving the claimed invariance $\tau(\pi'_c(s \cdot v)) = \tau(\pi'_c(v))$.

To see π_c is smooth, first note π'_c is smooth by (8.6) and (6.6). Moreover, $\tau\colon R_{\mathbb{C}}(F_n, S) \to X_{\mathbb{C}}(F_n, S)$ is smooth since it corresponds to taking traces of matrices. Hence, $\tau \circ \pi'_c$ in the diagram is smooth. The map σ is a submersion by Theorem 7.1, and so π_c is smooth, as claimed. □

We can now prove the theorems stated at the beginning of this section.

Proof of Theorem 8.4 To connect $\mathcal{X}^{c,\mathrm{red}}(F_n, S)$ and $X^c_{U_0}(F_n, S)$, note that Lemma 8.7 gives a diffeomorphism $\overline{\mathcal{R}}^0_n \to R^c_{U_0}(F_n, S)$ by restricting π'_c. Recall that $\widehat{G}_0 \cong \mathrm{Aut}(U_0) \cong \mathrm{Sim}(\mathbb{E}^2)$ acts on both $\overline{\mathcal{R}}^0_n$ and $R^c_{U_0}(F_n, S)$. We now show π'_c is $\widehat{G}_0$-equivariant, using that the action of $\widehat{G}_0$ on $\overline{\mathcal{R}}^0_n$ is generated by those of G_0 and $\mathbb{R}_{>0}$. In the proof of Lemma 8.8, we showed $\overline{\mathcal{R}}^0_n \to R^c_{U_0}(F_n, S)$ is G_0-equivariant. The element of $\widehat{G}_0$ corresponding to $s \in \mathbb{R}_{>0}$ is

$$\psi_0\left(\begin{pmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{pmatrix}\right) \quad \text{in } U'_0,$$

and these act equivariantly since conjugating by T_u in (8.11) is the same as conjugating by $\left(\begin{smallmatrix}\sqrt{u} & 0\\ 0 & 1/\sqrt{u}\end{smallmatrix}\right)$. Thus $\pi'_c\colon \overline{\mathcal{R}}^0_n \to R^c_{U_0}(F_n, S)$ is $\widehat{G}_0$-equivariant.

As $\pi'_c\colon \overline{\mathcal{R}}^0_n \to R^c_{U_0}(F_n, S)$ is a diffeomorphism, the $\widehat{G}_0$ action on $R^c_{U_0}(F_n, S)$ is free and proper. Hence, $X^c_{U_0}(F_n, S)$ is a smooth manifold diffeomorphic to $\overline{\mathcal{R}}^0_n/\widehat{G}_0 = \mathcal{X}^{c,\mathrm{red}}(F_n, S)$ by the map $\widetilde{\pi}^{\mathrm{red}}_c$ induced by π'_c. The final claim of the theorem now follows from commutativity of (8.9). □

Proof of Theorem 8.1 By Lemma 8.8, we have a smooth $\pi_c\colon \mathcal{Y} \to X^c(F_n, S)$, which is surjective by Lemma 8.7. Momentarily deferring the proof that π_c is proper, we next show that π_c gives diffeomorphisms $\mathcal{Y}_- \cong X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S)$ and $\mathcal{Y}_+ \cong X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n, S)$. For $t < 0$ there is a commutative square

(8.12)
$$\begin{array}{ccc} \overline{\mathcal{R}}^t_n & \xrightarrow{\ \pi'_c\ } & R^{c,\mathrm{irr}}_{U_t}(F_n, S) \\ \big\downarrow \sigma & & \big\downarrow \tau \\ \mathcal{Y}_- & \xrightarrow{\ \pi_c\ } & X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S) \end{array}$$

which sits inside the commutative square (8.9). The map π'_c is a diffeomorphism by Lemma 8.7 and it is equivariant with respect to the action of G_t. The maps σ and τ are projections to their quotients by the actions of G_t (the latter by Lemma 3.4 and the fact that U'_t is the conjugate of $\mathrm{SL}^{\pm}_2(\mathbb{R})$ corresponding to U_t), and both are smooth submersions. It follows that π_c is a diffeomorphism. The argument for $t > 0$ is the same, and the case $t = 0$ follows from Theorem 8.4 and the identification $\mathcal{X}^{c,\mathrm{red}}(F_n, S) \cong \mathcal{C}^{\pm}_n(\mathbb{E}^2)$.

Finally, we must check that π_c is proper. By Theorem 7.4 and the discussion in the proof of Lemma 8.7, near $\mathcal{Y}_0 = \pi_c^{-1}(X^{\mathrm{red}}(F_n, S))$ the map π_c is modeled on the projection

$$P^{n-2}(\mathbb{C}) \times [-1, 1] \to (P^{n-2}(\mathbb{C}) \times [-1, 1])/(P^{n-2}(\mathbb{C}) \times \{0\}),$$

which is proper as the domain is compact. Away from $\mathcal{Y}_0$, the map π_c is a diffeomorphism. Combining these two pictures, we see π_c is proper, as needed. □

9 The punctured sphere

9.1 Introduction

Let S_n be the sphere with n punctures. We use the presentation

$$\pi_1(S_n) = \langle s_1, \dots, s_n \mid s_1 s_2 \cdots s_n = 1\rangle,$$

where the s_i are a fixed set of loops representing the conjugacy classes of the boundary circles of S_n. As before, we take $S = \{s_1, \dots, s_n\}$, and consider the character variety $X^c(S_n, S) := X(S_n, S) \cap X^c(F_n, S)$; as S will be fixed throughout this section, we omit it and simply use $X^c(S_n)$ and analogous notation.

The aim of this section is to construct a resolution of $X^c(S_n)$. As in the free group case, it turns out that $X^{c,\mathrm{irr}}(S_n)$ is a smooth manifold (see Section 9.21 below), and $X^c(S_n)$ is usually singular along $X^{c,\mathrm{red}}(S_n)$. To resolve $X^c(S_n)$, we make use of our resolution $\mathscr{X}^c(F_n) = \mathscr{Y}$ from Theorem 8.1; to reduce clutter, we will use π rather than π_c for the map $\mathscr{X}^c(F_n) \to X^c(F_n)$. The main result of this section is:

9.2 Theorem *For $c \in (-2,2)$ and an even $n \geq 4$, there is a smooth closed codimension 3 submanifold $\mathscr{X}^c(S_n) \subset \mathscr{X}^c(F_n)$ such that $\pi(\mathscr{X}^c(S_n)) = X^c(S_n)$ and $\pi\colon \mathscr{X}^c(S_n) \to X^c(S_n)$ is a diffeomorphism away from $X^{c,\mathrm{red}}(S_n)$. Moreover, the closure of $\mathscr{X}^{c,\mathrm{irr}}(S_n)$ in $\mathscr{X}^c(F_n)$ is all of $\mathscr{X}^c(S_n)$.*

For odd n, the picture is slightly more complicated, and we do not pin it down completely as it is not needed in the rest of the paper; see Section 9.26 for details. For $n < 4$, there is no need to resolve $X^c(S_n)$ as you can check that it is empty when $n = 1$ and a single point when n is 2 or 3.

As with Theorem 8.4 for free groups, we can understand $\mathscr{X}^{c,\mathrm{red}}(S_n) := \mathscr{X}^c(S_n) \cap \mathscr{X}^{c,\mathrm{red}}(F_n)$ in terms of certain U_0 representations. Specifically, taking $X^c_{U_0}(S_n)$ to be the subset of $X^c_{U_0}(F_n)$ corresponding to representations of $\pi_1(S_n)$, we will see below that $\mathscr{X}^c(S_n)$ is a closed codimension 1 subset of $X^c_{U_0}(S_n)$.

For the rest of this section, we fix $c = 2\cos\alpha \in (-2,2)$ and some $n \geq 4$; unless explicitly specified, n can be either even or odd. Although $\mathscr{X}^c(F_n) \cong \mathscr{Y}$ for all $c \in (-2,2)$, the subset $\mathscr{X}^c(S_n)$, and indeed its topology, will depend on c. To begin, we consider the map $F\colon \overline{\mathscr{R}}_n \to \mathrm{SL}_2\mathbb{C}$ given by

$$F(v) = \prod_{i=1}^{n} A(\alpha, t(v), v_i) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

In other words, the map F is the composition of $\pi'_c\colon \overline{\mathscr{R}}_n \to R^c_{\mathbb{C}}(F_n)$ from Section 8 with the evaluation map $\rho \mapsto \rho(s_1 s_2 \cdots s_n)$. Although the definition of F depends on the trace c and the entries F_{ij} are functions of v, this is not reflected in the notation. Note that $F^{-1}(I)$ is the set of $v \in \overline{\mathscr{R}}_n$ whose representation $\pi'_c(v)$ is in $R^c_{\mathbb{C}}(S_n)$. Also, as all $A(\alpha, t(v), v_i)$ are in $U_{t(v)}$, the matrix $F(v)$ is in $U_{t(v)}$ as well. For $g \in G_{t(v)}$, equation (8.10) gives that $F(g \cdot v) = g \cdot F(v)$. Similarly, for $u \in \mathbb{C}^\times$, equation (8.11) implies $F(u \cdot v) = T_u F(v) T_u^{-1}$. Thus the subset $F^{-1}(I)$ is invariant under the actions of G_t and $\mathbb{C}^\times$. Naively, we might expect that the resolution $\mathscr{X}^c(S_n)$ should be the quotient of $F^{-1}(I)$ by these actions, but this is incorrect: the part of $F^{-1}(I)$ where $t = 0$ is too big, as we now explain.

9.3 Reducible characters

The reducible character $\chi^c_{[\epsilon]} \in X^c(F_n)$ corresponding to $[\epsilon] \in \{\pm 1\}^n / \pm\mathbb{1}$ is in $X^c(S_n)$ precisely when $\prod e^{i\epsilon_j\alpha} = 1$. We distinguish two kinds of reducible characters: the *balanced* ones are those where $\sum \epsilon_j = 0$ and the rest are *unbalanced*. This distinction is relevant as every balanced $\chi^c_{[\epsilon]} \in X^c(F_n)$ is in $X^c(S_n)$ for all values of c, but an unbalanced $\chi^c_{[\epsilon]}$ is in $X^c(S_n)$ for at most finitely many c. As c is fixed in this section, we will write $\chi_{[\epsilon]}$ in place of $\chi^c_{[\epsilon]}$.

Let $\Pi = \pi \circ \sigma \colon \overline{\mathcal{R}}_n \to X^c(F_n)$ be the composition of maps from (8.9). The preimage $\Pi^{-1}(\chi_{[\epsilon]})$ for $\chi_{[\epsilon]} \in X^c(F_n)$ consists of points of the form $(0, v_1, \ldots, v_n) \in \overline{\mathcal{R}}_n$, where $v_j = (x_j, y_j, z_j)$ with all $z_j = \epsilon_j$ or all $z_j = -\epsilon_j$. Setting $\omega_j = y_j - i x_j$, equation (8.6) gives

$$A(\alpha, 0, v_j) = \begin{pmatrix} e^{i z_j \alpha} & \omega_j \sin\alpha \\ 0 & e^{-i z_j \alpha} \end{pmatrix}.$$

For $\chi_{[\epsilon]}$ not in $X^c(S_n)$, it follows that $\Pi^{-1}(\chi_{[\epsilon]})$ is disjoint from $F^{-1}(I)$. For $\chi_{[\epsilon]}$ that are in $X^c(S_n)$, on $\Pi^{-1}(\chi_{[\epsilon]})$ we have $F(0, v_1, \ldots, v_n) = \left(\begin{smallmatrix} 1 & F_{12} \\ 0 & 1 \end{smallmatrix}\right)$, so the equation $F(v) = I$ is a (real) codimension 2 condition on $\overline{\mathcal{R}}_n^0$, specifically $F_{12} = 0$. (While it is not clear a priori, we show $F^{-1}(I) \cap \Pi^{-1}(\chi_{[\epsilon]})$ is nonempty for $\chi_{[\epsilon]}$ in $X^c(S_n)$ whenever n is even in Proposition 9.18.) In contrast, for $t \neq 0$, the equation $F(v) = I$ turns out to be a codimension 3 condition on $\overline{\mathcal{R}}_n^t$, since $\dim U_t = 3$ (see Proposition 9.24 for details). We now turn to resolving this discrepancy by imposing an additional condition when $t = 0$.

9.4 Defining the resolution

Given $\epsilon \in \{\pm 1\}^n$, let

$$Z_\epsilon = \{v \in \overline{\mathcal{R}}_n^0 \mid v_j = (x_j, y_j, \epsilon_j) \text{ for all } j\},$$

so $\Pi^{-1}(\chi_{[\epsilon]}) = Z_\epsilon \cup Z_{-\epsilon}$. For $\chi_{[\epsilon]} \in X^c(S_n)$, we define

$$V_\epsilon = \{v \in \overline{\mathcal{R}}_n \mid \operatorname{sign}(z_j) = \epsilon_j \text{ for all } j \text{ and } \operatorname{Re}(F_{11}(v)) > 0\},$$

so V_ϵ is an open neighborhood of Z_ϵ in $\overline{\mathcal{R}}_n$. Observe that the set V_ϵ is preserved by the $\mathbb{C}^\times$ action on $\overline{\mathcal{R}}_n$.

We have local coordinates $(t, \omega_1, \ldots, \omega_n)$ on V_ϵ where

$$\omega_j = y_j - i x_j \quad \text{and} \quad z_j = \epsilon_j (1 - t|\omega_j|^2)^{1/2}.$$

When $t = 0$, note that an $\omega \in \mathbb{C}^n$ gives a point in Z_ϵ — that is, some $v_i \neq \pm v_j$ — if and only if ω is not in the complex subspace generated by ϵ; hence, we can view Z_ϵ as $\mathbb{C}^n \setminus \langle \epsilon \rangle$.

It is easy to see that F_{11} is a real analytic function of $(t, \omega_1, \ldots, \omega_n)$, and, as we observed above, $F_{11}(0, \omega_1, \ldots, \omega_n) \equiv 1$. It follows that the function $(F_{11} - 1)/t$ is well defined and analytic on V_ϵ. The function $f_\epsilon \colon V_\epsilon \to \mathbb{C} \times \mathbb{R}$ defined by

$$f_\epsilon = (F_{12}, \operatorname{Im}(F_{11} - 1)/t)$$

will play a key role in what follows.

We define $\overline{\mathcal{R}}^c(S_n) \subset \overline{\mathcal{R}}_n$ to be the union of $F^{-1}(I) \cap \overline{\mathcal{R}}_n^{t \neq 0}$ with $f_\epsilon^{-1}(0,0) \subset V_\epsilon$ for all ϵ with $\chi_{[\epsilon]} \in X^c(S_n)$, and begin by showing:

9.5 Lemma *The set $\overline{\mathcal{R}}^c(S_n)$ is contained in $F^{-1}(I)$ and is equal to it inside $\overline{\mathcal{R}}_n^{t \neq 0}$.*

Proof Fix ϵ with $\chi_{[\epsilon]} \in X^c(S_n)$. For $v \in V_\epsilon$, we have $f_\epsilon(v) = (0,0)$ implies $F(v) = I$, so $f^{-1}(0,0)$ is indeed contained in $F^{-1}(I)$. It remains to show $f_\epsilon^{-1}(0,0) \cap V_\epsilon^{t\neq 0} = F^{-1}(I) \cap V_\epsilon^{t\neq 0}$. For $v \in V_\epsilon^{t\neq 0}$, we have $f_\epsilon(v) = (0,0)$ if and only if $F_{11}(v) \in \mathbb{R}_{>0}$ and $F_{12}(v) = 0$. Since $F(v) \in U_{t(v)}$, these conditions hold if and only if $F(v) = I$. □

The key to proving Theorem 9.2 will be:

9.6 Proposition *The subset $\overline{\mathcal{R}}^c(S_n)$ is a closed codimension 3 submanifold of $\overline{\mathcal{R}}_n$. It is invariant under the $\mathbb{C}^\times$ action on $\overline{\mathcal{R}}_n$ and each G_t action on $\overline{\mathcal{R}}_n^t$.*

The proof divides into two cases depending on whether t is zero and takes the bulk of the rest of this section. If $v \in \overline{\mathcal{R}}^c(S_n)$ has $t(v) \neq 0$, then $\Pi(v)$ is in $X^{c,\mathrm{irr}}(S_n)$. Following the work of [Lin 1992; Heusener 2003] in the case of SU_2 representations, we will show in Proposition 9.24 that $F^{-1}(I)$ is a submanifold near v. Otherwise, if $v \in \overline{\mathcal{R}}^c(S_n)$ has $t(v) = 0$, then $v \in Z_\epsilon$ for some ϵ. In this case, we will show in Proposition 9.16 that f_ϵ is a submersion at v.

9.7 f_ϵ is a submersion

Building towards the proof of Proposition 9.16, we begin by describing the behavior of f_ϵ under the $\mathbb{C}^\times$ action.

9.8 Lemma *If $f_\epsilon(v) = (a,b)$, then $f_\epsilon(u\cdot v) = (ua, |u|^2 b)$ for all $u \in \mathbb{C}^\times$.*

Proof Recall that $u\cdot(t, v_1, \dots, v_n) = (t/|u|^2, u\cdot v_1, \dots, u\cdot v_n)$. Using (8.11), we compute

$$\begin{aligned} F(u\cdot(t, v_1, \dots, v_n)) &= \prod_{i=1}^n A(\alpha, t/|u|^2, u\cdot v_i) = \prod_{i=1}^n T_u A(\alpha, t, v_i) T_u^{-1} \\ &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_{11}(v) & F_{12}(v) \\ F_{21}(v) & F_{22}(v) \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} F_{11}(v) & uF_{12}(v) \\ u^{-1}F_{21}(v) & F_{22}(v) \end{pmatrix}. \end{aligned}$$

So $f_\epsilon(u\cdot v) = \big(uF_{12}(v), \mathrm{Im}(F_{11}(v) - 1)/(|u|^{-2}t)\big)$, as needed. □

When $t = 0$, we can explicitly describe the function f_ϵ. Recall that we have local coordinates $(t,\omega) := (t, \omega_1, \dots, \omega_n)$ on V_ϵ.

9.9 Lemma *There is a linear function $L\colon \mathbb{C}^n \to \mathbb{C}$ and a nondegenerate sesquilinear form $B\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that $f_\epsilon(0,\omega) = (L(\omega), \mathrm{Im}\, B(\omega,\omega))$.*

We could guess that L is linear and B is sesquilinear from Lemma 9.8, but we need the computation below to check that B is nonsingular.

Proof We expand $A(\alpha, t, v_j)$ as a power series in t:

$$A(\alpha, t, v_j) = \begin{pmatrix} \cos\alpha + i z_j \sin\alpha & \omega_j \sin\alpha \\ -t\bar{\omega}_j \sin\alpha & \cos\alpha - i z_j \sin\alpha \end{pmatrix} = \begin{pmatrix} \zeta_j & \omega_j \sin\alpha \\ 0 & \zeta_j^{-1} \end{pmatrix} + t\sin\alpha \begin{pmatrix} -\frac{1}{2} i\epsilon_j |\omega_j|^2 & 0 \\ -\bar{\omega}_j & \frac{1}{2} i\epsilon_j |\omega_j|^2 \end{pmatrix} + \cdots,$$

where $\zeta_j = e^{i\epsilon_j\alpha}$. Expanding F to first order in t, we find that

$$F(t, \omega_1, \dots, \omega_n) = \prod_{j=1}^{n} A(\alpha, t, \omega_j) = A_0 + (t\sin\alpha)A_1 + \cdots,$$

where

$$A_0 = \begin{pmatrix} \xi & \sin\alpha \sum c_j\omega_j \\ 0 & \xi^{-1} \end{pmatrix}, \quad \xi = \prod_{l=1}^{n} \zeta_l \quad \text{and} \quad c_j = \prod_{1\le l<j} \zeta_l \prod_{j<l\le n} \zeta_l^{-1}.$$

The hypothesis that $\chi_{[\epsilon]} \in X^c(S_n)$ implies that $\xi = 1$ and so $c_j = \zeta_j \prod_{1\le l<j} \zeta_l^2$. Taking $L(\omega) = (\sin\alpha)\sum_{j=1}^{n} c_j\omega_j$, we see that the first component of f has the desired form.

We further compute that the upper-left entry of the matrix $(\sin\alpha)A_1$, which is $(F_{11}-1)/t$ at $(0,\omega)$, is given by

$$B(\omega, \omega) = \sin\alpha \sum_{j=1}^{n} -\tfrac{i}{2}\epsilon_j \zeta_j^{-1} |\omega_j|^2 - \sin^2\alpha \sum_{k<j} b_{jk}\bar{\omega}_j\omega_k,$$

where $b_{jk} = \prod_{1\le l<k} \zeta_l \prod_{k<l<j} \zeta_l^{-1} \prod_{j<l\le n} \zeta_l = \zeta_j\zeta_k \prod_{k\le l\le j} \zeta_l^{-2}$. The form B is nonsingular, since its corresponding matrix is lower-triangular with nonzero diagonal entries. □

9.10 Lemma *There is a Hermitian form H on* $\ker L$ *such that $B(\omega, \omega) = iH(\omega, \omega)$ for $\omega \in \ker L$.*

Proof From the proof of the last lemma, we have

$$B(\omega, \omega) = \lim_{t\to 0} \frac{F_{11}(t, \omega) - 1}{t} = \left.\frac{\partial F_{11}}{\partial t}\right|_{(0,\omega)}. \tag{9.11}$$

For a fixed value of ω, consider the path $\gamma(t) = (t, F(t, \omega))$ in $\mathbb{R} \times \mathrm{SL}_2\mathbb{C}$. The image of γ is contained in the submanifold $\widetilde{U} = \{(t, g) \mid g \in U_t\} \subset \mathbb{R} \times \mathrm{SL}_2\mathbb{C}$. It is easy to see that $T\widetilde{U}|_{(0,I)} = (0, \mathfrak{u}_0) \oplus \langle(1, \mathbf{0})\rangle$.

If $\omega \in \ker L$, then $F(0, \omega) = I$, so $\gamma'(0) = \bigl(1, \frac{\partial F}{\partial t}\big|_{(0,\omega)}\bigr) \in T\widetilde{U}|_{(0,I)}$. Hence, $\frac{\partial F}{\partial t}\big|_{(0,\omega)} \in \mathfrak{u}_0$, which implies that $B(\omega, \omega) = \frac{\partial F_{11}}{\partial t}\big|_{(0,\omega)}$ is purely imaginary for all $\omega \in \ker L$. In other words, $H := -iB|_{\ker L}$ is a sesquilinear form with the property that $H(\omega, \omega)$ is real for all $\omega \in \ker L$. Such a form must be Hermitian, proving the lemma. □

For $a \in \mathbb{C}$, the matrix $\tau_a = \bigl(\begin{smallmatrix} 1 & ia/2 \\ 0 & 1 \end{smallmatrix}\bigr) \in U_0$ acts on $Q_0 = \mathbb{E}^2$ by translation, with an easy computation showing that $\psi_0(\tau_a) \cdot (0, \omega) = (0, \omega + a\epsilon)$ in our local coordinates. We next show that L and H are translation-invariant.

9.12 Lemma *For all $a \in \mathbb{C}$ and $\omega \in \mathbb{C}^n$, one has $L(\omega + a\epsilon) = L(\omega)$. If $\omega \in \ker L$, then also $H(\omega + a\epsilon, \omega + a\epsilon) = H(\omega, \omega)$.*

Proof Recall from Section 9.1 that $F(g\cdot v) = g\cdot F(v)$ for $g \in U_{t(v)}$. Taking $t = 0$ and $g = \psi_0(\tau_a)$, we see that $F(0, \omega + a\epsilon) = \tau_a F(0, \omega)\tau_a^{-1}$. As $F(0, \omega) = \left(\begin{smallmatrix} 1 & F_{12} \\ 0 & 1 \end{smallmatrix}\right)$, it commutes with τ_a. Thus $F(0, \omega + a\epsilon) = F(0, \omega)$, giving the first relation, since $L(\omega) = F_{12}(0, \omega)$.

For the second relation, suppose that $\omega \in \ker L$, so $F(0, \omega) = I$. By (9.11), we must show that $\frac{\partial F_{11}}{\partial t}$ takes the same value at $(0, \omega)$ and $(0, \omega + a\epsilon)$. Since $F_{11}(0, \eta) \equiv 1$, the tangent space $T\overline{\mathscr{R}}_n^0$ is contained in $\ker(dF_{11})$. Hence, if $\gamma\colon \mathbb{R} \to \mathbb{R} \oplus \mathbb{C}^n$ is given by $\gamma(t) = (t, \omega(t))$, where $\omega(0) = \omega$, then

(9.13)
$$\frac{\partial F_{11}}{\partial t}\bigg|_{(0,\omega)} = \frac{d}{dt} F_{11}(\gamma(t))\Big|_{t=0}.$$

Choose a smooth family $\tau_a(t) \in U_t$ satisfying $\tau_a(0) = \tau_a$ and consider the path $\gamma_a(t) = \psi_t(\tau_a(t))\cdot(t, \omega)$. Then

(9.14)
$$\begin{aligned}
&\frac{d}{dt} F(\gamma_a(t))\Big|_{t=0} \\
&= \frac{d}{dt} F\big(\psi_t(\tau_a(t))\cdot(t, \omega)\big)\Big|_{t=0} \\
&= \frac{d}{dt} \tau_a(t) F(t, \omega) \tau_a(t)^{-1}\Big|_{t=0} \\
&= \tau_a'(0) F(0, \omega)\tau_a(0)^{-1} - \tau_a(0) F(0, \omega)\tau_a(0)^{-1}\tau_a'(0)\tau_a(0)^{-1} + \tau_a(0)\Big(\frac{d}{dt} F(t, \omega)\Big|_{t=0}\Big)\tau_a(0)^{-1} \\
&= \tau_a\Big(\frac{\partial F}{\partial t}\Big|_{(0,\omega)}\Big)\tau_a^{-1},
\end{aligned}$$

where the last equality uses the fact that $F(0, \omega) = I$. In the proof of Lemma 9.10, we saw that $\frac{\partial F}{\partial t}\big|_{(0,\omega)} \in \mathfrak{u}_0$ for $\omega \in \ker L$; in particular, it is upper triangular. Hence, the top-left entry of the matrix in the final line of (9.14) is given by $\frac{\partial F_{11}}{\partial t}\big|_{(0,\omega)}$. By (9.13), the top-left entry in the initial line of (9.14) is $\frac{\partial F_{11}}{\partial t}\big|_{(0,\omega + a\epsilon)}$. Thus (9.14) gives the desired equality $H(\omega, \omega) = H(\omega + a\epsilon, \omega + a\epsilon)$. □

9.15 Lemma *The nullspace $N = \{\eta \in \ker L \mid H(\eta, \omega) = 0$ for all $\omega \in \ker L\}$ of H is $\langle \epsilon \rangle$.*

Proof Lemma 9.12 gives $L(\epsilon) = L(0) = 0$ and so $\epsilon \in \ker L$. Moreover, that lemma implies $H(\epsilon, \epsilon) = H(0, 0) = 0$ and also that

$$H(\omega, \omega) = H(\omega + a\epsilon, \omega + a\epsilon) = H(\omega, \omega) + 2\operatorname{Re}(aH(\epsilon, \omega))$$

for all $\omega \in \ker L$ and $a \in \mathbb{C}$. Cancelling the $H(\omega, \omega)$ terms, we get that $\operatorname{Re}(aH(\epsilon, \omega)) = 0$. As this holds for all $a \in \mathbb{C}$, we must have $H(\epsilon, \omega) = 0$ for each $\omega \in \ker L$ and so $\epsilon \in N$. For the converse, note that H is the restriction of $-iB$ to $\ker L$. Since B is nonsingular and $\ker L$ has codimension 1 in $\mathbb{C}^n$, the nullspace N is at most 1-dimensional, proving $N = \langle \epsilon \rangle$. □

9.16 Proposition *If $v \in Z_\epsilon$ satisfies $f_\epsilon(v) = (0, 0)$, then $f_\epsilon|_{Z_\epsilon}$ is a submersion at v.*

Proof Suppose $v = (0, \omega_0)$ in local coordinates. We will identify $T_v Z_\epsilon$ with $\mathbb{C}^n$. As $f_\epsilon(v) = (0,0)$, we have $L(\omega_0) = 0$. For all $\omega \in \ker L$, Lemma 9.10 gives $f_\epsilon(0,\omega) = (0, H(\omega,\omega))$. Thus, for $\eta \in \ker L$,

$$df_\epsilon\big|_v(\eta) = (0, H(\omega_0,\eta) + H(\eta,\omega_0)) = (0, 2\operatorname{Re} H(\omega_0,\eta)).$$

So long as ω_0 is not in N, there will be some $\eta \in \ker L$ for which $df_\epsilon\big|_v(\eta) = (0,1)$. Lemma 9.15 gives that $\omega_0 \in N$ if and only if $\omega_0 = a\epsilon$, which happens if and only if $v_i = \pm v_j$ for all i and j. Since $v \in Z_\epsilon \subset \overline{\mathcal{R}}_n$, the latter is not the case, so $(0,1) \in \operatorname{im} df_\epsilon\big|_v$. By considering tangent vectors $\eta \notin \ker L$, we see that $\operatorname{im} df_\epsilon\big|_v$ contains vectors of the form (a,b) with $a \neq 0$. We conclude that $f_\epsilon|_{Z_\epsilon}$ is a submersion. □

We set $M_\epsilon = f_\epsilon^{-1}(0,0) \cap Z_\epsilon$, which is a smooth submanifold by Proposition 9.16.

9.17 Lemma *The submanifold M_ϵ is in the closure of $\overline{\mathcal{R}}^c(S_n) \setminus \overline{\mathcal{R}}_n^0$. The $\mathbb{C}^\times$ action leaves M_ϵ invariant. For $g \in G_0$, we have $g \cdot M_\epsilon = M_\epsilon$ when g is in the identity component of G_0 and $g \cdot M_\epsilon = M_{-\epsilon}$ otherwise.*

Proof Take K_ϵ to be the full $f_\epsilon^{-1}(0,0)$ in V_ϵ, so $M_\epsilon = K_\epsilon \cap Z_\epsilon$. By Proposition 9.16, K_ϵ is a smooth submanifold of V_ϵ near M_ϵ, and moreover the codimension of K_ϵ in V_ϵ is the same as that of M_ϵ in Z_ϵ, which is three. It follows that M_ϵ is a codimension one submanifold of K_ϵ, and hence the closure of $K_\epsilon \setminus M_\epsilon$ includes M_ϵ. This proves the first claim.

The statement for the $\mathbb{C}^\times$ action follows immediately from Lemma 9.8. For the rest, first recall that the identity component of G_0 is isomorphic to $\operatorname{Isom}^+(\mathbb{E}^2)$, and so is generated by rotations about the origin, which are part of the $\mathbb{C}^\times$ action, and translations, corresponding to $\psi_0(\tau_a)$ for $a \in \mathbb{C}$. As Lemma 9.12 implies that $\psi_0(\tau_a) \cdot M_\epsilon = M_\epsilon$, it follows that $g \cdot M_\epsilon = M_\epsilon$ for all g in the identity component of G_0, as claimed.

Knowing this, it suffices to check $g \cdot M_\epsilon = M_{-\epsilon}$ for a single g in the other component of G_0. We use the element C from Section 6.9. From the relation $F(C \cdot v) = C \cdot F(v)$, we see that if $v \in Z_\epsilon$ and $f_\epsilon(v) = (a,b)$, then $C \cdot v \in Z_{-\epsilon}$ and $f_{-\epsilon}(C \cdot v) = (\bar{a}, -b)$. So $C \cdot M_\epsilon = M_{-\epsilon}$, as desired. □

Let $\widehat{M}_{[\epsilon]}$ be the quotient of $M_\epsilon \cup M_{-\epsilon}$ by the simultaneous actions of $\mathbb{R}_{>0}$ and G_0, so $\widehat{M}_{[\epsilon]} \subset \mathcal{Y}$. Equivalently, it is the quotient of $M_\epsilon = \overline{\mathcal{R}}^c(S_n) \cap Z_\epsilon$ by the simultaneous actions of $\mathbb{R}_{>0}$ and the identity component of G_0. Note that $\widehat{M}_{[\epsilon]}$ is contained in $\pi^{-1}(\chi_{[\epsilon]})$, which is diffeomorphic to $P^{n-2}(\mathbb{C})$ by Lemma 5.3. We will later define $\mathcal{X}^c(S_n)$ so that $\mathcal{X}^c(S_n) \cap \pi^{-1}(\chi_{[\epsilon]}) = \widehat{M}_{[\epsilon]}$. We conclude this subsection by giving an explicit description of $\widehat{M}_{[\epsilon]}$:

9.18 Proposition *Suppose $\chi_{[\epsilon]} \in X^c(S_n)$. There are linear subspaces $V_1 \subset V_2 \subset \mathbb{C}^{n-1}$, where V_2 has dimension $n-2$, such that $\widehat{M}_{[\epsilon]}$ can be identified with the boundary of a tubular neighborhood of the projective space $P(V_1)$ in the larger projective space $P(V_2)$, which itself sits in $P^{n-2}(\mathbb{C})$. In particular, $\widehat{M}_{[\epsilon]}$ is compact. Moreover, the manifold $\widehat{M}_{[\epsilon]}$ is nonempty, except possibly when n is odd and $\epsilon = \pm\mathbb{1}$. If $\chi_{[\epsilon]}$ is balanced (so $n = 2m$), then $\dim V_1 = m-1$.*

Proof Recall from Section 9.4 that, in our local coordinates, $Z_\epsilon = \mathbb{C}^n \setminus \langle\epsilon\rangle$. The submanifold $M_\epsilon = \{\omega \in \mathbb{C}^n \setminus \langle\epsilon\rangle \mid L(\omega) = 0 \text{ and } H(\omega,\omega) = 0\}$ is invariant under translation by any element of $\langle\epsilon\rangle$. We identify $\mathbb{C}^{n-1}$ with $\mathbb{C}^n/\langle\epsilon\rangle$. Taking this quotient has the same effect as normalizing so that the first point ω_1 is 0, which absorbs all the G_0 action except the part contained in $\mathbb{C}^\times$. Setting $V_2 = \ker L/\langle\epsilon\rangle$, the form H descends to a Hermitian form H' on V_2 which is nondegenerate by Lemma 9.15. Then $\widehat{M}_{[\epsilon]}$ is the quotient of $M_\epsilon/\langle\epsilon\rangle = \{\omega \in V_2 \setminus \{0\} \mid H'(\omega,\omega) = 0\}$ by the action of $\mathbb{C}^\times$.

With respect to an appropriate basis of V_2, the form H' is

$$H'(\eta,\eta) = -|\eta_1|^2 - \cdots - |\eta_k|^2 + |\eta_{k+1}|^2 + \cdots + |\eta_{n-2}|^2,$$

where k is the index of H'. Take $V_1 = \{(\eta_1, \eta_2, \ldots, \eta_k, 0, \ldots, 0)\}$; we leave it as an exercise to see that $\widehat{M}_{[\epsilon]}$ is the boundary of a tubular neighborhood of $P(V_1)$ in $P(V_2)$.

Next, the manifold $\widehat{M}_{[\epsilon]}$ is nonempty provided V_1 is neither the trivial subspace nor all of V_2; equivalently, we need to show H' is not definite, ie there exists a nonzero $\eta \in V_2$ with $H'(\eta,\eta) = 0$. (Recall from the Section 9.1 that $n \geq 4$ so $\dim V_2 \geq 2$.) First, suppose $[\epsilon]$ is not $[\mathbb{1}]$. By cyclically permuting the generators s_i, we can assume $\epsilon_1 = 1$ and $\epsilon_2 = -1$. In the formulae used to prove Lemma 9.9, we have $c_1 = c_2 = e^{i\alpha}$ and $b_{21} = 1$, and hence $\widetilde{\eta} = (1, -1, 0, \ldots, 0)$ is in $\ker L$ with $H(\widetilde{\eta}, \widetilde{\eta}) = 0$. As $\widetilde{\eta} \notin \langle\epsilon\rangle$, this gives the desired element $\eta \in V_2$ showing that H' is indefinite.

Second, suppose $\epsilon = \mathbb{1}$ and set $\zeta = e^{i\alpha}$. Then each $\zeta_l = \zeta$, and so $\zeta^n = 1$ with $\zeta \neq \pm 1$ since the trace $c \neq \pm 2$. We then calculate $c_j = \zeta^{2j-1}$ and $b_{jk} = \zeta^{2(k-j)}$. So the vectors $\widetilde{v}_j = \zeta^2 e_j - e_{j+1}$ for $1 \leq j \leq n-2$ together with ϵ form a basis for $\ker L$. Setting $H'' = (-2/\sin\alpha)H'$, consider the associated Hermitian matrix W with respect to the induced basis $v_1, \ldots, v_{n-2}$ of V_2, that is, $H''(v,w) = w^* W v$. A calculation finds the only nonzero entries of W are $\zeta + \zeta^{-1} = 2\cos\alpha$ along the diagonal, $-\zeta$ immediately above the diagonal, and $-\zeta^{-1}$ immediately below. Using Lemma 9.20 below with $q = \zeta$ and $m = n-2$, we have

$$\det W = \frac{\zeta^{n-1} - \zeta^{-(n-1)}}{\zeta - \zeta^{-1}} = \frac{\zeta^{-1} - \zeta}{\zeta - \zeta^{-1}} = -1 \quad \text{as } \zeta^n = 1 \text{ and } \zeta \neq \pm 1.$$

As $\det W < 0$, the form H'' must be indefinite when n is even. Thus we have proved $\widehat{M}_{[\epsilon]}$ is nonempty for all $\chi_{[\epsilon]} \in X^c(S_n)$, except possibly when n is odd and $\epsilon = \pm\mathbb{1}$.

For the last claim, when $\chi_{[\epsilon]}$ is balanced, it is part of a continuous family of reducible characters obtained by varying α. The corresponding Hermitian forms are all nondegenerate and form a continuous family, and so they have the same index. We compute this index in the limit as $\alpha \to 0$. From the formulae in the proof of Lemma 9.9, we have $\lim_{\alpha\to 0}(L/\sin\alpha) = L_0$, where $L_0(w) = \sum_j \omega_j$, and $\lim_{\alpha\to 0}(B/\sin\alpha) = B_0$, where $B_0(w) = \sum_j -\frac{i}{2}\epsilon_j|\omega_j|^2$. Then $H_0 := -iB_0$ is a Hermitian form of index $m = \frac{n}{2}$. Since $\langle\epsilon\rangle$ is contained in the nullspace of $H_0|_{\ker L_0}$, the form on $\ker L_0/\langle\epsilon\rangle$ induced by H_0 has index $m-1$, as needed. □

9.19 Remark When n is odd, the form H'' with matrix W can be (negative) definite. Setting $\zeta_n = e^{2\pi i/n}$, then for $n = 5$ the form is definite when ζ is either ζ_5^2 or ζ_5^3. More generally, experiment strongly suggests that W is definite for n odd if and only if $c = \zeta_n^{(n-1)/2} + \zeta_n^{(n+1)/2} = \cos\frac{\pi(n-1)}{n}$. Indeed, regardless of the parity of n, it seems that the number of negative eigenvalues of H'' for ζ_n^k with $k < \frac{n}{2}$ is exactly $2k-1$.

Finally, here is the lemma needed for the proof of the previous proposition:

9.20 Lemma *Let J_m be the $m \times m$ matrix with entries in $\mathbb{Z}[q^{\pm 1}]$ whose nonzero entries are $q + q^{-1}$ along the diagonal, $-q$ immediately above the diagonal, and $-q^{-1}$ immediately below. Then* $\det J_m = q^m + q^{m-2} + \cdots + q^{-(m-2)} + q^{-m} = (q^{m+1} - q^{-(m+1)})/(q - q^{-1})$.

Proof We induct on m, with the base cases of $m = 1$ and $m = 2$ being easy checks. In the illustrative case of $m = 4$, setting $\bar{q} = q^{-1}$, we have

$$J_4 = \begin{pmatrix} q+\bar{q} & -q & 0 & 0 \\ -\bar{q} & q+\bar{q} & -q & 0 \\ 0 & -\bar{q} & q+\bar{q} & -q \\ 0 & 0 & -\bar{q} & q+\bar{q} \end{pmatrix}.$$

Thus, expanding on the first row and using the known values of $\det J_l$, we get

$$\det J_m = (q + \bar{q}) \det J_{m-1} - \det J_{m-2} = \frac{q^{m+1} - \bar{q}^{m+1}}{q - \bar{q}},$$

as needed to complete the induction and hence the proof. □

9.21 Irreducible characters

Our goal in this subsection is to show that $F^{-1}(I)$ meets $\bar{\mathcal{R}}_n^{t \neq 0}$ as a smooth codimension 3 submanifold. The argument follows the one given in [Heusener 2003], but with the group U_t in place of $\mathrm{SU}_2(2)$. We start by briefly recalling some properties of U_t from Section 6.5. Its Lie algebra $\mathfrak{u}_t$ consists of matrices of the form

$$\begin{pmatrix} ix & w \\ -t\bar{w} & -ix \end{pmatrix}, \quad \text{where } x \in \mathbb{R} \text{ and } w \in \mathbb{C}.$$

Both U_t and $\mathfrak{u}_t$ are contained in

$$V_t = \left\{ \begin{pmatrix} z & w \\ -t\bar{w} & \bar{z} \end{pmatrix} \,\middle|\, z, w \in \mathbb{C} \right\},$$

which we regard as a vector space over $\mathbb{R}$ rather than $\mathbb{C}$. There is a symmetric bilinear form on V_t given by $\langle u_1, u_2 \rangle = \mathrm{tr}(u_1 u_2)$; it is nondegenerate for $t \neq 0$. Note $\mathfrak{u}_t = I^{\perp}$ with respect to this form, and the orthogonal projection $p\colon V_t \to \mathfrak{u}_t$ takes the tracefree part, with $p(C) = C - \frac{1}{2}(\mathrm{tr}\, C)I$. The restriction of $\langle\,\cdot\,,\cdot\,\rangle$ to $\mathfrak{u}_t$ is a scalar multiple of our standard bilinear form from Section 6.1, and hence a scalar multiple of the Killing form.

Fix $t \neq 0$. We will show that the restriction of F to $\overline{\mathcal{R}}_n^t$ is a submersion. This map can be factored as a composition

$$\overline{\mathcal{R}}_n^t \xrightarrow{\boldsymbol{A}} (U_t)^n \xrightarrow{m} U_t, \quad \text{where } \boldsymbol{A}(t, v_1, \dots, v_n) = (A(\alpha, t, v_1), \dots, A(\alpha, t, v_n))$$

and m is the multiplication map. Recall that $\overline{\mathcal{R}}_n^t$ is defined by the open condition that $v_i \neq \pm v_j$ for some $i \neq j$, and therefore each vector v_k in a given $v \in \overline{\mathcal{R}}_n^t$ can be moved independently in $\overline{Q}_t$ on a small scale. Hence, $\operatorname{im} \boldsymbol{A}$ will be an open subset of $(U_t^c)^n$, where $U_t^c = \{g \in U_t \mid \operatorname{tr} g = c\}$. For $g \in U_t$, we identify $T_g U_t$ with $\mathfrak{u}_t$ via right multiplication, ie the map which sends $u \in \mathfrak{u}_t$ to $ug \in T_g U_t$.

9.22 Lemma *For $t \neq 0$ and any $c \in (-2, 2)$, the subset U_t^c is a smooth submanifold of U_t, with $T_g U_t^c = p(g)^\perp$ for each $g \in U_t^c$.*

Proof Identifying $T_g U_t$ with $\mathfrak{u}_t$ as above, for $u \in \mathfrak{u}_t$ we compute

$$d\operatorname{tr}_g(u) = \frac{d}{ds}\Big|_{s=0} \operatorname{tr}(e^{su} g) = \operatorname{tr}\Big(\frac{d}{ds}\Big|_{s=0} e^{su} g\Big) = \operatorname{tr}(ug) = \langle u, g\rangle = \langle u, p(g)\rangle.$$

Since $g \neq \pm I$, we have $p(g) \neq 0$. As $t \neq 0$, the bilinear form on $\mathfrak{u}_t$ is nondegenerate, so there is $u \in \mathfrak{u}_t$ with $\langle u, p(g)\rangle \neq 0$. Thus $\operatorname{tr}\colon U_t \to \mathbb{R}$ is a submersion at g and $T_g U_t^c = \ker(d\operatorname{tr}_g) = p(g)^\perp$, as needed. □

9.23 Lemma *Suppose $\boldsymbol{g} = (g_1, \dots, g_n) \in (U_t^c)^n$. If $p(g_i)$ and $p(g_j)$ are linearly independent for some i, j, the map $dm_{\boldsymbol{g}}\colon T_{\boldsymbol{g}}(U_t^c)^n \to \mathfrak{u}_t$ is surjective.*

The proof is essentially the same as that of [Heusener 2003, Lemma 3.1], but we give it here for completeness.

Proof If we identify $T_{\boldsymbol{g}} U_t^n$ with $\mathfrak{u}_t^n$ via the map which sends $(u_1, \dots, u_n) \in \mathfrak{u}_t^n$ to $(u_1 g_1, \dots, u_n g_n) \in T_{\boldsymbol{g}} U_t^n$, the derivative $dm\colon \mathfrak{u}_t^n \to \mathfrak{u}_t$ at $\boldsymbol{g}$ is given by

$$dm(u_1, \dots, u_n) = \sum_{j=1}^n g_1 \cdots g_{j-1} u_j (g_1 \cdots g_{j-1})^{-1} = \sum_{j=1}^n \operatorname{Ad}(g_1 \cdots g_{j-1}) \cdot u_j.$$

By Lemma 9.22, $dm(T_{\boldsymbol{g}}(U_t^c)^n)$ is spanned by the sum of the subspaces $\operatorname{Ad}(g_1 \cdots g_{j-1})(p(g_j)^\perp)$. Fix an index j where $p(g_j)$ and $p(g_{j+1})$ are linearly independent, and set $h = g_1 \cdots g_{j-1}$. To see dm is onto, as $\dim \mathfrak{u}_t = 3$, it suffices to show that the 2-dimensional subspaces

$$\operatorname{Ad}(h)(p(g_j)^\perp) \quad \text{and} \quad \operatorname{Ad}(hg_j)(p(g_{j+1})^\perp)$$

of $\mathfrak{u}_t$ are distinct. Applying $\operatorname{Ad}(g_j^{-1} h^{-1})$, we equivalently examine $\operatorname{Ad}(g_j^{-1})(p(g_j)^\perp)$ and $p(g_{j+1})^\perp$. As Ad preserves the bilinear form, we have

$$\operatorname{Ad}(g_j^{-1})(p(g_j)^\perp) = \big(\operatorname{Ad}(g_j^{-1})(p(g_j))\big)^\perp = p(g_j)^\perp.$$

As the form is nondegenerate with $p(g_j)$ and $p(g_{j+1}$ linearly independent, it follows that $p(g_j)^\perp \neq p(g_{j+1})^\perp$, as needed to show m is a submersion at $\boldsymbol{g}$. □

We can now prove the main result of this subsection:

9.24 Proposition *The intersection $F^{-1}(I) \cap \overline{\mathcal{R}}_n^{t\neq 0}$ is a smooth closed codimension 3 submanifold of $\overline{\mathcal{R}}_n^{t\neq 0}$.*

Proof Consider the submanifold $\widetilde{U} = \{(t,g) \mid g \in U_t\} \subset \mathbb{R} \times \mathrm{SL}_2\mathbb{C}$, and define a map $\widetilde{F} : \overline{\mathcal{R}}_n^{t\neq 0} \to \widetilde{U}$ by $\widetilde{F}(v) = (t(v), F(v))$. We claim that $\widetilde{F}$ is a submersion. For $v \in \overline{\mathcal{R}}_n$, set $g_i = A(\alpha, t, v_i)$. By Lemma 6.14, we have $p(g_i) = (\cos\alpha)v_i$. By definition, a point $v \in \overline{\mathcal{R}}_n$ has $v_i \neq \pm v_j$ for some j, and hence $p(g_i)$ and $p(g_j)$ are linearly independent. Lemma 9.23 thus shows that $d\widetilde{F}(T_v\overline{\mathcal{R}}_n^t) = \{0\} \times T_g U_t$, where $g = F(v)$. By considering a path of the form $\gamma(t) = (t, \gamma_1(t), \dots, \gamma_n(t))$ in $\overline{\mathcal{R}}_n$, we see that $\mathrm{im}(d\widetilde{F})$ contains a vector of the form $(1, *)$. Hence, $\mathrm{im}(d\widetilde{F})$ has dimension 4 and $\widetilde{F}$ is a submersion.

To conclude the proof, let $\widetilde{I} = \{(t, I) \mid t \in \mathbb{R}\}$. Now $\widetilde{I}$ is a smooth closed codimension 3 submanifold of $\widetilde{U}$, so $F^{-1}(I) \cap \overline{\mathcal{R}}_n^{t\neq 0} = \widetilde{F}^{-1}(\widetilde{I})$ is a smooth codimension 3 submanifold of $\overline{\mathcal{R}}_n^{t\neq 0}$. □

9.25 Putting it all together

We can now prove the results stated at the beginning of this section.

Proof of Proposition 9.6 Recall from Section 9.4 that $\overline{\mathcal{R}}^c(S_n)$ is the union of $F^{-1}(I) \cap \overline{\mathcal{R}}_n^{t\neq 0}$ with all $f_\epsilon^{-1}(0,0)$, where the union is taken over those ϵ with $\chi_{[\epsilon]} \in X^c(S_n)$. By Lemma 9.5, we know $\overline{\mathcal{R}}^c(S_n) \subset F^{-1}(I)$ and $\overline{\mathcal{R}}^c(S_n) \cap \overline{\mathcal{R}}_n^{t\neq 0} = F^{-1}(I) \cap \overline{\mathcal{R}}_n^{t\neq 0}$. To show that $\overline{\mathcal{R}}^c(S_n)$ is a codimension 3 submanifold of $\overline{\mathcal{R}}_n$, we must check that, for each $v \in \overline{\mathcal{R}}^c(S_n)$, there is an open subset $O_v \subset \overline{\mathcal{R}}_n$ containing v such that $\overline{\mathcal{R}}^c(S_n) \cap O_v$ is a codimension 3 submanifold of O_v. This follows from Proposition 9.24 when $t(v) \neq 0$ using $O_v = \overline{\mathcal{R}}_n^{t\neq 0}$ and from Proposition 9.16 when $t(v) = 0$ by taking O_v to be some V_ϵ.

Turning now to showing that $\overline{\mathcal{R}}^c(S_n)$ is closed, the main worry is that it could have a limit point in a component of $\overline{\mathcal{R}}_n^0$ corresponding to ϵ where $\chi_{[\epsilon]}$ is not in $X^c(S_n)$. As $X^c(S_n)$ is a closed subset of $X^c(F_n)$, for each ϵ with $\chi_{[\epsilon]} \notin X^c(S_n)$ we choose an open set $W_\epsilon \subset X^c(F_n)$ disjoint from $X^c(S_n)$. Let $V_\epsilon = \Pi^{-1}(W_\epsilon)$, so $Z_\epsilon \subset V_\epsilon$ and $V_\epsilon \cap \overline{\mathcal{R}}^c(S_n) = \varnothing$. For the other ϵ with $\chi_{[\epsilon]}$ in $X^c(S_n)$, we continue to define V_ϵ as in Section 9.4. Then $\overline{\mathcal{R}}_n$ has an open cover consisting of $\overline{\mathcal{R}}_n^{t\neq 0}$ together with the V_ϵ's. If V is one of the open sets in this cover, then $\overline{\mathcal{R}}^c(S_n) \cap V$ is closed in V. It follows that $\overline{\mathcal{R}}^c(S_n)$ is closed in $\overline{\mathcal{R}}_n$.

Finally, regarding the group actions, as noted in Section 9.1, the subset $F^{-1}(I)$ is preserved by the actions of $\mathbb{C}^\times$ and the G_t. Those actions also preserve $\overline{\mathcal{R}}_n^{t\neq 0}$, and hence also $F^{-1}(I) \cap \overline{\mathcal{R}}_n^{t\neq 0} = \overline{\mathcal{R}}^c(S_n) \cap \overline{\mathcal{R}}_n^{t\neq 0}$. The rest, namely $\overline{\mathcal{R}}^c(S_n) \cap \overline{\mathcal{R}}_n^0$, is the union of the submanifolds M_ϵ from Lemma 9.17, and that lemma gives the needed invariance. □

Proof of Theorem 9.2 We define $\mathcal{X}^c(S_n) = \sigma(\overline{\mathcal{R}}^c(S_n))$. Combining Lemma 2.2, Theorem 7.1, and Proposition 9.6, we see that $\mathcal{X}^c(S_n)$ is a closed codimension 3 smooth submanifold of $\mathcal{X}^c(F_n)$. To compute

$\pi(\mathscr{X}^c(S_n))$, first observe that it is $\pi\big(\sigma(\overline{\mathscr{R}}^c(S_n))\big)$ which is equal to $\tau\big(\pi'_c(\overline{\mathscr{R}}^c(S_n))\big)$ by Lemma 8.8. By Lemma 9.5, the subset $\overline{\mathscr{R}}^c(S_n)$ is contained in $F^{-1}(I)$, and, as discussed in Section 9.1, the subset $\pi'_c(F^{-1}(I))$ is contained in $R^c_{\mathbb{C}}(S_n)$. Hence, $\pi(\mathscr{X}^c(S_n)) \subset \tau(R^c_{\mathbb{C}}(S_n)) = X^c_{\mathbb{C}}(S_n)$. As $\pi(\mathscr{X}^c(F_n)) = X^c(F_n)$, taking intersections yields $\pi(\mathscr{X}^c(S_n)) \subset X^c(S_n)$. To show equality, we consider the cases of $X^{c,\mathrm{irr}}(S_n)$ and $X^{c,\mathrm{red}}(S_n)$ separately.

Given $\chi \in X^{c,\mathrm{irr}}(S_n)$, pick a corresponding representation $\rho\colon \pi_1(S_n) \to U_t$ where $t = \pm 1$. Viewing the domain of ρ as F_n, Lemma 8.7 gives a unique $v \in \overline{\mathscr{R}}^t_n$ with $\pi'_c(v) = \rho$. As ρ is in $R^c_{\mathbb{C}}(S_n)$, we have $v \in F^{-1}(I)$ and so $v \in \overline{\mathscr{R}}^t(S_n)$. Then $\pi(\sigma(v)) = \chi$, as needed. If instead $\chi_{[\epsilon]} \in X^{c,\mathrm{red}}(S_n)$, consider $\widehat{M}_{[\epsilon]}$ from Proposition 9.18. By this proposition and the discussion immediately before it, we have $\widehat{M}_{[\epsilon]} = \mathscr{X}^c(S_n) \cap \pi^{-1}(\chi_{[\epsilon]})$, and $\widehat{M}_{[\epsilon]}$ is nonempty since we are requiring that n is even. In particular, we have $\chi_{[\epsilon]} \in \pi(\mathscr{X}^c(S_n))$. So $\pi(\mathscr{X}^c(S_n)) = X^c(S_n)$, as claimed.

Next, the claim that the map $\pi\colon \mathscr{X}^c(S_n) \to X^c(S_n)$ is a diffeomorphism away from $X^{c,\mathrm{red}}(S_n)$ is immediate from the analogue for F_n in Theorem 8.1. Finally, to see that the closure of $\mathscr{X}^{c,\mathrm{irr}}(S_n)$ in $\mathscr{X}^c(F_n)$ is $\mathscr{X}^c(S_n)$, start by noting that $\mathscr{X}^{c,\mathrm{red}}(S_n) = \bigcup \widehat{M}_{[\epsilon]}$, where the union is over $\chi_{[\epsilon]} \in X^c(S_n)$. The result now follows from the first statement in Lemma 9.17. □

9.26 Odd number of punctures

In proving Theorem 9.2, the only place the parity of n was used was Proposition 9.18, in the special case that $\chi_{[\epsilon]} \in X^c(S_n)$ for $\epsilon = \mathbb{1}$. Thus Theorem 9.2 still holds for odd n provided we exclude the finitely many c of the form $2\cos\frac{2\pi k}{n}$ for $1 \le k < \frac{n}{2}$. For context, note that for a fixed odd n, any reducible $\chi_{[\epsilon]} \in X^c(S_n)$ is necessarily unbalanced, and hence there are only finitely many $c \in (-2,2)$ for which $X^{c,\mathrm{red}}(S_n)$ is nonempty. Thus, for most c, no resolution is necessary as $X^c(S_n) = X^{c,\mathrm{irr}}(S_n)$.

Moreover, given Remark 9.19, we conjecture that Theorem 9.2 holds if we exclude the single value $c = 2\cos\frac{\pi(n-1)}{n}$. For that specific c, it turns out that $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ is empty (apply [Biswas 1998, Theorem A] with $j = 0$), and experimental evidence suggests $X^{c,\mathrm{red}}(S_n) = \{\chi_{[\mathbb{1}]}\}$. It seems plausible that the corresponding $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ does not limit on $X^{c,\mathrm{red}}(S_n)$ and might even be empty; either way, Theorem 9.2 would hold modulo the isolated point $X^{c,\mathrm{red}}(S_n)$ not being in the image of $\mathscr{X}^c(S_n)$.

10 Varying the trace

Up to now, we have focused on character varieties where each of our preferred generators have the same fixed trace $c \in (-2,2)$. We now consider the effect of varying c, starting with the free group. Recall from Section 3.5 that $\mathrm{tr}\colon X(F_n, S) \to \mathbb{R}$ is the map sending χ to $\chi(s_1)$, or equivalently to any $\chi(s_i)$. We will show:

10.1 Theorem *For each $n \geq 2$, there is a smooth $(2n-2)$-dimensional manifold $\mathscr{X}(F_n, S)$ and a smooth map $\pi : \mathscr{X}(F_n, S) \to X(F_n, S)$ with the following properties: First, the map $\mathrm{Tr} = \mathrm{tr} \circ \pi : \mathscr{X}(F_n, S) \to \mathbb{R}$ is a submersion, with $\mathrm{Tr}^{-1}(c) \cong \mathscr{X}^c(F_n, S)$ for $c \in (-2, 2)$ and $\mathrm{Tr}^{-1}(c) \cong X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S)$ otherwise. Second, the subset $\mathscr{X}^{\mathrm{irr}}(F_n, S) := \pi^{-1}(X^{\mathrm{irr}}(F_n, S))$ is dense in $\mathscr{X}(F_n, S)$ and π restricts to a diffeomorphism $\mathscr{X}^{\mathrm{irr}}(F_n, S) \to X^{\mathrm{irr}}(F_n, S)$.*

For the punctured sphere, the situation is more complicated. Recall from Section 9.3 that for balanced ϵ, the reducible character $\chi^c_{[\epsilon]}$ is in $X^c(S_n)$ for all c, but for unbalanced ϵ, the character $\chi^c_{[\epsilon]}$ is in $X^c(S_n)$ only at isolated c. It turns out that we can fit the resolutions of the balanced reducibles into a family, but not the resolutions of the unbalanced reducibles. Our solution to this problem is to excise the unbalanced reducibles from our moduli space. More formally, we define

$$X_{\mathrm{bal}}(F_n, S) = X(F_n, S) \setminus \left\{ \chi^c_{[\epsilon]} \mid \textstyle\sum \epsilon_i \neq 0 \right\}$$

and $\mathscr{X}_{\mathrm{bal}}(F_n, S) = \pi^{-1}(X_{\mathrm{bal}}(F_n, S))$, so $\mathscr{X}_{\mathrm{bal}}(F_n, S)$ is an open submanifold of $\mathscr{X}(F_n, S)$. Likewise, we take $X_{\mathrm{bal}}(S_n, S) = X_{\mathrm{bal}}(F_n, S) \cap X(S_n, S)$ and $\mathscr{X}_{\mathrm{bal}}(S_n, S) = \mathscr{X}(S_n, S) \cap \mathscr{X}_{\mathrm{bal}}(F_n, S)$, as well as $\mathscr{X}^c_{\mathrm{bal}}(S_n, S) = \mathscr{X}^c(S_n, S) \cap \mathscr{X}_{\mathrm{bal}}(S_n, S)$, etc. We will show:

10.2 Theorem *For each even $n \geq 4$, there is a smooth closed codimension 3 submanifold $\mathscr{X}(S_n, S)$ of $\mathscr{X}_{\mathrm{bal}}(F_n, S)$ satisfying $X^{\mathrm{irr}}(S_n, S) \subset \pi(\mathscr{X}(S_n, S)) \subset X_{\mathrm{bal}}(S_n, S)$ with the following properties: The map $\mathrm{Tr}\colon \mathscr{X}(S_n, S) \to \mathbb{R}$ is a submersion, and $\mathrm{Tr}^{-1}(c) = \mathscr{X}^c_{\mathrm{bal}}(S_n, S)$ for $c \in (-2, 2)$; otherwise $\mathrm{Tr}^{-1}(c) = X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n, S)$. Also, the subset $\mathscr{X}^{\mathrm{irr}}(S_n, S) := \pi^{-1}(X^{\mathrm{irr}}(S_n, S))$ is dense in $\mathscr{X}(S_n, S)$ and π restricts to a diffeomorphism $\mathscr{X}^{\mathrm{irr}}(S_n, S) \to X^{\mathrm{irr}}(S_n, S)$.*

The generating set S will remain fixed throughout this section, so we drop it from the notation, writing $\mathscr{X}(F_n)$ for $\mathscr{X}(F_n, S)$, etc.

10.3 The free group

The space $\mathscr{X}(F_n)$ will be constructed as the union of two sets B and C. Let $B = (-2, 2) \times \mathscr{Y}$, and define $\pi_B\colon B \to X(F_n)$ by $\pi_B(c, y) = \pi_c(y)$, where π_c is the map constructed in Theorem 8.1. As in Lemma 8.8, we have a commutative square

$$\begin{array}{ccc} (-2,2) \times \overline{\mathscr{R}}_n & \xrightarrow{\ \pi'\ } & R_{\mathbb{C}}(F_n) \\ \big\downarrow{\scriptstyle \mathrm{id} \times \sigma} & & \big\downarrow{\scriptstyle \tau} \\ (-2,2) \times \mathscr{Y} & \xrightarrow{\ \pi_B\ } & X(F_n) \end{array}$$

where the topmost map is defined by $\pi'(c, v) = \pi'_c(v)$, that is,

$$\pi'(c, v)(s_i) = A\left(\cos^{-1}\left(\tfrac{1}{2}c\right), t(v), v_i\right),$$

and so π' is smooth by (8.6). As τ is also smooth, so is $\tau \circ \pi'$. As the left-hand arrow is a submersion, π_B is smooth.

Next, we define $C = X^{\mathrm{irr}}(F_n)$, which is a smooth manifold by Corollary 3.7. Let $\pi_C : C \to X(F_n)$ be the inclusion. We now define $\mathscr{X}(F_n) = (B \amalg C)/\sim$, where $b \sim c$ if $\pi_B(b) = \pi_C(c)$. We give $\mathscr{X}(F_n)$ the quotient topology, so the maps π_B and π_C combine to give a continuous map $\pi : \mathscr{X}(F_n) \to X(F_n)$. Finally, we let $i_B : B \to \mathscr{X}(F_n)$ and $i_C : C \to \mathscr{X}(F_n)$ be the maps induced by inclusion. We define $B^{\mathrm{irr}} = \pi_B^{-1}(X^{\mathrm{irr}}(F_n))$, which Theorem 8.1 shows is

$$(-2,2) \times (\mathscr{Y} \setminus \mathscr{C}_n^{\pm}(\mathbb{E}^2)) = (-2,2) \times (\mathscr{C}_n(S^2) \amalg \mathscr{C}_n^{\pm}(\mathbb{H}^2)).$$

Next, notice that $\operatorname{im} \pi_B \cap \operatorname{im} \pi_C \subset X^{\mathrm{irr}}(F_n)$ is $\bigcup\{X^{c,\mathrm{irr}}(F_n) \mid -2 < c < 2\}$. Since π_C is injective and π_B is injective on $\pi_B^{-1}(\operatorname{im} \pi_C) = B^{\mathrm{irr}}$, the maps i_B and i_C are also injective. Since i_B and i_C are injective, we view B and C as subsets of $\mathscr{X}(F_n)$.

Proof of Theorem 10.1 We first describe the smooth structure on $\mathscr{X}(F_n)$. The pieces B and C are both smooth manifolds, so we define $f : \mathscr{X}(F_n) \to \mathbb{R}$ to be smooth if and only if $f|_B$ and $f|_C$ are smooth. In order for this to make sense, we must check that the transition function $i_C^{-1} \circ i_B = \pi_C^{-1} \circ \pi_B$ is a diffeomorphism from B^{irr} onto its image. As π_C is the inclusion of the open subset $C = X^{\mathrm{irr}}(F_n)$ into $X(F_n)$, this amounts to analyzing π_B restricted to B^{irr}. As noted, we have $\pi_B(B^{\mathrm{irr}}) = \bigcup\{X^{c,\mathrm{irr}}(F_n) \mid -2 < c < 2\}$, and the latter is open in $X(F_n)$. Theorem 8.1 then gives that $\pi_B|_{B^{\mathrm{irr}}}$ is injective. As $\operatorname{tr}(\pi_B(c,y)) = c$, $d\pi_B\bigl(\frac{\partial}{\partial c}\bigr)$ is not in the image of each $T(\{c\} \times \mathscr{Y})$ under $d\pi_B$, so Theorem 8.1 also gives that $d\pi_B$ is an isomorphism at each point in B^{irr}, making $\pi_B|_{B^{\mathrm{irr}}}$ a diffeomorphism onto its image. We conclude that $\mathscr{X}(F_n)$ admits a smooth structure with respect to which i_B and i_C are open embeddings. The maps $\pi|_B = \pi_B$ and $\pi|_C = \pi_C$ are smooth, so π is smooth.

Next, we check that $\mathscr{X}(F_n)$ is Hausdorff. The pieces B and C are Hausdorff open subsets which cover $\mathscr{X}(F_n)$, so it suffices to show that if $x \in B \setminus C$ and $y \in C \setminus B$, then x and y can be separated by open sets in $\mathscr{X}(F_n)$. Considering the map $\pi : \mathscr{X}(F_n) \to X(F_n)$, we see $\pi(x) \neq \pi(y)$ since $\pi(x)$ is reducible but $\pi(y)$ is irreducible. As $X(F_n)$ is Hausdorff and π is continuous, we can separate x from y by the preimages of open sets in $X(F_n)$ separating $\pi(x)$ from $\pi(y)$. Thus $\mathscr{X}(F_n)$ is Hausdorff.

Turning now to the map $\operatorname{Tr} = \operatorname{tr} \circ \pi : \mathscr{X}(F_n) \to \mathbb{R}$, note that it satisfies $\operatorname{Tr}(c,x) = c$ for $(c,x) \in B$ and is given by the function tr on C. As $\operatorname{Tr}|_B$ is thus a submersion and $\operatorname{Tr}|_C$ is also by Corollary 3.8, Tr is a submersion. We leave the statements about $\operatorname{Tr}^{-1}(c)$ to you, but point out that for c outside $(-2,2)$ one has $X^{c,\mathrm{irr}}(F_n,S) = X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$ since $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n,S)$ is empty by Lemma 3.10. Finally, π maps $\mathscr{X}^{\mathrm{irr}}(F_n) := C$ diffeomorphically onto $X^{\mathrm{irr}}(F_n)$ by construction, and the density of $\mathscr{X}^{\mathrm{irr}}(F_n)$ in $\mathscr{X}(F_n)$ follows from the density of $\mathscr{X}^{c,\mathrm{irr}}(F_n)$ in $\mathscr{X}^{c}(F_n)$ for $c \in (-2,2)$, which was noted after (8.2). □

10.4 Properness of projections

We now consider the smooth map $\pi : \mathscr{X}(F_n) \to X(F_n)$. Its image is

$$X_\circ(F_n) := \{\chi \in X(F_n) \mid \chi \in X^{\mathrm{irr}}(F_n) \text{ or } \operatorname{tr}(\chi) \in (-2,2)\}.$$

We let $\pi_\circ : \mathscr{X}(F_n) \to X_\circ(F_n)$ be the map obtained by restricting the range of π.

10.5 Proposition *The map $\pi_\circ\colon \mathscr{X}(F_n) \to X_\circ(F_n)$ is proper.*

The proof makes use of the following:

10.6 Lemma *Suppose $f\colon \mathscr{W} \to W$ is continuous, where $W = U_0 \cup U_1$ is a metric space and the U_i are open. If both restrictions $\hat{f}_i\colon f^{-1}(U_i) \to U_i$ are proper, then so is f.*

Proof Let $C_i = W - U_i$, so C_i is closed and $C_0 \cap C_1 = \varnothing$. We consider the maps $d_i\colon W \to [0,\infty)$ given by $d_i(x) = d(x, C_i) = \inf\{d(x,y) \mid y \in C_i\}$, which are continuous. As C_i is closed in W, one has $d_i(x) = 0$ if and only if $x \in C_i$. Since the C_i are disjoint, $d_0(x) + d_1(x) > 0$ for all $x \in W$. Hence, if we define

$$g(x) = \frac{d_0(x)}{d_0(x) + d_1(x)},$$

we have $0 \le g(x) \le 1$. For each i, it follows that $x \in C_i$ if and only if $g(x) = i$.

Now let $K \subset W$ be compact, and let $K_0 = g^{-1}\bigl(\bigl[\frac{1}{2}, 1\bigr]\bigr) \cap K$, $K_1 = g^{-1}\bigl(\bigl[0, \frac{1}{2}\bigr]\bigr) \cap K$. Then $K_i \subset U_i$ is a closed subset of a compact set, hence compact, and $K = K_0 \cup K_1$. Since each $\hat{f}_i$ is proper, $f^{-1}(K) = \hat{f}_1^{-1}(K_1) \cup \hat{f}_2^{-1}(K_2)$ is a union of compact sets, hence compact. Thus f is proper. □

Proof of Proposition 10.5 The sets $U_1 := \operatorname{tr}^{-1}(-2,2)$ and $U_2 := X^{\mathrm{irr}}(F_n)$ form an open cover of $X_\circ(F_n)$ with $\pi_\circ^{-1}(U_1) = B$ and $\pi_\circ^{-1}(U_2) = C$. So, by Lemma 10.6, it is enough to check that $\pi_i := \widehat{\pi}_{\circ,i}$ is proper for $i = 1,2$. Now $\widehat{\pi}_{\circ,2}\colon C \to U_2$ is proper as it is a homeomorphism, and so it remains to consider $\widehat{\pi}_{\circ,1}$, which we abbreviate to π.

To show that $\pi\colon B \to U_1$ is proper, suppose $K \subset U_1$ is compact. By continuity of $\operatorname{tr}\colon X_\circ(F_n) \to \mathbb{R}$, there is a closed interval $[a,b] \subset (-2,2)$ such that $\operatorname{tr}(K) \subset [a,b]$. Inside B, consider $Z = \operatorname{Tr}^{-1}([a,b]) = [a,b] \times \mathscr{Y}$, and consider the smooth submanifold $M := \pi^{-1}(X^{\mathrm{red}}(F_n)) \cap Z$, which is $[a,b] \times \mathscr{C}_n^{\pm}(\mathbb{E}^2)$ by Theorem 7.1, and so compact by Lemma 5.3.

Let $D_1 \subset Z$ be a closed tubular neighborhood of M, which we identify with the product $M \times [-1,1]$ (using Theorem 7.4), and let D_2 be Z with $M \times \bigl(-\frac{1}{2}, \frac{1}{2}\bigr)$ removed. Then D_1 and D_2 are closed subsets of B, with $\pi(D_1)$ and $\pi(D_2)$ being closed subsets of U_1 which cover K. Now let $K_i = K \cap \pi(D_i)$. Then K_1 and K_2 are closed subsets of K, hence compact, and $K = K_1 \cup K_2$. The preimage $\pi^{-1}(K_1)$ is a closed subset of the compact set D_1, hence compact. The restriction of π to D_2 is a homeomorphism onto its image, so $\pi^{-1}(K_2)$ is compact. Hence, $\pi^{-1}(K) = \pi^{-1}(K_1) \cup \pi^{-1}(K_2)$ is compact. It follows that $\pi = \widehat{\pi}_{\circ,1}$ is proper and hence so is $\pi_\circ$. □

10.7 The punctured sphere

From now on, we assume n is even and construct the submanifold $\mathscr{X}(S_n) \subset \mathscr{X}_{\mathrm{bal}}(F_n)$ from submanifolds $B' \subset B$ and $C' \subset C$, where B and C were used to define $\mathscr{X}(F_n)$. First, we define B', which is inside the open subset $B_{\mathrm{bal}} = B \cap \mathscr{X}_{\mathrm{bal}}(F_n)$ of B:

10.8 Lemma *The set* $B' = \{(c, y) \in B \mid y \in \mathscr{X}^c_{\mathrm{bal}}(S_n)\}$ *is a closed smooth codimension* 3 *submanifold of* B_{bal}. *The map* $\mathrm{Tr}\colon B' \to \mathbb{R}$ *is a submersion.*

Proof Recall from Section 9.4 the subsets Z_ϵ and $\overline{\mathcal{R}}^c(S_n)$ of $\overline{\mathcal{R}}_n$. We define

$$\overline{\mathcal{R}}_{n,\mathrm{bal}} = \overline{\mathcal{R}}_n \setminus \bigcup \bigl\{ Z_\epsilon \mid \epsilon \in \{\pm 1\}^n \text{ and } \textstyle\sum \epsilon \neq 0 \bigr\},$$

so that $\overline{\mathcal{R}}_{n,\mathrm{bal}} = \Pi_c^{-1}(X^c_{\mathrm{bal}}(F_n))$ for all $c \in (-2, 2)$. Let $\overline{\mathcal{R}}(F_n) = (-2, 2) \times \overline{\mathcal{R}}_{n,\mathrm{bal}}$, and define $\overline{\mathcal{R}}(S_n) = \{(c, v) \in \overline{\mathcal{R}}(F_n) \mid v \in \overline{\mathcal{R}}^c(S_n)\}$. We will first show that $\overline{\mathcal{R}}(S_n)$ is a closed smooth codimension 3 submanifold of $\overline{\mathcal{R}}(F_n)$, and that the restriction of Tr to $\overline{\mathcal{R}}(S_n)$ is a submersion.

Consider the function $G\colon \overline{\mathcal{R}}(F_n) \to \mathrm{SL}_2\mathbb{C}$ given by

$$G(c, v) = \prod_{i=1}^{n} A\bigl(\cos^{-1}\bigl(\tfrac{1}{2}c\bigr), t(v), v_i\bigr),$$

so that the restriction to the slice $c \times \overline{\mathcal{R}}_{n,\mathrm{bal}}$ is the function F from Section 9.

Let $\widehat{W} = (-2, 2) \times \overline{\mathcal{R}}_n^{t \neq 0}$, and, for each balanced ϵ, let

$$\widehat{V}_\epsilon = \{(c, v) \in \overline{\mathcal{R}}(F_n) \mid \operatorname{sign} z_j = \epsilon_j \text{ for all } j \text{ and } \operatorname{Re}(G_{11}(c, v)) > 0\}.$$

Then $\widehat{W}$ together with the $\widehat{V}_\epsilon$'s form an open cover of $\overline{\mathcal{R}}(F_n)$. Recall that $\widetilde{U} = \{(t, g) \mid g \in U_t\} \subset \mathbb{R} \times \mathrm{SL}_2\mathbb{C}$, and define $\widetilde{G}\colon \widehat{W} \to \widetilde{U}$ by $\widetilde{G}(c, v) = (t(v), G(c, v))$. Then $\overline{\mathcal{R}}(S_n) \cap \widehat{W} = G^{-1}(I) = \widetilde{G}^{-1}(\widetilde{I})$, where $\widetilde{I} = \{(t, I) \mid t \in \mathbb{R}\}$. We showed in the proof of Proposition 9.24 that the restriction of $\widetilde{G}$ to each slice $c \times \overline{\mathcal{R}}_{n,\mathrm{bal}}$ is a submersion, so $\overline{\mathcal{R}}(S_n) \cap \widehat{W}$ is a closed codimension 3 submanifold of $\widehat{W}$, and $\mathrm{Tr}\colon \overline{\mathcal{R}}(S_n) \cap \widehat{W} \to \mathbb{R}$ is a submersion.

As in Section 9, we have local coordinates $(c, t, \omega_1, \dots, \omega_n)$ on $\widehat{V}_\epsilon$. The function G_{11} is real analytic on $\widehat{V}_\epsilon$, and the fact that ϵ is balanced implies that $G_{11}(c, 0, v) \equiv 1$. It follows that $g_\epsilon = (G_{12}, \operatorname{Im}(G_{11} - 1)/t)$ is real analytic on $\widehat{V}_\epsilon$. Its restriction to the slice $c \times V_\epsilon$ is the function f_ϵ from Section 9, so $g^{-1}(0, 0) = \overline{\mathcal{R}}(S_n) \cap \widehat{V}_\epsilon$. We showed in Section 9.21 that f_ϵ is a submersion, and hence g_ϵ is as well, so $\overline{\mathcal{R}}(S_n) \cap \widehat{V}_\epsilon$ is a smooth closed submanifold of $\widehat{V}_\epsilon$. Now $\mathrm{Tr}\colon \widehat{V}_\epsilon \to \mathbb{R}$ is a submersion, being projection onto the first coordinate c. As g_ϵ is a submersion restricted to each fiber of Tr, as that is some f_ϵ, it follows that $\mathrm{Tr}\colon \overline{\mathcal{R}}(S_n) \cap \widehat{V}_\epsilon \to \mathbb{R}$ is also a submersion. This completes our proof that $\overline{\mathcal{R}}(S_n)$ is a smooth submanifold of $\overline{\mathcal{R}}(F_n)$ of codimension 3 and that $\mathrm{Tr}\colon \overline{\mathcal{R}}(S_n) \to \mathbb{R}$ is a submersion.

Now B_{bal} is the quotient of $\overline{\mathcal{R}}(F_n)$ by the simultaneous actions of the G_t and $\mathbb{R}_{>0}$, and B' is the image of $\overline{\mathcal{R}}(S_n)$ under this quotient, so the statement of the lemma follows from Lemma 2.2. □

10.9 Lemma *The subset* $C' := X^{\mathrm{irr}}(S_n)$ *is a closed codimension* 3 *submanifold of* $C = X^{\mathrm{irr}}(F_n)$, *and* $\mathrm{tr}\colon C' \to \mathbb{R}$ *is a submersion.*

Proof Recall from Section 3.2 that

$$X^{\mathrm{irr}}(F_n) = X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \amalg X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n) = X^{\mathrm{irr}}_{U_1}(F_n) \amalg X^{\mathrm{irr}}_{U_{-1}}(F_n).$$

Similarly, $X^{\mathrm{irr}}(S_n) = X^{\mathrm{irr}}_{U_1}(S_n) \amalg X^{\mathrm{irr}}_{U_{-1}}(S_n)$. We will show that the claims of the lemma hold for $X^{\mathrm{irr}}_{U_{-1}}(S_n)$; the proofs for $X^{\mathrm{irr}}_{U_1}(S_n)$ are identical.

Consider the map $\mu\colon R^{\mathrm{irr}}_{U_{-1}}(F_n) \to U_{-1}$ given by $\mu(\rho) = \rho(s_1 \cdots s_n)$. Applying Lemmas 9.22 and 9.23, we see that the restriction of μ to each $R^{\mathrm{irr},c}(F_n)$ is a submersion. It follows that μ is a submersion, and so $R^{\mathrm{irr}}_{U_{-1}}(S_n) = \mu^{-1}(I)$ is a closed codimension 3 submanifold of $R^{\mathrm{irr}}_{U_{-1}}(F_n)$.

The projection $\tau\colon R^{\mathrm{irr}}_{U_{-1}}(F_n) \to X^{\mathrm{irr}}_{U_{-1}}(F_n)$ is a submersion by Lemma 3.9, and $R^{\mathrm{irr}}(S_n)$ is a union of fibers of τ. Applying Lemma 2.2, we conclude that $X^{\mathrm{irr}}_{U_{-1}}(S_n) = \tau(R^{\mathrm{irr}}_{U_{-1}}(S_n))$ is a smooth closed submanifold of $X^{\mathrm{irr}}_{U_{-1}}(F_n)$.

For the last claim, note that $\mathrm{tr}\colon R^{\mathrm{irr}}_{U_{-1}}(F_n) \to \mathbb{R}$ is a submersion by Corollary 3.8 and Lemma 3.9. The restriction of μ to the fibers of tr is a submersion, so $\mathrm{tr}\colon R^{\mathrm{irr}}_{U_{-1}}(S_n) \to \mathbb{R}$ is a submersion. This map factors as the composition $R^{\mathrm{irr}}_{U_{-1}}(S_n) \xrightarrow{\tau} X^{\mathrm{irr}}_{U_{-1}}(S_n) \xrightarrow{\mathrm{tr}} \mathbb{R}$, so $\mathrm{tr}\colon X^{\mathrm{irr}}_{U_{-1}}(S_n) \to \mathbb{R}$ is a submersion as well. □

Proof of Theorem 10.2 Theorem 9.2 gives that $B' \cap C = C' \cap B = \mathrm{tr}^{-1}(-2,2) \cap X^{\mathrm{irr}}(S_n)$. We define $\mathscr{X}(S_n) = B' \cup C'$, so $\mathscr{X}(S_n) \cap B = B'$ and $\mathscr{X}(S_n) \cap C = C'$. Since B and C form an open cover of $\mathscr{X}(F_n)$, Lemmas 10.8 and 10.9 give that $\mathscr{X}(S_n)$ is a closed submanifold of $\mathscr{X}_{\mathrm{bal}}(S_n, S)$ of codimension 3, and also $\pi(\mathscr{X}(S_n))$ is the union of $\bigcup_{c\in(-2,2)} X^c_{\mathrm{bal}}(S_n)$ with $X^{\mathrm{irr}}(S_n)$ which is a subset of $X_{\mathrm{bal}}(S_n)$ as claimed. The claims about Tr also follow immediately from Lemmas 10.8 and 10.9. Finally, the density of $\mathscr{X}^{\mathrm{irr}}(S_n)$ in $\mathscr{X}(S_n)$ follows from the last sentence of Theorem 9.2. □

10.10 Remark As discussed in Section 9.26, we do not determine the complete analogue of Theorem 9.2 when the number of punctures n is odd. However, the claims of Theorem 10.2 are effectively trivial for odd n: every reducible is unbalanced, and hence $X_{\mathrm{bal}}(S_n, S) = X^{\mathrm{irr}}(S_n, S)$, so one can simply define $\mathscr{X}(S_n, S)$ as $X^{\mathrm{irr}}(S_n, S)$.

11 Orientations

In this section, we orient the spaces constructed in the previous sections: $\mathscr{X}^c(F_n)$, $\mathscr{X}(F_n)$, $\mathscr{X}^c(S_n)$ and $\mathscr{X}(S_n)$. These spaces contain the SU_2 character varieties used in [Heusener 2003] as open subsets. We compare our orientations with his, as this will be needed in Section 15.3.

11.1 The free group

By Corollary 7.9 and Theorem 8.1, we know that $\mathscr{X}^c(F_n) \cong X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \cong \mathscr{C}_n(S^2)$. We defined an orientation on the latter space in Section 5.7. To compare this orientation with Heusener's, we explain how to orient $X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ from the perspective of character varieties, following [Heusener 2003].

First, we orient $\mathrm{SU}_2 = U_1$ via Remark 6.10. For all $c \in (-2,2)$, we orient the conjugacy class $R^c_{\mathrm{SU}_2}(F_1) = \mathrm{tr}^{-1}(c)$ so that the submersion $\mathrm{tr}\colon \mathrm{SU}_2 \setminus \{\pm I\} \to (-2,2)$ is compatibly oriented in the sense of Section 2.3, where $(-2,2)$ has the *opposite* of the standard orientation (this seemingly odd choice will be explained in the proof of Proposition 11.2). Give $R^c_{\mathrm{SU}_2}(F_n) = (R^c_{\mathrm{SU}_2}(F_1))^n$ the product orientation, which then orients its open subset $R^{c,\mathrm{irr}}(F_n)$. Now orient $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ by the requirement that $\mathrm{SO}_3 \to R^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \to X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ be compatibly oriented, where SO_3 is oriented as $\mathrm{SU}_2/\{\pm I\}$.

Furthermore, we orient $X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ using

$$X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \to X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \xrightarrow{\mathrm{tr}} (-2,2),$$

where $(-2,2)$ again has the reversed orientation, the map tr is a submersion by Corollary 3.8, and the orientations on the fibers vary smoothly by construction.

11.2 Proposition *The orientation on $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ is opposite to that of* [Heusener 2003]. *Also, the diffeomorphism $\pi_c\colon \mathscr{C}_n(S^2) \to X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ from Theorem 8.1 is orientation-preserving, where $\mathscr{C}_n(S^2)$ is oriented as in Section 5.7. Finally, the map $(-2,2)\times\mathscr{C}_n(S^2) \to X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ that sends (c,p) to $\pi_c(p)$ is an orientation-preserving diffeomorphism, where $(-2,2)$ has the reversed orientation.*

Proof For the first claim, our construction matches [Heusener 2003], except there conjugacy classes in SU_2 are parametrized by $\alpha \in (0,\pi)$, where $c = 2\cos\alpha$ and $(0,\pi)$ has the usual orientation. However, this corresponds to the reversed orientation we used on $(-2,2)$. Second, we must account for the differences in orientation conventions noted in Remark 2.4. The sequence used to orient $R^c_{\mathrm{SU}_2}(F_1)$ gives the same orientation as Heusener, since the fiber is even-dimensional. However the second sequence gives the opposite of Heusener's orientation, since both the fiber SO_3 and base $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ are odd-dimensional, proving the first claim.

For the second claim, the key is to show that the diffeomorphism of $\overline{Q}_1 = S^2$ to $R^c_{\mathrm{SU}_2}(F_1)$ that sends $v \mapsto A(\alpha,1,v)$ with $\alpha = \cos^{-1}\frac{c}{2}$ is orientation-preserving; this will suffice as the constructions of orientations on $\mathscr{C}_n(S^2)$ and $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ are then strictly analogous. Recall that $A(\alpha,t,v) = \exp(\alpha\varphi_t(v))$. Now the exponential map restricts to an orientation-preserving diffeomorphism from $\{v \in \mathfrak{u}_1 \mid 0 < B_1(v) < \pi^2\}$ to $U_1 \setminus \{\pm I\}$ which sends $\alpha\overline{Q}_1 = \{v \in \mathfrak{u}_1 \mid B_1(v) = \alpha^2\}$ to $R^c_{\mathrm{SU}_2}(F_1)$. Since the orientation of the sphere $\overline{Q}_1$ from Section 6.1 is the same as orienting it with the outward normals, this matches the orientation on $R^c_{\mathrm{SU}_2}(F_1)$ which comes from increasing α. So the map $\overline{Q}_1 \to R^c_{\mathrm{SU}_2}(F_1)$ is orientation-preserving, as needed to prove the second claim.

The final claim follows easily as $X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ was oriented using the submersion $\mathrm{tr}\colon X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \to (-2,2)$ where $(-2,2)$ has the reversed orientation. □

To orient $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$, we follow the same steps as for SU_2. First, fix the orientation on $\mathrm{SL}_2\mathbb{R}$ coming from its identification to $U_{-1} = \mathrm{SU}_{1,1}$. There is a submersion $\mathrm{tr}\colon (\mathrm{SL}_2\mathbb{R} \setminus \{\pm I\}) \to \overleftarrow{\mathbb{R}}$, where $\overleftarrow{\mathbb{R}}$ denotes

$\mathbb{R}$ with the reversed orientation, and we orient $R^c_{\mathrm{SL}_2\mathbb{R}}(F_1) := \mathrm{tr}^{-1}(c)$ as its fiber. By definition, the singular points $\pm I$ are not in $R^{\pm 2}_{\mathrm{SL}_2\mathbb{R}}(F_1)$. Unlike for SU_2, where each conjugacy class is a sphere, the topology of $R^c_{\mathrm{SL}_2\mathbb{R}}(F_1)$ depends on c, but this makes no difference at this stage. Now orient $R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$ as an open subset of the product $(R^c_{\mathrm{SL}_2\mathbb{R}}(F_1))^n$. (When $c = \pm 2$, recall that for $\rho \in R^c_{\mathrm{SL}_2\mathbb{R}}(F_n)$, the $\rho(s_i)$ are all conjugate in $\mathrm{SL}_2\mathbb{C}$. Hence, if $\rho \in R^{\pm 2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$, $\rho(s_i) \neq \pm I$.) Orient $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$ by the fiber bundle

$$\mathrm{PSL}^{\pm}_2(\mathbb{R}) \to R^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n) \to X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n),$$

where the orientation on $\mathrm{PSL}^{\pm}_2(\mathbb{R}) = \mathrm{SL}^{\pm}_2(\mathbb{R})/\{\pm I\}$ comes from the left-invariant extension to $\mathrm{SL}^{\pm}_2(\mathbb{R})$ of the one on $\mathrm{SL}_2\mathbb{R}$. Finally, orient $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$ by $\mathrm{tr}\colon X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n) \to \overleftarrow{\mathbb{R}}$, which is a submersion by Corollary 3.8.

11.3 Proposition *For $c \in (-2, 2)$, the diffeomorphism $\pi_c\colon \mathscr{C}^{\pm}_n(\mathbb{H}^2) \to X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}$ from Theorem 8.1 is orientation-preserving, where $\mathscr{C}^{\pm}_n(\mathbb{H}^2)$ is oriented as in Section 5.7.*

Proof As in the proof of Proposition 11.2, the key is to check that the diffeomorphism from $\overline{Q}_{-1}$ to $R^c_{\mathrm{SL}_2\mathbb{R}}(F_1)$ that sends $v \mapsto A(\alpha, 1, v) = \exp(\alpha\varphi_{-1}(v))$, where $\alpha = \cos^{-1}\frac{c}{2}$, is orientation-preserving. Recall from Section 6.1 that both sheets of $\overline{Q}_{-1}$ are oriented with respect to normals that point away from the origin in $\mathfrak{u}_{-1}$. For any $\beta \in (0, \pi)$, the exponential map gives a diffeomorphism from $\beta\overline{Q}_{-1}$ to $R^{2\cos\beta}_{\mathrm{SL}_2\mathbb{R}}(F_1)$; here, $\beta\overline{Q}_{-1}$ denotes the dilation by ordinary scalar multiplication of vectors, not the $\mathbb{R}_{>0}$ action from earlier. Orient each $\beta\overline{Q}_{-1}$ as we did $\overline{Q}_{-1}$; hence, the dilation $\overline{Q}_{-1} \to \beta\overline{Q}_{-1}$ is orientation-preserving. Since $R^{2\cos\beta}_{\mathrm{SL}_2\mathbb{R}}(F_1)$ was oriented using $\mathrm{tr}\colon R_{\mathrm{SL}_2\mathbb{R}}(R_1) \to \overleftarrow{\mathbb{R}}$, ie with respect to increasing β, and the map $\exp\colon \mathfrak{u}_{-1} \to U_{-1}$ is orientation-preserving by definition, the diffeomorphism from $\beta\overline{Q}_{-1}$ to $R^{2\cos\beta}_{\mathrm{SL}_2\mathbb{R}}(F_1)$ is orientation-preserving. Therefore the map from $\overline{Q}_{-1}$ to $R^c_{\mathrm{SL}_2\mathbb{R}}(F_1)$ that sends $v \mapsto A(\alpha, 1, v) = \exp(\alpha v)$ is the composition of two orientation-preserving maps and hence orientation-preserving, as needed. □

11.4 Theorem *The manifold $\mathscr{X}(F_n)$ is orientable, and we take its preferred orientation to be the one compatible with its open subset $X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$. Then the open subset $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$ has the reverse of its standard orientation. The submersion $\mathrm{Tr}\colon \mathscr{X}(F_n) \to \overleftarrow{\mathbb{R}}$ gives a preferred orientation to each $\mathscr{X}^c(F_n)$, which is the standard orientation on $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ but the reverse of it for $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$.*

Proof Recall from Section 10.3 that $\mathscr{X}(F_n)$ is the union of open submanifolds $B = (-2, 2) \times \mathscr{Y}$ and $C = X^{\mathrm{irr}}(F_n) = X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \amalg X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$. Both B and C are orientable, but the orientability of $\mathscr{X}(F_n)$ is not immediate as $B \cap C$ is disconnected. Orient B by giving $(-2, 2)$ the reversed orientation and $\mathscr{Y}$ the orientation compatible with the one on $\mathscr{C}_n(S^2)$. Now orient C by giving $X^{\mathrm{irr}}_{\mathrm{SU}_2}(F_n)$ and $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n)$ their standard and reversed orientations, respectively. By Proposition 7.10, the inclusions of $\mathscr{C}_n(S^2)$ and $\mathscr{C}^{\pm}_n(\mathbb{H}^2)$ into $\mathscr{Y}$ are orientation-preserving and -reversing, respectively. Combining Propositions 11.2 and 11.3 with the fact that the submersion $\mathrm{tr}\colon X^{\mathrm{irr}}(F_n) \to \overleftarrow{\mathbb{R}}$ is compatible with the orientations on $X^{c,\mathrm{irr}}(F_n)$, we see that the restrictions of the two orientations to $B \cap C$ agree. This gives the desired global orientation on $\mathscr{X}(F_n)$, and the claims about $\mathscr{X}^{c,\mathrm{irr}}(F_n)$ are immediate from the construction. □

11.5 The punctured sphere

To begin, we orient the manifold $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ as follows. By Lemma 9.23, the map $\mu\colon R^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n) \to \mathrm{SU}_2$ where $\mu(\rho) = \rho(s_1 \cdots s_n)$ is a submersion, so we orient $\mu^{-1}(I) = R^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ using our usual conventions. Now orient $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ by $\mathrm{SO}_3 \to R^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n) \to X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$. Finally orient $X^{\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ by $\mathrm{tr}\colon X^{\mathrm{irr}}_{\mathrm{SU}_2}(S_n) \to \overleftarrow{\mathbb{R}}$, which is a submersion by Lemma 10.9, and the just constructed orientation on the fibers $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$, noting that said orientation varies continuously in c. Our standard orientations on $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ and $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ are analogous.

11.6 Remark The orientations just constructed on $R^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ and $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ are the opposites of those used in [Heusener 2003]. For $R^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$, this is because $R^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ and SU_2 are both odd-dimensional, so the convention difference discussed in Remark 2.4 matters. This difference persists to $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ since in both cases $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ is even-dimensional and hence the convention for orienting submersions is irrelevant.

Unlike in the free group case, we can orient $\mathscr{X}^c(S_n)$ so that both $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ and $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ inherit their standard orientations:

11.7 Theorem *The manifold $\mathscr{X}^c(S_n)$ is orientable, and we take its preferred orientation to be the one compatible with its open subset $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$. The open subset $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ then also has its standard orientation.*

Proof We imitate the construction of the standard orientation on $X^{c}_{\mathrm{SU}_2}(S_n)$ in the setting of our construction of $\mathscr{X}^c(S_n)$ from Section 9. We proceed in three steps.

Step 1 (orient $\overline{\mathscr{R}}_n$) Recall from Section 6.1 that each $\overline{Q}_t$ is oriented so that $\langle u_1, u_2\rangle$ is a positive basis for $T_v(\overline{Q}_t)$ exactly when $\langle v, u_1, u_2\rangle$ is a positive basis for $\mathbb{R}^3$. There is a submersion $\overline{\mathscr{R}}_n \to \mathbb{R}$ whose fibers are $\overline{Q}_t^n$; we give $\overline{Q}_t^n$ the product orientation and use the fibration

$$\overline{Q}_t^n \to \overline{\mathscr{R}}_n \xrightarrow{\;t\;} \mathbb{R} \tag{11.8}$$

to orient $\overline{\mathscr{R}}_n$.

Step 2 (orient $\overline{\mathscr{R}}^c(S_n)$) Recall from the proof of Proposition 9.24 that $\overline{\mathscr{R}}^{c,t\neq 0}(S_n) := \widetilde{F}^{-1}(\widetilde{I})$, where $\widetilde{F}\colon \overline{\mathscr{R}}_n^{t\neq 0} \to \widetilde{U}$ is a submersion. The normal bundle $\nu_{\widetilde{I}}$ is the bundle over $\mathbb{R}$ whose fibers are $\mathfrak{u}_t \subset T\widetilde{U}$; we orient each $\mathfrak{u}_t$ as described in Remark 6.10, which gives smoothly varying orientations on the fibers of $\nu_{\widetilde{I}}$.

Over $\overline{\mathscr{R}}^{c,t\neq 0}(S_n)$, the short exact sequence

$$0 \to T\overline{\mathscr{R}}^{c,t\neq 0}(S_n) \to T\overline{\mathscr{R}}_n^{t\neq 0} \xrightarrow{\;d\widetilde{F}\;} \nu_{\widetilde{I}} \to 0 \tag{11.9}$$

can be used to orient $\overline{\mathcal{R}}^{c,t\neq 0}(S_n)$. You might think that we should extend this orientation to all of $\overline{\mathcal{R}}^c(S_n)$, but this is actually not possible. To see this, consider the function $h\colon \widetilde{U} \to \mathbb{C}\times\mathbb{R}$ given by $h(t,A) = (A_{12}, \operatorname{Im} A_{11})$. We have $h^{-1}(0,0) = \widetilde{I} \cup -\widetilde{I}$, where $-\widetilde{I}$ is everything of the form $(t,-I)$; thus $\nu_{\widetilde{I}}$ is one connected component of $h^*(T_0(\mathbb{R}\times\mathbb{C}))$. So the orientation on $\overline{\mathcal{R}}^{c,t\neq 0}(S_n)$ induced by (11.9) is the same as that induced by the submersion $\overline{\mathcal{R}}^{c,t\neq 0}(S_n) \to \overline{\mathcal{R}}_n^{t\neq 0} \xrightarrow{h\circ\widetilde{F}} \mathbb{R}\times\mathbb{C}$. But, on the open set V_ϵ from Section 9.3, the subset $\overline{\mathcal{R}}^c(S_n)\cap V_\epsilon$ is cut out by a submersion $f_\epsilon\colon V_\epsilon \to \mathbb{C}\times\mathbb{R}$, which satisfies

$$f_\epsilon = (F_{12}, f_{11}), \quad \text{where } h\circ\widetilde{F} = (F_{12}, t f_{11}).$$

Consequently the induced orientation on $f_\epsilon^{-1}(0,0)$ will agree with the orientation induced by (11.9) for $t>0$, but will be its opposite for $t<0$ since f_ϵ differs from $h\circ\widetilde{F}$ by scaling the second coordinate of $\mathbb{C}\times\mathbb{R}$ by t.

To orient $\overline{\mathcal{R}}^c(S_n)$, we give V_ϵ the orientation induced by our orientation on $\overline{\mathcal{R}}_n$, and the orientation on $\mathbb{C}\times\mathbb{R}$ that pulls back to our standard orientation on $\mathfrak{u}_t$ under h^*. Then the submersion f_ϵ induces an orientation on $f_\epsilon^{-1}(0,0)$. We give $\widetilde{F}^{-1}(\widetilde{I})\cap\overline{\mathcal{R}}_n^{t>0}$ the orientation coming from (11.9), and $\widetilde{F}^{-1}(\widetilde{I})\cap\overline{\mathcal{R}}_n^{t<0}$ the opposite orientation. The discussion in the previous paragraph shows that these orientations are all compatible and together give a global orientation on $\overline{\mathcal{R}}^c(S_n)$.

Step 3 (orient $\mathcal{X}^c(S_n)$) Now we can orient $\mathcal{X}^c(S_n)$. We use the orientation on $\mathfrak{u}_t$ to orient G_t as in Remark 6.10, give $\mathbb{R}_{>0}$ the positive orientation, and give $\mathbb{R}_{>0}\times G_t$ the product orientation. We will orient $\mathcal{X}^c(S_n)$ using the exact sequence

(11.10) $$0 \to T_e(\mathbb{R}_{>0}\times G_t) \to T_v\overline{\mathcal{R}}^c(S_n) \to T_p\mathcal{X}^c(S_n) \to 0,$$

where $t = t(v)$, $e = (0, I)$ and $\sigma(v) = p$. As the orientations on $T_v\overline{\mathcal{R}}^c(S_n)$ and $T_e(\mathbb{R}_{>0}\times G_t)$ vary continuously, even as t changes, the main issue is that, for $t<0$, the fibers $\mathbb{R}_{>0}\times G_t$ of the submersion $\sigma\colon \overline{\mathcal{R}}^c(S_n) \to \mathcal{X}^c(S_n)$ are disconnected. So, for $t<0$, we must find v, v' belonging to two different components of $\sigma^{-1}(p)$ where the orientations on $T_p\mathcal{X}^c(S_n)$ induced by the exact sequences

$$0 \to T_e(\mathbb{R}_{>0}\times G_t) \to T_v\overline{\mathcal{R}}^c(S_n) \to T_p\mathcal{X}^c(S_n) \to 0,$$
$$0 \to T_e(\mathbb{R}_{>0}\times G_t) \to T_{v'}\overline{\mathcal{R}}^c(S_n) \to T_p\mathcal{X}^c(S_n) \to 0$$

are the same.

To do this, consider

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which is in every G_t. Its action on each $\mathfrak{u}_t$ is orientation-preserving, and corresponds to complex conjugation, as noted in Section 6.9. We successively deduce that $C\colon \overline{Q}_t \to \overline{Q}_t$ and $C\colon \overline{\mathcal{R}}_n \to \overline{\mathcal{R}}_n$ are orientation-preserving. Since $F(C\cdot v) = C\cdot F(v)$, as noted in Section 9.1, we see that $C\colon \overline{\mathcal{R}}(S_n) \to \overline{\mathcal{R}}(S_n)$

and is orientation-preserving. Taking $v' = C \cdot v$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_e(\mathbb{R}_{>0} \times G_t) & \longrightarrow & T_v\overline{\mathcal{R}}^c(S_n) & \longrightarrow & T_p\mathcal{X}^c(S_n) & \longrightarrow & 0 \\ & & \big\downarrow \mathrm{id}\times\mathrm{Ad}_C & & \big\downarrow dC & & \big\downarrow \mathrm{id} & & \\ 0 & \longrightarrow & T_e(\mathbb{R}_{>0} \times G_t) & \longrightarrow & T_{v'}\overline{\mathcal{R}}^c(S_n) & \longrightarrow & T_p\mathcal{X}^c(S_n) & \longrightarrow & 0 \end{array}$$

The two leftmost vertical maps are orientation-preserving, so the right-hand vertical map is as well. In other words, the two induced orientations on $T_p\mathcal{X}^c(S_n)$ are the same. We conclude that (11.10) determines an orientation on $\mathcal{X}^c(S_n)$.

Having defined an orientation on $\mathcal{X}^c(S_n)$, it remains to check that the orientations it induces on its open subsets $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ and $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ agree with their standard orientations. First, we consider the case $t > 0$. For the standard orientation of $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$, we have the decomposition as oriented vector spaces

(11.11) $$T\overline{Q}^n_1 = T(R^c_{\mathrm{SU}_2}(F_1)^n) = TU_1 \oplus T(R^c_{\mathrm{SU}_2}(S_n)) = TU_1 \oplus [T(X^c_{\mathrm{SU}_2}(S_n)) \oplus TU_1].$$

For the orientation of $\mathcal{X}^c(S_n)$, we have $T\overline{\mathcal{R}}^c_n = \mathbb{R} \oplus T\overline{Q}^n_t$, where the $\mathbb{R}$-factor corresponds to the t-coordinate. Again as oriented vector spaces, we have, from (11.10),

$$\begin{aligned} T\overline{\mathcal{R}}^c_n &= TU_t \oplus T\overline{\mathcal{R}}^c_n(S_n) = TU_t \oplus (T\mathcal{X}^c(S_n) \oplus T(\mathbb{R}_{>0} \times G_t)) \\ &= -T(\mathbb{R}_{>0}) \oplus TU_t \oplus T\mathcal{X}^c(S_n) \oplus TU_t \end{aligned}$$

since $\dim \mathcal{X}^c(S_n) = 2n-6$ and $\dim U_t = 3$. Now, as $s \cdot \overline{Q}_t = \overline{Q}_{t/s^2}$ and $t > 0$, increasing s actually decreases t. Hence,

$$T\overline{\mathcal{R}}^c_n = -T(\mathbb{R}_{>0}) \oplus T\overline{Q}^n_t.$$

Comparing, we conclude that $T\overline{Q}^n_t = TU_t \oplus T\mathcal{X}^c(S_n) \oplus TU_t$, and, taking $t = 1$, we see from (11.11) that $\mathcal{X}^c(S_n)$ and $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ have matching orientations, as desired.

The case of $t < 0$ is very similar, except that now $T\overline{\mathcal{R}}^c_n = T(\mathbb{R}_{>0}) \oplus T\overline{Q}^n_t$ and

$$T\overline{\mathcal{R}}^c_n = T(\mathbb{R}_{>0}) \oplus TU_t \oplus T\mathcal{X}^c(S_n) \oplus TU_t$$

since the orientation on this part of $\mathcal{X}^c(S_n)$ comes from that of $\widetilde{F}^{-1}(\widetilde{I}) \cap \overline{\mathcal{R}}^{t<0}_n$ which was given the opposite orientation in Step 2. The net effect is to insert a $-$ sign at one point and remove it from another, so we again find that the induced orientation agrees with the standard one. ◻

11.12 Corollary *The manifold $\mathcal{X}(S_n)$ is orientable, with preferred orientation that matches the standard ones on both $X^{\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ and $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$. The submersion $\mathrm{Tr}\colon \mathcal{X}(S_n) \to \overleftarrow{\mathbb{R}}$ is compatible with the preferred orientations on $\mathcal{X}^c_{\mathrm{bal}}(S_n) \subset \mathcal{X}^c(S_n)$ from Theorem 11.7.*

Proof Recall from Section 10.7 that $\mathcal{X}(S_n) = B' \cup C'$. The map $\mathrm{Tr}\colon B' \to (-2,2)$ is a submersion by Lemma 10.8, so there is a short exact sequence

$$0 \to T\mathcal{X}^c_{\mathrm{bal}}(S_n) \to TB' \to T(-2,2) \to 0,$$

which we use to orient B'; as usual, we give $(-2,2)$ the reversed orientation. (Note $\mathscr{X}^c_{\mathrm{bal}}(S_n)$ is an open subset of $\mathscr{X}^c(S_n)$, and inherits an orientation from it.) We give both components of $C' = X^{\mathrm{irr}}_{\mathrm{SU}_2}(S_n) \amalg X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ our standard orientations. The orientation on $X^{\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ is induced by the short exact sequence

$$0 \to X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n) \to X^{\mathrm{irr}}_{\mathrm{SU}_2}(S_n) \to T(-2,2) \to 0$$

and similarly for $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$. By Theorem 11.7, the orientations on $\mathscr{X}^c(S_n)$, $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_n)$ and $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_n)$ are compatible. It follows that the orientations on B' and C' are compatible and determine a global orientation on $\mathscr{X}(S_n)$ which has the desired properties. □

12 Definition of the invariant

12.1 Braids and plat closures

We are now ready to define the total Lin invariant of a knot $K \subset S^3$. Following Heusener [2003], we represent knots as plat closures of braids. Our conventions for braids are given in Figure 4, which also describes the identification of the braid group B_n with $\mathrm{MCG}(D_n)$, the mapping class group of the n-punctured disk. In turn, there is a homomorphism from $\mathrm{MCG}(D_n)$ to $\mathrm{MCG}(S_n)$ induced by capping off the boundary to D_n with a disk to form S_n. If $\beta \in B_n$, we abuse notation and write β for its image φ_β in $\mathrm{MCG}(S_n)$. Recall a homomorphism $\varphi\colon \Gamma \to \Lambda$ of finitely generated groups has an induced regular real algebraic map $\varphi^*\colon X(\Lambda) \to X(\Gamma)$ given by $[\rho] \mapsto [\rho \circ \varphi]$. If S and T are finite generating sets of Γ and Λ, respectively, where φ takes each $s_i \in S$ to a conjugate of some $t_j \in T$ or its inverse, then we also have induced maps $\varphi^*\colon X^c(\Lambda, T) \to X^c(\Gamma, S)$ for each $c \in \mathbb{R}$. Hence, $\beta_*\colon \pi_1(S_n) \to \pi_1(S_n)$ induces a map $\beta^*\colon X^c(S_n, S) \to X^c(S_n, S)$ for each $c \in \mathbb{R}$.

Given a braid $\beta \in B_{2m}$, we can form its plat closure $\widehat{\beta}$ as shown in Figure 4. Given a knot $K \subset S^3$, we choose β with $\widehat{\beta} = K$. As shown in Figure 5, this presents the exterior of K as the union of two genus m handlebodies glued together along the $2m$–punctured sphere S_{2m}:

$$M_K = S^3 \setminus \nu(K) = H_1 \cup_{S_{2m}} H_2.$$

Here, the surface S_{2m} is identified with the boundary of H_1 and $x \in \partial H_1$ is glued to $\beta(x) \in \partial H_2$. The inclusion $S_{2m} \to M_K$ induces a surjection $\pi_1(S_{2m}) \to \pi_1(M_K)$ and hence an injection of $X^c(K) = X^c(M_K, \{\mu\})$ into $X^c(S_{2m}, S)$.

Similarly, if T_k is the set of generators for $\pi_1(H_k)$ shown in Figure 5, the inclusion $j_k\colon S_{2m} \to H_k$ induces an injection $j_k^*\colon X^c(H_k, T_k) \to X^c(S_{2m}, S)$. We let $L_k^c \subset X^c(S_{2m}, S)$ be the image of j_k^*. Then $X^c(K)$ can be identified with the intersection $L_1^c \cap L_2^c$ as in the original construction of Casson's invariant [Akbulut and McCarthy 1990].

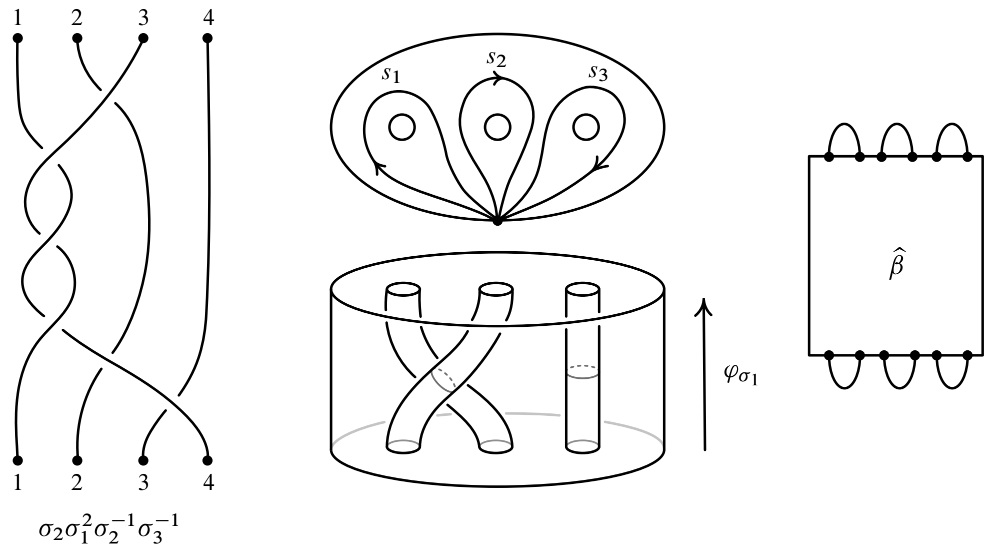

Figure 4: Our conventions for braids, including σ_i versus σ_i^{-1}, that left-to-right in the braid word corresponds to top-to-bottom in the picture, and that the generators s_i of $\pi_1(S_n)$ correspond to clockwise loops. Moreover, the induced mapping class $\varphi_\beta \in \mathrm{MCG}(D_n)$ of β is the map that pushes curves from the bottom to top; hence, $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta$, so $B_n \to \mathrm{MCG}(D_n)$ is a homomorphism rather than an antihomomorphism (here functions act on the left as usual); in particular, $\varphi_{\sigma_1}(s_1) = s_2$ and $\varphi_{\sigma_1}(s_2) = s_2^{-1}s_1s_2$.

More precisely, we index the standard generators $s_1, \dots, s_{2m}$ of $\pi_1(S_{2m})$, as shown in Figures 4 and 5, so that s_{2i-1} and s_{2i} are meridians of the i^{th} arc in the lower plat. The generators shown in Figure 5 determine isomorphisms

$$\ell_k \colon \pi_1(H_k) \to F_m = \langle t_1, \dots, t_m \rangle.$$

If we define $\hat{\jmath}_k = \ell_k \circ j_{k*}$, then $\hat{\jmath}_1(s_{2i-1}) = t_i$ and $\hat{\jmath}_1(s_{2i}) = t_i^{-1}$, so $\hat{\jmath}_1$ induces a map $\hat{\jmath}_1^* \colon X^c(F_m, T) \to X^c(S_{2m}, S)$ and $\operatorname{im} \hat{\jmath}_1^* = \operatorname{im} j_1^* = L_1^c$. We have $\hat{\jmath}_2 = \hat{\jmath}_1 \circ \beta_*$, so $L_2^c = \operatorname{im} \hat{\jmath}_2^* = \beta^*(L_1^c)$. To simplify the notation, we let $L^c = L_1^c$, so $L_1^c \cap L_2^c = L^c \cap \beta^*(L^c)$, and write $i = \hat{\jmath}_1^*$. To summarize, we have established:

12.2 Proposition *For a knot $K = \widehat{\beta}$ and each $c \in \mathbb{R}$, the inclusion $S_{2m} \to M_K$ gives an injection $X^c(K) \to X^c(S_{2m}, S)$ whose image is $L^c \cap \beta^* L^c$.*

We would like to define the total Lin invariant $h^c(K)$ to be the algebraic intersection number $\langle L^c, \beta^* L^c \rangle$, but there are two problems with doing so. The first is that $L^c \cap \beta^*(L^c)$ may not be compact, and the second is that $X^c(S_{2m}, S)$ is not a manifold. We more or less punt on the first issue: we will impose hypotheses on K which prevent it from happening. For the second problem, we restrict to $c \in (-2, 2)$ and replace $X^c(S_{2m}, S)$ with the resolution $\mathscr{X}^c(S_{2m}, S)$ constructed in Section 9. This requires us to construct smooth maps $i' \colon \mathscr{X}^c(F_m, T) \to \mathscr{X}^c_{\text{bal}}(S_{2m}, S)$ and $\beta^* \colon \mathscr{X}^c(S_{2m}, S) \to \mathscr{X}^c(S_{2m}, S)$ that are

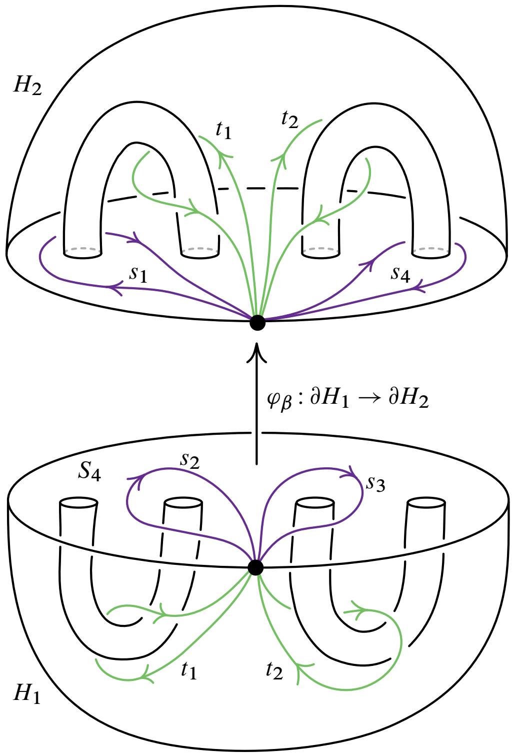

Figure 5: Our conventions for the fundamental group of the exterior $M_K = H_1 \cup_{\varphi_\beta} H_2$ of the plat closure $K = \widehat{\beta}$. Here the canonical copy of S_{2m} is the one on the boundary of H_1. For clarity, only half of the generators of $\pi_1(S_4)$ are drawn on each copy of S_4.

"resolutions" of the corresponding maps on character varieties. Letting $\mathscr{L}^c = i'(\mathscr{X}^c(F_m, T))$, we will then define $h^c(K) = \langle \mathscr{L}^c, \beta^* \mathscr{L}^c \rangle$. As described in Theorem 12.5 below, the invariant $h^c(K)$ can be interpreted as a signed count of representations of $\pi_1(M_K)$ with $\mathrm{tr}_\mu = c$ into certain double covers of $\mathrm{Isom}^+(S^2)$, $\mathrm{Isom}^+(\mathbb{E}^2)$ and $\mathrm{Isom}^+(\mathbb{H}^2)$.

12.3 Induced maps of resolutions

For any of our resolutions $\mathscr{X} \xrightarrow{\pi} X$ in Theorems 8.1, 9.2, 10.1, and 10.2, define a decomposition of $\mathscr{X}$ into $\mathscr{X}_+ \amalg \mathscr{X}_0 \amalg \mathscr{X}_-$ by $\mathscr{X}_+ = \pi^{-1}(X^{\mathrm{irr}}_{\mathrm{SU}_2})$, $\mathscr{X}_0 = \pi^{-1}(X^{\mathrm{red}})$ and $\mathscr{X}_- = \pi^{-1}(X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}})$. In particular, $\mathscr{X}^{\mathrm{irr}} = \mathscr{X}_+ \amalg \mathscr{X}_-$ and $\mathscr{X}^{\mathrm{red}} = \mathscr{X}_0$. In each case π gives diffeomorphisms $\mathscr{X}_+ \to X^{\mathrm{irr}}_{\mathrm{SU}_2}$ and $\mathscr{X}_- \to X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}$. For $c \in (-2, 2)$, each $\mathscr{X}^c$ is contained in some $\mathscr{X}^c(F_n) \cong \mathscr{Y}$ and this decomposition corresponds to $\mathscr{Y} = \mathscr{Y}_+ \amalg \mathscr{Y}_0 \amalg \mathscr{Y}_-$ from Theorem 7.1. In turn, viewing $\mathscr{Y}$ as a quotient of $\overline{\mathscr{R}}_n$, this corresponds to the regions $\overline{\mathscr{R}}_n^{t>0}$, $\overline{\mathscr{R}}_n^{t=0}$ and $\overline{\mathscr{R}}_n^{t<0}$, respectively.

For the case of F_n with its standard generators $S = \{s_1, \ldots, s_n\}$, Theorem 8.4 gives for each $c \in (-2, 2)$ a preferred diffeomorphism $\widetilde{\pi}_c^{\mathrm{red}}$ from $\mathscr{X}_0^c(F_n, S)$ to $X_{U_0}^c(F_n, S)$. Combined with the identifications provided by π, for $c \in (-2,2)$ we have a fixed decomposition of sets (not spaces) given by

$$\mathscr{X}^c(F_n,S) = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n,S) \amalg X^c_{U_0}(F_n,S) \amalg X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S).$$

For $c \notin (-2,2)$, we define $\mathscr{X}^c(F_n,S) = X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n,S)$, which matches Theorem 10.1.

Now suppose $T = \{t_1,\dots,t_m\}$ is the standard generating set for F_m and $\varphi\colon F_n \to F_m$ an epimorphism with each $\varphi(s_i)$ for $s_i \in S$ conjugate to some element of T or its inverse. Because φ is onto, $\varphi^*\colon R_{\mathbb{C}}(F_m,T) \to R_{\mathbb{C}}(F_n,S)$ takes each irreducible SU_2 or $\mathrm{SL}_2\mathbb{R}$ representation to another such representation and the same for those U_0 representations that are not fully reducible. Thus, we can take the induced maps on the character variety pieces of the above decompositions of $\mathscr{X}^c(F_m,T)$ and $\mathscr{X}^c(F_n,S)$ to construct a *set-theoretic* map $\varphi^*\colon \mathscr{X}^c(F_m,T) \to \mathscr{X}^c(F_n,S)$ for each $c \in \mathbb{R}$. These combine to give a set-theoretic map $\varphi^*\colon \mathscr{X}(F_m,T) \to \mathscr{X}(F_n,S)$. By our construction and Theorem 8.4, we have a commutative diagram

(12.4)
$$\begin{array}{ccc} \mathscr{X}(F_m,T) & \xrightarrow{\ \varphi^*\ } & \mathscr{X}(F_n,S) \\ \big\downarrow \pi & & \big\downarrow \pi \\ X(F_m,T) & \xrightarrow{\ \varphi^*\ } & X(F_n,S) \end{array}$$

For the φ relevant here, we show in Lemma 12.8 that the top φ^* is in fact smooth.

We will also consider epimorphisms where F_n is replaced by $\pi_1(S_n)$ and/or F_m is replaced by $\pi_1(S_m)$. Because $\mathscr{X}^c_0(S_n,S)$ is a proper subset of $X^c_{U_0}(S_n,S)$ for $c \in (-2,2)$, there is something to check to see that there is a map $\varphi^*\colon \mathscr{X}^c(S_m,T) \to \mathscr{X}^c(S_n,S)$, but this will hold in our cases of interest. For the map $\mathscr{X}(S_m,T) \to \mathscr{X}(S_n,S)$, one must also verify the correct behavior with regards to unbalanced reducibles.

Note that our induced maps on resolutions inherit functoriality from their component maps on character varieties: for example if $\alpha\colon F_n \to F_m$ and $\beta\colon F_m \to F_\ell$ are suitable epimorphisms then $(\beta\circ\alpha)^* = \alpha^*\circ\beta^*$.

We can now state the analogue of Proposition 12.2 for our resolved picture:

12.5 Theorem *Suppose $K = \widehat{\beta}$ for $\beta \in B_{2m}$ and fix $c \in (-2,2)$. The induced map of the epimorphism $\hat{\jmath}_1\colon \pi_1(S_{2m}) \to F_m$ is a well-defined smooth proper embedding $i'\colon \mathscr{X}^c(F_m,T) \to \mathscr{X}^c(S_{2m},S)$ whose image $\mathscr{L}^c$ is thus a smooth closed submanifold. Moreover, the map induced by the action of β on $\mathscr{X}^c(S_{2m},S)$ is a well-defined diffeomorphism $\beta^*\colon \mathscr{X}^c(S_{2m}) \to \mathscr{X}^c(S_{2m})$. At the level of character varieties, the inclusion $S_{2m} \to M_K$ induces an injection of*

$$\mathscr{X}^c(K) := X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(K) \amalg X^c_{U_0}(K) \amalg X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$$

into $\mathscr{X}^c(S_{2m},S)$, whose image is exactly $\mathscr{X}^c(\beta) := \mathscr{L}^c \cap \beta^*\mathscr{L}^c$. Finally, $\mathscr{L}^c$ is contained in $\mathscr{X}^c_{\mathrm{bal}}(S_{2m},S)$.

The proof of Theorem 12.5 will be given in Section 12.17, after establishing the smoothness of the induced maps on various resolutions.

12.6 The braid group action

The braid group B_n acts on the n-punctured disk D_n. With the conventions of Figure 4, the induced action on $\pi_1(D_n) = F_n$ is given by

(12.7) $$\sigma_i \cdot s_j = \begin{cases} s_{i+1} & \text{if } j = i, \\ s_{i+1}^{-1} s_i s_{i+1} & \text{if } j = i+1, \\ s_j & \text{if } j \neq i, i+1 \end{cases}$$

The action of $\beta \in B_n$ preserves our preferred generators S up to conjugacy, and thus induces a regular real algebraic map $\beta^* \colon X(F_n, S) \to X(F_n, S)$ as discussed above. This is an isomorphism of real algebraic sets since $(\beta^{-1})^*$ is the inverse to β^*. Turning to S_n, as $\sigma_i \cdot (s_1 s_2 \cdots s_n) = s_1 s_2 \cdots s_n$ from (12.7), the action of B_n on F_n descends to an action on $\pi_1(S_n)$. Thus, for $\beta \in B_n$, we have an induced automorphism $\beta^* \colon X(S_n, S) \to X(S_n, S)$, which can be viewed as the restriction of β^* on $X(F_n, S)$ to its subset $X(S_n, S)$. We next show the induced maps on resolutions are as expected:

12.8 Lemma *For each $\beta \in B_n$, the induced map β^* on $\mathscr{X}(F_n, S)$ is a diffeomorphism that preserves the submanifolds $\mathscr{X}^c(F_n, S)$, $\mathscr{X}_{\mathrm{bal}}(F_n, S)$, $\mathscr{X}^c(S_n, S)$ and $\mathscr{X}(S_n, S)$, where for the last two we are assuming that n is even. Hence, β^* is well defined on each of these five resolutions, and in each case it is a diffeomorphism.*

Proof As the induced maps in Section 12.3 are functorial, it suffices to prove the lemma for each generator σ_i of B_n. On the open submanifold $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S)$ of $\mathscr{X}(F_n, S)$, the map σ_i^* is a real algebraic automorphism, which you can easily check preserves the claimed submanifolds. So it suffices to understand σ_i^* on the subset $B = (-2,2) \times \mathscr{Y}$ of $\mathscr{X}(F_n, S)$, which we accomplish by lifting to $(-2,2) \times \overline{\mathscr{R}}_n$.

Motivated by (12.7), define $\overline{\Phi}_i \colon (-2,2) \times \overline{\mathscr{R}}_n \to (-2,2) \times \overline{\mathscr{R}}_n$ by

(12.9) $$\overline{\Phi}_i(c, t, v_1, \ldots, v_n) = (c, t, v_1, \ldots, v_{i-1}, v_{i+1}, \rho(-v_{i+1}) \cdot v_i, v_{i+2}, \ldots, v_n),$$

where $\rho(-v_{i+1}) = \psi_t\bigl(A\bigl(\cos^{-1}\frac{c}{2}, t(v), -v_{i+1}\bigr)\bigr)$. This is a smooth map as $A(\alpha, t, v)$ is a smooth function of (α, t, v) and G_t acts smoothly on $\overline{Q}_t$. Its inverse is given by

$$\overline{\Phi}_i^{-1}(c, t, v_1, \ldots, v_n) = (c, t, v_1, \ldots, v_{i-1}, \rho(v_i) \cdot v_{i+1}, v_i, v_{i+2}, \ldots, v_n),$$

which is also smooth. Hence, $\overline{\Phi}_i$ is a diffeomorphism on $(-2,2) \times \overline{\mathscr{R}}_n$.

Let $\pi' \colon (-2,2) \times \overline{\mathscr{R}}_n \to R_{\mathbb{C}}(F_n, S)$ be as in Section 10.3. Using (8.10) and (8.11), you can check that $\overline{\Phi}_i$ is equivariant with respect to the actions of $\mathbb{R}_{>0}$ and G_t. Hence, it descends to a diffeomorphism $\Phi_i \colon (-2,2) \times \mathscr{Y} \to (-2,2) \times \mathscr{Y}$, which we next show is σ_i^*. Again using (8.10) and (8.11), we see that the following commutes:

(12.10) $$\begin{array}{ccc} (-2,2) \times \overline{\mathscr{R}}_n & \xrightarrow{\overline{\Phi}_i} & (-2,2) \times \overline{\mathscr{R}}_n \\ \big\downarrow \pi' & & \big\downarrow \pi' \\ R_{\mathbb{C}}(F_n, S) & \xrightarrow{\sigma_i^*} & R_{\mathbb{C}}(F_n, S) \end{array}$$

To check $\Phi_i = \sigma_i^*$, we fix $c \in (-2,2)$ and study the restriction of Φ_i to each of $\mathscr{Y}_+$, $\mathscr{Y}_0$ and $\mathscr{Y}_-$ in turn. In the case of $\mathscr{Y}_-$, which maps to $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_n, S)$ under π_c, by restricting $\overline{\Phi}_i$ to $\overline{\mathscr{R}}^t_n$ for some $t < 0$, we can combine (8.12) and (12.10) to see that $\Phi_i = \sigma_i^*$ on $\mathscr{Y}_-$. The argument for $\mathscr{Y}_+$ is the same using $t > 0$ and $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_n, S)$. Similarly, $\mathscr{Y}_0$ is handled by considering $t = 0$ and $X^c_{U_0}(F_n, S)$; see also the proof of Theorem 8.4. Thus we have shown $\Phi_i = \sigma_i^*$ and hence σ_i^* is a self-diffeomorphism of $\mathscr{X}(F_n, S)$.

It remains to show σ_i^* preserves the listed submanifolds. For $\mathscr{X}^c(F_n, S)$, this is immediate from the definition of σ_i^*. The case of $\mathscr{X}_{\mathrm{bal}}(F_n, S)$ follows from its definition at the start of Section 10 and (12.7). For $\mathscr{X}^c(S_n, S)$, the only issue is whether σ_i^* preserves $\mathscr{X}^c_0(S_n, S)$ inside $\mathscr{X}^c_0(F_n, S)$, as the former is a proper subset of $X^c_{U_0}(S_n, S)$. By Theorem 9.2, $\mathscr{X}^c_0(S_n, S)$ is the intersection of the closure of $\mathscr{X}^{c,\mathrm{irr}}(S_n, S)$ with $\mathscr{X}^c_0(F_n, S)$; since σ_i is a diffeomorphism preserving $\mathscr{X}^{c,\mathrm{irr}}(S_n, S)$, it preserves $\mathscr{X}^c_0(S_n, S)$. The final case of $\mathscr{X}(S_n, S)$ is the union of various intersections of the previous submanifolds as described in Theorem 10.2, and hence also preserved. □

12.11 Lemma *Both $\beta^*\colon \mathscr{X}^c(F_n, S) \to \mathscr{X}^c(F_n, S)$ and $\beta^*\colon \mathscr{X}^c(S_n, S) \to \mathscr{X}^c(S_n, S)$ are orientation-preserving.*

Proof It is enough to check this for the actions of the elementary braids σ_i. To do this, we go back and look at the short exact sequences used to define the orientations in Section 11. The restriction of $\overline{\Phi}_i$ in (12.9) to $\{c\} \times \overline{\mathscr{R}}_n$ satisfies $t(\overline{\Phi}_i(v)) = t(v)$, so $\overline{\Phi}_i$ acts trivially on the $T\mathbb{R}$ factor in (11.8). The action of $\overline{\Phi}_i$ on the fiber $T\,\overline{Q}^n_t$ is the composition of a coordinate permutation $(v_1, \dots, v_n) \mapsto (v_1, \dots, v_{i+1}, v_i, \dots, v_n)$ and a rotation on one factor: $(v_1, \dots, v_n) \mapsto (v_1, \dots, v_i, \rho(-v_i) \cdot v_{i+1}, \dots, v_n)$. The permutation is orientation-preserving since $\overline{Q}_t$ is 2-dimensional, and the rotation is orientation-preserving since it is homotopic to the identity. It follows that the action of $\overline{\Phi}_i$ on $\overline{\mathscr{R}}_n$ is orientation-preserving. Finally, $\overline{\Phi}_i$ is equivariant with respect to the actions of $\mathbb{R}_{>0}$, and G_t, as observed in the proof of Lemma 12.8. Hence, the action of σ_i^* on $\mathscr{X}^c(F_n, S)$ is orientation-preserving.

To see that the same statement holds for $\mathscr{X}^c(S_n, S)$, we must also consider the exact sequence (11.9). From the definition of $\overline{\Phi}_i$, a direct computation shows that $F \circ \overline{\Phi}_i = F$. Referring to (11.9), we thus see that the action of $\overline{\Phi}_i$ on $\overline{\mathscr{R}}^{c,t\neq 0}(S_n, S)$ is orientation-preserving. As above, the actions of $\mathbb{R}_{>0}$ and G_t are equivariant with respect to $\overline{\Phi}_i$, so the action of σ_i^* on $\mathscr{X}^{c,\mathrm{irr}}(S_n, S)$ is orientation-preserving as well. Finally, $\mathscr{X}^{c,\mathrm{irr}}(S_n, S)$ is a dense open subset of $\mathscr{X}^c(S_n, S)$, so the action on $\mathscr{X}^c(S_n, S)$ is orientation-preserving. □

12.12 The character variety of the handlebody

Recall the epimorphism $i = \hat{j}_1^*$ from $\pi_1(S_{2m})$ to F_m coming from the inclusion of S_{2m} into ∂H_1. Let $\tilde{\imath}\colon F_{2m} \to F_m$ be the lift of i defined by $\tilde{\imath}(s_{2i-1}) = t_i$ and by $\tilde{\imath}(s_{2i}) = t_i^{-1}$. By Section 12.3, $\tilde{\imath}$ has an induced map $\tilde{\imath}^*\colon \mathscr{X}^c(F_m, T) \to \mathscr{X}^c(F_{2m}, S)$ for each $c \in \mathbb{R}$.

12.13 Proposition *For each $c \in (-2,2)$, the induced map $\tilde{\imath}^*$ is a smooth proper embedding with image contained in $\mathscr{X}^c_{\mathrm{bal}}(S_{2m}, S)$. Thus i induces a well-defined smooth proper embedding $i'\colon \mathscr{X}^c(F_m, T) \to \mathscr{X}^c_{\mathrm{bal}}(S_{2m}, S)$.*

Proof We first construct a map $j'\colon \overline{\mathscr{R}}_m \to \overline{\mathscr{R}}_{2m}$ and show it induces $\tilde{\imath}^*$ after we take quotients by the $\mathbb{R}_{>0}$ and G_t actions. Specifically, define j' by

$$j'(t, v_1, \dots, v_m) = (t, v_1, -v_1, \dots, v_m, -v_m). \tag{12.14}$$

The map j' is the restriction of a smooth embedding $\mathbb{R} \times (\mathbb{R}^3)^m \to \mathbb{R} \times (\mathbb{R}^3)^{2m}$ to $\overline{\mathscr{R}}_m$ and $\overline{\mathscr{R}}_{2m}$, and so is also a smooth embedding. Since $A(\alpha, t, -v) = A(\alpha, t, v)^{-1}$, we see that the following diagram commutes:

$$\begin{array}{ccc} \overline{\mathscr{R}}_m & \xrightarrow{\ j'\ } & \overline{\mathscr{R}}_{2m} \\ \big\downarrow \pi'_c & & \big\downarrow \pi'_c \\ R^c_{\mathbb{C}}(F_m, T) & \xrightarrow{\ \tilde{\imath}^*\ } & R^c_{\mathbb{C}}(F_{2m}, S) \end{array} \tag{12.15}$$

It is easy to see that j' is equivariant with respect to the actions of $\mathbb{R}_{>0}$ and G_t, and so it descends to a smooth embedding $\mathscr{X}^c(F_m, T) \to \mathscr{X}^c(F_{2m}, S)$ by Lemma 2.2. By restricting j' to various $\overline{\mathscr{R}}^t_m$ as in the proof of Lemma 12.8, you can see that this map $\mathscr{X}^c(F_m, T) \to \mathscr{X}^c(F_{2m}, S)$ must be $\tilde{\imath}^*$. Hence, $\tilde{\imath}^*$ is a smooth embedding.

To understand the image of $\tilde{\imath}^*$, first note that, for the map $F\colon \overline{\mathscr{R}}_{2m} \to \mathrm{SL}_2\mathbb{C}$ from Section 9.1, we have

$$F(j'(v)) = \prod_{i=1}^{m} A(\alpha, t, v_i) A(\alpha, t, -v_i) = I.$$

Thus $j'(\overline{\mathscr{R}}^{t\neq 0}_m) \subset F^{-1}(I) \cap \overline{\mathscr{R}}^{t\neq 0}_{2m}$. Since the closure of $F^{-1}(I) \cap \overline{\mathscr{R}}^{t\neq 0}_{2m}$ is $\overline{\mathscr{R}}^c(S_{2m}, S)$ by Lemma 9.17, we have $j'(\overline{\mathscr{R}}_m) \subset \overline{\mathscr{R}}^c(S_{2m}, S)$. Also, in the notation of Section 9.4, if $v \in Z_\epsilon$ then $j'(v) \in Z_{\epsilon'}$, where $\epsilon' = (\epsilon_1, -\epsilon_1, \epsilon_2, \dots, \epsilon_n, -\epsilon_n)$ is balanced. Combining, we see that the image of $\tilde{\imath}^*$ is contained in $\mathscr{X}^c_{\mathrm{bal}}(S_n, S)$, as claimed.

It remains to show that $\tilde{\imath}^*$ is proper. Now

$$\begin{array}{ccc} \mathscr{X}^c(F_m, T) & \xrightarrow{\ \tilde{\imath}^*\ } & \mathscr{X}^c(F_{2m}, S) \\ \big\downarrow \pi & & \big\downarrow \pi \\ X^c(F_m, T) & \xrightarrow{\ \tilde{\imath}^*\ } & X^c(F_{2m}, S) \end{array}$$

commutes as (12.15) does, the left-hand projection is proper by Theorem 8.1, and the bottom $\tilde{\imath}^*$ is proper by Lemma 3.1. It follows that the top $\tilde{\imath}^*$ is proper, completing the proof. □

12.16 Proposition *The induced map $\tilde{\imath}^*\colon \mathscr{X}(F_m, T) \to \mathscr{X}(F_{2m}, S)$ is smooth, and its image is contained in $\mathscr{X}(S_{2m}, S)$. Thus i induces a well-defined map $i'\colon \mathscr{X}(F_m, T) \to \mathscr{X}(S_{2m}, S)$. Moreover, i' is a smooth proper embedding.*

Proof Recall from Section 10.3 that $\mathscr{X}(F_m, T) = B \cup C$, where $B \cong (-2,2) \times \mathscr{Y}$ and $C \cong X^{\mathrm{irr}}(F_m, T)$ are open in $\mathscr{X}(F_m, T)$, as well as the analogous decomposition of $\mathscr{X}(S_{2m}, S)$ as $B' \cup C'$ from Section 10.7. The map $\tilde{\imath}^*$ restricted to C has image contained in $C' \cong X^{\mathrm{irr}}(S_{2m}, S)$ by construction, and is smooth as it is a regular real algebraic map. Smoothness of $\tilde{\imath}^*$ on B follows as in the proof of Proposition 12.13 by using the map $(-2,2) \times \overline{\mathscr{R}}_m \to (-2,2) \times \overline{\mathscr{R}}_{2m}$ sending (c, v) to $(c, j'(v))$, and this also shows that $\tilde{\imath}^*$ is an embedding. The claim about the image of $\tilde{\imath}^*$ is an immediate consequence of Proposition 12.13.

To show that i' is proper, recall from just before Proposition 10.5 the definitions of $X_\circ(F_m, T)$ and $\pi_\circ \colon \mathscr{X}(F_m, T) \to X_\circ(F_m, T)$ and the analogous notions for S_{2m}. Then

$$\begin{array}{ccc} \mathscr{X}(F_m, T) & \xrightarrow{\ i'\ } & \mathscr{X}(S_{2m}, S) \\ \big\downarrow{\scriptstyle \pi_\circ} & & \big\downarrow{\scriptstyle \pi_\circ} \\ X_\circ(F_m, T) & \xrightarrow{\ i^*\ } & X_\circ(S_{2m}, S) \end{array}$$

commutes by (12.4). To see i' is proper, note that the left-hand copy of $\pi_\circ$ is proper by Proposition 10.5 and i^* is proper by applying Lemma 3.1 to $i^* \colon X(F_m, T) \to X(S_{2m}, S)$ and then analyzing its restriction. □

12.17 Understanding the intersection

We can now give:

Proof of Theorem 12.5 Proposition 12.13 gives the claims about $i' \colon \mathscr{X}^c(F_m, T) \to \mathscr{X}^c(S_{2m}, S)$ and $\mathscr{L}^c$, including the final claim that $\mathscr{L}^c \subset \mathscr{X}^c_{\mathrm{bal}}(S_{2m}, S)$. The claim about β^* is immediate from Lemma 12.8. By definition all the induced maps on our resolutions respect the various decompositions $\mathscr{X} = \mathscr{X}_+ \amalg \mathscr{X}_0 \amalg \mathscr{X}_-$, which in turn correspond to SU_2, U_0 and $\mathrm{SL}_2\mathbb{R}$ representations. Applying the proof of Proposition 12.2 to each piece of the induced decomposition $\mathscr{L}^c = \mathscr{L}^c_+ \amalg \mathscr{L}^c_0 \amalg \mathscr{L}^c_-$, we see that the inclusion $S_{2m} \to M_K$ induces the promised bijection of $\mathscr{X}^c(K)$ with $\mathscr{X}^c(\beta)$. □

By Proposition 12.16, $\mathscr{L} := i'(\mathscr{X}(F_m, T))$ is a closed $(2m-2)$-dimensional submanifold of the $(4m-5)$-dimensional manifold $\mathscr{X}(S_{2m}, S)$. Recall from Theorem 10.2 the map $\mathrm{Tr} \colon \mathscr{X}(S_{2m}, S) \to \mathbb{R}$, where $\mathrm{Tr}^{-1}(a)$ is $\mathscr{X}^a_{\mathrm{bal}}(S_{2m}, S)$ for $a \in (-2,2)$ and $X^{a,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_{2m}, S)$ otherwise. For uniformity of notation, we now define $\mathscr{X}^a_{\mathrm{bal}}(S_{2m}, S) := \mathrm{Tr}^{-1}(a)$ for $|a| \geq 2$. If $[a,b] \subset \mathbb{R}$ is an interval, we let $\mathscr{X}^{[a,b]}(S_{2m}, S) = \mathrm{Tr}^{-1}([a,b]) \subset \mathscr{X}(S_{2m}, S)$. It is a (noncompact) manifold with boundary $\mathscr{X}^a_{\mathrm{bal}}(S_{2m}, S) \amalg \mathscr{X}^b_{\mathrm{bal}}(S_{2m}, S)$. We also set $\mathscr{L}^{[a,b]} = \mathscr{L} \cap \mathscr{X}^{[a,b]}(S_{2m}, S)$. The composition $\mathrm{Tr} \circ i'$ is the usual trace map on $\mathscr{X}(F_m, T)$, hence a submersion. Thus $\mathscr{L}^{[a,b]}$ is a closed proper submanifold of $\mathscr{X}^{[a,b]}(S_{2m}, S)$.

We say $\beta \in B_{2m}$ is real representation small if its plat closure $K = \hat{\beta}$ is real representation small in the sense of Section 3.13.

12.18 Lemma *If β is real representation small, the intersections $\mathscr{L}^c \cap \beta^* \mathscr{L}^c$ for $c \in [-2,2]$ and $\mathscr{L}^{[-2,2]} \cap \beta^* \mathscr{L}^{[-2,2]}$ are compact.*

Proof As $\mathscr{L}^c \cap \beta^*\mathscr{L}^c$ is a closed subset of $\mathscr{L}^{[-2,2]} \cap \beta^*\mathscr{L}^{[-2,2]}$, it suffices to prove that the latter is compact. Setting $\mathscr{X}(\beta) := \mathscr{L} \cap \beta^*\mathscr{L}$, our goal is to prove that $\mathscr{X}^{[-2,2]}(\beta)$ is compact. Throughout, we will view $\mathscr{X}(\beta)$ as a subset of $\mathscr{X}(F_{2m}, S)$ rather than $\mathscr{X}(S_{2m}, S)$.

Recall from Section 10.4 the surjection $\pi_\circ\colon \mathscr{X}(F_n, T) \to X_\circ(F_n, T)$. Then (12.4) gives that $\pi_\circ(\mathscr{L}) = L \cap X_\circ(F_{2m}, S)$ and $\pi_\circ(\beta^*\mathscr{L}) = (\beta^*L) \cap X_\circ(F_{2m}, S)$. Let $Z = \pi_\circ(\mathscr{X}(\beta))$. By Proposition 12.2, $Z \subset X(K)$, where we have identified $X(K)$ with a subset of $X(S_{2m}, S) \subset X(F_{2m}, S)$ via the map induced by $S_{2m} \hookrightarrow M_K$.

Now Z is not all of $X(K)$, as $X(K)$ contains the line of reducible representations χ^c for all $c \in \mathbb{R}$ by Section 3.15, while Z is contained in $X_\circ(F_{2m}, S)$, which omits all reducibles with trace not in $(-2, 2)$. However, Z does contain $X^{\mathrm{irr}}(K) = L^{\mathrm{irr}} \cap \beta^* L^{\mathrm{irr}}$. Moreover, by Theorem 12.5,

$$Z = X^{\mathrm{irr}}(K) \cup \{\chi^c \mid c \in (-2,2) \text{ and } X^c_{U_0}(K) \neq \varnothing\}.$$

The space $X^c_{U_0}(K)$ is nonempty exactly when there is a representation $\rho\colon \pi_1(M_K) \to U_0$ with $\mathrm{tr}_\mu(\rho) = c$ which is not fully reducible; this is equivalent to having a reducible representation $\pi_1(M_K) \to \mathrm{SL}_2\mathbb{C}$ with nonabelian image and character χ^c. Thus, by Section 3.15, we have $Z = X^{\mathrm{irr}}(K) \cup \{\chi^c \mid c \in D_K \cap (-2,2)\}$, where recall D_K is finite. Equivalently, since $\pm 2 \notin D_K$ as $\Delta_K(\pm 1)$ is an odd integer, we have $Z = X^{\mathrm{irr}}(K) \cup \{\chi^c \mid c \in D_K \cap [-2,2]\}$. Now Lemma 3.16 implies that any limit point of $X^{\mathrm{irr}}(K)$ in $X(K)$ is χ^c for some $c \in D_K$ and hence $Z^{[-2,2]}$ is closed in $X^{[-2,2]}(K)$. As K is real representation small, $X^{[-2,2]}(K)$ is compact by definition. Hence, so is its closed subset $Z^{[-2,2]}$, which is equal to $\pi_\circ(\mathscr{X}^{[-2,2]}(\beta))$.

From the form of $\hat{\jmath}_1\colon \pi_1(S_{2m}) \to F_m$, any reducible representation in L is balanced. Thus, $Z^{[-2,2]} \subset X(K) \subset L$ in fact lies in $X_{\mathrm{bal},\circ}(F_{2m}, S)$. As $\pi_\circ\colon \mathscr{X}(F_{2m}, S) \to X_\circ(F_{2m}, S)$ is proper by Proposition 10.5, we see that $\pi_\circ^{-1}(Z^{[-2,2]})$ is both compact and a subset of $\mathscr{X}_{\mathrm{bal}}(F_{2m}, S)$. Now $\mathscr{L}$ and $\beta^*\mathscr{L}$ are closed in $\mathscr{X}(S_{2m}, S)$ by Theorem 12.5, so their intersection $\mathscr{X}(\beta)$ is closed in $\mathscr{X}(S_{2m}, S)$. As the latter is closed in $\mathscr{X}_{\mathrm{bal}}(F_{2m}, S)$ by Theorem 10.2, $\mathscr{X}(\beta)$ is closed in $\mathscr{X}_{\mathrm{bal}}(F_{2m}, S)$. So $\mathscr{X}^{[-2,2]}(\beta)$ is a closed subset of the compact set $\pi_\circ^{-1}(Z^{[-2,2]})$, and is hence compact. □

So that we can break up the intersection number defining $h^c(\beta)$ into "local" pieces to prove invariance in Section 13, we note the following:

12.19 Lemma *If β is real representation small, the intersections $\mathscr{L}^c \cap \beta^*\mathscr{L}^c$ for $c \in [-2, 2]$ have finitely many connected components.*

Proof For $c = \pm 2$, this follows because $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) = \mathscr{L}^c \cap \beta^*\mathscr{L}^c$ is a real semialgebraic subset of the real semialgebraic set $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_{2m}, S) = \mathscr{X}^c_{\mathrm{bal}}(S_{2m}, S)$ and real semialgebraic sets have finitely many connected components. For $c \in (-2, 2)$, consider the decomposition $\mathscr{X}^c(K) = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(K) \amalg X^c_{U_0}(K) \amalg X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ of $\mathscr{L}^c \cap \beta^*\mathscr{L}^c$ from Theorem 12.5. Each part of the decomposition is real semialgebraic, and the inclusion of it into $\mathscr{X}^c_{\mathrm{bal}}(S_{2m}, S)$ is continuous; therefore $\mathscr{L}^c \cap \beta^*\mathscr{L}^c$ has finitely many connected components, as needed. □

12.20 The total Lin invariant

We orient $\mathscr{X}(F_m, T)$ and $\mathscr{X}(S_{2m}, S)$ as in Section 11 and give $\mathscr{L}$ the orientation inherited from $\mathscr{X}(F_m, T)$. The slices $\mathscr{X}^c(F_m, T)$, $\mathscr{X}^c_{\mathrm{bal}}(S_{2m}, S)$ and $\mathscr{L}^c$ are then oriented using the submersions Tr, where the range $\mathbb{R}$ has the reversed orientation; see Theorem 11.4 and Corollary 11.12. When β is real representation small, Lemma 12.18 and Theorem 2.9 show that, for each $c \in [-2, 2]$, there is a well-defined intersection number $\langle \mathscr{L}^c, \beta^* \mathscr{L}^c \rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})}$ and so we set

$$h^c(\beta) := (-1)^{|\beta|} \langle \mathscr{L}^c, \beta^* \mathscr{L}^c \rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})},$$

where $|\beta|$ is the number of crossings in β. Our setup now allows us to easily show $h^c(\beta)$ does not depend on c:

12.21 Theorem *If β is real representation small, $h^c(\beta) = h^{c'}(\beta)$ for all $c, c' \in [-2, 2]$.*

Proof We apply Theorem 2.10 with $M = \mathscr{X}^{[-2,2]}(S_{2m})$, $A = \mathscr{L}^{[-2,2]}$, $B = \beta^*(\mathscr{L}^{[-2,2]})$ and $\pi = \mathrm{Tr}$. □

Given the theorem, if $\beta \in B_{2m}$ is real representation small, we define the *total Lin invariant* $h(\beta)$ as $h^c(\beta)$ for any $c \in [-2, 2]$. Next, we define the SU_2 and $\mathrm{SL}_2\mathbb{R}$ Lin invariants. As in Section 12.17, we write

$$\mathscr{L}^c = \mathscr{L}^c_- \amalg \mathscr{L}^c_0 \amalg \mathscr{L}^c_+.$$

From Theorem 12.5, we know that $\mathscr{L}^c_+ \cap \beta^*(\mathscr{L}^c_+) = X^c_{\mathrm{SU}_2}(K)$ and similarly for $\mathrm{SL}_2\mathbb{R}$. Moreover, if $\mathscr{L}^c \cap \beta^*(\mathscr{L}^c)$ is compact and $c \notin D_K$ (so $X^c_{U_0}(K)$ is empty), then both $\mathscr{L}^c_+ \cap \beta^*(\mathscr{L}^c_+)$ and $\mathscr{L}^c_- \cap \beta^*(\mathscr{L}^c_-)$ will be compact. Hence, if β is real representation small and $c \in [-2, 2] \setminus D_{\widehat{\beta}}$, we define

$$h^c_{\mathrm{SL}_2\mathbb{R}}(\beta) = (-1)^{|\beta|} \langle \mathscr{L}^c_-, \beta^*(\mathscr{L}^c_-) \rangle_{\mathscr{X}^c_-(S_{2m})} \quad \text{and} \quad h^c_{\mathrm{SU}_2}(\beta) = (-1)^{|\beta|} \langle \mathscr{L}^c_+, \beta^*(\mathscr{L}^c_+) \rangle_{\mathscr{X}^c_+(S_{2m})}.$$

12.22 Theorem *If β is real representation small, then $h(\beta) = h^c_{\mathrm{SL}_2\mathbb{R}}(\beta) + h^c_{\mathrm{SU}_2}(\beta)$ for all $c \in [-2, 2] \setminus D_{\widehat{\beta}}$. Moreover, $h^c_{\mathrm{SL}_2\mathbb{R}}(\beta)$ and $h^c_{\mathrm{SU}_2}(\beta)$ are constant on each connected component of $[-2, 2] \setminus D_{\widehat{\beta}}$. Finally, $\pm 2 \notin D_{\widehat{\beta}}$ and $h^{\pm 2}_{\mathrm{SU}_2}(\beta) = 0$ and $h^{\pm 2}_{\mathrm{SL}_2\mathbb{R}}(\beta) = h(\beta)$.*

Proof To compute $h^c(\beta)$ as per Theorem 2.9, we perturb $\mathscr{L}^c$ to be transverse to $\beta^*\mathscr{L}^c$ by an isotopy supported in an arbitrarily small neighborhood of their intersection. As $c \notin D_{\widehat{\beta}}$, we have $X^c_{U_0}(\beta) = \varnothing$, and by Theorem 12.5 the initial intersection of $\mathscr{L}^c$ with $\beta^*\mathscr{L}^c$ is disjoint from $\mathscr{X}^c_0(S_{2m}, S)$. Thus we can perturb $\mathscr{L}^c$ to a transverse $\mathscr{L}'$ without changing the intersection with $\mathscr{X}^c_0(S_{2m}; S)$. We then use $\mathscr{L}'$, $\mathscr{L}'_-$ and $\mathscr{L}'_+$ to compute $h^c(\beta)$, $h^c_{\mathrm{SL}_2\mathbb{R}}(\beta)$ and $h^c_{\mathrm{SU}_2}(\beta)$, respectively. Each point of $\mathscr{L}' \cap \beta^*\mathscr{L}^c$ contributes ± 1 to $h^c(\beta)$ and the same to exactly one of $h^c_{\mathrm{SL}_2\mathbb{R}}(\beta)$ and $h^c_{\mathrm{SU}_2}(\beta)$. This proves the first claim.

For the second statement, suppose $[a, b]$ is disjoint from $D_{\widehat{\beta}}$. Then $\mathscr{L}^{[a,b]}$ is disjoint from $\mathscr{X}^{[a,b]}_0(S_{2m}, S)$ so we can study $\mathscr{L}^{[a,b]}_-$ and $\mathscr{L}^{[a,b]}_+$ separately using the proof of Theorem 12.21 to see that $h^c_{\mathrm{SU}_2}(\beta)$ and $h^c_{\mathrm{SL}_2\mathbb{R}}(\beta)$ are constant on $[a, b]$.

Finally, we observed in Section 3.15 that $\pm 2 \notin D_K$. By Theorem 10.2, we have $\mathscr{X}^{\pm 2}(S_{2m}, S) = X^{\pm 2, \mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_{2m}, S)$, so $h^{\pm 2}_{\mathrm{SU}_2}(\beta) = 0$ and $h^{\pm 2}_{\mathrm{SL}_2\mathbb{R}}(\beta) = h(\beta)$ by definition. □

13 Proof of invariance

The goal of this section is to prove:

13.1 Theorem *Suppose $\beta \in B_{2m}$ and $\beta' \in B_{2m'}$ with $\widehat{\beta}$ and $\widehat{\beta}'$ isotopic to the same knot K. If K is real representation small, then $h(\beta) = h(\beta')$. Moreover, for all c in $[-2,2] \setminus D_K$, we have $h^c_{\mathrm{SL}_2\mathbb{R}}(\beta) = h^c_{\mathrm{SL}_2\mathbb{R}}(\beta')$ and $h^c_{\mathrm{SU}_2}(\beta) = h^c_{\mathrm{SU}_2}(\beta')$.*

Thus, when K is real representation small, we define the *total Lin invariant* of K by $h(K) := h(\beta)$, where β is any braid with $\widehat{\beta} = K$, and similarly $h^c_{\mathrm{SL}_2\mathbb{R}}(K) := h^c_{\mathrm{SL}_2\mathbb{R}}(\beta)$ and $h^c_{\mathrm{SU}_2}(K) := h^c_{\mathrm{SU}_2}(\beta)$. Combined with Theorem 12.22, Theorem 13.1 gives Theorems 1.1 and 1.2 from the introduction.

To set up the proof, recall that $h^c(\beta) := (-1)^{|\beta|}\langle \mathscr{L}^c, \beta^*\mathscr{L}^c\rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})}$. This set has finitely many connected components by Lemma 12.19. Thus we can write $h^c(\beta)$, in terms of local intersection numbers as in (2.11):

(13.2)
$$\langle \mathscr{L}^c, \beta^*\mathscr{L}^c\rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})} = \sum_Z \langle \mathscr{L}^c, \beta^*\mathscr{L}^c\rangle|_Z,$$

where Z runs over the set of connected components of $\mathscr{L}^c \cap \beta^*\mathscr{L}^c$. If Z is a connected component of $\mathscr{X}^c(\beta)$, we define

$$n_{Z,\beta} = (-1)^{|\beta|}\langle \mathscr{L}^c, \beta^*\mathscr{L}^c\rangle|_Z.$$

We will prove that the $n_{Z,\beta}$ are invariants of the underlying knot K; to make this precise, we use the setup from Section 12.1 and Proposition 12.2. Given a knot $K = \widehat{\beta}$ for some $\beta \in B_{2m}$, let M_K be the exterior of K. The inclusion $S_{2m} \to M_K$, where $S_{2m} = \partial H_1$, induces an epimorphism $\pi_1(S_{2m}) \to \pi_1(M_K)$. Concretely, using the standard generators of $\pi_1(S_{2m}) = \langle s_1, \dots, s_{2m} \mid w_{2m} = 1\rangle$, where $w_{2m} = s_1 \cdots s_{2m}$, we get a group presentation for $\pi_1(M_K)$, namely

$$\Pi_\beta = F_{2m}/\langle w_{2m}, H, \beta_*^{-1}(H)\rangle, \quad \text{where } H \text{ is the kernel of } F_{2m} \to \pi_1(H_1).$$

By construction, Π_β comes equipped with a preferred map $i_\beta\colon F_{2m} \to \Pi_\beta$, as well as an isomorphism $\Pi_\beta \to \pi_1(M_K)$ induced by $D_{2m} \hookrightarrow S_{2m} \hookrightarrow M_K$.

Motivated by Theorem 12.5, for $c \in [-2,2]$ we define

$$\mathscr{X}^c(\Pi_\beta) := X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(\Pi_\beta, S) \amalg X^c_{U_0}(\Pi_\beta, S) \amalg X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Pi_\beta, S).$$

When $c = \pm 2$, since Π_β is generated by S, by Lemma 3.10 and its analogue for U_0, the sets $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(\Pi_\beta, S)$ and $X^c_{U_0}(\Pi_\beta, S)$ are empty; thus $\mathscr{X}^c(\Pi_\beta) = X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Pi_\beta, S)$ when $c = \pm 2$.

As in Section 12.3, we have an induced map $i_\beta^*\colon \mathscr{X}^c(\Pi_\beta) \to \mathscr{X}^c(F_{2m}, S)$ at the level of sets only, not topological spaces. For $c \in (-2,2)$, Theorem 12.5 implies i_β^* is injective and its image is $\mathscr{X}^c(\beta) := \mathscr{L}^c \cap \beta^*\mathscr{L}^c$; the same holds for $c = \pm 2$, where $\mathscr{X}^c_{\mathrm{bal}}(S_{2m}, S) = X^c_{\mathrm{SL}_2\mathbb{R}}(S_{2m}, S)$ by Proposition 12.2.

If $\varphi\colon \Pi_{\beta'} \to \Pi_\beta$ is an isomorphism which takes the conjugacy class of a meridian in $\Pi_{\beta'}$ to the conjugacy class of a meridian (or its inverse) in Π_β, we get a bijection $\varphi^*\colon \mathscr{X}^c(\Pi_\beta) \to \mathscr{X}^c(\Pi_{\beta'})$ by considering the induced maps on character varieties. (In fact, any isomorphism φ must have this property by [Gordon and Luecke 1989]; here, this will be obvious without appealing to such deep results.) We will prove:

13.3 Theorem *Suppose $\beta \in B_{2m}$ and $\beta' \in B_{2m'}$ with $\widehat{\beta} = \widehat{\beta}' = K$ and $c \in (-2, 2)$. Then there is an isomorphism $\varphi\colon \Pi_{\beta'} \to \Pi_\beta$ and a homeomorphism $\psi\colon \mathscr{X}^c(\beta) \to \mathscr{X}^c(\beta')$ such that there is a commuting square*

$$\begin{array}{ccc} \mathscr{X}^c(\Pi_\beta) & \xrightarrow{\;i_\beta^*\;} & \mathscr{X}^c(\beta) \\ \big\downarrow \varphi^* & & \big\downarrow \psi \\ \mathscr{X}^c(\Pi_{\beta'}) & \xrightarrow{\;i_{\beta'}^*\;} & \mathscr{X}^c(\beta') \end{array}$$

If in addition K is real representation small and Z is a connected component of $\mathscr{X}^c(\beta)$, then $n_{Z,\beta} = n_{\psi(Z),\beta'}$.

Assuming Theorem 13.3, we can show:

Proof of Theorem 13.1 For all c in the open interval $(-2, 2)$, we combine the formula (13.2) with Theorem 13.3 to get $h^c(\beta) = h^c(\beta')$, and then Theorem 12.21 gives $h(\beta) = h(\beta')$. Provided c is, further, not in D_K, we also get the corresponding statements for $h^c_{\mathrm{SU}_2}$ and $h^c_{\mathrm{SL}_2\mathbb{R}}$. The case of $c = \pm 2$ then follows from the last claim of Theorem 12.22. □

Theorem 13.3 extends to $c = \pm 2$ as well:

13.4 Theorem *Suppose $\beta \in B_{2m}$ and $\beta' \in B_{2m'}$ with $\widehat{\beta} = \widehat{\beta}' = K$ and $c = \pm 2$. Then the conclusions of Theorem 13.3 hold.*

13.5 Plat moves

Our proof of Theorem 13.3 follows that of Heusener [2003]. (Lin's original proof [1992] is similar but uses braid closures.) The basic idea is this: first show that if $\widehat{\beta} = \widehat{\beta}'$, then β and β' are related by a sequence of moves, and then check invariance under these moves. The first step was accomplished by Birman and Hilden:

13.6 Theorem [Birman 1976; Hilden 1975] *If $\widehat{\beta}$ and $\widehat{\beta}'$ are isotopic, then β and β' are related by a sequence of the following moves and their inverses:*

- ***Type I*** *Replace $\beta \in B_{2m}$ by $\alpha\beta$ or $\beta\alpha$, where $\alpha \in B_{2m}$ is one of the **type I braids** $\sigma_1, \sigma_2\sigma_1^2\sigma_2$, or $\sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}$ for $1 \le j \le m-1$.*
- ***Type II*** *Replace $\beta \in B_{2m}$ by $\beta\sigma_{2m} \in B_{2m+2}$.*

These moves are illustrated in [Heusener 2003, Figure 3].

13.7 Invariance under type I moves

Suppose $\beta' = \alpha\beta$, where α is a type I braid. Then $\alpha_*(H) = H$, and so

$$\Pi_{\beta'} = F_{2m}/\langle w_{2m}, H, \beta_*^{-1}\alpha_*^{-1}H\rangle = F_{2m}/\langle w_{2m}, H, \beta_*^{-1}H\rangle = \Pi_\beta$$

and the isomorphism $\varphi\colon \Pi_{\beta'} \to \Pi_\beta$ is induced by $1_{F_{2m}}$. On the other hand, if $\beta' = \beta\alpha$, then $\Pi'_\beta = F_{2m}/\langle w_{2m}, H, \alpha_*^{-1}\beta_*^{-1}H\rangle$, and the isomorphism φ is induced by the map $\alpha_*\colon F_{2m} \to F_{2m}$ since $\alpha_*(w_{2m}) = w_{2m}$.

Next, we construct the homeomorphism ψ for a given $c \in (-2,2)$. Our isomorphism $\varphi\colon \Pi_{\beta'} \to \Pi_\beta$ is induced by an isomorphism $\widehat{\varphi}\colon F_{2m'} \to F_{2m}$, where each $\widehat{\varphi}(s_i)$ is conjugate to some s_j or its inverse. Thus, by Section 12.3, there is a well-defined set map $\widehat{\varphi}^*\colon \mathscr{X}^c(F_{2m}) \to \mathscr{X}^c(F_{2m'})$ with a commutative square

$$\begin{array}{ccc} \mathscr{X}^c(\Pi_\beta) & \xrightarrow{i_\beta^*} & \mathscr{X}^c(F_{2m}) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \widehat{\varphi}^*} \\ \mathscr{X}^c(\Pi_{\beta'}) & \xrightarrow{i_{\beta'}^*} & \mathscr{X}^c(F_{2m'}) \end{array} \tag{13.8}$$

so $\widehat{\varphi}^*$ maps $\mathscr{X}^c(\beta) = \operatorname{im} i_\beta^*$ bijectively to $\mathscr{X}^c(\beta') = \operatorname{im} i_{\beta'}^*$. Thus, if we define $\psi = \widehat{\varphi}^*$, we get a commutative square as in Theorem 13.3. When $\beta' = \alpha\beta$, the map $\widehat{\varphi}^*$ is the identity, and hence ψ is a homeomorphism; when $\beta' = \beta\alpha$, this follows from:

13.9 Proposition *If $\alpha \in B_{2m}$ is a braid of type I and $c \in (-2,2)$, the map $\alpha^*\colon \mathscr{X}^c(S_{2m}, S) \to \mathscr{X}^c(S_{2m}, S)$ is a diffeomorphism which fixes $\mathscr{L}^c$ setwise. The map α^* reverses the orientation on $\mathscr{L}^c$ when $\alpha = \sigma_1$; otherwise, it preserves it.*

Proof Recall from (12.14) that $i'\colon \mathscr{X}^c(F_m, T) \to \mathscr{X}^c(S_{2m}, S)$ is induced by $j'\colon \overline{\mathscr{R}}_m \to \overline{\mathscr{R}}_{2m}$, where $j'(v_1, \dots, v_m) = (v_1, -v_1, \dots, v_m, -v_m)$. Similarly, the map α^* on $\mathscr{X}^c(S_{2m}, S)$ is induced by the map $\alpha^*\colon \overline{\mathscr{R}}_{2m} \to \overline{\mathscr{R}}_{2m}$ given explicitly by the formulae in the proof of Lemma 12.8. In each case, we check that $\alpha^* \circ j' = j' \circ \Phi_\alpha$ for some diffeomorphism $\Phi_\alpha\colon \overline{\mathscr{R}}_m \to \overline{\mathscr{R}}_m$.

To start, from (12.9) we have

$$\sigma_1^*(v_1, -v_1, \dots, v_m, -v_m) = (-v_1, v_1, v_2, -v_2, \dots, v_m, -v_m),$$

so we can take $\Phi_{\sigma_1}(v_1, \dots, v_m) = (-v_1, v_2, v_3, \dots, v_m)$. The map $\overline{Q}_t \to \overline{Q}_t$ given by $v \mapsto -v$ is orientation-reversing, so Φ_{σ_1} is orientation-reversing. The map Φ_{σ_1} commutes with the actions of $\mathbb{R}_{>0}$ and the G_t, so it descends to an orientation-reversing diffeomorphism $\varphi_{\sigma_1}\colon \mathscr{L}^c \to \mathscr{L}^c$, which satisfies $\sigma_1^* \circ i' = i' \circ \varphi_{\sigma_1}$.

Similarly, we compute

$$(\sigma_2\sigma_1^2\sigma_2)^*(v_1, -v_1, v_2, -v_2, \dots, v_m, -v_m) = (v'_1, -v'_1, v_2, -v_2, \dots, v_m, -v_m),$$

where $v_1' = \rho(-v_2) \cdot v_1$, and

(13.10) $$(\sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j})^*(v_1, -v_1, \dots, v_m, -v_m) = (v_1, -v_1, \dots, v_{j+1}, -v_{j+1}, v_j, -v_j, \dots, v_m, -v_m),$$

so we take $\Phi_{\sigma_2\sigma_1^2\sigma_2}(v_1, \dots, v_m) = (v_1', v_2, \dots, v_m)$ and

$$\Phi_{\sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}}(v_1, \dots, v_m) = (v_1, \dots, v_{j+1}, v_j, \dots, v_m).$$

To see these Φ preserve orientation, note that the first map is effectively a rotation in the first $\overline{Q}_t$-coordinate (so homotopic to the identity) and the second interchanges two such 2-dimensional coordinates; see the proof of Lemma 12.11 for complete details. □

13.11 Corollary *Theorem 13.3 holds when β and β' are related by a type I move.*

Proof We have already constructed φ and ψ giving the commutative square, so it remains to check the last statement. If Z is a component of $\mathscr{X}^c(\beta)$, we write $Z' = \psi(Z)$. Suppose $\alpha = \sigma_1$, and let $\beta' = \alpha\beta$. Then we have

$$\langle \mathscr{L}^c, (\alpha\beta)^*(\mathscr{L}^c)\rangle|_{Z'} = \langle \mathscr{L}^c, \beta^*\alpha^*(\mathscr{L}^c)\rangle|_{Z'} = -\langle \mathscr{L}^c, \beta^*(\mathscr{L}^c)\rangle|_Z$$

by Proposition 13.9. Since $|\alpha\beta| = |\beta| + 1$, we see that $n_{Z',\beta'} = n_{Z,\beta}$. Similarly, if $\beta' = \beta\alpha$,

$$\langle \mathscr{L}^c, (\beta\alpha)^*(\mathscr{L}^c)\rangle|_{Z'} = \langle \mathscr{L}^c, \alpha^*\beta^*(\mathscr{L}^c)\rangle|_{Z'} = \langle (\alpha^*)^{-1}\mathscr{L}^c, \beta^*(\mathscr{L}^c)\rangle|_Z = -\langle \mathscr{L}^c, \beta^*(\mathscr{L}^c)\rangle|_Z,$$

where in the second step we have used the fact that $\alpha^* \colon \mathscr{X}^c(S_{2m}, S) \to \mathscr{X}^c(S_{2m}, S)$ is an orientation-preserving diffeomorphism. Hence, $n_{Z',\beta'} = n_{Z,\beta}$.

The proof for other α of type I is exactly the same, except that $|\alpha\beta| \equiv |\beta| \bmod 2$ and $\alpha^* \colon \mathscr{L}^c \to \mathscr{L}^c$ is orientation-preserving, so, rather than a pair of canceling signs, we get no signs at all. □

13.12 Invariance under the type II move

Given $\beta \in B_{2m}$, let $\beta' = \beta\sigma_{2m} \in B_{2m+2}$. If $\Pi_\beta = \langle s_1, \dots, s_{2m} \mid R\rangle$, where R is the set of relators, then

$$\Pi_{\beta'} = \langle s_1, \dots, s_{2m}, s_{2m+1}, s_{2m+2} \mid R, s_{2m+1}s_{2m+2}, s_{2m}s_{2m+2}\rangle.$$

We define $\widehat{\varphi} \colon F_{2m+2} \to F_{2m}$ by $\widehat{\varphi}(s_i) = s_i$ for $i \le 2m$, by $\widehat{\varphi}(s_{2m+1}) = s_{2m}$, and by $\widehat{\varphi}(s_{2m+2}) = s_{2m}^{-1}$, so $\widehat{\varphi}$ induces an isomorphism $\varphi \colon \Pi_{\beta'} \to \Pi_\beta$. As in the previous subsection, we define $\psi = \widehat{\varphi}^*$, where $\widehat{\varphi}^*$ is as in (13.8); the restriction of ψ to $\mathscr{X}^c(\beta)$ will give the commutative square required by Theorem 13.3.

We now establish the properties of ψ needed to complete the proof of Theorem 13.3. For ease of notation, we write $\mathscr{L}_1 = \mathscr{L}^c$ and $\mathscr{L}_2 = \beta^*(\mathscr{L}^c)$ in $\mathscr{X}^c(S_{2m})$, and similarly $\mathscr{L}_1' = \mathscr{L}^c$ and $\mathscr{L}_2' = (\beta')^*(\mathscr{L}^c)$ in $\mathscr{X}^c(S_{2m+2})$.

13.13 Proposition *For each $c \in (-2,2)$, the map $\psi\colon \mathcal{X}^c(F_{2m}) \to \mathcal{X}^c(F_{2m+2})$ is a smooth embedding. The image X of $\mathcal{X}^c(S_{2m})$ under ψ is contained in $\mathcal{X}^c(S_{2m+2})$ and satisfies the following properties:*

1. $\mathcal{L}'_\ell \cap X = \psi(\mathcal{L}_\ell)$ *for* $\ell = 1,2$.
2. $\mathcal{L}'_1 \cap \mathcal{L}'_2 \subset X$.
3. *The normal bundle ν of X in $\mathcal{X}^c(S_{2m+2})$ contains oriented 2-dimensional subbundles U_1, U_2 such that $\nu = -U_1 \oplus U_2$ and $U_\ell|_{\psi(\mathcal{L}_\ell)} = \nu_{\mathcal{L}'_\ell/\psi(\mathcal{L}_\ell)}$ as oriented bundles for $\ell = 1,2$.*

Proof Let $\Psi\colon \overline{\mathcal{R}}_{2m} \to \overline{\mathcal{R}}_{2m+2}$ be given by

$$\Psi(t, v_1, \dots, v_{2m}) = (t, v_1, \dots, v_{2m}, v_{2m}, -v_{2m}).$$

Now Ψ is equivariant with respect to the actions of $\mathbb{R}_{>0}$ and the G_t, so it descends to a smooth embedding $\mathcal{X}^c(F_{2m}) \to \mathcal{X}^c(F_{2m+2})$. It is easy to see that this map agrees with the map ψ constructed at the beginning of the subsection. From Section 9.4, recall that $\overline{\mathcal{R}}^c(S_{2m}) \subset \overline{\mathcal{R}}_{2m}$ is defined using $F_{2m}\colon \overline{\mathcal{R}}_{2m} \to \mathrm{SL}_2\mathbb{C}$ and similarly $\overline{\mathcal{R}}^c(S_{2m+2}) \subset \overline{\mathcal{R}}_{2m+2}$ via the analogous F_{2m+2}. We next show $Y = \Psi(\overline{\mathcal{R}}^c(S_{2m}))$ is contained in $\overline{\mathcal{R}}^c(S_{2m+2})$, and moreover $Y = \Psi(\overline{\mathcal{R}}_{2m}) \cap \overline{\mathcal{R}}^c(S_{2m+2})$. Since $F_{2m+2} \circ \Psi = F_{2m}$, for $v \in \overline{\mathcal{R}}_{2m}$, if $F_{2m}(v) = I$ then $F_{2m+2}(\Psi(v)) = I$ as well. By Lemma 9.5, this shows Ψ takes $\overline{\mathcal{R}}^c(S_{2m}) \setminus \overline{\mathcal{R}}^{t\neq 0}_{2m}$ into $\overline{\mathcal{R}}^c(S_{2m+2}) \setminus \overline{\mathcal{R}}^{t\neq 0}_{2m+2}$. The locus where $t = 0$ is then covered by continuity of Ψ and Lemma 9.17, so $Y \subset \overline{\mathcal{R}}^c(S_{2m+2})$ and hence $X \subset \mathcal{X}^c(S_{2m+2})$. The stronger claim that $Y = \Psi(\overline{\mathcal{R}}_{2m}) \cap \overline{\mathcal{R}}^c(S_{2m+2})$ follows from $F_{2m+2} \circ \Psi = F_{2m}$ by examining the local functions f_ϵ in Section 9.4.

Now we verify each of the three properties:

(1) Define $\mathcal{N}_\ell = \sigma^{-1}(\mathcal{L}_\ell)$ and $\mathcal{N}'_\ell = \sigma^{-1}(\mathcal{L}'_\ell)$, and then let $i'_1, i'_2\colon \overline{\mathcal{R}}_{m+1} \to \overline{\mathcal{R}}_{2m+2}$ be the inclusions whose images are $\mathcal{N}'_1$, $\mathcal{N}'_2$. If $u = (u_1, \dots, u_m)$ and $u' = (u_1, \dots, u_{m+1})$, we have $i'_1(u') = (u_1, -u_1, \dots, u_{m+1}, -u_{m+1})$. It is easy to see that $i'_1(u')$ is in Y if and only if $u_{m+1} = -u_m$. It follows that $\mathcal{N}'_1 \cap Y = \Psi(\mathcal{N}_1)$, and hence that $\mathcal{L}'_1 \cap X = \psi(\mathcal{L}_1)$.

Next, suppose that $i_2(u) = \beta^*(i_1(u)) = (v_1, \dots, v_{2m})$. Then

$$\begin{aligned} i'_2(u') &= (\beta')^*(u_1, -u_1, \dots, -u_m, u_{m+1}, -u_{m+1}) \\ &= \sigma^*_{2m}(v_1, \dots, v_{2m}, u_{m+1}, -u_{m+1}) \\ &= (v_1, \dots, v_{2m-1}, u_{m+1}, \rho(-u_{m+1}) \cdot v_{2m}, -u_{m+1}). \end{aligned}$$

Hence, $i'_2(u') \in Y$ if and only if $\rho(-u_{m+1}) \cdot v_{2m} = u_{m+1}$. This occurs if and only if $v_{2m} = u_{m+1}$, so $\mathcal{N}'_2 \cap Y = \{(v_1, \dots, v_{2m}, v_{2m}, -v_{2m}) \mid v = i_2(u) \text{ with } u \in \overline{\mathcal{R}}_{2m}\} = \Psi(\mathcal{N}_2)$. It follows that $\mathcal{L}'_2 \cap X = \psi(\mathcal{L}_2)$.

(2) We use the descriptions of $\mathcal{N}'_1$ and $\mathcal{N}'_2$ from property (1). If

$$(v_1, \dots, v_{2m-1}, u_{m+1}, \rho(-u_{m+1}) \cdot v_{2m}, -u_{m+1}) \in \mathcal{N}'_1,$$

then $\rho(-u_{m+1}) \cdot v_{2m} = u_{m+1}$. This can only happen if $v_{2m} = u_{m+1}$. Hence, $\mathcal{N}'_1 \cap \mathcal{N}'_2 \subset Y$, which implies $\mathcal{L}'_1 \cap \mathcal{L}'_2 \subset X$.

(3) Consider the projection $p\colon \overline{\mathcal{R}}_1 \to \mathbb{R}$ that sends (t, v) to t, and let $\widetilde{V} = \ker dp$. Our standard orientation on $\overline{Q}_t$ makes $\widetilde{V}$ into an oriented 2-dimensional vector bundle over $\overline{\mathcal{R}}_1$. If $\iota\colon \overline{\mathcal{R}}_1 \to \overline{\mathcal{R}}_1$ is given by $\iota(t, v) = (t, -v)$, then $d\iota\colon \widetilde{V} \to \widetilde{V}$ is an orientation-reversing isomorphism. Note that the actions of G_t and $\mathbb{R}_{>0}$ on $\overline{\mathcal{R}}_1$ extend to actions on $\widetilde{V}$.

Define $\widehat{Y} = \Psi(\overline{\mathcal{R}}_{2m})$ and let $\rho_i : \overline{\mathcal{R}}_{2m+2} \to \overline{\mathcal{R}}_1$ be given by $\rho_i(v) = (t(v), v_i)$. Thus $\widehat{Y}$ is the locus where

$$\rho_{2m}(v) = \rho_{2m+1}(v) = \iota(\rho_{2m+2}(v)).$$

Note that each ρ_i is equivariant with respect to the actions of the G_t and $\mathbb{R}_{>0}$.

Now let $V = \rho_{2m}^*(\widetilde{V})|_{\widehat{Y}}$, and consider the bundle maps

$$\widetilde{\alpha}\colon T\overline{\mathcal{R}}_{2m+2}|_{\widehat{Y}} \to \widetilde{V} \oplus \widetilde{V} \quad \text{and} \quad \alpha\colon T\overline{\mathcal{R}}_{2m+2}|_{\widehat{Y}} \to V \oplus V$$

given by $\widetilde{\alpha} = (d\rho_{2m+1} - d\rho_{2m}, d(\iota\rho_{2m+2}) - d\rho_{2m})$ and $\alpha = \rho_{2m}^*\widetilde{\alpha}$. Since the ρ_i are equivariant with respect to the actions of the G_t and $\mathbb{R}_{>0}$, so are $\widetilde{\alpha}$ and α.

Let $\hat{\nu}$ be the normal bundle of $\widehat{Y}$ in $\overline{\mathcal{R}}_{2m+2}$. You can check that α is surjective and $\ker\alpha = T\widehat{Y}$, so, ignoring orientations, $\hat{\nu} \cong V \oplus V$. We now show this isomorphism is orientation-reversing. To see this, note that if $v' = \Psi(v) \in \widehat{Y}$, then

$$T_{v'}\overline{\mathcal{R}}_{2m+2} = T_{v'}\widehat{Y} \oplus T_{v_{2m}}\overline{Q}_t \oplus T_{-v_{2m}}\overline{Q}_t$$

as an oriented vector space, where $\widehat{Y}$ is oriented as the image of $\overline{\mathcal{R}}_{2m}$. Since the dimension of $T\,\overline{Q}_t \oplus T\,\overline{Q}_t$ is even, $\hat{\nu}|_{v'} \cong T_{v_{2m}}\overline{Q}_t \oplus T_{-v_{2m}}\overline{Q}_t$ as oriented vector spaces. The map

$$\alpha\colon T_{v_{2m}}\overline{Q}_t \oplus T_{-v_{2m}}\overline{Q}_t \to V_{v'} \oplus V_{v'}$$

is given by $(\boldsymbol{w}_1, \boldsymbol{w}_2) \mapsto (\boldsymbol{w}_1, d\iota(\boldsymbol{w}_2))$, which is orientation-reversing. Finally, α is equivariant with respect to the actions of the G_t and $\mathbb{R}_{>0}$, so our identification $\hat{\nu} \cong -V \oplus V$ respects the actions of these groups.

We define $V_1 \subset V \oplus V$ to be the image of the morphism $V \to V \oplus V$ given by $\boldsymbol{v} \mapsto (\boldsymbol{v}, \boldsymbol{v})$, and let $V_2 = V \oplus 0 \subset V \oplus V$. Then V_1 and V_2 are equivariant 2-dimensional subbundles of $V \oplus V$ with the orientation induced by the orientation on V, and $V_1 \oplus V_2 \cong V \oplus V$ as oriented bundles.

We claim that $\nu_{\mathcal{N}'_\ell/\mathcal{N}_\ell} = V_\ell|_{\mathcal{N}_\ell} \subset \hat{\nu}|_{\mathcal{N}_\ell}$ as oriented equivariant vector bundles, where we have identified $\mathcal{N}_\ell$ with $\Psi(\mathcal{N}_\ell)$ to reduce clutter. To see this except for the orientation, first note that, on $\mathcal{N}'_1$, $\rho_{2m+1} = \iota\rho_{2m+2}$, so $\alpha(T\mathcal{N}'_1) = V_1$. Similarly, $\iota\rho_{2m+2} = \rho_{2m}$ on $\mathcal{N}'_2$, so $\alpha(T\mathcal{N}'_2) = V_2$ as unoriented vector spaces. For the orientations, observe that, for $u \in \overline{\mathcal{R}}_m$ and $u' \in \overline{\mathcal{R}}_{m+1}$ as in the proof of property (1), we have

$$T_{u'}\overline{\mathcal{R}}_{m+1} \cong T_u\overline{\mathcal{R}}_m \oplus T_{u_{m+1}}\overline{Q}_t$$

as oriented vector spaces, so, for $v = i_\ell(u)$ and $v' = i'_\ell(u')$, we have

$$T_{v'}\mathcal{N}'_\ell = T_v\mathcal{N}_\ell \oplus d i'_\ell(T_{u_{m+1}}\overline{Q}_t).$$

A computation shows that $\alpha(di_1'(\boldsymbol{w})) = (\boldsymbol{w},\boldsymbol{w})$ for $\boldsymbol{w} \in T_{u_{m+1}}\overline{Q}_t$, which is an orientation-preserving map to $(V_1)_{v'}$. Similarly, you can check that $\alpha(di_2'(\boldsymbol{w})) = (-R(\boldsymbol{w}),0)$, where R is rotation clockwise by 2α, ie $R = \left(\begin{smallmatrix}\cos 2\alpha & \sin 2\alpha\\ -\sin 2\alpha & \cos 2\alpha\end{smallmatrix}\right)$, which is again orientation-preserving.

Having studied $\widehat{Y} \subset \overline{\mathcal{R}}_{2m+2}$, we now turn our attention to $Y \subset \overline{\mathcal{R}}^c(S_{2m+2})$. The orientation on $\overline{\mathcal{R}}^c(S_n)$ was determined using the map $\widetilde{F}_n : \overline{\mathcal{R}}_n \to \widetilde{U}$ from (11.9), which is a submersion for $t \neq 0$. Observe that $\widetilde{F}_{2m+2} \circ \Psi = \widetilde{F}_{2m}$, so we are in the situation of Corollary 2.6, with $M' = \overline{\mathcal{R}}_{2m+2}$, $M = \widehat{Y}$, $X' = \widetilde{U}$, $X = \widetilde{I}$, $f = \widetilde{F}_{2m+2}$, $N' = \overline{\mathcal{R}}^c(S_{2m+2})$ and $N = Y$. Define $V_\ell' = V_\ell|_Y$. Applying Corollary 2.6 and noting that the dimension of $V_1' \oplus V_2'$ is even, we see that

$$\nu_{\overline{\mathcal{R}}^c(S_{2m+2})/Y} = \hat{\nu}|_Y = -V_1' \oplus V_2'$$

as oriented bundles. Since $\mathcal{N}_\ell \subset Y$, we have $\nu_{\mathcal{N}_\ell'/\mathcal{N}_\ell} = V_\ell|_{\mathcal{N}_\ell} = V_\ell'|_{\mathcal{N}_\ell}$.

Finally, let ν be the normal bundle of X in $\mathcal{X}^c(S_{2m+2})$. The actions of $\mathbb{R}_{>0}$ and the G_t on Y and $\overline{\mathcal{R}}^c(S_{2m+2})$ are free and extend to actions on $\nu_{\overline{\mathcal{R}}^c(S_{2m+2})/Y}$ which preserve the subbundles V_1' and V_2'. Passing to the quotient and using Corollary 2.7, we obtain subbundles $U_1, U_2 \subset \nu$ with $U_1 \oplus U_2 = -\nu$ and $U_\ell|_{\mathcal{L}_\ell} = \nu_{\mathcal{L}_\ell'/\psi(\mathcal{L}_\ell)}$. This completes the proof of property (3) and hence the proof of the proposition. □

13.14 Corollary *Theorem 13.3 holds when β and β' are related by a move of type II.*

Proof Let $\mathcal{L}_\ell$, $\mathcal{L}_\ell'$ be as above. By Proposition 13.13, the hypotheses of Lemma 2.13 are satisfied with the opposite of the usual orientation on $\mathcal{X}^c(S_{2m+2})$. As $\dim U_1 = 2$ is even, we get

$$\langle \mathcal{L}_1, \mathcal{L}_2 \rangle|_Z = \langle \psi(\mathcal{L}_1), \psi(\mathcal{L}_2) \rangle|_{\psi(Z)} = -\langle \mathcal{L}_1', \mathcal{L}_2' \rangle|_{\psi(Z)}.$$

Since $|\beta'| = |\beta| + 1$,

$$n_{Z,\beta} = (-1)^{|\beta|}\langle \mathcal{L}_1, \mathcal{L}_2 \rangle|_Z = (-1)^{|\beta'|}\langle \mathcal{L}_1', \mathcal{L}_2' \rangle|_{\psi(Z)} = n_{\psi(Z),\beta'},$$

proving the corollary. □

We end this section with the proofs of its two main technical theorems.

Proof of Theorem 13.3 If the conclusion of Theorem 13.3 holds for the pairs (β,β') and (β',β''), it then holds for (β,β'') as well. Thus the general case of Theorem 13.3 follows from Theorem 13.6 and Corollaries 13.11 and 13.14. □

Proof of Theorem 13.4 For concreteness, we do the case $c = 2$. Since $\mathcal{X}^2(\Pi_\beta) = X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Pi_\beta)$ and $\mathcal{X}^2(\beta) \subset \mathcal{X}^2_{\mathrm{bal}}(S_{2m}, S) = X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_{2m}, S)$, the needed homeomorphism ψ and commutative square exist just from the maps on $\mathrm{SL}_2\mathbb{R}$ character varieties. So it remains to show $n_{Z,\beta} = n_{\varphi(Z),\beta'}$ when K is real representation small. Rather than tackle this directly, it is more expedient to prove it "by continuity" from Theorem 13.3 and Corollary 2.12.

13.15 Claim *For any knot K, there is an $a < 2$ such that $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(K)$ is empty for all $c \in [a, 2]$.*

To see this, first note that $X_{\mathrm{SU}_2}(K)$ is compact as it is the image of a closed subset of $(\mathrm{SU}_2)^{2m}$, which implies that the closure C of $X^{\mathrm{irr}}_{\mathrm{SU}_2}(K)$ in $X_{\mathrm{SU}_2}(K)$ is compact. Let χ be the character in C where tr has its maximum, say $a' = \mathrm{tr}(\chi)$. If χ is reducible, then by Lemma 3.16 we have $a \in D_K$ and so $a' < 2$ as $2 \notin D_K$. If instead χ is irreducible, then $a' < 2$ by Lemma 3.10. So any a with $a' < a < 2$ satisfies the claim.

Let a satisfy the claim and further assume $a > \max D_K$. Then $\mathscr{X}^{[a,2]}(\Pi_\beta) = X^{[a,2],\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(\Pi_\beta)$ and hence $\mathscr{X}^{[a,2]}(\beta)$ is contained in $\mathscr{X}^{[a,2]}_{-}(S_{2m}, S) = X^{[a,2],\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_{2m}, S)$. Hence, $\mathscr{L}^{[a,2]} \cap \beta^*\mathscr{L}^{[a,2]}$ is a real semialgebraic set with finitely many connected components, which are also its path components. Since we are working with character varieties only, we also immediately have a "parametrized" version ψ_c of ψ in Theorem 13.3 which is still a homeomorphism. Next, increase a so that the components of the closed set $\mathscr{L}^{[a,2]} \cap \beta^*\mathscr{L}^{[a,2]}$ correspond bijectively with $\mathscr{L}^2 \cap \beta^*\mathscr{L}^2$. For a connected component Z of $\mathscr{X}^2(\beta)$, let W be the component of $\mathscr{L}^{[a,2]} \cap \beta^*\mathscr{L}^{[a,2]}$ that contains it. By Corollary 2.12, we have $n_{Z,\beta} = \langle \mathscr{L}^c, \beta^*\mathscr{L}^c\rangle|_W$ for all $c \in [a, 2]$ and similarly for $n_{\psi(Z),\beta'}$. Applying Theorem 13.3 for any c in $[a, 2)$ proves the theorem. □

14 Parabolics and the extended Lin invariant

In this section, we define the extended Lin invariant $\tilde{h}(K)$, which counts the heights of parabolic representations $\pi_1(S^3 - K) \to \widetilde{\mathrm{SL}_2\mathbb{R}}$. We begin by discussing parabolic representations, and then review some facts about the $\mathrm{SL}_2\mathbb{R}$ character varieties of knot complements and their boundaries. Finally, we discuss the translation extension locus introduced in [Culler and Dunfield 2018] and use it to define $\tilde{h}(K)$.

14.1 Parabolic representations

Let $\rho\colon \pi_1(T^2) \to \mathrm{SL}_2\mathbb{R}$ be a representation, where T^2 is the 2-torus. Since $\pi_1(T^2) \cong \mathbb{Z}^2 \cong \langle \mu, \lambda\rangle$ is abelian, $\rho(\mu)$ and $\rho(\lambda)$ must belong to the same maximal abelian subgroup of $\mathrm{SL}_2\mathbb{R}$. Up to conjugation by an element of $\mathrm{SL}_2\mathbb{R}$, there are three such:

$$T_e = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in [0, 2\pi) \right\} \quad \text{(elliptic)},$$

$$T_p = \left\{ \begin{pmatrix} \pm 1 & t \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \quad \text{(parabolic)},$$

$$T_h = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \;\middle|\; a \in \mathbb{R}^\times \right\} \quad \text{(hyperbolic)}.$$

We call the representation *elliptic*, *parabolic* or *hyperbolic* accordingly. Note that the center $\{\pm I\}$ of $\mathrm{SL}_2\mathbb{R}$ is contained in all three subgroups; a representation whose image in contained in $\{\pm I\}$ is called *central*.

Now suppose $K \subset S^3$ is a knot with $M_K = S^3 \setminus \nu(K)$ its exterior. We say that $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ is elliptic, parabolic, hyperbolic or central according to its restriction to $\pi_1(\partial M_K)$. We will always use generators μ, λ for $\pi_1(\partial M_K)$, where μ is a meridian for K and λ a homological longitude. We begin with a simple observation:

14.2 Lemma *If a nontrivial representation $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ has $\mathrm{tr}_\mu\, \rho = \pm 2$, then ρ is parabolic.*

Proof Suppose $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ with $\mathrm{tr}_\mu\, \rho = \pm 2$. Since $\rho(\mu)$ and $\rho(\lambda)$ belong to the same maximal abelian subgroup of $\mathrm{SL}_2\mathbb{R}$, either $\rho(\mu)$ is central or the representation ρ is parabolic. If $\rho(\mu)$ is central and $\rho(\lambda)$ is not, then ρ descends to a nontrivial representation $\rho\colon \pi_1(M_K(\mu)) \to \mathrm{PSL}_2\mathbb{R}$, a contradiction as $M_K(\mu) = S^3$. □

14.3 Corollary *If K is real representation small and $h(K) \neq 0$, then $\pi_1(M_K)$ has an irreducible parabolic representation into $\mathrm{SL}_2\mathbb{R}$. Moreover, $2h(K)$ is a signed count of conjugacy classes of such representations, where the signs and multiplicities come from the Casson–Lin picture.*

Proof By Theorem 12.22, we have $2 \notin D_K$ and $h^2_{\mathrm{SL}_2\mathbb{R}}(K) = h(K)$. So $h^2_{\mathrm{SL}_2\mathbb{R}}(K)$ is nonzero, and in particular $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is nonempty by Theorem 12.5. By Lemma 14.2, each element of $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ comes from an irreducible *parabolic* representation, completing the proof of the first claim. As K is real representation small, the sets $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ and $X^{-2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ are finite, so the second claim is immediate from the definitions and the fact that $h^{-2}_{\mathrm{SL}_2\mathbb{R}}(K)$ is also equal to $h(K)$. □

14.4 The $\mathrm{SL}_2\mathbb{R}$ character variety of T^2

We are principally interested in $X_{\mathrm{SL}_2\mathbb{R}}(K) = X_{\mathrm{SL}_2\mathbb{R}}(M_K, \{\mu\})$, where K is a knot in S^3, and its image under the map $i^*\colon X_{\mathrm{SL}_2\mathbb{R}}(K) \to X_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)$ coming from the inclusion $i\colon M_K \to \partial M_K$. To this end, we first describe $X_{\mathrm{SL}_2\mathbb{R}}(T^2)$.

We define the *elliptic locus* $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ to be the subset of $X_{\mathrm{SL}_2\mathbb{R}}(T^2)$ in the image of the elliptic representations, and similarly for the parabolic and hyperbolic loci.

14.5 Lemma *The locus $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ is the pillowcase orbifold $T^2/\{x \sim x^{-1}\}$, which is the flat orbifold with underlying space S^2 and four orbifold points of order 2. It is shown in Figure 6.*

Proof By conjugation, every point in $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ is the image of a representation $\rho\colon \pi_1(T^2) \to T_e$. Such representations are parametrized by $T_e \times T_e \cong T^2$. Now if $A, A' \in T_e$, one has $\mathrm{tr}\, A = \mathrm{tr}\, A'$ if and only if $A' = A^{\pm 1}$. It follows that $\rho, \rho'\colon \pi_1(T^2) \to T_e$ have the same character if and only if $\rho' = \rho^{\pm 1}$, where $\rho^{-1}(x) = \rho(x)^{-1}$, proving the lemma. □

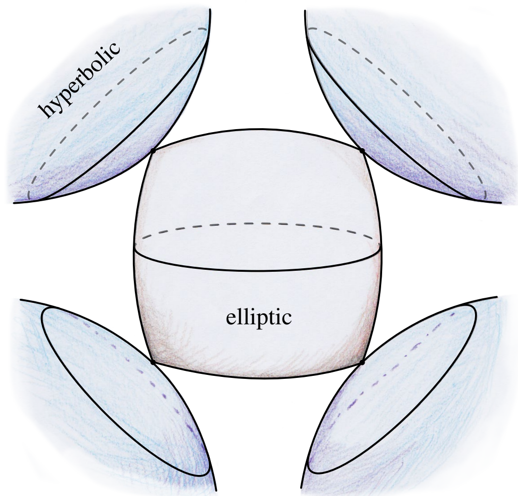

Figure 6: The $\mathrm{SL}_2\mathbb{R}$ character variety $X_{\mathrm{SL}_2\mathbb{R}}(T^2)$ of the 2-torus. The central pillowcase is the elliptic locus $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$, with the rest the hyperbolic locus $X^{\mathrm{hyp}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$; they meet at the four corners of the pillowcase, which is the parabolic locus.

We will use the following explicit system of coordinates on $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$: Let $\mu^*, \lambda^* \in \mathrm{Hom}(\pi_1(T^2), \mathbb{R}) = H^1(T^2; \mathbb{R})$ be the basis that is algebraically dual to $\{\mu, \lambda\}$, and let $\overline{\mu}^*$ and $\overline{\lambda}^*$ be their images in $\mathrm{Hom}(\pi_1(T^2), \mathbb{R}/2\mathbb{Z})$. Identifying T_e with $\mathbb{R}/2\mathbb{Z}$ via $\theta \mapsto \theta/\pi$, we get coordinates on $\mathrm{Hom}(\pi_1(T^2), T_e)$, and hence on the pillowcase. We usually normalize so that the $(\overline{\mu}^*, \overline{\lambda}^*)$-coordinates on the pillowcase are in $[0,1] \times [-1,1]$. (The reason we identify T_e with $\mathbb{R}/2\mathbb{Z}$ rather than the seemingly more natural $\mathbb{R}/\mathbb{Z}$ is to match the conventions of [Culler and Dunfield 2018], where the focus is on $\mathrm{PSL}_2\mathbb{R}$ rather than $\mathrm{SL}_2\mathbb{R}$; note also that the matrix $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ gives an anticlockwise rotation about the point i in the upper-halfspace model of $\mathbb{H}^2$ through angle 2θ.)

A similar argument shows that the *hyperbolic locus* $X^{\mathrm{hyp}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ consists of four distinct copies of $\mathbb{R}^2/\{x \sim -x\}$ (see also [Gao 2022a]); we view the latter space as $\mathbb{R}^2$ with a single orbifold point of order 2. The elliptic and hyperbolic loci intersect at the parabolic locus, which consists of the four corners of the pillowcase, as illustrated in Figure 6.

For the hyperbolic and elliptic loci of $X_{\mathrm{SL}_2\mathbb{R}}(T^2)$, the preimage of a character consists of either one or two conjugacy classes of representations into $\mathrm{SL}_2\mathbb{R}$. In contrast, the preimage of each parabolic character contains a whole circle's worth of different conjugacy classes, in addition to the conjugacy class of a central representation.

14.6 The character variety of a knot complement

We return to the setting of the exterior M_K of $K \subset S^3$, where μ and λ are its meridian and longitude. Now $X_{\mathrm{SL}_2\mathbb{R}}(K)$ decomposes as $X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}}(K) \cup X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$, and we discuss the two parts separately.

As in the case of $\mathrm{SL}_2\mathbb{C}$ detailed in Section 3.12, the characters in $X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}}(K)$ are precisely those coming from $\rho\colon \pi_1(M_K)\to H$, where H is a maximal abelian subgroup of $\mathrm{SL}_2\mathbb{R}$. Thus, to understand $X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}}(K)$, it suffices to study abelian representations. Any such representation factors through $H_1(M_K)\cong\mathbb{Z}$, so $X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}}(K)\cong X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}}(U)$, where U is the unknot. Arguing as in Lemma 14.5 shows that the set of elliptic reducible representations $X^{\mathrm{ell,red}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is homeomorphic to $S^1/\{x\sim x^{-1}\}\cong[0,1]$. Also, for the inclusion $i\colon\partial M_K\to M_K$, we have

$$i^*(X^{\mathrm{ell,red}}_{\mathrm{SL}_2\mathbb{R}}(K))=\{\chi\in X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)\mid \overline{\lambda}^*(\chi)=0\}.$$

Next, we consider the irreducible part of the character variety. For the unknot U, we have $\pi_1(M_U)\cong\mathbb{Z}$, so every representation is abelian and hence $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(U)=\varnothing$. For a more interesting example, we consider the trefoil.

14.7 The positive trefoil

Let $K=T(2,3)$ be the positive (right-handed) trefoil knot, which is the plat closure of $\sigma_1^3\in B_2$. Then M_K is Seifert fibered, and we have a presentation

$$\pi_1(M_K)=\langle x,y,f\mid [x,f]=[y,f]=1,\ x^2=f=y^3\rangle,$$

where f is the class of the Seifert fiber. Suppose $\rho\colon\pi_1(M_K)\to\mathrm{SL}_2\mathbb{R}$ is irreducible. Since f is central and ρ is irreducible, $\rho(f)$ must be central. If $\rho(f)=I$, then $\rho(x)^2=I$, which implies $\rho(x)=\pm I$. This implies that ρ is reducible, contradicting our assumption. Hence, $\rho(f)=-I$.

Since $\rho(x)^2=\rho(y)^3=-I$, both $\rho(x)$ and $\rho(y)$ are elliptic. The set of elliptic 1-parameter subgroups can be identified with $\mathbb{H}^2$ by taking a subgroup to its unique common fixed point. Under this identification, the action of $\mathrm{SL}_2\mathbb{R}$ on such subgroups by conjugation becomes its action on $\mathbb{H}^2$ by isometries. Hence, after conjugation, we can assume that $\rho(x)$ and $\rho(y)$ respectively fix i and ti in $\mathbb{H}^2$ for some $t>1$; equivalently,

$$\rho(x)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\quad\text{and}\quad \rho(y)=\begin{pmatrix}\cos\varphi & -t\sin\varphi\\ t^{-1}\sin\varphi & \cos\varphi\end{pmatrix}$$

for some $\theta,\varphi\in(-\pi,\pi]$ and $t\in[1,\infty)$. As $\rho(x)^2=\rho(y)^3=-I$, we must have $\theta=\pm\frac{\pi}{2}$ and $\varphi=\pm\frac{\pi}{3}$. Moreover, if $\theta=\pm\frac{\pi}{2}$, $\varphi=\pm\frac{\pi}{3}$ and $t\in[1,\infty)$, the above formula determines a representation $\rho_{\theta,\varphi,t}\colon\pi_1(M_K)\to\mathrm{SL}_2\mathbb{R}$. The representations $\rho_{\theta,\varphi,t}$ and $\rho_{-\theta,-\varphi,t}$ are not conjugate in $\mathrm{SL}_2\mathbb{R}$, but they are conjugate in $\mathrm{SL}_2\mathbb{C}$, so they have the same character. Hence, $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ consists of two arcs parametrized by $t\in(1,\infty)$. Each arc limits on the reducible locus as $t\to1$.

To describe their image in $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)$ under i^*, note there are curves on ∂M_K that are Seifert fibers, and that $[f]=6\mu+\lambda$. Since an irreducible ρ satisfies $\rho(f)=-I$, the image $i^*(X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K))$ lies on the image of the line given by the equation $6\mu^*+\lambda^*=1$. We have $\mu=xyf^{-1}$, so if $\rho_t=\rho_{\pi/2,\pi/3,t}$, we compute $\operatorname{tr}\rho_t(\mu)=\frac{\sqrt{3}}{2}(t+t^{-1})$. At $t=1$, the representation ρ_1 is reducible and $\operatorname{tr}_\mu\rho_1=\sqrt{3}=2\cos\frac{\pi}{6}$. The trace increases monotonically as t increases, until, at $t=\sqrt{3}$, we have $\operatorname{tr}\rho_{\sqrt{3}}(\mu)=2$ and ρ_t is

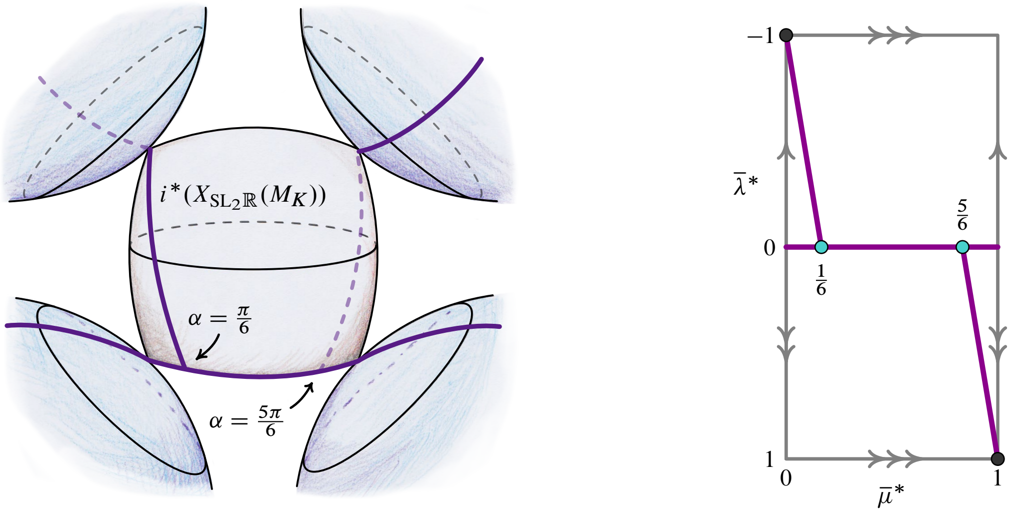

Figure 7: The image $X_{\mathrm{SL}_2\mathbb{R}}(M_K)$ inside $X_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)$ for the trefoil knot K. Right: the full picture, where $i^*(X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}})$ is the approximately horizontal curve. Left: $i^*(X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(M_K))$ in our $(\bar\mu^*, \bar\lambda^*)$-coordinates on the pillowcase.

parabolic. For larger values of t, the representation ρ_t is hyperbolic, and, as $t \to \infty$, the character of ρ_t limits to an ideal point of $X_{\mathrm{SL}_2\mathbb{R}}(K)$ corresponding to the vertical essential annulus in M_K. The arc given by $\rho'_t = \rho_{\pi/2,-\pi/3,t}$ is very similar, except now $\operatorname{tr} \rho'_t(\mu) = -\frac{\sqrt{3}}{2}(t + t^{-1})$, so at $t = 1$ we have $\operatorname{tr}_\mu \rho_1 = -\sqrt{3} = 2\cos\frac{5\pi}{6}$ and the trace decreases monotonically with t. The character variety $X_{\mathrm{SL}_2\mathbb{R}}(K)$ is shown in Figure 7.

14.8 General picture

In general, $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ has expected dimension 1 although the actual dimension may be larger. If K is small, Lemma 3.14 implies that $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ has dimension 1, and that K is real representation small. As in the case of the trefoil, $X^{\mathrm{ell,irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is typically noncompact and limits to both parabolic and reducible characters. If χ is a limit point of the latter type with $\operatorname{tr}_\mu \chi = c$, then $c \in D_K$ by Lemma 3.16. For the trefoil $K = T(2,3)$, the roots of Δ_K are $e^{\pm i\pi/3}$, and so $D_K = \{\sqrt{3}\}$, corresponding to the reducible representations ρ_1, ρ'_1 identified above.

We can interpret $h^c_{\mathrm{SL}_2\mathbb{R}}$ in terms of $X_{\mathrm{SL}_2\mathbb{R}}(K)$. For $\alpha \in [0,\pi]$, let V_α be the vertical line/circle in the pillowcase where the $\bar\mu^*$-coordinate is α/π. For $c = 2\cos\alpha$ not in D_K, the invariant $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ is a count (with signs and multiplicities) of the intersection of V_α with $i^*(X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K))$. When $X_{\mathrm{SL}_2\mathbb{R}}(K)$ is transversely cut out, one can show (as Heusener [2003] did for SU_2) that the multiplicities can be used to orient the arcs of $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ and make this a precise relationship.

For example, if K is the positive trefoil, Figure 7 shows that $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is empty for $\alpha \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, so $h^c_{\mathrm{SL}_2\mathbb{R}}(K) = 0$ in this range. For $\alpha \in \left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right]$, $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ contains a single point, which suggests (but does not prove) that $h^c_{\mathrm{SL}_2\mathbb{R}}(K) = 1$ for such α. We will show that this is indeed the case in Example 15.6.

14.9 Symmetry

The character variety in Figure 7 is symmetric under π-rotation around the vertical line through the middle of the pillowcase. This symmetry is present for any K as we now explain, following [Culler and Dunfield 2018, Lemma 6.1], and extends to the resolved setting, as we show in Lemma 14.10.

For a space Y, a representation η of $\pi_1(Y)$ to the center $\{\pm I\}$ of $\mathrm{SL}_2\mathbb{C}$ is determined by its character, and the resulting *central characters* are classified by $H^1(Y;\mathbb{Z}/2)$ by viewing $\mathbb{Z}/2$ as $\{\pm I\}$. The group of central characters acts on $R_{\mathbb{C}}(Y)$ by $(\eta\cdot\rho)(g) = \eta(g)\rho(g)$, and this descends to an action of $H^1(Y;\mathbb{Z}/2)$ on $X_{\mathbb{C}}(Y)$ and $X(Y)$.

When $Y = M_K$, the group $H^1(M_K;\mathbb{Z}/2)$ has a unique nonzero element $\hat{\mu}^*$. There is thus an involution $b\colon X(K) \to X(K)$ given by $b(\chi) = \hat{\mu}^*\cdot\chi$. If $i^*(\chi) = (m,\ell)$, then

$$i^*(b(\chi)) = i^*(\hat{\mu}^*\cdot\chi) = i^*(\hat{\mu}^*)\cdot i^*(\chi) = (m+1,\ell) = (1-m,-\ell).$$

This symmetry is clearly visible in Figure 7.

If $\chi \in X^c(K)$, then $b(\chi) \in X^{-c}(K)$, and b restricts to maps $X^{c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) \to X^{-c,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ and $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(K) \to X^{-c,\mathrm{irr}}_{\mathrm{SU}_2}(K)$. The same $\hat{\mu}^*$ action also gives $b\colon X^c_{U_0}(K) \to X^{-c}_{U_0}(K)$. Putting this together, we get a map $\varphi\colon \mathscr{X}^c(K) \to \mathscr{X}^{-c}(K)$, which we use to investigate the effect of this symmetry on the local intersection numbers n_Z. Specifically, we next show the following, which is needed for Proposition 15.11:

14.10 Lemma *For $c \in [-2,2]$, the map $\varphi\colon \mathscr{X}^c(K) \to \mathscr{X}^{-c}(K)$ is a diffeomorphism. If Z is a component of $\mathscr{X}^c(K)$, then $n_{\varphi(Z)} = n_Z$.*

Note that if we use the local intersection multiplicities to orient the arcs of $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$, as suggested in Section 14.8, then the action of b on $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is orientation-reversing because b preserves the orientation of $X_{\mathrm{SL}_2\mathbb{R}}(T^2)$ but reverses the orientations of the vertical circles V_α.

Proof First, assume that $c \in (-2,2)$. Given a plat diagram of K corresponding to a braid $\beta \in B_{2m}$, let $j : S_{2m} \to M_K$ be the inclusion of the splitting 2-sphere. Then $j^*(\hat{\mu}^*)([s_i]) = -I$ for $i = 1,\dots,2m$. Consider the map $\Phi\colon \overline{\mathscr{R}}_{2m} \to \overline{\mathscr{R}}_{2m}$ given by $\Phi(v) = -v$. From Section 8.5, we have

$$\pi'_{-c}(\Phi(v))(s_i) = A(\pi-\alpha, t(v), -v_i) = -A(\alpha, t(v), v_i), \tag{14.11}$$

so $\pi'_{-c}(\Phi(v)) = j^*(\hat{\mu}^*)\cdot\pi'_c(v)$. This identity also shows that $F_{-c}(\Phi(v)) = (-1)^{2m}F_c(v) = F_c(v)$, so $\Phi\colon \overline{\mathscr{R}}^c(S_{2m}) \to \overline{\mathscr{R}}^{-c}(S_{2m})$. Finally, it is easy to check that Φ commutes with the actions of G_t and $\mathbb{R}_{>0}$, so it descends to a smooth map $\mathscr{X}^c(S_{2m}) \to \mathscr{X}^{-c}(S_{2m})$, which you can check is φ. Note φ is a diffeomorphism since interchanging the roles of c and $-c$ builds φ^{-1}.

Let $\mathscr{L}_0^c = \mathscr{L}^c$ and $\mathscr{L}_1^c = \beta^*(\mathscr{L}^c)$. It is immediate from the definition that $\varphi(\mathscr{L}_0^c) = \mathscr{L}_0^{-c}$ as sets. Referring to (12.9) and using (14.11) again, we see that φ is equivariant with respect to the actions of B_{2m} on $\mathscr{X}^c(S_{2m})$ and $\mathscr{X}^{-c}(S_{2m})$; the key point is that $\psi_t(-A) = \psi_t(A)$. Hence, $\varphi(\mathscr{L}_1^c) = \mathscr{L}_1^c$ as sets.

The local multiplicity $n_Z = (-1)^{|\beta|}\langle \mathscr{L}_0^c, \mathscr{L}_1^c\rangle|_Z$, so to prove the lemma we must show $\langle \mathscr{L}_0^c, \mathscr{L}_1^c\rangle|_Z = \langle \mathscr{L}_0^{-c}, \mathscr{L}_1^{-c}\rangle|_{\varphi(Z)}$. To do this, we examine the effect of φ on the orientations of $\mathscr{X}^c(S_{2m})$, $\mathscr{L}_0^c$ and $\mathscr{L}_1^c$. The antipodal map on $\overline{Q}_t$ is orientation-reversing, so $\Phi\colon \overline{\mathscr{R}}_{2m} \to \overline{\mathscr{R}}_{2m}$ multiplies the orientation by a factor of $(-1)^{2m} = 1$. As we observed above, $F_{-c}(\Phi(v)) = F_c(v)$. Referring to (11.9), we see that $\Phi\colon \overline{\mathscr{R}}^c(S_{2m}) \to \overline{\mathscr{R}}^{-c}(S_{2m})$ also has degree 1. Finally, Φ commutes with the actions of G_t and $\mathbb{R}_{>0}$, so $\varphi\colon \mathscr{X}^c(S_{2m}) \to \mathscr{X}^{-c}(S_{2m})$ has degree 1 as well.

A similar calculation shows that $\varphi\colon \mathscr{L}_i^c \to \mathscr{L}_i^{-c}$ has degree $(-1)^m$. So

$$\langle \mathscr{L}_0^c, \mathscr{L}_1^c\rangle|_Z = (-1)^m(-1)^m\langle \mathscr{L}_0^{-c}, \mathscr{L}_1^{-c}\rangle_{\varphi(Z)} = \langle \mathscr{L}_0^{-c}, \mathscr{L}_1^{-c}\rangle_{\varphi(Z)},$$

as desired. The remaining case of $c = \pm 2$ now follows "by continuity" as in the proof of Theorem 13.4. □

14.12 The translation extension locus

We next turn to studying representations to $\widetilde{\mathrm{SL}_2\mathbb{R}}$ in the spirit of [Culler and Dunfield 2018]. Recall from our Section 4 the basic properties of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ and the space $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$ that encodes representations $\Gamma \to \widetilde{\mathrm{SL}_2\mathbb{R}}$ modulo conjugation by $\widetilde{\mathrm{SL}_2\mathbb{R}}$. We also use from Section 4 the character variety $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma) = R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)/\!\!/\mathrm{SL}_2\mathbb{R}$, which is a branched double cover of our usual $X_{\mathrm{SL}_2\mathbb{R}}(\Gamma) = R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)/\!\!/\mathrm{SL}_2^{\pm}(\mathbb{R})$. We start with some simple examples.

14.13 Example Suppose $\Gamma = \pi_1(T^2) = \mathbb{Z}^2$. Then $\widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ is a double cover of $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ branched over the set of central characters, which is the same as the set of parabolic points. In contrast, the map $\widetilde{X}^{\mathrm{hyp}}_{\mathrm{SL}_2\mathbb{R}}(T^2) \to X^{\mathrm{hyp}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ is one-to-one. We conclude that $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ consists of a torus (the branched double cover of the pillowcase) with four cones (the hyperbolic components) attached at the four parabolic points. Coordinates on $\widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ are given by pairs $\overline{\mu}^*, \overline{\lambda}^*$ with $-1 \le \overline{\mu}^*, \overline{\lambda}^* \le 1$. The generator $a\colon \widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2) \to \widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ of the branched covering group corresponds to flipping over the copy of $\mathbb{H}^2$ that $\mathrm{SL}_2\mathbb{R}$ acts on, in particular reversing the rotation direction of each elliptic element; thus, in our coordinates we have $a(\overline{\mu}^*, \overline{\lambda}^*) = (-\overline{\mu}^*, -\overline{\lambda}^*)$. There is also a map $b\colon \widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(T^2) \to \widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(T^2)$ given by multiplication by the central character where $\mu \mapsto -I$ and $\lambda \mapsto I$; in coordinates, we have $b(\overline{\mu}^*, \overline{\lambda}^*) = (\overline{\mu}^* + 1, \overline{\lambda}^*)$. (Caution: unlike a, the map b is not a covering transformation for $\widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2) \to X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2)$, but rather permutes the fibers of the map $\widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(T^2) \to \widetilde{X}^{\mathrm{ell}}_{\mathrm{PSL}_2\mathbb{R}}(T^2)$.)

Next, we consider $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(T^2)$. As T^2 is a $K(\Gamma, 1)$, we have $H^1(\Gamma) = H^1(T^2) = \mathbb{Z}^2$. The Euler class of the trivial representation is 0 and $R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ is connected, so $e(\rho) = 0$ for all $\rho \in R_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$. It follows that $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(T^2)$ is a $\mathbb{Z}^2$ cover of $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(T^2)$, ie its universal cover. We will focus on the elliptic component of this space, which we view as $\mathbb{R}^2$, with coordinates μ^*, λ^*. Another viewpoint on these coordinates is described in [Culler and Dunfield 2018, Section 3.5]; namely, one defines

$$\mathrm{trans}\colon \widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(T^2) \to H^1(T^2;\mathbb{R}) \quad \text{by } [\rho] \mapsto \mathrm{trans} \circ \rho.$$

Concretely, in our (μ^*, λ^*)-coordinates, this sends $[\rho]$ to $\big(\mathrm{trans}(\rho(\mu)), \mathrm{trans}(\rho(\lambda))\big)$. In particular, we have a homeomorphism $\widetilde{X}^{\mathrm{ell}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(T^2) \xrightarrow{\mathrm{trans}} H^1(T^2;\mathbb{R})$.

The preimages of the parabolic points form a lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, and the covering group of the previous paragraph corresponds to the subgroup $(2\mathbb{Z})^2 \subset \mathbb{Z}^2$. There is a short exact sequence

$$1 \to \mathbb{Z} \to Z(\widetilde{\mathrm{SL}_2\mathbb{R}}) \to Z(\mathrm{SL}_2\mathbb{R}) \to 1.$$

The extension is nontrivial with $Z(\widetilde{\mathrm{SL}_2\mathbb{R}}) = \mathbb{Z}$. As before, the set of central characters

$$H^1(\mathbb{Z}^2; Z(\widetilde{\mathrm{SL}_2\mathbb{R}})) \cong \mathbb{Z}^2$$

acts on $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\mathbb{Z}^2)$; this is the action of the full lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ by translation.

14.14 Example Now suppose $\Gamma = \pi_1(M_K)$. Any nonparabolic point of $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(K)$ has two distinct preimages in $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(K)$, as does any irreducible parabolic by Lemma 4.2. However, the parabolic reducible characters have only a single preimage. We denote the covering action of $\mathbb{Z}/2$ on $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(K)$ by a as well; it satisfies $i^* \circ a = a \circ i^*$, so the image of $i^*(\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(K))$ in $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)$ is determined by the image of $i^*(X_{\mathrm{SL}_2\mathbb{R}}(K))$ in $X_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)$.

Now we consider lifts to $\widetilde{\mathrm{SL}_2\mathbb{R}}$. We have $H^2(\Gamma) = H^2(K) = 0$, so every representation $\Gamma \to \mathrm{SL}_2\mathbb{R}$ lifts to $\widetilde{\mathrm{SL}_2\mathbb{R}}$, and $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\Gamma)$ is a covering space of $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$ with deck group $H^1(K) = \mathbb{Z}$ by Theorem 4.3. We are primarily interested in the elliptic part of $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$. The closure of $i^*(\widetilde{X}^{\mathrm{ell}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K))$ in $\widetilde{X}^{\mathrm{ell}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\partial M_K) \cong H^1(\partial M_K;\mathbb{R})$ is the *translation extension locus* $\mathrm{EL}_{\widetilde{G}}(M_K) \subset H^1(\partial M_K;\mathbb{Z})$ studied in [Culler and Dunfield 2018]. (Whenever M_K is real representation small, one simply has $i^*(\widetilde{X}^{\mathrm{ell}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)) = \mathrm{EL}_{\widetilde{G}}(M_K)$.) Figure 8 shows the translation extension locus of $T(2,3)$; it is entirely determined by $X^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(\Gamma)$, since the latter space is connected.

Lemma 6.1 of [Culler and Dunfield 2018] shows that the maps a and b, which are automorphisms of $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(K)$, lift to automorphisms $\tilde{a}$, $\tilde{b}$ of $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$. The maps $\tilde{a}$, $\tilde{b}$ generate an action of the infinite dihedral group D_∞ on $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$. Their action on $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(T^2)$ is given by $\tilde{a}(\mu^*, \lambda^*) = (-\mu^*, -\lambda^*)$ and $\tilde{b}(\mu^*, \lambda^*) = (\mu^* + 1, \lambda^*)$. This D_∞ action is evident in Figure 8. For any knot, the translation extension locus is a finite union of analytic arcs and isolated points, whose quotient under D_∞ is a finite graph [ibid., Theorem 4.3].

14.15 The extended Lin invariant

We now use the translation extension locus to define a refinement of the invariant $h(K)$ that also encodes the values of λ^* for lifts of $\chi \in X^2_{\mathrm{SL}_2\mathbb{R}}(K)$ to $\widetilde{\mathrm{SL}_2\mathbb{R}}$. For this, we need:

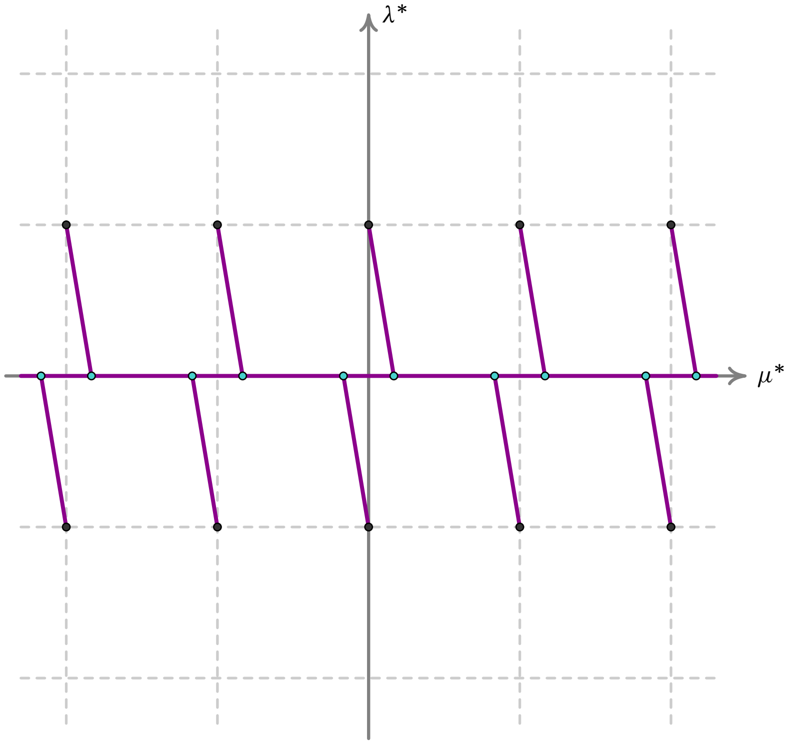

Figure 8: The translation extension locus $i^*(\widetilde{X}^{\mathrm{ell}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K))$ for the positive trefoil. The related set $i^*(\widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(K))$ in the torus $\widetilde{X}^{\mathrm{ell}}_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)$ can be visualized by taking the four squares touching the origin and identifying the sides of the larger square they form by the action of $(2\mathbb{Z})^2$.

14.16 Lemma *A nonparabolic* $\chi \in X^{\mathrm{ell,irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ *has a unique preimage* $\widetilde{\chi} \in \widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ *satisfying* $\mu^*(\widetilde{\chi}) = \overline{\mu}^*(\chi)$*, where the latter is normalized to be in* $[0,1]$*. In contrast, if* χ *is parabolic, there are two such preimages* $\widetilde{\chi}_\pm$*, which satisfy* $\lambda^*(\widetilde{\chi}_-) = -\lambda^*(\widetilde{\chi}_+)$*.*

Proof Throughout, you may find the illustration for the trefoil in Figure 8 helpful. By Lemma 4.2, there are two preimages $\chi_\pm$ of χ in $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(K)$. Using $-1 \le \overline{\mu}^*, \overline{\lambda}^* \le 1$ as our fundamental domain for $\widetilde{X}_{\mathrm{SL}_2\mathbb{R}}(T^2)$, we have $\chi_- = a(\chi_+)$ and $\overline{\mu}^*(\chi_-) = -\overline{\mu}^*(\chi_+)$.

If χ is elliptic, take χ_+ to be the one with $\overline{\mu}^*$ in $(0,1)$. Then $\overline{\mu}^*(\chi_-)$ is in $(-1,0)$. Now the lifts of $\widetilde{\chi}_\pm$ to $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ have $\mu^*(\widetilde{\chi}_\pm)$ in $\overline{\mu}^*(\chi_\pm) + 2\mathbb{Z}$. So there is a unique lift of χ with $\mu^* = \overline{\mu}^*(\chi)$, namely a lift of χ_+, proving the lemma in this case.

Suppose next that χ is parabolic with $\chi(\mu) = 2$. Then there is one lift $\widetilde{\chi}_+$ of χ_+ to $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ where $\mu^*(\widetilde{\chi}_+) = 2k$ for each $k \in \mathbb{Z}$. As the same is true for χ_-, we see there are exactly two lifts $\widetilde{\chi}_+$ and $\widetilde{\chi}_-$ of χ to $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ with $\mu^* = 0$. We have $\tilde{a} \cdot \widetilde{\chi}_+ = \widetilde{\chi}_-$, so $\lambda^*(\widetilde{\chi}_-) = -\lambda^*(\widetilde{\chi}_+)$. The final case of a parabolic with $\chi(\mu) = -2$ is the same, except now we have $\mu^*(\widetilde{\chi}_\pm) = 1$. □

Suppose X is a component of $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$. By Lemma 14.2, X is composed entirely of parabolic representations. It follows that $\overline{\mu}^*$ and $\overline{\lambda}^*$ are integral (and hence constant) on X; the same is true for μ^* and λ^* on each component of the preimage of X in $\widetilde{X}_{\widetilde{\mathrm{SL_2\mathbb{R}}}}(K)$. Applying Lemma 14.16, we see that X has two preimages $\widetilde{X}_\pm$ in $\widetilde{X}_{\widetilde{\mathrm{SL_2\mathbb{R}}}}(K)$ which satisfy $\mu^*|_{\widetilde{X}_\pm} = \overline{\mu}^*|_X$, and that $\lambda^*|_{\widetilde{X}_+} = -\lambda^*|_{\widetilde{X}_-}$.

14.17 Definition If K is real representation small, the *extended Lin invariant* of K is

$$\tilde{h}(K) = \sum_X n_X \cdot (t^{\lambda^*(\widetilde{X}_+)} + t^{\lambda^*(\widetilde{X}_-)}) \in \mathbb{Z}[t, t^{-1}],$$

where the sum runs over components of $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$, and n_X is well defined by Theorem 13.4.

We will see in Example 15.6 that $\tilde{h}(T(2,3)) = t + t^{-1}$, which is plausible from Figure 8. Informally, the coefficient of t^i in $\tilde{h}(K)$ is a signed count of arcs exiting the translation extension locus at the parabolic of height i. The sign is determined both by the orientation of the arc and which side of the line $\mu^* = 0$ it exits from.

14.18 Lemma *For a real representation small knot K, the invariant $\tilde{h}$ has the following properties*:

1. $\tilde{h}(K)|_{t=1} = 2h(K)$.
2. *$\tilde{h}(K)$ is invariant under the involution of $\mathbb{Z}[t^{\pm 1}]$ sending t to t^{-1}.*
3. *Define $\deg \tilde{h}(K)$ to be the maximum of $|e|$ where t^e appears in $\tilde{h}(K)$. Then $\deg \tilde{h}(K) \le 2g(K) - 1$, where $g(K)$ is the Seifert genus of K.*
4. *If K is fibered and hyperbolic, the inequality in part* (3) *is strict.*

Proof Item (1) is immediate from the definition, and (2) follows from the fact that $\lambda^*(\widetilde{\chi}_-) = -\lambda^*(\widetilde{\chi}_+)$ by Lemma 14.16. Next, (3) follows from the Milnor–Wood inequality in the form of [Culler and Dunfield 2018, Proposition 6.5]. Finally, for (4), we use the argument of [Calegari 2006, Section 3.5] as follows: If $\rho\colon \pi_1 M_K \to \mathrm{PSL}_2\mathbb{R}$ corresponds to the putative t^{2g-1} term of $\tilde{h}(K)$, then its restriction to the fiber F must lie in the component of $X_{\mathrm{PSL}_2\mathbb{R}}(F)$ corresponding to the Teichmüller space of F. Hence, the bundle monodromy φ leaves invariant that hyperbolic structure on F. This forces φ to have finite order in the mapping class group of F, which contradicts that M_K is hyperbolic. □

15 Properties of the invariant

In this section, we collect some basic properties of $h(K)$. These include its behavior under mirroring, its relationships with the original Lin invariant and the Levine–Tristram signature, and its parity. The relationship with the signature allows us to compute $h(K)$ for the first time in Example 15.6.

15.1 Mirrors

The invariant $h(K)$ behaves nicely under taking the mirror image:

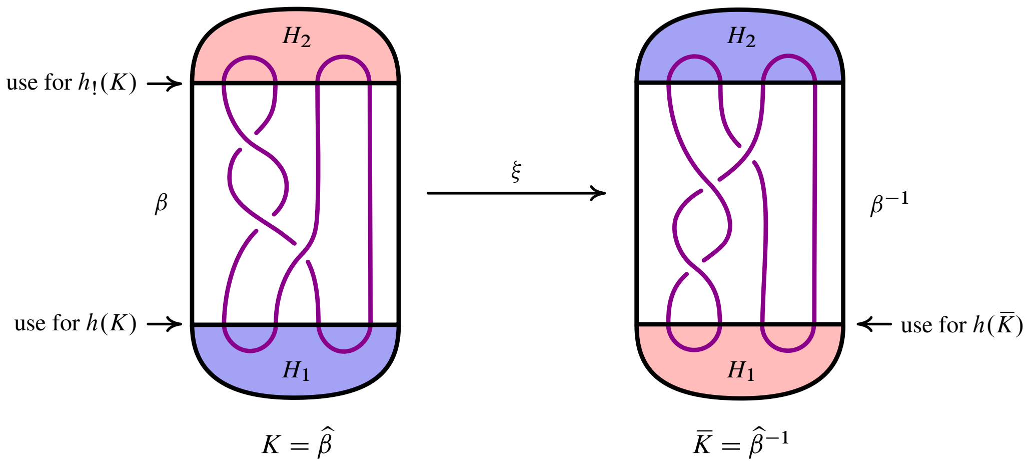

Figure 9: The homeomorphism $\xi\colon M_K \to M_{\overline{K}}$ is reflection in the plane of the page followed by π-rotation about the horizontal axis. This takes the copy of S_{2m} used to compute $h_!(K)$ (left) to the one used to compute $h(\overline{K})$ (right).

15.2 Proposition *If K is real representation small, then so is its mirror $\overline{K}$ and $h(\overline{K}) = -h(K)$. Moreover, $\tilde{h}(\overline{K}) = -\tilde{h}(K)$.*

Proof First, the knot $\overline{K}$ is real representation small as there is an isomorphism from $\pi_1(M_{\overline{K}})$ to $\pi_1(M_K)$ sending meridians to meridians. Second, note that if $K = \widehat{\beta}$, then $\overline{K} = \widehat{\beta}^{-1}$, and so for $c \in [-2,2] \setminus D_K$ we have

$$\begin{aligned} h(\overline{K}) &= (-1)^{|\beta^{-1}|}\langle \mathscr{L}^c, (\beta^{-1})^*\mathscr{L}^c\rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})} \\ &= (-1)^{|\beta|}\langle \beta^*\mathscr{L}^c, \mathscr{L}^c\rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})} && \text{(by functoriality and Lemma 12.11)} \\ &= -(-1)^{|\beta|}\langle \mathscr{L}^c, \beta^*\mathscr{L}^c\rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})} && \text{(since } \dim \mathscr{L}^c = 2m-3 \text{ is odd)} \\ &= -h(K). \end{aligned}$$

The calculation above is underlaid by a particular isomorphism from $\pi_1(M_{\overline{K}})$ to $\pi_1(M_K)$. While we have always identified S_{2m} with ∂H_1 as in Figure 5, we could have instead viewed it as ∂H_2, and defined

$$\mathscr{X}^c_!(\beta) = ((\beta^{-1})^*\mathscr{L}^c) \cap \mathscr{L}^c \quad \text{and} \quad h_!(K) = (-1)^{|\beta|}\langle (\beta^{-1})^*\mathscr{L}^c, \mathscr{L}^c\rangle_{\mathscr{X}^c_{\mathrm{bal}}(S_{2m})}.$$

This is illustrated in Figure 9, left. As subsets of $\mathscr{X}^c_{\mathrm{bal}}(S_{2m})$, note that $\beta^*(\mathscr{X}^c_!(\beta)) = \mathscr{X}^c(\beta)$. Moreover, the previous calculation shows $h_!(K) = h(K)$. Now consider the homeomorphism ξ from M_K to $M_{\overline{K}}$ shown in Figure 9. This induces an isomorphism $\xi_*\colon \pi_1(M_K) \to \pi_1(M_{\overline{K}})$, which takes s_i in ∂H_2 for M_K to s_i in ∂H_1 for $M_{\overline{K}}$. Note here that the orientations of the s_i are preserved, not reversed, so if we take s_1 as the meridian in both cases, we get $\xi_*(\lambda) = \overline{\lambda}^{-1}$. This shows that $\mathscr{X}^c_!(\beta) = \mathscr{X}^c(\beta^{-1})$ as subsets of $\mathscr{X}^c_{\mathrm{bal}}(S_{2m})$, as in Proposition 12.2. We then have $h_!(K) = -h(\overline{K})$ since both are computing the intersection numbers of $\mathscr{L}^c$ and $(\beta^{-1})^*\mathscr{L}^c$, just in the opposite order. To extend this to $\tilde{h}_!(K) = -\tilde{h}(\overline{K})$, given a connected component X of $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$, we have to show $\overline{X} = \xi^*(X)$ contributes the same to $-\tilde{h}(\overline{K})$ as

X does to $\tilde{h}_!(K)$. Now $X = \overline{X}$ as subsets of $\mathscr{X}^2_{\mathrm{bal}}(S_{2m})$, so the contributions of the local intersection numbers match. Now consider the two preimages $\widetilde{X}_\pm$ of X in $\widetilde{X}_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$. We want to compare $\lambda^*(\widetilde{X}_\pm)$ and $\overline{\lambda}^*(\widetilde{X}_\pm)$. Since $\lambda^*(\widetilde{X}_+) = -\lambda^*(\widetilde{X}_-)$ and $\overline{\lambda}^*(\widetilde{X}_+) = -\overline{\lambda}^*(\widetilde{X}_-)$ by Lemma 14.16, as $\xi_*(\lambda) = \overline{\lambda}^{-1}$ we see that the corresponding terms of $-\tilde{h}(\overline{K})$ and $\tilde{h}_!(K)$ match, completing the proof. □

15.3 Relation with the signature

For $\alpha \in (0, \pi)$ with $\Delta_K(e^{2i\alpha}) \neq 0$, we let $h'_\alpha(K)$ be the Lin invariant as defined in [Heusener 2003], where it is denoted by $h^{(\alpha)}(K)$. We will show:

15.4 Proposition *When K is real representation small and $c \notin D_K$, then $h^c_{\mathrm{SU}_2}(K) = h'_\alpha(K)$, where $c = 2\cos\alpha$.*

This has the following important consequence:

15.5 Corollary *For $c \in [-2, 2] \setminus D_K$, one has $h^c_{\mathrm{SU}_2}(K) = -\frac{1}{2}\sigma_K(e^{2i\alpha})$, where σ_K is the Levine–Tristram signature function.*

This follows from the corresponding statement for h'_α, which was proved by Herald [1997b] using gauge theory, and by Heusener and Kroll [1998] following Lin's original proof [1992] for $\alpha = \frac{\pi}{2}$; see also [Heusener 2003, Section 5]. Here, our convention for the signature is that positive knots have negative signature. Note that [Lin 1992; Heusener and Kroll 1998] use the opposite convention, so their statements have no minus sign.

15.6 Example Let K be the positive trefoil, so $\sigma_K(e^{2i\alpha}) = -2$ for $\alpha \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and is 0 for $\alpha \in \left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 1\right]$. We saw in Section 14.7 that $h^c_{\mathrm{SL}_2\mathbb{R}}(K) = 0$ for $\alpha \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, so $h(K) = 0 + 1 = 1$. Applying the relation with the signature again, we see that $h^c_{\mathrm{SL}_2\mathbb{R}}(K) = 1$ for $\alpha \in \left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 1\right]$, as we expected from looking at Figure 7. We summarize the full picture in Figure 10.

Proof of Proposition 15.4 For $\beta \in B_{2m}$, recall that

$$h^c_{\mathrm{SU}_2}(\hat{\beta}) = (-1)^{|\beta|}\langle L_1, L_2\rangle_X,$$

where $X = \mathscr{X}^c_+(S_{2m}) = X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(S_{2m})$, $L_1 = \mathscr{L}^c_+$ and $L_2 = \beta^*(L_1)$. Heusener's $h'_\alpha(\hat{\beta})$ is defined as the intersection number of the same objects but with possibly different orientations and global sign. If X', L'_1 and L'_2 denote these manifolds with Heusener's orientation, one has

$$h'_\alpha(\hat{\beta}) = (-1)^m \langle L'_1, L'_2\rangle_{X'}.$$

Thus to complete the proof, we must compare our orientations with Heusener's. We have already observed in Remark 11.6 that $[X] = -[X']$. The relation between $[L_k]$ and $[L'_k]$ is more complicated, and depends on β, as we now detail.

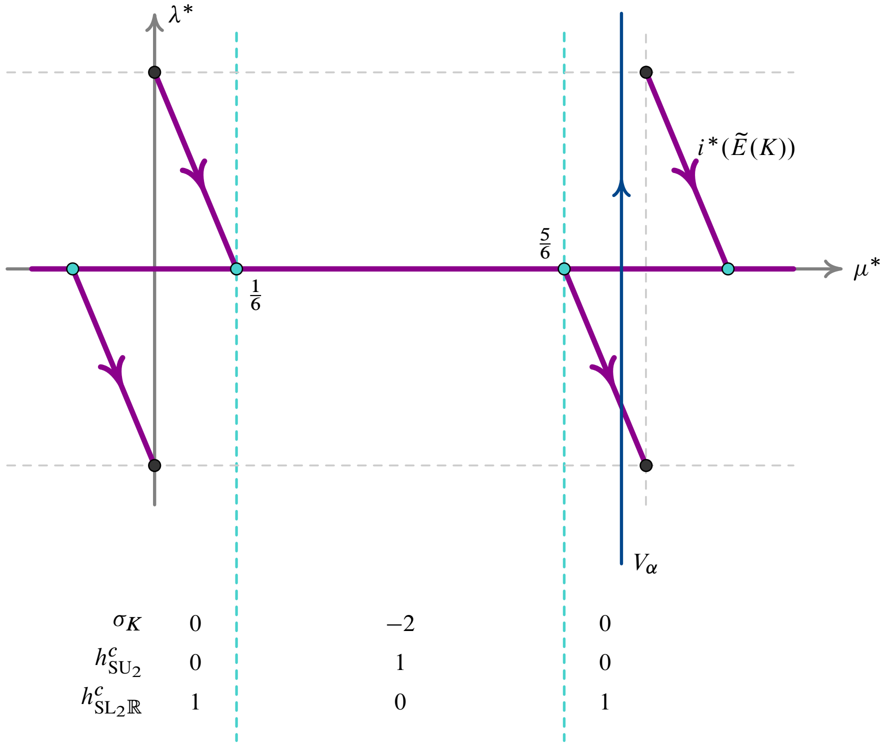

Figure 10: For the positive trefoil K, this figure shows a portion of $i^*(\widetilde{E}(K))$ with its irreducible characters oriented according to Section 14.8; in particular $\langle i^*(\widetilde{E}(K)), V_\alpha\rangle = h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ for $c = 2\cos\alpha$. The orientation is derived from Theorem 12.22 and Corollary 15.5, which allow us to compute $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ and $h^c_{\mathrm{SU}_2}(K)$ from $\sigma_K(e^{2i\alpha})$. Note that this orientation is preserved by the translations in the D_∞ action but reversed for rotations, as is consistent with Lemma 14.10.

The orientations on $[L_k]$ and $[L'_k]$ are defined by choosing a set of generators T_k for $\pi_1(H_k)$ and using T_k to identify $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(H_k, T_k)$ with $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_m, T)$. (Our orientation on $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_m, T)$ agrees with Heusener's, but this is a bit of a moot point. The orientation on $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_m, T)$ appears twice (once for L_1 and once for L_2), so we would get the same answer even if the two orientations on $X^{c,\mathrm{irr}}_{\mathrm{SU}_2}(F_m, T)$ were different.)

While our choice of generators for $\pi_1(H_1)$ is independent of β, Heusener orients the knot $\widehat{\beta}$ and uses generators of $\pi_1(H_1)$ that are compatible with that orientation. For related reasons, while we always have $[L_2] = (\beta^*)_*[L_1]$, it turns out that $[L'_2]$ is $(\beta^*)_*[L'_1]$ or $-(\beta^*)_*[L'_1]$, depending on β.

To make the comparison easy, we will work with particularly nice β. Specifically, given K, we claim there is a β with an orientation on $\widehat{\beta}$ such that:

(1) Every overbridge is oriented right to left, and every underbridge is oriented left to right.

(2) The corresponding permutation $\overline{\beta}$ fixes all odd numbers and acts on the evens as the *inverse* of the cycle $(2\ 4\ 6\ 2m)$.

Starting from any β and any orientation on $\widehat{\beta}$, part (1) is easy to ensure by adding a twist to reverse each bridge with the wrong orientation. To arrange (2), begin by walking along $\widehat{\beta}$ in the prescribed orientation. As you go, number the overbridges $1, 2, \ldots, m$ and do the same for the underbridges. Now slide the overbridges in front or behind each other to get a new braid β' such that the overbridge numbers appear in order from left to right; repeat this process for the underbridges. You can now check that (2) must hold, so from now on we assume β has both these properties.

Condition (1) means that in [Heusener 2003, Section 4] we have all $\epsilon_\ell^{(i)} = +1$, and in this situation the construction of [Heusener 2003] matches ours on the nose. In particular, $[L_1] = [L_1']$ and $[L_2] = [L_2']$. Since, as noted, $[X'] = -[X]$, to complete the proof we simply need to show that $(-1)^{m+1} = (-1)^{|\beta|}$. Since each braid generator corresponds to a transposition in the symmetric group, the latter is the sign of the permutation $\overline{\beta}$. As $\overline{\beta}$ is an m-cycle, it has sign $(-1)^{m+1}$, as needed. □

15.7 Geometry at simple roots

At a simple root of $\Delta_K(t)$ on the unit circle, the equivariant signature σ_K must jump by ± 2. The sign of this jump is related to the local geometry of the character variety by part (3) of the following lemma, which we thank Chris Herald for explaining to us:

15.8 Lemma *Suppose a knot K has a simple root $e^{i2\omega_0}$ of $\Delta_K(t)$ for $0 < \omega_0 < \pi$. Then there is a smooth arc $\gamma\colon [0,1] \to \widetilde{E}(K)$ such that*

(1) *$i^*(\gamma(1)) = (\omega_0/\pi, 0)$ with $\gamma(1)$ a lift of $\chi^c \in X^{\mathrm{red}}_{\mathrm{SL}_2\mathbb{R}}(K)$, where $c = 2\cos\omega_0$.*

(2) *$i^* \circ \gamma\colon [0,1] \to \widetilde{V}(K)$ is a smooth embedding.*

If in addition the derivative $(\mathrm{tr}_\lambda \circ \gamma)'(0)$ is nonzero, then we have:

(3) *If $\sigma_K(e^{i2\omega})$ jumps by -2 as ω increases past ω_0, then $\gamma\big([0,1)\big)$ lies entirely above the horizontal axis. If instead $\sigma_K(e^{i2\omega})$ jumps by $+2$, then $\gamma\big([0,1)\big)$ lies entirely below the horizontal axis.*

15.9 Remark We do not know of an example of a simple root of $\Delta_K(t)$ on the unit circle where $(\mathrm{tr}_\lambda \circ \gamma)'(0) = 0$. Indeed, it seems likely this never happens, and so (3) always holds. (For multiple roots, it does arise; see the upper-right example in [Culler and Dunfield 2018, Figure 10].) The question is subtle in that for other $\alpha \in \pi_1(\partial M_K)$ the derivative $(\mathrm{tr}_\alpha \circ \gamma)'(0)$ can vanish: for the trefoil, $\mathrm{tr}_\alpha \circ \gamma \equiv 2$ for $\alpha = 6\mu + \lambda$, as noted in Section 14.7.

Proof of Lemma 15.8 The idea follows [Herald and Zhang 2019], which adds [Herald 1997b; 1997a] to the results of [Heusener et al. 2001] (see also [Heusener and Porti 2005]) to strengthen [Culler and Dunfield 2018, Lemma 7.3]. To start, from [Heusener et al. 2001, Corollary 1.3], one has a smooth arc $\overline{\gamma}\colon [-1,1] \to X(K)$ where $\overline{\gamma}(0) = \chi^c$ with $\overline{\gamma}(t)$ in $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ for $t < 0$ and in $X^{\mathrm{irr}}_{\mathrm{SU}_2}(K)$ for $t > 0$. As argued in [Herald and Zhang 2019, page 2818], their Corollary 6 allows us to assume that $\mathrm{tr}_\lambda\, \overline{\gamma}(t) = 2$ only for $t = 0$. Additionally, [Culler and Dunfield 2018, Lemma 7.3(4)] — or, more accurately, the fact that

$\iota^*\colon H^1(M_K;\mathfrak{sl}_2(\mathbb{C})_{\rho^+}) \to H^1(\partial M_K;\mathfrak{sl}_2(\mathbb{C})_{\rho^+})$ is injective, which follows from [ibid., equation 7.4] — implies there is some $\alpha \in \pi_1(\partial M_K)$ such that, after possibly restricting the domain of $\overline{\gamma}$, $\mathrm{tr}_\alpha \circ \overline{\gamma}$ is a smooth embedding.

Lifting to $\widetilde{E}(K)$ now gives all the claims except for the part of (3) about which side of the horizontal axis $\gamma\big([0,1)\big)$ lies on. This follows from the corresponding behavior on the SU_2 side, which is understood by [Herald 1997b, Proposition 4.2; 1997a, Corollary 3]; see the remark immediately after the former. The hypothesis on $(\mathrm{tr}_\lambda \circ \gamma)'(0)$ ensures that if $\overline{\gamma}$ lies above the reducible line on the SU_2 side, it lies below on the $\mathrm{SL}_2\mathbb{R}$ side, and vice versa. Rather than trace through the various conventions, we can justify our claimed "sign rule" that a negative jump in $\sigma_K(e^{i2\omega})$ means γ is above the axis by looking at the positive trefoil in Figure 10. □

15.10 Parity of *h*

The map $b\colon X^c(K) \to X^{-c}(K)$ introduced in Section 14.9 preserves $X^0(K)$ setwise. The fixed points of b on $X^{0,\mathrm{irr}}(K)$ were classified in [Nagasato and Yamaguchi 2012, Corollary 1]: they are all in $X^{0,\mathrm{irr}}_{\mathrm{SU}_2}(K)$ and consist of characters of binary–dihedral representations. We now use this to show:

15.11 Proposition *If K is a small knot, then* $h(K) \equiv \frac{1}{2}\sigma(K) \equiv \frac{1}{2}(\det K - 1) \bmod 2$.

Proof By Lemma 3.14, as K is small, the set $X^{0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is finite. As noted, b has no fixed points on $X^{0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$, so that set can be partitioned into pairs $\{\chi, b(\chi)\}$ with $\chi \neq b(\chi)$. Lemma 14.10 tells us that the local intersection numbers n_χ and $n_{b(\chi)}$ are equal. It follows that $h^0_{\mathrm{SL}_2\mathbb{R}}(K)$ is even, and hence that $h(K) = h^0(K)$ is congruent to $h^0_{\mathrm{SU}_2}(K) = -\frac{1}{2}\sigma(K)$ modulo 2, establishing $h(K) \equiv \frac{1}{2}\sigma(K) \bmod 2$.

Continuing this idea, only the binary–dihedral characters can contribute to $h^0_{\mathrm{SU}_2}(K)$, and there exactly are $\frac{1}{2}(\det K - 1)$ such [Klassen 1991]. If each binary–dihedral character contributes ± 1 to $h^0_{\mathrm{SU}_2}(K)$, we would get that $h^0_{\mathrm{SU}_2}(K) \equiv \frac{1}{2}(\det K - 1) \bmod 2$, implying our second claim. We do not know if this holds, but for any knot one has $\frac{1}{2}\sigma(K) \equiv \frac{1}{2}(\det K - 1) \bmod 2$ by [Murasugi 1965, Theorem 5.6], completing the proof. □

Next, we relate the parity of $h(K)$ to the $\mathrm{SL}_2\mathbb{C}$ character variety, leading to Corollary 15.13, which shows there are ideal points that are limits of $\mathrm{SL}_2\mathbb{R}$ characters when $\frac{1}{2}\sigma(K)$ is odd. Fix a braid β with $\widehat{\beta} = K$. As in Section 12, this determines a splitting $M_K = H_1 \cup_{S_{2m}} H_2$. We define $X^{\mathrm{irr}}_{\mathbb{C}}(\beta)$ to be the intersection of the two smooth varieties $L_{j,\mathbb{C}} = i_j^*(X^{\mathrm{irr}}_{\mathbb{C}}(H_j, T))$ for $j = 1, 2$ inside the smooth variety $X^{\mathrm{irr}}_{\mathbb{C}}(S_{2m}, S)$. Note that this intersection may be nonreduced, ie the components of $X^{\mathrm{irr}}_{\mathbb{C}}(\beta)$ may have multiplicities; if it is, we view it as a nonreduced scheme rather than passing to the reduction. (As reduced schemes, $X^{\mathrm{irr}}_{\mathbb{C}}(\beta) = X^{\mathrm{irr}}_{\mathbb{C}}(K)$, but it is not clear that the multiplicity of each component is an invariant of K.)

When K is small, every component C of $X^{\mathrm{irr}}_{\mathbb{C}}(K)$ is a complex curve and $\mathrm{tr}_\mu\colon C \to P^1(\mathbb{C})$ is nonconstant (see Lemma 3.14). We define $d(K, \beta)$ to be the (total) degree of the map $\mathrm{tr}_\mu\colon X^{\mathrm{irr}}_{\mathbb{C}}(\beta) \to P^1(\mathbb{C})$; this is a weighted sum of all the Culler–Shalen seminorms of μ [Boyer and Zhang 1998], which is related in turn to the A-polynomial [Boyer and Zhang 2001, Section 8]. We conjecture that $d(K,\beta)$ depends only on K but do not prove this here. We next show:

15.12 Proposition *If K is a small knot, then $d(K,\beta) \equiv h(K) \bmod 2$.*

Proof For $c \in \mathbb{C}$, let $L^{c,\mathrm{irr}}_{j,\mathbb{C}} = L^{\mathrm{irr}}_{j,\mathbb{C}} \cap \mathrm{tr}_\mu^{-1}(c)$. (This should also be the scheme-theoretic intersection, but $\mathrm{tr}_\mu \colon L^{\mathrm{irr}}_{j,\mathbb{C}} \to \mathbb{C}$ is a submersion by Lemma 3.6, so it is the same as the setwise intersection.) Then

$$d(K,\beta) = \langle L^{c,\mathrm{irr}}_{1,\mathbb{C}}, L^{c,\mathrm{irr}}_{2,\mathbb{C}} \rangle_{X^{c,\mathrm{irr}}_{\mathbb{C}}(S_{2n})}.$$

Now, if $A_\mathbb{C}, A'_\mathbb{C} \subset B_\mathbb{C}$ are the $\mathbb{C}$ points of algebraic sets defined over $\mathbb{R}$, by [Fulton 1984, Chapter 13] their real points $A_\mathbb{R}$, $A'_\mathbb{R}$ and $B_\mathbb{R}$ satisfy

$$\langle A_\mathbb{C}, A'_\mathbb{C} \rangle_{B_\mathbb{C}} \equiv \langle A_\mathbb{R}, A'_\mathbb{R} \rangle_{B_\mathbb{R}} \bmod 2.$$

In our case, the real parts of $L^{c,\mathrm{irr}}_{j,\mathbb{C}}$ and $X^{c,\mathrm{irr}}_{\mathbb{C}}(S_{2n})$ are the spaces we have been referring to as $L^{c,\mathrm{irr}}_j$ and $X^{c,\mathrm{irr}}(S_{2n})$. Hence,

$$d(K,\beta) \equiv \langle L^{c,\mathrm{irr}}_1, L^{c,\mathrm{irr}}_2 \rangle_{X^{c,\mathrm{irr}}(S_{2n})} \equiv h(K) \bmod 2. \qquad \square$$

Recall that the affine algebraic set $X^{\mathrm{irr}}_\mathbb{R}(K)$ and the map tr_μ can be extended to a projective variety $\widehat{X}^{\mathrm{irr}}_\mathbb{R}(K)$ and a map $\mathrm{tr}_\mu \colon \widehat{X}^{\mathrm{irr}}_\mathbb{R}(K) \to P^1(\mathbb{R})$. The points in $\mathrm{tr}_\mu^{-1}(\infty)$ are called *real ideal points*. We just showed that the mod 2 degree of this map is given by $h(K)$. If the mod 2 degree is nonzero, $\mathrm{tr}_\mu^{-1}(\infty)$ must be nonempty. Hence, we deduce:

15.13 Corollary *If K is a small knot with $\sigma(K) \equiv 2 \bmod 4$, or equivalently $\det K \equiv 3 \bmod 4$, then $\widehat{X}^{\mathrm{irr}}_\mathbb{R}(K)$ contains a real ideal point.*

Unsurprisingly, there are also knots with real ideal points where $\sigma(K) \not\equiv 2 \bmod 4$, for example the figure-eight knot, which has $\sigma(K) = 0$. Moreover, real ideal points are extremely common, even outside the context of exteriors of knots in S^3, raising:

15.14 Question *Does every small knot in S^3 have a real ideal point?*

An initial search using [Culler 2006] suggests that the answer might well be yes.

16 Applications to left orderings

16.1 Left orderings

Let G be a nontrivial group. We say that G is *left-orderable* (or, for short, that G is LO), when there is a total order on G which satisfies $gx > gy$ whenever $x > y$. (By convention, the trivial group is not left-orderable.) The study of left-orderable 3-manifold groups dates back to [Boyer et al. 2005] and was given fresh impetus by the L-space conjecture.

16.2 L-space conjecture [Boyer et al. 2013; Juhász 2015] *If Y is a closed prime orientable 3-manifold, then the following are equivalent*:

1. Y *is not a Heegaard Floer L-space*, *ie* $\widehat{HF}_{\mathrm{red}}(Y) \neq 0$.
2. $\pi_1(Y)$ *is left-orderable.*
3. Y *admits a coorientable taut foliation.*

For brevity, when $\pi_1(Y)$ is LO we will call Y itself LO.

In this section, we use the techniques of this paper to prove many Y are LO, using the following theorem:

16.3 Theorem [Boyer et al. 2005] *If Y is a prime 3-manifold with a nontrivial homomorphism $\pi_1(Y) \to \widetilde{\mathrm{SL}_2\mathbb{R}}$, then $\pi_1(Y)$ acts faithfully on $\mathbb{R}$, and hence Y is LO.*

In particular, if $\rho\colon \pi_1(Y) \to \mathrm{SL}_2\mathbb{R}$ is a nontrivial homomorphism whose Euler class $e(\rho) \in H^2(Y;\mathbb{Z})$ is zero, then ρ lifts to $\widetilde{\rho}\colon \pi_1(Y) \to \widetilde{\mathrm{SL}_2\mathbb{R}}$, where Theorem 16.3 applies. We will also consider the target group $\mathrm{PSL}_2\mathbb{R}$, where again any $\widehat{\rho}\colon \pi_1(Y) \to \mathrm{PSL}_2\mathbb{R}$ has an Euler class $e(\widehat{\rho}) \in H^2(Y;\mathbb{Z})$ that obstructs lifting $\widehat{\rho}$ to $\widetilde{\mathrm{SL}_2\mathbb{R}}$.

We begin with results showing cyclic branched covers are LO (Theorem 16.7). We then compute $h(K)$ for alternating knots (Section 16.10), which in turn strengthens the results on branched covers (Remark 16.9). The rest of the section is devoted to showing certain Dehn surgeries on K are LO; key results there include Theorems 16.21, 16.22 and 16.27.

16.4 Branched covers

Let $\Sigma_n(K)$ be the n-fold cyclic branched cover of a prime knot K. The manifold $\Sigma_n(K)$ is obtained by Dehn filling the n-fold cyclic cover $\widetilde{M}_{K,n}$ of M_K along the curve $\widetilde{\mu}_n$ which is the n-fold cyclic cover of μ. Moreover, $\Sigma_n(K)$ is the n-fold cyclic orbifold cover of the orbifold (S^3, K_n), where the knot K is labeled by $\mathbb{Z}/n$. The orbifold fundamental group $\pi_1(S^3, K_n)$ is isomorphic to $\pi_1(M_K)/\langle \mu^n \rangle$, and $\pi_1(\Sigma_n(K))$ is the kernel of the homomorphism $\pi_1(S^3, K_n) \to \mathbb{Z}/n$ where $\mu \mapsto 1$.

16.5 Remark It follows from the orbifold sphere theorem (see eg [Maillot 2003, Theorem 10.2]) that if $\Sigma_n(K)$ is reducible, then the orbifold (S^3, K_n) is reducible. Consequently, when K is prime, so are all $\Sigma_n(K)$.

Now suppose $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ is a representation with $\mathrm{tr}(\rho(\mu)) = 2\cos(k\pi/n)$ for some integer k with $0 < k < n$. Then ρ is elliptic and satisfies $\rho(\mu^n) = \pm I$, so ρ descends to a representation $\widehat{\rho}\colon \pi_1(S^3, K_n) \to \mathrm{PSL}_2\mathbb{R}$. We let $\widehat{\rho}_n$ be the restriction of $\widehat{\rho}$ to $\pi_1(\Sigma_n(K))$. The next result has been much used to show branched covers are LO:

16.6 Lemma [Hu 2015, Theorem 3.3] *If $\widehat{\rho}$ is irreducible*, *then $\widehat{\rho}_n$ is nontrivial and $e(\widehat{\rho}_n) = 0$.*

Let $A_K = \{\alpha \in [0,\pi] \mid 2\cos\alpha \in D_K\}$ and consider the corresponding partition

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < \pi \quad \text{of } [0,\pi].$$

The signature function $\sigma_K(e^{2i\alpha})$ is constant on the interiors of the intervals of this partition. Define w_K to be the minimum length of any of these intervals, including $[0,\alpha_1]$ and $[\alpha_k,\pi]$. Our main result on branched covers is:

16.7 Theorem *If a prime knot K is real representation small and $\sigma_K(\omega)$ is not identically 0, then $\Sigma_n(K)$ is LO for all $n > \pi/w_K$.*

Proof Taking $c = 2\cos\alpha$, we can view $h^c_{\mathrm{SL}_2\mathbb{R}}(K)$ as a function of α in $[0,\pi]\setminus A_K$ rather than of c in $[-2,2]\setminus D_K$; in the new setting, we will write $h^\alpha_{\mathrm{SL}_2\mathbb{R}}(K)$ and use other similar notation. By Theorem 12.22 and Corollary 15.5, we then have

(16.8) $$h^\alpha_{\mathrm{SL}_2\mathbb{R}}(K) = h(K) - h^\alpha_{\mathrm{SU}_2}(K) = h(K) + \tfrac{1}{2}\sigma_K(e^{2i\alpha}).$$

As σ_K is nonconstant, no matter what $h(K)$ is there will be an interval I in our partition where $h^\alpha_{\mathrm{SL}_2\mathbb{R}}(K)$ is nonzero on the interior of I.

As $n > \pi/w_K$, the interval I has length greater than π/n. Thus the interior of I contains a point $\alpha_0 = \pi k/n$ where k is an integer. Thus $h^{\alpha_0}_{\mathrm{SL}_2\mathbb{R}}(K)$ is nonzero, which means that $X^{c_0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is nonempty for $c_0 = 2\cos\alpha_0$. In particular, there is an irreducible $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ which induces an irreducible $\hat{\rho}\colon \pi_1(S^3, K_n) \to \mathrm{PSL}_2\mathbb{R}$. By Lemma 16.6, we have a nontrivial representation $\hat{\rho}_n : \pi_1(\Sigma_n(K)) \to \mathrm{PSL}_2\mathbb{R}$ with $e(\hat{\rho}_n) = 0$. As $\Sigma_n(K)$ is prime by Remark 16.5, it is LO by Theorem 16.3. □

16.9 Remark For any concrete signature function σ_K, one can often refine the result of Theorem 16.7, especially if one knows $h(K)$. For example, the alternating knot $K15a78855$ has

$$\Delta_K = 8t^6 - 21t^5 + 27t^4 - 27t^3 + 27t^2 - 21t + 8,$$

which is an irreducible polynomial all of whose roots are on the unit circle. The resulting partition is roughly $[0, 0.139, 0.398, 0.963, 2.178, 2.743, 3.002, 3.141]$ and $\frac{1}{2}\sigma_K$ has values $0, -1, -2, -3, -2, -1, 0$ on the corresponding open intervals. We checked K is small using the method of [Dunfield et al. 2022], so Theorem 16.7 applies for $n \geq 23$. However, by Corollary 16.12 below, $h(K) = -\frac{1}{2}\sigma(K) = 3$. Hence, from (16.8), we see that $h^\alpha_{\mathrm{SL}_2\mathbb{R}}(K)$ is nonzero on every interval except the middle one, $[0.963, 2.178]$. As Δ_K has no cyclotomic factors, no point of the form $k\pi/n$ is in our partition. Thus, $h^\alpha_{\mathrm{SL}_2\mathbb{R}}(K)$ is valid and nonzero for $\alpha = \pi/n$ whenever $\alpha < 0.963$, that is, for $n \geq 4$. As $\Sigma_2(K)$ is not LO, Theorem 16.7 settles the question of which $\Sigma_n(K)$ are LO except for $n = 3$.

16.10 Alternating knots

The results of the previous section can be used to compute $h(K)$ for many knots.

16.11 Proposition *Suppose a prime knot K is real representation small. If $\Sigma_2(K)$ is not LO, then $h(K) = -\frac{1}{2}\sigma(K)$.*

Proof Suppose $[\rho] \in X^{0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$. By Lemma 16.6, $\hat{\rho}_2 \in R_{\mathrm{PSL}_2\mathbb{R}}(\Sigma_2(K))$ is nontrivial and has $e(\hat{\rho}_2) = 0$. Since $\Sigma_2(K)$ is prime and not LO, this contradicts Theorem 16.3. Hence, $X^{0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) = \varnothing$, and so $h^0_{\mathrm{SL}_2\mathbb{R}}(K) = 0$. Since $0 \notin D_K$, as noted in Section 3.15, by Theorem 12.22 and Corollary 15.5 we have $h(K) = h^0_{\mathrm{SU}_2}(K) = -\frac{1}{2}\sigma_K(-1) = -\frac{1}{2}\sigma(K)$, as claimed. □

If the L-space conjecture holds, the hypothesis that $\Sigma_2(K)$ is not LO is equivalent to it being an L-space, a condition which has been intensively studied. In particular, the Ozsváth–Szabó spectral sequence [2005b] implies that if K has thin Khovanov homology, then $\Sigma_2(K)$ is an L-space. The equivalence of conditions (1) and (2) of the L-space conjecture have been checked for many such knots, including alternating knots [Boyer et al. 2013]. Hence, we have:

16.12 Corollary *If a prime knot K is real representation small and alternating, then $h(K) = -\frac{1}{2}\sigma(K)$.*

16.13 Dehn surgeries

Let $M_K(\alpha)$ be the Dehn filling of M_K along slope $\alpha = p\mu + q\lambda$. In this subsection, we give criteria for showing $M_K(\alpha)$ is LO. The basic strategy follows [Culler and Dunfield 2018], using the translation extension locus which is built from representations $\pi_1(M_K) \to \widetilde{\mathrm{SL}_2\mathbb{R}}$, as discussed in Section 14.12. Throughout, we freely use the notation from Section 14, and for brevity we set $\widetilde{E}(K) := \widetilde{X}^{\mathrm{ell}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ and $\widetilde{V}(K) := \widetilde{X}^{\mathrm{ell}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\partial M_K)$. Recall the translation extension locus of K is the image $i^*(\widetilde{E}(K))$ inside $\widetilde{V}(K)$, where i^* is induced by the inclusion $i\colon \partial M_K \to M_K$. (Provided K is real representation small, $i^*(\widetilde{E}(K))$ is precisely the locus $\mathrm{EL}_{\widetilde{G}}(M_K) \subset H^1(\partial M_K; \mathbb{R})$ from [loc. cit.], as discussed earlier in Section 14.12.) The new ingredient compared to [loc. cit.] is using the tools in this paper to show that $i^*(\widetilde{E}(K))$ contains "long arcs" which give rise to explicit ranges of LO fillings; in contrast, the main theorems of [loc. cit.] only give orders on arbitrarily small intervals of fillings. (For context, note Figure 24 shows a case where a small interval is the best one can do.)

Define $\widetilde{L}_\alpha$ in $\widetilde{V}(K)$ to be those $\widetilde{\chi}$ where $\widetilde{\rho}(\alpha) = 1$ for some $\widetilde{\rho}$ with $[\widetilde{\rho}] = \widetilde{\chi}$; in our (μ^*, λ^*)-coordinates from Section 14.12, this is the line of slope $-\alpha$ passing though the origin. Let $\widetilde{L}^\circ_\alpha \subset \widetilde{L}_\alpha$ be the subset of nonparabolic points. In this language, [ibid., Lemma 4.4] becomes:

16.14 Lemma *If $\widetilde{L}^\circ_\alpha$ intersects $i^*(\widetilde{E}(K))$, then $M_K(\alpha)$ admits a nontrivial representation*

$$\rho\colon \pi_1(M_K(\alpha)) \to \widetilde{\mathrm{SL}_2\mathbb{R}}.$$

Consequently, if $M_K(\alpha)$ is prime, then it is LO.

Proof A nonparabolic $\widetilde{\chi} \in \widetilde{V}(K)$ is in $\widetilde{L}_\alpha$ if and only if $\widetilde{\rho}(\alpha) = 1$ for all $\widetilde{\rho} \in R_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(\partial M_K)$ representing $\widetilde{\chi}$. In particular, if $\widetilde{L}^\circ_\alpha$ intersects $i^*(\widetilde{E}(K))$, we get a nontrivial representation $\pi_1(M_K(\alpha)) \to \widetilde{\mathrm{SL}_2\mathbb{R}}$; the last conclusion now comes from Theorem 16.3. □

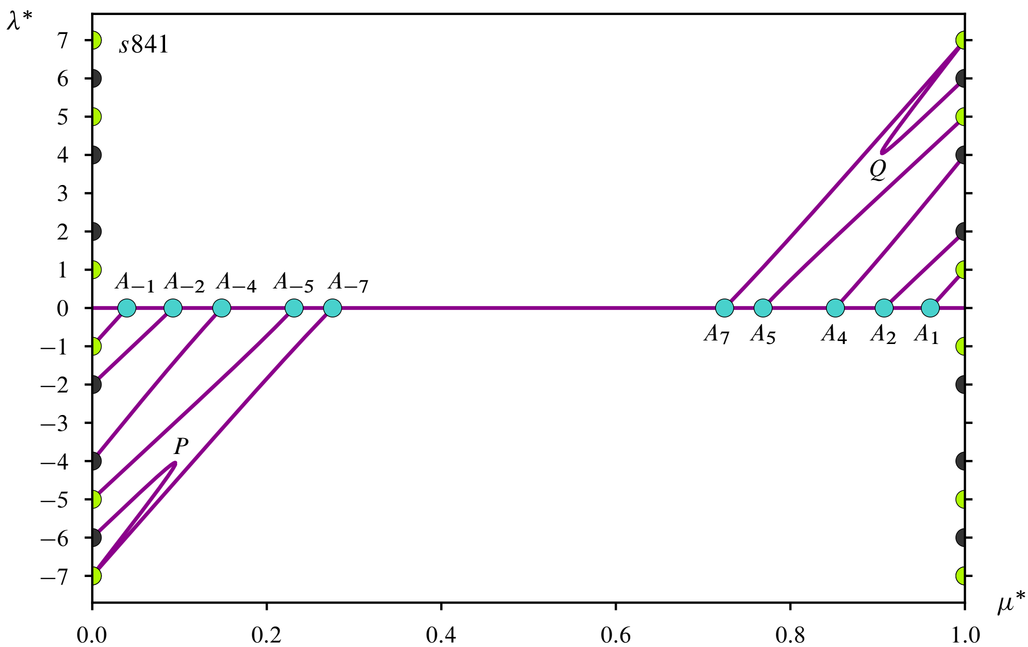

Figure 11: For the knot exterior $s841$, the set $i^*(\widetilde{E}(K))$ consists entirely of images of good arcs. There are 10 of type A, labeled near the endpoint of their reducible representation, and two of type B, specifically P of type B_{p_1,p_2} for $p_1 = (0,-7)$ and $p_2 = (0,-6)$ and Q of type B_{q_1,q_2} for $q_1 = (1,7)$ and $q_2 = (1,6)$. Picture adapted from [Culler and Dunfield 2018, Figure 7].

16.15 Good arcs

We say that $\gamma\colon [0,1] \to \widetilde{E}(K)$ is a *good arc* if the endpoint $\gamma(0)$ is parabolic, the other endpoint $\gamma(1)$ is either parabolic or reducible, the interior contains only elliptic irreducibles, and finally $i^*(\gamma(0)) \neq i^*(\gamma(1))$. A good arc γ is of type A_k for $k \in \mathbb{Z}$ if it starts at a parabolic χ with $\lambda^*(\chi) = k$ and ends at a reducible. It is of type B_{p_1,p_2} if its endpoints are parabolics χ_1, χ_2 with $i^*(\chi_i) = p_i \in \mathbb{Z}^2$, where we have identified the lattice of parabolic representations in $\widetilde{V}(K)$ with $\mathbb{Z}^2$. See Figures 11 and 12 for the images of some good arcs in the translation extension locus in $\widetilde{V}(K)$. We show in Lemmas 16.19 and 16.20 that many such good arcs lead to large intervals where we can apply Lemma 16.14. Before continuing, we note:

16.16 Lemma *Suppose γ is a good arc. Then the vertical lines where $\mu^* = k$ for $k \in \mathbb{Z}$ can meet $i^*(\gamma)$ only at its endpoints.*

Proof Since $\pi_1(S^3) = 1$, Lemma 16.14 gives the claim for the line $\widetilde{L}_\infty = \{\mu^* = 0\}$. The claim for the other vertical lines follows from the D_∞ action discussed in Example 14.14. □

16.17 Lemma *Suppose that K is real representation small. If $h(K) \neq 0$, then $\widetilde{E}(K)$ contains either an arc of type A_k, or an arc of type B_{p_1,p_2} where $p_1 = (0, y_1)$ and $p_2 = (1, y_2)$. If $\deg \tilde{h}(K) > 0$, then $\widetilde{E}(K)$ contains either an arc of type A_k with $k \neq 0$, or an arc of type B_{p_1,p_2} with $p_1 = (0, y_1)$ with $y_1 \neq 0$.*

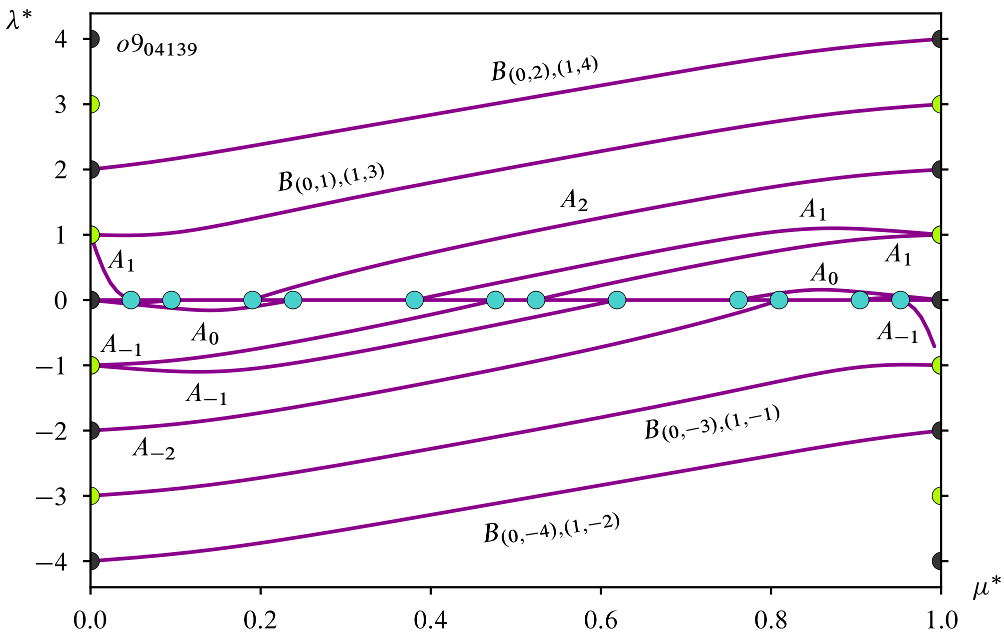

Figure 12: For the knot exterior $o9_{04139}$, the set $i^*(\widetilde{E}(K))$ again consists entirely of good arcs. The specific types are all labeled, with the caveat that there are actually four arcs of type A_0, only two of which are large enough to see clearly. Picture adapted from [Culler and Dunfield 2018, Figure 8].

Proof If W is a path component of $\mathscr{X}^{[-2,2]}(K)$, let $n_W^c = \sum_Z n_Z$, where Z runs over the set of components of $W \cap \mathscr{X}^c(K)$. By Corollary 2.12, $n_W := n_W^c$ is independent of $c \in [-2,2]$. Since $h(K) \neq 0$, we can choose some component W with $n_W \neq 0$. Since $n_W^2 = n_W^{-2} \neq 0$, the component W contains a parabolic χ_1 with $\mathrm{tr}_\mu\, \chi_1 = 2$ and a parabolic χ_2 with $\mathrm{tr}_\mu\, \chi_2 = -2$. Let γ be a path in W between them. By restricting to a subarc if necessary, we may assume that the interior of γ consists entirely of nonparabolic points, retaining the property that $\mathrm{tr}_\mu\, \chi_1 = 2$ and $\mathrm{tr}_\mu\, \chi_2 = -2$. If γ is contained in $X_{\mathrm{SL}_2\mathbb{R}}(K)$, then γ lifts to an arc $\widetilde{\gamma}$ of type B_{p_1,p_2}. Using the D_∞ action, specifically the index-two subgroup that preserves traces in $\mathrm{SL}_2\mathbb{R}$, we can assume $p_1 = (0, y_1)$. Then $p_2 = (k, y_2)$, where k must be odd. Lemma 16.16 forces $k = \pm 1$; if $k = -1$, act by $\tilde{a} \in D_\infty$ to rotate $i^*(\gamma)$ about $(0,0)$ so that $k = 1$, as desired. If instead γ is not contained in $X_{\mathrm{SL}_2\mathbb{R}}(K)$, consider the first intersection point of γ with $\mathscr{X}_0(K)$. This initial segment of γ then lifts to an arc of type A_k. This proves the first claim.

For the second claim, let $\tilde{n}_W = \sum_Z n_Z (t^{\lambda^*(\widetilde{Z}_+)} + t^{\lambda^*(\widetilde{Z}_-)})$, where the sum runs over components of $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ contained in W, as in Definition 14.17. For $\chi \in X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$, consider the two lifts $\widetilde{\chi}_\pm$ to $\widetilde{X}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ where $\mu^*(\widetilde{\chi}_\pm) = 0$. These have $|\lambda^*(\widetilde{\chi}_+)| = |\lambda^*(\widetilde{\chi}_-)|$, and we denote this common value by $|\lambda^*(\widetilde{\chi})|$ without reference to a particular lift. As $\sum_W \tilde{n}_W = \tilde{h}(K)$, we can choose some W with $\deg \tilde{n}_W \geq \deg \tilde{h}(K) > 0$. Let $\chi_1 \in W$ satisfy $\mathrm{tr}_\mu\, \chi_1 = 2$ and $|\lambda^*(\widetilde{\chi}_1)| = \deg \tilde{n}_W$.

If $n_W \neq 0$, we choose γ in W joining χ_1 to χ_2 with $\mathrm{tr}_\lambda(\chi_2) = -2$. Restricting to a subarc, we may assume the interior of γ consists entirely of nonparabolic points, retaining $\mathrm{tr}_\mu(\chi_1) = 2$ and $|\lambda^*(\widetilde{\chi}_1)| = \deg \tilde{n}_W$

and $p_1 \neq p_2$, but perhaps not $\operatorname{tr}_\mu \chi_2 = -2$. As before, either γ is entirely contained in $X^{\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$, or there is a path from χ_1 to a reducible. Lifting to $\widetilde{E}(K)$ now proves the second claim in the case $n_W \neq 0$.

Finally, if $n_W = 0$, note that $\tilde{n}_W|_{t=1} = 2n_W = 0$, so W must contain some other parabolic χ_2 with $\operatorname{tr}_\mu(\chi_2) = 2$ and $|\lambda^*(\tilde{\chi}_2)| \neq |\lambda^*(\tilde{\chi}_1)|$. After possibly changing the χ_i, we can find an arc γ in W joining some χ_1 and χ_2 which satisfies the following:

(1) $\operatorname{tr}_\mu \chi_1 = 2$ and $\operatorname{tr}_\mu \chi_2 = \pm 2$;

(2) $|\lambda^*(\tilde{\chi}_1)| = \deg \tilde{n}_W$;

(3) $p_1 \neq p_2$; and

(4) the interior of γ consists of nonparabolic points.

The second claim now follows just as in the previous cases. □

The set of Dehn filling slopes $S(K)$ of M_K is the projective space of $H_1(\partial M_K, \mathbb{Z})$. It is identified with $\mathbb{Q} \cup \{\infty\}$ by our choice of basis $\langle \mu, \lambda \rangle$ for $H_1(\partial M_K, \mathbb{Z})$. We define

$$S_{\mathrm{irr}}(K) = \{\alpha \in S(K) \mid M_K(\alpha) \text{ is irreducible}\},$$
$$S_{\mathrm{LO}}(K) = \{\alpha \in S(K) \mid \pi_1(M_K(\alpha)) \text{ is left-orderable}\},$$
$$S_{\widetilde{\mathrm{SL_2\mathbb{R}}}}(K) = \{\alpha \in S(K) \mid \widetilde{L}^\circ_\alpha \text{ intersects } i^*(\widetilde{E}(K))\}.$$

By Lemma 16.14, we know that $S_{\widetilde{\mathrm{SL_2\mathbb{R}}}}(K) \cap S_{\mathrm{irr}}(K) \subset S_{\mathrm{LO}}(K)$. As $i^*(\widetilde{E}(K))$ contains the horizontal axis (as the image of the reducible representations) and the 0-surgery is irreducible by [Gabai 1987], the slope 0 is in all three of these sets.

Suppose γ is a good arc joining χ_1 to χ_2. Let $p_i = i^*(\chi_i) \in \widetilde{V}(K)$. Suppose $p_1, p_2 \neq (0,0)$ and at most one p_i is on the line $\mu^* = 0$. In this case set

$$I^\circ(\gamma) = \{\alpha \in S(K) \mid L^\circ_\alpha \text{ meets the interior of the line segment from } p_1 \text{ to } p_2\}.$$

When $p_i = (0, y_i)$ with y_1 and y_2 nonzero, we define $I^\circ(\gamma) = S(K) \setminus \{\infty\}$ when the y_i have opposite signs and $I^\circ(\gamma)$ to be empty otherwise. Finally, when either p_1 or p_2 is $(0,0)$, we take $I^\circ(\gamma)$ to be empty. We next show:

16.18 Lemma *For a good arc γ, one has $I^\circ(\gamma) \subset S_{\widetilde{\mathrm{SL_2\mathbb{R}}}}(K)$.*

Proof Whenever α is in $I^\circ(\gamma)$, the definitions give that p_1 and p_2 are strictly separated from each other by the line L_α. It follows that L_α meets $i^*(\gamma)$ at some point in the latter's interior, which is not parabolic by the definition of a good arc, and hence Lemma 16.14 applies. □

16.19 Lemma *If $\widetilde{E}(K)$ contains a good arc of type A_k for $k \neq 0$, then either $(-\infty, |k|)$ or $(-|k|, \infty)$ is contained in $S_{\widetilde{\mathrm{SL_2\mathbb{R}}}}(K)$.*

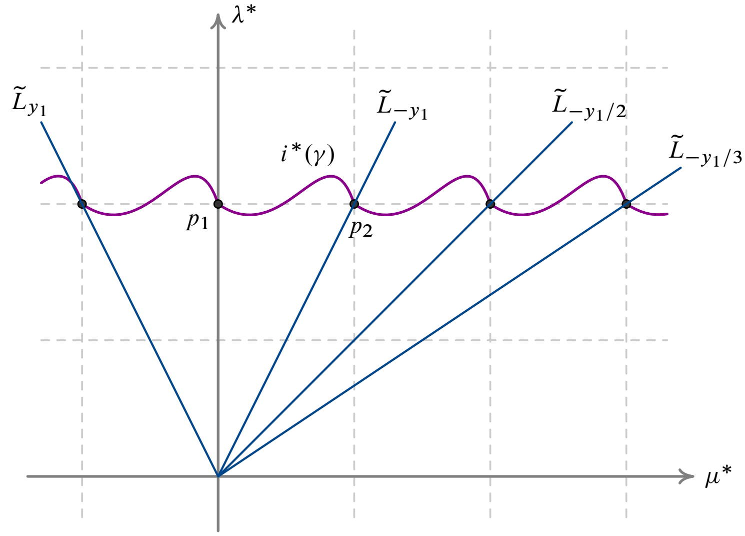

Figure 13: The situation of the proof of Lemma 16.20(1), showing translates of $i^*(\gamma)$ for a good arc γ of type B_{p_1,p_2} with $y_1 = y_2 = 2$. The $\widetilde{L}_\alpha$ drawn are among the few that are not guaranteed to meet $i^*(\widetilde{E}(K))$.

Before proving this, we note [Culler and Dunfield 2018, Figure 5] shows that the condition $k \neq 0$ is crucial, as that example has an A_0 arc which only yields $[-0.36, 3.6)$ in $S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$.

Proof Since $i^*(\widetilde{E}(K))$ is invariant under translation by $(1,0)$, we can assume that the good arc γ of type A_k starts at the point $(0,k)$ and ends at $(x,0)$ for some $x \neq 0$ (since $\pm 2 \notin D_K$). Further, since $i^*(\widetilde{E}(K))$ is invariant under the rotation $\tilde{a} \in D_\infty$ about the origin, we can assume $k > 0$. Suppose $x > 0$, so $I^\circ(\gamma) = (-\infty, 0)$. Next, consider the translate of γ by $(-1,0)$, which we denote by γ_{-1}. Its endpoints are $(-1,k)$ and $(x-1,0)$, so $I^\circ(\gamma_{-1}) = (0,k)$. The union of these two sets gives the promised range since $0 \in S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$ for all K. The argument when $x < 0$ is symmetric. □

16.20 Lemma *Suppose $\widetilde{E}(K)$ has a good arc of type B_{p_1,p_2} with $p_i = (x_i, y_i)$. Assume further that at least one y_i is nonzero. Then*:

(1) *If $y_1 = y_2 \neq 0$, then $\mathbb{Q} \setminus \{y_1/n \mid n \in \mathbb{Z}\} \subset S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$.*

(2) *If $x_1 = x_2$, then $(-1,1) \subset S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$.*

(3) *Otherwise, either $\left(-\infty, \frac{1}{2}\right)$ or $\left(-\frac{1}{2}, \infty\right)$ is contained in $S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$.*

Proof By Lemma 16.16, after translation we can assume $i^*(\gamma)$ is contained in the strip $0 \leq \mu^* \leq 1$. In case (1), we have $p_1 = (0,k)$ and $p_2 = (1,k)$ for some $k \neq 0$, and we can arrange that $k > 0$ using the D_∞ action, resulting in the situation shown in Figure 13. The translate of γ by $(n,0)$ then has I° equal to $\left(\frac{k}{n}, \frac{k}{n+1}\right)$ or $\left(\frac{k}{n+1}, \frac{k}{n}\right)$ depending on the sign of n. The union of these I° covers all of $\mathbb{Q}$ except 0 and those points of the form $\frac{k}{n}$ with $n \in \mathbb{Z}$, completing the proof of case (1).

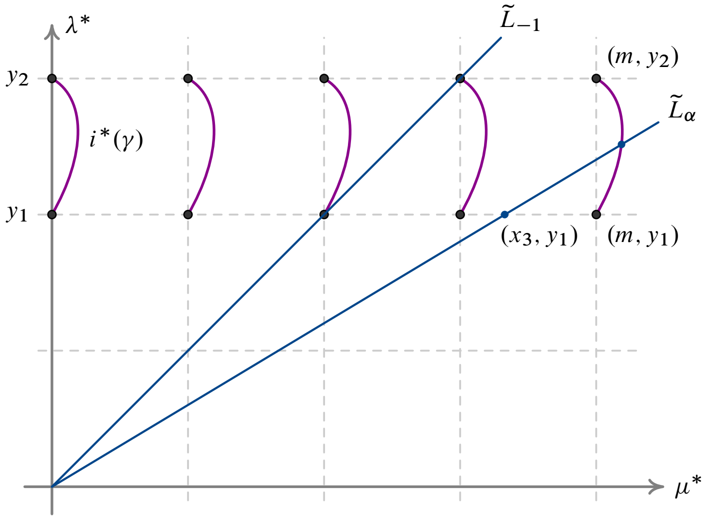

Figure 14: The situation of the proof of Lemma 16.20(2), showing translates of $i^*(\gamma)$ for a good arc γ of type B_{p_1,p_2} with $x_1 = x_2$.

In case (2), if y_1 and y_2 have opposite signs, we are done as $I^\circ(\gamma)$ is all of $S_{\widetilde{\mathrm{SL}_2\mathbb{R}}} K$ except ∞. Otherwise, use the D_∞ action and possibly interchange p_1 and p_2 to assume $x_1 = x_2 = 0$ and $0 \le y_1 < y_2$, giving the situation in Figure 14. If $0 < |\alpha| < y_2 - y_1$, consider the point where $\widetilde{L}_\alpha$ intersects the line $\lambda^* = y_1$. For concreteness, suppose this point is (x_3, y_1) with $x_3 > 0$ (equivalently, assume $\alpha < 0$). If m is the smallest integer with $x_3 < m$, then $\widetilde{L}_\alpha^\circ$ will have to meet the translate of γ joining (m, y_1) to (m, y_2). As $y_2 - y_1 \ge 1$, we have shown $(-1, 1) \subset S_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$.

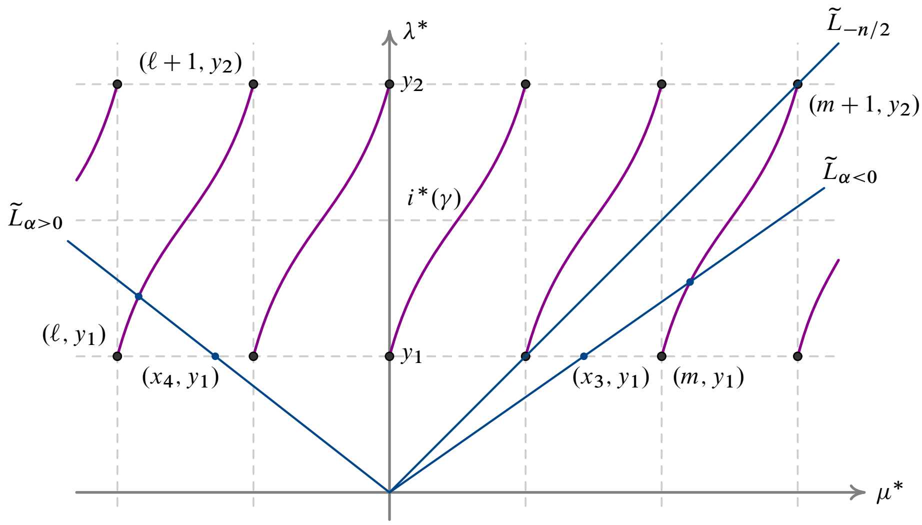

Figure 15: The situation of the proof of Lemma 16.20(3) when $y_1 \ge 0$, showing that $\widetilde{L}_\alpha^\circ$ meets a translate of $i^*(\gamma)$ for any $\alpha \in \left(-\frac{n}{2}, \infty\right)$, where $n = y_2 - y_1$.

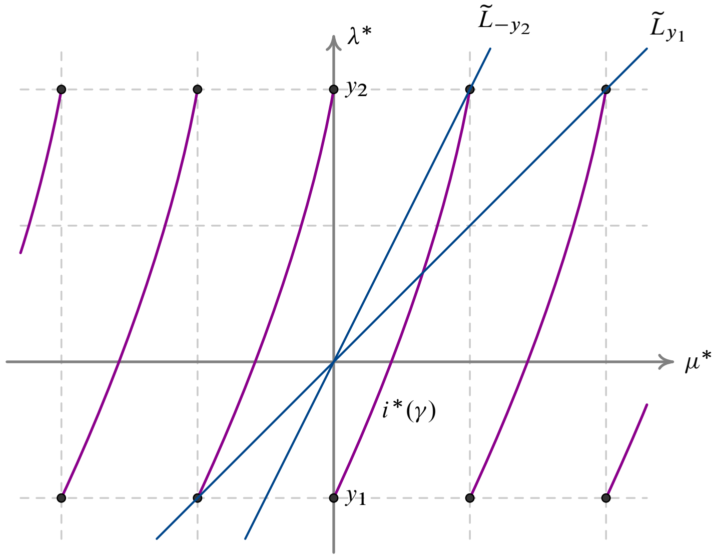

Figure 16: The situation of the proof of Lemma 16.20(3) when $y_1 < 0$, showing that $\widetilde{L}_\alpha^\circ$ meets a translate of $i^*(\gamma)$ for any $\alpha \in (-m, \infty)$, where $m = \max(-y_1, y_2)$.

Next, in case (3), we can assume that $p_1 = (0, y_1)$ and $p_2 = (1, y_2)$. Now $y_1 \neq y_2$, and let us assume $y_2 > y_1$ as the other case can be reduced to this one by reflection across the vertical axis. Moreover, we can also assume $y_2 \geq 0$, since, if not, we apply the element of D_∞ that rotates around $\left(\frac{1}{2}, 0\right)$.

First suppose $y_1 \geq 0$ and set $n = y_2 - y_1$, giving the situation of Figure 15. If $-\frac{n}{2} < \alpha < 0$, then $\widetilde{L}_\alpha$ has positive slope and consider the point (x_3, y_1) where $\widetilde{L}_\alpha$ meets the line $\lambda^* = y_1$. If m is the smallest integer where $x_3 < m$, then $\widetilde{L}_\alpha^\circ$ will meet the translate of $i^*(\gamma)$ joining (m, y_1) to $(m+1, y_2)$. Hence, $\left(-\frac{n}{2}, 0\right) \subset S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$. If $\alpha > 0$, let (x_4, y_1) again be the point where $\widetilde{L}_\alpha$ meets the line $\lambda^* = y_1$. If ℓ is the largest integer where $\ell < x_4$, then $\widetilde{L}_\alpha^\circ$ will meet the translate of $i^*(\gamma)$ joining (ℓ, y_1) to $(\ell+1, y_2)$. In particular, we have $\left(-\frac{n}{2}, \infty\right) \subset S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$, as needed.

The final subcase is when $y_1 < 0$, which is shown in Figure 16. There, you can see that $\widetilde{L}_\alpha^\circ$ meets a translate of $i^*(\gamma)$ for any $\alpha \in (-m, \infty)$, where $m = \max(-y_1, y_2) \geq 1$. This completes the proof of case (3), and hence the lemma. □

16.21 Theorem *If* $\deg \tilde{h}(K) > 0$, *then* $A \cap S_{\mathrm{irr}}(K) \subset S_{\mathrm{LO}}(K)$, *where* A *is one of the sets* $\left(\infty, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, \infty\right)$, $(-1, 1)$ *or* $\mathbb{Q} \setminus \left\{\frac{1}{n} \mid n \in \mathbb{Z}\right\}$.

Proof Just combine Lemmas 16.14, 16.17, 16.19 and 16.20. □

These techniques apply to many 2-bridge knots:

16.22 Theorem *If* K *is a 2-bridge knot with* $\sigma(K) \neq 0$, *then* $\widetilde{E}(K)$ *contains an arc of type* A_k *for some* $k \neq 0$. *Moreover, either* $(-\infty, 1)$ *or* $(-1, \infty)$ *is in* $S_{\mathrm{LO}}(K)$.

The positive trefoil knot K from Section 14.7 is a 2-bridge knot and has $\sigma(K) = -2$. There, the locus $i^*(\widetilde{E}(K))$ consists of the orbit of a single A_1 arc under the D_∞ action. This shows that the final conclusion of Theorem 16.22 cannot be strengthened to $(-\infty, \infty) \subset S_{\mathrm{LO}}(K)$ using the present methods. Of course, the -1 Dehn surgery on K is the Poincaré homology sphere, which is not LO, but the same pattern holds for the knot 5_2, where every Dehn surgery is expected to be LO.

Proof of Theorem 16.22 As 2-bridge knots are small [Hatcher and Thurston 1985], K is real representation small. As K is alternating, we have $h(K) = -\frac{1}{2}\sigma(K) \neq 0$ by Corollary 16.12. By Lemma 16.17, there is a good arc γ of either type A_k or of type B_{p_1,p_2} with $p_1 = (0, y_1)$ and $p_2 = (1, y_2)$. In the latter case, we would have $X^{0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) \neq \varnothing$, but this is impossible since $\Sigma_2(K)$ is a lens space and so not LO (compare with the proof of Proposition 16.11). So we have a good arc of type A_k. By [Riley 1972, Theorem 2], if $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ is a parabolic representation then $\operatorname{tr}\rho(\lambda) = -2$. This says that any arc in the translation extension locus $i^*(\widetilde{E}(K))$ ending on a parabolic must do so at an odd height. Thus $k \neq 0$ and Lemma 16.19 applies. Finally, every Dehn surgery on K is prime by [Delman 1995; Naimi 1997], allowing us to pass from $S_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ to $S_{\mathrm{LO}}(K)$ unimpeded. □

16.23 Remark It is natural to ask whether Theorem 16.22 can be extended to other small Montesinos knots or to other small alternating knots. In both cases one has $h(K) = -\frac{1}{2}\sigma(K)$ by Proposition 17.2 and Corollary 16.12. Unlike for 2-bridge knots, in these broader settings, parabolics with $\mathrm{tr}_\lambda = 2$ and hence even height can occur. For Montesinos knots, the $(-2, 3, 7)$ pretzel has several parabolics at even heights (see Figure 17). For small alternating knots, Goerner's data [2014] includes nine such examples: $9_{34} = K9a28$, $10_{100} = K10a104$, $10_{108} = K10a119$, $10_{103} = K10a105$, $10_{104} = K10a118$, $10_{106} = K10a95$, $10_{102} = K10a97$, $10_{101} = K10a45$ and $11_{337} = K11a330$. However, we have not seen any parabolics of height 0 for alternating or Montesinos knots, so it is possible that the same conclusion as in Theorem 16.22 holds in these cases.

16.24 Knots with lens space surgeries

In this last subsection, we give strong constraints on $i^*(\widetilde{E}(K))$ when K has a lens space surgery or, more generally, a Dehn surgery $M_K(\alpha)$ with few $\mathrm{PSL}_2\mathbb{R}$ representations. We begin with an easy lemma that nonetheless explains a great deal of the striking structure of $i^*(\widetilde{E}(K))$ for the knots shown in [Culler and Dunfield 2018, Figures 3 and 4]. (The first of those figures is incorporated here as Figure 17.) To state it, for a slope $\alpha = p\mu + q\lambda$ and an $n \in \mathbb{Z}$, we define $\widetilde{L}_{\alpha,n}$ to be the line in $\widetilde{V}(K)$ of slope $-\alpha$ that meets the horizontal axis at $\left(\frac{n}{p}, 0\right)$. We also take $\widetilde{L}^\circ_{\alpha,n}$ to be its subset of nonparabolic points where $\lambda^* \neq 0$, ie remove $\mathbb{Z}^2$ and $\left(\frac{n}{p}, 0\right)$ from $\widetilde{L}_{\alpha,n}$.

16.25 Lemma *Let K be any knot where $X^{\mathrm{irr}}_{\mathrm{PSL}_2\mathbb{R}}(M_K(\alpha))$ is empty. Then $i^*(\widetilde{E}(K))$ is disjoint from $\bigcup\{\widetilde{L}^\circ_{\alpha,n} \mid n \in \mathbb{Z}\}$.*

Figure 17 shows just how constraining Lemma 16.25 is when there are multiple lens space surgeries.

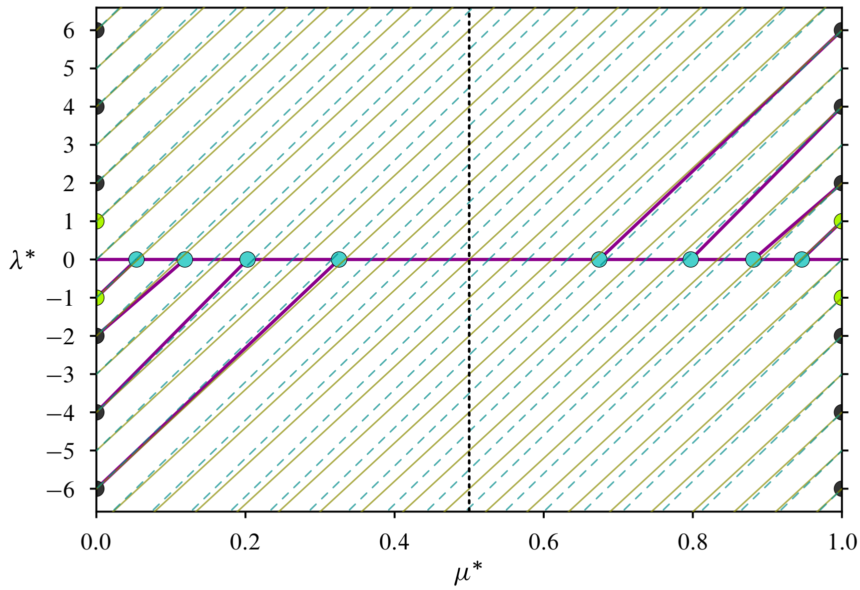

Figure 17: $i^*(\widetilde{E}(K))$ for the $(-2,3,7)$ pretzel knot, whose exterior is small. Surgery along the slopes -18 and -19 both yield lens spaces, forcing $i^*(\widetilde{E}(K))$ to avoid two families of lines per Lemma 16.25: those of slope 18 (solid lines) and slope 19 (dashed lines). The roots of Δ_K on S^1 are simple and are indicated by the blue dots on the line $\lambda^* = 0$. Because of the locations of the roots, these lines rule out any arcs joining two distinct reducibles. From Proposition 17.2, we know $i^*(\widetilde{E}(K))$ is disjoint from the vertical line $\mu^* = \frac{1}{2}$. Consequently, an arc starting at a particular reducible has only one or two possible parabolic ending points, basically dictating the picture shown (see the proof of Theorem 16.27 for details). In particular, there must be an arc from the reducible at $\approx (0.6744, 0)$ to the parabolic at $(1, 6)$, and so $(-6, \infty) \subset S_{\widetilde{\mathrm{SL_2}\mathbb{R}}}(K)$. Here, $h(K) = -\frac{1}{2}\sigma(K) = -4$, and the signature function jumps by $+2$ at the first four roots of Δ_K and then by -2 at the remaining four. Picture adapted from [Culler and Dunfield 2018, Figure 3].

Proof The line $\widetilde{L}_{\alpha,n}$ is exactly the locus where $\mathrm{trans}_\alpha(\widetilde{\chi}) = n$, where trans_α is the translation number function of Corollary 4.7. Hence, if $\widetilde{\chi}$ is in both $\widetilde{L}^\circ_{\alpha,n}$ and $i^*(\widetilde{E}(K))$, then we get an irreducible elliptic representation $\widetilde{\rho}\colon \pi_1(M_K) \to \widetilde{\mathrm{SL_2}\mathbb{R}}$ where $\widetilde{\rho}(\alpha)$ is in the center of $\widetilde{\mathrm{SL_2}\mathbb{R}}$. Quotienting $\widetilde{\mathrm{SL_2}\mathbb{R}}$ by its center gives an irreducible representation $\rho\colon \pi_1(M_K(\alpha)) \to \mathrm{PSL_2}\mathbb{R}$, which is a contradiction. □

16.26 Lemma *Let W be a component of $X^{[-2,2],\mathrm{irr}}_{\mathrm{SL_2}\mathbb{R}}(K)$, and suppose that $c, c' \in [-2,2] \setminus D_K$. If $n^c_W \neq n^{c'}_W$, then there is a path in $X_{\mathrm{SL_2}\mathbb{R}}(K)$ which starts in either W^c or $W^{c'}$ and is contained in $W^{[c,c']}$ except for its final endpoint, which is a reducible character χ with $\mathrm{tr}_\mu(\chi) \in D_K \cap [c,c']$.*

Proof Since we can compute n^c_W and $n^{c'}_W$ by summing over the connected components of Z of $W^{[c,c'],\mathrm{irr}}(K)$, let Z be such a component with $n^c_Z \neq n^{c'}_Z$. Let $\overline{Z}$ be the component of $\mathscr{X}^{[c,c']}(K)$ that contains Z. As $n^c_{\overline{Z}} = n^{c'}_{\overline{Z}}$, it follows that $\overline{Z} \neq Z$ and so $\overline{Z}$ meets $\mathscr{X}^{[c,c']}_0(K)$. As $n^c_Z \neq n^{c'}_Z$, at least one of them must be nonzero, say $n^c_Z \neq 0$; in particular, Z^c is nonempty. Thus there is a path in $\overline{Z}$ starting in Z^c

and ending in $\overline{Z} \cap \mathscr{X}_0^{[c,c']}(K)$. Taking an initial segment of this path and using the map $\mathscr{X}(K) \to X(K)$ gives the desired path by Lemma 3.16. □

16.27 Theorem *Suppose a prime knot K is real representation small and that $M_K(p\mu + \lambda)$ is a lens space for some $p > 0$. Suppose that for some $n = 1, 2, \ldots, p$ there is a unique $x_0 \in \left[\frac{n-1}{p}, \frac{n}{p}\right]$ with $\Delta_K(e^{2\pi i x_0}) = 0$, that the corresponding root of Δ_K is simple, and that $x_0 \neq \frac{n-1}{p}, \frac{n}{p}$. Then $\widetilde{E}(K)$ contains an arc γ of type A_k, where k is one of $n-1, n, n-1-p$, or $n-p$, and where the other endpoint of $i^*(\gamma)$ is $(x_0, 0)$. If $n \notin \{1, p\}$, then $(-\infty, 1) \subset S_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$.*

A similar statement holds when $M_K(-p\mu + \lambda)$ is a lens space by taking the mirror. Also, the hypothesis of a lens space filling in Theorem 16.27 can be replaced with the requirement that $X^{\mathrm{irr}}_{\mathrm{PSL}_2\mathbb{R}}(M_K(p\mu + \lambda))$ be empty.

16.28 Remark We conjecture that the hypothesis on the Alexander polynomial in Theorem 16.27 always holds when $M_K(p\mu + \lambda)$ is a lens space. By Greene's lens space realization theorem [2013], the set of possible Δ_K's coincides with the set of Alexander polynomials of Berge knots. For fixed p, there are only finitely many possibilities, and the hypothesis on their roots holds for $p \leq 1000$ (which is some 5265 distinct Δ_K). In fact, with the exception of $p = 5$, there is always a root leading to the conclusion that $(-\infty, 1) \subset S_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$, indeed usually tens or even hundreds of such roots. Here, it is unknown whether having a lens space surgery means K is real representation small, and there are such K that are not small [Baker 2005].

Proof of Theorem 16.27 Let $\omega = \frac{(n-1)\pi}{p}$ and $\omega' = \frac{n\pi}{p}$, as well as $c = 2\cos\omega$ and $c' = 2\cos\omega'$. By hypothesis, $c, c' \notin D_K$, and there is a unique root $e^{2i\theta}$ of Δ_K with $\theta \in [\omega, \omega']$. Let $\widetilde{\chi}_\theta$ be the point in $X^{\mathrm{red}}_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ with $i^*(\widetilde{\chi}_\theta) = (\theta/\pi, 0)$. Now, if $\widetilde{\gamma}$ is a path in $\widetilde{E}(K)$ starting at $\widetilde{\chi}_\theta$ whose interior consists of nonparabolic characters, then $i^*(\widetilde{\gamma})$ is contained in the shaded region R shown in Figure 18 and meets ∂R only if the other endpoint of $\widetilde{\gamma}$ is a parabolic. In the latter case, the parabolic must have height $n-1, n, n-p$ or $n-p-1$, and we can apply Lemma 16.19 to the arc to conclude $(-\infty, \min(n-1, n-p)) \subset S_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ provided $n \notin \{1, p\}$. Thus, to prove the theorem it suffices to find such a $\widetilde{\gamma}$.

Since $e^{2i\theta}$ is a simple root of Δ_K, we have $\sigma_K(e^{2i\omega}) - \sigma_K(e^{2i\omega'}) = \pm 2$. Using Corollary 15.5 and Theorem 12.22, we compute

$$\pm 1 = h^c_{\mathrm{SU}_2}(K) - h^{c'}_{\mathrm{SU}_2}(K) = h^{c'}_{\mathrm{SL}_2\mathbb{R}}(K) - h^c_{\mathrm{SL}_2\mathbb{R}}(K) = \sum_W (n^{c'}_W - n^c_W), \tag{16.29}$$

where the sum is over the path components W of $X^{[-2,2],\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$. Let W be any such component with $n^c_W \neq n^{c'}_W$, and consider the path given by Lemma 16.26, which must end at the reducible character χ_θ where $\mathrm{tr}_\mu = 2\cos\theta$.

If the closure $\overline{W}$ of W in $X^{[-2,2]}_{\mathrm{SL}_2\mathbb{R}}(K)$ contains a reducible $\chi_{\theta'}$ other than χ_θ, let γ be a path in W joining χ_θ to $\chi_{\theta'}$. Taking the lift $\widetilde{\gamma}$ as before, note that $i^*(\widetilde{\gamma})$ must exit R to get to $i^*(\widetilde{\chi}_{\theta'})$. As it can only do

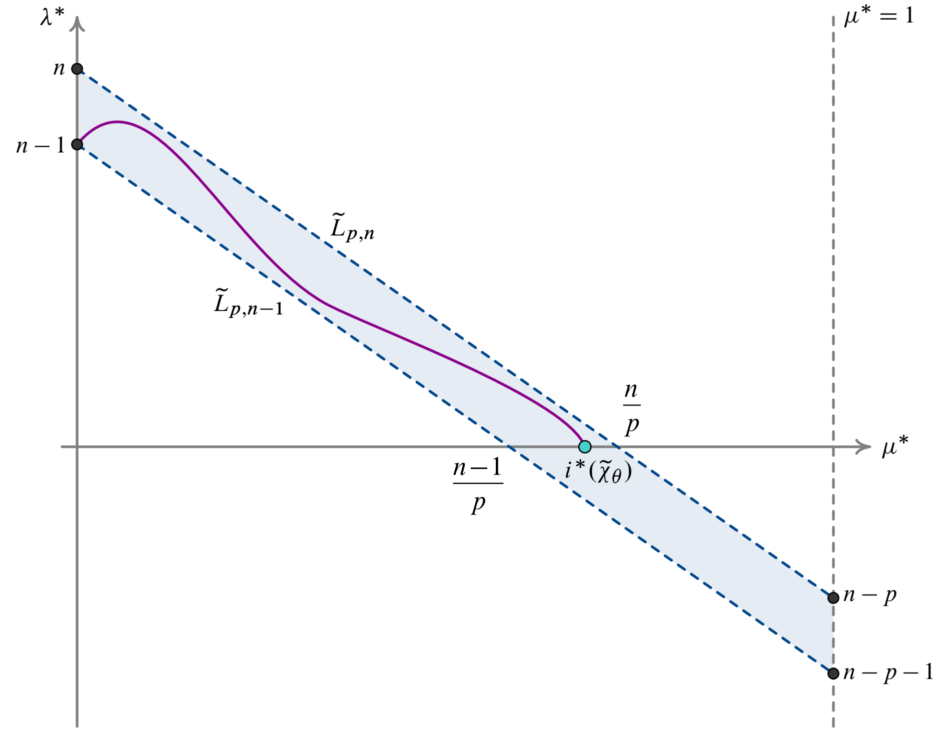

Figure 18: A good arc ending at $\widetilde{\chi}_\theta$ must have image lying in the shaded region.

so at one of the parabolic corners of R, an initial segment of $\widetilde{\gamma}$ is the path we seek. So now assume $\overline{W} = W \cup \{\chi_\theta\}$, which means n^a_W can only change at $a = 2\cos\theta$. In particular, $n^2_W = n^c_W$ and $n^{-2}_W = n^{c'}_W$. As $n^c_W \neq n^{c'}_W$, at least one of n^2_W and n^{-2}_W is nonzero. In particular, W contains a parabolic character, so $\overline{W} = W \cup \{\chi_\theta\}$ contains a path γ from χ_θ to a parabolic character whose interior consists of irreducible nonparabolic characters. The lift of γ starting at $\widetilde{\chi}_\theta$ is thus the path $\widetilde{\gamma}$ we seek, proving the theorem. □

17 Computations and conjectures

In this final section, we compute h for some additional classes of small knots, including Montesinos knots, torus knots and 98.7% of those with at most 11 crossings. We use h to give a new proof of Riley's conjecture on parabolic representations of 2-bridge knots. Finally, we make some conjectures about the behavior of the extended Lin invariant $\tilde{h}(K)$ for 2-bridge knots and knots with lens space surgeries.

17.1 Montesinos knots

In Proposition 16.11, we showed that $h(K) = -\frac{1}{2}\sigma(K)$ whenever K is real representation small and $\Sigma_2(K)$ is not LO. In fact, there are many knots which satisfy $h(K) = -\frac{1}{2}\sigma(K)$ even though $\Sigma_2(K)$ is LO. The Montesinos knots provide a good class of such examples. Oertel [1984] showed a Montesinos knot is small if and only if it has at most three rational tangles. (A Montesinos knot with one or two rational tangles is 2-bridge, and hence alternating.) We will show:

17.2 Proposition *For any Montesinos knot $K = K(p_1/q_1, p_2/q_2, p_3/q_3)$, we have $X^{0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) = \varnothing$ and $h(K) = -\frac{1}{2}\sigma(K)$.*

The branched double covers of these knots are Seifert fibered spaces over the disk with three exceptional fibers, many of which are LO and admit nontrivial representations to $\mathrm{SL}_2\mathbb{R}$. As in Section 16.4, we write (S^3, K_n) to denote the orbifold, where the knot K is labeled by the cyclic group $\mathbb{Z}/n$.

17.3 Lemma *If $\rho\colon \pi_1(S^3, K_n) \to \mathrm{PSL}_2\mathbb{R}$ is irreducible, then so is its restriction to $\pi_1(\Sigma_n(K))$.*

Proof Set $\Gamma = \pi_1(S^3, K_n)$ and $\Lambda = \pi_1(\Sigma_n(K))$. We will prove the contrapositive: for any representation $\rho\colon \Gamma \to \mathrm{PSL}_2\mathbb{R}$, if $\rho(\Lambda)$ is reducible then so is $\rho(\Gamma)$. We consider the various possible restrictions of ρ to Λ from most to least degenerate. To begin, if $\rho(\Lambda)$ is trivial, then ρ factors through $\Gamma/\Lambda = \mathbb{Z}/n$. As every finite subgroup of $\mathrm{PSL}_2\mathbb{R}$ is reducible, $\rho(\Gamma)$ is reducible.

Suppose next that $\rho(\Lambda)$ is reducible but nontrivial. The fixed-point set F of $\rho(\Lambda)$ on $P^1(\mathbb{C})$ must be either one or two points, since, if there were more, then $\rho(\Lambda)$ would be trivial. Since Λ is normal in Γ, the full group $\rho(\Gamma)$ leaves F invariant. Thus, if F is a single point, then $\rho(\Gamma)$ too is reducible. So assume F is two points, and let L be an oriented geodesic joining them, which may not lie in $\mathbb{H}^2$. The stabilizer of L in $\mathrm{PSL}_2\mathbb{C}$ is the abelian group $\mathbb{C}^\times$, so the restriction $\rho|_\Lambda$ factors through $\Lambda^{\mathrm{ab}} = H_1(\Sigma_2(K); \mathbb{Z})$. As K is a knot, the latter is finite of odd order. In particular, $\rho(\Lambda)$ is finite and moreover so is $\rho(\Gamma)$ since $[\Gamma : \Lambda] = n$. Thus, again, $\rho(\Gamma)$ must be reducible. □

Proof of Proposition 17.2 Set $\Gamma = \pi_1(S^3, K_2)$ and $\Lambda = \pi_1(\Sigma_2(K))$ and suppose that some $\rho\colon \Gamma \to \mathrm{PSL}_2\mathbb{R}$ is irreducible. As discussed in [Oertel 1984], the orbifold (S^3, K_2) is Seifert fibered with quotient orbifold Q being a triangle with mirrored sides and vertices labeled by dihedral groups of order $2q_i$. Hence, if f represents a regular fiber, we have

$$1 \to \langle f \rangle \to \Gamma \to \pi_1(Q) \to 1.$$

While $\langle f \rangle$ is not central in Γ, it is in Λ since $\Sigma_2(K)$ is Seifert fibered over the orbifold $S^2(q_1, q_2, q_3)$ with *orientable* fibers. Thus $\rho(f)$ centralizes $\rho(\Lambda)$, which is irreducible by Lemma 17.3; by Lemma 3.11, $\rho(f) = 1$. Thus, on all of Γ, the representation ρ factors through

$$\pi_1(Q) = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^{q_1} = (xz)^{q_2} = (yz)^{q_3} = 1 \rangle.$$

However, any two elements of $\mathrm{PSL}_2\mathbb{R}$ of order two are either equal or have product a nontrivial hyperbolic element. This forces any representation of $\pi_1(Q)$ to $\mathrm{PSL}_2\mathbb{R}$ to be reducible, a contradiction. So $X^{0,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) = \varnothing$ and $h(K) = -\frac{1}{2}\sigma(K)$, as claimed. □

17.4 Knots with small crossing number

There are 801 nontrivial prime knots with at most 11 crossings, of which 601 are small [Burton et al. 2013]. All but possibly 8 of these 601 satisfy $h = -\frac{1}{2}\sigma$. Indeed, some 85.9% of these knots are either Montesinos or alternating [Livingston and Moore 2021; Castellano-Macías and Owad 2022]. For the 85 knots that are not in those classes, the question of whether $\Sigma_2(K)$ is LO is usually known from [Dunfield 2020b] or can be determined using those methods. If $\Sigma_2(K)$ is not LO, then $h(K) = -\frac{1}{2}\sigma(K)$ by Proposition 16.11. In this way, we can show that for small knots with ≤ 11 crossings, $h(K) = -\frac{1}{2}\sigma(K)$ for all but possibly the knots $10_{161} = 10n31$, $11n96$, $11n111$, $11n116$, $11n135$, $11n143$, $11n145$ and $11n183$. For these eight knots, $\Sigma_2(K)$ is LO for all but possibly $\Sigma_2(11n143)$ and $\Sigma_2(11n145)$ by [loc. cit.]; we expect $\Sigma_2(11n143)$ and $\Sigma_2(11n145)$ to be LO as they are not L-spaces. (All eight $\Sigma_2(K)$ have coorientable taut foliations by the techniques of [loc. cit.].)

17.5 Torus knots

The knots for which we computed h in earlier sections all satisfy $h(K) = -\frac{1}{2}\sigma(K)$. We now turn to torus knots and show they exhibit more interesting behavior. Let $K = T(p,q)$ be the positive (p,q) torus knot. We can compute $X_{\mathrm{SL}_2\mathbb{R}}(M_K)$ using the same method as for the trefoil in Section 14.7. The exterior of $K = T(p,q)$ is Seifert fibered and

$$\pi_1(M_K) = \langle x, y, f \mid x^p = f = y^q \rangle,$$

where f is the class of the Seifert fiber. If $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$ is irreducible, $\rho(f)$ is central, so $\rho(x)^p = \rho(y)^q = \pm I$. As in Section 14.7, we may assume that

$$\rho(x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} \cos\varphi & -t\sin\varphi \\ t^{-1}\sin\varphi & \cos\varphi \end{pmatrix}$$

For a fixed t, there are $2(p-1)(q-1)$ choices of θ and φ which result in irreducible representations. The representations for (θ,φ) and $(-\theta,-\varphi)$ have the same character, so $X_{\mathrm{SL}_2\mathbb{R}}(M_K)$ consists of $(p-1)(q-1)$ arcs, which are obtained by varying t. Each arc tends to a reducible character as $t \to 1$, and its image in $X_{\mathrm{SL}_2\mathbb{R}}(\partial M_K)$ lies on a line of slope $-pq$, since $\rho(f) = \pm I$ for all ρ on a given arc and $[f] = (pq)\mu + \lambda$ in $H_1(\partial M_K)$.

The Alexander polynomial of K is

(17.6)
$$\Delta_K(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)},$$

which has $(p-1)(q-1)$ roots; specifically, for $\zeta_{pq} = e^{2\pi i/pq}$, the roots are $t = \zeta_{pq}^n$ for $1 \leq n < pq$ with p and q not dividing n. Each root is simple, and the number of arcs and roots is the same. Hence, arguing as in Theorem 16.27, we see that the character variety has a single arc ending at each point on the reducible line with $\bar{\mu}^* = \frac{n}{pq}$, for n as before. Lifting to $\widetilde{E}(K)$, we see that the translation extension locus has the form shown in Figure 19. In order to describe $i^*(\widetilde{E}(K))$ precisely, we must determine

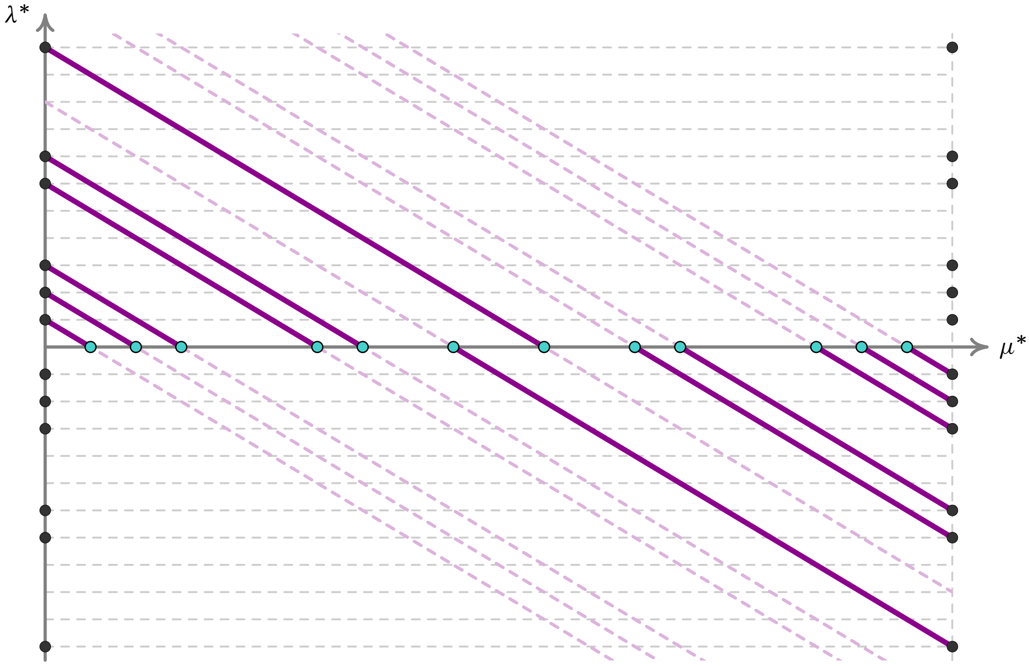

Figure 19: The basic structure of $i^*\big(\widetilde{E}(T(p,q))\big)$: it consists of arcs of slope $-pq$ joining a parabolic point to one of the $(p-1)(q-1)$ points of the form $\big(\frac{n}{pq},0\big)$ with neither p nor q dividing n. In particular, exactly one segment of the line of slope $-pq$ through $\big(\frac{n}{pq},0\big)$ in the region $0 \le \mu^* \le 1$ is part of $i^*\big(\widetilde{E}(T(p,q))\big)$. The complete picture is thus determined by which parabolics of the form $(0,k)$ for $k>0$ are ends of such arcs; these are characterized by Proposition 17.7. Here, the plot is that for $T(4,5)$.

whether the arc ending at a given root of the Alexander polynomial lies below the reducible line or above it. Equivalently, it suffices to determine the set

$$\Gamma(K) = \{n \in \mathbb{N} \mid \text{an arc of } \widetilde{E}(K) \text{ ends at a parabolic of height } n\}.$$

The arc ending at $\bar{\mu}^* = \frac{n}{pq}$ will thus lie above the reducible line if and only if $n \in \Gamma(K)$.

17.7 Proposition *For the torus knot $K = T(p,q)$, one has $\Gamma(K) = \mathbb{N} \setminus \Gamma_{p,q}$, where $\Gamma_{p,q}$ is the additive semigroup generated by p and q.*

Proof All the roots of Δ_K are simple, so we can apply Lemma 15.8. The arc γ considered in part (1) of the lemma can be taken to have image a line segment of slope $-pq$ in the translation extension locus, so it satisfies $(\mathrm{tr}_\lambda \circ \gamma)'(0) \neq 0$. We conclude that the segment which limits to the reducible at $\big(\frac{n\pi}{pq},0\big)$ lies above the reducible line if the jump in $\sigma_K(\omega)$ at ζ_{pq}^n is -2 and below it if the jump is $+2$.

Litherland [1979] gave a simple formula for the sign of the jump in σ_K at ζ_{pq}^n. It is

(17.8) $$(-1)^{\lfloor a/q\rfloor + \lfloor b/p\rfloor + \lfloor n/pq\rfloor}, \quad \text{where } n = ap+bq \text{ for } a,b \in \mathbb{Z}.$$

If $0 \le n < pq$, there is a unique $\hat{n} \in \mathbb{Z}$ with $\hat{n} \equiv n \mod pq$ and $\hat{n} = \hat{a}p + \hat{b}q$ with $0 \le \hat{a} < q$ and $0 \le \hat{b} < p$. Moreover, the sign in (17.8) is unchanged if we replace (a, b, n) by $(\hat{a}, \hat{b}, \hat{n})$. Hence, the jump at ζ_{pq}^n is positive if $\hat{n} < pq$ and negative if $\hat{n} > pq$. But $0 \le n < pq$ is in $\Gamma_{p,q}$ if and only if $n = \hat{n}$, which is equivalent to $\hat{n} < pq$. It follows that $\Gamma(K) = \{n \in \mathbb{N} \mid n \notin \Gamma_{p,q}\}$. □

17.9 Corollary *For $K = T(p,q)$, we have $h(K) = g(K) = \frac{1}{2}(p-1)(q-1)$ and*

$$\tilde{h}(K) = \sum_{i \notin \Gamma_{p,q}} (t^i + t^{-i}).$$

In particular, $\deg \tilde{h}(K) = 2g(K) - 1$ *and we have equality in Lemma 14.18(3).*

Proof The formula for $\tilde{h}$ follows from Proposition 17.7 and the fact that all arcs of $\widetilde{E}(K)$ have the same left-to-right orientation by Lemma 15.8; compare Figures 10 and 19. It is straightforward to show $\mathbb{N} \setminus \Gamma_{p,q}$ has $\frac{1}{2}(p-1)(q-1)$ elements, the largest of which is $pq - p - q$, so the other two statements follow from the formula for $\tilde{h}(K)$. □

This picture gives us a nice geometric interpretation of the signature defect $g(T(p,q)) + \frac{1}{2}\sigma(T(p,q))$: it is the number of arcs in $i^*(\widetilde{E}(K))$ which cross the line $\mu^* = \frac{1}{2}$. Since $\deg \tilde{h} = pq - p - q$ and the slope of each arc is $-pq$, the signature defect is 0 precisely when $1 - \frac{1}{p} - \frac{1}{q} < \frac{1}{2}$, ie when $\Sigma_2(K) = \Sigma(2, p, q)$ is Seifert fibered over a positively curved orbifold. Recalling that $h(K) = -\frac{1}{2}\sigma(K)$ when K is Montesinos, we get an alternative proof of the fact that $T(p,q)$ is Montesinos exactly when $(p,q) = (2, 2n+1)$, $(3,4)$ or $(3,5)$.

17.10 Berge knots

Many more examples of knots with $h(K) \neq -\frac{1}{2}\sigma(K)$ are provided by Berge knots, and, more generally, by L-space knots. A knot $K \subset S^3$ is an *L-space knot* when some positive Dehn surgery on K is an L-space. Positive torus knots, and, more generally, any knot with a positive lens space surgery, such as the Berge knots, are L-space knots. For an L-space knot K, the surgery $K(p\mu + q\lambda)$ is an L-space if and only if $\frac{p}{q} \ge 2g(K) - 1$ by [Ozsváth and Szabó 2005a]. The L-space conjecture tells us to expect the same relation to hold for left-orderings.

When K has a lens space surgery of slope $p > 0$, the form of $i^*(\widetilde{E}(K))$ is constrained by Lemma 16.25 and Theorem 16.27. In particular, the lines of slope $-p$ through the parabolics intersect $i^*(\widetilde{E}(K))$ only at points that are reducible or parabolic. Moreover, whenever we have a unique root of the Alexander polynomial in the interval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, we get an arc of type A_k in $\widetilde{E}(K)$. For such knots, the pictures of [Culler and Dunfield 2018] suggest that $i^*(\widetilde{E}(K))$ is similar to that of a torus knot, in that it is composed entirely of arcs of type A_k. Numerical computations of $\tilde{h}(K)$ for about 150 small Berge knots exhibit some striking behavior, which we now describe.

The *Milnor torsion* of a knot K is the Laurent series $\tau_M(K) := \Delta_K(t)/(1-t)$ normalized so that $\tau_M(K) = \sum_{n\geq 0} a_n t^n$ with $a_n \neq 0$. We say a Laurent series $\sum_{n\geq 0} a_n t^n$ is *good* if $a_n \in \{0,1\}$ for all n. If K is an L-space knot, then the Milnor torsion is good [Ozsváth and Szabó 2005a]. We write $\tilde{h}_+(K)$ for the "nonnegative exponent" part of $\tilde{h}(K)$; ie if $\tilde{h}(K) = \sum_{i\geq 0} c_i(t^i + t^{-i})$, then $\tilde{h}_+(K) = \sum_{i\geq 0} c_i t^i$. Based on the computations discussed below, we posit:

17.11 Conjecture *If K is a small Berge knot, then*

(1) $h(K) = \frac{1}{2} r(K)$, *where $r(K)$ is the number of roots of $\Delta_K(t)$ on the unit circle*;

(2) $\tilde{h}_+(K)$ *is good*; *and*, *moreover*,

(3) $\tilde{h}_+(K) + \tau_M(K)$ *is good.*

In other words, the coefficients of $\tilde{h}_+(K)$ must all be 1 and fall into the "gaps" of the Milnor torsion $\tau_M(K) = \sum_{n\geq 0} a_n t^n$ given by those n where $a_n = 0$. The symmetry of the Alexander polynomial implies that $a_n = 1 - a_{2g-1-n}$, where $g = g(K)$ is the Seifert genus of K. (Recall that Berge knots are fibered [Ni 2007], so $\deg \Delta_K(t) = 2g$.) Hence, $a_n = 1$ for $n \geq 2g$, and, if $\tau_M(K)$ is good, precisely g of the $2g$ coefficients between a_0 and a_{2g-1} will be 0. Conjecture 17.11 predicts that $\frac{1}{2} r(K)$ of these $g(K)$ gaps should be filled by the nonzero coefficients of $\tilde{h}_+(K)$.

When $K = T(p,q)$ is a positive torus knot, all the roots of $\Delta_K(t)$ lie on the unit circle, so $r(K) = 2g(K)$. In addition, from (17.6) we have

$$\tau_M(T(p,q)) = \sum_{i\in\Gamma_{p,q}} t^i \quad \text{and so} \quad \tilde{h}_+(K) + \tau_M(K) = \sum_{i\geq 0} t^i$$

by Corollary 17.9. Thus Conjecture 17.11 holds in this case, and moreover all of the gaps in τ_M are filled.

In contrast, if K is a hyperbolic Berge knot, some gaps in τ_M must remain empty for the following reason: We have $a_{2g-1} = 1 - a_0 = 0$, so the highest gap is always at height $2g(K) - 1$. As every Berge knot is fibered [Ni 2007], by Lemma 14.18(4) we have $\deg \tilde{h}(K) < 2g(K) - 1$, and the highest gap must remain unfilled.

As a consequence, if K is a hyperbolic L-space knot, the approach of constructing left-orderings for Dehn surgeries K via $\mathrm{SL}_2\mathbb{R}$ representations should not be completely effective at proving the L-space conjecture for these manifolds. Indeed, the best expected bound on the range of left-orderable surgery slopes coming from Lemma 16.19 says that Dehn fillings of slope $< \deg \tilde{h}(K)$ should be LO. In contrast, the L-space conjecture says that we should expect all fillings of slope $< 2g(K) - 1$ to be LO. This happens even for the $(-2,3,7)$ pretzel knot of Figure 17, where $S_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K) = (-6,\infty)$ [Varvarezos 2021] but we expect $S_{\mathrm{LO}}(K) = (-9,\infty)$.

The slopes in this gap provide an interesting test case for the L-space conjecture.

17.12 Question *For some hyperbolic Berge knot K, can we show that $K(\alpha)$ is LO for all rational $\alpha \in (2g(K)-1-\epsilon, 2g(K)-1)$, where $\epsilon > 0$?*

The best results in this direction are due to [Krishna 2020; Hu 2023], whose combined work shows that the $(2,-3,-7)$ pretzel knot has a sequence of such slopes converging to 9.

We arrived at Conjecture 17.11 by considering the 158 small hyperbolic Berge knots whose exteriors can be triangulated with at most nine ideal tetrahedra [Dunfield 2020a]. The knots in this sample have small hyperbolic volumes (all less than 7.1), but their genus is generally quite large: the median genus is 50.5. For these knots, a collection of parabolic representations has been rigorously computed by Goerner [2014] using a method based on ideal triangulations and Ptolemy coordinates; however, these lists may not be complete due to limitations of this approach [Goerner and Zickert 2018]. We computed the heights of these representations using [Culler and Dunfield 2015] to find their possible contributions to $\tilde{h}_+(K)$; eg in Figure 17 the heights would be $\{1,2,4,6\}$. For all 158 knots, the parabolic heights are all distinct and each one lies in a gap in $\tau_M(K)$, which is consistent with parts (2) and (3) of Conjecture 17.11.

The fact that the parabolic heights fall into the gaps in $\tau_M(K)$ for even one of the larger of these knots is remarkable, since on average 73% of the gaps were filled by some parabolic height. For example, the manifold $o9_{03188}$, which is one of the most complicated in the sample, has an $L(317,121)$ filling, Seifert genus 139 and 99 parabolics. If we imagine that the parabolic heights were randomly distributed in the interval $[0, 2g(K)-1]$, the probability that they all landed on a gap would be $2^{-99} \approx 1.6 \times 10^{-30}$. Even if we restrict to the interval between 1 and 219 (the largest parabolic height), where there are 122 gaps and 97 nongaps, the probability of all the heights landing in the gaps is less than 10^{-23}.

Turning to the count $r(K)$, the ratio $\frac{r(K)}{2g(K)}$ had mean 0.74 and was always in the interval $[0.60, 0.78]$; hence, a majority of the roots of $\Delta_K(t)$ were on the unit circle for all these knots. For comparison, define $h_G(K)$ to be the naive (unsigned) count of parabolic $\mathrm{SL}_2\mathbb{R}$ representations from [Goerner 2014]. As mentioned above, this may be an undercount of the actual number. In all cases, we have $h_G(K) \leq \frac{1}{2}r(K)$ with equality in 133 cases. The sum of all the $h_G(K)$ was 6479, whereas the sum of all $\frac{1}{2}r(K)$ was 6532. The largest difference $\frac{1}{2}r(K) - h_G(K)$ observed was 4. The manifold $o9_{03188}$ mentioned is one of two such examples, as there $h_G(K) = 99$ but $\frac{1}{2}r(K) = 103$. The simplest example where the counts differ is $v1392$ where $h_G(K) = 14$ but $\frac{1}{2}r(K) = 16$. Computations of $i^*(\widetilde{E}(K))$ using [Culler and Dunfield 2015] are consistent with a pair of parabolic representations being missed. Indeed, using the alternative triangulation `kLvLAPQkaedgijhijijnxsvepnxxti_abba` for $v0220$, we found there are in fact 16 distinct parabolic representations as predicted by $r(K)$.

Finally, for 62 of the 158 manifolds, we have a plot generated by [Culler and Dunfield 2015] as part of [Culler and Dunfield 2018]; these all consist of nonintersecting arcs of type A_k for $k \neq 0$. For a more complicated example than Figure 17, see [ibid., Figure 4] for the plot for $v0220$, which has $h_G(K) = \frac{1}{2}r(K) = 36$ with 47 gaps in τ_M and $i^*(\widetilde{E}(K))$ consisting of 72 arcs, making it a median-complexity example in the overall sample. Using Lemma 15.8, the pictures provide strong evidence

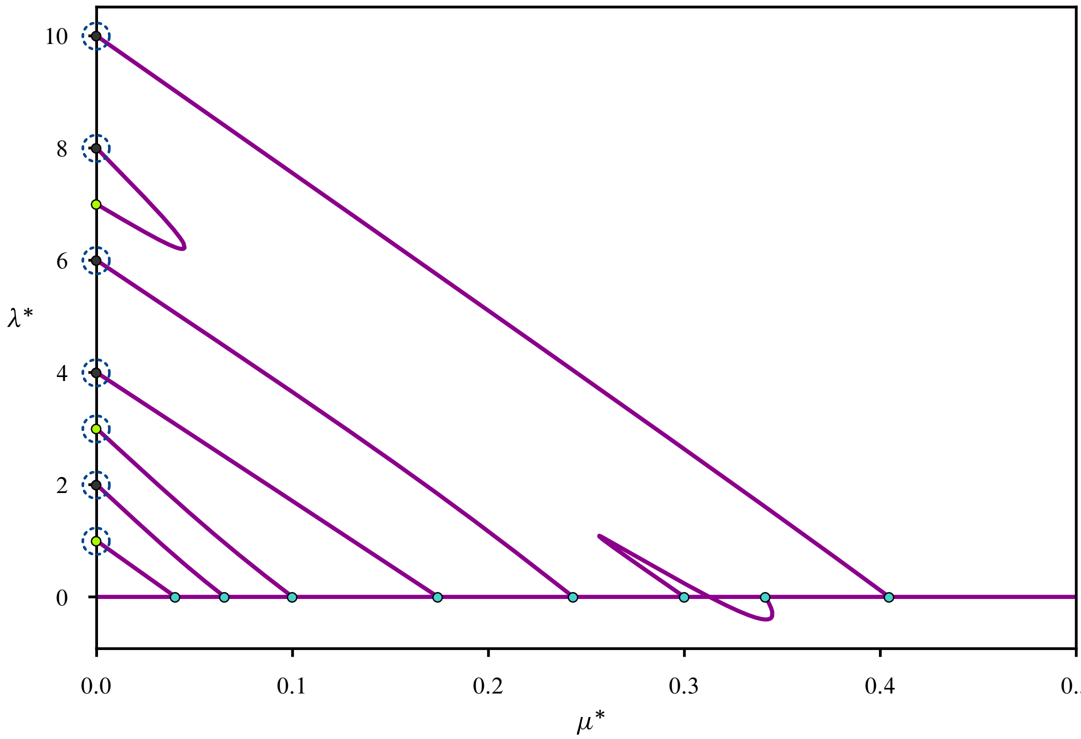

Figure 20: The manifold $t09882$ is the exterior of the mirror of an L-space knot K, half of whose $i^*(\widetilde{E}(K))$ is shown here. The lighter dots on the vertical axis correspond to parabolics that are Galois conjugates of the holonomy representation of the hyperbolic structure on M; the darker dots on that axis are other parabolics. A dotted circle about a parabolic point indicates a gap in τ_M; further gaps at heights 12 and 17 are not shown. Here, $\Delta_K = (t^4 - t^3 + t^2 - t + 1)(t^{14} - t^{12} + t^7 - t^2 + 1)$ has 16 roots on the unit circle. The knot K is not a Berge knot, having no lens space surgeries, and appears to satisfy none of the conclusions of Conjecture 17.11, having $h(K) = 6 < 8 = \frac{1}{2}r(K)$ and $\tilde{h}_+(K)$ not good since the coefficients t^7 and t^8 should have opposite signs.

that all parabolics from [Goerner 2014] contribute to $\tilde{h}_+(K)$ with positive signs, further confirming Conjecture 17.11.

17.13 Remark The pattern described in Conjecture 17.11 holds for many, but not all, L-space knots; see Figures 20 and 21 for two examples where it appears not to hold. Caveat: the qualification in the last sentence is because pictures produced by [Culler and Dunfield 2015] are not completely rigorous; see [Culler and Dunfield 2018, Section 5.1] for details.

17.14 2-bridge knots

If K is 2-bridge, it is alternating and small, so $h(K) = -\frac{1}{2}\sigma(K)$ by Corollary 16.12. Using Corollary 14.3, we give a new proof of the following result, which was conjectured by Riley [1972; 1984] and proved recently by Gordon [2017]:

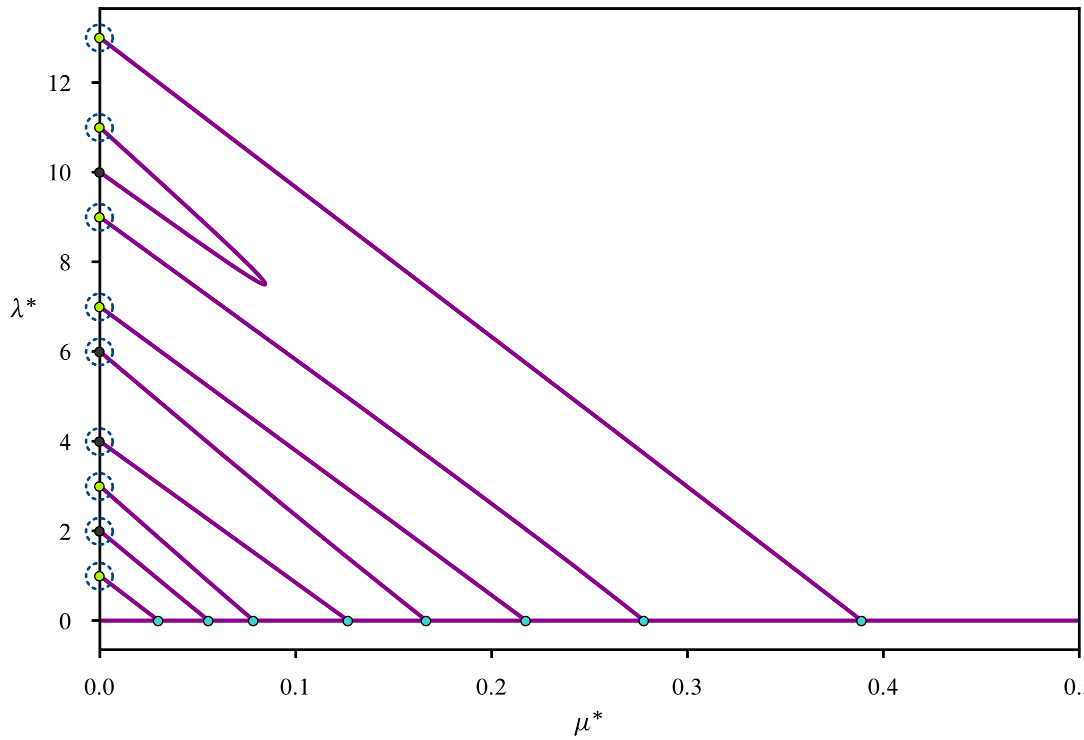

Figure 21: The manifold $t08114$ is the exterior of the mirror of an L-space knot K, half of whose $i^*(\widetilde{E}(K))$ is shown here using the same conventions as Figure 20. Here, $\Delta_K = (t^2-t+1)(t^6-t^3+1)(t^{16}-t^{14}+t^{11}-t^9+t^8-t^7+t^5-t^2+1)$ has sixteen roots on the unit circle and τ_M has additional gaps at heights 15, 18, and 23. The knot K appears to satisfy only part (1) of Conjecture 17.11, having $h(K) = 8 = \frac{1}{2}r(K)$ but $\tilde{h}_+(K)$ is not good since the coefficients t^{10} and t^{11} should have opposite signs.

17.15 Theorem (Riley conjecture) *If K is a 2-bridge knot, then $\pi_1(M_K)$ admits at least $|\sigma(K)|$ nonconjugate irreducible parabolic representations into* $\mathrm{SL}_2\mathbb{R}$.

The *Riley polynomial* $p_K(y)$ plays a central role in [Riley 1972; Gordon 2017]. It is a 1-variable polynomial whose roots determine the parabolic representations of a 2-bridge knot K. Riley proved that p_K has only simple roots, which is a key ingredient in the proof below.

Proof We will show that $\pi_1(M_K)$ admits at least $|h(K)| = \frac{1}{2}|\sigma(K)|$ conjugacy classes of representations ρ with $\mathrm{tr}_\mu(\rho) = 2$; the statement then follows from the bijection $b\colon X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) \to X^{-2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ discussed in Section 14.9.

Since K is 2-bridge, we can write $K = \widehat{\beta}$, where K is a 4-strand braid. We choose β so that we can orient K with all the underbridges and overbridges coherently oriented from left to right. Our usual setup leads us to consider the character varieties $L_i = X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(H_i) \subset X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_4)$, where $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_4)$ is a real 2-dimensional surface (as in Figure 3, left), and each L_i is a smooth curve in it. Then $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K) = L_1 \cap L_2$

and $h(K) = \pm\langle L_1, L_2\rangle$. To prove the theorem, it suffices to show that L_1 is transverse to L_2, since, if so, each point of $L_1 \cap L_2$ contributes ± 1 to $h(K)$ and hence $|h(K)| \leq \#(L_1 \cap L_2)$.

Here is an outline of our strategy for proving that $L_1 \cap L_2$ is transverse. Consider the representation varieties $N_i = R^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(H_i) \subset R^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_4) \subset R^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_4)$. Riley [1972] defines a map $\gamma\colon \mathbb{R}_{>0} \to N_1$ such that

(1) $\tau \circ \gamma\colon \mathbb{R}_{>0} \to X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(S_4)$ is an embedding whose image is $L_1' \subset L_1$, and

(2) if $p \in L_1 \cap L_2$, then $p \in L_1'$.

Since L_1 is 1-dimensional, it is enough to show that for all $p = \tau \circ \gamma(u)$ in $L_1 \cap L_2$, one has $T_p L_1 \not\subset T_p L_2$. By the path lifting properties of [Culler and Dunfield 2018, Lemma 2.11], this is equivalent to $\gamma'(u) \notin T_{\gamma(u)} N_2$. To show the latter, we will define a map $f : R^{2,\mathrm{irr}}(F_4) \to \mathbb{R}$ such that

(3) $f(N_2) = \{0\}$, and

(4) $f \circ \gamma(u) = u p_K(u^2)$, where p_K is the Riley polynomial.

Since all the roots of p_K are simple [Riley 1972, Theorem 3], it will follow that $\gamma'(u) \notin T_q N_2$ whenever $q = \gamma(u) \in N_2$.

It remains to define the maps γ and f and check claims (1)–(4). To start, let $C \subset \mathrm{SL}_2\mathbb{R}$ be the set of noncentral matrices of trace 2. It has two connected components corresponding to whether a parabolic element rotates (really translates) its invariant horocircles in $\mathbb{H}^2$ clockwise or anticlockwise. Conjugating by $\mathrm{SL}_2\mathbb{R}$ preserves each component of C, but the two components are exchanged by the action of the nonidentity component of $\mathrm{SL}_2^{\pm}(\mathbb{R})$. We can identify $Z = R^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_2)$ and $Y = R^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_4)$ with open subsets of C^2 and C^4, respectively, so that N_1 is the image of the map $i\colon Z \to Y$ given by $i(A, B) = (A, A^{-1}, B, B^{-1})$.

Following Riley, for $u \in \mathbb{R}_{>0}$ we define

$$\gamma(u) = (A(u), A(u)^{-1}, B(u), B(u)^{-1}), \quad \text{where } A(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ B(u) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}.$$

The image of γ is contained in N_1, so $\tau \circ \gamma\colon \mathbb{R}_{>0} \to L_1$. The map $\tau \circ \gamma$ is easily seen to be injective by considering $\mathrm{tr}(A(u)B(u)) = 2 - u^2$, and the same calculation shows its derivative is injective as well. This establishes claim (1).

Next, we consider the image L_1' of $\tau \circ \gamma$. Recall that C has two connected components that can be interchanged by $\mathrm{SL}_2^{\pm}(\mathbb{R})$. Consequently, Z has four connected components and $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_2)$ has two, where the components of $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(F_2)$ correspond to whether the invariant horoballs of the two generators are rotated in the same or opposite directions. The image L_1' is contained in the former component since conjugating by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2\mathbb{R}$ interchanges $A(u)$ and $B(u)$; moreover, it is all of that component by [ibid., Lemma 2]. Turning to (2), recall β was chosen so that the underbridges are oriented left to right, so the generators t_1 and t_2 of $\pi_1(H_1)$ are both positively oriented meridians for K; in particular, they are

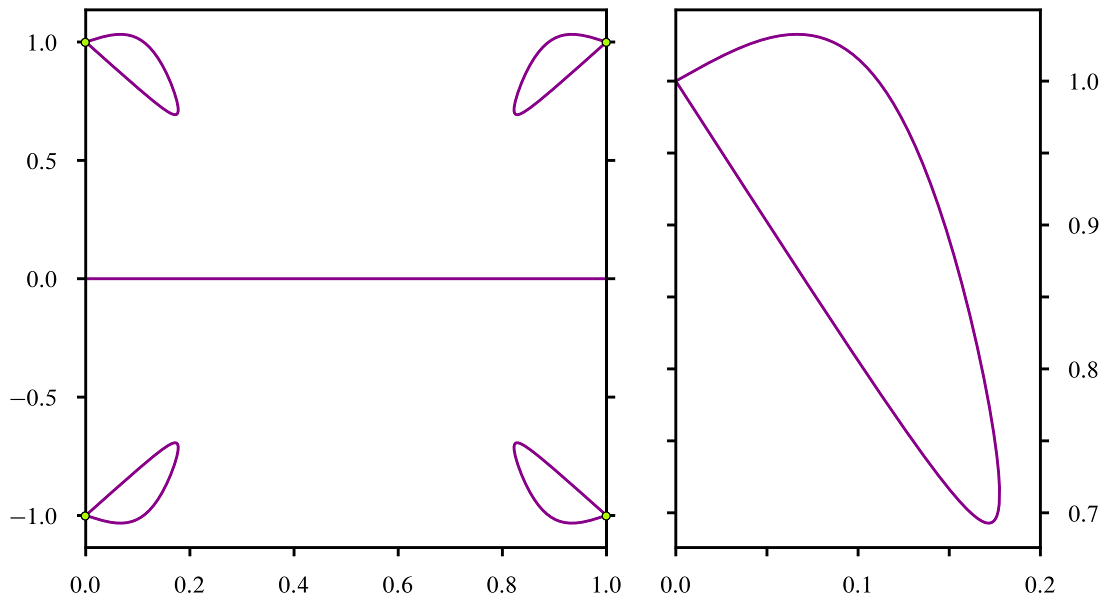

Figure 22: The 2-bridge knot $K_{41/9} = K10a107 = 10_{17}$ is an example where Conjecture 17.16 requires cancellation of distinct parabolics when computing $\tilde{h}(K)$. Here $\sigma_K = 0$, but $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ consists of four points, all at height ± 1. Assuming everything is transversely cut out, either orientation on the lobe of $i^*(\widetilde{E}(K))$ shown at right results in $\tilde{h}(K) = 0$, as predicted. Plot made using [Culler and Dunfield 2015].

conjugate in $\pi_1(M_K)$. Thus any $\rho \in N_1 \cap N_2$ gives rise to $\pi_1(M_K) \to \mathrm{SL}_2\mathbb{R}$, where $\rho(t_1)$ and $\rho(t_2)$ are conjugate in $\mathrm{SL}_2\mathbb{R}$, implying that the corresponding point in $L_1 \cap L_2$ is in L_1'. This proves claim (2).

A plat diagram representing K determines a Heegaard splitting of the complement, and hence a presentation of $\pi_1(M_K)$ with one generator for each underbridge and one relation for each overbridge. The relations are redundant: any one relation is a consequence of the others. If the plat diagram is chosen so that all of the underbridges and overbridges are coherently oriented, this presentation is closely related to the presentation obtained by starting with $\pi_1(S_{2n})$ and adding the relations $s_{2i-1} = s_{2_i}^{-1}$ and $\beta_*^{-1}(s_{2i-1}) = \beta_*^{-1}(s_{2i}^{-1})$. To be precise, if we use the first set of relations to eliminate the generators s_{2i}, we get the presentation described above up to the operation of cyclically permuting the elements in each relator. In the case of a coherently oriented 2-bridge knot $K_{p/q}$ with q odd, the resulting presentation is a 2-generator 1-relator presentation $\pi_1(M_K) = \langle a, b \mid r(a,b) \rangle$, where $r(a,b) = waw^{-1}b^{-1}$ and $w = w(a,b)$ is determined by p/q. A precise formula for $w(a,b)$ is given in [ibid., Proposition 1].

We now explain how this relates to $N_1 \cap N_2$. Suppose $q = (A, A^{-1}, B, B^{-1}) \in N_1$. Then $(\beta^*)^{-1}(q) = (C_1, C_2, C_3, C_4)$, where each C_i is a word in A and B determined by β. The condition that $q \in N_2$ is equivalent to the relations $C_1 = C_2^{-1}$ and $C_3 = C_4^{-1}$, and, by the discussion above, each of these are equivalent to the relation $r(A, B)$. It follows that there are words $v_1 = v_1(A, B)$ and $v_2 = v_2(A, B)$ such that $v_1 C_1 v_2 = wA$ and $v_1 C_2^{-1} v_2 = Bw$.

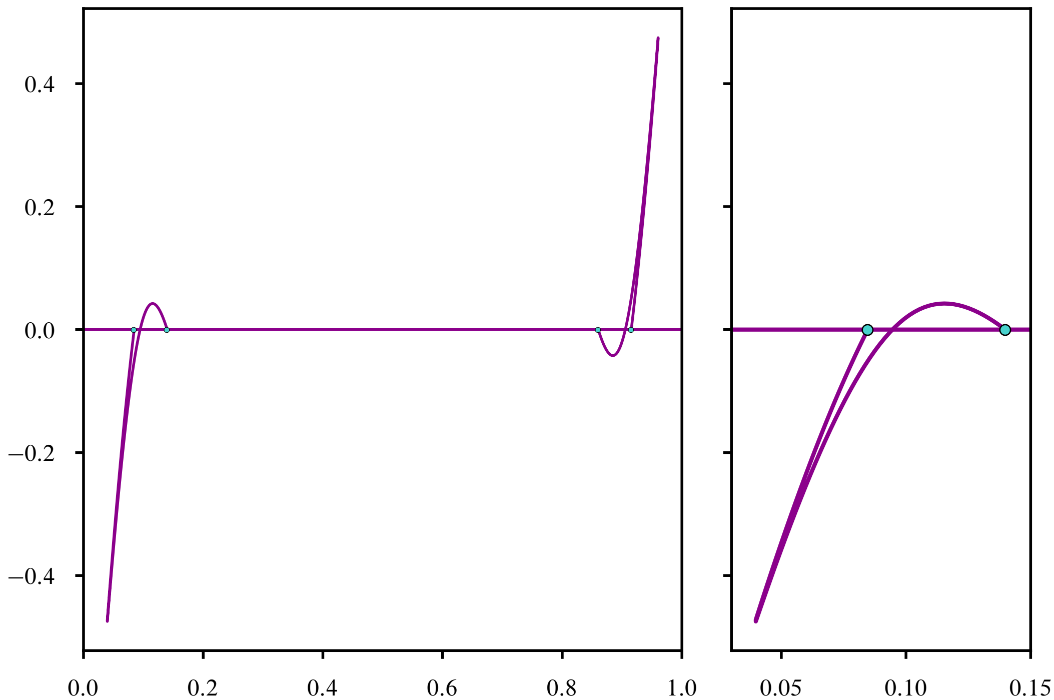

Figure 23: The 2-bridge knot $K_{61/21} = K14a2459$ is an example where $i^*(\widetilde{E}(K))$ contains an arc that starts and ends at reducibles but whose interior consists of irreducibles. While such arcs are extremely common in $X_{\mathrm{SU}_2}(K)$, this seems to be the first such observed in $X_{\mathrm{SL}_2\mathbb{R}}(K)$. Here, $\Delta_K = 5t^4 - 15t^3 + 21t^2 - 15t + 5$, which has all roots on the unit circle. Moreover, $h(K) = 0$ as $X^{2,\mathrm{irr}}_{\mathrm{SL}_2\mathbb{R}}(K)$ is empty, and $\sigma_K = 0$ except in intervals between each pair of close roots where it is 2. Lemma 15.8 forces the arc to cross the horizontal axis. In the closeup at right, the two regions enclosed by $i^*(\widetilde{E}(K))$ must have equal area: the derivative of the Chern–Simons invariant/Seifert volume/Godbillon–Vey invariant is essentially $\eta = x\,dy - y\,dx$ (see [Khoi 2003]), so, as $d\eta = 2\,dx\,dy$, the loop created from the small segment of the axis between the two roots and the curved parts of $i^*(\widetilde{E}(K))$ must have signed area 0. Plot made using [Culler and Dunfield 2015].

Suppose $q = (A_1, A_2, A_3, A_4) \in Y$. If $(\beta^*)^{-1}(q) = (c_1, c_2, c_2, c_4)$, where the c_i are words in A_1, A_2, A_3, A_4, we define $F : C^4 \to M_{2\times 2}(\mathbb{R})$ by

$$F(q) = v_1(A_1, A_3)(c_1 - c_2^{-1})v_2(A_1, A_3).$$

Then, if $q \in N_2$, we have $c_1 = c_2^{-1}$, so $F(q) = 0$. We define $f(q)$ to be the upper right entry of the matrix $F(q)$; claim (3) follows immediately. Finally, we compute

$$F(\gamma(u)) = v_1(u)(C_1(u) - C_2^{-1}(u))v_2(u) = w(u)A(u) - B(u)w(u),$$

where we write $v_1(u) = v_1(A(u), B(u))$ etc. Riley [1972, Theorem 2] computes the quantity on the right and shows it has the form

$$\begin{pmatrix} 0 & up_K(u^2) \\ up_K(u^2) & 0 \end{pmatrix},$$

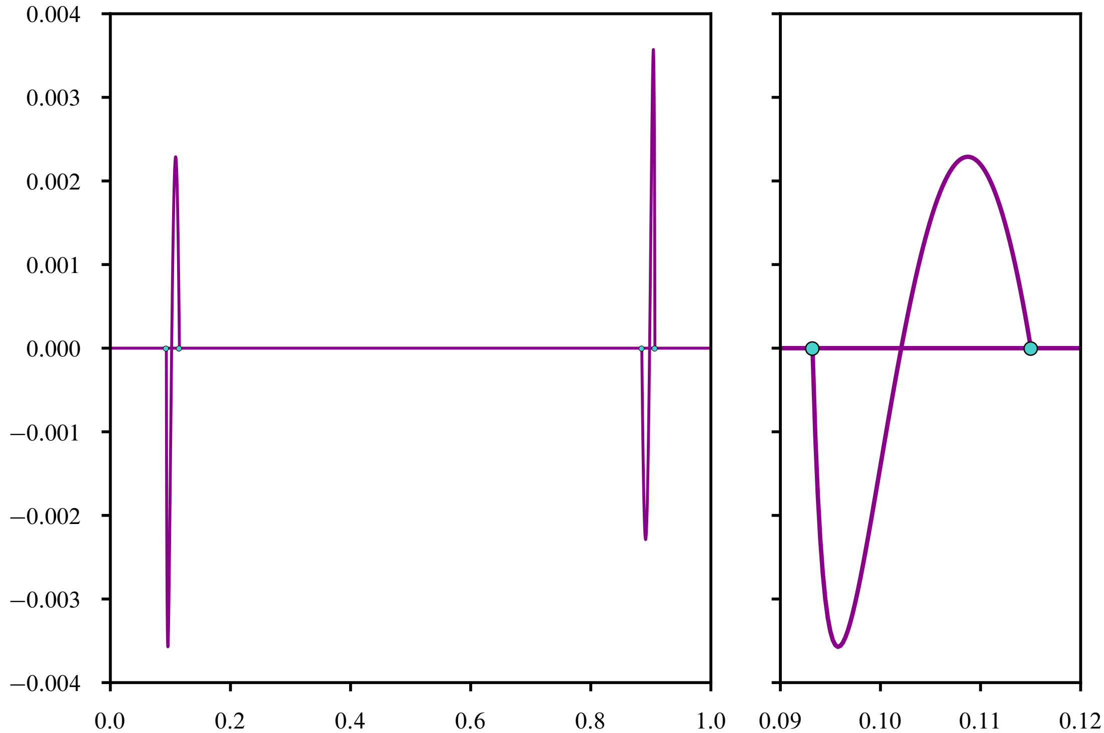

Figure 24: The 2-bridge knot $K_{77/20} = K12a380$ is another example where $i^*(\widetilde{E}(K))$ contains an arc that starts and ends at reducibles, but whose interior consists of irreducibles (compare Figures 20 and 23). It is remarkable how small $S_{\widetilde{\mathrm{SL}_2\mathbb{R}}}(K)$ is here, contained in $[-0.022, 0.038]$. Here, $\Delta_K = (2t^2 - 3t + 2)(3t^2 - 5t + 3)$ which has all roots on the unit circle. Moreover, $h(K) = 0$ as $X^2_{\mathrm{SL}_2\mathbb{R}}(K)$ is empty, and $\sigma_K = 0$ except in intervals between each pair of close roots, where it is 2. Plot made using [Culler and Dunfield 2015].

where $p_K(y)$ is the Riley polynomial. This proves claim (4) and hence completes the proof of the theorem. □

The torus knot $T(p,q)$ is 2-bridge only if $p = 2$. If $K = T(2, 2n+1)$, Corollary 17.9 tells us that $\tilde{h}(K) = t^{-2n+1} + t^{-2n+3} + \cdots + t^{2n-3} + t^{2n-1}$. In general, we conjecture that if K is 2-bridge, then $\tilde{h}(K) = \tilde{h}(T(2, 2n+1))$, where $-2n = \sigma(K) = \sigma(T(2, 2n+1))$. In other words:

17.16 Enhanced Riley conjecture *If K is a 2-bridge knot, we conjecture that*

$$\tilde{h}(K) = -\frac{t^{\sigma(K)} - t^{-\sigma(K)}}{t - t^{-1}}.$$

Note that this is compatible with Riley's theorem [1972] that if K is 2-bridge, any parabolic representation $\rho\colon M_K \to \mathrm{SL}_2\mathbb{R}$ satisfies $\operatorname{tr}\rho(\lambda) = -2$; equivalently, any parabolic must have an odd height in the translation extension locus. By using the Riley polynomial to find all the parabolic $\mathrm{SL}_2\mathbb{R}$ representations of a 2-bridge knot $K_{p/q}$, we checked numerically that the conjecture holds "modulo 2" for all 2-bridge knots $K_{p/q}$ with $p < 500$. That is, the number of parabolics at an odd height i is odd if $|i| < |\sigma(K)|$ and is even otherwise. There are 12 929 knots in this range after accounting for $(p,q) \sim (p, \pm q^{\pm 1})$. More than 20% have strictly more than $\sigma(K)$ parabolics, and so would require cancellation when computing $\tilde{h}(K)$ in order for Conjecture 17.16 to hold. A simple example of this is shown in Figure 22. A complicated example is the 32-crossing knot $K = K_{479/29}$ where $\sigma = 2$ and so we expect $\tilde{h}_+ = t$. In fact, there are actually 55 parabolics that contribute to $\tilde{h}_+$; the *unsigned* count of these parabolics is $21t + 16t^3 + 10t^5 + 6t^7 + 2t^9$, which is indeed equal to t modulo 2.

Although alternating and Montesinos knots also satisfy $h(K) = -\frac{1}{2}\sigma(K)$, Conjecture 17.16 does not extend to either class of knots: Goerner's tables [2014] of parabolic representations contain knots in each of these classes that admit parabolic representations with $\operatorname{tr} \rho(\lambda) = 2$. Finally, we note that although for small values of p the translation extension locus of $K_{p/q}$ contains only arcs of type A_k (joining reducibles to parabolics), there are 2-bridge knots for which the translation extension locus provably contains arcs joining two reducibles. Two such examples are shown in Figures 23 and 24; we found them by looking for knots where the number of real roots of the Riley polynomial is smaller than the number of roots of the Alexander polynomial on the unit circle.

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